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LOGIC, EPISTEMOLOGY, AND THE UNITY OF SCIENCE 17

Procedural Semantics for Hyperintensional Logic

Foundations and Applications of Transparent Intensional Logic

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Logic, Epistemology, and the Unity of Science

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VOLUME 17

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ISBN 978-90-481-8811-6 e-ISBN 978-90-481-8812-3
DOI 10.1007/978-90-481-8812-3
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2010927470

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

We wish to dedicate this book to the memory of Pavel Tichý

Preface

This book is about Transparent Intensional Logic, the brainchild of Pavel Tichý. Three books and around 100 papers on Transparent Intensional Logic have till now seen the light of day since the mid-1960s. So why a book of more than 500 pages now? For two reasons.

Firstly, Transparent Intensional Logic is a theory without something like a textbook. Now this is not an actual textbook, if a textbook is a patient introduction garnished with exercises and solutions; nor is it a teach-yourself-in-a-week manual for the uninitiated-but-curious. But we, the three authors, have striven to write an accessible one-stop survey of Transparent Intensional Logic that may be read by advanced students of logic, semantics, linguistics, informatics, computer science, and kindred disciplines.

Secondly, logical semantics is a field progressing by leaps and bounds, and much has happened since Tichý put out his first and only book in 1988. We thought it was about time for us to assemble in one place the most important extensions, improvements and applications stemming from the last several years that address issues not dealt with either at all or only cursorily by Tichý. We have also made a point of flagging various unsettled issues in the theory's edifice and of indicating the general direction in which we expect solutions are most likely to be found.

This book is, if you like, a snapshot of Transparent Intensional Logic as it looks in early 2010, and makes no claim to being the 'mature', let alone 'ultimate', statement of the theory. If the theory keeps evolving at its current pace, another update will be called for within the next 5–10 years. At the same time, a both methodological and philosophical constraint that is dear to us is that the applications we present should not be ad hoc. Rather they must fall out of an existing theory; and if a particular application calls for amendment of the foundations then it must be thoroughly justified. We like to think of Transparent Intensional Logic as an open-ended theory with a cast-iron core. The execution of the project informing Transparent Intensional Logic—a fully compositional procedural semantics applying indiscriminately to all logico-semantic contexts—is itself an open-ended process.[1]

The book treats of topics familiar from contemporary formal semantics, but devotes special attention to some topics that generally tend to be dealt with only in passing. They include, inter alia, notional attitudes, knowing whether, concepts (understood rigorously and non-mentalistically), attitudes *de re* and anaphora in hyperintensional contexts. Besides, the extensive treatment of anaphora found in this book represents a major step forward for the development of Transparent Intensional Logic, which had so far barely dealt with this linguistic device. The addition opens up new fragments of natural language to analysis. Another vastly

[1] Of course, the theory's detractors would also want to *execute* the project, but then in the sense of eliminating rather than implementing it.

developed notion would be *requisite*, which underpins our intensional essentialism (in terms of a priori relations-in-extension between intensions). The jewel in the crown, however, must be the extremely detailed and principled elaboration of the *de dicto*/*de re* dichotomy. The dichotomy is at the heart of Transparent Intensional Logic, because it pretty much does the work that is done by reference shift in most other theories. Without a fully-fledged theory of *de dicto*/*de re*, the project of a *transparent* intensional logic would remain a pipe dream.

For historical background, Tichý's 1968 paper 'Smysl a procedura' (reprinted as 'Sense and procedure' in Tichý (2004)) marks the inception of Transparent Intensional Logic. There he says that, '[T]he relation between sentences and procedures is of a semantic nature; for sentences are used to record the results of performing particular procedures' (2004, p. 80). Twenty years later he was to publish his critical study of Frege's logic, where his early ideas of procedural semantics and of semantics as being a priori were transformed into an elaborate theory whose *leitmotiv* is the profound and carefully argued conviction that an expression represents 'a definite intellectual journey to an entity' (1988, p. 284). This conviction explains why syntax and semantics are developed in tandem. Transparent Intensional Logic is an interpreted formal syntax (in the shape of its 'language of constructions'), a feature it shares with a proof-theoretic semantics such as Per Martin-Löf's and which sets it apart from a model-theoretic semantics such as Richard Montague's. The simultaneous development of syntax and semantics is one reason why in this book philosophical discussion and technical details are not segregated into entirely separate chapters. Another reason is that Transparent Intensional Logic is a case of philosophical logic, which consists in the application of logical techniques to philosophical problems. Practicing philosophical logic requires continuous coordination between logic and philosophy, and so it would both be inconvenient and contrary to the spirit of the enterprise of philosophical logic to attempt to treat logic and philosophy separately. Logic, as Transparent Intensional Logic understands it, is a calculus, to be sure, but not only. Logic is the noble art of inference, and who wishes to draw valid inferences will need a tool for doing so. This tool is an array of procedures, or instructions or prescriptions, detailing how to proceed when drawing inferences. We identify these procedures with linguistic meanings. Therefore, since this book is about logic it is about semantics.

Tichý began demonstrating the expressive power of Transparent Intensional Logic from the 1970s through the 1990s after emigrating ('defecting', in the parlance of the Czechoslovak Socialist Republic) with his family from Czechoslovakia to New Zealand, where he eventually became Professor of Logic in the University of Otago at Dunedin. In 1974–1976 he worked out a system of atemporal intensional logic based on the simple theory of types, but the manuscript, Introduction to Intensional Logic, was not published. The main principles of Transparent Intensional Logic based on the ramified hierarchy of types were laid down in his 1988 monograph, *The Foundations of Frege's Logic*, while demonstrating its puzzle-solving mettle by solving an impressive range of semantic problems in numerous papers appearing in significant and widely read journals. During Tichý's quarter of a century in New Zealand, as well as after his death in 1994, a group of logicians and philosophers had begun to appreciate the assets and potential of Transparent Intensional Logic and continued working in two directions. Much energy has gone into making the theory more widely known, alerting students and peer researchers to the possibilities offered by Transparent Intensional Logic, both as foundations and applications go. At the same time the theory has seen continued development and application to further topics. Two monographs in English (*Concepts and Objects,* 1998, and *Conceptual Systems*, 2004, both by Pavel Materna), several monographs in Czech and numerous articles in Czech, Slovak and international journals have appeared over the years and contributed to logic, philosophy, linguistics, and computer science.

Two approaches to writing are common. One approach provides a rich background in the shape of discussion, criticism and comparison with kindred and rival theories and makes a minor contribution. The upside is that the selflocation of the new contribution is clear and its virtues explicit. The downside is that the informed reader will have to plough through piles of familiar material before getting to the point. The other, bolder, approach offers generous helpings of new material against a sketched background. The upside is that the informed reader gets to the several new points fairly quickly. The downside of this manner of exposition is that it discharges a good deal of comparative work onto the reader, and perhaps also evokes the impression that the theory were conceived in a conceptual vacuum. We have opted for boldness, though. Our primary goal is to present a particular theory and defend it, while rectifying, amending and expanding it whenever and wherever we saw fit. The comparisons and discussions we have inserted serve both to illustrate our theory better (by describing the less known by the better known) and to demonstrate what we argue to be its superiority.

Acknowledgments

Writing this book together was fun. It was also rather a workout. Fun the way squash or handball is said to be fun, as Quine once commented after reading David Kaplan's 60-page paper 'Opacity'. We are indebted to a large group of people for stimulating discussions, advice, and encouragement, as well as to institutions for funding along the way. Among our favourite discussion partners are the Slovak logicians and philosophers helping develop TIL, notably Pavel Cmorej, František Gahér, Marián Zouhar and the Czech researchers Jiří Raclavský and Petr Kuchyňka, both using TIL to solve semantic problems. Moreover, we wish to thank several researchers who are familiar with or even well-versed in TIL without necessarily subscribing to the theory, including Gabriel Sandu, Andrew Holster, Jaroslav Peregrin, Vladimír Svoboda, Petr Kolář, Jan Štěpán and Vladimír Janák. Their questions and objections alerted us to problems which we might have otherwise neglected. It is also interesting to note that TIL—originally developed as a theory within philosophical logic—has turned out to be of particular interest to computer scientists, not least the members of a team led by Zdenko Staníček as well as some computer linguists, especially Aleš Horák. Marie Duží is grateful to her students Martina Čihalová, Nikola Ciprich, Michal Košinár, Marek Menšík, Jaroslav Müller, and the researchers involved in the Research Laboratory of Intelligent Systems (VSB-Technical University, Ostrava), whose comments contributed to the improvement of the text. Bjørn Jespersen would like to thank several of his colleagues at the Section of Philosophy at Delft University of Technology for their interest in Transparent Intensional Logic, not least Maarten Franssen.

Finally, all three of us wish to thank two referees for Springer-Verlag for comments and recommendations that led us to reconsider and especially rephrase various portions of the book, not least the first two chapters. We also wish to thank Springer-Verlag for believing in the book, and Floor Oosting and Ingrid van Laarhoven for smooth handling.

Marie Duží and Pavel Materna were supported by the Grant Agency of the Czech Republic (Project No. 401/04/2073, *Transparent Intensional Logic: Systematic Exposition*, and Project No. 401/07/0451, *Semantization of Pragmatics*) and the Czech Academy of Sciences (Project No. 1ET101940420, *Logic and Artificial Intelligence for Multi-agent Systems* within the programme *Information Society*).

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March, 2010

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1
A programme of general semantics

1.1 The programme in outline

Transparent Intensional Logic is a logical theory developed with a view to logical analysis of sizeable fragments of primarily natural language.

It is an unabashedly Platonist semantics that proceeds top-down from structured meanings to the entities that these meanings are modes of presentation of. It is a theory that, on the one hand, develops syntax and semantics in tandem while, on the other hand, keeping pragmatics and semantics strictly separate. It disowns *possibilia* and embraces a fixed domain of discourse. It rejects individual essentialism without quarter, yet subscribes wholeheartedly to intensional essentialism. It denies that the actual and present satisfiers of empirical conditions (possible-world intensions) are ever semantically and logically relevant, and instead replaces the widespread semantic actualism (that the actual of all the possible worlds plays a privileged semantic role) by a thoroughgoing anti-actualism. And most importantly, it unifies unrestricted referential transparency, unrestricted compositionality of sense, and all-out hyperintensional individuation of senses and attitudes in one theory.

The way we understand the enterprise of logical analysis of (natural) language, it is neither eliminative nor reductive, but selective. The analysis selects particular features of language, leaving all the remaining untouched and unscathed. We obviously acknowledge the pragmatic categories of (act of) assertion, language acquisition, communication, speaker's intention, etc. And we acknowledge no less the full range of pragmatic paraphernalia that keep natural language lubricated and running, including non-verbal winks and nods, hints and clues. But while they exist in their own right, they are immaterial to the project of, ideally, isolating all, and only, logically salient features of (natural) language. So we blot out what is in effect the vast bulk of natural language in order to zoom in on the remaining fragment and blow it large, as it were, with a view to studying it in more detail.

Yet the very name of our theory, 'Transparent Intensional Logic', is likely to strike one as being an oxymoron, like 'roaring silence'. How can there possibly be a *logic* that is *intensional* and at the same time *transparent*? Is not any intensional logic one which fails to heed various laws of extensional logic, such as referential transparency, substitution of identicals, and compositionality? Certainly, if 'intensional' is synonymous with 'non-extensional', then any logic is indeed intensional which fails to comply with one or more of the principles and rules of extensional logic. But 'intensional' may also mean—and this is the notion of intensionality germane to Transparent Intensional Logic—that the logic in question comes with an ontology of intensional entities and the means to logically manipulate these entities.

M. Duží et al., *Procedural Semantics for Hyperintensional Logic*, Logic, Epistemology, and the Unity of Science 17, DOI 10.1007/978-90-481-8812-3_1,

Transparent Intensional Logic flouts none of the principles of extensional logic and is, insofar, an extensional logic.

The underlying project is to operate with only one semantics for all kinds of logical-semantic context while adhering to the compositionality principle throughout. This universal semantics is obtained by developing a semantic theory for the hardest case (to wit, hyperintensional attitude contexts) and extending it to all the other cases (i.e., 'generalising from the hardest case'). Less-hard cases demand less logical and semantic sophistication, and irrelevant subtleties are weeded out by installing cruder principles of individuation and substitution.

Referential transparency is the phenomenon that any term or expression—when used in a communicative act—expresses the same entity as its meaning and denotes the same entity as its denotation (or 'semantic value') regardless of the embedding context. This means rejecting so-called reference shift across the board. Instead the 'shifts' that reference-shift is intended to trigger are brought about by distinguishing between two different ways in which the meaning of a word may occur relative to a logical-semantic context, namely with either supposition *de dicto* or *de re*. The point is that the a priori relation between word and sense fixes a sense, which exhausts the function of the word. The so fixed sense may consequently be subjected to logical manipulation, for instance, by being made to occur with supposition *de dicto* or else *de re*.

Transparent Intensional Logic also contains the resources to distinguish in a principled manner between functions and their values. This is because the underlying logic is a (typed) lambda calculus (equipped with partial functions). Church's logic of functions has been around for 70-odd years now, and is well-integrated into logical lore. Our main departure from how Church understood his calculus is that, in Transparent Intensional Logic, the terms for functional abstraction and functional application do not denote functions and functional values, respectively. Instead they denote multiple-step structures specifying how to form functions and functional values, respectively. We conceive of these structures as procedures whose products are either functions or functional values. Our theory's own word for such structured procedures is *construction*.

Intuitively, constructions are procedures, of one or more steps, for inputting and outputting entities. Tichý often likens constructions to calculations.[1,2] Just as

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[1] The idea of linguistic sense as a calculation will be familiar not least from Moschovakis' work on constructive semantics. See Section 1.5 for discussion.

[2] Muskens, in (2005, p. 474, n. 2), interprets constructions as 'procedures that can be used to compute [Fregean] references]' (ibid., p. 474), which is basically on the right track. We agree, with one proviso, with Muskens' characterisation of a computational, or procedural, interpretation of Fregean sense: 'If senses are a certain kind of algorithms, then two senses are identical if the corresponding algorithms are. While identity of algorithms itself is a non-trivial problem, this at least gives something to start with' (Ibid.). The proviso is that constructions are allowed to be non-finitary. With this proviso in mind, we subscribe to the general 'propositions-as-algorithms picture' that Muskens sketches in (ibid., pp. 487ff). For an introduction to how reference-fixing along Fregean lines works in Martin-Löf's type theory, see Primiero (2004) and (2008).

an arithmetic calculation takes numbers, processes them and yields other numbers, so constructions are, semantically speaking, calculations whose results may be, for instance, truth-values, truth-conditions, individuals, numbers, sets, properties, as well as other calculations. It is important not to confuse *procedures* (calculations) with the agent-, world-, and time-indexed *processes* of executing the procedures (i.e., individual cases of actual calculating) or with their *products* (results, output) or with the symbolic encoding of a computer *programme* in a programming language. *Construction* is the single most important notion of Transparent Intensional Logic, and its defining feature *par excellence*. It is anchored to an older notion of function as being more than a mere mapping from one set into another:

> In the 1920s, when [the lambda calculus and combinatory logic] began, logicians did not automatically think of functions as sets of ordered pairs, with domain and range given, as they are trained to do today. Throughout mathematical history, right through to modern computer science, there has run another concept of functions, less precise but strongly influential; that of a function as an operation-process (in some sense) which may be applied to certain objects to produce other objects. Such a process can be defined by giving a set of rules describing how it acts on arbitrary input-objects (The rules need not produce an output for every input.) (Hindley and Seldin, 1986, p. 44).

The constructions of Transparent Intensional Logic are intended precisely as such 'operation-processes' that receive an input and deliver an output (or in well-defined cases fail to deliver an output). The historical resources that Tichý acknowledges are first and foremost Frege's notions of sense (*Sinn*) and unsaturated function (*ungesättigte Funktion*), as opposed to modern-day functions, which are extensionally individuated mappings (akin to Frege's *Wertverläufe*), but also Russell's (not all-too crisp) notion of proposition. Tichý's objectualist take on 'operation-processes' may be seen in part as linguistic structures transposed into an objectual key; operations, procedures, structures are not fundamentally and inherently syntactic items, but fully-fledged, non-linguistic entities residing in a Platonic realm.

Still, the two most common misconceptions of constructions are that they are functions or formulae. True, functions (conceived of as mappings) are constructible by any of the different kinds of construction that their recursive definition enumerates (Definition 1.2); but constructions are distinct from what they construct (in particular, those constructions that construct nothing are still something). Especially, constructions are not what Tichý would begin to call 'determiners', which are just possible-world intensions.[3] Functions-in-extension (i.e., mappings) are set-theoretic entities; constructions not. Church-style functions-in-intension are much closer to constructions. But though functions-in-intension are construed, in mathematical logic and computer science, as rules, these are not clearly defined, whereas constructions are.[4] And, to be sure, constructions are encodable in artificial

[3] We have avoided the term 'determiner' in this book, because it is already in use in linguistics where it has a somewhat different meaning; e.g., articles are determiners.

[4] For functions-in-intension as rules or 'codes' for rules, see Mitchell (1990, p. 371) or Church (1941).

symbolic notation, but constructions are distinct from the formulae they are cloaked in. Formulae are linguistic entities; constructions not.

Apart from recasting the lambda-calculus procedurally, another departure from Church is that Transparent Intensional Logic includes not only a simple type theory but also a ramified type theory. The ramified type hierarchy serves to organise the constructions, together with functions with domain or range in constructions. Constructions offer a worked-out, positive answer to the open question of just how 'hyper' hyperintensions are. The leading principle is that any two procedurally isomorphic hyperintensions are identical. It turns out, though, that there are cases when two procedurally isomorphic constructions are just that—two constructions and not one. So a slightly coarser principle of individuation than the constructional one is called for to preserve the idea of hyperintensional individuation in terms of procedural individuation. Pavel Materna has introduced, in 1998, a rigorous notion of *concept* that identifies as one concept any two procedurally isomorphic constructions. This notion of concept has been incorporated into Transparent Intensional Logic, which therefore operates with four measures of individuation; extensional, intensional, conceptual, and constructional. Hyperintensional individuation is, in the final analysis, conceptual individuation. But since concepts are themselves constructions, we shall often speak in terms of constructions.

Having adumbrated the very basic ideas underlying Transparent Intensional Logic, here is how we locate Transparent Intensional Logic within the current landscape of formal semantics. Once the foundations of formal semantics seemed to have been firmly established. What remained to do was working out the subtleties of their applications to various problems concerning meaning and reference. David Kaplan puts it eloquently in this way:

> During the Golden Age of Pure Semantics we were developing a nice homogenous theory, with language, meanings, and entities of the world each properly segregated and related one to another in rather smooth and comfortable ways. This development probably came to its peak in Carnap's *Meaning and Necessity* (from 1947). Each *designator* has both an intension and an extension. Sentences have truth-values as extensions and propositions as intensions, predicates have classes as extensions and properties as intensions, terms have individuals as extensions and *individual concepts* as intensions, and so on. The intension of a compound is a function of the intensions of the parts and similarly the extension (except when intensional operators appear). There is great beauty and power in this theory (1990b, pp. 13–14).

However, Kaplan points out that already then there was trouble in paradise:

> [T]here remained some nagging doubts: proper names, demonstratives, and quantification into intensional contexts[5] (ibid., p. 14).

[5] Saarinen (1982, p. 131) offers the same list of trouble-makers, adding logical omniscience. As an aside, it is interesting to note that whereas epistemology has been preoccupied with skepticism (the spectre of knowing too little or nothing at all), epistemic logic has been preoccupied with omniscience (the spectre of knowing too much).

And Carnap himself observed in 1947 the problem of how to logically handle what Cresswell was later to dub ‘hyperintensional’ contexts:

> Hyperintensional contexts are simply contexts which do not respect logical equivalence (1975, p. 25).

Carnap asks whether a context might be neither extensional nor intensional, answering in the affirmative:

> Although [the sentences] ‘D’ and ‘D″ have the same intension, namely, the L-true or necessary proposition, and hence the same extension, namely, the truth-value truth, their interchange transforms the [belief-reporting sentence ‘John believes that D’] into the [belief-reporting sentence ‘John believes that D″], which does not have the same extension, let alone the same intension, as the first (1947, pp. 53–4).

So attitudes must be added to the list of nagging doubts, as soon as we are not content with holding, heroically but irrationally, that any two logically equivalent propositions (or whatever else plays the role of attitude *relata*) may always be validly substituted when figuring as complements of attitudes.

The over-all goal driving hyperintensional attitude logic is to avail ourselves of epistemic operators that are, in Dretske’s wording, at most ‘semi-penetrating’ (see Dretske, 1977). For instance, it may be true that you know that if it is raining then the street gets wet and that you know that it is raining; but not that you, thereby, also know that the street gets wet. Or, since we favour relations over operators, the relation of knowing obtaining between knowers and hyperpropositions needs to have the effect of being at most ‘semi-penetrating’. Much research in epistemic logic since Hintikka (1962) has centred on which restrictions to impose, and how to impose them, particularly with a view to solving the problem of logical omniscience. The solution we offer relates agents to constructions and equips each agent with one or more rules of inference that they are able to apply flawlessly to any appropriate set of premises. This way we are able to calculate the *inferable* knowledge of agents relative to their intelligence (*in casu*, their inferential capacities), and their individual inferable knowledge will be a proper subset of pieces of knowledge of all the constructions that are consequences of those already explicitly known by individual agents.

Nonetheless, despite the nagging doubts, formal semantics continued to blossom as a research discipline, really taking off in the late 1950s and early 1960s thanks to the advent of possible-world semantics. Kripke offered a semantics (several, in fact) for C.I. Lewis’ naked modal syntax from the late 1910s. And Montague would soon afterwards develop an intensional logic based on Tarski-style semantics enriched with possible worlds by means of which to analyse large fragments of natural language. Kripke says that

> The main and the original motivation for the ‘possible worlds analysis’ — and the way it clarified modal logic — was that it enabled modal logic to be treated by the same set theoretic techniques of model theory that proved so successful when applied to extensional logic. It is also useful in making certain concepts clear (1980, p. 59, n. 22).

Kripke does not mention here which concepts he has in mind, but it seems safe to assume that they must be the notoriously elusive intensional entities like propositions, properties, relations-in-intension, individual concepts, magnitudes, etc. Possible-world semantics can tell us what an intension is and when any two intensions are identical. An intension is a function whose domain is made up of possible worlds, and *qua* functions intensions are individuated extensionally. If f, g range over intensions and w over possible worlds then if f, g return the same values for the same arguments then $f = g$. That is, co-intensionality is the principle of individuation of intensions:

$$\forall fg\ (\forall w\ (f_w = g_w) \supset f = g).$$

The upside of defining propositions and other intensions extensionally is that they become logically manageable and that we help ourselves to a clear notion, thanks to the fact that the logic of (total) functions is well understood.

Properly speaking, though, intensions are not functions, but pre-theoretic entities that are modelled intra-theoretically as functions. This slight correction is important to forestall an objection due to George Bealer. He launches in several places (variations of) what we would call 'the argument from aroma' (e.g., 1982, p. 90). The aroma of coffee is a property (an intension), but certainly not a mapping (mappings having no aroma); hence, properties are not mappings. We do not literally identify intensions with world-defined mappings—though for technical convenience we do identify the modelling and what is so modelled. The purpose of intensions is to capture empirical variability; such-and-such is the case, but might not have been the case, and vice versa. We construe intensions as functions from possible worlds to chronologies of entities, chronologies being functions from times to entities (including other intensions). Mathematics and logic, on the other hand, have no need for empirical variability; hence, they have no need for intensions.

Meanwhile, during the ascent of the model theory of possible-world semantics, in the non-model-theoretic quarters Heyting had long before formulated a constructivist semantics for mathematical language, Dummett would later extend, in a usually informal manner, constructivism to natural-language discourse, and Martin-Löf would put forward a detailed constructive type theory for mathematical language. Yet constructivism has so far not succeeded in framing a fully-fledged semantics for natural language, no least because it is far from obvious what the natural-language counterpart of a mathematical proof (-object) would be.[6]

Despite their initial success, the multifarious theories based on model theory eventually ground to a halt over the old problem of how to logically analyse attitudes. For the downside of intensions as mappings is that, though propositions and other intensions may have been logically murky prior to possible-world semantics,

[6] See Ranta (1994) for an application of Martin-Löf's type theory to natural-language discourse.

the corresponding notions were also somewhat richer.[7] The clarity of possible-world intensions comes at the price of impoverishing these notions. Put uncharitably, possible-world intensions are intensionality on the cheap. Though one of its champions, Kripke is alert to various shortcomings of possible-world semantics. Thus, after a remark on attitudes he vents an afterthought with far-reaching implications:

> How this relates to the question what 'propositions' are expressed by these [attitude-reporting] sentences [and] whether these 'propositions' are objects of knowledge and belief…are vexing questions. I have no 'official doctrine' concerning them, and in fact I am unsure that the apparatus of 'propositions' does not break down in this area. … Of course there may be more than one notion of 'proposition', depending on the demands we make of the notion (1980, p. 21; ibid., p. 21, n. 21).

The impasse over no least attitudes has lead to the re-discovery of so-called *structured* propositions. Kaplan may well have pioneered their revival in a 1970 talk that appeared as (1990b) when he urged that the analysis of 'John is tall' should include two components:

> [T]he property expressed by the predicate ['is tall'], and the individual John. That's right, John himself, trapped in a proposition (1990b, p. 13).

Along the same lines, Cresswell called for

> [An] analysis of propositions which assumes that they are structured entities…The most fully worked out account of structured meanings within a possible-worlds framework is that presented by David Lewis [in (1972)] (1975, p. 78).

Unfortunately, manoeuvring within a set-theoretic paradigm such as model theory, the only avenue open to Kaplan and Lewis was to identify structure with ordered *n*-tuples (or at least to model them as such). Tuples are a non-starter, for the simple reason that they are simple while structures are complex. Complexes have parts arranged in a particular way while sets only have elements.[8] The most a set can offer is a sequential ordering of its elements. So ⟨Is_Tall, John⟩, or ⟨John, Is_Tall⟩, is not a structured proposition. An additional objection is that either of these two two-tuples merely enumerates a property and an individual without specifying that the former is predicated of the latter. This is tantamount to the standard 'laundry list' objection that the items on the 'list' fail to hook up with one another so as to integrate into a whole, that is, it is left unexplained how sense atoms combine into one molecule. Yet another objection would be that ordered *n*-tuples most likely cannot do some of what propositions are intuitively expected to do. In particular, it is not clear in what sense a tuple can be said to be a truth-bearer

[7] Three cases in point would be Fregean *Sinn* and Russellian *propositions*, and also Bolzanian *Sätze an sich*; see Materna (1998, 2004a).

[8] See also Simons (2007, §8): 'A complex whole is an object with more than one proper part, such that the parts are related together in the whole in a determinate way. This way of their being together in the whole is the *structure* of the whole.' Hence, 'a musket is not a *sum* of parts: it is a structured whole of parts put together in a certain way' (ibid., §7).

(i.e., something capable of being true/false) or an attitude relatum (i.e., something known/believed/hoped, etc., to be true/false).

Transparent Intensional Logic agrees with Cresswell, Kaplan, Richard and others that the meaning of a sentence must match, more or less, the structure of the sentence:

> [I]f the structure of propositions is as fine-grained as the structure of sentences, then it is hard to give to propositions any content but in terms of something analogous to sentence-like structured objects (Chierchia, 1989, p. 131).

For what other structure could arguably be a serious candidate? None that leaps readily to mind; especially not if, as in Transparent Intensional Logic, it is required that a logical analysis must treat of all, and only, those entities denoted in the analysandum. This constraint is called the *Parmenides Principle*, a forerunner of which would be Carnap's principle of subject-matter (1947, §24.2, §26.)

Apparently, mainstream analytic philosophy of language has bumped up against serious shortcomings in its foundations, with no obvious remedy in sight. True, when propositions are identified whenever materially equivalent or co-extensional, we have what we need for extensional logic, which validates the substitution of any two propositions having the same truth-value. And when propositions are identified whenever logically equivalent or co-intensional, we have what we need for intensional logic, which validates the substitution of any two propositions having, or being, the same truth-condition. But taking it to the third level of hyperintensions has seemed so far an insurmountable obstacle.

Little wonder, then, that much of what passes for analytic philosophy of language nowadays is shot through with semantic minimalism or even nihilism and an over-emphasis on pragmatic notions such as assertion, (act of) utterance, understanding, communication, language acquisition, etc. The glory days of Golden Age Semantics seem buried in the dim and distant past, with little hope of resurrection.

However, running alongside the mainstream of theories following in the slipstream of Kripke, Kaplan, Montague, etc., and the parallel mainstream of Dummett-style proof-theoretic semantics, we find a small group of lesser-known, worked-out theories of hyperintensional logic. These include, inter alia, George Bealer's, Edward Zalta's—and Pavel Tichý's. Tichý's is a theory that comes with a (very) 'big' semantics and a (very) 'small' pragmatics. The central concerns are only those a priori features of language that lend themselves to description and analysis in a purely logical manner. Thus, Tichý's theory is distinct both from those that 'pragmatize' their semantics and those that 'semanticize' their pragmatics. It observes a strict demarcation between semantics and pragmatics; so since even very sophisticated attitudes are to be analysed strictly semantically, it is obvious why a 'big' semantics is wanted. But whereas semantic and pragmatics are kept apart, semantics and syntax are developed in parallel. This turns the syntax of Transparent Intensional Logic into an *interpreted* one. We do not proceed as in model-theoretic semantics, in which first a lexicon and a set of rules of formation are introduced, followed by a syntax, and topped off with a semantics (interpretation). In particular, in TIL no expression may be introduced without typing the construction it expresses as its sense, which entails a typing of the entity that it denotes.

The puzzle-solving mettle of Transparent Intensional Logic comes at a high ontological price, due to its infinite hierarchy of higher-order entities; but it excels at parsimony in another respect. It contains but four essential constructions. They are called *Trivialization*, *Variable*, *Composition* (originally: *Application*), and *Closure* (originally: *Abstraction*).[9] These four key constructions can be divided into two groups of two. Composition and Closure are computation-like constructions; namely, the ('downward') application of a function to an argument, and the ('upward') formation of a function, respectively. The other two, Variable and Trivialization, provide the first two with input in each their way and independently of each other. Variables provide their values relative to a valuation function; Trivializations provide the entities they Trivialize by presenting them directly. The fact that constructions may themselves be Trivialized holds the key to how we obtain hyperintensional attitudes, by being able to distinguish between *using* and *mentioning* constructions.[10] These four constructions correspond to the syntax of a lambda calculus whose terms are variables, constants, applications and abstractions. Trivializations match constants, by picking out definite entities in just one step. The unusual ontological status of Variables should be underlined; they are objectual and not linguistic entities. The assignment of an entity to a Variable *x* does not relate this entity to a piece of language, unlike '*x*', but completes an open construction that subsequently constructs a definite entity.

Constructions are arranged in a ramified, higher-order type theory that is based on a simple type theory of first-order objects. The simple type theory, when used for natural-language analysis, spans four ground types (individuals, truth-values, possible worlds, and reals doubling as times) and types of partial functions defined over them. The typing does not apply to linguistic entities, as in categorial grammar (cf. Montague, Leśniewski, Ajdukiewicz, Cresswell), but to abstract objects such as functions, truth-values, and higher-order entities, as in the constructivist type theory of Martin-Löf. Our bi-dimensional type theory fixes the objective relations among this multi-layered multitude of abstract entities. It thus enables the semanticist to control whether the input is type-theoretically internally coherent and whether the right type of output follows, so as to prevent categorial mismatches.

Transparent Intensional Logic eschews *possibilia* (possible worlds arguably the only exception). Instead the theory operates with a constant domain for all worlds

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[9] It turns out, however, that we occasionally also need a fifth and a sixth construction, called *Execution* and *Double Execution*. Furthermore, the application of Transparent Intensional Logic to database theory has prompted two more constructions; one for constructing ordered *n*-tuples and another for constructing projections; see Duží (1992).

[10] See Section 2.6.

and times. What varies are the values that (non-constant) intensions have in different worlds and at different times, and not the domains that different worlds and times have. The theory also rejects individual essentialism; no individual bears any purely non-constant property by any sort of necessity (including the enigmatic 'metaphysical' necessity). This is not to say, though, that we reject essentialism across the board; far from it. Taking a lead from a 1979 paper by Tichý, we have built up an essentialist theory, according to which relations of conceptual necessity obtain between various kinds of intension. The result is *intensional essentialism*, which says, roughly, that, necessarily, if *x* is a/the *F* then *x* is also a/the *G*, because being a/the *G* is in the essence of being a/the *F*. Intensional essentialism comes in handy, for instance, when spelling out the *de dicto*/*re* ambiguities besetting, e.g., 'Necessarily, the King of Bhutan is a king'. Taken *de dicto*, it is true, for there is a necessary, a priori link between the intensions *the King of Bhutan* and *being a king*—you cannot have the former without also having the latter. Taken *de re*, it is false, for nothing of a logical or conceptual nature forces whatever individual is the King of Bhutan to be a king. It is neither true nor false, if there happens to be no King of Bhutan, for then there is nobody of whom it would be either true or false that *he* is a king.

Tichý began developing Transparent Intensional Logic simultaneously with Montague's, around the mid-1960s, both attempting to get as much logical and semantic mileage as possible out of the possible-world paradigm. One tenet informing this project was that a natural language such as English is largely on an equal footing with the formal logical language in which it is analysed. This is a strong common point to share, and a major departure from the thoroughly suspicious attitude toward natural language that Russell, Frege, and Church, to mention but a few, championed. But Tichý and Montague parted company over some of the tenets that should inform the logical analysis of natural language. The most important difference is probably over whether natural language is permeated by shifts of reference (in the Fregean sense) and, if so, whether it should be replicated in the formal language in which the logical analysis is couched. Two other noteworthy differences between Tichý's TIL and Montague's IL are these. First, thanks to so-called explicit intensionalization and temporalization (see Section 2.4), TIL makes a fine-grained analysis of the *de re*/*de dicto* difference possible. For now, explicit intensionalization consists in explicit mention of variables ranging over worlds and times in the logical syntax proper. Moreover, each TIL analysis is fully compositional so that the 'Church-Rosser diamond' (the Koh-I-Noor of the lambda-calculi) holds, unlike IL.[11] Second, due to its hyperintensional procedural semantics, TIL offers a principle of individuation finer than logical equivalence, so that equivalent expressions may have different meanings. This feature enables us to analyse hyperintensional attitudes in an adequate manner (see Chapter 5).

[11] For further comparison of TIL and Montague's IL, see Section 2.4.3.

It is vital to appreciate just how deep the issue of reference shift runs. Here is how we would rationally reconstruct how referential obliqueness came to be a theme pervading contemporary philosophy of language. What has become known as 'Frege's puzzle' can be summarised as follows. Historically, the puzzle turns essentially on judgements (*Urteile*). Frege's question is whether the *judgement* $\vdash Fa$ is identical to the *judgement* $\vdash Fb$ in case $a = b$. (' $\vdash$' is Frege's *Urteilsstrich*, judgement stroke, and not the symbol of validity.) For instance, is the judgement *the proposition (Gedanke) that the Morning Star is a heavenly body illuminated by the sun is true* identical to the judgement *the proposition that the Evening Star is a heavenly body illuminated by the sun is true*, in case the Morning Star is the same heavenly body as the Evening Star? Frege's answer is in the negative due to the manifest difference in epistemic value (*Erkenntniswert*) between the two judgements. Yet as an extensionalist logician (*Umfangslogiker*), Frege would have expected an answer in the affirmative. Hence his puzzlement. Frege's puzzle deals with the acquisition of knowledge by making judgements and the difference, puzzling at first, between knowing that the Morning Star is an *F* and knowing that the Evening Star is an *F*, even though the Morning Star is identical to the Evening Star. There are two things to know, not just one, and one may know the one without knowing the other.

However, the modern Anglo-Saxon reception of Frege has tended to neglect the differences between judgements and propositions in Frege, speaking of propositions only. Phrased in terms of propositions, the puzzle is why the proposition *that the F is the G* conveys non-trivial information, if true, while the proposition *that the F is the F* fails to. Or in terms of attitudes, an agent may believe the latter without believing the former and without being guilty of inconsistency or irrationality. In order to solve the puzzle, Frege attempts first to apply universal transparency to the puzzle, assuming that 'the *F*' and 'the *G*' refer to the same individual *a*. Call this 'Millian universal transparency'; 'Millian' because a singular term refers to an object, not a connotation, and because its reference is not mediated by a connotation.[12] Any account of the non-triviality of the former proposition is blocked, since it reduces to the triviality that *a* is self-identical. So, Millian universal transparency must be abandoned. Still two options apart from Millian universal transparency are open; Fregean systematic contextualism and Fregean universal transparency.

Frege, for extensionalist reasons, opted for contextualism. Tichý goes for universal transparency. The basic 'trick' behind the transparency of Transparent Intensional Logic is to universalise Frege's anomaly. Thus, universal transparency is obtained by means of universal obliqueness. If every context is oblique, or if every context is 'straight' (*gerade*), then it is pointless to uphold the distinction between oblique and straight context. Not that it would be a distinction without a difference, but the object under scrutiny—natural language—would fail to exemplify

[12] We are neglecting Mill's actual psychologistic semantic theory here.

the distinction. So Tichý takes Frege's semantics reserved for a marginal case and elevates it to the semantics for the universal case.

Interestingly, Transparent Intensional Logic agrees verbally with what Donald Davidson says about 'semantic innocence', that words maintain their meanings and denotations across shifts of context.[13] But Davidson's so-called 'paratactic' approach maintains that expressions invariably denote extensional entities, whereas our 'hypotactic' approach maintains that (empirical) expressions invariably denote intensional entities. Tichý was adamant from the outset that natural language does not display shifts of reference and that, even if it had, there should be no room for it in a logical formalism. The rationale for the latter claim is that, as already Frege himself emphasized, logical notation must be unambiguous. Logical notation must disambiguate language and not perpetuate ambiguities.[14] The rationale for the former claim is, briefly, that if the terms and expressions of natural language were to denote extensional entities (like individuals, truth-values, sets) then successful communication would require of speakers and hearers that they knew which possible world was the actual one.

The gist of the argument is this. Intensions are conditions satisfiable by possible worlds (and whatever other empirical indices we care to add, such as times). If empirical terms denoted the actual-world satisfiers of these conditions, then one must know which world is actual to know which entity is being so denoted. Successful communication would require not only understanding the meanings of terms and expressions, but also knowing their actual values. But then the empirically omniscient would have no epistemic need for communication, for they would already know everything there was to know; whereas the empirically non-omniscient would never know whom or what was being talked about.[15] We mortal language-users do possess much empirical knowledge (neglecting for now the challenges posed by radical scepticism), but even if mankind were to pool together all its current knowledge, what could be identified would at most be an equivalence class of possible worlds, of which the actual world would be a member. Hence it should not be a (mostly tacit) requirement that we be able to identify the actual world. At the same time, though, we do know that we live in the actual world and we do make our empirical assertions about the actual world. We agree with this portion of David Lewis' 'indexical' theory of actuality. But this is not to

[13] See Davidson (1968).

[14] As Muskens says, 'Why does [Montague's] IL show such exotic behaviour; why do Leibniz's Law, Universal Instantiation and Lambda Conversion not hold under the normal conditions? Because the logic was explicitly designed to reflect certain opacity phenomena in natural language' (1989, p. 10).

[15] In a recent comparison of Tichý and Zalta, Sierszulska says correctly that, '[K]nowing all the values of the [intensions] … would be the same as knowing all the facts … The proper analysis of a proposition cannot assume such [empirical, as opposed to logical] omniscience, and it stops at the point where all the possibilities are taken into account, but none is specified' (2006, p. 491).

say that we know of one particular possible world that it is actual, for this is exactly what we cannot know for want of empirical omniscience.

An additional point is that the widespread idea that empirical terms denote extensions fails to keep the factual relation between an intension and its world-relative satisfier apart from the semantic relation between an empirical expression and its denotation. Transparency is underpinned by an anti-actualist semantics founded upon a sharp demarcation between denotation and reference. The denotation relation holds a priori between a word and the entity (if any) identified by the meaning of the word (or *meanings* if the word is ambiguous, and *meaning* if unambiguous, at the level of logical analysis). Of course, it is a historically contingent fact that a configuration of letters of some alphabet and/or sounds constitutes a *word* of a language and expresses one meaning rather than another or none at all. Diachronically, such configurations may criss-cross in and out of a language and enter into different semantic relations at different points in time. Synchronically, however, the semantic relations characterising a certain language are fixed for any given point in time. When we use an expression in a communicative act we communicate its sense. The same configuration of letters or sounds might have had wildly different senses, since the relation between term/expression and sense is wholly arbitrary and not inherent. Only this fact is irrelevant to logic and semantics. It falls to linguistics and not logic or formal semantics to associate terms with senses. The starting-point of logical analysis of language presupposes both that the word/sense relations are in place and that the speakers of the language under scrutiny master these relations.[16]

As this book shows, this choice of starting-point dictates our analysis of, e.g., ‘Hesperus is Phosphorus’ and ‘Cicero is Tully’. If the terms are names of individuals, then the sentences merely express the self-identity of an individual bearing two names. What is to be known concerns not a worldly but a linguistic matter, then. But if the terms are instead names of individuals-in-intension—what Church, Carnap, Kaplan and others call ‘individual concepts’ and we call ‘individual offices or roles’—then what is to be known does concern a worldly matter; namely, that two differently named individuals-in-intension contingently coincide in the same individual (-in-extension), which or who bears neither name. As logical semanticists we adjudicate neither way. We enumerate the various possible semantic analyses of, e.g., ‘Hesperus is Phosphorus’ and ‘Cicero is Tully’, and chart their presuppositions and consequences.

The *denotation* of an empirical term is always an intension. The *reference* relation holds a posteriori between an empirical word and the value, if any, of its denotation at

[16] Tichý puts the point succinctly in a 1966 paper; ‘We assume, of course, a normal linguistic situation, in which communication proceeds between two people, both of whom understand the language. Logical semantics does not deal with other linguistic situations’ (2004, p. 55, n. 1). Likewise, C.A. Anderson says about Church’s Alternative (0): ‘Sense is what is known when the language is understood. In accordance with this, the intensional semantical rules should state essential facts about the semantics, the mastery of which constitutes (ideal) competence with the language. These may include the rules of synonymy [.]’ (1998, p. 163).

the actual world at the present moment. So, while an empirical word may lack a reference, it never lacks a denotation. For instance, the term 'The King of France' lacks a reference in the actual world at the present time. Yet the term has a denotation, namely the individual-in-intension (individual office) *the King of France*. The semantics of empirical words is such that no such word can be 'empty' in the sense of failing to pick out an entity; for it invariably picks out an intension. Whether a given intension lacks an extension at the actual world is a factual rather than semantic question. In the case of non-empirical words, the extra-semantic relation of reference drops out, since non-empirical terms and expressions do not pick out anything relative to empirical indices. They denote what—if anything—is constructed by their respective senses. The qualification 'if anything' is important, since some non-empirical words fail to denote. For instance, whereas there is a construction of the largest prime, there is no number for this construction to construct. Still the term 'the largest prime' is meaningful and has a meaning to contribute to a compound meaning, like the one of the sentence, 'The largest prime is odd'. But the sentence fails to denote, because 'the largest prime' fails to.

So there is this one difference between empirical and non-empirical expressions. But let us stress the reason why both kinds of expression spring from the same source. *All* expressions, without exception, denote what is constructed by their senses. It is just that there are non-empirical cases where the sense fails to construct something for the relevant word to denote. The overarching semantic idea pertaining equally to mathematical and natural language is that sense is a calculation or procedure, while at the same time observing a thoroughgoing demarcation between these two compartments of language. Natural language descends from a calculation to an intension. Mathematical language descends either from a calculation to an extension or a lower-order calculation. The semantics of natural language demands an intensional intermediary between sense and (possible) extension due to the inherent anti-actualism informing Transparent Intensional Logic. The semantics of a natural-language term or expression terminates in the calculation of an intension. The sense is a manner of calculating the given intension so as to be able to arrive at its value at any world and time of evaluation. The semantics *does not* terminate in a calculation of the *actual and present value* of an intension, let alone in the value itself (if indeed any). There can be no final semantic, a priori step from intension to actual and present value on pain of reinstating empirical omniscience as a prerequisite for successful communication among non-omniscient language-users. The denotation is the same for all worlds and times, so words denoting intensions qualify as rigid designators. What varies is the reference; non-constant intensions do not return the same values at all worlds and times. But the reference relation is factual, a posteriori and extra-semantic; unlike the denotation relation, which is a priori and intra-semantic.

1.1.1 Semantic schemas

We are placing our procedural semantics within the general Fregean programme of explicating sense (*Sinn*) as the *mode of presentation* (*Art des Gegebenseins*) of the entity (*Bedeutung*) that a sense determines.[17] Muskens correctly points out that, ‘The idea was provided with extensive philosophical justification in Tichý (1988)’ and that ‘[Tichý’s] notion of senses as *constructions* essentially captures the same idea’ (2005, p. 474).

So our starting-point is Frege’s well-known semantic diagram (FSD). This diagram is frequently accepted as one of the foundations of modern semantics. To explain why a true sentence of the form ‘$a = b$’ can be informative, unlike a sentence of the form ‘$a = a$’, Frege introduced an entity standing between an expression and the object denoted (*bezeichnet*) by the expression. He named this entity *Sinn* (sense) and explained the informative character of the true ‘$a = b$’-shaped sentences by saying that ‘a’ and ‘b’ denote one and the same object but differ in expressing (*ausdrücken*) distinct senses. Thus FSD can be visualized as in Fig. 1.1.

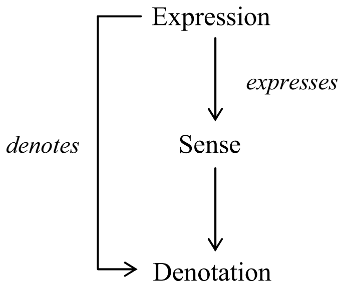

Fig. 1.1 Frege’s semantic diagram (FSD)

So far, so good. The problem, though, is that Frege never defined *sense*. All he says is that it is a ‘mode of presentation’ (*Art des Gegebenseins*) of the denotation. The frequent interpretation of sense in contemporary semantics has it that sense is an *intension*. Thus, Kirkham says:

[17] We know we are cutting corners here by paraphrasing ‘Bedeutung’ as ‘entity’. We are doing so in order not to get bogged down in the ongoing discussion of how best to render ‘Bedeutung’. The standard translation has been ‘reference’, but this does not do justice to Frege’s idiosyncratic distinction between ‘Sinn’ and ‘Bedeutung’, which are more or less synonymous nouns in ordinary German, barring idiomatic usage; e.g., ‘sinnlos’ and ‘bedeutungslos’ are certainly not synonymous adjectives. The best verbatim translation would have been ‘meaning’, to be contrasted with ‘sense’. But the idea of Frege being the *meaning* of ‘Frege’ sits very poorly indeed on the ears. Besides, ‘Bedeutung’ comes with a suggestion of pointing at an entity—‘deuten auf’—that ‘meaning’ lacks. Fortunately, we can afford to be offhand about ‘Bedeutung’, since we are so strongly biased toward *Sinn*.

> Since the seminal work of Gottlob Frege (1892a) it has been a *commonplace* [italics ours] that the meaning of an expression has at least two components: the sense and the reference. The sense of an expression is often called the *connotation* or the *intension* of the expression, and the reference is often called the *denotation* or *extension* of the expression. The extension of an expression is the object or set of objects referred to, pointed to, or indicated by, the expression. ... The extension of 'the morning star' is a certain planet, Venus. The extension of a *predicate* is the set of all objects to which the predicate truly applies. The extension of 'red' is the set of all red things. The extension of 'vertebrate with a liver' is the set of all vertebrates with a liver (1992/1997, p. 4).

'Intension' can be interpreted in various ways. In the quotation above it is used as in Montague's theory, viz. as *the intension of* an expression. At the same time contemporary possible-world semantics takes intensions to be functions whose domain is made up of possible worlds. According to this view, an expression possesses an intension and an extension;[18] the former corresponding to Frege's 'Sinn', the latter to Frege's 'Bedeutung'.[19,20]

The intuition behind this interpretation is at first sight attractive. This can be shown by the classical Fregean example of 'The Morning Star' vs. 'The Evening Star'.[21] The senses of these expressions are distinct according to Frege. Now if we connect with either of these expressions an intension then the result is this: the sense of 'The Morning Star' is another possible-world intension than the sense of 'The Evening Star', but the value of both intensions in the actual world at the present moment is one and the same object—Venus, as it happens.

Of course, aspersions have been cast upon this view independently of the criticism that TIL had raised much earlier. For instance, van Lambalgen and Hamm say:

> In formal semantics for natural language it is not common practice to associate algorithms to expressions. …it is usually assumed that all one needs is the intension of an expression, defined as a function which maps a possible world into an extension of the expression in that possible world. It seems to us that this picture of meaning is too static, and by and large cognitively irrelevant (2004, p. 7).

As we argued above, the interpretation of *sense* as *intension* and *denotation* as *extension* in the case of empirical expressions (like 'The Morning Star', 'The Evening Star') is counterintuitive. Already Carnap (1947), knew that a logical analysis cannot provide the contingent values of intensions. If intensions are functions from possible worlds (and times, as in TIL) then we could logically determine the value of an intension in the actual world only if we knew which of the possible worlds is the actual one. On any rational explication of the notion of possible world, this knowledge cannot be a priori; therefore, determining the value of an

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[18] See also Carnap (1947).

[19] Church (1956) has 'denotation'.

[20] Originally, Tichý also held to the view that Fregean sense may be explicated as a possible-world intension; cf. (1986a, p. 253, 2004, p. 651).

[21] See Section 3.3.

intension in the actual world must always be a matter of factual experience (rather than of logic).

The relation between an intension and its actual/present extension is beyond logical semantics. The spirit of TIL requires that the terms 'denotation' and 'reference' be semantically kept separate, at least in the case of empirical expressions. What is denoted are intensions, whereas the *value* of such an intension in the actual world (and at the present time) is called the *reference* (of the respective expression). Thus reference is not a matter of logical semantics, being ascertainable via experience only. The necessity of this decision is intuitively clear, as soon as we agree that logical analysis cannot contain any empirical elements. Consider an FSD where the expression is an empirical sentence. For Frege such a sentence denotes its truth-value. Take the sentence 'Mars contains water'. If denoting is (as it should be) a logical relation then we could derive its actual truth-value. Then why send probes to Mars?

Accepting the view that empirical expressions denote possible-world intensions, the 'The Morning/Evening Star' problem might seem to be heading for a solution. The question, however, arises: do we need the notion of sense as a semantic category at all? Prevailing logical theories are *denotational* and *set-theoretic*:

> [T]he meanings, it should be stressed once more, are the semantic objects in the model, i.e., the individuals, properties, propositions, second-order properties and so on that we associate with the expressions. The logical expressions serve to represent these but are not to be confused with them (Gamut, 1991, p. 218).

We shall show that denotational and other set-theoretic approaches are too coarse-grained. Theories based on standard logic run together the meanings of terms and expressions that are classically equivalent, even if they are evidently not strictly synonymous. For an example, consider the two sentences

(1) 'Bill walks',
(2) 'Bill walks and whales are mammals'.

Intuitively, (1) and (2) do not have the same meaning. Standardly, however, the meaning of (1) will be a certain set of possible worlds (the worlds in which Bill walks) and the meaning of (2) will be the intersection of this set with the set in which all whales are mammals. Since we presuppose full linguistic competence in language-users, sentences like 'No bachelor is married' and 'Whales are mammals' come out analytically true, i.e., true only in virtue of their meaning. Provided that we understand the meanings of the predicates 'is a whale' and 'is a mammal' as used in current English, when learning that whales are mammals we do not acquire factual information bearing on the state of the world. If you know that the individual before you is a whale, you need not examine the world in order to get to know that the individual is a mammal. Instead, an analytically true sentence is true in all possible worlds. Hence if the meaning of (2) is a certain set of possible worlds, then it is the same set as the set of worlds in which Bill walks.

Therefore, (1) and (2) are predicted to be synonymous, which obviously they are not.[22]

This inaccuracy might seem not to be that important, though. After all, the theory gives a correct prediction of the relation of *entailment* here. The two sentences entail each other, and this fact correctly follows from Montague-like set-theoretical theories. So why not embrace co-entailment, although more coarse-grained than strict synonymy, as a good approximation to meaning and synonymy in natural language? Here is why not. Natural language is rich enough to express the differences in the meanings of co-entailing sentences. Attitudes are typical examples. One can easily believe that Bill walks without believing that Bill walks and that whales are mammals. Though 'All whales are mammals' denotes a constant intension, the sentence is far from being meaningless.[23] Only a more fine-structured notion of meaning than co-entailment will capture the meaning of 'All whales are mammals'. But, of course, if an empirical expression denotes an intension then what would its sense be? And, furthermore, what would the sense of a *mathematical* expression be?[24]

Consider, e.g., the expression

$$\text{‘}(2 \times 2) - 3\text{’}$$

It will probably be agreed that this expression denotes the number 1. But why is that? What is its sense? This problem is eloquently formulated by Tichý:

> If the term '(2 × 2) – 3' is not diagrammatic of anything, in other words, if the numbers and functions mentioned in this term do not themselves combine into any whole, then the term is the only thing which holds them together. The numbers and functions hang from it like Christmas decorations from a branch. The term, the linguistic expression, thus becomes more than a way of *referring* to an independently specifiable subject matter: it becomes *constitutive* of it. An arithmetical finding must, on this approach, be construed as a finding *about* a linguistic expression. ... But since an expression is always part of a particular notational system, our theorist must construe the arithmetician as being concerned specifically with a definite notation (1988, p. 7).

Now if we wish to retain Frege's idea that between an expression and its denotation there is some abstract entity (*Sinn*) serving as intermediary then such an entity cannot be a possible-world intension. In the empirical case an intension is what the expression *denotes*; in the non-empirical case, either no intensions are needed or they are always going to be constant functions. Possible-world intensions serve the purpose of modelling empirical variability, and are out of place in mathematics. Yet there are theories that attempt to account for, e.g., inconsistent beliefs and absurd

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[22] See Section 2.1.2 for another aspect of this problem, and Section 2.2.1 for the definitions of *synonymy* and *equivalence*.

[23] This is not to say that it would have *empirical* information to offer; see Section 5.4.

[24] Note that the first place in Frege (1892a) where he introduces the notion of sense is not the famous one involving 'The Morning Star' and 'The Evening Star', but one involving the medians of a triangle. Here we chose a still simpler example.

objects (like round squares) by introducing a parallel logical space of logically impossible worlds (cf. Priest, 1992). But just as little as the number five belongs to the domain of possible worlds and just as little as mathematical sentences are evaluated at possible worlds, so round squares should not be assigned to the domain of any impossible world. The very idiom of worlds, whether possible or impossible, is out of place, as soon as non-empirical objects like numbers and figures are involved. We will show that terms like 'round square' and 'the greatest prime' are not meaningless expressions and that we can handle them without the category of impossible worlds.[25] So, which kind of entity can play the role of sense and possibly be captured by logical analysis?

The example of a simple arithmetical expression shows that the sense should be an extra-linguistic entity, whose existence would explain the connection between an expression and the object denoted. As we have already pointed above, we have such an entity at hand. It is *the* key notion of TIL, the one of *construction*. Our neo-Fregean semantic schema is the adjusted version of FSD as visualized by Fig. 1.2.

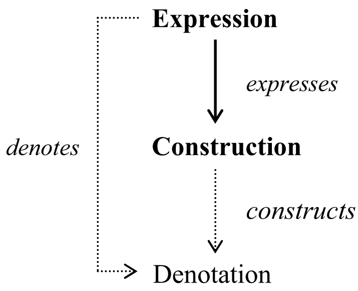

Fig. 1.2 TIL semantic schema

The most important relation in this schema is between an expression and what is expressed by it: its meaning, i.e., a construction. Once we exactly define *construction*, we can logically examine it; we can investigate what (if anything) the construction constructs, what is entailed by it, etc. Thus constructions are semantically primary, denotations secondary. Once a construction is explicitly given, the entity (if any) it constructs is already implicitly given, but will have to be teased out by means of logical analysis. As a limiting case, the logical analysis may reveal that the construction fails to construct anything; we will say that it is *improper*.

It might be tempting to say that the references of empirical terms and expressions were tertiary. But they are not. The preceding discussion of denotation versus reference served to make the point that the relation of denotation is intra-semantic and the relation of reference extra-semantic. Given a denotation, logical

[25] They express empty concepts, the former identifying an empty class of geometrical figures, the latter identifying no number at all. See Section 2.2.

analysis cannot tease out its reference. So there is no room for reference in our semantic schema.

As for *terminology*, Tichý himself did not use Fregean expressions; he did not refer (at least in his mature works, in particular in his 1988) to constructions as 'meanings'. So he did not use the term 'concept' in our sense.[26] Further, Tichý's final semantic schema, in (1988, p. 224), reduces all semantic relations to *denotation*; what is denoted is, without exception, a construction. True, as mentioned above, from the semantic point of view a construction is primary and the product of the construction secondary. Thus the above semantic schema of Fig. 1.2 is *impure*. Our *pure semantic schema* (Fig. 1.3) comes down to this (see also Section 3.2.1):

$$\begin{tikzcd}
\text{Expression } E \arrow[d, "\textit{expresses}"] \\
\text{Construction (the sense of } E\text{)}
\end{tikzcd}$$

Fig. 1.3 TIL pure semantic schema

We have one, methodological, reason for not going along with Tichý's final schema. TIL is a procedural semantics and as such opposed to denotational semantics. So Tichý's final schema represents a hybrid between the procedural and the denotational approaches, by having terms directly denote procedures without a procedure being a stepping-stone between term and entity. Moreover, according to well-entrenched terminology, 'denotation' is reserved for a relation between terms and set-theoretic entities, yet procedures are none such. Hence our preference for a three-tiered impure semantic schema to make the relation between what is expressed and what is denoted explicit, and a pure semantic schema to go with our procedural semantics. So we say that expressions express their meanings and denote (or fail to denote) entities identified (constructed) by the respective construction. The impure semantic schema must help us achieve the goal of this book, which is to assign constructions to expressions as their meanings and the products of the constructions as their denotations. This is also to say that being impure does not detract from a semantic schema's standing.

The viability of the thesis that empirical terms and expressions denote intensions presupposes that we possess of a means to obtain an extension from an intension. For surely we do not want to end up claiming that the sentence, 'The King of Bhutan is a benign ruler' ascribes the property of being a benign ruler to the intension *The King of Bhutan*. Two standard options are in circulation in the literature; a special extensionalization operation/operator or functional application. We use functional application, so we have no need for an operation/operator earmarked

[26] The general idea that concepts are procedures was, however, advanced by Tichý already in 1968 and 1969. We will deal with concepts (i.e. closed constructions in normal form) as *procedural meanings* in Section 2.2.

specially for extensionalization. Nor do we need a special operation/operator of predication, as functional application fits the bill. Hence, the logical analysis of 'The King of Bhutan is a benign ruler' will contain multiple instances of functional application; one from *The King of Bhutan* to an individual and another from *being a benign ruler* to a set. An additional instance of functional application takes the set to a truth-value, according as the individual is a benign ruler or not. If no individual is forthcoming, nor is a truth-value.[27]

The anti-actualism permeating Transparent Intensional Logic is what motivates *explicit intensionalization* and *temporalization*. The syntactic form of explicit intensionalization and temporalization consists in lambda abstraction over variables ranging over possible worlds and instants of time:

$$\lambda w \lambda t\, [\ldots w \ldots t \ldots].$$

Any formula matching this schema is to be read as follows: In any possible world (λw), at any time (λt), evaluate $[\ldots w \ldots t \ldots]$.

A Closure such as the above may be completed in this or that manner. Whichever way, though, the Closure will be a construction of a denotation (intension), which, if defined at the particular world and time of evaluation, will yield a reference. In other words, our semantics is *top-down*, from structured senses to empirical conditions. From this point there is an extra-semantic transition from empirical conditions to satisfiers (if any). As is seen, explicit intensionalization and temporalization operates with a set of worlds, whereas semantic actualism operates with one particular world. Still, the assertion that the sun is shining is obviously not to the effect that the sun is shining in some possible world or other. Rather the assertion is targeted at the actual world. And that is just the point—the link from possible-world propositions to the actual world is not mediated semantically, but pragmatically. It is by asserting a proposition (by assertorically uttering a sentence denoting it) that a speaker anchors the proposition to the actual world. Communication about matters empirical proceeds on the understanding that assertions are assertions about the actual world and the present time. Propositions (or any other types of intension) are not in and by themselves anchored to the actual world or the present time. Consider again the example of the King of Bhutan being a benign ruler. In case a truth-value is forthcoming, it is abstracted over to obtain a function from worlds and times to truth-values. Such a function is a proposition, and the assertion that the King of Bhutan is a benign ruler is to the effect that the proposition thus asserted is true in a set of possible worlds that includes the actual world at the present time.

Having introduced explicit intensionalization and temporalization, here is, briefly, how Trivialization helps us to a notion of hyperintensional attitudes. If an agent is related to $\lambda w \lambda t\, [\ldots w \ldots t \ldots]$, then the agent is related to what this Closure constructs, i.e. an intension, typically a proposition (in the case of 'propositional attitudes') or

[27] See Section 2.4.2.

else an individual role or a property (in the case of 'notional attitudes'). We say that the Closure occurs *used*, because it is used to yield an entity different from itself, namely the entity it constructs. But the whole Closure may itself be constructed, in this manner:

$$^0[\lambda w \lambda t\,[\ldots w \ldots t \ldots]].$$

We say that the entire Closure $[\lambda w \lambda t\,[\ldots w \ldots t \ldots]]$ occurs *mentioned*, because it itself is the object of discourse. Recalling the semantic schema of Fig. 1.2, the Closure is now in the position of denotation, whereas the Trivialization $^0[\lambda w \lambda t\,[\ldots w \ldots t \ldots]]$ is in the position of a construction that constructs the Closure. What the agent is related to is no longer what the Closure constructs, but the Closure itself (i.e., a procedure and not its product). Whereas empirical attitudes come in two variants, intensional and hyperintensional, mathematical attitudes are invariably hyperintensional. For instance, the attitude of *calculating* relates an individual to a Composition (rather than the outcome of the Composition). So the relevant construction must again be Trivialized: $^0[\ldots]$.

In general, since Closures and Compositions are hyperintensionally individuated, substitution of attitude *relata* will be much more restrictive than is the case with attitude logics based on set-theoretic modal logic.

The rejection of reference shift by no means implies that Tichý was blind to various both subtle and entrenched distinctions in logic. Only he accommodates them differently. Tichý claims that empirical terms and expressions exhaust their role by expressing a sense and denoting the intension that the sense yields. This holds for all contexts, such that empirical terms and expressions denote intensions and not extensions, whatever sort of semantic context they are embedded in. Once an intension has been picked out by a word, the word has fulfilled its task, and the so denoted intension can be logically manipulated. The intension may be either extensionalized or not. If extensionalized, it yields its value, if any, at the given world and time of evaluation. If un-extensionalized, it yields itself. The distinction between extensionalized and un-extensionalized intensions concerns two different ways of using (as opposed to mentioning) constructions as constituents of larger constructions. Constituent constructions occur with supposition *de dicto* or *de re*. Briefly, if *de dicto*, the so constructed intension is not extensionalized. If *de re*, it is. If the constructions do not construct intensions, then the *de dicto*/*de re* distinction is the distinction between the either intensional* or extensional* supposition that a constituent construction can occur with. Intensional* and extensional* are not the same as intensional and extensional, as the latter pair is used in possible-world semantics. The former pair applies to all constructions; the latter exclusively to constructions of intensions. When a constituent construction occurs with extensional* supposition, then the so constructed function is applied to an argument in order to obtain the corresponding value, if any. This way a property becomes attributable to a functional value. When occurring with intensional* supposition, then the so constructed function is not applied. This way a property becomes attributable to the function itself.[28]

All in all, the particular use that Transparent Intensional Logic makes of the distinction between *de dicto* and *de re* substitutes reference shift. It is of vital importance to the project of Transparent Intensional Logic that a very sophisticated and detailed conception of supposition *de dicto*/*re* be in place. Elaborating this conception has been the focus of intense research the last some years, and in this book we present the most elaborate conception to this day.[29]

In a wider context, the typed universe of Transparent Intensional Logic, with its ever-ascending hierarchy of constructions, can be seen in part as a counter-reaction to the frugal ontologies propagated by Quine and a host of others, not least under the banner of nominalism. Quine combines his pragmatism-flavoured nominalism with an extensionalist conception of semantics, according to which only extensional entities are ever denoted. Quine's final verdict on denotation is unfavourable to modalities and attitudes, not to extensionalism; Tichý draws the opposite conclusion.

One of many ways of summing up this clash is as the clash between *bottom-up* and *top-down* approaches to semantic analysis. The parallel clash over ontology is then the clash between an approach that starts out with concrete particulars and stays as close as possible to *terra firma* and an approach that starts out with abstract modes of presentation and only introduces concrete particulars in their capacity as whatever is presented in a particular manner. To express the difference metaphorically, if the former approach to semantics and ontology is *terrestrial*, the latter approach is *celestial*. So, tongue-in-cheek, whereas Isaac Newton founded a modern celestial mechanics, Tichý founded a modern *celestial semantics*.

1.2 The top-down vs. bottom-up approach to logical semantics

1.2.1 The bottom-up approach

In its broadest sense, logic is the science of correct reasoning and the art of argumentation.

Today's logic is *formal* logic. This is to say that logic investigates the *validity* of arguments irrespective of what the premise(s) and the conclusion of a given argument *mean*. It is quite another issue whether the premise(s) and the conclusion form a *sound* argument; i.e., whether the premises are *true*. The notion of truth presupposes the notion of meaning. And in order to reason we have to understand particular sentences. Since we understand a sentence by knowing its *meaning*, we

[28] Moreover, intensional* supposition is *dominant* with respect to the extensional* one. For details, see Section 2.6.

[29] This marks an advance over Tichý's stance as expounded in 1986a and 1988 (§41).

need to know what the premises of an argument mean. We agree with Frege that drawing inferences must be from sound arguments, since the point of inferring is to obtain new knowledge (the conclusion) from old knowledge (the premises). Thus analysis of language (i.e., discovering the meanings of particular expressions) is a necessary precondition for reasoning.

Historically, many logical systems developed from the simplest cases to increasingly more complicated ones. Beginning in ancient times with the logic of Aristotle and the Stoics, currently characterised as fragments of first-order predicate logic and propositional logic, respectively, many specialised logical systems have since emerged. These include, inter alia, modal logic, epistemic logic, doxastic logic, deontic logic, fuzzy logic, paraconsistent logic, many-valued logic, provability logic, temporal logic, and intuitionistic logic. How is that possible, though? Isn't there just one logic? Yes and no. In the broadest sense, there is just one logic. In a much more narrow sense, there are many logical theories of this or that. Beginning with atomic sentences, propositional logic specialises in how to compose atomic sentences into compound ones. Predicate logic investigates the structure of atomic sentences with quantifiers. If you add modalities you enter the sphere of modal logic. If you add other operators like epistemic or doxastic ones, still other logics emerge. Thus it is natural to start with the simple cases first. Let us consider some examples.

(1) 'Some prime numbers are even.'
(2) 'Some odd numbers are even.'
(3) 'Some clever students are lazy.'

If analyzed in first-order predicate logic, one formula analyses all three sentences:

$$\exists x\, (P(x) \wedge Q(x)).$$

As it stands, the formula is neither true nor false. It is only a syntactically well-formed formula, which cannot be evaluated unless and until meanings have been assigned to P and Q and a functional range to x, it is just a pattern for applying particular symbolic inference rules. Thus we can infer, e.g., the formulae '$\exists x\, P(x)$' and '$\exists x\, Q(x)$'.

In order to decide whether the formula is true or false, we have to interpret it first. On some interpretations it is true, on others it is false. Interpreting P, Q over the universe of numbers as the set of prime numbers and even numbers, respectively, it come out true. Interpreting the same symbols as representing odd numbers and even numbers, it comes out false. And interpreting the symbols P, Q, e.g., as a set of clever students and lazy students, respectively, over some universe of individuals, it is either true or else false according as these sets share a non-empty intersection.

This sort of analysis is worrisome. First, why do all the above sentences receive one and the same analysis? Sentence (1) is analytically and provably (hence, necessarily) true, whereas sentence (2) is analytically and provably (hence, necessarily) false. Sentence (3) is only contingently true, and so requires empirical inquiry to establish its actual truth-value. The formula is true on some interpretations and false on others. Second, in what way does such a translation of a perfectly well-understandable natural-language sentence into a symbolic formula make its *meaning* clear?

Consider further the sentences

(4) 'No bachelor is married.'
(5) 'No bachelor is rich.'

The identical formula analyzing both sentences would be

$$\forall x\, (P(x) \supset \neg Q(x)),$$

or equivalently,

$$\neg\exists x\, (P(x) \wedge Q(x)).$$

While (4) is analytically true, (5) is contingently true or false. Since neither formula is logically valid, one may again wonder how it is possible that two so semantically different sentences lend themselves to one and the same logical analysis (whether the analysis be $\forall x\, (P(x) \supset \neg Q(x))$ or $\neg\exists x\, (P(x) \wedge Q(x))$).

The standard answer is that it is not the point of first-order predicate logic to deal with *empirical sentences* like (3) and (5). This logic was designed for the purpose of mathematical reasoning. First-order predicate logic was designed to prove theorems, not to spell out what theorems mean, so as long as (1) and (2) have the same consequences, there is no need to assign different formulae to them.

But first-order predicate logic is standardly used to analyse empirical sentences. This practice creates a mismatch between the analytic tool and what is to be analysed. The analyses above are too coarse-grained, as well as being ambiguous. These difficulties would be neglectable if we could always infer the correct consequences from the premises. Unfortunately, we cannot. An up-dated puzzle of old shows why:

Necessarily, 8 is greater than 5
The number of planets equals 8

Necessarily, the number of planets is greater than 5.

We just used Leibniz's law of substitution of identicals to infer from true premises a false conclusion. Paradox! Modal logic sorts out the fallacy, though:

$$\frac{\begin{array}{c}\Box\, G(8, 5)\\ n(p) = 8\end{array}}{\Box\, G(n(p), 5).}$$

The conclusion is not derivable, just as we desired. '$G(8, 5)$' occurs within the scope of a modal operator, and we must not substitute co-extensional terms into contexts governed by a modal operator. But we are left in the dark as to *why* not. A rule is required that suspends the applicability of Leibniz's Law in precisely circumscribed cases. Without such a rule available to us, blocking an argument such as this remains ad hoc. As with solutions ad hoc in general, while they may succeed in alerting us to the fact *that* there is a problem, they fail to show *how* to solve the problem. Little logical insight is to be garnered from a mere ban on substituting into modal contexts.

Another problem concerning this solution is what the *meaning* of the modal operator $\Box$ is. Obviously, it is not a property of the truth-value **T**, though '$(8 > 5)$' denotes **T**. One may grant that the 'language' of modal logic is a handy shorthand and still suspect that it hardly provides a transparent analysis. Furthermore, the following fallacies cannot be blocked by modal logic:

$$\frac{\begin{array}{c}\text{John McCain wanted to become the President of the USA}\\ \text{Barack Obama is the President of the USA}\end{array}}{\text{John McCain wanted to become Barack Obama.}}$$

$$\frac{\begin{array}{c}\text{Oedipus sought the murderer of his father}\\ \text{Oedipus is the murderer of his farther}\end{array}}{\text{Oedipus sought Oedipus.}}$$

We have to switch to a system of some intensional logic in order to render the fact that 'to become' and 'to seek' establish intensional contexts that are not to be substituted into. If B is an attitudinal operator, the shared analysis is

$$\frac{\begin{array}{c}B(a, f(b))\\ a = f(b)\end{array}}{B(a, a).}$$

Again, the undesirable substitution is said to be blocked, because the substitution of 'a' for '$f(b)$' in a context preceded by B is banned. But *why* and *how*? What is the meaning of the operator B? Obviously, B does not stand for a relation between two individuals; an individual cannot become another individual, unless it would somehow bizarrely alter its identity. Yet '$f(b)$' does denote an individual.

In general, a ban on substitution will cure the symptom, but not the disease. Addressing the underlying problem requires formulating a non-circular, independently motivated rule to regulate substitution in intensional contexts.

Another fallacy is this famous example calling for deontic logic:

The letter ought to be delivered
If the letter is delivered, then it is delivered or burnt

The letter ought to be delivered or burnt.

O a deontic operator, the argument goes into

$$\frac{\begin{array}{c}O(d(a)) \\ d(a) \supset (d(a) \vee b(a))\end{array}}{O(d(a) \vee b(a)).}$$

O blocks the undesirable application of *modus ponendo ponens*—somehow. However, consider this variant:

The letter ought to be written and delivered
If the letter is written and delivered, then it is delivered

The letter ought to be delivered.

$$\frac{\begin{array}{c}O(w(a) \wedge d(a)) \\ w(a) \wedge d(a) \supset d(a)\end{array}}{O(d(a)).}$$

Why it is that this time around O does not block the application of *modus ponens*? What is the meaning of O? What does the operator operate on? Certainly not on a truth-value; the property of being ordered has to be ascribed to a *proposition*, not to a truth-value. Thus, though the standard version of deontic logic is an extensional first-order logic, it should actually be an intensional logic.

However, none of the standard logics deal with the problem of *existence*, since existence is simply assumed. Consider Russell's classical example:

The King of France does not exist.

As the King of France does not exist, it is not true that the King of France is bald. And since it is not true that the King of France is bald, the King of France is not bald. Since the King of France is not bald, it follows that there is somebody who is the King of France and who is not bald. Finally, from this it follows that the King of France exists.

What went wrong? First-order logic can provide no diagnosis of the fallacy involved. The formula corresponding to both the sentence ‘It is not true that the King of France is bald’ and the sentence ‘The King of France is not bald’ is ‘$\neg B(k(a))$’. The only standard answer would be that ‘$k(a)$’ is not a well-formed term, because it is non-denoting. But what is, in fact, needed to block Russell’s argument from going through is a logic of *partial* functions. Only this involves a departure from a logic that tolerates only total functions.

This example mixes existence and modality:

$$\frac{\text{Necessarily, the King of France is a king}}{\text{The King of France is necessarily a king.}}$$

The premise is (necessarily) true if read *de dicto*. The conclusion is (necessarily) false or else undefined if read *de re*. So the argument is invalid. But the notation of modal logic analyses both the premise and the conclusion as ‘$\Box P(k(a))$’, which does not render the difference between necessity *de dicto* and necessity *de re*. So the invalidity of the argument is obfuscated by the notation.

This is not to say that modal logic cannot distinguish, in general, between necessity *de dicto* and *de re*; of course, it can. For instance, it easily manages to distinguish between necessitating a consequence and necessitating a consequent, as in

$$\frac{\Box\ \forall x\ (x \text{ is the King of France } \supset x \text{ is a king})}{\forall x\ (x \text{ is the King of France} \supset \Box\ (x \text{ is a king})).}$$

The argument comes out invalid, because it trades a premise sporting necessity *de dicto* for a conclusion sporting necessity *de re*. So that is good. What is not good is that this argument is an analysis of another pair of sentences than {‘Necessarily, the King of France is a king’, ‘The King of France is necessarily a king’}, namely {‘Necessarily, for all x, *if* x is the King of France *then* x is a king’, ‘For all x, *if* x is the King of France *then*, necessarily, x is a king’}. These two pairs are nowhere near to being equivalent, not least because the second pair incorporates implication and universal quantification, and the first one does not. The second argument simply does not qualify as a logical analysis of the first pair of sentences and is insofar irrelevant.

A*ttitudes* are another notorious troublemaker. They force us to switch to some epistemic, doxastic, etc., logic. Here is a standard example.

Charles believes that if it is raining then the street is wet
(If it is raining then the street is wet) iff
(if the street is not wet then it is not raining)

Charles believes that if the street is not wet then it is not raining.

A case can be made for the validity of this argument, as well as for its invalidity. If Charles' attitude concerns an empirical state-of-affairs then his attitude is not sensitive to whether its complement (what is believed) is a proposition or its contraposition. If, on the other hand, his attitude concerns a particular way of conceptualising or presenting an empirical state-of-affairs, then there are strong reasons for blocking the argument. One thing is to believe one conceptualisation or presentation of a state-of-affairs, quite another thing is to believe another such conceptualisation. *Ex hypothesi*, Charles agrees to the first conceptualisation, but he may dissent from, or have no opinion about, the one occurring as complement in the conclusion.

However, consider another example:

Charles knows that Thelma is happy

Charles knows that (Thelma is happy and whales are mammals).

It may be the case that the first sentence is true whereas the second is false. Yet the standard possible-world semantics of epistemic logic yields the result that the second sentence must be true as well, 'Thelma is happy' and 'Thelma is happy and whales are mammals' being analytically equivalent. This is due to the fact that the proposition that whales are mammals is the necessary proposition *TRUE*, which takes the truth-value **T** for all possible worlds and times. Provided (as we are supposing) we understand the meaning of 'is a whale' and 'is a mammal' as these predicates are used in current English, if an individual is known to be a whale, we need not (empirically) examine the state of the world in order to get to know that the individual is a mammal.

In the standard notation of epistemic logic, the premise and the conclusion above become

$$\frac{K_a\, H(b)}{K_a\, [H(b) \wedge \forall x\, (W(x) \supset M(x))].}$$

But in the epistemic systems based on Kripkean possible-world semantics, this variant of *epistemic closure* holds:

$$\text{If } (M, w) \models K_a\, \varphi \text{ and } (\varphi \models \psi)\text{, then } (M, w) \models K_a\, \psi.$$

If *a* knows an empirical proposition, then *a* also knows everything logically implied by it. And *a* immediately knows all analytical truths as well, because they follow from the empty set of assumptions; or semantically put, they are true in every possible world.

Hence, when knowing that Thelma is happy, Charles is bound to know that Thelma is happy and that whales are mammals. And he is bound to know all mathematical truths as well, because they are analytically true, hence either true throughout all logically possible worlds or true independently of worlds altogether.

Here is an example demonstrating the difference between beliefs *de dicto* and *de re*:

‘Charles believes that the King of France is a king.’

‘Charles believes of the King of France that he is a king.’

Whereas the first sentence may be true, the second sentence cannot be true, as long as there is no King of France. The standard advice is to turn to doxastic logic:

$$B_b P[k(a)] \qquad (\textit{de dicto})$$
$$\lambda x\, B_b P[x]\, k(a) \qquad (\textit{de re}).$$

Again, worrisome questions arise. β-reduction converts the two analyses into one and the same formula. Why aren’t we allowed to execute *the* basic computational rule of the λ-calculi in this case? The standard answer would be, ‘Because the term ‘$k(a)$’ is non-denoting’. But how can we know that the term is non-denoting and, thus, not well-formed? On another interpretation the same term will be a perfectly well-formed term. It does not seem right that the vicissitudes of the empirical world should make a difference as to whether a term is well-formed.

Or for a variant analysis:[30]

$$B_b P[k(a)] \qquad (\textit{de dicto})$$
$$(\exists x)\, (x = k(a) \wedge B_b P[k(a)] \qquad (\textit{de re}).$$

Where does the existential quantifier come from in the *de re* case? There is no trace of it in the original sentence. How can the logical forms of two similar sentences differ so radically? Hintikka and Sandu propose in 1996 a remedy by means of Independence Friendly first-order logic:

> Independence Friendly (IF) first-order logic deals with a frequent and important feature of natural language semantics. Without the notion of *independence*, we cannot fully understand the logic of such concepts as belief, knowledge, questions and answers, or the *de dicto* vs. *de re* contrast (1996, p. 173).

[30] See Hintikka and Sandu (1989).

They solve the *de dicto* case as above, and propose the *de re* solution with the independence indicator '/':

$$B_b\, P[k(a)\ /\ B_b].$$

This is certainly a more plausible analysis, closer as it is to the syntactic form of the original sentence. Furthermore, the independence indicator indicates the essence of the matter; there are two *independent* questions: 'Who is the King of France (if *k(a)* is interpreted as the King of France)?' and 'What does Charles think of that person?'. Of course, Charles needs to have a relation of 'epistemic intimacy' (cf. Chisholm, 1976) to a certain individual, but he need not make the connection between this person and the office of King of France (though the ascriber must). Still, the semantics of '$/B_b$' is not pellucid, which tells against it as a tool suitable for logical analysis. We will show that informational independence can be precisely captured by means of TIL's explicit intensionalization and temporalization without invoking any new non-standard operators.[31]

We consider it a non-negotiable datum to be respected by any viable attitude logic that attitudes *de dicto* and *de re* do not turn out to be equivalent. But it won't suffice for a given theory of attitude logic to simply point out the non-equivalence and ban conversion, again because a ban must be backed up by a logical insight into *why* conversion will fail to preserve equivalence. The following example serves to motivate the non-equivalence between attitudes *de dicto* and *de re*:

'Charles believes that the President of Zimbabwe is an absolute despot.'
'Charles believes of the President of Zimbabwe that he is an absolute despot.'

These two sentences do not denote the same proposition, for their truth-conditions differ. Charles might have read in a reliable newspaper, and so have come to believe, that the President of Zimbabwe is an absolute despot, thus making the first sentence true. However, Charles may have no idea as to who the President of Zimbabwe is, nor whether this particular individual is a despot. In such a situation the second sentence is not true. Or, another scenario is imaginable: Charles is acquainted with someone who happens to be the President of Zimbabwe, and Charles believes that his acquaintance is a despot, without having the slightest idea that this person is the President of Zimbabwe. In such a situation the second sentence is true and the first false.

Regrettably, the standard notation of doxastic logic deployed above does not reveal the difference in meaning between these two sentences. If '$k(a)$' is a denoting term, then the two formulae come out equivalent. The only way out of this predicament seems to be to heed the advice not to use the β-rule here, because the variable x occurs within the scope of the doxastic operator 'B'. The fact *that* x occurs within the scope of B is unquestionably the source of the trouble. But *why*

[31] See Section 5.1.2.

does x's occurrence within the scope of B invalidate β-transformation? This is the question that the logical semanticist must answer.

Qualms about substitution within attitude contexts motivate the need to ascend from intensional logic to *hyperintensional* logic. Here is an example in which it is indisputable that hyperintensional attitude complements are called for.

Charles calculates 2+5
2+5 = 7

Charles calculates 7.

It is no option to relate Charles to possible-world intensions. Their granularity is far too crude for them to figure as complements in mathematical attitudes. Thus, Charles would be related to a constant function from possible worlds and instants of time to a number. This grossly misrepresents what the activity of calculating is all about, which is to apply arithmetic operations to numbers. Finer granularity that would block the undesirable derivation would relate Charles to the expression '2+5'. Yet Charles cannot be related to a piece of mathematical *notation.* The argument does not say what syntactic transformation Charles performs in order to calculate the sum of 2 and 5. In the case at hand Charles calculates 2+5 by applying the addition function to the pair of numbers (2, 5). Besides, the conclusion is either false or nonsensical, depending on what sense can imaginably be made of calculating an individual number. Yet also this argument has the airs of a valid argument.

All the arguments above are *puzzles*. If there is a definition of *puzzle*, it is that a puzzle is an argument that takes premises individually considered true to conclusions that are indisputably false or else nonsensical. Hence, a puzzle threatens to trade (seeming) truths for either falsehoods or nonsense. In general, puzzles flow from two different sources. Either the logical form of one or more premises is ill-understood, or an otherwise valid rule of inference is applied outside its domain. (Of course, a puzzle may well flow from both sources.) The solution to a puzzle consists, thus, in blaming either the analysis of one or more of the premises or the rule of inference (or both). If one blames the rule of inference, one thereby claims to have discovered that, in the cases at hand, Leibniz's Law is valid only in some contexts. If one blames the analysis of the premises, one thereby claims to have discovered that Leibniz's Law does not apply, because the argument in question fails to have the appropriate logical form for it to apply. Our strategy throughout is to find fault, not with Leibniz's Law, but with how one or more premises of a given argument are logically analysed. The logical forms of the premises of the arguments above (as well as those of many others considered in this book) will turn out to be somewhat more complicated than predicted by first-order logic. This is in itself hardly a revolutionary claim; but what is innovative about our approach is that it offers an exact calibration of the degree of complexity of particular premises and conclusions.

If we start with first-order predicate logic (FOL), then what we have is a system that is broadly known, well-researched and profoundly elaborated. There are sound and complete calculi for this logic, such that all the logically valid formulas of FOL are provable. Though the system is not decidable, it is partially decidable: if a formula is logically true then there are algorithms that would answer Yes in a finite number of steps when inputting such a formula. The language of FOL has become the language of mathematics. Attractive mathematical theories have been couched in this language, and their properties are well-known.

But, there is only so much one can use FOL to. The shortcomings of FOL can be briefly summarised as follows. First, it is an *extensional* system. Though this is in itself no shortcoming, this fact does not make it possible to distinguish between analytical and empirical expressions. The difference is that the reference of the latter is dependent on modal and/or temporal parameters. Thus there is a need for an intensional system in the vein of possible-world semantics.

Second, FOL is a *first-order* system. This fact does not make it possible to systematically distinguish between ascribing a property to a function as a whole (like in 'Sinus is a periodic function') and ascribing a property to a particular functional value (as in '$\sin(\pi) = 1$'). Another example: 'Charles is incorruptible' versus 'Being incorruptible is an honourable property'. We need a higher-order system.

Third, FOL is a system working with *total* functions only. However, in order to work with empty concepts and functions not returning values at some arguments, as well as the problems of empirical (non)existence, and value gaps in mathematics, what is needed is a logic of partial functions.

Fourth, FOL is a system whose universe is always *one-sorted*, while allowing one sort to be replaced by another. However, one needs to be able to distinguish distinct types of entities that the system talks about. There is certainly a categorial difference between an individual role such as The King of Bhutan and any of the extensions of this intension, which are individuals. Similarly, there is certainly a categorial difference between a numerical function and any of its arguments or values, which are numbers. Thus, one is better off switching to many-sorted logics. And if, moreover, one needs to distinguish between modal and temporal parameters, as in 'The President of the USA might not have been a president' and 'The President of the USA is often a Republican', one needs to switch to modal logics, temporal logics, etc.

Thus we need increasingly expressive logical systems—only to realize sooner or later that there is always something missing. Today, as a result, we have ended up with a sprawling tree whose branches are particular logics. Certainly, no single logic can render all the features of natural language. Furthermore, these individual logics are well elaborated from the formal point of view. Starting with an alphabet, grammatical rules determine a set of well-formed formulae. Having thus defined the syntax of a formal language, we choose a subset of the set of well-formed formulae as axioms, and specify the rules of inference by choosing a finite set of sequences of formulae. Finally, the so defined theory is investigated for its interesting mathematical/logical properties. We ask whether a theory is consistent and, thus, has a model, whether it is complete, whether the underlying calculus is complete, etc. As a result, instead of natural language we find ourselves studying the formal language itself.

This is unquestionably an interesting and legitimate task of logic and mathematics. Indeed, some of the greatest achievements of twentieth-century logic and mathematics are meta-mathematical, including meta-logical, insights into the properties of particular sets of well-formed formulae (*wff*'s). Yet you may ask: How does such a translation of a natural-language sentence into a shorthand formula contribute to the *analysis* of the sentence? In what way does it cultivate our *reasoning*? The answer would be, 'By following the formal axioms and rules of a given theory you obtain the logical consequences of its axioms'. But then one has to correctly *interpret* the theory in order to use it to solve a particular problem. Moreover, *which* particular theory should an agent apply in this or that case, and *how* should the resulting formulae be interpreted?

Still, if this panoply of logics is indispensable for something beginning to look like a full theory of natural language, and if the individual logics are technically precise, do we not have, as working logicians, all we need to go about our business of logically analysing fragments of natural language? Yes and no. We do have some logic or other available for almost all particular kinds of context involving particular problematic expressions. But what we do not have is an overarching, unitary logic.

Imagine one is building up a multi-agent system of autonomous, intelligent agents who are to communicate by exchanging messages, and who make decisions based on the content of these messages. Each message may concern a particular problem; thus the agents would have to keep switching between logical systems. They would have to combine modal logics, epistemic logics, temporal logics, provability logics, and so on and so forth. But inter-translatability forms a stumbling-block, since the same connectives may not preserve meaning when switching between logics. Agents may end up speaking at cross purposes.

Thus, in our opinion, in a multi-agent world of the Semantic Web, information and communication technologies (ICT), artificial intelligence (AI), and other such facilities, there is a pressing need for a *universal framework* informed by one philosophical logic making all the semantically salient features of natural language explicit. Consequently, such a universal logical framework would and should make a fine-grained logical analysis of relevant premises possible to create a platform for an ideal inference machine that neither over-infers (yielding consequences not entailed by the premises) nor under-infers (failing to yield consequences entailed by the premises).

The ambition of TIL is to provide such a universal framework. The purpose of this book is to display the framework in all its might. The TIL 'language of constructions' is not a formal language of non-interpreted terms. It is formal, if by 'formal' we mean rigorously defined and employing a special notation. But the individual terms and the entire language are themselves not the subject of our study. Rather the terms of the 'language of constructions' unambiguously encode logical constructions, and these extra-linguistic procedures are the ultimate subject matter of our study.

1.2.2 The top-down approach

We mentioned in Section 1.1 that TIL generalises from the hardest case and obtains the less-hard cases by lifting various restrictions that apply only higher up. This way of proceeding is opposite to how semantic theories tend to be built up. As we illustrated in Section 1.2.1, the standard approach consists in beginning with atomic sentences, proceeding to molecular sentences formed by means of truth-functional connectives or by quantifiers, and from there to sentences containing modal operators and, finally, attitudinal operators.

Thus, to use a simple case for illustration, once a vocabulary and rules of formation have been laid down, a semantics gets off the ground by analysing an atomic sentence as follows:

(1) 'Charles is happy'
Fa

And further upwards:

(2) 'Charles is happy, and Thelma is grumpy'
$Fa \wedge Gb$

(3) 'Somebody is happy'
$\exists x\ (Fx)$

(4) 'Possibly, Charles is happy'
$\Diamond\ (Fa)$

(5) 'Thelma believes that Charles is happy'
$\mathrm{B}b\ (Fa)$.

In non-hyperintensional (including non-procedural) theories of formal semantics, attitudinal operators are swallowed by the modal ones, typically with '$\Box$' standing for knowledge and '$\Diamond$' for belief (as in the so-called modal logic of knowledge and belief). But when they are not, we have three levels of granularity: the coarse level of truth-values, the fine-grained level of truth-conditions (propositions, truth-values-in-intension), and the hyper-fine-grained level of hyperpropositions (propositional constructions).

TIL operates with these three levels of granularity (in fact, adding a fourth level of granularity, slightly coarser than that pertaining to constructions, in terms of *concepts*; see Section 2.2). We start out by analysing sentences from the uppermost end, furnishing them with a hyperintensional semantics, and working our way downwards, furnishing even the lowest-end sentences (as well as non-sentential expressions) with a hyperintensional semantics. That is, the sense of an atomic sentence such as ‘Charles is happy’ is a hyperproposition, i.e., a propositional *construction*, due to the trickle-down effect of our top-down approach. Likewise, the sense of ‘1+2=4’ is a *construction* of a truth-value.

Our motive for working top-down is pivoted on anti-contextualism: any given term or expression expresses the same construction as its sense in whatever sort of context the term or expression is embedded within. As for denotation, in the case of non-denoting expressions (mathematical expressions expressing improper constructions) it holds that such an expression does not denote anything in any context. Further, some terms, like indexicals, express only what we call ‘pragmatically incomplete meanings’[32] and, therefore, only denote relative to a valuation, being insofar sensitive to which context they are embedded in. All remaining terms do denote, though, and have context-insensitive denotations.

Furthermore, the sentence ‘Charles is happy’ is an intensional context, in the sense that its logical analysis must involve reference to empirical parameters, in this case both possible worlds and instants of time. One reason is because Charles is only *contingently* happy; i.e., he is only happy at some worlds and only sometimes. The other reason is because the *analysans* must be capable of figuring as an argument for functions whose domain is made up of propositions rather than truth-values. Construing ‘Fa’ as a name of a truth-value works only in the case of extensional contexts like (1) and (2). It won’t work in modal contexts like (4), since truth-values are not the sort of thing that can be possible. Nor will it work in a *hyperintensional* context of knowing or believing, since truth-values are not the sort of thing that can be known or believed. The sentence ‘Charles is happy’ is a hyperintensional context, as soon as Thelma’s art of believing relates her to a hyperproposition.

A logical syntax cannot tolerate ambiguous terms. The historical culprit for the notation found in the *analysantes* of (3), (4) and (5) must, in our view, be the conception of modalities due to the original syntax of ‘$\Box$’, ‘$\Diamond$’, which treats ‘$\Box$’, ‘$\Diamond$’ as being syntactically on a par with ‘$\neg$’; both ‘$\neg p$’ and ‘$\Box p$’ are well-formed formulae. This makes for handy notation, but it remains implicit that the argument of $\neg$ is a truth-value of p and the argument of $\Box$, p itself, i.e., the entire function. If ‘K’ (denoting an epistemic operator) is introduced as a notational variant of ‘$\Box$’ we get formulae like ‘Kp’, and we are allowed to generate strings like, ‘$\neg p \wedge K\neg p$’, where the extension/intension ambiguity of the notation is manifest. Moreover, if K is a hyperintensional operator, and $\Box$ an intensional operator, then we are in for three-way ambiguity as in, ‘$(\Box p \rightarrow p) \wedge Kp$’.

Tichý also bemoans the inherent ambiguity of the syntax of modal logic:

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[32] See Section 3.4. Though *incomplete* is, strictly speaking, a privative modifier, such that an incomplete meaning would not be a meaning, by ‘pragmatically incomplete meaning’ we intend, stipulatively, a meaning that is an open construction with free variables.

> [T]he modal logician keeps us in the dark…about [the logical type of □]. His axioms are framed in terms of p's and q's – as in '$\Box(p\supset q)\supset(\Box p\supset\Box q)$' – but it is entirely unclear what these variable-letters are meant to range over. The fact that they combine with truth-functional connectives like '⊃' might suggest that they range over the truth-values. This, however, is hardly compatible with their combinableness with '□' (Tichý, 1988, p. 279).

And he notes elsewhere that

> [S]tandard first-order logic is only capable of dealing with propositional constructions *de re*: negation, conjunction, alternation and the like. Propositional construction[s] *de dicto*, especially modal, probabilistic, epistemic, deontic, subjunctive, and causal constructions, are far beyond the reach of first[-]order logic. All attempts to force such constructions on to the Procrustean bed of first-order idiom are, in my view, doomed to failure (Tichý, 1978a, p. 10; 2004, p. 258).

It is worth dwelling on the topic of typing for a minute. Our perhaps pedantic-seeming harping on notational tidiness is grounded in a contentual issue of wide-ranging importance; namely, what we just said, that a logical syntax cannot tolerate ambiguous terms.

On our diagnosis, a bottom-up approach to modalities and attitudes is bound, it seems, to acquiesce in ambiguous notation and context-sensitive reference shift. This amounts, in effect, to operating with several semantic theories, one for each sort of semantic context. A top-down approach holds out the prospect of one semantic theory for all sorts of semantic context. The methodology consists in starting out on the top floor with a hyperintension and then either staying there or, if the semantic analysis requires it, taking the lift down and getting off either at the floor of intensions or at the floor of extensions.[33] Since we start out at the top, we start out with *constructions*, which we define next.

1.3 Foundations of TIL

1.3.1 Functional approach

The fundamental notions in terms of which a system is built up cannot be defined *in* the system itself, but must be understood prior to the theory and are introduced into the theory as primitives. So, for example, predicate logics are built up in terms of sets and relations. By contrast, the fundamental notion for TIL is the one

[33] We are making a simplification here to get the top-down picture clear. As a matter of fact, there are several floors of hyperintensions, intensions and extensions to get off at. In particular, while you always start out at the top, at *a* level of hyperintensions, there are going to be floors of hyperintensions above the floor you are on. Furthermore, the floor you get off at may itself be one of hyperintensions (though a floor one level down from where you started out). On the other hand, the vast bulk of empirical cases that we analyse in this book conform to the picture of starting out with a hyperintension, descending to the intension it presents and then descending from intension to extension. 'Charles is happy' would be a case in point.

of *function*.[34] This seemingly banal fact *is* important. Functions—unlike relations or sets—are *procedure-friendly* in the following sense:

(i) for any n-ary function *qua* mapping $M_1 \times \ldots \times M_n \rightarrow N$ there is an abstract procedure (often called *abstraction*) that produces at every n-tuple of elements of $M_1,\ldots,M_n$, respectively, at most one member of N;
(ii) the reverse procedure *applies* the mapping $M_1 \times \ldots \times M_n \rightarrow N$ to a particular n-tuple of elements of $M_1,\ldots,M_n$, respectively, and produces either nothing (if the mapping is undefined at that tuple) or the value of the mapping at that tuple.

Moreover, contemporary mathematics and logic define functions as *mappings*; i.e., as a special kind of set. The principle of extensionality is what guarantees this set-theoretical character of functions. Where *f, g* are functions the Principle says:

$$\forall x_1 \ldots x_n\, (f(x_1,\ldots,x_n) = g(x_1,\ldots,x_n)) \supset f = g.$$

On the other hand, as it is documented, e.g., in Tichý,

> Originally functions were understood as particular ways or methods of proceeding from numbers to numbers, i.e., as incomplete numerical constructions (1988, p. 3).

So

> [I]n order to properly grasp the modern notion of function one must keep it strictly apart from the notion of schematic calculation. ... one must always remember that the method is extraneous to the function itself (ibid).

Indeed, any function *qua* mapping can be constructed in infinitely many ways. Not distinguishing functions from methods is a source of many wrong turns in semantics, as will be shown when applying TIL to puzzle-solving.

Another reason for preferring functions to relations is *partiality*. A partial function f may return no value at some n-tuples. The corresponding relation R_f is the set of $(n+1)$-tuples, i.e., the subset of the respective Cartesian product. But among the $(n+1)$-tuples that are elements of the *complement* relation, one is not able to distinguish those which do not belong to the relation R_f (due to the fact that the respective entity is not a value of f at the argument) from those at which the function is undefined.

A simple example. Let f be a function that maps $M = \{a, b, c, d\}$ onto $N = \{\alpha, \beta, \gamma\}$ as follows: $a \rightarrow \beta$, $b \rightarrow \gamma$, $d \rightarrow \alpha$; at argument c function f is undefined. The respective relation R_f contains three of the twelve possible couples: $\{\langle a, \beta\rangle, \langle b, \gamma\rangle, \langle d, \alpha\rangle\}$. Now, although we know that, e.g., $\neg R_f(a, \gamma)$ and $\neg R_f(c, \alpha)$, the difference

[34] Also Montague (1974a), together with other semanticists, has opted for the functional approach and adopted a typed λ-calculus for his logical analysis of natural language.

between *f* being defined at *a* and undefined at *c* is lost. We cannot deduce whether the value of *f* exists at *c* or not.[35]

Finally, the functional approach is connected with the idea that any logical analysis of natural language should obey *compositionality*, which comes down to explaining the semantic behaviour of compounds in terms of the semantic behaviour of their components.[36] Obviously, our concern with partiality is part of a wider concern with compositionality. A term that has no reference (as opposed to denotation) affects the semantic behaviour of the compound it is part of. The challenge for a theory like ours which wishes to heed both the partiality constraint and the compositionality constraint becomes how to avoid that the semantic analysis of a compound comes to a standstill if one or more constituents contribute nothing at the level of denotation or reference. The way we tackle the challenge is, not surprisingly, by having non-referring terms contribute something at another level. *All* terms contribute a sense to the compounds they are constituents of; but some terms contribute only a sense.

The reasons just outlined explain why we are using a Frege-Church-style function/argument logic. The philosophical as well as logical advantage of a logic based on functions is that it can model interlocking logical structures in terms of functional dependencies. Functional dependencies are modelled by how the value of one function becomes the argument of another function, or how a function applicable to some particular argument is handed down by another function. A logic of functions is erected on the idea that one operation typically presupposes that another operation has already been executed so as to provide something to work with. As mentioned *ad* (i) and (ii) above, the functional operations are two in number—*application* and *abstraction*—of which the former 'descends' from a function to a value, while the latter 'ascends' to a function from other entities (perhaps including other functions). It is a key characteristic of the logic we are advocating that the outcome of the execution of an operation may itself be an operation. Otherwise the machinery would grind to a halt far too soon.

Our functional approach affects also how we think of language. We adhere to the Fregean tenet that every sentence contains at least one functor. For instance, we construe predicates as functors.[37] Predicates denote functions whose argument(s) must be picked out by some other expression(s) of the sentence. For instance, in 'Charles is happy', 'is happy' is the functor and 'Charles' the argument expression.

So why not settle for functions as meanings? For several reasons, each of which is conclusive. First, functions are too crudely individuated to qualify as hyperintensions. Functions are extensionally individuated, so possible-world intensions are

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[35] We will deal with partiality in detail in Sections 2.6 and 2.7, where the need for partial functions is demonstrated together with a specification of inference rules for working with them.

[36] See Section 2.1.2 and also Tichý (1988, p. 287).

[37] —which is to say that we adhere to 'the Fregean doctrine that predicates name functions', as Bealer says (1982, p. 89).

individuated up to co-intensionality. Second, the operations of abstraction and application are exterior to functions and cannot be captured in terms of functions. Functions are not themselves procedures; functions can, and do, instead figure as input and output of procedures. Third, functions are set-theoretic entities and so cannot have parts. So it is not obvious how the account of compositionality, including partiality, is supposed to proceed. Fourth, functions cannot figure as modes of presentation. For sure, one can attempt to strain the notion of function and make it play the role of mode of presentation. But who wants a poor man's modes of presentation? Functions are sets, so it takes some charity to accept that the Cartesian product $A \times B$ would qualify as a presentation of, say, the mapping of a particular argument $a \in A$ onto a particular value $b \in B$. Any such correspondence between a and b records merely the fact *that a* is mapped onto b, but not *how*. Countless many procedures for mapping a onto b can be reconstructed; but none in particular. Yet a key reason for introducing modes of presentation is that there may be two or more clearly circumscribed modes of presentation of the same thing.

To anticipate a possible misunderstanding, note that in the semantics of mathematics, the terms 'function-in-intension' and 'function-in-extension' are sometimes used. For instance, Church (1941) broaches the question under which circumstances two functions are to be considered the same. He says:

> The most immediate and, from some points of view, the best way to settle this question is to specify that two functions f and g are the same if they have the same range of arguments and, for every element a that belongs to this range, (fa) is the same as (ga). When this is done we shall say that we are dealing with *functions in extension.*
>
> It is possible, however, to allow two functions to be different on the ground that the *rule* of correspondence is *different in meaning* in the two cases although always yielding the same result when applied to any particular argument. When this is done we shall say that we are dealing with *functions in intension*. The notion of difference in meaning between two rules of correspondence is a vague one, but, in terms of some system of notation, it can be made exact in various ways. We shall not attempt to decide what the true notion of difference in meaning is but shall speak of functions in intension in any case where a more severe criterion of identity is adopted than for functions in extension. There is thus not one notion of function in intension, but many notions; involving various degrees of intensionality (1941, pp. 2–3; emphasis ours).

Function-in-extension corresponds to the modern notion of function as a mapping, and function-in-intension could arguably correspond to our notion of *construction of a function.* However, since the notion of function-in-intension is a vague one, and obviously dependent on the formal system in which the meaning of the correspondence rule is captured, we will *not* use the term 'function-in-intension'. There is no reason for us to trade the crisp notion of construction (of a function) for the vague one of function-in-intension. But vague though it may be, its vagueness is in part owed to the 'various degrees of intensionality' that Church wants his overarching notion of function-in-intension to encompass.

Two degrees are minimally required to get the programme of a general semantics for natural-language discourse off the ground. The first is the degree made available by possible-world semantics, which individuates its intensions up to logical equivalence (cf. Church's functions-in-extension). The second is the hyperintensional degree. Only there is no such thing as *the* hyperintensional degree. Both Church and Cresswell define hyperintensionality negatively as any individuation finer than logical equivalence. The question, then, is how austere or how lax a degree of individuation we as semanticists need to impose when analysing a piece of language. We neither want the possible need for very fine-grained hyperintensionality to outstrip our logical resources to meet the need, nor do we want to arbitrarily impose just one degree of hyperintensionality.

So what we do is take the TIL constructions and have them serve as the most fine-grained hyperintensions available to us. If one imagines a hierarchy of hyperintensions, with the most fine-grained ones at the top, then one moves down the hierarchy by forming equivalence classes of more fine-grained hyperintensions and obliterating the differences among their individual members. This is how we arrive at our rigorous notion of *concept*. *Concepts* have a particular degree of hyperintensionality, and this degree seems, by and large, to be what we are looking for. What we are looking for are higher-order objects that satisfy the following criterion of hyperintensional individuation: any two hyperintensions are identical exactly when they are procedurally indistinguishable. The idea of procedure that guides us is, in general, that a procedure prescribes what to do to what entity or entities in what order to obtain what sort of entity. It seems natural to us to hold, then, that two expressions are synonymous just in case their respective meanings prescribe one and the same procedure. We find it hard to imagine what might be the semantic or logical import of a principle of hyperintensional individuation finer than procedural individuation.[38] Procedural individuation is pretty fine-grained, anyway. Yet the research project of laying down just how hyper hyperintensionality is must respect the fact, *vide* Church, that hyperintensionality is an open-ended cluster concept. What TIL contributes to this project is an intuitive principle of individuation (a procedural one) and higher-order entities that satisfy the principle (constructions, especially concepts), together with the possibility of a hierarchy of hyperintensions with constructions at the top.

The verdict is that functions are no good, if we want to assign hyperintensionally individuated, structured procedures to terms and expressions as their meanings. Constructions, in contrast, do the trick.

[38] In particular, we are not going to draw distinctions that reflect notational differences that are not backed up by abstract procedural differences. So *Mates' puzzle* is not a puzzle for us; see Section 5.1.

1.3.2 Constructions and types

Constructions are procedures, or instructions, specifying how to arrive at less-structured entities. Qua procedures, constructions are structured, unlike set-theoretical objects, which are devoid of structure. Qua abstract, extra-linguistic entities, constructions are reachable only via a verbal definition. The 'language of constructions' is a modified hyperintensional version of the typed λ-calculus, where Montague-like λ-terms denote, not the functions constructed, but the constructions themselves. The modification is extensive. Church's λ-terms form part of his *simple* type theory, whereas our λ-terms belong to a *ramified* type theory. Constructions *qua* procedures operate on input objects (of any type, even constructions of any order) and yield as output (or, in well-defined cases, fail to yield) objects of any type. This way constructions construct *partial* functions.

When claiming that constructions are algorithmically structured, we mean the following. A construction *C* consists of one or more particular steps, or *constituents*, that are to be individually executed in order to execute *C*. The entities a construction operates on are not constituents of the construction. Similarly as the constituents of a computer program are its subprograms, so the constituents of a construction are again constructions. Thus on the lowest level of non-constructions, the objects that constructions work on have to be supplied by other (albeit trivial) constructions. The constructions hosting these trivial constructions may occur not only as constituents to be executed, but also as entities that still other constructions operate on. Therefore, one should not conflate *using* constructions as constituents of Composed constructions (where a Composed construction is what results from applying the operation of composition/application to a construction) and *mentioning* constructions that enter as input entities into Composed constructions. So we must distinguish strictly between *using* and *mentioning* constructions. We will deal with the use/mention distinction in Section 2.6; for now just briefly this. The constituents of a construction *C*, which are to be individually executed in order to execute *C*, are *used in C*. On the other hand, the entities (constructions or non-constructional objects) a construction *C* operates on are *mentioned in C*. Mentioning is, in principle, achieved by *using* atomic constructions. A construction *C* is atomic if it does not contain any other construction as a used subconstruction (a 'constituent of *C*') than *C*. There are two atomic constructions that supply entities (of any type) on which complex constructions operate: *Variables* and *Trivializations.*

Variables are constructions that construct an object dependently on *valuation*: they *v-construct*, where *v* is the parameter of valuation. With the important difference that we construe variables as extra-linguistic objects and not as expressions, our theory of variables is otherwise identical to Tarski's. Thus, in TIL variables construct objects of the respective types dependently on valuation in the following way. For each type α there are countably infinitely many variables x_1, x_2, …. The members of α (unless α is a singleton) can be organised in infinitely many infinite sequences. Let the sequences be given (as one is allowed to assume in a realist semantics). The valuation v takes a sequence $\langle s_1, s_2, \dots \rangle$ and assigns s_1 to the variable x_1, s_2 to the variable x_2; and so on.[39]

When X is an object of any type (including a construction), the Trivialization of X, denoted '^0X', constructs X without the mediation of any other constructions. ^0X is the unique atomic construction of X that does not depend on valuation: it is a primitive, non-perspectival mode of presentation of X.

The other constructions are *compound*, as they consist of other constituents apart from themselves. These are *Composition*, *Closure*, *Execution* and *Double Execution.* Composition is the procedure of applying a function f to an argument A to obtain the value (if any) of f at A. Closure is the procedure of constructing a function by abstracting over variables; i.e., the procedure of abstracting, or extracting, a function from a context, as when abstracting $\lambda x(\varphi\, x)$ from $\varphi\,(a)$. Finally, higher-order constructions can be used twice over as constituents of Composed constructions. This is achieved by the construction called *Double Execution*, which we are going to need later. (Tichý adds also a simple construction called *Execution*, see Definition 1.2.)

TIL constructions, as well as the entities they construct, all receive a type. Thus TIL has a liberal ontology, accommodating both intensions of whatever degree n whose values are intensional entities of degree n–1, as well as constructions of whatever order $m > 1$ that construct entities of order m–1. Intensions may come in different orders, due to type rising, and in different degrees. An intension is higher-order if its range is made up of higher-order entities. For instance, a relation-in-intension relating individuals to constructions, as in the case of hyperintensional attitudes, is higher-order. An intension is first-order, but of a higher degree than zero, if its range is made up of first-order intensions; i.e., any such intensions as do not include constructions. For instance, *the tallest mountain* is of degree one, because its (world- and time-relative) values are themselves extensional entities (individuals), while *the most characteristic property of a war criminal* is of degree two, because its values are themselves intensional entities of degree one. Extensional entities also come in different orders. For instance, the set of all n-order constructions with some particular property is an extensional n-order entity.[40]

The definitions proceed inductively. First, we define simple types of order 1; second, constructions operating on types; finally, the whole ontology of entities as organised into a ramified hierarchy of types.

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[39] Tichý (1988) devotes an entire chapter to variables, explaining their objectual role as constructions; for details see (1988, pp. 47–62).

[40] The *degree* of a first-*order* entity corresponds roughly to an order in predicate logics. For instance, in order to ascribe properties to individual properties in predicate logic, we need to work within second-order logic. However, in TIL, properties of individuals are 1st-order objects of degree 1. Properties of properties of individuals are 1st-order objects of degree 2; and so on.

Definition 1.1 (*types of order 1*) Let B be a *base*, where a base is a collection of pair-wise disjoint, non-empty sets. Then:

(i) Every member of B is an elementary *type of order 1 over B.*
(ii) Let α, β_1, ..., β_m ($m > 0$) be types of order 1 over B. Then the collection $(\alpha\beta_1...\beta_m)$ of all m-ary partial mappings from $\beta_1 \times ... \times \beta_m$ into α is a functional *type of* order *1 over B.*
(iii) Nothing is a *type of order 1 over B* unless it so follows from (i) and (ii). □

Remark. For the purposes of natural-language analysis we choose the so-called *objectual base* described and motivated in the following Section 1.4. The objectual base B consists of the following atomic types:

ο	the set of truth-values {**T, F**};
ι	the set of individuals (the universe of discourse);
τ	the set of real numbers;
ω	the set of logically possible worlds (the logical space).

TIL is an open-ended system. The above objectual base {ο, ι, τ, ω} was chosen, because it is apt for natural-language analysis, but in the case of mathematics a (partially) distinct base would be appropriate; for instance, the base consisting of natural numbers, of type ν, and truth-values. The derived functional types would then be defined over {ν, ο}.

Remark. An object O belonging to a type α is an α-object, denoted 'O/α'.

Remark. α-sets of elements of type α are modelled by their characteristic functions. Thus they are $(\omicron\alpha)$-objects. For instance, a set of individuals is an object of type $(\omicron\iota)$, a set of real numbers is an object of type $(\omicron\tau)$, a set of couples of real numbers (i.e., a binary relation over reals) is an object of type $(\omicron\tau\tau)$.

Example 1.1 *Types of extensional mathematical objects (non-constructions)*

- *Prime* is the set of prime numbers. It is an object of type $(\omicron\nu)$.
- The factor set of sets of numbers that have the same remainder when dividing by 5 is an object of type $(\omicron(\omicron\nu))$.
- Binary functions defined on reals, like +, –, ×, :, are objects of type $(\tau\tau\tau)$.
- Binary relations-in-extensions on reals, like >, <, having the same remainder when dividing by 5 with an integer quotient, are objects of type $(\omicron\tau\tau)$.

Definition 1.2 (*construction*)

(i) The *variable* x is a *construction* that constructs an object O of the respective type dependently on a valuation v; it v-constructs O.
(ii) Where X is an object whatsoever (an extension, an intension or a *construction*), ${}^0X$ is the *construction Trivialization.* It constructs X without any change.
(iii) The *Composition* $[X\ Y_1 \ldots Y_m]$ is the following *construction*. If X *v-constructs* a function f of a type $(\alpha\beta_1 \ldots \beta_m)$, and $Y_1, \ldots, Y_m$ *v-construct* entities $B_1, \ldots, B_m$ of types $\beta_1, \ldots, \beta_m$, respectively, then the *Composition* $[X\ Y_1 \ldots Y_m]$ *v-constructs* the value (an entity, if any, of type α) of f on the tuple-argument $\langle B_1, \ldots, B_m \rangle$. Otherwise the *Composition* $[X\ Y_1 \ldots Y_m]$ does not *v-construct* anything and so is *v-improper.*
(iv) The *Closure* $[\lambda x_1 \ldots x_m\ Y]$ is the following *construction*. Let $x_1, x_2, \ldots, x_m$ be pairwise distinct variables v-constructing entities of types $\beta_1, \ldots, \beta_m$ and Y a construction v-constructing an α-entity. Then $[\lambda x_1 \ldots x_m\ Y]$ is the *construction* λ-*Closure* (or *Closure*). It *v-constructs* the following function $f/(\alpha\beta_1 \ldots \beta_m)$. Let $v(B_1/x_1,\ldots,B_m/x_m)$ be a valuation identical with v at least up to assigning objects $B_1/\beta_1, \ldots, B_m/\beta_m$ to variables $x_1, \ldots, x_m$. If Y is $v(B_1/x_1,\ldots,B_m/x_m)$-improper (see iii), then f is undefined on $\langle B_1,\ldots,B_m \rangle$. Otherwise the value of f on $\langle B_1,\ldots,B_m \rangle$ is the α-entity $v(B_1/x_1,\ldots,B_m/x_m)$-constructed by Y.
(v) The *Execution* ${}^1X$ is the *construction* that either v-constructs the entity v-constructed by X or, if X v-constructs nothing, is v-improper.
(vi) The *Double Execution* ${}^2X$ is the following *construction*. Let X be any entity; the *Double Execution* ${}^2X$ is *v-improper* (yielding nothing relative to v) if X is not itself a construction, or if X does not v-construct a construction, or if X v-constructs a v-improper construction. Otherwise, let X v-construct a construction X' and X' v-construct an entity Y. Then ${}^2X$ v-constructs Y.
(vii) Nothing is a *construction*, unless it so follows from (i) through (vi). □

Remark. That a variable x constructs an entity dependently on valuation v will be referred to as 'v-constructing'. That a variable x v-constructs entities of a type α will be referred to as 'ranging over α', denoted by '$x \rightarrow_v \alpha$'.

Remark. ${}^1X$ is the procedure of executing X. Thus if X is a construction then the execution of ${}^1X$ consists in executing X. However, if X is not a construction then ${}^1X$ is the abortive construction whose input is X and whose output is nothing. A non-construction cannot be executed. Thus if X is a v-improper construction or a non-construction, ${}^1X$ is v-improper. Similarly, ${}^2X$ is the instruction to execute X and go on and execute the result. Thus ${}^2X$ is a v-improper construction if X is a v-improper construction or if the object v- constructed by X is a v-improper construction or a non-construction.

Remark. In principle, also *Triple Execution* could be defined, as could any other multiple *Execution*. But, pragmatically speaking, we as practising TIL-ians have had no need so far for Executions beyond Double Execution. And, methodologically speaking, we observe the constraint that the different kinds of construction should not be multiplied beyond what we know to be necessary. But, should Triple (or whatever) Execution turn out to be indispensable, it will be defined and added to the open-ended recursive definition of *construction*.

Remark. We use the terms 'mapping' and 'function' synonymously. By 'partial mapping' we mean a mapping that associates every argument (of the respective type) with at most one value (of the respective type); a total function is then a limiting case of the former; namely, a mapping that associates every argument with just one value. By 'properly partial mapping' we mean a partial mapping that is not total.

Remark. The names of constructions are written with upper-case first letters, to distinguish them from regular English words. The exception is 'variable', since it is already a well-established technical term in logical and mathematical literature.

Remark. Outer brackets of Closure will be omitted whenever no confusion can arise. We will say that a construction C *constructs* an entity E if C v-constructs E for all valuations v. Similarly, we will say that a construction C is *improper* if C is v-improper for all valuations v.

Definition 1.3 (*subconstruction*) Let C be a construction. Then

(i) C is a *subconstruction of* C.
(ii) If C is ${}^0X$, ${}^1X$ or ${}^2X$ and X is a construction then X is a *subconstruction of* C.
(iii) If C is $[X\,X_1 \ldots X_n]$ then $X, X_1, \ldots, X_n$ are *subconstructions of* C.
(iv) If C is $[\lambda x_1 \ldots x_n\ Y]$ then Y is a *subconstruction of* C.
(v) If A is a *subconstruction of* B and B is a *subconstruction of* C then A is a *subconstruction of* C.
(vi) A construction is a *subconstruction of* C only if it so follows from (i) to (v). □

Above we warned against the confusion that might arise from not distinguishing two ways in which a subconstruction D of a construction C may occur. The two ways were *using* D as a constituent of C and *mentioning* D by means of another constituent of C. Constructions are *used* in extensional or intensional contexts, and *mentioned* in hyperintensional (i.e., *constructional* or *conceptual*) contexts. These three kinds of context and the difference between using and mentioning constructions will be rigorously defined in Section 2.6, once more notions have been defined. Now we only briefly characterise the use/mention distinction.

> Let D be a subconstruction of a construction C. Then an occurrence of D *is mentioned in* C if the execution of C does not include the execution of D. Otherwise the occurrence of D *is used in* C *as a constituent.*

The simplest way to mention a construction C is by using the Trivialization of C. Thus in the Trivialization ${}^0[{}^0\!+\ {}^0 2\ x]$ the Composition $[{}^0\!+\ {}^0 2\ x]$ is not used; it is mentioned by using its Trivialization ${}^0[{}^0\!+\ {}^0 2\ x]$, which constructs $[{}^0\!+\ {}^0 2\ x]$ independently of valuation. The variable x is not free for substitution in ${}^0[{}^0{+}\ {}^0 2\ x]$, as it is bound by the outer Trivialization. Thus we define:

Definition 1.4 (*free variable, bound variable, open/closed construction*) Let C be a construction with at least one occurrence of a variable ξ.

(i) Let C be ξ. Then the *occurrence of ξ in C* is *free*.
(ii) Let C be ^{0}X. Then every *occurrence of ξ in C* is *^{0}bound* ('Trivialization-bound').
(iii) Let C be $[\lambda x_1...x_n\ Y]$. Any *occurrence of ξ* in Y that is one of x_i, $1 \leq i \leq n$, is *λ-bound* in C unless it is *^{0}bound in Y*. Any *occurrence of ξ in Y* that is neither *^{0}bound* nor *λ-bound* in Y is *free* in C.
(iv) Let C be $[X\ X_1...X_n]$. Any *occurrence of ξ* that is *free*, *^{0}bound*, *λ-bound* in one of $X, X_1,...,X_n$ is, respectively, *free, ^{0}bound, λ-bound in C*.
(v) Let C be ${}^1X$. Then any *occurrence of ξ* that is *free, ^{0}bound, λ-bound* in X is, respectively, *free, ^{0}bound, λ-bound in C*.
(vi) Let C be ${}^2X$. Then any *occurrence of ξ* that is *free, λ-bound* in a constituent of C is, respectively, *free, λ-bound in C*. If an occurrence of ξ is *^{0}bound* in a constituent ${}^0D$ of C and this occurrence of D is a constituent of X' *v*-constructed by X, then if the occurrence of ξ is free, λ-bound in D it is *free, λ-bound in C*. Otherwise, any other occurrence of ξ in C is *^{0}bound* in C.
(vii) An *occurrence* of ξ is *free*, *λ-bound*, *^{0}bound* in C only due to (i)–(vi).

A construction with at least one occurrence of a free variable is an *open construction.* A construction without any free variables is a *closed construction.* □

TIL has two kinds of binding, either by Trivialization or by lambda. In both cases variables behave in harmony with the general principle that a bound variable is not free for substitution. The distinction between ^{0}-binding and λ-binding can be best illuminated as follows. Consider the following Closures (variables x, y *v*-constructing elements of type τ):

(a) $\lambda x\ [{}^0{\leq}\ x\ {}^0 0]$

(b) $\lambda y\ [{}^0{\leq}\ y\ {}^0 0]$.

The Closures (a), (b) are equivalent in that they construct the same class of numbers. The variables x, y are λ-bound in (a), (b). By contrast, consider

(c) ${}^0[\lambda x\ [{}^0{\leq}\ x\ {}^0 0]]$

(d) ${}^0[\lambda y\ [{}^0{\leq}\ y\ {}^0 0]]$.

The Trivializations (c), (d) are not equivalent, since they construct distinct (albeit equivalent) constructions. The variables x, y are ^{0}bound in (c), (d) according to the points (ii) and (iii) in Definition 1.4. Note, however, that x has only *one* occurrence in (a), as well as in (c), the former occurrence being λ-bound, the latter ^{0}bound. Similarly, there is one occurrence of y in (b) and in (d). 'λx', 'λy' are not improper symbols; they denote instructions to abstract over occurrences of x, y, respectively. And even if there is no occurrence of x, as in $\lambda x\ [^0{+}\ ^01\ ^01]$, the instruction specified by 'λx' is a one-step instruction. For instance, the Closure

$$\lambda x\ [^0{+}\ ^01\ ^01]$$

does not construct the number 2, but the constant function, of type $(\tau\tau)$, that associates the value 2 with all arguments.

Concerning point (vi) of Definition 1.4, consider $^2(^0x)$, which is the Double Execution of the Trivialization of x. If $x \to_v \tau$ and x v-constructs the number 1, then $^2(^0x)$ v-constructs what is v-constructed by the result of executing ^0x, i.e., by x. Thus $^2(^0x)$ v-constructs the number 1 and it is equivalent to x. In general, $^2(^0x)$ v-constructs what x v-constructs. Hence x is *free* in $^2(^0x)$. However, the Double Execution $^2(^0(^0x))$ constructs what ^0x constructs, namely the variable x; the variable x is thus ^{0}bound in $^2(^0(^0x))$.

Definition 1.5 (*congruency and equivalence of constructions*) Let C, $D/*_n \to \alpha$ be constructions, and $\approx_v/(o*_n*_n)$, $\approx/(o*_n*_n)$ binary relations between constructions of order n. Using infix notation $^0C \approx_v {}^0D$, $^0C \approx {}^0D$, we define:

C, D are *v-congruent*, $^0C \approx_v {}^0D$, iff either C and D v-construct the same α-entity, or both C and D are v-improper;

C, D are *equivalent*, $^0C \approx {}^0D$, iff C, D are v-congruent for all valuations v. □

Corollaries.

If C, D are *identical*, $^0C =_* {}^0D$, then C, D are equivalent, $^0C \approx {}^0D$, but not vice versa.

If C, D are equivalent, $^0C \approx {}^0D$, then C, D are v-congruent, $^0C \approx_v {}^0D$, but not vice versa.

Remark. Recall that C, D are *identical*, $^0C =_* {}^0D$, if C and D are exactly the same procedure. Thus, for instance, though $^0[\lambda x\ [^0{+}\ x\ ^01]] \approx {}^0[\lambda y\ [^0{+}\ y\ ^01]]$, the two constructions are not identical. They construct one and the same function *Successor*/$(\tau\tau)$, i.e.,

$$\lambda x\ [^0{+}\ x\ ^01] =_{(\tau\tau)} \lambda y\ [^0{+}\ y\ ^01],$$

but in two different ways, because x, $y \to \tau$ are two different procedures. Different variables are not even equivalent and may be only v-congruent. On the other hand, if 'is sky-blue' and 'is azure' denote one and the same property of individuals, then not only ^{0}*Sky-blue* $=_{(((o\iota)\tau)\omega)}$ ^{0}*Azure*, but also ^{00}*Sky-blue* $=_*$ ^{00}*Azure*, i.e., ^{0}*Sky-blue* is identical to ^{0}*Azure.*

Types: $=_{(\tau\tau)}/(o(\tau\tau)(\tau\tau))$; $=_{(((o\iota)\tau)\omega)}/(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; $=_*/(o*_1*_1)$; *Sky-blue*, *Azure*$/(o\iota)_{\tau\omega}$.

Example 1.2 *Equivalent and v-congruent constructions.*

(a) Let $v(5/x,1/y)$ be a valuation identical to v at least up to assigning the number 5 to the variable x and the number 1 to the variable y. Then the constructions $[{}^0{+}\ x\ {}^01] \rightarrow_v \tau$, $[\lambda x\ [{}^0{+}\ x\ y]\ {}^05] \rightarrow_v \tau$, $[{}^0Succ\ x] \rightarrow_v \tau$, are $v(5/x,1/y)$-congruent, because they $v(5/x,1/y)$-construct the number 6.
Types: $+/(\tau\tau\tau)$; $x, y/*_1{\rightarrow_v}\tau$; $Succ/(\tau\tau)$, the successor function.

(b) The constructions $[{}^0Divide\ {}^05\ x] \rightarrow_v \tau$, $[{}^0Square\_root\ [{}^0{-}\ [x\ {}^05]]] \rightarrow_v \tau$ are $v(0/x)$-congruent, because they are $v(0/x)$-improper.
Types: *Divide*$/(\tau\tau\tau)$, the division function; $x/*_1{\rightarrow}\tau$; *Square_root*$/(\tau\tau)$, the positive square root function.

(c) The constructions $[{}^0{+}\ {}^05\ {}^01] \rightarrow \tau$, $[\lambda x\ [{}^0{+}\ x\ {}^01]\ {}^05] \rightarrow \tau$, $[{}^0Succ\ {}^05] \rightarrow \tau$, are equivalent. They construct the number 6;

(d) The constructions $[{}^0Divide\ x\ {}^00] \rightarrow_v \tau$, $[{}^0Square\_root\ [{}^0{-}\ [{}^00\ {}^05]]] \rightarrow \tau$ are equivalent, because they are v-improper for every valuation v.

In TIL—as also in Montague Grammar—*quantifiers* denote functions of type $(o(o\alpha))$, α an arbitrary type. Quantifiers are not 'improper symbols', 'syncategorematic signs', and suchlike. Note that TIL quantifiers do not bind variables. '$\forall x$', '$\exists y$' are shorthand for '$\forall\lambda x$', '$\exists\lambda y$', so the binding is done exclusively by λ.

The phenomenon of λ-binding arises due to λ-abstraction, i.e., Closure. The semantics of a formula of the form '$\forall x\ A$' is in TIL deciphered as $[{}^0\forall\ \lambda xA]$, x v-constructing ('ranging over') objects of type α and A (v-)constructing a truth-value.

Quantifiers are thus defined as follows.

Definition 1.6 (*quantifiers* $\forall$ *and* $\exists$, *singulariser Sing*) The *quantifiers* $\forall^\alpha$, $\exists^\alpha$ are type-theoretically polymorphic total functions of type(s) $(o(o\alpha))$ defined as follows:

The *universal quantifier* $\forall^\alpha$ is a function that associates a class C of α-elements with **T** if C contains all elements of the type α, otherwise with **F**. The *existential quantifier* $\exists^\alpha$ is a function that associates a class C of α-elements with **T** if C is a non-empty class, otherwise with **F**.

The *singulariser* $Sing^\alpha$ is a partial, type-theoretically polymorphic function of type(s) $(\alpha(o\alpha))$ that associates a class C with the only α-element of C if C is a singleton, otherwise the function $Sing^\alpha$ is undefined. □

If $A \to o$ and $x \to_v \alpha$, we will often use the abbreviated notation

$$\text{‘}\forall x\, A\text{’, ‘}\exists x\, A\text{’ and ‘}\iota x\, A\text{’}$$

instead of

$$\text{‘}[{}^0\forall^\alpha\, \lambda x\, A]\text{’, ‘}[{}^0\exists^\alpha\, \lambda x\, A]\text{’, ‘}[{}^0Sing^\alpha\, \lambda x\, A]\text{’,}$$

respectively, when no confusion can arise.

Remark. Classes of elements of type α are modelled by their characteristic functions, of type $(o\alpha)$. Hence there are *several empty classes*, of types $(o\alpha_1)$, $(o\alpha_2)$, etc., and not just one empty class *simpliciter*. Moreover, due to partiality there may be different kinds of emptiness; the respective characteristic function can be either *false* at a given argument or *undefined*. We can even obtain a degenerate class by using a function undefined at all arguments. An example would be the class of numbers that are equal to the result of dividing the number two by zero, constructed by $\lambda x\, [{}^0{=}\, x\, [{}^0\!:\, {}^02\; {}^00]]$.

Example 1.3 *Mathematical constructions*

(a) The function +, defined on the natural numbers (of type ν), is not a construction. It is a mapping of type $(\nu\, \nu\nu)$, i.e., a set of triples, the first two members of which are natural numbers, while the third member is their sum. The simplest construction of this mapping is ${}^0{+}$ (See Definition 1.2, (ii)).
(b) The function + can be constructed by infinitely many equivalent, yet distinct constructions; for instance, the following Closures are equivalent by constructing the same function +:
$\lambda xy\, [{}^0{+}\, x\, y]$, $\lambda yx\, [{}^0{+}\, x\, y]$, $\lambda xz\, [{}^0{+}\, x\, z]$, $\lambda xy\, [{}^0{+}\, [{}^0{-}\, [{}^0{+}\, x\, y]\, y]\, y]$ (See Definition 1.2 (iii) and (iv)).
(c) The Composition $[{}^0{+}\; {}^02\; {}^05]$ constructs the number 7, i.e., the value of the function + (constructed by ${}^0{+}$) at the argument $\langle 2, 5\rangle$ constructed by ^{02} and ^{05} (See Definition 1.2 (iii)).
Note that the numbers 2, 5 are not constituents of this Composition, nor is the function +. Instead, the Trivialisations ${}^0{+}$, ^{02}, ^{05} are the constituents of the Composition $[{}^0{+}\; {}^02\; {}^05]$.
(d) The Composition $[{}^0{+}\, x\; {}^01]$ v-constructs the successor of any number x.
Note that the number 1 is not a constituent of this Composition. Instead, the Trivialisation ^{01} is a constituent; the other two constituents are ${}^0{+}$, x.
(e) The Closure $\lambda x\, [{}^0{+}\, x\; {}^01]$ constructs the successor function (See Definition 1.2 (iv)). The successor function can be constructed by infinitely many constructions, the simplest one of which is the Trivialisation of the function: ${}^0Succ$. Thus $\lambda x\, [{}^0{+}\, x\; {}^01]$ and ${}^0Succ$ are equivalent by constructing the same function. Yet the Trivialization ${}^0Succ$ is not a finitary, executable procedure. It is a one-step procedure producing an *infinite mapping* as its product. On the other hand, the Closure $\lambda x\, [{}^0{+}\, x\; {}^01]$ is an easily executable procedure. The instruction to execute this procedure can be decomposed into the following steps:

Take any number x and the number 1*; apply the function* + *to the couple of numbers obtained at the previous step; abstract from the value of x.*

(f) The Composition of this closure with $^0 5$, i.e., $[\lambda x\ [^0{+}\ x\ ^0 1]\ ^0 5]$, constructs the number 6 (See Definition 1.2 (iii)).

(g) The Composition $[^0{:}\ x\ ^0 0]$ does not *v*-construct anything for any valuation of *x*; it is *v-improper* for any valuation *v* (See Definition 1.2 (iii)). We will say 'improper', for short.

(h) The closure $\lambda x\ [^0{:}\ x\ ^0 0]$ is not improper, as it constructs something, even though it is only a degenerate function, *viz.* one undefined at all its arguments (See Definition 1.2 (iv)).

(i) If *x* is a variable *v*-constructing real numbers of type τ, then the Compositions $[^0\exists\lambda x\ [^0{>}\ x\ ^0 5]]$, $[^0\forall\lambda x\ [^0{>}\ x\ ^0 5]]$ construct the truth-value **T** and **F**, respectively, because the class of real numbers greater than 5 constructed by the Closure $\lambda x\ [^0{>}\ x\ ^0 5]$ is not empty, but is not the whole type τ.

(j) If $Sing^\tau/(\tau(o\tau))$ is a singularizer, then the following construction (the meaning of 'the greatest prime') is *v*-improper for all valuations *v*, i.e., improper: $[^0 Sing^\tau\ \lambda x\ [^0{\wedge}\ [^0 Prime\ x]\ [^0\forall\lambda y\ [^0{\supset}\ [^0 Prime\ y]\ [^0{\geq}\ x\ y]]]]]]$, or for short, $\iota x\ [[^0 Prime\ x] \wedge \forall y\ [[^0 Prime\ y] \supset [^0{\geq}\ x\ y]]]$.

So much for examples for now. As mentioned above, constructions can not only be *used* to construct objects of a lower-order type, they can also be *mentioned* by other constructions. Constructions can in this manner themselves serve as input/output objects, on which higher-order constructions operate. However, within the simple hierarchy of types, as defined in Definition 1.1, there is no type to be assigned to constructions themselves. For instance, the Composition $[^0{+}\ ^0 2\ ^0 5]$ constructs the number 7, an entity of type τ (or ν, depending on the choice of objectual base). But when Charles calculates 2+5, he is related to the Composition $[^0{+}\ ^0 2\ ^0 5]$ and not to its product 7. What is then the type of the activity of *calculating*? It is a relation (-in-intension) of an individual to the respective construction itself. And this constructional type has to be of a higher order than the type of its product.

Typical examples of hyperintensional contexts are attitude reports involving mathematical knowledge and belief. For instance, in

> 'Charles believes that arithmetic is recursively axiomatizable
> and that Gödel proved it'

the meanings of 'that arithmetic is recursively axiomatizable' and 'that Gödel proved it' are only mentioned, because Charles does not believe the truth-value **F**. Instead, he believes that the meaning of the embedded clause yields **T.** In other words, Charles is related to a *construction* of **F**.

At its most fundamental level, the formal ontology of TIL is bi-dimensional. One dimension is made up of constructions, while the other dimension encompasses non-constructions. The ontology of entities of TIL organised in a *ramified* *hierarchy of types* enables us to logically handle *structured meanings* as higher-order, hyperintensional, abstract entities, thus avoiding inconsistency problems stemming from the need to mention these entities within the theory itself. Any higher-order entity can be safely, not only *used*, but also *mentioned* within the theory.

On the ground level of the type-hierarchy, there are entities unstructured from the algorithmic point of view belonging to a *type of order* 1. Given an *objectual base* of atomic types, molecular complexity is increased by the induction rule for forming partial functions. Where α, $\beta_1,\ldots,\beta_n$ are types of order 1, the set of partial mappings from $\beta_1 \times\ldots\times \beta_n$ to α, denoted '$(\alpha\ \beta_1\ldots\beta_n)$', is a type of order 1 as well. (See Definition 1.1.)

Constructions that construct entities of order 1 are constructions of order 1. They belong to a type of order 2, denoted '$*_1$'. The type $*_1$ serves as a base for the induction rule: any collection of partial functions, of type $(\alpha\ \beta_1\ldots\beta_n)$, involving $*_1$ in their domain or range is a *type of order* 2. Constructions belonging to a type $*_2$, which construct entities of order 1 or 2, and partial functions involving such constructions, belong to a *type of order* 3; and so on *ad infinitum*.

Definition 1.7 (*ramified hierarchy of types*) Let B be a base. Then:

$\mathbf{T_1}$ (*types of order* 1) defined by Definition 1.1.

$\mathbf{C}_n$ (*constructions of order n*)

(i) Let x be a variable ranging over a type of order n. Then x is a *construction of order n over B*.
(ii) Let X be a member of a type of order n. Then ^{0}X, ^{1}X, ^{2}X are *constructions of order n over B*.
(iii) Let $X, X_1, \ldots, X_m$ $(m > 0)$ be *constructions of order n over B*. Then $[X\, X_1 \ldots X_m]$ is a *construction of order n over B*.
(iv) Let $x_1, \ldots, x_m, X$ $(m > 0)$ be *constructions of order n over B*. Then $[\lambda x_1 \ldots x_m\, X]$ is a *construction of order n over B*.
(v) Nothing is a *construction of order n over B* unless it so follows from $\mathbf{C}_n$ (i) to (iv).

$\mathbf{T}_{n+1}$ (*types of order n* + 1)
Let $*_n$ be the collection of all constructions of order n over B.
(i) $*_n$ and every type of order n are *types of order n* + 1.
(ii) If $0 < m$ and α, $\beta_1,\ldots,\beta_m$ are *types of order n* + 1 *over B*, then $(\alpha\ \beta_1 \ldots \beta_m)$ (see T_1 (ii)) is a *type of order n* + 1 *over B*.
(iii) Nothing is a *type of order n* + 1 *over B* unless it so follows from $\mathbf{T}_{n+1}$ (i) and (ii). □

***Example 1.4** Entities of higher-order types*

(a) The constructions
$^0+$, $[^0+\ x\ ^01]$, $\lambda x\ [^0+\ x\ ^01]$, $[\lambda x\ [^0+\ x\ ^01]\ ^05]$, $[^0:\ x\ ^00]$, $\lambda x\ [^0:\ x\ ^00]$, all mentioned in Example 1.3, construct objects of types of order 1. They are constructions of order 1 (see Definition 1.7, C_n), and belong, thus, to the type $*_1$ (see Definition 1.7, T_{n+1}); i.e., to the type of order 2 (See Definition 1.7, T_{n+1} (i)).

(b) Let *Improper* be the set of constructions of order 1 that are *v*-improper for all valuations *v*; then *Improper* is an object belonging to $(o*_1)$, the type of order 2 (See Definition 1.7, T_{n+1} (ii)).

(c) The Composition $[^0Improper\ ^0[^0:\ x\ ^00]]$ is a member of $*_2$, the type of order 3. It constructs the truth-value **T**. The constituent $^0[^0:\ x\ ^00]$ of this Composition is a member of $*_2$; it is an atomic *proper* construction that constructs $[^0:\ x\ ^00]$, a member of $*_1$. It is atomic, because the construction $[^0:\ x\ ^00]$ is not used here as a constituent but only mentioned as an input object.

(d) Let *Arithmetic* be a set of unary arithmetic functions defined on natural numbers, making *Arithmetic* an entity of type $(o(\nu\nu))$, and let $x \rightarrow_v \nu$. Then the Composition $[^0Arithmetic\ [\lambda x\ [^0+\ x\ ^01]]]$ belonging to $*_1$, the type of order 2, constructs **T** (an entity of type o, the type of order 1), because the Closure $[\lambda x\ [^0+\ x\ ^01]]$ constructs the unary function *Successor*, and this function is arithmetic. It belongs to the set *Arithmetic*.

(e) The Composition $[^0Arithmetic\ ^2c]$ *v*-constructs the truth-value **T** if *c* *v*–constructs, for instance, the Closure $[\lambda x\ [^0+\ x\ ^01]]$. The Double Execution ^2c then *v*-constructs what is *v*-constructed by this Closure; namely, the arithmetic successor function. The Composition $[^0Arithmetic\ ^2c]$ is an object belonging to $*_3$, the type of order 4; the variable *c* *v*-constructing the Closure of type $*_1$ is an entity of type $*_2$, the type of order 3. Since Double Execution increases the order of a construction (see Definition 1.7., C_n (ii) and T_{n+1} (i)), ^2c belongs to $*_3$, the type of order 4. Therefore, the Composition $[^0Arithmetic\ ^2c]$ belongs to $*_3$, the type of order 4. This exemplifies the phenomenon of *type raising*.

Note that every construction *C* belongs to $*_n$, so that *C* is an entity of a type of order $n > 1$, and (*v*–) constructs an entity belonging to a type α of a lower order. We will use the notation '$C/*_n \rightarrow_v \alpha$'. For instance, '$x/*_1 \rightarrow_v \tau$' reads 'The variable *x* belongs to the type $*_1$ and *v*-constructs reals'. For the variable *c* of the above example we write '$c/*_2 \rightarrow_v *_1$'.

Typing not only enables us to avoid vicious-circle problems, it also makes it possible to avoid another kind of 'improperness'. If *X* is not a construction of order *n* ($n \geq 1$), then ^1X does not construct anything and so is improper; if *X* is not a construction of order *n* ($n \geq 2$), then ^2X is improper; finally, if $X, X_1, \ldots, X_n$ are not constructions of types according to Definition 1.2 (iii), then $[X\ X_1 \ldots X_n]$ does not construct anything and so is improper. If a construction *C* is type-theoretically improper, then it does not *v*-construct an entity of any type α due to *wrong typing*.

The notion of construction is both the most important and most misunderstood of all TIL notions. This is little wonder, considering the fact that the model-theoretic paradigm of doing semantics continues to be overwhelmingly dominant and set theory continues to be the background theory of most analytic ontology. Constructivist logicians and computer scientists, in contrast, tend to find it easier to tap into TIL. Again, this is little wonder, since constructivists have their own notion of construction and computer scientists are trained in reasoning in procedures. Perhaps a Platonic dialogue (*sans comparaison*!) is as good a means as any to lay to rest the most common misconceptions of the Platonist notion of construction. Imagine the following dialogue taking place between a TILian and a non-TILian during a coffee break at a conference:

Question: Are constructions formulae of some type logic?
Answer: No!
Q: Are they equivalence classes of such formulae?
A: No!
Q: Are they denotations of closed formulae?
A: No!
Q: So what are they?
A: They are what Definition 1.2 says they are.
Q: Sure, I understand the formalities of your definition, but saying what the particular constructions construct you're not saying what they *are*!
A: So an informal, pre-theoretical characterisation is what you're after? Well, the fundamental idea is that of *abstract procedure*.
Q: Procedures are set in time, so how can they be abstract, as constructions are supposed to be?
A: The execution of a procedure (or algorithm, if you like) is a time-consuming process, all right, whereas the procedure itself is beyond time and space.
Q: So what about your symbolic language, the 'language of constructions'—why do you not simply say that its expressions are constructions?
A: These expressions serve only to represent, or encode, constructions; as expressions they cannot construct anything. What is important about expressions is only what they mean and not their syntactic shapes.
Q: But constructions outside time and space can construct something? How can abstract objects *do* anything?
A: They don't *do* anything, for sure. But *agents* can *execute* them. We do this sort of thing every day when executing algorithms or following instructions. When agents execute constructions, they follow an intellectual path that is already laid out. Agents, or any of their artefacts, do not construct constructions. This is why TIL is a realist and not an idealist theory.
Q: But you could do it like Montague did—translating expressions of natural language into the language of intensional logic, and then interpreting the result in the standard manner. What you achieve by using your constructions you would

get using a meta-language. So it seems like your superstructure of higher-order objects is not needed at all.[41]

A: Okay, this calls for a longer reply. Montague's and other intensional logics interpret the expressions of their language in terms of functions. However, from our perspective these mappings are only the products of the respective procedures. In terms of conceptual priority, there is an instance preceding functions. Montague does not make it possible to *mention* the procedures as objects *sui generis* or to make a shift to hyperintensions. Yet we do need a hyperintensional semantics. Procedures—our constructions—can be not only executed in order to obtain a product but also talked about in their own right, by using other higher-order constructions. It is not by chance that mathematicians did not always use the term 'function' in its contemporary sense, as standing for mappings, which are mere set-theoretic objects. Functions were previously thought of as *calculation procedures*. Also, the original interpretation of the terms of lambda calculus was procedural. For instance, Barendregt says,

> [I]n this interpretation the notion of a function is taken to be intensional, i.e., as an algorithm (1997, p. 184).

We would say, '... is taken to be *hyperintensional*, i.e., as an algorithm', because the term 'intensional' is currently reserved for mappings from possible worlds (if not among proof-theoretic semanticists, then at least among model-theoretic semanticists). Besides, our approach to semantic analysis is simpler and more direct. We do not pair expressions from, say, English off with symbols stemming from an artificial symbolism, interpret this symbolism and then couch our analysis in terms of what these symbols mean. Rather we pair English words and phrases off with their meanings straightaway, using our 'language of constructions' to encode these meanings. TIL does not need a meta-language, since we have a ramified type hierarchy instead.[42]

Q: You don't have a meta-language? That's somewhat unusual in modern logic.

A: It is. Yet we do have a parallel notion of *using* and *mentioning*, only what is used and mentioned are constructions and not words (though, of course, we're also able to quote words, by means of quotation marks). But let me quote Tichý on why TIL does not need a meta-language. Look:

> The whole linguistic outlook of modern logic and metamathematics, the preoccupation with symbols and strings of symbols as objects of study, results from the parsimonious decision to dispense with all entities other than first-order ones […]. The mathematician averts his eyes from constructions, which constitute his real subject matter, and looks at pieces of notation instead. This approach may satisfy his craving for ontological economy, but let it not be thought that it *simplifies* matters. If a range of entities is studied obliquely by means of proxies, rather than directly, the cognitive situation is complicated by the gratuitous intrusion of the proxy relation.

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[41] See Montague (1974a).

[42] For a more detailed comparison of Tichý's TIL with Montague's IL, see Section 2.4.3.

> There is no *intrinsic* relation between a formula and the construction it represents. Hence if anything said about the formula is to have a bearing on things mathematical, the relation of the formula as a whole, or of its constituents, to mathematical objects must be explicitly stipulated. In order to put a stipulation into words, one has to *name* entities of both kinds: the mathematical objects and the linguistic expressions corresponding to them. Hence the need for a metalanguage, distinct and separate from the original notation in question. But the metalinguistic expressions themselves signify constructions. One thus faces a choice: one can either acquiesce in these higher-order constructions, or one can ignore them too and look instead at the meta-meta-expressions corresponding to them. If the first option is chosen the question arises why the same treatment cannot be applied at the bottom level, thus avoiding the original linguistic detour as well. And if the second option is taken one is obviously caught in an infinite regress of ever higher metalanguages (1988, p. 71).

Q: But that direct route to meanings comes with a completely *objectual* vision of logic, right?

A: Right. To get your head around TIL, don't think in terms of language-meets-language; think in terms of language-meets-reality. This reality is the Platonic realm of realist logic and semantics. In fact, what we're studying, at the end of the day, is not language, whether natural or artificial, but the simple and complex objects populating this realm. Language is a gateway, even if it's of independent interest. TIL *is* a philosophy of language, it's just that we think one can't, ultimately, study language by means of language.

Q: Okay, so that's why you replace other people's upper-level languages, or metalanguages, by a sphere of upper-level abstract objects?

A: Exactly. That's what TIL is pretty much all about. Leşniewski and Tarski were good Polish nominalists, so they wouldn't dream of admitting higher-level objects. Instead they erected higher-level languages. We're Platonists, on the other hand, so we agree with Frege that a third realm must be acknowledged. Only we're actually telling you what's in that realm.

Q: Constructions?

A: Constructions!

1.4 Possible-world intensions vs. extensions

1.4.1 Epistemic framework

TIL operates with a single *procedural semantics*, as explained above. TIL constructions are, without exception, assigned to expressions as their structured meanings. But within this one semantics TIL observes a strict demarcation between two kinds of subsidiary semantics: one for logical and mathematical languages and another for empirical languages, whether colloquial or scientific. The demarcation hinges not on formal vs. natural, but on empirical vs. non-empirical. The defining difference is that empirical languages incorporate an element of *contingency* that the non-empirical ones lack. Empirical languages must be able to denote *empirical conditions* that may or may not be satisfied. Non-empirical languages have no need for an additional category of expressions for empirical conditions. Roughly, the semantics for non-empirical languages is simpler, because the intensional level has been lopped off; yet also more complicated, because constructions constructing constructions (for instance, variables of type $*_{n+1}$ constructing constructions of type $*_n$), rather than intensions, are often needed.

For instance, the predicate 'is a student' does not denote each individual that is a student, nor a class of students. Rather, it denotes a *property* of individuals, the 'populations' of which are particular sets of individuals depending on particular states-of-affairs. To master 'is a student' is not to know a particular set of individuals; rather, it is to know how, for any state-of-affairs, to determine whether a given individual satisfies the condition of being a student. We model these empirical conditions as possible-world *intensions* which are functions with domain in possible worlds and values in chronologies of elements of a given type α. Thus we distinguish between the *hyperintension* (i.e., a construction of an intension *I*) assigned to an empirical expression *E* as its meaning, and the possible-world *intension I* denoted by *E*. However, as soon as we introduce what we shall call an *epistemic framework* for a given empirical language, the procedural semantics of the language operates in the same way as in the case of mathematical language. This is so because the epistemic framework assigned to a language confines what can possibly be talked about within that language.

In order to specify the objectual base of TIL over which an infinite ramified hierarchy of types is defined (see Definition 1.7), we must explicate the category of possible worlds. To this end we first need to explain the informal, pre-theoretical epistemic framework of a given empirical language.[43] First of all, the main methodological principle of TIL-based logical analysis of natural language has been formulated in Tichý as follows:

> To explicate a system of intuitive, pre-theoretical, notions is to assign to them, as surrogates, members of the functional hierarchy over a definite objectual base. Relations between the intuitive notions are then represented by the mathematically rigorous relationships between the functional surrogates (1988, pp. 194–95).

Everybody has a pre-theoretical understanding concerning reality, according to which there are things doing things and doing things to other things. A first approximation of a theoretical explication of this intuition would amount to saying that reality consists of individuals exemplifying properties and occurring in relations. By even just beginning to offer an explication along these lines, the formal semanticist has embarked upon the enterprise of providing a logical surrogate of reality. This surrogate is not supposed to render reference to reality superfluous; instead it must run in parallel to reality. The surrogate is the framework within which a semantic theory is stated. The things, in the widest possible sense, which

[43] For further background, see Tichý (1988, pp. 177–200).

are represented by a surrogate in the framework are the things that can possibly be talked about in some given language *L*. The overall project of TIL is (nothing less than) the explication of the framework underlying natural language, so *L* is not a particular national language, but any natural language.

Any successful linguistic communication between language-users makes use of a shared framework.[44] Tichý says,

> Communication between speakers and their audiences can only succeed on the basis of a shared logical space (1988, p. 201).
> To tell someone that Ali is sick I must somehow draw his attention to the [propositional] construction $\lambda w \lambda t\ [{}^0 Sick_{wt}\ {}^0 Ali]$. Communication is exchange of linguistic constructions over [an objectual base] (1986a, p. 264, 2004, p. 662).

To account for the expressive power of a given language shared by a community of language-users, Tichý introduces the concept of *epistemic framework* and the concepts of intensional and objectual bases affiliated with it. The goings-on of extra-theoretical reality make up the pre-theoretic *intensional base*, and the intensions defined over an *objectual base* attempt to capture them intra-theoretically. They do so by means of assignments to the functions defined over the objectual base. Tichý calls the totality of these assignments an 'explication' of the intensional base by means of the objectual base. An epistemic framework is then an intensional base garnished with an explication.

For instance, the string 'Ali is sick' presupposes, in order to have the sense it has in English, that it belongs to a language interpreted over an epistemic framework that comes with individuals, properties and a vehicle of predication.

Why is it important to point out that successful communication presupposes a shared epistemic framework common to all the parties to a discourse? Because the framework reconstructs the range of expressions a speaker or hearer can possibly make sense of. An expression which falls outside the purview of the framework is without sense (i.e., strictly speaking, not an expression at all, but a string of letters or sounds).[45,46]

The pre-theoretically understood elements of the *objectual base B* may in principle be pretty much whatever. But for the purposes of natural-language analysis, it has turned out that the elements must include, at least, *truth-values*, *individuals*, *times*, and *possible worlds*. Formally, $B = \{o, \iota, \tau, \omega\}$, each element of which is a non-empty set and disjoint from any of the three other sets. These four kinds of

[44] This holds no less for communication between solitary language-users and themselves in the form of inner soliloquies, as ought to be uncontroversial as far as philosophical theses go. We also tend to think that unverbalized thinking is impossible without the use of (a non-private) language; but we are not broaching this issue here.

[45] Cmorej calls such a string a 'semi-expression' in his 2005 discussing the thesis that semantics is a priori.

[46] In a wider philosophical context, the notion of epistemic framework might be of use to *hermeneutics*; e.g., with respect to Gadamer-like melting-together (*Verschmelzung*) of two or more different epistemic frameworks. We have not attempted to take the notion into this direction, though.

object are all non-functions, and cannot be defined (though characterised) within TIL. They are, in a word, logically primitive relative to *B*. However, the functions arising from *B* by combining elements drawn from it can be defined within TIL; this is required if we wish to display functional dependencies in accordance with our functional approach.

Explication of pre-theoretical intuitions consists, by and large, in offering an analysis of how α-objects are functionally dependent on β-objects. In hyperintensional contexts the analysis becomes more involved, since it must be spelt out how the relevant function(s) is (are) constructed. However, not everything can be either a function or a construction. Some objects serve as functional arguments or values without themselves being functions;[47] they are the 'rock-bottom' objects of the given epistemic framework. The elements of the members of *B* serve as arguments for intensions, and cannot be analysed within TIL without incurring circularity. It is particularly important that a state-of-affairs which is said to obtain at some world *W* not be conceived as a function from worlds to (chronologies of) extensions, but as entities being atomic relative to the given epistemic framework. The objectual base *B*, for its part, can be thought of as being among the fundamental ontological assumptions—or 'ontological commitments', if you like—of TIL.

A most important part of the explication is the interpretation of possible worlds. It goes as follows:

> By an *intension/time* I shall understand an ordered couple consisting of a member of the intensional base and a moment of time. A *determination system* is then an assignment which assigns to (some) intension/times unique objects over $\{\iota, o, \tau, \omega\}$ in such a way that if the type corresponding to the intension is $\xi_{\tau\omega}$[48] then the object assigned to the intension/time is ξ. Briefly, a determination system specifies one combinatorial possibility as to what objects are determined…by what intensions at what times. Now to interpret the basic category ω is to assign to each of its members a unique determination system (Tichý, 1988, p. 199).

The notion of epistemic framework is indispensable within TIL—as well as within any other theory of philosophical logic directed toward natural-language analysis—as it regulates the relationship between artificial and natural language. If the intensional base was skipped and the starting-point was the objectual base instead, TIL would be exactly what Tichý takes other intensional systems to task for being; namely, nothing but a logical game. In the form of a rhetorical question, what would be the purpose of defining an infinity of functions of type(s) $\alpha_{\tau\omega}$ if they were not somehow anchored to (fragments of) reality external to the system? In brief, intensions are not to be made sense of by means of another language that natural-language terms are translated into, but by being paired off with the pre-theoretical grasp of reality we all have to the effect that things do things to things.

[47] To be sure, in mathematics we can *model* them as zero-arity functions. But this hardly *makes* them functions.

[48] See below; it is the type of a function $(\omega \rightarrow (\tau \rightarrow \xi))$ for a type ξ.

Still, any explication will have to cut corners in order to match, at least to some tolerable degree, extra-systematic reality and so cannot be expected to cut it cleanly by the joints. Fine-tuning an explication will come down to making the type-theoretic analysis more sophisticated or, more drastically, adding new types to *B*. In fact, the latter has already happened more than once. Whereas the original type-theories included only individuals and truth-values (or only individuals, if truth-values were just individuals), every possible-world semantics with a type-theory will have to add worlds as a type (if only in the half-hearted manner of Montague). Later on, when Tichý realised the need for numbers and times as an independent type, type τ was added. Kaplan (1975) is another possible-world semanticist with the same simple type theory as TIL, since also his intensions are defined over times as well.

In a nutshell, the enterprise of logical analysis of natural language has as its ultimate (and extremely ambitious) goal the exhaustive explication of the intensional base underlying natural language; i.e., its epistemic framework *in totō*. As Frege famously said in a not all-too dissimilar connection, *dahin gelangen wir nie*. So the goal is of a regulative nature. In what follows the *epistemic framework* that TIL assigns to natural language is described.[49]

Universe of discourse (type ι). The members of the universe are *individuals.* The individuals are *bare individuals*. This means that all the properties possessed by an individual necessarily are, roughly, trivial. In Section 1.4.2 we will explain in which sense some properties are trivial. For now, trivial properties are either *constant* functions (i.e., properties that have a constant extension—a set of individuals—as value in all possible worlds and times) or *partially constant* functions (whose extension varies for some possible worlds/times) with a *constant subset* of their possible extensions. All purely non-constant properties (without a non-empty constant subset of all possible extensions) are had by an individual only contingently. A bare individual is, then, what remains if one abstracts from all its non-trivial properties. From a logico-semantic point of view, a bare individual is simply a peg on which to hang properties. Another important feature of the universe is that it is one in number; there are no other universes/domains in other possible worlds, so there are no *possibilia* ('possible individuals').

Truth-values (type ο). There are just two truth-values, **T** and **F**. So TIL is a bivalent logic and insofar classical. TIL comes with truth-value gaps, however, and is insofar not classical. Any abstract objects can serve as surrogates, but we have to interpret them, so we say that **T** is the truth-value True and **F** the truth-value False.

Times or *real numbers* (type τ). The easy interpretation is described in Tichý (1988, p. 199); choosing the origin 0 of the time scale and a specific duration of time between 0 and the time represented by 1, we get the result that every real number will represent a unique instant of time, and vice versa. In TIL time forms a *continuum*. Alternatively, times could have been paired off with natural numbers,

[49] As of early 2010.

making times *discrete* instead. And, in fact, discrete times will often suffice for the purposes of analysis of natural language. But in order to avoid that the times we are analysing should outstrip our capacity to model them (to avoid running out of time(s), as it were), we are playing safe and modelling times as continuous straightaway.

Possible worlds (type ω). Consider an intensional base (relative to a given language). Every member of the intensional base conjugated with a time singles out some object, and every *possible world* is interpreted as specifying 'one combinatorial possibility as to what objects are [singled out]… by what intensions (i.e., members of the intensional base) at what times' (Tichý, 1988, p. 199).

This construal of possible worlds is distinct from many other conceptions, not least D. Lewis', according to which all possible worlds are concrete and actual *sub specie aeternitatis* (see his 1986). Nor are our possible worlds sets of formulae or Carnap-style state descriptions. Our construal is *Tractarian* in that it takes possible worlds as collections of *states-of-affairs* rather than of objects. Possible worlds, as we understand them, are the maximal consistent sets of chronologies of possible states-of-affairs.[50]

1.4.2 Intensions and extensions

The previous section provided the philosophy of intensions. In this section their logic follows.

Definition 1.8 ((α-)*intension*, (α-)*extension*) (*α-*)*intensions* are members of a type $(\alpha\omega)$: functions from possible worlds to the arbitrary type α; (*α-*)*extensions* are objects of the type α, where α is not equal to $(\beta\omega)$ for any β; i.e., extensions are α-objects that are not functions from possible worlds. □

Remark. Intensions are frequently functions of the type $((\alpha\tau)\omega)$, i.e., functions from possible worlds to *chronologies* of type α (in symbols: $\alpha_{\tau\omega}$), where an α-chronology is a function of type $(\alpha\tau)$.

Remark. It is a noteworthy upshot of our general top-down approach that extensional entities are defined negatively and in terms of intensional entities; namely, as those objects that are not intensions. In case of an ordinary language extensional entities are of logical and semantic interest only insofar as they figure as values (or in the values) of intensions.

[50] Also Hintikka seems to accept this conception, but his possible worlds are *epistemic*, dependent on particular language-users (See, e.g., Hintikka and Hintikka, 1989).

We will use variables w, w_1, w_2,… as v-constructing elements of type ω (possible worlds), and t, t_1, t_2, … as v-constructing elements of type τ (times). If C v-constructs an α-intension, the frequently used Composition of the form $[[C\ w]\ t]$, v-constructing the intensional descent, or extensionalization, of an α-intension, will be abbreviated as 'C_{wt}'.

Intensions may come in different *orders*, due to type raising, and in different *degrees*.

An intension is a *higher-order* entity if its range is made up of higher-order entities. For instance, a relation-in-intension relating individuals to constructions, as in the case of hyperintensional attitudes, is higher-order. E.g., *Believe**, *Know** are entities of type $(o\iota *_n)_{\tau\omega}$, i.e., entities belonging to a type of order $n+1$, $n \geq 1$.

An intension is first-order, but of a *higher degree* than zero, if its range is made up of first-order intensions; i.e., any such intensions as do not include constructions. For instance, *the tallest_mountain*/$\iota_{\tau\omega}$ is of degree 1, because its (world- and time-relative) values are themselves extensional entities (individuals), while *the most characteristic property of a war criminal* is an entity of type $((o\iota)_{\tau\omega})_{\tau\omega}$, i.e. an intension of order 1 and degree 2, because its values are themselves intensional entities of degree 1 (properties of individuals).

Extensional entities also come in different orders. For instance, the *set* of all n-order constructions with some particular property is an extensional n-order entity of type $(o *_n)$.

Some important kinds of intension are:

Proposition/$o_{\tau\omega}$. They are denoted by empirical (declarative) sentences. Propositions are truth-values-in-intension.

Property of members of a type α, or simply α-*property*/$(o\alpha)_{\tau\omega}$.[51] General terms (some nouns intransitive verbs, adjectives) usually denote properties, mostly of individuals. Properties are sets-in-intension.

Relation-in-intension/$(o\beta_1\ldots\beta_m)_{\tau\omega}$. For example, transitive empirical verbs and attitudinal verbs denote such relations. If omitting $_{\tau\omega}$, we get the type $(o\beta_1\ldots\beta_m)$ of *relation-in-extension* (to be found mainly in mathematics and logic).

α*-role*/α*-office*/$\alpha_{\tau\omega}$, $\alpha \neq o$, $\alpha \neq (o\beta)$, $\alpha \neq (o\beta_1\ldots\beta_m)$, frequently $\iota_{\tau\omega}$; often denoted by the concatenation of a superlative and a noun ('the highest mountain'). An individual role corresponds to what Church (1956) calls an 'individual concept'. This word could cause misunderstandings, since *concept* in TIL is no intension, so we shan't use it.[52] Individual offices are individuals-in-intension.

[51] Remember that collections, sets, classes of α-objects are members of type $(o\alpha)$; TIL handles classes (subsets of a type) as characteristic functions. Similarly, relations (-in-extension) are of type(s) $(o\beta_1\ldots\beta_m)$.

[52] For the theory of concepts, see Section 2.2.

Example 1.5 *Types of intensional objects*

- 'Being happy', or 'is happy', denotes a property of individuals/$(o\iota)_{\tau\omega}$. Given a possible world and a time, we are given the class of individuals that are happy at that world/time pair.
- 'The President of the Czech Republic' denotes an individual office, a.k.a. individual role/$\iota_{\tau\omega}$. Given a possible world and a time, we are given the individual, if any, who occupies the office, or plays the role, of President of the Czech Republic at that world/time pair. At some world/time pairs, there is no such individual (the function being properly partial).
- 'The King of France is happy' denotes a proposition/$o_{\tau\omega}$. If $\langle w, t\rangle$ is such a pair of worlds and times where the role of King of France is occupied by an individual X and X is happy at $\langle w, t\rangle$ then the proposition is true at $\langle w, t\rangle$. If X is not happy at $\langle w, t\rangle$ the proposition is false at $\langle w, t\rangle$. If the office of King of France is not occupied at $\langle w, t\rangle$ (as in the actual world now), the proposition lacks a truth-value.[53]
- 'Calculating' denotes an attitude of an individual to a construction, i.e., a relation-in-intension that is a higher-order intension of type $(o\iota*_n)_{\tau\omega}$.
- 'Knowing* (explicitly)' denotes an attitude of an individual to a construction, i.e., a relation-in-intension of a higher-order type $(o\iota*_n)_{\tau\omega}$.
- 'Knowing (implicitly)' denotes an attitude of an individual to a proposition, i.e., a relation-in-intension that is a higher-degree intension of the first-order type $(o\iota o_{\tau\omega})_{\tau\omega}$.[54]

For an example of the distinction between mathematical and ordinary language, consider the sentence

'The number of the planets is 8.'

This sentence does not denote a truth-value, but a proposition/$o_{\tau\omega}$, and its meaning is a construction of the denoted proposition, namely a *hyperproposition*:

$$\lambda w\lambda t\ [{}^0{=}\ [{}^0Number\_of\ {}^0Planet_{wt}]\ {}^08].$$

Types: *Number_of*/$(\tau(o\iota))$: the cardinality function that returns the number of elements of an $(o\iota)$-set; *Planet*/$(o\iota)_{\tau\omega}$; $=/(o\tau\tau)$; $8/\tau$.

The denoted proposition is an empirical truth-condition that is satisfied only by those worlds and times at which the number of planets is 8.[55] Provided these are post-Plutonic times then (for all that is commonly known) there are exactly eight planets in the Solar system. If so, then it is a contingent truth. If not, then it is a

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[53] See Section 3.1 dealing with definite descriptions.

[54] See Section 5.1 dealing with propositional attitudes.

[55] We are presupposing—naïvely, as it happens—the existence of a definition of the property of planethood that will decide unequivocally for any celestial body in our solar system whether it is a planet.

contingent falsehood. The example demonstrates that 'The number of the planets' cannot be a name of 8, nor that 'The number of the planets is 8' can be a name of the truth-value **T** (or **F**, for that matter). For then the semantic naming relation would fluctuate in accordance with either astronomical facts or our presumed knowledge of such facts.

What does denote a truth-value is the sentence

'The number of elements in {Mercury, Earth, …, Neptune} is 8.'

It denotes the truth-value **T**/ο, and its meaning is the Composition

$$[{}^0{=}\ [{}^0\mathit{Number\_of}\ {}^0S]\ {}^08].$$

Type: S = {Mercury, …, Neptune}/(οι).

Whatever, if any, the planets of a solar system may be, it is a mathematical truth that the set {Mercury, Earth, …, Neptune} has 8 elements. Making an inventory of the planets of a solar system does not consist in counting the number of elements in sets of planets. It consists in applying the empirical condition of being a planet to the celestial bodies of the solar system in question.

1.4.2.1 Classification of empirical properties

In Chapter 4 we will explain in detail how two intensions may be conceptually related in such a way that having one necessitates having the other as well. When there is such necessitation, we say that one intension is *essential* of the other. It is intensions, and not extensions such as individuals, that are the bearers of essential properties. Instead our individuals are 'bare' in the sense that no non-trivial intension is necessarily true of them.

However, it remains at this point in time an open issue whether it is possible that a ι-object may lack all non-trivial properties at some $\langle w, t\rangle$. If this is possible, then such an individual will be 'bare' in a more dramatic sense than just not possessing any non-trivial properties necessarily (which is already considered dramatic enough in several quarters).[56]

Consider three ways of analysing 'the man without properties' (example courtesy of Robert Musil).

First analysis:

$$\lambda w\lambda t\ [{}^0\mathit{Sing}\ \lambda x\ [[{}^0\mathit{Man}_{wt}\ x] \wedge [\forall p\ \neg[p_{wt}\ x]]]].$$

Second analysis:

$$\lambda w\lambda t\ [{}^0\mathit{Sing}\ \lambda x\ [\forall p\ \neg[p_{wt}\ x]]].$$

[56] Now we are using 'trivial' and 'non-trivial' intuitively. By 'trivial' we do not mean epistemically trivial. Once we explain what is meant by 'trivial', we will use rigorous terms instead.

Third analysis:

$$\lambda w \lambda t\ [{}^0 Sing\ \lambda x\ [\forall p\ [[p_{wt}\ x] \supset [{}^0 Triv\ p]]]].$$

Types: *Sing*/(ι(οι)): the singulariser function that associates a singleton *S* with its only member and is otherwise undefined;[57] $\forall/(o(o(o\iota)_{\tau\omega}))$: the general quantifier over ι-properties; $Man/(o\iota)_{\tau\omega}$; $p \to (o\iota)_{\tau\omega}$; $x \to \iota$; $Triv/(o(o\iota)_{\tau\omega})$: the class of trivial ι-properties.

The first analysis is a construction of a ι-office occupiable by any individual who has the property of being a man and at the same time no properties at ⟨*w*, *t*⟩. Since *Man* is in the domain of *p*, the conjuncts cannot both be true.

The second analysis is a construction of a ι-office occupiable by any individual who has no properties at ⟨*w*, *t*⟩. Since every individual has the property of being self-identical, this office is necessarily vacant. Hence both constructions construct the 'impossible' ι-office, which is necessarily vacant. (Similarly, the property of being an *x* such that *x* has no *p* is paradoxical, since it is in the range of *p*.)

The third analysis is a construction of an ι-office occupiable by an individual that does not have any non-trivial properties. The question is whether this office is ever occupied. The answer will depend on how restrictive or how liberal a notion of *non-trivial* ι-property is used; i.e., what the class *Triv* is taken to be. It certainly contains all *constant* properties, i.e., the properties that have a constant set of individuals as a value at all ⟨*w*, *t*⟩. One of them is self-identity, which every individual necessarily possesses. However, should we take on board Cambridge-like properties in the vein of *being an x such that x is the same height as Kim Jong Il*? Whatever height Comrade Kim may have at this or that ⟨*w*, *t*⟩, it is necessary that he have exactly the same height as Kim Jong Il. The trick is to index a property to a specific individual *a*, such that, necessarily, *a* must have that property, without using a trivial, constant property such as being self-identical. *Being the same height as Kim Jong Il* is a contingent property, for it is not a constant function. Not all individuals have the same height as Kim Jong Il at all worlds and times, so the sets that are its extensions at various ⟨*w*, *t*⟩-pairs will not always have the same members. But, due to the indexing, one individual can always be relied upon to be in whatever set is the extension at whatever ⟨*w*, *t*⟩; to wit, Kim himself. So the intension *being the same height as Kim Jong Il* is insofar *partially constant*. The property has an *essential core*: namely, the set {Kim Jong Il}.[58] Similarly, the contingent, i.e. non-constant, property *being the same age as a or b* has the essential core {*a*, *b*}. All individuals but *a*, *b* have this property contingently; only *a*, *b* have it necessarily. If the intension is non-trivial, its non-triviality is 'partial' or 'impure'; and if trivial, then its triviality is also impure. We will call such a property 'partially constant'.

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[57] See Definition 1.6.

[58] The term 'essential core' was coined by Pavel Cmorej (1996). See also Cmorej (1988, 2006).

Intensions that have constant values in all worlds and times are certainly trivial. However, as explained above, some non-constant, contingent properties can also be necessarily applicable or inapplicable to some individuals (though not to just any individual), and are in some sense also trivial. Thus the characterisation of the class *Triv* has to be extended. The general direction in which to look for an answer is indicated by Tichý's distinction between *primary* and *parasitic* properties.

> A change in a thing clearly consists in the acquisition or loss of a property. But if any property is as good as any other, we get the odd result that a thing cannot change without every other thing changing as well. Suppose object *X* becomes red and consider another object, *Y*. *Y* will be spatially related to *X* in a definite way; suppose it is 50 miles due south from *X*. Then as *X* acquires redness, *Y* acquires the property of being 50 miles due south from a red object. This change in *Y*, however, is obviously a phoney change, because the property of being 50 miles due south from a red object is a phoney, parasitic property. It is a property which will not figure in the specification of a possible world. To specify a possible world, one has to specify, *inter alia,* where each object is and what colour it is. Once all this has been fixed, there is no need to specify which objects have the property of being 50 miles due south from red objects; for all this has been implicitly specified already. While the extension of redness is part of what makes a world the world it is, the extension of the property of being 50 miles due to south from a red object is not. It is a parasitic property, a mere logical shadow cast by genuine—or, as we will say, *primary*—properties like being red and being at a certain place. For a thing to change, it must acquire or lose not any arbitrary property, but a primary one. We have seen that the possible worlds of a logical space are generated as distributions of the attributes in the intensional base through things. It is thus natural to identify primary properties, relations, etc. with those which correspond to the members of the intensional base
> (Tichý, 1980b, p. 271, 2004, p. 419).

As explained in Section 1.4.1, every language is based on a definite universe of discourse (i.e., a collection of *individuals*) and an *intensional base*, which is the collection of primary intensions[59] that the given language has predicates for. The objectual base (o, ι, τ, ω), together with a definite interpretation of o, τ, ω, forms an epistemic framework. Possible worlds are then possible chronologies of distributions of members of the intensional base over individuals.

Hence primary properties are certainly *contingent*, *non-constant* and thus non-trivial. No individual has a primary property of the intensional base necessarily, i.e., in all $\langle w, t\rangle$. So there is no non-empty constant subset of the possible extensions of a primary property.

Some of the derivative properties parasitic upon the *primary properties* are also contingent, like the above property *being* 50 *km due south from a red object.* It is a contingent fact that an object *X* possesses at some time the property *being red.* This fact implies infinitely many facts where *derivative* properties play a role; for example, an object *Y* that happens to be 50 km due south from *X* gets the derivative property *being* 50 *km due south from a red object.* And *Y* does not have this

[59] The distinction between 'primary' vs. 'secondary' intensions is not to be confused with some other distinctions like, e.g. Evans' 'deep' vs. 'superficial' intensions or what also goes under the name 'primary and secondary intensions' in two-dimensional semantics. See Evans (1977).

property of *logical* necessity. However, Y necessarily has the derivative property of *not* being 50 km due south from *itself*.[60]

Note that the ‘derivateveness’ of a property does not concern a construction of the property. Any property can be constructed in infinitely many ways. Rather, it concerns necessary dependencies between the respective facts and thus properties as well. For instance, the fact that an individual a is this or that age is logically contingent. But there is a necessary correlation between a being 50 and a not being younger than 30. It is impossible that a be 50 and at the same time younger than 30. As we will explain in Chapter 4, there are so-called *requisite* relations between intensions. On the other hand, there are no such dependencies between primary properties of the intensional base; the respective basic facts are independent, parallel to the *Tractarian* conception of *Tatsachen*.[61]

As explained above, non-constant, contingent properties with an essential core are *partly constant*. They are *essential of some* individuals, namely of those belonging to the relevant essential core. All other individuals contingently have, or do not have, these properties. Hence, if P is a partly constant property, then there are at least two world/time pairs $\langle w, t\rangle$, $\langle w', t'\rangle$, such that P_{wt} is not the same set as $P_{w't'}$. There is, however, a constant subset of the varying extensions of P, namely the essential core of P.[62]

Our hypothesis is that *partly constant properties* with an essential core are parasitic on *reflexive* relations-in-intension, where a reflexive relation-in-intension is an entity $R/(o\iota\iota)_{\tau\omega}$ such that, necessarily, its value in $\langle w, t\rangle$ is a reflexive relation-in-extension:

$$\forall w \forall t\ [\forall x\ [{}^0R_{wt}\ x\ x]].$$

The relations of being the same height as some individuals, of being of the same age, of not being 20 years older than, etc., can serve as examples. Of course, since *being the same age as* is necessarily reflexive, an individual a cannot be a different age than a, unless a would, bizarrely, lose its identity.

On the other hand, *purely constant properties* are functions having the same set of individuals as value in all worlds w at all times t.[63] Thus if P is a purely constant property, the set P_{wt} is the same in all $\langle w, t\rangle$, and it is the essential core of P. Every individual belonging to P_{wt} has P at all $\langle w, t\rangle$, and every individual not belonging to P_{wt} lacks P at every $\langle w, t\rangle$. The essential core of a purely constant

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60 We do not consider here subatomic particles of quantum physics, of course. After all, Heisenberg’s uncertainty principle has a negligible effect on objects of macroscopic scale.

61 The claim that there are no dependencies between primary properties of the intensional base requires qualification, however. Consider *being red* and *being blue*. Neither is parasitic upon the other, but at the same time they are dependent, by being defined in terms of their respective positions in a spectrum.

62 Cmorej (2006) calls these properties *partly essential*.

63 Cmorej (2006) calls these properties *essential.*

property P is either equal to the whole universe or is a proper subset of the universe. An example of the former would be the property of being self-identical, constructed by $\lambda w \lambda t\ \lambda x\ [x = x]$; examples of the latter would be the properties of being identical to a particular individual a, $\lambda w \lambda t\ \lambda x\ [x = a]$, being identical to an individual a or b, $\lambda w \lambda t\ \lambda x\ [[x = a] \vee [x = b]]$, being identical to neither a nor b, $\lambda w \lambda t\ \lambda x\ [\neg[x = a] \wedge \neg[x = b]]$; etc.

To sum up, a property P belongs to the class *Triv* iff P has a non-empty essential core *EC*. Individuals belonging to *EC* have P necessarily. So the property P is essential of the elements of *EC*. Properties with a non-empty essential core are either purely constant or partly constant. The former are constant intensions and the latter contingent.

Now we can classify individual properties according to different criteria into the following categories.

Partiality criterion:

- *Purely partial properties.* A property P is *purely partial* iff there is a world w and a time t at which P has no extension: $[{}^0P_{wt}]$ is v-improper.[64]
- *Partial properties.* A property P is *partial* iff there is a world w and a time t at which the characteristic function v-constructed by ${}^0P_{wt}$ is purely partial; equivalently, there is an individual a such that $[{}^0P_{wt}\ {}^0a]$ is v-improper.

For instance, the property of having stopped smoking is partial. If *StopSmoking*$/(o\iota)_{\tau\omega}$ is this property, then $[{}^0StopSmoking_{wt}\ x]$ v-constructs **T** if individual x used to smoke and stopped smoking, **F** if x used to smoke and did not stop smoking. Finally, $[{}^0StopSmoking_{wt}\ x]$ is v-improper if x never smoked.

Now let P be a property that is not purely partial. Then we can further apply the

Criterion of contingency or non-contingency:

- A property P is *constant* (or *non-contingent*) iff P has the same extension in all worlds and times, where the extension is defined as follows:

$$[\iota c\ \forall w \forall t\ [c = {}^0P_{wt}]], \text{ where } c/*_1 \to_v (o\iota).$$

If a property P is constant, then its extension is its essential core.

The property of being *self-identical*, constructed by $\lambda w \lambda t\ \lambda x\ [x = x]$, $x \to \iota$, is an example of a constant property; the essential core of this property is the set of all individuals. An example of a constant property with an empty essential core is the property of not being identical with itself, $\lambda w \lambda t\ \lambda x\ [x \neq x]$.

The property of being identical to a or b, constructed by $\lambda w \lambda t\ \lambda x\ [[x = a] \vee [x = b]]$, is another example of a constant property. Its essential core is the set $\{a, b\}$. The other individuals necessarily lack this property.

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[64] We add this category just for completeness. Purely partial properties are bizarre properties like the one defined as follows: $\lambda w \lambda t\ \iota c\ [[c = \varnothing] \wedge \neg\ [c = \varnothing]]$, where $c/*_1 \to_v (o\iota)$.

- A property P is *non-constant* (or *contingent*) iff there are at least two distinct extensions of P. In other words, there are world/time pairs $\langle w_1, t_1\rangle$, $\langle w_2, t_2\rangle$ such that ${}^0P_{w1t1} \neq {}^0P_{w2t2}$.

If P is a non-constant (contingent) property, then we can further distinguish between a partially constant and a purely contingent property:

– A non-constant property P is *partially constant* iff there is a non-empty essential core of P. The essential core of a non-constant property P is defined as follows:

$$\iota c\ [\exists x\ [cx] \wedge [c = \lambda x\ [\forall w \forall t\ [{}^0P_{wt}x]]]], \text{ where } c/*_1 \rightarrow (o\iota).$$

Obviously, the essential core of a non-constant property P is the smallest non-empty subset of all the possible extensions of P.

If P is a contingent property with a non-empty essential core, then P is *partially contingent*; or equivalently, *partially constant*. We have decided in favour of the latter characterization in order to stress that P is constant with respect to *some* individual(s) and contingent with respect to others.

For example, the property of being of the same height as a or b is constant with respect to a and b. Its essential core is the set $\{a, b\}$. The other individuals contingently have this property or contingently lack it. It seems that all partially constant properties are based on a reflexive relation. But we are not going to assume, let alone attempt to prove that this is so, we treat it only as a hypothesis.

– A property P is *purely contingent* (or *purely non-constant*) iff P is neither constant nor partially constant. In other words, there is no non-empty essential core of P.

As examples of purely contingent properties, think of *being happy, weighing 88 kg*.

Our *individual anti-essentialism* thus qualifies as a 'modest' one:[65]

If an individual a has a property P necessarily (i.e., at all w, t), then P has a non-empty essential core *Ess* and the individual a is an element of *Ess* (i.e., P is a constant or partly constant function). Formally,

$$\forall p\ [[\exists x\ \forall w \forall t\ [p_{wt}\, x]] \supset [[{}^0Constant\ p] \vee [{}^0Partially\_constant\ p]]]$$

[65] The idea of *modest anti-essentialism* owes much to Pavel Cmorej.

where $x \rightarrow \iota$; $p \rightarrow (o\iota)_{\tau\omega}$; *Constant, Partially_Constant*/$(o(o\iota)_{\tau\omega})$ are the classes of constant or partially constant properties, respectively.

Figure 1.4 illustrates particular kinds of properties (*Ess* is here the essential core of *P*).

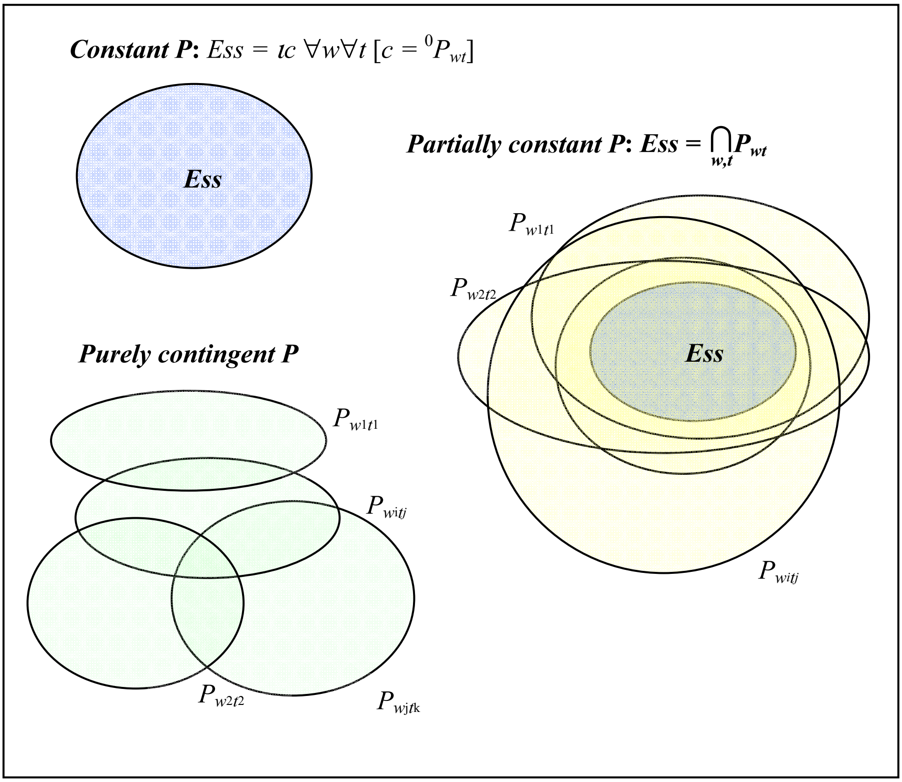

Fig. 1.4 Schema of constant, partially constant/contingent, and purely contingent properties

Now we are in a position to answer the question raised at the outset of this section of whether it is possible that an individual may lack all purely contingent (non-constant) properties at some $\langle w, t\rangle$. The answer is No. To show why, we use an example of a more outlandish property than *being the same height as King Jong Il*, namely, the property *being self-identical and the time is T* (for instance, noon on April 1, 2010).[66] One of its constructions is (T/$(o\tau)$ being some fixed interval of times)

$$\lambda w \lambda t\, \lambda x\, [[x = x] \wedge [{}^0T\, t]].$$

[66] This example is due to Pavel Cmorej.

(The construction $[{}^0T\ t]$ suffices, because it is immaterial how the proposition that the time is 12 o'clock on April 1, 2010 is constructed.) An individual satisfies this property if it is self-identical and the time is T when it is tested for self-identity. The time is not always T, so the property is not constant. But each x is self-identical. Hence, each individual has such properties, and there are no strictly bare individuals. However, as explained above, such a phoney property is derivative and not a member of the intensional base.

Apart from dividing properties into constant and non-constant, partly constant and purely contingent, there is another criterion, according to which properties divide into *empirical* and *analytical*. An empirical property is a property P such that for *no* individual a is it decidable a priori whether P applies to a. It must always be established a posteriori. On the other hand, an analytic property P' is decidable a priori for all individuals. Obviously, purely constant properties are analytic, and purely contingent properties are empirical. Partly constant/contingent properties should be decidable analytically a priori with respect to the individuals belonging to the essential core. Of course, we do not need experience in order to decide whether an individual a is the same age as a or b.[67]

A note on *self-predication*. Muskens cites 'Having fun is fun' as an example of self-predication (2005, p. 485). We do not think it qualifies as one, though. The first occurrence of 'fun' is as a noun and the second as an adjective (like 'funny'). Better examples of apparent self-predication would be, 'Being nice is nice' and 'It is fine to be fine'. A type-theoretic analysis shows that the two respective occurrences of 'nice' and 'fine' denote entities of different types. One occurrence denotes entities of type $(o(o\iota)_{\tau\omega})_{\tau\omega}$, which are empirical properties of ι-properties. The other occurrence denotes ι-properties/$(o\iota)_{\tau\omega}$. If $F/(o\iota)_{\tau\omega}$ and $F^*/(o(o\iota)_{\tau\omega})_{\tau\omega}$, the analysis is $\lambda w \lambda t\ [{}^0F^*_{wt}\ {}^0F]$.

Self-predication is never an option in TIL, unlike what type-free logics like Bealer's allow for.

[67] However, as Cmorej points out in 1988, it is an open question whether there are properties that are partly constant in a less obvious way, for which the respective essential core would be decidable only a posteriori. The thoughts on how to categorize properties arose from a discussion with Cmorej in 2005.

1.4.2.2 The part-whole relation

In Section 1.4.2.1 above, we broached the thesis of *modest individual anti-essentialism*:

> If an individual *I* has a property *P* necessarily (i.e., at all worlds and times), then *P* has a non-empty essential core *Ess* and *I* is an element of *Ess* (i.e., *P* is a constant or partly constant function).

There is, however, a frequently voiced objection to individual anti-essentialism. If, for instance, Tom's only car is disassembled into its elementary physical parts, then Tom's car no longer exists; hence, the property of being a car is essential of the individual referred to by 'Tom's only car'. Our response to the objection is this. First, what is *denoted* (as opposed to *referred to*) by 'Tom's only car' is not an individual, but an individual office, which is an intension having occasionally different individuals, and occasionally none, as values in different possible worlds at different times. Whenever Tom does buy a car, it is not logically necessary that Tom buy some one particular car rather than any other. Second, the individual *referred to* as 'Tom's only car' does not cease to exist even after having been taken apart into its most elementary parts. It has simply lost some properties, among which the property of being a car, the property of being composed of its current parts, etc, while acquiring some other properties. Suppose somebody by chance happened to reassemble the parts so that the individual would regain the property of being a car. Then Tom would have no right to claim that this individual was *his* car, in case it was allowed that the individual had ceased to exist. Yet Tom should be entitled to claim the reassembled car as his.[68] Therefore, when disassembled, Tom's individual did not cease to exist; it had simply (unfortunately) obtained the property of completely disintegrating into its elementary physical parts. So much for modest individual anti-essentialism.

The second thesis we are going to argue for is this. A material entity that is a mereological sum of a number of parts, such as a particular car, is—from a logical point of view—a simple, hence unstructured individual. Only its design, or *construction*, is a complex entity, namely a structured procedure. This is to say that a car is not a structured whole that organizes its parts in a particular manner. Tichý says:

> [A] *car* is a simple entity. But is this not a *reductio ad absurdum*? Are cars not complex, as anyone who has tried to fix one will readily testify?
>
> No, they are not. If a car were a complex then it would be legitimate to ask: Exactly how complex is it? Now how many parts does a car consist of? One plausible answer which may suggest itself is that it has three parts: an engine, a chassis, and a body. But an equally plausible answer can be given in terms of a much longer list: several spark plugs, several pistons, a starter, a carburettor, four tyres, two axles, six windows, etc. Despite

[68] As Tichý argues in 1987, where he uses the example of a watch being 'repaired' by a watchmaker in such a way as to become a key.

> being longer the latter list does not overlap with the former: neither the engine, nor the chassis nor the body appears on it. How can that be? How can an engine, for example, both be and not be a part of one and the very same car?
>
> There is no mystery, however. It is a commonplace that a car can be *decomposed* in several alternative ways. … Put in other words, a car can be *constructed* in a very simple way as a mereological sum of three things, or in a more elaborate way as a mereological sum of a much larger set of things (1995, pp. 179–80).

It is a contingent fact that this or that individual consists of other individuals and thereby creates a mereological sum. Importantly, *being a part of* is a relation between *individuals*, not between intensions. There can be no inheritance or implicative relation between the respective properties ascribed to a whole and its individual parts. It is vital not to confuse the *requisite* relation, which obtains between intensions, with the *part-whole* relation, which obtains between individuals. The former relation obtains of necessity (e.g., necessarily, any individual that is an elephant is a mammal), while the latter relation obtains contingently.[69] Logically speaking, any two individuals can enter into the part-whole relation. One possible combination has Saturn a part of Socrates (or vice versa). There will be restrictions on possible combinations, but these restrictions are anchored to nomic necessity (provided a given possible world at which a combination of individuals is attempted has laws of nature at all).[70] One impossible combination would have the largest mountain on Saturn be a part of π (or vice versa). Why impossible? Because of wrong typing: the arguments of the part-whole relation must be individuals (i.e., entities of type ι), but the largest mountain on Saturn is an individual office while π is a real number.

Still, *which parts are essential* for an individual in order to have a *property P*? The property of having an engine is essential for the property of being a car, because something designed without an engine does not qualify as a car, but at most as a toy car, which is not a car. The answer to the question which parts are essential in order to have a property *P* is, in the car/engine example, that the *property* of having an engine is a *requisite* of the *property* of being a car. What is necessary is that a car, *any* car, should have an engine. It is even necessary that it should have a particular kind of engine, where *being a kind of engine* is a property of a property of individuals. What is not necessary is that any car should have some one particular engine belonging to a particular kind of engine: *mutatis mutandi*, any two members of a particular kind of engine will be mutually replaceable.[71] Thus the relation *Part_of* is of type $(o\iota\iota)_{\tau\omega}$.

The sort of unrestricted mereological combinations that we are adumbrating and advocating gives rise to a more fundamental problem that Cmorej takes on in

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[69] The full logic of requisites is set out in Chapter 4.

[70] See Duží (2007) for a discussion of *wharrots*. A wharrot is an individual consisting of a carrot and a whale. Unless further restrictions are laid down, wharrots exist as soon as whales and carrots do. (We are indebted to Maarten Franssen for the example of wharrots.)

[71] This problem is connected with the analysis of property modification, including *being a malfunctioning P*, dealt with in Section 4.4.

1988.[72] The problem is this. If a composition of a physical individual is contingent and allows parts to be replaced or lost, then *which unique part of such an individual is essential for the individual's identity?* Cmorej argues that the assumption of variable composition of a mereological sum leads to absurd consequences. Let us briefly summarise his arguments.

Cmorej presents two puzzling thought experiments. The first puzzle can be called, 'Did, or did not, an individual have the property *P*?'; the second, 'Where is the individual?'

Here is the first puzzle. Imagine an individual *X* that has the property *P*. The property *P* is stipulated to be *penetrating*, which means that, necessarily, if *X* has *P* then all its parts have *P*.

Formally, *P* is *penetrating* iff

$$\forall w \forall t\ \forall x\ [[{}^0P_{wt}x] \supset \forall y\ [[{}^0Part\_of_{wt}\, y\, x] \supset [{}^0P_{wt}y]]].$$

Types: $P/(o\iota)_{\tau\omega}$; $Part\_of/(o\iota\iota)_{\tau\omega}$; $x, y \rightarrow \iota$.

For instance, the property of weighing less than 50 kg is penetrating. An individual cannot weigh less than 50 kg if some of its parts weigh more than 50 kg.

Let *X* have a penetrating property *P* at time t_1. During the time interval $\langle t_1, t_2\rangle$, $t_1 < t_2$, *X* loses all its *proper* parts, as well as the property *P*, so that at t_2 *X* does not have *P* anymore, and *X* also does not contain any *proper* parts that *used* to have *P*.[73] Now the question is whether at t_2 we can truly ascribe to *X* the property of having had *P*. Cmorej uses a past-tense operator ***Pt*** that is applied to the proposition that *X* has *P*, forming the proposition that *X* had *P* in the past. Thus the operator denotes a property *Pt* of propositions, *Pt* of type $(oo_{\tau\omega})_{\tau\omega}$, which is defined as follows: Let $p \rightarrow o_{\tau\omega}$ be a variable *v*-constructing a proposition. Then

$${}^0Pt = \lambda w \lambda t\ \lambda p\ \exists t'\ [[t' < t] \wedge p_{wt'}].$$

Intuitively, the answer should be in the affirmative. It is true at t_2 that *X* used to have *P*, because *what is done cannot be undone* (as Macbeth learnt the hard way). But how are we to evaluate the truth-conditions of the proposition constructed by $\lambda w \lambda t\ [{}^0Pt_{wt}\ \lambda w \lambda t\ [{}^0P_{wt}\, X]]$ at t_2? When evaluating the proposition constructed by $\lambda w \lambda t\ [{}^0P_{wt}X]$, we must consider all the parts of *X*, because *P* is penetrating. Cmorej argues that, similarly, when evaluating the truth-conditions of $\lambda w \lambda t\ [{}^0Pt_{wt}\ \lambda w \lambda t\ [{}^0P_{wt}X]]$ at t_2, we must take into account the parts that *X* consists of *at time* t_2. But, there is no trace of *P* in *X* at t_2; no proper part of *X* used to have *P*. This is peculiar, indeed. Could *X* have been, for instance, inside a room, or in a magnetic field, or submerged into a liquid, if there is not even a tiny proper part of *X* to which the respective property could have been ascribed? Hardly. Thus Cmorej comes to the conclusion that $\lambda w \lambda t\ [{}^0Pt_{wt}\ \lambda w \lambda t\ [{}^0P_{wt}\, X]]$ is, at t_2, both true (according to the principle that what is

[72] See also Geach (1972, pp. 215–16) for the related problem of 'the cat on the mat'.

[73] A proper part of *X* is an individual *Y* such that *Y* is a part of *X* and $Y \neq X$.

done cannot be undone) and false, because none of its parts used to have the property P. Contradiction!

First, however, we disagree with Cmorej's argument on grounds of analogy. He argues that when evaluating whether 'The world champion of 100 m race used to be a smoker' we examine the *current* world champion, not any of the previous ones. Of course, we have to examine the individual that *currently* and *actually* plays the role of world champion of 100 m sprint race—but we should examine his/her *history*. Though the current champion may have stopped smoking, we should ask whether he/she *previously smoked.* Similarly, when asking whether X used to have P we have to examine the *history* of X, which includes the proper parts that X used to consist of. We have to ask which parts X consisted of *in the past*, and whether any of these parts previously used to have P in the interval $\langle t_1, t_2\rangle$.

Thus we must use the *Past* function, which we will define in Section 2.5.2. Simplifying a bit, the result of applying *Past* to the proposition constructed by $\lambda w\lambda t\ [{}^0P_{wt}X]$ and to the interval $\langle t_1, t_2\rangle$ referring to the past is this:

$$\lambda w\lambda t\ \exists t'\ [[t' < t] \wedge [t_1 \leq t' \leq t_2] \wedge [{}^0P_{wt'}X]].$$

Evaluating the truth-conditions in a world w at a time t comes down to empirically searching for the truth-value v-constructed by $\exists t'\ [[t' < t] \wedge [t_1 \leq t' \leq t_2] \wedge [{}^0P_{wt'}X]]$. In other words, we have to examine the history of X in the interval $\langle t_1, t_2\rangle$ *preceding* time t.

But, secondly, there is another, more alarming question. If no current proper part of X can help us examine the history of X, how are we to examine its history at all? We need to abstract from all the current proper parts of X, as well as all *their* properties, and consider only the properties that the *bare* individual X used to have. What, then, determines the *numerical identity* of the *bare* individual X?

This problem ties in with the second puzzle. The second puzzle is this. Imagine that a person a owns a golden fountain pen (i.e., a pen, all of whose parts are golden) and a person b owns a pen that looks exactly like a's, except that it is not made of gold but of fool's gold (i.e., all its parts being made of fool's gold). Moreover, b's pen and all its parts function in exactly the same way as a's pen and its parts and, so, are functionally equivalent. At time t_1 a's pen is located at the place L_a and b's pen at the place L_b. During the time interval $\langle t_1, t_2\rangle$ b gradually replaces, part by part, the *proper* parts of a's pen by the *proper* parts of b's pen, so that at t_2 all the *proper* parts of a's pen are located at L_b and all the *proper* parts of b's pen are located at L_a. As a result, a's pen and b's pen look and function in the same way at t_2 as they did at t_1, except that a's pen is made of fool's gold and b's pen is made of gold.

The conclusion of the thought experiment has an air of plausibility. Yet we are not convinced that a's pen is made of fool's gold and b's pen is made of gold. To see why, imagine that the interval $\langle t_1, t_2\rangle$ is very short and that *all* the parts have been interchanged *at once*. Wouldn't most people be inclined to say that b simply *stole* a's pen and replaced it by his junk pen? We would, at least. Furthermore, even if the swap was performed part by part, how could *all* the proper parts of a's pen be transferred from L_a to L_b without the whole individual being *ipso facto* transferred?

Hence the questions arise: Where is a's pen and where is b's pen at t_2? Which of the pens is golden at t_2? There are two mutually incompatible answers:

(i) a's pen is located at L_a and is made of fool's gold, whereas b's pen is located at L_b and is made of gold; b did not steal a's pen, b only drastically lowered the value of a's pen.
(ii) a's pen is located at L_b and is made of gold, whereas b's pen is located at L_a and is made of fool's gold; b stole a's pen, and replaced it by his pyrites pen.

Now let someone unaware of the swap examine the two pens at t_2. In *both cases* the result of the examination would be as follows. The pen located at L_a is made of fool's gold, because *all* its parts are made of fool's gold, whereas the pen located at L_b is golden, because *all* its parts are made of gold. Since the examiner is unaware of the swap, he naturally assumes that the golden pen at L_b is a's pen. Consequently, the variant *ad* (i) will seem impossible to the examiner.

Cmorej thus arrives at the conclusion that the assumption of unrestricted variation of an individual's composition is unacceptable. In other words, given an individual X, the property of being a part of X must be essential of X. Hence, for any individual X it must hold that the property constructed by

$$\lambda w \lambda t\, \lambda y\, [{}^0Part\_of_{wt}\, y\, X]$$

is an *essential* property of X, i.e., a constant function. But at the same time this property is, intuitively, *empirical*, for we cannot know a priori which parts X consists of.

What are we to make of Cmorej's conclusion that some properties of X are both essential and empirical? We wish to reject it. Here is why. A consequence of Cmorej's conclusion is that X would consist of the very same parts in each world w at each time t. This would mean that the material composition of X must be constant, such that each time X loses some part and obtains a new one, a *new* individual X' comes into being. As a result, the universe of discourse would have to vary accordingly. Moreover, we could not a priori distinguish between individuals X, X', X'', X''', etc. For instance, your cells are continuously being renewed, yet your numerical identity should certainly not hinge on one particular pool of cells. You would not be the same individual in the morning as the one who went to bed the night before. This is certainly untenable as a criterion of numerical individuation of individuals.

As the above thought experiments show, if we embrace variable composition of a mereological sum, then we face the problem of the identity of individuals. To dramatize the problem, imagine that somebody is gradually stealing proper parts of your car (rather than stealing the whole car in one go). If the thief steals one molecule he has not stolen your car. If he steals the steering wheel, he has not stolen your car. If he steals all four wheels, he has not stolen your car. But if the thief steals *all* proper parts of your car, wouldn't you say that he had stolen your car? Of course, you would, and so would your insurance company (hopefully). The car thief has committed diachronic theft, as it were, the same way an embezzler may gradually drain an account. If one goes along with our view, the question *which* part is essential of your car's identity turns out to be ill-posed.

This example suggests that the only way out is to say that *no proper* physical part is essential of your car (or of any other concrete individual). But this is to say that an individual may lose all its proper physical parts without losing its identity, making the identity of an individual a purely *abstract object*. A bare individual is an abstract object of a transcendental nature, and Cmorej's proposed proof that bare individuals do not exist is correct, because existence is a property of *intensions*, namely the property of being instantiated or being occupied. As we showed in Section 1.4.2.1, we cannot specify the property of *not* having any properties. We can only *abstract away* the properties an individual has. We must presuppose pre-theoretically that there is a fixed domain of individuals whose identity is given to us a priori, regardless of whether we are able to determine *which* particular individual we are examining on some occasion. Within our theory, individuals are *logically primitive* relative to a base *B* (see Section 1.4.1).

1.4.2.3 The top-down approach to semantics revisited

In Section 1.2 we critically examined the standard bottom-up way of analysing terms and expressions. We adduced the following five examples and explained why their standard analyses are too coarse-grained:

(1) 'Charles is happy'
Fa

And further upwards:

(2) 'Charles is happy, and Thelma is grumpy'
$Fa \wedge Gb$

(3) 'Somebody is happy'
$\exists x\ (Fx)$

(4) 'Possibly, Charles is happy'
$\Diamond\ (Fa)$

(5) 'Thelma believes that Charles is happy'
$\mathrm{B}b\ (Fa)$.

Now we have the tools to analyze these sentences in a fine-grained way. As we explained above, we aim at assigning propositional *constructions* to the analysed sentences. We are going to illustrate the method of analysis by analysing first the sentences (1) and (2). Our method consists in three steps.

First, we assign types to the objects mentioned by the sentences: *Charles*, *Thelma*/ι; *Happy*, *Grumpy*/$(o\iota)_{\tau\omega}$; ∧/(οοο).

Second, by Composing constructions of these objects (here, Trivializations) we aim at constructing the propositions denoted by (1) and (2), respectively:

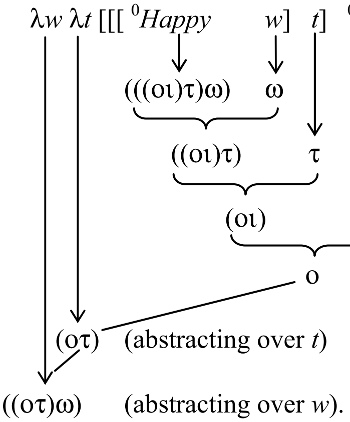
(1′) $\lambda w \lambda t\ [{}^0\mathit{Happy}_{wt}\ {}^0\mathit{Charles}]$.

(2′) $\lambda w \lambda t\ [{}^0\wedge\ [\lambda w \lambda t\ [{}^0\mathit{Happy}_{wt}\ {}^0\mathit{Charles}]]_{wt}\ [\lambda w \lambda t\ [{}^0\mathit{Grumpy}_{wt}\ {}^0\mathit{Thelma}]]_{wt}]$.

Note that our uniform semantics works smoothly top-down and back up again, involving all three kinds of context, to wit, hyperintensional, intensional and extensional. The Closures $[\lambda w \lambda t\ [{}^0\mathit{Happy}_{wt}\ {}^0\mathit{Charles}]]$ and $[\lambda w \lambda t\ [{}^0\mathit{Grumpy}_{wt}\ {}^0\mathit{Thelma}]]$ construct the propositions that Charles is happy and that Thelma is grumpy, respectively. However, propositions are not arguments of the right type for truth-value functions. They are intensional objects and have to be extensionalized first in order to yield an extension. That is, the proposition that Charles is happy has to be subjected to intensional descent: $\lambda w \lambda t\ [{}^0\mathit{Happy}_{wt}\ {}^0\mathit{Charles}]_{\boldsymbol{wt}}$.

The Composition $[\lambda w \lambda t\ [{}^0\mathit{Happy}_{wt}\ {}^0\mathit{Charles}]]_{wt}$ is a construction *v*-constructing a truth-value; i.e., the type of the value of the proposition constructed by $\lambda w \lambda t\ [{}^0\mathit{Happy}_{wt}\ {}^0\mathit{Charles}]$ at $\langle w, t\rangle$. Similarly, $\lambda w \lambda t\ [{}^0\mathit{Grumpy}_{wt}\ {}^0\mathit{Thelma}]$ constructs the proposition that Thelma is grumpy, and its Composition with $\langle w, t\rangle$, as in $[\lambda w \lambda t\ [{}^0\mathit{Grumpy}_{wt}\ {}^0\mathit{Thelma}]]_{wt}$, *v*-constructs the value (of type ο) of this proposition at $\langle w, t\rangle$. So a conjunction receives two truth-values as input, yielding a third as output. Finally, we need to abstract from the values of *w*, *t* in order to construct the proposition that Charles is happy and Thelma is grumpy.

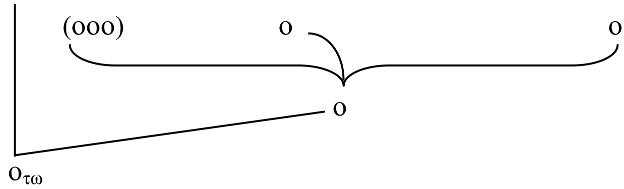
Third, via type-theoretical checking we verify that the individual constructions have been combined in a type-theoretically coherent way:



The Composition $[^0Happy_{wt}\ ^0Charles]$ *v*-constructs **T**, according as the individual constructed by $^0Charles$ (i.e., Charles) belongs to the extension of the property *Happy* (*v*-constructed by $^0Happy_{wt}$) at a given $\langle w, t\rangle$. Abstraction over the values of w, t constructs a proposition/$o_{\tau\omega}$. In other words, the sense of 'Charles is happy' is a procedure the evaluation of which in any world w (λw) at any time t (λt) consists in checking whether Charles has the property of being happy at that $\langle w, t\rangle$-pair.

The type-theoretical checking of $[\lambda w\lambda t\ [^0Grumpy_{wt}\ ^0Thelma]]$ proceeds in the same way. Finally, we check the whole (2′).

$$\lambda w\lambda t\ [^0\wedge\ \ [\lambda w\lambda t\ [^0Happy_{wt}\ ^0Charles]]_{wt}\ \ [\lambda w\lambda t\ [^0Grumpy_{wt}\ ^0Thelma]]_{wt}]$$



We have just verified that what this Closure constructs is a proposition, which is the right type of object to be denoted by a sentence.

Now we are going to analyse (3), (4) and (5) along the same lines. Quantifiers were defined in Definition 1.6. Thus the analysis of sentence (3) is this:

(3′) $\lambda w\lambda t\ [^0\exists^\iota\ \lambda x\ [\lambda w\lambda t\ [^0Happy_{wt}\ x]]_{wt}]$.

The Closure $[\lambda x\ [\lambda w\lambda t\ [^0Happy_{wt}\ x]]_{wt}]$ *v*-constructs the set of individuals instantiating the property *Happy* at $\langle w, t\rangle$. $\exists^\iota$ is here a function of type $(o(o\iota))$ inputting the set just constructed and outputting a truth-value, according as the set is empty or not. Finally, by abstracting over the values of w, t we construct the proposition that somebody is happy.

The analysis of sentence (4) depends on the type of possibility ascribed to the proposition that Charles is happy. If possibility is understood as *logical possibility* then $\Diamond^L$ is a function of type $(oo_{\tau\omega})$: the class of logically possible propositions. In such a case we have:

(4′) $\lambda w\lambda t\ [^0\Diamond^L\ [\lambda w\lambda t\ [^0Happy_{wt}\ ^0Charles]]]$.

This construction constructs the trivial proposition *TRUE*. It is certainly logically possible that Charles be happy; in the possible-world idiom, there is a world w and a time t at which Charles has the property of being happy. Thus logical possibility can be defined by the following construction:

$$\lambda p\ [^0\exists^\omega\ \lambda w\ [^0\exists^\tau\ \lambda t\ p_{wt}]]$$

or for short,

$$\lambda p\ [\exists w\exists t\ p_{wt}].$$

Types: $p \rightarrow_v o_{\tau\omega}$; $\exists^{\omega}/(o(o\omega))$; $\exists^{\tau}/(o(o\tau))$.

This definition yields (by performing equivalent β-reductions):

$$(4'') \qquad \lambda w \lambda t\ [\exists w' \exists t'\ [{}^0 Happy_{w't'}\ {}^0 Charles]].$$

Obviously, (4″) constructs *True*. A more natural analysis can be obtained by construing *empirical possibility* $\Diamond^{em}$ as a *property of propositions*, an $(oo_{\tau\omega})_{\tau\omega}$-object. This yields

$$(4''') \qquad \lambda w \lambda t\ [{}^0 \Diamond^{em}{}_{wt}\ [\lambda w \lambda t\ [{}^0 Happy_{wt}\ {}^0 Charles]]].$$

This Closure constructs the contingent proposition that Charles' being happy is possible at the given $\langle w, t\rangle$ of evaluation.

In Section 1.2.2 we claimed that logical syntax cannot tolerate ambiguous terms. We explained that the handy notation of modal logics found in the *analysantes* of (3), (4) and (5) treats '$\Box$', '$\Diamond$' as being syntactically on a par with truth-functional connectives like '$\neg$', both '$\neg p$' and '$\Box p$' being well-formed formulae. Also we are allowed to generate strings like '$(\Box p \rightarrow p) \wedge Kp$', '$K$' standing for *knowing*. However, since what is necessary is not a truth-value but a proposition, and what is known is not a truth-value but a hyperproposition (in the case of explicit knowledge, see Section 5.1.2), we face here three-way ambiguity mixing together an extensional, an intensional and a hyperintensional context.

Now our context-*in*variant semantics begins to pay off. We need not analyse 'Charles is happy' any differently, nor are we forced to hold that '*Fa*', hitherto denoting a truth-*value*, now denotes a truth-*condition* (proposition) instead.

Similarly, when analysing (5), the meaning of 'Charles is happy' is the same as above, namely the Closure $\lambda w \lambda t\ [{}^0 Happy_{wt}\ {}^0 Charles]$, and we get

$$(5') \qquad \lambda w \lambda t\ [{}^0 Believe_{wt}\ {}^0 Thelma\ \lambda w \lambda t\ [{}^0 Happy_{wt}\ {}^0 Charles]].$$

This is the analysis of attitudes germane to classical possible-world semantics, according to which the object of an attitude is a proposition. Thus, *Believe* is a function of type $(o\iota o_{\tau\omega})_{\tau\omega}$. Again it is now paying off that 'Charles is happy' was paired off with a proposition straightaway, despite the fact that in (2) we need two truth-values as functional arguments.

If we analyse 'to believe' in (5) as a case of *explicit belief*, then *Believe** is a function of type $(o\iota *_n)_{\tau\omega}$. An agent, Thelma in our case, is now related to a hyperproposition. Again, it is paying off that 'Charles is happy' was analysed as expressing a hyperproposition, viz. the above Closure $\lambda w \lambda t\ [{}^0 Happy_{wt}\ {}^0 Charles]$. Thelma is related to this very Closure, which can be constructed most directly by its Trivialization. We obtain

$$(5'') \qquad \lambda w \lambda t\ [{}^0 Believe^*{}_{wt}\ {}^0 Thelma\ {}^{\mathbf{0}}[\lambda w \lambda t\ [{}^0 Happy_{wt}\ {}^0 Charles]]].$$

Sometimes it is said that the value of an intension in a possible world and at a time is an extension. As a general claim this is not true, however, because, as was pointed out above, there are intensions of a higher *degree* and of a higher *order*. Examples of the latter would be hyperintensional attitudes like *Believe**, *Know**, *Calculate*, all of type $(o\iota*_n)_{\tau\omega}$. As an example of a higher-degree intension, consider, for instance, the expression 'Einstein's favourite proposition'. This definite description obviously does not *refer* rigidly: in some equivalence classes of worlds/times Einstein will favour one proposition, in another equivalence class he will favour another proposition, and in yet another equivalence class he will favour none at all. So the type of the *denotation* of 'Einstein's most favourite proposition' is $(o_{\tau\omega})_{\tau\omega}$: a 2nd-*degree proposition*, the type of whose values is $o_{\tau\omega}$.

Type-theoretical analysis, which is the first part of our logical analysis of natural language (see Section 2.1), consists in associating types with meaningful expressions.[74] As competent users of our native language we know which expressions are empirical and we should be able to find the adequate type. (Montague's associating categories and then types with particular classes of expressions corresponds to this stage of logical analysis of natural language.) Sometimes the situation is not immediately clear, though. For instance, compare 'colour' and 'colour of'. The empirical character of the latter is obvious. What may be less obvious is what sort of intension it denotes. Now, it denotes an intension whose type is $((o\iota)_{\tau\omega}\,\iota)_{\tau\omega}$: in any world/time the outcome of applying this function to an individual is at most one colour (black, red, blue, etc.; i.e., a property of individuals). But from the fact that particular colours are properties, and so intensions, it does not follow that 'colour' denotes an intension. Actually, whereas asserting that an object is blue involves uttering an empirical sentence that denotes a contingent proposition, asserting that blue is a colour involves uttering an analytically true sentence denoting the proposition *True*. This is because 'colour' denotes a *set* (rather than a property) of properties (colours). Black, red, blue, etc., are colours at all worlds and times, or rather independently of worlds and times. What varies are their extensions at various world/time pairs. Thus the type of the entity *Colour* is $(o(o\iota)_{\tau\omega})$: the word 'colour' denotes an extension. Whether a property belongs to this *set* of properties is true or false independently of empirical facts.

Another example of a 2nd-*degree intension* would be *the highest US executive office*. This role is occupied by individual offices, currently by the office of US

[74] As this point about typing also shows, TIL requires that the objects that are to be logically manipulated be typed and defined before any (possible) axiomatization. Of course, proposing some axioms involves running a risk, for it could be objected that the chosen axioms do not truly describe the nature of the objects. But this risk is only what characterises scientific work when carried out in a realist manner, according to which axioms do not prescribe what the objects of a domain are, but instead try to describe some properties that are ontologically and conceptually prior to the axioms. Analogously, 'Poincaré, like Kronecker, thought one does not have to define the whole numbers or construct their properties on an axiomatic foundation. Our intuition precedes such a structure' (Kline, 1980, p. 233).

President. So the type of the intension denoted by 'The highest US executive office' is $(\iota_{\tau\omega})_{\tau\omega}$.

If the type of the values of an intension is, say, $(o_{\tau\omega})_{\tau\omega}$, then the type of that intension has got to be $((o_{\tau\omega})_{\tau\omega})_{\tau\omega}$ to form a 3rd-degree intension. The rule for forming higher-degree intensions is straightforward: whenever a world index *w* is added as an argument to an intension of degree *n*, the degree of the resulting intension is *n*+1. Adding only a temporal index *t* won't suffice, since intensions are, strictly speaking, defined as functions from possible worlds. But adding *t* next to adding *w* may be called for to capture the temporal variability of the value distribution of a particular higher-degree intension.

To illustrate, the analysis of 'Einstein's most favourite proposition' is as follows. Einstein may have favoured many propositions, so the type of *Favour_prop_of (somebody)* needs to be $((oo_{\tau\omega})\iota)_{\tau\omega}$: A function that associates, dependently on $\langle w, t\rangle$, an individual with the set of propositions the individual favours at $\langle w, t\rangle$. The Composition $[{}^0Favour\_prop\_of_{wt}\ {}^0Einstein]$ *v*-constructs the set of propositions that Einstein favours at $\langle w, t\rangle$. Which of these propositions is the most favourite one depends again on the circumstances at $\langle w, t\rangle$. Thus the type of *The_Most* turns out to be $(o_{\tau\omega}(oo_{\tau\omega}))_{\tau\omega}$. A function of this type associates, dependently on $\langle w, t\rangle$, a set of propositions with a proposition, to wit, the most favoured one of them all. Thus the Composition

$$[{}^0The\_Most_{wt}\ [{}^0Favour\_prop\_of_{wt}\ {}^0Einstein]]$$

v-constructs the proposition that is Einstein's most favourite one at a given $\langle w, t\rangle$. Finally, by abstracting over the values of *w*, *t*, we construct the propositional role of Einstein's most favourite proposition:

$$\lambda w\lambda t\ [{}^0The\_Most_{wt}\ [{}^0Favour\_prop\_of_{wt}\ {}^0Einstein]].$$

To illustrate the distinction between 'colour' and 'colour of', we analyse the sentences

(6) 'The colour of Charles' most favourite shirt is green'

and

(7) 'Charles' most favourite colour is green'.

To trim the notation, let π be the type of an individual property, i.e. $(o\iota)_{\tau\omega}$. Then the types of entities that receive mention in (6) and (7) are:

$Charles/\iota$; $Favour\_of^1/((o\iota)\iota)_{\tau\omega}$; $Favour\_of^2/((o\pi)\iota)_{\tau\omega}$; $Colour\_of/(\pi\iota)_{\tau\omega}$; $Colour/(o\pi)$; $Shirt\_of/((o\iota)\iota)_{\tau\omega}$; $Green/\pi$; $Most^1/(\iota(o\iota))_{\tau\omega}$; $Most^2/(\pi(o\pi))_{\tau\omega}$.

The definite description 'Charles' most favourite shirt' denotes the individual office $ChFS/\iota_{\tau\omega}$ occupiable by the shirt, if any, that Charles happens to favour the most at some world/time of evaluation. Thus a coarse-grained analysis of (6) is

(6′) $$\lambda w \lambda t\, [[^0Colour\_of_{wt}\ ^0ChFS_{wt}] = {}^0Green].$$

On the other hand, the definite description 'Charles' most favourite colour' denotes the property office $ChFC/\pi_{\tau\omega}$ occupiable by the property, if any, that happens to be Charles' most favourite. A coarse-grained analysis of (7) is

(7′) $$\lambda w \lambda t\, [^0ChFC_{wt} = {}^0Green].$$

Now, in order to refine the above analyses, we define the entities *ChFS* and *ChFC* in terms of the simpler entities the sentences talk about, i.e., *Shirt_of*, $Favour\_of^1$, $Favour\_of^2$. The individual office *ChFS* is defined as follows ($x \rightarrow \iota$):

$$\lambda w \lambda t\, [^0Most^1{}_{wt}\ \lambda x\, [[[^0Shirt\_of_{wt}\ ^0Charles]\ x] \wedge [[^0Favour\_\ of^1{}_{wt}\ ^0Charles]\ x]]].$$

The Closure $\lambda x\, [[[^0Shirt\_of_{wt}\ ^0Charles]\ x] \wedge [[^0Favour\_\ of^1{}_{wt}\ ^0Charles]\ x]]$ *v*-constructs the set of individuals that are Charles' favourite shirts at $\langle w, t\rangle$; $Most^1{}_{wt}$ selects from this set the individual, if any, that is the most favourite one at $\langle w, t\rangle$.

The individual office *ChFC* is defined as follows ($p \rightarrow \pi$):

$$\lambda w \lambda t\, [^0Most^2{}_{wt}\ \lambda p\, [[^0Colour\ p] \wedge [[^0Favour\_of^2{}_{wt}\ ^0Charles]\ p]]].$$

The Closure $\lambda p\, [[^0Colour\ \ p] \wedge [[^0Favour\_of^2{}_{wt}\ ^0Charles]\ p]]$ *v*-constructs the set of properties which belong to the set of colours (the first conjunct) and are Charles' favourite properties (the second conjunct). The application of $Most^2$ to this set yields the property selected from this set, if any, namely the one that is the most favourite one at $\langle w, t\rangle$.

By substituting these definitions into (6′) and (7′), we get these fine-grained analyses of (6) and (7):

(6″) $$\lambda w \lambda t\, [[^0Colour\_of_{wt}\ [^0Most^1{}_{wt}\ \lambda x\, [[[^0Shirt\_of_{wt}\ ^0Charles]\ x] \wedge [[^0Favour\_of^1{}_{wt}\ ^0Charles]\ x]]]] = {}^0Green].$$

(7″) $$\lambda w \lambda t\, [[^0Most^2{}_{wt}\ \lambda p\, [[^0Colour\ p] \wedge [[^0Favour\_of^2{}_{wt}\ ^0Charles]\ p]]] = {}^0Green].$$

1.4.3 Logical objects

In this section we specify those important extensions that are classified as *logical objects* in TIL. We are aware of the problem of determining which objects are logical and which are extra-logical. For our purposes, we consider as logical objects only the extensions defined in this Section 1.4.3, i.e., truth-functions, quantifiers, singularizers, identities, and the functions *Sub* and *Tr*.

(a) *Truth-functions.* Unary (negation, $\neg$), type ($\omicron\omicron$); binary ($\wedge$, $\vee$, $\supset$, etc.), type ($\omicron\omicron\omicron$). TIL is a classical logic in that it works with just two truth-values, **T**, **F**. This does not mean that every sentence of natural language must be true or false, though. Since some truth-bearers are neither true nor false, TIL has adopted *partial* functions, which associate with each argument *at most* one value. Thus the sentence

'The King of France is bald'

denotes a properly partial proposition (of type $\omicron_{\tau\omega}$) that lacks a truth-value in, among others, the actual world at the present time. And the sentence

'The greatest prime is even or not even'

does not denote a truth-value, because 'the greatest prime' expresses an improper construction (see Example 1.3 (j)).

Remark. The third and further values in so-called many-valued logics cannot be construed as *truth*-values. They can be interpreted in various other ways (uncertainty and fuzziness being the most famous cases). The way TIL handles partiality bears similarities to Bochvar's three-valued logic (see Bochvar, 1939), where the 'third value' associated with one variable is the reason why it must be associated with the entire complex formula. Thus, if phrased in TIL jargon, if p v-constructs **T** and a construction Q constructs the third value, then the disjunction ($p \vee Q$) gets **T** in Łukasiewicz, the 'third value' in Bochvar, and no value in TIL. The matrices of Bochvar's three-valued logic will coincide with the matrices of a theory like TIL, which operates with three options: **T**, **F**, neither ('gap').

The following Table 1.1 is a TIL matrix of truth-functions and their Composition with truth-values and truth-value gaps. By the sign '$\bot$' we do not mark a third value, but a truth-value gap. P, Q are constructions v-constructing truth-values, and P, Q may be v-improper. The sign '*' marks rather peculiar rows, to be explained below.

According to Definition 1.2 (iii), the Composition $[X\ X_1 \ldots X_m]$ is v-improper whenever one or more of the constructions X, X_1, …, X_m are v-improper. This is in accordance with the compositionality constraint: once a construction X_i does not supply an object on which the construction X is to operate, the whole Composition fails to v-construct anything, making it v-improper. In this way partiality is being propagated upwards. This holds also for the Compositions of the constructions of truth-functions. Thus, e.g., $[{}^0\vee\ P\ Q]$ is v-improper if P or Q is v-improper.

Table 1.1 TIL matrix of truth-values

P	Q	$[{}^0\wedge P\ Q]$	$[{}^0\vee P\ Q]$	$[{}^0\supset P\ Q]$	$[{}^0\equiv P\ Q]$	
1	1	1	1	1	1	
1	0	0	1	0	0	
0	1	0	1	1	0	
0	0	0	0	1	1	
1	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	*
0	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	*
$\perp$	1	$\perp$	$\perp$	$\perp$	$\perp$	*
$\perp$	0	$\perp$	$\perp$	$\perp$	$\perp$	*
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	

Rows marked by ‘*’ might seem peculiar. Aren’t we used to a disjunction being true iff at least one disjunct is true? Aren’t we used to an implication being true iff the antecedent is false or the consequent true? Imagine a situation in which Charles does not smoke. Ostensibly, in such a situation we may truly claim the following:

‘Charles stopped smoking or he never smoked.’

Alas, analysing the sentence in this careless way yields a construction of a proposition that goes *undefined* at such $\langle w, t\rangle$ pairs at which Charles never smoked:

$$\lambda w\lambda t\ [{}^0\vee\ [\lambda w\lambda t\ [{}^0StopSmoking_{\mathrm{wt}}\ {}^0Ch]_{wt}\ [\lambda w\lambda t\ [{}^0NeverSmoked_{wt}\ {}^0Ch]_{wt}].$$

Types: *StopSmoking*, *NeverSmoked*/$(o\iota)_{\tau\omega}$; *Ch(arles)*/ι.

The problem is created by the proposition constructed by $[\lambda w\lambda t\ [{}^0StopSmoking_{wt}\ {}^0Ch]$ that does not have a truth-value at those $\langle w, t\rangle$ at which Charles never smoked. Hence the Composition $[\lambda w\lambda t\ [{}^0StopSmoking_{wt}\ {}^0Ch]_{wt}$ is v-improper for any such $\langle w, t\rangle$ pairs. This is due to the fact that the proposition that Charles stopped smoking comes with the *presupposition* that he used to smoke.[75] Thus the first argument of the function $\vee$ is missing and so the application fails. For this reason the Closure constructs a proposition that has truth-value gaps at those $\langle w, t\rangle$ pairs at which Charles never smoked.

A remedy is within reach, fortunately. In those cases where an extensionalized proposition enters as argument of a truth-function, we should use the totalizing

[75] *Presupposition* will be defined in Definition 1.14

propositional property $True/(oo_{\tau\omega})_{\tau\omega}$, which returns **T** for those $\langle w, t\rangle$ pairs at which the argument proposition is true, and **F** in all the remaining cases. The resulting analysis is:

$$\lambda w\lambda t\ [{}^0\vee\ [{}^0True_{wt}\ \lambda w\lambda t\ [{}^0StopSmoking_{wt}\ {}^0Ch]]\ [\lambda w\lambda t\ [{}^0NeverSmoke_{wt}\ {}^0Ch]_{wt}].$$

Gloss: 'It is true that Charles stopped smoking, or he never smoked'.

We will discuss the problem of partial functions and truth-value gaps in more details in Sections 2.6 and 2.7.[76]

(b) *Quantifiers.* The standard universal ($\forall^\alpha$) and existential ($\exists^\alpha$) quantifiers were defined in Definition 1.6. They are not 'improper symbols' for TIL; rather, they are type-theoretically polymorphous total functions of a type $(o(o\alpha))$ for the given type α, so they are classes of classes.[77]

- The universal quantifier $\forall^\alpha$ is the class of those classes that are not proper subclasses of α, so $\forall$ is a singleton.
- The existential quantifier $\exists^\alpha$ is the class of all non-empty subclasses of the class α.

Some sentences cannot be literally analysed using these standard quantifiers, unless we reformulate them. For instance,

(8) 'Some students are clever'

and

(9) 'All students are lazy'

can be analysed in the standard way as follows:

$$\lambda w\lambda t\ [{}^0\exists\lambda x\ [{}^0\wedge\ [{}^0Student_{wt}\,x]\ [{}^0Clever_{wt}\,x]]]$$

and

$$\lambda w\lambda t\ [{}^0\forall\lambda x\ [{}^0\supset\ [{}^0Student_{wt}\,x]\ [{}^0Lazy_{wt}\,x]]].$$

Types: $\forall, \exists/(o(o\iota))$; $\wedge, \supset/(ooo)$; *Student*, *Clever*, *Lazy*$/(o\iota)_{\tau\omega}$; $x/*_1 \to \iota$.

However, the above sentences (8) and (9) do not mention conjunction and implication. Thus these analyses are not in accordance with the principle of subject matter, which says, roughly, that each subconstruction of a given meaning of an

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[76] See also Duží (2003a).

[77] By 'type-theoretically polymorphous functions' we mean a set of functions that are defined and thus behave in the same way, independently of their type. For instance, any member of the set of functions *Cardinality* associates a finite class with the number of its elements. Hence this definition is polymorphous; there are actually infinitely many cardinality functions, one for each type: $Card_1/(\tau(o\iota))$—the number of a set of individuals, $Card_2/(\tau(o\tau))$—the number of a set of numbers, etc., which we indicate by using a type variable α in the type of $Cardinality/(\tau(o\alpha))$.

expression E has to be assigned to a meaningful subexpression of E as its meaning. In other words, each subconstruction of the meaning assigned to E must construct an object denoted by a subexpression of E.[78] Therefore, these constructions are the meanings of different, albeit equivalent, sentences, namely 'There are individuals who are students and who are lazy' and 'It holds for all individuals x that if x is a student then x is lazy'.

In order to analyse sentences like (8) and (9) literally, in accordance with the principle of subject matter, we must use another type of quantifier, for example *All*, *Some*, and *No*, which are known as *restricted quantifiers.* These are type-theoretically polymorphous functions of type $((o(o\alpha))(o\alpha))$, defined as follows:

- All^{α} is the function which associates a class A of α-objects with the class of all those classes that contain A as a subset.
- $Some^{\alpha}$ is the function which associates a class A of α-objects with the class of all those classes that have a non-empty intersection with A.
- No^{α} is the function which associates a class A of α-objects with the class of all those classes that have an empty intersection with A.

All^{ι} and $Some^{\iota}$ of type $((o(o\iota))(o\iota))$ enable us to analyse (8) and (9) as expressing the Closures

(8′) $\lambda w \lambda t\ [[{}^0Some^{\iota}\ {}^0Student_{wt}]\ {}^0Clever_{wt}]$

and

(9′) $\lambda w \lambda t\ [[{}^0All^{\iota}\ {}^0Student_{wt}]\ {}^0Lazy_{wt}]$.

The Composition $[{}^0All^{\iota}\ {}^0Student_{wt}]$ v-constructs the set M of those sets of individuals which contain the population of students at a given $\langle w, t\rangle$ as a subset. The Composition $[[{}^0All^{\iota}\ {}^0Student_{wt}]\ {}^0Lazy_{wt}]$ v-constructs **T** for those $\langle w, t\rangle$ at which the set of individuals who are lazy at $\langle w, t\rangle$ belongs to M. In other words, it v-constructs **T** for a given $\langle w, t\rangle$ if the population of students is a subset of the population of lazy individuals at that $\langle w, t\rangle$. Abstraction over the values of w, t constructs the proposition that all students are lazy. It takes **T** at those $\langle w, t\rangle$ pairs at which all students are lazy.

For a mathematical example, consider the sentence

'It holds for all numbers that if the number is a prime then it is odd'.

The construction expressed by this sentence constructs **F**:

$$[{}^0\forall\ \lambda x\ [{}^0\supset [{}^0Prime\ x][{}^0Odd\ x]]].$$

The class constructed by

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[78] This principle, and its relevance to semantic analysis, is discussed in Section 2.1.

$$\lambda x\ [^0\!\supset [^0 Prime\ x][^0 Odd\ x]]$$

is not the class of all real numbers, of course, because the Composition

$$[^0\!\supset [^0 Prime\ x][^0 Odd\ x]]$$

$v(2/x)$-constructs the truth-value **F**.

Types: $\forall/(o(o\tau))$; $\supset/(ooo)$; *Prime, Odd*$/(o\tau)$.

Similarly, the construction expressed by 'No prime number is even' constructs **F**:

$$[[^0 No\ ^0 Prime]\ ^0 Even].$$

The type of *No* is here $((o(o\tau))(o\tau))$.

The class of even numbers does not belong to the class of all those classes that have an empty intersection with the class of prime numbers. On the other hand, the construction expressed by 'All primes greater than 2 are odd' constructs **T**:

$$[[^0 All\ \lambda x\ [^0\!\wedge [^0 Prime\ x]\ [^0\!>x\ 2]]]\ ^0 Odd].$$

The type of *All* is here $((o(o\tau))(o\tau))$.

The set of numbers constructed by $\lambda x\ [^0\!\wedge [^0 Prime\ x]\ [^0\!>x\ 2]]$ is a subset of the set of odd numbers. Note that, for instance, the last sentence (and its corresponding meaning) is equivalent to, 'It holds for all numbers that if the number is a prime greater than 2, then it is odd', the analysis of which is:

$$[^0\forall\ \lambda x\ [^0\!\supset [^0\!\wedge [^0 Prime\ x]\ [^0\!>x\ 2]]\ [^0 Odd\ x]]].$$

The class of numbers constructed by $\lambda x\ [^0\!\supset [^0\!\wedge [^0 Prime\ x]\ [^0\!>x\ 2]]\ [^0 Odd\ x]]$ is the whole type τ.

(c) *Singulariser*. The function *Sing* was defined in Definition 1.6. If a construction C v-constructs a singleton whose only member is a then $[^0 Sing\ C]$ v-constructs a. Otherwise (i.e., if C v-constructs an empty class or a class containing more than one element) $[^0 Sing\ C]$ is v-improper (See Definition 1.2 (iii)).

Remark. Often the abbreviated notation '$\iota x\ A$' will be preferred to '$[^0 Sing\ [\lambda x\ A]]$'.

Examples: The analysis of 'The only even prime number' is the Composition $(x \rightarrow \tau)$

$$[\iota x\ [^0\!\wedge [^0 Even\ x]\ [^0 Prime\ x]]].$$

It constructs the number 2, because the class of numbers constructed by

$$\lambda x\ [{}^0\wedge\ [{}^0Even\ x]\ [{}^0Prime\ x]]$$

is the singleton {2}.

The analyses of 'The only man to ever run 100 m in less than 9 s', 'The only man to ever run 100 m in less than 10 s' are the respective constructions of ι-offices:

$$\lambda w\lambda t\ \iota x\ [{}^0\wedge\ [{}^0Man_{wt}\ x][{}^0<\ [{}^0Run\_in_{wt}\ x\ {}^0100]\ {}^09]],$$
$$\lambda w\lambda t\ \iota x\ [{}^0\wedge\ [{}^0Man_{wt}\ x][{}^0<\ [{}^0Run\_in_{wt}\ x\ {}^0100]\ {}^010]].$$[79]

Types: $x \to \iota$; $Man/(o\iota)_{\tau\omega}$; $</(o\tau\tau)$; $Run\_in/(\tau\ \iota\tau)_{\tau\omega}$: an empirical function that assigns to an individual and a number (the distance in metres) the number (of seconds) that it takes the given individual to run the respective distance.

Both offices are currently vacant, because the ι-class *v*-constructed by

$$\lambda x\ [{}^0\wedge\ [{}^0Man_{wt}\ x][{}^0<\ [{}^0Run_{wt}\ x\ {}^0100]\ {}^09]]$$

is empty in the actual world now, and the class *v*-constructed by

$$\lambda x\ [{}^0\wedge\ [{}^0Man_{wt}\ x][{}^0<\ [{}^0Run_{wt}\ x\ {}^0100]\ {}^010]]$$

is a multi-element class. Its elements are, in 2009, Jim Hines, Ronnie Ray Smith, Charles Greene, Steve Williams, Eddie Hart, Reynaud Robinson, Silvio Leonard, Carl Lewis, Maurice Greene, Asafa Powell, Usain Bolt, and others.

(d) *Identity*. The type-theoretically polymorphic function = of type $(o\alpha\alpha)$, occasionally with an index pointing to the type α, is *identity*. We have, e.g.,

$$[{}^0=_\tau [{}^0+\ {}^07\ {}^05]\ {}^012],$$
$$[{}^0\neg\ [{}^0=_{*1}\ {}^0[{}^0+\ {}^07\ {}^05]\ {}^{00}12],$$
$$[{}^0=_{(o\tau)}\ \lambda x\ [{}^0\geq x\ {}^00]\ \lambda x\ [{}^0\neg[{}^0< x\ {}^00]]]$$

(all constructing **T**).

(e) *Sub* and *Tr* functions. In Definition 1.4 we specified two ways of binding variables in TIL, λ-binding and ^{0}binding. In both cases, a bound variable is not free for substitution, which brings technical trouble with it. To appreciate what sort of trouble, here are two examples of reckless deriving.

[79] We are disregarding here the problem of physical units.

Types: $B^*/(\omicron\iota^*_1)_{\tau\omega}$; $B/(\omicron\iota\omicron_{\tau\omega})_{\tau\omega}$; $F/(\omicron\iota)_{\tau\omega}$; a/ι; $x/^*_1 \rightarrow_v \iota$; $C/^*_1 \rightarrow_v \iota$.

$$(\mathrm{A}_1) \qquad \frac{\lambda w\lambda t\ [{}^0B^*_{wt}\ {}^0a\ {}^0[\lambda w\lambda t\ [{}^0F_{wt}\ C]]]}{\lambda w\lambda t\ [{}^0\exists\lambda x\ [{}^0B^*_{wt}\ {}^0a\ {}^0[\lambda w\lambda t\ [{}^0F_{wt}\ x]]]].}$$

$$(\mathrm{A}_2) \qquad \frac{\lambda w\lambda t\ [{}^0B_{wt}\ {}^0a\ [\lambda w\lambda t\ [{}^0F_{wt}\ C]]]}{\lambda w\lambda t\ [{}^0\exists\lambda x\ [{}^0B_{wt}\ {}^0a\ [\lambda w\lambda t\ [{}^0F_{wt}\ x]]]].}$$

Why are the conclusions no good? The occurrence of x in

$${}^0[\lambda w\lambda t\ [{}^0F_{wt}\ x]]$$

of the conclusion of (A_1) is ^{0}bound, so the variable x is *mentioned* and not used, hence not available for manipulation. It is, as it were, shielded from $\exists$ by the first Trivialization in ${}^0[\lambda w\lambda t\ [{}^0F_{wt}\ x]]$. In other words, x and C occur in (A_1) in a *hyperintensional context*.[80] A linguistic parallel would be to attempt to quantify into a quotational context, where the quotation marks would have an analogous shielding effect.

The argument (A2) is also invalid, for similar though slightly different reasons. Although the occurrence of x in

$$[\lambda w\lambda t\ [{}^0F_{wt}\ x]]$$

of the (A_2)-conclusion is free, the conclusion is not entailed by the premise. There are $\langle w, t\rangle$-pairs at which the proposition constructed by the premise is true, while the proposition constructed by the conclusion is false. The construction C v-constructing individuals occurs in the *intensional* context of $[\lambda w\lambda t\ [{}^0F_{wt}\ C]]$; thus C may be v-improper while the Closure $[\lambda w\lambda t\ [{}^0F_{wt}\ C]]$ is always proper (see Definition 1.2 (iv)) and the Composition $[{}^0B_{wt}\ {}^0a\ [\lambda w\lambda t\ [{}^0F_{wt}\ C]]]$ may v-construct **T** even if there is no C.

A parallel would be to attempt to quantify into an intensional or a hyperintensional context. For instance, from the truth of

Charles believes that Santa Claus is generous

[80] *Intensional* and *hyperintensional context* were characterized in Section 1.3, and will be formally defined in Section 2.6 together with valid rules for inferring existence. Here just briefly: a hyperintensional context is one in which constructions are mentioned, whereas an intensional context is one in which constituents are used with intensional (or *de dicto*) supposition.

we cannot validly infer that Santa Claus exists.[81] What we can infer is that there is an individual *office* (a.k.a. individual *role*) such that Charles believes that its occupant is generous.

However, sometimes we do need to quantify and/or substitute into a hyperintensional or intensional context; for instance, when analysing *de re* attitudes or sentences with anaphoric reference.[82] The solution is to substitute for the variable x the Trivialization of the entity v-constructed by the respective construction C instead of substituting the construction C itself. To this end, we need the functions Sub^n and Tr^α, which make variables amenable to manipulation by, first, untying them from the context they occur in and, second, substituting Trivialization of an appropriate entity for them.

Let X, Y, Z be constructions of order n, Y a variable. Then the function $Sub_n/(*_n\ *_n\ *_n\ *_n)$ is a mapping which, when applied to $\langle X, Y, Z\rangle$, returns the construction that is the result of correctly substituting X for Y in Z. Correct substitution will be defined in Definition 2.22. For now it suffices to say that a substitution is correct if no free variable occurring in X becomes bound in the resulting construction. Thus, for instance, the Composition

$$[{}^0Sub_1\ {}^{00}2\ {}^0x\ {}^0[{}^0{+}\ x\ {}^01]]$$

constructs the result of substituting ^{02} for x into $[{}^0{+}\ x\ {}^01]$, so the result is the Composition $[{}^0{+}\ {}^02\ {}^01]$. Therefore, the Composition $[{}^0Sub_1\ {}^{00}2\ {}^0x\ {}^0[{}^0{+}\ x\ {}^01]]$ is equivalent to ${}^0[{}^0{+}\ {}^02\ {}^01]$, both constructing as they do the Composition $[{}^0{+}\ {}^02\ {}^01]$:

$$[{}^0Sub_1\ {}^{00}2\ {}^0x\ {}^0[{}^0{+}\ x\ {}^01]] =_{*1} {}^0[{}^0{+}\ {}^02\ {}^01]].$$

Next, let α be a type of order n, a an object of type α. Then $Tr_\alpha/(*_n\ \alpha)$ is a function which, when applied to a, returns the Trivialization of a.[83]

Note that there is an essential difference between using Trivialization and applying the Tr_α function. For instance, whereas ^{03} constructs the number 3, the Composition $[{}^0Tr_\tau\ {}^03]$ constructs the construction ^{03}. Whereas the Trivialization ${}^0x$ binds the variable x and constructs just x, the variable x is free in the Composition $[{}^0Tr_\tau\ x]$, which v-constructs the Trivialization of the number that v assigns to x. For instance, $[{}^0Tr_\tau\ x]$ $v(2/x)$-constructs the construction ^{02}.

To illustrate the application of the *Sub* function, consider the schematic Composition

$$[{}^0Sub_1\ [{}^0Tr_\iota\ {}^0A_{wt}]\ {}^0y\ {}^0[\ldots y\ldots]].$$

Types: $A/\iota_{\tau\omega}$; $y \rightarrow_v \iota$; a/ι.

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[81] For the propositional attitudes of knowing and believing, see Sections 5.1 and 5.3.

[82] For attitudes and anaphoric sentences, see Chapter 5 and Section 3.5, respectively.

[83] See Tichý (1988, pp. 74–5).

This Composition either v-constructs the construction $[\ldots{}^0a\ldots]$, in case ${}^0A_{wt}$ v-constructs a, or is v-improper, in case ${}^0A_{wt}$ is v-improper.

We will often omit the lower-index when using the polymorphic functions Sub_n and Tr_α, writing simply '*Sub*' and '*Tr*', when the typing is obvious.

We will deal with quantifying into intensional and hyperintensional contexts in Section 5.3. To get a first feel for how TIL approaches quantifying in, consider again the above example

$$\frac{\text{Charles believes that Santa Claus is generous}}{\text{There is an office such that Charles believes that its occupant is generous.}}$$

We will analyse Charles' attitude as one of explicit belief, which is a relation-in-intension of an individual to a hyperproposition (a propositional construction). First, type-theoretical analysis:

$Charles/\iota$; $Believe/(o\iota{*_1})_{\tau\omega}$; $Santa\_Claus/\iota_{\tau\omega}$: an individual office; $Generous/(o\iota)_{\tau\omega}$; $\exists/(o(o\iota_{\tau\omega}))$.

The premise says that Charles explicitly believes that Santa Claus is generous. To construct the proposition, we have to ascribe the property of being generous to the occupant of the office of Santa Claus. To this end we use the Trivialization of the office and its intensional descent, ${}^0Santa\_Claus_{wt}$, which v-constructs the individual (if any) that plays the role of Santa Claus at the $\langle w, t\rangle$-pair of evaluation. The proposition is now constructed by the Closure $\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]$. Since Charles bears the relation of explicit belief to this *construction*, we must *mention* it by means of Trivialization. The analysis of the premise is

$$\text{(P)} \qquad \lambda w\lambda t\ [{}^0Believe_{wt}\ {}^0Charles\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]]].$$

Now, we cannot frivolously derive that Santa Claus exists, of course, for the office of Santa Claus is not occupied. But we can derive that there is such an office. Here is how. Let variable $r/{*_1}$ v-construct individual offices, of type $\iota_{\tau\omega}$. Then for any $\langle w, t\rangle$ such that the Composition

$$[{}^0Believe_{wt}\ {}^0Charles\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]]]$$

v-constructs **T**, the Composition

$$[{}^0\exists\lambda r\ [{}^0Believe_{wt}\ {}^0Charles\ [{}^0Sub\ [{}^0Tr\ r]\ {}^0r\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ r_{wt}]]]]]$$

v-constructs **T** as well. To show this, let $v(Santa\_Claus/r)$ be a valuation identical to v up to assigning the office *Santa_Claus* to the variable r. Then $[{}^0Tr\ r]$ $v(Santa\_Claus/r)$-constructs ${}^0Santa\_Claus$, and

$$[{}^0Sub\ [{}^0Tr\ r]\ {}^0r\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ r_{wt}]]]$$

$v(Santa\_Claus/r)$-constructs the Closure

$$[\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]].$$

So the Composition

$$[{}^0{=_{*1}}\ [{}^0Sub\ [{}^0Tr\ r]\ {}^0r\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ r_{wt}]]]\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]]]$$

$v(Santa\_Claus/r)$-constructs **T**. Hence the class of individual offices v-constructed by the Closure

$$\lambda r\ [{}^0Believe_{\mathrm{wt}}\ {}^0Charles\ [{}^0Sub\ [{}^0Tr\ r]\ {}^0r\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ r_{wt}]]]]$$

is not empty. The analysis of the conclusion entailed by the premise (P) is then:

(C) $\lambda w\lambda t\ [{}^0\exists\lambda r\ [{}^0Believe_{wt}\ {}^0Charles\ [{}^0Sub\ [{}^0Tr\ r]\ {}^0r\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ r_{wt}]]]]]$.

For a mathematical example, consider the sentence

(10)
'There is a number x such that dividing any number y by x is improper'.

If objects of higher-order types were not admitted, we would have no means to analyse this true sentence. The procedure of dividing y by x is improper for some number x, because it does not yield a product for some x, namely 0.

Let $Div/(\tau\tau\tau)$ be the function of dividing and $Improper/(o*_1)$ the class of constructions of order 1 that are v-improper for any valuation v. Finally, let the variables x, y range over the type τ. Then to express that dividing y by x is improper amounts to expressing the Composition

$$[{}^0Improper\ {}^0[{}^0Div\ y\ x]].$$

Now, we cannot recklessly quantify over x and y, because x, y are ^{0}bound here. There is a way out, however. We use *Sub* and *Tr* to pre-process, as it were, the Composition

$$[{}^0\mathit{Improper}\ {}^0[{}^0\mathit{Div}\ y\ x]]$$

to make it construct **T**. First, by means of *Tr*, we untie *x* and *y*, and then substitute the resulting Trivialization of the numbers *v*-constructed by *x* and *y* into the Composition $[{}^0\mathit{Div}\ y\ x]$. Here is how:

$$[{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ x]\ {}^0x\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ x]]].$$

Note that in this Composition *x* and *y* are free for manipulation; the result is a *construction*, *in casu* the procedure of applying the division function to the numbers *v*-constructed by *x* and *y*. Now we want to express that this construction is improper for some number *v*-constructed by *x* and for all numbers *v*-constructed by *y*. The resulting analysis is thus

(10′) $[{}^0\exists\lambda x\ [{}^0\forall\lambda y\ [{}^0\mathit{Improper}\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ x]\ {}^0x\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ x]]]]]].$

To see that this Composition constructs **T**, it suffices to realise that the Composition

$$[{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ x]\ {}^0x\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ x]]]$$

behaves as follows. It $v(0/x)$-constructs the construction $v(0/x)$-constructed by $[{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ x]]$, i.e. by $[{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ {}^00]]$. The latter $v'(0/x)$-constructs a $v'(0/x)$-improper construction for any valuation $v'(0/x)$ identical to $v(0/x)$ up to assigning any number to *y*. For instance, for $v(0/x,1/y)$ we obtain the construction $[{}^0\mathit{Div}\ {}^01\ {}^00]$. For $v(0/x, 2/y)$ we obtain the construction $[{}^0\mathit{Div}\ {}^02\ {}^00]$; and so on. Thus, the class $v(0/x)$-constructed by

$$\lambda y\ [{}^0\mathit{Improper}\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ x]\ {}^0x\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ x]]]]$$

is the whole type τ, and the Composition

$$[{}^0\forall\lambda y\ [{}^0\mathit{Improper}\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ x]\ {}^0x\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ x]]]]]$$

$v(0/x)$-constructs **T**. Therefore, the class of numbers constructed by

$$\lambda x\ [{}^0\forall\lambda y\ [{}^0\mathit{Improper}\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ x]\ {}^0x\ [{}^0\mathit{Sub}\ [{}^0\mathit{Tr}\ y]\ {}^0y\ {}^0[{}^0\mathit{Div}\ y\ x]]]]]$$

is non-empty (because its element is the number 0), and the Composition (10′) constructs the truth-value **T**.

1.5 Constructions as structured meanings

1.5.1 Structured meanings

The contemporary mainstream method of logically analyzing expressions of a natural language consists in building up an artificial language and defining some rules of translation that make it possible to find for every expression of the given language its translated counterpart in the artificial language. The latter is unambiguous (unlike the former) and is interpreted in a model in the usual way.[84]

Tichý calls this method *formalization*. Formalization itself, if thought of as a means to make ideas precise, is indispensable. The method deployed by TIL to make ideas precise is a method of *direct* analysis. The notion of construction enables us to justify this direct transition from expressions to their meanings.

In a wider perspective, an important difference between Tichý and Montague is preceded by a famous difference Schopenhauer saw between himself and Kant. Schopenhauer said that,

> [Kant] is comparable to a person who measures the height of a tower from its shadow; but I am like one who applies the measuring rod directly to the tower itself.[85] (1819, p. 555.)

Montague, like other model-theoretic ('Tarskian') semanticists, translates natural-language phrases into shapes belonging to a pure syntax which are subsequently valuated. Tichý translates natural-language phrases into a likewise artificial symbolism. But TIL's symbolism is importantly different from IL's. TIL's 'language of constructions' is an *interpreted* formalism, so syntax and semantics work in tandem. The syntax of the λ-terms of TIL is provided by the existing λ-calculus, while the formalism is inherently interpreted, because its λ-terms are introduced as terms denoting constructions. The TIL analysis of a natural-language expression does not tell us which expression belonging to some other language it is synonymous with. Instead it tells us which its sense is. Montague's approach to analysis is indirect, Tichý's direct. The TIL λ-terms are in themselves of no interest and serve only as gateways or stepping-stones to non-linguistic entities, namely senses (constructions).[86] The only way to talk about senses is to avail oneself of terms denoting them. But the only task that the symbolic 'language of constructions' has to fulfil is to denote (atomic and compound) constructions. Metaphorically,

[84] See, e.g., Gamut (1991) or Montague (1974d).

[85] The original German text can be found in the *Anhang: Kritik der Kantischen Philosophie* in the 2nd Book of (1819), and goes,'[Kant] ist demjenigen zu vergleichen, der die Höhe des Thurmes aus dessen Schatten mißt, ich aber dem, welcher den Maaßstab unmittelbar anlegt.'

[86] Cf. Russell, who famously talked about thinking about logical objects for 2 s every 6 months, the rest of the time thinking about notation (1953, p. 185).

the symbols are transparent in the sense that we look through them to look at the constructions they denote.[87]

By way of illustration, the TIL analysis of

'1 + 2 = 3'

is the Composition

$$[{}^0{=}\ [{}^0{+}\ {}^01\ {}^02]\ {}^03].$$

The term '$[{}^0{=}\ [{}^0{+}\ {}^01\ {}^02]\ {}^03]$' denotes the sense of '1 + 2 = 3', i.e. the procedure of applying the identity function to two arguments, the first being the result of applying the plus function to 1 and 2, the second argument being the number 3. In general terms, a logical analysis of a given language consists in establishing such pairs of expressions and constructions. The *code* function underlying a given natural language, at a given phase of its historical development, will have been cracked, once all the expressions of the language have been paired off with constructions. That meanings are conceptually prior to their encoding in a language is summed up thus:

> The notion of a *code* [our emphasis] presupposes that prior to, and independently of, the code itself there is a range of items to be encoded in it. Hence…meanings cannot be conceived of as products of the language itself. They must be seen as logical rather than linguistic structures, amenable to investigation quite apart from their verbal embodiments in any particular language. To investigate logical constructions in this way is the task of logic. The linguist's brief is to investigate how logical constructions are encoded in various vernaculars (Tichý, 1994b, pp. 804–05).

Coupling all the expressions of an actual natural language with constructions would be no mean achievement for field linguistics. Logical analysis does not aspire to crack the code for *all* the expressions of a language, but it must aspire to be able to crack the code of *any* expression. When the idealization is made that language-users are perfectly competent, the idealization amounts to the language-users mastering every ⟨expression, construction⟩ pair of a given language.

TIL, and so its adjacent conception of logical analysis of natural language, is strictly opposed to any theories that maintain that meanings are produced by language.

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[87] Tichý suggested construing his λ-formalism as an iconography or pictorial script (see especially 1988, p. 224). This construal is buttressed by a strict enforcement of the principle of subject matter, which in turn might suggest something like a homomorphism between the set of λ-terms and the set of constructions of a given order (though an isomorphism is excluded, since there are more constructions of a given order than there are λ-terms). However, we have not attempted to develop this sketchy idea of iconography into a theory of λ-terms as something like *logical pictures* of constructions, mainly because the project of logical analysis of language does not need it and because any such theory would have to be embedded within the vast discussion on perfect languages, the expressive power of pictures, etc. For a discussion of the notion of pictorial script (without reference to TIL), see Jespersen and Reintges (2008).

Language, instead, is a code, and a code is a mapping of linguistic entities into non-linguistic entities. The latter are not inherently meanings, but become meanings in virtue of the code. That is, entities existing conceptually prior to pieces of language are made to serve in the office of linguistic meaning.[88] The idea of code is squarely incompatible with all theories that share with Quine the view that taking language as a code for certain objective operations means being a naïve advocate of 'the myth of the museum'.[89]

Constructions, then, are the primary subject-matter of our logical study. Their encoding in particular languages is of secondary importance. How constructions are encoded is fixed by sets of linguistic conventions, and field linguistics studies a posteriori the conventions germane to different languages at particular stages in their historical development. But the properties of and relations between particular constructions are a priori. For instance, the argument

$$\frac{\begin{array}{c}\text{There is no } x \text{ such that } x \text{ is a prime greater than 2 and } x \text{ is even}\\ \text{5 is a prime greater than 2}\end{array}}{\text{5 is not even}}$$

is logically valid independently of in which natural language these constructions are encoded. If we choose Czech instead of English, the encoding of the above valid argument would trivially be different:

$$\frac{\begin{array}{c}\text{Neexistuje } x \text{ takové, že } x \text{ je prvočíslo větší než 2 a } x \text{ je sudé}\\ \text{5 je prvočíslo větší než 2}\end{array}}{\text{5 není sudé číslo.}}$$

Yet the underlying constructions are *identical*. For instance, ${}^{00}\mathit{Prime} =_{*1} {}^{00}\mathit{Prvočíslo}$, ${}^{00}\mathit{Even} =_{*1} {}^{00}\mathit{Sudé}$. What is Trivialized is not a symbol, but an object (here, the set of prime numbers and the set of even numbers). In fact, an identical construction is what two synonymous expressions (whether of the same language or different languages) owe their synonymy to.[90]

The very *term* 'construction' is not entirely felicitous, connected as it is with many potentially misleading connotations, chief among which are the ones of mental procedure and constructivist proof (–object).[91] However, TIL constructions and those of intuitionism/constructivism share some noteworthy common ground.

[88] —where 'office' is used as in normal English and not as in TIL.

[89] See Materna (2004b).

[90] More precisely, synonymous expressions express a common concept; see Section 2.2.

[91] In choosing the term 'construction', Tichý was inspired by geometry 'where we speak of various constructions of, say, the center of a circle, using rule and compass' (1986b, p. 514, 2004, p. 601).

For instance, verbally, at least, we agree with the intuitionist Fletcher when he says,

> If one had to define constructions in general, one would surely say that a type of construction is specified by some *atoms* and some *combination rules* of the form "Given constructions $x_1, \ldots, x_k$ one may form the construction $C(x_1,\ldots,x_k)$, subject to certain conditions on $x_1, \ldots, x_k$" (1998, p. 51).

TIL constructions are in themselves abstract, objective procedures. When made to serve as meanings, they are procedures detailing how to arrive at denoted entities.

What in part characterizes semantic realism is exactly that 'thoughts [*in casu* constructions. *Our insertion*] are independent of their expression in any language' (Tichý, 1988, p. vii). Yet, although TIL is semantic realism with a vengeance, TIL fails to qualify as such according to Dummett's entrenched definition of semantic realism. According to Dummett, realism construes sentential meanings as truth-conditions, while Dummett's own proof-theoretic anti-realism is cast in terms of assertability conditions. To qualify as realism in Dummett's sense, since empirical truth-conditions are possible-world propositions, TIL would have to construe propositions as the senses of empirical sentences; but we have argued at length why we are not pursuing this tack. One tenet, though, that TIL shares with realism as Dummett understands it is that truth-conditions obtain or fail independently of human cognitive means to establish which way they go. It is evident, however, that Dummett's conception of realism is too narrow to capture TIL, or indeed any other realist theory based on a procedural rather than truth-conditional semantics.[92]

Thus one of the advocates of procedural semantics, W.A. Woods, sums up

> two extreme interpretations of procedural semantics – a black-box approach in which the internal structure of a meaning function is inaccessible (only the input-output relations are available), and a low-level detail approach in which every detail of the operation of the meaning function procedure is considered a 'part of the meaning'. The former gives rise to a sense of equivalence between meaning functions that is too weak ..., in that it counts as equivalent meaning functions whose input-output relations are the same (in all possible situations) regardless of the means by which those extensions are determined [thus identifying, e.g., tautologies]. The low-level detail interpretation is at the opposite extreme of this spectrum. Its sense of equivalence is so strong that it counts two meaning functions as different if they differ in any detail of their operation regardless of the extent to which they effectively do the same thing. The notion of abstract procedure that is required for the characterization of meaning functions appears to lie somewhere between these extremes – providing a degree of internal structure that is considered significant, while leaving certain low-level details unspecified (or specified with suitable don'-care conditions) (1981, p. 329).

When assigning a *construction* to an expression as its meaning, we specify *procedural know-how*, which must not be confused with the respective *performatory know-how*. Distinguishing performatory know-how from procedural know-how,

[92] For the notion of procedural semantics, see Johnson-Laird (1977) and Woods (1981). For a defence of denotational semantics against procedural semantics, see Fodor (1975).

Rescher says of the latter that a knower x 'knows how A is done in the sense that x can spell out instructions for doing A' (2005, p. 6). Thus,

> x knows that people swim by moving their arms and legs in a certain cycle of rhythmic motions. But, of course, x can know how A is done without being able to do A—that is, without x having the performatory skills that enable x to do A. (For instance, x may know that a certain result is produced when a text is translated from one language to another without actually knowing how to make such a translation.) (ibid., p. 7).

If linguistic meaning is procedural, then to know what a given expression of a given language means is to possess procedural know-how. Linguistic competence is to know what particular procedure is encoded by an expression and how to execute the procedure. It is not required of the linguistically competent either that they should execute the procedure or even have the performatory know-how to do so.

For instance, to know what '1 + 2' means is to understand the instruction to add 1 and 2. It does not include either actually adding 1 and 2 (whether by following a procedure or by luck) or possessing the skill to do so. Similarly, we do understand the formulation of the Goldbach Conjecture (i.e., we do know the meaning of 'All positive even integers ≥ 4 can be expressed as the sum of two primes') without being able to execute the instruction in order to obtain the respective truth-value. In other words, we know the following construction without knowing what this construction constructs:[93]

$$\forall x \left[[[{}^0\!Even\ x] \wedge [{}^0\!> x\ {}^0 2]] \supset \exists yz\ \mathbf{[}[{}^0\!Prime\ y] \wedge [{}^0\!Prime\ z] \wedge [x = [{}^0\!+ y\ z]]\mathbf{]}\right].$$

Types: ν (the type of natural numbers); $\forall/(o(o\nu))$; $Even/(o\nu)$; $Prime/(o\nu)$; x, y, $z/*_1 \rightarrow \nu$.

Constructions are structured from the algorithmic point of view. We will now illustrate the way in which they are so structured.

Let us again consider a simple arithmetical expression, say,

'7 + 5'.

Bearing in mind that language is a code, we see that the above expression can be construed as encoding the meanings of particular simple subexpressions, but—and this is most important—also *the way these particular meanings combine* to form the meaning of the whole expression. In other words, the meaning **M** of the whole expression

M('7 + 5')

is not the same as the set of meanings of particular subexpressions of E, here

[93] The respective hypothesis expresses an ineffective procedure.

$$\{\mathbf{M}(‘7’), \mathbf{M}(‘5’), \mathbf{M}(‘+’)\}.$$

(Remember Tichý’s metaphor of ‘Christmas decorations hanging from the branch’.) In general, constructions are abstract procedures that integrate particular subprocedures (‘steps’) into one whole. A mere set of meanings could not integrate individual meanings into the meaning of a molecule. Constructions consist of parts that are themselves constructions. So since constructions are procedures, one could equally well say that procedures consist of parts that are themselves procedures. The meaning of ‘7 + 5’ is the procedure $[^0+ \ ^07 \ ^05]$ decomposable into constituents as follows:

(1) ^{07}: identify the number 7
(2) ^{05}: identify the number 5
(3) $^0+$: identify the function +
(4) $[^0+ \ ^07 \ ^05]$: apply the product of step (3) to the products obtained at steps (1) and (2), respectively, in order to obtain the value of the function at this pair of arguments.

At least since Frege’s days there have been logicians who strove to avail themselves of fine-grained and structured meanings.[94] Analytic philosophy of language has pretty much since its inception been characterised in part by this quest. For instance, Russell’s structured propositions were not unlike our constructions. Unlike sets, they consisted of *parts*, but some of these parts were (due to Russell’s theory of acquaintance) concrete particulars. This leads to consequences that do not tally with our intuitive use of the term ‘proposition’; for instance, that propositions must be mind-friendly. Thus, we would definitely side any day with Frege against Russell over whether Mont Blanc can be in any sensible way part of anything deserving the name ‘proposition’. Moreover, language-users understand many sentences without being acquainted with the concrete particulars that the sentences talk about by means of abstract objects. The parts of a procedure have to be other procedures and cannot be the objects themselves, though the procedure may lead up to a non-procedure as its final output. A procedure (including any procedure figuring as a constituent subprocedure) is a presentation of an object rather than a presented object. But when knowing a procedure we need not know its output before actually executing it. We need to be acquainted with the procedure first before being able to execute it so as to arrive at the result. And some procedures may even fail to provide an output. A procedure is a different object than its product (if any), which is why exhaustive knowledge of the procedure does not include knowledge of its product. One thing is to know what to do (to know the procedure), quite another thing is to actually execute the procedure, and yet another thing is to know and understand what sort of object, if any, is the output.

[94] For example, see Sundholm (1994) on Frege’s epistemological motivations for a fine-grained individuation of *Gedanken*.

As pointed out in Section 1.1, Carnap (1947) rightly recognised that his intensions cannot handle all cases of synonymy and attempted to define the concept of intensional isomorphism. Church (1954) launched a counterexample involving two intensionally isomorphic sentences, one of which can be easily believed and the other not. A criticism of Carnap's attempt can be also found in Tichý (1988, pp. 8–9), where it is pointed out that the notion of intensional isomorphism is too dependent on a particular choice of notation. The structured character of meaning was later urged by David Lewis (1972), where non-structured intensions are generated by finite, ordered trees. This idea of 'tree-like' meanings obviously influenced George Bealer's idea of 'intensions of the second kind' in his (1982).[95]

The idea of structured meaning was propagated also by M.J. Cresswell (1975) and (1985), in which meaning is defined as an ordered *n*-tuple. Cresswell would construe the meaning of the above expression as a triple, viz.,

$$\langle \mathbf{M}(\text{‘}+\text{’}), \mathbf{M}(\text{‘}7\text{’}), \mathbf{M}(\text{‘}5\text{’})\rangle.$$

That this is far from being a satisfactory solution is shown in Tichý (1994a) and Jespersen (2003). In brief, these tuples are again set-theoretic entities structured at most from a mereological point of view, by having elements or parts (though one balks at calling elements 'parts', since sets, including tuples, are not complexes). Besides, tuples are of the wrong making to serve as truth-bearers and objects of attitudes, since a tuple cannot be true or be known, hoped, etc., to be true. The above tuple is 'flat' from the procedural or algorithmic point of view. The *way* of combining particular parts together is missing here. For instance, the instruction to *apply* the function *plus* to a particular argument could have been one such way. It is to no avail to add the operation of application to a tuple to somehow create propositional unity, since the operation would merely be an element alongside other elements.[96] Moreover, the procedure specifying a function remains the same when other arguments are supplied as input for the function to be applied to. It is uncontroversial that tuples are set-theoretic objects; and all sets, unlike procedures, are algorithmically simple, have no 'input/output gaps', and are flat mappings.

[95] Van Heijenoort attempts to interpret Fregean *Sinn* in terms of trees. He suggests (1977, pp. 99–100) that the Fregean *Sinn* of a formula T is to be identified with a tree T', whose semantic structure will be isomorphic to the syntactic structure of T. The suggestion is *prima facie* appealing, not least because the diagrammatic structure of trees is in the vicinity of the syntactic structure of Frege's *Begriffsschrift* notation. However, as Van Heijenoort himself points out, 'a tree is a mapping… Thus, in Fregean terms, a tree would be the object that is the *Werthverlauf* of a certain function. This conclusion may seem quite odd.' Indeed it does. But, worse, if Fregean *Sinn* is to be sliced in terms of cognitive significance rather than merely logical equivalence, then a mapping won't do as *analysans* due to the crude individuation of mappings.

[96] See Cocchiarella (2003, p. 51) for a recent statement of this objection. For a philosophical and historical discussion of propositional unity, see Gaskin (2008).

We agree with Moschovakis' idea of meaning as algorithm (see Moschovakis (1994, 2006), van Lambalgen and Hamm (2004)). In Moschovakis (2006) the meaning of a term *A* is 'an (abstract, idealized, not necessarily implementable) algorithm which computes the denotation of *A*.' (2006, p. 27; see also 1994).[97] The later version (2006) works with a formal language that extends the typed λ-calculus and so can accommodate, *per* Montague, reasonably large fragments of natural language. Moschovakis outlines his conception thus:

> The starting point … [is] the insight that a correct understanding of programming languages should explain the relation between a program and the algorithm it expresses, so that the basic interpretation scheme for a programming language is of the form
>
> (50) program *P* → algorithm(*P*) → den(*P*).
>
> It is not hard to work out the mathematical theory of a suitably abstract notion of algorithm which makes this work; and once this is done, then it is hard to miss the similarity of (50) with the basic Fregean scheme for the interpretation of a natural language,
>
> (51) term *A* → meaning(*A*) → den(*A*).
>
> This suggested at least a formal analogy between algorithms and meanings which seemed worth investigating, and proved after some work to be more than formal: when we view natural language with a programmer's eye, it seems almost obvious that we can represent the meaning of a term *A* by the algorithm which is expressed by *A* and which computes its denotation (ibid., p. 42).

In modern jargon, TIL belongs to the paradigm of *structured meaning*. However, Tichý does not reduce structure to set-theoretic sequences, as do Kaplan and Cresswell. Nor does Tichý fail to explain how the sense of a molecular term is determined by the senses of its atoms and their syntactic arrangement, as Moschovakis objects to 'structural' approaches in (2006, p. 27).

The notion of TIL construction is bound to elude the followers of holistic theories (Quine, the later Wittgenstein, etc.). In fact, the idea of construction is an anti-holistic idea, supposing as it does that the meaning of an expression can be in principle *composed* from the meanings of its subexpressions.

TIL is opposed to various nominalist trends in contemporary philosophy, not least their misuse of Occam's razor. Tichý' succinctly sums up the lie of the land:

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[97] Moschovakis' notion of algorithm borders on being too permissive, since algorithms are normally understood to be effective. (See Cleland (2002) for discussion.) Tichý separates algorithms sharply from constructions: 'The notion of construction is…correlative not with the notion of algorithm itself but with what is known as a particular algorithmic *computation*, the sequence of steps prescribed by the algorithm when it is applied to a particular input. But not every construction is an algorithmic computation. An algorithmic computation is a sequence of *effective* steps, steps which consist in subjecting a manageable object…to a feasible operation. A construction, on the other hand, may involve steps which are not of this sort' (1986b, p. 526 2004, p. 613).

> [T]he vision informing 20th century philosophy has been aptly described as one of a desert landscape. Philosophers behave as if in expectation of an ontological tax collector to whom they will owe the less the fewer entities they declare. The metaphysical purge is perpetrated under a banner emblazoned with Occam's Razor. But Occam never counselled ontological genocide at all cost. He only cautioned against multiplying entities *beyond necessity.* His Razor is thus in full harmony with the complementary principle, known as Menger's Comb, which cautions against trying to do with less what requires more. The two methodological precepts are just two sides of the same coin
> (1995, p. 175, 2004, p. 875).

Thus one should bear in mind that there is a complementary warning in the shape of *Menger's comb*. Another pointed criticism of the abuse of Occam's razor is this:

> To satisfy the constraints of ontological parsimony, one should add as few objects as possible in a nonarbitrary way. But with abstract objects, the only way to add as few objects as possible in a nonarbitrary way is to add them all! … Platonized naturalism acknowledges that a maximal ontology of abstracta is the simplest because a plenum is not an arbitrary selection from some larger class (Linsky and Zalta, 1995, p. 552).

Morale: logical analysis of natural language must take the form of a procedural semantics in order to succeed. So, in keeping with Menger's comb, nothing less than a 'maximal ontology of abstracta' is going to be plentiful enough to contain procedures as fully-fledged entities.

1.5.1.1 Analytic vs. logical

There has been a long philosophical dispute concerning the definition of analytic truth and the relation between analytic and synthetic truths. The distinction goes as far back as Leibniz, at least. For now it is sufficient to adopt the explication that an analytically true sentence is true solely in virtue of its meaning. Since we presuppose full linguistic competence in language-users, sentences like 'No bachelor is married', 'Whales are mammals', and also mathematical sentences like 'The problem of logical validity is not decidable in first-order predicate logic' come out analytically true. Provided that we understand the meanings of the predicates 'is a whale' and 'is a mammal' as used in current English, when learning that whales are mammals we do not acquire information bearing on the state of the world. If you know that the individual before you is a whale, you need not examine the world in order to get to know that the individual is a mammal.

Our procedural semantics enables us to easily define the difference between analytically and logically true sentence, as well as the difference between analytically and logically valid argument. Recall that *TRUE* is the proposition that takes value **T** in all worlds at all times.

Definition 1.9 (*analytically true sentence*) A mathematical *sentence* is *analytically true* iff it expresses a construction constructing the truth-value **T**. A *sentence* involving empirical expressions is *analytically true* iff it expresses a construction constructing the proposition *TRUE.*

Yet the literal analysis of the sentence 'No bachelor is married' does not reveal the fact that it is analytically true.

The types are: *Bachelor*, *Married*/$(o\iota)_{\tau\omega}$; *No*/$((o(o\iota))(o\iota))$: the quantifier that assigns to a given set *M* the set of those sets of individuals which have an empty intersection with *M.*

Thus the synthesis is:

(*) $\lambda w\lambda t\ [[{}^0No\ {}^0Bachelor_{wt}]\ {}^0Married_{wt}]$.

Type-checking:

$$\begin{array}{lccc}
\lambda w\lambda t & [[{}^0No & {}^0Bachelor_{wt}] & {}^0Married_{wt}] \\
\downarrow & \underbrace{((o(o\iota))(o\iota)) \quad\quad} & \underbrace{(o\iota)} & \downarrow \\
\downarrow & \underbrace{(o(o\iota)) \qquad\qquad} & & \underbrace{(o\iota)} \\
o_{\tau\omega} & \longleftarrow & o &
\end{array}$$

This Closure constructs a proposition, as it should, but it is not obvious that the so constructed proposition is identical to *TRUE*.[98]

On the other hand, the sentence 'It is not true that there is an individual *x* such that *x* is not married and *x* is a man and *x* is married' is also analytically true; but not only that: it is also *logically* true, as its analysis shows:

(**) $\lambda w\lambda t\ [\forall w\forall t\ [\neg\exists x\ [\neg[{}^0Married_{wt}\,x] \wedge [{}^0Man_{wt}\,x] \wedge [{}^0Married_{wt}\,x]]]]$.

Since the Composition $[\neg\exists x\ [\neg[{}^0Married_{wt}\ x] \wedge [{}^0Man_{wt}\ x] \wedge [{}^0Married_{wt}\ x]]]$ obviously and provably *v*-constructs **T** for any valuation *v*, the generalisation

$$[\forall w\forall t\ [\neg\exists x\ [\neg[{}^0Married_{wt}\,x] \wedge [{}^0Man_{wt}\,x] \wedge [{}^0Married_{wt}\,x]]]]$$

constructs **T**. Therefore, the proposition constructed by the above Closure is the proposition *TRUE*.

But, which of the two equivalent constructions (*), (**) should be assigned to 'No bachelor is married' as its meaning? Provided the predicate 'is a bachelor' is a

[98] This three-step analysis anticipates Section 2.1.1.

semantically simple expression, the *literal meaning* of this sentence is (*).[99] Thus we define:

Definition 1.10 (*literal meaning of an expression*) Let E be an expression whose semantically simple subexpressions are $S_1, \dots, S_n$, and let $S_1, \dots, S_n$ denote the objects $X_1, \dots, X_m$. Let C_E be a construction that is assigned to E as its meaning such that there is no closed subconstruction of C_E constructing an object that is not denoted by a subexpression of E. Then C_E is the *literal meaning of E* iff ${}^0X_1, \dots, {}^0X_m$ are all closed subconstructions of C_E constructing the objects $X_1, \dots, X_m$, respectively.

Definition 1.10 imposes the constraint that the objects that receive mention by *simple* meaningful subexpressions should be constructed by their Trivialisations. If the expression E is semantically simple, then the Trivialisation of the denoted object is assigned to E as its literal meaning. On the other hand, if E is semantically complex, then the Trivialisations of objects denoted by simple subexpressions of E are combined into a complex construction assigned to E as its literal meaning in the manner complying with the set-theoretical conditions imposed by E.[100]

In order to define the notion of logical truth, we must first define the notion of *literal logical form*:

Definition 1.11 (*literal logical form of an expression*) Let C_E be the literal logical analysis of E, whose subconstructions construct (by Trivialisation) the extra-logical objects $X_1, \dots, X_n$, X_i/α_i. Let $V_1 \to \alpha_1, \dots, V_n \to \alpha_n$ be variables not occurring in C_E. Then the *literal logical form* (*LLF*) of E is the construction LC_E that differs from C_E only in replacing all occurrences of ${}^0X_i$ by V_i.

It is important to note that according to Definition 1.11 only Trivialisations of *extra-logical* objects are replaced by type-theoretically appropriate variables in order to obtain the literal logical form of the relevant expression. Construction of logical objects like truth-functions and quantifiers are left unchanged.[101] Thus the literal logical form of a sentence corresponds to a formula of a formal language. The formulae of a formal language are associated with their models by means of an interpretation of special non-logical symbols. A formula is then logically true if it is true on every interpretation.

As we explained at the outset of this section, we do not translate sentences of a natural language into a formal language with a view to interpreting this language. Instead, by means of 'the language of constructions' we directly examine constructions expressed by natural-language sentences. Yet there is a similarity with

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[99] The other option amounts to conceiving 'is a bachelor' as a semantically complex expression. See also Section 2.2.1.

[100] See Section 2.1 for the method of semantic analysis.

[101] See Section 1.4.3 for the list of logical objects.

the formal approach. If a sentence is logically true, it is true in virtue of its logical form, regardless of any particular extra-logical objects receiving mention in the sentence.[102] For instance, the sentence 'No number is even and not even' is logically true, unlike the sentence 'No number is even and odd', which is only analytically true. The literal logical form assigned to the former is

$$\neg\exists x\,[[E\,x] \wedge \neg[E\,x]],$$

whereas the literal logical form assigned to the latter is

$$\neg\exists x\,[[E\,x] \wedge [O\,x]].$$

Types: $x \to \tau$; $E, O \to (o\tau)$.

The construction $\neg\exists x\,[[E\,x] \wedge \neg[E\,x]]$ v-constructs **T** for *all* valuations of the variable E, whereas the construction $\neg\exists x\,[[E\,x] \wedge [O\,x]]$ v-constructs **F** for some valuations of variables E and O. These are those valuations for which E and O v-construct sets with a non-empty intersection.[103]

Thus Definition 1.11 enables us to easily define *logically true sentence*.

Definition 1.12 (*logically true sentence*) A mathematical *sentence S* is *logically true* iff the *LLF* of *S* v-constructs the truth-value **T** for every valuation v. A *sentence S* involving empirical expressions is *logically true* iff the *LLF* of *S* v-constructs the proposition *TRUE* for every valuation v.

Obviously, any logically true sentence is analytically true. It is a well-known fact that the converse does not hold, as indeed the 'bachelor' example showed. The same holds also for mathematical sentences, as showed by the above mathematical example. For another mathematical example, the sentence T_1

T_1 'If 2 < 5 and 5 < 11 then 2 < 11'

is analytically, but not logically, true. The LLF of T_1 is ($L \to (o\tau\tau)$, $k, m, n \to \tau$):

[102] This problem was tackled as early as in 1837 by Bolzano, who introduced a modern method of variation of (objective) representations ('Vorstellungen an sich') and defined *generally valid* sentences *with respect to representations* $r_1,\dots,r_m$ such that the sentence remains true if these representations are changed or varied (See 1837, §§147–48).

[103] Similarly, the formula '$\neg\exists x\,[E(x) \wedge \neg E\,(x)]$' of first-order predicate logic is true on every interpretation assigning a subset of the universe to the symbol 'E', whereas there are interpretations of 'E' and 'O' on which the formula '$\neg\exists x\,[E(x) \wedge O(x)]$' is false, viz. those interpretations that assign non-disjoint sets to the symbols 'E' and 'O'.

T_1' $[[[L\ k\ m] \wedge [L\ m\ n]] \supset [L\ k\ n]]$.[104]

There is a valuation v such that the antecedent v-constructs **T** and the consequent **F**. (For instance, the valuation v that assigns the relation $\neq$ to the variable L, and the numbers 2, 5, 2 to variables k, m, n, respectively.) For the same reason, even the sentence T_2 is not logically true:

T_2 'If $2 < 5$ and $5 < 11$ and if $<$ is transitive then $2 < 11$'.

Though T_2 specifies a more detailed procedure than T_1, it leaves it open what is the definition of the transitive relation. *LLF* of T_2 is (the variable $T \rightarrow (\text{o}\ (\text{o}\tau\tau))$ v-constructing a class of binary relations)

T_2' $[[[L\ k\ m] \wedge [L\ m\ n] \wedge [T\ L]] \supset [L\ k\ n]]$,

which is not the form of a logically true sentence. Only when we explicitly define the class of transitive binary relations by

$$\lambda r\ \forall x \forall y \forall z\ \big[[r\ x\ y] \supset [[r\ y\ z] \supset [r\ x\ z]]\big]$$

is the logically true sentence T_3 obtained:

T_3 'If $2 < 5$ and $5 < 11$ and if $\forall x \forall y \forall z\ (x < y \supset (y < z \supset x < z))$ then $2 < 11$'.

Additional types: $r \rightarrow (\text{o}\tau\tau)$; $x, y, z \rightarrow \tau$.
The LLF of T_3 is the form of a logically true sentence:

T_3' $[[[L\ k\ m] \wedge [L\ m\ n] \wedge \forall x \forall y \forall z\ \big[[L\ x\ y] \supset [[L\ y\ z] \supset [L\ x\ z]]\big]] \supset [L\ k\ n]]$.

These definitions make it possible to easily define the difference between analytically and logically valid arguments. For instance, the following argument is analytically, but not logically, valid:

$$\frac{\text{No bachelor has ever been married}}{\text{Whales are mammals.}}$$

Since both the premise and the conclusion are analytically true sentences, the argument is analytically valid; there is no possible world w and time t at which the premise would be true and the conclusion false. Similarly, the following mathematical argument is analytically, but not logically, valid:

[104] For the sake of simplicity we are now omitting the symbol of Trivialization of logical objects and using the standard notation of quantifiers and infix notation for the truth-functions.

No prime number greater than 2 is even;
9 is not a prime number greater than 2

---

9 is not even.

Since every true mathematical sentence is true *only* in virtue of its meaning, there is no world/time pair at which the premises were true and the conclusion false. Any argument with premises $S_1,\ldots,S_n$ and conclusion S corresponds to a conditional sentence of the form 'If S_1 and … and S_n then S'. If the argument is analytically valid, then there is no possible world w and time t such that the premises would be true and the conclusion false. Hence, the conditional sentence is analytically true. And vice versa, if the conditional sentence is analytically true, the corresponding argument is analytically valid. Thus we define:

Definition 1.13 (*analytically/logically valid argument*) Let $S_1, \ldots, S_n$ be premises and S the conclusion of an argument A, and let S_A be the respective implicative statement of the form 'If S_1 and … and S_n then S'. Then

(i) *A is analytically valid* iff S_A is analytically true.
(ii) *A is logically valid* iff S_A is logically true.

For instance, the following argument is not only analytically, but also logically valid:

There is no x such that x is a prime number greater than 2 and x is even;
5 is a prime number greater than 2

---

5 is not even.

The literal analysis of the premises and the conclusion is as follows:

$$\frac{\begin{array}{c}[{}^0\neg\ [{}^0\exists\lambda x\ [{}^0\wedge\ [{}^0\wedge\ [{}^0Prime\ x]\ [{}^0>\ x\ {}^02]]\ [{}^0Even\ x]]]] \\ [[{}^0\wedge\ [{}^0Prime\ {}^05]\ [{}^0>\ {}^05\ {}^02]]\end{array}}{[{}^0\neg\ [{}^0Even\ {}^05]].}$$

Types: $\exists/(o(o\tau))$; *Prime*, *Even*$/(o\tau)$, $>/(o\tau\tau)$; $5, 2/\tau$; $x/*_1\rightarrow\tau$.

And the corresponding literal logical form is:

$$\frac{\begin{array}{c}[{}^0\neg\ [{}^0\exists\lambda x\ [{}^0\wedge\ [{}^0\wedge\ [P\ x]\ [R\ x\ a]]\ [E\ x]]]]; \\ [[{}^0\wedge\ [P\ b]\ [R\ b\ a]]\end{array}}{[{}^0\neg\ [E\ b]].}$$

Now it is easy to prove that the corresponding implicative sentence is logically true (to make this fact easier to see, we are again using standard infix notation without Trivialisation for logical connectives):

$$[\neg\exists x\, [[P\, x] \wedge [R\, x\, a] \wedge [E\, x]] \wedge [[P\, b] \wedge [R\, b\, a]]] \supset \neg[E\, b] =$$
$$\Big[\forall x\Big[[[P\, x] \wedge [R\, x\, a]] \supset \neg[E\, x]\Big] \wedge \Big[[P\, b] \wedge [R\, b\, a]\Big]\Big] \supset \neg[E\, b]$$

Variables: P, $E/*_1\rightarrow(o\tau)$; $R/*_1\rightarrow(o\tau\tau)$; a, $b/*_1\rightarrow\tau$.

As we have argued in Section 1.2, an argument is valid or invalid in virtue of the meanings of its premises and conclusion. Therefore, the type of the *entailment relation* obtaining between the set of premises and the conclusion of an argument is $(o(o*_n)*_n)$. It is a relation-in-extension between a set of constructions (the meanings of the premises) and a construction (the meaning of the conclusion).[105] Thus the entailment relation can be defined as follows:

Let $S_1, \ldots, S_n$ be the premises and S the conclusion of an argument involving the empirical expressions $S_1, \ldots, S_n$, S thus expressing the propositional constructions $\boldsymbol{C_1}, \ldots, \boldsymbol{C_n}, \boldsymbol{C} \rightarrow o_{\tau\omega}$. Then $S_1,\ldots, S_n$ *entail* S if $\{C_1, \ldots, C_n\} |= C$. As a corollary of definition 1.13, this is so iff

$$\forall w \forall t\, [[[{}^0True_{wt}\, \boldsymbol{C_1}] \wedge \ldots \wedge [{}^0True_{wt}\, \boldsymbol{C_n}]] \supset [{}^0True_{wt}\, \boldsymbol{C}]].$$

$True/(oo_{\tau\omega})_{\tau\omega}$ is the propositional property of being true at $\langle w, t\rangle$.

Let $S_1, \ldots, S_n$ be the premises and S the conclusion of a mathematical argument, $S_1, \ldots, S_n$, S thus expressing the truth-value constructions $\boldsymbol{C_1}, \ldots, \boldsymbol{C_n}, \boldsymbol{C} \rightarrow o$. Then $S_1, \ldots, S_n$ *entail* S if the set of constructions $\boldsymbol{C_1}, \ldots, \boldsymbol{C_n}$ entails the construction $\boldsymbol{C}$. As a corollary of definition 1.13, this is so iff

$$[[[{}^0True^*\, {}^0\boldsymbol{C_1}] \wedge \ldots \wedge [{}^0True^*\, {}^0\boldsymbol{C_n}]] \supset [{}^0True^*\, {}^0\boldsymbol{C}]].$$

$True^*/(o*_n)$ is the function that, when applied to a truth-value construction C, returns the value **T** if C v-constructs **T**, otherwise **F**.

Remarks.

(a) *Empirical case.*
Since the propositions denoted by the premises and the conclusion of a valid argument may lack a truth-value in some world w at a time t, we have to use the propositional property *True*.

[105] See Tichý (1988, p. 235).

(b) *Mathematical case.*
Since the premises or conclusion of a mathematical argument may express *v*-improper constructions, we need to use the function *True**.[106]
If partiality were not involved, then the Composition $[^0True^*\ ^0\boldsymbol{C}]$ would be equivalent to $[^{20}\boldsymbol{C}]$ or simply to $\boldsymbol{C}$.

1.5.2 Supposition de dicto and de re vs. reference shift

The term 'transparent' in 'transparent intensional logic' is to be interpreted in an *anti-contextualistic* manner. The point is that various alternative approaches lead to a seemingly necessary limitation of the compositionality principle. 'Oblique contexts' are standardly cited as a motive for restraining the principle. Intentional contexts are typical instances of 'oblique contexts'. Example: Since it was Sir Walter Scott who wrote the novels *Waverley* and *Ivanhoe*, Frege would have held that the definite descriptions

'The author of *Waverley*'

and

'The author of *Ivanhoe*',

denoted Sir Walter Scott. Evidently, the sentence

'Charles believes that the author of *Waverley* is a poet'

can be true whereas the sentence

'Charles believes that the author of *Ivanhoe* is a poet'

can be false at the same time. Frege wanted to observe compositionality, which would be obviously violated if 'The author of *Waverley*' denoted the same individual as 'The author of *Ivanhoe*'; the truth-value of both sentences would necessarily be the same. Wishing to save compositionality, Frege made the semantics of an expression depend on the linguistic context in which it is embedded. In atomic and molecular contexts 'The author of *Waverley*' and 'The author of *Ivanhoe*' both denote Sir Walter Scott, but in 'oblique contexts' like the one above both descriptions denote what in atomic and molecular contexts (e.g., 'The author of *Waverley* is happy and the Sun is shining') is their sense. Compositionality is

[106] Note also that due to the ramified hierarchy of types, no inconsistency problems arise when introducing truth predicates like *True* and *True**. In our higher-order typed approach there is no need to use disquotation like True('walks(Bill)') $\Leftrightarrow$ walks(Bill) and a hierarchy of meta-languages with their established grounded truths. The sentence 'Bill walks' is true in world *w* at time *t* if the proposition constructed by $\lambda w\lambda t\ [^0Walk_{wt}\ ^0Bill]$ takes value $\mathbf{T}$ in *w* at *t*.

saved (the expressions possessing distinct senses); the price exacted is *contextualism*.

The price is very high indeed. No expression can denote an object, unless a particular kind of context is provided. Yet such a solution is far from being natural. There are cases of real ambiguity, witness homonymous expressions. Which of the denotations is relevant in such cases (e.g., 'is a bank') can be detected by a particular context (cf. 'A bank was robbed' vs. 'A woman walks along the banks of the Dnepr'), but would anybody say that 'The author of *Waverley*' were another such case of homonymy? Hardly; unless, of course, their intuitions had been warped by Fregean contextualism. Furthermore, expressions can be embedded within other expressions to various degrees; consider the sentence

'Charles knows that Tom believes that the author of *Waverley* is a poet.'

The expression 'The author of *Waverley*' should now denote the 'normal' sense of the 'normal sense' of itself. Adding still further layers of embedding sets off an infinite hierarchy of senses, which is to say that 'The author of *Waverley*' has the potential of being infinitely ambiguous. This seems plain wrong, and is first and foremost an awkward artefact of Frege-Churchian semantics.

One well-known form of contextualism consists in distinguishing two kinds of context. In one kind ('referential context') a definite description refers to the object that satisfies the uniqueness condition, in the other context a definite description denotes something else. The problem with the distinction between two kinds of semantic context is that their definition is *circular*. Someone who propounds it wants to say that the descriptive term refers to the object that occupies the respective individual office in the respective kind of context. But this kind of context is defined just *via* the way the term is supposed to function in such a context:

Q: When is a context extensional?
A: A context is extensional if it validates the rules of (i) substitution of co-referential singular terms and (ii) existential generalisation.
Q: And when are (i), (ii) valid?
A: These rules are valid if all the contexts they are applied to are extensional.

Hence, the notions of extensional context and the validity of (i), (ii) are interdefined, the respective *definiendum* and *definiens* presupposing one another. This argument, which Tichý merely drops in passing,[107] is a potent one. In general the obvious move is to either define the semantic notion of extensional context (partly) in terms of the logical notion of the validity of one or more rules or else define the logical notion (partly) in terms of the semantic one. But to do either, it is required that the respective *definiens* be already determinate.

[107] See Tichý (1986a, p. 256, 2004, p. 654).

In this book we proceed in the following manner:[108] We first define the occurrence of a meaning-endowed constituent with extensional and intensional *supposition*, respectively. Thus we speak of extensional contexts in which constructions occur with extensional supposition, and of intensional contexts in which constructions occur with intensional supposition. Then we go on to prove that the rules (i) and (ii) are valid in extensional contexts.

Besides, even if reference shift is embraced, it is insufficient to let 'the *F*' denote a *Sinn* in an oblique context. If *a* believes that *the F* is a *G* then 'the *F*' denotes a *Sinn*—but *a* does not believe that some *Sinn* is a *G*. For instance, if Charles believes that the author of *Ivanhoe* is a Dutchman then Charles does not believe that the *Sinn* of 'The author of *Ivanhoe*' is a Dutchman. The advocates of reference shift need to explain how, in an oblique context, the *Sinn* of a term is to descend to an entity capable of being a Dutchman. In other words, what is needed is an account of extensionalization, or intensional descent.

The way out of the circle consists in (disambiguated) expressions denoting objects *independently of context*. In our example we say that 'The author of *Waverley*' *never* denotes the individual Sir Walter Scott; it *always* denotes the individual office that an individual must occupy to be the author of *Waverley*.

In TIL we construe this office as an ι-intension of type $\iota_{\tau\omega}$; a function from possible worlds and times to the *universe* (the set of individuals) . In a so-called 'direct' context (*oratio recta*) like

'The author of *Waverley* is a poet'

we predicate the respective property of whomever individual (if any) occupies this office in the given world/time of evaluation. Thus the truth-value of the proposition denoted by the sentence at the given $\langle w, t\rangle$ depends only on the particular individual who occupies the office at that $\langle w, t\rangle$; it is irrelevant who occupies it at worlds/times other than $\langle w, t\rangle$. In an 'oblique' context (*oratio obliqua*) we do not use the office in this manner, we just mention it, and the truth-value of the proposition is dependent on the occupancy of the office in all worlds at all times. The former case is known as using the definite description 'The author of *Waverley*' with *de re* supposition, the latter as using it with *de dicto* supposition. *Its meaning and denotation are, however, the same in both cases*.

Thus the meaning of 'The author of Waverley' is a construction of an individual office:

$$\lambda w\lambda t\ [{}^0Author\_of_{wt}\ {}^0Waverley] \rightarrow \iota_{\tau\omega}.$$

[108] See Section 2.6.

Types: $Author\_of/(\iota\iota)_{\tau\omega}$; $Waverley/\iota$.[109]

The meaning of 'The author of *Waverley* is a poet' is the propositional construction

$$\lambda w\lambda t\, [{}^0Poet_{wt}\; \lambda w\lambda t\, [{}^0Author\_of_{wt}\; {}^0Waverley]_{wt}] \rightarrow o_{\tau\omega}.$$

Additional type: $Poet/(o\iota)_{\tau\omega}$.

The meaning of 'Tom believes that the author of Waverley is a poet' is a construction of another proposition:

$$\lambda w\lambda t\, [{}^0Believe_{wt}\; {}^0Tom\; [\lambda w\lambda t\, [{}^0Poet_{wt}\; \lambda w\lambda t\, [{}^0Author\_of_{wt}\; {}^0Waverley]_{wt}]]] \rightarrow o_{\tau\omega},$$

(if the sentence is construed as expressing an implicit belief), or alternatively

$$\lambda w\lambda t\, [{}^0Believe^*_{wt}\; {}^0Tom\; {}^{\mathbf{0}}[\lambda w\lambda t\, [{}^0Poet_{wt}\; \lambda w\lambda t\, [{}^0Author\_of_{wt}\; {}^0Waverley]_{wt}]]] \rightarrow o_{\tau\omega},$$

(if the sentence is construed as expressing an explicit belief).[110]

Additional types: $Believe/(o\iota o_{\tau\omega})_{\tau\omega}$: a relation(-in-intension) of an individual to a proposition; $Believe^*/(o\iota *_n)_{\tau\omega}$: a relation(-in-intension) of an individual to a hyperproposition, i.e. a propositional construction; Tom/ι.

Finally, the meaning of 'Charles knows that Tom believes that the author of *Waverley* is a poet' is again a construction of a proposition. Implicit knowledge first:

$$\lambda w\lambda t\, [{}^0Know_{wt}\; {}^0Charles\; [\lambda w\lambda t\, [{}^0Believe_{wt}\; {}^0Tom\; [\lambda w\lambda t\, [{}^0Poet_{wt}\; \lambda w\lambda t\, [{}^0Author\_of_{wt}\; {}^0Waverley]_{wt}]]]]].$$

Explicit knowledge:

$$\lambda w\lambda t\, [{}^0Know^*_{wt}\; {}^0Charles\; {}^{\mathbf{0}}[\lambda w\lambda t\, [{}^0Believe_{wt}\; {}^0Tom\; [\lambda w\lambda t\, [{}^0Poet_{wt}\; \lambda w\lambda t\, [{}^0Author\_of_{wt}\; {}^0Waverley]_{wt}]]]]].$$

Additional types: $Know/(o\iota o_{\tau\omega})_{\tau\omega}$: a relation(-in-intension) of an individual to a proposition; $Know^*/(o\iota *_n)_{\tau\omega}$: a relation(-in-intension) of an individual to a propositional construction; $Charles/\iota$.

Our top-down approach furnishing all the expressions with a hyperintensional semantics—i.e., assigning *constructions* (of intensions) to (empirical) expressions as their meanings in all kinds of context—makes it possible to adhere to the

[109] To assign the type ι to a novel is a crass philosophical simplification, of course; here it is logically innocuous, since we are not going to draw inferences.

[110] 'Propositional' attitudes divide into relations (-in-intension) to propositions/$o_{\tau\omega}$ and propositional constructions/$*_n \rightarrow o_{\tau\omega}$. The former are often called *implicit* attitudes, the latter *explicit* attitudes. We will deal with propositional attitudes in detail in Section 5.1.

compositionality principle. In a word, compositionality is saved without resorting to contextualism.

In TIL, there is no such contextual thing as the *intension/extension of an expression*. Instead every expression either denotes an extension or an intension, independently of contextual embedding. What is dependent on context is the *supposition*, which comes in a *de dicto* and a *de re* variant. In general, *empirical expressions* denote non-constant intensions. We will rigorously define the *de dicto*/*de re* distinction in Section 2.7. Now we explicate the difference only informally.

Compare the following sentences:

(S_1) 'The President of the Czech Republic is an economist.'

(S_2) 'The President of the Czech Republic is eligible.'

First, neither sentence talks about Václav Klaus, though the office of President of the Czech Republic is currently (2010) occupied by Klaus. The individual named 'Václav Klaus' does not receive mention here. Instead, both sentences talk about the individual *office* denoted by 'The President of the Czech Republic'. The definite description 'The President of the Czech Republic' never *denotes* the individual (if any) that occupies the office; it only contingently *refers to* a particular individual. We language-users understand the expression in exactly the same way regardless of the embedding context. Moreover, we understand it even if we do not know which individual occupies the office in the actual world at time *t,* and we do understand it even with respect to such a state of affairs $\langle w, t\rangle$ at which *no* individual is occupying the office. Hence the definite description 'The President of the Czech Republic' denotes the office $PresCR/\iota_{\tau\omega}$ itself, and its meaning is a construction of that office:

$$\lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0CR] \rightarrow \iota_{\tau\omega}.$$

Types: $Pres\_of/(\iota\iota)_{\tau\omega}$; CR/ι.

Yet there is a substantial difference between how the meaning of 'The President of the Czech Republic' occurs in (S_1) and (S_2). The property of being an economist cannot be ascribed to an office but only to an individual. On the other hand, the property of being eligible can only be ascribed to the *office* itself. That the President is eligible means that the presidency acquires a holder by election. It would appear as though (S_1) were about, inter alia, the individual occupying the office *PresCR*, anyway. But 'The President of the Czech Republic' is used here as a pointer to an individual, so the office must be extensionalized via application to the values of *w, t* to provide an individual:

$$[\lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0CR]]_{\boldsymbol{wt}} \rightarrow_v \iota$$

This Composition *v*-constructs relative to a world/time parameter the individual (if any) occupying the office at the given $\langle w, t\rangle$. (Remember that *denotation* is a semantic relation a priori between expressions and entities, and *reference* an extra-semantic, factual relation between expressions and world-time relative entities.)

Thus the analysis of (S_1) comes down to this construction:

(S_1') $\lambda w\lambda t\ [^0Economist_{wt}\ [\lambda w\lambda t\ [^0Pres\_of_{wt}\ ^0CR]]_{wt}]$

Additional type: *Economist*/$(o\iota)_{\tau\omega}$.

Individuals can be economists, but they cannot be eligible; individual offices can. Though a particular individual, say Klaus, can be elected *for a* presidential *office*, the individual itself is not eligible. (If individuals were eligible, it would mean that one could become a particular individual by election: a fascinating thought, perhaps.) Instead, the office is currently eligible by the Czech Parliament; but the office could be hereditary, or eligible by referendum. *Eligible* is of type $(o\iota_{\tau\omega})_{\tau\omega}$, and the analysis of ($S_2$) is this:

(S_2') $\lambda w\lambda t\ [^0Eligible_{wt}\ [\lambda w\lambda t\ [^0Pres\_of_{wt}\ ^0CR]]]$.

We say that the meaning of 'The President of the Czech Republic' is used with supposition *de re* in (S_1') and supposition *de dicto* in (S_2'). However, the meaning of 'The President of the Czech Republic', namely the Closure $\lambda w\lambda t\ [^0Pres\_of_{wt}\ ^0CR]$, remains the same. Again, the shift concerns neither the meaning nor the denotation, but only the *supposition* with which the (same) meaning is used.

The proposition constructed by (S_1') takes the value **T** at those $\langle w, t\rangle$ at which the individual that occupies *PresCR* belongs to the class of individuals that instantiate the property of being an economist, and **F** if the individual does not belong to the class. It might seem that in such a state-of-affairs where there is no President of the Czech Republic the proposition should be false. (This would be the Russellian tack.) However, if it was so, the proposition that the President of the Czech Republic is *not* an economist would have to be true, which would in turn entail that there were indeed a President of the Czech republic.[111] In other words, that the President of the CR is an economist not only *entails* but also *presupposes* that the President of the CR exists. Remember that our logic is one of *partial* functions. Once a constituent—$\lambda w\lambda t\ [^0Pres\_of_{wt}\ ^0CR]_{wt}$ in our case—of a Composed construction is *v*-improper, the whole Composition is *v*-improper, and the function (here, a proposition) constructed by the respective Closure is undefined at its argument (See Definition 1.2). Therefore in those states of affairs where *PresCR* is vacant, the proposition has *no truth-value.*

On the other hand, the proposition denoted by (S_2) may be false even in the states-of-affairs lacking a President of the Czech Republic. Its truth-value does not depend on the occupancy of *PresCR* in those states-of-affairs. In particular, we

[111] See Strawson (1950).

cannot substitute a construction of the current occupant of the office. For if we could do this, we could deduce, absurdly, that Klaus is eligible.

In Section 1.1 we argued that empirical expressions rigidly *denote* intensions. Later we added that empirical expressions non-rigidly *refer* to particular values of the intensions denoted by them. However, there are expressions that never refer to an extension. For instance, when we claim that the President of the USA is eligible, we should, properly speaking, say that the *office* of President of the USA is eligible. Eligibility is a property of the office (of type $(o\iota_{\tau\omega})_{\tau\omega}$). The expression 'The office of the President of the USA' (or 'The American presidency' for short) never refers to an individual. It rigidly denotes the office itself and can be used only with *de dicto* supposition.[112] Similarly, the predicate 'is happy' denotes a property of individuals (*Happiness*/$(o\iota)_{\tau\omega}$) and when used (in the *de re* way) in order to be predicated of an individual it refers at $\langle w, t\rangle$ to a particular class of individuals. However, 'happiness' rigidly denotes the property *Happiness* but cannot be predicated of individuals. It can be used only in the *de dicto* way, like in the sentence 'Happiness is Charles' ultimate goal in life'. In general, the intensional semantics of TIL enables us to say that some empirical expressions like 'happiness', 'the American presidency', 'the proposition that G.W. Bush is the President of the USA', etc., which rigidly denote intensions, are *names* given to those entities by a linguistic convention. They are rigid designators *de jure* and they never non-rigidly refer to particular extensions.[113]

The *de dicto/de re* distinction can be summarized as follows:

De dicto supposition:
A construction $C_E \rightarrow \alpha_{\tau\omega}$ (and derivatively the subexpression E whose meaning C_E is) occurring in the analysis C_S of a sentence S is used with *de dicto* supposition in C_S iff the truth-value of the proposition v-constructed by C_S in a world w at a time t does not depend only on the particular value of the α-intension I_E v-constructed by C_E at *this particular* $\langle w, t\rangle$. Rather, it depends on the whole I_E. In other words, the intension I_E is a *dictum* and is *not used* to point to a value.

De re supposition:
There is *de re* supposition when the reference of E (namely, the α-value, the *res*, v-constructed by C_{Ewt}) of the denoted α-intension I_E comes into play. The truth-value of the proposition denoted by S in a world w at a time t depends on the value of the α-intension I_E denoted by E at *this particular* $\langle w, t\rangle$, while the values of I_E at other $\langle w', t'\rangle$ are irrelevant.

This preliminary characterization could serve almost as a definition, though not quite. According to it, the sentence S alone would be in *de re* supposition in itself, which is not so. The sentence talks about (denotes) the whole *dictum*—a

[112] More precisely, its meaning occurs always intensionally, see Section 2.6.2, in particular Definition 2.20.

[113] See Zouhar (2009), where he deals with the Kripkean distinction between rigid designators *de jure* and *de facto.*

proposition—and never its reference (*res*)—its truth-value in the actual world-time. The sentences (S_1), (S_2), and the constructions (S_1'), (S_2'), respectively, occur with *de dicto* supposition in themselves.

Note that in a compound sentence particular clauses may occur with *de re* as well as with *de dicto* supposition. Consider the following example:

(S_3) 'If the President of the Czech Republic is a playwright then the President of the Czech Republic is Václav Havel.'

An analysis of the antecedent and consequent sentences yields the following propositional constructions, respectively:

(Ca) $\lambda w\lambda t\ [{}^0Playwright_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$

(Cb) $\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]$.

Additional types: $Playwright/(o\iota)_{\tau\omega}$; $Havel/\iota$.

However, the propositional connective '$\supset$' (implication) denotes a truth-function of type (ooo); it must be applied to truth-values and cannot be applied to propositions. Thus the propositions constructed by (Ca), (Cb) have to undergo intensional descent, and the truth-value (in w at t) of the proposition denoted by (S_3) does depend on the truth-values of these propositions at the same particular $\langle w, t\rangle$:

$$(S_3')\qquad \lambda w\lambda t\ [{}^0\supset [\lambda w\lambda t\ [{}^0Playwright_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]]_{wt}\\ [\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]]_{wt}].$$

Both sentences and their meanings (Ca), (Cb) occur with supposition *de re* in (S_3), (S_3'), respectively. Again, at those $\langle w, t\rangle$ at which *PresCR* is vacant, the sentence (S_3) does not have a truth-value. The fact is even more evident if we consider the β-reduced construction ($S_{3\beta}$) equivalent to (S_3'):

$$(S_{3\beta})\qquad \lambda w\lambda t\ [[{}^0Playwright_{wt}\ [{}^0Pres\_of_{wt}\ {}^0CR]] \supset [[{}^0Pres\_of_{wt}\ {}^0CR] = {}^0Havel]].$$

At those worlds and times where the Presidency is vacant, the construction $[{}^0Pres\_of_{wt}\ {}^0CR]$ fails to construct an occupant of *PresCR*. Due to the definition of Composition, both Composed subconstructions of ($S_{3\beta}$), namely $[{}^0Playwright_{wt}\ [{}^0Pres\_of_{wt}\ {}^0CR]]$ and $[[{}^0Pres\_of_{wt}\ {}^0CR] = {}^0Havel]$, are also v-improper. Thus the construction of the implication function $\supset$ does not receive an argument to work on, and it also fails to v-construct a truth-value. The proposition constructed by (S_3') is undefined for those worlds and times at which the Presidency goes vacant. This is so because (S_3) comes with an *existential presupposition*: for (S_3) to take a truth-value at a given $\langle w, t\rangle$, the President of the Czech Republic has to exist at that $\langle w, t\rangle$. Again, (S_3) not only entails but also presupposes the existence of the President of the Czech Republic.

Remark. This kind of a β-reduction has been called in Duží (2003a, b, 2004) *β_i-reduction* ('i' meaning 'innocuous'). It consists simply in substituting variables for variables (of the same type), in our case *w*, *t* for *w*, *t.* Since a variable can never be *v*-improper, such a reduction is always an equivalent transformation. In this sense it is 'innocuous'. However, in a logic of partial functions like TIL it must be taken into account that a simple 'syntactic version' of the β-reduction rule is generally *not* valid. We will deal with the problem in Section 2.7.

1.5.2.1 Two principles *de re*

Existential presupposition is a special case of presupposition. For instance, the sentence 'Charles stopped smoking' not only entails that Charles previously smoked, but also presupposes it. One cannot stop doing something that one has not previously done. Strawson's test makes this clear. Being asked whether you stopped smoking, you are not entitled to give a Yes/No answer unless you previously smoked.

To define the notion of presupposition, we make use of the three propositional properties *True*, *False*, and *Undef*, all of type $(o o_{\tau\omega})_{\tau\omega}$. They are defined as follows.[114] Let *P* be a propositional construction ($P/*_n \to o_{\tau\omega}$). Then

$[{}^0True_{wt}P]$ *v*-constructs the truth-value **T** iff P_{wt} *v*-constructs **T**, otherwise **F**.

$[{}^0False_{wt}P]$ *v*-constructs the truth-value **T** iff $[\neg P_{wt}]$ *v*-constructs **T**, otherwise **F**.

$[{}^0Undef_{wt}P]$ *v*-constructs the truth-value **T** iff $[[\neg[{}^0True_{wt}P]] \wedge [[\neg{}^0False_{wt}P]]]$ *v*-constructs **T**, otherwise **F**.

Hence $[{}^0Undef_{wt}P] = [[\neg{}^0True_{wt}P] \wedge [\neg{}^0False_{wt}P]]$.

Note that, e.g., $[\neg[{}^0True_{wt}P]]$ is not equivalent to $[{}^0False_{wt}P]$, though our logic is bivalent. We do not work with a third truth-value. If $[{}^0Undef_{wt}P]$ *v*-constructs **T**, then P_{wt} is *v*-improper, and the proposition P constructed by *P* does not have *any* truth-value at $\langle w, t\rangle$.[115]

Now we define:

Definition 1.14 (*presupposition*) Let *P*, $Q \to o_{\tau\omega}$ be constructions constructing propositions P, Q. Then Q is a *presupposition* of P iff the truth of Q at $\langle w, t\rangle$ is a necessary condition for P having a truth-value at $\langle w, t\rangle$:

$$\forall w \forall t\, [[{}^0True_{wt}P] \vee [{}^0False_{wt}P]] \supset [{}^0True_{wt}Q]].$$

[114] Now we use this convention: '*P*' for a construction of a proposition, 'P' for the proposition *v*-constructed by *P*.

[115] Cf. Table 1.1: truth-value matrix, Section 1.4.3.

Corollary. Q is a *presupposition* of P iff *Q* is entailed both by *P* and *non-P*. If Q is not true at $\langle w, t\rangle$, then P is undefined at $\langle w, t\rangle$:

$$\forall w \forall t\ [\neg[{}^0True_{wt}\,Q] \supset [{}^0Undef_{wt}\,P]].$$

One should not confuse the notion of presupposition with the notion of *commitment*, for the latter is weaker than the former. In order to exactly determine the difference, we recall the definition of the entailment relation. Let *P, Q* be propositional constructions as above. Then the *P entails Q* ($P \models Q$) iff

$$\forall w \forall t\ [[{}^0True_{wt}\,P] \supset [{}^0True_{wt}\,Q]].$$

We will often use the notation '($P \models Q$)' instead of '$[{}^0{\models}\ {}^0P\ {}^0Q]$'. Note that *P*, *Q* must be Trivialized, since these very constructions, rather than the propositions they construct, are the arguments of $\models$.

Schematically, the difference between presupposition and commitment is this. Let *non-P* be a propositional construction of the form $\lambda w \lambda t\ [\neg P_{wt}]$. Then

(i) Q is a *presupposition* of P iff ($P \models Q$) and ($non\text{-}P \models Q$)
(ii) Q is a *commitment* of P iff ($P \models Q$) and neither ($non\text{-}P \models Q$) nor ($non\text{-}P \models non\text{-}Q$)

An example of commitment would be, for instance:

'Ground zero was visited by the Pope in April of 2008.'

The sentence is multiply ambiguous. The ambiguity concerns the supposition with which the definite descriptions 'ground zero' and 'the Pope' occur, where 'ground zero' goes short for 'the ground zero in New York City'.[116] On one reading both occur with *de re* supposition. In such a case the sentence presupposes that both ground zero and the Pope exist *now.* Yet there are other readings. Among them is the reading on which 'ground zero' occurs *de re* and 'the Pope' occurs *de dicto* with respect to the temporal parameter.[117] In such a case the sentence presupposes the existence of ground zero, but not of the Pope now. It only entails that the Pope *existed* in April 2008. Hence, if it were true that

'Ground zero was *not* visited by the Pope in April 2008',

one could *not* deduce that the Pope exists now or existed in April, 2008. That ground zero was not visited by the Pope in April of 2008 might have been either because the office of Pope was vacant at the time or that the Pope did exist but its

[116] It is interesting to note that '[the] ground zero [of New York City]' has now been elevated to the status of proper name, which requires capitalizing both words, as in 'Ground Zero'. Many sites are ground zero, but only one is Ground Zero, relative to the status that current American English has bestowed upon 'Ground Zero'. In journalese 'Ground Zero' refers to one particular ground zero. So if the Pope visits the NYC ground zero then the New York Times et al. are likely to write 'The Pope to visit Ground Zero'.

[117] We will deal with temporal *de dicto* vs. *de re* cases in Section 2.5.2.3.

occupant was not among the visitors of ground zero. This goes to show why commitment is weaker than presupposition.

We are now able to formulate *the first principle de re*:

> ***Principle of existential presupposition.*** If a construction C of an α-office $C/\alpha_{\tau\omega}$ occurs with *de re* supposition in the propositional construction P, then the proposition constructed by P has the *presupposition* that C exist (that the α-office C be occupied): $\lambda w \lambda t\ [{}^0 Exist_{wt}\ C]$, $Exist/(o\iota_{\tau\omega})_{\tau\omega}$.

The office of President of the Czech Republic is certainly a *properly* partial function: there are worlds/times at which the President of the Czech Republic does not exist; for instance, in the actual world and at all times before 1993. However, if $[{}^0 =$ is true, then so is the proposition that the President of the Czech Republic exists. In Section 2.3 we show that existence can be analysed as a property of intensions, in this case of individual offices, $Exist/(o\iota_{\tau\omega})_{\tau\omega}$. Hence the following argument is valid:

$$\frac{\lambda w \lambda t\ [{}^0 Economist_{wt}\ \lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]_{wt}]}{\lambda w \lambda t\ [{}^0 Exist_{wt}\ \lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]].}$$

Similarly, the President of the Czech Republic *not* being an economist entails the existence of the President of the Czech Republic.

Since the property of existence (in the sense of occupancy of an office) can be defined by means of the existential quantifier ($x \to \iota$; $r \to \iota_{\tau\omega}$; $=_\iota/(o\iota\iota)$),

$$\lambda w \lambda t\ \lambda r\ [{}^0\exists\ \lambda x\ [{}^0 =_\iota x\ r_{wt}]],$$

the conclusion can be equivalently expressed by the construction

$$\lambda w \lambda t\ [{}^0\exists\ \lambda x\ [{}^0 =_\iota x\ \lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]_{wt}]].$$

Valid logical forms of the arguments are thus easily obtained by existential generalisation:

$$\frac{\lambda w \lambda t\ [P_{wt}\ r_{wt}]}{\lambda w \lambda t\ [{}^0\exists\ \lambda x\ [{}^0 =_\iota x\ r_{wt}]]} \qquad \frac{\lambda w \lambda t\ \neg[P_{wt}\ r_{wt}]}{\lambda w \lambda t\ [{}^0\exists\ \lambda x\ [{}^0 =_\iota x\ r_{wt}]].}$$

Additional type: $(P \to (o\iota)_{\tau\omega})$:[118]

Of course, if the proposition constructed by the premise takes value **T** at $\langle w, t\rangle$ then the individual occupying at $\langle w, t\rangle$ the office constructed by $\lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]$

[118] See Section 1.5.1 for details on the notion of logical form.

belongs to the class *v*-constructed by ${}^0Economist_{wt}$. Hence the office of President must be occupied at $\langle w, t\rangle$, and the conclusion is true at $\langle w, t\rangle$. In other words, the argument *is truth-preserving* from premises to conclusion.

However, due to partiality, a valid argument may fail to be *falsity-preserving* from conclusion to premises.[119] If at $\langle w, t\rangle$ the conclusion is false, then it does not mean that at least one of the premises is false at $\langle w, t\rangle$. For, if the office is not occupied at a particular world W and a particular time T, then the construction $\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}$ is *v-improper* for the valuation assigning W to w and T to t. Therefore, the Composition in which the construction of the office occurs *de re,* namely

$$[{}^0Economist_{wt}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$$

is also *v(W/w, T/t)*-improper, and the proposition P constructed by the premise has *no truth-value* at this $\langle W, T\rangle$. It is neither true nor false, because in the absence of a President at $\langle W, T\rangle$ there is no fact of the matter as to whether the President is an economist at $\langle W, T\rangle$. The proposition P is a properly partial function, because it has truth-value gaps. In order that P have a truth-value, the President of the Czech Republic has to exist; P comes with an *existential presupposition*.

Remember that we do not introduce a third truth-value in order to handle partiality. Thus we do not follow Muskens' theory of *partial possible worlds* or Barwise and Perry's *situation semantics*, nor do we introduce partiality whenever it might seem to be technically convenient.[120] TIL is a Platonist semantics, ideally aiming at cutting reality at its joints, as the saying goes. Propositions simply *are* true, false or neither, independently of our 'allowing' them to be so, and they are never both true and false. (There is no room for paraconsistent truth-value gluts in TIL.)

For example, Muskens (1995, pp. 42–50) introduces four combinations of truth-values: **T** = 'true and not false', **F** = 'false and not true', **N** = 'neither true nor false' and **B** = 'both true and false', in order to handle synonymy in terms of co-entailment. In Muskens' partial logic, the sentences

(1) 'John walks'

and

(2) 'John walks and Bill talks or does not talk'

are not equivalent, though 'Bill talks or does not talk' is a classical tautology and as such denotes the necessary proposition true in all possible worlds. According to Muskens, the reason is because in a situation where Mary sees John but not Bill, the sentence 'Mary sees John walk' can be true or false, unlike the sentence 'Mary

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[119] A valid argument need not be truth-preserving from conclusion back up to its premises, either; namely, if the argument is unsound.

[120] See Muskens (1995), Barwise and Perry (1983).

sees John walk and Bill talk or not talk' which is undefined in a situation where Mary does not see Bill. Thus (2) does not follow from (1).[121]

We disagree on this point. If Mary does not see Bill at all, then, of course, she cannot see him talk or doing anything else, which does *not mean* (*contra* Muskens) that '[T]he sentence "Bill talks" will be undefined, that is, neither true nor false, in the part of the world that is seen by her'. Nor does it mean that as a consequence the sentence 'Bill talks or doesn't talk' and 'John walks and Bill talks or doesn't talk' are both undefined in that situation as well. Sentences (1) and (2) *are* equivalent (as they denote the same proposition), and the sentence 'Bill talks or does not talk' is a tautology, independently of whether Mary knows it.[122] Note that Muskens uses classical entailment to argue that (2) does not follow from (1). But (2) does follow from (1), independently of Mary's cognitive abilities and *independently of situations*. And (1) and (2) are true or false, dependently on states-of-affairs, but independently of Mary's seeing that they are. There is no reason to introduce partiality here.

According to Muskens, co-entailment in a partial theory will be a better approximation to synonymy than classical co-entailment is. In our opinion, Muskens is in effect modelling our cognitive abilities, and his theory can be treated as a cognitive theory. The new 'truth-values' he introduces, namely **N** and **B**, are actually not (objective) truth-values of propositions, but, say, subjective degrees of *knowledge* in a particular situation. We can even introduce infinitely many such 'truth-values', for instance, an interval between 0 and 1, to map 'degrees of preciseness of measurement', or 'degrees of our conviction in the truth', or any other (subjective) degrees, and build up fuzzy logics, etc. We can even introduce new (objectively correct) *inference rules* within our logic that would better map the relation of logical consequence. Still, the relation of co-entailment, or co-denotation, will always be just an *approximation* to synonymy, and a counter-example could always be found. Notoriously well-known ones are attitudinal sentences (see Chapter 5). No intensional semantics can properly handle synonymy, because its finest individuation is equivalence. We need a hyperintensional semantics to properly handle synonymy[123] and to construe meaning as an algorithmically structured *procedure.*

Now we are going to explain the second principle *de re*, namely the principle of substitution of co-*referential* expressions. First, what does it mean that the truth-value at $\langle w, t \rangle$ of a proposition *depends* on the value of another intension? Consider again the sentence

(S_1) 'The President of the Czech Republic is an economist'

[121] See Muskens (1995, pp. 1–3).

[122] The semantics of proper names is simplified here, allowing 'Bill' to be simply a label of an individual. See, however, Section 3.2. Moreover, on the TIL conception, there are no non-existing individuals: we work with a constant domain of individuals.

[123] For the definition of synonymy, see Section 2.2, Definition 2.10.

and its analysis:

(S_1') $\lambda w \lambda t\ [{}^0Economist_{wt}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$.

'The President of the Czech Republic' occurs *de re* in (S_1), as does the occurrence of the construction $\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$ in (S_1'). If the President is Václav Klaus, then (S_1) and this additional premise entail that Václav Klaus is an economist; hence the following argument is valid:

$$\frac{\begin{array}{c}\lambda w \lambda t\ [{}^0Economist_{wt}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}] \\ \lambda w \lambda t\ [{}^0{=}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}\ {}^0Klaus]\end{array}}{\lambda w \lambda t\ [{}^0Economist_{wt}\ {}^0Klaus].}$$

Similarly, if the President is the husband of Livie Klausová then (S_1) and this additional premise entail that the husband of Livie Klausová is an economist ($Husband\_of/(\iota\iota)_{\tau\omega}$):

$$\frac{\begin{array}{c}\lambda w \lambda t\ [{}^0Economist_{wt}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}] \\ \lambda w \lambda t\ [{}^0{=}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}\ \lambda w \lambda t\ [{}^0Husband\_of_{wt}\ {}^0Livie]_{wt}]\end{array}}{\lambda w \lambda t\ [{}^0Economist_{wt}\ \lambda w \lambda t\ [{}^0Husband\text{-}of_{wt}\ {}^0Livie]_{wt}].}$$

This is no surprise, of course, because Leibniz's law of substitution law is uncontroversially valid in these cases, and the following is the schema of a valid argument:

$$\frac{\begin{array}{c}\lambda w \lambda t\ [\ldots\ C\ \ldots] \\ \lambda w \lambda t\ [{}^0{=}\ C\ D]\end{array}}{\lambda w \lambda t\ [\ldots.\ D \ldots].}$$

The principle of substitution of co-*referential* expressions is an instance of Leibniz's Law.

Tichý formulates the principle as follows.

> Let 'X', 'Y' denote individual offices. Let '…Y…' be a sentence arising from sentence '…X…' by putting the term 'Y' for some *de re* occurrences of 'X' in '…X…'. Then the argument
>
> $$\frac{\begin{array}{l}X \text{ at } \langle W, T\rangle \text{ is } Y \text{ at } \langle W, T\rangle \\ \ldots X \text{ at } \langle W, T\rangle \ldots\end{array}}{\ldots Y \text{ at } \langle W, T\rangle \ldots}$$
>
> is valid. (1978a, p. 9, 2004, p. 257).

The rationale behind the substitution is that what is predicated of the occupant of X at $\langle w, t\rangle$ is what is predicated of the occupant of Y at $\langle w, t\rangle$ on condition of co-occupation of X and Y at $\langle w, t\rangle$. That is, even though '…X at $\langle w, t\rangle$…' and '…Y at

$\langle w, t\rangle$…' may have different truth-conditions, their truth-values coincide at every $\langle w, t\rangle$ at which 'X at $\langle w, t\rangle$ is Y at $\langle w, t\rangle$' expresses a truth.

Hence *the second principle de re* is the following:

> ***Principle of substitution of co-referential expressions.*** If an expression E occurs in a sentence S with *de re* supposition, then the substitution (*salva veritate*) of a co-referential expression E' for the occurrence of E in S is valid.

The corresponding rule of substitution *de re* is then:

> ***Rule of substitution of v-congruent constructions.*** Let $C \rightarrow \alpha_{\tau\omega}$, $D \rightarrow \alpha_{\tau\omega}$ and let C_{wt}, D_{wt} be *v*-congruent constructions (i.e., $C_{wt} = D_{wt}$) and let $S(D/C)$ be a construction that arises from S by substituting D for one or more *de re* occurrences of C in S. Then S_{wt} and $S(D/C)_{wt}$ are *v*-congruent as well (i.e., $S_{wt} = S(D/C)_{wt}$).

For another example, the denoted office can be a second-degree office (an office of an individual office), like, for instance *the highest executive office of the USA*. The following argument is valid:

The highest executive office of the USA is the President, not the King
The highest executive office of the USA is the most respectable office in the USA

The most respectable office of the USA is the President, not the King.

Type-theoretical analysis: $HEO/(\iota_{\tau\omega})_{\tau\omega}$: the highest executive office of the USA; $MRO/(\iota_{\tau\omega})_{\tau\omega}$: the most respectable office of the USA; $PresUSA$, $KingUSA/\iota_{\tau\omega}$; $=_{\iota\tau\omega}/(o\iota_{\tau\omega}\iota_{\tau\omega})$.

Synthesis:

$$\frac{\begin{array}{c}\lambda w\lambda t\,[[{}^0\!=_{\iota\tau\omega}\,{}^0HEO_{wt}\,{}^0PresUSA] \wedge [\neg[{}^0\!=_{\iota\tau\omega}\,{}^0HEO_{wt}\,{}^0KingUSA]]]\\ \lambda w\lambda t\,[{}^0\!=_{\iota\tau\omega}\,{}^0HEO_{wt}\,{}^0MRO_{wt}]\end{array}}{\lambda w\lambda t\,[[{}^0\!=_{\iota\tau\omega}\,{}^0MRO_{wt}\,{}^0PresUSA] \wedge [\neg[{}^0\!=_{\iota\tau\omega}\,{}^0MRO_{wt}\,{}^0KingUSA]]].}$$

Since ${}^0HEO$ and ${}^0MRO$ occur with *de re* supposition in the premises (unlike the constituents ${}^0PresUSA$, ${}^0KingUSA$), the substitution *salva veritate* is valid.

A classical puzzle from around 1970 due to Barbara Partee can also be resolved by sorting out the interplay between *de dicto* and *de re* supposition.[124] Partee's puzzle is this:

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[124] For discussion, see Yagisawa (2001), Moschovakis (2006, p. 43), and Partee (2005, p. 43).

The temperature is 90°F
The temperature is rising

90°F is rising.

The argument seems at first blush to invite a smooth substitution of '90°F' for 'the temperature' in the context '…is rising…' by Leibniz's Law. Yet the conclusion is indisputably either false or nonsensical. Partee did intend, however, to come up with a flawed argument to make a particular point within a particular discussion at the time to do with so-called intensional positions for singular terms to occur in, such that these positions would be distinct from (overtly) modal contexts. And her argument obviously is flawed. The challenge that her argument presents is to construct a logical analysis that will block the inference. Here is how we go about this.

As always, we begin with a type-theoretical analysis of the objects mentioned by the premises: *Temperature*/$\tau_{\tau\omega}$: a magnitude;[125] *Rising*/$(o\tau_{\tau\omega})_{\tau\omega}$: a property of a magnitude; $=_\tau/(o\tau\tau)$; $90/\tau$.

(P_1) $\lambda w \lambda t\ [{}^0{=_\tau}\ {}^0Temperature_{wt}\ {}^090]$
(P_2) $\lambda w \lambda t\ [{}^0Rising_{wt}\ {}^0Temperature]$

The diagnosis of the invalidity of the argument is now straightforward. The Trivialization ${}^0Temperature$ occurs *de re* in (P_1), but *de dicto* in (P_2). In other words, the object of predication in (P_2) is the entire function *Temperature* rather than its particular value. So the substitution of the construction ^{090} for ${}^0Temperature$ into (P2) would be invalid.

1.5.2.2 Interplay between *de dicto* and *de re*

Consider now another sentence:

(S_4) 'If the President of the Czech Republic is a playwright then Charles believes that the President of the Czech Republic is Václav Havel.'

An adequate analysis of the consequent has to respect the fact that Charles can believe that the President is Václav Havel even if the President is instead Václav Klaus, or even if the President does not exist. Charles may simply not be up on

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[125] It is understood that the temperature is not just any temperature (of something), but a particular temperature, and most likely the temperature at the location of whoever says the temperature is rising.

Czech public affairs. Thus the meaning of the clause expressed by the consequent is ($Believe/(o\iota o_{\tau\omega})_{\tau\omega}$):[126]

(S_{4emb})
$\lambda w\lambda t\ [{}^0Believe_{wt}\ {}^0Charles\ \lambda w_1\lambda t_1\ [\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]_{w1t1} = {}^0Havel]]$.

The Closure $\lambda w_1\lambda t_1\ [\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]_{w1t1} = {}^0Havel]$ occurs *de dicto* in (S_{4emb}). Also the Closure $\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]$ occurs *de dicto* in (S_{4emb}), even though it is Composed with w_1, t_1, which triggers intensional descent of the office *PresCR*. The truth-value of the proposition constructed by (S_{4emb}) at a particular $\langle w, t\rangle$ may well depend on *PresCR* being occupied at worlds other than w or at times other than t.

The sentence (S_4) expresses the construction:

(S_4') $\lambda w\lambda t\ [{}^0\supset [\lambda w\lambda t\ [{}^0Playwright_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]]_{wt}$
$[\lambda w\lambda t\ [{}^0Believe_{wt}\ {}^0Charles\ [\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]]]]_{wt}]$.

The construction $\lambda w\lambda t\ [{}^0Playwright_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$ is used with *de re* supposition in (S_4'), and so is the first occurrence of $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$. The construction $\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]$ is used with *de dicto* supposition in (S_4'), and so is the second occurrence of $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$.

This goes to show that the *de dicto* context is *dominant* over the *de re* context. In the Closure $\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]$ the construction of the presidency, viz. $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$, occurs with *de re* supposition, such that the individual value of the office at a given $\langle w, t\rangle$-pair of evaluation is the object of predication, whereby the values of the office at $\langle w', t'\rangle$-pairs other than the $\langle w, t\rangle$-pair of evaluation become irrelevant. By contrast, the occurrence of the Closure $\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]$ in (S_{4emb}), as well as the second occurrence of the Closure $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$ in (S_4'), is intensional, i.e. with *de dicto* supposition. This is so, because in (S_{4emb}) the whole proposition that the President of the Czech Republic is Havel is the object of predication. Thus it is not so that the individual values of the presidency at $\langle w', t'\rangle$-pairs other than the $\langle w, t\rangle$-pair of evaluation are irrelevant.

Tichý sums it up thus:

> In general, a *de re* constituent of D is a *de re* constituent of any application in which D appears as a *de re* constituent; a *de re* constituent of D is a *de dicto* constituent of any application in which D appears as a *de dicto* constituent. A *de dicto* constituent is a *de dicto* constituent of any application in which D appears as a (*de re* or *de dicto*) constituent. Briefly, *de dicto* is the dominant one of the two suppositions (1988, p. 217).

Examples of sentences with 'the *F*' occurring with *de re* supposition:

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[126] We conceive of *believing* as a relation-in-intension between an individual and a proposition here, making believing an implicit attitude. See, however, Chapter 5. In order to mark the scope of particular λ-bindings of variables w and t we use numerical subscripts here.

- simple sentences: ‘The F is a G’.
- modalities: ‘The F is necessarily a G’.
- attitudes: ‘The F is believed by Charles to be a G’.

Modalities will be resumed in Chapter 4 and attitudes in Chapter 5.

Simple sentences of the form ‘The F is a G’ as dealt with above are, however, ambiguous between *de re* and *de dicto* readings. Consider, for instance, the sentence

‘Kurt Gödel’s most favourite argument is analytically valid.’

On its *de re* reading the sentence has the existential presupposition that there be exactly one argument that is Gödel’s favourite. If Gödel favoured more arguments to the same degree or if he had no one favourite argument, the sentence would have no truth-value. The reading *de dicto* mentions a necessary condition to be satisfied by an argument in order to qualify as Gödel’s favourite argument. The *de dicto* reading can be loosely paraphrased as

‘Being analytically valid is indispensable for an argument to be Gödel’s most favourite one.’

The truth-condition of this sentence does not require that Gödel have a favourite argument.

Types of the objects mentioned by the sentence:
Argument/$*_n$: a hyperproposition (a construction of a proposition);[127]
Gödel’s favourite argument/$*_{n\tau\omega}$: a constructional office (an office occupiable by constructions of order n);
Favour_arg_of/$((o*_n)\iota)_{\tau\omega}$: an empirical function assigning a set of arguments to an individual;
Most/$(*_n\ (o*_n))_{\tau\omega}$: an empirical function associating a set of arguments with an argument, the most favourite one;
Analytical/$(o*_n)$: the class of analytically valid arguments;
Indispensable/$(o(o*_n)*_{n\tau\omega})_{\tau\omega}$: a relation (-in-intension) between a class of arguments and a constructional office.

Now the Closure $\lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_arg\_of_{wt}\ {}^0\text{Gödel}]] \to *_{n\tau\omega}$ constructs the constructional office, and we have:

(a) *de re* reading:

$$\lambda w\lambda t\ [{}^0Analytical\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0Gödel]]_{wt}]$$

(b) *de dicto* reading (rephrased):

$$\lambda w\lambda t\ [{}^0Indispensable_{wt}\ {}^0Analytical\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0Gödel]]].$$

[127] For details on *arguments*, see Sections 1.5.1 and 5.4.

Let $Occ^*/(o*_{n\tau\omega})_{\tau\omega}$ be the property of a constructional office of being occupied. The relation of being indispensable can be defined as follows:

$$[{}^0Indispensable_{wt}\ C\ H] = [[{}^0Occ^*_{wt}\ H] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [C\ H_{wt}]]].$$

Types: $C \rightarrow (o*_n)$, $H \rightarrow *_{n\tau\omega}$.

Finally, using this refinement, the *de dicto* reading of the sentence expresses the construction:

(c) *de dicto* reading:

$$\lambda w\lambda t\ [[{}^0Occ^*_{wt}\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0Gödel]]] \supset$$
$$[{}^0True_{wt}\ \lambda w\lambda t\ [{}^0Analytical\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0Gödel]]_{wt}]]].$$

Another example of the ambivalence of simple sentences of the form 'The *F* is a *G'* is the sentence

'The King of France is a king.'

On its *de re* reading it expresses the construction ($King/(o\iota)_{\tau\omega}$; $King\_of/(\iota\iota)_{\tau\omega}$; $France/\iota$)

$$\lambda w\lambda t\ [{}^0King_{wt}\ \ \lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]_{wt}],$$

β-reducible to

$$\lambda w\lambda t\ [{}^0King_{wt}\ [{}^0King\_of_{wt}\ {}^0France]],$$

both of which construct a proposition that has no truth-value in the actual world now (as well as in any of the world/time at which the King of France does not exist). The *de re* reading of the sentence comes with the existential presupposition that the King of France exist. In those worlds/times at which the King of France exists, the proposition is true. Hence, on its *de re* reading the sentence does not express an analytically true proposition, though one that *almost* is. It does not denote the proposition *TRUE*, but a properly partial proposition that is true at some $\langle w, t\rangle$, and undefined at all the rest (hence nowhere and never false).

On its *de dicto* reading the sentence rather expresses a necessary relation between the property of being a king and the office of King of France. Necessarily, whenever somebody or other occupies the office of King of France, that individual is a king. (Or in plain English, if you are the king of something, then you are a king.) We call such a relation between intensions a *requisite*. Here the property of being a king is a requisite of the office of King of France, such that every occupant must have the relevant property. Thus the analysis of the *de dicto* reading of the above sentence is

$$[{}^0Requisite\ {}^0King\ \lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]].$$

Additional type: $Requisite/(o(o\iota)_{\tau\omega}\iota_{\tau\omega})$.

Each office may have indefinitely many such requisites. For instance, the office of President of the USA has the properties of being born in the United States, being above 35 years of age, etc., as its requisites. The set of all the requisites of an office is called its *essence,* and the office is fully characterised by its essence.[128]

A broader problem arises when we consider the *context* in which a particular construction occurs. We tackled the problem above, when we analysed the sentence (S_4) and concluded that the *de dicto* context is the dominant one of the two suppositions.

Now we are going to show that there are *three* contexts: hyperintensional (constructional), intensional (*de dicto*) and extensional (*de re*). Of these three the hyperintensional context is dominant over both the intensional and the extensional context, and the intensional context is dominant over the extensional context.

Consider again the sentence

(S_4) 'If the President of the Czech Republic is a playwright then Charles believes that the President of the Czech Republic is Václav Havel.'

Above we analysed Charles's belief as a relation-in-intension of an individual to a proposition. However, an alternative belief relation is an option. When belief is explicit belief, the believer enters into a relation-in-intension to a hyperproposition. Where $Believe^*/(o\iota *_n)_{\tau\omega}$, we have:[129]

$$(S_{4emb*})\quad \lambda w\lambda t\ [{}^0Believe^*_{wt}\ {}^0Charles\ {}^0[\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]]].$$

Now it no longer holds that the Closure $\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]$ is used with *de dicto* supposition in (S_{4emb*}), because it is *not used as a constituent* of (S_{4emb*}). It is *mentioned* here. Moreover, its constituents are *mentioned* in (S_{4emb*}) as well.

For this reason we must distinguish between *using* a construction as a constituent of another construction and *mentioning* a construction. If a construction is used as a constituent, it can be used in two different ways: intensionally or extensionally. The three kinds of context are as follows:[130]

- *Hyperintensional context*: the sort of context in which a construction is not used to *v*-construct an object. Instead, the *construction* itself is an argument of another function; the construction is just *mentioned.*

Example: 'Charles calculates 2+5' expresses as its meaning the Closure

[128] For more on requisites and essence, see Chapter 4.

[129] See Chapter 5 for details on propositional attitudes.

[130] Here we only briefly characterize the three contexts. Precise definitions will be provided in Section 2.6. Note that the notions 'intensional' and 'extensional' are used here in a broader sense than in possible-world semantics. To distinguish these notions from possible-world intension and extension, we will often add the asterisk '*' when talking about (hyper-) intensional/extensional occurrence of a construction.

$$\lambda w \lambda t\ [{}^0 Calculate_{wt}\ {}^0 Charles\ {}^0[{}^0{+}\ {}^0 2\ {}^0 5]].$$

The Composition $[{}^0{+}\ {}^0 2\ {}^0 5]/{*}_1$ is not used to construct the number 7 here. Instead, it is an argument of the function $Calculate/(\omicron\iota{*}_1)_{\tau\omega}$. Thus $[{}^0{+}\ {}^0 2\ {}^0 5]$ occurs in the *hyperintensional* context of $\lambda w \lambda t\ [{}^0 Calculate_{wt}\ {}^0 Charles\ {}^0[{}^0{+}\ {}^0 2\ {}^0 5]]$.

- *Intensional context*: the sort of context in which a construction is used to *v*-construct a function and *not a particular value* of the function. Moreover, the construction does not occur within another hyperintensional context.

Example: 'Sinus is a periodical function' expresses the Composition

$$[{}^0 Periodical\ {}^0 Sinus],$$

where $Periodical/(\omicron(\tau\tau))$ is the class of periodical functions of type $(\tau\tau)$; $Sinus/(\tau\tau)$.

${}^0 Sinus$ occurs in the *intensional* context of the Composition $[{}^0 Periodical\ {}^0 Sinus]$. It is not Composed with a τ-argument in order to construct a value of the sinus function. Instead the function is just mentioned, as it must be if a property is to be predicated of it.

On the other hand, 'Charles knows that sinus is periodical' expresses the construction $\lambda w \lambda t\ [{}^0 Know{*}_{wt}\ {}^0 Charles\ {}^0[{}^0 Periodical\ {}^0 Sinus]]$, $Know{*}/(\omicron\iota{*}_1)_{\tau\omega}$. Here the Composition $[{}^0 Periodical\ {}^0 Sinus]$ occurs hyperintensionally; therefore also all its subconstructions, including ${}^0 Sinus$, occur in a hyperintensional context.

In the empirical case, intensional constructions usually occur in intensional contexts. Consider 'Charles wants to become the President of the USA'. Charles is related here to the presidential *office*; he wants to occupy it. Thus the analysis comes down to this:

$$\lambda w \lambda t\ [{}^0 Want\_to\_become_{wt}\ {}^0 Charles\ \lambda w \lambda t\ [{}^0 President\_of_{wt}\ {}^0 USA]].$$

Types. $Want\_to\_become/(\omicron\iota\iota_{\tau\omega})_{\tau\omega}$; $President\_of/(\iota\iota)_{\tau\omega}$; $Charles, USA/\iota$;

The whole Closure occurs intensionally; it is not used to *v*-construct the truth-value of the so constructed proposition. Moreover, the construction of the presidency, namely $\lambda w \lambda t\ [{}^0 President\_of_{wt}\ {}^0 USA]$, occurs intensionally (i.e., with *de dicto* supposition) in the intensional context of the whole Closure.

- *Extensional context*: the sort of context in which a construction of a function is used to construct a particular value of the function at a given argument, and the construction does not occur within another intensional or hyperintensional context.

Example: '$\sin(\pi) = 0$' expresses the Composition $[[{}^0 Sinus\ {}^0\pi] = {}^0 0]$, where ${}^0 Sinus$ occurs extensionally; the Composition is used to construct the value of the sinus function at the argument π.

As mentioned above, constructions of intensions usually occur intensionally; if occurring extensionally, then they usually *v*-construct a particular value of an intension. For instance, $[\lambda w \lambda t\ [{}^0President\_of_{wt}\ {}^0USA]]_{wt}$ *v*-constructs an individual; the Closure $\lambda w \lambda t\ [{}^0President\_of_{wt}\ {}^0USA]$ occurs extensionally, since the so constructed office is extensionalized.

However, in the Closure

$$\lambda w \lambda t\ [{}^0Republican_{wt}\ [\lambda w \lambda t\ [{}^0President\_of_{wt}\ {}^0USA]]_{wt}]$$

(which is the meaning of 'The President of the USA is a Republican') the construction of the presidency occurs extensionally (i.e., with *de re* supposition), but in the intensional context of the whole Closure.

The topics of *de dicto*/*de re* supposition and hyperintensional, intensional and extensional contexts are resumed in Section 2.6.

1.5.3 Important entities and notational conventions: summary

Below follows a summary of the main features of our semantic schemas which we introduced in Section 1.1, as well as the main notational conventions. In this chapter we defined, among others, *construction*, *ramified hierarchy of types*, important extensions like *quantifiers* and the notion of *literal meaning* of an expression. We also illustrated how constructions are assigned to semantically self-contained expressions, whereby an expression invariably expresses a construction as its meaning. Whenever an expression does have a *denotation*, the denotation can be any entity of the ontology of TIL:

- an α-*intension* (an object of type α_ω, typically $\alpha_{\tau\omega}$) when the expression is *empirical*;
- an α-*extension*, i.e., an α-object, where $\alpha \neq (\beta\omega)$ for any β;
- a *construction* of type $*_n$, when the expression is mathematical or logical.

Empirical expressions invariably denote α-intensions. The sense of the sentence 'Charles is a bachelor' is a procedure for evaluating, in any possible world at any time, the truth-conditions of this sentence. The sense is the Closure $\lambda w \lambda t\ [{}^0Bachelor_{wt}\ {}^0Charles]$. The denotation of this sentence is the proposition $P/o_{\tau\omega}$ constructed by this construction. P is true in a subset of logical space; namely, at those worlds and times at which Charles has the property of being a bachelor. If the sentence is true *simpliciter*, then the pair made up of the actual world and the current time is a member of this subset. The reference of this sentence (its truth-value) is beyond the purview of the a priori discipline of logical semantics. (See Sections 1.1 and 2.4.1 for the details of the argument from omniscience in favour of anti-actualism.)

Mathematical expressions denote α-extensions. But even in this case the respective extension is only of secondary semantic interest. What is of primary semantic interest is the respective construction. This is especially clear in the case of expressions lacking denotation, like ‘the greatest prime’. Mathematicians had to first understand the expression, i.e., to know the respective instruction detailing *how* to seek the product; only then were they able to prove that there is no product of the procedure expressed by the expression:

$$\iota x\ [^0\wedge\ [^0Prime\ x]\ \forall y\ [^0\supset\ [^0Prime\ y]\ [^0\geq x\ y]]].$$

We now recapitulate the most important entities and notational conventions occurring throughout the book.

- An arbitrary object X of the arbitrary type α is an *α-object*, denoted ‘X/α’.
- The notation for the type $((\alpha\tau)\omega)$ of α-intensions is abbreviated ‘$\alpha_{\tau\omega}$’.
- The constant proposition that takes value **T** in all possible worlds at all times will be referred to as ‘*TRUE*’.
- The propositional properties of being true, false, undefined are the functions $True/(o o_{\tau\omega})_{\tau\omega}$, $False/(o o_{\tau\omega})_{\tau\omega}$, $Undef/(o o_{\tau\omega})_{\tau\omega}$, respectively.
- Every construction C belongs to $*_n$: C is an entity of a type of order $n > 1$, and (v-) constructs an entity (if any) belonging to a type α of a lower order. That a construction C v-constructs an α-object will be denoted ‘$C/*_n \rightarrow_v \alpha$’, or sometimes ‘$C \rightarrow_v \alpha$’. For instance, ‘$x/*_1 \rightarrow_v \tau$’ reads, ‘The variable x belongs to the type $*_1$ and constructs reals relative to a valuation.’
- If a construction C v-constructs an α-object a independently of valuation, we simply say that C constructs a and write ‘$C \rightarrow \alpha$’.
- We often write ‘$\forall x\ A$’, ‘$\exists x\ A$’, ‘$\iota x\ A$’, instead of ‘$[^0\forall^\alpha\ \lambda x\ A]$’, ‘$[^0\exists^\alpha\ \lambda x\ A]$’, ‘$[^0Sing^\alpha\ \lambda x\ A]$’, respectively, when it is not urgent to highlight typing and lambda-binding.
- We also often use infix notation without Trivialization when using constructions of the truth-functions $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\equiv$ (equivalence) and negation ($\neg$), and when using a construction of an identity relation.
- Variables w, w_1, w_2, … v-construct elements of type ω (possible worlds), and t, t_1, t_2, … v-construct elements of type τ (times ordered in a continuum).
- If C v-constructs an α-intension, the frequently used Composition of the form $[[C\ w]\ t]$, v-constructing the *intensional descent* (a.k.a. *extensionalization*) of an α-intension, is abbreviated ‘C_{wt}’.

2
Foundations of semantic analysis

2.1 A logical method of semantic analysis

In this chapter we introduce the foundations of our method of *logical analysis of natural language* (*LANL*).

2.1.1 The Parmenides principle

We begin by summarising the notions defined in the introductory Chapter 1 that we have at our disposal:

- an *objectual base* with elementary (atomic) types $\omicron$ = {**T**, **F**}, ι = the universe of discourse, τ = times/reals, ω = possible worlds, together with all the functional types definable over the base;
- *constructions*: higher-order entities which are abstract procedures that are assigned to the expressions of a given language, whether natural or artificial, and which make up their structured meanings; the constructions are *Variable*, *Trivialization*, *Composition*, *Closure*, *Execution*, and *Double Execution*;
- *a ramified hierarchy of types* that makes it possible to not only *use* constructions as constituents of other constructions, but also to *mention* constructions in a hyperintensional context in order to manipulate them.

One reason why the study of constructions is important is because constructions are assigned to expressions of a natural language as their meanings. To make explicit what we argue to be the meaning of a given expression, namely the construction encoded by the expression, is a non-trivial task of the general programme of logical analysis of natural language. Logical semantics differs from linguistic semantics (which is an empirical discipline) as well as from that sort of formal semantics that was invented for the purpose of interpreting formal systems.[1] Logical analysis of language must address the questions, *What do we talk about?*[2] and, *How do we talk about it*?

When doing this sort of analytic work, we heed a constraint put forward by Frege:

[1] —formal semantics being the sort of thing that Tarski propagated and Leśniewski objected to. For details, see Betti (2008).

[2] See Tichý (1975, 2004, pp. 205–20).

M. Duží et al., *Procedural Semantics for Hyperintensional Logic*, Logic, Epistemology, and the Unity of Science 17, DOI 10.1007/978-90-481-8812-3_2,

> It is simply not possible to speak about an object without somehow denoting or naming it[3] (1884, p. 60).

In an unpublished study Tichý called this principle the *Parmenides Principle* (***PP***), alluding to Parmenides' doctrine of being. Some examples to fix ideas:

'The biggest planet is smaller than the Sun.'
'The highest mountain is in Asia.'
'The President of the USA is a Democrat.'
'The Mayor of Warsaw is a friend of the richest man in Poland.'

Does the first sentence talk about Jupiter, the second about Mount Everest, and the third about Barack Obama? Further, if we do not know who the Mayor of Warsaw or the richest man of Poland is, are we then barred from fully understanding the fourth sentence? ***PP*** answers all these questions in the negative, and rightly so. Provided we have succeeded in showing that the biggest planet is another sort of object than Jupiter the planet, that the highest mountain is another sort of object than Mount Everest, and that the President of the USA is another sort of object than Barack Obama, then we have also succeeded in showing that it is possible to understand these sentences without knowing to which objects particular descriptions refer in the actual world now.[4] Further, we hope to have shown that the fourth sentence offers a self-contained piece of information, *viz*. that the Mayor of Warsaw—whoever he or she may be, if any—is a friend of the richest man in Poland—whoever he may be, if any.

Indeed, elementary logic is sufficient to prove that, e.g., 'Jupiter is smaller than the Sun' does not follow from the first sentence. Whenever the impression that it does follow arises, it is because we happen to know which planet is the biggest one, and so omit the obvious premise 'The biggest planet is Jupiter'. This premise, however, is obviously not logically trivial and is necessary for deriving the conclusion. In elementary logic, the full argument looks like this:

(1) The F is smaller than a
(2) the F is b

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(3) b is smaller than a.

Absent (2), the substitution of 'b' for 'the F' is not valid.

Thus ***PP*** confirms the basic distinction between names and definite descriptions. In our sentences we talk about the biggest planet, the highest mountain, the President of the USA, the Mayor of Warsaw, and the richest man in Poland, respectively. We do *not* talk about Jupiter, Mount Everest, Barack Obama, Hanna

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[3] The German original goes, 'Überhaupt ist es nicht möglich von einem Gegenstand zu sprechen, ohne ihn irgendwie zu bezeichnen oder benennen.'

[4] Assuming that 'Jupiter', 'Mount Everest' and 'Barack Obama' each names a numerically specific individual. For proper names, see Section 3.2.

Gronkiewicz-Waltz, or Zygmunt Solorz-Żak. Therefore, from the point of view of LANL, empirical expressions do not talk about their extensions in any particular state-of-affairs $\langle w, t \rangle$; they talk about α-intensions of type $\alpha_{\tau\omega}$.[5]

The *Parmenides Principle* can be condensed into:

> If an expression *E* talks about an object *X* then some subexpression of *E* denotes *X*.

This principle gives rise to the following constraint imposed on the analysis of an expression *E*:

Do not add *anything that is not talked about by* E.

For LANL, as carried out by TIL, the implication is this:

> An *admissible analysis* of an expression *E* is a construction *C* such that no closed subconstruction of *C* constructs an object that *E* does not talk about.

To adduce a simple example, an admissible analysis of the sentence 'Venus is a planet' is the Closure

$$\lambda w \lambda t\ [{}^0Planet_{wt}\ {}^0Venus]$$

but *not*, e.g., the Closure

$$\lambda w \lambda t\ [[{}^0Planet_{wt}\ {}^0Venus] \wedge [[{}^0{+}\ {}^01\ {}^01] = {}^02]].$$

Types: $Planet/(o\iota)_{\tau\omega}$; $Venus/\iota$; $1, 2/\tau$; $+/(\tau\tau\tau)$; $=/(o\tau\tau)$.

Though both Closures are equivalent by constructing one and the same proposition, the latter is *not* an admissible analysis of 'Venus is a planet', because the sentence does not talk about the function +, the numbers 1 and 2 or identity between numbers.

To further exemplify this requirement, let us return to the sentence, 'The biggest planet is smaller than the Sun'. To begin, we assign types to the objects talked about:

BP (the biggest planet)$/\iota_{\tau\omega}$; *Smaller (_than)*$/(o\iota\iota)_{\tau\omega}$; *the Sun*$/\iota$.

Second, we combine constructions (Trivializations) of these objects in order to construct the proposition denoted by the whole sentence:

$$\lambda w \lambda t\ [{}^0Smaller_{wt}\ {}^0BP_{wt}\ {}^0Sun].$$

[5] There are cases where the temporal parameter has been blotted out, e.g. in the case of laws of nature. Then the denoted intension is of type α_{ω}.

The constraint has been respected: ^{0}*Smaller* is assigned to ‘smaller than’, ^{0}*BP* to ‘the biggest planet’, and ^{0}*Sun* to ‘the sun’. Nothing has been added. (In particular, ^{0}Jupiter has not been added.)

In general, heeding this constraint is far from trivial. Consider the following sentence:

‘Some students are bald.’

A standard predicate-logical analysis would use the existential quantifier:

$$\lambda w \lambda t\, [\exists x\, [[^{0}Student_{wt}\, x] \wedge [^{0}Bald_{wt}\, x]]].$$

Types: $Bald/(o\iota)_{\tau\omega}$; $Some = \exists/(o(o\iota))$; $\wedge/(ooo)$; $x \rightarrow \iota$.

But our sentence does not contain any expression (e.g., ‘and’) whose meaning would be the construction of conjunction, $^{0}\wedge$. Thus the above construction corresponds to another sentence, namely

‘There are individuals who are students and who are bald.’

However, we can easily satisfy the constraint dictated by ***PP***. In Section 1.4.3 we defined another type of quantifier, *All* and *Some*, both of type $((o(o\alpha))(o\alpha))$, α any type. *All* applied to a class A of α-objects returns the class of such classes whose subclass is A. *Some* applied to a class A of α-objects returns the class of such classes whose intersection with A is non-empty. Then the following construction can be assigned to the sentence ‘Some students are bald’ as its meaning:

$$\lambda w \lambda t\, [[^{0}Some\ ^{0}Student_{wt}]\ ^{0}Bald_{wt}].$$

Some and *All* make it possible to analyse a given sentence so that the respective construction does not contain a construction of conjunction or implication where the sentence does not contain subexpressions denoting these truth-functions.

Thus when performing some analytic task falling within LANL, we adhere to the constraint on natural-language analysis dictated by ***PP*** and pursue an *admissible* and *ideal analysis* of an expression E. Such an analysis is a construction C such that C uses, as its constituents, constructions of *just* those objects that receive mention in E, i.e., the objects denoted by subexpressions of E. The principle is central to our general three-step *method of logical analysis* of language:

(i) *Type-theoretical analysis.* Assign types to the objects mentioned, i.e., *only* those that are denoted by subexpressions of E, and do not omit any semantically self-contained subexpression of E.
(ii) *Synthesis*. Combine constructions of these objects so as to construct the object D denoted by E.

(iii) *Type checking*. Use the assigned types for *control* so as to check whether the various types are compatible and, furthermore, produce the right type of object in the manner prescribed by the analysis.

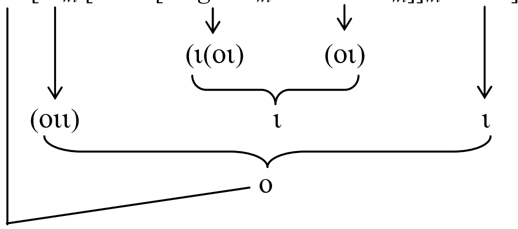
Example of analysis. We are going to analyse the sentence,

‘The highest mountain is in Asia’.

(i′) *Highest*/$(\iota(o\iota))_{\tau\omega}$: an empirical function that, relative to worlds and times, associates a set of individuals with an individual; namely, the highest one; *Mountain*/$(o\iota)_{\tau\omega}$; *HM* (the Highest Mountain)/$\iota_{\tau\omega}$; *In*/$(o\iota\iota)_{\tau\omega}$; *Asia*/$\iota$.[6] The whole sentence denotes a proposition/$o_{\tau\omega}$.

(ii′) $[{}^0Highest_{wt}\ {}^0Mountain_{wt}] \rightarrow_v \iota$ (the individual that is the highest one in the ‘population’ of mountains at $\langle w, t\rangle$)
$\lambda w\lambda t\ [{}^0Highest_{wt}\ {}^0Mountain_{wt}] \rightarrow \iota_{\tau\omega}$ (*HM*)
$[\lambda w\lambda t\ [{}^0Highest_{wt}\ {}^0Mountain_{wt}]]_{wt} \rightarrow_v \iota$ (the occupant of *HM* at $\langle w, t\rangle$)
$[In_{wt}\ [\lambda w\lambda t\ [{}^0Highest_{wt}\ {}^0Mountain_{wt}]]_{wt}\ {}^0Asia] \rightarrow_v o$
$\lambda w\lambda t\ [In_{wt}\ [\lambda w\lambda t\ [{}^0Highest_{wt}\ {}^0Mountain_{wt}]]_{wt}\ {}^0Asia] \rightarrow o_{\tau\omega}$.

(iii′) $\lambda w\lambda t\ [In_{wt}\ [\lambda w\lambda t\ [{}^0Highest_{wt}\ {}^0Mountain_{wt}]]_{wt}\ {}^0Asia]$



Abstracting over *t*: $(o\tau)$
Abstracting over *w*: $((o\tau)\omega)$; i.e., $o_{\tau\omega}$.

Step (iii′) depicts the construction, *via* Closure, of the proposition that the highest mountain is in Asia. This Closure is the meaning expressed by ‘The highest mountain is in Asia’, and the Closure constructs the proposition that the sentence denotes.

2.1.2 The compositionality constraint

The point (i) of the method of analysis mentioned in the preceding subsection is of key importance. Our approach obeys the principle of *compositionality*. A rough characterization of compositionality says that the syntactic operations which

[6] Here we use an innocuous type-theoretical simplification.

derive an expression from its subexpressions must be matched by semantic operations which derive the meanings of the expressions from the meanings of its subexpressions. If the theory of so-called autonomous syntax were to be followed, it would hardly be possible to respect this constraint; for what would be the criterion, compatible with autonomous syntax, defining the subexpressions of a given expression? Fortunately, the principle of compositionality is in principle incompatible with autonomous syntax. As Gamut says,

> [L]ogical grammar, with its principle of compositionality of meaning, goes straight against the autonomy of syntax so cherished in the generative tradition. … And that means, at least in principle, that semantic considerations may influence the syntax, thus breaching the supposed autonomy of syntax (1991, p. 141).

Similarly as Montague, TIL rejects the 'pure syntax' approach to logical analysis. Tichý argues:

> [I]t would be … in vain to ask an autonomous syntactician what the term 'constituent' means. He certainly cannot say that a constituent is an expression which is complete in that it refers all by itself to a definite entity, in contrast to an incomplete expression which refers only in combination with some other expressions. For that… would amount to leaving the domain of autonomous syntax. The term 'constituent' (or 'phrase') is apparently not to be burdened with any pre-theoretical meaning at all: a constituent is simply whatever the grammarians' theory brands as such in any particular case.
>
> [I]t would be equally idle to ask what governs the distribution of the mother/daughter relation. Why is it, for example, that 'slowly' is a sister of 'works' but not of 'Fred'? The syntactician cannot explain it by pointing out the obvious fact that 'slowly' stands for an activity modifier, i.e. for a mapping which takes activities to activities, and that the activity named by the **VP** 'slowly works' is the value of that mapping at the argument named by 'works'. In brief, he cannot say that 'slowly' is a sister of 'works' because the entities they stand for are related as a mapping and its argument (2004, p. 807).

TIL couples *subexpressions* of the analysed expression E with the respective subconstructions of the analysis of E. Now we need to say more about the principle of *compositionality* that we claim to be obeying. We begin with a definition taken over from Szabó (2005, p. 5).

Definition 2.1 (*Compositionality*) Let E be a set of expressions, m a meaning-assignment, M a set of 'available' meanings, and let F be a k-ary syntactic operation on E. Then m is *F-compositional* if there is a k-ary partial function G on M such that whenever $F(e_1,\dots,e_k)$ is defined, $m(F(e_1,\dots,e_k)) = G(m(e_1),\dots,m(e_k))$.

It turns out that F is a function that composes an expression from what we refer to as *subexpressions.* Its syntactic character is, however, not 'autonomously syntactic', as the quotations by Gamut and Tichý brought out, so the terminal nodes of the well-known annotated linguistic trees are meaningful expressions, and the dependencies that link these nodes are based on objective, extra-syntactic, functional relations.[7]

Now it might seem that the problem of compositionality were a simple one, as indeed Horwich (1997) tries to argue. Allegedly, the condition of compositionality is fulfilled as soon as the meaning of an expression is unambiguously determined by the meanings of the subexpressions. Well, this is easy to accomplish in case 'meaning' is artificially defined as interpretation in an artificial (formal) language. For then the correspondence between syntax and semantics is automatically ensured. As soon as we turn to natural language, the situation radically changes, since a consequence of the spontaneous development of natural languages is that the way they encode their logical foundations becomes extremely complicated, many not only lexical but also syntactic ambiguities disguising their logical structure. This explains why logical structure has to be *discovered* and *made explicit* by logical analysis.

Further, what is, or is not, compositional is *meaning* rather than language.[8] In general, 'meaning' in such definitions is a 'generic rather than specific term' (Sandu and Hintikka, 2001, p. 49). Sandu and Hintikka propose speaking 'of the different semantic attributes of an expression'. Now we will examine *reference*,[9] *denotation* and *construction* as candidates for semantic attributes and show that only the last one can guarantee such a disambiguation of a given natural language so as to make it possible for compositionality to hold. To this end we need to define *synonymy* with respect to a particular meaning attribute:

Definition 2.2 (*synonymy with respect to a meaning attribute m*) Let m be a meaning attribute. Then an expression E is *m-synonymous with* an expression E' iff $m(E) = m(E')$. □

Claim 2.1 (*PS*: *Principle of substitutability*) Let m be compositional. If E is m-synonymous with E', then so is any expression F with an expression F' where F and F' differ just in that F contains E as a subexpression whereas F' contains E' in the same position. □

Proof follows from Definitions 2.1 and 2.2.

First, we show that natural languages are not reference-compositional (if their complexity is even vaguely similar to that of English). The sentence

---

[7] See also some general remarks in the beginning of Jackendoff (1990).

[8] This conflation of a language and its semantics may have been what led Pietroski to claim '[T]hat there are reasons for thinking that natural languages violate substitutivity (without violating compositionality)' (1996, p. 346).

[9] It should be clear now that *reference*, as a contingent value of a denoted intension, is a factual and, therefore, not an a priori semantic notion. Yet for completeness we choose to consider reference as well.

‘The Moon is larger than the Sun’

is referentially synonymous with the sentence

‘$2 > 3$’.

According to ***PS*** the sentence

‘Charles believes that the Moon is larger than the Sun’

should be referentially synonymous with the sentence

‘Charles believes that $2 > 3$’.

But this evidently does not hold.

Second, consider now *denotation* instead of *reference*. It is easy to show that also denotation fails as a meaning attribute with respect to which natural languages would be compositional. As an example, consider the sentences

‘Venus is a planet’

and

‘Venus is a planet and mammals are vertebrates’.

The sentence ‘Venus is a planet’ denotes a proposition that is true in a set S of world/times. ‘Mammals are vertebrates’ is true in all possible worlds and times.[10] It denotes the proposition *TRUE*. Since the intersection of S with the set of all possible worlds and times is S, the two sentences are denotationally synonymous. They co-denote the same *proposition* and would be synonymous were co-intensionality the criterion of synonymy. Thus according to ***PS*** the following sentences should be also denotationally synonymous:

‘Charles knows that Venus is a planet’

and

‘Charles knows that Venus is a planet and that mammals are vertebrates’.

Yet this evidently does not hold, as already argued.

[10] The sentence is analytically true, because the property of being a vertebrate is a *requisite* of (is implied by) the property of being a mammal. See Section 4.1.

Third, if *m* is the construction expressed by an expression, the above counterexamples apparently proving that *m* is not compositional do not apply. In other words, if natural-language expressions are hyperintensionally (thus, according to TIL, constructionally) synonymous, then ***PS*** is unassailable by such counterexamples. Again, this does not mean that the problem would turn out to be a simple one. Many ambiguities of natural language need to be charted and explained, and the search for the best analysis (see below) is one of the methods making explicit a given meaning constrained by compositionality.

A consequence of the foregoing considerations is that what we construe in TIL as the meaning of an expression *E* is the construction that is provably the best literal meaning of *E*.[11] This construal of meaning makes it possible to define the notions of *equivalence*, *synonymy* and *co-referentiality* of expressions. To do so we must, however, define a slightly more coarse-grained notion of meaning than that of construction. The resulting notion is called *concept*. The relevant definitions will be introduced in Section 2.2.

2.1.3 Better and worse analyses

Frege's constraint above is incomplete, failing as it does to include the complementary constraint that nothing is to be *omitted*. So the other half of the Parmenides principles should read,

Do not omit *anything talked about by* E.

The principle that arises by adding this constraint to ***PP*** will be called ***PP'***. This principle is not satisfied by our analysis of the sentence, 'The biggest planet is smaller than the Sun'. The analysis we offered was the Closure

$$(C_1) \qquad \lambda w \lambda t\ [{}^0Smaller_{wt}\ {}^0BP_{wt}\ {}^0Sun].$$

Types: *BP* (the biggest planet)/$\iota_{\tau\omega}$; *Smaller (_than)*/$(o\iota\iota)_{\tau\omega}$; *the Sun*/$\iota$.

The sentence contains the subexpressions 'the biggest' and 'planet'. No subconstruction of the above construction matches either of them. The new, refined, construction will contain subconstructions matching both of them. As always, we first assign types to the objects denoted by these subexpressions:

(The) *Biggest*/$(\iota(o\iota))_{\tau\omega}$: the function that selects, dependently on a given world-time, from a set of individuals the biggest one, if any; *Planet*/$(o\iota)_{\tau\omega}$.

Now we have to construct the office *BP* by Composing the constructions of these two objects:

[11] See Definition 1.10.

$$\lambda w \lambda t\, [{}^0\!Biggest_{wt}\ {}^0\!Planet_{wt}] \to \iota_{\tau\omega}.$$

Substituting this compound construction for the Trivialization ${}^0BP$ into (C_1) yields the refined analysis of the sentence:

(C_2) $\lambda w \lambda t\, [{}^0\!Smaller_{wt}\ [\lambda w \lambda t\, [{}^0\!Biggest_{wt}\ {}^0\!Planet_{wt}]]_{wt}\ {}^0\!Sun]$.

Now (C_2) reveals that our sentence talks about

(a) the proposition *that the biggest planet is smaller than the Sun,*
(b) the relation *smaller than,*
(c) the function *the biggest,*
(d) the property *planet,*
(e) the individual office *the biggest planet,*
(f) the individual *the Sun.*

However, (C_1) does not reveal that the sentence also talks about (c) and (d).

What happens when we exploit the fact that $[\lambda w \lambda t\, [{}^0\!Biggest_{wt}\ {}^0\!Planet_{wt}]]_{wt}$ *v*-constructs the same individual as $[{}^0\!Biggest_{wt}\ {}^0\!Planet_{wt}]$? We get the construction

(C_3) $\lambda w \lambda t\, [{}^0\!Smaller_{wt}\ [{}^0\!Biggest_{wt}\ {}^0\!Planet_{wt}]\ {}^0\!Sun]$.

This time (C_3) fails to reveal that the sentence talks about (e), because $[{}^0\!Biggest_{wt}\ {}^0\!Planet_{wt}]$ goes straight to the individual that is the biggest planet at $\langle w, t\rangle$, skipping the detour around the individual office.

We will return to the comparison of (C_1), (C_2), and (C_3) later. For now we wish to consider the possibility of using ***PP′*** to answer the following fundamental question:

> Let *an admissible analysis of an expression E* be any such construction as is assigned to the expression *E* as its meaning in accordance with ***PP***. Is it possible in principle to find for every expression of a given language its *best analysis*?

According to which criteria are we able to say that an analysis of an expression is better than another analysis of the same expression? These two, mutually dependent, criteria seem the obvious place to start:

***Criterion* 1** A construction *C* is *a worse analysis of E than C′* iff the set of valid inferences based on *C* is a proper subset of such a set based on *C′*.

***Criterion* 2** A construction *C* is *a worse analysis of E than C′* iff *C′* is more fine-grained than *C*; i.e., if the set of subexpressions of *E* to which a construction (as meaning) has been assigned by *C* is a proper subset of those subexpressions of *E* to which a construction (as meaning) has been assigned by *C′*.

It might seem that these criteria were equivalent, except that the first criterion is not effective. An example shows, however, that the connection between these criteria is not that straightforward. Let us apply ***PP′*** in order to find particular admissible analyses of the sentence

‘A young girl is admired by the richest mathematician.’

The sentence uses the passive voice because we want its topic to be a young girl and its focus to be the property of being admired by the richest mathematician. Had the active voice been used instead, it would not be unambiguous what its focus was.[12]

Here is the list of objects this sentence talks about:

(a) the whole proposition *that a young girl is admired by the richest mathematician*
(b) a—*Some*/((ο(οι))(οι)): applied to a class P it returns the class of those classes which share with P at least one member
(c) *Young*/$((οι)_{τω}\ (οι)_{τω})$: a modifier which, when applied to a property, returns another property
(d) *Girl*/$(οι)_{τω}$
(e) young girl—*YG*/$(οι)_{τω}$
(f) *Admire*/$(οιι)_{τω}$
(g) the richest—*Rich*/$(ι(οι))_{τω}$: the function that dependently on world and time selects an individual from a set of individuals, namely the richest one
(h) mathematician—*Math*/$(οι)_{τω}$
(i) the richest mathematician—*RM*/$ι_{τω}$
(e) being admired by the richest mathematician—*ARM*/$(οι)_{τω}$

The following constructions are admissible analyses of the sentence:

C′1 $\quad {}^0(\textit{A young girl is admired by the richest mathematician})$

C′2 $\quad λwλt\ [[{}^0Some\ {}^0YG_{wt}]\ {}^0ARM_{wt}]$

C′3 $\quad λwλt\ [[{}^0Some\ {}^0YG_{wt}]\ [λx\ [{}^0Admire_{wt}\ {}^0RM_{wt}\ x]]]$

C′4 $\quad λwλt\ [[{}^0Some\ [{}^0Young\ {}^0Girl]_{wt}]\ [λx\ [{}^0Admire_{wt}\ {}^0RM_{wt}\ x]]]$

C′5 $\quad λwλt\ [[{}^0Some\ {}^0YG_{wt}]\ [λx\ [{}^0Admire_{wt}\ [{}^0Rich_{wt}\ {}^0Math_{wt}]\ x]]]$

C′6 $\quad λwλt\ [[{}^0Some\ [{}^0Young\ {}^0Girl]_{wt}]\ [λx\ [{}^0Admire_{wt}\ [{}^0Rich_{wt}\ {}^0Math_{wt}]\ x]]]$

C′7 $\quad λwλt\ [[{}^0Some\ {}^0YG_{wt}]\ [λx\ [{}^0Admire_{wt}\ λwλt\ [{}^0Rich_{wt}\ {}^0Math_{wt}]_{wt}\ x]]]$

C′8 $\quad λwλt\ [[{}^0Some\ [{}^0Young\ {}^0\mathrm{Girl}]_{wt}]$
$\qquad\qquad [λx\ [{}^0Admire_{wt}\ λwλt\ [{}^0Rich_{wt}\ {}^0Math_{wt}]_{wt}\ x]]]$

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[12] For more on topic-focus articulation, see Duží (2009).

C′9 $\lambda w\lambda t\, [[{}^0Some\; {}^0YG_{wt}]\; \lambda w\lambda t\, [\lambda x\, [{}^0Admire_{wt}\; {}^0RM_{wt}\; x]]_{wt}]$

C′10 $\lambda w\lambda t\, [[{}^0Some\; [{}^0Young\; {}^0Girl]_{wt}]\; \lambda w\lambda t\, [\lambda x\, [{}^0Admire_{wt}\; {}^0RM_{wt}\; x]]_{wt}]$

C′11 $\lambda w\lambda t\, [[{}^0Some\; {}^0YG_{wt}]\; \lambda w\lambda t\, [\lambda x\, [{}^0Admire_{wt}\; [{}^0Rich_{wt}\; {}^0Math_{wt}]\; x]]_{wt}]$

C′12 $\lambda w\lambda t\, [[{}^0Some\; [{}^0Young\; {}^0Girl]_{wt}]$
$\lambda w\lambda t\, [\lambda x\, [{}^0Admire_{wt}\; [{}^0Rich_{wt}\; {}^0Math_{wt}]\; x]]_{wt}]$

C′13 $\lambda w\lambda t\, [[{}^0Some\; {}^0YG_{wt}]\; \lambda w\lambda t\, [\lambda x\, [{}^0Admire_{wt}\; \lambda w\lambda t\, [{}^0Rich_{wt}\; {}^0Math_{wt}]_{wt}\; x]]_{wt}]$

C′14 $\lambda w\lambda t\, [[{}^0Some\; [{}^0Young\; {}^0\mathrm{Girl}]_{wt}]$
$\lambda w\lambda t\, [\lambda x\, [{}^0Admire_{wt}\; \lambda w\lambda t\, [{}^0Rich_{wt}\; {}^0Math_{wt}]_{wt}\; x]]_{wt}]$

Now which objects does the sentence talk about according to the particular analyses?

C′1 the proposition
C′2 the proposition, *Some*, *YG*, *ARM*
C′3 the proposition, *Some*, *YG*, *Admire*, *RM*
C′4 the proposition, *Some*, *Young*, *Girl*, *YG*, *Admire*, *RM*
C′5 the proposition, *Some*, *YG*, *Admire*, *Rich*, *Math*
C′6 the proposition, *Some*, *Young*, *Girl*, *YG*, *Admire*, *Rich*, *Math*
C′7 the proposition, *Some*, *YG*, *Admire*, *Rich*, *Math*, *RM*
C′8 the proposition, *Some*, *Young*, *Girl*, *YG*, *Admire*, *Rich*, *Math*, *RM*
C′9 the proposition, *Some*, *YG*, *Admire*, *RM*, *ARM*
C′10 the proposition, *Some*, *Young*, *Girl*, *YG*, *Admire*, *RM*, *ARM*
C′11 the proposition, *Some*, *YG*, *Admire*, *Rich*, *Math*, *ARM*
C′12 the proposition, *Some*, *Young*, *Girl*, *YG*, *Admire*, *Rich*, *Math*, *ARM*
C′13 the proposition, *Some*, *YG*, *Admire*, *Rich*, *Math*, *RM*, *ARM*
C′14 the proposition, *Some*, *Young*, *Girl*, *YG*, *Admire*, *Rich*, *Math*, *RM*, *ARM.*

It is readily seen that the set {C′1,…,C′14} can be ordered by the relation ≤ defined as follows:

Let A, A′ be analyses of *E*. Then A ≤ A′ iff either A is worse than A′ according to Criterion 2 or A, A′ are identical ($^0\mathrm{A} =_{*n} {}^0\mathrm{A}'$).

In Materna and Duží (2005) it was proved that the relation ≤ is a partial ordering on the set of admissible analyses of *E* inducing a complete lattice. The least element of the lattice is the Trivialization of the entity denoted by *E* (the worst analysis of *E*). The best analysis is then the greatest element of the lattice.

In our example C′1 is the worst analysis. Actually, it barely qualifies as an analysis, for it simply black-boxes the logical structure of the sentence. The best analysis is C′14:

C′14 $\lambda w\lambda t\, [[{}^0Some\; [{}^0Young\; {}^0Girl]_{wt}]$
$\lambda w\lambda t\, [\lambda x\, [{}^0Admire_{wt}\; \lambda w\lambda t\, [{}^0Rich_{wt}\; {}^0Math_{wt}]_{wt}\; x]]_{wt}]$

We would like to be able to draw, in virtue of our analysis, various conclusions from the sample sentence. It can be shown, even by means of this simple example, that some analyses support valid deductions, and others do not. Thus, for instance, the conclusions

‘There is a girl whom the richest mathematician admires’;

‘There is a girl whom a mathematician admires’;

‘There is a young girl whom a mathematician admires’

which are entailed by the sample sentence, are supported by C′14, whereas C′1 does not support anything and the other analyses support some, but not all, of the conclusions that C′14 supports. For instance, the first conclusion is not supported by C′1, C′2, C′3, C′5, C′7, C′9, C′11 and C′13.

For instance, using C′14 as an assumption, it is easy to prove the third conclusion. To this end we need to utilise two facts. First, each richest mathematician is a mathematician. Thus we apply the following rule:

(R) $[y = [{}^0Rich_{wt}\, {}^0Math_{wt}]] \vdash [{}^0Math_{wt}\, y]$.[13]

Second, the restricted quantifier *Some* can be defined by means of the quantifier $\exists$ in this way:

$$Some = \lambda c\, \lambda d\, \exists x\, [[c\, x] \wedge [d\, x]].$$

Types: $x, y \to \iota$; $c, d \to (o\iota)$.

Now in any world w at any time t at which the proposition constructed by C′14 takes value **T**, the following proof steps v-construct **T**:

1. $[[{}^0Some\, [{}^0Young\, {}^0\mathrm{Girl}]_{wt}]\, \lambda w \lambda t\, [\lambda x\, [{}^0Admire_{wt}\, \lambda w \lambda t\, [{}^0Rich_{wt}\, {}^0Math_{wt}]_{wt}\, x]]_{wt}]$
   assumption
2. $[[{}^0Some\, [{}^0Young\, {}^0\mathrm{Girl}]_{wt}]\, [\lambda x\, [{}^0Admire_{wt}\, [{}^0Rich_{wt}\, {}^0Math_{wt}]\, x]]$
   β_{i}-reduction
3. $\exists x\, [[[{}^0Young\, {}^0\mathrm{Girl}]_{wt}\, x] \wedge [{}^0Admire_{wt}\, [{}^0Rich_{wt}\, {}^0Math_{wt}]\, x]]$
   by Definition of *Some*
4. $\exists x\, [[[{}^0Young\, {}^0\mathrm{Girl}]_{wt}\, x] \wedge \exists y\, [[y = [{}^0Rich_{wt}\, {}^0Math_{wt}]] \wedge [{}^0Admire_{wt}\, y\, x]]]$
   $\exists$-generalisation
5. $\exists x\, [[[{}^0Young\, {}^0\mathrm{Girl}]_{wt}\, x] \wedge \exists y\, [[{}^0Math_{wt} y] \wedge [{}^0Admire_{wt}\, y\, x]]]$
   application of (R)

---

[13] The property of being a mathematician is a requisite of the office of the richest mathematician, see Section 4.1. Thus for any y it holds that whenever the left-hand side Composition v-constructs **T**, the right-hand side v-constructs **T** as well.

Thus from C′14 the meaning of 'There is a young girl such that some mathematician admires her' is validly inferable.

This example illustrates the connection between Criteria 1 and 2. The better (i.e., more detailed) analysis according to Criterion 2 makes it possible to infer more conclusions, and is thus better according to Criterion 1. We have suggested, however, that the claim that these criteria are equivalent would be a simplification. The hypothesis that whatever is better according to Criterion 1 is also better according to Criterion 2 is plausible (although it is doubtful whether this could actually be proved); but there are cases where a more fine-grained construction does not increase the number of inferences, or, still worse, where we get more than one candidate for the best analysis, which seems to contradict what has been proven in Materna and Duží (2005). Such cases show that our definition of Criterion 2 calls for qualification.[14] To make the problem crisper, we will reconsider the previous example of the sentence

'The biggest planet is smaller than the Sun.'

Our second analysis was

C2 $$\lambda w \lambda t\ [{}^0\mathit{Smaller}_{wt}\ [\lambda w \lambda t\ [{}^0\mathit{Biggest}_{wt}\ {}^0\mathit{Planet}_{wt}]]_{wt}\ {}^0\mathit{Sun}]$$

The sentence was said to talk about *Smaller*, *Biggest*, the *Biggest Planet* (*BP*), *the Sun* (and, of course, the proposition itself). But it seems that ***PP′*** demands including two additional objects. These objects are the property of being smaller than the Sun, which is predicated of the biggest planet, and the property of some *y* that the biggest planet is smaller than *y*, which is ascribed to the Sun.[15]

Let us construct the first property:

$$\lambda w \lambda t\ [\lambda x\ [{}^0\mathit{Smaller}_{wt}\ x\ {}^0\mathit{Sun}]].$$

Application of this property to the biggest planet yields

C2′ $$\lambda w \lambda t\ [\lambda w \lambda t\ [\lambda x\ [{}^0\mathit{Smaller}_{wt}\ x\ {}^0\mathit{Sun}]]_{wt}\ [\lambda w \lambda t\ [{}^0\mathit{Biggest}_{wt}\ {}^0\mathit{Planet}_{wt}]]_{wt}].$$

Obviously, C2 ≤ C2′ (as *per* Criterion 2), because C2′ arises from C2 by β-expansion.[16]

However, we must be careful. In the logic of partial functions β-reduction is not in general an equivalent transformation. We will deal with this problem in

[14] The proof mentioned in Materna and Duží (2005) implicitly presupposed this qualification.

[15] We are glossing over various subtleties due to topic-focus articulation. See Duží (2009).

[16] The β-expansion as known from the λ-calculi: $[P\ a\ b] \mid\!-\ [\lambda x\ [P\ x\ b]\ a]$, where $P \rightarrow (o\iota\iota)$, a, b, $x \rightarrow \iota$.

details in Sections 2.6 and 2.7. Here just briefly. β-reduction yields a non-equivalent construction in those cases when a substitution of a *v*-improper construction for a variable occurring in an intensional (*de dicto*) context, is involved. But C2′ is not such a case. Both the construction of the property of being smaller than the Sun, viz. $\lambda w\lambda t\ [\lambda x\ [{}^0Smaller_{wt}\ x\ {}^0Sun]]$, and the construction of the office of the biggest planet, viz. $\lambda w\lambda t\ [{}^0Biggest_{wt}\ {}^0Planet_{wt}]]$, occur with *de re* supposition in C2′. Hence, the β-reductions of C2′ yield equivalent constructions, among which are these:

$$\lambda w\lambda t\ [\lambda x\ [{}^0Smaller_{wt}\ x\ {}^0Sun]\ [{}^0Biggest_{wt}\ {}^0Planet_{wt}]]$$

$$\lambda w\lambda t\ [{}^0Smaller_{wt}\ [{}^0Biggest_{wt}\ {}^0Planet_{wt}]\ {}^0Sun]].$$

Thus C2 and C2′ are equivalent and C2′ seems to be also an admissible analysis of the sentence 'The biggest planet is smaller than the Sun'.

Moreover, there is another analysis C2″ such that C2 ≤ C2″, namely:

C2″ $\lambda w\lambda t\ [\lambda w\lambda t\ [\lambda y\ [{}^0Smaller_{wt}\ [\lambda w\lambda t\ [{}^0Biggest_{wt}\ {}^0Planet_{wt}]]_{wt}\ y]]_{wt}\ {}^0Sun]$.

Even worse, there is no admissible analysis C of the sentence such that C2′ ≤ C and C2″ ≤ C. So it would seem, paradoxically, as if the sentence had *two* best analyses. Yet this is not so. The sentence whose meaning is supposed to be C2′ is distinct from (albeit equivalent to) the sentence

'The biggest planet is smaller than the Sun'.

It is the sentence

'Being smaller than the Sun is a property of the biggest planet'.

Similarly the sentence whose meaning is C2″ is the sentence

'Being such that the biggest planet is smaller than it is a property of the Sun'.

Hence the best analysis of the sentence 'The biggest planet is smaller than the Sun' is the construction C2.

Now one could object that—***PP′*** notwithstanding—we could get a still better (because finer) analysis in such a way that we would replace some semantically simple expressions by the *definientes* of the respective definitions, where these expressions are supposed to be *definienda*. In our last example we could consider the possibility of defining *planet* in terms of *celestial body with such and such properties*. Our answer makes the analysis be the best one with respect to the literal meaning as defined in Definition 1.10: the *literal analysis* of an expression *E* is such an admissible analysis of *E* in which the objects that receive mention by simple meaningful subexpressions of *E* are constructed by their Trivialisations. In fact, our examples—to the extent that they were examples of the best analysis—were always examples working with literal analyses.

PP′ is a principle that we should obey. However, sometimes this is impossible unless the input expression is subjected to an equivalent reformulation. The reason is because natural languages, due to their spontaneous development, frequently use abbreviations (detectable by means of definitions) that leave out some semantically relevant information, which would elude a literal analysis. A literal analysis would, in any such case, be a shallow analysis and a far cry from being the best analysis.

To say that the replacement of a simple expression by a definition is a refinement is risky, though. It would be a refinement only if the resulting expression was equivalent to the original one (see the definition of *refinement* in Section 5.4). In a language involving mathematics, where linguistic definitions assigning complex meanings to syntactically simple expressions are frequent (e.g., 'prime' denoting the set of numbers with exactly two factors), such a refinement is possible. However, in the most interesting cases of empirical expressions (cf. the 2006 Prague redefinition of *planet*) we use a Carnapian *explication* rather than a definition proper, and then equivalence is surely not guaranteed. For instance, the mentioned definitional decomposition in terms of *celestial body with such and such properties* means that we have accepted a new *conceptual system*; i.e., a new set of simple concepts which are Trivialisations of non-constructions.[17] Thus we would find ourselves faced with diachronic analyses: a new (stage of the) language comes into being, for an explication—unlike a definition proper—involves in general a non-conservative extension of the language. We can apply the notion of conceptual system and make the analysis resulting from the application of an explication the best analysis with respect to the new conceptual system; but then we are comparing *two* languages, or two stages of the same language, both of which are a far more complicated matter.[18] Thus we will mostly apply the method of unveiling the *literal* meanings of expressions.

2.2 Concepts as procedural meanings

2.2.1 Concepts and synonymy

When comparing contemporary and traditional textbooks, one is likely to come away with the impression that contemporary logic is no more interested in studying concepts. True, already in 1837 the psychologistic tradition of construing

[17] For the definition of *conceptual system*, see Section 2.2, Definition 2.14.

[18] For further discussion, see Materna (2004a).

concepts as a sort of mental objects (and thus of nil interest to logic) was dealt a serious blow by Bolzano, who worked out, in his *Wissenschaftslehre*, a systematic realist theory of concepts. In Bolzano concepts are construed as objective entities endowed with structure. But his ingenious work was not well-known at the time when modern logic was founded by Frege and Russell. Thus the first theory of concepts that was recognized as being compatible with modern, entirely anti-psychologistic logic was Frege's (1891) and (1892b). We will show that TIL makes it possible to construe concepts as logical objects and avoid, at the same time, the problems hampering the Fregean conception. In so doing we will draw on insights from Bolzano.

Frege's theory, as found in 1891 and 1892b, construes concepts as a kind of *function*. A Fregean concept is a function whose arguments are objects (*Gegenstände*) and whose values are truth-values. This definition seems to be rather intuitive. The concept of *dog* could be such a function: for such objects as are dogs its value would be **T**, for all other objects it would be **F**. Yet this conception is open to several objections.

First, there are concepts that cannot be conceived of in this way, since they are *not general*. Such singular concepts—like THE RICHEST MAN, THE HIGHEST MOUNTAIN, THE PRESIDENT OF THE USA, THE SUM OF 3 AND 3, THE SUCCESSOR FUNCTION, etc., etc.[19]—cannot be represented in Frege's theory, since each of them has to be replaced by the respective singleton, so we would get sets, the only member of which would be the respective object. But to claim that the richest man is married is not to claim that the respective set is married.

Second, the Fregean definition of concept is strongly counterintuitive in *empirical* cases, as his (characteristic) function of the respective set determines distinct sets for distinct populations. In a word, no temporally sensitive intensionality is forthcoming; as soon as some dog dies the concept of dog changes; i.e., during the development of the populations of dogs there are as many distinct concepts as there are distinct populations. This cannot be right, for concepts ought not to be susceptible to empirical vicissitudes.

Third, the way Frege defines function is *not unambiguous*. This has been convincingly shown in Tichý (1988), where Frege's oscillation between taking functions as mappings and something like rules for forming mappings is documented. Roughly, this oscillation is between functions-in-extension and functions-in-intension.[20] The former interpretation is open to the additional objection that the concept of any class *C* would be identical with *C* (as far as the characteristic function of a class does not essentially differ from the class itself). Our natural intuition associated with the use of the expression 'concept' suggests, however, that there may be more distinct concepts of one and the same class. This is an important point: we would expect concepts to be *ways to an object* rather than the object

[19] Capitals indicate that the expression represents a concept (concepts being whatever).

[20] See Section 1.3.1 for discussion of the notion of function-in-intension, especially as to how it relates to constructions.

itself. For a time-honoured example, the same set of geometrical figures can be equally well conceptualized as a set of *triangular figures* (*triangles*) or a set of *trilateral figures.* Put differently, concepts are reasonably expected to be *conceptualizations* of objects rather than the conceptualized objects themselves. Yet Frege consistently locates concepts, in his well-known schema, on the level of *Bedeutung* and not on the level of *Sinn*. As we point out below, Church anticipated this objection when replacing Frege's definition of concept by his own proposal.

Fourth, in Frege (1892b) the following problem is addressed. If concepts are Fregean functions, i.e., 'unsaturated entities', then the expression ('Begriffswort') that denotes a concept should never stand in the position of grammatical subject. A grammatical subject should stand for an *object* (*Gegenstand*), whereas concepts are functions; hence, a concept is no object from the viewpoint of Frege's dichotomy between *Gegenstand* and *Funktion (Begriff).*

As Frege's contemporary opponents pointed out,[21] the problem is that there are counterexamples to the dichotomy. Consider the sentence

'The concept of horse is a zoological concept.'

Any case of *mentioning* concepts seems to refute Frege's claim. Frege's answer is interesting, though. If the respective *Begriffswort* stands in subject position it no more denotes a concept: it denotes an object![22]

Well, this solution is possible as soon as we distinguish between Frege's notion of function as an unsaturated entity and his notion of *Wertverlauf.* The former is far from being clear but *might* be taken to be a notion of procedure or function-in-intension, as suggested above, whereas the latter seems to correspond to the notion of *function as a mapping* or function-in-extension. Yet the intuition that a concept remains a concept whether *used* or *mentioned* (to use TIL terminology) has much to be said for it, since it seems that Frege's concept of horse vacillates between being a concept and being an object only relative to a flawed theory of concepts.

In his (1956) Church tries to adhere to Frege's principles of semantics, but comes to realize that Frege's explication of the notion of concept is untenable. Concepts should be located on the level of Fregean *sense*—in fact, as Church maintains, the *sense* of an expression *E* should be a *concept* of what *E* denotes. Consequently, concepts should be associated not only with general ('predicate-like') expressions (as was the case with Frege), but with any kind of expression, since all kinds of expression are associated with a sense. Even sentences express concepts; in the case of empirical sentences the concepts are concepts of propositions ('proposition' as understood by Church, as a concept of a truth-value, and not as understood in this book, as a function from possible worlds to (functions from times to) truth-values).

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[21] Not least Benno Kerry, whose criticism is responded to in Frege (1892b).

[22] See Frege (1892b).

The degree to which 'intensional' entities, and so concepts, should be fine-grained was of the utmost importance to Church.[23] When summarising Church's heralded Alternatives of constraining intensional entities, Anderson (1998, p. 162) canvases three options considered by Church. Senses are identical if the respective expressions are (A0) 'synonymously isomorphic', (A1) mutually λ-convertible, (A2) logically equivalent. (A2), the weakest criterion, was refuted already by Carnap (1947), and would not be acceptable to Church, anyway. (A1) is surely more fine-grained. However, partiality throws a spanner in the works: β-reduction is not guaranteed to be an equivalent transformation as soon as partial functions are involved (see Section 2.7). The alternative (0) arose from Church's criticism of Carnap's notion of intensional isomorphism and is discussed in Anderson (1980) Carnap proposed intensional isomorphism as a criterion of the identity of belief. Roughly, two expressions are intensionally isomorphic if they are composed from expressions denoting the same intensions *in the same way*.

Church (1954) constructs an example of expressions that are intensionally isomorphic according to Carnap's definition (i.e., expressions that share the same structure and whose parts are necessarily equivalent), but which fail to satisfy the principle of substitutability.[24] The problem Church tackled is made possible by Carnap's principle of tolerance (which itself is plausible). We are free to introduce into a language *syntactically simple* expressions which denote the same intension in *different ways* and thus fail to be synonymous. Yet they are intensionally isomorphic according to Carnap's definition. Church used as an example of such expressions two predicates P and Q, defined as follows: $P(n) = n < 3$, $Q(n) = \exists xyz\ (x^n + y^n = z^n)$, where x, y, z, n are positive integers. P and Q are necessarily equivalent, because for all n it holds that $P(n)$ if and only if $Q(n)$. For this reason P and Q are intensionally isomorphic, and so are the expressions '$\exists n\ (Q(n) \wedge \neg P(n))$' and '$\exists n\ (P(n) \wedge \neg P(n))$'. Still one can easily believe that $\exists n\ (Q(n) \wedge \neg P(n))$ without believing that $\exists n\ (P(n) \wedge \neg P(n))$.[25]

Church's conception of Alternative (0), as reproduced in Anderson (1980, p. 221), amounts to this:

> Synonymous isomorphism can be properly defined as holding between closed well-formed formulas (*cwffs*) (of the same type) A and B if A can be obtained from B by a sequence of (zero or more) steps consisting of (1) replacement of constants by *cwffs* with which they are stipulated to be synonymous, and *vice versa*, or (2) alphabetic change of bound variables.

Church's Alternative (1) characterizes synonymous expressions as those that are λ-convertible.[26] But, Church's λ-convertability includes also β-conversion,

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[23] Now we are using Church's terminology; in TIL concepts are hyperintensional entities.

[24] See also Materna (2007).

[25] Criticism of Carnap's intensional isomorphism can be also found in Tichý (1988, pp. 8–9), where Tichý points out that the notion of intensional isomorphism is too dependent on the particular choice of notation.

[26] See Church (1993, p. 143).

which goes too far due to partiality. Church also considered Alternative (1′) that includes η-conversion. In TIL we define synonymy on the basis of *procedural isomorphism* (see Definition 2.3 below) which would be closer to Alternative (1′) without unlimited β-conversion.

Church's concept is a *way* to the denotation rather than a special kind of denotation. There are not only general concepts, and more concepts can identify one and the same object (notice that Church says that *the* sense is *a* concept of the denotation (ibid., p. 6)). What would we, as realists, say about this connection between *sense* and *concept*? Accepting, as we do, Church's version as an intuitive one, we claim that

senses are concepts.

Can we, however, claim the converse? This would be:

concepts are senses.

A full identification of senses with concepts would presuppose that every concept were the meaning of some expression. But then we could hardly explain the phenomenon of historical evolution of language, first and foremost the fact that new expressions are introduced into a language and other expressions vanish from it. Thus with the advent of a new ⟨expression, meaning⟩ pair a new concept would have come into being. Yet this is unacceptable for a realist: concepts, *qua* logical entities, are *abstract* entities and, therefore, cannot come into being or vanish. Therefore, concepts outnumber expressions; some concepts are yet to be discovered and encoded in a particular language while others sink into oblivion and disappear from language, which is not to say that they would be going out of existence. For instance, before inventing computers and introducing the noun 'computer' into our language(s), the procedural design of a computer architecture that von Neumann made explicit was already around. The fact that in the nineteenth century we did not use (electronic) computers, and did not have a term for them in our language, does not mean that the concept (*qua* procedure) did not exist. In the dispute over whether concepts are discovered or invented we come down on the side of discovery.

The view that *meanings must be structured* has been gaining ground as of late.[27] TIL accommodates this view by conceiving of meanings as constructions. Are we prepared to identify concepts with constructions?

Not quite. Compare the expressions

'my father'

and

'father'.

[27] See, e.g., Cresswell (1975, 1985) and Moschovakis (1994).

No great problem arises as for the latter expression: we would not hesitate to associate it with a concept. As for the former expression, we would, of course, hesitate: there is a pragmatic parameter present ('my') and we would probably speak of a concept only after identifying a speaker so as to establish a link from that individual to the individual who is the speaker's father. When analysing expressions of this kind, we must take into account also indexical factors. We need to let the respective construction contain a *free variable*, because the meaning of 'my father' is *pragmatically incomplete* (see Section 3.4).

Then, however, we cannot say that the meaning of 'my father' is a procedure which produces (as a construction that constructs, or a concept that conceptualizes) a definite object. The free variable awaits valuation, and (as shown in Section 3.4) the meaning of an indexical expression constructs a definite object only after the situation- or context-dependent valuation has done its job. We will identify concepts only with procedures which produce definite objects *independently* of pragmatic factors. A concept must not contain free variables, so a concept is a *closed* construction. Our first preliminary characterisation of a concept is:

Concepts are closed constructions.

It is hopefully obvious at this point why a closed construction is a plausible candidate for being a concept. A closed construction is a structured, abstract procedure which, when executed, yields an object, or in well-defined cases fails to. Therefore, we can say that every concept is a closed construction. However, and this is important, constructions are a bit too fine-grained from the procedural point of view. Some closed constructions differ so slightly that they are *virtually* identical. In a natural language we cannot even render their distinctness, which is caused by the role of λ-bound variables that lack a counterpart in natural language.

Compare these two constructions of the set of positive numbers:

$$\lambda x\ [{}^0\!>\ x\ {}^0 0]$$

and

$$\lambda y\ [{}^0\!>\ y\ {}^0 0].$$

Barring professional jargon, the procedural difference between them cannot be distinguished in an ordinary natural language. We will say that they are *procedurally isomorphic.* A similar case is the ontological counterpart of η-reduction in the λ-calculus. Compare the η-equivalent constructions

$${}^0Believe;\ \ \lambda w\ [{}^0Believe\ w];\ \ \lambda w \lambda t\ {}^0Believe_{wt};\ \ \lambda w \lambda t\ \lambda xy\ [{}^0Believe_{wt}\, x\ y].$$

Though the number of steps to be executed is increasing, the additional instructions to Compose a construction with variables and abstract over these same variables make for insignificant differences in terms of what procedure is prescribed. All four constructions share the common property of constructing the object *Believe* without the mediation of any other closed constructions. Again, the four constructions above are procedurally isomorphic.[28]

These considerations motivate the following definition.

Definition 2.3 (*procedural isomorphism*) Let C, D be constructions. Then C, D are *α-equivalent*, denoted '${}^0C \approx_\alpha {}^0D$', $\approx_\alpha/(o*_n*_n)$, iff they *v*-construct the same entity and differ at most by using different λ-bound variables. C, D are *η-equivalent*, denoted '${}^0C \approx_\eta {}^0D$', $\approx_\eta/(o*_n*_n)$, iff one arises from the other by η-reduction or η-expansion. C, D are *procedurally isomorphic* iff there are constructions $C_1,\ldots, C_n$ $(n > 1)$ such that ${}^0C = {}^0C_1$, ${}^0D = {}^0C_n$, and each C_i, C_{i+1} are either α- or η-equivalent. □

Examples. ${}^0[\lambda x\, [{}^0\!> x\ {}^00]] \approx_\alpha {}^0[\lambda y\, [{}^0\!> y\ {}^00]]$; ${}^0[\lambda xy\, [{}^0\!+ x\ y]] \approx_\eta {}^{00}\!+$.
(Types: $+/(\tau\tau\tau)$; $x, y \to \tau$.)

Still it is a problem how *concept* is to be defined. A closed construction is a good candidate, but it is a bit too fine-grained. Materna (1998, p. 96) defined an equivalence relation over the set of closed constructions, dubbing it the relation of *quasi-identity* ($Quid/(o*_n*_n)$). This relation is induced by α- and η- transformations: the closed constructions C, C' are *quasi-identical* (*Quid*-related) iff they are either identical or procedurally isomorphic.

Since *Quid* is reflexive, symmetric and transitive, it defines an equivalence class. Materna, also in 1998, identified a *concept* with the respective equivalence class. A *concept generated by* a closed construction C was the set constructed by $\lambda c\ [{}^0Quid\ c\ {}^0C] \to (o*_n)$, c ranging over closed constructions. The drawback, however, of this solution is obvious: a concept was construed as a *set*, an outcome that is in direct opposition to concepts being structured procedures and not mere set-theoretic entities. This problem can be overcome by exploiting the difference between using and mentioning constructions (see Section 2.6). Briefly, a construction C is used in a construction D (in order to construct an object) if C itself is not an object on which another construction operates. Otherwise—i.e., if C is an object on which another construction D' operates—C is mentioned in D. Since particular members of a *Quid* class are procedurally isomorphic, it does not matter which of them is used when we need to construct an object. What is relevant about the construction is not the construction itself but only what it constructs. Thus when using a concept we may use any member of the respective *Quid*-equivalence

[28] Slightly different definitions of procedural isomorphism are thinkable. We are also considering whether it might be philosophically wise to adopt several notions of procedural isomorphism. It is not at all improbable that several degrees of hyperintensional individuation are called for, depending on exactly which sort of hyperintensional context happens to be analyzed. This pluralistic approach ties in with our considerations in Section 1.3.1 regarding Church's open-ended characterization of *function-in-intension* (See Jespersen (2010) for the latest statement of our position).

class. However, when mentioning a concept we are talking about the whole class: we construct the whole class by generating it from any representative of the mentioned class.

The solution that Horák puts forward in 2002 is based on exploiting the *Quid* relation to define a *normalization* procedure resulting in the unique *normal form* of a *construction* C: $NF(C)$. If this procedure is applied to a closed construction C, the result, $NF(C)$, is the *simplest* member of the *Quid* equivalence class generated by C. The simplest member is defined as the alphabetically first, non-η-reducible construction. For every closed construction C it holds that $NF(C)$ is the *concept induced by* C, the other members of the same equivalence class *pointing to* this concept. In this manner Horák's solution makes it possible to define *concepts* as *normalized closed constructions*. Their type is always $*_n$, $n \geq 1$.

For instance, the following constructions are procedurally isomorphic and thus belong to the same *Quid* class (a Materna-style concept of the successor function):

$$\lambda x\,[{}^0{+}\ x\ {}^01];\ \lambda y\,[{}^0{+}\ y\ {}^01];\ \lambda z\,[{}^0{+}\ z\ {}^01];\ \lambda x\,[\lambda x\,[{}^0{+}\ x\ {}^01]\ x];\ \lambda y\,[\lambda x\,[{}^0{+}\ x\ {}^01]\ y].$$

The normal form of these constructions is $\lambda x\,[{}^0{+}\ x\ {}^01]$. Thus, $\lambda x\,[{}^0{+}\ x\ {}^01]$ is a Horák-style concept of the successor function.

Since Horák's solution is more plausible than the previous solution offered by Materna, we adopt this definition:

Definition 2.4 (*concept*) A *concept* is a closed construction in its normal form.

So, in general, the meaning of an expression is a construction. If an expression contains indexicals its meaning is an open construction; the meaning of a non-indexical expression is a concept.

Having decided in favour of construing concepts as closed constructions, we can define some special categories of concepts. First:

Definition 2.5 (*simple concept*) Let X be an object that is not a construction. Then ${}^0X$ is a *simple concept of* X. Let $x/*_n \rightarrow \alpha$ be a variable. Then $[\lambda x\ x]$ is a *simple concept of the identity function* of type $(\alpha\alpha)$. □

Every worthwhile theory of procedures must eschew infinite regress. Some procedures, and so some concepts, must figure as primitive and be understood pre-theoretically, thus defying definition. The notion of conceptual system (see Section 2.2.3) is based on the difference between simple and compound concepts. According to Definition 2.5, a simple concept constructs an object without drawing upon other concepts.

An important class of simple concepts contains concepts of the form ${}^0X$, where X is a 1st-order object. When conceptually analysing a given area of interest, we must choose an initial collection of simple concepts that are intuitively understood and can't and won't be further refined within the conceptual system relative to which they are simple. Of course, the level of simplicity depends on the analysed area. For instance, ^{0}*Car*, ^{0}*Road*, ^{0}*Junction* might be simple concepts of the formal ontology of a standard traffic system. However, the ontology of a traffic police system might be much more detailed, using instead *ontological definitions* (roughly, definitions of entities rather than of words) Composed of much finer simple concepts, like ^{0}*Motor_vehicle*, ^{0}*Amphibious_vehicle*, ^{0}*Disabled_car*, ^{0}*T_Junction*, ^{0}*Y_Junction*, ^{0}*Traffic_circle*, ^{0}*High_Way*, ^{0}*Lane*, ^{0}*Road_element*, etc.[29]

The category of simple mathematical concepts is not without theoretical problems. To illustrate the nature of these problems, consider the simple concepts ^{0}*Prime* and ^{0}*Natural_number*. They both construct an infinite class of numbers, though in a not particularly illuminating way. Furthermore, no one can execute an instruction consisting in directly accessing and delivering an actual infinity in a finitary manner. We need a more fine-grained *definition* of these objects, which comes down to a compound concept of them. The definition of *prime number* is well-known. However, since Hilbert mathematicians had been devoting much effort to developing an analytic theory that would fully define the set of natural numbers in a finitary way, until Gödel proved the futility of this endeavour. In Duží and Materna (2004) we characterised the difference between synthetic and analytic concepts a priori in such a way that the former are closed constructions involving actual infinity and the latter potential infinity at most. Thus simple mathematical concepts of infinite classes come out synthetic concepts a priori. In order to make good use of them in mathematics, mathematicians aim at discovering one or more of their compound analytical equivalents that define potentially infinite classes.[30] However, as stated above, some concepts must be chosen as primitive and pre-theoretically understood, which means that they are not susceptible to further refinement.

Moreover, it is impossible to find analytical equivalents to concepts of non-recursive functions.[31] The conceptual specification of a non-recursive function is in principle ineffective. A simple example can be found in (Kleene, 1952, p. 317). Let R be a binary effectively computable relation. Let the function $\lambda x\, \varepsilon y\, \mathrm{R}(x,y)$ be defined as follows:

$\varepsilon y\, \mathrm{R}(x, y) =$ (i) the least number y such that $\mathrm{R}(x, y)$ if $\exists y\, \mathrm{R}(x, y)$,

(ii) 0 otherwise.

Obviously, unless it holds that $\forall x\, \exists y\, \mathrm{R}(x, y)$, the function $\lambda x\, \varepsilon y\, \mathrm{R}(x, y)$ is not recursive. The above definition does not provide an effectively executable

[29] The topic of definition will be dealt with in Section 2.2.2, where *ontological definition* is defined.

[30] We will deal with the problem of mathematical concepts in Section 3.2.1.

[31] Classically, recursive functions are *mappings* for which there is an algorithm (Turing machine and suchlike) that determines for any argument the value. Intuitionistically, recursive functions are the algorithms themselves.

prescription of obtaining a value of this function, although the definition is precise and unambiguous.

Second, a procedure producing nothing is no less a procedure for it, just like a road to nowhere is still a road, only one lacking a destination or terminal point. The same holds for concepts: some concepts fail to conceptualise anything, yet are no less concepts for it. This stance on empty concepts falls quite naturally out of our general top-down approach and realism *ante rem.* Empty concepts are needed as meanings of those mathematical expressions that lack a denotation. They express a closed construction that is improper. Thus we define:

Definition 2.6 (*strictly empty concept*) A concept C is *strictly empty* iff C is improper. □

Example. The concept THE GREATEST REAL NUMBER is strictly empty; the Composition $[{}^0Sing\ \lambda x\ [\forall y\ [{}^0\geq x\ y]]]$ is improper, because the class constructed by $\lambda x\ [\forall y\ [{}^0\geq x\ y]]$ is empty.

We shall say that a class K is *empty* if its characteristic function, of type $(o\alpha)$ or $(o\beta_1\ldots\beta_n)$, is not true at any argument (i.e., its value is either **F** or *undefined* at all arguments).

Definition 2.7 (*quasi-empty concept*) A *concept* is *quasi-empty* iff it constructs an empty class. □

Consider the concept EVEN PRIME NUMBERS GREATER THAN 2. It constructs the empty class of numbers:

$$\lambda x\ [[{}^0Even\ x] \wedge [{}^0Prime\ x] \wedge [{}^0> x\ {}^02]].$$

The distinction between strictly empty and quasi-empty concepts is bound up on the fact that empty classes are objects (*qua* classes) whereas there is no such thing as ‘empty particulars’ (like numbers and individuals).

It might seem that simple concepts could be neither strictly nor quasi-empty, because emptiness arises due to the application of a function f at an argument a at which f is undefined. However, while indeed no simple concept can be strictly empty, it can be quasi-empty. Quasi-empty simple concepts arise because an empty α-class $\varnothing_\alpha/(o\alpha)$ is an object, and as such can be Trivialised. Thus ${}^0\varnothing_\alpha$ is a quasi-empty simple concept.[32]

On the other hand, a Composition or another construction involving a Composition can be improper. For instance, ‘dividing 5 by 0’ expresses a strictly empty concept, to wit $[{}^0{:}\ {}^05\ {}^00]$. Only this concept is not simple.

The other source of improperness can be wrong typing or Double Execution. For instance, the Double Execution ${}^2[{}^1[{}^0+\ {}^02\ {}^05]]/*_3$ is improper (though well-typed),

[32] We will deal with improper constructions and non-existence in Section 2.3.

because $^1[^0{+}\ ^02\ ^05]/{*_2}$ constructs what $[^0{+}\ ^02\ ^05]/{*_1}$ constructs, which is the number 7. This Double Execution prescribes that this number should be executed, which is impossible; only constructions can be executed.[33] It might be helpful to spell out in prose how to read $^2[^1[^0{+}\ ^02\ ^05]]$. Read it thus: execute the Composition $[^0{+}\ ^02\ ^05]$ twice over; i.e., execute the result of the single Execution $^1[^0{+}\ ^02\ ^05]$ that consists in the application of the function + at the argument $\langle 2, 5\rangle$.

Third, *empirical* expressions denote non-constant α-intensions and as such express non-empty concepts. By 'non-constant α-intension' we mean a function f of type $\alpha_{\tau\omega}$ such that there are world/time pairs $<w, t>$, $<w', t'>$ for which it holds that $[[f\,w]\ t] \neq [[f\,w']\ t']$.[34]

Definition 2.8 (*empirical concept*) C is an *empirical concept* iff it constructs a non-constant intension. □

A non-empirical, non-empty concept constructs either an extension or a constant intension or another (lower-order) construction. It cannot construct a non-constant intension. Examples of a constant intension being constructed by a non-empirical concept would be the one denoted by 'Whales are mammals' or 'Bachelors are men' (cf. Definition 1.9).

Claim 2.2 No simple concept is strictly empty.

Proof. The Trivialisation of an entity X of any type constructs X and is thus never improper. The Closure $[\lambda x\ x]$ constructs an identity function, and is thus not improper.

Claim 2.3 No empirical concept is strictly empty.

Proof follows directly from Definition 2.8.

In general, there are quasi-empty empirical concepts. The existence of quasi-empty empirical concepts is due to intensions being *partial functions.* Thus there are non-constant intensions of type $(o\omega)$ which are empty classes of possible worlds. For instance, let P be a class of type $(o\omega)$ such that for no possible world w does the Composition $[^0P\ w]$ v-construct **T**, while there are possible worlds w_1, w_2 such that $[^0P\ w_1]$ constructs **F** and $[^0P\ w_2]$ is improper. Then the simple concept ^0P is an empirical quasi-empty concept.

Definition 2.9 (*empirically empty concept*) C is an *empirically empty concept at* $\langle w, t\rangle$ iff the intension it constructs either (a) lacks a value *at* $\langle w, t\rangle$ or (b) its value *at* $\langle w, t\rangle$ is an empty class. □

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[33] Execution and Double Execution raise the type order; see Definition 1.7, C_n (ii) and T_{n+1} (i).

[34] As we showed in Section 1.4.2, some empirical expressions denote non-constant functions having an *essential core*.

Examples. The concept expressed by 'The King of France'—i.e., $\lambda w\lambda t\ [^0King\_of_{wt}\ ^0France] \rightarrow \iota_{\tau\omega}$—is empirically empty in the actual world now, because the office of King of France is vacant. The concept ^0Moa is empirically empty in the actual world now, because the population of Moa birds is empty (Types: $King\_of/(\iota\iota)_{\tau\omega}$; $France/\iota$; $Moa\ /(o\iota)_{\tau\omega}$).

It may come to pass that a concept *C* is empirically empty in all worlds *w* at all times *t.* Russell's barber ('Bertie the barber', perhaps?) falls under clause (a) in Definition 2.9., for no world and no time boasts a barber who shaves all and only those who do not shave themselves.[35] An example of (b) would be the property of being King of France without being a king. There is a noteworthy difference between (a) and (b). (a) offers only the distinction between an office being occupied or vacant, whereas (b) offers the distinction between a characteristic function being **T**, **F** or undefined. Therefore, whenever *C* is a concept of a necessarily vacant office, *C* is not an empirical concept and constructs a constant function; namely, a degenerate office, which goes vacant at all worlds and times. Whether *C* constructs such an office can be established a priori, since if it does it is an analytic truth that it does. Not so with (b). It must be established individually for any given world/time pair whether the characteristic function yields **F** or *undefined* depending on whether the King of France exists in a world *w* at a time *t,* thus the concept of the property of being King of France without being a king is an empirical concept.

So far *concept* and various subsidiary notions have been defined. Now we are in a position to define three key relations obtaining between expressions. The point of all three definitions is to lay down the exact calibration of synonymy, equivalence and co-reference.

In what follows we will consider only expressions with a complete meaning, i.e., expressions without indexicals. As we will show in Section 3.4.1, expressions with indexicals have what we call a pragmatically incomplete meaning, which is an open construction. A situation of utterance must complete their meaning by providing a valuation of the free variable(s) occurring in the respective construction. Due to their being incomplete, any two incomplete meanings cannot be compared as for identity and equivalence; only co-reference is an option. Once values have been assigned to the free variables relative to a context, the resulting constructions can indeed be compared, but then they are no longer open constructions, hence no longer pragmatically incomplete senses.

First we recapitulate the principles of logical analysis as presented above. An admissible analysis of an expression *E* is a construction *C* that complies with the Parmenides principle; there is no closed subconstruction of *C* that constructs an object that does not receive mention by *E.* In other words, each closed subconstruction of *C* constructs an object denoted by a subexpression of *E*; or in case

[35] Thus Russell's barber is a *degenerate* individual office, vacant as it is in all worlds and at all times.

there is no such object, the subconstruction complies with the type-theoretical conditions imposed by *E*. There are many admissible analyses of *E*, which are more or less fine-grained. Among these our task is to select the best analysis, which is the most fine-grained one. To make this selection possible, we introduced a method of analysis. This method lays down the rule that the constructions of all the objects mentioned by *E* are to be combined so that *C* *v*-constructs the object *D* (if any) denoted by *E*.

However, our method of analysis does not prescribe the form of the relevant subconstructions that are combined into the meaning of *E*. The method imposes only the demand that particular closed subconstructions should construct the objects that receive mention in the analysed expression. But these objects can again be constructed in more or less fine-grained ways. In order to obtain a *unique* analysis better than all the rest, we introduced in Definition 1.10 the notion of literal meaning. This definition imposes the demand that the objects that receive mention by semantically *simple* meaningful subexpressions (lexica) should be constructed by their respective Trivialisations.

Here is an example. The application of this method to 'the richest bachelor' yields the Closure

$$\lambda w \lambda t\ [{}^0 Richest_{wt}\ {}^0 Bachelor_{wt}]$$

Types: $Richest/(\iota(o\iota))_{\tau\omega}$; $Bachelor/(o\iota)_{\tau\omega}$.

Yet we may still refine this Closure by means of an ontological definition of the property *Bachelor*.[36] If the meaning of 'is a bachelor' is 'is an unmarried man' *ex definitione*, then

$$\lambda w \lambda t\ [{}^0 Richest_{wt}\ \lambda x\ \neg[[{}^0 Married\ {}^0 Man]_{wt}\, x]]$$

is also an admissible analysis of 'the richest bachelor'.
Additional types: $x \rightarrow \iota$; $Married/((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; $Man/(o\iota)_{\tau\omega}$.

The decision as to which of the two Closures is to qualify as the literal analysis of 'the richest bachelor', and thus the concept expressed by this expression, depends on the vernacular being analysed. Either 'is a bachelor' is a semantically *simple* expression, or else it is a semantically *complex* expression, because it is an abbreviation of 'is an unmarried man'. In the former case the literal meaning of 'is a bachelor' is ${}^0 Bachelor$; in the latter case, $\lambda w \lambda t\ \lambda x\ \neg[[{}^0 Married\ {}^0 Man]_{wt}\, x]$. Since we do not analyse different vernaculars diachronically but only synchronically, our method yields in both cases the unique literal analysis *C* of an expression *E*. Such a construction *C* is assigned to the expression *E* as its literal meaning, and we shall say that *E expresses the concept NF*(*C*).

The definitions of *synonymous*, *equivalent* and *co-referential expressions* are now as follows.

[36] See Section 5.4, Definition 5.5 of *meaning refinement*.

Definition 2.10 (*synonymous expressions*) Expressions E_1, E_2 are *synonymous* iff they both express one and the same concept. □

Definition 2.11 (*equivalent expressions*) Expressions E_1, E_2 are *equivalent* iff they denote one and the same object. □

Definition 2.12 (*co-referential expressions*) Expressions E_1, E_2 are *co-referential* iff they denote intensions whose values are the same in the actual world at the present time. □

Remark. Obviously, synonymous expressions are equivalent, whereas merely equivalent expressions are not synonymous, and equivalent expressions are co-referential, whereas merely co-referential expressions are not equivalent.

Remark. Whereas *reference*, without qualification, is reference in the actual world at the present time, it is an option to qualify *reference* with respect to a particular world and time, as in 'E refers to X in W_n at T_m', which presupposes an ordering of worlds and times to furnish n, m with values.

Remark. In Definition 2.2 (in Section 2.1) we defined *synonymy with respect to a meaning attribute m.* Thus Definition 2.10 defines *synonymy* in terms of *concept* being construed as m. Definition 2.11 corresponds to 'synonymy with respect to denotation' conceived as m, and finally Definition 2.12 corresponds to 'synonymy with respect to reference' conceived as m. Needless to say, we adhere to *synonymy* as synonymy with respect to *concept*.

For example, the simple English predicates 'is azure' and 'is sky-blue' are *synonymous*. The concept expressed by either of them is the Trivialization of the property of being azure; i.e., of the property of being sky-blue, regardless of the name used for this property. Thus ${}^0Azure$ and ${}^0Sky\_blue$ are not two merely equivalent constructions, but one and the same construction. In the case of complex expressions, synonymy is often a matter of a syntactic reformulation that is not semantically reflected, as in 'Charles wants Peter to go away' as opposed to 'Charles wants that Peter should go away'. In mathematics, synonymy is often introduced by a definition that assigns the meaning of a complex expression to a newly introduced abbreviation (usually a simple expression).[37] For instance, definitions like 'π is the ratio of a circle's circumference to its diameter', 'Factorization is the decomposition of a number into a product of other numbers', or 'An irrational number is a number which cannot be expressed as a fraction m/n, m and n

[37] We will deal with definitions in Section 2.2.2.

integers, *n* non-zero' make the respective *definiendum* and *definiens* synonymous.[38]

Examples of *equivalent* expressions would be 'is higher' versus 'is neither smaller nor equally high', 'is a father' vs. 'is a male parent', 'It is not true that smoking is forbidden and drinking is allowed' vs. 'Smoking is not forbidden or drinking is not allowed', etc.

Note that all true mathematical sentences are equivalent, and similarly for all false mathematical sentences. For a mathematical example, consider, for instance, the mathematical constant 'π'. As we show in Section 3.2.1, its meaning depends on the mathematical vernacular in use. In one vernacular 'π' is a semantically *complex* expression synonymous with 'the ratio of a circle's area and its radius squared', its literal meaning being

$$[\iota x\, \forall y\, [x = [{}^0\mathit{Ratio}\, [{}^0\mathit{Area}\ y]\ [{}^0\mathit{Square}\, [{}^0\mathit{Radius}\ y]]]]].$$

In another vernacular 'π' is synonymous with 'the ratio of a circle's circumference to its diameter', its literal meaning being

$$[\iota x\, \forall y\, [x = [{}^0\mathit{Ratio}\, [{}^0\mathit{Circumference}\ y]\ [{}^0\mathit{Diameter}\ y]]]].$$

And in still another vernacular its literal meaning is a π-calculating algorithm. It is also thinkable that 'π' is a semantically *simple* expression, whose literal meaning is ${}^0\pi$. Thus the concept expressed by 'π' is one of those literal meanings depending on which particular mathematical vernacular is under scrutiny. All these semantic variants of 'π' are equivalent, but, of course, not synonymous.

As for *co-referential* expressions, any two actually and presently true empirical sentences are co-referential. Another example of co-referential expressions is Frege's famous 'Morning Star' vs. 'Evening Star' example. This example creates this sort of puzzle only if at least one of the terms is not a *proper* proper name: if they were both descriptively naked names, then from a semantic point of view it would make no sense for Frege to raise the question, 'Why is the judgement that the Evening Star is the Morning Star more informative than the judgement that the Evening Star is the Evening Star?'. The question would then not concern the semantics of two terms, but the linguistic competence of language users. The question would be whether a language user masters two distinct proper names belonging to a given language. The way to keep Frege's puzzle afloat is to treat at least one of the terms 'The Morning Star' and 'The Evening Star' as a hidden description. At least one of the terms needs to denote some condition that an object (individual) has to fulfil in order to occupy the individual office denoted by the hidden definite description. In the interest of parity, both terms may be construed as hidden definite descriptions denoting two separate individual offices. The path to the respective conditions could be something like THE BRIGHTEST CELESTIAL BODY IN

[38] For the semantics of mathematical constants, see Section 3.2.1.

THE MORNING/EVENING SKY. So either expression expresses a concept and at least one denotes a condition (an intension of type $\iota_{\tau\omega}$). Venus is not the shared co-denotation of these two terms. If only one term denotes an office, then the other term denotes Venus. If both terms denote each their own office, then neither of them denotes Venus. It so happens, given the actual celestial scheme of things, that Venus figures as *referent* of both expressions. So two options: either we have an instance of 'The *F* is the *G*' or an instance of '*a* is the *F*'. In the first case the two terms are *co-referential*. In neither case are the terms equivalent.[39]

Frege's schema contains Venus on the level of denotation (*Bedeutung*). So for Frege 'The Evening Star', 'The Morning Star' and 'Venus' would qualify as equivalent expressions according to Definition 2.11.

2.2.2 Concepts and definitions

Consider the compound concept expressed by the predicate 'is a natural number with exactly two factors'. This concept constructs something by means of other concepts, like ^0Factor. In this way this concept *defines* something. *What* is defined? The set of natural numbers which is standardly known as 'prime numbers'. Such compound concepts which define extra-linguistic entities will be called 'ontological definitions'. But what are the respective *definiendum* and *definiens* here? The answer we offer is that two different kinds of definition need to be considered. One is called 'verbal definition', the other 'ontological definition'. Roughly, the former serves to introduce a new term into an old vocabulary, while the latter defines an extra-linguistic entity by means of other concepts. Our equational verbal definition is used to introduce the predicate 'is prime' into an existing arithmetical vocabulary by means of already understood arithmetic terms like 'is a factor'. An ontological definition of *Prime* is used to define a particular set of numbers. The ontological definition is conceptually prior to the equational verbal definition of 'is a prime'; the former defines an extra-linguistic entity denoted by a compound predicate, which 'is a prime' is a shorthand for.

2.2.2.1 Ontological definition

An ontological definition is a compound concept that is not strictly empty. Such a definition defines the extra-linguistic entity constructed by the concept. Here is an example of an ontological definition:

$$\lambda x\, [^0{=}\, [^0Card\, \lambda y\, [^0Div\, x\, y]]\, ^02]].$$

[39] For more details, see Section 3.3.

This concept constructs the set of natural number having exactly two factors and is, therefore, an ontological definition of this set.

Here is another example. Consider

$$\lambda f\,[\forall x\,[\neg[\exists y\,[{}^0{=}\,y\,[f x\,{}^0 0]]]]].$$

Types: $f \rightarrow (\tau\tau\tau)$; $x, y \rightarrow \tau$; $=/(o\tau\tau)$.

Here we have *defined* a class of functions (division being a member of this class, for example). Yet no *definiendum* has been used (nor is any needed). What has been defined is the class of those binary functions of real numbers that are undefined on any pair $\langle a, 0\rangle$, a a number.

However, not every compound concept is an ontological definition of something, because some compound concepts are strictly empty. THE GREATEST PRIME defines nothing. By contrast, EVEN PRIME NUMBERS GREATER THAN 2 does define something; namely, the empty set of numbers. An empty set is not much, but still something. Quasi-empty ontological definitions are definitions of empty sets (relative to a type).

Definition 2.13 (*ontological definition*) Let C be a compound concept constructing an object a. Then C is an *ontological definition of* the object a.

Every ontological definition of an object a defines a by means of other concepts. For instance, the set of primes was defined above in terms of the concepts of cardinality and division. But this set can be defined in terms of alternative concepts, for instance, the concept of multiplication and existential quantification, because the relation *being divisible by* can itself be defined:

$${}^0Div = \lambda xy\,\exists z\,[x = [{}^0Mult\; y\; z]].$$

Types: $x, y, z/*_1 \rightarrow \tau$; $Mult$(iply)/$(\tau\tau\tau)$.

Theoretically, we could define particular objects in infinitely many ways, in terms of increasingly finer definitions. However, there is a bottom level at which we have to terminate the process of defining on pain of circularity. This bottom level is determined by the *conceptual system* underlying a given fragment of a given language (see Section 2.2.3).

2.2.2.2 Equational verbal definition

Here is an example of an equational verbal definition:

‘is a fortnight’ $=_{\text{df}}$ ‘is a 2-week period’.

Verbal definitions are expressions associating a concept with a new expression.[40] Its general form is the schema

$$A =_{df} \Phi(B_1, \ldots, B_m)$$

where A is a placeholder for a new symbol (mostly, but not necessarily, a simple one), called *definiendum*, Φ a syntactic function whose application to $B_1,\ldots,B_m$ creates a complex expression called *definiens*, and $B_1,\ldots,B_m$ existing expressions of the given language. This schema corresponds to the Russellian formulation that

> A definition is a declaration that a certain newly introduced symbol or combination of symbols is to mean the same as certain combination of symbols of which the meaning is already known (Whitehead and Russell, 1964, p. 11).

What is important about this formulation is that the *definiendum* is to be a simple symbol (or such a combination of symbols that contains only one semantically self-contained symbol, as is well-known[41]) and the *definiens*, on the contrary, a complex expression.

A Russellian definition is not a declarative sentence: rather it is an abbreviating stipulation. It cannot be true or false. So the situation can be described from the viewpoint of TIL as follows:

> *Definiens expresses a construction, say, C, which (unless improper) constructs an object, O, say. The definition is the procedure of letting* definiendum *mean the same as* definiens, *i.e., letting its meaning be C and its denotation O.*

This does mean, of course, that *definiendum* receives its meaning *via definiens*, so that it becomes not only *equivalent* but *synonymous* with *definiens*. Any definition of this kind expands and enriches the given language, but only as an abbreviation, complying thus with the requirement that definitions must be *conservative* extensions of languages. Any sentence *S* formulated in the language containing a *definiens* A is equivalent to the sentence that arises from *S* by replacing A by the respective *definiendum*. The verbal definition stipulates that the *meaning* (i.e. not only the denotation) of the *definiendum* be identified with the meaning of the *definiens.* Thus, for instance, the meaning of 'is a fortnight' is identical with the meaning of 'is a 2-week period'. The *definiens* is, in logical prose, 'is the set of time intervals whose duration is 2 weeks':

$$\lambda c\ [[{}^0 Duration\_of\ c] = {}^0 2W]$$

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[40] Here we are not analyzing Carnap-style *explications*. Nor will an Aristotelian (hence Scholastic, hence essentialist) theory of definition be discussed here; see Materna and Petrželka (2008).

[41] If *definiendum* contained more such symbols (distinct from variables and auxiliary symbols), the resulting expression would no more fulfill the role of definition, and in some cases a contradiction would arise.

Types: $c/*_1 \rightarrow (o\tau)$: the variable c ranging over time intervals, which are sets of times; $Duration\_of/(\tau(o\tau))$: the function that associates a time interval with its duration; $2W/\tau$: the number denoted by '2 weeks'.

When working out analyses on the basis of a *literal analysis* (see Definition 1.10), we associate semantically simple expressions with simple concepts. We should, however, be mindful of the fact that it does not mean that the meaning of every syntactically simple expression is a simple concept. Syntactically simple expressions can be frequently understood as *definienda*, so that their meaning is given by a *definiens*. Thus the meaning of a syntactically simple expression may well be a compound concept. The predicate 'is a bachelor' is one such example; 'is a fortnight' another.[42] On the other hand, a syntactically complex expression can be semantically simple. This is frequently the case with *idioms* like, for instance, 'is French chips', 'kicks the bucket'.

2.2.3 Conceptual system

In general, conceptual systems are a tool by means of which to characterise and categorize the expressive force of a vernacular and compare the expressive power of two or more vernaculars.[43] In this book we need the notion of conceptual system to do two things for us. First, it must delimit the domain of objects that a given language offers the linguistic resources to talk about. Second, it must fix the limit up to which we can refine, in a non-circular manner, the ontological definitions of the objects within the domain of a given language. We apply the notion of conceptual system to three topics; namely, the Parmenides principle (see Section 2.1.1), mathematical constants (see Section 3.2.1), and analytic information (see Section 5.4).

A conceptual system is a set of concepts, some of which must be simple. Simple concepts were defined as Trivializations of non-constructional entities of types of order 1 (cf. Definition 2.5). A system's compound concepts are exclusively derived from its simple concepts. Each conceptual system is unambiguously individuated in terms of its set of simple concepts.

The definition of *conceptual system* is this.

Definition 2.14 (*conceptual system*) Let a finite set ***Pr*** of simple concepts $C_1,\ldots,C_k$ be given. Let ***Type*** be an infinite set of types induced by a finite base (e.g., $\{\iota, o, \tau, \omega\}$ or $\{o, \nu\}$). Let ***Var*** be an infinite set of variables, countably infinitely many for each member of ***Type***. Finally, let ***C*** be an inductive definition of constructions. In virtue of ***Pr***, ***Type***, ***Var*** and ***C***, an infinite class ***Der*** is defined as

[42] See Materna (1999).

[43] The theory of conceptual systems was first introduced in Materna (1998, Chapters 6 and 7) and further elaborated on in Materna (2004a).

the transitive closure of all the closed compound constructions derivable from ***Pr*** and ***Var*** using the rules of ***C***, such that:

(i) every member of Der is a compound concept;
(ii) if C $\in$ Der, then every subconstruction of C that is a simple concept is a member of ***Pr***.

The set of concepts ***Pr*** $\cup$ ***Der*** is a *conceptual system* derived from ***Pr.*** The members of ***Pr*** are the *primitive concepts*, and the members of ***Der*** the *derived concepts*, of the given *conceptual system*.

Remark. As is seen, ***Pr*** unambiguously determines ***Der***. The *expressive power* of a given (stage of a) language ***L*** is then determined by the set ***Pr*** of the conceptual system underlying ***L***.

Every conceptual system delimits a domain of objects that can be conceptualized by the resources of the system. There is the correlation that the greater the expressive power, the greater the domain of objects that can be talked about in ***L***. Yet ***Pr*** can be extended into ***Pr′*** in such a way that ***Pr′*** is no longer logically independent (the way the axioms of an axiomatic system may be mutually independent). Independency means here that ***Der*** does not contain a concept *C* equivalent to *C′* of ***Pr***, unless *C′* is a subconstruction of *C*.

An example of a, minuscule, independent system would be ***Pr*** = $\{{}^0Succ, {}^0 0\}$, where $Succ/(\nu\nu)$, $0/\nu$. Due to transitive closure, there is a derived concept of the function $+/(\nu\nu\nu)$ defined as follows ($f \rightarrow (\nu\nu\nu)$):

$$\iota f\, \forall x\, [[[f\, x\, {}^0 0] = x] \wedge \forall y\, [[f\, x\, [{}^0Succ\; y]] = [{}^0Succ\, [f\, x\, y]]]].$$

This concept is not equivalent to any primitive concept of the system. However, among the derived concepts of this system there is, for instance, the compound concept of the sum 0+0,

$$[\iota f\, \forall x\, [[[f\, x\, {}^0 0] = x] \wedge \forall y\, [[f\, x\, [{}^0Succ\; y]] = [{}^0Succ\, [f\, x\, y]]]]\; {}^0 0\; {}^0 0],$$

which is equivalent to ${}^0 0$. Yet the system is independent, because the primitive concept ${}^0 0$ is a subconstruction of the above compound concept.

An example of a, likewise minuscule, dependent system would be ***Pr***$_1$ = $\{{}^0\neg, {}^0\wedge, {}^0\vee\}$. In this system either ${}^0\wedge$ or ${}^0\vee$ is superfluous because, e.g., disjunction can be defined by the compound concept $\lambda pq\, [{}^0\neg\, [{}^0\wedge\, [{}^0\neg p][{}^0\neg q]]]$, which is equivalent to ${}^0\vee$. The simple concept ${}^0\vee$ is not a subconstruction of the compound concept $\lambda pq\, [{}^0\neg\, [{}^0\wedge\, [{}^0\neg p][{}^0\neg q]]]$. To obtain independent systems, omit either ${}^0\wedge$ or ${}^0\vee$. This will yield either ***Pr***$_2$ = $\{{}^0\neg, {}^0\wedge\}$ or ***Pr***$_3$ = $\{{}^0\neg, {}^0\vee\}$.

Thus, the set of primitive concepts of an independent system contains no superfluous concepts and is insofar minimal. ***Pr***$_1$ was an example of a system containing a superfluous element. However, it should be possible to take an independent system and add one or more concepts to it and still keep the system independent. When such interesting extensions are made, the expressive power of the new system increases. To show how this works, first we define *proper extension of a system S* as individuated by ***Pr***. A proper extension of *S* is simply defined as a system *S′* individuated by ***Pr′*** such that ***Pr*** is a proper subset of ***Pr′***. An interesting extension is one that preserves the independency of the initial system.

The definition of conceptual system does not require that the system's ***Pr*** contain concepts of logical or mathematical operations. However, any conceptual system intended to underpin a language possessing even a minimal amount of expressive power of any interest must contain such concepts. Otherwise there will be no means to combine the non-logical concepts of the system, whether that system be mathematical, empirical or a mix of both. Let 'LM-part of *S*' denote the portion of logical/mathematical concepts of *S*, and 'E-part of *S*' denote the portion of empirical concepts of *S*.

Proper extensions of *S* come in two variants, essential and non-essential. A *proper non-essential extension* S′ of S is defined as follows: the LM-part of S $\subset$ the LM-part of S′ and the E-part of S = the E-part of S′. A *proper essential extension* S′ of S is defined as follows: the LM-part of S = the LM-part of S′ and the E-part of S $\subset$ the E-part of S′. It may happen that both the LM-part and the E-part of the system are extended. Then we simply talk about an *extension* of S.

Here is an example. Let S be assigned to a language L as its conceptual system. Let $\boldsymbol{Pr}_{\mathrm{L}} = \{{}^0Parent, {}^0Male, {}^0Female, {}^0\neg, {}^0\wedge, {}^0\forall, {}^0=\}$. An element of $\boldsymbol{Der}_{\mathrm{L}}$ is the concept of the relation-in-intension *sisterhood*; to wit,

$$\lambda w\lambda t\ [\lambda xy\ \exists z\ [[[{}^0Parent_{wt}\, z\, x] \wedge [\,{}^0Parent_{wt}\, z\, y]] \wedge [{}^0Female_{wt}\, x]]]].$$

Types: *Male*, *Female*/$(o\iota)_{\tau\omega}$; *Parent*/ $(o\iota\iota)_{\tau\omega}$; the types of the logical objects are obvious.

In general, when the speakers of ***L*** find that the object defined by a compound concept is frequently needed, they are free to introduce, via a linguistic convention, a new expression co-denoting this object. Whenever this happens, an equational verbal definition (see Section 2.2.2) sees the light of day. For instance, the speakers may decide to introduce the relational predicate 'is a sister of' to co-denote the relation-in-intension defined by some compound concept encompassing various logical concepts and empirical concepts such as *Parent* and *Female*, as done above.

To exemplify the definitions of non-essential and essential extension, take $\boldsymbol{Pr}_{\mathrm{L}}$ again. Its LM-part contains the primitive logical concepts ${}^0\neg$, ${}^0\wedge$, ${}^0\forall$, ${}^0=$, and the E-part contains the primitive empirical concepts ${}^0Male$, ${}^0Female$, and ${}^0Parent$. Then consider two extensions of S_{L}:

(a) The LM-part of S'_{L} adds to the LM-part of S_{L} the singularizers ${}^0Sing^{(o\iota)}$ and ${}^0Sing^{\iota}$, the E-part remaining unchanged. This is a non-essential extension.

(b) The LM-part of S''_L = the LM-part of S_L, E-part adds ^{0}*Ancestors_of*. This is an essential extension.

The upshot of the non-essential extension is that the expressive power of S'_L is greater than that of S''_L. This is because the addition of logical concepts makes it possible to make new combinations among the non-logical concepts. Therefore, the domain of objects constructible in S'_L is a proper superset of the domain of S''_L.

We first show that the function $Ancestors\_of/((o\iota)\iota)_{\tau\omega}$ is definable in S'_L. The respective construction is ($c \rightarrow (o\iota)$, $x, y, z, u \rightarrow \iota$):

$$\lambda w\lambda t\, [\lambda x\, [^0Sing^{(o\iota)}\, \lambda c\, [^0\wedge\, [^0\forall^\iota\lambda y\, [^0\supset [^0Parent_{wt}\, y\, x][c\, y]]]$$
$$[^0\forall^\iota\lambda z\, [^0\supset [c\, z][\, ^0\forall^\iota\lambda u\, [^0\supset [^0Parent_{wt}\, u\, z][c\, u]]]]]]]]$$

or in infix notation,

$$\lambda w\lambda t\, \lambda x\, \iota c\, [\forall y\, [[^0Parent_{wt}\, y\, x] \supset [c\, y]] \wedge$$
$$\forall z\, [[c\, z] \supset \forall u\, [[^0Parent_{wt}\, u\, z] \supset [c\, u]]]].$$

Gloss: 'The set c of the ancestors of x is recursively defined thus: All parents of x belong to c, and if z belongs to c then the parents of z belong to c as well.'

In S''_L individual roles of type $\iota_{\tau\omega}$ or functions of type $(\iota\iota)_{\tau\omega}$ like *Father_of*, *Mother_of*, called 'singular attributes', cannot be defined, because in S''_L we have no singularizers at our disposal. By contrast, in S'_L we can define, for instance, $Father\_of/(\iota\iota)_{\tau\omega}$ and $Mother\_of/(\iota\iota)_{\tau\omega}$ in virtue of the primitive concepts ^{0}*Parent*, ^{0}*Male*, ^{0}*Female* and $^0Sing^\iota$:

$$^0Father\_of = \lambda w\lambda t\, \lambda x\, [^0Sing^\iota\, \lambda y\, [[^0Parent_{wt} y\, x] \wedge [^0Male_{wt} y]]];$$

$$^0Mother\_of = \lambda w\lambda t\, \lambda x\, [^0Sing^\iota\, \lambda y\, [[^0Parent_{wt} y\, x] \wedge [^0Female_{wt} y]]].$$

Further refinement of these definitions within S'_L is impossible, there being no further suitable primitive concepts at our disposal within S'_L (nor within S''_L, of course).

So much for the logic of conceptual systems. We next consider a philosophical application. In general, we define intensions, e.g., properties, by means of empirical ontological definitions. *Taxonomies* are a case in point. Lay people use predicates like 'is a dog' and 'is a cat' without knowing the exact biological taxonomy and may still qualify as competent language-users. They (we) have a simple concept of dog and a simple concept of cat. Possessing these simple concepts, people competently apply an intuitive criterion in order to decide whether this or that individual is a dog (cat). However, at some point biologists introduced the definition of the property of being a dog as an animal belonging to the phylum *Chordate*, the class *Mammal*, the order *Carnivorous*, the family *Canidae*, the genus *Canis* and the species *Canis Familiaris*. From this point onwards it is no longer an empirical fact that beasts ever instantiating the so defined property are or were mammals. We need not empirically investigate particular instances of this property in a given state of the world in order to learn that, necessarily (because *ex definitione*), whatever individual happens to be a dog co-instantiates the property of being a mammal. Since the inception of the definition, it has been an *analytical* truth that dogs are mammals.[44]

By introducing a definition of a property, we assign a much crisper meaning to a hitherto much vaguer expression. A new, precisely defined concept (*explicans*) is introduced in place of one which is familiar but insufficiently precise (*explicandum*).[45] The biologists' concept of *dog* is as crisp as their concepts of *Chordate*, *Canis*, etc., since the logical operations involved in the compound concept neither add to nor detract from the crispness of the new concept.

It is worth pointing out that the Trivialization of the property of being a dog that lay people rely on may not be equivalent with the Closure constructing the biologists' property *dog*. Lay people and biologists are likely to agree on paradigmatic instances of doghood, but may well differ over limiting cases. In TIL jargon, lay people have a rudimentary conceptual system guiding their use of their predicate 'is a dog', while biologists draw upon an elaborate conceptual system to guide their use of their predicate 'is a dog'.

2.3 Empirical and mathematical existence

We adhere to the Fregean tenet that existence is a property of *Begriffe* rather than of *Gegenstände*. We are going to show that non-trivial existence cannot be ascribed to entities of elementary (atomic) types like individuals or numbers; all entities of elementary types trivially exist. Instead, whenever existence is nontrivial, it concerns entities of functional types like individual offices, properties, or functions in general, or entities of higher-order types.

Let c be an arbitrary concept. What do we mean when we say that c exists or fails to? In general, it means that c is, or is not, empty. In Section 2.2 we distinguished three kinds of emptiness as applicable to concepts: strict emptiness, quasi-emptiness and empirical emptiness.[46] We are going to exploit these definitions when investigating particular kinds of (non-)existence.

[44] See Definition 1.9.

[45] See Carnap (1950).

[46] See Definitions 2.5, 2.6, and 2.8.

2.3.1 Existence and extensions

Consider the sentences

'The number two exists'

and

'Venus exists.'[47]

What might they mean?. Does the *number* two possess a special property called 'existence' (as Kant framed the problem)? And, what would the claim that a certain *individual* (like Venus) exists mean?

If the sentences are analysed as ascribing existence to a number and an individual, we get, respectively,

$$[{}^0\exists^\tau\ \lambda x\ [{}^0{=}\ x\ {}^0 2]],$$

and

$$[{}^0\exists^\iota\ \lambda y\ [{}^0{=}\ y\ {}^0 Venus]].$$

Types: $\exists^\tau/(o(o\tau))$; $\exists^\iota/(o(o\iota))$; $2/\tau$; $Venus/\iota$; $x \rightarrow \tau$; $y \rightarrow \iota$.

Both constructions construct **T**, because the class containing the number 2 is not empty, just like the class containing the individual Venus is not. Put in another way, the simple concepts ${}^0 2$, ${}^0 Venus$ are incapable of being empty. The property of existence ($y, y_1 \rightarrow \iota$; $x, x_1 \rightarrow \tau$)

$$\lambda w \lambda t\ \lambda y\ [{}^0\exists^\iota\ \lambda y_1\ [{}^0{=}\ y_1\ y]] \rightarrow (o\iota)_{\tau\omega}$$

$$\lambda w \lambda t\ \lambda x\ [{}^0\exists^\tau\ \lambda x_1\ [{}^0{=}\ x_1\ x]] \rightarrow (o\tau)_{\tau\omega}$$

predicated of an individual or of a number trivially and provably yields **T**.

TIL does not rule out the ascription of existence to extensional entities like 2 and Venus. But there is little reason to do so. The ascription of *non*-trivial existence would be ruled out, so any ascription would come out trivially true. False ascriptions would not be an option, for there could not be a bearer of the property of non-existence. A test deployed to ascertain whether a given entity exists would pre-empt the result: by getting hold of the entity to check it for existence the question of its existence will already have been settled. Conversely, if no entity can be gotten hold of, it is *ipso facto* settled that there is no bearer of non-existence.[48] If ascription of existence, or non-existence, is to be of any cognitive value, then it needs to be possible to make a false ascription. What exists must be capable of not

[47] Venus the planet and not Venus the goddess.

[48] Cf. Carnap's 'internal' vs. 'external' questions in 1952.

existing, and *vice versa*. The only way we can imagine achieving this is by making other entities than those of an atomic type figure as the bearers of the properties of existence and non-existence. Below we flesh out this idea.

Consider the sentences

'Prime numbers exist',

and

'Even prime numbers greater than two exist'

together with their respective meanings,

$$[{}^0\exists^\tau\ {}^0Prime]$$

and

$$[{}^0\exists^\tau \lambda x\ [[{}^0Even\ x] \wedge [{}^0Prime\ x] \wedge [{}^0{>}\ x\ {}^02]].$$

The assertions made by these two sentences seem to be epistemologically open. If true, they are not trivially so; if false, then not trivially so. When evaluating the truth-value of the first sentence, we are not rehearsing the trivial fact that the simple concept ${}^0Prime$ is not strictly empty. Rather, we are inquiring whether ${}^0Prime$ is quasi-empty. If quasi-empty, it follows by definition that the class of primes is empty. Similarly, when evaluating the truth-value of the second sentence, we are inquiring whether the compound concept $\lambda x\ [[{}^0Even\ x] \wedge [{}^0Prime\ x] \wedge [{}^0{>}\ x\ {}^02]]$ is quasi-empty. If quasi-empty, it follows by definition that the class of even primes greater than two is empty.

Though the first construction obviously constructs **T**, and the second **F**, the proofs of these facts are not trivial, unlike the proof that the number 2 exists. Proving the latter two requires ontological definitions of *Prime*, *Even*, etc. In general, to prove existence in mathematics is to prove that a concept is not strictly or quasi-empty. In the specific case at hand, to prove existence is to prove that a concept of a class is not quasi-empty. If the example is that the largest prime exists then to prove non-existence is to prove that a concept of a number is strictly empty.

Now we come to the second problem connected with the existence of extensions; to wit, the values of a *function.* Above we argued that it makes little sense to ask whether an individual or a number exists; any such question is trivially answered in the affirmative. Now we are going to show that in the non-empirical case there is still another way of reasonably ascribing existence. This time non-trivial existence is rooted in partiality. Since we work with partial functions, we can ask whether a given function *f* has a value at a specific argument *a*. If it does not, then the application of *f* at *a* fails to produce anything; the application is an improper construction (an improper Composition) and the respective concept is strictly empty.

For instance, it is a fact of mathematical practice that the expressive force of 'The sinus of π exists' is not to the effect that the number zero exists. Similarly, when claiming that the cotangent of π does not exist, we are not claiming that some non-existing number fails to exist. We simply express the fact that the sinus function takes a value at the argument π, and that the cotangent function does not.

The latter mathematical fact can be easily proved. Since the cotangent function can be defined as the ratio of the sinus and cosinus functions, *cot*, *cos*, *sin*/(ττ), we have

$$[{}^0cot\ \zeta] = [{}^0\mathbf{:}\ [{}^0cos\ \zeta]\ [{}^0sin\ \zeta]].$$

Now since the value of sinus at π is zero, the right-hand side is $v(\pi/\zeta)$-improper, and so is the left-hand side. In other words (non-) existence is expressed as a mathematical property of a *concept*:

$$[{}^0Improper\ {}^0[{}^0cot\ \pi]],\ [{}^0Improper\ {}^0[{}^0\mathbf{:}\ [{}^0cos\ \pi]\ [{}^0sin\ \pi]]],$$

where *Improper*/($o*_1$) is the class of constructions of order 1 *v*-improper for any valuation *v;* the other types are obvious.

Similarly, when mathematicians were proving that the greatest prime does not exist, they were not using 'the greatest prime' as a meaningless term. How would they be able to prove a true claim if they had not understood what was to be proved in the first place? They proved that the procedure expressed by 'the greatest prime' is a blind alley not leading to any output. That is, the concept of the greatest prime is strictly empty and the respective construction is improper:

$$[{}^0Improper\ {}^0[\iota x\ [[{}^0Prime\ x] \wedge \forall y\ [[{}^0Prime\ y] \supset [{}^0\geq x\ y]]]]].$$

Types: $x, y/*_1\rightarrow\tau$; *Improper*/($o*_1$); *Prime*/(oτ); $\iota/(\tau(o\tau))$: singularizer.

In general, non-existence is a property of mathematical concepts. If $C/*_1\rightarrow(\beta\alpha)$ is a concept of function $f/(\beta\alpha)$ and $a/*_n\rightarrow\alpha$ is a construction of an argument of *f*, then the non-existence of the value of *f* at *a* is analysed as follows:

$$[{}^0Improper\ {}^0[C\ a]],$$

where *Improper*/($o*_n$).

Using existential quantifiers, non-existence construed as strict emptiness can be equivalently transformed into the emptiness of a (singleton) class:

$$\neg[{}^0\exists\lambda x\ [x = [C\ a]],$$

where $x\rightarrow\beta$, $\exists/(o(o\beta))$.

Thus we have:

$$\neg[{}^0\exists\lambda x\ [x = [{}^0cot\ \pi]]],$$

encoded in mathematatese as ‘The cotangent of π does not exist’, and

$$\neg[{}^0\exists\lambda x\ [x = [\iota x\ [[{}^0Prime\ x] \wedge \forall y\ [[{}^0Prime\ y] \supset [{}^0\geq x\ y]]]]]],$$

encoded as ‘The greatest prime does not exist’.

To sum up, in the case of extensions the sentence ‘*X* does not exist’ means that the concept expressed by ‘*X*’ is strictly or quasi-empty. If the former, then the meaning of ‘*X*’ is an improper construction. If the latter, then the class constructed by the meaning of ‘*X*’ is an empty class. Therefore, any such claim concerning extensions is a non-empirical claim.

2.3.2 Existence and intensions

As soon as we find ourselves in the empirical sphere, intensions crop up. It was established in Section 2.2.1, following Claim 2.3, that empirical concepts are never strictly empty. We are going to show here that non-triviality hinges on whether an empirical concept is empirically empty.

Provided Václav Klaus is an individual just like Venus, the question of whether Klaus exists pre-empts the answer and so makes little sense to pose. On the other hand, the question whether the President of the Czech Republic exists does not pre-empt the answer. Similarly, the questions of whether zebras exist or unicorns exist do not pre-empt the answer.

Our thesis is that these questions really pose the question whether an empirical concept is empirically empty at the $\langle w, t\rangle$ of evaluation. If empty, the concept in question constructs, by definition, an intension which at the world and time of evaluation either lacks a value altogether or yields an empty class. If the concept of President of the Czech Republic is empirically empty, it constructs, by definition, an individual office which at the world and time of evaluation lacks a value. If the concept of being a zebra is empirically empty, it constructs, by definition, a property which at the world and time of evaluation returns the empty class of individuals.

When we dress these two cases up in type-theoretic cloaking, existence turns out to be a polymorphous property of intensions.

Case (i): existence as a property of an α-*office*/$\alpha_{\tau\omega}$ is the property $Exist^{\alpha}/(o\alpha_{\tau\omega})_{\tau\omega}$ of being occupied at a given $\langle W, T\rangle$ pair defined as follows. Let $X/*_n \rightarrow \alpha_{\tau\omega}$. Then $[{}^0Exist^{\alpha}{}_{wt}X]$ $v(W/w,\ T/t)$-constructs **T** iff the office constructed by X is occupied at $\langle W, T\rangle$; otherwise **F**.

For instance, the sentences

‘The President of the Czech Republic exists’ and

'Pegasus does not exist'

express the following respective constructions:

$$\lambda w \lambda t\ [{}^0Exist^{\iota}_{wt}\ [\lambda w \lambda t\ [{}^0President\_of_{wt}\ {}^0CR]]]$$

and

$$\lambda w \lambda t\ \neg[{}^0Exist^{\iota}_{wt}\ {}^0Pegasus].$$

Types: $Exist^{\iota}/(o\iota_{\tau\omega})_{\tau\omega}$; $President\_of/(\iota\iota)_{\tau\omega}$; CR/ι; $Pegasus/\iota_{\tau\omega}$.

The above constructions construct propositions taking the value **T** in the actual world throughout 2009.

When saying that *X* exists (instead of saying that *X* exists *at a given world/time pair*), what we mean is that the office *X* (denoted by '*X*') is occupied by an individual in the *actual* world at the *present* time.[49] Thus we simply say that the President of the Czech Republic exists and that Pegasus does not.

Note that we analyse 'Pegasus' as denoting an individual *office* rather than a particular individual. If 'Pegasus' denoted an individual we would be thrown back to the problems trivializing the ascription of existence to Venus.[50] However, examples like 'Pegasus does not exist but might have' seem to serve as arguments for associating various possible worlds with distinct universes. Accordingly, some worlds would boast the individual Pegasus, while other worlds would not; hence the number of individuals may differ in distinct worlds. This line of reasoning is flawed due to a misinterpretation of the semantics of 'Pegasus'. This term seems to denote an individual (due to its grammatical form), but whoever understands the term derives their understanding from a description like 'the winged horse'.[51] 'Pegasus' is a shorthand for 'the winged horse' (perhaps with some further qualifications), so the ascription of non-existence to Pegasus is synonymous with the ascription of non-existence to the winged horse. The universe of individuals is the same for all possible worlds, but in some worlds an individual plays the role, while at the remaining worlds none does. Thus the above sentence 'Pegasus does not exist but might have' expresses the construction:[52]

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[49] See, however, Section 2.4.1 on our inability to semantically determine which of the possible worlds is actual.

[50] Recall that our universe of discourse is fixed and independent of possible worlds. See Section 1.4.1.

[51] We are here abstracting from the fact that 'Pegasus' might just as well be construed as a so-called fictional name. For fictional names, see Section 3.2.

[52] See also Section 3.1 for an analysis of definite descriptions and Section 4.6 for an analysis of counterfactuals.

$$\lambda w \lambda t\ [\neg[{}^0 Exist^{\iota}_{wt}\ {}^0 Pegasus] \wedge \exists w' \exists t'\ [{}^0 Exist^{\iota}_{w't'}\ {}^0 Pegasus]].$$

Note that the second conjunct would be trivially true if 'Pegasus' denoted an individual. If it did then Pegasus would not be an individual office, and the concept of Pegasus would not be an empirical one.

The property $Exist^{\alpha}$ can be defined using an existential quantifier by an ontological definition as follows:

$$\lambda w \lambda t\ \lambda r\ [{}^0\exists \lambda x\ [x = r_{wt}]],$$

where $r/*_n \to \alpha_{\tau\omega}$; $\exists/(o(o\alpha))$; $x/*_n \to \alpha$.

Thus the analysis of the above sentences can be equivalently transformed as follows:

$$\lambda w \lambda t\ [{}^0 Exist^{\iota}_{wt}\ [\lambda w \lambda t\ [{}^0 President\_of_{wt}\ {}^0 CR]]] =$$
$$\lambda w \lambda t\ [{}^0\exists \lambda x\ [x = [\lambda w \lambda t\ [{}^0 President\_of_{wt}\ {}^0 CR]]_{wt}]];$$

$$\lambda w \lambda t\ \neg[{}^0 Exist^{\iota}_{wt}\ {}^0 Pegasus] = \lambda w \lambda t\ \neg[{}^0\exists \lambda x\ [x = {}^0 Pegasus_{wt}]].$$

Case (ii): existence as a property of an α-*property*/$(o\alpha)_{\tau\omega}$ is the property $Exist^{(o\alpha)}/(o(o\alpha)_{\tau\omega})_{\tau\omega}$ of being instantiated at a given $\langle W, T\rangle$ pair defined as follows. Let $X/*_n \to (o\alpha)_{\tau\omega}$. Then $[{}^0 Exist^{(o\alpha)}_{wt}\ X]$ $v(W/w, T/t)$-constructs **T** iff the class $v(W/w, T/t)$-constructed by X_{wt} is not empty; otherwise **F**.

Examples. The sentences 'Zebras exist', 'Unicorns do not exist' express the following constructions:

$$\lambda w \lambda t\ [{}^0 Exist^{(o\iota)}_{wt}\ {}^0 Zebra],$$

$$\lambda w \lambda t\ \neg[{}^0 Exist^{(o\iota)}_{wt}\ {}^0 Unicorn].$$

Types: $Exist^{(o\iota)}/(o(o\iota)_{\tau\omega})_{\tau\omega}$; $Zebra$, $Unicorn/(o\iota)_{\tau\omega}$.

The property $Exist^{(o\alpha)}$ can be defined as follows ($x \to \alpha$, $p \to (o\alpha)_{\tau\omega}$):

$$\lambda w \lambda t\ \lambda p\ [{}^0\exists\ \lambda x\ [p_{wt}\, x]].$$

Thus the above constructions can be equivalently transformed into a quantified form:

$$\lambda w \lambda t\ [{}^0 Exist^{(o\iota)}_{wt}\ {}^0 Zebra] = \lambda w \lambda t\ [{}^0\exists \lambda x\ [{}^0 Zebra_{wt}\, x];$$

$$\lambda w \lambda t\ \neg[{}^0 Exist^{(o\iota)}_{wt}\ {}^0 Unicorn] = \lambda w \lambda t\ \neg[{}^0\exists \lambda x\ [{}^0 Unicorn_{wt}\, x].$$

Notice that claims concerning existence in the case of intensions are empirical claims, expressed by means of sentences denoting non-constant intensions. Their truth-values depend on the state of the world, unlike what is the case with the existence of extensions (reducible to a quantified form not involving *w*, *t*).[53]

One may wonder, when ascribing either existence or non-existence to an object, with which *supposition* the respective construction occurs. In order to provide an answer, we have to generalise our preliminary characterisation of the *de dicto/de re* distinction as presented in Section 1.1.1. Consider, for instance, the following true ascriptions of non-existence:

(i) 'Pegasus does not exist'

(ii) 'Unicorns do not exist'

Above we set out the principles of analysis of such sentences. Let $Pegasus/\iota_{\tau\omega}$; $Unicorn/(o\iota)_{\tau\omega}$; $Exist^{\iota}/(o\iota_{\tau\omega})_{\tau\omega}$; $Exist^{(o\iota)}/(o(o\iota)_{\tau\omega})_{\tau\omega}$. Then the sentences express the following respective constructions:

(i′) $\lambda w\lambda t\ [\neg[{}^0Exist^{\iota}{}_{wt}\ {}^0Pegasus]]$

(ii′) $\lambda w\lambda t\ [\neg[{}^0Exist^{(o\iota)}{}_{wt}\ {}^0Unicorn]]$.

Exist$^{\iota}$ is the property of an individual office of being occupied; *Exist*$^{(o\iota)}$ is the property of a property of individuals of having a non-empty extension. Obviously, the constructions ^{0}*Pegasus* and ^{0}*Unicorn* occur with *de dicto* supposition in (i′) and (ii′), respectively, because non-occupancy and emptiness of a population are ascribed to the office *Pegasus* and the property *Unicorn*, respectively. These intensions do not undergo intensional descent.

Equivalent analyses of (i), (ii) can be obtained by using $\exists^{\iota}$ $(x, y \to \iota)$:

(i″) $\lambda w\lambda t\ [\neg[{}^0\exists^{\iota}\ \lambda x\ [x = {}^0Pegasus_{wt}]]]$

(ii″) $\lambda w\lambda t\ [\neg[{}^0\exists^{\iota}\ \lambda y\ [{}^0Unicorn_{wt}\,y]]]$.

Do ^{0}*Pegasus* and ^{0}*Unicorn* occur with *de dicto* or *de re* supposition in (i″), (ii″)? It might seem as though they occurred *de re*, because the values of *Pegasus* and *Unicorn*, respectively, in worlds/times other than those of evaluation are irrelevant to the values of the so constructed propositions. However, (i″) and (ii″) do not come with an existential presupposition. Instead, the respective sentences express non-existence. This is due to the fact that the *classes* *v*-constructed by $\lambda x\ [x = {}^0Pegasus_{wt}]$, $\lambda y\ [{}^0Unicorn_{wt}\ y]$ are the objects of predication here; non-emptiness is an attribute of a class, not of its elements. In Section 2.6 we will define the notion of an *intensional occurrence*. Roughly, a constituent *D* of a construction *C* occurs intensionally in *C* if *D* *v*-constructs a function that is not

[53] The present approach to existence makes it possible to avoid Meinongian or neo-Meinongian solutions (see, e.g., Zalta, 1988a, b, 1989, 1997). On this point, see Tichý (1987, 2004, pp. 709–49).

applied to its argument within C. Thus $\lambda x\ [x = {}^0Pegasus_{wt}]$, $\lambda y\ [{}^0Unicorn_{wt}\,y]$ occur intensionally in $[{}^0\exists^\iota\ \lambda x\ [x = {}^0Pegasus_{wt}]]$ and $[{}^0\exists^\iota\ \lambda y\ [{}^0Unicorn_{wt}\,y]]$. For this reason ${}^0Pegasus$ and ${}^0Unicorn$ do not occur with supposition *de re* in (i″), (ii″), respectively.

To summarise, non-trivial existence and non-trivial non-existence are never ascribed to atomic entities like individuals or numbers. Our semantics readily incorporates non-existence without introducing so-called impossible possible worlds, non-existing individuals (*possibilia*) and other peculiarities that extensionalist approaches find themselves saddled with. Properly partial functions simply have value gaps; there are empty concepts; and constructions may be v-improper for some valuations v or may v-construct empty sets.

2.4 Explicit intensionalization and temporalization

Here we argue why actualism makes for an unsatisfactory semantic theory and present explicit intensionalization and temporalization as our alternative.

2.4.1 Anti-actualism

By ‘explicit intensionalization’ we mean, following Franz Guenthner, ‘[e]xplicit mentioning of possible worlds in the syntax’.[54] The difference, in a word, is between operating with one particular, privileged possible world and operating with a set of possible worlds, respectively. Likewise, presentism is rejected and replaced by explicit temporalization, which does not privilege a particular time but operates instead with a set of times.

By ‘actualism’ we do not mean the metaphysical claim, according to which everything that exists is actual. Instead by ‘actualism’ we intend the semantic claim, according to which the actual of all the possible worlds is implicitly or explicitly the locus at which truth-bearers are evaluated. We also consider a variant of actualism, according to which the actual world is the point in logical space at which designations are assigned to designators. Nor do we mean by ‘presentism’ the metaphysical claim, according to which only entities existing at the present moment may be said to exist at all. Instead we mean the semantic claim, according

[54] Guenthner, incidentally, dismisses the method, unfortunately without motivation. The full passage runs, ‘If such [intensionalized] languages also contain the lambda-operator, it will be possible to have expressions which denote […] the intensional values of expressions. For instance, λi (Pj, i) is a name of the set of worlds in which the object denoted by j has the property P [i, a world-variable]. Explicit mentioning of possible worlds in the syntax is probably not the ideal way of doing intensional semantics for natural languages, and we shall not discuss this method further here (cf. Tichý, 1971, pp. 292ff)’ (2004, pp. 132–37) (Guenthner, 1978, p. 44).

to which the present of all times is implicitly or explicitly the point at which truth-bearers are evaluated. We do agree, of course, that truth-bearers are to be evaluated in the actual world at the present moment. However, we take issue with actualism over the following general point. It is a fact that true propositions hold in a *superset* of world/time pairs containing the actual world and the present moment. We certainly know a lot about the actual world; but everything we know is not unique of the actual world; so we cannot identify the actual world but only an equivalence class counting the actual world as a member. Therefore, evaluation in the actual world at the present time is a matter of empirical inquiry and not a matter of logical semantics. For this reason no pair belonging to that superset should be singled out as enjoying a privileged status in a theory of logical semantics.

Strictly speaking, nothing in itself precludes explicit intensionalization from including an explicit reference to the actual world. As formulated so far, explicit intensionalization is simply a semantic method based on a syntax containing symbols for points in logical space—and the actual world is one such point. However, as we prefer to understand explicit intensionalization, the method is restricted to *variables* ranging over possible worlds, which may then be bound in a variety of ways. Similarly, just as we deploy variables ranging over possible worlds to analyze modally parameterized contexts, so we deploy variables ranging over times to analyze temporally parameterized contexts. The latter is called explicit temporalization. The presence in the syntax of variables ranging over possible worlds and instants of time leads to a two-sorted notation in our 'language of constructions' with the denotations of 'w_0, w_1, w_2, …, w_n, …' ranging over possible worlds, and the denotations of 't_0, t_1, t_2, …, t_n, …' ranging over times. Since variables are constructions, the availability of variables ranging over worlds and variables ranging over times offers us the possibility of constructing worlds and times by means of variables.

Actualism admittedly thrives on a very natural intuition. When using ordinary language, we put forward our claims with the intention that they be taken to be about how things actually are. If someone claims that it is raining, then they wish to be talking about the actual weather, and their claim is either a hit or a miss, depending on whether it is actually raining. In terms of pragmatics, ordinary language is fundamentally anchored to the actual world. But the question then arises whether this pragmatic commitment should be explicitly reflected in logical semantics. Actualists say Yes, anti-actualists such as we say No. The question for the actualist is only how the commitment is to be reflected. We will briefly discuss two variants in this section. One variant inserts into its syntax a constant or an operator standing for the actual world, which serves to take an intension to its actual extension. The other variant brings out the technique of scope distinction. No mention is made of the actual world; instead the fact that the definite description 'the *F*' takes wide scope over the modal operator $\Diamond$ is supposed to single out the *F* in the actual world. Thomas E. Patton characterises accurately this second variant thus:

> Within the scope of '◇', itself treated as a possible worlds quantifier '$\exists W$', the [definite] description is treated as if it contained a free possible worlds variable 'W'. Outside the scope of '◇', it is treated as if it contained instead a constant 'G' naming the real world (1997, p. 255).

The first variant is represented by, e.g., Patton (1997) and Einheuser (2005). Patton forms formulae such as

$$\text{‘}\exists W\,(\neg\mathrm{M}((dy)(\mathrm{M}ys\mathrm{G}))sW)\text{’}$$

in which 'G' is a constant for the actual world, as known from the standard structure ⟨G, K, R⟩, where K is a logical space such that G∈K and R an accessibility relation such that R⊆K×K (Patton, ibid., p. 255). The formula reads, on Patton's interpretation, 'There is some possible world at which Smith's unique murderer at the actual world does not murder Smith'. Einheuser considers formulae such as

$$\text{‘}(@x\!: Fx)\ (Gx)\text{’}.$$

This formula is intended to express that the unique F at the world being considered as actual is a G (ibid., p. 371).

As for the second variant, in his polemics against Kripkean rigid designators Dummett sets out to demonstrate that a sentence containing what is in effect a so-called rigidified definite description (i.e., a definite description whose denotation has been fixed at the actual world and remains the same for all possible worlds at which the denoted object exists) is equivalent to a sentence containing in its place a Kripkean rigid designator. Says Dummett,

> Sentences containing 'Deutero-Isaiah' demand to be understood in such a way that, in regimenting them, the term that represents 'Deutero-Isaiah', namely 't', is given the widest possible scope. Hence 'Deutero-Isaiah might [not have been a prophet]' must be regimented as '$\lambda x\ [\Diamond\neg F(x)](t)$', which is true; it does not admit the regimentation '$\Diamond\neg F(t)$' (1981, p. 577).

Dummett's reason for not admitting the latter regimentation is obvious. β-conversion would make a wide-scope context β-equivalent to a narrow-scope context, although Dummett claims that they do not share the same truth-conditions:

$$\frac{\lambda x\ [\Diamond\neg F(x)](t)}{\Diamond\neg F(t)}\ \beta\text{-conv}$$

This outcome would undermine Dummett's attempt to make a definite description having a wider scope than a modal operator equivalent to a Kripkean proper name. So to save his proposal, Dummett needs to impose a ban on β-conversion:

> It is, perhaps, inappropriate to use the λ-notation to construct predicate abstracts which do not satisfy the principle of λ-conversion […]; but this is only a point of notational etiquette (ibid., p. 576).

However, since Dummett does not give the logic of his alternative predicate abstracts, the logical underpinning of his theory of rigidified definite descriptions remains obscure and his proviso seems ad hoc.[55]

This shortcoming of Dummett's proposal does not suggest, of course, that every actualist semantics with an implicit, or understood, reference to the actual world must defy β-conversion or be otherwise ad hoc. But it does suggest that, in general, there may be a lack of expressive power in the actualist analysis of a given natural-language expression, since without tampering with this or that rule the analysis renders the analysed sentence equivalent to another sentence that it is obviously not equivalent with.[56]

Here follow our two objections to actualism. One turns on empirical omniscience, the other on factual vacuity (and the consequent loss of contingency as modal profile).

The omniscience objection. This objection goes *via* the notion of Plantinga book (cf. Plantinga, 1974, §4.2). Briefly, for every possible world there is exactly one 'Plantinga book', which records all and only those propositions that are true at that world. What is unique about the actual possible world is that

> [A]ll the members of its world-story (the set of all the propositions that are true in it) are true, whereas the stories of all the other possible worlds have false propositions among their members (Adams, 1974, pp. 225–26).

That is, the Adams world-story, or Plantinga book, of the actual world is the only world-story that contains the truth, the whole truth and nothing but the truth about the empirical universe.

To get an objection to our objection out of the way right away, the omniscience objection is bound to be wasted on anyone who believes that we could simply use a complex deictic term such as 'this world' to pinpoint the actual world as in, 'But of course we know which world is the actual one. This world!'. We have so far never encountered this remark in print, though it is often voiced in discussion. But, one can use '…this…' only to demonstrate concrete particulars in one's immediate vicinity, so it is far from clear what, if anything, is being demonstrated by means of 'this world'. Q*ua* possible world the actual world is an abstract object, so it simply cannot be pointed at. This is so, whether worlds be totalities of things or of states-of-affairs (to use the *Tractarian* dichotomy). 'This world' is a linguistic monstrosity that does not rival 'w_{666}' as a means to pick out a particular possible world relative to a numbered sequence of possible worlds.

The underlying conception of possible worlds is the standard one in terms of maximal consistent sets of states-of-affairs (formulae, facts, etc.), according to which worlds are individuated in terms of their distribution of truths and falsehoods. Thus, it is a prerequisite that possible worlds be *Tractarian* in the sense of

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[55] See Section 2.7 for further discussion of the validity of the β-rule.

[56] For extensive discussion and criticism of Dummett (1981) and Patton (1997), see Jespersen (2005).

totalities of states of affairs or possible-world propositions, and not of objects. We are assuming throughout this book that they are.[57] The objection, now, is this:

> If the knowledge of the actual world was one of the preconditions for grasping the message carried by an utterance, communication would be pointless. For if one did not possess the knowledge, the message would escape him. And if one did possess it, the message could not enlighten him (Tichý, 1975, pp. 92–3; 2004, pp. 219–20).

If a given proposition is true, and if you are empirically omniscient, you will be told something you knew already: you already know the proposition to be mentioned in the Plantinga book of the actual world. If the proposition is true, and you are not empirically omniscient, you will be told something you do not fully understand. Why? Since you do not know *which* of the Plantinga books is the book of the actual world, you will not know, for instance, which particular individual is asserted to belong to which set in, for instance, 'The actual Pope is a member of the set of actual Germans', 'At the actual world, the Pope is a German'. On the other hand, if you know which world is actual you *ipso facto* know that Joseph Ratzinger is a member of some particular 85-odd million-membered set of all and only Germans of the actual world. In formal terms, being empirically omniscient is to know for each intension what its value is at the actual world (at whatever moment happens to be present).[58]

The argument from omniscience can be further developed in the following way. The following two-step procedure is the backbone of actualism:

(I) Identify the actual world.
(II) Identify the extension of the relevant intension at the world identified at step (I).

Imagine you find yourself in the library housing all the Plantinga books describing the entire logical space and are asked to pick the only book that contains the entire truth about all matters empirical. Which do you pick? None. For want of omniscience, you do not know which book is the book of the world that happens to be the actual one. But once you do hold the book of the actual world in your hand, looking up the value of this or that intension at that world is just like looking up a number in a phone book. This, then, is the catch built into actualism: those who would want the book of the actual world cannot have it, and those who could have it do not need it. This explains why actualism demands too much and offers too little, and so it is either inoperative or redundant. Therefore, actualist semantics cannot be the actual semantics of any human language.

We can run the same kind of argument for terms ('designators', in the rigid-designation literature) and what they denote instead of intensions and extensions. Assume, with actualism, that the actual world is the locus at which denotations are assigned to terms (designators). Since we cannot identify the actual world, we

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[57] See Section 1.4, as well as Tichý (1988, §36), for motivation.

[58] *Logical* omniscience is no prerequisite for identifying the actual world. Sections 5.1.2 and 5.1.5 discuss logical omniscience.

cannot know a priori what are the designations of the designators of any given language, so we cannot know who or what we as speakers or hearers are talking about when a designator is being used. The argument (1) through (4) spells out this point.

Assume that

(1) w_n is the actual world
(2) at w_n, *a* is the inventor of the zip.

The truth-value of (1) cannot be established a priori, but only by possessing empirical omniscience, so we are nowhere near to knowing whether (1) is true. We can still decide the truth-value of (2), by checking the Plantinga book of w_n, but if (1) is false (contrary to the assumption just made) the truth-value of (2) becomes irrelevant; however, we cannot know whether it is.

Now add (3), which is a semantic convention introduced by actualism and ascertainable a priori in virtue of being a convention:

(3) 'the inventor of the zip' is a rigid designator of whoever is the inventor of the zip at the actual world.

The conclusion of the argument whose premises are (1), (2), (3) is

(4) 'the inventor of the zip' is a rigid designator of *a*.

The argument is obviously valid, but we cannot know whether it is sound as well, so we cannot know whether the conclusion is true. Hence, as speakers we cannot know whether we are entitled to use 'the inventor of the zip' as a rigid designator of *a*. Nor, if a speaker nonetheless uses 'the inventor of the zip', can we, as hearers, know a priori who is being designated. This predicament extends to all rigid designators of any language that goes via (1), so it is hard to see any justification for such terms.

A likely actualist objection would be this. We all know it is true that Scott is the author of *Waverley*, together with tons of other truths, so we certainly have managed to descend from several intensions to their actual extensions. But no. Getting to know that Scott is the author of *Waverley* cannot be formally rendered as applying the individual office of author of *Waverley* to the actual world to obtain an individual. The actual world is such that it has a unique inventor of the zip, and that individual is none other than Whitcomb Judson. Anyone keeping up on the inception and evolution of the zip or of two-dimensional modal logic knows this. So they (we) know *something* about the actual world. And we know a multitude of other things about the actual world; for instance, that Joseph Ratzinger is the Pope or what penguins eat. But the snag is that everything we know about the actual world is something we also know about other possible worlds. If you know that Whitcomb Judson is the inventor of the zip then what you are related to—the proposition that Whitcomb Judson is the inventor of the zip—is the multi-membered *set* of worlds at which it is true that Whitcomb Judson is the inventor. It is not true only at the actual world that Judson is the inventor of the zip. If—and this is already a mind-bending assumption—we humans would compile all the empirical facts known to us at some point in time, then mankind could, at most, identify an *equivalence class* of possible worlds; namely, all and only those worlds that are indistinguishable as far as this set of truths is concerned. The actual world would be a member of this set, for sure, but we would have no means to single it out. Only in the fullness of time shall the class have been winnowed down to a singleton, whose member will be the actual world, and only then shall we be able to identify the actual world.

The vacuity objection. Let *Prop* be the proposition that it is raining. If *Prop* is true at the actual world, then it is true at all worlds that *Prop* is true at the actual world. If *Prop* is false at the actual world, then it is true at all worlds that *Prop* is false at the actual world. And if *Prop* lacks a truth-value at the actual world, then it is true at all worlds that *Prop* lacks a truth-value at the actual world. But whereas the proposition *At the actual world Prop* takes the same truth-value (if any) at all worlds, *Prop* does not, hence *Prop* and *At the actual world Prop* are manifestly distinct propositions.[59] *Prop* is true or false, according as it is raining or not at the actual world, while *At the actual world Prop* is true or false independently of any worlds, i.e., necessarily true or necessarily false.[60] The actual-world indexed proposition is thus seen to lack the empirical and contingent dimensions that the non-indexed proposition has.[61] The latter is empirical and contingent, because it is not true at all possible worlds. The general formulation of the objection is that appending a constant for a possible world (and not just the actual possible world) to a term or expression abolishes its modal profile of contingency:

> This is … why any 'world-indexed' [intension] is trivial: the … function does not depend for its values on its arguments (Tichý, 1972, p. 92, 2004, p. 187).

Apparently, though, a particular form of actualism found in two-dimensional modal logic is immune at least to the omniscience objection. Two-dimensional modal logic operates with two possible worlds, one of which is 'considered as actual' and the other as counterfactual. Since it is a stipulation that some possible world w_n is to play the role of actual world, and since our stipulations are knowable a priori, it is knowable that w_n is the actual world. Thus, to track down the

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[59] This holds if 'at the actual world' is construed as referring to the one world that is the totality of all empirical facts. If, however, the actual world is construed (as in TIL) as the identity function of type ($\omega\omega$), then the phrase 'at the actual world' does not add anything to *Prop* and *At the actual world Prop* is identical with *Prop.*

[60] Davies and Humberstone (1980, p. 2) and Davies (1981, p. 222) enshrine the factual vacuity of such truths in an axiom of their logic of actuality:

(A4) $A\sigma \rightarrow \Box A\sigma$.

The relevant necessity of (A4)-truths is 'superficial' in Gareth Evans' sense.

[61] Cf. Quine's so-called eternal sentences: If *Sent* is true at time *t*, then it is true at all times that *Sent* is true at *t*.

designation of ‘the inventor of the zip’ at the world considered as actual, it is both necessary and sufficient to run the following argument:

(1*) w_n is considered as actual
(2*) at w_n, a is the inventor of the zip
(3*) ‘the inventor of the zip’ is a rigid designator of whoever is the inventor of the zip at the world considered as actual

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(4*) ‘the inventor of the zip’ is a rigid designator of a.

The truth-value of (1*) is ascertained by knowing, a priori, whether w_n is being stipulated to be actual. The truth-value of (2*) is ascertained by checking, a priori, whether it is true of w_n that its unique inventor of the zip is a. (3*) is a semantic convention introduced by this sort of actualism, and as such knowable a priori. (4*) follows *via* obvious substitution.

However, it would seem that this form of actualism reduces to a semantic game that *can* provide a viable semantics for, say English terms like ‘the inventor of the zip’ and English sentences like, ‘The inventor of the zip invented the zip’, but cannot reveal anything about the actual semantics of either. The problem is that the notion of considering a possible world as actual simulates an ability (the ability to identify the actual world) that we are nowhere close to possessing. Call this *the simulation objection*. This objection is not a decisive one, like the omniscience or vacuity objections, but rather the methodological one that ‘considering a world as actual’ is too much of an over-idealisation. However, also actualism based on considering a possible world as actual is susceptible to the vacuity objection. If w_n is the world being considered as actual, as in (1*), then if it is true at w_n that it is raining then it is true at all the other possible worlds in the model in which w_n is considered as actual that it is raining at w_n.

For more detailed comments, let us dwell for a minute on the actualist semantics outlined in Einheuser (2005), which revolves around the notion of considering a world as actual. So-called @-rigid designators have their designations fixed at whatever possible world happens to be considered as actual, in the sense of being the point at which designations are assigned to rigid designators. The question is how a semantics for @-rigid definite descriptions is supposed to work in details. (Call a semantics for @-rigid designators ‘@ctualist’.) It turns out that the answer is less than obvious. The problem is that the operator $@x$ is not defined separately but only in context; yet, ‘for any predicate F, the term ($@x$: Fx) refers to the unique F in the world considered as actual’ (ibid., p. 369). Unlike Russell’s iota terms, however, Einheuser’s @-rigid definite descriptions are supposed to be genuine designators. What Einheuser offers is only an equivalence between ‘($@x$: Fx)(Gx)’ and a Russell-style quantificational paraphrase (ibid., p. 371). This equivalence fails to explain how ‘($@x$: Fx)’ works, since ‘($@x$: Fx)’ lacks a counterpart in the paraphrase. Consequently, it becomes murky how the denotation of ‘($@x$: Fx)(Gx)’ is composed from the denotations of its parts, for how does the denotation of ‘(@*x*: *Fx*)’ contribute to the denotation of ‘(@*x*: *Fx*)(*Gx*)’? What is more, the absence of a semantics specifically for ‘(@*x*: *Fx*)’ obscures the fact that in order for ‘(@*x*: *Fx*)’ to designate an individual, a world (to wit, the world being considered as actual) needs to be selected before an individual at that world may be selected as its designation. The syntax is not of much help, though, since in ‘(@*x*: *Fx*)’, ‘@*x*’ binds a variable ranging over individuals and not worlds. Worlds, to be sure, do receive mention in the clause (i.e., the Russell-style right-hand side of the equivalence) for ‘(@*x*: *Fx*)(*Gx*)’, and it is obvious that ‘@*x*’ is intended to read something like, ‘the unique individual x at the world w_n considered as actual’. But the logical link between @*x* and worlds goes undefined in Einheuser’s @ctualism. One may suspect that at least part of the reason why this link is unaccounted for is because @*x* rolls two operations into one. One operation is to select a world, and the other to select an individual at that world. @*x* gets to bind a variable ranging over individuals (rather than worlds), since ‘*F*’ is a predicate applicable to individuals (rather than worlds). Hence the neglected mention of worlds.

The general negative point we wish to make is that the explicit-actual-world semantics of Patton and Einheuser possesses too much expressive power, while Dummett’s implicit-actual-world semantics possesses too little. So the morale is clear. We need to exclude step (I) (of identifying the actual world) from our natural-language semantics. How should we do this?

Enter *explicit intensionalization*. It is a method of logical analysis of natural language that extends beyond explicitly modal contexts boasting locutions like ‘possibly’ and ‘necessarily’. In fact, it applies across the board to *any* empirical expression, including atomic sentences. E.g., the advocate of explicit intensionalization argues that the sentence ‘It is raining’ is an intensional context in the sense that its analysis involves reference to possible worlds (and also times). For it is only at some worlds that it is true that it is raining, so the truth-value of the proposition that it is raining ought to be parameterized to worlds. And it is only sometimes that it is raining, so the truth-value of the proposition that it is raining ought to be parameterized to times as well. ‘It is raining’ induces a two-way *modal* context due to this dependency on both worlds and times. The way to make the modal profile of ‘It is raining’ explicit in prose is to add the sentential modifier ‘contingently’ to engender, ‘Contingently, it is raining’.

Both the modal and the temporal parameterization ought to be reflected in the syntax of the semantic theory within which ‘It is raining’ is analyzed for a fuller account of the conditions under which it is true that it is raining. If we intensionalize the semantics of ‘It is raining’ and any other empirical sentence by including possible worlds in its semantic analysis, we are getting nearer to a robust alternative to actualism. Possible-world semantics is able to distinguish systematically between *conditions* and their *satisfiers* by means of the distinction between intensions and their values (which may themselves be intensions, though of a lower degree; see Section 1.4). An intension is a condition that is satisfied, or fails to be satisfied, relative to a world (and perhaps also an instant of time). The semantic competence that language-users possess never concerns an intension’s world-relative satisfiers, *a fortiori* not its satisfier in the actual world at the present moment. The semantic competence we possess, when do we possess it, is that we know how to empirically test a given intension for its value at a given world and time of evaluation.

Possible-world semantics is custom-built for us humans lacking empirical omniscience. Though we are far from knowing all the actual satisfiers of the various conditions, nothing of an epistemic nature bars us from being able to apply the conditions and having our discourse revolve around them. What explicit intensionalization does is to make the satisfiers vanish from the logical-semantic realm altogether and to focus instead on the conditions.

The obvious technical tool to step back from actual extensions to entire intensions is to λ-abstract over logical space (i.e., over variables ranging over ω). Something akin to world abstraction will be familiar from Montague's IL, but TIL goes one critical step further. As C. Anthony Anderson says,

> $^\wedge A_\alpha$ is the intensional analog of $\lambda x_\beta A_\alpha$— it is functional abstraction on possible worlds. If we had variables of type s we could write $\lambda x_s A_\alpha$ instead of $^\wedge A_\alpha$ (1984, p. 361).

But IL fails to include an independent category of worlds, s not itself being a type. The most we can say is that if α is a type then $(s\alpha)$ is also a type. Hence we cannot say that x in $\lambda x_\beta A_\alpha$ ranges over type s, hence abstraction over worlds is not an option.[62] TIL goes one step further than IL by treating possible worlds as a ground type in their own right. Thus TIL is in this respect what is called a 'two-sorted theory' in the Montague literature.[63]

Explicit intensionalization inserts terms for possible-world variables directly into the logical syntax. We can thus directly construct possible-world intensions. Where $w \rightarrow \omega$ and $t \rightarrow \tau$, the following form (broached in Section 1.1) essentially characterizes the syntax of explicit intensionalization and temporalization:

$$\lambda w \lambda t\, [\ldots w \ldots . t \ldots].$$

In the parlance of TIL, this Closure is a construction constructing a possible-world intension. Alternative, and perhaps more suggestive, characterizations would be that Closure is a hyperintensionally individuated, algorithmically structured mode of presentation of a function from logical space to a function from times to entities, or that it is a procedure whose product is a condition to be satisfied by world/time pairs. For instance, in terms of the condition/satisfier jargon, a possible-world proposition is a condition that a $\langle w, t\rangle$ pair satisfies if and only if the pair makes the proposition true.

Observe how $[\lambda w\, A_w]$, A a construction of an arbitrary intension, in TIL is a far cry from being an 'intensional analog' of $^\wedge A$ in Montague. However, explicit

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[62] For criticism of Montague's implicit intensionalisation, see Section 2.4.3. See also Tichý (1994b, pp. 70ff, 2004, pp. 831ff) for criticism of Montague's $^\wedge$.

[63] See Gamut (1991, §5.8).

intensionalization is in principle an option for any intensional logic whose ontology includes possible worlds. As Thomas E. Zimmermann points out,

> [E]ven if abstraction from and quantification over possible worlds is made explicit, no additional objects in the original ontology become definable. …[T]he only essential difference between IL and Ty2 [two-sorted type theory], then, lies in the choice of 'admissible' types, i.e. in the question of whether indices should be allowed to figure as the values of functions (1989, §4.3).

Interestingly, Tichý himself originally had actualist leanings. In his (1971, p. 285) he introduces '*M*' as a constant for the actual world. Leaving out times, the syntax would have instead been

$$[\lambda w\ [\ldots w\ldots]]_M.$$

For instance, the construction of the actual truth-value of the proposition that it is raining would have been

$$[\lambda w\ [{}^0Rain^*{}_w]]_M.$$

where $Rain^*/(o\omega)$.[64] However, as Tichý was to realise in his following paper (1972), it is a step like the following that must be expelled from semantics:

> Apply the function obtained *via* abstraction over *w* to the value of *M* to obtain the truth-value of *Rain* at *M*.

But, if the explicit intensionalists eject any dependence on the actual world from their semantics, how does 'It is raining' become an assertion about the *actual* world and the *present* moment? If no anchoring to the actual world or the present moment is built into the semantics of the sentence, then how does a speaker succeed in making a claim about the actual world and the present moment? Obviously, the sentence should not be taken to mean merely that it is raining at *some* world and time.

The solution TIL offers relegates the issue of commitment to the actual world and the present moment to the *pragmatics* department. Pragmatics introduces the empirical, a posteriori dimension (which semantics lacks) by introducing language-users who make their assertions from the vantage point of a particular world and a particular moment. The notion of *assertion* of propositions is what bridges between propositions and the actual world.[65]

The sentence 'It is raining' is in and by itself no claim about the actual world or the present moment. Indeed, a sentence is not a claim at all. Instead it is a necessary (though in itself insufficient) vehicle for making an assertion about the actual world and the present moment. In order to make an assertion, what is required,

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[64] If *Rain* is of type $o_{\tau\omega}$ then $[\lambda w\lambda t\ [{}^0Rain_{wt}]]_M$ constructs the chronology of times at which it rains at the actual world.

[65] Cf. Tichý: 'Any successful assertion carries out two separate tasks: (i) it draws the audience's attention to a particular proposition and (ii) it makes it clear that the proposition is being judged to be true.' (Tichý, 1988, p. 162).

apart from a sentence denoting a proposition, is an *act* of assertion to the effect that the proposition denoted by the sentence is true.[66] Sentences do not make assertions; people do. An assertion involves reference to a proposition that partitions the logical space into those worlds at which the proposition is true and the rest, where it is not. This second subset of worlds are those where the proposition is false; or, if the proposition is only partially defined, the second subset divides into those worlds where the proposition is false and those where it is neither true nor false. The assertion is to the effect that the actual world is a member of the set of worlds where the proposition denoted by the asserted sentence is true. The question of whether the assertion is a hit or a miss is external to the assertion and is a task to be settled by empirical inquiry.

Still, the actualist may insist, a non-paradoxical proposition is true at infinitely many worlds. Yet what we are after is truth at the unique possible world that is actual. It is not enough to assert that the proposition is true; it must be made explicit that the proposition is asserted to be *actually* true. Tichý disagrees:

> [I]t is often suggested that by asserting a sentence one refers to a specific *world*, namely the actual one. What we talk about, it is often said or tacitly presupposed, is the actual world. Both views are misconceived. It is a pragmatic *presupposition* of communication that the propositions speakers refer to in affirmative discourse are offered as *true*. […] It is also pragmatically understood that truth means *truth in the actual world*. There is thus no need to *refer* to that world (1988, p. 197).

That is, truth *simpliciter* is truth *actualiter*. Even so, the actualist may be adamant that a sentence like ‘It is raining’ is an elliptical way of writing ‘At the actual world, it is raining’, or that the assertion that it is raining is an elliptical way of asserting that it is raining at the actual world. Tichý points out that if the reference to the actual world is made part and parcel of what a sentence says, or of what is asserted to be true in an act of assertion, then the vacuity objection kicks in.

With ‘Actually’ and kindred locutions expunged from semantics, as well as from the pragmatics of making assertions, is there a niche still left open for them? There is; ‘actually’ belongs to *rhetoric*. Terms like ‘actually’, ‘really’, ‘as a matter of fact’, etc., add colour and relief to a claim by sharpening contrasts and providing emphasis, as in ‘Now, you may all have thought that Carla was an Italian—but, she’s actually Swiss!’

We recommend not letting ‘actual(ly)’ and other references to the actual world make it to the final semantic analysis. Should one insist on matching ‘actual(ly)’ with a semantic counterpart, our only suggestion would be to pair it off with the identity function defined over the logical space. On this analysis of ‘Actually, Carla is Swiss’ the relevant types are $Actually/(\omega\omega)$; $Carla/\iota$; $Swiss/(o\iota)_{\tau\omega}$. Now, the sentence is evidently true in a world w if Carla belongs to the population of Swiss people in the world which is actual in w. This world is the value of the function *Actually* in w, i.e., $[{}^0Actually\ w]$. And the world that is actual in w is none

[66] See Martin-Löf (2001) for emphasis on the distinction between a (constructivist) proposition (a set of proof-objects) and the act of judging that the proposition is true.

other than w itself. The extension of *Swiss* in w at time t is then constructed by $[[{}^0Swiss\ [{}^0Actually\ w]]\ t]$, and the sentence 'Actually, Carla is a Swiss' will express the Closure

(*) $\lambda w \lambda t\ [[[{}^0Swiss\ [{}^0Actually\ w]]\ t]\ {}^0Carla]$.

But since *Actually* is an identity function, the proposition so constructed is the same as the proposition constructed by $\lambda w \lambda t\ [[[{}^0Swiss\ w]\ t]\ {}^0Carla]$, or using abbreviated notation,

(**) $\lambda w \lambda t\ [{}^0Swiss_{wt}\ {}^0Carla]$.

Thus (*) and (**) are equivalent Closures. In this case the semantic analysis (*) contains the semantically redundant constituent $[{}^0Actually\ w]$ endowed only with rhetoric import.[67]

2.4.2 Predication as functional application

We are going to argue that the logic of predication is functional application, just as it is of extensionalization.[68] We disagree that a special operation of predication is required, as in Bealer's intensional logic. We show how extensionalization saves from a type-incongruity objection the claim that empirical predicates denote properties, a claim which Colin McGinn argues for in 2000 (Chapter 3).

Now, it might seem tempting to cut the Gordian knot of what predicates denote by denying that they denote at all. But we would recommend against such a move. Suppose we wish to determine the denotation of a compound expression, in which there is at least one occurrence of a non-denoting predicate. Then the principle of compositionality entails that the compound lacks a denotation, too. However, we are far from convinced that, say, 'Mary is happy' fails to denote (In fact, we hold that it denotes a proposition). Alternatively, one might relinquish compositionality and still have 'Mary is happy' denote, while the predicate 'is happy' did not. But in our view compositionality is a condition *sine quā non* for any formal semantics.[69]

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[67] Lewis (1970), famously argues that 'actual' is an indexical and as such an expression whose reference is a function of the context of utterance, *in casu* the entire world at which the sentence '…actual…' is uttered. Tichý objects to this in 1975 (§6), 2004 (pp. 217–220). He points out, among other, that as long as the parties to the discourse cannot identify the reference of 'the actual world' they cannot fully grasp what proposition is denoted by '…actual…'.

[68] This section is based in part on material appearing in Jespersen (2008a).

[69] See Section 2.1.2.

In McGinn's sample sentence, 'Russell is bald', two entities are picked out, Russell the man and baldness the property. The predicate 'is bald' neither denotes, plurally, each and every bald individual nor, singularly, the set of all bald individuals. His general reason is that the extensions of the properties constitute no additional semantic level alongside the properties themselves: 'Extensions will no longer be in the picture' (ibid., p. 63), he says, continuing,[70]

> A predicate refers to a property with many instances; a name refers to an object with many properties: that is all. The meaning of each category of terms stops at its ordinary reference without reaching out further into the non-semantic world of property instantiation. Extensions of both kinds are fixed by the facts of the world, not by the meaning of the terms. They are extra-semantic items (ibid., pp. 65–6).
>
> Predicates are also rigid designators for me, as they cannot be if taken to designate their extensions, since these vary from world to world [and from time to time, cf. p. 59]. I say that 'red' designates the property of redness in every possible world, as 'Bertrand Russell' designates Bertrand Russell in every possible world. Here again names and predicates are semantically analogous (ibid., p. 67, n. 11).

We agree entirely with McGinn's claim that predicates are rigid designators denoting properties. Unfortunately, McGinn provides little by way of argument in its favour. What he offers amounts to the claims that the thesis 'meshes naturally with speakers' understanding' and that 'we know antecedently that names denote objects and predicates denote properties' (ibid., p. 57, pp. 58–9, resp.). We think a cogent argument is available. The modal argument by Tichý that we are deploying in Section 3.3 shows that, to take McGinn's example, only if 'is bald' denotes the property of being bald can the contingency of 'Russell is bald' (or rather of the proposition it denotes) be guaranteed.

Hence, modal (and arguably also temporal) variability must be built into the semantics, as was also argued in Section 2.4.1. Non-triviality, or contingency, can be restored if 'is bald' does not designate a set, but designates instead a property with *different* world/time relativized sets as its extensions.[71] The benefit is that since properties are sets-in-intension, the predicate 'is bald' does not single out a set, but instead a function whose values are sets.

But now, the tenet that (empirical) predicates denote properties brings out a serious drawback of the thesis: the denotation of a predicate is not directly attributable to an individual! This is serious, because predication would simply be rendered impossible. To see why McGinn's thesis, without appropriate theoretical embedding, renders predication impossible, consider the general form of the truth-condition that McGinn assigns to 'Russell is bald', or any other sentence in which

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[70] McGinn says 'reference' where we would say 'denotation'.

[71] The thesis that predicates denote properties turns them into Kripke-style rigid designators. This is only appropriate, though, since properties are world-invariant: baldness is the same property in all worlds (and at all times), whereas it has different extensions. Both McGinn (ibid., p. 59, p. 67, n. 11), as we saw, and Tichý (1986a, p. 255) embrace this consequence. See also Zouhar (2009).

a property is predicated of an individual. The truth-condition is that individual *a* has property *P*, and McGinn casts it as the ordered pair $\langle a, P\rangle$ (ibid., p. 63). The problem is: how are *a*, *P* to correlate with one another in such a way that *P* is predicated of *a*? It would seem that they just cannot. Put syntactically, 'the result of juxtaposing an [intensional] abstract and an individual constant does not form a well-formed expression.' (Bealer and Mönnich, 1984, p. 237.) Analogously, the concatenation of the two English words 'Russell', 'baldness' in 'Russell baldness' does not constitute a sentence of English. Put objectually, our intensionalist thesis lacks what its foremost extensionalist rival has, namely the compatibility between the two extensional entities *a* and $\{\ldots\}$ thanks to the relation $\in$ of set membership; '$a \in \{\ldots\}$' is a well-formed expression of the syntax of set theory. However, no intensional counterpart of $\in$ is available that would make feasible the predication of *P* of *a*, on the assumption made above that *P* is a function from worlds and times to sets.[72] It is probably telling, then, that McGinn in fact sidesteps the issue of how *P* is to be predicated of *a* by simply offering the two-membered sequence $\langle a, P\rangle$ lacking a third member to trigger the predication of *P* of *a*. Mere sequences of individuals and properties are incapable of 'setting up a great chain of interlocking objects and properties' (McGinn, ibid., p. 63).

Notice, in passing, that the problem of how several atoms form one compound is nothing other than Russell's old problem of propositional unity; namely, the problem of the logic and semantics of predication, as when predicating baldness of Russell. As Davidson rightly states, 'The 'problem of predication' is the problem of the unity of the proposition.' (Quoted from Gaskin 2008, p. 25, n. 113.)

Fortunately, a solution is readily available. What we need to do is to make the denotation of a predicate *indirectly* attributable to the denotation of a singular term. The predicate will denote a property, which is then extensionalized so as to yield a set.

In this section we restrict our treatment to McGinn's sentence, 'Russell is bald', which is an instance of singular predication. We do not discuss whether generic predication like 'The raven is black' (i.e., 'All ravens are black') also requires extensionalization and, if so, which particular form it might take.[73]

Notice that when an $(o\iota)_{\tau\omega}$-entity is applied to ω- and τ-entities, the resulting entity is of type $(o\iota)$. Such extensional entities are *sets* of individuals. It is essential to our solution that sets not be treated as primitive, but instead construed as functions, so that we can feed the notion of set into the logic of functions. A set is a function that takes all and only its members to **T** and non-members to **F**.[74] Such a construal will be familiar from Frege, whose concepts (*Begriffe*) are also

[72] Strictly speaking, *P* is a function from worlds to chronologies of sets of individuals, and not a binary function from pairs of worlds and times (See Section 1.4).

[73] However, see Section 1.4 for an explanation of how universal quantification works in TIL, and Chapter 4 for an explanation of how 'Whales are mammals' may be analysed in terms of *requisites*.

[74] —or is undefined, as the case may be.

characteristic functions.[75] Functions, on the other hand, figure as primitive entities in our framework.[76]

Let us apply this framework to 'Russell is bald'. Suppose we assign the type $(o\iota)_{\tau\omega}$ to baldness and ι to Russell. Then the technical problem arises how baldness and Russell must be arranged for baldness to be predicated of Russell. As already mentioned, what we should *not* do is attempt to apply baldness directly to Russell. For then incongruity ensues. Instead, baldness demands to be applied to a world and then to a time before it can be predicated of Russell. But then it is not baldness, of type $(o\iota)_{\tau\omega}$, that gets applied to Russell, but an *extension* of baldness. Only when arriving at an extensional entity of type $(o\iota)$ have we arrived at the right sort of thing to apply to Russell.

But we just argued that in attributing baldness to Russell, two entities are involved: Russell and a *property*. Yet here we are talking about a *set*. Isn't our attempt to add a missing piece to McGinn's project in fact taking it into the direction he just objected to? Despite immediate appearances, the answer is No. The reason is because *empirical properties can be predicated of individuals only relative to worlds and times*. Hence, in formal terms, *predicating* baldness of Russell can be nothing other than applying to Russell the extension that the ⟨world, time⟩-extensionalized property of baldness returns at a given world and time of evaluation.

The proceedings can be explained type-theoretically. After P of type $(o\iota)_{\tau\omega}$ has been applied to w of type ω, P_w of type $(o\iota)_\tau$ demands to be applied to t of type τ. Only then does an entity of the appropriate type emerge, *viz*., a set of individuals of type $(o\iota)$. It comes in handy now that sets are functions in our framework. We execute our third functional application by applying the characteristic function to an individual. The result of the application is an o-object. The truth-value is **T**, if the individual is a member of the set, and **F**, if not. The resulting truth-value, formally, is abstracted over to obtain a function from logical space to chronologies of truth-values.

Thus, the solution to the incongruity problem consists in extensionalization of P via functional application of P to w and then of this result to t to yield a set, of which a either is or is not an element.

The Closure expressed by 'Russell is bald' is

$$\lambda w \lambda t\, [{}^0P_{wt}\, {}^0a].$$

The truth-condition so constructed is verbally in agreement with the truth-condition McGinn states for simple sentences involving singular predication:[77]

[75] See also McGinn, ibid., p. 67.

[76] See Section 1.3.

[77] '*Fa*' was changed into '*Pa*'.

> The truth-conditions of simple subject-predicate sentences are given [as follows]: a sentence of the form '*Pa*' is true if and only if the object referred to by the name has the property referred to by the predicate (ibid., p. 53).

However, above we saw McGinn offer $\langle a, P \rangle$ as the logical form of how 'the object referred to by the name has the property referred to by the predicate'. We advance $\lambda w \lambda t\ [{}^0P_{wt}\ {}^0a]$ as a rival to $\langle a, P \rangle$. We already objected that $\langle a, P \rangle$ fails to indicate how P is to be predicated of a. This lacuna leaves an open flank in McGinn. But, McGinn might argue, once this has been taken care of, his theory of predication can do without introducing the complicating factor of extensionalization. This he would be in a position to argue, because it is compatible with his thesis that predicates denote properties to construe properties as *primitive* rather than as functions. This option is admittedly a tempting one. For one thing, it renders the w, t indices superfluous and reduces the number of logical steps in one go. No less importantly, the construal avoids incongruity, for one simply defines those primitive properties as being directly applicable to individuals. But, as we try to show below, construing properties as primitive sheds little light on the logic of predication, which is why we come set against the construal.

Another apparently attractive alternative to construing properties as $(o\iota)_{\tau\omega}$-entities would be as *propositional functions*. If properties are (modelled as) functions from worlds to functions from times to sets of entities, as we suggest, then they are equivalent to, e.g., functions from entities to propositions, where propositions are functions from worlds to functions from times to truth-values. Propositional functions defined over individuals would be of type $(\iota \to o_{\tau\omega})$, i.e. $(o_{\tau\omega}\iota)$. Let the property of baldness be the propositional function of being an x such that x is bald. If this propositional function is applied to Russell as argument, the functional value is the proposition that Russell is bald. Modelling properties as propositional functions is not without its attractions. For it does away with the incongruity problem in one fell swoop. No need to groom the property before predicating it of an individual; just apply the propositional function as is to an individual and obtain a proposition in return. Tempting though the construal of properties as propositional functions may appear, there are two reasons for resisting its lure, one general and the other more specific.

The general reason is the concern to maintain the uniformity of the system of intensions. Properties would no longer be intensional entities defined on possible worlds, unlike all the other intensions. Instead they would be defined on, for instance, individuals. This sort of argument fails, of course, to impress anyone who rejects that intensions are functions from logical space, but ought to strike everyone as being ad hoc: why would properties have a wholly different type of argument than all the other intensional entities? Surely, if we are able to maintain a principled, unified, general theory of intensions then we ought *a fortiori* to do so. Linked with this top-down, methodological argument is one owing to Bealer, which may be summarised as, 'So properties are propositional functions? But then what are propositions?' In particular, are propositions extensionally individuated or hyperintensionally individuated 0-ary intensions? In Bealer's words,

> How is one to develop a theory of the *other* type of intension? This job will require some new kind of logical machinery, machinery not used in the original propositional-function approach … This new logical machinery is likely to be very much like that used in the algebraic approach [Bealer's own] to intensional entities, which is the main competitor to the propositional-function approach. If so, what is gained by not using an algebraic approach to both types of intension [property, proposition] from the start? (1989, p. 10.)

We agree, except that the 'new logical machinery' is just as likely to be that offered by the rival possible-world approach. Interestingly, though, the propositional-function approach could, in fact, level an argument from theoretic uniformity against us. A few remarks to set the stage. In general, a property is predicated of something. But something may also be predicated of a property. For instance, the property of being attractive may be predicated of the property of being bald. On the assumption that properties are propositional functions, attractiveness must be a propositional function that takes another propositional function to a proposition. Therefore, if baldness, B, is of type $(o_{\tau\omega}\iota)$ then attractiveness, A, must be of type $(o_{\tau\omega}(o_{\tau\omega}\iota))$. If we retain functional application as the logic of predication (but leave out Trivialization), the analyses of 'Russell is bald' (Russell, R, still of type ι) and 'Baldness is attractive' turn out as follows:

$$[B\ R]$$

$$[A\ B]$$

B remains unaltered in both cases. Not the form of B (i.e., B as opposed to B_{wt}) but only its position in $[X\ Y]$ determines whether B occurs as subject or object of predication. So the propositional-function approach offers a uniform account of attribution of properties to individuals and properties, whereas a theory such as ours needs to extensionalize B in the case of 'Russell is bald'. This is admittedly a point in favour of the propositional-function approach. But the simplicity of the logical forms of the two sentences above is detrimental to their ability to capture not only modal but also temporal modalities, as well as the interplay between the two. For an example, consider the non-equivalent sentences, 'Frequently, my neighbour is sick' and 'My neighbour is frequently sick' as found in Tichý (1986a, pp. 261–63, propositions L^2 and L^4, resp.). Their respective analyses require explicit (λ-bound) w and t variables, which are nowhere to be found in $[B\ R]$ and $[A\ B]$.[78]

The specific reason for eschewing propositional functions is that it is not entirely obvious how contingency is supposed to be captured. For a concrete example, consider Aczel (1980). Aczel's proposition that the propositional function f is true of a ought not to reduce to a being a member of the *set* $\lambda x f(x)$, as in $a \in \lambda x f(x)$ (ibid., p. 31). For then the proposition $f(a)$ is insufficient for the purposes of

[78] See Section 2.5 where we deal with the interplay between modal and temporal modalities, using another example.

modelling contingently satisfied truth-conditions (cf. Tichý's modal argument in Section 3.3.1).[79]

So perhaps we still ought to consider inserting occurrences of *w*, *t*; only where? Consider the β-reduced form $f(a)$ of $(\lambda x f(x), a)$. Will $\lambda w \lambda t(f_{wt}(a))$ do? Will $\lambda w \lambda t(f(a))_{wt}$? Neither, in case propositions are of type $o_{\tau\omega}$. The stumbling block in the first case is that *f* can be extensionalized only after having been applied to *a* (where it is assumed that *a* is of a type appropriate for *f*). Since f_{wt} is ill-typed, $\lambda w \lambda t(f_{wt}(a))$ is no option.

The second suggestion fares ostensibly better. If *a* is of type ι and *f* of type ($o_{\tau\omega}$ ι) then *f* applied to *a* yields an $o_{\tau\omega}$-object. So we have our proposition. When extensionalized, it yields a truth-value, which is abstracted over by $\lambda w \lambda t$, again spawning an $o_{\omega\tau}$-object. Technically, $\lambda w \lambda t(f(a))_{wt}$ works. Philosophically, it is somewhat peculiar. The problem is that the addition of *w*, *t* is gratuitous: $f(a)$ is already a proposition, so what is the point of $\lambda w \lambda t(f(a))_{wt}$? Well, it might be rejoined, the point is that the latter makes *w*, *t* explicit while the former fails to. But this merely goes to show, in our view, that an entity of type ($o_{\tau\omega}$ι) such as *f* is out of place in a framework that comes with explicit *w*, *t* variables. Which is to say that propositional functions are, at the very least, at odds with an intensional type theory whose propositions are of type $o_{\tau\omega}$. On the other hand, propositions of type $o_{\tau\omega}$ and properties of type $(o\iota)_{\tau\omega}$ walk hand in hand, as soon as we avail ourselves of a vehicle of extensionalization.

Let us summarize. The predicate 'is bald' denotes the property of being bald. The predicate picks out an intensional entity that must undergo extensionalization to render it applicable to individuals so that baldness may be predicated of individuals. Extensionalization takes the form of the logical operation of functional application. In TIL parlance, the sentence 'Russell is bald' expresses a construction of the proposition that Russell is bald and counts among its constituents three occurrences of Composition:

[1] $[{}^0Bald\ w]$: the application of Baldness/$(o\iota)_{\tau\omega}$ to $w/*_1 \rightarrow \omega$ to obtain $(o\iota)_{\tau}$, a chronology which inputs instants of time and outputs the respective sets of bald people at those particular times.

[2] $[[{}^0Bald\ w]\ t]$: the application of the chronology obtained at [1] to $t/*_1 \rightarrow \tau$ to obtain a set of individuals/(οι).

[3] $[[[{}^0Bald\ w]\ t]\ {}^0Russell]$: the application of the set obtained at [2] to Russell/ι to obtain a truth-value/ο.

The truth-value obtained in [3] is parameterized to worlds and times by means of two instances of Closure to obtain a proposition: $\lambda w \lambda t\ [[[{}^0Bald\ w]\ t]\ {}^0Russell]$; or, in our abbreviated notation, $\lambda w \lambda t\ [{}^0Bald_{wt}\ {}^0Russell]$.

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[79] In all fairness, though, it is obvious from Aczel (1980) that he is concerned only with mathematical propositions (what TIL would call mathematical constructions of truth-values).

The third Composition [3], is the predication of Baldness of Russell. The availability of a set for the operation of predication is functionally dependent on a property having undergone extensionalization in the two preceding steps. Steps [1] and [2] can be rolled into one step, if we either eliminate one of the indices or roll worlds and times into pairs, as in a Montague-like two-sorted logic. Conversely, it is also an option to add a third (fourth, …) index, which will also require extensionalization. Even a pruned-down logic of predication must, however, contain two steps: first extensionalization, then predication.

Having set aside propositional functions, we turn now to primitive properties, as advocated by Bealer. Our outline of Bealer's theory is based on Bealer (1979, 1993), and Bealer and Mönnich (1984). Bealer provides his formal analysis of the predication inherent in a sentence like, 'Russell is bald', within the framework of an intensional algebraic model $M = \langle D, K, \tau \rangle$. D is the union of denumerably many disjoint subdomains, such that D_0 is the subdomain of propositions and D_1 the subdomain of properties (the two kinds of intensional entities we need here), while D_{-1} is the subdomain of individuals. K is a set of extensionalization functions.[80] The semantics of the extensionalization functions $\partial \in K$ is such that they assign the following possible extensions to individuals, propositions and properties:

$$x \in D_{-1} \rightarrow \partial(x)=x$$

$$x \in D_0 \rightarrow \partial(x)=n \text{ for } n \in \{0, 1\}$$

$$x \in D_1 \rightarrow \partial(x) \subseteq D.$$

τ is a set of truth-functional connectives and other operations. The set includes, inter alia, the operation pred_s of singular predication, which is, in the present case, defined as $D_1 \times D_{-1} \rightarrow D_0$. The semantics of pred_s is the following, if the quantificational range of y is restricted to D_{-1}:

$$\forall x \in D_1\ \forall y \in D_{-1}\ \forall \partial \in K\ (\partial(\text{pred}_s\langle x, y\rangle) = 1 \leftrightarrow y \in \partial(x)).$$

Finally, an interpretation function I assigns a value to the individual constant 'a', $I('a') = a \in D_{-1}$, and to the predicate 'P', $I('P') = P \in D_1$.

Singular predication of unary predicates satisfies the following truth-condition:

$$\partial(\text{pred}_s\langle P^1, a_1\rangle) = 1 \leftrightarrow a_1 \in \partial(P^1).$$

The extension of the proposition $\text{pred}_s\langle P^1, a_1\rangle$ is identical to the truth-value 1 iff the individual a_1 is an element of the extension of the property P^1. That is, the

[80] One of the extensionalization functions is G which 'tells us the actual extension of the elements of D' (1993, p. 25). G is comparable to Montague's $^\vee$ and susceptible to the same confutation that Tichý provides in (1988, pp. 151ff) of Montague's 'downer', that the identification of the function $^\vee$ requires empirical omniscience.

result of predicating a property of an individual is a proposition that is true iff the individual is in the extension of the property.

Assume that the interpretation function I has assigned the unary property $Baldness \in D_1$ to 'B' and $Russell \in D_{-1}$ to 'r'. Then the formal semantics of the predication of baldness of Russell is straightforward:

$$\partial(\text{pred}_s\langle B, r\rangle) = 1 \leftrightarrow r \in \partial(B).$$

That is, the proposition that Russell is bald is true iff Russell is in the extension of the property of Baldness that the semantic interpretation has assigned to 'B'.

One syntactic difference between Bealer's $\partial(\text{pred}_s\langle B, r\rangle) = 1 \leftrightarrow r \in \partial(B)$ and our $\lambda w \lambda t\ [{}^0Bald_{wt}\ {}^0Russell]$ immediately stands out. Whereas we write '$\ldots{}^0Bald_{wt} \ldots$' to indicate the extensionalization of *Bald*, Bealer does not write '$\ldots\partial(B)\ldots$' to signal extensionalization of B. Bealer uses property-to-set extensionalization only when stating the set-membership condition and not also prior to predicating B of r. In $\partial(\text{pred}_s\langle B, r\rangle)$, pred_s is a binary function that inputs an intensional and an extensional entity and outputs a new intensional entity. In fact, pred_s *must* be binary, since the property is not itself an operation: it is a functional argument that is not itself a function.

At the same time, there are alternative options available to TIL that exclude extensionalization of *Baldness* but preserve the intensional character of predication and are as such closer to Bealer's stance:

$$\lambda w \lambda t\ [{}^0Pred_{wt}\ {}^0Bald\ {}^0Russell]$$

and

$$\lambda w \lambda t\ [{}^0Instantiate_{wt}\ {}^0Russell\ {}^0Bald].$$

Types: $Pred/(o\ (o\iota)_{\tau\omega}\ \iota)_{\tau\omega}$; $Instantiate/(o\ \iota\ (o\iota)_{\tau\omega})_{\tau\omega}$.

However, what cripples either of these constructions as analyses of 'Russell is bald' is that they are analyses of other sentences; namely, 'Baldness is predicated of Russell' and 'Russell instantiates baldness', respectively. The two analyses are a far cry from qualifying as a literal analysis of 'Russell is bald'.

The reason why properties are logically inert in Bealer is because he construes properties, relations, and propositions (PRP's) as *primitive*, irreducibly intensional objects.[81] In general, that an entity e or an operation o is primitive relative to a system s is to say that e or o is not explicated within s. Instead e or o is used to explicate or define other entities or operations within s. The system lays down how e or o behaves technically, but any understanding of what e or o is in the first place must be obtained outside s.

[81] Bealer's M is *intensional*, because some elements of D defy the axiom of extensionality: they are necessarily co-extensional, yet not co-intensional. This is a hyperintensional notion of intensionality exceeding the notion of intensionality characteristic of possible-world semantics.

Bealer's system, in this sense, is his algebraic structure M. Since the PRP's are primitive within M, Bealer needs to add to his logic the operation $\text{pred}_s \in \mathcal{T}$ and earmark it specifically for singular predication.[82] But also pred_s is primitive by not being an instance of a more general operation that is either defined or explicated within M. Similarly, the extensionalization functions are also introduced into M as primitive operations rather than being instances of a more general operation. The operation of extensionalization is not needed to prevent incongruity, since $x \in D_1$ is immediately compatible with $y \in D_{-1}$. But the operation is primitive relative to M, and must be so, because the framework is designed to lack indices to figure as input for extensionalization. That PRP's are primitive is probably *the* leading philosophical idea informing Bealer's intensional algebra. For without this idea, his intensional logic could not treat PRP's as individuals, and the logic would fail to qualify as first-order. The price exacted for setting up a first-order intensional logic, on the other hand, is an abundance of primitive operations.

Consequently, $\partial(\text{pred}_s\langle B, r\rangle) = 1 \leftrightarrow r \in \partial(B)$ involves the primitive operations ∂, pred_s and the primitive entities B, r. Bealer's formal semantics of predication comes down to how some primitive operations operate on some primitive entities to generate a new primitive entity. The theory cannot tell us what predication and extensionalization are. Neither can it tell us what a property or an individual is. Nor is M designed to do any of this. It is all something we are supposed to understand pre-theoretically, or intuitively.

Since both extensionalization and predication are primitive, we cannot study the functional dependencies obtaining among these operations and the property, the individual, and the proposition. In particular, Bealer is in no position to say that predication is the application of a property to an individual, the value of which application is a proposition. Predication is instead a matter of applying the operation of predication to ⟨property, individual⟩ to obtain a proposition. But how does a property get predicated directly of an individual? Bealer's semantics tells us *what* pred_s does, by providing a truth-condition for the proposition that emerges from the predication. But to understand its truth-condition we must understand what extensionalization is; otherwise '$r \in \partial(B)$' will be meaningless to us. M tells us (by means of recursion, of which a fragment was reproduced above) what extensionalization does. But again, '$\partial(B)$' simply records the fact *that B* has been extensionalized: it does not tell us *how*. That is, the backtracking stops at ∂ and pred_s. If we do not already understand, pre-theoretically, what extensionalization is, $\partial(\text{pred}_s\langle B, r\rangle) = 1 \leftrightarrow r \in \partial(B)$ is not an informative analysis of 'Russell is bald' or of any other instance of singular predication. And if we do not understand, pre-theoretically, what predication is, we shall not understand the logical operation pred_s.

[82] We are leaving out of consideration Bealer's independently motivated introduction of the operation of predication, which has to do with his project of establishing a logic that is both *intensional* and *first-order*.

To see why Bealer's choice of primitives for his formal system may pose a philosophical-methodological problem, consider Bealer's view of the connection between intuition and formal rules:

> [O]ur intuitive grasp of these operations [including predication and extensionalization] can be codified by means of appropriate elementary rules (1993, p. 24).

However, it would appear to be open to doubt whether we actually have enough of a firm intuitive grasp of such 'techno-logical' notions as predication and extensionalization so as to suppose they can be introduced as primitive operations into a system of formal semantics. This is our general philosophical-methodological objection to Bealer's theory of predication. And even if we did understand, pre-theoretically, what predication and extensionalization are, then since Bealer's PRP's are not operations or conditions, $pred_s$ and ∂ need to be added as separate, primitive operations. But then they cannot be subsumed under one overarching primitive operation. However, we believe the following methodological guideline has something to be said for it: the fewer primitive notions (including operations) a theory is furnished with, the better. For then the formal theory is able to presuppose fewer notions (operations) be understood pre-theoretically and can instead elucidate a higher number of notions (operations). This way there is less of a risk of taxing our pre-theoretical intuitions beyond capacity. This is our foremost reason for holding that predication and extensionalization are not two functions, but two cases of Composition. Therefore, we also hold that we are better off with properties as functions rather than as functional arguments only.

If we treat properties and sets as functions, we can study how we form a new entity, e.g., a truth-value or a truth-condition, by means of Composition. The philosophy and logic of how the descent from intensions to extensions and the ascent (*via* Closure) from extensions to intensions work then become internal to the framework.

2.4.3 Montague's implicit intensionalization

Montague's Intensional Logic (IL) steers a middle course between actualism and explicit intensionalization. Montague introduces linguistic IL types as follows:[83]

i) e and t are IL types
ii) If α and β are IL types, then $(\alpha\beta)$ is an IL type
iii) If α is an IL type, then $(s\alpha)$ is an IL type.

[83] For details see Montague (1974a), and also Gamut (1991, pp. 117–138) or Muskens (1989, pp. 6–24).

Thus we have e and t as basic types, e the type for entities and t for truth-values. Clause (ii) is a rule for forming functional types for mappings from α to β.[84] Clause (iii) makes it possible to form an intensional type $(s\alpha)$ from a type α. However, s is itself not a type. In IL there are no expressions referring to elements of s; in particular, there are no variables ranging over s. Expressions of any intensional type $(s\alpha)$ are interpreted in terms of functions mapping possible worlds to elements of the interpretation domain corresponding to the type α.

Terms of the IL language are defined in the usual inductive way. For each IL type α, the terms of type α are:

- *Constants* and *variables* of the type α
- *Formulae* of type t (atomic φ, ψ, molecular $\neg\varphi$, $\varphi \wedge \psi$, and universal $\forall x\varphi$)
- *Identity ($A = B$),* where A, B are terms of the same type.
- *Application (AB)* of A of type $(\alpha\beta)$ to B of type α is a term of type β.
- *λ-abstraction $\lambda x(A)$* is a term of type $(\alpha\beta)$, where A is of type β, x of type α.

However, since there are no variables of type s, terms denoting intensions of type $(s\alpha)$ cannot be defined by λ-abstraction. Similarly, intensional descent of an intension to a particular world cannot be defined by application. Instead, Montague introduces two operators $^\wedge$ (read 'cap' or 'up') and $^\vee$ (read 'cup' or 'down'). In order to imitate λ-abstraction and application, respectively, two special types of terms are defined:

- If A is a term of type α, then $^\wedge(A)$ is a term of type $(s\alpha)$.
- If A is a term of type $(s\alpha)$, then $^\vee(A)$ is a term of type α.

Thus, for instance, if φ is a term of type t, $^\wedge\varphi$ is a term referring to a function from possible worlds to truth-values, i.e., a proposition; if A is a term of type e, $^\wedge(A)$ refers to a function from possible worlds to individuals, i.e., an individual concept (what TIL calls an individual office).

Due to the lack of variables ranging over possible worlds, also modalities can only be accommodated by means of an additional operator earmarked especially for modalities. Thus $\Box$ stands for necessity:

- If φ is a formula then $\Box\varphi$ is a formula.

And moreover, since there is no type for times, there are also special temporal operators F and P standing for 'future' and 'past', respectively:

- If φ is a formula, then $F\varphi$ and $P\varphi$ are formulae.

Intensions are thus modelled as functions from possible worlds and times to a type α. However, there is no means of handling temporal and modal parameters separately.

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[84] Note that Montague uses left-to-right notation for functional types, unlike TIL.

Nonetheless, at first glance IL seems to be a both elegant and simple theory of natural-language semantics. For one thing, IL is an extensional logic, since the axiom of extensionality is valid:

$$\forall x\,(Ax = Bx) \rightarrow A = B.$$

This is a good thing. However, the price exacted for this simplification of the language is too high. In particular, the law of universal instantiation, lambda conversion (β-rule) and Leibniz's Law do not generally hold, all of which is rather unattractive. Restrictions must be imposed to obtain valid instances of the laws. No wonder that these restrictions concern the operators ^, □, F and P, because these operators actually imitate the scope of λ-abstraction. However, the respective variables of abstraction are sorely missed. Thus, for instance, β-transformation

$$\lambda x\,(A)B = [B/x]A$$

is valid, provided the substitution of B for x happens without collision (B being free for x in A), *and*

(a) no free occurrences of x in A lie within the scope of the operators ^, □, F and P, or
(b) B is 'modally closed' (i.e., built up from variables and terms of the form $^\wedge A$, $\Box A$ using only connectives, quantifiers and the λ-operator).

Of course, all these restrictions make the logic much less transparent than would be desirable. Worse, even if we employ this restricted version of lambda conversion, IL does *not* validate the Church-Rosser 'diamond'. It is a well-known fact that an ordinary typed λ-calculus will have this property. Given a term $\lambda x(A)B$ (the *redex*), we can simplify the term to the form $[B/x]A$, and *the order in which we reduce particular redeces does not matter*. The resulting term is uniquely determined up to α-renaming of variables.

Unfortunately, IL fails to satisfy the Church-Rosser property even if the redeces are restricted to satisfy conditions (a) and (b) above. Muskens adduces an example taken from Friedman and Warren (1980):

(1) $\lambda x\,(\lambda y\,(^\wedge y = f(x))x)c$

where x, y are variables of type α, c a constant of type α and f a variable of type $(\alpha(s\alpha))$. We can reduce the term in two different ways, neither of which can be further reduced:

(2) $\lambda y\,(^\wedge y = f(c))c$

(3) $\lambda x\,(^\wedge x = f(x))c.$

The reason typically cited for IL displaying such a deviant behaviour is that the logic has been designed to reflect opacity phenomena of natural language. But we agree with Muskens that this is actually a serious deficiency:

> [T]o take an example, the sentences *Sara is Miss America* and *Necessarily Sara is Sara* do not entail the sentence *Necessarily Sara is Miss America*: and so, it is argued, (10) [Leibniz's law] should not be valid … According to this view, the inelegancies of IL should be weighed against the fact that the logic truly reflects natural language in an important respect.
>
> But there is an objection against this argument that derives from the nature of our trade. We are doing semantics, that is, we are after a theory of meaning for natural language. The subject matter of our theory, then, is *meanings* (or *intensions* if you like the word better) and identity in our theory is most naturally interpreted as identity of meaning, or *synonymy*. From this it follows that we should not formalise the sentence *Sara is Miss America* as an identity statement; the word *is* simply does not express synonymy, it expresses *coreference* and the sentence expresses the fact that the two names happen to have the same bearer in the present situation. Opacity phenomena in natural language are no counterexamples to Leibniz's Law, they merely illustrate that the forms of the verb *be* do not express identity of meaning (although they express identity of reference) (1989, pp. 10–11, 1995, pp. 24–25).

In effect, the operators ^, □, F and P are equipped with hidden 'ghost' variables. For want of possible-world variables, the reduction of (2) and (3) would not be free of collision, which can be easily shown if we replace the cap operator by variables and λ-abstraction:

(1′) $\lambda x\,(\lambda y\,(\lambda w\,(y) = f(x))x)c_w$

where c is now a term of type $(s\alpha)$ and c_w of type α.

There are two ways of reducing, namely either the outer redex first,

(2′) $\lambda y\,(\lambda w(y) = f(c_w))c_w)$

or the inner redex first,

(3′) $\lambda x\,(\lambda w(x) = f(x))c_w$.

But this time the terms (2′) and (3′) can be further reduced via α-transformation, which makes it possible to avoid collision of variables:

(4) $\lambda w^*(c_w) = f(c_w)$.

Note, however, that even though the above reduction is without collision, the resulting terms on the left-hand and right-hand side of (4) are *not* equivalent if involving partial functions, since λ-*abstraction is always defined* whereas $f(c_w)$ may be undefined.[85]

There is a variant of type theory which can replace IL when doing Montagovian semantics. From a formal point of view, we can introduce a two-sorted variant known as TY_2. The adjustment actually consists in introducing the fixed type *s* interpreted as world-time couples, and variables *i*, *j*, … to play the role of world/time indices. The operators $^\wedge$, $^\vee$, and $\Box$ are then reduced to λ-abstraction, application and universal quantification, respectively.

TY_2 has more plausible properties than IL, but is still too restricted so as to capture the semantics of natural language in a logically perspicuous way.

First, we need to be able to manipulate worlds and times separately. As we will see in Section 2.5, a constructed intension can undergo, in one context, descent with respect to a time parameter, in another context with respect to a modal parameter, and in still another context with respect to both parameters.

Second, the functions of TY_2 are restricted to unary total functions. But we need to be able to work with partial functions, unless we rest content with an unmanageable explosion of domains. It is neither possible to restrict the logical space in an ad hoc way so as to avoid working with non-referring terms like 'the King of France', nor philosophically plausible, though technically possible, to introduce so-called impossible possible worlds counting the 'existing' King of France in their domain.

Moreover, functions typically have more than one argument. Usually we are told that *n*-ary functions can be *represented* by unary composite functions. Schönfinkel (1924) observed that there is a one-to-one isomorphic correspondence between *n*-ary functions and certain unary composite functions. For instance, a two-argument function $f/(\iota\ \iota\iota)$ from couples of individuals to individuals can be represented by an unary function $f_1/((\iota\iota)\iota)$ mapping individuals to functions from individuals to individuals.[86] However, this isomorphism breaks down when *partial* functions are involved, as Tichý showed in 1982.[87] One and the same partial multi-argument function may correspond to more than one unary function. Here is Tichý's example (slightly paraphrased):

Let a/ι be an individual and $f/(\iota\ \iota\iota)$ a two-argument function defined as follows:

$$f(x, y) = y \text{ for } x \neq a, \text{ and } f(a, y) \text{ is undefined.}$$

Two unary functions f_1, f_2, both of type $((\iota\iota)\iota)$, correspond to f:

$$f_1(x) = \lambda y(y) \text{ for } x \neq a, \text{ and } f_1(a) \text{ is undefined.}$$

---

[85] The proof of the non-equivalence of the above β-reduction schema (even if no collision of variables arises) in case of involving partial functions is provided in Section 2.7 together with rules of substitution. Different types of identities will be dealt with in Section 3.3.

[86] Now we are using the TIL right-to-left notation for types.

[87] See 2004, (pp. 467–8).

$$f_2(x) = \lambda y(y) \text{ for } x \neq a,$$
and $f_2(a)$ is defined as the degenerate function of type $(\iota\iota)$;
i.e., the function undefined at all its ι-arguments.

Clearly, $f_1 \neq f_2$.

In Section 2.7 we provide examples of an application of a partial binary function to an argument using the non-equivalent 'syntactic' form of the β-rule ('by name') where we actually meet with a similar problem. The resulting reduced construction is not equivalent to the original non-reduced one. The former *v*-constructs a function of the f_2 form, the latter a function of the f_1 form. Therefore, we formulate a general valid β-rule ('by value') whose execution reduces the original construction to an equivalent one.

A final, and more general, objection to IL is that it fails to accommodate *hyperintensionality*, as indeed any formal logic interpreted set-theoretically is bound to. Only when embracing an algorithmic/procedural semantics are we able to handle hyperintensional meanings of natural-language expressions. Any theory of natural-language analysis needs a hyperintensional (preferably procedural) semantics in order to render synonymy in natural language accurately, as well as to adequately analyse hyperintensional attitude reports (see Chapter 5).

In global terms, without constructions TIL is an anti-contextualist (i.e., transparent), explicitly intensional modification of IL. With constructions, TIL rises above the model-theoretic paradigm and joins instead the paradigm of hyperintensional logic, structured meanings and procedural semantics.

2.5 Modal and temporal interplay

We turn now to studying the interplay between worlds and times; i.e., the interplay between modal and temporal variability. We are studying the interplay, within one world, between two times, between a time and an interval, and between two intervals. We are not studying the interplay, across two worlds, between two times and the other combinations just mentioned, because we do not want to assume that time behaves uniformly throughout logical space. The central aim of the inquiry is to pin down the semantics of temporal modifiers such as *Frequently*, as they occur in, e.g., 'Henry VIII's wife is frequently sick'.

We argued in Section 2.4 that all empirical terms and expressions are intensional in the sense of involving a modal parameter and often also a temporal one. In short, a logical analysis of an empirical expression must take into account *modal* and frequently also *temporal variability*. Let us briefly recapitulate these two points.

Modal variability. Let E be an empirical expression. Let T be a time and o the object that E refers to in a given world W at T. E empirical, o is only contingently the reference of E in W at T. So there must be at least one possible world w' such that E does not refer to o in w' at T.

For an example, consider the sentence 'Warsaw is the capital of Poland'. Let T be the present time. The sentence refers in the actual world at T to the truth-value **T**. Even so, the fact that the sentence is true at T does not justify the view that it would be true necessarily. It holds even for T that *it might have been otherwise*; in other words, there is at least one world w' where the sentence is not true at time T. In general, what is true might have been false; what is false might have been true. Since TIL includes truth-value gaps, there are cases where what lacks a truth-value might have had a truth-value and what has a truth-value might have lacked one.

Temporal variability. Let W be the actual world, and let o be the object that E refers to in W at a moment t. E is an empirical expression, so there is no logical guarantee that o is referred to by E in W at any other time t'.

For instance, 'Warsaw is the capital of Poland' is true at the actual world at, say, noon on August 1, 2009. At the same world it was, however, false at all times when the capital of Poland was another city than Warsaw (e.g., Cracow, as it happened), and it lacked a truth-value whenever no city was the capital of Poland.

When characterizing modal variability, we used the phrase, 'There must be at least one possible world such that...'. We did not use the analogous phrase, 'There must be at least one time such that...' in the case of temporal variability. Instead we said, 'There is no logical guarantee that…'. Why this distinction? The reason is because there are empirical expressions whose reference is 'eternal' at the given world. For example, the sentence

'At noon on September 15, 2004 it is raining in Prague'

if true at the given world, is true at the given world eternally. This is so because a specific time of reference has been fixed. If we disregard tenses and interpret the sentence as the implication 'If it is noon on September 15, 2004 then it is raining in Prague', the analysis of the sentence is this. Let $T/(o\tau)$ be a given time interval (e.g., around noon on September 15, 2004); $Rain\_in/(o\mu)_{\tau\omega}$: a property of a place of type μ; $Prague/\mu$:[88]

$$\lambda w \forall t\, [[{}^0Tt] \supset [{}^0Rain\_in_{wt}\, {}^0Prague]].$$

The Composition $[[{}^0Tt] \supset [{}^0Rain\_in_{wt}\, {}^0Prague]]$ v-constructs **T** for every valuation v, provided the Composition $[{}^0Rain\_in_{wt}\, {}^0Prague]$ $v(t'/t)$-constructs **T**, where t' is any time belonging to the interval T. For all other times the Composition v-constructs **T** due to the definition of implication. Thus, the truth-value that the proposition denoted by the sentence takes in a given world w does not depend on the time at which the sentence is being evaluated.

[88] μ is a placeholder for the type of the specification of a particular place on Earth (whatever its exact type may turn out to be); for instance, specification by means of GPS coordinates. See also Section 5.2.

But the eternality of such expressions does not mean that they would not be empirical. We have explained in Chapter 1 that empirical expressions denote non-constant intensions, and the type of an intension is typically $\alpha_{\tau\omega}$. However, the existence of 'eternal sentences' shows that there are non-trivial intensions lacking a temporal factor; their type is $(\alpha\omega)$, $\alpha \neq (\beta\tau)$.

Where $p \rightarrow o_{\tau\omega}$, the Closure

$$\lambda w\, \lambda p\, [\forall t\, p_{wt}]$$

constructs the propositional property of holding eternally.[89] The Closure

$$\lambda w\, \lambda p\, [\exists t\, p_{wt}]$$

constructs the propositional property of holding at least sometimes.

2.5.1 Supposition de dicto with respect to temporal parameters

In Section 1.5.2 we explicated the traditional categories *de re* and *de dicto* within TIL. To recall these two distinct suppositions, consider the sentences

(1) 'Henry's wife was born in Düsseldorf'

(2) 'Catherine is going to become Henry's wife'.

As always, we begin with a *type-theoretical analysis*: *Henry*, *Catherine*, *Düsseldorf*/ι; (the/a) *Wife_of*/$(\iota\iota)_{\tau\omega}$; *Henry's_wife*/$\iota_{\tau\omega}$; *Born_in*/$(o\iota\iota)_{\tau\omega}$; (going to) *Become*/ $(o\iota\iota_{\tau\omega})_{\tau\omega}$: the relation-in-intension of an individual to an office that the individual is going to occupy.

Synthesis: the definite description 'Henry's wife' invariably denotes the office *Henry's_wife* and has the same meaning in all sentences, including (1) and (2). The office is constructed by combining the constituents ^{0}*Wife_of*, ^{0}*Henry* into the Composition [^{0}*Wife_of*$_{wt}$ ^{0}*Henry*] and abstracting over w, t to obtain $\lambda w \lambda t$ [^{0}*Wife_of*$_{wt}$ ^{0}*Henry*]. What varies is not the meaning of 'Henry's wife' (or synonymously 'the wife of Henry') but the supposition with which it occurs. Whereas in sentence (1) the office *Henry's_wife* is used as a pointer to an unspecified individual, whereby only its value at the world and time of evaluation matters, sentence (2) can be true even if the office goes vacant at that world and time. The entire office matters, because Catherine is going to occupy it, regardless of the individual, if any, who occupies it at the world and time at which it is true that Catherine is going to (but does not yet) occupy the office. Thus the meaning of

[89] In Section 4.5 we shall argue that such a Closure constructs nomological necessity.

'Henry's wife', $\lambda w \lambda t\ [{}^0\mathit{Wife\_of}_{wt}\ {}^0\mathit{Henry}]$, is used with *de re* supposition in (1) and with *de dicto* supposition in (2). The analyses are as follows.[90] Sentence (1) ascribes the property of being born in Düsseldorf to whatever individual is Henry's wife. The property is constructed thus $(x \rightarrow \iota)$:

$$\lambda w \lambda t\ \lambda x\ [{}^0\mathit{Born\_in}_{wt}\, x\ {}^0\mathit{Düsseldorf}].$$

First, the property has to be extensionalized, by Composing its construction with w and t:

$$\lambda w \lambda t\ \lambda x\ [{}^0\mathit{Born\_in}_{wt}\, x\ {}^0\mathit{Düsseldorf}]_{wt},$$

or equivalently,

$$\lambda x\ [{}^0\mathit{Born\_in}_{wt}\, x\ {}^0\mathit{Düsseldorf}].$$

Afterwards, the result of the extensionalization is to be applied to the individual who happens to play the role of Henry's wife:

(1′) $\lambda w \lambda t\ [\lambda x\ [{}^0\mathit{Born\_in}_{wt}\ x\ {}^0\mathit{Düsseldorf}]\ \lambda w \lambda t\ [{}^0\mathit{Wife\_of}_{wt}\ {}^0\mathit{Henry}]_{wt}]$

(2′) $\lambda w \lambda t\ [{}^0\mathit{Become}_{wt}\ {}^0\mathit{Catherine}\ \lambda w \lambda t\ [{}^0\mathit{Wife\_of}_{wt}\ {}^0\mathit{Henry}]]$.

In (1′) the construction $\lambda w \lambda t\ [{}^0\mathit{Wife\_of}_{wt}\ {}^0\mathit{Henry}]$ occurs extensionally, being Composed with the left-most w and t of evaluation, whereas in (2′) this construction occurs intensionally, with *de dicto* supposition. Note that the latter holds both with respect to the modal (λw) and temporal (λt) factor.

Now we wish to show how the two modal parameters, when coupled with explicit intensionalization, make it possible to analyze expressions whose meaning occurs *de dicto* only with respect to one of the temporal/modal factors.[91] Such examples are run-of-the-mill; e.g., 'My next-door neighbour has frequently been sick.'[92] However, the phrase 'my next-door neighbour' is ambiguous between denoting an individual office and a property, so we prefer 'The wife of Henry VIII', which unambiguously denotes an office against a cultural background of monogamy. The sample sentence is

(3) 'The wife of Henry VIII is frequently sick'.

---

[90] Tenses are still disregarded. See, however, Section 2.5.2.

[91] As for explicit intensionalization, TIL stands alone: no other logico-semantic theory uses variables for possible worlds and times in a systematic way. As for the ability to handle modal and temporal parameters as separate factors, one may find some places in the standard literature where both factors are handled simultaneously (in particular in connection with counterfactuals; see, e.g. Gamut, 1991, § 2.5).

[92] See Tichý (1986a, pp. 260ff, 2004, pp. 658ff.)

The sentence is ambiguous between interpretations *de re* and *de dicto*, and so is compatible with two different scenarios.

Scenario I (*de re*). Whoever is Henry's wife has the property of being frequently sick. If there is no such individual as Henry's wife, the proposition denoted by (3), as well as that denoted by 'Henry's wife is *not* frequently sick', has no truth-value. And if Anne Boleyn is Henry's wife, then it follows that Anne is frequently sick. For instance, it is compatible with this scenario that during the last 10 years she was sick once a week (poor woman!). Which does not, of course, mean that during this period the office *Henry's_wife* is occupied by one and the same individual. Henry might have divorced several times during this period and married a string of women. However, only the occupant (if any) of the office matters when evaluating the truth-value of the sentence, namely the occupant at the $\langle w, t\rangle$-pair at which the proposition is being evaluated. So the meaning of 'Henry's wife' occurs *de re*.

Scenario II (*de dicto*). Now the meaning of 'Henry's wife is sick' occurs *de dicto*. It no longer matters who, if anybody, is Henry's wife. What does matter is whether the proposition that Henry's wife is sick is frequently true. Imagine the entire time span during which Henry was married: more often than not did Henry find himself married to a sick woman (poor man!). This scenario leaves it open whether the property of being sick was distributed over one, two or multiple women. One variant of the scenario is that Henry has a 10-year career as a husband (possibly with interruptions), having been married to five women and currently being on his sixth. He was married to his first wife for 9 years, while he has consumed a rapid succession of wives during the last year. The last four wives were all in stellar shape, and so is the sixth one, while the first wife was sick most of the time. Another variant of the scenario is this. During the last 10 years six women were Henry's wife. Each of them was sick once a week, but only when being Henry's wife. Henry's current wife is Catherine. It does not follow that Catherine has been frequently sick, but it does follow that it is frequently the case that Henry's wife is sick. All the occupants of the office *Henry's wife* matter, not just the current one.

Since sentence (3) is two-way ambiguous, it has to be analyzed such that two constructions are associated with it as its two different meanings. Moreover, these two constructions need to be non-equivalent, since the two scenarios are associated with two distinct truth-conditions.

Type-theoretical analysis:

Henry/ι; *Henry's_wife*/$\iota_{\tau\omega}$; *Sick*/$(o\iota)_{\tau\omega}$; *Frequent* is a temporal modifier: for any time t it returns the class of such intervals as are frequent with respect to t, so the type of *Frequent* is $((o(o\tau))\tau)$.[93]

[93] The obvious, and inherent, vagueness of this modifier is irrelevant to our present purposes, since we merely need to type *Frequent* as a *temporal* modifier.

The office *Henry's wife* is constructed by $[\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]$. The property of being frequently sick is constructed as follows:

$$\lambda w \lambda t\ [\lambda x\ [{}^0Frequent_t\ \lambda t'[{}^0Sick_{wt'}\,x]]].$$

Gloss: $\lambda t'\ [{}^0Sick_{wt'}\,x]$ *v*-constructs the period in which *x* is sick. If it is the case, at *t*, that *x* has been frequently sick in this period, then $[{}^0Frequent_t\ \lambda t'[{}^0Sick_{wt'}\,x]]$ *v*-constructs **T**; otherwise **F**. The property is constructed by Abstracting first over *x* and then over *w* and *t*.
Remark. Renaming the variable *t* for *t′* in the inner Closure is illustrative, but not necessary.

Synthesis of the de re *reading*:
The property constructed above must be applied to whomever (if anybody) plays the role of Henry's wife:

(3_{re}) $\quad \lambda w \lambda t\ [\lambda w \lambda t\ [\lambda x\ [{}^0Frequent_t\ \lambda t'[{}^0Sick_{wt'}x]]]_{wt}\ \lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]_{wt}].$

Gloss: In any world *w* (λw) at any time *t* (λt) check whether the occupant of the office *Henry's wife* ($\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]_{wt}$) instantiates at that $\langle w, t\rangle$ the property of being frequently sick $\lambda w \lambda t\ [\lambda x\ [{}^0Frequent_t\ \lambda t'[{}^0Sick_{wt'}\,x]]]_{wt}$.

The Composition $[\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]_{wt}$ occurs *de re* in (3_{re}), because the truth-value at the $\langle w, t\rangle$ pair of evaluation of the proposition constructed by (3_{re}) depends only on the ι-value (if any) at the $\langle w, t\rangle$ pair of evaluation of the office constructed by the Closure $[\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]$. This is why $\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]$ need to occur extensionally in (3_{re}). If there is no wife of Henry VIII, the so constructed proposition has no truth-value. And if Henry's wife is Jane, or Catherine, or whichever other lady, then the proposition takes the value **T** or **F**, according as said lady is frequently sick.

(3_{re}) can be equivalently β-reduced to

$(3_{re}')$ $\quad \lambda w \lambda t\ [\lambda x\ [{}^0Frequent_t\ \lambda t'[{}^0Sick_{wt'}x]]\ [{}^0Wife\_of_{wt}\ {}^0Henry]].$

Further β-reduction of $(3_{re}')$ by substituting $[{}^0Wife\_of_{wt}\ {}^0Henry]$ for *x* would, however, *not* be an equivalent transformation. Here is why. In Section 1.5.2.2 we characterised three kinds of context (hyperintensional, intensional, extensional) and explained why a higher context is dominant over a lower one. Here we encounter the problem of an intensional context dominating an extensional one. The sub-Closure $\lambda t'\ [{}^0Sick_{wt'}x]$ *v*-constructs an interval, an (οτ)-object. The interval occurs intensionally in $(3_{re}')$, because this function is not applied to its τ-argument. We say that the context of $\lambda t'[{}^0Sick_{wt'}\,x]$ is τ-*generic*.[94] Thus the constituent $x/*_1 \rightarrow \iota$ occurs in the τ-generic intensional context of the Closure $\lambda t'[{}^0Sick_{wt'}\,x]$, and $\lambda t'$

[94] See Section 2.7 for details.

$[{}^0Sick_{wt'}\ x]$ occurs in $(3_{re}')$ with *de dicto* supposition with respect to the *temporal* parameter.

If we now substitute the Composition $[{}^0Wife\_of_{wt}\ {}^0Henry]$ for the variable x, we draw its extensional *de re* occurrence into the (τ-generic) intensional context of $\lambda t'$ $[{}^0Sick_{wt'}\ x]$. The occurrence of $[{}^0Wife\_of_{wt}\ {}^0Henry]$ becomes *de dicto* due to the dominancy of the intensional context, and a non-equivalent construction (3_{redex}) arises:

(3_{redex}) $\lambda w \lambda t\ [{}^0Frequent_t\ \lambda t'[{}^0Sick_{wt'}[{}^0Wife\_of_{wt}\ {}^0Henry]]]$.

The problem is created by the fact that $[{}^0Wife\_of_{wt}\ {}^0Henry]$ is v-improper for some valuations v. In those $\langle w, t\rangle$ pairs where a particular lady is the wife of Henry, the proposition takes **T** or **F** exactly as the proposition constructed by (3_{re}) does. However, the two propositions differ with respect to those $\langle w, t\rangle$ pairs where Henry is single. Let $\langle W, T\rangle$ be a pair at which Henry is single. The Closure $\lambda t'$ $[{}^0Sick_{wt'}\ [{}^0Wife\_of_{wt}\ {}^0Henry]]$ now $v(W/w, T/t)$-constructs a degenerate interval: the characteristic function of this interval is undefined at every argument. This is due to the fact that the Composition $[{}^0Sick_{wt'}\ [{}^0Wife\_of_{wt}\ {}^0Henry]]$ is $v(W/w, T/t)$-improper. Since the degenerate interval cannot belong to the class of such intervals as are frequent with respect to t, the Composition $[{}^0Frequent_t\ \lambda t'[{}^0Sick_{wt'}$ $[{}^0Wife\_of_{wt}\ {}^0Henry]]]$ v-constructs **F**, and the proposition constructed by the reduced Closure above takes **F** (whereas the proposition constructed by (3_{re}) is undefined).

Notice that it is necessary to use another variable $t' \rightarrow_v \tau$ in the inner Closure $\lambda t'[{}^0Sick_{wt'}\ x]$. Otherwise the substitution of $[{}^0Wife\_of_{wt}\ {}^0Henry]$ for the variable x into this Closure would cause variable t to collide, as it would become bound in this inner Closure.

Synthesis of the de dicto *reading*.

In order to disambiguate the sentence, we may rephrase (3) as

'Frequently, Henry's wife is sick'.

Now the frequency of being sick concerns Henry's matrimony rather than a particular spouse of his. What is meant is that the *proposition* that Henry's wife is sick frequently takes the value **T**. This proposition is constructed by

$$\lambda w \lambda t\ [{}^0Sick_{wt}\ [\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]_{wt}]$$

or in its β- and α-equivalent form,

$$\lambda w' \lambda t'\ [{}^0Sick_{w't'}\ [{}^0Wife\_of_{w't'}\ {}^0Henry]].$$

In order to *v*-construct the interval in which the proposition is **T** (in the world *w* of evaluation), we apply the proposition to *w*:

$$[\lambda t'[{}^0Sick_{wt'}\ [{}^0Wife\_of_{wt'}\ {}^0Henry]]].$$

By applying the modifier *Frequent* (at the time *t* of evaluation) to the so constructed interval, we *v*-construct **T** or **F** according as the interval is frequent or not; recall that *Frequent* is a function of type $((o(o\tau))\tau)$:

$$[{}^0Frequent_t\ [\lambda t'[{}^0Sick_{wt'}\ [{}^0Wife\_of_{wt'}\ {}^0Henry]]]].$$

Finally, Abstraction over the values of *w*, *t* constructs the proposition that it is frequently the case that Henry's wife is sick:

$$(3_{dicto}) \qquad \lambda w\lambda t\ [{}^0Frequent_t\ [\lambda t'[{}^0Sick_{wt'}\ [{}^0Wife\_of_{wt'}\ {}^0Henry]]]].$$

Gloss: In any world *w* (λw) at any time *t* (λt) check whether the periods of sickness of Henry's wives ($[\lambda t'[{}^0Sick_{wt'}\ [{}^0Wife\_of_{wt'}\ {}^0Henry]]]$) are frequent at *t*.

The Composition $[{}^0Wife\_of_{wt'}\ {}^0Henry]$ occurs in a τ-generic intensional context of the Composition $[{}^0Frequent_t\ [\lambda t'[{}^0Sick_{wt'}\ [{}^0Wife\_of_{wt'}\ {}^0Henry]]]]$. Thus in (3_{dicto}) the Composition $[{}^0Wife\_of_{wt'}\ {}^0Henry]$ occurs intensionally, with *de dicto* supposition with respect to the temporal parameter *t′*. In fact, the proposition constructed by (3_{dicto}) takes **T** or **F** even at those $\langle w, t\rangle$ pairs at which Henry has no wife and the proposition that Henry's wife is sick has been frequently true. If Jane or Catherine, or whichever other lady, is the current wife of Henry VIII, we cannot infer that this particular lady is frequently sick, because the whole office matters, not only its current value (cf. Scenario II).

Note that the three constructions (3_{re}), (3_{redex}) and (3_{dicto}) are not equivalent. They construct different propositions, as explained above. Of these (3_{re}) is the analysis of the *de re* reading of sentence (3), and (3_{dicto}) is the analysis of the *de dicto* reading of (3), or rather of its rephrased variant, 'Frequently, Henry's wives are sick'. The construction (3_{redex}) cannot be assigned as an analysis to either of these sentences. If we did not handle modal (ω) and temporal (τ) parameters separately, we would not be able to explain the differences embodied in the two analyses.

2.5.2 Tenses and truth-conditions

Up till now we did not take *tenses* into account. For instance, 'Henry's wife was born in Düsseldorf' was analysed in the previous section as denoting the proposition that Henry's wife *has* the property of being born in Düsseldorf. The analysis was

(1′) $\lambda w \lambda t\ [\lambda x\ [{}^0Born\_in_{wt}\ x\ {}^0Düsseldorf]\ [\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]_{wt}]$.

This Closure constructs a proposition that lacks a truth-value nowadays, since no lady can play the role of wife of Henry VIII.[95] Thus the Composition $[\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]_{wt}$ is v-improper for the actual world and the present time, and so is the Composition $[\lambda x\ [{}^0Born\_in_{wt}\ x\ {}^0Düsseldorf]\ [\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]_{wt}]$. Hence, the proposition constructed by (1′) lacks a truth-value. Yet one might object that the sentence is true, because one of Henry's wives *was* indeed born in Düsseldorf, namely Anne of Cleves. We will deal with this case at the end of this section.

The sentence 'The wife of Henry VIII is frequently sick' is not true nowadays, either. If read *de re*, it is neither true nor false for want of a unique wife of Henry VIII. Read *de dicto*—'Frequently, Henry's wife is sick'—it is false, because nowadays it is not the case that the proposition that Henry's wife is sick is frequently true, since no-one is Henry's wife nowadays.

Consider, however, the *simple past* of the *de dicto* variant of (3), namely

(3p) 'It was frequently the case that Henry's wife was sick'.

Nowadays the sentence might be true, and presumably is. Thus, there is undeniably a difference in truth-condition between a sentence in the present tense and in the past tense. This fact has been observed by numerous logicians, and many variants of so-called *temporal logic* have been developed. The term 'temporal logic' is broadly used to cover all approaches to the representation of the temporal dimension within a logical framework. More narrowly, it is also used to refer to a particular modal system of temporal propositional logic that Arthur Prior introduced in (1957, 1962 and 1967) under the name 'tense logic'.

The logical language of Prior's tense logic contains, in addition to the usual truth-functional operators, four modal operators whose intended meanings are:

P 'It has at some time been the case that …'
F 'It will at some time be the case that …'
H 'It has always been the case that …'
G 'It will always be the case that …'.

[95] The particular *individual* that was born in Düsseldorf and used to play the role of Henry's wife has, of course, had this property ever since September 22, 1515, even though the respective lady lost the property of walking the face of the earth long ago. Yet the sentence does not mention Lady Anne of Cleves, or the date of her birth. The sentence is about the respective *office*, which is presently vacant.

P and *F* are known as the *weak tense operators*, while *H* and *G* are known as the *strong tense operators*. Prior developed a formal system of tense logic with axioms like

$Gp \rightarrow Fp$	'What will always be will be';
$G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq)$	'If *p* will always imply *q*, then if *p* will always be the case, so will *q*';
$Fp \rightarrow FFp$	'If it will be the case that *p*, it will be the case that it will be that *p*';
$\neg Fp \rightarrow F\neg Fp$	'If it will never be that *p* then it will be that it will never be that *p*'.

Similarly for the past operators *P*, *H*; e.g., $Hp \rightarrow Pp$, 'What has always been has been'.

Subsequently, systems of temporal logic have been further developed by computer scientists, notably Zohar Manna and Amir Pnueli,[96] and widely used for formal verification of programs and for encoding temporal knowledge within artificial intelligence. These logics are undeniably simple, elegant and logically convenient. However, simplicity and convenience do not always go hand in hand with *logical adequacy*, and the systems just mentioned fall prey to this problem.

The approach of the modality-based propositional tense logics just sketched is claimed to be applicable also as a framework within which to define the semantics of temporal expressions in natural language. However, despite the great applicability of tense logic in the semantics of programming languages, they suffer a major drawback when applied to the semantics of natural language. The drawback is their inability to adequately analyze sentences that require a specification of a *point of reference.*

To highlight the problem, consider

(4) 'Catherine of Aragon *has been* sick'

and

(5) 'Catherine of Aragon *was* sick'.

Tense logic assigns to these two sentences the same truth-conditions, formally expressed in terms of *P*(*A*), *P* being the (weak) 'Past' operator.[97] Yet while (4) has

[96] See Manna and Pnueli (1992, 1995).

[97] —presumably applicable to a proposition; however, since tense logic is a *propositional* modal logic, the arguments of the truth-functions are truth-values. Thus there is the same category mistake as in ordinary modal logic (see Section 1.2.2 for details).

a self-contained meaning,[98] (5) is incomplete in isolation. A sentence like (5) needs a *point of reference*. In order to fully understand the sentence, one must know at *what particular time* Catherine was sick. (5) provokes the question, '*When* was Catherine of Aragon sick?' The answer must provide a point in time.

2.5.2.1 Simple past

The point of reference is often an interval, as in

(6) 'Catherine of Aragon *was* sick throughout 1530'.

In general, a definite instant or interval must be cited, directly or indirectly, as the time *when* Catherine was sick. But this point of reference cannot be rendered when using the operator *P*, because its argument is a proposition. The simple past is best seen as the time-dependent relation *Past* between a class of intervals and a point of reference. The reason is because it is *intervals* that divide into those throughout which Catherine was sick without interruption and those throughout which she was not. 'Throughout' thus denotes the world-dependent function *Through* that takes a proposition *Prop* to the class of those intervals contained in the chronology of *Prop*; i.e., in the set $Prop_w$. Hence, the type of *Through* is $((o(o\tau))o_{\tau\omega})_\omega$. Especially, given a particular world *W*, *Through* takes the proposition that Catherine is sick to the set of intervals which are, at *W*, distinguished by Catherine's uninterrupted sickness. Thus the Composition

$$[{}^0Through_w\ \lambda w\lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]]$$

v-constructs a class of intervals during which Catherine is sick.

This is a first approximation to the truth-conditions of (6). The second obvious point is this. The sentence (6) speaks about two things: the underlying proposition that Catherine is sick and the point of reference, viz. the year 1530. The sentence may be true at any time in 2009, or even at any time after 1530, but *before* 1530 it was not true. Actually, the sentence 'Catherine of Aragon *was* sick throughout 1530' was neither true nor false before 1530. The sentence in the past tense not only *implies* but also *presupposes* that the time of reference should lie in the past. What (6) means is that the year 1530 is a *past* interval and that the interval is a member of the value taken by *Through* at the proposition that Catherine is sick.

[98] One may object that since Catherine of Aragon is not alive anymore, it is out of place to use the present perfect when talking about her in A.D. 2009. Yet the sentence has a complete meaning, and if evaluated sometime between 1485 and 1536, the sentence might have been true. In 2009 it lacks a truth-value, because the proposition it denotes presupposes that the individual in question should still be alive. Thus in 2009 it would be more appropriate to use the sentence, 'Catherine of Aragon *had* frequently *been* sick'.

The definition of *Past* is as follows. It is a time-dependent function that takes a class of ο-chronologies (the intervals in which a given proposition is true) together with an (implicit or explicit) interval serving as point of reference and returns **T, F** or no value, according as the interval serving as point of reference belongs to the respective class of ο-chronologies *and* precedes the time T at which the proposition denoted by the sentence is being evaluated. Thus, *Past* is typed as $((o(o(o\tau))(o\tau))\tau)$.

Our resources up to now yield the following coarse-grained analysis of (6):

(6′) $\lambda w\lambda t\ [{}^0Past_t\ [{}^0Through_w\ \lambda w\lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]]\ {}^0Y1530]$.

Types: $Past/((o(o(o\tau))(o\tau))\tau)$; $Through/((o(o\tau))o_{\tau\omega})_\omega$; $Sick/(o\iota)_{\tau\omega}$; $Catherine/\iota$; $Y1530/(o\tau)$: the year 1530.

The analysis can be made more fine-grained, though. To achieve this, we need to come up with compound constructions equivalent with ${}^0Past$ and ${}^0Through$. Let $p \rightarrow_v o_{\tau\omega}$, $c \rightarrow_v (o\tau)$: the interval serving as point of reference. The definition of *Through* is straightforward:

$$^0Through = \lambda w\ \lambda p\ \lambda c\ \forall t\ [[c\ t] \supset p_{wt}].$$

The Composition $\forall t\ [[c\ t] \supset p_{wt}]]$ specifies the condition that the interval c be contained in the chronology of the proposition p. This condition can be expressed as the construction $[{}^0\subset c\ p_w]$; or in infix notation without Trivialisation of $\subset$: $[c \subset p_w]$; $\subset/(o(o\tau)(o\tau))$. The application of *Through* to the proposition that Catherine is sick yields these constructional equivalences:

$$[{}^0Through_w\ \lambda w\lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]] =$$

$$\lambda c\ \forall t\ [[c\ t] \supset [{}^0Sick_{wt}\ {}^0Catherine]] =$$

$$\lambda c\ [c \subset \lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]].$$

For instance, if in a world W Catherine was sick without interruption during January 1528, as well as during the year 1530, then the class of the intervals {January 1528, year 1530} is a subset of the class of intervals (W/v)-constructed by $\lambda c\ [c \subset \lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]]$.

We now turn to defining *Past*. Let $s \rightarrow_v (o(o\tau))$ be the class of ο-chronologies of a given proposition; $b \rightarrow_v o$, a variable ranging over truth-values; $Sing/(o(oo))$, the function *singularizer* that takes a singleton containing just one truth-value to this truth-value, and is otherwise undefined. The relation *Past* is then defined as follows:[99]

[99] In order to make our encoding of constructions easier to read, we now use infix notation like ‘$t_1 < t_2$’ to stand for the Composition $[{}^0< t_1\ t_2]$.

$$^{0}Past = \lambda t\, \lambda sc\, [^{0}Sing\, \lambda b\, [\forall t'[[c\, t'] \supset [t' < t]] \wedge b{=}[s\, c]]].$$

$\forall t'[[c\, t'] \supset [t' < t]]$ specifies the condition that the interval c serving as point of reference precede time t. If the condition is not fulfilled, then the Composition v-constructs **F** and the class of truth-values v-constructed by

$$\lambda b\, [\forall t'[[c\, t'] \supset [t' < t]] \wedge b{=}[s\, c]]$$

is empty. Hence, no truth-value is returned by the function *Sing*. On the other hand, if $\forall t'[[c\, t'] \supset [t' < t]]$ v-constructs **T** then the class v-constructed by $\lambda b\, [\forall t'[[c\, t'] \supset [t' < t]] \wedge b{=}[s\, c]]$ is a singleton containing **T** or **F**, according as c is a member of s.

Suppose we apply *Past*, first, to a time t and, second, to the class of intervals v-constructed by $\lambda c\, [c \subset \lambda t\, [^{0}Sick_{wt}\, ^{0}Catherine]]$ and to the point of reference *Y1530*. Then we get

$$[^{0}Past_{t}\, \lambda c\, [c \subset \lambda t\, [^{0}Sick_{wt}\, ^{0}Catherine]]\, ^{0}Y1530] =$$
$$[^{0}Sing\, \lambda b\, [\forall t'[[^{0}Y1530\, t'] \supset [t' < t]] \wedge$$
$$b = [\lambda c\, [c \subset \lambda t\, [^{0}Sick_{wt}\, ^{0}Catherine]]\, ^{0}Y1530]]].$$

For instance, in the world W as above this Composition (W/w)-constructs **T** in 2009, because the year 1530 precedes any moment of time in the year 2009 and the year 1530 belongs to the class (W/v)-constructed by $\lambda c\, [c \subset \lambda t\, [^{0}Sick_{wt}\, ^{0}Catherine]]$.

As a result, the refined analysis of (6) is the following.[100]

(6″) $\lambda w \lambda t\, \iota b\, [\forall t_1\, [[^{0}Y1530\, t_1] \supset [t_1 < t]] \wedge$
$b = [\lambda c\, [c \subset \lambda t_2\, [^{0}Sick_{wt2}\, ^{0}Catherine]]\, ^{0}Y1530]]$.

This Closure can still be β-reduced to the equivalent Closure

$$\lambda w \lambda t\, \iota b\, [\forall t_1\, [[^{0}Y1530\, t_1] \supset [t_1 < t]] \wedge b{=}[^{0}Y1530 \subset \lambda t_2\, [^{0}Sick_{wt2}\, ^{0}Catherine]]]$$

or equivalently (replacing $\subset$ by its logical definition),

$$\lambda w \lambda t\, \iota b\, [\forall t_1[[[^{0}Y1530\, t_1] \supset [t_1 < t]] \wedge b{=}\forall t_2\, [[^{0}Y1530\, t_2] \supset [^{0}Sick_{wt2}\, ^{0}Catherine]]].$$

The values of the proposition constructed by these analyses in a particular world W at a particular time T are:

(a) no value, if $T \leq$ December 31, 1530, 24:00;

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[100] For the sake of simplicity, we now write 'ιb' instead of '$^{0}Sing\, \lambda b$' and for the sake of clarity we rename the t variables.

(b) **T**, if the whole year 1530 precedes T (i.e., $T >$ December 31, 1530, 24:00) and Catherine was sick at all times during 1530;
(c) **F**, if $T >$ December 31, 1530, 24:00 and Catherine was not sick at all times during 1530.

We analysed (6) as lacking a truth-value till the end of 1530. In order for (6) to have a truth-value, the *whole* interval serving as point of reference must precede the time of evaluation. Tichý (1980a) analyses the simple past tense together with frequency adverbs like 'throughout', 'once', 'twice', 'at most once', and 'often', in such a way that the interval c of reference has a non-empty intersection with the past. Tichý's definition of *Past** is

$$^{0}Past^{*} = \lambda t\, \lambda sc\, [^{0}Sing\, \lambda b\, [^{0}\exists \lambda t'[[c\, t'] \wedge [t' < t]] \wedge b{=}[s\, \lambda t'[[c\, t'] \wedge [t' < t]]]]].$$

Here the Closure $\lambda t'[[c\, t'] \wedge [t' < t]]$ specifies the intersection I of the interval c of reference and the past with respect to time t. If this set is empty, the whole conjunction takes **F** and the set of truth-values λb […] is empty. This leaves *Sing* undefined. Otherwise *Sing* returns **T** or else **F,** according as this non-empty intersection I is a member of s.

This truth-condition is debatable, though. The 'otherwise' clause seems problematic. To see why, let (6) be evaluated on November 30, 1530. Is it true to say that Catherine was sick throughout that year? We think not, for Catherine may recover in December. She may well be sick from the beginning of the year up to and including November 30, but if she recovers between December 1 and New Year's Eve, it is not true that she was sick throughout the year. So on November 30, or any other day before the end of the year, let no man claim that Catherine was sick throughout the year.[101]

The same problem crops up in a different guise when '(at least) twice' replaces 'throughout'. If Catherine was sick twice or more between January 1, 1530 and November 30, 1530, then the sentence is true on November 30, despite the fact that the year is still not over. Yet, if she was not, then it does matter that the year is not over yet. In this case, on November 30, or any other time before the end of the year, there is no fact of the matter as to whether Catherine was sick at least twice in 1530.

How are we to analyse sentence (7)?

(7) 'Catherine of Aragon was sick at least once before the year 1530.'

[101] We part company with Tichý because he takes 'Catherine was sick throughout 1530' to be synonymous with 'Catherine has been sick ever since the beginning of 1530' (cf. 1980a, p. 347, 2004, p. 379). Furthermore, he takes 'Catherine was sick twice in 1530' to be synonymous with 'Catherine has been sick at least twice since the beginning of 1530' (cf. ibid., pp. 347–8, pp. 379–80).

Here the point of reference is specified in another manner than in (6). In (7) the point of reference is *any time before 1530*. To analyse 'before the year 1530', we have to define the type of the object denoted by 'before'. Given a time t and a τ-class c, the time t is prior to c if t is prior to every element of c. Thus *Before*/(οτ(οτ)) receives the definition

$$^{0}Before = \lambda tc\ [\forall t'[c\ t'] \supset t<t'].$$

Variables: $t, t' \rightarrow \tau$; $c \rightarrow (o\tau)$.

The definition of 'at least once' is easy. It is of the same type as 'throughout', i.e., *At_least_once*/$((o(o\tau))o_{\tau\omega})_{\omega}$, and receives the definition

$$^{0}At\_least\_once = \lambda w\ \lambda p\ \lambda c\ \exists t\ [[c\ t] \wedge p_{wt}].$$

The truth-condition is that a proposition p be true at least once in a world w in an interval c if there is at least one time t in c at which p is true in w.

For instance, the Composition $[^{0}At\_least\_once_{w}\ \lambda w\lambda t\ [^{0}Sick_{wt}\ ^{0}Catherine]]$ *v*-constructs the class *S*/(ο(οτ)) of intervals in which Catherine is sick at least once.

An admissible analysis of (7) is thus the Closure

(7′) $\lambda w\lambda t\ [^{0}Past_{t}\ [^{0}At\_least\_once_{w}\ \lambda w\lambda t\ [^{0}Sick_{wt}\ ^{0}Catherine]]$
$\lambda t\ [^{0}Before\ t\ ^{0}Y1530]].$

The substitution of the above definitions of *At_least_once* and *Before* for the respective Trivialisations and the renaming of the second, third and fourth variable t for t_1, t_2, t_3, respectively, produces

(7″) $\lambda w\lambda t\ [^{0}Past_{t}\ \lambda c\ \exists t_1\ [[c\ t_1] \wedge [^{0}Sick_{wt1}\ ^{0}Catherine]]$
$\lambda t_2[\forall t_3[^{0}Y1530\ t_3] \supset t_2<t_3]].$

The substitution of the above definition of *Past*, i.e.

$$\lambda t\ \lambda sc\ [^{0}Sing\ \lambda b\ [\forall t'[[c\ t'] \supset [t' < t]] \wedge b=[s\ c]]]$$

for $^{0}Past$, coupled with some simplifying equivalent transformations, produces the refined analysis of (7):

$$\lambda w\lambda t\ [\lambda sc\ [^{0}Sing\ \lambda b\ [\forall t'\ [[c\ t'] \supset [t' < t]] \wedge b=[s\ c]]]$$
$$\lambda c\ \exists t_1\ [[c\ t_1] \wedge [^{0}Sick_{wt1}\ ^{0}Catherine]]\ \lambda t_2[\forall t_3[^{0}Y1530\ t_3] \supset t_2<t_3]].$$

This construction is equivalent to

(7‴) $\lambda w \lambda t\ \iota b\ [\forall t'\ [[\forall t_3[{}^0Y1530\ t_3] \supset t' {<} t_3] \supset [t' < t]] \wedge$
$b = \exists t_1[[\forall t_3[{}^0Y1530\ t_3] \supset t_1 {<} t_3] \wedge [{}^0Sick_{wt1}\ {}^0Catherine]]]$.

The equivalence is obtained via β-conversion by substituting the Closure $\lambda c\ \exists t_1\ [[c\ t_1] \wedge [{}^0Sick_{wt1}\ {}^0Catherine]] \rightarrow_v (o(o\tau))$ for s and $\lambda t_2[\forall t_3[{}^0Y1530\ t_3] \supset t_2 {<} t_3] \rightarrow_v (o\tau)$ for c.

(7‴) constructs a proposition whose distribution of truth-values with respect to a world W and a time T is:

(a) no truth-value, if there is a time t' preceding any time of the year 1530 and $t' \geq T$ (in other words, the time T of evaluation < January 1st, 1530, 00:00);
(b) **T**, if T comes after any time before the beginning of the year 1530 (i.e., $T \geq$ January 1st, 1530, 00:00), and Catherine was sick in W at least once before the year 1530;
(c) **F**, if T comes after any time before the year 1530 (i.e., $T \geq$ January 1st, 1530, 00:00), and Catherine was not sick in W at least once before the year 1530.

In Section 2.5.1 we analysed the temporal modifier expression ‘frequently’ as denoting the function *Frequent* of type $((o(o\tau))\tau)$; i.e. the function that, given a time t, returns the class of intervals frequent with respect to t. The frequency adverb ‘often’ denotes a world-dependent function that assigns to a proposition the class of intervals in which the proposition is frequently true. Thus, *Often* is of type $((o(o\tau))o_{\tau\omega})_\omega$, and we have the means to define it.

The analysis of

(8) ‘Catherine of Aragon was often sick in 1530’

is

(8′) $\lambda w \lambda t\ [{}^0Past_t\ [{}^0Often_w\ \lambda w \lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]]\ {}^0Y1530]$.

Substituting again the definition of *Past* for ${}^0Past$, we get

(8″) $\lambda w \lambda t\ [\lambda sc\ [{}^0Sing\ \lambda b\ [\forall t'\ [[c\ t'] \supset [t' < t]] \wedge b{=}[s\ c]]]$
$[{}^0Often_w\ \lambda w \lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]]\ {}^0Y1530]$.

β-conversion—consisting in substituting $[{}^0Often_w\ \lambda w \lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]] \rightarrow_v (o(o\tau))$ for s and ${}^0Y1530 \rightarrow (o\tau)$ for c—transforms (8″) into the equivalent construction

(8‴) $\lambda w \lambda t\ [{}^0Sing\ \lambda b\ [\forall t'\ [[{}^0Y1530\ t'] \supset [t' < t]] \wedge$
$b{=}[[{}^0Often_w\ \lambda w \lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]]\ {}^0Y1530]]]$.

If we want to further refine the analysis of (8), we face the problem of how to define *Often.* A proposition p is *often true* in a world w if the chronology of p in w,

$[p\ w]$, belongs to the class of intervals c that are frequent for p; i.e., the intervals that are frequent with respect to any time t belonging to c. For instance, Catherine's sickness is frequent with respect to a time t if Catherine has been sick, say, five times per months up until t. Thus we have

$$^0Often = \lambda w\ \lambda p\ \lambda c\ \forall t\ [[c\ t] \supset [^0Frequent_t\ \lambda t' p_{wt'}]].$$

Types: $Often/((o(o_\tau))o_{\tau\omega})_\omega$; $Frequent/(o(o\tau))$; $p \rightarrow o_{\tau\omega}$; $c \rightarrow (o\tau)$; $t, t' \rightarrow \tau$.

The Composition $[^0Often_w\ \lambda w \lambda t\ [^0Sick_{wt}\ ^0Catherine]]$ v-constructs the class of intervals in which Catherine's sickness is frequent; i.e.,

$$\lambda c\ \forall t\ [[ct] \supset [^0Frequent_t\ [\lambda t'[^0Sick_{wt'}\ ^0Catherine]]]].$$

Substitution of the latter for the former into (8''') produces

$$\lambda w \lambda t\ [^0Sing\ \lambda b\ [\forall t'[[^0Y1530\ t'] \supset [t' < t]] \wedge$$
$$b{=}[\lambda c\ \forall t\ [[ct] \supset [^0Frequent_t\ [\lambda t'[^0Sick_{wt'}\ ^0Catherine]]]]\ ^0Y1530]]].$$

Using β-conversion, and renaming different variables t, t' for the sake of clarity, we can still simplify our analysis along the lines of the equivalent (8^{iv}):

(8^{iv}) $\lambda w \lambda t\ \iota b\ [\forall t_1[[^0Y1530\ t_1] \supset [t_1 < t]] \wedge$
$b{=}\ \forall t_2\ [[^0Y1530\ t_2] \supset [^0Frequent_{t2}\ [\lambda t_3[^0Sick_{wt3}\ ^0Catherine]].$

(8^{iv}) constructs the proposition that takes, in a particular world W at a particular time T, truth-values as follows:

(a) no truth-value, if there is a time t_1 belonging to the year 1530 and $t_1 \geq T$ (the first conjunct);
(b) **T**, if T comes after any time of the year 1530 (the first conjunct), and Catherine's sickness is frequent with respect to any time of the year 1530 (the second conjunct);
(c) **F**, if T comes after any time of the year 1530 (the first conjunct), and Catherine's sickness is not frequent with respect to any time of the year 1530 (the second conjunct).

2.5.2.2 Present perfect

So far we dealt only with the simple past tense. *Present-perfect* sentences mostly also contain an indication of *when* something happened. For instance, if we transport ourselves back to 1530, we can say that Catherine of Aragon *has been* sick *since the beginning of 1530*, or *throughout the last 2 weeks*. The present perfect is a function *PrPf* of the same type as *Past*, i.e., $PrPf/((o(o(o\tau))(o\tau))\tau)$. The chronology of relations between a class of intervals and an interval is another, however. For *PrPf* to obtain between a class $S/(o(o\tau))$ and an interval $C/(o\tau)$ at time T/τ, C must be an interval which runs from the past up to T (and possibly beyond); otherwise *PrPf* comes out undefined. Let C' be a subinterval of C that extends until time T but not beyond. In order for the relation *PrPf* to obtain at T at the tuple argument $\langle S, C\rangle$, the subinterval C' must be an element of S. C' can then be defined by $\lambda t_2\,[[{}^0C\ t_2] \wedge t_2 \leq {}^0T]]$, and we have:[102]

$$[[{}^0PrPf\ {}^0T]\ {}^0S\ {}^0C] = \iota b\ [\exists t_1\ [\lambda t_2\ [t_1 < t_2 \leq {}^0T] \subset {}^0C] \wedge b = [{}^0S\ \lambda t_2\ [[{}^0C\ t_2] \wedge [t_2 \leq {}^0T]]]].$$

Thus *PrPf* can be defined as

$${}^0PrPf = \lambda t\ \lambda sc\ \iota b\ [\exists t_1\ [\lambda t_2\ [t_1 < t_2 \leq t] \subset c] \wedge b = [s\ \lambda t_2\ [[c\ t_2] \wedge [t_2 \leq t]]]].$$

For instance, the sentence

(9) ‘Catherine of Aragon has been sick at least once in 1530’

expresses the Closure

(9′) $\lambda w \lambda t\ [{}^0PrPf_t\ [{}^0At\_least\_once_w\ \lambda w \lambda t\ [{}^0Sick_{wt}\ {}^0Catherine]]\ {}^0Y1530]$.

Types and the definition of ${}^0At\_least\_once$ as above.

Note that (9) has no truth-value at the present time, unlike the sentence ‘Catherine of Aragon *was* sick at least once in 1530’, which is true or false in 2009 depending on her health back then. The crux is that the present perfect, but not the simple past, presupposes that the point of reference include the present time. Moreover, the present perfect operates on that portion of the point of reference that is not located in the future, whereas the simple past works on its past part.

The general verdict is that those tense logics that do not distinguish between present perfect and simple past do not make it possible to analyse tenses in a sufficiently fine-grained way.

What happens if the point of reference is not explicitly specified? Tichý, in (1980a), offers a solution to Chomsky’s (1972) puzzle concerning the difference between sentences like

‘Einstein has visited Princeton’

and

‘Princeton has been visited by Einstein’.

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[102] We again use the infix notation ‘$[t_1 < t_2 \leq {}^0T]$’ instead of ‘$[[{}^0< t_1\ t_2] \wedge [{}^0\leq t_2\ {}^0T]]$’, similarly for set-theoretical inclusion ‘$\subset$’. Moreover, we write ‘ιb...’ instead of ‘$[{}^0Sing\ \lambda b\ ...]$’.

Arguably, the first sentence presently lacks a truth-value, eliciting as it does the rejoinder, '*Who* has visited Princeton? Einstein's been dead for years, you know.' At the same time it is uncontroversial that the second sentence is true. Hence the puzzle, for normally one would expect active and passive locutions to be equivalent. Tichý's explanation is this. Both sentences indicate the point of reference indirectly. Though they are grammatically well-formed English sentences, their meaning is incomplete. The first sentence should be understood as, 'Einstein has visited Princeton *during his lifetime*', and the second sentence as, 'Princeton has been visited by Einstein *during its history*'. Since Einstein's lifetime came to an end in 1955, the first sentence denotes a proposition that will lack a truth-value ever after. In contrast, the point of reference of the second sentence does include the present time (since Princeton is still around) and so the denoted proposition does return a truth-value (**T**, as it happens).[103]

2.5.2.3 Temporal *de dicto* vs. *de re*

At the outset of Section 2.5.2 we promised to also analyse sentences like, 'Henry's wife was born in Düsseldorf', and 'Henry's wife was frequently sick' versus 'Frequently, Henry's wife was sick'. Now we are going to make good on this promise, which means that we are going to deal with the *temporal de dicto* and *temporal de re* together with the simple past. As explained above, the sentence

(10) 'Henry's wife was born in Düsseldorf'

is incomplete in isolation, because we always use the simple past when we say *when* something happened. Thus it must be associated with a certain past-time point of reference, as in

(11) 'Henry's wife was born in Düsseldorf on September 22, 1515'.

Now, the sentence (11) is ambiguous. Its ambiguity is again pivoted on *de dicto* vs. *de re*.

(*a*) Analysis *de re*.

(11) can be taken to mean that whoever is currently Henry's wife has the property of having been born in Düsseldorf on September 22, 1515. On this *de re* reading the sentence denotes a proposition lacking a truth-value in 2009, because (as explained above) the office of Henry's wife is vacant in 2009. Let *Sep22*/$(o\tau)$ be the interval of September 22, 1515. Then the property of having been born in Düsseldorf on September 22, 1515 is constructed by

[103] These two sentences differ in their topic-focus articulation. For instance, Duží (2009) shows that sentences differing only as to their topic-focus articulation have different meanings, and proposes a logical analysis of this phenomenon.

$$\lambda w \lambda t\ \lambda x\ [{}^0Past_t\ \lambda c\ \exists t'[[c\ t'] \wedge [{}^0Born\_in_{wt'} x\ {}^0Düsseldorf]]\ {}^0Sep22].$$

This construction constructs a property whose characteristic function for any individual x in W at T is evaluated as follows:

- no truth-value, if T is prior to September 22, 1515. Of course, it cannot be reasonably said of any individual x whether x has the property of having been born on September 22, 1515, in Düsseldorf *before* this date;
- **T**, if time T comes after September 22, 1515, and the date of September 22, 1515 is an element of the class of intervals c which contain times t' at which x was born;
- **F**, if time T comes after September 22, 1515, and the date of September 22, 1515 is *not* an element of the class of intervals c which contain times t' at which x was born.

If we apply this property to the individual who happens to play the role of Henry's wife, we obtain this analysis:

$$\lambda w \lambda t\ [[\lambda w \lambda t\ \lambda x\ [{}^0Past_t\ \lambda c\ \exists t'\ [[c\ t'] \wedge$$
$$[{}^0Born\_in_{wt'} x\ {}^0Düsseldorf]]\ {}^0Sep22]]_{wt}\ [\lambda w \lambda t\ [{}^0Wife\_of_{wt}\ {}^0Henry]]_{wt}].$$

This construction can still be β-reduced to the equivalent analysis:

(11*re*) $\lambda w \lambda t\ [\lambda x\ [{}^0Past_t\ \lambda c\ \exists t'[[ct'] \wedge$
$[{}^0Born\_in_{wt'} x\ {}^0Düsseldorf]]\ {}^0Sep22]\ [{}^0Wife\_of_{wt}\ {}^0Henry]].$

The proposition constructed by (11*re*) has no truth-value in 2009 due to the fact that $[{}^0Wife\_of_{wt}\ {}^0Henry]$ is *v*-improper. Further β-reduction (substituting $[{}^0Wife\_of_{wt}\ {}^0Henry]$ for x) would not be an equivalent transformation, because we would be drawing the extensional occurrence of $[{}^0Wife\_of_{wt}\ {}^0Henry]$ into the (οτ)-intensional context of the Closure $\lambda c\ \exists t'[[ct'] \wedge [{}^0Born\_in_{wt}\ x\ {}^0Düsseldorf]]$. Since a Closure is never *v*-improper, the resulting construction would construct a proposition without a truth-value gap.

(*b*) Analysis *de dicto.*

The other disambiguated reading of (11) requires us to evaluate the proposition *Henry's wife was born on September 22, 1515 in Düsseldorf* at a time in the past. There are past times at which the proposition yields **T**, other times at which it yields **F**, and still other times at which it yields no value.

Before Henry VIII first got married in 1509, the proposition had no truth-value, because nobody was Henry's wife. After Henry's death in 1547 the proposition had no truth-value either, because his widow (Catherine Parr, as it happened) was no longer his wife. Between these two dates there was an interval during which the proposition was true, because he happened to have a wife that was born on September 22, 1515 in Düsseldorf.

Before we analyse the *de dicto* reading of (11), let us consider another variant of this *de re/de dicto* dichotomy, namely the sentence

(12) 'Henry's wife was beheaded on February 13, 1542.'

If understood *de re* as expressing the proposition that Henry's *current* wife was beheaded on February 13, 1542, then obviously the sentence cannot be true. On the other hand, if the proposition is evaluated in the past then it occasionally takes **T**, because as a matter of historical fact the woman married to Henry on February 13, 1542 (though not throughout the entire day) was beheaded on February 13, 1542. Catherine Howard happened to be this hapless lady. Let *Feb13*/(οτ) be the interval of February 13, 1542. Then the analysis of the *de dicto* variant of (12) consists in applying the function *v*-constructed by $Past_t$ to the set of o-chronologies of the proposition that Henry's wife is beheaded and to the interval *Feb13*:

(12*d*) $\lambda w\lambda t\ [{}^0Past_t\, \lambda c\, \exists t'[[ct'] \wedge [{}^0Beheaded_{wt'}\, [{}^0Wife\_of_{wt'}\, {}^0Henry]]]\ {}^0Feb13].$

The proposition constructed by (12*d*) takes **T** in 2009, because 2009 comes after February 13, 1542, and because there were times *t′* during that day when the individual who played, *at that time*, the role of Henry's wife was beheaded.

The scenario of the *de dicto* reading of (11) is a bit different. Chroniclers back in Henry VIII's days enjoyed a linguistic privilege that we do not have. They could report, using the present prefect, what we must report using the past perfect. They could write, 'Henry's wife *has* since September 22, 1515 *had* the property of being born in Düsseldorf'. In our analysis of this sentence occurring in a chronicle we must apply *Past* to the underlying proposition *Henry's wife has…had the property of being born in Düsseldorf.* So we must apply *Past* to the result of the application of *PrPf* to the underlying proposition *Henry's wife is born on September 22, 1515 in Düsseldorf.* Let *Sep22*/(οτ) be as above. Then the analysis of the chroniclers' present-perfect version is:

(PrP) $\lambda w\lambda t\ [\lambda x\ [{}^0PrPf_t\ [{}^0Through_w\ \lambda w\lambda t\ [{}^0Born\_in_{wt}\ x\ {}^0Düsseldorf]]\ {}^0Sep22]$
$[{}^0Wife\_of_{wt}\ {}^0Henry]].$

Through is part of the analysis, because what is wanted is the uninterrupted interval between September 22, 1515 and the time that was present for the chroniclers.

Now we need to apply *Past* to the proposition constructed by the above Closure (PrP). If we reformulate (11) in order to state its truth-conditions explicitly, we have:

‘It was once the case that the individual who was Henry’s wife during a certain period *HW*/(οτ) of her life had possessed he property of having been born in Düsseldorf since September 22, 1515.’

If *Past* is applied to the proposition constructed by (PrP), the result is

(11*d*) $\lambda w \lambda t\ [{}^0 Past_t\ [\lambda c\ \exists t'[[c\ t'] \wedge [\lambda x\ [{}^0 PrPf_{t'}\ [{}^0 Through_w\ \lambda w \lambda t\ [{}^0 Born\_in_{wt}\ x\ {}^0 D\ddot{u}sseldorf]]\ {}^0 Sep22]\ [{}^0 Wife\_of_{wt}\ {}^0 Henry]]\ \lambda t'[{}^0 After_{t'}\ {}^0 Sep22]].$

After is defined as $\lambda tc\ \forall t_1\ [[ct_1] \supset [t{>}t_1]]$. Thus the Closure $\lambda t'[{}^0 After_{t'}\ {}^0 Sep22] = \lambda t' \forall t_1[[{}^0 Sep22\ t_1] \supset [t'{>}t_1]]$ constructs the interval stretching from September 22, 1515 up to whatever time t' prior to whatever is the present time t for the one who evaluates (11*d*).

2.5.2.4 Future tenses

The operators controlling simple future and future perfect can be made to mirror *Past* and *PrPf*, respectively:

$${}^0 Future = \lambda t\ \lambda sc\ [{}^0 Sing\ [\lambda b\ [\forall t'[[c\ t'] \supset [t' > t]] \wedge b{=}[s\ c]]];$$

$${}^0 FutPf = \lambda t\ \lambda sc\ \iota b\ [\exists t_1\ [\lambda t_2\ [t{<}t_2{\leq}t_1] \subset c] \wedge b{=}[s\ \lambda t_2\ [[c\ t_2] \wedge [t{<}\ t_2]]]].$$

Similarly as with sentences in past tense, a sentence in future tense would denote a proposition that is either undefined or takes **T** or **F** at the time of evaluation. For instance, the sentence

(13) ‘Charles will go to the theatre on August 21, 2009’

would denote a proposition that takes **T** or **F** before the specified date, according as Charles goes to the theatre on that date, and is undefined if the time of evaluation comes after the beginning of the specified date.

However, if we evaluate the proposition just on August 21, 2009 before midnight, then the proposition would still be true or false, according as Charles will be going to the theatre, say at two minutes to midnight. This suggests that *Future* is not the operator we want to control simple future when analysing (13). The correct truth-condition of (13) presupposes that the point of reference (here, August 21, 2009) should have a non-empty intersection with the interval stretching from the time of evaluation into the future. Above we found fault with Tichý’s *Past* operator, but it arguably correctly mirrors the *Future* operator. Tichý’s *Past* operator is this:

$${}^0 Past' = \lambda t\ \lambda sc\ [{}^0 Sing\ \lambda b\ [{}^0 \exists \lambda t'[[c\ t'] \wedge [t' < t]] \wedge b{=}[s\ \lambda t'[[c\ t'] \wedge [t' < t]]]]].$$

The definition of the matching *Future′* function, of type ((ο(ο(οτ))(οτ))τ), is this:

$$^0Future' = \lambda t\, \lambda sc\, [^0Sing\, \lambda b\, [^0\exists\lambda t'[[c\, t'] \wedge [t' > t]] \wedge b{=}[s\, \lambda t'[[c\, t'] \wedge [t' > t]]]]].$$

Let *August21* be the interval of August 21, 2009. Then the analysis of (13) is this:

(13′) $\lambda w\lambda t\, [^0Future'_t\, [\lambda c\, \exists t'\, [[c\, t'] \wedge [^0Go\_Theatre_{wt'}\, ^0Charles]]\, ^0August21]$.

Types: *Go_Theatre*/$(\omicron\iota)_{\tau\omega}$: the property of going to the theatre; *Charles*/ι; *August21*/(οτ); *c*→(οτ); the other types as above.

Now consider

(14) ‘Charles will go to the theatre tomorrow.’

The sentence is now devoid of reference to any specific interval such as August 21, 2009. Instead, the adverb ‘tomorrow’ denotes a chronology of days, which is the function *Tomorrow* taking a time *t* to the day immediately succeeding the day that contains *t. Tomorrow* receives the type ((οτ)τ), and the analysis of (14) becomes

(14′) $\lambda w\lambda t\, [^0Future_t\, [\lambda c\, \exists t'\, [[c\, t'] \wedge [^0Go\_Theatre_{wt'}\, ^0Charles]]\, ^0Tomorrow_t]$.

Note that we are now using the alternative function *Future* and not Tichý’s *Future’*. The latter fits points of reference that are specified absolutely, like *August 21, 2009*. The former fits points of reference specified relative to the time of evaluation, like *tomorrow*. Here the whole interval serving as point of reference must be in the future.

We now shift from simple future to future perfect. The sample sentence is

(15) ‘Charles will have received his promotion by August 21, 2009.’

Its meaning is

(15′) $\lambda w\lambda t\, [^0FutPf_t\, [\lambda c\, \exists t'\, [[c\, t'] \wedge [^0ReceiveP_{wt'}\, ^0Charles]]\, ^0August21]$.

Types: *ReceiveP(romotion)*/$(\omicron\iota)_{\tau\omega}$; the other types as above.

The remaining tenses can be reconstructed from the above analyses. The uniform schema is that the relevant construction must contain the following subconstructions. Firstly, a construction of the tense function, like *Future*, *Past* or *PrPf.* Secondly, a construction of the frequency function, like *Often* or *Twice*. Thirdly, a construction of the underlying proposition, like *Henry is beheaded*. Fourthly and finally, a construction of the interval serving as point of reference, such as *August 21, 2009.*

2.6 Three kinds of context

In Section 1.5.2 we explained why we eschew shift of reference and how we obtain transparency. The leading idea behind transparency, to quote Tichý, is that

> [T]he Fregean sense/reference semantics, which in many different guises still dominates the field, is the product of a failure to distinguish supposition from reference (1988, p. 216).

So shift between supposition *de dicto* and *de re* replaces shift of reference between 'sense' and 'reference' (or 'denotation', in TIL jargon). What TIL calls 'denotation' is context-invariant (unlike the references of empirical expressions), but the construction expressed by a given word may occur either mentioned or used with either supposition *de dicto* or *de re*. We disentangle the *metasemantic* issue of how denotation is fixed and the *semantic* issue of what is so fixed from the *logical* issue of what sort of contribution a construction occurring as a constituent within a larger construction makes.

In this section we show in great technical detail how the *de dicto*/*de re* distinction falls out of a more general distinction between *use* and *mention*.[104]

The *use-mention* distinction is traditionally understood as the distinction between using an *expression* (or any piece of language) and mentioning it using a meta-language. Says the *Stanford Encyclopaedia of Philosophy*:[105]

> Starting with Frege, the semantics (and pragmatics) of quotation has received a steady flow of attention over the last one hundred years. It has not, however, been subject to the same kind of intense debate and scrutiny as, for example, both the semantics of definite descriptions and propositional attitude verbs. Many philosophers probably share Davidson's experience: 'When I was initiated into the mysteries of logic and semantics, quotation was usually introduced as a somewhat shady device, and the introduction was accompanied by a stern sermon on the sin of confusing the use and mention of expressions' (Davidson, 1979, p. 79).

In written language, mentioned words or phrases often appear between quotation marks or in italics. Used words or phrases, being more common than mentioned ones, do not have any typographic distinction. Making a statement mention itself is an interesting way of producing semantic paradoxes. Violation of the use-mention distinction can produce sentences that sound and seem similar to the original, but have an entirely different meaning.

However, in this book we are not going to analyse the semantics of quotation, which is not to say that it is not an interesting topic in the philosophy of language. Instead, we analyse the semantics of using expressions in a communicative act. Thus when we use quotation, we do so only to mark the expression under logical scrutiny.

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[104] The results presented in this section are due in part to Petr Kuchyňka.

[105] Retrieved from http://www.science.uva.nl/~seop/entries/quotation/ (November 27, 2008).

From the logical point of view, there is a less-examined, albeit not less-interesting, phenomenon to do with paradoxes and misconceptions arising from running together different ways in which a meaningful expression can be *used*. Though there is a lot of dispute on using/mentioning *expressions*, little attention has been paid to the distinction between using and mentioning *entities*, which are denoted when expressions are used. The examples in, e.g., Gamut (1991, pp. 203–04) illustrate the problems arising from different ways of using expressions. To adduce one, consider the following (obviously invalid) argument, which is structurally similar, though not identical, to the Partee puzzle analyzed in Section 1.5.2:

The temperature in Amsterdam equals the temperature in Prague
The temperature in Amsterdam is increasing

The temperature in Prague is increasing.

In which way can an *entity* be used or mentioned? To explain the notion informally and metaphorically, one should realize first of all that there is an essential difference between the way of using the definite description 'the temperature in Amsterdam' in the first and the second premise. In the first premise the (empirical) *function*, namely the *magnitude TA* denoted by 'the temperature in Amsterdam', is not only talked about but also *used* to point to its current actual value (whatever it may be). The premise says that this value equals the current value of another magnitude *TP* (denoted by 'the temperature in Prague'). However, the second premise ascribes the property of being increasing to the whole magnitude *TA*: the *function TA* itself is thus not used as a pointer to its current value but is only *mentioned*, in order to figure as the subject of predication. In Section 1.5.2 we showed that the meaning of 'the temperature' is used (occurs) with *de re* supposition in the meaning of the first premise, while in the second premise it occurs with *de dicto* supposition. Thus we cannot replace the meaning of 'the temperature in Amsterdam' by the meaning of 'the temperature in Prague' in the second premise. The principle of substitution of co-referential expressions is valid only in the *de re* case.[106]

The above example illustrates one kind of fallacy arising from not distinguishing between two ways of *using* a construction expressed by an expression; a construction occurs either with *de dicto* or *de re* supposition. In this section we will show that the *de dicto*/*de re* dichotomy is a special case of a broader dichotomy between using a construction either intensionally* or extensionally*.

Another kind of fallacy arises from not respecting the difference between using and mentioning a construction embedded within another construction. Here is an example:

[106] See Section 1.5.2.1 for the two principles *de re*.

(Calc) Charles calculates 2 + 5
2 + 5 = 7
———————————
Charles calculates 7.

The conclusion is obviously unreasonable, and probably even nonsensical, for how could Charles be calculating anything in the absence of an arithmetical operation? Again, there is a substantial difference between using the term ‘2 + 5’ in the first and the second premise. This time the distinction does not consist in talking about a function by ascribing a property to it and using a function as a pointer to its value. Rather, the first premise expresses Charles’ relation(-in-intension) to the very procedure of calculating 2 + 5. Charles is busy executing the procedure, and the *procedure*, which is the meaning of ‘2 + 5’, is *mentioned* in the first premise. The evaluation of the truth-conditions expressed by the first premise does not include the execution of the procedure of adding 2 and 5; this is something Charles is responsible for. On the other hand, in the second premise the *procedure* of adding 2 and 5 is *used* to identify the number 7.

How to solve these apparent paradoxes? In general, TIL offers a fine-grained analysis of premises that neither makes it possible to over-infer (which leads to paradoxes) nor under-infer (which leads to lack of inferential knowledge). It is thus of critical importance to distinguish between *using* constructions as constituents of compound constructions and *mentioning* constructions that enter as input into compound constructions. As we stated in Section 1.3, the latter is, in principle, achieved by using atomic constructions, namely Trivializations and Variables. Moreover, the rich ontology of TIL makes it possible to further distinguish between *two* basic *ways* in which a *construction* can be *used as a constituent* of another construction; namely, as *used with intensional* (or *de dicto*) *supposition* and *used with extensional* (or *de re*) *supposition*.

The intensional/extensional supposition in which a constituent can occur concerns not only constructions of intensions/extensions as defined in Section 1.4 and as used in possible-world semantics. In its most general formulation, the dichotomy between intensional and extensional supposition applies to constructions of *all* functions and not only of intensions understood as mappings from possible worlds to (chronologies of) a type α. As a matter of fact, the terms ‘intension’ and ‘extension’ were originally used in mathematics and computer science in a broader sense than is currently the case in what nowadays goes under the name of ‘intensional logic’. In this broader sense, an ‘[i]ntension is a definition of a set by mentioning a defining property.’[107] Similarly, in computer science intensional attributes are usually understood as being the attributes of an entire set or function (like cardinality, non-emptiness, periodicity, etc.), whereas extensional attributes are attributes of particular elements of a set or of particular functional values. We

[107] Eric W. Weisstein, ‘Intension’, retrieved from http://mathworld.wolfram.com/Intension.html (February 19th, 2007).

shall say that a constituent construction occurs with *extensional supposition* when it constructs a function and the so constructed function is applied to an argument in order to obtain the corresponding value, if any. This way a property becomes attributable to a functional value. When a constituent construction occurs with *intensional supposition*, the so constructed function is not applied. This way a property becomes attributable to the function itself. Thus we shall also speak about *intensional* and *extensional context* in this broader sense. To avoid confusing the broad notions of intension/extension with the narrow possible-world notions, we shall say that constructions occur intensionally or extensionally, whereas the dichotomy without reference to an occurrence of a construction concerns only the possible-world notions.

Let us consider some contexts of discourse. At the linguistic level an expression E can be either used or mentioned. If E is mentioned then it is so by another expression E' used in the context. This is the case of 'meta-language' and we are not going to deal with the problems of meta-language here. The cases we are interested in concern distinct ways in which E can be used in order to communicate its meaning, i.e., the construction C_E expressed by E.

The expression E can be used in such a way that its meaning C_E is either mentioned* or used*. The former case is similar to mentioning expressions in a meta-language. If we mention an expression, like 'cow' in 'Cow' has three letters', the expression is not used to communicate its meaning. Similarly, if the *meaning* C_E of an expression E is mentioned*, it does not serve to identify the entity to be talked about; instead, C_E itself is talked about. For instance, in the sentence

'*Cow* is a general concept that identifies the property of being a cow'

the first use of 'cow' only mentions* the concept[108] of cow, i.e., the meaning of 'cow' (here the Trivialization ${}^0Cow$), whereas the second occurrence of 'cow' uses* the meaning of 'cow', namely ${}^0Cow$, to identify the property $Cow/(o\iota)_{\tau\omega}$. Though obvious, it is important to stress that if C_E is mentioned* then it is so by means of another used* construction.

For instance, the analysis of '*Cow* is a general concept' would be as follows. If $General\_C/(o*_1)$ is a class of closed constructions of order 1 constructing properties of individuals, then $[{}^0General\_C\ {}^{00}Cow]$ constructs **T.** Here ${}^{00}Cow$ is used* to construct ${}^0Cow$, which is only mentioned*. Where $Identify\_P/(o*_1(o\iota)_{\tau\omega})$ is a relation between a construction of order 1 and a property of individuals constructed by this construction, the analysis of the whole sentence is the Composition

$$[[{}^0General\_C\ {}^{00}Cow] \wedge [{}^0Identify\_P\ {}^{00}Cow\ {}^0Cow]].$$

Note that, where = is the identity relation between ι-properties, an equivalent analysis can be obtained by means of Double Execution:

[108] In TIL concepts are defined as normalized closed constructions. See Section 2.2 for details.

$$[[^0General\_C\ ^{00}Cow] \wedge [^0{=}\ ^{200}Cow\ ^0Cow]].$$

Gloss: The simple concept ^0Cow belongs to the class *General_C* of general concepts and the result of the execution of this concept is identical to the property *Cow*.

The construction $^{200}Cow/*_3 \rightarrow (o\iota)_{\tau\omega}$ is the procedure of executing (i.e., using*) the construction ^{00}Cow twice over: (i) first execute $^{00}Cow/*_2{\rightarrow}*_1$ to construct the concept $^0Cow/*_1{\rightarrow}(o\iota)_{\tau\omega}$, and (ii) then execute $^0Cow/*_1{\rightarrow}(o\iota)_{\tau\omega}$ to construct the property $Cow/(o\iota)_{\tau\omega}$. This property is, however, not predicated of any individual; it is only talked about here. Thus the last occurrence of ^0Cow is used* with *de dicto* supposition (whereas its first occurrence is mentioned*).

To complete the example in order to illustrate the *de re* use* of a construction, consider the sentence

'*Cow* is a general concept that identifies the property of being a cow, and Milka is a cow'.

Obviously, in the second conjunct 'cow' is used to express its meaning ^0Cow, which in turn is used* to identify the property *Cow*, which is predicated of the individual $Milka/\iota$. The sentence thus expresses the Closure

$$\lambda w\lambda t\ [[[^0General\_C\ ^{00}Cow] \wedge [^0Identify\_P\ ^{00}Cow\ ^0Cow]] \wedge [^0Cow_{wt}\ ^0Milka]].$$

The last occurrence of ^0Cow is used* here with *de re* supposition.

Hence, if C_E is used* then it is used in order to produce an output (functional) entity *f* (if any).[109] There are again two distinct ways in which an occurrence of C_E can be used*, namely with *de dicto* (or in general *intensional*) supposition, or with *de re (*or in general *extensional*) *supposition.* If C_E is proper (i.e., if it *v*-constructs an output *f*), then the so constructed function *f* is just mentioned or also used by C_E. Roughly, in the former case we talk only about the whole function *f* without applying it to any of its arguments. The meaning C_E is used* with *intensional* (in particular—if *f* is an intension—*de dicto*) supposition. In the latter case the mentioned function *f* is furthermore used in order to point to its value at an argument, and this is the case of using* C_E with *extensional supposition* (or—if *f* is an intension—with *de re* supposition). Though again obvious, it is important to stress that no non-construction can be used to construct anything. *Any constituent whatsoever of a construction is itself a construction*. This is in keeping with the compositionality principle, according to which what a construction constructs is exclusively a function of what its subconstructions construct plus how the subconstructions interact via various operations.

[109] The function *f* is generally of arity $n \geq 0$. An elementary entity *e* is then conceived of as a function of arity 0.

The totality of this interaction is the logical structure of the construction that the relevant subconstructions are constituents of.

Figure 2.1 illustrates using/mentioning entities at the three different levels:

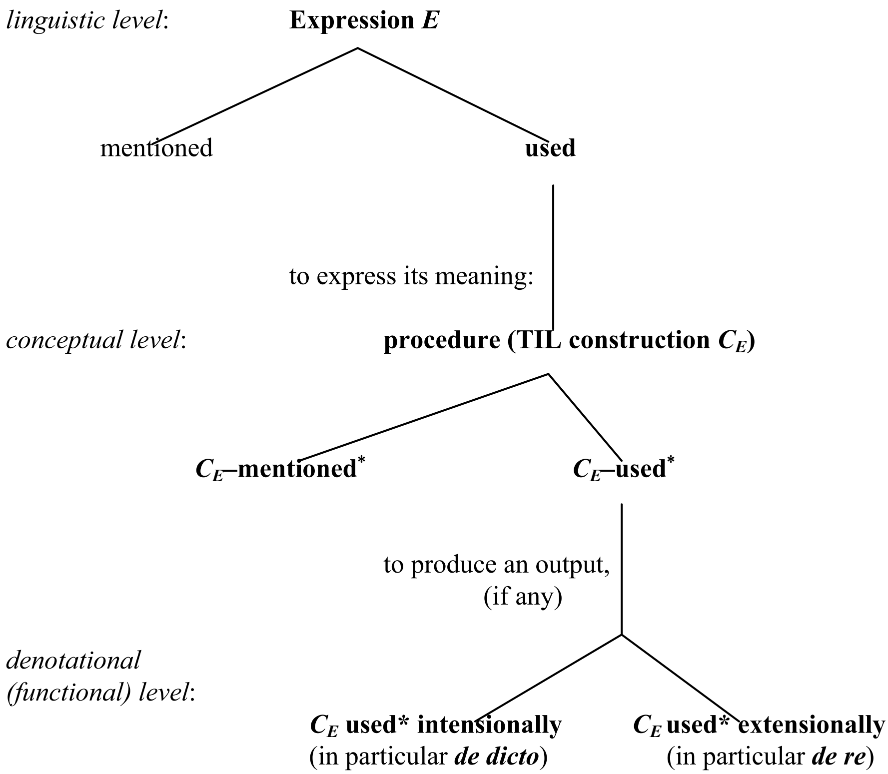

Fig. 2.1 Using/mentioning entities

The principle of compositionality of constructions was also heeded prior to Tichý (1988), in which Trivialization was first introduced, but only in a somewhat contrived manner. In papers published prior to 1988, entities were their own constructions,[110] which runs counter to the principle that constructions are distinct from what they construct.[111] It is also barely comprehensible how Mont Blanc, the very mountain, could possibly be a constituent of a construction alongside abstract constructions and cooperate with them in forming an abstract object such as a construction. Furthermore, there is nothing procedural about a mountain, yet constructions are primarily

[110] See, e.g., Tichý (1986b, p. 525, 2004, p. 612).

[111] However, Tichý always conceived of meaning as a procedure; cf. his pioneering papers (1968, 1969).

explicated as being procedures, so again there is no room for concrete objects in constructions (And also there is no room for non-constructional abstract objects like numbers, sets, functions as constructional constituents, as we argued in Section 1.3). A mountain would stick out like a sore thumb, since our acquaintance with mountains must be perceptual, for sure, though conceptually mediated to enable us to perceive something *as* a mountain; but then mountains are not 'mind-friendly'. In a word, intellectual acquaintance with abstract procedures is both necessary and sufficient for understanding what a given expression means.

As Fig. 2.1 makes clear, we strictly distinguish between particular semantic levels and different ways of using/mentioning entities. In Section 1.5.2.2 we informally characterized the three ways of using the meaning of an expression by distinguishing between (1) *hyperintensional context* of mentioning* constructions, (2) *intensional context* of using* constructions intensionally, and (3) *extensional context* of using* constructions extensionally. In the following paragraphs we are going to analyze (1) through (3) rigorously. In Section 2.7 we set out the rules of substitution pertaining to particular kinds of context. These definitions are the most technically complicated of all to be found in this book. They demonstrate that our philosophical project of distinguishing between three kinds of logical context is technically feasible. However, readers content with intuitively grasping the principles may want to skip over the following highly technical passages.

2.6.1 Using and mentioning constructions

To characterise the use/mention distinction at the conceptual level of constructions, it must be taken into account (a) that a construction C can be mentioned* only within some other construction D that operates on C by using another subconstruction C' of D; (b) that C itself has, therefore, to be constructed by C'; and (c) that it is necessary to define this distinction for *occurrences* of constructions, because one and the same construction C can be used* in D and at the same time serve as an input/output object for another subconstruction C' of D that operates on C.

As we said already, a construction C consists of particular steps (i.e., *constituents of* C) that are to be executed individually in order to execute the compound C. These constituents operate on input objects (either non-constructions or mentioned* constructions of a lower order). The distinction between using* and mentioning* constructions is characterised as follows, with a definition following afterwards.

Using*/mentioning* constructions. Let C be a subconstruction of a construction D. Then an occurrence of *C is mentioned* in D* if the execution of D does not involve the execution of this occurrence of C. Otherwise, an occurrence of *C is used* in D as a constituent.*

Following the example (*Calc*) of Charles' calculation, the analyses of premises P_1, P_2 are:

P_1: (a) $[^0{+}\ ^02\ ^05]$ $/*_1, \rightarrow \tau$
(b) $^0[^0{+}\ ^02\ ^05]$ $/*_2, \rightarrow *_1$ (the Composition $[^0{+}\ ^02\ ^05]$)
(c) $[^0Calc_{wt}\ ^0Charles\ ^0[^0{+}\ ^02\ ^05]]$ $/*_2, \rightarrow_v o$
(d) $\lambda w \lambda t\, [^0Calc_{wt}\ ^0Charles\ ^0[^0{+}\ ^02\ ^05]]$ $/*_2, \rightarrow o_{\tau\omega}$

P_2: (a) $[^0{+}\ ^02\ ^05]$ $/*_1, \rightarrow \tau$
(b) ^{07} $/*_1, \rightarrow \tau$
(c) $^0{=}$ $/*_1, \rightarrow (o\tau\tau)$
(d) $[^0{=}\ [^0{+}\ ^02\ ^05]\ ^07]$ $/*_1, \rightarrow o$.

Types: *Charles*/ι; *Calc*(ulate)/$(o\iota*_1)_{\tau\omega}$; $+/(\tau\tau\tau)$; 2, 5, 7/τ; $=/(o\tau\tau)$.

Now, it is obvious that the Closure $[^0{+}\ ^02\ ^05]$, which constructs a number, cannot be substituted for the Trivialization $^0[^0{+}\ ^02\ ^05]$, which constructs a first-order Composition, in the P_1-constituents *ad* (b), (c), (d). Such a substitution would constitute a type-theoretical category mistake, attempting as it would to substitute an entity of one type for an entity of another type. Calculating is not a relation (-in-intension) between an individual and a particular number; rather it is a relation (-in-intension) between an individual and a *construction* of a number. This is why *Calc* is an object of type $(o\iota*_1)_{\tau\omega}$. By calculating 2 + 5, Charles is related to a construction of the number seven, namely $[^0{+}\ ^02\ ^05]/*_1$. He is trying to find out which number is constructed in this way. Thus $^0Calc_{wt}$ *v*-constructs an entity of type $(o\iota*_1)$, rather than of type $(o\iota\tau)$. This goes to show that the occurrence of the construction $[^0{+}\ ^02\ ^05]$ is *mentioned** in the P_1-constituents *ad* (b), (c), (d) by another constituent of P_1, namely the Trivialization $^0[^0{+}\ ^02\ ^05]$, whereas it is *used** in P_2. In order to evaluate, for a state of affairs $\langle w, t\rangle$, the truth-conditions specified by P_1, one must execute the steps *ad* (b), (c), (d), but not (a). P_1 has nothing to do with whether Charles succeeds in executing step (a).

The particular *execution* steps specified by P_1 (i.e., the constituents of P_1) are as follows. In order to obtain a truth-value for any particular $\langle w, t\rangle$, do the following:

(1) ^{0}Charles: take the individual Charles
(2) $^0[^0{+}\ ^02\ ^05]$: take the construction $[^0{+}\ ^02\ ^05]$
(3) ^{0}*Calc*: take the relation-in-intension of calculating
(4) $[^0Calc_{wt}\ ^0Charles\ ^0[^0{+}\ ^02\ ^05]]$: check whether the entities identified at steps 1 and 2 occur at $\langle w, t\rangle$ in the relation obtained at step 3.

The argument (Calc), in Section 2.6, has an invalid logical form,[112] which is generated by substituting variables (ranging over the respective types) for Trivializations of extra-logical entities into P_1 and P_2.

[112] For details on logically valid arguments and the notion of *logical form*, see Section 1.5, Definitions 1.11 and 1.13, or Duží and Materna (2005).

The relevant variables are $P \to (o\iota*_1)_{\tau\omega}$ for ${}^0Calc$, $X \to \iota$ for ${}^0Charles$, $C_1 \to \tau$ for ^{02}, $C_2 \to \tau$ for ^{05}, $C_3 \to (\tau\tau\tau)$ for ${}^0+$, and $C_4 \to \tau$ for ^{07}:

$$\begin{array}{ll} [P_{wt}\, X\, {}^0[C_3\, C_1\, C_2]] & \\ [C_3\, C_1\, C_2] =_\tau C_4 & =_\tau/(o\tau\tau) \\ \hline [P_{wt}\, X\, C_4]. & \end{array}$$

The form is invalid, because in the first premise the variables C_1, C_2 and C_3 are ^{0}bound by Trivialization. Valid logical forms, on the other hand, would be:

$$\begin{array}{ll} [P_{wt}\, X\, C] & C, D \to \tau \\ C =_\tau D & =_\tau/(o\ \tau\tau) \\ \hline [P_{wt}\, X\, D] & \end{array}$$

and

$$\begin{array}{ll} [P_{wt}\, X\, {}^0C] & C, D/*_n \\ {}^0C =_{*n} {}^0D & =_{*n}/(o*_n*_n) \\ \hline [P_{wt}\, X\, {}^0D]. & \end{array}$$

We see no reason to challenge the unrestricted validity of Leibniz's Law of substitution (except for quotational contexts), and TIL has the resources to validate the Law in any sort of context, as we are now going to show.

It might seem that the use/mention distinction could be fully characterised by the points A, B:

A. An occurrence of the construction C_E is *mentioned* in* the construction C iff it is constructed by another subconstruction C' of C; C_E is then *mentioned* in C by C'*.

B. Otherwise, the occurrence of the construction C_E is *used* in C* as a constituent; i.e., if C_E is not constructed by a subconstruction of C.

So far, so good—and if this were the end of the use/mention story, it would be an easy one. Unfortunately, the characterization is too simple.

The simplification concerns two points. First, an occurrence of a construction can be *mentioned** indirectly by being a constituent of another subconstruction which is itself mentioned* in C. This fact is due to the dominancy of a higher-order conceptual context over a lower-order denotational context. Second, a construction can be executed twice over by means of Double Execution, which may undo the effect of mentioning*. For instance, though the Composition $[{}^0+\ {}^02\ {}^05]$ is constructed by (i.e., mentioned* in) the Trivialization ${}^0[{}^0+\ {}^02\ {}^05]$, the Double

Execution ${}^{20}[{}^{0}+\ {}^{0}2\ {}^{0}5]$ constructs the number 7, and both ${}^{0}[{}^{0}+\ {}^{0}2\ {}^{0}5]$ and $[{}^{0}+\ {}^{0}2\ {}^{0}5]$ are used* in ${}^{20}[{}^{0}+\ {}^{0}2\ {}^{0}5]$.

Concerning indirect mentioning*, consider

'Charles knows that dividing six by three makes two and that dividing six by zero is improper.'

Observe that if we wanted to analyse this sentence in any standard logic (including Montague's intensional logic, which lacks constructions or something akin to them) we would not have the tools to analyse it. We would be forced to switch to some kind of linguistic metamathematics. Says Tichý:

> If objects of higher-order are admitted, the need for metamathematics disappears and the mathematician need no longer be portrayed as a part-time linguistician. The notions and results of mathematics can be re-interpreted and integrated into mathematics proper. One need not, for example, ascend to the formal mode of speech to say that dividing six by zero produces nothing (1988, p. 72).

Let *Improper* be the class of constructions of order 1 that are v-improper for any valuation v. Then $Improper/(o*_1)$ belongs to a type of order 2. When knowing that dividing six by three makes two, Charles is not related to the truth-value **T**, but to a *construction* (belonging to $*_2$) of the value **T**. Therefore, *Know(ing)** is here a relation-in-intension of an individual to a construction of order 2, hence an $(o\iota*_2)_{\tau\omega}$–object.

Types: 0, 2, 3, $6/\tau$; $Div/(\tau\tau\tau)$; $Improper/(o*_1)$; $Know*/(o\iota*_2)_{\tau\omega}$.

The analysis of the complement clause 'that dividing six by three makes two and that dividing six by zero is improper' is:

(Em) $$\big[[[{}^{0}Div\ {}^{0}6\ {}^{0}3] = {}^{0}2] \wedge [{}^{0}Improper\ {}^{0}[{}^{0}Div\ {}^{0}6\ {}^{0}0]]\big].$$

The construction (Em) constructs **T**, and its subconstructions, with type assignments, are as follows:

(a) $\big[[[{}^{0}Div\ {}^{0}6\ {}^{0}3] = {}^{0}2] \wedge [{}^{0}Improper\ {}^{0}[{}^{0}Div\ {}^{0}6\ {}^{0}0]]\big]/*_2$
(b) $[[{}^{0}Div\ {}^{0}6\ {}^{0}3] = {}^{0}2]/*_1$
(c) $[{}^{0}Improper\ {}^{0}[{}^{0}Div\ {}^{0}6\ {}^{0}0]]/*_2$
(d) $[{}^{0}Div\ {}^{0}6\ {}^{0}3]/*_1$
(e) ${}^{0}Improper/*_2$
(f) ${}^{0}[{}^{0}Div\ {}^{0}6\ {}^{0}0]/*_2$
(g) ${}^{0}6$, ${}^{0}3$, ${}^{0}2$, ${}^{0}\wedge$, ${}^{0}{=}$ (all of type $*_1$)
(h) $[{}^{0}Div\ {}^{0}6\ {}^{0}0]/*_1$
(i) ${}^{0}Div$, ${}^{0}6$, ${}^{0}0$ (all of type $*_1$).

The constructions (a), (b), (c), (d), (e), (f), and the constructions *ad* (g) are *used** as *constituents* of (Em). The construction (h) is not used* in (Em) as a constituent because it is improper. If it were used*, then due to compositionality (Em) would itself be improper (see Definition 1.2). Since (Em) does construct something, namely **T**, (h) is only mentioned* in (Em) by using the constituent (f). As a result, due to the dominancy of the higher-order context of mentioning*, (h) is mentioned* also in (f) and in (c). The constructions ^0Div, ^{06}, ^{00} occur as constituents of (h), but they are mentioned* in (f), and consequently in (c) and in (a), i.e., in (Em). However, the first occurrences of ^0Div and ^{06} are used* in (d), (b), (a).

The analysis of the whole sentence is:

$$(C) \qquad \lambda w\lambda t\, [^0Know^*_{wt}\, ^0Charles \\ ^0[[[^0Div\ ^06\ ^03] = {^0}2] \wedge [^0Improper\ ^0[^0Div\ ^06\ ^00]]]].$$

Now all the occurrences of the constructions (a)–(i) are mentioned* in (C), i.e., none of them is a constituent of (C). The context of Charles's knowing is *hyperintensional* (or *conceptual* in TIL jargon), and hyperintensional (i.e., higher-order) contexts are seen to be dominant over lower-order functional (intensional/extensional) contexts. The constituents used* in (C) are:

$[^0Know^*_{wt}\ ^0Charles\ ^0[[[^0Div\ ^06\ ^03] = {^0}2] \wedge [^0Improper\ ^0[^0Div\ ^06\ ^00]]]]$
$[[^0Know^*\ w]t]$
$[^0Know^*\ w]$
$^0Know^*$
w
t
$^0Charles$
$^0[[[^0Div\ ^06\ ^03] = {^0}2] \wedge [^0Improper\ ^0[^0Div\ ^06\ ^00]]]$.

The second problem mentioned above—the problem of indirectly using* a construction by means of Double Execution—can be demonstrated by means of, for instance,

$$^{20}[^0Div\ ^06\ ^00].$$

It is the following two-step procedure:

(i) execute $^0[^0Div\ ^06\ ^00]$ to obtain $[^0Div\ ^06\ ^00]$
(ii) execute the result obtained at (i) to obtain the number resulting from dividing 6 by 0.

Though the constituent $^0[^0Div\ ^06\ ^00]$ does not fail to construct an entity in (i), the second execution, (ii), is improper due to using* rather than mentioning* the improper construction $[^0Div\ ^06\ ^00]$.

As a result, though the construction $[^0Div\ ^06\ ^00]$ is constructed (and thus mentioned*) in $^0[^0Div\ ^06\ ^00]$, the occurrence of $[^0Div\ ^06\ ^00]$ is *used** as a constituent of $^{20}[^0Div\ ^06\ ^00]$.

To put it in simple terms, Double Execution suppresses the effect of Trivialization. More generally, Double Execution decreases the level of a context. However, a context can be *hyper ... hyperintensional*. For instance, the Double Execution $^2[^0[^0[^0Div\ ^06\ ^00]]]$, or $^{200}[^0Div\ ^06\ ^00]$ for short, is proper, as the occurrence of $[^0Div\ ^06\ ^00]$ is *mentioned** in it by the constituent $^0[^0Div\ ^06\ ^00]$, and it constructs the Composition $[^0Div\ ^06\ ^00]$. The particular execution steps are:

(i) execute $^0[^0Div\ ^06\ ^00]$ to obtain $[^0Div\ ^06\ ^00]$
(ii) execute $^{00}[^0Div\ ^06\ ^00]]$ to obtain $^0[^0Div\ ^06\ ^00]$
(iii) execute $^{200}[^0Div\ ^06\ ^00]]]$,
 i.e., execute the result obtained at step (ii) to obtain $[^0Div\ ^06\ ^00]$.

Let $Proper/(o*_n)$ be the class of proper closed constructions of order n. Then the following constructions construct **T**:

(1) $[^0Improper\ ^{020}[^0Div\ ^06\ ^00]]$
(2) $[^0Proper\ ^{0200}[^0Div\ ^06\ ^00]]$
(3) $[^0Proper\ ^0[\lambda x\ [^0Div\ x\ ^00]]]$
(4) $[^0Proper\ ^{020}[\lambda x\ [^0Div\ x\ ^00]]]$.

The constructions (3) and (4) construct **T** because the Closure $\lambda x\ [^0Div\ x\ ^00]$ is not improper, as a Closure never is. It constructs a degenerate function, which is undefined at all its arguments. Note that the variable $x \rightarrow_v \tau$ is ^{0}bound rather than λ-bound[113] in (3) and (4): it occurs mentioned*.

It may also happen that a construction C is v-mentioned* or v-used* in another construction C', though not explicitly encoded by C'.

For instance, ^2c v-constructs the number 7 for a valuation v that assigns the Composition $[^0+\ ^02\ ^05]$ to the variable c ($c/*_2 \rightarrow_v *_1$). The variable c is used* in ^2c, and $[^0+\ ^02\ ^05]$ is then $v([^0+\ ^02\ ^05]/c)$-used* in ^2c. However, $[^0+\ ^02\ ^05]$ is not a subconstruction[114] of ^2c, and so is not a constituent of it. Note that what is v-constructed by ^2c may also depend on what v assigns to variables other than c. For instance, the construction ^2c is $v([^0Div\ ^02\ x]/c, 0/x)$-improper for a valuation assigning $[^0Div\ ^02\ x]$ to c and 0 to x ($x \rightarrow_v \tau$).

Hence, Double Execution (as well as Execution) does not bind variables. For instance, for one and the same variable c, $c/*_2 \rightarrow_v *_1$, the constructions ^0c, ^1c, ^2c are three different constructions belonging to type $*_3$: ^0c v-constructs c entirely independently of v; ^1c v-constructs whatever construction is assigned by v to c; ^2c v-constructs whatever entity (if any) is v-constructed by the construction which v assigns to c. Unlike Trivialization, which is an operation of *mentioning*, Execution and Double Execution are operations of *using*. If a variable is mentioned* in C then it is not free for substitution. It may be ^{0}-bound or v-mentioned* by another variable used* as a constituent of C.

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113 See Definition 1.4.

114 See Definition 1.3.

Example 2.1 *Quantifying into a hyperintensional context.* Consider the argument

$$\text{(A)}\quad \frac{\text{Charles believes that dividing five by zero makes zero}}{\text{There is a number } x \text{ such that Charles believes that dividing } x \text{ by zero makes zero.}}$$

The argument is obviously valid. The analysis of the complement clause ‘that dividing five by zero makes zero’ is

$$[^0\!= [^0Div\ ^05\ ^00]\ ^00].$$

This Composition is improper.[115] Yet if Charles believes that dividing five by zero makes zero, then there is something that Charles believes: he believes, erroneously, that the above Composition constructs **T**. Thus Charles’ relation of believing is here a relation-in-intension to a construction, and we have a case of explicit $Believe^*/(o\iota{*_1})_{\tau\omega}$.

The analysis of the premise of (A) thus comes down to this:

$$\lambda w\lambda t\ [^0Believe^*_{wt}\ ^0Charles\ ^0[^0\!= [^0Div\ ^05\ ^00]\ ^00]].$$

Now we might be tempted to simply apply the rule of existential generalisation in order to infer the conclusion

$$\lambda w\lambda t\ \exists x\ [^0Believe^*_{wt}\ ^0Charles\ ^0[^0\!= [^0Div\ x\ ^00]\ ^00]].$$

Types: $x \rightarrow_v \tau$; $0/\tau$; $\exists/(o(o\tau))$; $Charles/\iota$; $0, 5/\tau$; $Believe^*$, Div as above.

Alas, such a conclusion is *not* entailed by the premise. The reason is this. Variable x occurs *mentioned**; it is ^{0}bound and therefore not free for λ-binding, and the truth-value v-constructed by the Composition

$$[^0Believe^*_{wt}\ ^0Charles\ ^0[^0\!= [^0Div\ x\ ^00]\ ^00]]$$

is the same as the truth-value $v(5/x)$-constructed by this Composition. We have to *pre-process* the closed construction $^0[^0\!= [^0Div\ x\ ^00]\ ^00]$ first (that is, make the variable x free for manipulation) and only then can we quantify over x. To this end we apply the functions $Sub/({*_1}{*_1}{*_1}{*_1})$ and $Tr/({*_1}\tau)$ introduced in Section 1.4.3.

Applying Tr to x gives as a result the Trivialization of the number v-constructed by x. For instance, $[^0Tr\ x]$ $v(5/x)$-constructs ^{05}. The variable x is now *used**, and thus free, in $[^0Tr\ x]$.

[115] See Definition 1.2.

The second step in pre-processing the Trivialization ${}^0[{}^0\!= [{}^0Div\ x\ {}^00]\ {}^00]]$ consists in substituting the product v-constructed by $[{}^0Tr\ x]$ for x into the Composition constructed by this Trivialization:

$$[{}^0Sub\ [{}^0Tr\ x]\ {}^0x\ {}^0[{}^0\!= [{}^0Div\ x\ {}^00]\ {}^00]].$$

Note that the first occurrence of x is now used* (and thus free) here, whereas the second and third occurrences are ^{0}bound, and thus mentioned*. This Composition $v(1/x)$-constructs the construction $[{}^0\!= [{}^0Div\ {}^01\ {}^00]\ {}^00]$, $v(2/x)$-constructs the construction $[{}^0\!= [{}^0Div\ {}^02\ {}^00]\ {}^00]$, and so on. The resulting analysis of the conclusion is

$$\lambda w \lambda t\ \exists x\ [{}^0Believe*_{wt}\ {}^0Charles\ [{}^0Sub\ [{}^0Tr\ x]\ {}^0x\ {}^0[{}^0\!= [{}^0Div\ x\ {}^00]\ {}^00]]].$$

Indeed, if the premise of (A) is true, then the class of numbers v-constructed by

$$\lambda x\ [{}^0Believe*_{wt}\ {}^0Charles\ [{}^0Sub\ [{}^0Tr\ x]\ {}^0x\ {}^0[{}^0\!= [{}^0Div\ x\ {}^00]\ {}^00]]]$$

is not empty, because it contains at least the number 5. Thus the conclusion is true as well.

The above examples illustrate the fact that an occurrence of a subconstruction D in a construction C need not be executed in order to execute C when D is Trivialized or D is a subconstruction of a construction D' that is Trivialized. Thus, roughly speaking, D is mentioned* in C if C is of the form $[\ldots\ {}^0[\ldots D\ldots]\ \ldots]$. On the other hand, D is *used* as a constituent of* C if C is of one of the forms $[\ldots D\ldots]$, $[\ldots{}^1[\ldots D\ldots]\ \ldots]$, $[\ldots{}^2[\ldots D\ldots]\ldots]$.

To put these considerations on a more solid ground, we define:

Definition 2.15 (*construction mentioned* vs. used* as a constituent*) Let C be a construction and D a subconstruction of C.

(i) If D is identical to C (i.e., ${}^0C = {}^0D$) then the occurrence of D is used* as a constituent of C.
(ii) If C is identical to $[X_1\ X_2\ldots X_m]$ and D is identical to one of the constructions $X_1, X_2,\ldots, X_m$, then the occurrence of D is used* as a constituent of C.
(iii) If C is identical to $[\lambda x_1\ldots x_m\ X]$ and D is identical to X, then the occurrence of D is used* as a constituent of C.
(iv) If C is identical to ${}^1X$ and D is identical to X, then the occurrence of D is used* as a constituent of C.
(v) If C is identical to ${}^2X$ and D is identical to X, or ${}^0D$ occurs as a constituent of X and this occurrence of D occurs as a constituent of Y v-constructed by X, then the occurrence of D is used* as a constituent of C.
(vi) If an occurrence of D is used* as a constituent of an occurrence of C' and this occurrence of C' is used* as a constituent of C, then the occurrence of D is used * as a constituent of C.

(vii) If an occurrence of a subconstruction D of C is not used* as a constituent of C then the occurrence of D is mentioned* in C.
(viii) No occurrence of a subconstruction D of C is used*/mentioned* in C unless it so follows from (i) to (vii). □

Remark. If a construction D is mentioned* in C then all the variables occurring in D are ^{0}bound in C. *Proof* follows from Definitions 1.4 and 2.15: if D is mentioned* in C then there is a construction D' such that $^0D'$ is, and D' is not, used* as a constituent of C, and D is a subconstruction of D'.

As we mentioned above, a construction D can be also v-used* or v-mentioned* in another construction C. This happens due to Double Execution. Though this is a bit peculiar, because in this case D is not a subconstruction of C, we define v-using* and v-mentioning* for completeness.

Definition 2.16 ***(v-mentioned* and v-used* constructions)*** Let C be a construction whose constituent is an occurrence of a construction ^2X. Let X v-construct a construction Y and let D be a subconstruction of Y.

(i) If an occurrence of D is used* in Y and not used* in C, then the *occurrence of D is v-used* in C.*
(ii) If an occurrence of D is mentioned* in Y or in a construction that is v-used* in C then the *occurrence of D is v-mentioned* in C.*
(iii) No occurrence of a subconstruction D of C is v-used*/v-mentioned* in C unless it so follows from (i) to (ii). □

Example 2.2 *Using vs. mentioning constructions.*

Let $C = {}^2[{}^0Sub\ [{}^0Tr\ {}^0\surd x]\ {}^0y\ {}^0\big[{}^0Deg\ \lambda z\ [{}^0\mathbf{:}\ z\ y]\big]]$.

Types: $\surd/(\tau\tau)$: the *positive* square-root function, that is, the function that gives the absolute value of the square root of a given number; $Deg/(o(\tau\tau))$: the class of degenerate functions of type $(\tau\tau)$; $\mathbf{:}/(\tau\tau\tau)$: the division function; x, y, $z/*_1\rightarrow_v\tau$; $Sub/(*_1*_1*_1*_1)$; $Tr/(*_1\tau)$.

Then:
(a) C is *used** in C due to (i) of Definition 2.15.
(b) $[{}^0Sub\ [{}^0Tr\ {}^0\surd x]\ {}^0y\ {}^0[{}^0Deg\ \lambda z\ [{}^0\mathbf{:}\ z\ y]]]$ is *used** in C due to (v) of Definition 2.15.
(c) ${}^0Sub$, $[{}^0Tr\ {}^0\surd x]$, ${}^0Tr$, ${}^0\surd x$, ${}^0\surd$, x, ${}^0y$, ${}^0[{}^0Deg\ \lambda z\ [{}^0\mathbf{:}\ z\ y]]$ are *used** in C due to (ii) and (vi) of Definition 2.15.
(d) $[{}^0Deg\ \lambda z\ [{}^0\mathbf{:}\ z\ y]]$, ${}^0Deg$, $\lambda z\ [{}^0\mathbf{:}\ z\ y]$, $[{}^0\mathbf{:}\ z\ y]$, ${}^0\mathbf{:}$, z, y are *mentioned** in ${}^0[{}^0Deg\ \lambda z\ [{}^0\mathbf{:}\ z\ y]]$ due to (vii) of Definition 2.15.
(e) $[{}^0Deg\ \lambda z\ [{}^0\mathbf{:}\ z\ y]]$ is *not used** as a constituent of C, though it might seem that due to Double Execution it were so. Yet due to the application of *Sub* and *Tr*, this Composition is being pre-processed. Thus only for some valuations v_i does it hold that the Composition resulting from this pre-processing is v-used*. Here is why.

According to (v) of Definition 2.15, in order to be a constituent of C the Composition $[^0Deg\ \lambda z\ [^0\!:\ z\ y]]$ would have to be a constituent of a construction v-constructed by the constituent *ad* (b). Let, for instance, v be a valuation assigning the number 9 to x. Then the Composition

$$[^0Sub\ [^0Tr\ ^0\surd x]\ ^0y\ ^0[^0Deg\ \lambda z\ [^0\!:\ z\ y]]]$$

$v(9/x)$-constructs the Composition $[^0Deg\ \lambda z\ [^0\!:\ z\ ^03]]$. Though the Composition $[^0Deg\ \lambda z\ [^0\!:\ z\ ^03]]$ is neither *used** as a constituent of C nor *mentioned** in C, it is $v(9/x)$-*used** in C (according to Definition 2.16 (i)). When evaluating C for a valuation $v(n/x)$, $n < 0$, the construction $^0\surd x$ is v-improper and so is the construction *ad* (b); i.e., the substitution fails to produce a product. Thus the execution of C does not involve the execution of $[^0Deg\ \lambda z\ [^0\!:\ z\ y]]$, and the latter is not a constituent of C.

Example 2.3. Let $C' = {}^2[^0\iota\ \lambda c\ [^0\!=c\ ^0[^0Deg\ \lambda z\ [^0\!:\ z\ y]]]]$.

Types: $c/*_2 \rightarrow_v *_1$; the other types as above.

Let $C^* = [^0\iota\ \lambda c\ [^0\!=c\ ^0[^0Deg\ \lambda z\ [^0\!:\ z\ y]]]]$. Then C^* constructs

$$[^0Deg\ \lambda z\ [^0\!:\ z\ y]].$$

Thus the Trivialization $^0[^0Deg\ \lambda z\ [^0\!:\ z\ y]]$ is a constituent of C^* and $[^0Deg\ \lambda z\ [^0\!:\ z\ y]]$ is a constituent of the construction v-constructed by C^*. Therefore, $[^0Deg\ \lambda z\ [^0\!:\ z\ y]]$ is *used** in C' due to (v) of Definition 2.15.

Typical cases of mentioning* are sentences expressing hyperintensional attitudes, which are attitudes to the *meaning* of the embedded clause, like the premise and conclusion of the argument (A) of Example 2.1 above. We will tackle attitude reports in Chapter 5, together with some more complicated cases involving substitution and Double Execution.

2.6.2 Intensional and extensional occurrence of constituents

In Section 2.6.1 we defined the distinction between using* a construction as a constituent of another construction and mentioning* a construction within another construction. Now we are going to deal with two kinds of context in which a constituent can be *used**, namely intensional and extensional contexts. In Section 1.5.2 two ways of using a construction of an intension, namely with supposition *de dicto* and *de re*, were characterised. We discussed two principles *de re*: the principle of substitution of v-congruent constructions and the principle of existential presupposition. We also mentioned the dominancy of *de dicto* contexts over *de re* contexts. Now we are going to generalise these two kinds of supposition in which a constituent can occur to the case of a construction of a function of any type and not only of an intension.

The goal of this section is to define two ways in which a constituent can occur: intensionally and extensionally. In the next section we then use these results to generalise the two principles *de re* to (a) general rules of valid substitution in any context, and (b) rules of inferring existence/non-existence.

We begin by recapitulating the characterizations of the three kinds of context provided at the end of Section 1.5.2.

- *Hyperintensional context*: the kind of context in which a construction is not used to *v*-construct a function (or a value). Instead the *construction* itself is the argument of another function; the construction is merely *mentioned**. Only in a hyperintensional context can a construction figure as subject of predication.
- *Intensional context*: the kind of context in which a construction *C* is *used** intensionally to *v*-construct a function *rather than a particular value* of the function.

  Example. Consider the Composition $[{}^0\mathit{Arithmetic}\ {}^0\mathit{Square\_root}]$.

  Types: $\mathit{Arithmetic}/(o(\tau\tau))$: the class of arithmetic functions of type $(\tau\tau)$; $\mathit{Square\_root}/(\tau\tau)$.

  ${}^0\mathit{Square\_root}$ is used *intensionally* within the Composition $[{}^0\mathit{Arithmetic}\ {}^0\mathit{Square\_root}]$. It is not composed with a τ-argument in order to construct an absolute value of the square root function. The subject of predication is not a value of the square root function but this very function.

- *Extensional context*: the kind of context in which a construction *C* of a function is used extensionally as an instruction to apply the function in order to *v*-construct a particular value of the function.

  Example.
  'The square root of 4 = 2' expresses the Composition $[{}^0{=}\ [{}^0\mathit{Square\_root}\ {}^04]\ {}^02]$, where ${}^0\mathit{Square\_root}$ occurs extensionally; the Composition is used to construct the value of the square root function at 4. Also, in the previous example, ${}^0\mathit{Arithmetic}$ is used extensionally.

The details, however, are somewhat more involved. The basic idea is that a 'higher' context is dominant over a 'lower' one. Thus, for instance, in the meaning of the sentence, 'The square root is an arithmetic function' the Trivialization ${}^0\mathit{Square\_root}$ occurs intensionally, and extensionally in the meaning of 'The square root of 4 equals 2'. However, if included in a hyperintensional context, the respective constructions are both mentioned*. For instance, in the meaning of 'Charles believes that the square root is an arithmetic function' or 'Charles believes that the square root of 4 equals 2' the construction ${}^0\mathit{Square\_root}$ is mentioned*, as the respective analyses reveal:

$$\lambda w \lambda t\ [{}^0\mathit{Believe}_{wt}\ {}^0\mathit{Charles}\ {}^0[{}^0\mathit{Arithmetic}\ {}^0\mathit{Square\_root}]];$$

$$\lambda w \lambda t\ [{}^0\mathit{Believe}_{wt}\ {}^0\mathit{Charles}\ {}^0[{}^0{=}\ [{}^0\mathit{Square\_root}\ {}^04]\ {}^02].$$

Using a construction of an intension either with *de dicto* or *de re* supposition is closely connected with the intensional or extensional occurrence of a constituent. Consider, for instance,

(1) ‘The Pope is a German’

(2) ‘Joseph Ratzinger became the Pope on April 19, 2005’.

Sentence (1) expresses the construction

(1′) $\lambda w\lambda t\ [{}^0German_{wt}\ {}^0Pope_{wt}]$

whereas (2) expresses

(2′) $\lambda w\lambda t\ [{}^0Past_t\ \lambda c\ \exists t'[[c\ t'] \wedge [{}^0Become_{wt'}\ {}^0Ratzinger\ {}^0Pope]]\ {}^0April19]$.

Types: $German/(o\iota)_{\tau\omega}$; $Pope/\iota_{\tau\omega}$; $Past/(o(o(o\tau))(o\tau))_\tau$; $Become/(o\iota\iota_{\tau\omega})_{\tau\omega}$; $Ratzinger/\iota$; $April19/(o\tau)$; $c \rightarrow (o\tau)$; $t, t' \rightarrow \tau$.

The Trivialization ${}^0Pope \rightarrow \iota_{\tau\omega}$ occurs *extensionally* in $[{}^0German_{wt}\ {}^0Pope_{wt}]$, because it is used* to *v*-construct the value of an ι-office. Thus ${}^0Pope$ occurs with *de re* supposition in (1′), as only the value *v*-constructed by $[{}^0German_{wt}\ {}^0Pope_{wt}]$ matters, the other values being irrelevant. On the other hand, in (2′) ${}^0Pope$ occurs *intensionally*. The constituent ${}^0Pope$ is used to construct an ι-office rather than a particular value. The whole papal office matters in the truth-conditions of the proposition constructed by (2′) rather than just a particular value. Thus ${}^0Pope$ occurs with *de dicto* supposition in (2′). Thus for (1) the two principles *de re* are valid. If Ratzinger is the Pope, then from (1) we may validly infer that Ratzinger is a German, whereas we cannot validly infer from (2) that Ratzinger became Ratzinger. Moreover (1) not only implies but even presupposes that the Pope should exist, unlike (2).

In Section 2.3 we set out the principles of analysis of sentences ascribing existence or non-existence to entities. One may wonder in which context or with which supposition constructions to do with existence occur. Consider, for instance, the following true ascriptions of non-existence:

(i) ‘Pegasus does not exist.’

(ii) ‘Water sprites do not exist.’

(iii) ‘The greatest prime does not exist.’

Let $Pegasus/\iota_{\tau\omega}$; $Water\_sprite/(o\iota)_{\tau\omega}$; $Exist^{\iota}/(o\iota_{\tau\omega})_{\tau\omega}$; $Exist^{(o\iota)}/(o(o\iota)_{\tau\omega})_{\tau\omega}$; $Exist^*/(o*_1)$; $x, y \rightarrow \tau$; $Prime/(o\tau)$; $\geq/(o\tau\tau)$. Then the sentences express the following constructions:

(i′) $\lambda w\lambda t\ [\neg[{}^0Exist^{\iota}{}_{wt}\ {}^0Pegasus]]$

(ii′) $\lambda w\lambda t\ [\neg[{}^0Exist^{(o\iota)}{}_{wt}\ {}^0Water\_sprite]]$

(iii′) $\neg[{}^0Exist^*\ {}^0[{}^0\iota\ \lambda x\ [[{}^0Prime\ x] \wedge {}^0\forall\lambda y\ [[{}^0Prime\ y] \supset [{}^0\geq x\ y]]]]]$.

$Exist^{\iota}$ is the property of an individual office of being occupied; $Exist^{(o\iota)}$ is the property of an individual property of being instantiated; $Exist^{*}$ types the class of proper constructions of order 1, and is the property of a first-order construction of being proper.

As we explained in Section 2.3.2, the constructions ${}^0Pegasus$ and ${}^0Water\_sprite$ are used* as constituents, occurring *de dicto* in (i′) and (ii′), respectively, because lack of occupancy and instantiation are ascribed to the office of *Pegasus* and the property *Water sprite*, respectively, rather than to any values at some particular $\langle w, t\rangle$.

Concerning the meaning of 'the greatest prime' in (iii), can we sensibly speak of its *de dicto* supposition? No; we have defined *de dicto* supposition only for constituents constructing intensions. In case of non-empirical concepts we speak of intensional/extensional occurrence, which we are going to define below. However, the Composition $[{}^0\iota\ \lambda x\ [[{}^0Prime\ x] \wedge {}^0\forall\lambda y\ [[{}^0Prime\ y] \supset [{}^0\geq x\ y]]]]$ occurs neither intensionally nor extensionally in (iii′), because it is *mentioned** and so is not a constituent of (iii′). It occurs hyperintensionally in (iii′). What is a constituent of (iii′) is its Trivialization. So (iii′) has only three constituents, the other two being ${}^0\neg$ and ${}^0Exist^{*}$.

Equivalent analyses of (i), (ii) can be obtained by using the definition of $Exist^{\iota}$ and $Exist^{(o\iota)}$, respectively, using the existential quantifier $\exists^{\iota}/(o(o\iota))$. Let variables $x, y \rightarrow \iota$. Then (i′), (ii′) are equivalent to (i″), (iii″), respectively:

(i″) $\lambda w\lambda t\ [\neg[{}^0\exists^{\iota}\ \lambda x\ [x = {}^0Pegasus_{wt}]]]$

(ii″) $\lambda w\lambda t\ [\neg[{}^0\exists^{\iota}\ \lambda y\ [{}^0Water\_sprite_{wt}\, y]]]$.

We shall say that the constituents

$$\lambda x\ [x = {}^0Pegasus_{wt}]$$

and

$$\lambda y\ [{}^0Water\_sprite_{wt}\, y]$$

occur with $(o\iota)$-*intensional supposition* in the respective Compositions of (i″), (ii″); consequently, ${}^0Pegasus$, ${}^0Water\_sprite$ occur in the *intensional context* of (i″), (ii″).

Definition 2.15 distinguishes rigorously between a hyperintensional context, in which constructions are mentioned*, and a context in which constructions are used* as constituents. We have shown that in a hyperintensional context constructions do not operate as constituents to be executed. Rather they are just objects that other constituents operate on. Above we also illustrated the fact that constituents can be used* either intensionally (*de dicto*) or extensionally (*de re*), and that only if used* extensionally does partiality become an issue. The definitions that follow serve to provide these considerations with a solid underpinning. Having defined the difference between using* a construction as a constituent and mentioning* a construction in Definition 2.15, we are now going to define the two ways in which a constituent can be used*: intensionally and extensionally. There are two main reasons for distinguishing sharply between intensional and extensional occurrences of a constituent *D* within a construction *C*. First, if *D* occurs extensionally then a construction *D′* *v*-congruent with *D* can be validly substituted for *D*.[116] On the other hand, if *D* occurs intensionally then only an equivalent construction can be validly substituted for *D*. Second, only if *D* occurs extensionally can *D* as well as *C* be *v*-improper.

One general problem we face stems from the fact that we work with properly partial functions. Therefore, the instruction to apply, e.g., the positive square-root function to a negative argument is an improper construction (in the domain of real numbers).

Another type of improperness is due to type-theoretic mismatch. For instance, where $Student/(o\iota)_{\tau\omega}$ and $5/\tau$, the sentence 'The number five is a student' is not reasonable. Predicating an individual property of a number yields neither **T** not **F**: $[{}^0Student_{wt}\ {}^05]$ is *v*-improper for any valuation *v*. The sentence does have a meaning—namely, $\lambda w\lambda t\ [{}^0Student_{wt}\ {}^05]$—but this Closure constructs an impossible proposition, being an empty subset of logical space. Similarly, an attempt to execute a non-constructional entity always fails: ${}^1Student$, ${}^2Student$, ^{15}, ^{25} are all improper Executions.[117]

Thus when a constituent is used* extensionally, it may be improper for some valuations *v* and this fact must be taken into account. Partiality makes itself felt in extensional contexts. According to Definition 1.2, *v*-improperness arises due to a Composition $[X\ X_1 \ldots X_n]$ *v*-constructing an entity of type α by Composing a constituent *X* *v*-constructing a function $f/(\alpha\ \beta_1 \ldots \beta_n)$ with constituents $X_1 \rightarrow_v \beta_1, \ldots, X_n \rightarrow_v \beta_n$, in case *f* is undefined at the respective argument. In such a case *X* occurs with extensional supposition.

The definitions that follow proceed inductively with respect to the complexity of a construction. Thus we first define *atomic construction*.

Definition 2.17 (*atomic construction*) A *construction C is atomic* if *C* does not contain any other constituent but itself. □

Corollary. A construction *C* is atomic if *C* is
(i) a variable; or
(ii) a Trivialization ${}^0X$, where *X* is an entity of any type, even a construction; or

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[116] See Definition 1.5 and Claim 2.4 below.

[117] An investigation into the notion of *nonsense* within the confines of procedural semantics would probably cite 'Five is a student' and 'Charles is a prime number' as limiting cases at the 'soft' end of the spectrum of degrees of nonsense. Such an investigation must await another occasion. However, the ability to describe and classify different kinds of nonsense is a touchstone for any theory of linguistic sense, such as ours, since a theory must be capable of conceptualizing not only the success cases, but also various failed cases.

(iii) an Execution ^1X or a Double Execution ^2X, where X is an entity of a type of order 1, i.e., a non-construction.

An atomic construction of kind (i) or (ii) is v-proper for any valuation v. An atomic construction ^1X, ^2X, of kind (iii), is v-improper for any valuation v. In this case ^1X or ^2X does not v-construct anything, and $^1X \rightarrow \alpha$, $^2X \rightarrow \alpha$, for any type α, would constitute a type-theoretic mismatch.

Now, in order to define the intensional/extensional occurrence of a constituent within a compound construction, we proceed in three stages, of which the first two definitions are auxiliary for the third. First we define the *intensional/extensional supposition* of a constituent. Roughly, a constituent C v-constructing a function $f/(\alpha\beta)$ occurs with extensional (intensional) supposition if C is (not) Composed with a constituent $D \rightarrow_v \beta$ in order to v-construct the α-value of f. Second, in order to capture the dominancy of a higher intensional context over a lower extensional one, we define the notion of *generic intensional context*. Third, we define the *intensional/extensional occurrence* of a constituent.

Definition 2.18 (*intensional/extensional supposition*)

(i) Let C be an atomic construction and let D be C, $D \rightarrow_v (\beta_1 \ldots \beta_n)$, $n \geq 1$. Then *D occurs in C with* $(\beta_1 \ldots \beta_n)$*-intensional supposition*.

(ii) Let C be a Closure of the form $[\lambda x_1 \ldots x_m\, X]$, $x_1 \rightarrow_v \beta_1, \ldots, x_m \rightarrow_v \beta_m$, $X \rightarrow_v \alpha$. Then:
   1. If D is C then *D occurs in C with* $(\alpha\beta_1 \ldots \beta_m)$*-intensional supposition*.
   2. If D is a constituent of X then *D occurs in C with the same supposition* as does D in X.

(iii) Let C be a Composition of the form $[X\, Y_1 \ldots Y_m]$, $m \geq 1$, and $X \rightarrow_v (\alpha\beta_1 \ldots \beta_m)$, $Y_1 \rightarrow_v \beta_1, \ldots, Y_m \rightarrow_v \beta_m$. Then:
   1. If D is C then *D occurs in C with* α*-intensional supposition*.
   2. If D is X then *D occurs in C with* $(\beta_1, \ldots, \beta_m)$*-extensional supposition*.
   3. If D is a constituent of X that is not equal to X or if D is a constituent of Y_i $(1 \leq i \leq m)$ then *D occurs* in *C with* the same *supposition* as does D in X, Y_i, respectively.

(iv) Let C be ^1X or ^2X, X a construction. Then the constituents of *X occur in C* with the same *supposition* as they do in X.

(v) Let C be ^1X or ^2X, X an entity of a type of order 1, and let D be C. Then D occurs in C with *extensional supposition.*

(vi) Nothing else *occurs in C with intensional/extensional supposition* unless it so follows from (i) to (v). □

Corollary. A constituent D occurs with extensional supposition in C, if C is a Composition $[D\, Y_1 \ldots Y_m]$, or a (Double) Execution $^1[D\, Y_1 \ldots Y_m]$, $^2[D\, Y_1 \ldots Y_m]$, or D is identical to C of the form ^1X or ^2X, where X is an entity of a type of order 1.

We have seen that, for instance, the Composition

$$[{}^0\!: {}^05\ {}^00]$$

is improper, because the division function is undefined at $\langle 5, 0\rangle$, and the constituent ${}^0\!:$ occurs in this Composition with extensional supposition. If, *per impossibile*, the Composition were proper, we could deduce that there were a value of the division function at $\langle 5, 0\rangle$. On the other hand, Closures like

$$\lambda x\ [{}^0\!: {}^05\ {}^00]$$

$$\lambda x\ [{}^0\!: x\ {}^00]$$

are proper, constructing as they do a degenerate function for any valuation of x. Though the constituent ${}^0\!:$ still occurs with extensional supposition in the Compositions $[{}^0\!: {}^05\ {}^00]$, $[{}^0\!: x\ {}^00]$, it now occurs within the scope of λ-abstraction. As Tichý says, λ-abstraction constitutes a *generic intensional context* (1988, p. 204). When a Closure is Composed with its constituents, improperness becomes again an issue, the context becoming extensional and the Composition improper:

$$[\lambda x\ [{}^0\!: x\ {}^00]\ {}^05].$$

The generic intensional level can be a multiple one, as in

$$[\lambda x\ [\lambda y\ [{}^0\!: x\, y]]],$$

and in order to decrease an intensional context to the extensional level, two nested Compositions are needed:

$$[\lambda x\ [\lambda y\ [{}^0\!: x\, y]\ {}^00]\ {}^05].$$

This composition is improper again, unlike

$$[\lambda x\ [\lambda y\ [{}^0\!: x\, y]\ {}^00]],$$

$$[\lambda x\ [\lambda y\ [{}^0\!: x\, y]]\ {}^05].$$

The former constructs the degenerate function of dividing x by zero. The latter constructs the function of dividing 5 by some number y, because the context is still intensional.

Hence a Closure increases the intensional level, whereas a Composition decreases it, and we need to trace the level of intensionality. To this end we use types of arguments to which a given Closure is *hospitable*.[118]

[118] Here we use the term ‘hospitable’ in a different sense than Tichý (1986a, pp. 261–63, 2004, pp. 659–61).

For instance, if $C \rightarrow \alpha$, $x \rightarrow \gamma$ and $y \rightarrow \delta$, the Closure $\lambda y\ [\lambda x\ C] \rightarrow ((\alpha\gamma)\delta)$ is (at least) doubly generic, being hospitable both to a δ-argument and to a γ-argument. We will say that the context of $[\lambda x\ [C\ D]]$ is (γ)-*generic*, and the context of $[\lambda y\ [\lambda x\ [C\ D]]]$, $(\delta\gamma)$-*generic*. Before defining these phenomena rigorously, we schematize the interplay between Compositions and Closures in Table 2.1.

Table 2.1 Generic intensional context

Constituent	Type of constructed entity	Context
$[X_0X]$	α_0	Non-generic
$\lambda x_1\ [X_0X]$	$(\alpha_0\alpha_1)$	(α_1)-generic
$[\lambda x_1\ [X_0X]\ X_1]$	α_0	Non-generic
$\lambda x_2\ [\lambda x_1\ [X_0X]]$	$((\alpha_0\alpha_1)\alpha_2)$	$(\alpha_2\alpha_1)$-generic
$[\lambda x_2\ [\lambda x_1\ [X_0X]]\ X_2]$	$(\alpha_0\alpha_1)$	(α_1)-generic
$[\lambda x_2\ [\lambda x_1\ [X_0X]\ X_1]]$	$(\alpha_0\alpha_2)$	(α_2)-generic
$[\lambda x_2\ [\lambda x_1\ [X_0X]\ X_1]\ X_2]$	α_0	Non-generic

Types: $X \rightarrow \alpha$; $X_0 \rightarrow (\alpha_0\alpha)$; $X_1, x_1 \rightarrow \alpha_1$; $X_2, x_2 \rightarrow \alpha_2$.

The last problem we must address when tracing the intensional level of a generic context stems from the fact that we work with *n*-ary functions, which are applied to tuple-arguments. As we showed in Section 2.4.3, Schönfinkel's reduction of *n*-ary functions to unary composite functions does not go through when properly partial functions are involved. Thus we have to distinguish between

$$[\lambda x_2\ [\lambda x_1\ X]] \rightarrow ((\alpha\alpha_1)\alpha_2)) \text{ and } [\lambda x_1 x_2\ X] \rightarrow (\alpha\alpha_1\alpha_2),$$

because such Closures are not equivalent.

To overcome this difficulty, we now introduce a *tuple* type. Let $\alpha_1,\ldots,\alpha_m$ be types; then the *tuple* $(\alpha_1,\ldots,\alpha_m)$, which is the Cartesian product of types $\alpha_1, \ldots, \alpha_m$, is a type as well. Of course, tuple types could be defined as molecular functional types. Note, however, that the tuple type $(\alpha_1,\ldots,\alpha_m)$ is not identical to the functional type $(\alpha_1\ldots\alpha_m)$. The latter is a mapping $(\alpha_2,\ldots,\alpha_m) \rightarrow \alpha_1$.

Thus we shall say that the context of $[\lambda x_2\ [\lambda x_1\ X]]$ is $(\alpha_2\alpha_1)$-*generic*, whereas the context of $[\lambda x_1 x_2\ X]$ is (α_1,α_2)-*generic*. When increasing the generic intensional level, hospitality to tuple arguments and single arguments can be combined. Thus, for instance, if X is $[\lambda x\ C]$, $x \rightarrow \alpha$, the context of $[\lambda x_1 x_2\ [\lambda x\ C]]$ is $((\alpha_1,\alpha_2)\alpha)$-generic, the context of $[\lambda x_1[\lambda x_2\ [\lambda x\ C]]]$ is $(\alpha_1\alpha_2\alpha)$-generic, and the context of $[\lambda x_1\ [\lambda x_2\, x\ C]]$ is $(\alpha_1(\alpha_2,\alpha))$-generic.

Definition 2.19 (*generic/non-generic context*)

(i) Let C be an atomic construction. Then *C occurs in the non-generic context of C.*

(ii) Let C be a Closure of the form $[\lambda x_1 \ldots x_m\, X]$; $x_1 \to_v \gamma_1, \ldots, x_m \to_v \gamma_m$.

(a) If D is C and X occurs in a non-generic context of X then *D occurs in the* $(\gamma_1,\ldots,\gamma_m)$*-generic intensional context of C.*

(b) If D is C and X occurs in a (β)-generic intensional context of X for some type β then *D occurs in the* $((\gamma_1,\ldots,\gamma_m)\beta)$*-generic intensional context of C.*

(c) If D is a constituent of X and D occurs in a non-generic context of X then *D occurs in the* $(\gamma_1,\ldots,\gamma_m)$*-generic intensional context of C.*

(d) If D is a constituent of X and D occurs in a β-generic intensional context of X for some type β then *D occurs in the* $((\gamma_1,\ldots,\gamma_m)\beta)$*-generic intensional context of C.*

(iii) Let C be a Composition of the form $[X\, Y_1 \ldots Y_m]$; $Y_1 \to_v \gamma_1, \ldots, Y_m \to_v \gamma_m$.

(a) If X occurs in a generic intensional context of X then

- if D is a constituent of X and D occurs in a $(\gamma_1,\ldots,\gamma_m)$-generic context of X then *D occurs in the non-generic (extensional) context of C*, and
- if D is a constituent of X and D occurs in a $((\gamma_1,\ldots,\gamma_m)\beta)$-generic context of X for some β then *D occurs in* the (β)*-generic (intensional) context of C.*

(b) If X occurs in a non-generic extensional context of X then X occurs in a non-generic extensional context of C and the constituents of X *occur in C* in the same context as they do in X.

(c) If D is C then the context in which *D occurs in C* is the same as the context in which X occurs in C.

(d) The contexts in which the constituents of Y_i ($1 \leq i \leq m$) *occur in C* are the same as the contexts in which they occur in Y_i.

(iv) Let C be ${}^1X$ or ${}^2X$. Then the constituents of X *occur in C* in the same context as they do in X.

(v) Nothing else *occurs in a generic/non-generic context of C* unless it so follows from (i) to (iv).

Definition 2.20 (*intensional/extensional occurrence*) If D occurs with an intensional supposition or in a generic context of C, then *D occurs intensionally in C.* If D occurs with extensional supposition (and) in a non-generic context of C, then *D occurs extensionally in C.*

Corollaries. Let a construction D v-construct a function $f/(\alpha\, \beta_1 \ldots \beta_n)$. Then:

- If D occurs intensionally in C, then D occurs with $(\alpha\, \beta_1 \ldots \beta_n)$-intensional supposition or in a generic intensional context of C. Thus what C v-constructs depends on the whole f rather than on its particular α-value at a $(\beta_1,\ldots,\beta_n)$-parameter; the function f is then merely *mentioned* by the occurrence of D in C.

- If D occurs extensionally in C, then D occurs with ($\beta_1 \ldots \beta_n$)-extensional supposition in a non-generic extensional context of C. Thus what C v-constructs depends only on the particular α-value of f (the other α-values being irrelevant). The function f is not only mentioned but also *used to point to its α-value* by the occurrence of D in C.
- If, however, D occurs with ($\beta_1 \ldots \beta_n$)-extensional supposition in a generic intensional context of C, then although f is used to point to its α-value by the occurrence of D, then what C v-constructs may well depend on values other than the value to which f is used to point to. It is in this way that *an intensional context is dominant over an extensional context.*

It is hopefully clear at this point how to determine whether an occurrence of a constituent is intensional or extensional. Table 2.2 schematizes the intensional/extensional occurrence of constituents as introduced by Table 2.1

Table 2.2 Intensional/extensional occurrence

Constituent C	Intensionally in C	Extensionally in C
$[X_0X]$	X, $[X_0X]$	X_0
$\lambda x_1\ [X_0X]$	$\lambda x_1[X_0X]$, $[X_0X]$, X, X_0	*None*
$[\lambda x_1\ [X_0X]\ X_1]$	$[\lambda x_1[X_0X]\ X_1]$, X_1	$\lambda x_1\ [X_0X]$
$\lambda x_2\ [\lambda x_1\ [X_0X]]$	$\lambda x_2[\lambda x_1[X_0X]]$, $\lambda x_1[X_0X]$, $[X_0X]$, X	*None*
$[\lambda x_2\ [\lambda x_1\ [X_0X]]\ X_2]$	$[\lambda x_2\ [\lambda x_1\ [X_0X]]\ X_2]$, $[\lambda x_1[X_0X]]$, X_2	$\lambda x_2\ [\lambda x_1\ [X_0X]]$
$[\lambda x_2\ [\lambda x_1\ [X_0X]\ X_1]]$	$[\lambda x_2[\lambda x_1[X_0X]X_1]]$, $[\lambda x_1[X_0X]X_1]$, X_1	*None*
$[\lambda x_2\ [\lambda x_1\ [X_0X]\ X_1]\ X_2]$	$[\lambda x_2\ [\lambda x_1\ [X_0X]\ X_1]\ X_2]$, X_1, X_2	$\lambda x_2[\lambda x_1[X_0X]\ X_1]$, $\lambda x_1[X_0X]$

Example 2.4 *Intensional/extensional occurrence.*

(a) Let $C = [\lambda x\ [\lambda y\ [{}^0Div\ y\ x]]\ [{}^0\surd z]]$; $x, y, z \to \tau$; $Div/(\tau\tau\tau)$: the division function;$\surd/(\tau\tau)$: the positive square-root function. Then:

^0Div occurs with extensional supposition in the non-generic extensional context of $[{}^0Div\ y\ x]$. Therefore, ^0Div occurs with extensional supposition in the (τ)-generic intensional context of $[\lambda y\ [{}^0Div\ y\ x]]$ and in the ($\tau\tau$)-generic intensional context of $\lambda x\ [\lambda y\ [{}^0Div\ y\ x]]$. Thus, ^0Div occurs with extensional supposition in the (τ)-generic intensional context of C, while $^0\surd$ occurs with extensional supposition in the *non-generic extensional* context of C, i.e., extensionally. Therefore, for $n < 0$, construction C is $v(n/z)$-improper, because $[{}^0\surd z]$ is $v(n/z)$-improper.

(b) Let $C_\beta = [\lambda y\ [{}^0Div\ y\ [{}^0\surd z]]]$. Then:

^0Div occurs with extensional supposition in the (τ)-generic intensional context of C_β. $^0\surd$ occurs with extensional supposition in the (τ)-*generic intensional* context of C. Therefore, C_β is *v-proper* for all valuations v, so C and C_β are *not equivalent.*

(c) Let $C' = [[{}^0Trans\ \lambda x\ [{}^0Div\ {}^03\ x]]\ {}^00]$, where $Trans(position)/((\tau\tau)(\tau\tau))$ is the function that associates a function $f/(\tau\tau)$ with a function $g/(\tau\tau)$ such that $[f\,x] =_\tau [g\,x]$ for all x at which f is defined; otherwise $[g\,x] = 0$.

Then C' constructs 0, because the Closure $\lambda x\ [{}^0Div\ {}^03\ x]$ occurs intensionally in C′, with $(\tau\tau)$-intensional supposition in the (τ)-generic intensional context of the Composition C'. The function constructed by $\lambda x\ [{}^0Div\ {}^03\ x]$ is merely mentioned in C' and its partiality does not cause the whole Composition C' to be improper. C' is proper, because the transformed function g (the value of *Trans* at the function f constructed by $\lambda x\ [{}^0Div\ {}^03\ x]$) is applied to 0. Since f is not defined at 0, g by definition returns the value 0.

(d) Let $C = [\lambda y\ [\lambda x\ [{}^0{+}\ x\ \mathrm{y}]]\ {}^01]$; $x, y \rightarrow \nu$; $+/(\nu\nu\nu)$.

C constructs the successor function of type $(\nu\nu)$. The constituent ${}^0{+}$ occurs with extensional supposition in the non-generic extensional context of $[{}^0{+}\ x\ y]$, in the (ν)-generic intensional context of $[\lambda x\ [{}^0{+}\ x\ y]]$ and in the $(\nu\nu)$-generic intensional context of the Closure $\lambda y\ [\lambda x\ [{}^0{+}\ x\ y]]$. This Closure $\lambda y\ [\lambda x\ [{}^0{+}\ x\ y]]$ occurs in the (ν)-generic intensional context of C with extensional supposition, and ${}^0{+}$ occurs with extensional supposition in the (ν)-generic intensional context of C. The whole Composition C occurs with $(\nu\nu)$-intensional supposition in the (ν)-generic intensional context of C.

(e) Let $C = [[\lambda y\ [\lambda x\ [{}^0{+}\ x\ y]]\ {}^01]\ {}^03]$.

Now C constructs the number 4. $[\lambda y\ [\lambda x\ [{}^0{+}\ x\ y]]\ {}^01]$ occurs with extensional supposition in the non-generic extensional context of C. ${}^0{+}$ occurs with extensional supposition in the non-generic extensional context of C.

Now we are ready to provide a rigorous definition of the difference between occurrences of constituents with supposition *de dicto* and *de re*. This distinction concerns exclusively constructions of intensions (i.e., functions from possible worlds) and is as such a special case of the general one set out above concerning any construction used* as a constituent.

Definition 2.21 (*supposition de dicto/de re*) Let C be a construction that is not a Closure of the form $\lambda w \lambda t\ C'$ and let $D \rightarrow \alpha_{\tau\omega}$, $D' \rightarrow (\alpha\omega)$, $D'' \rightarrow (\alpha\tau)$ be constituents of C.

I. Occurrence with *de re* supposition

(i) *D' occurs in C with (ω-)de re supposition* if D' occurs with extensional supposition in a non-generic context of C.

(ii) *D'' occurs in C with (τ-)de re supposition* if D'' occurs with extensional supposition in a non-generic context of C.

(iii) *D occurs in C with ($\omega\tau$-)de re supposition* if D occurs in a Composition $[D\ W]$ for some $W \rightarrow \omega$ with (ω-)*de re* supposition in a non-generic context of C, and the Composition $[D\ W]$ itself occurs with (τ-)*de re* supposition in a non-generic context of C.

(iv) If D occurs with (τ-)*de re*, (ω-)*de re* or ($\omega\tau$-)*de re* supposition in C then D *occurs with* (τ-)*de re*, (ω-)*de re* or *de re supposition*, respectively, *in* $\lambda w \lambda t\ C$, too.

(v) Nothing else occurs with *de re supposition* unless it so follows from I. (i) to (iv).

II. Occurrence with *de dicto* supposition

(i) If D occurs in C with ($\alpha\tau$)-intensional supposition (for some type α) or D occurs in a (τ)-generic context of C then D occurs in C *with (τ-)de dicto supposition*.

(ii) If D occurs in C with ($\alpha\omega$)-intensional supposition (for some type α) or D occurs in an ω-generic context of C then D occurs in C *with (ω-)de dicto supposition*.

(iii) If D occurs in C with (($\alpha\tau$)ω)-intensional supposition (for some type α) or D occurs in an ($\omega\tau$)-generic intensional context of C then D occurs in C *with ($\omega\tau$-)de dicto supposition*.

(iv) If D occurs with (τ-)*de dicto*, (ω-)*de dicto* or *($\omega\tau$-)de dicto supposition* in C then D *occurs with* (τ-)*de dicto*, (ω-)*de dicto* or *de dicto supposition*, respectively, *in* $\lambda w \lambda t\ C$, too.

(v) Nothing else occurs with *de dicto supposition* unless it so follows from II. (i) to (iv). □

Remark. Instead of 'D occurs with *de dicto*/*de re* supposition' we shall often say, 'D occurs *de dicto*/*de re*' for short.

We hope it is clear by now that the *de dicto*/*de re* occurrence of a constituent $D \rightarrow \alpha_{\tau\omega}$ in $\lambda w \lambda t\ C$ is induced by an intensional/extensional occurrence of D in C. The above definitions are thus rigorous formulations of the preliminary characterizations introduced in Section 1.5.2. To illustrate, let us examine again the examples adduced in Section 1.5.2. First, we examine the familiar example of the King of France. In Section 1.5.2 we showed that the sentence, 'The King of France is a king' has two readings, one *de dicto* and the other *de re*. On its *de re* reading the sentence comes with the existential presupposition that the King of France exist. On its *de dicto* reading it instead expresses a necessary relation between the property of being a king and the individual office of King of France.

Example 2.5 *Intensional/extensional occurrence of a constituent.*

(a) Consider the constructions expressed by the *de re* reading of the sentence, 'The King of France is a king' ($King/(o\iota)_{\tau\omega}$; $King\_of/(\iota\iota)_{\tau\omega}$; $France/\iota$):

(1) $\lambda w \lambda t\ [{}^0King_{wt}\ \lambda w \lambda t\ [{}^0King\_of_{wt}\ {}^0France]_{wt}]$,

or β-reduced,

(2) $\lambda w \lambda t\ [{}^0King_{wt}\ [{}^0King\_of_{wt}\ {}^0France]]$.

Let C be the Composition $[{}^0King_{wt}\ \lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]_{wt}]$, and C' the reduced Composition $[{}^0King_{wt}\ [{}^0King\_of_{wt}\ {}^0France]]$.

The constructions ${}^0King$, $[{}^0King_w]$ occur with (ω-)*de re*, (τ-)*de re* supposition, respectively, in the non-generic contexts of C and C'. Thus, ${}^0King$ occurs with *de re* supposition in C and C', as well as in the Closures (1) and (2). ${}^0King_{wt}$ occurs with (ι-) extensional supposition in (1) and (2).

The Closure $\lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]$ occurs with ($\iota_{\tau\omega}$)-intensional supposition in the ($\omega\tau$)-generic context of itself, and with (ω-)extensional supposition in the (τ)-generic context of $[\lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]w]$. The latter Composition occurs with ($\iota\tau$)-intensional supposition in the (τ)-generic context of itself. Therefore, $\lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]$ and $[\lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]w]$ occur with (ω-) and (τ-)extensional supposition, respectively, in the non-generic context of C. Thus, $\lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]$ occurs with *de re* supposition in (1).

The constructions ${}^0King\text{-}of$, $[{}^0King\_of\ w]$ occur with (ω-) and (τ-) extensional supposition, respectively, in the non-generic context of $[{}^0King\_of_{wt}\ {}^0France]$. Thus, ${}^0King\_of$ occurs in (2) with *de re* supposition. And since the context of the Composition $[\lambda w[\lambda t\ [{}^0King\_of_{wt}\ {}^0France]\ t]\ w]$ is non-generic, ${}^0King\_of$ occurs also in (1) with *de re* supposition.

If some particular $\langle W, T\rangle$ has no King of France to offer, the constituent ${}^0King_{wt}$, which occurs with extensional supposition in C and in C', does not receive an argument due to the *v(W/w,T/t)*-improperness of ${}^0King\_of$; thus both C and C' come out *v(W/w,T/t)*-improper.

(b) Now consider the *de dicto* reading of 'The King of France is a king' and its analysis:

(3) $[{}^0Req\ {}^0King\ \lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]]$,

where $Req/(o(o\iota)_{\tau\omega}\iota_{\tau\omega})$ is the requisite relation as defined for ordered pairs of individual offices and properties of individuals.[119] The Composition (3) occurs in the non-generic context of itself with (o)-intensional supposition. ${}^0King$ occurs within itself and in (3) with $(o\iota)_{\tau\omega}$-intensional supposition, i.e., *de dicto*. $\lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]$ occurs within itself and in (3) with $\iota_{\tau\omega}$-intensional supposition, i.e., *de dicto*. ${}^0King\_of$ occurs in the ($\omega\tau$)-generic context of (3) and occurs, thus, with supposition *de dicto* in (3).

The only constituent that occurs extensionally in (3) and might, thus, seem to be a potential source of improperness is ${}^0Req$. But Req is a *total* relation; either a property is a requisite of an office or it is not. In this case the property of being a king is indeed a requisite of the office of the King of France. Thus (3) constructs **T**.

[119] See Section 4.1, Definition 4.3.

(c) Let $C = [\neg[^0\exists^\iota \lambda x\, [x = {}^0Pegasus_{wt}]]]$.

C occurs in the non-generic extensional context of itself with (o)-intensional supposition. $\lambda x\, [x = {}^0Pegasus_{wt}]$ occurs with (oι)-intensional supposition in the ι-generic context of itself and also of C. Though the constituent ${}^0Pegasus \rightarrow \iota_{\tau\omega}$ occurs with (ωτ-)extensional supposition in C, it does not occur with supposition *de re* in C, because the context of C in which ${}^0Pegasus$ occurs is (ι)-generic. Therefore, ${}^0Pegasus$ does not occur extensionally in C, and should *Pegasus* go vacant at $\langle W, T\rangle$, C will not be improper.

The constituent ${}^0\exists^\iota$ occurs in itself with (o(oι))-intensional supposition, and with (oι-) extensional supposition in the non-generic context of C. The constituent ${}^0\neg$ occurs in itself with (oo)-intensional supposition and in C with (o-)extensional supposition. Since the existential quantifier $\exists^\iota$ and negation are total functions, no improperness can arise and C *v*-constructs either **T** or **F** (**T** in the actual world, as far as we know).

Let $C' = [\neg[^0Exist^\iota_{wt}\, {}^0Pegasus]]$.

The constituent ${}^0Pegasus$ occurs in C′ with $(\iota_{\tau\omega})$-intensional supposition, hence *de dicto*. The constituent ${}^0Exist^\iota$ occurs in itself with $(o\iota_{\tau\omega})_{\tau\omega}$-intensional supposition, and it occurs in C' extensionally, thus with supposition *de re*. Similarly as above, no improperness can arise and C' *v*-constructs either **T** or **F** (**T** in the actual world, as far as we know).

Thus C and C' are equivalent.

(d) Consider the Closure

(4) $\lambda w \lambda t\, [^0Seek_{wt}\, {}^0Ch\, \lambda w_1 \lambda t_1\, [^0Lok_{w1t1}\, \lambda w_2 \lambda t_2\, [^0King\text{-}of_{w2t2}\, {}^0France]_{wt}]]$,

where $Seek/(o\iota\mu_{\tau\omega})_{\tau\omega}$; Ch/ι; $Lok/(\mu\iota)_{\tau\omega}$); μ is a placeholder for the type of the specification of a particular place on Earth, whatever its exact type may turn out to be.[120]

Gloss of (4): 'Charles is seeking the location of the King of France.'[121]

The question now arises whether $\lambda w_2 \lambda t_2\, [^0King\_of_{w2t2}\, {}^0France]$ occurs in (4) with supposition *de dicto* or *de re*. $\lambda w_2 \lambda t_2\, [^0King\_of_{w2t2}\, {}^0France]$ occurs with (ωτ-)extensional supposition. It is Composed with the variables w, t (the attributer's perspective). But this Closure occurs intensionally in (4), with supposition *de dicto*, because it occurs in the (ωτ)-generic context of the Composition

$[^0Seek_{wt}\, {}^0Ch\, \lambda w_1 \lambda t_1\, [^0Lok_{w1t1}\, \lambda w_2 \lambda t_2\, [^0King\_of_{w2t2}\, {}^0France]_{wt}]]$.[122]

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[120] See also Section 3.5.1 for the type μ.

[121] For the notional attitude of *seeking*, see Section 5.2.2.

[122] At this point we note that Definition 2.21 specifies a slightly different notion of the *de dicto/de re* distinction than the one defined in Duží (2004). The dominance of the *de dicto* supposi-

Thus (4) is not an adequate analysis of the *de re* reading of the sentence 'Charles is looking for the King of France', when understood to be about seeking the location of the King of France.

(e) Recall the example of the second-degree office *HEO* (the highest executive office of the USA) from Section 1.5.2. Let us check the occurrence of its construction in "The highest executive office of USA is the President, not the King". In Section 1.5.2 we claimed that the following argument is valid:

The highest executive office of the USA is the President, not the King
The highest executive office of the USA is
the most respectable office in the USA

The most respectable office of the USA is the President, not the King.

The analysis of the premises and the conclusion is

$$\lambda w\lambda t\ [[{}^0{=}_{\iota\tau\omega}\ {}^0HEO_{wt}\ {}^0Pres\_of\_USA] \wedge [\neg[{}^0{=}_{\iota\tau\omega}\ {}^0HEO_{wt}\ {}^0King\_of\_USA]]]$$
$$\lambda w\lambda t\ [{}^0{=}_{\iota\tau\omega}\ {}^0HEO_{wt}\ {}^0MRO_{wt}]$$

$$\lambda w\lambda t\ [[{}^0{=}_{\iota\tau\omega}\ {}^0MRO_{wt}\ {}^0Pres\_of\_USA] \wedge [\neg[{}^0{=}_{\iota\tau\omega}\ {}^0MRO_{wt}\ {}^0King\_of\_USA]]].$$

Types: $HEO/(\iota_{\tau\omega})_{\tau\omega}$: the highest executive office of the USA; $MRO/(\iota_{\tau\omega})_{\tau\omega}$: the most respectable office of the USA; $Pres\_of\_USA$, $King\_of\_USA/\iota_{\tau\omega}$; $=_{\iota\tau\omega}/(o\iota_{\tau\omega}\iota_{\tau\omega})$.

The constituents ${}^0HEO$, ${}^0MRO$ occur extensionally; to wit, with ($\omega\tau$-)*de re* supposition in the non-generic context of the Compositions

$$[{}^0{=}_{\iota\tau\omega}\ {}^0HEO_{wt}\ {}^0Pres\_of\_USA],$$
$$[\neg[{}^0{=}_{\iota\tau\omega}\ {}^0HEO_{wt}\ {}^0King\_of\_USA]],$$
$$[{}^0{=}_{\iota\tau\omega}\ {}^0HEO_{wt}\ {}^0MRO_{wt}],$$
$$[{}^0{=}_{\iota\tau\omega}\ {}^0MRO_{wt}\ {}^0Pres\_of\_USA],$$
$$[\neg[{}^0{=}_{\iota\tau\omega}\ {}^0MRO_{wt}\ {}^0King\_of\_USA]].$$

Thus these constituents occur with *de re* supposition both in the premises and the conclusion. They are used to point to an individual office. If the second premise constructs **T**, the constituents are *v*-congruent and thus mutually intersubstitutable. Hence the argument is indeed valid.

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tion following from the notion of generic intensional context was not strictly enforced in Duží (2004). A special theorem was then needed to reinstate the *de dicto* dominance.

For completeness, though irrelevant to the validity of the above argument, the constituents ${}^0HEO_{wt}$, ${}^0Pres\_of\_USA$, ${}^0King\_of\_USA$ occur with $(\iota_{\tau\omega})$-intensional supposition in the premises and the conclusion, because they v-construct individual offices, figuring as the arguments of $=_{\iota\tau\omega}$.

(f) For another empirical example, consider the analysis of

'The President of the USA knows that John McCain wanted to become President of the USA.'

Obviously, if the President of the USA is Barack Obama, or the husband of Michelle Obama, we want to be able to validly infer that Barack Obama, or the husband of Michelle Obama, knows that John McCain wanted to become President of the USA, but *not* that John McCain wanted to become Barack Obama, or the husband of Michelle Obama. Hence the first occurrence of 'President of the USA' is extensional (*de re*), whereas the second is not.[123] In Section 5.1 we will discuss in detail the two kinds of *knowing* already broached; namely, as a propositional attitude, of type $K/(o\iota o_{\tau\omega})_{\tau\omega}$, or as a hyperpropositional attitude, of type $K^*/(o\iota *_n)_{\tau\omega}$. Hence, there are two readings of the sentence, and two admissible analyses to match.

Types: *Pres_of* (something)/$(\iota\iota)_{\tau\omega}$; *USA*, *McCain*/$\iota$; *Become*/$(o\iota\iota_{\tau\omega})_{\tau\omega}$: the relation-in-intension of an individual to an individual office whose occupant the individual becomes; *Want*(to)/$(o\iota(o\iota)_{\tau\omega})_{\tau\omega}$: the relation-in-intension of an individual to an individual property that the individual wants to obtain.[124]

(A_1) $$\lambda w\lambda t\, [{}^0K_{wt}\, \lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]_{wt}\, [\lambda w\lambda t\, [{}^0Want_{wt}\, {}^0McCain\ \lambda w\lambda t\, \lambda x\, [{}^0Become_{wt} x\, \lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]]]]].$$

(A_2) $$\lambda w\lambda t\, [{}^0K^*_{wt}\, \lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]_{wt}\, {}^0[\lambda w\lambda t\, [{}^0Want_{wt}\, {}^0McCain\ \lambda w\lambda t\, \lambda x\, [{}^0Become_{wt} x\, \lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]]]]].$$

(i) The *first occurrence* of $\lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]$ in (A_1) and (A_2).

The Closure $\lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]$ is a constituent both of (A_1) and (A_2). It occurs extensionally in the outmost Compositions of (A_1) and (A_2), because it occurs with $(\omega\tau)$-extensional supposition in the non-generic context of these Compositions. Therefore, it occurs with supposition *de re* both in (A_1) and (A_2). Any v-congruent construction v-constructing the same individual as does the Composition $[\lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]_{wt}]$ can be substituted *salva veritate* for the latter. And if the construction $\lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0USA]_{wt}$ is v-improper, both propositions v-constructed by (A_1) and (A_2) lack a truth-

[123] Recall that an *expression* occurs *de dicto*/*de re* in a sentence if its *meaning* occurs with *de dicto*/*de re* supposition in the construction expressed by the sentence.

[124] Sentences reporting wishes will be analyzed in detail in Section 5.2.2.

value, because the respective Compositions of (A_1) and (A_2) come out v-improper as well.

(ii) The *second occurrence* of $\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0USA]$ in (A_1) and (A_2).

The constituent

$[\lambda w \lambda t\ [{}^0Want_{wt}\ {}^0McCain\ \lambda w \lambda t\ \lambda x\ [{}^0Become_{wt}\ x\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0USA]]]]$ occurs in the outmost Composition of (A_1) with ($o_{\tau\omega}$)-intensional supposition; i.e., it occurs in (A_1) with supposition *de dicto*. Thus all its constituents, including the second occurrence of $\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0USA]$, occur in the ($\omega\tau$)-generic context of the outmost Composition of (A_1). Thus they occur intensionally here, and with supposition *de dicto* in (A_1).

The construction

$[\lambda w \lambda t\ [{}^0Want_{wt}\ {}^0McCain\ \lambda w \lambda t\ \lambda x\ [{}^0Become_{wt}\ x\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0USA]]]]$ is not a constituent of (A_2). It is mentioned* in (A_2). Thus all its subconstructions are mentioned* in (A_2), occurring hyperintensionally.

For this reason, no construction v-congruent with $[\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0USA]]_{wt}$ can be validly inserted into the intensional context of (A1) or the hyperintensional context of (A_2).

(g) For a host of mathematical examples, consider the following constructions (a)–(i), where $Div/(\tau\tau\tau)$ is the division function; $Deg/(o(\tau\tau))$ the class of *degenerate* unary *functions* undefined at all their τ-arguments; $\surd/(\tau\tau)$: the positive square-root function; $Square/(\tau\tau)$: the square function; $Inverse/(o(\tau\tau)(\tau\tau))$: the relation of being mutually inverse between unary functions; $x, y, z \to \tau$. Then

(a) $[{}^0Div\ x\ y] \to_v \tau$

${}^0Div$ occurs with extensional supposition in the non-generic context of (a); thus (a) is $v(0/y)$-improper.

(b) $\lambda x\ [{}^0Div\ x\ y] \to_v (\tau\tau)$

${}^0Div$ occurs with extensional supposition in the (τ)-generic context of (b); thus the Closure is not v-improper for any valuation v. Instead (b) $v(0/y)$-constructs a degenerate function.

(c) $\lambda xy\ [{}^0Div\ x\ y] \to (\tau\tau\tau)$

${}^0Div$ occurs with extensional supposition in the (τ,τ)-generic context of (c); the Closure is not v-improper for any valuation v.

(d) $\lambda x\ [{}^0Div\ x\ {}^00] \to (\tau\tau)$

${}^0Div$ occurs as in (b); (d) constructs a degenerate function.

(e) $[{}^0Deg\ \lambda z\ [{}^0Div\ z\ {}^00]] \to o$

${}^0Div$ occurs as in (b); ${}^0Deg$ occurs with extensional supposition in the non-generic context of (e); $\lambda z\ [{}^0Div\ z\ {}^00]$ occurs with ($\tau\tau$)-intensional supposition in the (τ)-generic context of (e); thus (e) is not v-improper for any v.

(f) $[{}^0Deg\ \lambda z\ [{}^0Div\ z\ [{}^0\surd x]]] \to_v o$

${}^0Div$, ${}^0Deg$ occur as in (e); $\lambda z\ [{}^0Div\ z\ [{}^0\surd x]]$ occurs with ($\tau\tau$)-intensional supposition in the (τ)-generic context of (f); ${}^0\surd$ occurs with extensional supposition in the (τ)-generic context of (f); thus (f) is not v-improper for any v.

(g) $[\lambda y\ [{}^0Deg\ \lambda z\ [{}^0Div\ z\ y]]\ [{}^0\surd x]] \to_v o$
${}^0Div$ occurs with extensional supposition in the (τ)-generic intensional context of (g); ${}^0Deg$ occurs with extensional supposition in the non-generic context of (g); $\lambda z\ [{}^0Div\ z\ y]$ occurs with ($\tau\tau$)-intensional supposition in the (τ)-generic context of (g); ${}^0\surd$ occurs with extensional supposition in the non-generic context of (g); thus (g) is $v(n/x)$-improper for all $n < 0$.

(h) $[{}^0Inverse\ {}^0\surd\ {}^0Square] \to o$
${}^0Inverse$ occurs with extensional supposition, ${}^0\surd$ and ${}^0Square$ occur with ($\tau\tau$)-intensional supposition, all three of them in the non-generic context of (h).

This completes our exposition of what it means for a construction to occur either used* (intensionally/extensionally) or mentioned* within a construction. In the next section we utilise these results in order to define valid rules of inference for TIL, in particular the rules of substitution in various kinds of context.

2.7 TIL as a hyperintensional, partial, typed lambda-calculus

From the formal point of view, TIL is a hyperintensional, partial, typed λ-calculus. The qualification ‘hyperintensional’ is a nod to the fact that the elements of TIL’s ‘language of constructions’ are not interpreted as the functions denoted by them, but as *constructions* of the denoted functions.

In Section 2.4.3 we made a comparison between TIL and Montague’s IL and the latter’s two-sorted variant TY_2. We characterised TIL is an explicitly intensional and temporal modification of IL enriched with a hyperintensional, procedural semantics. In virtue of its transparency and anti-contextualism, TIL is a system in which the Axiom of Extensionality, Leibniz’s Law and other classical axioms and rules of extensional logics are valid, as far as the occurrences of constituents constructing total functions in extensional contexts go. However, as we have been arguing in this book, we need to work with properly partial functions and not only intensional but also hyperintensional contexts as well. In particular, although the β-rule is not in general valid in the logic of partial functions, we do need a universal rule regulating β-transformation, since this is a fundamental computational rule of the λ-calculi that cannot remain lawless. So we have to somehow handle the application of a partial function to an argument which may fail to deliver a value. And if an application does not fail, we must be able to deduce that the function has a value at the respective argument.

Partiality, as we know all too well, is a complicating factor. Both ordinary predicate logic and lambda logic can be modified so as to allow ‘undefined terms’. Non-constructive extensions of standard first-order predicate logic are known as *Logic of Partial Terms* (*LPT*) and *Logic of Partial Terms with Definite Descriptions* (*LPD*).[125] Logics of this family belong to so-called *negative free logics*. Free logics were developed in the 1960s to handle anomalies arising from the inclusion of non-denoting singular terms in mathematics and related areas, such as mathematical computer science.[126] There are two categories of free logics, positive and negative. *Positive free logics* allow atomic formulas containing non-denoting terms to be true. On the other hand, in a *negative free logic* any atomic formula containing a non-denoting singular term comes out false. In *LPT* there is no domain of so-called non-existing entities as found in the semantics of many positive free logics. Moreover, *LPT* is a strict variant of free logic, because it endorses only functions that are *strict* at every argument. An n-ary function f is said to be strict at its i-th argument iff $f(t_1,\ldots,t_n)$ is undefined whenever t_i is undefined. An n-ary predicate P is said to be strict at its i-th argument iff $P(t_1,\ldots,t_n)$ is false if t_i is undefined. TIL departures from *LPD* in the latter regard. Since the TIL approach is strictly functional and compositional, the application of a property to an undefined argument yields no value. As for intuitionistic logic, partiality has been studied in particular by D. Scott.[127] So-called partial combinatory algebras are studied as a framework for an abstract specification of the theory of computation or recursion.[128] Whereas total combinatory algebras, in which the operation is total on the carrier set, have been extensively studied and applied within the model theory of the lambda calculi and combinatory logic, partial combinatory algebras are less well-established in this respect.[129] However, the application of a function in the lambda calculi has always been total. E. Moggi (1988) would appear to have been the first to advance a definition of a partial lambda calculus. Moggi (1988) investigates various formal systems for reasoning about partial functions with a particular emphasis on the lambda calculi.

Solomon Feferman (1995) introduces axioms (λ_p) for *Partial Lambda Calculus (LPT)* as follows. (The predicate ‘↓’ in ‘t↓’ means that the term t is ‘defined’; i.e., it denotes an element of the universe; $t \simeq s$ is (weak) congruency of terms; i.e., if t or s is defined then they denote the same value, $t = s$):

> [T]he terms are generated from variables as usual by application and abstraction, i.e. if s, t are terms then st is a term and $\lambda x.t$ is a term. … [w]e introduce the axioms λp for *Partial Lambda Calculus* directly as follows:

[125] See, for instance, Gumb (2000).

[126] See, for instance, Lambert (1991).

[127] See, for instance, Scott (1979).

[128] See, for instance, Bethke and Klop (1996).

[129] By ‘carrier set’ we mean what it means in algebra theory. An algebra is a tuple like $\langle S, \#\rangle$, where S is the carrier set and # an *n-ary* operation defined over S.

(λp)

(i) $\lambda x.t\downarrow$

(ii) $(\lambda x.t(x))y \simeq t(y)$.

The axiom (ii) corresponds to -reduction in the ordinary ('total') lambda calculus, but the limitation on instantiation in LPT restricts its application to:

$$s\downarrow \rightarrow (\lambda x.t(x))s \simeq t(s) \qquad \text{(ibid., p. 11).}$$

M.J. Beeson (2004), introduces a partial lambda calculus in which *Ap* (application) is not necessarily total, but he does not have strictness for *Ap*:
$(\lambda y.a)b \simeq a$ whether or not $b\downarrow$. On this point he says:

> [W]e could have formulated a rule "strict(β)" that would require deducing $r\downarrow$ before concluding $Ap(\lambda x.t, r) \simeq t[x{:=}r]$, but not requiring strictness corresponds better to the way functional programming languages evaluate conditional statements. Note that $\lambda x.t$ is defined, whether or not t is defined (ibid., §8).

Since β-reduction is *the* fundamental computational rule of the λ-calculi, it is desirable to give reasons for the above restriction to 'denoting terms' or non-strictness. There is a satisfactory reason for the restriction to denoting terms: such a 'syntactic' β-reduction 'by name' is generally not an equivalent transformation in the logic of partial functions. As shown above in Example 2.4 (a), (b), it may happen that the non-reduced construction is *v*-improper whereas the reduced one is *v*-proper, which may lead to inconsistencies with respect to deducing existence.

However, there is no satisfactory reason for non-strictness. The instruction to apply a function to an argument, i.e., the Composition $[X\ Y_1 \ldots Y_n]$, is always strict: if one of X, Y_i is *v*-improper, the Composition must be *v*-improper.

A standard example of non-strictness is the case of the 'if-then-else' connective. For instance, when executing the instruction

$$\text{if } (0{=}0) \text{ then } (x{=}0), \text{ else } (x{=}1/0)$$

we end up with the result $x{=}0$, because the instruction to divide 1 by 0 is not executed.[130] Thus we are told that the 'if-then-else' connective is not a strict function. Sure, if we carelessly translate the instruction into propositional logic as

$$[(0{=}0) \supset (x{=}0)] \wedge [\neg(0{=}0) \supset (x{=}1/0)]$$

we end up with an improper formula, because the term '1/0' fails to denote, unless we specify some non-strict truth-conditions for 'if-then-else'.

Yet in TIL we have the means to specify a 'strict' instruction that complies with compositionality while at the same time taking into account that the instruction

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[130] We avoid, of course, the (inconvenient but occasionally accepted) fiction that 1/0 be some real number, though it does not matter which, because nothing can be proved of it, anyway. We say instead that the division function is *undefined* at the pair ⟨1, 0⟩, thereby inducing a value gap.

to divide 1 by 0 is not executed if the 'else-condition' is not satisfied. Here we show how.

The instruction encoded by 'If P ($\rightarrow$ o) then C_1 ($\rightarrow \alpha$), else C_2 ($\rightarrow \alpha$)' behaves as follows:

(a) If P v-constructs **T** then execute C_1 (and return the result of type α, provided C_1 is not v-improper).
(b) If P v-constructs **F** then execute C_2 (and return the result of type α, provided C_2 is not v-improper).
(c) If P is v-improper then it fails to produce the result.

Hence, *if-then-else* is seen to be a function of type ($\alpha o *_n *_m$), and its definition is an instruction that decomposes into two phases. First, select a construction to be executed on the basis of a specific condition. Second, execute the selected construction; hence Double Execution. Let $p \rightarrow o$, c, c_1, $c_2 \rightarrow *_n$, ${}^2c \rightarrow \alpha$. The function is then defined as follows ($=_*/(o *_n *_n)$):

$$^0\textit{if-then-else} =_{df} \lambda p\, c_1\, c_2\ {}^2[^0\textit{Sing}\ \lambda c\ [[p \supset [c =_* c_1]] \wedge [\neg p \supset [c =_* c_2]]]].$$

Thus the Composition $[^0\textit{if-then-else}\ P\ C\ D]$ comes down to

$${}^2[^0\textit{Sing}\ \lambda c\ [[P \supset [c =_* C]] \wedge [\neg P \supset [c =_* D]]]].$$

Types: $P \rightarrow o$ constructs the condition of the choice between the execution of C or D, $C/*_n$, $D/*_n$; $c \rightarrow *_n$; $\iota/(*_n(o*_n))$: the singularizer function that associates a singleton set of constructions with the only construction that is an element of this singleton, and is otherwise (i.e., if the set is empty or many-valued) undefined.

First, the Composition $[^0\textit{Sing}\ \lambda c\ [[P \supset [c =_* C]] \wedge [\neg p \supset [c =_* D]]]$ is the procedure that realises the choice between C and D. If P v-constructs **T** then the variable c v-constructs the *construction* C, and if P v-constructs **F** then the variable c v-constructs the *construction* D. In either case, the set constructed by

$$\lambda c\ [[P \supset [c={}^0C]] \wedge [\neg P \supset [c={}^0D]]]$$

is a singleton and the singularizer ι returns as its value either the construction C or the construction D. Second, the selected construction is executed.

When rejecting 'non-strictness', one might accept the restriction to 'denoting terms', whose counterpart in TIL would be v-proper constituents. But this is too restrictive in the logic of partial functions, because we cannot *analytically* specify *which* valuations a given constituent is v-proper for. For instance, if *Mayor_of* is of type $(\iota\iota)_{\tau\omega}$ and *Dunedin* of type ι, then the evaluation of the constituent $[^0\textit{Mayor\_of}_{wt}\ {}^0\textit{Dunedin}]$ amounts to an *empirical* investigation into who, if any, is the Mayor of Dunedin.

Hence, since neither non-strictness nor the restriction to denoting terms is plausible, we are going to define a 'strict β-rule' as outlined above by Beeson. So, in what follows we are going to define a generally valid objectual version of 'β-reduction by value'. First, we define *collisionless substitution*. It is the substitution of a construction D for a variable x free in another construction C. If a conflict of variables might arise by λ-binding, it is prevented by α-renaming the λ-bound variables:

Definition 2.22 (*collisionless substitution*) Let x be a variable and C, D any kinds of construction. If x is not free in C then *the result of substituting D for x in C* is C. Assume now that x is free in C. Then:

(a) If C is x then *the result of substituting D for x in C* is D. If C is ${}^1X$ or ${}^2X$ then *the result of substituting D for x in C* is ${}^1Y$, ${}^2Y$, where Y is the result of substituting D for x in X.
(b) If C is $[X\, X_1 ... X_m]$ then *the result of substituting D for x in C* is $[Y\, Y_1 ... Y_m]$, where $Y, Y_1,...,Y_m$ are the results of substituting D for x in $X, X_1,...,X_m$, respectively.
(c) Let C be of the form $[\lambda x_1 ... x_m\, Y]$; for $1 \le i \le m$, let $y_i = x_i$ if x_i is not free in D, and otherwise the first variable v-constructing entities of the same type as x_i, not occurring in C, not free in D, and distinct from $y_1,...,y_{i\text{-}1}$. Then *the result of substituting D for x in C* is $[\lambda y_1 ... y_m\, Z]$, where Z is the result of substituting D for x in the result of substituting y_i for x_i $(1 \le i \le m)$ in Y.

To simplify the formulations and proofs of the following claims, let us make the following notational agreement. Let $C, D_1,\ldots,D_n$ be arbitrary constructions, $x_1,\ldots,x_n$ variables. Then, for $1 \le i \le n$, '$C(D_i/x_i)$' will stand for the result of substituting D_i for x_i in C.

Example 2.6 *Collisionless substitution.*

Ad (b) Let $C = [{}^0Div\ x\ y]$ and let $D = {}^05$; $Div/(\tau\tau\tau)$; $x, y \to \tau$; $5/\tau$.
Then $C(D/x) = [{}^0Div\ {}^05\ y]$.

Ad (c) Let $C = \lambda y\ [{}^0Div\ x\ y]$ and let $D = [{}^0{+}\ y\ {}^01]$.
Then $C(D/x) = \lambda y\ [{}^0Div\ [{}^0{+}\ y_1\ {}^01]\ y]$.
Let $C = [{}^0B_{wt}\ a\ [\lambda w \lambda t\ {}^0P_{wt}\ x]]$ and let $D = {}^0Q_{wt}$.
Then $C(D/x) = [{}^0B_{wt}\ a\ [\lambda w \lambda t\ {}^0P_{wt}\ {}^0Q_{w*t*}]]$.
Types: $x \to \iota$; $B/(o\iota o_{\tau\omega})_{\tau\omega}$; $a \to \iota$; $P/(o\iota)_{\tau\omega}$; $Q/\iota_{\tau\omega}$.

Entry (c) of Definition 2.22 specifies the technique known as *renaming variables*, so that no variable free in D becomes bound in C.

Lemma *(Replacement of free variables)* Let C be a construction, $x_1,\ldots,x_m$ distinct variables, $y_1,\ldots,y_m$ distinct variables v-constructing entities of the same respective types as $x_1,\ldots,x_m$, such that, for $1 \le i \le m$, either $y_i = x_i$ or y_i does not occur in C. Then C $v(d_i/x_i)$-constructs c iff $C(y_i/x_i)$ $v(d_i/y_i)$-constructs c.

Proof is obvious.[131]

[131] See also Tichý (1988, p. 288).

Claim 2.4 (*Compensation Principle*) Let C be a construction. Then for any valuation v and a construction D, if D v-constructs an entity d then $C(D/x)$ v-constructs an entity c iff C $v(d/x)$-constructs c. □

Remark. Tichý put forward and proved a similar claim for constructions of order 1.[132] Here we generalized his claim to hold for constructions of order $n \geq 1$.

In order to simplify the proof of Claim 2.4, which is a proof by induction over the complexity of the construction C, we now introduce the notion of *rank of complexity* of a construction.

Let C be any construction. Then:

(i) If C is an atomic construction, then the *rank* of C is 1.
(j) If C is of the form $[X\,X_1 \ldots X_m]$ then the *rank* of C is $r+1$, where r is the greatest among the *ranks* of $X, X_1, \ldots, X_m$.
(k) If C is $[\lambda x_1 \ldots x_m Y]$ then the *rank* of C is $r+1$, where r is the *rank* of Y.
(l) If C is of the form ${}^1X$ and X is of *rank* r, then C is of *rank* $r+1$.
(m) If C is of the form ${}^2X$ and X is of *rank* r, then C is of *rank* $r+2$.

Proof of Claim 2.4. If x is *not* free in C then $C(D/x) = C$ and the Claim is valid. Assume, therefore, that x is free in C. Now we prove the Claim by induction on the rank of C. First, assume that $C(D/x)$ v-constructs c. Since x is neither λ- nor ^{0}bound, x is used* as a constituent of C, and according to Definitions 2.15 and 2.17 one of the following options obtains.

- If C is atomic, then x is free in C only if $x = C$. Then it is x that $v(d/x)$-constructs d; moreover, $C(D/x) = D$ and D v-constructs d *ex hypothesi*.
- Assume now as an induction hypothesis that any C of rank less than or equal to r satisfies the Claim, and consider a C of rank $r+1$. Then:
  (a) If C is of the form $[X_0\, X_1 \ldots X_m]$ then by Definition 2.22 $C(D/x)$ is of the form $[X_0(D/x)\, X_1(D/x) \ldots X_m(D/x)]$. Then by Definition 1.2 (iii), there are f and $d_1,\ldots,d_m$ such that $X_0(D/x)$ v-constructs f and, for $1 \leq i \leq m$, $X_i(D/x)$ v-construct d_i and f takes value c at $\langle d_1,\ldots,d_m\rangle$. By the induction hypothesis, X_0 $v(d/x)$-constructs f and X_i $v(d/x)$-construct d_i. Thus by Definition 1.2 (iii), C $v(d/x)$-constructs c.
  (b) If C is of the form $[\lambda x_1 \ldots x_m Y]$ then by Definition 2.22 $C(D/x)$ is of the form $[\lambda y \ldots y_m\; Y(y_1/x_1,\ldots,y_m/x_m)(D/x)]$, where, for $1 \leq i \leq m$, either $y_i = x_i$ or y_i does not occur in C and is not free in D. Let C $v(d/x)$-construct a function c' and $C(D/x)$ v-construct a function c.
  We will show that $c = c'$. Let c' take entities $d_1,\ldots, d_m$ to e. Then the construction Y $v(d/x)(d_1/x_1,\ldots,d_m/x_m)$-constructs e. Now by the Lemma above, $Y(y_1/x_1,\ldots,y_m/x_m)$ $v(d/x)(d_1/y_1,\ldots,d_m/y_m)$-constructs, and therefore also $v(d_1/y_1,\ldots,d_m/y_m)(d/x)$-constructs, e. Since y_i are not free in D, D

[132] See Tichý (1988, p. 75).

$v(d_1/y_1,\ldots, d_m/y_m)$-constructs e. Hence, by the induction hypothesis, $Y(y_1/x_1,\ldots,y_m/x_m)(D/x)$ $v(d_1/y_1,\ldots, d_m/y_m)$-constructs e, which goes to show that c, c' are one and the same function.

(c) If C is of the form ${}^1X$ then by Definition 2.22 $C(D/x)$ is of the form ${}^1X(D/x)$. By Definition 1.2 (v), ${}^1X(D/x)$ v-constructs what $X(D/x)$ v-constructs. Since by the induction hypothesis, if $X(D/x)$ v-constructs c then X $v(d/x)$-constructs c.

(d) If C is of the form ${}^2X$ then by Definition 2.22 $C(D/x)$ is of the form ${}^2X(D/x)$. By Definition 1.4, x is free in ${}^2X(D/x)$ if either x is free in $X(D/x)$, or x is ^{0}bound in a constituent ${}^0Y$ of ${}^2X(D/x)$ and x is free in Y and Y is a constituent of what is v-constructed by $X(D/x)$. If x is free in $X(D/x)$, then by the induction hypothesis, if $X(D/x)$ v-constructs Z then X $v(d/x)$-constructs Z, and again by the induction hypothesis, if $Z(D/x)$ v-constructs c then Z $v(d/x)$-constructs c. If x is free in a constituent of Z v-constructed by $X(D/x)$, then by the induction hypothesis, if $Z(D/x)$ v-constructs c then Z $v(d/x)$-constructs c.

The converse argument—i.e., if C $v(d/x)$-constructs c then $C(D/x)$ v-constructs c—is proven along similar lines.

In its syntactic formal version, β-reduction is not generally valid. A valid rule leads from a construction C to a construction D, such that C and D are equivalent.[133] Our objectual counterpart of a *defined term* would be a *v-proper construction*, so the objectual version of the formal 'syntactic β-rule by name' can be formulated as follows ($x_i \rightarrow \beta_i$; $D_i \rightarrow \beta_i$; for $1 \leq i \leq m$; $Y \rightarrow \alpha$):

$$[[\lambda x_1 \ldots x_m\ Y]\ D_1 \ldots D_m] \vdash Y(D_i/x_i)$$

where $Y(D_i/x_i)$ is the result of collisionless substitution of D_i for x_i ($1 \leq i \leq m$) in Y.

In order for the rule to be valid, D_i must be v-proper for $1 \leq i \leq m$, or Y must occur in a non-generic extensional context. The constructions $[[\lambda x_1 \ldots x_m\ Y]\ D_1 \ldots D_m]$, $Y(D_i/x_i)$ are v-congruent for those valuations v for which all D_i are v-proper. If one or more D_i are v-improper for some v, then it cannot be ruled out that the result of substituting D_i for x_i in Y will v-construct an entity c, whereas $[[\lambda x_1 \ldots x_m\ Y]\ D_1 \ldots D_m]$ is v-improper. To demonstrate the non-equivalence of the β-reduced construction with the original one, this example will suffice:

C: $[\lambda x\ [{}^0Deg\ \lambda z\ [{}^0Div\ z\ x]]\ [{}^0\surd y]]$

Y: $[{}^0Deg\ \lambda z\ [{}^0Div\ z\ x]]$

D: $[{}^0\surd y]$.

[133] For the definition of *equivalent construction*, see Definition 1.5.

The Composition $[{}^0\surd y]$ v-constructs a number k for a valuation v assigning a positive number or zero to y, and is $v(n/y)$-improper for $n < 0$. The result of substituting $[{}^0\surd y]$ for x in Y is

$$Y({}^0\surd y/x): \qquad [{}^0Deg\ \lambda z\ [{}^0Div\ z\ [{}^0\surd y]]].$$

The v-constructed truth-values differ for valuations that assign a negative number to y, as illustrated in the following example:

	$Y({}^0\surd y/x)$	C
$v(y) > 0$	**F**	**F**
$v(y) = 0$	**T**	**T**
$v(y) < 0$	**T**	$\perp$

As is seen, ${}^0[\lambda x\ [{}^0Deg\ \lambda z\ [{}^0Div\ z\ x]]\ [{}^0\surd y]] \approx_v {}^0[{}^0Deg\ \lambda z\ [{}^0Div\ z\ [{}^0\surd y]]]$ for some but not all v; hence, the constructions are not equivalent. By substituting $[{}^0\surd y]$ for x in Y, we draw a construction occurring in the non-generic extensional context of C into the (τ)-generic intensional context of Y. Since a higher intensional context is dominant over a lower extensional context, the result is not equivalent.

Yet why would it matter? After all, $Y({}^0\surd y/x)$ truly reports whether the resulting function is degenerate. If no conclusion might validly be inferred from C, unlike from $Y({}^0\surd y/x)$, it would indeed not matter. However, this is not so. From C we can validly infer that ${}^0\surd y$ does not exist for $y < 0$, whereas from $Y({}^0\surd y/x)$ we cannot—which is a logical defect. If $Y({}^0\surd y/x)$ v-constructs **T** then we can infer that the resulting function is degenerate, but we cannot infer *which* construction is the culprit. We cannot validly infer whether it is degenerate due to the division function or due to the positive square-root function. To put it metaphorically, the reduced construction $Y({}^0\surd y/x)$ is a procedure 'biased' by the reduction, because if ${}^0\surd y$ is v-improper it is futile to call the ${}^0Div$ procedure. The latter does not obtain an argument to operate on.

The above considerations motivate the following claim about β-reduction 'by name':

Claim 2.5 (*β-reduction 'by name'*) Let x_i $(1 \le i \le m)$ occur in an α-generic context of Y, and let D_i occur in a non-generic extensional context of the Composition $[[\lambda x_1 \ldots x_m\ Y]\ D_1 \ldots D_m]$. Then if, for some i and v, D_i is v-improper then the β-reduction 'by name'

$$[[\lambda x_1 \ldots x_m\ Y]\ D_1 \ldots D_m] \vdash Y(D_i/x_i)$$

is *not a valid rule.* Otherwise, β-reduction 'by name' is a valid rule.

Proof. If D_i is v-proper for all i and v, the proof of the 'otherwise' clause follows from the Compensation Principle, Claim 2.4. So let D_i be v-improper instead. Then the Composition $[[\lambda x_1 \ldots x_m\ Y]\ D_1 \ldots D_m]$ is v-improper according to Definition 1.2

(iii) If x_i occur in an α-generic context of Y then there is a constituent of Y of the form of a Closure; let [λy Y'] be such a constituent. Further, let [λz Z] be the result of substituting D_i for x_i in [λy Y']. Then [λz Z] is not v-improper for any v according to Definition 1.2 (iv).

For an application, one place where the phenomenon of invalid β-reduction crops up is *de re* attitudes. Consider the sentence

(1) 'The President of the Czech Republic is believed by Charles to be an economist.'

The logical form of the sentence consists in attributing the property *BCh* (i.e., being believed by Charles to be an economist) to the individual (if any) who occupies the office of President of the Czech Republic. Hence, 'the President of the Czech Republic' occurs *de re* here. The sentence is ambiguous, as it may concern either an implicit belief, $B/(o\iota o_{\tau\omega})_{\tau\omega}$, or an explicit belief, $B^*/(o\iota *_1)_{\tau\omega}$. For the sake of simplicity, let us consider the implicit belief B.[134]

Types: *Pres(ident_of_something)*/$(\iota\iota)_{\tau\omega}$; *CR*, *Ch(arles)*/ι; *Econom(ist)*/$(o\iota)_{\tau\omega}$; $x \rightarrow \iota$.

The property *BCh* is constructed by

$$\lambda w\lambda t\ [\lambda x\ [{}^0B_{wt}\ {}^0Ch\ \lambda w\lambda t\ [{}^0Econom_{wt}\, x]]].$$

The whole sentence expresses the Closure

(1′) $\lambda w\lambda t\ [\lambda w\lambda t\ [\lambda x\ [{}^0B_{wt}\ {}^0Ch\ \lambda w\lambda t\ [{}^0Econom_{wt}\, x]]]_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$.

$\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$ occurs with extensional supposition in the non-generic context of the Composition

$$[\lambda w\lambda t\ [\lambda x\ [{}^0B_{wt}\ {}^0Ch\ \lambda w\lambda t\ [{}^0Econom_{wt}\, x]]]_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}].$$

Hence, it occurs *de re* in (1′). But the question arises whether (1′) could be equivalently β-reduced to the construction expressed by the *de dicto* attitude, which is

(2) $\lambda w\lambda t\ [{}^0B_{wt}\ {}^0Ch\ \lambda w\lambda t\ [{}^0Econom_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]]$.

The answer is No; (1′) is *not* equivalent to (2). We can only β-reduce (1′) to (1″):

(1″) $\lambda w\lambda t\ [\lambda x\ [{}^0B_{wt}\ {}^0Ch\ \lambda w\lambda t\ [{}^0Econom_{wt}\, x]]\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$.

[134] Explicit belief, explicit knowledge, etc., are dealt with in detail in Chapter 5.

Further β-reducing would result in

(3) $\lambda w \lambda t\ [{}^0B_{wt}\ {}^0Ch\ \lambda w_1 \lambda t_1\ [{}^0Econom_{w1t1}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]]$.

The construction (3) is not equivalent to any of the above constructions (1′) and (2). Again, whereas the argument

$$\frac{\lambda w \lambda t\ [\lambda w \lambda t\ [\lambda x\ [{}^0B_{wt}\ {}^0Ch\ \lambda w \lambda t\ [{}^0Econom_{wt}\ x]]]_{wt}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]}{\lambda w \lambda t\ [{}^0Exist^t{}_{wt}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]]}$$

is *valid*, the argument

$$\frac{\lambda w \lambda t\ [{}^0B_{wt}\ {}^0Ch\ \lambda w_1 \lambda t_1\ [{}^0Econom_{w1t1}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]]}{\lambda w \lambda t\ [{}^0Exist^t{}_{wt}\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]]}$$

is *invalid.*

As we said above, β-reduction is the fundamental rule of the λ-calculi, and its restriction to non-recursively defined cases of v-properness would be a serious defect of the calculus. Fortunately, it turns out to be feasible to formulate a generally valid computational rule. In TIL, β-reduction is the rule for computing the value of a (perhaps properly) partial function v-constructed by $[\lambda x_i\ Y]$ at an argument v-constructed by D_i. The invalid rule above is reminiscent of the programming technique of calling a subprocedure D_i 'by name': the subprocedure itself is substituted for the 'local variable' x_i in the 'procedure body' Y. It is well-known among programmers that this technique can have undesirable side-effects, unlike the technique of calling a subprocedure 'by value'. The idea is simple: execute the subprocedure D_i first, and then, if the first step does not fail, substitute the construction of the result ('the value') for x_i.

To specify the rule rigorously, we invoke the functions $Tr_{\beta i}$ and Sub_n introduced in Section 1.4.3. $Tr_{\beta i}/(*_n\beta_i)$ is the mapping which takes a β_i-entity and returns its Trivialization, while $Sub_n/(*_n*_n*_n*_n)$ is the mapping which takes a construction C_1, a variable x, and a construction C_2 to the construction C_3, where C_3 is the result of substituting C_1 for x in C_2.

Claim 2.6 (*valid β-reduction* '*by value*') The Composition

(Ap) $[[\lambda x_1 ... x_m\ Y]\ D_1 ... D_m]$

that realizes the *application* of the function v-constructed by $[\lambda x_1 ... x_m\ Y]$ at an argument v-constructed by $D_1,...,D_m$ is equivalent to the computationally reduced Double Execution

($Ap\beta$)

$$^{2}\Big[{}^{0}Sub_n\ [{}^{0}Tr_{\beta 1}\ D_1]\ {}^{0}x_1\ {}^{0}\Big[{}^{0}Sub_n\ [{}^{0}Tr_{\beta 2}\ D_2]\ {}^{0}x_2 \ldots\ [{}^{0}Sub_n\ [{}^{0}Tr_{\beta m}\ D_m]\ {}^{0}x_m\ {}^{0}Y]]\ \ldots\ \Big]\Big]$$

(for $1 \le i \le m$, $x_i \rightarrow_v \beta_i$, $D_i \rightarrow_v \beta_i$, $Y \rightarrow_v \alpha$).

Proof. We are to prove that (Ap), ($Ap\beta$) are v-congruent for all valuations v.

(a) According to the definitions of Closure, Composition and Double Execution, (Ap) is v-improper iff for some i ($1 \le i \le m$) D_i is v-improper, and the same holds for ($Ap\beta$).

(b) Let $D_1,\ldots, D_m$ v-construct the entities $d_1/\beta_1,\ldots,d_m/\beta_m$, respectively. Suppose that $[\lambda x_1\ldots x_m\ Y]$ v-constructs a function f that takes $d_1,\ldots,d_m$ to an entity d/α. Then by Definition 1.2 (iv), Y $v(d_1/x_1,\ldots,d_m/x_m)$-constructs the entity d, and the Composition (Ap) v-constructs d. Now we prove that ($Ap\beta$) v-constructs d as well. By the definitions of Sub_n and Tr, the Composition

$$\Big[{}^{0}Sub_n\ [{}^{0}Tr_{\beta 1}\ D_1]\ {}^{0}x_1\ {}^{0}\Big[{}^{0}Sub_n\ [{}^{0}Tr_{\beta 2}\ D_2]\ {}^{0}x_2 \ldots\ [{}^{0}Sub_n\ [{}^{0}Tr_{\beta m}\ D_m]\ {}^{0}x_m\ {}^{0}Y]]\ \ldots\ \Big]\Big]$$

v-constructs the construction $Y({}^{0}d_1/x_1,\ldots,{}^{0}d_m/x_m)$. By Claim 2.4, $Y({}^{0}d_1/x_1,\ldots,{}^{0}d_m/x_m)$ v-constructs d iff Y $v(d_1/x_1,\ldots,d_m/x_m)$-constructs d. Hence, by the definition of Double Execution, ($Ap\beta$) v-constructs d.

So much for β-conversion. Next, we illustrate another important fact pertaining to properly partial functions; namely, that the *De Morgan laws are not valid* in their *simple form.* Consider the claims (E) and (G):

(E) There exists a pair of natural numbers such that their ratio is not a rational number.

(G) It is not true that for all the pairs of natural numbers it holds that their ratio is a rational number.

Though (E) and (G) are a verbal application of the De Morgan laws, they are not equivalent, (E) being false (there is no such pair of natural numbers whose ratio would be irrational), and (G) being true (it is not true that all pairs of natural numbers belong to the set of pairs of numbers whose ratio is a rational number; the pairs $\langle k, 0\rangle$ do not).

In constructional cloaking:

(E′) $[{}^{0}\exists\ \lambda m\ n\ [[{}^{0}Nat\ m] \wedge [{}^{0}Nat\ n] \wedge \neg[{}^{0}Rat\ [{}^{0}\mathbf{:}\ m\ n]]]]$

(G′) $[\neg[{}^{0}\forall\ \lambda m\ n\ [[[{}^{0}Nat\ m] \wedge [{}^{0}Nat\ n]] \supset [{}^{0}Rat\ [{}^{0}\mathbf{:}\ m\ n]]]]]$.

Types: $Rat/(o\tau)$: the class of rational numbers; $Nat/(o\tau)$: the class of natural numbers; $m, n \rightarrow \tau$; $\mathbf{:}/(\tau\tau\tau)$.

Now obviously (E′) constructs **F**, whereas (G′) constructs **T**.

The Composition

$$[[{}^0Nat\ m] \wedge [{}^0Nat\ n] \wedge \neg[{}^0Rat\ [{}^0\mathbf{:}\ m\ n]]]$$

is $v(0/n)$-improper and v-constructs **F** for all other valuations. Hence the class constructed by $\lambda m\ n\ [[{}^0Nat\ m] \wedge [{}^0Nat\ n] \wedge \neg[{}^0Rat\ [{}^0\mathbf{:}\ m\ n]]]$ is not non-empty, and so ${}^0\exists$ returns **F**.

On the other hand, the Composition

$$[[[{}^0Nat\ m] \wedge [{}^0Nat\ n]] \supset [{}^0Rat\ [{}^0\mathbf{:}\ m\ n]]]$$

is $v(0/n)$-improper and v-constructs **T** for all other valuations. Hence the class constructed by $\lambda m\ n\ [[[{}^0Nat\ m] \wedge [{}^0Nat\ n]] \supset [{}^0Rat\ [{}^0\mathbf{:}\ m\ n]]]$ is not the whole type τ; therefore, ${}^0\forall$ returns **F** and the entire (G′) constructs **T**.

However, the *De Morgan laws are generally valid*. They simply lay down the classical rules of transformation controlling the behaviour of classical negation of classical conjunction, disjunction and quantification.

So our logic should specify valid De Morgan laws, even when properly partial functions are involved. To this end we make use of *True**, *False**, *Undef**/$(o*_1)$, defined as follows $(C/*_1)$, to obtain totality:

- $[{}^0True^*\ {}^0C]$ v-constructs **T** iff C v-constructs **T**, otherwise $[{}^0True^*\ {}^0C]$ v-constructs **F**;
- $[{}^0False^*\ {}^0C]$ v-constructs **T** iff C v-constructs **F**, otherwise $[{}^0False^*\ {}^0C]$ v-constructs **F**;
- $[{}^0Undef^*\ {}^0C]$ v-constructs **T** iff C is v-improper, otherwise $[{}^0Undef^*\ {}^0C]$ v-constructs **F**.

For all valuations and all first-order constructions the following holds:

$$\neg[{}^0True^*\ {}^0C] = [{}^0False^*\ {}^0C] \vee [{}^0Undef^*\ {}^0C];$$

$$\neg[{}^0False^*\ {}^0C] = [{}^0True^*\ {}^0C] \vee [{}^0Undef^*\ {}^0C];$$

$$[{}^0Undef^*\ {}^0C] = \neg[{}^0True^*\ {}^0C] \wedge \neg[{}^0False^*\ {}^0C].$$

Now a valid application of the De Morgan laws to (E) and (G) would be, for instance:

(E″) $\neg[{}^0\exists\lambda mn[[{}^0Nat\ m] \wedge [{}^0Nat\ n] \wedge$
$[{}^0False^*\ [{}^0Sub_1\ [{}^0Tr_\tau\ m]\ {}^0m\ [{}^0Sub_1\ [{}^0Tr_\tau\ n]\ {}^0n\ {}^0[{}^0Rat\ [{}^0\!:\ m\ n]]]]]]].$

(G″) $[\forall\lambda mn\ [[[{}^0Nat\ m] \wedge [{}^0Nat\ n]] \supset$
$\neg[{}^0False^*\ [{}^0Sub_1\ [{}^0Tr_\tau\ m]\ {}^0m\ [{}^0Sub_1\ [{}^0Tr_\tau\ n]\ {}^0n\ {}^0[{}^0Rat\ [{}^0\!:\ m\ n]]]]]]].$

2.7.1 Substitution and Leibniz's Law

As mentioned in Section 2.6, the definitions of extensional, intensional and hyperintensional occurrence were introduced in order to define valid inference rules for TIL in its capacity as a hyperintensional logic of partial functions. Now we have all the ingredients required to present these rules, and this section sets out all the details.

Once the difference between mentioning* and using* a construction, and the difference between using* a construction either intensionally or extensionally, have both been defined, the specification of the rules is smooth sailing. Verbally, they can be formulated as follows.

- ***Improperness***. A construction C v-constructing an entity of a type α can be v-improper only due to a constituent D occurring extensionally in C; i.e., with an extensional supposition in a non-generic context of C (see Definitions 2.18 and 2.19).

  Proof. If D is mentioned* in C then D is not executed when executing C, and D itself is an object that C operates on. If D occurs in an α-generic context of C then D is a constituent of a Closure, which is not v-improper for any v. If D occurs in a non-generic context of C with a β-intensional supposition for some β, then D is not Composed with any other subconstituents of C, thus D is not v-improper for any v.

  Remark. Improperness stems from using Composition, which is the procedure of applying a function f to an argument; either f has a value gap, or Composition C does not obtain an argument to operate on because some of the constituents of C are v-improper. In this way partiality is strictly propagated upwards.

- ***Existence***. If a construction C is v-proper then all its constituents D_i occurring with (γ_i-)extensional supposition in a non-generic context of C are v-proper. In other words, the respective γ_i-values *exist.*

- ***Leibniz's law of substitution***.
  A collisionless replacement of the *v-congruent* constructions D, D' in C is valid for extensionally occurring constituents; i.e., for D occurring with extensional supposition in a non-generic context of C.

  A collisionless replacement of the *equivalent* constructions D, D' in C is valid for all constituents of C.

  A collisionless replacement of the *procedurally isomorphic* constructions D, D' in C is valid for all subconstructions of C.

Moreover, for v-proper constituents occurring extensionally, the classical extensional rules of inference (as in the sequent calculus) are valid.

To formulate the rules more rigorously, we now use the following *notational conventions*:

'$C(y)$' stands for a construction with a free variable y;

'$C(D/y)$' stands for the result of a collisionless substitution of D for y in C;

'$C(D'/D)$' stands for the result of the collisionless replacement of D by D' in C: construction C differs from $C(D'/D)$ only in replacing an occurrence of the subconstruction D of C by the construction D', and no occurrence of a variable free in D' is bound in C;

'*v-Improper*(A)/*v-Proper*(A)' stands for the construction A being v-improper/v-proper for a valuation v;

'*Improper*(A)/*Proper*(A)' stands for the construction A being v-improper/v-proper for all valuations v;

'$^0C \approx_v {}^0C'$ stands for constructions C, C' being *v-congruent*;

'$^0C \approx {}^0C'$ stands for constructions C, C' being *equivalent*;

'$^0C = {}^0C'$ stands for constructions C, C' being *identical*.

Note that if $^0C = {}^0C'$ then $^0C \approx {}^0C'$, and if $^0C \approx {}^0C$' then $^0C \approx_v {}^0C'$, but not *vice versa*, since traffic in that direction would be going from coarser to finer distinctions.

Rules.

(a) ***Extensionality***. The rules of the sequent calculus are valid for derivations operating on the v-proper constituents of constructions.[135]

(b) ***Properness***. Let C be a construction such that no subconstruction D of C occurs with extensional supposition in a non-generic context of C. Then *Proper*(C). In particular, *Proper*($[\lambda y\ C(y)]$), *Proper*(^0D), *Proper*(x), for any variable x.

(c) ***Improperness***. Let $D \rightarrow_v (\beta\gamma_1\ldots\gamma_n)$ be a constituent of C occurring with *extensional supposition in a non-generic context* of C. Then there are some $Y_1,..,Y_m$ such that $[D\ Y_1\ldots Y_m]$ is a constituent occurring in a non-generic context of C, and the following rule is valid:

$$\frac{v\text{-}Improper([D\ Y_1\ldots Y_m])}{v\text{-}Improper(C).}$$

[135] See Tichý (1982).

In particular:

$$\frac{v\text{-}Improper(D)}{v\text{-}Improper([\lambda y\ C(y)\ D]).}$$

$$\frac{v\text{-}Improper(D)}{v\text{-}Improper({}^2[{}^0Sub\ [{}^0Tr\ D]\ {}^0y\ {}^0C(y)]).}$$

(d) ***Existence***. Let $D \rightarrow_v (\beta\gamma_1\ldots\gamma_n)$ occur with extensional supposition in a non-generic context of C. Then there are some $Y_1,\ldots,Y_n$ such that $[D\ Y_1\ldots Y_n]$ is a constituent occurring in a non-generic context of C, and the following rule is valid, where $Exist/((o(\beta\gamma_1\ldots\gamma_n))\ \gamma_1\ldots\gamma_n)$ is the function that associates a $(\gamma_1,\ldots,\gamma_n)$-entity with the class of functions that are defined at this argument:[136]

$$\frac{v\text{-}Proper(C)}{[[{}^0Exist\ Y_1\ldots Y_n]\ D].}$$

The function v-constructed by D is defined at the argument v-constructed by $Y_1,\ldots,Y_n$. Hence, $[[{}^0Exist\ Y_1\ldots Y_n]\ D] = [{}^0\exists\ \lambda b\ [[D\ Y_1\ldots Y_n] = b]]$,
where $b \rightarrow_v \beta$.

(e) ***Rules of substitution (Leibniz's Law).***

The extensional rule of substitution. Let $D \rightarrow (\alpha\ \beta_1\ldots\beta_m)$, $m \geq 1$, be a constituent of C and let D occur with extensional supposition in a non-generic context of C. Further, let D_i', D_i be constituents of D, and let $C(D_i'/D_i)$ be the result of collisionless substitution of D_i' for D_i in C. Then:

$$\frac{{}^0D_i \approx_v {}^0D_i'}{{}^0C \approx_v {}^0C(D_i'/D_i).}$$

The intensional rule of substitution. Let $D \rightarrow (\beta_1\ldots\beta_m)$, $m \geq 1$, be a constituent of C and let D occur with $(\beta_1\ldots\beta_m)$-intensional supposition or in a γ-generic context of C for some γ. Then:

$$\frac{{}^0D \approx {}^0D'}{{}^0C \approx {}^0C(D'/D).}$$

[136] For introduction of a tuple type like $(\gamma_1,\ldots,\gamma_n)$, see Section 2.6.2.

The hyperintensional rule of substitution. Let the occurrence of D be mentioned* in C, and let $=_{isom}/(o*_n*_n)$ be the relation of *procedural isomorphism* holding between constructions.[137] Then, provided no collision of variables arises:

$$\frac{{}^0D =_{isom} {}^0D'}{{}^0C =_{isom} {}^0C(D'/D).}$$

Remark. As we showed in Section 2.2.1, identity of constructions is too strong a criterion of procedural identity, hence of synonymy. This explains why

$$\frac{{}^0D = {}^0D'}{{}^0C = {}^0C(D'/D)}$$

is *not* our rule of hyperintensional substitution.

Remark. Throughout this book we shall often use the terms 'extensional context' instead of 'non-generic extensional context', 'intensional context' instead of 'α-generic intensional context', and 'hyperintensional context' for a context in which a construction is mentioned*.

Examples.

Ad the extensional rule of substitution: Let $C = [\lambda xy\ [{}^0\times y\ [{}^0+ x\ {}^01]]\ {}^03\ {}^05]$.

The constructions ${}^0+ \to (\tau\tau\tau)$ and ${}^0\times \to (\tau\tau\tau)$ (multiplication) occur with (τ,τ)-extensional supposition in the non-generic extensional context of C. Therefore, the substitution of $[{}^0+ [{}^0\times\ x\ y]\ y]$ for $[{}^0\times\ y\ [{}^0+ x\ {}^01]]$ in C is valid, and this identity holds:

$$[[\lambda xy\ [{}^0\times y\ [{}^0+ x\ {}^01]]\ {}^03\ {}^05] = [[\lambda xy\ [{}^0+ [{}^0\times x\ y]\ y]\ {}^03\ {}^05].$$

Let $C = [{}^0Happy_{wt}\ {}^0Pope_{wt}]$.

Types: $Happy/(o\iota)_{\tau\omega}$; $Pope/\iota_{\tau\omega}$; $Ratzinger/\iota$.

The Trivialization ${}^0Pope$ occurs with $(\omega\tau)$-*de re* supposition in C. Therefore, if $[{}^0= {}^0Ratzinger\ {}^0Pope_{wt}]$ v-constructs **T** then the substitution *salva veritate* of ${}^0Ratzinger$ for ${}^0Pope_{wt}$ in C is valid.[138]

Ad the intensional rule of substitution: Let $C = [{}^0Become_{wt}\ {}^0Ratzinger\ {}^0Pope]$.

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[137] See Definition 2.3.

[138] See the *Rule of substitution of v-congruent constructions* (*de re*) in Section 1.5.2.

Types: $Become/(o\ \iota\ \iota_{\tau\omega})_{\tau\omega}$; $Bishop\_of\_Rome/\iota_{\tau\omega}$.

The Trivialization ^0Pope occurs with $(\iota_{\tau\omega})$-intensional supposition in C (*de dicto*). If $[^0\!=\ ^0Bishop\_of\_Rome\ ^0Pope]$ constructs **T** then the substitution of $^0Bishop\_of\_Rome$ for ^0Pope in C is valid:

$$[^0Become_{wt}\ ^0Ratzinger\ ^0Pope] = [^0Become_{wt}\ ^0Ratzinger\ ^0Bishop\_of\_Rome].$$

Ad the hyperintensional rule of substitution: Let $C = [^0Calculate_{wt}\ ^0Charles\ ^0[^0\!+\ ^05\ ^07]]$.

Types: $Calculate/(o\iota{*}_1)_{\tau\omega}$; $Charles/\iota$.

The occurrence of the Composition $[^0\!+\ ^05\ ^07]$ is mentioned* in C. Therefore, if $^0[^0\!+\ ^05\ ^07] = {}^0[^0plus\ ^0five\ ^0seven]$ then the substitution of $^0[^0plus\ ^0five\ ^0seven]$ for $^0[^0\!+\ ^05\ ^07]$ in C is valid:

$$[^0Calculate_{wt}\ ^0Charles\ ^0[^0\!+\ ^05\ ^07]] = [^0Calculate_{wt}\ ^0Charles\ ^0[^0plus\ ^0five\ ^0seven]].$$

We conclude this otherwise technical section with some philosophical considerations on the relation between logic, language and reality. As semantic realists and logical Platonists, we are convinced that logic should assist in unearthing the objective structures underlying the expressions of a given language. A piece of language serves to point to a logical construction beyond itself, which is its sense. In order to reflect ‘gaps in reality’ faithfully (i.e., to obtain a counterpart of Bolzano’s *Gegenstandslosigkeit*), TIL adopts *properly partial functions* and *improper constructions.*

From a Platonist point of view, part of the task of a logician must be to adequately model the semantic features of (fragments of) a given language even at the cost of incurring technical complications. This explains why we are not going to join the game of playing fast and loose with existing logical symbols in order to define new ad hoc connectives and ‘entailment relations’ so as to either preserve or invalidate this or that commonly accepted law. Instead, in the remainder of this book we are going to deploy methods that overcome these technical complications and are at the same time in full accordance with the principles of TIL as specified in Chapter 1.

A comprehensive catalogue of constructions must also include those that construct partial functions and those that fail to construct anything. Improper constructions are the empty concepts expressed by those mathematical expressions that fail to denote.[139] Since empirical terms invariably denote intensions, such terms are never ‘non-denoting’, and so there is always a particular entity being talked about at the receiving end of the denotation relation. Similarly, some mathematical terms denote functions, and if they fail to denote then this is due to

[139] See Section 2.3.

using a function undefined at a given argument. Thus in both cases partiality arises only when using the operation of application of a function to an argument in a futile attempt to point to a non-existing value.

Intensions, modelled as partial functions, form part of the fabric of reality. But does anything, in and by itself, require adopting partial functions in order to reflect 'gaps' in reality? For instance, Strawson's well-known remarks on ontological presupposition are substantial only if one has already decided that intensions do enjoy an objective status and must not be missing from an exhaustive inventory of reality. There might just as well have been a King of France in the actual world now, only there is none. If we modelled this situation by a total function, we would be simply forced to supply something or other as a value of the function. Only what? It is not clear to us how such an analysis would comply with the constraint that it must not make it possible to deduce any untoward consequences.

By way of comparison, in first-order and higher-order classical logic there are two kinds of expression; terms and formulae. Terms are used to denote values in a mathematical structure or model, and formulae are used to make assertions about these values concerning their properties and relations to other values. The semantics of classical formal logic comes with an *existence commitment* to the effect that terms always have a denotation. Functors are interpreted as denoting *total* functions such that the combination of these symbols with argument terms always denotes the value of the function at its argument. If the function were partial the term would lack a denotation and so fail to be well-defined. It is a brute fact, however, that there are partial functions in mathematics, for instance, the function of division (unless we unnaturally restrict the domain of the function), so we need to be able to denote and logically manipulate them.

Observe that if 'the greatest prime' is dismissed as senseless, the principle of compositionality is violated. The sentence 'The greatest prime does not exist' expresses a truth, so it must be endowed with sense. If 'the greatest prime' is senseless, then this definite description makes no contribution to '...the greatest prime...', in which case the sense of '..the greatest prime...' is not composed of the senses of all its self-contained particles. Since we consider the principle of compositionality non-negotiable (see Section 2.1.2), the outcome that the sense of '...the greatest prime...' would not depend on the sense of 'the greatest prime' amounts to a *reductio*, in our view, of deeming 'the greatest prime', or any other self-contained term, senseless.

We do not dispute for a moment that it is tempting, and definitely less of a headache, to simply decree that, e.g., 'the greatest prime' is a senseless term than it is to sort out the technical difficulties stemming from partial functions. Furthermore, many commonly accepted classical laws have to be modified in order to come out valid in a logic embracing partial functions. For instance, $(A \vee \neg A)$ is not a tautology and $(A \wedge \neg A)$ is not a contradiction, as they may fail to take a truth-value. Other examples would include the De Morgan laws of negation for quantified formulae, the equivalence of β-transformation (λ-conversion), the reduction of *n*-ary functions to unary ones,[140] the equivalence of co-entailing formulae, the equivalence of relational and functional approaches, etc.

In a word, we opt for partiality for philosophical rather than technical reasons. Partiality causes technical pain and philosophical gain. This is why our semantics is designed to run smoothly even with partial functions and improper constructions.

[140] See Section 2.4.3, or Tichý (1982, pp. 59–60, 2004, pp. 467–8), on Schönfinkel reduction.

3
Singular reference and pragmatically incomplete meaning

This chapter details how TIL analyses terms like ‘Charles’, ‘π’, ‘the tallest mountain’, ‘the largest prime’, and ‘it’. These terms are self-contained semantic units and must therefore have a construction assigned to them as their meaning; only which one? We finish by outlining how updating works within a dynamic discourse involving singular terms.

3.1 Definite descriptions

Consider the two sentences

(A) ‘Bill Gates is married.’

(B) ‘The richest man is married.’

The truth-conditions of (A) and (B) are distinct. That they are so should not be influenced by the fact that Bill Gates happens to be the richest man (as of 2009). The point is that ‘Bill Gates’ is a proper name[1] and so we cannot suppose that in distinct possible worlds this name would identify distinct individuals. Independently of any particular theory of proper names, it should be granted that a *proper* proper name (as opposed to a definite description grammatically masquerading as a proper name) is a rigid designator of a numerically particular individual. On the other hand, ‘the richest man’ as a (definite) *description* does offer an *empirical* criterion that both enables and forces us to establish which individual, if any, plays the role of the richest man at a particular world/time pair. If a pair $\langle W,T\rangle$ is such that Bill Gates is married at $\langle W,T\rangle$ and the man who is the richest man at $\langle W,T\rangle$ is not married at $\langle W,T\rangle$, then (A) is, and (B) is not, true at $\langle W,T\rangle$.

We can demonstrate this claim by associating (A) and (B) with two non-equivalent constructions. Let the types be: *BillGates*/ι; *Married*, *Man*/$(o\iota)_{\tau\omega}$; *Richest*/$(\iota(o\iota))_{\tau\omega}$: the empirical function that, dependently on states-of-affairs, associates a class of individuals with at most one individual, namely the richest one.

(A′) $\lambda w\lambda t\ [{}^0Married_{wt}\ {}^0BillGates]$

(B′) $\lambda w\lambda t\ [{}^0Married_{wt}\ [{}^0Richest_{wt}\ {}^0Man_{wt}]]$,

[1] For a discussion of the semantics of proper names, see Section 3.2.

M. Duží et al., *Procedural Semantics for Hyperintensional Logic*, Logic, Epistemology, and the Unity of Science 17, DOI 10.1007/978-90-481-8812-3_3,

or equivalently,

(B″) $\lambda w \lambda t\ [{}^0Married_{wt}\ [\lambda w \lambda t\ [{}^0Richest_{wt}\ {}^0Man_{wt}]]_{wt}]$.

The distinction between names and descriptions is of crucial importance due to their vastly different logical behaviour. This distinction is explicitly respected in TIL (a) type-theoretically (names lacking intensional character), and (b) by equipping analyses of empirical definite descriptions with the empirical indices w, t.

The contemporary discussion of the distinction between names and descriptions was triggered by Russell (1905). The relevant place is where Russell proposes eliminating the descriptive operator 'ι'. Where 'P' and 'A' are one-place predicates, Russell's standard formulation is

(Rus) $P(\iota x\ Ax) \equiv \exists x\ (Ax\ \&\ \forall y\ (Ay \supset y = x)\ \&\ Px)$.

Russell's idea is that 'ι' possesses neither a self-contained meaning nor a denotation and that every context containing 'ι' can be replaced by an equivalent context lacking 'ι'. Below follows a set of comments outlining how we position ourselves in the contemporary debate on names and descriptions.

(a) *Frege vs. Russell*. Frege's conception of definite descriptions is referential; Russell's, quantificational. Frege assigns a dual, context-sensitive semantics to definite descriptions, while Russell argues that this sort of expressions must be done away with in the final logical analysis. We agree with Frege that definite descriptions are vehicles of reference. We find that Russell goes too far when arguing that they are syncategorematic expressions devoid of a semantics of their own. But we agree with Russell that definite descriptions do not denote the objects (if any) that they uniquely describe (even if we do not at all sympathise with his reasons for claiming so). Frege holds the same view, though only with regard to definite descriptions occurring in what he calls oblique (*ungerade*) contexts. In such contexts they denote what is in straight (*gerade*) contexts their sense, while in straight contexts they denote the unique objects, if any, they uniquely describe. Contextualism forces itself upon Frege because of his extensionalist semantics for straight contexts, which he himself acknowledges to fail to apply to oblique contexts. Despite their differences, a noteworthy feature shared by Frege's and Russell's conceptions is this. In oblique contexts Frege's definite descriptions denote a sense, which may be pre-theoretically construed as something like a *condition* satisfiable by the sort of objects that the descriptions denote in straight contexts. The gist of Russell's quantificational analysis of '$P(\iota xAx)$' is that there is exactly one thing possessing the two properties A, P. (While there may be more than one P-object, there is to be exactly one A-object.) This analysis may likewise be construed pre-theoretically as forming the *condition* of being the unique thing that is both an A and a P. This is a very inspiring feature, because it suggests what kind of thing a definite description denotes, as soon as this is not to be whatever (if anything) it uniquely describes. In TIL this feature translates into the tenet that what is semantically salient about a definite description is its uniqueness clause, which is a condition, rather than what (if anything) satisfies it. To be more specific, the tenet is that empirical definite descriptions denote intensions (namely, offices or roles), which are the theoretical counterparts of pre-theoretically understood empirical conditions with a built-in uniqueness clause. In the case of mathematical definite descriptions, constructions figure as conditions while their denotations are the entities (if any) which are so constructed. Whether intensions or constructions figure as conditions, the principle that the semantic relation of denotation is a priori is heeded.

However, despite this common feature, Frege's and Russell's theories are inherently heterogeneous. Writing down the construction underlying the schema (Rus) in the empirical case brings out the fundamental distinction between Frege's and Russell's views. Let $P, A \rightarrow (o\iota)_{\tau\omega}$; $\iota/(\iota(o\iota))$; $x, y \rightarrow \iota$. Then:

(Rus')

Left-hand side:

$$\lambda w \lambda t\, [P_{wt}\, [\iota x\, [A_{wt} x]]].$$

Right-hand side:

$$\lambda w \lambda t\, [\exists x\, [[A_{wt} x] \wedge [\forall y\, [[A_{wt} y] \supset [y = x]]] \wedge [P_{wt} x]]].$$

Both sides construct *propositions* and not truth-values. Russell's insight (as opposed to Frege's) is that no individual makes any semantic or logical contribution to the analysis of definite descriptions.

(b) *Referential vs. attributive use*. Holding, as TIL does, that pragmatic problems are altogether different from semantic problems, referential use in Donnellan's sense is irrelevant to semantics. Our analyses terminate in constructions, so semantics affords no means to obtain the values of the particular intensions (as constructed by the constructions cited as meanings) in the actual world at the present time. So TIL is not able to accommodate Donnellan's bifurcation; nor ought any theory in the business of *logical* analysis of natural language to be able to do so, since the bifurcation can be upheld only in the sphere of pragmatics. In particular, Donnellan's famous example in 1966 of the man over there drinking martini is to be explained in terms of the pragmatics of communicative situations. The meaning of the phrase 'the man over there drinking martini' is an open construction which only v-constructs the individual office occupiable by whatever unique man is drinking martini. In order to be able to execute the construction, the parameter of valuation v must be added by a situation of utterance. Thus the phrase in and by itself does not denote an office prior to evaluation, it only denotes one in a given situation. If there is no such individual who at the given $\langle w, t\rangle$ and in the given situation occupies the office, then the phrase does not *refer to* anything (in our stipulative sense of *reference*), whereas the *speaker intends* to identify an individual.

'The man over there drinking martini' is a vehicle of reference that has recourse to such pragmatic factors as background knowledge shared by speaker and audience, gestures (like a nod in a particular direction), and perhaps a bit of charitable guesswork on the hearer's behalf. Absent such factors, the expressive power of 'the man over there drinking martini' is too feeble to enable the hearer to fix the speaker's reference.[2]

(c) *Eliminability of* 'ι'. (Rus') fails to apply, as soon as functions are allowed to be properly partial. We show this for two cases.

(c_1) *The Strawsonian case*. In TIL the functions corresponding to descriptive operators are of the polymorphous type $(\alpha(o\alpha))$ and not total. If the set that is the argument of ι is a singleton then ι returns the α-object that is the unique member of the set. Otherwise ι is *undefined*.[3] Thus the well-known proposition that the King of France is bald lacks a truth-value at such world-time pairs where there is no King of France: the set of Kings of France is empty at such worlds-times and so ι comes out undefined when applied to it. This result is in harmony with Strawson's criticism and, we might be so bold as to suppose, with people's untutored linguistic intuition as well. If that proposition were false (it cannot be true at such worlds/times) then its negation would have to be true. The King of France would not be bald, entailing that the King of France exists, thus colliding with the fact that there is no King of France.

Schiffer argues that Russell's theory cannot accommodate referential uses of definite descriptions, as it leaves it indeterminate what the entity intended by the speaker is (2005, p. 1179). But nor can Russell's theory, according to Schiffer, accommodate attributive uses, as it admits two interpretations of 'the *A*' in 'The *A* is a *P*', either as a quantifier phrase or a singular term. Instead Schiffer agrees with Frege that 'the *A*' is always a singular term, but adds that truth-gaps are acceptable.

(c_2) *Existential commitment and expressivity.* Suppose Charles is thinking about the Golden Mountain. He can do so if he thinks about the individual office[4] constructed by

$$\lambda w \lambda t\ \iota x\ [x = [[{}^0Golden_{wt}\, x] \wedge [{}^0Mountain_{wt}\, x]]].$$

Types: *Golden*, *Mountain*/$(o\iota)_{\tau\omega}$; $x \rightarrow_v \iota$.

The predicate corresponding to the left-hand side of (Rus') will denote the property that is had by a *y* being thought about by Charles; i.e.,

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[2] See Section 3.4.

[3] Recall that 'ι' is abbreviated notation for '*Sing*', which denotes the function *singularizer*; see Definition 1.6. in Section 1.4.3.

[4] The following consideration holds even for the well-conceivable case that Charles is thinking about the *property* of being a golden mountain, perhaps wondering if there is any such property (as maybe the properties of being golden and being a mountain could not be co-instantiated).

$$\lambda w \lambda t\ \lambda y\ [{}^0 Think_{wt}\ {}^0 Charles\ y]$$

where $Think/(o\iota\iota_{\tau\omega})_{\tau\omega}$; $y \rightarrow_v \iota_{\tau\omega}$.[5] The left-hand side of (Rus') would be

$$\lambda w \lambda t\ [\lambda y\ [{}^0 Think_{wt}\ {}^0 Charles\ y]\ \lambda w \lambda t\ \iota x\ [x = [[{}^0 Golden_{wt} x] \wedge [{}^0 Mountain_{wt} x]]]].$$

What could Russell do with his right-hand side? He cannot distinguish between what we would call supposition *de dicto* and *de re*, for want of an equivalent mechanism, so he would use the existential quantifier to bind individual variables:

$$\lambda w \lambda t\ [\exists x\ [[{}^0 Golden_{wt} x] \wedge [{}^0 Mountain_{wt} x]] \wedge$$
$$\forall y\ [[[{}^0 Golden_{wt} y] \wedge [{}^0 Mountain_{wt} y]] \supset [y = x]] \wedge [{}^0 Think_{wt}\ {}^0 Charles\ x]]]].$$

For this right-hand side to come out true there must be a golden mountain (!). Thus in the worlds/times where there are no golden mountains the right-hand side of (Rus') would be false, whereas Charles can think about the office even in such world/time pairs.

A similar objection applies in all cases involving a construction of an office occurring *de dicto*, as in 'Charles wants to become the President of the USA'. Russell's solution involves the existence of the President of the USA, whereas Charles may well want to become President even if there is none.

(d) *Incomplete descriptions*. Most phrases of the form 'The *A* is a *P*', *P* some empirical property, can be conceived of as incomplete descriptions. The phrase 'The dog is dangerous' is obviously pragmatically incomplete in that it needs some contextual amendment: otherwise it would possess a truth-value only in a $\langle w, t \rangle$ having exactly one dog.[6] (We do not intend 'The dog is dangerous' to be synonymous with 'All dogs are dangerous'.)

(e) *Mathematical descriptions*. So far we have handled only empirical descriptions, as most problems with descriptions concern just those. In the case of mathematical descriptions, the question of whether referential, as opposed to attributive, use is possible does not arise. As an example consider the sentence

'The least prime number is even.'

Let ν be the type of natural numbers, and the other types as follows: (the)*Least*/$(\nu(o\nu))$; *Prime*/$(o\nu)$; *Even*/$(o\nu)$. We get

(C) $[{}^0 Even\ [{}^0 Least\ {}^0 Prime]]$.

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[5] This means that the construction of the office occurs *de dicto*; Charles is not thinking about whatever individual might be the occupant of the office, but about this office itself. For details on the *de dicto/de re* distinction, see Section 1.5.2.

[6] See Section 3.4.

This Composition obviously constructs the truth-value **T**.[7] What about the following sentence?

‘2 is even.’

An analysis of this sentence would be

(D) $[{}^{0}Even\ {}^{0}2]$.

(C) and (D) are simply two equivalent constructions. No problems analogous to those from the empirical sphere arise.

Another example:

‘The greatest prime is even.’

This time our sentence lacks a truth-value. *Greatest* is obviously of the same type as *Least* in the previous example, so we get

(E) $[{}^{0}Even\ [{}^{0}Greatest\ {}^{0}Prime]]$.

The important difference between the sentence having and lacking a truth-value is not visible, the logical form being the same. The Composition (E) reflects, however, the fact that even when a sentence lacks a truth-value we *understand* the sentence: we know which procedure is to be executed. The fact that in this case the procedure would lead nowhere is given by the nature of the respective mathematical concepts.

3.2 Proper names

The formal semantics of TIL requires that every expression belonging to natural language that does not play an exclusively syntactic role must express a construction as its meaning and denote whatever is so constructed. The requirement presents us with an awkward problem in the case of *proper* proper names; that is, those so-called ordinary proper names whose semantics cannot be reduced to the semantics of any other sort of expression (typically definite descriptions) and which serve to pick out one numerically specific individual. Absent this requirement, however, a sentence containing an occurrence of a proper name would, due to the compositionality constraint, fail to express a sense and so would also fail to denote a proposition. Which cannot be right. In Section 3.3.1 it is shown how

[7] Throughout this book we mostly analyze simple *expressions* as expressing *simple concepts*. Thus the above analyses are *literal meanings* of the analyzed expressions. See Definition 1.10, Section 1.5.1. Though irrelevant here, a definition of *Prime* would help us to *refined* constructions of, e.g., the functions *Even* and *Least*.

'Hesperus' and 'Phosphorus' may fruitfully be construed as denoting individual offices; and this approach may no doubt be extended to several other ordinary proper names. But we may still have good reasons to preserve an irreducible category of proper names, for there are occasions when we wish to talk about some one particular individual (whatever may be true of this individual) rather than about whatever individual (if any) something is true of, relative to a given universe of discourse.

This prompts the question of which sort of construction to assign to *proper* proper names as their meaning. As we just suggested, the problem cannot be circumvented by simply declaring that names have no meaning at all, being mere 'labels', 'tags', or whatnot. For the meaning of a compound in which such a name occurs is a function of the meanings of its atoms, including names. Without assigning a construction to 'Charles', the sentence

'Charles is happy'

will elude semantic analysis. All we would have would be

$$\lambda w \lambda t\, [{}^0H_{wt} \ldots].$$

For lack of an argument for ${}^0H_{wt}$ the analysis would be nonsensical.

A second reason for assigning senses to proper names is that understanding the sense of a name is what enables a language-user to intellectually identify or select the bearer of the name. In keeping with our procedural semantics, identification or selection must take the form of executing a procedure whose product is the bearer.

The only two candidate constructions are Trivializations of individuals and variables ranging over individuals. This gives us either

$$\lambda w \lambda t\, [{}^0H_{wt}\, {}^0Charles]$$

or

$$\lambda w \lambda t\, [{}^0H_{wt}\, x].$$

The former pairs 'Charles' off with a construction directly of Charles the individual. The latter renders 'Charles' analogous to the occurrence of 'he' as in 'He is happy'. In Section 3.4 such an occurrence is paired off with a free occurrence of x.

There is a link between these two possible interpretations of 'Charles':

$$\frac{\begin{array}{c}\lambda w \lambda t\, [{}^0H_{wt}\, x] \\ \lambda w \lambda t\, [{}^0{=}\, x\, {}^0Charles]\end{array}}{\lambda w \lambda t\, [{}^0H_{wt}\, {}^0Charles].}$$

This sort of argument is needed in order to turn an open construction into a closed construction, which can then be evaluated for its truth-value at a $\langle w, t\rangle$.

What recommends the free-variable analysis is that often the mere name will not carry enough information to identify any particular individual. In a conversational context it will have to be settled, one way or the other, which particular individual is the denotation of a particular use of a particular name, whenever a name is just a string of characters formed from a vocabulary and not an ordered pair of such a string and a construction. The Achilles' heel of the free-variable analysis is that it cannot stand alone. 'Charles', as it occurs in 'Charles is happy' when this sentence is embedded in a particular conversational context, picks out one particular individual, and the only way to present this individual directly is by means of a Trivialization of him. Hence the second premise in the argument above. This suggests that the sense of a *proper* proper name is a Trivialization of an individual. To understand a name will then amount to knowing the numerical identity of the Trivialized individual.[8]

This construal of proper names offers a solution to the 'Cicero'/'Tully' puzzle. Let the meaning of 'Cicero' be ^{0}Cicero and the meaning of 'Tully', ^{0}Tully. Then if ^{0}Cicero and ^{0}Tully Trivialize the same individual, ^{0}Cicero and ^{0}Tully will be one and the same Trivialization, though encoded linguistically in two different manners, as '^{0}Cicero' and '^{0}Tully'. ^{0}Cicero and ^{0}Tully will be intersubstitutable in any sort of context, since anything may always be substituted for itself. The Closures

$$\lambda w\lambda t\ [{}^0B^*_{wt}\ {}^0a\ {}^0[\lambda w\lambda t\ [{}^0H_{wt}\ {}^0Cicero]]]$$
$$\lambda w\lambda t\ [{}^0B^*_{wt}\ {}^0a\ {}^0[\lambda w\lambda t\ [{}^0H_{wt}\ {}^0Tully]]]$$

are identical, $B^*/(o\iota{*_1})_{\tau\omega}$ ('to believe* constructionally' or explicitly, i.e., being related to the literal meaning of the embedded clause; see Section 1.5.1, Definition 1.10). It is irrelevant whether a, the attributee, knows either of the *words* 'Cicero' and 'Tully', since the belief ascription concerns (among other) what is constructed by ^{0}Cicero/^{0}Tully (to wit, Cicero the man) and not either of the names 'Cicero' or 'Tully' (see Section 5.1.1 for Mates' puzzle). Thus, to know that Cicero is Tully is only to know that Cicero is self-identical. Knowing that Tully is called both 'Cicero' and 'Tully' is something entirely different, concerning as it does linguistic competence with two English words. It is non-trivial to know that 'Cicero' and 'Tully' are synonymous (and therefore co-denoting). Only this is an a priori fact about English and not an a posteriori fact about empirical reality. If you wish to turn it into a discovery about empirical reality that Cicero was Tully, it is necessary to construe at least one of 'Cicero', 'Tully' as a name of an individual office, along the lines of the analysis of 'Hesperus is Phosphorus' in Section 3.3.1.

Ordinary proper names also occur in fictional literature. The sentence

'Sherlock Holmes is happy'

[8] For further discussion, see Jespersen (2000).

is not to be paired off with the closed construction

$$\lambda w \lambda t\ [{}^0H_{wt}\ {}^0a]$$

where a/ι. This would namely make it either true or false ('truth-apt') whether Holmes is happy. But there is no fact of the matter as to whether Holmes is happy. Nor should there be, as soon as we wish to uphold a demarcation between fact and fiction. The meaning of so-called fictional names such as 'Sherlock Holmes' is instead construed as a free variable ranging over individuals. The meaning of the sentence is the open construction

$$\lambda w \lambda t\ [{}^0H_{wt}\ x].$$

This analysis is analogous to the analysis of 'He is happy' up to this point. But the difference is that the step from open to closed construction is never made. It is, as it were, left hanging in the air *which* individual is Sherlock Holmes. For all the analysis says, *any* individual is a possible value of x. This allows both the author's and the reader's imagination free rein to identify Sherlock Holmes with any particular individual (e.g., the author or reader himself or herself) or no-one in particular. This analysis also removes the need for a parallel pseudo-universe of fictional entities as denotations of fictional names. The semantics of 'Sherlock Holmes' is that it expresses a free variable ranging over individuals as its sense, but fails to denote, since a free variable does not construct anything (until a valuation assigns a value to it). So 'Sherlock Holmes is happy' has a meaning, but lacks a denotation (a truth-condition) and, therefore, a reference (a truth-value). The attractive outcome is that fictional discourse may be meaningful without thereby lending itself to making assertions about factual matters.

A seemingly more tricky case is provided by occurrences in fiction of names familiar from extra-fictional discourse, such as 'London' as it occurs in, 'Sherlock Holmes lives in London'. But Conan Doyle's novels are not drama-documentaries about London. We speak sloppily, and misleadingly, when we say that the novels are set in London, if by this we mean that they literally take place in London. Rather we are to *imagine* the plots unfolding in London (and Yorkshire and Switzerland and wherever else). Also 'London' as it occurs in the novels expresses as its sense a free variable ranging over individuals (on the simplifying, but probably innocuous, assumption that cities are mere individuals). The analysis of 'Sherlock Holmes lives in London' is, therefore, this open construction ($Live\_in/(o\iota\iota)_{\tau\omega}$):

$$\lambda w \lambda t\ [{}^0Live\_in_{wt}\ x\ y].$$

But the respective values of x, y cannot just have any properties the reader cares to imagine. If x lives in y then x must be a person and y a house/village/town/city/country. It means that a sentence attributing the relation *Live_in* to a pair $\langle x, y\rangle$ of individuals comes with a presupposition, namely that x should be a person and y a venue[9].

$$\forall x \forall y\ [^0 Presupposition\ \lambda w \lambda t\ [[^0 Person_{wt}\ x] \wedge [^0 Venue_{wt}\ y]]$$
$$\lambda w \lambda t\ [^0 Live\_in_{wt}\ x\ y]]]]].$$

Types: $Presupposition/(o\ o_{\tau\omega}\ o_{\tau\omega})$; $Person$, $Venue/(o\iota)_{\tau\omega}$; $x \rightarrow \iota$; $y \rightarrow (o\iota)_{\tau\omega}$.

Gloss: 'For all *x, y*, in order that the proposition that *x* lives in *y* have a truth-value, the proposition that *x* is a person and *y* a venue has to be true.'

This means that when reading that Sherlock Holmes lives in London, the reader must *imagine* a person living in a city. Furthermore, to *read* the novels the reader must adapt his or her images to the predicates that the author uses to describe Sherlock Holmes and London. Within these two constraints, the reader is free to build up his or her own images of Sherlock Holmes and London.[10]

The use of free variables is what underpins poetic licence, enabling artists to separate a string like 'Amerika' from the pair ⟨'Amerika', ^{0}America⟩ and assign instead a free occurrence of z to it as its meaning to form the pair ⟨'Amerika', z⟩. This is what both enabled and entitled Franz Kafka to use the string 'Amerika' in his novel *Amerika* when conjuring up scenes from an imaginary country that at the best of times bears only superficial resemblance to the country that German-speaking Kafka knew as 'Amerika' without thereby making any claims about America. Only on an overly naïve interpretation of Kafka's 'Amerika', disregarding the fictional status of *Amerika*, would its sense be taken to be ^{0}America.[11]

3.2.1 Mathematical constants

Consider numerical constants like '1' and 'π'. What is their semantics? Since our general procedural semantics correlates sense and denotation as procedure and product, the resulting theory bears similarities to Moschovakis' as based on algorithm and value. At the same time we are in stark opposition to Kripke's unrealistic realist contention that the semantics of 'π' consists in nothing other than 'π' rigidly denoting π. For sure, 'π' does denote π—indeed, 'π' qualifies as a strongly rigid

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[9] For the definition of presupposition, see Section 1.5.2, Definition 1.14.

[10] To settle just how free the reader is—for instance, as concerns inconsistent images of Sherlock Holmes and London—will include a discussion of *conceivability*, which we are not going to broach here.

[11] We do not pretend to have put forward anything like a semantics for fictional terms and expressions other than ordinary proper names. For instance, we have at this point nothing to say about the semantics of predicates or definite descriptions as they occur in fiction, or about the sense in which it seems somehow true (-in/about-fiction) to say that Sherlock Holmes' sidekick is Dr Watson and false that Sherlock Holmes plays the tuba. See, however, Tichý (1988, §49).

designator of π (cf. Kripke 1980, p. 48)—but there is substantially more to the semantics of 'π' than merely the denotation relation. In this section we focus on 'π', since our general top-down strategy is to develop a semantics for the hardest (or a very hard) case and then generalise downwards to increasingly less hard cases from there.

In outline, our procedural semantics says that 'π' expresses as its sense a procedure whose product is π. The procedure is, as a matter of mathematical convention, a *definition* of π and the product is, as a matter of mathematical fact, the (transcendental) *number* so defined. For comparison, '1' expresses as its sense the procedure consisting in applying the successor function to 0 once and denotes whatever (natural) number emerges as the product of this procedure.

The upside of a procedural semantics for 'π' is that to *understand*, as a reader or hearer, and to exercise *linguistic competence*, as a writer or speaker, one must merely understand a particular numerical definition and need not know which number it defines. Procedural semantics, whether realist or idealist, construes sense as an *itinerario mentis* abstracting from the itinerary's destination. Making the denotation of a numerical constant irrelevant to understanding and linguistic competence is not pressing in the case of '1', but it is so in the case of 'π'. The downside, however, is that at least two equivalent, but obviously distinct, definitions of π are vying for the role as *the* sense of 'π'. One is *the ratio of a circle's area and its radius squared*; the other is *the ratio of a circle's circumference to its diameter*. They are equivalent, because the same number is harpooned by both definitions. But the procedures are conceptually different, so they should not both be assigned to 'π' as its sense on pain of installing homonymy. This kind of predicament has become historically famous. Frege says, in analytic philosophy's single most notorious footnote:

> Solange nur die Bedeutung dieselbe bleibt, lassen sich diese Schwankungen des Sinnes ertragen, wiewohl auch sie in dem Lehrgebäude einer beweisenden Wissenschaft zu vermeiden sind und in einer vollkommenen Sprache nicht vorkommen dürften.
> (1986b, n. 2, p. 42.)

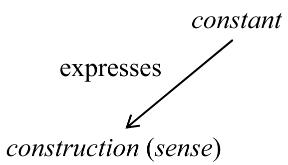
We shall suggest a solution to this predicament. The crust of the solution is to relegate each definition of π to individual *conceptual systems*.[12] Since an interpreted sign such as 'π' is a pair whose elements are a character (in this case the Greek letter 'π') and a sense, there will be as many such pairs as there are conceptual systems defining π. Disambiguation of 'π'-involving discourse will consist in making explicit which particular π-defining system should supply the sense of a token of the character 'π'.

A related predicament, which we shall also address, is whether 'π' is best construed as a *name* for π or as a shorthand for a *definite description*. If a name, the sense of 'π' will, in our semantics, be the Trivialisation $^0\pi$, i.e., the primitive procedure consisting in the instruction to obtain, or access, π in one step. The procedure

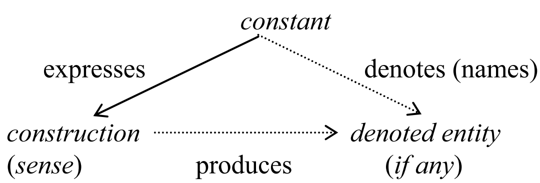
[12] See Definition 2.14, Section 2.2.3.

will not tell us *how* to obtain π, but only *that* π is to be obtained. This does not sit well with π being something as complicated as a transcendental number. But it does sit well with 'π' being itself a primitive, or simple, character not disclosing any information about its denotation. So at least on a naïve *literal analysis* (see Section 1.5.1, Definition 1.10), 'π' should be paired off with a non-complex sense. If 'π' is a definite description (in disguise), the sense of 'π' will, in our semantics, be a complex procedure consisting in the instruction to manipulate various mathematical operations and concepts in order to define a number. Only the problem, as we just pointed out, is, *which* procedure? Is it the instruction to calculate the ratio of a circle's area and its radius squared, or is it the instruction to calculate the ratio of a circle's circumference and its diameter, or is it some yet other instruction? Whichever it may be, though, the grammatical constant 'π' will be *synonymous* with the definite description 'the ratio…' chosen. The problem of homonymy does not rear its head in case the sense of 'π' is $^0\pi$, for then 'π' is only *equivalent* (co-denoting) with a particular definition. In fact, since all the variants of definitions co-denote the same number, 'π' will be equivalent with all such descriptions.

Whether 'π' be a name or a disguised definite description, it holds that its denotation needs to be *defined* and that an *algorithm* is required to bridge between definition and number. By showing how to calculate π, the algorithm shows, *ipso facto*, what the denotation of 'π' is. Our underlying semantic schema comes in two variants, one pure, the other impure. The pure one is



The relation a priori of expressing as obtaining between constant and sense exhausts the pure semantics of the constant. Only its sense is semantically salient, so a semantic analysis of 'π' must make its sense explicit. However, as soon as a procedure is explicitly given, its product (if any) is implicitly given, for the relation from procedure to product is an internal one: a procedure can have at most one product, and that product is invariant. The pure schema depicts a constant expressing its sense and not also what the constant denotes. An impure schema includes not only constant and sense, but also denotation:



The construction will produce its product independently of any algorithm; this is why the relation between construction and product is an internal one. But for epistemological reasons we will need some way or other of calculating its product to learn what it is, so we need a π-calculating algorithm to show us what number satisfies whatever π-defining condition. Such an algorithm will, *ipso facto*, reveal to us what the denotation of 'π' is. The number 3.14159… which is π is itself no player in the pure semantics of 'π'. The value of π is just whatever number rolls out as the value of the given procedure. The number 3.14159…is itself of little mathematical interest and of no semantic import. The *properties* of π, by contrast, are of great interest; e.g., whether π is normal in some base; and establishing that π is transcendental (and not merely rational) was a major mathematical achievement.

An algorithm may appear in one of two capacities. Either it is an intermediary between the definition and the number so defined: then the algorithm (whichever it is) is no player in the pure semantics of 'π'. Or an algorithm is the very sense of 'π': then the algorithm is a player in the pure semantics of 'π'. Our procedural semantics allows that a π-calculating algorithm may itself be elevated to playing the role of sense of 'π'. In such a case 'π' will have as its sense one particular way of calculating π. An algorithm is a particular kind of procedure and can as such figure as a linguistic sense relative to a procedural semantics.

In the former case, if the definition is a *condition* then the algorithm will calculate the *satisfier* of the condition. Full competence with respect to the definition *the ratio…* will yield knowledge of a condition to be satisfied by a real number, but will not yield knowledge of which number satisfies it. So the definition is, strictly speaking, a definition of something for a number to be; namely, the ratio of two geometric proportions. Hence three players need to be kept separate in the impure semantics of 'π': constant, sense, and number. If an algorithm is a sense then the sense is an effective mathematical procedure calculating π. Otherwise the sense is a logical procedure defining π in a non-effective way. Hence, if the sense defines π as the ratio between the area of a circle and its radius squared, a matching algorithm must calculate this ratio. Full linguistic competence with respect to 'π' neither presupposes, nor need involve, knowledge of how to calculate π. What competence consists in depends on whether the sense of 'π' is an atomic or a compound construction. If atomic, competence requires knowing which transcendental real 3.14159… is π. If compound, competence requires understanding the concept *the ratio of*, as well as either the concepts *the area of*, *the radius of*, *the square of*, or the concepts *the circumference of* and *the diameter of*, together with knowledge of how to mathematically manipulate them. A school child will understand such a complex procedure; it takes a professional mathematician to develop and comprehend a π-calculating algorithm. The task facing the mathematician is to come up with an algorithm equivalent with the sense of the definition defining the given ratio.

In the latter case, where an algorithm is the sense of ‘π’, full linguistic competence with respect to ‘π’ is to understand a definition of π and, again, not of the number so defined. But since the algorithm is now not an intermediary between definition and number, linguistic competence will be harder to come by, since the sense of ‘π’ is now likely to involve much more complicated mathematical notions than just, say, those of ratio, area, and circumference, such as the limit of an infinite series.

Assume now that the truth-condition of ‘…π…’ requires π to exist as an independent, abstract entity. Assume, further, that we can have no epistemic access to entities that we can have no causal interaction with. Then next stop is Benacerraf's dilemma as formulated for π: we do not know what number is π; yet we want to dub π ‘π’ in order to talk about π in ‘…π…’.[13] So how is ‘π’ to be introduced into mathematese? Moreover, now that ‘π’ has actually been introduced into standard mathematical vocabulary and been in use for 300 years, what would a realist (as opposed to constructivist or otherwise idealist) construal of its semantics look like?

As language-users we can *baptise* an abstract entity *E* ‘*E*’, as well as *use* ‘*E*’ competently, provided the following two conditions are met.

First condition. In order to *introduce* ‘*E*’ into mathematese, we must have a complex *procedure P* at our disposal, such that the unique output of *P* is the entity *E*, making the procedure *P* an ontological definition of *E*. An ontological definition of *E* is a closed construction of *E* different from ^0E.[14] Two examples of ontological definition of the real number π would be the right-hand sides of the equivalences

$$^0\pi = \iota x\,[\forall y\,[x = [^0Ratio\,[^0Area\ y]\,[^0Square\,[^0Radius\ y]]]]];$$
$$^0\pi = \iota x\,[\forall y\,[x = [^0Ratio\,[^0Circumference\ y]\,[^0Diameter\ y]]]].$$

Types: π/τ; $x/*_1 \rightarrow \tau$; $y/*_1 \rightarrow \gamma$; $Ratio/(\tau\tau\tau)$; *Area*, *Radius*, *Circumference*, *Diameter*/$(\tau\gamma)$; *Square*/$(\tau\tau)$; $=/(o\tau\tau)$; γ is here the type of geometrical figures, whatever it may be.

The sense of any *program* computing π is going to be an algorithm *equivalent* to, but *not synonymous* with, ontological definitions such as *the ratio of the circumference of a circle to its diameter* or *the ratio between a circle's area to its radius squared*. To competently use ‘π’ is to know at least one of these definitions.

Second condition. In order to be able to *use* ‘*E*’, we must not kick off the definition(s) of *E*; for we need to know that the sense of ‘*E*’ is *equivalent* to, though *not synonymous* with, the respective definition(s). For instance, we can use ‘is a prime’, provided we know at least one of the possible definitions of the set of primes. That is, pretending that the three equivalencies below exhaust the possible

[13] See Benacerraf (1973).

[14] See Definition 2.13, Section 2.2.2.

definitions of the set of prime numbers, we must know at least one of them to qualify as competent with respect to 'is a prime'.

$^0Prime = \lambda x\ [[^0Cardinality\ \lambda y\ [^0Divide\ y\ x]] = {}^02]$;

$^0Prime = \lambda x\ [[x \neq {}^01] \wedge \forall y\ [[^0Divide\ y\ x] \supset [[y=x] \vee [y = {}^01]]]]$;

$^0Prime = \lambda x\ [[x \neq {}^01] \wedge \neg\exists y\ [y > {}^01] \wedge [x \neq y] \wedge [^0Divide\ y\ x]]]$.

In other words, we can baptise the set of primes 'is a prime', 'is a prôtos', 'is an euthymetric', 'is a rectilinear', or whatever other predicate may have been used, but without a complex procedure yielding the set as output, these concatenations of letters are semantically void and futile.

Similarly for the introduction of 'π' via an ontological definition of π. Any algorithm computing π is going to be equivalent to, but not synonymous with, ontological definitions such as *the ratio of the circumference of a circle to its diameter* or *the ratio between a circle's area to its radius squared*. To master 'π' is to know at least one of these definitions (and not the number).

It may be illustrative to compare our realist procedural semantics to Kripke's realist denotational semantics. Central to the latter is the distinction between *fixing the reference* and *giving the meaning/a synonym*. One of Kripke's illustrations is this:

> ['π'] is not being used as *short* for the phrase 'the ratio of the circumference of a circle to its diameter'… It is used as a *name* for a real number, which in this case is necessarily the ratio of the circumference of a circle to its diameter (1980, p. 60).

Kripke's semantics for 'π' is simple (simplistic, as it turns out):

$$\text{`}\pi\text{'} \xrightarrow[\text{rigidly designates}]{\hspace{6em}} \pi$$

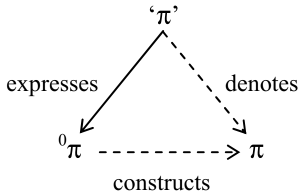
The description 'the ratio…' serves to single out the unique ratio shared by all circles, after which that number is baptised 'π'. The description is subsequently kicked off and so does not form part of the semantics proper of Kripke's 'π'. This is problematic. Nobody knows of some one particular real that it is π. So nobody knows of some one particular real that it is the reference of 'π'. So it is obscure what linguistic competence with respect to 'π' would consist in. Note that it is not an option to say that 'π' designates whatever real is the ratio of a circle's circumference to its diameter, for this uniqueness condition forms no part of Kripke's semantics for 'π'.[15] Kripke's introduction of 'π' is impeccable, and his 'π' does

[15] The Kripkean can have recourse to some causal theory of reference in the case of words for empirical entities like tigers, lemons and gold. But Benacerraf's second horn (the one that concerns knowledge and reference) blocks this avenue in the case of abstract entities like numbers. We hypothesise that Kripkean rigid designation cannot possibly be extended to numerical constants and other terms denoting abstract entities.

denote π. But we cannot use his ‘π’ to denote π, nor can we understand anyone else’s use of ‘π’, since we cannot know which particular transcendental number is π. In sum, Kripke’s ‘π’ has been severed radically from any humanly possible linguistic practice, so it is inoperative.[16]

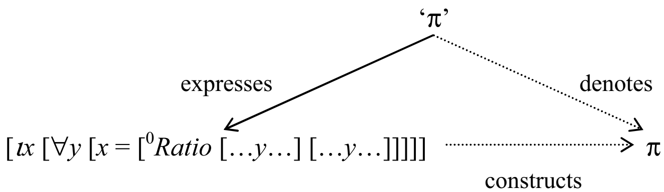
In the idiom of procedural semantics, Kripke focuses entirely on the *product* at the expense of the *procedure*. As a matter of mathematical fact, 3.14159… is π, but why introduce a non-descriptive name when that name severs the link between condition/procedure and satisfier/product? It seems that on Kripke’s semantics it will be a discovery, and not a convention, that π is the ratio of a circle’s circumference to its diameter (the template of the discovery being that *a* is the *F*). If so, it also seems that Kripke’s ‘π’ misconstrues mathematical practice.

Some π-producing procedure must figure in the semantics of ‘π’; but how? TIL faces a dilemma of its own, as we saw above. On the one hand, a literal analysis of ‘π’ would dictate that the sense of ‘π’ be $^0\pi$, yielding the schema



The advantage of this construal is that what looks like a constant *is* a constant (and not a definite description masquerading as one). However, this is too close to Kripke’s ‘π’ for comfort. We would be reinstating the problem that the semantics of ‘π’ pairs no mathematical condition off with ‘π’. To master ‘π’, $^0\pi$ would suffice. But, of course, this Trivialization merely instructs us to construct π and is silent on how to construct it.

On the other hand, not least epistemic concerns dictate that the sense of ‘π’ ought to be an *ontological definition* of π, yielding the schema



This makes ‘π’ a shorthand term synonymous with ‘the ratio…’, and its sense is an ontological definition of π. The advantage of this construal is that it pairs a mathematical condition off with ‘π’; but again, which one? There is no criterion to help decide which of the possible ontological definitions should be *the* sense of

[16] See Kripke (2008).

'π'. It would be arbitrary to select one and assign it as sense; but assigning them all would introduce homonymy.

It would seem evident that a language-user needs to know at least one definition of π in order to use and understand 'π'. If we go with the Trivialization-based analysis of 'π', the first step toward enhancing it is to make the logico-semantic fact that $^0\pi$ is *equivalent* with $[\iota x\, [\forall y\, [x = [^0Ratio\, [\ldots y\ldots]\, [\ldots y\ldots]]]]]$ part of the semantics of 'π'. $^0\pi$ is indifferent to how π is constructed by this or that compound construction, so as far as equivalence goes, any compound π-construction is as good as any.

'π' may be *introduced* as equivalent with

$$[\iota x\, [\forall y\, [x = [^0Ratio\, [\ldots y\ldots]\, [\ldots y\ldots]]]]],$$

or

$$[\iota x\, [\forall y\, [x = [^0Ratio^*\, [\ldots y\ldots]\, [\ldots y\ldots]]]]],$$

or any other compound π-constructing construction. *Understanding* is another matter. One thing is to understand $[\iota x\, [\forall y\, [x = [^0Ratio\, [\ldots y\ldots]\, [\ldots y\ldots]]]]]$; another thing is to understand $[\iota x\, [\forall y\, [x = [^0Ratio^*\, [\ldots y\ldots]\, [\ldots y\ldots]]]]]$. One may well know that 'π' is equivalent to this Composition without knowing, *ipso facto*, that it is equivalent to that Composition.

It is hopefully clear by now that both causal theory of reference and denotational semantics are neither here nor there as a theory of terms for abstract entities such as numbers. So we are putting forward a procedural semantics as a rival theory in order not to get gored by Benacerraf's horns or turning linguistic competence with mathematical constants into an enigma. We suggest, in the final analysis, that the semantics of 'π' ought to be that it is shorthand for, and therefore synonymous with, a definite description expressing a definition of π and denoting the number so defined. But for each definition$_n$ of π there is going to be a pair ⟨'π', definition$_n$ (π)⟩. So how do we handle the resulting homonymy? *Schwankungen des Sinnes* are neither here nor there in a regimented language such as mathematese. Our solution consists in relegating different definitions of π to different π-defining conceptual systems.

Relative to a particular conceptual system, a pair ⟨'π', definition$_n$ (π)⟩ is an unambiguous assignment of exactly one definition of π to 'π', provided the conceptual system is *independent* (as described in Section 2.2.3). Consequently, 'π' is not ambiguous, for this character must always be given together with a particular definition of π drawn from a particular conceptual system. The appearance of ambiguity arises only when two or more conceptual systems are invoked in the course of a discourse in which tokens of 'π' occur.

The upshot of our solution is that there are several π-denoting constants sharing the same first element, 'π'. So when two mathematicians are both deploying tokens of 'π', there is a risk of them talking at cross purposes, until and unless they compare notes and, in case of invoking different conceptual systems, come to agree on the same definition of π in the interest of synonymy. Yet the mathematical results they may have individually obtained with respect to π are bound to be equivalent, for any two definitions of π are bound to converge in the same number. The problem, after all, was always to do with *Schwankungen des Sinnes* and never *Schwankungen der Bedeutung*.

The more general morale we extract is that abstract entities cannot be dealt with without ontologically defining them first. Therefore, complex procedures are indispensable in the semantics of names for abstract entities.

3.3 Identities involving descriptions and names

TIL is a typed logic, so the identity relation = is of the polymorphous type $(o\alpha\alpha)$. There is no such thing as being identical to something *simpliciter*; there is only being identical to an α-entity, α an arbitrary type. So, unlike type-free logics such as Bealer's, we cannot express that *everything* is self-identical. What we can express is that every α-entity is self-identical, that every β-entity is self-identical, and so on for each particular type. Two random instances of the type $(o\alpha\alpha)$ would be $(o\iota\iota)$, the self-identity of an individual, and $(o(o\iota*_1)_{\tau\omega}(o\iota*_1)_{\tau\omega})$, the self-identity of a relation-in-intension between an individual and a first-order construction.

Here we take a closer look at various identity sentences culled from natural language. It would seem that the open-ended, seven-membered list below shall be able to cover a wide range of such sentences. TIL makes it possible to express that a particular individual bearing two names is self-identical; that a particular ι-Trivialization is self-identical (equivalently, that two different names are synonymous); that a particular individual is identical to the occupant of an office; that the occupant of one office is identical to the occupant of another office; that some particular individual is identical to the value of an attribute/$(\iota\iota)_{\tau\omega}$[17]; that the value of one attribute is identical to the value of another attribute; and that, necessarily, the occupant of one office is the occupant of another office.

Here is the list.

(1) $[{}^0a = {}^0b]$
(2) $[{}^{00}a =' {}^{00}b]$
(3) $\lambda w\lambda t\ [{}^0a = {}^0A_{wt}]$
(4) $\lambda w\lambda t\ [{}^0A_{wt} = {}^0B_{wt}]$
(5) $\lambda w\lambda t\ [{}^0a = [{}^0C_{wt}\ {}^0b]]$
(6) $\lambda w\lambda t\ [[{}^0C_{wt}\ {}^0a] = [{}^0D_{wt}\ {}^0b]]$
(7) $[{}^0Req_2\ {}^0A\ {}^0B]$.

[17] Attributes are here construed as empirical functions of type $(\alpha\beta)_{\tau\omega}$; α, β arbitrary types. *Father_of*, *Mother_of*/$(\iota\iota)_{\tau\omega}$, *Colour_of*/$((o\iota)_{\tau\omega}\iota)_{\tau\omega}$ are examples of attributes.

Types: $=/o\iota\iota$; $='/(o*_1*_1)$; $a, b/\iota$; A, B $(A \neq B)/\iota_{\tau\omega}$; C, D $(C \neq D)/(\iota\iota)_{\tau\omega}$; $Req_2/(o\iota_{\tau\omega}\iota_{\tau\omega})$ (Section 4.1 explaining the subscript in 'Req_2').

Remark. It is also an option that C, D share the same argument: $[[^0C_{wt}\ ^0a] = [^0D_{wt}\ ^0a]]$. Alternatively, the argument of an attribute may be the value of an ι-office: $[^0C_{wt}\ ^0A_{wt}]$. Or a pair of attributes may be arranged in a requisite relation. This opens up the possibility of further (obvious) combinations.

Example 3.1 '*Leningrad is St Petersburg*', '*Praha is Prague*', '*Den Bosch is 's Hertogenbosch*'.

There are two options: either (1) or (2). (1) attributes self-identity to an individual bearing two different names and is, therefore, trivially true or trivially false. Since semantics, as TIL understands it, is a priori, (1) is knowable a priori, as it requires only linguistic competence to establish whether it is true. (2) says that ^0a is the same Trivialization as ^0b, attributing self-identity to the meanings of '*a*', '*b*'; i.e., that '*a*', '*b*' are synonymous. Whether (1) or (2), it makes no difference if the two names belong to two different languages, as with 'Prague' and 'Praha'. It constitutes a linguistic, and not empirical, discovery that 'Prague' and 'Praha' are synonymous expressions. (1), (2) are logically related, (1) trivially following from (2) and (2) from (1). We do not accept an interpretation to the effect that, say, 'Den Bosch is 's Hertogenbosch' would mean that Den Bosch is (also) called 's Hertogenbosch'.[18] This interpretation would require amending the intensional base {ο, ι, τ, ω} so as to include *linguistic types*. This is formally feasible, of course; but there is a philosophical-methodological reason not to make the amendment. Such a 'meta-linguistic' solution, as it is commonly dubbed in the literature, runs counter to the TIL tenet that semantics is a priori. It is not consonant with the tenet to include expressions *qua* expressions, or linguistic items, into a logico-semantic analysis. Expressions exhaust their role by expressing constructions. Expressions are gateways to constructions, which are the objects of logico-semantic study. The study of expressions (their grammar, etymology, etc.) belongs to linguistics. For instance, TIL is geared to logico-semantic analysis and cannot, without thoroughgoing alteration, analyse a linguistic sentence like, 'The English word 'word' is a monosyllabic word of Germanic origin'. The tenet is based on the assumptions that a logico-semantic analysis is a synchronic snapshot of the ⟨expression, meaning⟩ pairs of a given language (English, Dutch, and Czech, as it happens) at a given

[18] A real-life case would be registered trademarks involving not only a logo but also a name encodable in plain lettering as well, like 'Budweiser'. As the on-going legal battle over the string 'Budweiser' has shown, uniqueness does matter. Allegedly the US market is not big enough for two syntactically indistinguishable brand names, since name recognition is part of brand recognition. So the battle turns on over whether American Budweiser or Czech Budweiser will occupy the office *the unique beer named 'Budweiser' on the American market*. A TIL construction of that office would involve linguistic types, which is fine, since the string 'Budweiser' is obviously part of the analysandum. What we are opposed to is the trick of shifting the denotation of a name from an entity beyond the name to the name itself, as in the 'Den Bosch'—''s Hertogenbosch' case.

point in time, and that full linguistic competence with a language is tantamount to knowing the finite set of such pairs and the grammatical rules for combining atoms into molecules.

Since we are assuming that speakers are fully competent language-users, we disagree with Kripke when he claims that

> You certainly *can*, in the case of ordinary [non-Russellian] proper names, make quite empirical discoveries that… Hesperus is Phosphorus, though we thought otherwise. We can be in doubt as to whether Gaurisanker is Everest or Cicero is in fact Tully.
> (1971, p. 143.)

The only empirical discoveries our speakers make concern facts about the world and not facts about language. This stance is, of course, in stark opposition to the naturalistic conception of language made popular by Quine. So we also disagree that Kripke does right in agreeing with Quine when the latter says that

> When…we discover that we have tagged the same planet twice, our discovery is empirical… (ibid., p. 141.)

If 'Hesperus' and the rest are names of individuals, then we would be discovering empirically that Hesperus is self-identical and doubting whether Cicero is self-identical. It is not clear to us how self-identity might be established empirically or what it would mean to doubt anything's or anyone's self-identity. On the other hand, if 'Hesperus' and the rest are names of individual offices, then it must be established empirically whether Hesperus and Phosphorus happen to share the same occupant (see Section 3.3.1 for a worked-out analysis) and it may be rationally doubted whether the Cicero and Tully offices happen to share the same occupant.[19]

It might be tempting to analyse cases involving *pseudonyms* along the same lines. However, we suggest that they are best analysed as instances of (3).

Example 3.2 '*Samuel Langhorne Clemens is Mark Twain.*'

(3) says that *a* occupies the office denoted by *N*, where the office named by *N* is defined in terms of certain achievements, like authoring certain books. This analysis makes it analytically necessary that Mark Twain should pen *Huckleberry Finn*, or whatever else may define the office of Mark Twain. Similarly, it will be analytically necessary that Shakespeare should write *Richard III* (and all the rest traditionally attributed to Shakespeare). What constitutes a historical discovery will then instead be who occupied the Shakespeare office. It may even be a historical discovery that the Shakespeare office was occupied by two or more different

[19] On a charitable reading, Quine (and Kripke) perhaps had in mind another kind of empirical discovery, not concerning the state of the world but of the language in question. This would make a *linguistic* discovery of Quine's empirical discovery that two different names have been tagged to Venus. But this empirical linguistic discovery belongs to linguistics rather than semantics. Schematically, the formal semanticist lays down the rule that *if* two names NAME 1, NAME 2 co-denote *then* a sentence in which NAME 1, NAME 2 flank the identity sign expresses that the shared denotatum is self-identical. The field linguist instead establishes whether the antecedent is true of two actual names.

individuals at different points in time. In this respect there is something close to what Kripke calls a 'logical fate' hanging over Mark Twain and William Shakespeare (cf. 1980, p. 77.)

Example 3.3 '*Angela Merkel is the Bundeskanzlerin of Germany.*'

(3) is the right choice if we wish to express that some particular individual (denoted by a *proper* proper name) is, contingently, the occupant of some particular office (*in casu*, the *Bundeskanzleramt*). A deeper analysis would be (5), however, in order to also mention Germany in the analysis. The analysis is then

$$\lambda w \lambda t\ [{}^0a = [{}^0Bundeskanzlerin\_of_{wt}\ {}^0Germany]]$$

Types: $Bundeskanzlerin\_of/(\iota\iota)_{\tau\omega}$; $Germany/\iota$.

Example 3.4 '*The President of Turkmenistan is the Prime Minister of Turkmenistan*'.

This can be analysed in two different ways, one *de dicto* in terms of requisites, and the other *de re* in terms of contingent coincidence of two offices (see Section 1.5.2 for the corresponding dual *de dicto*/*re* analysis of, 'The King of France is a king'). The analysis *de dicto,* in accordance with (7):

$$[{}^0Req_2\ \lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0Turk]\ \lambda w \lambda t\ [{}^0PM_{wt}\ {}^0Turk]]$$

Types: $Req_2/(o\ \iota_{\tau\omega}\ \iota_{\tau\omega})$; $Pres\_of$, $PM\_of/(\iota\iota)_{\tau\omega}$: president of something, prime minister of something; $Turk/\iota$: Turkmenistan.

The analysis *de re*, in accordance with (6):

$$\lambda w \lambda t\ [[{}^0Pres\_of_{wt}\ {}^0Turk] = [{}^0PM_{wt}\ {}^0Turk]].$$

The analysis *de re* is a case of *co-reference*; i.e., of contingent co-occupation of two offices at a $\langle w, t \rangle$.

For a slightly more complicated case, consider

Example 3.5 '*Napoleon is the first Emperor of France.*'

$$\lambda w\ [{}^0a = [{}^0First_w\ [\lambda x\ \exists t\ [{}^0Emperor_{wt}\ {}^0France] = x]]].$$

Types: $First/(\iota(o\iota))_\omega$; $Emperor/(\iota\iota)_{\tau\omega}$; $France/\iota$; $x \to \iota$.

At a world w, $\lambda x\ \exists t\ [[{}^0Emperor_{wt}\ {}^0France] = x]$ v-constructs the set of the individuals who were, are or will be Emperor of France in w. The function *First* then, dependently on worlds, picks out the first one from this set. If we wanted to make a still finer analysis, we would have to take into account the semantics of '*First*'; it picks out the individual who plays the role of Emperor of France at a time t' such that for all other times t when the role of Emperor is occupied it holds that $t' \leq t$. But the literal analysis would be the one above. On this analysis the proposition denoted by the sentence is, at a given world w, eternally true, or eternally false or eternally without a truth-value. Here we do not take into account the obvious semantic distinction between 'Napoleon *is* the first Emperor of France', 'Napoleon *will be* the first Emperor of France' and 'Napoleon *was* the first Emperor of France'. In other words, we are not analysing here the semantics of verbs with different grammatical tenses.[20]

For an instance of (6), consider

Example 3.6 'a*'s wife is* b*'s mother.*'

Assuming monogamy, the current state of bio-technology, and an atemporal copula so that it does not matter whether b's mother has passed away, a will have at most one wife and b exactly one mother. If a has no wife, then it is neither true nor false that a's wife is b's mother, since the identity relation will lack an argument. If a does have a wife, then it is true or else false that that individual happens to be the same as b's mother.

$$\lambda w \lambda t\ [[{}^0 Wife\_of_{wt}\ {}^0 a] = [{}^0 Mother\_of_{wt}\ {}^0 b]].$$

Type: (the)$Mother\_of$; (the) $Wife\_of / (\iota\iota)_{\tau\omega}$.

Remark. Example 3.6 should not be confused with the *predication*

'b's mother is a's sister.'

Individual a may have more than one sister, so the construction is

$$\lambda w \lambda t\ [[{}^0 Sister\_of_{wt}\ {}^0 a]\ [{}^0 Mother\_of_{wt}\ {}^0 b]].$$

$Sister\_of/((o\iota)\iota)_{\tau\omega}$: given a $\langle w, t\rangle$ and an individual, we are given the set of that individual's sisters.[21] However, we can obtain uniqueness and so a case of *identity* by using a singularizer *Sing* of type $(\iota(o\iota))$:

$$\lambda w \lambda t\ [{}^0 Sing\ [{}^0 Sister\_of_{wt}\ {}^0 a] = [{}^0 Mother\_of_{wt}\ {}^0 b]].$$

This construction corresponds to the sentence, 'b's mother is the sister of a'.

[20] For details on tenses, see Section 2.5.2, or Tichý (1980a, b).

[21] Russell noted already in 1905, 'Now *the,* when it is strictly used, involves uniqueness; we do, it is true, speak of "*the* son of So-and-so" even when So-and-so has several sons, but it would be more correct to say "*a* son of So-and-so".' (1953, p. 44). So the English sentence 'Bertha's mother is the sister of Alfie' need not imply that Alfie would have one sister only.

3.3.1 Hesperus is Phosphorus: co-occupation of individual offices

In this section we discuss how to make it non-trivial that Hesperus is Phosphorus. Our solution is that 'Hesperus is Phosphorus' means that two individual offices—one named 'Hesperus', the other 'Phosphorus'—are co-occupied by the same individual at a given $\langle w, t\rangle$ of evaluation. The solution also makes it plain why it is vital to distinguish *denotation* from *reference*. Since *reference*, as we understand it, is extra-semantic and factual, while *denotation* is semantic and a priori, it is not an empirical fact that 'Hesperus' and 'Phosphorus' co-denote (for they never do), whereas it is an empirical fact that they co-refer (namely, when being co-occupied).[22]

Bealer (2004) discusses what it is we learn (i.e., get to know) when learning that Hesperus is Phosphorus. The conclusion of his discussion is that the best answer direct reference theory can offer is inconsistent. Bealer summarizes direct reference theory in these two tenets (tenet I, ibid., p. 575, tenet II, ibid., p. 576):[23]

Tenet 1. If 'a' and 'b' co-refer then '$a = a$' and '$a = b$' are synonymous.

Tenet 2. If $a = b$ then $(a = a) = (a = b)$.

He then raises the question:

> *How*, if $a = b$, can the proposition that $a = a$ and the proposition that $a = b$ be different? (Ibid., p. 575)

Well, they just cannot, if $a = a$ and $a = b$ are the proposition that a is self-identical. Yet what you learnt in astronomy class was certainly not that Venus, under whatever name, is self-identical. But this outcome is inescapable if one subscribes to Kripkean names and Russellian propositions. As Russell observed long ago,

> [I]f…'*c*' is a name for Scott, then the proposition [expressed by "Scott is the author of *Waverley*"] will become simply a tautology. It is at once obvious that if '*c*' were 'Scott' itself, 'Scott is Scott' is just a tautology. But if you take any other name which is just a

[22] Tichý points out that, according to Kripke, 'Hesperus is Phosphorus' comes out *necessarily* true, because it expresses that Phosphorus (Hesperus) is self-identical, and a posteriori because it is an empirical fact that 'Hesperus' and 'Phosphorus' co-refer. (1983, pp. 232–33; 2004, pp. 514–15). Kripke's observation concerning a posteriority is correct all right, but in our view also irrelevant to semantics. The semantic analysis must terminate at the *denotations* of 'Hesperus', 'Phosphorus', and not make the extra step from denotation to reference.

[23] The formulation of tenet 1 seems problematic. This is so because 'names do not contribute anything to the *meaning* of a sentence over and above their *reference*' (ibid., p. 575) (italics inserted). But we are far from sure that all direct reference theorists do, or must, hold that sentences have meanings at all. What is more, how can a *reference* contribute anything to a *meaning*? It would seem to fly in the face of any principle of semantic compositionality that the meaning of a sentence is a function of the meaning of some of its constituents together with the reference of its remaining constituents. For a critique of Bealer's critique of direct reference theory, see Jespersen and Zouhar (ms.).

> name for Scott, then if the name is being used *as* a name and not as a description, the proposition will still be a tautology. (1953, p. 245.)

This is sufficient to convince us that direct reference theory is a non-starter and that a more sophisticated alternative is called for. The alternative we are proposing takes its lead from the observation that

> When we are told that tan 45° = cot 45° …we learn something about the tangent and cotangent functions, not about the number one, which *is* the common value of those functions at 45°. (Tichý, 1986a, p. 254; 2004, p. 652.)

Let us agree, to begin with, that one constraint must guide our analysis, namely that it must be a discovery entirely due to astronomy that Hesperus is Phosphorus.[24] It ought to be neither logically nor analytically true that Hesperus is Phosphorus.[25] In particular, any viable theory must avoid that linguistic competence would suffice to establish whether Hesperus is Phosphorus. Otherwise the question whether Hesperus is Phosphorus would be prejudged. For all one learnt when learning that Hesperus is Phosphorus, the relevant individual might be Mercury, the moon, or any other celestial body visible from earth. It must require astronomical investigation to establish which of the candidate celestial bodies is the right one. Once the astronomers have done so, they will be sharing an additional, and logically independent, piece of information with you by adding that the co-extension that Hesperus and Phosphorus share in the actual world at the present moment is Venus. (Here we are assuming that ‘Venus’ does not name an individual office but an individual, which happens to be a planet.)

If the analysis of ‘Hesperus is Phosphorus’ is hedged in by the constraint that only astronomy will determine whether Hesperus is Phosphorus, then according to our proposal at least one of ‘Hesperus’ and ‘Phosphorus’ must denote an individual office on pain of perpetuating the self-identity analysis, according to which ‘*a* is *b*’ can only mean that the co-referent of ‘*a*’, ‘*b*’ is self-identical. However, since ‘Hesperus’ and ‘Phosphorus’ ought to belong to the same semantic category to avoid arbitrarily making one a name of an individual office and the other a name of an individual, they must both denote an individual office, though emphatically not the same one on pain of reinstalling the self-identity analysis.

The following three tenets summarise our proposal.

---

[24] This constraint is in keeping with Bealer’s claim that ‘[n]early everyone agrees that the following at least seems intuitively obvious: It is not possible to know a priori that Hesperus = Phosphorus.’ (*Ibid.*, p. 576.) But the analysis we offer is obviously incompatible with the widespread construal of the proposition that Hesperus is Phosphorus as being both (‘metaphysically’) necessary and not knowable a priori.

[25] So an analysis in terms of requisites – $[{}^0\!Req\ {}^0\!Hesperus\ {}^0\!Phosphorus]$, $Req/(o(\iota_{\tau\omega}\iota_{\tau\omega}))$ – is out of the question. (For *requisite*, see Section 4.1.) This would be philosophically and astronomically unreasonable, anyway. It ought to be conceptually and nomologically possible that a celestial body should satisfy the condition of being the brightest body in the evening sky without thereby satisfying the condition of being the brightest body in the morning sky, or the other way around. That is, these two conditions must be independent.

Tenet 1 (*individual offices*). 'Hesperus', 'Phosphorus' rigidly denote intensions (individual offices).

Tenet 2 (*co-extensionality*). 'Hesperus = Phosphorus' expresses the contingent co-extensionality of two named intensions coinciding in one (anonymous) individual, not the necessary self-identity of an individual bearing two names.

Tenet 3 (*contingency*). The contingency of the proposition that Hesperus is Phosphorus must be made explicit in the logical analysis of the sentence 'Hesperus is Phosphorus'. This is achieved by means of explicit intensionalization and temporalization (see Section 2.4).

The sought-after modality of contingency is acquired by relativizing the co-extensionality between the intensions Hesperus and Phosphorus to worlds and times. At some, but not all, worlds and times Hesperus and Phosphorus share the same extension. At some, but not all, worlds and times within this set of world-time pairs the shared extension is Venus. At other world-time pairs it is Mercury, or Titan, or UB313, or whatever else the universe may have in supply. In general, what makes a context modal is not exclusively the explicit presence of modal operators or modal expressions like 'necessary' and 'possible'. It is enough that the context concerns contingent truths and falsehoods.

The problem with neglecting the difference between empirical and non-empirical is that the specifically *empirical modality of contingency* is swept under the carpet. Yet the informational non-triviality of 'a is b', if true, can, at least in our opinion, be accounted for only in terms of its being contingently true.[26] When we know that $a = b$ then we know something that might have failed to be the case but in fact is the case. Therefore, if we fail to incorporate reference to contingency into our logical analysis of empirical sentences, our analysis is bound to be botched. This is evidenced by the direct reference analysis of 'Hesperus is Phosphorus', which can be nothing other than the self-identity analysis.

The fact that contingency is obviously pivoted on intensionality is at loggerheads with the prevalent tendency to treat 'a is b' ('a', 'b' empirical terms) as being on a par with non-empirical identity sentences. Our analysis is also at variance with the pre-theoretic conception of 'ordinary' proper names as (non-descriptive) names of individuals that not least direct reference theory has sought to underpin theoretically. In this book we do not engage in a large-scale confrontation with the various arguments that have been advanced in favour of this conception of 'ordinary' proper names. But the specific morale of our discussion of how 'Hesperus is Phosphorus' can be rescued from triviality is that the direct reference construal of 'Hesperus', 'Phosphorus' yields the wrong result. The failure of direct reference theory in this department is not a cogent argument for our approach, of course;

[26] The account of the informational value of the mathematical proposition expressed by 'a^* is b^*', 'a^*', 'b^*' mathematical terms, cannot be cast in terms of contingency, and must be altogether different. As for the informativeness of mathematical sentences, see Section 5.4.

non-triviality can be achieved along alternative routes. But the case of ‘Hesperus is Phosphorus’ is a suitable platform for broaching one such route that has as yet not received its fair share of attention. The idea, again, is that ‘Hesperus’, ‘Phosphorus’ denote two distinct individual roles also when flanking the ‘is’ of identity in atomic sentences.

We claimed above that these two pieces of information are logically independent: (a) that Hesperus is Phosphorus and (b) that Venus is the shared extension of these two intensions. For comparison, consider how we do *not* intuitively construe a non-empirical claim such as ‘$7 + 5 = 4 \times 3$’, namely as claiming the self-identity of the number 12. It seems an intuitively appealing idea that what it claims is the coincidence in an anonymous number of the outcomes of two named operations, namely the operations of adding 7 to 5 and multiplying 4 by 3. That $7 + 5 = 12$ and that $4 \times 3 = 12$ constitute two further pieces of (analytic) information (see Section 5.4).

If we are agreed that the number 12 should play no role in the semantic analysis of ‘$7 + 5 = 4 \times 3$’, then, by analogy, Venus, or any other concrete celestial body, should likewise drop out of the picture. A second argument for leaving the actual extension out of the semantic analysis is Tichý’s modal argument to the effect that ‘the Morning Star’ does not denote Venus, or in general, that a definite description does not denote its actual *descriptum*.

> Those who hold that [‘The Morning Star is a planet’] does treat of Venus, the celestial body, will probably agree with one another that what [this sentence] *says* about the celestial body is … that it is a planet. It is easily seen, however, that [the sentence] might be true *without* that body’s being a planet. For consider a world in which Mars instead of Venus is the brightest celestial body one can see in the morning sky and in which Venus fails to be a planet. Clearly there are possible worlds of this sort. But in any such world [‘The Morning Star is a planet’] comes out true. Surely a sentence cannot come out true in a state of affairs where what it says is not the case. Hence what [the sentence] says cannot be to the effect that Venus is a planet (1975, p. 87; 2004, p. 214).

That is, if it is true that Venus is a planet, then it might have been false. If it is false that Venus is a planet, then it might have been true. Hence, it is a contingent truth or falsehood that Venus is a planet. The semantics in terms of which we analyse ‘Venus is a planet’ cannot confine itself to set membership, for the following is trivially true and trivially false, respectively: $a \in \{a, \ldots\}$, $a \notin \{b, c, d\}$, and $a \in \{b, c, d\}$, $a \notin \{a\ldots\}$. Let C be the set of all and only those individuals that are actually (and presently) a planet. Then consider a world (and a time) at which Venus is not a planet:

> In such a world (as in any world) it is true that Venus is a member of C (i.e. of the class consisting of Mercury, Venus, ..., and Pluto), yet (3) [“Venus is a planet”] is false. Now surely a sentence cannot be false in a state of affairs where what it says is the case. Consequently, what (3) [“Venus is a planet”] says cannot be to the effect that Venus is a member of C (or any other class) (1975, p. 83; 2004, p. 210).

Thus, according to Tichý, what *is* relevant to the truth-condition denoted by, ‘The Morning Star is a planet’ is the individual office *the Morning Star*; i.e., the condition of being the brightest body in the morning sky. The truth-condition of the sentence is that whatever celestial body is the brightest in the morning sky should be a planet. The analogy between the definite description 'the Morning Star' and the 'ordinary' proper name 'Phosphorus' is that neither denotes an individual, both denoting an individual office instead. So Tichý's modal argument above applies equally to 'Phosphorus' (and 'Hesperus'). If 'the Morning Star' and 'Phosphorus' are introduced as two names, in the same language, of the same individual office then one of them is redundant. That a definite description and an 'ordinary' proper name co-denote (even rigidly) the same entity is, of course, at odds with what Kripke and various proponents of direct reference have claimed; but it is a quite natural possibility within a semantic theory allowing both definite descriptions and 'ordinary' proper names to denote intensional entities.

However, one way of attempting to reinstate individuals as the denotations (as opposed to references) of definite descriptions would be to turn to the notion of *flexible designation*. Roughly, what a term denotes is then a function not only of linguistic *fiat* but also of the index at which the term is used (various subtleties of multi-dimensional semantics aside). Thus, at w 'the Morning Star' denotes Venus, while at w' 'the Morning Star' denotes Mars, say, since Mars is the brightest heavenly body in the morning sky at w'. (Similarly, if 'Phosphorus' is declared a flexible designator then it denotes Venus at w but Mars at w'.) The problem with flexible designation, though, is that it turns the designation relation into a part factual one. World-relative facts will in part determine a semantic property of flexible designators; namely, what their denotation is at a given world. Consequently, it takes not only knowledge of a linguistic convention but also knowledge of a world-relative fact to know what individual is predicated to be a planet in, 'The Morning Star is a planet'. Furthermore, a semantic theory boasting flexible designation ends up offering the self-identity analysis of sentences in the vein of, 'The F is the G'. If 'the F' and 'the G' both denote Mars at w' then the sentence just expresses that Mars is self-identical. This gets the modal profile of 'The F is the G' wrong. On the other hand, a pure and a priori semantics is available both for definite descriptions and 'ordinary' proper names by having them denote individual offices, since this relation between term and *denotatum* is independent of empirical indices.[27]

If, as we suggest, one goes for a pure semantics, what would the relevant portions of such a semantic theory look like? Let H, P be individual offices. Two scenarios involving some specific individual i would be

- individual i = individual i
- i-under-H = i-under-P.

However, an alternative, rival, scenario would be

- $[{}^0H_{WT} = {}^0P_{WT}]$.

[27] See Tichý (1986a, p. 255; 2004, p. 653) and Section 3.1 for the claim that also definite descriptions are rigid designators.

Where the first two scenarios include ('bare') individuals and individuals-as-occupants-of-offices, respectively, the third scenario includes offices-plus-extensionalization.[28] This scenario does not name the occupant of H_{wt}, P_{wt}, which is this or that particular individual. The only entities are H, P, T, W, Composition, and the identity relation.

If we allow that the offices may not be occupied at all worlds and times, we end up with the following five possibilities.

- At $\langle W, T\rangle$, $H_{WT} = P_{WT}$
- At $\langle W, T\rangle$, $H_{WT} \neq P_{WT}$
- At $\langle W, T\rangle$, there is an H_{WT} but no P_{WT}
- At $\langle W, T\rangle$, there is no H_{WT} but a P_{WT}
- At $\langle W, T\rangle$, there is neither an H_{WT} nor a P_{WT}.

Which of these five actually and presently obtains is a contingent matter, and one that must be settled a posteriori by astronomical research.

Let us compare the semantic analyses of 'Hesperus is Phosphorus' offered by direct reference theory and TIL. First, if 'Hesperus', 'Phosphorus' are Kripkean proper names denoting individuals, the analysis in TIL guise becomes

$$[{}^0H' = {}^0P'].$$

The analysis is cast in terms of functional application of the identity relation to H' and P' to obtain a truth-*value* as the product of this Composition. The truth-value is **T**, since this is the self-identity analysis.

Second, if 'Hesperus', 'Phosphorus' denote individual offices, the analysis becomes:

$$\lambda w \lambda t\ [{}^0H_{wt} = {}^0P_{wt}].$$

The so constructed truth-condition is that H_{wt} and P_{wt} should be one and the same celestial body for a given choice of values for $\langle w, t\rangle$ as points of evaluation.

It may be illuminating to compare Bealer's approach to ours. Bealer's question was this: if $a = b$, how can the proposition that $a = a$ be different from the proposition that $a = b$? Restricting ourselves to the empirical case, our answer was because our a and b are not individuals but two distinct individual offices alias conditions to be satisfied by individuals. Trivializations of these two individual

[28] The idea of operating directly with specified individual offices and only indirectly with unspecified individuals (in TIL, via extensionalisation) is one of the three approaches that Aloni considers in her (2005). She both rejects operating with 'bare individuals' and 'ways of specifying [bare] individuals' (i.e., individual offices), opting for 'individuals specified in one determinate way' (see, for instance, p. 27). Her stance appears to square with the second scenario adumbrated above. However, despite the length of her paper, we are still not sure we fully understand the idea of identifying an individual-under-a-description with an individual-under-a-different-description. In particular, it is not clear how the self-identity analysis is to be avoided.

offices are constituents of the Closure $\lambda w \lambda t\ [{}^0H_{wt} = {}^0P_{wt}]$, as are variables ranging over worlds and times, together with a Trivialization of the identity relation defined over individuals. The other construction would be $\lambda w \lambda t\ [{}^0H_{wt} = {}^0H_{wt}]$. The so constructed proposition is true at all worlds and times at which H is occupied, lacks a truth-value at those at which it is vacant, and is never false. These two constructions contain the same number of occurrences of subconstructions, but not entirely the same constituents. $\lambda w \lambda t\ [{}^0H_{wt} = {}^0H_{wt}]$ contains two occurrences of Trivialization of the same individual office, whereas $\lambda w \lambda t\ [{}^0H_{wt} = {}^0P_{wt}]$ contains two occurrences of a Trivialization of two different individual offices. A Trivialization of Venus (or of any other particular celestial body) is a component of neither construction. Thus, in direct reference parlance, these two propositional constructions are no 'singular propositions', in contrast to the status direct reference theory would bestow upon its ⟨Hesperus, =, Phosphorus⟩.[29]

Bealer does not attempt to answer his own question in 2004. However, we may get enough of an impression of the tack of his answer from his (1993, 1998). We will briefly sketch it to compare Bealer's intensional logic with Tichý's, as far as the analysis of 'Hesperus is Phosphorus' is concerned.

Part of Bealer's grand-scale project of establishing a logic that is both first-order and hyperintensional is to devise a semantics, according to which

> [P]roper names do not have Fregean senses, and predicates do not have Fregean references or Millian denotations. Nevertheless, a sentence like 'Cicero is a person' does have a meaning not shared with 'Tully is a person' and 'Tully is a person' has a meaning not shared with 'Cicero is a person'. (1993, p. 43.)

Bealer wishes to accommodate both the alleged intuition that 'Cicero is a person' means something different from 'Tully is a person' as well as Kripke's claim that 'Cicero = Tully' expresses a proposition that is simultaneously ('metaphysically') necessary and knowable a posteriori only. To this end Bealer introduces what he dubs 'non-Platonic modes of presentation', which encompass what he calls 'intentional naming trees', 'causal naming chains' (1993, pp. 35ff), 'living names' (1998, pp. 16ff) and 'conventional naming practices' (1993, p. 36). For instance, two such practices P, P' may present the same individual, Cicero/Tully, but do so in two different ways: $P_{\text{'Cicero'}} \neq P'_{\text{'Tully'}}$.

Bealer discusses very briefly the Hesperus/Phosphorus case in (1993, p. 45, 1998, pp. 28–29). We are to imagine that the inception of the non-Platonic mode of presentation $P_{\text{'Phosphorus'}}$ comes after the inception of $P'_{\text{'Hesperus'}}$. The solution is that

> [T]he relevant non-Platonic modes of presentation are different: the … newly instituted practice is different from [the] standing practice $P_{\text{'Hesperus'}}$ … Accordingly, descriptive predications involving the new non-Platonic mode of presentation … result in propositions that are different from those which result from descriptive predications

[29] We are here following the direct reference practice of encoding what this theory considers 'structured propositions' as ordered n-tuples.

> involving instead [the] standing non-Platonic mode of presentation [$P_{\text{‘Hesperus’}}$]. (1993, p. 45.)

Bealer and Tichý both make use of the resources of intensional logic by descending from modes of presentation to extensions such as individuals rather than ‘giving’ individuals straightaway. Thus, the fine-grained content of ‘Hesperus is Phosphorus’ is both in Bealer and Tichý to do with two different intensions converging in the same entity.

However, we have two objections. The first is this. We feel uneasy about the sort of modes of presentation Bealer invokes. To get off the ground, Bealer's proposal requires that the logical operation of *descriptive predication*, pred_d, may operate in part on historical chains of linguistic practice and the like to form (hyper-) propositions.[30] Bealer in effect ‘semanticizes’ his non-Platonic modes of presentation by making them denizens of the subdomain D_1 of the domain D of one of his intensional algebras. Otherwise he would not be in a position to claim that the solutions he offers to various puzzles are ‘purely semantical’ (1993, p. 43).[31] Although it is trivial that Cicero = Tully, it is (supposedly) not trivial that $P_{\text{‘Cicero’}}$, $P'_{\text{‘Tully’}}$ present the same individual. However, the sort of non-Platonic modes of presentation that Bealer invokes, such as naming chains and naming practices, undeniably belong to the pragmatics department of semiotics. This is to say that Bealer somewhat strains the notion of semantics so as to include also certain pragmatic entities.

The second objection is this. All the elements of D are to be thought of as *primitive*, irreducible items (cf. 1993, p. 25). Indeed, they must be, for otherwise the very project of erecting a (hyper-) intensional *first*-order logic would be a non-starter. But the philosophical price exacted is that we must possess a firm pre-theoretic grasp of those non-Platonic modes of presentation, among other. We are not sure our grasp is firm enough for us to understand in sufficient theoretical detail what, e.g., $P_{\text{‘Hesperus’}}$ might be. We would have much preferred an intra-theoretic explanation.[32]

We find that these two objections provide enough reason not to include non-Platonic modes of presentation into the domain D of a Bealer-style intensional algebra. If one is reluctant to include them, then Bealer's solution grinds to a halt, since pred_d needs them as arguments.

Bealer shares the conviction with direct reference theory that ‘Cicero’ and ‘Tully’, ‘Hesperus’ and ‘Phosphorus’ are mere labels of individuals. Hence, any differences between ‘Hesperus’ and ‘Phosphorus’, or ‘Cicero’ and ‘Tully’, must be located somewhere other than in the semantics of the reference relation to steer

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[30] If $x, y \in D_1$ then pred_d predicates x of y. See Bealer (1998, p. 14).

[31] Bealer's account of why his solutions are not metalinguistic ‘in any of the standard senses’ needs in our view to be made clearer in order to become part of his theory. At this point we need to content ourselves with ‘a type of proposition that is ‘metalinguistic without being metalinguistic’’ (1993, fn. 62).

[32] See also Section 2.4.2 for a similar objection to Bealer's theory of predication.

clear of triviality. This is the same tack as followed by direct reference theory, which also has two options. The difference between 'Hesperus' and 'Phosphorus' is either to do with these two words having different orthographic shapes or with their being guided by two different sets of pragmatic rules for their correct use. Bealer explores both avenues in (1993, pp. 35ff, 1998, pp. 16ff). A third option, which is neither based on syntax nor pragmatics but on semantics, is not available, namely that 'Hesperus' and 'Phosphorus' would denote two different entities to begin with. Bealer is, at the end of the day, closer to direct reference theory than to TIL, as far as the analysis of 'Hesperus is Phosphorus' goes. This is little wonder, after all, since Bealer explicitly has his mind set on a Russellian semantics, just as direct reference theory.[33] But then, while it is clear that Tichý's intensional logic is in a position to offer a purely semantic explanation of the non-triviality of 'Hesperus is Phosphorus', it is far from clear this holds for Bealer's.[34]

Our solution is neo-Fregean, insofar as it has recourse to individual offices and accounts for the non-triviality of '$a = b$' in terms of the identity of the respective extensions of a and b rather than in terms of identity between a and b construed as extensions. But our solution is only broadly neo-Fregean, because we eschew reference shift. Nonetheless, it would seem that Bealer is anticipating a position similar to ours when saying,

> The two propositions differ because those two senses differ (the concept of being $A \neq$ the concept of being B). So goes the Fregean solution to … Frege's puzzle (2004, p. 573).

Though not quite. We balk at labelling our solution 'Fregean', or even 'neo-Fregean', in any narrow sense for the simple reason that Frege's own solution to his famous 1892 puzzle is a half-solution at most, although this point tends to be overlooked. Here is why. If you know that Hesperus is a planet then you definitely do not know that the usual *sense* of 'Hesperus' is a planet. Yet this is exactly the upshot of Frege's shifting the reference from an individual to a sense without accompanying the shift with some means of bending the sense toward a celestial body within sentences whose reference is a thought (*Gedanke*) and not a truth-value. Frege's semantics provides the wrong sort of subject of predication in the subclauses that Frege considers.[35]

If making empirical expressions denote intensions instead of extensions is the first half of the solution, the other half is making the so denoted intensions descend to extensions. If H is the individual office of Hesperus, then H_{wt} is an individual,

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[33] See Bealer (1993, pp. 40ff).

[34] Interestingly, Bealer says that, 'I myself defended a form of direct reference theory…but have since abandoned it in favour of a *semantical* account.' A reflection of this change in orientation is that the proposition that Hesperus is Phosphorus counts as knowable a priori in 1993 and as knowable only a posteriori in 2004. (See Bealer 1982, pp. 161–66, for his pro-Russellian, anti-Fregean stance on 'ordinary proper names' at the time.) Yet the paper he cites as where he pursues his new, 'purely semantical' orientation is none other than his 1998.

[35] See Frege (1986b, pp. 51–4). Bealer touches upon the problem of mismatch between property and subject of predication in passing (1993, p. 34).

namely the celestial body that is the extension of H at $\langle w, t\rangle$. Notice, though, that while the first half concerns semantics—assigning a denotation to 'Hesperus'—the other half is a logical matter: identifying a logical operation that will take intensions to their extensions.[36] While 'Hesperus' invariably refers to the individual office H and exhausts its purpose by picking out its denotation, H may, or may not, be extensionalized. This was the difference between H_{wt} and H, respectively. Whether it does is a matter of whether the abstract entity H has, or has not, been subjected to extra-semantic, logical manipulation in the form of extensionalization. One of the essential purposes of a logically perspicuous notation is, therefore, to flag whether H occurs extensionalized or not. This way we steer clear of an ambiguous notation in which 'H' refers to an individual in one sort of context and to an individual office in another sort of context.

By way of summary, our analysis of 'Hesperus is Phosphorus' is:

$$\lambda w \lambda t\ [{}^0H_{wt} = {}^0P_{wt}],$$

where H, P are two different individual offices, which are extensionalized in order to pick up the individual (if any) that occupies the respective offices at a given $\langle w, t\rangle$ of evaluation. The Trivializations of the offices, namely ${}^0H$, ${}^0P$, are the respective senses of 'Hesperus' and 'Phosphorus'. Venus (or any other specific celestial body) is no part of the semantic analysis.[37]

On a polemic note, if you go along with the general drift of our analysis of 'Hesperus is Phosphorus', the answer to the direct reference theorist Jonathan Berg's rhetorically intended question, 'But does anybody ever explicitly mention notions?' (1999, p. 463) is straightforward: 'Everybody does it all the time!'

[36] A logical operation taking intensions to extensions should not be conflated with an empirical operation whereby an agent executes such a logical operation. The former operation qualifies as a *procedure*; the latter, as a *process*, which is the actual execution of a procedure by an agent relative to a world and a time. What is intended above is a *logical* operation.

[37] Tichý himself offers an alternative analysis of 'Hesperus is Phosphorus' in 1983. His analysis is in effect a two-dimensionalist one: the same sentence may *express* (or *denote*, in mature TIL parlance) one proposition and be *associated* with another. The expressed proposition is that Venus is self-identical; hence, a necessary and a priori one. The associated proposition is that 'Hesperus is Phosphorus' is a true sentence of English; hence, a contingent and a posteriori one. The analysis allows 'Hesperus', 'Phosphorus' to be two names of the same individual, which some may consider an asset of the analysis. However, the notion of associated proposition remains intuitive in 1983 and does not re-appear in later works, so it smacks of adhockery. Besides, the analysis is superfluous, since the one we present above is more in the spirit of TIL. All that is needed is the proposition *denoted* by 'Hesperus is Phosphorus', because 'Hesperus', 'Phosphorus' denote two different individual offices and not the same individual. (For comments on Tichý as a very early, 'very strong two-dimensionalist', see Soames, 2005 pp. 171ff.)

3.4 Pragmatically incomplete meanings

TIL is thoroughly anti-contextualistic, which may seem to be unrealistic at least when dealing with anaphoric terms or sentences containing indexicals. In this section we show that the analysis of sentences containing indexicals is, indeed, compatible with the anti-contextualism of TIL. Sentences containing anaphoric reference will be analysed in the same spirit in Section 3.5.

As far as indexicals are concerned, would it not be true to say that the *meaning* of sentences like 'The man over there is drinking beer', 'I am hungry', etc., depends on the context that is the situation of utterance in which the truth-conditions of such sentences are to be evaluated? No, it would not. It is an old truth, for sure, that the empirical *evaluation* of sentences in a given situation of utterance belongs to the realm of pragmatics rather than (logical) semantics. However, at the same time, as argued in this book, the meaning of a sentence should make it possible to evaluate the proposition denoted by the sentence in any state of affairs. This holds, *provided*, of course, that there is a proposition to evaluate in the first place. However, sentences containing indexicals like 'over there', 'I', 'he', etc., do not express a *closed* construction constructing a proposition susceptible to being evaluated in any state of affairs. This is just because indexicals are what might be called 'pragmatic gaps'. Sentences containing such 'gaps' are not pragmatically complete, in the sense that a value to be supplied to fill the gap by the state of affairs serving as point of evaluation has not been supplied. Thus from the logical point of view, these pragmatic gaps are to be paired off with free variables that do not construct an entity, but only *v*-construct one. Only after valuation has been supplied by a situation of utterance assigning values to these free variables is a proposition susceptible to evaluation obtained. For this reason we will assign an *open* construction with one or more free variables to sentences containing indexicals as their meaning, and we will say that such sentences have a *pragmatically incomplete meaning.*[38]

All semantic analyses undertaken so far in this book have been couched within *pure semantics*. We did not need to study events like *utterances* of expressions. A brief recapitulation of our pure semantics might be helpful now to make clear how pragmatically incomplete meanings fit into the bigger picture.

The meaning of an unambiguous expression E is a *construction* C expressed by E. If E is an empirical expression, then C *v*-constructs an α-intension, i.e., a function of type $\alpha_{\tau\omega}$ denoted by E.[39] In particular, if E is an empirical sentence S then the meaning of S is a construction C_P of a *proposition* P of type $o_{\tau\omega}$. The construction

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[38] Since 'incomplete' denotes a privative modifier (see Section 4.4), an incomplete meaning is not a meaning. However, by 'pragmatically incomplete meaning' we do mean a meaning, though one that is pragmatically incomplete.

[39] Recall that we reserve the term 'denote' for the a priori relation between E and an α-intension and the term 'refer' for the a posteriori relation between E and the α-value (if any) of the intension in the actual world at the present moment.

C_P is an instruction of how to evaluate the truth-conditions denoted by the sentence for any state of affairs $\langle w, t\rangle$. So C_P makes it in principle possible to determine the value of P (if any) at any $\langle w, t\rangle$ pair. Which truth-value, if any, the denoted proposition has in particular circumstances is not a matter of a priori logical investigation; rather it is a matter of a posteriori empirical investigation. Disclosing the expressed construction is a matter of logic, hence a priori. The process of *executing* the procedure at $\langle w, t\rangle$ is in turn a posteriori.

Frege's semantic schema was essentially modified in Section 1.1.1 by

(a) letting *constructions* play the role of Fregean *Sinn*;
(b) distinguishing between *denotation* and *reference* (in the case of empirical expressions);
(c) letting *intensions* (in the case of empirical expressions) play the role of *Bedeutung* (so that intensions are *denoted*).

Further, the Parmenides principle is a vital step towards finding for every meaningful expression its best literal analysis (see Section 2.1).

Yet natural language contains an important class of expressions where what is denoted is dependent on contexts of utterance. The members of this class are expressions that contain *indexicals*. These are mostly pronouns; for example, personal pronouns ('I', 'you', 'they', etc.), demonstratives ('this', 'those', etc.), possessive pronouns ('my', 'their', etc.), as well as some adverbs (e.g., 'here', 'there', 'now'[40]). Clearly, the construction expressed by an expression that contains indexicals cannot be evaluated if such an expression is simply written on the blackboard, say, without being wrapped within a context of use to provide determinate references. This goes to show that empirical expressions containing indexicals have a *pragmatically incomplete* meaning. We propose construing the meaning of an empirical expression containing indexicals as an *open construction*. An intension is, then, constructed only after a valuation of the free variable(s) has been provided by the situation of utterance. The valuation fills in the pragmatic gaps and thus closes the open construction so that it constructs an intension. (Or, properly speaking, the open construction is *replaced* by a closed construction.) Thus the *denotation* of an expression E_I containing indexicals *is* context-dependent, which, however, does not make the *meaning* of E_I context-dependent. The open construction is context-*in*variably assigned to E_I as its meaning, which complies with our anti-contextualistic stance.

For instance, the sentence

'He is a logician'

[40] In systems equipped with explicit temporalisation (such as TIL), 'now' does not have an indexical character. Instead 'now' denotes the identity function of type ($\tau\tau$) taking every instant of time to itself, since the time that is present at t is t itself.

has a pragmatically incomplete meaning and expresses, thus, the following open construction with a free variable *he* $\rightarrow_v \iota$; *Logician*/$(o\iota)_{\tau\omega}$:

$$\lambda w \lambda t\ [{}^0 Logician_{wt}\ he].$$

This Closure only *v-constructs* a proposition. It is not possible to evaluate the truth-condition of the sentence unless and until a value of the parameter *he* has been provided by a context.

The context can be one of two kinds: *a pragmatic context* (a situation of utterance[41]) or a *linguistic-discourse context*, which is the case of anaphora (see Section 3.5). If the sentence is uttered in a situation where a hearer succeeds, in whatever manner, in identifying the particular individual Charles, then the *pragmatic meaning* of the sentence *in that situation* is the closed construction

$$\lambda w \lambda t\ [{}^0 Logician_{wt}\ {}^0 Charles].$$

On the other hand, the sentence

'Charles is a logician'

has a complete meaning, and expresses the closed construction $\lambda w \lambda t\ [{}^0 Logician_{wt}\ {}^0 Charles]$ independently of contextual embedding.[42]

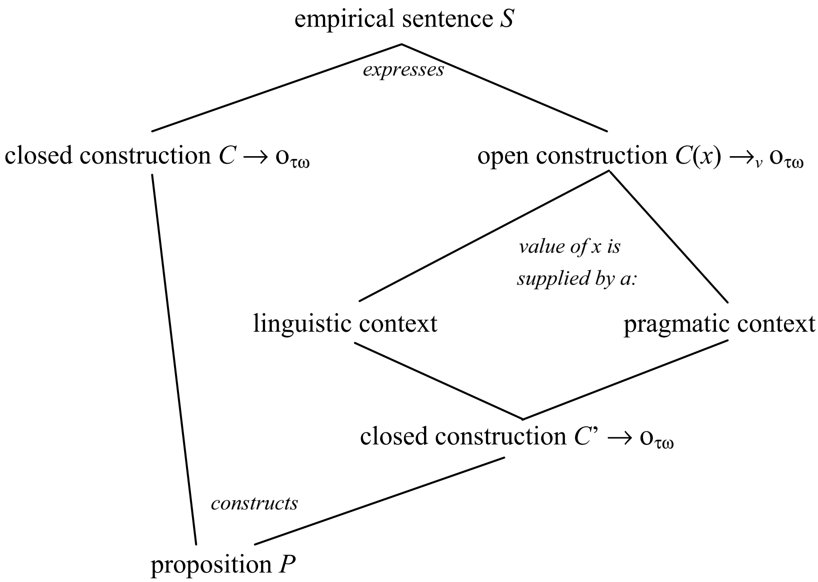
The two sentences are not equivalent; the meaning of the former is an open construction whereas the meaning of the latter is a closed one. In another situation of utterance or in another linguistic context, the variable *he* may *v*-construct another individual.[43] Thus the sentences are only *v*(*Charles*/*he*)-congruent.

The following Fig. 3.1 sums up our conception of the semantics and pragmatics of empirical sentences. By '$C(x)$' we denote an open construction with the free variable *x*. For the sake of simplicity, we consider only one free variable here. Remember that by '$C \rightarrow_v o_{\tau\omega}$' we mean that the construction *C* *v*-constructs a proposition, whereas by '$C \rightarrow o_{\tau\omega}$' we mean that *C* constructs a proposition independently of valuation.

[41] By 'pragmatic context' we mean only a situation of utterance. Hence we do not take into account, e.g., the interrogative, imperative, emotional and other intentions of a speaker, as well as other pragmatic aspects to do with the speaker's reasons for making a particular utterance. See Materna et al. (1976).

[42] 'Charles' is paired off with ^{0}*Charles*; see Section 3.2.

[43] For details, see Materna (1998, pp. 115–21).



Outside the scope of logic: empirical (a posteriori) evaluation of the proposition P at $\langle w, t \rangle$, resulting in True, False, or no value at all.

Fig. 3.1 The semantics and pragmatics of empirical sentences

3.4.1 Indexicals

To get the ball rolling, consider the sentence

(1) 'This hat is blue'.

Following our method of analysis informed by the Parmenides principle, we first assign types to the entities that the sentence talks about. The first attempt might be this one:

Types: *Blue*, *Hat*/$(o\iota)_{\tau\omega}$; *This_Hat* $\rightarrow_v \iota_{\tau\omega}$.

Such a type assignment comes down to the following coarse-grained schematic analysis:

$$\lambda w \lambda t\ [{}^0 Blue_{wt}\ This\_Hat_{wt}].$$

What speaks against this attempt is that we cannot write down '^{0}*This_Hat*', because the indexical term 'this hat' does not denote an individual office; it has a pragmatically incomplete meaning and so does not denote anything. (Remember that it is internal to a construction what it constructs.) There is no definite individual office to be Trivialized, and the construction *This_Hat*, whatever it may be, only *v*-constructs an individual office.

Above we explained that the meaning of (1) is an open construction containing the free variable *this*. Only what type of entity is *v*-constructed by this variable? The answer is that what is constructed is a property of individuals that should pragmatically complete the description of the hat in question. To make the situation clearer, let us rephrase the sentence as

'The only individual with this property and the property of being a hat is blue'.

This yields the additional types $Hat/(o\iota)_{\tau\omega}$; $this \rightarrow_v (o\iota)_{\tau\omega}$; $\iota/(\iota(o\iota))$: singularizer; $x \rightarrow \iota$. The individual office in question is *v*-constructed by

$$\lambda w \lambda t\ \iota x\ [[this_{wt}\ x] \wedge [{}^0Hat_{wt}\ x]].$$

Gloss: 'In any *w* at any *t*, pick up the only individual *x* that has at $\langle w, t\rangle$ *this* property and the property of being a hat.'

Thus the analysis of (1) is:

(1′) $\lambda w \lambda t\ [{}^0Blue_{wt}\ [\lambda w \lambda t\ \iota x\ [[this_{wt}\ x] \wedge [{}^0Hat_{wt}\ x]]]_{wt}]$,
or its β-reduced form,

(1″) $\lambda w \lambda t\ [{}^0Blue_{wt}\ \iota x\ [[this_{wt}\ x] \wedge [{}^0Hat_{wt}\ x]]]$.

Now imagine that (1) is uttered by our friend Charles in a situation where there is just one hat lying on the table in front of him, and Charles points at that hat. In such a situation one can agree or disagree with Charles, because the situation of utterance makes it possible to assign the property of lying on the table in front of Charles to the variable *this*. This assignment pragmatically completes the meaning of (1), and yields another sentence:

(1^P) 'The hat lying on the table in front of Charles is blue'.

Observe that (1) and (1^P) are not co-denoting, because they are not equivalent. In fact, they are not even co-referring. Since due to its pragmatically incomplete meaning (1) does not denote a proposition (unlike (1^P)), it cannot be said to refer to a truth-value at a given $\langle w, t\rangle$ of evaluation.

If $Lying\_on\_table/(o\iota)_{\tau\omega}$ is the property of lying on the table in front of Charles, then (1^P) expresses the closed construction

($1^{P\prime}$) $\lambda w \lambda t\ [{}^0Blue_{wt}\ \iota x\ [[{}^0Lying\_on\_table_{wt}\ x] \wedge [{}^0Hat_{wt}\ x]]]$.

Thus the situation *S* described above makes (1′) and ($1^{P\prime}$) *v*(*Lying_on_table*/*this*)-congruent.[44]

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[44] See Section 1.5.1, Definition 1.5.

Materna (1998, pp. 118–19) talks about the ‘pragmatic meaning’ and the ‘pragmatic denotation’ of an expression E_I containing indexicals *in a situation S.* Thus he would have said at the time that ($1^{P\prime}$) is the pragmatic meaning of (1) in the situation *S.* Accordingly, he would have said that (1′) and ($1^{P\prime}$) are co-referring in the situation *S.*

In this book, though, we are not going to adopt this terminology, nor are we including situations of utterance into our semantic theory. The latter would most probably amount to enriching the base of the TIL type hierarchy with an additional atomic type σ of situations. For instance, Montague (1974a, pp. 95–118) has among his *indices* not only possible worlds and times but also context-dependent indices for a speaker in a situation of utterance. The reason why we do not want to include the pragmatic meaning of a sentence *S* is that, strictly speaking, a pragmatic meaning is not a or the meaning of *S* at all. Rather, it is the meaning of another sentence. Thus we will not say that, for instance, ($1^{P\prime}$) is the pragmatic meaning of (1) in the situation *S*, because ($1^{P\prime}$) is *not* the meaning of (1), but of (1^{P}), and the two sentences are neither synonymous nor equivalent, for the reasons explained above. Instead we will say that ($1^{P\prime}$) is the pragmatic meaning *associated with* (1) *in the situation S.*

For comparison, one of the most elaborate theories of indexicals is David Kaplan’s, as set out in his 1978 and 1989. Kaplan’s conception shares some common ground with the functional approach of TIL. For example, what he calls *character* is a function from tuples of contextual parameters to contents, while what he calls *content* is a function from empirical parameters (‘circumstances of evaluation’) to individuals, truth-values or whatever the case may be. The general idea is that, e.g., a sentence containing indexicals will, relative to a context, express a particular proposition and that proposition may then be evaluated to obtain a truth-value. But a major difference between Kaplan and us is that the sort of proposition that an indexical-involving sentence picks out relative to a context is a so-called singular proposition, which counts among its constituents the individual referred to by the indexical. This makes one wonder whether the content of an indexical is an individual or an individual-in-intension (what TIL calls as an ‘individual office’ or ‘individual role’).[45] On the other hand, contents are supposed to be functions: contrast (1989, p. 523) with (ibid., p. 546). Where Γ is either a term or a formula, Kaplan writes ‘$\{\Gamma\}^{\mathsf{A}}{}_{cf}$’ for the content of Γ in the context *c* (under assignment *f* and in the structure A). The structure is an ordered *n*-tuple involving sets of contexts, $\mathcal{C}$, worlds, $\mathcal{W}$, individuals, $\mathcal{U}$, positions, $\mathcal{P}$, times, $\mathcal{T}$, as well as a function, $\mathcal{I}$, assigning intensions to predicates and functors. Hence, “If α is a term, $\{\alpha\}^{\mathsf{A}}{}_{cf}$ = that function which assigns to each $t \in \mathcal{T}$ and $w \in \mathcal{W}$, $|\alpha|_{cftw}$.” (Ibid., p. 546). More specifically,

[45] We are indebted to Marian Zouhar for alerting us to Kaplan’s apparent oscillation between function and functional value and for providing exact references.

Where Γ is either a term or a formula, *the Content of* Γ *in the context c (in the structure* A*) is Stable* iff for every assignment *f*, $\{\Gamma\}^{\mathsf{A}}_{cf}$ is a constant function (i.e., $\{\Gamma\}^{\mathsf{A}}_{cf}(t, w) = \{\Gamma\}^{\mathsf{A}}_{cf}(t', w')$, for all t, t', w, w' in A). (Ibid., p. 547.)

Here it is plain: contents are always functions, either constant or not. It would seem as though Kaplan is simply taking the liberty of identifying a constant function with its value, in order to uphold his theses that indexicals refer directly to individuals and that individuals are constituents of singular propositions. This oscillation, if that is what it is, between function and functional value would be symptomatic of the awkwardness of the combination of directly referring terms, singular propositions, and contents as functions.

It may also be illustrative to briefly compare our theory of indexicals to Castañeda's distinction between the speaker's *execution*, or *production*, of indexical reference and the hearer's *interpretation*, or *consumption*, of it.[46] In TIL, thanks to the pragmatic assignment of the proper value to the free occurrence of the pragmatic variable x in an open construction, the hearer is able to close the construction and obtain the propositional construction which is the meaning of the sentence as viewed from the speaker's perspective. The speaker intends to assert a proposition to be true when assertorically uttering, 'He is a logician', though leaving it to the hearer to assign to 'he' the referent that the speaker intended. The speaker must be able to spell out whom he or she intended by 'he', and there are various ways of doing so. Two would be to cite either a proper name of the individual or a definite description denoting an individual office that the individual occupies. Two other ways would be to either use a demonstrative (whether simply 'This one!' or 'That guy in the corner') or point at the individual. The reason for the requirement is that the speaker must, upon request, be able to display the closed construction that constructs the proposition which the speaker asserted to be true. In principle, any way of identifying an individual (whether a numerically specific individual or whatever individual satisfies some condition) goes, provided 'he' is matched by a construction (rather than a non-construction like a nod or other non-verbal, pragmatic vehicles of communication). Of course, if communication is to succeed, the speaker and the hearer need to be talking about the same individual. There are various routes leading up to the same individual, and speaker and hearer may well use different routes. However, nothing in the propositional construction that the hearer completes reflects the speaker's perspective, if x is replaced by a Trivialization of an individual (Trivialization being a non-perspectival mode of presentation; see Section 1.3). Perspectives are reinstalled, if a construction of an individual office is used. In case individual *a* co-occupies two different individual offices at $\langle w, t\rangle$ then speaker and hearer may identify *a* from these two different perspectives. Only this notion of perspective does not correspond to the notion of perspective that Castañeda operates with as regards indexicals. In TIL, since indexicals do not verbally or literally reflect perspectives, perspectives are not reflected semantically, either. This is a departure from Castañeda's perspectival,

[46] See Castañeda (1989) and Kapitan (2001, 2004).

dual-sense theory of indexicals. According to his theory, an indexical comes with a two-pronged sense, one prong being executive, the other interpretative. For instance, ‘I’ has an executive sense, which the speaker uses, and an interpretative one, which the hearer uses. Whatever the details of these two senses, the hearer uses the interpretative sense to track the individual who is the author of ‘I’ as (tokens of) ‘I’ occurs (occur) in ‘…I…’, while the speaker uses the executive sense for indexical self-identification.[47] Note, however, that unlike the hearer, the speaker has already fixed the references of ‘I’, ‘they’ ‘this’, ‘that’, etc., when ‘…I…’, ‘….they…’, etc., are uttered and does not interpret his or her own utterances. Interpreting one’s own utterances would be pretty much like putting the cart before the horse; for pieces of language are produced (by a speaker) before they are consumed (by a hearer), and producers are not consumers of their own products. At the same time, it may be helpful for speaker and hearer to ‘compare notes’ by realising how, on some particular occasion, the speaker identifies whatever he or she refers to by means of ‘that’, as in ‘That’s country-and-western music at its best!’ and how the hearer identifies it from his or her particular vantage point. As Tomis Kapitan writes,

> [F]ully successful communication with an indexical token requires both parties to utilize both meanings of the associated type, and it is this coordinated duality, in addition to the peculiar sorts of context-dependence, that distinguishes indexicals semantically (2001, p. 297).

This brief comparison is not intended to imply that TIL is eventually going to veer off into the general direction of Castañeda’s position. But an intensionalist theory such as Castañeda’s probably fits the edifice of TIL better than an extensionalist one like Kaplan’s. Going intensionalist in this manner would, in the parlance of TIL, amount to expressions containing indexicals assuming a dual meaning; one for the speaker, the other for the hearer. But such a reform would not be a straightforward undertaking. In particular, the notion of pragmatic meaning would have to be altered. For instance, the pragmatic meaning associated with ‘I am hungry’ when uttered by Albert Einstein is $\lambda w \lambda t\ [{}^0 Hungry_{wt}\ {}^0 Albert\_Einstein]$.[48] But ${}^0 Albert\_Einstein$ obliterates the differences that are bound to exist between

[47] Tichý analyses Castañeda’s (1968) example, ‘The Editor of *Soul* knows that he* is a millionaire’ in 1971, p. 290, 2004, p. 130. Tichý puts the difference between ‘…he*…’ and ‘…he…’ down to ‘The Editor of *Soul*’ occurring *de re* and *de dicto*, respectively. This analysis obviates the need for a first-person sense of ‘he*’. But it might be objected that the Editor of *Soul* does not identify himself as the Editor of *Soul* (cf. Castañeda, ibid., p. 441). There is a shift in perspective involved. On Tichý’s analysis the Editor must identify himself from a third-person perspective (along the lines of, ‘I am identical to whoever individual is the Editor of *Soul*’) to have any thoughts about himself. In Castañeda the Editor identifies himself via a first-person perspective. Tichý’s analysis gets the truth-condition, though arguably not the sense, of ‘…he*…’ right. Cf. Kapitan (1992, esp. p. 127).

[48] See Materna (1998, p. 120). Thus the pragmatic meaning associated in a given situation with an expression containing indexicals is a (TIL) concept arising from replacing free occurrences of variables.

how the hearer interprets this token of 'I' and *how* the speaker identifies himself. On the other hand, the substitution of ${}^{0}Albert\_Einstein$ for x in $\lambda w \lambda t\ [{}^{0}Hungry_{wt}\ x]$ fixes the denotation shared by speaker and hearer in successful communication.[49]

3.4.2 Indefinite descriptions

The problem of indefinite descriptions has been the subject of much dispute among philosophers and logicians just in connection with anaphoric reference. Neale characterizes indefinite descriptions as follows:

> The label 'incomplete description' is misleading. But we need to begin somewhere, so let us have some preliminary definitions. Let us say for the moment that a description is *proper* if, and only if, its nominal—or its superficial matrix in some standard system of representation—is true of exactly one thing, and *improper* otherwise. And let us say that an *improper* description is *empty* if it is true of nothing, and *incomplete* if it is true of more than one thing (2004, p. 32).

However, the condition of a description being 'proper', namely 'its nominal—or its superficial matrix in some standard system of representation—[being] true of exactly one thing' is not clear. From the point of view of TIL, there are two options. Either a description expresses *analytical uniqueness*, which means that in every state of affairs $\langle w, t\rangle$ there is *at most one* entity of which the description is true. Or a description can be *contingently true* of one entity at some $\langle w, t\rangle$, while at another $\langle w, t\rangle$ it is true of more entities (cf. Neale's *incomplete description*) or even none (cf. Neale's *empty description*). The former are *definite descriptions* that denote α-offices (for a type $\alpha \neq (o\beta)$ for any β); these were dealt with in Section 3.1. The latter are *indefinite descriptions* that denote $(o\alpha)$-properties; we are going to discuss them in this section.

Neale goes on to characterize in which sense a description can be *incomplete*:

> So what sorts of things have we really been attributing *incompleteness* to for the past sixty years? [R]emarks by Quine and Sellars … suggest we have been talking all along about incomplete *uses or utterances* of descriptions. Recall that they brought the suggestive word 'elliptical' into the debate in the course of sketching their own answers to the question the Russellian must answer. They talk of elliptical 'uses' (Quine) or elliptical 'utterances' (Sellars) of descriptions, and not of descriptions *per se* being elliptical. According to Sellars, an utterance of 'the table' will typically be elliptical for an utterance the speaker could have made of a richer description such as 'the table over here' or 'the table beside me'. The connection between ellipsis and incompleteness in Sellars's thinking manifests itself when he says (i) that 'in *ellipsis* the context completes the utterance and enables it to say something which it otherwise would not, different contexts enabling it to say different things,' (ii) that some 'utterances … are *not* complete and are only made complete by the context in which they are uttered,' and (iii) that 'statements which are non-elliptical … do not depend on their contexts for their *completion*'. Drawing

[49] Bjørn is indebted to Tomis Kapitan for discussion of Castañeda's theory (March 2007).

upon these early discussions, we might talk of incomplete 'utterances' of descriptions (Ibid., p. 36).

Here Neale talks about the difference that was introduced at the beginning of the previous section, namely the difference between a *pragmatically incomplete/complete meaning* (dependent/independent of a situation of utterance) and a *pragmatic meaning in a given situation of utterance*.

It ought to be obvious that the sentence 'The mountain is high' has a pragmatically incomplete meaning. It expresses an open construction with a free variable, as it does not express a complete instruction for evaluating truth-conditions in *any* empirical context $\langle w, t\rangle$. If the sentence is used out of context, one cannot evaluate its truth-condition, unless additional information is provided on *which mountain, among several other mountains*, is predicated to be high. If somebody asserts, out of context, that the mountain is high, the audience is entitled to an answer to the question '*Which* mountain is high?' Even if there happened to be just one mountain in the entire universe, the question would be legitimate, because the noun 'mountain' does not semantically reveal such a contingent uniqueness. This goes to show that in terms like 'the *F*', where '*F*' denotes a property of individuals rather than an individual office, the definite article 'the' functions as a demonstrative like 'this' in 'this *F*'. Roughly speaking, there are in principle (at least) two ways of using the definite article in English:

(a) The expression '*F*' of 'the *F*' denotes an office *F* of type $\alpha_{\tau\omega}$ (where $\alpha \neq (o\beta)$ for any type β). The description is analytically (hence, necessarily) *unique.* The value of the office *F* is *necessarily*, at *every* $\langle w, t\rangle$, *at most one object* of type α. Expressions like 'the Pope', 'the President of the USA', 'the highest mountain on earth' may serve as examples. In Slavic languages (such as Czech), which for the most part lack articles, this way of using 'the' does not correspond to any expression; instead the necessary uniqueness is determined by the *meaning* of '*F*'. Thus 'President České republiky' expresses a meaning determining uniqueness such that the Czech Republic can have at most one president at a time. Hence, the definite article is redundant. This is the case of *definite descriptions*.

(b) The expression '*F*' denotes a property *F* of type $(o\alpha)_{\tau\omega}$, which can *contingently at some* $\langle w, t\rangle$ pairs have a singleton as its value, while at other $\langle w, t\rangle$ pairs the value of *F* is of more than one element or the empty set. If there may be more than one *F* or none, the expression 'the *F*' has a pragmatically incomplete meaning. A sentence in which 'the *F*' is used does not denote a proposition; it has as yet no truth-condition to evaluate at *any* $\langle w, t\rangle$ pair unless an additional piece of information is provided that uniquely selects an α-object (an element of a many-valued population). In Slavic languages this way of using 'the' corresponds to using a definite pronoun (like 'ten', 'ta', 'to', in Czech). Hence the definite article is not redundant, as it signals the need for additional specification. This is the case of *indefinite descriptions*.

Sentences containing definite descriptions were analysed in Section 3.1. In this section we discuss indefinite descriptions. When analysing, for instance, the sentence

(2) 'The mountain is high'

we have got a case *ad* (b); i.e., the expression 'the mountain' serves as an indefinite description. Its meaning is thus an open construction with a free variable *the*, and the analysis of 'the mountain' obtains in the same way as the analysis of 'this hat' provided in Section 3.4.1. The sentence expresses the construction

(2′) $\lambda w \lambda t\ [{}^0High_{wt}\ \iota x\ [[the_{wt}\ x] \wedge [{}^0Mountain_{wt}\ x]]]$.

Types: $High$, $Mountain/(o\iota)_{\tau\omega}$; $x \rightarrow \iota$; $the \rightarrow (o\iota)_{\tau\omega}$; $\iota/(\iota(o\iota))$.

A valuation of the subsidiary parameter *the* must provide an additional property so that the set *v*-constructed by the construction $\lambda x\ [[the_{wt}\ x] \wedge [{}^0Mountain_{wt}\ x]]$ becomes a singleton.

When there is just one mountain on the skyline, (2′) is *v*(*Skyline*/*the*)-*congruent* with the construction ($Skyline/(o\iota)_{\tau\omega}$):

(2^P) $\lambda w \lambda t\ [{}^0High_{wt}\ \iota x\ [[{}^0Skyline_{wt}\ x] \wedge [{}^0Mountain_{wt}\ x]]]$

which is the *pragmatic meaning* associated with the sentence *in the described situation.* If the sentence occurs in a linguistic context, the article 'the' has an anaphoric character; it refers to the meaning of an antecedent expression that denotes a property.[50] Here we just outline the substitution method that serves to complete the meaning of an expression with anaphoric reference. The method was first encountered in Section 1.4.3. The meaning of the sentence

(3) 'There is just one mountain on the skyline and *the* mountain is high'

becomes

(3′) $\lambda w \lambda t\ [\exists y\ [[{}^0Mountain_{wt}\, y] \wedge [{}^0Skyline_{wt}\, y]] \wedge {}^2\mathbf{[}{}^0Sub\ {}^{\mathbf{00}}Skyline\ {}^0the$
$\mathbf{{}^0[}\lambda w \lambda t\ [{}^0High_{wt}\ \iota x\ [[{}^0Mountain_{wt}\ x] \wedge [the_{wt}\ x]]]\mathbf{]]}_{wt}]$.

The *Sub* function, here of type $(*_1 *_1 *_1 *_1)$, associates constructions C_1, C_2 and C_3 with the construction C which is the result of substituting C_1 for C_2 into C_3. Here the construction ${}^0Skyline$ is substituted for variable *the* into the Composition $\mathbf{[}\lambda w \lambda t\ [{}^0High_{wt}\ \iota y\ [[{}^0Mountain_{wt}\ y] \wedge [the_{wt}\ y]]]]$. As a result, the construction $\lambda w \lambda t$ $[{}^0High_{wt}\ \iota x\ [[{}^0Mountain_{wt}\ x] \wedge [{}^0Skyline_{wt}\ x]]]$ is returned, which must be executed in order to obtain a proposition; this explains the use of Double Execution. Finally, the so constructed proposition has to undergo intensional descent in order to yield a truth-value, which is the second argument of the conjunction.

[50] For anaphoric reference, see Section 3.5.

Note that (3′) is equivalent to (2^P), because the product of the Double Execution of the substitution is the proposition constructed by

$$\lambda w \lambda t\ [{}^0High_{wt}\ \iota x\ [[{}^0Mountain_{wt}\ x] \wedge [{}^0Skyline_{wt}\ x]]].$$

If this proposition has at a particular $\langle W, T\rangle$ pair a truth-value (**T** or **F**), then the class $v(W/w,T/t)$-constructed by $\lambda x\ [[{}^0Mountain_{wt}\, x] \wedge [{}^0Skyline_{wt}\, x]]$ is a singleton, which is a non-empty set. Thus the first conjunct of (3′) $v(W/w,T/t)$-constructs **T**. At those $\langle w, t\rangle$ pairs where the set of mountains on the skyline is not a singleton, the Composition $[{}^0Sing\ \lambda x\ [[{}^0Mountain_{wt}\ x] \wedge [{}^0Skyline_{wt}\ x]]]$ is v-improper and so is the entire conjunction of (3′). Thus the propositions constructed by (3′) and (2^P) are identical.

By contrast, the sentence

(4) ‘There is a mountain on the skyline *which* is high’

is not equivalent to (2) and (3). It is simply a case of anaphoric reference to a quantified variable:

(4′) $\lambda w \lambda t\ \exists x\ [[{}^0Mountain_{wt} x] \wedge [{}^0Skyline_{wt} x] \wedge {}^2\mathbf{[}{}^0Sub\ {}^\mathbf{0}x\ {}^0which$
$\mathbf{{}^0[}\lambda w \lambda t\ [{}^0High_{wt}\ which]\mathbf{]]}_{wt}]$

which is equivalent to

$$\lambda w \lambda t\ \exists x\ [[{}^0Mountain_{wt} x] \wedge [{}^0Skyline_{wt} x] \wedge [{}^0\mathrm{High}_{wt}\ x]].$$

Additional type: *which* $\rightarrow_v \iota$.

The proposition constructed by (4′) is false at those $\langle w, t\rangle$ pairs where there are no mountains on the skyline, and at those $\langle w, t\rangle$ pairs where there are some mountains on the skyline it is true or false, according as some of them are high.

An indefinite description can be combined with a pragmatic (indexical) variable, as is, for instance, the case in the sentence

(5) ‘*The boy* believes that *he* is immortal’.

The sentence has a pragmatically incomplete meaning due to the indefinite description ‘the boy’ that is assigned an open construction with the free variable *the*:

$$\lambda w \lambda t\ [{}^0Sing\ \lambda x\ [[the_{wt} x] \wedge [{}^0Boy_{wt} x]]],$$

or in abbreviated form:

$$\lambda w \lambda t\ \iota x\ [[the_{wt} x] \wedge [{}^0Boy_{wt} x]].$$

This construction must be substituted for the variable *he* into the meaning of ‘He is immortal’. Assuming that *Believe* is an intensional attitude, i.e. an attitude to a possible-world proposition, the analysis of the sentence comes down to this:

(5′) $\lambda w \lambda t\ [{}^0Believe_{wt}\ \iota x\ [[the_{wt}\,x] \wedge [{}^0Boy_{wt}\,x]]$
$\quad {}^2[{}^0Sub\ [{}^0Tr\ \iota x\ [[the_{wt}\,x] \wedge [{}^0Boy_{wt}\,x]]]\ {}^0he\ {}^0[\lambda w \lambda t\ [{}^0Immortal_{wt}\,he]]]]$.

Types: $Boy/(o\iota)_{\tau\omega}$; $Believe/(o\iota o_{\tau\omega})_{\tau\omega}$; $Immortal/(o\iota)_{\tau\omega}$; x, $he \to \iota$; $the \to (o\iota)_{\tau\omega}$.

Now if the construction $\iota x\ [[the_{wt}\,x] \wedge [{}^0Boy_{wt}\,x]]$ is v-improper, then the whole Composition $[{}^0Believe\ldots]$ is v-improper and the so v-constructed proposition is undefined. In another situation, if the construction $\iota x\ [[the_{wt}\,x] \wedge [{}^0Boy_{wt}\,x]]$ is v-proper, it v-constructs an individual. Let Charles be this individual. Then the function $Tr/(*_1\iota)$ takes Charles to his Trivialization, ${}^0Charles$. Finally, the *Sub* function applied to ${}^0Charles$, *he* and $[\lambda w \lambda t\ [{}^0Immortal_{wt}\ he]]$ returns $\lambda w \lambda t\ [{}^0Immortal_{wt}\ {}^0Charles]$, which is the *pragmatic* meaning associated with the embedded clause. This construction has to be executed in order to obtain the proposition to which Charles is related; hence Double Execution is called for. The *pragmatic* meaning associated with the whole sentence in this situation is then

$$\lambda w \lambda t\ [{}^0Believe_{wt}\ [{}^0Charles\ \lambda w \lambda t\ [{}^0Immortal_{wt}\ {}^0Charles].$$

If we analyzed the sentence as a hyperintensional attitude $Believe^*/(o\iota *_1)_{\tau\omega}$ to a propositional construction, we would simply omit the second step (after the substitution). Thus, we would not use Double Execution:

$$\lambda w \lambda t\ [{}^0Believe^*_{wt}\ \iota x\ [[the_{wt}\,x] \wedge [{}^0Boy_{wt}\,x]]$$
$$[{}^0Sub\ [{}^0Tr\ \iota x\ [[the_{wt}\,x] \wedge [{}^0Boy_{wt}\,x]]]\ {}^0he\ {}^0[\lambda w \lambda t\ [{}^0Immortal_{wt}\,he]]]].$$

Attitude sentences will be analysed in detail in Chapter 5, and sentences with anaphoric references in the next Section 3.5.

3.5 Anaphora and meaning

Here we take on sentences containing anaphoric reference to the meaning of an expression previously used in linguistic discourse (the antecedent of the anaphoric reference). In principle, there are two problems connected with the analysis of anaphoric sentences.

The first problem is how to combine the meaning of an antecedent with the meaning of the clause where the anaphorically referring pronoun is used. We encountered this problem in the previous Section 3.4 when analysing the sentences ‘There is a mountain on the skyline *which* is high’ and ‘The boy believes that *he* is immortal’. Our solution consisted in applying the substitution method. Thus in the analysis of the first sentence we substituted the existentially bound variable x for the free variable *which*, in order to predicate of some mountain x that it is high. Similarly, in the analysis of the second sentence we substituted the Composition

ιx [[$the_{wt}\,x$] $\wedge$ [$^0Boy_{wt}\,x$]], v-constructing a particular boy, for the variable *he* which is the meaning of the anaphoric pronoun 'he'.

The second problem is how to determine the antecedent of an anaphoric reference. The problem is a well-known hard nut of *linguistic* analysis, because the antecedent is often not unambiguously determinable. For instance, the sentence

'The boy and his daddy saw a dragon, and the boy thought that *he* was immortal'

is ambiguous. If the second clause stood alone, the anaphoric pronoun 'he' would unambiguously refer to the boy, but in this compound sentence it might refer to a dragon rather than the boy.

Thus it is often said that anaphora constitute a pragmatic problem rather than a problem of (logical) semantics. We agree that *logical* analysis cannot disambiguate the above sentence. Actually, logical analysis does not, and cannot, disambiguate any sentence in the sense of privileging one particular meaning. What a logical analysis does is enumerate all the unambiguous individual readings of an ambiguous sentence, or any other kind of expression. Our method of logical analysis can contribute to disambiguation in this manner: type-theoretical analysis of the entities that receive mention in the sentence and/or a specification of some of the requisites of these entities serve to unambiguously determine which of the possible meanings of a homonymous expression is used in a sentence.[51] Thus when analysing a sentence which is ambiguous by having *n* different meanings, we simply propose *n* different constructions as expressed by the sentence. As shown in Sections 1.5.2 and 2.6.2., one kind of logically interesting ambiguity feeds on the distinction between *de dicto* and *de re* readings.

Here it will be shown that the same kinds of disambiguation apply to sentences involving anaphoric reference. If the sentence is unambiguous, a type-theoretical analysis determines unambiguously the antecedent of the anaphoric reference, and we propose a method of *logically* analysing such a sentence. As outlined above, the method consists in substituting an appropriate construction for the anaphoric variable. Which construction is to be substituted is determined by the meaning of the antecedent and the type of the object which is the subject of predication in the embedded anaphoric clause. In other words, we perform a *semantic preprocessing* of the embedded anaphoric clause based on the *meaning* of the respective antecedent. In this sense anaphora *are* a *semantic* problem. For the sake of simplicity, we will presuppose that the antecedent is the first expression to the left of the anaphoric reference which denotes a type-theoretically appropriate object whose construction is to be substituted. Hence we will not address the pragmatic problem of disambiguation when the anaphoric reference is ambiguous. However, at the end of this section we outline how to *implement* our method in a way that takes into account the need to make other possible readings explicit as well.

[51] The notion of requisite has been introduced in Section 1.5.2. We will deal in details with requisites and the logic of intensions in Chapter 4, where requisites are defined in Section 4.1.

3.5.1 Semantic pre-processing of anaphora

As explained in Section 3.4, the sentence 'He is a logician' has a pragmatically incomplete meaning and so expresses the *open* construction $\lambda w \lambda t\ [{}^0Logician_{wt}\ he]$, where $Logician/(o\iota)_{\tau\omega}$; $he/*_1 \to \iota$. If the sentence is uttered in a situation where the speaker succeeds, in whatever manner, in identifying Charles, then the *pragmatic meaning* associated with the sentence *in this situation* is the construction $\lambda w \lambda t\ [{}^0Logician_{wt}\ {}^0Charles]$, which, though *v(Charles/he)*-congruent with the construction $\lambda w \lambda t\ [{}^0Logician_{wt}\ he]$, is not equivalent to the open construction. In another situation we may well obtain a different construction, because the variable *he* will *v*-construct another individual. Hence the *pragmatic meaning* associated with the sentence *in the given situation of utterance* is a closed construction, whereas *the meaning* of the sentence is the open construction.

If the sentence 'He is a logician' occurs in a *linguistic context*, does it also have an incomplete meaning? Since we advocate an anti-contextualist approach, the answer is Yes. The sentence has the same meaning in every context, which is to say that it expresses, always and in every context, one and the same *open* construction. However, when the sentence occurs in a linguistic context then we, as readers or hearers, are able to get to know only from the linguistic context what the anaphoric pronoun 'he' refers to. This is possible only if the whole sentence has a *complete* meaning. For instance, in the following sentence (1) the pronoun '*he*' refers to Charles:

(1) 'If Charles is rational, then *he* is a logician',

and to understand the sentence completely we do not need any situation of utterance. The sentence encodes a *complete procedure* for evaluating the truth-condition for any $\langle w, t\rangle$. Hence its meaning has to be a *closed construction* constructing a proposition without the mediation of pragmatic or empirical factors.

Note that the meaning of (1) is *not* construction (2′):

(2′) $\lambda w \lambda t\ [\lambda w \lambda t\ [{}^0Rational_{wt}\ {}^0Charles]_{wt} \supset \lambda w \lambda t\ [{}^0Logician_{wt}\ {}^0Charles]_{wt}]$

(or, β-reduced: $\lambda w \lambda t\ [[{}^0Rational_{wt}\ {}^0Charles] \supset [{}^0Logician_{wt}\ {}^0Charles]]$), because then (1) would be synonymous with

(2) 'If Charles is rational then *Charles* is a logician',

which it obviously is not.
Types: (Being) *Rational*, $Logician/(o\iota)_{\tau\omega}$; $Charles/\iota$.

The common objection to such a solution that the first occurrence of the name 'Charles' can denote a different individual than the second one can readily be set aside. The construction ${}^0Charles$ is a simple concept of the particular individual Charles, regardless of how, or whether, the individual is named. It constructs—in every context, without exception—one and the same individual.

But, there is a more serious objection. If (2′) were the meaning of (1), then the meaning of the embedded clause ‘he is a logician’ would in *this context* have to be the Closure $\lambda w \lambda t\ [{}^0Logician_{wt}\ {}^0Charles]$, rather than $\lambda w \lambda t\ [{}^0Logician_{wt}\ he]$. Otherwise we would have to give up the compositionality principle. In keeping with this principle, we hold that the meaning of (1) has to be derived in part from the meaning of ‘he is a logician’; and we have seen that this meaning is not the meaning of ‘Charles is a logician’.

It seems that we either have to give up the compositionality constraint or else the anti-contextualist transparency constraint. Yet our goal is to propose a solution that is in full accordance both with compositionality and transparency. Much is at stake. If no such solution is forthcoming, then TIL will turn out to be inapplicable to a key fragment of natural language.

A moment's reflection on the way we *understand* sentence (1) indicates where to look for a solution. Since the whole sentence has a complete meaning, a complete procedure for evaluating its truth-condition in *any* $\langle w, t\rangle$ is encoded. This means that as soon as we understand (1), we know that a *semantic pre-processing* of the anaphoric reference has been specified. The pre-processing must be specified *neither by a pragmatic factor nor* be performed at *the empirical level* of evaluation of reference a posteriori. The procedure of pre-processing the anaphoric reference must be specified at the *semantic level*, since the (sub-) procedure is a *constituent of the meaning* of the whole sentence. So as language-users we understand how an open Closure, $\lambda w \lambda t\ [\ldots he \ldots]$, is to be converted into a closed Closure, $\lambda w \lambda t\ [\ldots {}^0X \ldots]$, X the specific individual cited by the anaphoric pronoun ‘he’. The fact that we understand the sentence is evidence that also an open construction is a procedure. This fact, in turn, is further evidence that the concept of procedural semantics has much going for it.

In the present case the meaning of the antecedent ‘Charles’, i.e., the Trivialization ${}^0Charles$, is to be substituted for the variable *he*. This suggests to us that an anaphoric pronoun is a *semantic* abbreviation. Accordingly, the sentence encodes a two-phase procedure:

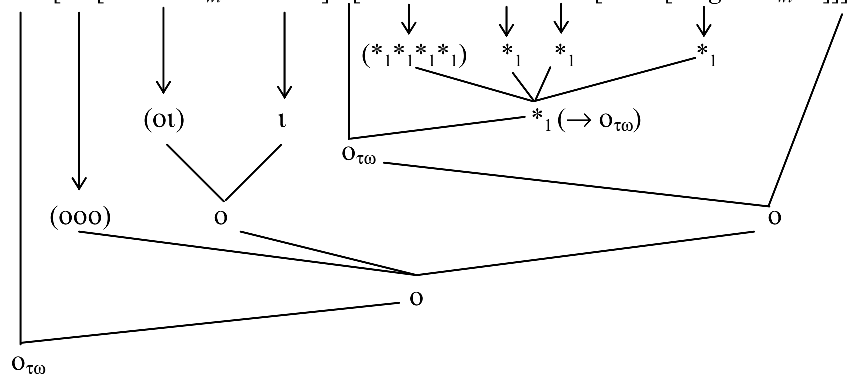
(i) pre-process the anaphoric reference by means of the meaning of the antecedent expression;
(ii) execute the adjusted meaning, which is the pre-processed construction.

To specify phase (i) we use the substitution function Sub_n introduced in Section 1.4.3. In the case of sentence (1) we have $n = 1$, hence $Sub_1/(*_1 *_1 *_1 *_1)$. The meaning of (1) is the Closure (1′):

$$(1') \qquad \lambda w \lambda t\ [[{}^0Rational_{wt}\ {}^0Charles] \supset {}^2[{}^0Sub\ {}^{00}Charles\ {}^0he\ {}^0[\lambda w \lambda t\ [{}^0Logician_{wt}\ he]]]_{wt}].$$

Since (1′) may appear rather complicated at first sight, we first run a type check (using prefix notation) and then show that (1′) is an adequate analysis meeting our three requirements of heeding compositionality, anti-contextualism and being a purely semantic solution.

$\lambda w \lambda t\, [{}^0\!\supset [{}^0Rational_{wt}\ {}^0Charles]\ {}^2[{}^0Sub\ {}^{00}Charles\ {}^0he\ {}^0[\lambda w \lambda t\, [{}^0Logician_{wt}\ he]]]_{wt}]$



The constituent (S) of (1′)[52]

(S) $[{}^0Sub\ {}^{00}Charles\ {}^0he\ {}^0[\lambda w' \lambda t'\, [{}^0Logician_{w't'}\ he]]] \rightarrow *_1$

constructs a *construction* of order 1, namely the one obtained by the substitution of ${}^0Charles$ for the variable *he* into the Closure $\lambda w' \lambda t'\, [{}^0Logician_{w't'}\ he]$. The result is the construction

(S') $\lambda w' \lambda t'\, [{}^0Logician_{w't'}\ {}^0Charles]$,

which constructs a proposition *P*. But an argument of the truth-function of implication ($\supset$) can be neither a propositional construction, nor a proposition, but must be a truth-value. Since (S) constructs the construction (S′), and (S′) constructs *P*, the execution steps have to be:

(a) execute (S) to obtain the propositional construction (S′),
(b) execute the result (S′) to obtain *P* (hence we need Double Execution of (S) to construct *P*),
(c) extensionalize *P* with respect to the external *w*, *t* in order to *v*-construct a truth-value:

$$[[{}^2[{}^0Sub\ {}^{00}Charles\ {}^0he\ {}^0[\lambda w' \lambda t'\, [{}^0Logician_{w't'}\ he]]]w]\ t] \rightarrow_v o.$$

This construction *v*-constructs the truth-value **T** at those $\langle w, t \rangle$, at which *Charles* is a logician, just as it should in accordance with the three requirements.

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[52] To make things clearer by displaying which of the occurring Closures λ-bind which variables, we α-renamed the *w*, *t* variables.

The meaning of a sentence containing a clause with an anaphoric reference is the procedure which is, in this case, a two-phase procedure, as specified by Double Execution.[53] The procedure comes down to this:

- first, execute the substitution based on the meaning of the antecedent for the anaphoric variable;
- second, execute the result (a propositional construction) again to obtain a proposition.

If $=_o/(ooo)$ is the identity of truth-values, then for any valuation v of variables w, t it holds that

$$^2[^0Sub\ ^{00}Charles\ ^0he\ ^0[\lambda w'\lambda t'\ [^0Logician_{w't'}\ he]]]_{wt} =_o$$
$$\lambda w'\lambda t'\ [^0Logician_{w't'}\ ^0Charles]_{wt} =_o [^0Logician_{wt}\ ^0Charles].$$

Hence constructions (1′) and (2′) are equivalent, yet the *meaning* of the sentence (1) is *not* the construction (2′), but (1′). In Section 3.5.2 we will show that it is not always possible to equivalently transform the meaning of an anaphoric sentence into the construction obtained after executing the substitution, because its execution may depend on a particular valuation *v*. This is another good reason for assigning a construction with an explicit specification of substitution to an anaphoric sentence as its meaning. Thus we have a unique method for analyzing sentences containing occurrences of anaphoric reference.

However, at this point the analysis might be objected to. We said that the way how we analyse expressions is in accordance with the Parmenides principle.[54] An adequate analysis of an expression *E* contains only constructions of those objects that receive mention in *E*. One may wonder, though, which subexpression of (1) expresses the instruction to perform the substitution (S).[55] Our answer is this. The sentence (1) is a semantic abbreviation, and its full, unpacked meaning expresses the semantic substitution. When unpacking the abbreviation, the sentence can be read as follows: ‘If Charles is rational then he (‘he’ referring to Charles) is a logician’.

At the beginning of this section we said that the type-theoretical analysis facilitates an assignment of a proper antecedent to the anaphorically referring term. This term expresses a variable for which the construction of a respective entity is substituted *via* the meaning of the antecedent expression. If the anaphoric variable *v*-constructs an α-entity, then the construction of an entity of the type α must be substituted. The type α can be any of the type hierarchy, even the type of a construction. However, until now we analysed only examples of substituting

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[53] In what follows we show that the second step (Double Execution) can be absent in an adequate analysis of an anaphoric sentence. This holds for cases where the meaning of the antecedent is *mentioned* in a hyperintensional context.

[54] For details, see Section 2.1.1.

[55] We are grateful to Jaroslav Peregrin for this remark.

individuals into an extensional context. Let us now analyze, by the method described above, some more examples of anaphoric reference as they occur in (A) a hyper-intensional context, (B) an intensional context, and (C) an extensional context. First up is (A):

(A) ‘5 + 7 = 12, and Charles knows *it*.’

The embedded clause ‘Charles knows it’ does not express Charles’ relation (-in-intension) to the truth-value **T**, but to the *procedure* of calculating the result of 5 + 7 = 12. Hence the pronoun ‘*it*’ refers anaphorically to the meaning of ‘5 + 7 = 12’, and *knowing* is here a relation-in-intension between an individual and a construction, in this case the Composition $[[^0{+}\ ^05\ ^07] = {}^012]$. The meaning of (A) is thus the closed construction

(A′) $\lambda w \lambda t\ [[[^0{+}\ ^05\ ^07] = {}^012] \wedge$
$^2[^0Sub\ ^{00}[[^0{+}\ ^05\ ^07] = {}^012]\ ^0it\ ^0[\lambda w \lambda t\ [^0Know^*_{wt}\ ^0Charles\ it]]]_{wt}]$

Types: $Know^*/(o\iota{*_1})_{\tau\omega}$; $Sub/(*_2*_2*_2*_2)$; $it/*_2 \rightarrow *_1$.

The meaning of the sentence ‘Charles knows it’ is the open construction

$$\lambda w \lambda t\ [^0Know^*_{wt}\ ^0Charles\ it].$$

The variable *it* is free here either for a pragmatic valuation (by the situation of utterance) or for a substitution of the meaning of the antecedent that is referred to in the linguistic context. The object—*what* is known by Charles—can be completed by a situation of utterance or by a linguistic context. If the sentence occurs within another linguistic context, then *Sub* substitutes a different construction for the variable *it*, namely the construction to which ‘*it*’ anaphorically refers. Next up is (B):

(B) ‘Charles sought the Mayor of Dunedin but (*he*) did not find *him*.’

Consider the *de dicto* reading of (B), which is that Charles’ search concerned the office of Mayor of Dunedin and not the location of its holder. Charles wanted to find out who the Mayor of Dunedin is, that is, who is the occupant of the individual office of Mayor. Thus *seeking* and *finding* are here relations-in-intension of an individual to an *individual office*, of type $(o\iota\iota_{\tau\omega})_{\tau\omega}$, and the context under scrutiny is an intensional one.[56]

The function *Sub* creates a new construction from constructions and so can easily be iterated. The *de dicto* analysis of (B) is:

[56] The attitudes of seeking and finding will be analyzed in details in Section 5.2.2, where other types of these attitudes will be examined as well. For the sake of simplicity, in this section we again disregard tenses. See, however, Section 2.5.2.

(B^d) $\lambda w \lambda t\, [[{}^0Seek_{wt}\, {}^0Ch\, \lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0D]] \wedge {}^2[{}^0Sub\, {}^{00}Ch\, {}^0he\, [{}^0Sub\, {}^0[\lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0D]]\, {}^0him\, {}^0[\lambda w \lambda t\, \neg[{}^0Find_{wt}\, he\, him]]]]_{wt}]$.

Types: *Seek*, *Find*/$(o\iota\iota_{\tau\omega})_{\tau\omega}$; *Ch*(*arles*)/$\iota$; *Mayor_of* (something)/$(\iota\iota)_{\tau\omega}$; *D*(unedin)/$\iota$; *he*/$*_1 \rightarrow \iota$; *him*/$*_1 \rightarrow \iota_{\tau\omega}$.

Again, the meaning of (B) is the closed construction (B^d), and the meaning of the embedded clause '*he* did not find *him*' is the open construction $\lambda w \lambda t\, \neg[{}^0Find_{wt}\, he\, him]$ with the two free variables *he* and *him*.

Of course, another refinement is thinkable. The variables *he* and *him*, ranging over individuals and individual offices, respectively, reduce the ambiguity of 'to find' by determining that here we are concerned with finding the occupant of an individual office. But the expressions 'he', 'him', or 'she', 'her', also indicate that the finder as well as the occupant of the sought office are male and female, respectively. Thus, e.g., a refined meaning of 'He found her' might be

$$\lambda w \lambda t\, [[{}^0Find_{wt}\, he\, her] \wedge [{}^0Male_{wt}\, he] \wedge [{}^0Female_{wt}\, her_{wt}]].$$

Additional types: *Male*, *Female*/$(o\iota)_{\tau\omega}$; *her*/$*_1 \rightarrow \iota_{\tau\omega}$.

The meaning of the *de dicto* reading of the sentence

'Charles sought the Mayor of Dunedin and he found her'

refined in the way just described is then

$$\lambda w \lambda t\, [[{}^0Seek_{wt}\, {}^0Ch\, \lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0D]] \wedge$$
$${}^2[{}^0Sub\, {}^{00}Ch\, {}^0he\, [{}^0Sub\, {}^0[\lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0D]]\, {}^0her\, {}^0[\lambda w \lambda t\, [[{}^0Find_{wt}\, he\, her] \wedge$$
$$[{}^0Male_{wt}\, he] \wedge [{}^0Female_{wt}\, her_{wt}]]]]]]_{wt}]$$

which is equivalent to

$$\lambda w \lambda t\, [[{}^0Seek_{wt}\, {}^0Ch\, \lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0D]] \wedge [{}^0Find_{wt}\, {}^0Ch\, \lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0D]]$$
$$\wedge [{}^0Male_{wt}\, {}^0Ch] \wedge [{}^0Female_{wt}\, [{}^0Mayor\_of_{wt}\, {}^0D]]].$$

Since such a refinement is obvious, we shall not make these additional specifications in the following analyses.

Now perhaps a more natural *de re* reading of the 'seeking clause' of (B) can be reformulated as

(B^r) 'Charles is looking for the Mayor of Dunedin (namely, his location)'.

This sentence is understood as uttered in a situation where Charles knows who the Mayor is, and is striving to locate this individual. Unlike the *de dicto* case, the sentence when understood *de re* comes with an *existential presupposition*: in order for (B^r) to have a truth-value, the Mayor must exist.[57] The object of Charles's search is now a μ-office, μ being the type of location/position.[58] The μ-office is *v*-constructed by $[\lambda w\lambda t\ [^0Loc\_of_{wt}\ him]]$. This time we must not substitute the *de re* occurrence of the construction $[^0Mayor\_of_{wt}\ ^0D]$. We would be drawing an extensional occurrence of $[^0Mayor\_of_{wt}\ ^0D]$ into the intensional context of $[\lambda w\lambda t\ [^0Loc\_of_{wt}\ him]]$, which is not a valid substitution.[59] Instead we must use the function *Tr* in order to substitute the Trivialization of the individual (if any) *v*-constructed by $[^0Mayor\_of_{wt}\ ^0D]$. The Composition $[^0Tr\ [^0Mayor\_of_{wt}\ ^0D]]$ fails to *v*-construct anything if $[^0Mayor\_of_{wt}\ ^0D]$ is *v*-improper (the Mayor failing to exist); otherwise it *v*-constructs the Trivialisation of the occupant of the office. By using the substitution technique, we can obtain the adequate analysis of (B^r):

$$\lambda w\lambda t\ [^0Look\_for_{wt}\ ^0Ch\ ^2[^0Sub\ [^0Tr\ [^0Mayor\_of_{wt}\ ^0D]]\ ^0him\ ^0[\lambda w\lambda t\ [^0Loc\_of_{wt}\ him]]]].$$

Additional types: $Look\_for/(o\iota\mu_{\tau\omega})_{\tau\omega}$; $Tr/(*_1\iota)$; $him\ /*_1 \rightarrow \iota$; $Loc\_of/(\mu\iota)_{\tau\omega}$.

When the clause '*He* did not find *him*' occurs in a different linguistic context, its meaning is the same. For instance, the *de dicto* reading of the sentence

(B_1) 'Whomever Charles is seeking, *he* is not finding *him*',

where *Seek* is again a relation to a ι-office, is analysed as

(B_1^d) $\lambda w\lambda t\ \forall z\ [[^0Seek_{wt}\ ^0Ch\ z] \supset\ ^2[^0Sub\ ^{00}Ch\ ^0he\ [^0Sub\ ^0z\ ^0him\ ^0[\lambda w\lambda t\ \neg[^0Find_{wt}\ he\ him]]]]_{wt}].$

Types: $z \rightarrow \iota_{\tau\omega}$; otherwise as above.

The construction (B_1^d) is again equivalent to the construction resulting from the substitution

$$\lambda w\lambda t\ \forall z\ [[^0Seek_{wt}\ ^0Ch\ z] \supset [\lambda w\lambda t\ \neg[^0Find_{wt}\ ^0Ch\ z]]_{wt}],$$

which is β-equivalent to

$$\lambda w\lambda t\ \forall z\ [[^0Seek_{wt}\ ^0Ch\ z] \supset \neg[^0Find_{wt}\ ^0Ch\ z]].$$

The meaning of (B_1) is, however, the construction (B_1^d), in which the semantic pre-processing of the anaphora is specified.

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[57] See Section 1.5.2, Definition 1.14.

[58] We do not specify this type. It can be, for instance, the GPS coordinates of an individual.

[59] The rules of valid substitution are found in Section 2.7.

An example of a relation-in-intension to an extensional entity is easily found and easily analyzed:

(C) 'Charles met the Mayor of Dunedin and *he* talked to *him*.'

Types: *Meet*, *Talk_to*/$(o\iota\iota)_{\tau\omega}$); *he, him* $\rightarrow \iota$.

The meaning of the embedded clause is again an open construction:

$$[\lambda w\lambda t\ [{}^0Talk\_to_{wt}\ he\ him]].$$

Now suppose that the sentence has been disambiguated into 'Charles met the Mayor of Dunedin and *he* (namely, Charles) talked to *him* (namely, the Mayor of Dunedin).' The substitution of the meaning of the first antecedent (${}^0Ch$) for the anaphoric variable *he* is not a problem. But, for the variable *him* we are to substitute the construction of that (unspecified) individual (if any) who is referred to by 'the Mayor of Dunedin' at $\langle w, t\rangle$. In other words, we need to substitute a construction of the individual (if any) *v*-constructed by the Composition $[{}^0Mayor\_of_{wt}\ {}^0D]$ into the construction $\lambda w\lambda t\ [{}^0Talk\_to_{wt}\ he\ him]$. There are two equivalent alternatives. The first alternative uses the function *Tr* to substitute the Trivialization of the individual *v*-constructed by the Composition $[{}^0Mayor\_of_{wt}\ {}^0D]$. The resulting analysis is

$$(C_1)\quad \lambda w\lambda t\ [[{}^0Meet_{wt}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]] \wedge$$
$${}^2[{}^0Sub\ {}^{00}Ch\ {}^0he\ [{}^0Sub\ [{}^0Tr\ [{}^0Mayor\_of_{wt}\ {}^0D]]\ {}^0him$$
$${}^0[\lambda w\lambda t\ [{}^0Talk\_to_{wt}\ he\ him]]]]_{wt}].$$

If there is no Mayor of Dunedin at a given $\langle w, t\rangle$, then $[{}^0Mayor\_of_{wt}\ {}^0D]$ is *v*-improper, and due to compositionality the whole Composition

$$(S_1)\quad [{}^0Sub\ {}^{00}Ch\ {}^0he\ [{}^0Sub\ [{}^0Tr\ [{}^0Mayor\_of_{wt}\ {}^0D]]\ {}^0him$$
$${}^0[\lambda w\lambda t\ [{}^0Talk\_to_{wt}\ he\ him]]]]$$

is *v*-improper. The so constructed proposition has a truth-value gap. Of course, if there is no Mayor of Dunedin then there is no Mayor of Dunedin to talk to, nor is there any Mayor of Dunedin not to talk to. On the other hand, if the Mayor is, e.g., Mr Taylor, then $[{}^0Tr\ [{}^0Mayor\_of_{wt}\ {}^0D]]$ *v*-constructs ${}^0Taylor$ and the result of the substitution is the construction $[\lambda w\lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ {}^0Taylor]]$. Yet, note that (C) does not mention Mr Taylor. The Closure $\lambda w\lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ {}^0Taylor]$ is not equivalent to (S_1), but only *v*-congruent.

The analysis (C_1) is thus equivalent to

$$(C')\quad \lambda w\lambda t\ [[{}^0Meet_{wt}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]] \wedge$$
$$[{}^0Talk\_to_{wt}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]]].$$

The second alternative analysis makes use of two things: (a) both ${}^0Meet_{wt}$ and ${}^0Talk_{wt}$ are constituents occurring extensionally in the Compositions $[{}^0Meet_{wt}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]]$, $[{}^0Talk\_to_{wt}\ he\ him]$, respectively; (b) the constructions $[{}^0Mayor\_of_{wt}\ {}^0D]$, *him* must be *v*-congruent in the analysis of (C). Thus we can substitute the former for the latter. In order to prevent collision of variables, we must rename the variables *w*, *t*.[60] The upshot is the analysis

(C_2) $$\lambda w\lambda t\ [[{}^0Meet_{wt}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]] \wedge {}^2[{}^0Sub\ {}^{00}Ch\ {}^0he\ [{}^0Sub\ {}^0[{}^0Mayor\_of_{wt}\ {}^0D]\ {}^0him\ {}^0[\lambda w'\lambda t'\ [{}^0Talk\_to_{w't'}\ he\ him]]]]_{wt}],$$

which is again equivalent to (C′).

Note that (C_1) differs from (C_2) only in using the function *Tr*, which is applied to the individual (if any) *v*-constructed by $[{}^0Mayor\_of_{wt}\ {}^0D]$. In (C_2) we substituted directly the Composition $[{}^0Mayor\_of_{wt}\ {}^0D]$ for *him* into the intensional context of the Closure $[\lambda w'\lambda t'\ [{}^0Talk\_to_{w't'}\ he\ him]]$. One may wonder whether such a substitution is correct. To be sure, while the Composition $[{}^0Mayor\_of_{wt}\ {}^0D]$ may be *v*-improper for some valuations *v* due to ${}^0Mayor\_of$ occurring *de re* in it, the above Closure is never *v*-improper. In Section 2.7 we warned against dragging a construction occurring *de re* into an intensional context on pain of ending up with a non-equivalent construction. Yet the two constructions just considered are equivalent. Here is why. The result of applying the *Sub* function twice is here the Closure

$$\lambda w'\lambda t'\ [{}^0Talk\_to_{w't'}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]].$$

Due to Double Execution, this Closure is *used* in (C_2) to *v*-construct a proposition. In the next step this proposition is subjected to intensional descent with respect to *w*, *t*, and the result is that the Closure occurs extensionally in (C_2). The Composition $\lambda w'\lambda t'\ [{}^0Talk\_to_{w't'}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]]_{wt}$ is equivalent to $[{}^0Talk\_to_{wt}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]]$. Thus the second conjunct of (C_2), namely

$${}^2[{}^0Sub\ {}^{00}Ch\ {}^0he\ [{}^0Sub\ {}^0[{}^0Mayor\_of_{wt}\ {}^0D]\ {}^0him\ {}^0[\lambda w'\lambda t'\ [{}^0Talk\_to_{w't'}\ he\ him]]]]_{wt}$$

is equivalent to

$$[{}^0Talk\_to_{wt}\ {}^0Ch\ [{}^0Mayor\_of_{wt}\ {}^0D]].$$

Hence both analyses are equivalent.

Anaphoric reference can occur not only in an embedded clause, but also be part of a sentence. Some further examples:

(D_1) ‘John loves *his* mother’

[60] See again the rules of valid substitution in Section 2.7.

(D_2) ‘Everybody loves *their* mother’.

The subexpression ‘loves his/their mother’ denotes a property $LM/(o\iota)_{\tau\omega}$, and a coarse-grained analysis of the above sentences is:

(D_1') $\lambda w\lambda t\ [{}^0LM_{wt}\ {}^0John]$

(D_2') $\lambda w\lambda t\ [\forall x\ [{}^0LM_{wt}\ x]]$.

Additional types: $John/\iota$; $x \rightarrow \iota$.

${}^0LM$ must be refined by Composing Trivializations of $Love/(o\iota\iota)_{\tau\omega}$ and $Mother\_of/(\iota\iota)_{\tau\omega}$; $y \rightarrow \iota$:

(LM) $\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\, y\ [{}^0Mother\_of_{wt}\, y]]$.

Replacing ${}^0LM$ by the construction (LM) we obtain these finer analyses of (D_1) and (D_2):

(D_1'') $\lambda w\lambda t\ [\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\, y\ [{}^0Mother\_of_{wt}\, y]]_{wt}\ {}^0John]$

(D_2'') $\lambda w\lambda t\ [\forall x\ [\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\, y\ [{}^0Mother\_of_{wt}\, y]]_{wt}\ x]]$.

And after β-reduction:

(D_{1red}'') $\lambda w\lambda t\ [\lambda y\ [{}^0Love_{wt}\, y\ [{}^0Mother\_of_{wt}\, y]]\ {}^0John]$

(D_{2red}'') $\lambda w\lambda t\ [\forall x\ [\lambda y\ [{}^0Love_{wt}\, y\ [{}^0Mother\_of_{wt}\, y]]\ x]]$.

Further β-reducing is valid, because ${}^0John$ and x are not v-improper for any v. However, in case of (D_{1red}'') the result

$$\lambda w\lambda t\ [{}^0Love_{wt}\ {}^0John\ [{}^0Mother\_of_{wt}\ {}^0John]]$$

will be an analysis of another sentence, in which the anaphoric reference is lost:

‘John loves John’s mother’.

Moreover, consider the following valid argument:

(D_3) ‘John loves his mother and *so does* Peter’;

hence

(D_4) 'John and Peter share a common property'.[61]

By means of this example we are going to show that the adequate analyses of the above sentences are the non-reduced constructions.

The antecedent of 'so does' in (D_3) is 'loves his mother', which denotes the property of individuals constructed by (LM). Thus the analysis of the embedded clause 'so does Peter' is an open construction with a free variable $so\_does/*_1 \rightarrow (o\iota)_{\tau\omega}$:

$$\lambda w\lambda t\ [so\_does_{wt}\ {}^0Peter].$$

To analyze (D_3), we must substitute the construction (LM) for the variable *so_does*:

(D_3') $\lambda w\lambda t\ [[[\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\ y\ [{}^0Mother\_of_{wt} y]]]_{wt}\ {}^0John] \wedge$
${}^2[{}^0Sub\ {}^0[\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\ y\ [{}^0Mother\_of_{wt} y]]]\ {}^0so\_does$
${}^0[\lambda w\lambda t\ [so\_does_{wt}\ {}^0Peter]]]_{wt}].$

The construction (D_3') is equivalent to the construction that emerges *after* semantic pre-processing:

(D_3'') $\lambda w\lambda t\ [[[\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\ y\ [{}^0Mother\_of_{wt} y]]]_{wt}\ {}^0John] \wedge$
$[\lambda w\lambda t\ [[\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\ y\ [{}^0Mother\_of_{wt} y]]]_{wt}\ {}^0Peter]]_{wt}] =_\beta$

(D_3''') $\lambda w\lambda t\ [[[\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\ y\ [{}^0Mother\_of_{wt} y]]]_{wt}\ {}^0John] \wedge$
$[[\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\ y\ [{}^0Mother\_of_{wt} y]]]_{wt}\ {}^0Peter]].$

The consequence that Peter and John share a common property,

$$\lambda w\lambda t\ [\exists p\ [[p_{wt}\ {}^0John] \wedge [p_{wt}\ {}^0Peter]]],$$

is now trivially derivable by existential generalisation. Obviously, the property *LM* constructed by $[\lambda w\lambda t\ \lambda y\ [{}^0Love_{wt}\ y\ [{}^0Mother\_of_{wt} y]]]$ is here the common property shared by John and Peter.

Note that if the reduced construction $\lambda w\lambda t\ [{}^0Love_{wt}\ {}^0John\ [{}^0Mother\_of_{wt}\ {}^0John]]$ were assigned to the first clause of (D_3) as its meaning, then there would be no construction of *LM* to be substituted for *so_does* into $\lambda w\lambda t\ [so\_does_{wt}\ {}^0Peter]$, and the consequence would not be directly derivable.

With the exception of (B^r), the above analyses containing the constituent ${}^0Sub$ were equivalent to the construction obtained *after* the execution of the substitution (and, as the case may be, after the execution of intensional descent). The meaning

[61] We include here only the 'strict reading' of (D_3), on which Peter loves his own mother, and exclude the 'sloppy reading', on which Peter loves John's mother. See Neale (2004, p. 63).

of the antecedent to which the anaphoric term refers has been (a) mentioned in a hyperintensional context, (b) used with *de dicto* supposition in an intensional context, or (c) used with *de re* supposition in an extensional context. The equivalence mentioned above is due to the fact that the respective substitutions are homogeneous. We inserted (a) a construction into a hyperintensional context, (b) an intension into an intensional context, or (c) an extension into an extensional context. According to the (constructional, intensional and extensional) rules introduced in Section 2.7.1, these substitutions are valid and thus result in equivalent constructions.

However, not all sentences containing anaphoric reference are this simple, as (B^r) illustrates. Problems may crop up when there is a need to substitute a construction of a lower-order entity into a higher-order context, namely of an extension into an intensional or hyperintensional context, or of an intension into a constructional context, because the higher-order context is dominant. This problem comes to the fore not least when analysing (propositional and notional) attitudes *de re* (see Chapter 5).

3.5.2 Donkey sentences

The following example is a variant of the well-known problem of Peter Geach's *donkey sentence*s:

(D) 'If somebody has got a new car then *he* often washes *it*.'

The analysis of the embedded clause '*he* often washes *it*', containing the anaphoric pronouns 'he' and 'it' is again an open construction with the two free variables *he* (who washes), *it* (what is washed), *he*, *it* $\rightarrow \iota$; *Wash*/$(o\iota\iota)_{\tau\omega}$:

$$\lambda w \lambda t\, [{}^0Wash_{wt}\ he\ it].$$

If we also want to analyze the frequency of washing, i.e., the meaning of 'often', then we use the function *Often*/$((o(o\tau))\tau)$.[62] The function *Often* associates each time *t* with a set of those time intervals (of type $(o(o\tau))$) that are frequent at *t* (for instance, once a week). The analysis of '*he* often washes *it*', is then

$$\lambda w \lambda t\, [{}^0Often_t\, \lambda t' [{}^0Wash_{wt'}\ he\ it]].$$

However, since rendering the frequency of washing does not influence how the problem of anaphora in donkey sentences is solved, we will use, for the sake of simplicity, the first construction.

[62] See Section 2.5, or Tichý (1986a, pp. 261–63) and Duží (2004).

The problem of donkey sentences consists first and foremost in discovering their *logical form*,[63] because it is far from clear how to understand them. Geach, (1962, p. 126), proposed a structure that can be rendered in 1st-order predicate logic as follows (*NC*, new car):

$$\forall x \forall y \, ((NC(y) \wedge Has(x, y)) \rightarrow Wash(x, y)).$$

However, Russell objected to this analysis that the expression 'a new car' is an *indefinite description*, something which does not come across in Geach's analysis. Hence Russell proposed an analysis that corresponds to this formula of 1st-order predicate logic:

$$\forall x \, (\exists y \, (NC(y) \wedge Has(x, y)) \rightarrow Wash(x, \mathbf{y})).$$

But the last occurrence of the variable y (marked in bold) is *free* in this formula and so out of the scope of the existential quantifier supposed to bind it.

Neale (1990) proposes a solution that combines both of the above proposals. On the one hand, the existential character of an indefinite description is saved (Russell's demand), and on the other hand, the anaphoric variable is bound by a general quantifier (Geach's solution). Neale introduces so-called *restricted quantifiers*[64]:

[every x: man x and [a y: new-car y](x owns y)]
([whe z: car z and x owns z] (x often washes z)).

The sentence (D) does not entail that if a man owns more than one new car then some of these cars are not washed by him. Hence we can reformulate the sentence into (D_1):

(D_1) 'Everybody who owns some new cars often washes *all of them* [each of the new cars he owns].'

However, the following sentence (D_2) obviously means something else:

(D_2) 'Everybody who owns some new cars often washes *some of them* [some, though not all, of the new cars he owns].'

The analysis of (D_1), which in principle corresponds to Geach's proposal, is

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[63] See Section 1.5.1, or Duží and Materna (2005).

[64] Neale (1990, p. 236). Neale takes it into account that the sentence is true even if a man owns *more than one* new car. To avoid singularity he thus claims that the description used in his analysis need not be singular (definite), but may be plural: his abbreviation 'whe *F*' stands for 'the *F* or the *F*s'.

(D_1') $\lambda w \lambda t\ \forall x \forall y\ [[[{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\, y]] \supset$
${}^2[{}^0Sub\ {}^0x\ {}^0he\ [{}^0Sub\ {}^0y\ {}^0it\ {}^0[\lambda w \lambda t\ [{}^0Wash_{wt}\ he\ it]]]]_{wt}]$,

which is β-equivalent to this construction after executing the substitution:

(D_1'') $\lambda w \lambda t\ \forall x \forall y\ [[[{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\, y]] \supset [{}^0Wash_{wt}\, x\, y]]$.

Types: $Own/(o\iota\iota)_{\tau\omega}$; $Wash/(o\iota\iota)_{\tau\omega}$; NC (being a new car)$/(o\iota)_{\tau\omega}$; $x, y, he, it \to \iota$; $\forall/(o(o\iota))$.

But then an objection due to Neale can be levelled against these analyses, namely that in the original sentence (D) the anaphoric pronoun ‘it’ stands *outside* the scope of the quantifier occurring in the antecedent. Moreover, Russell’s objection applies as well. To overcome these objections, we use a different type of quantifier. Apart from the common quantifiers $\forall, \exists/(o(o\iota))$ that operate on a set of individuals (returning **T** iff this set is the whole universe ($\forall$)/non-empty ($\exists$), respectively), we use quantifiers of another type, namely the restricted quantifiers *Some*, $All/((o(o\iota))(o\iota))$, which were introduced in Section 1.4.3.

To recapitulate, *Some* is the function that associates an argument (a set *S*) with the set of all those sets sharing a non-empty intersection with *S*. *All* is the function that associates an argument (a set *S*) with the set of all those sets containing *S* as a subset. For instance, the sentence ‘Some students are happy’ is analyzed as (*Student*, $Happy/(o\iota)_{\tau\omega}$)

$$\lambda w \lambda t\ [[{}^0Some\ {}^0Student_{wt}]\ {}^0Happy_{wt}].$$

Similarly, the sentence ‘All students are happy’ is analyzed as

$$\lambda w \lambda t\ [[{}^0All\ {}^0Student_{wt}]\ {}^0Happy_{wt}].$$

Back to the car washer. We first analyze the embedded clauses of (D_1), (D_2), namely:

(E_1): ‘*he* washes all of *them*’

(E_2): ‘*he* washes some of *them*’.

The anaphoric pronoun ‘*them*’ refers here to a *set of individuals*, *viz*. the set of new cars that a man owns. Thus we use the variable $them \to (o\iota)$ as the meaning of ‘them’. The analyses of (E_1), (E_2) are:

(E_1') $\lambda w \lambda t\ [[{}^0All\ them]\ \lambda it\, [\lambda w \lambda t\ [{}^0Wash_{wt}\ he\ it]]_{wt}]$,

(E_2') $\lambda w \lambda t\ [[{}^0Some\ them]\ \lambda it\, [\lambda w \lambda t\ [{}^0Wash_{wt}\ he\ it]]_{wt}]$

or, if β_i-reduced,

(E$_1''$) $\lambda w \lambda t\ [[{}^0 All\ them]\ \lambda it\ [{}^0 Wash_{wt}\ he\ it]]$

(E$_2''$) $\lambda w \lambda t\ [[{}^0 Some\ them]\ \lambda it\ [{}^0 Wash_{wt}\ he\ it]]$.

Now we need to substitute a construction of the set of new cars owned by the man for the variable *them*. Moreover, we have to substitute the variable x ('anybody') for the variable *he* ('*who* washes'), and then the pre-processed construction must undergo Double Execution. Finally, the so v-constructed proposition must undergo intensional descent to a truth-value in order to obtain the second argument for the connective $\supset$. To prevent collision of variables, we rename the internal variables w, t.

The analysis of (D$_1$):

(D$_1^A$) $\lambda w \lambda t\ [{}^0\forall \lambda x\ [[[{}^0 Man_{wt}\ x] \wedge [{}^0\exists \lambda y\ [[{}^0 NC_{wt}\ y] \wedge [{}^0 Own_{wt}\ x\ y]]]] \supset$
${}^2[{}^0 Sub\ {}^0[\lambda y\ [[{}^0 NC_{wt}\ y] \wedge [{}^0 Own_{wt}\ x\ y]]]\ {}^0 them\ [{}^0 Sub\ {}^0 x\ {}^0 he$
${}^0[\lambda w' \lambda t'\ [[{}^0 All\ them]\ \lambda it\ [{}^0 Wash_{w't'}\ he\ it]]]]]]_{wt}]]$.

Gloss: 'For every man x, if the man x owns some new cars then each of them [i.e., the new cars owned] is such that he [i.e., the man x] washes it.'

This construction can be viewed as the most adequate analysis of (D$_1$), because it meets Russell's requirement of an indefinite description in the antecedent, while the scope of $\exists$ does not exceed the antecedent. Now (D$_1^A$) is equivalent to the construction that would be obtained after pre-processing (i.e., execution of the respective substitutions):

(D$_1^{A'}$) $\lambda w \lambda t\ [{}^0\forall \lambda x\ [[[{}^0 Man_{wt}\ x] \wedge [{}^0\exists y\ [[{}^0 NC_{wt}\ y] \wedge [{}^0 Own_{wt}\ x\ y]]]] \supset$
$[[{}^0 All\ [\lambda y\ [[{}^0 NC_{wt}\ y] \wedge {}^0 Own_{wt}\ x\ y]]]]\ \lambda it\ [{}^0 Wash_{wt}\ x\ it]]]]$.

Gloss: 'For every man x, if the man x owns some new cars then each of these new cars is such that x washes it.'

The second possible reading of (D) is now analyzed in a similar way using *Some* instead of *All*:

(D$_2^A$) $\lambda w \lambda t\ [{}^0\forall \lambda x\ [[[{}^0 Man_{wt}\ x] \wedge [{}^0\exists \lambda y\ [[{}^0 NC_{wt}\ y] \wedge [{}^0 Own_{wt}\ x\ y]]]] \supset$
${}^2[{}^0 Sub\ {}^0[\lambda y\ [[{}^0 NC_{wt}\ y] \wedge [{}^0 Own_{wt}\ x\ y]]]\ {}^0 them\ [{}^0 Sub\ {}^0 x\ {}^0 he$
${}^0[\lambda w' \lambda t'\ [[{}^0 Some\ them]\ \lambda it\ [{}^0 Wash_{w't'}\ he\ it]]]]]]_{wt}]]$.

Gloss: 'For every man x, if the man owns some new cars then some of them [i.e., the new cars owned] is such that he [i.e., the man x] washes it.'

(D$_2^A$) is also equivalent to the construction that would be obtained after pre-processing:

$(\mathrm{D}_2{}^{A'})$ $\lambda w \lambda t\ [{}^0\forall \lambda x\ \big[[[{}^0Man_{wt}\ x] \wedge \big[{}^0\exists y\ [[{}^0NC_{wt}\ y] \wedge [{}^0Own_{wt}\ x\ y]]]] \supset$
$[[{}^0Some\ [\lambda y\ [[{}^0NC_{wt}\ y] \wedge [{}^0Own_{wt}\ x\ y]]]\big]\ \lambda it\ [{}^0Wash_{wt}\ x\ it]]]].$

As we pointed out above, it is not clear how to exactly understand (D). We thus offered several analyses to disambiguate it. Whether these readings are the only possible ones is not for us to decide. In our opinion the reading (D_1) is more plausible, and Neale considers only this one. However, our method makes it possible to easily analyse particular variants of donkey sentences like '… none of them …', '… most of them…', and suchlike.

It might be objected, however, that in the interest of disambiguation we actually analysed two variants of the original sentence (D). Therefore we are going now to supply a deeper analysis of (D). Gabriel Sandu (1997) formulates two principles that every compositional procedure for analysing natural language sentences should obey:

(a) there is a one-to-one mapping of the surface structure of a sentence of (a fragment of) English into its logical form which preserves the left-to-right ordering of the logical constants;
(b) the mapping preserves the nature of the lexical properties of the logical constants, in the sense that an indefinite is translated by an existential quantifier, etc.

Evidently, our analyses $(\mathrm{D}_1{}^A)$ and $(\mathrm{D}_2{}^A)$ obey these principles with respect to the glossed variants, but not with respect to the original sentence (D):

(D) 'If a man has got a new car then *he* often washes *it*.'

Regardless of the disambiguation concerning some/all new cars being washed, principle (b) is violated because 'a man' is analysed as 'every man'. In this respect the analyses $(\mathrm{D}_1{}^A)$, $(\mathrm{D}_2{}^A)$ deviate as much from the above principles as does an analysis couched in standard first-order logic:

$$\forall x \forall y\ ((Man(x) \wedge NC(y) \wedge Own(x, y)) \supset Wash(x, y))).$$

Whereas it is generally admitted that traditional first-order predicate logic is not a satisfactory tool for the analysis of natural-language sentences, dynamic predicate logic (DPL) is considered superior to other competing first-order theories of discourse semantics.

While referring for details to Kozen and Tiuryn (1990) and Sandu (1997), we briefly summarise how DPL analyses donkey sentences. From the syntactic point of view, DPL is a first-order predicate logic. The basic difference between the two concerns the semantics, in particular the scope of the existential quantifier and binding conventions. DPL is often characterised as a logic of programmes, for the interpretation of a DPL formula is a programme. Thus it might seem as though

DPL embedded a procedural semantics as found in TIL or Moschovakis. However, a DPL programme is understood as a set of pairs of assignments in a model *M*, where an assignment is a function from the set of variables to the universe of *M*. The model *M* is construed as the set of all the input/output pairs of the states of a computation. A formula is interpreted as a set of pairs of assignments; that is, as a programme. Therefore, the semantics of DPL is an enhanced version of the denotational semantics of modal logics, where the role of Kripke-like possible worlds is played by assignment functions. Roughly speaking, a pair $\langle i, j\rangle$ satisfies a formula φ if and only if the evaluation of φ with respect to the input state i results in the output state j.

Atomic formulae and formulae composed of negation, disjunction, implication or universal quantification are called 'tests'. When evaluated with the input assignment i, they only examine whether i satisfies the condition specified by the formula and, if so, do not change the assignment, and otherwise reject it. Existentially quantified formulas and conjunctions have a non-standard interpretation, since they pass on assignments of variables and their semantic bindings. The 'conjunction' of the programmes φ and ψ is not a commutative operation, but a sequence of programmes (i.e., something akin to progressive conjunction). Similarly, a formula '$(\varphi \wedge \psi)$' is interpreted as a sequence of programmes: φ, when evaluated on an initial assignment g, returns an output assignment h that serves as an input for ψ yielding an output assignment k. Similarly, a formula '$\exists xP(x)$' yields an output assignment $h(x)$ that may serve as an input assignment for a succeeding formula. Thus, as Sandu says (1997, p. 150), a formula '$\exists xP(x) \wedge Q(x)$' is interpreted, or rather 'computed', as follows:

$$||\exists xP(x) \wedge Q(x)|| = \{(g,h) \mid h[x]g \wedge h(x) \in F(P) \wedge h(x) \in F(Q)\},$$

where $h[x]g$ is an assignment which differs from the assignment g at most with respect to the value it assigns to x, while F is the interpretation function that assigns to the non-logical symbols of a formula the respective denotation in the model *M*.

The occurrence of x in the second conjunct '$Q(x)$' is thus 'syntactically free' and at the same time 'semantically bound'. The DPL approach to the problem of anaphora makes use of just this kind of non-standard binding. Thus the pair of sentences

'A man is walking. He whistles.'

receives in DPL the logical form

$$\exists x\ (Man(x) \wedge Walk(x)) \wedge Whistle(x).$$

The last occurrence of x, though syntactically free, is semantically bound by $\exists$. Similarly, the donkey sentence (D) has the DPL logical form

$$\exists x \exists y\ (Man(x) \wedge NC(y) \wedge Own(x, y)) \supset Wash(x, y).$$

Unfortunately, since non-standard binding applies only to DPL conjunction and existential quantification, this approach fails to generalize. In particular, it does not work for an anaphor whose antecedent contains functionally dependent quantifiers. Sandu (1997, p. 151) adduces the example

‘Every player chooses a pawn. *He* puts *it* on square one.’

The DPL logical form constructed as above would be

$$\forall x\ [Player(x) \supset \exists y\ [Pawn(y) \wedge Choose(x, y)]] \wedge Put(x, y, a).$$

But since the general quantifier does not pass on binding, the last occurrences of x and y are semantically free. Therefore, it is said that the DPL analysis has to be as the standard one in

$$\forall x\ [Player(x) \supset \exists y\ [Pawn(y) \wedge Choose(x, y) \wedge Put(x, y, a)]].$$

One pressing question is whether the anaphoric pronouns should be, in general, syntactically/semantically bound, and if so, another pressing question is whether this is to be in a standard or non-standard way. DPL does not provide an answer. But even if we put this fundamental question aside,

> The main question of anaphora is not, in our opinion, how to represent in the symbolism of some logic the anaphoric relation between a pronoun and its head, but to formulate general principles predicting when an anaphorical interpretation of a pronoun is possible and when it is not (Ibid., pp. 151–2).

Sandu further argues that

> One of the praised merits of DPL is that it preserves compositionality. In the game-theoretical tradition, compositionality is not a desired outcome. Hintikka (1991) has argued that to try to maintain compositionality is merely an attempt to enforce a paradigm which has already proved too narrow. The latest developments in GTS have led to a logic (i.e. the independence-friendly logic) which is non-compositional. The key idea on which this logic is based is the idea of informational independence, which *ipso facto* involves a violation of compositionality. For if a quantifier or a connective is independent of another, its interpretation depends on the latter one, which is located further out in the sentence in question, hence violating compositionality (Ibid., p. 152).

However, as we consider compositionality not only desirable but adamantly non-negotiable, we are not going to dispute the necessity of this principle. Suffice it to say that, of course, compositionality is ‘too narrow’ if we restrict ourselves to a first-order approach, or to an approach close to the first-order one, only slightly exceeding it, like GTS does. Our priorities are different, so we preserve compositionality by applying a higher-order logic.

In Section 3.5.1 we argued that anaphoric pronouns are bound by Trivialization and processed semantically by substitution based on the meaning of the antecedent. Thus our answer to Sandu’s questions is: if a pronoun is anaphoric then the substitution method can always be applied (as we illustrated by examples). To put our arguments on a still more solid ground, we now propose analyses of the sentences adduced by Sandu and Hintikka as examples where the compositional treatment allegedly fails.

First, as mentioned above, a literal compositional analysis of the sentence (D)

(D) 'If a man has got a new car then *he* (often) washes *it*' is called for. Here is how.

The analysis of the antecedent 'A man has a new car' is as follows:

(NC) $\lambda w \lambda t\ [{}^0\exists \lambda xy\ [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\ y]]]$.

Types: $\exists/(o(o\iota\iota))$; Man, $NC/(o\iota)_{\tau\omega}$; $Own/(o\iota\iota)_{\tau\omega}$.
Gloss: 'The set of couples <*man*(*x*), *new_car*(*y*)> such that *x owns y* is non-empty.'

The consequent '*he* washes *it*' expresses the open construction

$$\lambda w \lambda t\ [{}^0Wash_{wt}\ he\ it].$$

Types: $Wash/(o\iota\iota)_{\tau\omega}$; he, $it/*_1 \rightarrow \iota$.

The sentence (D) expresses that *if* the former is true, *then all* the pairs <*he*, *it*> which belong to the set of pairs mentioned by the former are such that *he washes it*. Using a variable $pairs/*_1 \rightarrow (o\iota\iota)$, and a quantifier $All^p/((o(o\iota\iota))(o\iota\iota))$, we have:

$$\lambda w \lambda t\ [[{}^0All^p\ pairs]\ \lambda he\ it\ [{}^0Wash_{wt}\ he\ it]].$$

The problem now consists in how to Compose the two constructions so as to construct the proposition denoted by (D). In order to obey the Parmenides principle, we must apply implication. To ensure that the pairs <*he*, *it*> belong to the respective set of *pairs* we need to apply the substitution method. Hence we substitute the construction of the set of pairs constructed by the Closure of (NC) for the variable *pairs*:

(D′) $\lambda w \lambda t\ [{}^0\exists \lambda xy\ [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\ y]] \supset$
${}^2[{}^0Sub\ {}^0[\lambda xy\ [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\ y]]]\ {}^0pairs$
${}^0[\lambda w \lambda t\ [{}^0All^p\ pairs]\ \lambda he\ it\ [{}^0Wash_{wt}\ he\ it]]_{wt}]]$.

The analysis can be simplified by removing the redundant η-expansion:

$$\lambda he\ it\ [{}^0Wash_{wt}\ he\ it] = Wash_{wt}$$

(D″) $\lambda w\lambda t\, [[{}^0\exists\lambda xy\, [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\, y]]] \supset$
$\quad {}^2[{}^0Sub\ {}^0[\lambda xy\, [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\, y]]]\ {}^0pairs$
$\quad {}^0[\lambda w\lambda t\, [{}^0All^p\ pairs]\ {}^0Wash_{wt}]_{wt}]]$.

To illustrate the adequacy of our analysis, imagine that at a given $\langle w, t\rangle$ there are five men, $M_1, \ldots, M_5$, and six cars, $C_1, \ldots, C_6$, related to each other as follows (h – has, w – washes):

	C_1	C_2	C_3	C_4	C_5	C_6
M_1		$h + w$		$h + w$		
M_2	$h + w$				h	
M_3			w			
M_4			$h + w$			
M_5						

For this $\langle w, t\rangle$ the Closure $\lambda xy\, [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\, y]]$ v-constructs the set H of pairs:

$$H = \{<M_1, C_2>,\ <M_1, C_4>,\ <M_2, C_1>, <M_2, C_5>, <M_4, C_3>\}.$$

The result of substitution, Double Execution and application of intensional descent in the consequent construction is equivalent to

$$[{}^0All\ \lambda xy\, [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\, y]]]\ \lambda he\ it\ [{}^0Wash_{wt}\ he\ it]]].$$

The constituent $[{}^0All\ \lambda xy\, [[{}^0Man_{wt}\, x] \wedge [{}^0NC_{wt}\, y] \wedge [{}^0Own_{wt}\, x\, y]]]$ v-constructs the set of sets of pairs containing H as a subset. Let it be H':

$$\begin{aligned} H' = \{&\{<M_1, C_2>,\ <M_1, C_4>,\ <M_2, C_1>, <M_2, C_5>, <M_4, C_3>\},\\ &\{<M_1, C_2>,\ <M_1, C_4>,\ <M_2, C_1>, <M_2, C_5>, <M_4, C_3>, <M_3, C_3>, <M_5, a>\},\\ &\{<M_1, C_2>,\ <M_1, C_4>,\ <M_2, C_1>, <M_2, C_5>, <M_4, C_3>, <M_3, b>, <M_5, a>\}, \ldots\}. \end{aligned}$$

The set v-constructed at $\langle w, t\rangle$ by $Wash_{wt}$ is the set of pairs $<he, it>$ such that *he washes it*:

$$Wash_{wt} = \{<M_1, C_2>,\ <M_1, C_4>,\ <M_2, C_1>, <M_3, C_3>, <M_4, C_3>\}.$$

Now the construction of the consequent v-constructs **T** if $Wash_{wt}$ is an element of H', which is not the case here. This is due to the fact that man M_2 has car C_5, but does not wash it. If he did, then the set $Wash_{wt}$ would be

$$\{<M_1, C_2>,\ <M_1, C_4>,\ <M_2, C_1>, <M_3, C_3>, <M_4, C_3>, <M_2, C_5>\}$$

and this set would be an element of H'.

As is seen, (D′) is fully compositional. Our constituents operate on constructions of sets of pairs of individuals, as well as on constructions of particular individuals, which is impossible within a first-order theory. In this respect Hintikka is right when claiming that the compositional treatment does not work; it does not work within a first-order framework. But as soon as we have a powerful higher-order system like TIL at our disposal, there is no need to give up compositionality.

Note that (D′) provides at the same time an explication of DPL's mechanism of passing on binding. As mentioned above, in DPL an existentially quantified formula yields an output assignment that may serve as an input assignment for a succeeding formula. Indeed, the antecedent of (D′), $^0\exists\lambda xy\ [[^0Man_{wt}\ x] \wedge [^0NC_{wt}\ y] \wedge [^0Own_{wt}\ x\ y]]$, yields an output assignment for the consequent: the set of pairs constructed by $\lambda xy\ [[^0Man_{wt}\ x] \wedge [^0NC_{wt}\ y] \wedge [^0Own_{wt}\ x\ y]]$ is substituted for the variable *pair* into $[^0All^p\ pairs]\ \lambda he\ it\ [^0Wash_{wt}\ he\ it]]$.

Thus the variable *pairs* is bound in (D′), but the binding is of another kind. It is not directly bound by the existential quantifier. Formally, the variable is bound by Trivialization; semantically, it is bound by the condition that the pairs of individuals it *v*-constructs must be those which belong to the set mentioned by the antecedent clause.

The other example in Sandu (1997) was

(P) 'Every player chooses a pawn. *He* puts *it* on square one.'

Obviously, 'he' and 'it' anaphorically refer to 'any player' and 'a pawn', respectively. However, in

(W) 'Every man walks. He whistles.'

the pronoun 'he' cannot be interpreted as anaphorically referring to 'every man'. Sandu's worries concern the lack of a universal method to determine when an anaphoric pronoun refers to an antecedent, and when not. The only answer we can give is that (P) is *understood* as being equivalent to

'Every player chooses a pawn and (*he*) puts *it* on square one',

unlike (W). The sentence

'Every man walks and whistles'

has obviously a different meaning than (W).

The respective analyses are:

(P′) $\lambda w\lambda t\ [[^0Every\ ^0Player_{wt}]$
$\lambda x\exists y\ [[^0Choose_{wt}\ x\ y] \wedge [^0Pawn_{wt}\ y] \wedge [^0Put_{wt}\ x\ y\ ^0Sq_1]]]$

(W′) first sentence: $\lambda w \lambda t\ [[{}^0\mathit{Every}\ {}^0\mathit{Man}_{wt}]\ {}^0\mathit{Walk}_{wt}]$;
second sentence: $\lambda w \lambda t\ [{}^0\mathit{Whistle}_{wt}\ \mathit{he}]$;
a pragmatically incomplete meaning.

Types: $\mathit{Every}/((o(o\iota))(o\iota))$ is a restricted quantifier; *Player*, *Man*, *Pawn*, *Walk*, $\mathit{Whistle}/(o\iota)_{\tau\omega}$; $\mathit{Choose}/(o\iota\iota)_{\tau\omega}$; $\mathit{Put}/(o\iota\iota\iota)$; Sq_1/ι; $\mathit{he}, x, y/{*_1}\rightarrow\iota$.

Note that in (P′) we do not need *Sub*. Yet an adequate analysis of (P) should heed the anaphoric status of the pronouns '*he*' and '*it*'. By applying the same method as above, we obtain an analysis involving *Sub*. First, the second sentence of (P) expresses the open construction ($\mathit{it}/{*_1}\rightarrow\iota$)

$$\lambda w \lambda t\ [{}^0\mathit{Put}_{wt}\ \mathit{he}\ \mathit{it}\ {}^0\mathit{Sq}_1].$$

The first sentence expresses

$$\lambda w \lambda t\ [[{}^0\mathit{Every}\ {}^0\mathit{Player}_{wt}]\ \lambda x \exists y\ [[{}^0\mathit{Choose}_{wt}\ x\ y] \wedge [{}^0\mathit{Pawn}_{wt}\ y]]].$$

The gloss is that the application of the restricted quantifier *Every* to the set of *Players* at a given $\langle w, t\rangle$ gives as a result the set $S/(o(o\iota))$ of supersets of Player_{wt}. Further, the application of S to the set v-constructed by $\lambda x\ \exists y\ [[{}^0\mathit{Choose}_{wt}\ x\ y] \wedge [{}^0\mathit{Pawn}_{wt}\ y]]$ returns **T** or **F**, according as the set of those who choose a pawn belongs to S.

Now, in order to analyze (P), x must be substituted for *he* and y for *it*:

(P″) $\lambda w \lambda t\ [[{}^0\mathit{Every}\ {}^0\mathit{Player}_{wt}]\ \lambda x \exists y\ [[{}^0\mathit{Choose}_{wt}\ x\ y] \wedge [{}^0\mathit{Pawn}_{wt}\ y] \wedge$
${}^2[{}^0\mathit{Sub}\ {}^0x\ {}^0\mathit{he}\ [{}^0\mathit{Sub}\ {}^0y\ {}^0\mathit{it}\ {}^0[\lambda w \lambda t\ [{}^0\mathit{Put}_{wt}\ \mathit{he}\ \mathit{it}\ {}^0\mathit{Sq}_1]]]]_{wt}]]$

The result of the Double Execution of the application of *Sub* is obtained as follows ($=/(ooo)$, the identity of truth-values):

$$ {}^2[{}^0\mathit{Sub}\ {}^0x\ {}^0\mathit{he}\ [{}^0\mathit{Sub}\ {}^0y\ {}^0\mathit{it}\ {}^0[\lambda w \lambda t\ [{}^0\mathit{Put}_{wt}\ \mathit{he}\ \mathit{it}\ {}^0\mathit{Sq}_1]]]]_{wt} = $$
$$ {}^2[{}^0[\lambda w \lambda t\ [{}^0\mathit{Put}_{wt}\ x\ y\ {}^0\mathit{Sq}_1]]]_{wt} = [\lambda w \lambda t\ [{}^0\mathit{Put}_{wt}\ x\ y\ {}^0\mathit{Sq}_1]]]_{wt} = [{}^0\mathit{Put}_{wt}\ x\ y\ {}^0\mathit{Sq}_1].$$

Thus the analysis (P′) is equivalent to (P″). The literal analysis of the disambiguated variant of the sentence (P) is (P″).

3.5.3 Dynamic discourse

In this section we outline a method for computing the complete meaning of anaphoric sentences. This is a method for implementing the substitution of an appropriate antecedent to accompany an anaphoric reference. Our method is similar to the one applied in general by Hans Kamp's Discourse Representation Theory

(DRT).[65] 'DRT' is an umbrella term for a collection of logical and computational linguistic methods developed for a dynamic interpretation of natural language, where each sentence is interpreted within a certain discourse, which is a sequence of sentences uttered by the same speaker. Interpretation conditions are given *via* instructions for updating the discourse representation. DPL, as described briefly above, is a logic belonging to this group of theories.[66] DRT as presented in Kamp (1981) addresses in particular the problem of anaphoric links crossing the sentence boundary. It is a first-order theory, and it is provable that the expressive power of the DRT language with negation is the same as that of first-order predicate logic.[67] Thus, actually only expressions denoting individuals (indefinite or definite noun phrases) can introduce so-called discourse referents, which are free variables that are updated when interpreting the discourse. Anaphoric pronouns are also represented by free variables linked to appropriate antecedent variables. There are various extensions of the basic theory which are now more or less assimilated to the existing formalism, in particular treatments of plurality and presupposition. For instance, the system of Brasoveanu (2007a, 2007b) deals with plural discourse reference to a quantificational dependency between sets of objects. The system is based on classical type logic that extends the compositional DRT of Muskens (1996). In principle, our approach to dynamic discourse representation is similar to that of Brasoveanu and Muskens.

Muskens proposes tackling explicit attitudes as attitudes to what he calls 'propositions', where a 'proposition' is a primitive entity individuated in a finer way than by co-entailment. Thus more 'propositions' can identify the same set of possible worlds. However, there is no hint of *what* kind of entity a 'proposition' is. Muskens draws upon Thomason's primitive type *p*, whose elements are hyperpropositions. Thomason (1980) defines the granularity of *p*-objects negatively (as being finer than logical equivalence), and he says nothing about the substance of *p*-objects.[68] Thus introducing hyperpropositions as primitives is to acknowledge the very need for entities with certain properties, but the theory is barred from saying much at all about them. TIL, unlike Thomason, has a substantial philosophical theory to tell in terms of hyperintensions as procedures, and this theory has, furthermore, been worked out in great technical detail in terms of TIL constructions. Moreover, it is obvious that co-entailment would be too crude a criterion for hyperpropositions, so we agree with Muskens on that point.

Since our semantics is procedural, hence hyperintensional and higher-order, not only individuals, but entities of any type, like properties of individuals, propositions, relations-in-intension, and even constructions (i.e., meanings of antecedent expressions), can be linked to anaphoric variables. Moreover, the thoroughgoing

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[65] For details, see Kamp (1981) and Kamp and Reyle (1993).

[66] See also Grenendijk and Stockhof (1991).

[67] See Eijck (2006, p. 666).

[68] See Jespersen (2010) for comments on Thomason's *p*.

typing of the universe of TIL makes it possible to *determine the respective type-theoretically appropriate antecedent*.

The specification of the implementation algorithm proposed here is imperative.[69] Similarly as in DRT, we update the list of potential antecedents, or rather the constructions expressed by them, in order to substitute the type-theoretically appropriate entities for anaphoric variables whenever needed. For each type, *viz.* ι, μ, $(o\iota)_{\tau\omega}$, $o_{\tau\omega}$, $(o\iota(o\iota)_{\tau\omega})_{\tau\omega}$, $(o\iota\iota)_{\tau\omega}$, $*_n$, etc., a list of *discourse referents* is formed. These discourse referents are free variables which serve a dual purpose. First, similarly as the variables of an imperative programming language, discourse referents function as memory cells to which a program stores objects in order to temporarily remember them. Thus each closed constituent of the meaning of a message becomes a temporal value of a type-theoretically appropriate discourse-referent variable. The method substitutes these values for anaphoric variables to complete the meanings of anaphoric clauses. Here our substitution method is applied so that discourse-referent variables serve their second purpose, *viz.* as ordinary constituents of the Composition $[{}^0Sub\ \ldots]$. The completed closed construction becomes in turn a new value of a discourse-referent variable of an appropriate type. In this way the discourse variables are gradually updated.

We now illustrate the method by a simple dialogue between three agents, *Adam*, *Berta* and *Cecil*. The agents communicate by exchanging messages of various kinds. Basic kinds are 'inform', 'query', 'reply' and 'order'. The *content* of a message is a sentence that is analysed using TIL and pre-processed by the substitution method. We use the sign ':=' to indicate the type of entities the constructions of which are being assigned to a discourse referent variable by the algorithm. From the logical point of view, these variables are of type $*_n$ and v-construct constructions of entities of the indicated type. For instance, the discourse-variable *ind* serves to keep track of individuals that receive mention in the dialogue. Thus we should write '$ind/*_n \rightarrow *_{n\text{-}1};\ {}^2ind \rightarrow \iota$'. Instead we write '$ind{:=}\iota$'. If the algorithm assigns to *ind* the Trivialization ${}^0Berta$, then *ind* v-constructs ${}^0Berta$, where $Berta/\iota$. The list of discourse-referent variables used in the dialogue is this:

- $ind{:=}\iota$, to keep track of individuals;
- $loc{:=}\mu$, to keep track of locations of the type μ;
- $pred{:=}(o\iota)_{\tau\omega}$, $prof{:=}(o\iota)_{\tau\omega}$, to keep track of individual properties; the former keeps track of properties denoted by simple predicates, the latter of properties denoted by complex predicates;
- $rel_1{:=}(o\iota(o\iota)_{\tau\omega})_{\tau\omega}$, to keep track of relations-in-intension of an individual to a property of individuals;
- $rel_2{:=}(o\iota\mu)_{\tau\omega}$, to keep track of relations-in-intension of an individual to a location;
- $rel_3{:=}(o\iota o_{\tau\omega})_{\tau\omega}$, to keep track of relations-in-intension of an individual to a proposition;
- $prop{:=}o_{\tau\omega}$, to keep track of propositions;
- $constr{:=}*_n$, to keep track of constructions.

[69] The algorithm was first proposed in Křetínský (2007).

Adam to Cecil: 'Berta is coming. ***She*** is looking for a parking space'.
'Inform' message content (first sentence):

$\lambda w \lambda t\ [{}^0Coming_{wt}\ {}^0Berta]$;

(Relevant) discourse variables updates:

$ind{:=}{}^0Berta;\ pred{:=}{}^0Coming;\ prop{:=}\ \lambda w \lambda t\ [{}^0Coming_{wt}\ {}^0Berta]$;

'Inform' message content (second sentence):

$\lambda w \lambda t\ {}^2[{}^0Sub\ ind\ {}^0she\ {}^0[{}^0Looking\_for_{wt}\ she\ {}^0Park\_Space]] \Rightarrow$
(is transformed into)
$\lambda w \lambda t\ [{}^0Looking\_for_{wt}\ {}^0Berta\ {}^0Park\_Space]$.

(Relevant) discourse variables updates:

$rel_1{:=}\ {}^0Looking\_for;\ pred{:=}{}^0Park\_Space;$
$prop{:=}\ \lambda w \lambda t\ [{}^0Looking\_for_{wt}\ {}^0Berta\ {}^0Park\_Space];$
$prof{:=}\ \lambda w \lambda t\ \lambda x\ [{}^0Looking\_for_{wt}\ x\ {}^0Park\_Space];$

Cecil to Adam: '***So*** am I.'
'Inform' message content:

$\lambda w \lambda t\ {}^2[{}^0Sub\ prof\ {}^0so\ {}^0[so_{wt}\ {}^0Cecil]] \Rightarrow$
$\lambda w \lambda t\ [{}^0Looking\_for_{wt}\ {}^0Cecil\ {}^0Park\_Space]$

(Relevant) discourse variables updates:

$ind{:=}{}^0Cecil;$

Adam to both: 'There is a car park with vacant slots at P_1'.
'Inform' message content:

$\lambda w \lambda t \exists x\ [[[{}^0Vac\ {}^0Car\_Park]_{wt}\ x] \wedge [{}^0At_{wt}\ x\ {}^0P_1]]$

(Relevant) discourse variables updates:

$loc{:=}{}^0P_1;\ pred{:=}[{}^0Vac\ {}^0Car\_Park];$
$prop{:=}\ \lambda w \lambda t\ [\exists x\ [[{}^0Vac\ {}^0Car\_Park]_{wt}\ x] \wedge [{}^0At_{wt}\ x\ {}^0P_1]]$

Cecil to Adam: 'I don't think ***so***. I have just been ***there***'.
'Inform' message content (first sentence):

$\lambda w \lambda t\ [{}^2[{}^0Sub\ prop\ {}^0so\ {}^0[\neg[{}^0Think_{wt}\ {}^0Cecil\ so]]] \Rightarrow$
$\lambda w \lambda t\ \neg[{}^0Think_{wt}\ {}^0Cecil$
$[\lambda w \lambda t\ [\exists x\ [[{}^0Vac\ {}^0Car\_Park]_{wt}\ x] \wedge [{}^0At_{wt}\ x\ {}^0P_1]]]]$,

'Inform' message content (second sentence):

$\lambda w \lambda t\ \exists t'[[t' \leq t] \wedge {}^2[{}^0Sub\ loc\ {}^0there\ {}^0[{}^0Been\_at_{wt'}\ {}^0Cecil\ there]]] \Rightarrow$
$\lambda w \lambda t\ \exists t'[[t' \leq t] \wedge [{}^0Been\_at_{wt'}\ {}^0Cecil\ {}^0P_1]]$.

Berta to Adam: 'What do you mean by 'car park with vacant slots'?'

'Query' message content:

$$\lambda w\lambda t\ [{}^0Unrecognized_{wt}\ {}^0[{}^0Vac\ {}^0Car\_Park]]$$

(Relevant) discourse variables updates:

$$constr := {}^0[{}^0Vac\ {}^0Car\_Park]$$

Adam to Berta: 'A car park with vacant slots is a parking lot some of whose parking spaces are not occupied'.

'Reply' message content:

$$[{}^0Refined_{wt}\ {}^0[{}^0Vac\ {}^0Car\_Park]] =$$
$${}^0[\lambda w\lambda t\ \lambda x\ [[{}^0Car\_Park_{wt}\,x] \wedge \exists y\ [[{}^0Part\_of_{wt}\ y\ x] \wedge \neg[{}^0Occupied_{wt}\,y]]]]$$

And so on.

Note that our hyperintensional procedural semantics makes it possible to easily specify and implement *agents' learning* by experience. The sort of agents we are considering here learn not only empirical facts, but also new *concepts*. They come equipped with a minimal ontology of primitive concepts and in the course of their life cycle enrich their ontology with new compound concepts. This is done in particular by messaging and consulting fellow agents. If an agent *a* does not have a concept *C* in his or her ontology, then *a* sends a query message announcing that the concept *C* has not been recognized by *a*. The appropriate reply provides a concept *C*' serving as an explication of *C*. To arrive at explications we use two functions that have concepts as arguments, *viz.* $Unrecognized/(o*_n)_{\tau\omega}$ and $Refined/(*_n*_n)_{\tau\omega}$. The former is a property of concepts (of not being known by an agent), the latter is a function that dependently on worlds and times returns a concept *C*' which is an explication of the argument concept. Thus we need to mention concepts as arguments and values, which means that the content of these messages must be hyperintensional.

In our example, upon receiving Adam's reply, Berta learns the refined meaning of the predicate 'is a car park with vacant slots', i.e., she updates her ontology by the respective compound construction defining the property of being a car park some of whose parking spaces are still vacant.

Moreover, our method makes it possible to work with *multi-lingual ontologies*. The content of an agent's knowledge is not a piece of syntax, but its meaning. And since a construction is what synonymous expressions (even of different languages) have in common, agents behave in the same way independently of the language in which their knowledge and ontology is encoded. For instance, if we throw some Czech in, the underlying constructions are *identical*:

$${}^0[{}^0Vac\ {}^0Car\_Park] = {}^0[{}^0Voln\acute{e}\ {}^0Parkovi\check{s}t\check{e}].$$

Of course, improvements of the above method are possible. For instance, in the above dialogue, for each type we kept track only of the *last* type-theoretically appropriate entity that had been mentioned. If we wanted to take into account possible ambiguities of the anaphoric references, we might store into the discourse-representation file a list of variables for each type, so as to be able to spell out more meanings of an ambiguous sentence, and thus to contribute further to its disambiguation.

3.6 Questions and answers

In the previous section we adduced an example of a dialogue in order to illustrate our implementation method of dynamic discourse representation and pre-processing of anaphora. In the dialogue, there was a 'query message' the content of which was analysed in the same way as the content of a corresponding 'inform message'. A question arises here, though. Is it plausible to analyse interrogative sentences in the same way as declarative ones? In this section we provide an answer.

There are many *logics of questions* (*interrogative* or *erotetic logics*).[70] The question, however, is whether it is necessary to build up specific systems in which to semantically analyze interrogative sentences and call each of them a *logic*. TIL answers this question in the negative.[71] Our principal tenet is that

> Logic investigates logical objects and ways they can be constructed. Its findings apply regardless of what people *do* with those objects: whether they exploit them in asserting, desiring, commanding, or questioning. (Tichý, 1978b, p. 278, 2004, p. 298.)

To motivate this stance, consider the declarative sentence

(1) 'Bill walks.'

and the corresponding interrogative sentence

(2) 'Does Bill walk?'

Tichý argues that the syntactic difference between these sentences 'reflects no difference in the *logic* of the two sentences' (ibid., p. 275/p. 295).[72] Instead the

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[70] See the overview in Harrah (2002).

[71] Here we confine ourselves to setting out our general approach to the semantics of interrogative sentences. Some consequences of this approach can be found in Materna (1981) and Materna, Hajičová and Sgall (1987). It might well prove fruitful to compare ours to the approaches offered by, e.g., Belnap et al. (See Harrah, 2002).

[72] He quotes the general claim advanced by Fitch that '[W]e do not need a special 'logic of imperative statements', 'logic of performative statements', and so on, as logic over and beyond, or basically different from the standard logic of propositions.' See Fitch (1971, p. 40).

difference between (1) and (2) is to do with the *pragmatic* use made of them. Thus, (1) is used to *assert* that Bill walks, while (2) is used to *ask* whether Bill walks.

Borrowing the terms 'concern' and 'topic' from Leonard (the former used for the pragmatic, the latter for the semantic aspect),[73] Tichý claims that (1) and (2) have the same topic but not the same concern. Logic is interested exclusively in topics. In the above example the topic is the proposition constructed by

$$\lambda w \lambda t\, [{}^0 Walk_{wt}\, {}^0 Bill].$$

Types: $Walk/(o\iota)_{\tau\omega}$; $Bill/\iota$.

Groenendijk and Stokhof (1994) advocate the thesis that interrogatives have a semantics of their own and so do not share their semantics with, e.g., indicatives. The difference in semantics is the one between truth-conditional content and so-called answerhood (cf. ibid., pp. 30ff). Their stance is at odds with the one expounded in Tichý (1978b), which they categorize as being a 'reductionist view' (ibid., p. 19). They level an objection against what they argue to be Tichý's position, which runs as follows:

> Consider 'John knows that Bill walks' and 'John knows whether Bill walks'. If the embedded interrogative and the embedded indicative really have the same semantic value, then each of these sentences should have the same value, too. If Bill walks, and John knows this, we might say that that is indeed the case: both are true. But if Bill does not walk, and John knows this, then they differ in value: in that case the first sentence is false, whereas the second is true. Such a simple example suffices to show that there are semantic differences between interrogatives and indicatives, and that the semantic content of interrogatives needs to be accounted for. (Ibid., p. 19.)

Tichý's position, however, is that no custom-built *semantics* for interrogatives, and no special erotetic *logic*, is needed. Propositions wrapped inside interrogatives, as in 'Is *P* true?', will suffice. So whether what John knows is that Bill walks or whether Bill walks, the semantic value embedded in the indicative or the interrogative is of the same *kind*, namely a (hyper-) proposition. It cannot be the same (hyper-) proposition, obviously, and their respective truth-values may well differ, but while Tichý's view does qualify as 'reductionist', it is not true to say that it requires that the complements be 'the same semantic value'.

To amplify the point, 'knowing that' and 'knowing whether' denote *different* relations-in-intension, and are for this reason assigned different (hyper-) propositions. Whereas 'knowing that *P*' not only implies, but also *presupposes* that *P* should be true, it is not so with 'knowing whether'. *Contra* Groenendijk and Stokhof, if Bill does not walk then the proposition denoted by 'John knows that Bill walks' has *no* truth-value. If the sentence is false then it is true that John does not know that Bill walks, which entails that Bill does walk. Thus if John knows that

[73] See Leonard (1959).

Bill does *not* walk then John neither knows that Bill walks, nor does John not know that Bill walks.[74]

First, we deal with sentences containing empirical expressions. We define:

Definition 3.1 (*topic of an interrogative empirical sentence)* The *topic* of an interrogative empirical sentence *S* is the intension denoted by *S*. □

Remark. The topic is an intension whose value in the actual world and present time the questioner would like to know. Thus what is usually called a 'question' is, properly speaking, just the topic of an interrogative sentence.

In the case of Yes/No interrogative sentences, the topic is a proposition. Any other kind of (essentially '*wh*-') interrogative sentence is connected with another kind of topic. In general, the topic is indicated by the type of the subject of an admissible answer.[75] If the type of the subject is α, then the topic of the interrogative sentence is an intension of type $\alpha_{\tau\omega}$. Here is a survey of several kinds of '*wh*-' interrogative sentences.

A. *Who is ...* ?

'Who is the father of the Pope?'

The syntactic means, namely the phrase 'Who is' and the question mark, do not possess any semantic significance. They are pragmatic indicators, instructing us what to do with the topic. In this case the topic is not a proposition. Since the type of the expected answer is ι, the topic is an *individual office* of type $\iota_{\tau\omega}$. The questioner wants to know the value of the intension denoted by

'The father of the Pope',

which is the individual office constructed by

$$\lambda w \lambda t\ [{}^0 Father\_of_{wt}\ {}^0 Pope_{wt}].$$

Types: $Father\_of/(\iota\iota)_{\tau\omega}$; $Pope/\iota_{\tau\omega}$.

We can use the phrase 'the father of the Pope' indicatively, e.g., when answering the question, 'Who is your favourite person?', or interrogatively, i.e., when wishing to know who occupies the office. Grammatical means then indicate particular kinds of use (full stop in the former case, 'who is' and question mark in the latter case).

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[74] See Section 5.1 for our analysis of 'knowing that' and 'knowing whether'.

[75] Groenendijk and Stokhof (1994) correctly classify Tichý (1978b) as a 'categorial' theory of questions: 'Tichý prefers to identify the category of an interrogative with that of its characteristic answers.' (Ibid., p. 54.)

B. *Which ... are ...* ?

'Which mountains are higher than Makalu?'

Since an admissible answer is a set of individuals, an $(o\iota)$-object, the topic is a *property of individuals*/$(o\iota)_{\tau\omega}$, namely *being a mountain higher than Makalu*, constructed by

$$\lambda w \lambda t\, \lambda x\, [[{}^0Mountain_{wt}\, x] \wedge [{}^0Higher\_than_{wt}\, x\, {}^0Makalu]].$$

Types: $Mountain/(o\iota)_{\tau\omega}$; $Higher\_than/(o\iota\iota)_{\tau\omega}$; $Makalu/\iota$.

C. ... *or*... ? (*Alternative questions*)

'Is Charles a composer or a dancer?'

Such sentences are ambiguous. They can be construed either as Yes/No questions or alternative questions.[76] If the former, then 'or' denotes an inclusive disjunction; we say 'Yes' if at least one of the alternatives holds and 'No' otherwise.

Now we are interested in the case of alternative questions, *viz.* the questions involving exclusive disjunction. Here the topic is a little bit more complex. The questioner wants to know *which* of the alternative propositions is the case. Since admissible answers are 'Charles is a composer' or 'Charles is a dancer', both denoting $o_{\tau\omega}$-objects, the topic is now a *propositional office*/$(o_{\tau\omega})_{\tau\omega}$, constructed by

$$\lambda w \lambda t\, \iota' p\, [p_{wt} \wedge [[p = \lambda w \lambda t\, [{}^0Composer_{wt}\, {}^0Ch]] \vee [p = \lambda w \lambda t\, [{}^0Dancer_{wt}\, {}^0Ch]]]]].$$

Types: $p/*_1 \rightarrow o_{\tau\omega}$; $=/(oo_{\tau\omega}o_{\tau\omega})$; $Ch(arles)/\iota$; $Composer$, $Dancer/(o\iota)_{\tau\omega}$; $\iota'/(o_{\tau\omega}(oo_{\tau\omega}))$.

The topic cannot be a proposition; for this would mean that when answering the question in a correct way we would be saying either 'Yes' or 'No', because to answer the question correctly means to determine the value of the topic in the actual world at the present time. Instead a correct answer will be one of the sentences 'Charles is a composer', 'Charles is a dancer'. 'Charles is both a composer and a dancer' is not an option, since the question is stipulated to be an *alternative* question requiring as an answer exactly one of the disjuncts and not their conjunction. Thus, the value of the topic in the actual world at the present time is a proposition. The sort of intension whose value at a world/time pair is a proposition is a propositional office/$(o_{\tau\omega})_{\tau\omega}$.

Other cases would be 'Why…?', 'How…?', 'When…', 'How long…?'. Similar cases can be reformulated so that the character of the topic is made clear; for example, 'What is the cause of ...?', 'What is the length of…?'.

Logical analysis of (empirical) interrogative sentences unveils constructions of the *topics* of the interrogative sentences. Since we are now dealing

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[76] The disambiguation can be realized on the phonetic level. The Yes/No case obtains if the pitch of the voice rises at the end and the alternative case if the pitch goes down.

with *empirical* questions, the interrogative sentences are empirical expressions and so it holds that

the topics of empirical interrogative sentences are non-constant intensions.[77]

Tichý introduced the terminological convention that the topic of an interrogative sentence is to be called a *question*, such that 'People's questions are ... propositions, individual offices, properties, and the like.' (Tichý 2004, p. 297.) Tichý concedes that this terminology may be objected to; for instance, we do not ask propositions but questions. But Tichý rebuts the objection by pointing out that one could then likewise insist that what one believes are *beliefs,* what one wishes are *wishes*, and what one conjectures are *conjectures,* not propositions. Yet what *a* believes to be the case, what *b* wishes to be the case, and what *c* conjectures to be the case may obviously be *one and the same* thing. So we propose the thesis that *questions are intensions* in the following particular sense: every intension may be used as the topic of an interrogative sentence. The notion of question is in this sense a *pragmatic* one.[78]

That the notion of interrogative sentence has to be distinguished from the notion of question is obvious. The interrogative sentences

'Who is the father of the Pope?'

and

'Wer ist der Vater des Papstes?'

are distinct, for sure, but the respective question is the same (viz. the topic constructed in our example above). Thus we can say that these two interrogative sentences share the same topic as well as the same meaning. To correctly translate an interrogative sentence *S* from one language *L* into another language *L′* means finding in *L′* an expression whose meaning constructs the same topic as does the meaning of *S* in *L* and to add, furthermore, the syntactic signals of the interrogative attitude in *L′*.[79]

A question can also be embedded in an indicative sentence. For instance, the above question can be embedded in

'Charles asked: Who is the father of the Pope?'

Then the topic of the question constructed by $\lambda w \lambda t\ [^0 Father\_of_{wt}\ {}^0 Pope_{wt}]$ is an argument of $Asked/(o\iota\iota_{\tau\omega})_{\tau\omega}$, relating an individual to an individual office, and the analysis comes down to:

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[77] Recall that empirical expressions denote non-constant intensions.

[78] See Materna et al. (1976, p. 177), '[T]his sphere of pragmatics, which deals with potential attitudes of potential language users and which directly manifests itself in the syntactic component of an ordered triple, is to be termed internal (pragmatic) indices. ... [I]n communicative situations, we have to introduce the notion of external pragmatics, characterizing the respective situation in which the given sentence has been uttered.'

[79] For instance, if *L* is English and *L*' is Spanish, then the correct translation of '…?' is '¿…?'.

$$\lambda w\lambda t\ [{}^0Asked_{wt}\ {}^0Charles\ \lambda w\lambda t\ [{}^0Father\_of_{wt}\ {}^0Pope_{wt}]].$$

Interrogative sentences, just like other kinds of expressions of natural language, can have a pragmatically incomplete meaning. Then the analysis is an open construction *v*-constructing the topic. In case of indexicals their analysis contains—as is the case with other indexicals (see Section 3.4.1—free variables whose valuation is given by a situation of utterance. The answer then depends on the same situation. For example, the interrogative sentence 'Who is this man?' depends for its correct answer on some one particular man being singled out as the one about whom the speaker wishes to know who he is. The correct answer is going to be his name or a definite description identifying him as the occupant of an individual office. In the case of anaphoric reference to a discourse, the meaning is completed by substitution based on the meaning of the antecedent phrase, as described in Section 3.5.3. For instance, the two atomic sentences of the discourse,

'The richest man in the world came to Prague on Monday.'

'Where does *he* come from?'

express the following constructions:

$$\lambda w\lambda t\ [\lambda x\ [{}^0Past_t\ \lambda c\ \exists t'\ [[ct'] \wedge [{}^0Come\_to_{wt'}\ [x\ {}^0Prague]]]\ {}^0Monday]\ {}^0RichestMan_{wt}]$$

and

$$\lambda w\lambda t\ \iota x\ {}^2[{}^0Sub\ {}^{00}RichestMan_{wt}\ {}^0he\ {}^0[{}^0Come\_from_{wt}\ he\ x]].$$

Types: $Past/((o(o(o\tau))(o\tau))\tau)$; $c/*_1\rightarrow(o\tau)$; $Come\_to/(o\iota\iota)_{\tau\omega}$; $Prague/\iota$; $Monday/(o\tau)$; $RichestMan/\iota_{\tau\omega}$; $he/*_1\rightarrow\iota$; $Come\_from/(o\iota\iota)_{\tau\omega}$.[80]

As outlined above, questions and answers are type-theoretically interlocked, namely in the following fashion: interrogative sentences denote questions, while true answers cite the values of these questions at the given $\langle w, t\rangle$ of evaluation. If no particular such pair is mentioned, it is assumed that the intended pair is the actual world and the present moment.

Definition 3.2 (*complete answer*) Let S be an interrogative sentence whose topic—and so the respective question Q—is of type $\alpha_{\tau\omega}$. Then a *complete answer* to the question Q is an expression that cites an object of type α. □

Remark. We use the neutral verb 'to cite' on purpose. A complete answer has to guide us to the value of the respective intension in the actual world at the present moment. But in general we cannot say in which manner it will do so. With the

[80] For an analysis of the simple past tense, see Section 2.5.2.

exception of Yes/No questions, where we can *denote* the respective object ('Yes', 'No' being names of **T**, **F**, respectively), we have no means to do so. In general, the object that is the actual value of an intension is not accessible as the *denotation* of an expression (with the possible exception of proper names). Empirical expressions denote intensions, never their actual values, so we often use indexicals and rely upon pragmatic factors for identification. For example, a complete answer to the question

'Who is the head of al-Qaida?'

has to cite an individual; it can do so by saying

'Osama bin Laden'

or

'This one'.

In the latter case we rely upon the given situation to unequivocally fix an individual.

If the cited α-object is the value of the respective intension in the actual world at the present time, then we say that the answer is *right*, otherwise *wrong*. Some particular cases will justify this notion of complete answer.

The type of Yes/No questions is $o_{\tau\omega}$. Hence the type of the object cited by an answer is o. To cite such an object is tantamount to saying 'Yes' or 'No'.

If the type of the question is $\iota_{\tau\omega}$, then the type of the object cited by an answer is ι. Citing an individual qualifies as an answer to the question. If the type of the question is $(o\iota)_{\tau\omega}$, then the type of the object cited by an answer must be $(o\iota)$. Thus citing a class of individuals (in the above example *ad* (B), the class of mountains that are actually higher than Makalu) counts as an answer to the question. A different class of mountains would also be an answer, only not the right one. The type of alternative questions is $(o_{\tau\omega})_{\tau\omega}$. To answer such questions is to cite a proposition.

The notion of *incomplete answer* is easily derivable from the notion of complete answer. Let the type of the object cited by a complete answer be α. Then an incomplete answer will offer (in some way or other) a *class* of α-objects (different from a singleton). The answer is right only if this class contains the object cited by the right complete answer. Notice, however, that an incomplete answer to a Yes/No question is uninformative. Such a question must be answered by 'Yes' or 'No'.

Examples. First, let a complete (right or wrong) answer to the question 'Who is Charles's father?' be 'Abraham'. An incomplete answer will be, e.g., 'Balthazar or Abraham'. If the complete answer was right then the incomplete answer would be right as well, for the cited class contains Abraham. Second, if the question is a property of individuals, then offering more than one class of individuals amounts to offering an incomplete answer. Thus the following schematic answers must be distinguished. Let the class cited by the *right complete* answer to the question Q be {A, B, C, D}. Citing the class {A, B, C, D, E} is a *wrong complete* answer to Q. Offering the classes {A, B, C, D} *or* {A, B, C} *or* {A, B, C, D, E} is a *right* *incomplete* answer to *Q*. Offering {A, B, C} *or* {A, B, C, D, E} is a *wrong incomplete* answer to *Q*.[81]

The term ‘question’ is mostly used just in the sense of empirical question. Questions in logic and mathematics are oftentimes *examinatorial* questions and rather more like imperatives—‘Prove Fermat’s last theorem!’, ‘Define the De Morgan laws!’—and they are answered correctly if the ordered task is fulfilled.[82] The examiner is not trying to get to know what the De Morgan’s laws (etc.) are, for these he or she already knows. Instead the examiner wants to get to know whether the examined student knows them. So the examiner might ask the non-examinatorial question, ‘Do you know the De Morgan Laws?’

A non-examinatorial question concerning mathematical/logical objects is a *construction* (rather than an intension) which is an analysis of the respective interrogative sentence divested, as the case may be, of interrogative phrases like ‘which is’, ‘which are’, etc. The correct answer denotes the object (if any) constructed by this construction. So, for example, the question asked by means of the interrogative sentence ‘Which are the roots of the equation $8x^2 + 8x + 2 = 0$?’ is the construction

$$\lambda x\ [{}^0{=}\ [{}^0\mathit{Add}\ [{}^0\mathit{Add}\ [{}^0\mathit{Mult}\ {}^08\ [{}^0\mathit{Power\_of}\ x]]\ [{}^0\mathit{Mult}\ {}^08\ x]]\ {}^02]\ {}^00].$$

Types: *Add*, *Mult*/(τττ): the functions of adding and multiplying, respectively; *Power_of*/(ττ); 0,2,8/τ; =/(οττ); $x \rightarrow \tau$.

The correct answer is the singleton {– ½}.

[81] To *react* to a question does not automatically mean to *answer* the question. Reactions which do not satisfy the definitions of complete/incomplete answers may be called *replies* or *responses*. For instance, punishing silence or a ‘I am not going to dignify that question with an answer’ would be replies and not answers. In everyday transaction we may well succeed in converting a reply into an answer, given a sufficient supply of background information and suchlike. This is still not to say that, e.g., silence on behalf of the one who was asked the question qualifies as an answer; the audience must still make explicit to themselves what the answer implied by the silence is, if indeed there is an answer to be teased out.

[82] See also Materna (1981).

4
Requisites: the logic of intensions

In Section 2.4.1 we argued in favour of semantic anti-actualism: the actual of all the possible worlds should play no semantic role. In this and the following sections we outline an essentialism that likewise accords no privileged status to the actual world by making the notion of essence independent of world and time and a priori instead.[1] At the same time we are arguing in favour of ontological actualism: all the individuals at the actual world are all the individuals there are at all the other possible worlds as well (hence, there are no merely possible individuals, or *possibilia*).

Our essentialism is based on the idea that since no purely contingent intension can be essential of any individual, essences are borne by intensions rather than by individuals exemplifying intensions.[2] That an intension has an essence means that a relation-in-extension obtains a priori between an intension and other intensions such that, necessarily, whenever an individual (an ι-entity) exemplifies the intension at some $\langle w, t\rangle$ then the same individual also exemplifies certain other intensions at the same $\langle w, t\rangle$. This relation is called the *requisite* relation.[3] We base our essentialism on the requisite relation and call our position *intensional essentialism*, couching as it does essentialism in terms of interplay between intensions, regardless of who or what exemplifies a given intension. This is in line with our general top-down approach from construction to intension and from intension to extension.

Let the property of being a mammal be related by the requisite relation to the property of being a whale. Then, necessarily, *if* the individual *a* is a whale at $\langle w, t\rangle$ *then a* is also a mammal at $\langle w, t\rangle$. It is an open question (epistemologically and ontologically speaking) *whether a* is a whale at $\langle w, t\rangle$. Establishing whether it is requires investigation a posteriori. On the other hand, establishing whether *a* must be a mammal in case *a* happens to be a whale is a priori, the requisite relation being in-extension and as such independent of what is true at any $\langle w, t\rangle$. Thus, there is a sense in which intensional essentialism qualifies as anti-essentialism: Robert Stalnaker labels as 'bare particular anti-essentialism' any theory (such as ours) which includes bare particulars and which claims that no empirical property is essential of any individual (1979, p. 344).

Intensional essentialism is technically an algebra of individually necessary and jointly sufficient conditions for having a certain intension. This makes it possible to *define* a given intension by means of other intensions. The essence of an intension is

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[1] This section and the next draw in part on material published as Jespersen and Materna (2002).

[2] By 'purely contingent intension' we mean an intension that is not constant and does not have an essential core. See Section 1.4.2.1 for the classification of empirical properties.

[3] Tichý first broached the notion of requisite in 1979, but abstained from further developing it in later works.

M. Duží et al., *Procedural Semantics for Hyperintensional Logic*, Logic, Epistemology, and the Unity of Science 17, DOI 10.1007/978-90-481-8812-3_4,

identical to its set of requisites. The $\langle w, t\rangle$-relative extensions of a given intension are irrelevant, as we said; but so are the various equivalent constructions of the intension.

4.1 Requisites defined

Here we set out the logic of requisites. The requisite relations *Req* are a family of relations-in-extension between two intensions, hence of the polymorphous type $(o\alpha_{\tau\omega}\beta_{\tau\omega})$, where possibly $\alpha = \beta$. Infinitely many combinations of *Req* are possible, but the following four are the philosophically relevant ones we wish to consider:

(1) $Req_1/(o\ (o\iota)_{\tau\omega}\ (o\iota)_{\tau\omega})$: an individual *property* is a requisite of another such *property*.
(2) $Req_2/(o\ \iota_{\tau\omega}\ \iota_{\tau\omega})$: an individual *office* is a requisite of another such *office*.
(3) $Req_3/(o\ (o\iota)_{\tau\omega}\ \iota_{\tau\omega})$: an individual *property* is a requisite of an individual *office*.
(4) $Req_4/(o\ \iota_{\tau\omega}\ (o\iota)_{\tau\omega})$: an individual *office* is a requisite of an individual *property*.

Partiality gives rise to the following complication both with respect to offices and properties. The requisite relation obtains for all worlds *w* and times *t*, and the values at $\langle w, t\rangle$ of particular intensions are irrelevant. Thus if an office *X* has the requisite intension *Y*, it is so no matter whether an office *X* is occupied or vacant at a given $\langle w, t\rangle$. For instance, even at those $\langle w, t\rangle$ where the office of King of France is vacant it is true that the property of being a king is a requisite of the office. Similarly, it is true at all $\langle w, t\rangle$ (including those where the office of President of USA is vacant) that the office of Commander-in-Chief is a requisite of the office of President of USA. Therefore, it does not suffice to add the antecedent condition that *X* be occupied. For, at a $\langle w, t\rangle$ where *X* is vacant, the antecedent condition is false, and so the intensional descent of *X* to $\langle w, t\rangle$ picks up *no* individual. In other words, the Compositions X_{wt}, $[{}^0Y_{wt} = {}^0X_{wt}]$ and $[{}^0Z_{wt}\ {}^0X_{wt}]$ will be *v*-improper ($Y/\iota_{\tau\omega}$; $Z/(o\iota)_{\tau\omega}$). The truth-functional connective of material implication ($\supset/(ooo)$) is such that when applied to a missing argument (a truth-value gap), the result is *v*-improper as well, making the Composition $[[{}^0Occ_{wt}\ {}^0X] \supset [{}^0Y_{wt} = {}^0X_{wt}]]$ *v*-improper for $\langle w, t\rangle$.[4] The whole *definiens* $\forall w \forall t\ [[{}^0Occ_{wt}\ {}^0X] \supset [{}^0Y_{wt} = {}^0X_{wt}]]$ will, thus, construct **F**! ($Occ/(o\iota_{\tau\omega})_{\tau\omega}$ is the property of an individual office of being occupied.)

A similar problem arises even in case of properties. The reason is because properties are isomorphic to characteristic functions, and these functions can also

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[4] In a programming language, one would say that $\supset$ is a *strict function* returning an error value for an error value: $\perp \rightarrow \perp$.

have truth-value gaps. For instance, the property of having stopped smoking comes with a bulk of requisites like, e.g., the property of being an ex-smoker. Thus, the predication of such a property Z of an individual a may also fail, causing $[{}^0Z_{wt}\ {}^0a]$ to be v-improper. The remedy is easy, fortunately—just use the propositional property of being true at $\langle w, t\rangle$: $True/(\text{o}\text{o}_{\tau\omega})_{\tau\omega}$. Given a proposition P, $[{}^0True_{wt}\ {}^0P]$ v-constructs **T** if P is true at $\langle w, t\rangle$; otherwise (i.e., if P is false or else undefined at $\langle w, t\rangle$) **F**.[5]

Now we are going to define the four above kinds of requisite relations.

Ad (1):

Definition 4.1 (*requisite relation between ι-properties*) Let X, Y be intensional constructions such that X, $Y/*_n \rightarrow (\text{o}\iota)_{\tau\omega}$; $x \rightarrow \iota$. Then

$$[{}^0Req_1\ Y\ X] = \forall w \forall t\ [\forall x\ [[{}^0True_{wt}\ \lambda w \lambda t\ [X_{wt}\, x]] \supset [{}^0True_{wt}\ \lambda w \lambda t\ [Y_{wt}\, x]]]]. \quad \square$$

Gloss *definiendum* as, 'Y is a requisite of X', and *definiens* as, 'Necessarily, at every $\langle w, t\rangle$, whatever x instantiates X at $\langle w, t\rangle$ also instantiates Y at $\langle w, t\rangle$.'

Example. *All whales are mammals*, provided the property of being a mammal is a requisite of the property of being a whale.[6]

Ad (2):

Definition 4.2 (*requisite relation between ι-offices*) Let X, Y be intensional constructions such that X, $Y/*_n \rightarrow \iota_{\tau\omega}$. Let $Occ/(\text{o}\iota_{\tau\omega})_{\tau\omega}$ be the property of an office of being occupied (or existing, as *existence* was defined in Section 2.3). Then

$$[{}^0Req_2\ Y\ X] = \forall w \forall t\ [[{}^0Occ_{wt}\ X] \supset [{}^0True_{wt}\ \lambda w \lambda t\ [X_{wt} = Y_{wt}]]]. \qquad \square$$

Gloss *definiendum* as, 'Y is a requisite of X', and *definiens* as, 'Necessarily, if X is occupied at some $\langle w, t\rangle$ then whoever occupies X at $\langle w, t\rangle$ also occupies Y at this $\langle w, t\rangle$.'

Remark. Due to partiality, the relation between offices may not be symmetric. If the office X is occupied, then the office Y is occupied as well, and X and Y are occupied by the same individual. If Y is not occupied then X is not occupied either. However, Y can be occupied, and X vacant, at some $\langle w, t\rangle$. If being X is a *sufficient* condition for being Y, whereas being Y is a *necessary* condition for being X, it follows that the set of world/time pairs at which Y is occupied is a superset of the set of world/time pairs at which X is occupied. Suppose we rank individual offices in terms of the ordering defined by the subset relation between sets of worlds and times at which they are occupied according to the rule that a rarely occupied office

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[5] See Section 1.4.3.

[6] We also often say that the property of being a whale *implies* the property of being a mammal; or, in the vernacular of computer science, that the concept of whale *subsumes*, or *contains*, the concept of mammal.

is higher up the hierarchy than a frequently occupied one. Then X is higher up than Y. One could also say that X is, in a quite literal sense, more *exclusive* than Y.

Example. *The President of the USA is the Commander-in-Chief.* The latter office is a requisite of the former, such that whoever is the President is also the Commander-in-Chief. However, it may happen that the presidency goes vacant, while somebody occupies the office of Commander-in-Chief.

Ad (3):

Definition 4.3 (*requisite relation between a ι-property and a ι-office*) Let X, Y be intensional *constructions* such that $X/*_n \rightarrow \iota_{\tau\omega}$ and $Y/*_n \rightarrow (o\iota)_{\tau\omega}$. Then

$$[{}^0Req_3\ \ Y\,X] = \forall w \forall t\ [[{}^0Occ_{wt}\,X] \supset [{}^0True_{wt}\,\lambda w \lambda t\ [Y_{wt}\,X_{wt}]]]. \qquad \Box$$

Example. *The King of France is a king.*

Remark. 'The King of France is a king' is *ambiguous* between two readings—one necessarily true, the other contingently without a truth-value—as Tichý points out (1979, p. 408, 2004, p. 360).[7] One is the requisite (i.e., *de dicto*) reading:

$$[{}^0Req_3\ {}^0King\ \lambda w \lambda t\ [{}^0King\_of_{wt}\ {}^0France]].$$

Types: $King/(o\iota)_{\tau\omega}$; $King\_of/(\iota\iota)_{\tau\omega}$; $France/\iota$. If true, it is necessarily so, regardless of whether or not some $\langle w, t\rangle$ lacks an occupant of $\lambda w \lambda t\ [{}^0King\_of_{wt}\ {}^0France]$.

The other reading is the *de re* reading:

$$\lambda w \lambda t\ [{}^0King_{wt}\ \lambda w \lambda t\ [{}^0King\_of_{wt}\ {}^0France]_{wt}].$$

If true, it is so only because somebody occupies $\lambda w \lambda t\ [{}^0King\_of_{wt}\ {}^0France]$ at $\langle w, t\rangle$ and its occupant is in the extension of *King* at $\langle w, t\rangle$.

Remark. When defining a requisite of an *office* X, the antecedent condition on X being occupied is required. Otherwise we shall have the following invalid argument on our hands (see Tichý, 1979, pp. 408ff, 2004, pp. 360ff).

$$\frac{P \text{ is a requisite of office } O}{\text{The occupant of } O \text{ instantiates } P.}$$

This inference pattern is fallacious,

> for the premise may be true even if O is vacant, in which case the conclusion, so far from being true, is vacuous (i.e., lacks a truth value). (Ibid., p. 408, p. 360, resp.)

[7] See Section 1.5.2.2 for the ambiguity between the two readings.

However, a valid inference rule can be obtained by adding an extra premise to the effect that the relevant office is occupied:

$$\frac{\begin{array}{c}P \text{ is a requisite of office } O\\ \text{Office } O \text{ is occupied}\end{array}}{\text{The occupant of } O \text{ instantiates } P.}$$

Ad (4):

Definition 4.4 (*requisite relation between a ι-office and a ι-property*) Let X, Y be intensional constructions such that $X/*_n \to (o\iota)_{\tau\omega}$ and $Y/*_n \to \iota_{\tau\omega}$. Then

$$[{}^0Req_4\ \ Y\,X] = \forall w \forall t\ [\forall x\ [{}^0True_{wt}\ \lambda w\lambda t\ [X_{wt}\ x]] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [Y_{wt} = x]]]]. \quad \square$$

Example. *God is omnipotent.* (That is, if somebody/something is omnipotent then he/she/it is God.)

Above we defined four types of requisite relation, namely Req_1, Req_2, Req_3, Req_4. While $Req_1/(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ and $Req_2/(o\iota_{\tau\omega}\iota_{\tau\omega})$ are homogeneous, Req_3, Req_4 are heterogeneous. Since the latter two do not have a unique domain, it is not sensible to ask what sort of ordering they are. Not so with the former two. We define them as *quasi-orders* (a.k.a. *pre-orders*) over $(o(o\iota)_{\tau\omega})$, $(o\iota_{\tau\omega})$, respectively, that can be strengthened to *weak partial orderings*. However, they cannot be strengthened to strict orderings on pain of paradox, since they would then both be reflexive and irreflexive. We wish to retain reflexivity, such that any intension having requisites will count itself among its requisites. Otherwise there will be worlds and times at which an office X is occupied and $X_{wt} \neq X_{wt}$, and worlds and times at which a property Y is instantiated and $\neg[[Y_{wt}\ x] \supset [Y_{wt}\ x]]$.

Claim 4.1 Req_1 is a quasi-order on the set of ι-properties.

Proof. Let X, $Y \to (o\iota)_{\tau\omega}$. Then Req_1 belongs to the class $QO/(o(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega}))$ of quasi-orders over the set of individual properties:

Reflexivity.
$$[{}^0Req_1\ X\,X] =$$
$$\forall w \forall t\ [\forall x\ [[{}^0True_{wt}\ \lambda w\lambda t\ [X_{wt} x]] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [X_{wt} x]]]]$$

Transitivity.
$$[[[{}^0Req_1\ Y\,X] \wedge [{}^0Req_1\ Z\,Y]] \supset [{}^0Req_1\ Z\,X]] =$$
$$\Big[\forall w \forall t\ \Big[\forall x\ [[{}^0True_{wt}\ \lambda w\lambda t\ [X_{wt} x]] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [Y_{wt} x]]] \wedge$$
$$[[{}^0True_{wt}\ \lambda w\lambda t\ [Y_{wt} x]] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [Z_{wt} x]]]\Big] \supset$$
$$\forall w \forall t\ [\forall x\ [[{}^0True_{wt}\ \lambda w\lambda t\ [X_{wt} x]] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [Z_{wt} x]]]]\Big]$$

In order for a requisite relation to be a weak partial order, it will need to be also anti-symmetric. The Req_1 relation is, however, not anti-symmetric. If properties X, Y are mutually in the Req_1 relation, i.e., if

$$[[{}^0Req_1\ Y\ X] \wedge [{}^0Req_1\ X\ Y]]$$

then at each $\langle w, t\rangle$ the two properties are truly ascribed to exactly the same individuals. This does not entail, however, that X, Y are identical. It may be the case that there is an individual a such that $[X_{wt}\ a]$ v-constructs **F** whereas $[Y_{wt}\ a]$ is v-improper. For instance, the following properties X, Y differ only in truth-values for those individuals who never smoked (let $StopSmoke/(o\iota)_{\tau\omega}$ be the property of having stopped smoking[8]). Whereas X yields truth-value *gaps* on such individuals, Y is false of them:

$$X = \lambda w \lambda t\ \lambda x\ [{}^0StopSmoke_{wt}\, x]$$

$$Y = \lambda w \lambda t\ \lambda x\ [{}^0True_{wt}\, \lambda w \lambda t\ [{}^0StopSmoke_{wt}\ x]].$$

In order to abstract from such an insignificant difference, we introduce the equivalence relation $Eq/(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ on the set of individual properties; $p, q \rightarrow (o\iota)_{\tau\omega}$; $=/(ooo)$:

$${}^0Eq = \lambda pq\ [\forall x\ [[{}^0True_{wt}\, \lambda w \lambda t\ [p_{wt}\ x]] = [{}^0True_{wt}\, \lambda w \lambda t\ [q_{wt}\ x]]]].$$

Now we define the Req_1' relation on the factor set of the set of ι-properties as follows. Let $[p]_{eq} = \lambda q\ [{}^0Eq\ p\ q]$ and $[Req_1'\ [p]_{eq}\ [q]_{eq}] = [Req_1\ p\ q]$. Then:

Claim 4.2 Req_1' is a weak partial order on the factor set of the set of ι-properties with respect to Eq.

Proof. It is sufficient to prove that Req_1' is well-defined. Let p', q' be ι-properties such that $[{}^0Eq\ p\ p']$ and $[{}^0Eq\ q\ q']$. Then

$$[Req_1'\ [p]_{eq}\ [q]_{eq}] = [Req_1\ p\ q] =$$

$$\forall w \forall t\ [\forall x\ [[{}^0True_{wt}\, \lambda w \lambda t\ [p_{wt}\, x]] \supset [{}^0True_{wt}\, \lambda w \lambda t\ [q_{wt}\, x]]]] =$$
$$\forall w \forall t\ [\forall x\ [[{}^0True_{wt}\, \lambda w \lambda t\ [p'_{wt}\, x]] \supset [{}^0True_{wt}\, \lambda w \lambda t\ [q'_{wt}\, x]]]] =$$

$$[Req_1'\ [p']_{eq}\ [q']_{eq}].$$

Now obviously the relation Req_1' is antisymmetric:

$$[[{}^0Req_1'\ [p]_{eq}\ [q]_{eq}] \wedge [{}^0Req_1'\ [q]_{eq}\ [p]_{eq}]] \supset [[p]_{eq} = [q]_{eq}].$$

Claim 4.3 Req_2 is a weak partial order defined on the set of ι-offices.

Proof. Let $X, Y \rightarrow \iota_{\tau\omega}$. Then the Req_2 relation belongs to the class $WO/(o(o\ \iota_{\tau\omega}\iota_{\tau\omega}))$ of weak partial orders over the set of individual offices.

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[8] We take the property of having stopped smoking as presupposing that the individual previously smoked. For instance, that Charles stopped smoking can be true or false only if Charles was once a smoker. Similarly for the property of having stopped whacking one's wife. For more on presuppositions, see Section 1.5.2.1, Definition 1.14.

Reflexivity. $[{}^0Req_2\, X\, X] = [\forall w \forall t\, [[{}^0Occ_{wt}\, X] \supset [{}^0True_{wt}\, \lambda w \lambda t\, [X_{wt} = X_{wt}]]]]$.

Antisymmetry. $[[[{}^0Req_2\, Y\, X] \wedge [{}^0Req_2\, X\, Y]] \supset [X = Y]] =$
$\Big[\forall w \forall t\, \Big[[[{}^0Occ_{wt}\, X] \supset [{}^0True_{wt}\, \lambda w \lambda t\, [X_{wt} = Y_{wt}]]] \wedge$
$[[{}^0Occ_{wt}\, Y] \supset [{}^0True_{wt}\, \lambda w \lambda t\, [X_{wt} = Y_{wt}]]]\Big] \supset [X = Y]\Big]$

Transitivity. $[[[{}^0Req_2\, Y\, X] \wedge [{}^0Req_2\, Z\, Y]] \supset [{}^0Req_2\, Z\, X]] =$
$\Big[\forall w \forall t\, \Big[[[{}^0Occ_{wt} X] \supset [{}^0True_{wt}\, \lambda w \lambda t\, [X_{wt} = Y_{wt}]]] \wedge$
$[[{}^0Occ_{wt}\, Y] \supset [{}^0True_{wt}\, \lambda w \lambda t\, [Y_{wt} = Z_{wt}]]]\Big] \supset$
$\forall w \forall t\, [[{}^0Occ_{wt}\, X] \supset [{}^0True_{wt}\, \lambda w \lambda t\, [X_{wt} = Z_{wt}]]]\Big]$.

Remark. Antisymmetry requires the consistent *identity* of the offices constructed by X, Y: $[X = Y]$. The two offices are identical iff at all worlds/times they are either co-occupied by the same individual or are both vacant: $\forall w \forall t\, [[{}^0True_{wt}\, \lambda w \lambda t\, [X_{wt} = Y_{wt}]] \vee [{}^0Undef_{wt}\, \lambda w \lambda t\, [X_{wt} = Y_{wt}]]] = \forall w \forall t\, \neg[{}^0False_{wt}\, \lambda w \lambda t\, [X_{wt} = Y_{wt}]]$.

This concludes our definition of the logic of the requisite relations. We turn now to the definition of essence as a set of requisites. Tichý considered only requisites of offices (though of offices of any degree):

> [T]he *requisite* of an office is any property such that, for any world *w* and time *t*, if *x* occupies the office in *w* at *t* then *x* instantiates the property in *w* at *t*
> (1979, p. 408, 2004, p. 360).

But the underlying idea readily generalises, in that the requisite of an intension Int_i is any Int_j, such that, for any $\langle w, t\rangle$, if x either instantiates or occupies Int_i then x instantiates or occupies Int_j at $\langle w, t\rangle$. Obviously, Tichý considered only essences of offices:

> [T]he conjunction of all [the requisites of an office] is fittingly called its *essence*. The essence of an office is thus a property such that the having of it by *x* in world *w* at time *t* is not only necessary but also sufficient for *x* to occupy the office in *w* at *t*. Whereas a requisite of an office is *part* of what it takes for something to occupy it, the *essence* is *all* it takes. An office can thus be defined by specifying its essence (Ibid).

Again, the underlying idea readily generalises, such that any Int_j can be defined by specifying its essence. However, unlike Tichý, we are not restricting requisites to properties suitable for the occupants of offices of degree n, $n \geq 1$: witness (1), (2), (4). Now, intensional essentialism simply says: specify the intensions that are the requisites of a given intension, pool those requisites, this will give you the essence of your intension. However, this drags type-theoretic complications along with it, since the requisites of Int_i may well be of different types. For instance, let $X/(o\iota)_{\tau\omega}$; $x \rightarrow \iota_{\tau\omega}$. Then we can formally specify all those of X's requisites that are of type $\iota_{\tau\omega}$:

$$\lambda x\, [{}^0Req_4\, x\, {}^0X].$$

But if X's requisites also count intensions of type $(o\iota)_{\tau\omega}$ then they need to be specified separately, $y \rightarrow (o\iota)_{\tau\omega}$:

$$\lambda y\ [{}^0Req_1\ y\ {}^0X].$$

The former is a construction of a set of ι-offices; the latter, a construction of a set of ι-properties. Thus, *contra* Tichý, we are banned from holding that the essence of an intension is 'the conjunction of all its requisites' (ibid.), if this would mean the *set* of all its requisites. There can be no such set, as soon as more than one characteristic function is involved, as with $\lambda x\ [\ldots x\ldots]$ and $\lambda y\ [\ldots y\ldots]$ above. We are *not* permitted to do what would ostensibly be the obvious thing to do; namely, forming their union:

$$\lambda x\ [{}^0Req_4\ x\ {}^0X] \cup \lambda y\ [{}^0Req_1\ y\ {}^0X].$$

This is off-limits, as the union would contain elements of more than one type. This predicament becomes evident if we attempt to type $\cup$. It must be a function from pairs of sets to sets, and the types of its arguments in this case are known: $(o\iota_{\tau\omega})$ and $(o(o\iota)_{\tau\omega})$, respectively. But the question of what the type of its value would be affords no answer.

However, there are two solutions possible, one arguably superior to the other. One solution makes the essence of an intension a pair whose first element is a set of $\iota_{\tau\omega}$-entities and whose second element is a set of $(o\iota)_{\tau\omega}$-entities. (It is obvious how to generalise this solution to cover any two homogeneous or heterogeneous combinations of requisites.) The other solution makes the essence of an intension a set of $(o\iota)_{\tau\omega}$-entities only, without thereby restricting the requisites to such entities.

Here is the definition of the essence of the intension $Y \rightarrow (o\iota)_{\tau\omega}$, according to which its essence is a heterogeneous pair of sets of intensions (of a set of ι-properties and a set of ι-offices, respectively), where $x_1 \rightarrow (o(o\iota)_{\tau\omega})$; $x_2 \rightarrow (o\iota_{\tau\omega})$; $c \rightarrow (o\iota)_{\tau\omega}$; $d \rightarrow \iota_{\tau\omega}$; $Essence_1'/((o(o(o\iota)_{\tau\omega})(o\iota_{\tau\omega}))(o\iota)_{\tau\omega})$.

$$[{}^0Essence_1'\ Y] = [\lambda x_1x_2\ [x_1 = \lambda c\ [{}^0Req_1\ c\ Y] \wedge [x_2 = \lambda d\ [{}^0Req_4\ d\ Y]]]].$$

A *pair* is here a relation-in-extension between two sets of arbitrary intensions; therefore, the polymorphous type of $Essence_1$ is $((o\ (o\gamma_{\tau\omega})\ (o\beta_{\tau\omega}))\ \alpha_{\tau\omega})$.

Here is the definition of the essence of $Y \rightarrow (o\iota)_{\tau\omega}$, according to which its essence is a set of ι-properties. That is, $Essence_2'/((o(o\iota)_{\tau\omega})(o\iota)_{\tau\omega})$ is a function from a ι-property to a set of ι-properties. If $p \rightarrow (o\iota)_{\tau\omega}$ then

$$[{}^0Essence_2'\ Y] = \lambda p\ [{}^0Req_1\ p\ Y].$$

The polymorphous type of *Essence*$_2$ is $((o(o\alpha)_{\tau\omega})\ \beta_{\tau\omega})$: given an arbitrary intension of type $\beta_{\tau\omega}$, *Essence*$_2$ returns the set of α-properties that are the requisites of this arbitrary intension. In general, if $Z \rightarrow \beta_{\tau\omega}$; $q \rightarrow (o\alpha)_{\tau\omega}$; $Req_n/((o\ (o\alpha)_{\tau\omega})\ \beta_{\tau\omega})$ then

$$[{}^0Essence_2\ Z] = \lambda q\ [{}^0Req_n\ q\ Z].$$

The reason why the definition of essence can be made homogeneous is because, given an arbitrary intension, there will always be a corresponding property. For instance, the ι-office *the tallest woman* will correspond to the property *being an x such that x is identical to the tallest woman*. If this office is A then the corresponding property is $(x \rightarrow \iota)$

$$\lambda w \lambda t\ [\lambda x\ [x = {}^0A_{wt}]].$$

Which of *Essence*$_1$, *Essence*$_2$ is preferable? *Essence*$_2$, in our view. It is more elegant, first of all, in that it makes it possible to define the essence of a given intension by means of one type of intension only. But there is also a substantial reason for preferring *Essence*$_2$. If we were to consider more requisite relations than just Req_1 through Req_4 we would need to specify an n-tuple of intensions, n>2, each element being of a different intensional type. It would remain indeterminate which particular intensional types to insert into the n-tuple and what the value of n would be. In particular, even if n were just countably infinite, it would be impossible to identify any appropriate construction of the tuple.

4.2 Intensional essentialism

Here we motivate intensional essentialism philosophically in opposition to extensional essentialism and its adjacent notion of metaphysical modality.

Intensional essentialism is opposed to standard contemporary essentialism, which is set within an extensional framework, according to which essential properties are borne by extensional entities such as individuals.[9] Extensional essentialism

[9] Nortmann distinguishes between what he calls *property essentialism* and *individual essentialism* (2002, pp. 8ff). The property essentialist inquires about the essence of those properties that he considers accidental in whatever bearers they may have. This inquiry contributes nothing to the question of what the nature of any bearer of the relevant property is, if the property is contingently borne by the bearer. If an individual a has the property F at time T then this only means that the following may be known a priori: *If* something has F at T *then* it has F throughout its existence. But whether a actually has F is something that can, in general, not be known a priori. (Ibid., pp. 26–27.) TIL comes close to qualifying as property essentialism in Nortmann's sense; though not entirely – we do not require that if a is an F then a must be an F from beginning to end of its cycle. First, we do not wish to exclude nomologically deviant worlds in which an F-object may shed F at some point without ending its cycle. This is to say that in such a world a

has received extensive attention in the vast literature on 'Aristotelian essentialism' following in the wake of the development of quantified modal logic, Kripke (1980) arguably being *the* modern classic.

Typical questions would be whether Socrates is essentially a human being or essentially Plato's teacher. We ask a different kind of question, such as whether being human is an essential property of any occupant of the property of being Plato's teacher (i.e., whether it is a requisite of this property).[10] The leading idea is that modality *de dicto* is based on a priori relations between intensions, while modality *de re* is based on bare particulars. (For modality *de dicto* and *de re*, see Section 4.6).

Our extensive reliance on intensional entities at the expense of extensional ones is 'pre-revolutionary' in the general sense that TIL has not joined the current orthodoxy ushered in by Kripke, Kaplan, etc., that began as a 'revolution' against Carnap, Church, etc. Simchen (2004, esp. pp. 528–40) provides a precise description of the change in perspective and priorities that the 'revolution' (as he terms it) brought about. Pre-revolutionary possibility was analytical possibility, which was simply a matter of consistency of the co-instantiation of intensions, with little concern for 'what *things* would have been like had they been different from the ways they are' (2006, p. 24; emphasis ours.) In keeping with this, 'the conditions [i.e., the 'purely qualitative manners of presenting portions of our surroundings', 2004, p. 543]…should be just as they are in the complete absence of any world to satisfy them' and 'the world [supplies] mere satisfiers for independently constituted conditions' (ibid., p. 530, p. 531, resp.). We agree wholeheartedly, TIL being ('Platonic') realism *ante rem*.[11] Kripke, by contrast, holds that

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may end its cycle as an elephant without thereby dying (but, e.g., becoming a different sort of mammal or something much more exotic). Second, our ι-objects are incapable of going out of existence, so *coming into and going out of existence* will often have to be recast as *being born and dying*, *being created and destroyed*, etc., and then only in an individual's capacity as an *F*-thing, a *G*-thing, etc.

[10] Bordering on morbidity, Kim Il Sung was made Eternal President of the Democratic People's Republic of Korea in 1998, 4 years after his death. If a dead human being is not a human being then it is not a requisite of this office to be a human being. Since the ontological status of deceased people is far from obvious (just as it is uncertain whether *deceased* is a privative modifier), it is far from obvious what it takes to occupy the office. This suggests that the office of Eternal President of the DPRK is ill-defined; so, strictly speaking, there may be no such office (but only a vacuous title with no office to back it up). Nor is it entirely clear what it actually means to say that Kim was made Eternal President after his death; for, assuming that a dead person is not a person, *who* acquired, in 1998, the property of being the occupant of the office of Eternal President? Colloquially, one would say that Kim did (as we just did a few lines up); but he died in 1994, so in what (non-ghoulish) sense was *he* around in 1998 to acquire any new properties at all? Our concern is with the exact requisites of an (alleged) office and the possibility of a deceased person (hence, probably non-person) occupying it.

[11] On a similar note, Sartre says, '[Essence] precedes existence for Leibniz, and the chronological order depends on the eternal order of logic' (1943, p. 469). The priority of essence over existence holds for complete individual offices; i.e., entire life-stories. The only dash of contingency is choosing one such office at the expense of all the rest. Once that choice is made, all the rest

> [W]e begin with the objects, which we *have*, and can identify, in the actual world. We can then ask whether certain things might have been true of the objects (1980, p. 53).

Transposed into the key of intensional essentialism, the conceptual order would be the other way around; we begin with the conditions (intensions) that we have and can identify regardless of any particular possible world. We can then ask whether certain conditions might have been satisfied by something (extensions) at this or that world.[12]

We make two negative claims. The first is that the predication

Necessarily, *a* is an *F*

is false, if *a* is an extensional entity and *F* a purely contingent property. That is, we reject individual essentialism. The second is that no purely contingent relation-in-intension between any two different individuals ever obtains of necessity. So

Necessarily, *a* has origin *o*

where, e.g., *a* is a wooden table and *o* a chunk of wood, is false. That is, we reject the thesis of the necessity of origin. This is not to say that it could not be made a requisite of some particular individual office that its bearer must have its material origin in either a specific individual or whatever occupies a specific office. It could; but then *necessity of origin* is no longer a relation-in-intension between two individuals, but a relation-in-extension between either a property and an office or between two offices. Whether *Necessarily, a is an F* or *Necessarily, a has origin o*, we find ourselves rejecting the category of so-called metaphysical modality due

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follows as a matter of necessity. We side with Leibniz against Sartre in saying that essence does precede existence. But we also deny that what an individual is and does throughout its life-span is a matter of unfolding an entire, pre-programmed individual office. One could, in principle, individuate any two individuals strictly in terms of what is true of either of them with respect to worlds and times. But such a principle of individuation would be of no use to us humans, since the respective sets of truths applying to two individuals are infinite and as such cannot be grasped in full by humans. If individuals would enter into our ontology only as values of intensions, especially of individual offices, we would never be entitled to Trivialize an individual *a*: ^0a. We would never get 'closer' to individuals than in terms of A_{wt}, $A/\iota_{\tau\omega}$.

[12] Whether *a* is an *F* or the *G*, for instance, is, logically speaking, irrelevant. What is relevant is only whether some individual or other is an *F* or the *G* at some $\langle w, t\rangle$ of evaluation, whether the same individual is both the *G* and the *H* at $\langle w, t\rangle$, etc., and not whether it is *a*, *b*, *c*, etc. This will come across as exceedingly cynical if *a* is a human being; for it may well be extremely relevant to *a* whether he or she is an *F* or the *G* or both the *G* and the *H* (etc.). We wish to emphasise, therefore, that our top-down approach from condition to satisfier combined with an exterior, or outside-in, perspective on individuals is no theory of the good (human) life, but adopted for strictly logical and semantic purposes, having 'little to do with how men (or men and animals) fare in [the actual world]', as Rescher says about Leibniz's struggle with squaring the well-being of rational creatures with God's choice of *the* possible world that will combine the fewest and simplest laws with the greatest multitude of phenomena (1986, p. 157).

to Kripke (1980), which is supposed to be the sort of modality a posteriori underlying the modifier *Necessarily* in both cases.

Consider this example from contemporary analytic metaphysics, which has taken on a life of its own, spawning a literature dedicated particularly to it. The example will prove helpful in discussing both, 'Necessarily, *a* is an *F'* and, 'Necessarily, *a* has origin *o*'.

Pointing at his wooden lectern in the auditorium, Kripke says:

> In the case of this table, we may not know what block of wood the table came from. Now could *this table* have been made from a completely *different* block of wood, or even of water cleverly hardened into ice [?] (1980, p. 113).

Kripke argues that, necessarily, that wooden table is wooden, and that, necessarily, its material origin is the block of wood it was actually hewn from. Our objection to individual-essentialist predication is its *circularity*. Our objection to necessity of origin is its *infinite regress*.

First, individual essentialism. Consider two individuals, *a* and *b*, of which we already know that they are both tables but only one is wooden while the other merely appears to be so. Our task is to decide whether it is *a* or it is *b* that is the wooden table of the two. Pick one of the tables and apply a (low-key) scientific procedure to check whether it is wooden. Let the outcome be that it is, indeed, wooden. If we know this, our knowledge is a posteriori, because we applied an empirical procedure, and of a *contingent* truth, because if we had checked the other table it would have been false that the inspected table was wooden. Our knowledge is insufficient to establish whether it is true that *a* is the wooden table:

> But if [it is not knowable *a priori* that *a* is wooden] it is hard to see how, on Kripke's theory, it can be knowable at all. For…if we do not know [that *a* is wooden] to start with, no amount of inspecting or testing a table will tell us that it is *a* rather than *b* that we are dealing with. Accordingly, no amount of inspecting and testing will tell us that *a* is wooden (Tichý, 1983, pp. 239–40, 2004, pp. 521–22.)

Semi-formally:

(1) the inspected table = the wooden table
(2) *a* = the inspected table
(3) *a* = the wooden table
(4) if *x* is the wooden table then *x* is wooden
(5) *b* = the inspected table

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(6) *a* is wooden.

The argument is valid, of course, but we cannot know it to be sound, as long as we do not know whether it is *a* or if it is *b* that is the inspected table. As long as we do not know whether (2) is true, we cannot know whether (3) is true; but then we cannot know whether it is (5) or its rival (*b* = the inspected table) that is true. The truth-value of (2) can be ascertained only if it is already known that being wooden is an essential property of every wooden table (understood *de re*), such that woodenness can be used to tell *a* from *b*, no matter whatever other properties *a*, *b* may have at a given $\langle w, t\rangle$. Remember that, according to Kripkean essentialism, if *b* fails to be wooden at one world *W* then *b* fails to be wooden at all other worlds accessible from *W*.

Tichý's verdict is that

> Kripke's individual essentialism … involves an epistemological circle. In order to establish that an object has an essential property, we have to inspect that object. But we cannot be sure that we are inspecting the right object unless we know that the object has that essential property. The Kripkean essentialist is thus saddled with the absurd conclusion that no particular table can be known to be wooden [.]
> (1983, p. 240, 2004, p. 522.)

The circularity objection readily extends from individuals to natural kinds. In order to establish whether all individuals belonging to a particular species share some particular essential property, we have to inspect such individuals (say, cats). But we cannot be sure that the individuals we are inspecting are cats, unless we know that the inspected individuals possess that essential property.[13] Remember that, according to Kripkean essentialism, if *x* is a cat at one world *W* then *x* is a cat at all worlds accessible from *W* at which *x* exists.

Then, necessity of origin. In its crudest form the thesis is that the binary relation *Origin* holding between two individuals *a*, *b*, such that *a* is the material origin of some artefact or organism *b*, obtains as a matter of 'metaphysical' necessity:

$$Origin\ \langle a, b\rangle \supset \Box\ Origin\ \langle a, b\rangle.$$

An object owes its origin to other objects, the way a child owes its origin to its parents or a statue owes its origin to a lump of bronze (say).[14] But those objects are also anchored to other objects, and so on backwards into the bottomless past.[15] A full description or comprehension of an individual's origin would include an amount of other things so vast, it could not possibly be surveyed. The notion of origin will be epistemologically and conceptually inoperative unless made manageable

[13] See Kripke (1980, pp. 125–27) regarding 'the actual cats that we *have*' *versus* demons masquerading as cats, where it is assumed that we would be able to know of something that it is a cat prior to knowing what the species-specific essence of cats is.

[14] Though a statue owes its origin to much more than just some lump of matter. A statue is an artistic artefact that also embodies an artistic idea which is materialized by means of a lump of matter. From the point of view of artistic idea, it matters little which lump of matter happens to embody Michelangelo's ideal male youth. Yes, the statue at *Accademia* in Florence is the original and the one in front of *Palazzo Vecchio* is a copy; but they manifest the same idea(l) of male youth. The bottom-line is that a statue is at the intersection of matter and idea, and is not reducible to a chunk of clay, marble, or stone.

[15] Berkovski notes that, 'The full specification of Napoleon's origin will be recursive. If the question is how we identify Letizia Bonaparte [Napoleon's mother], the same proof of origin is to be repeated for her, her own parent, and so forth.' (2005, p. 17.) Berkovski, however, fails to point out that the recursion is going to be infinite, unless terminated by *fiat*. (We thank Berkovski for permitting us to quote from his unpublished manuscript.)

by arbitrarily stipulating a point at which the backtracking were to end. This is not to say that the notion of origin might not underpin some form of essentialism. In fact, we can express the thesis by means of Req_2, since the relation between origin and destination (i.e., the resulting artefact or organism) is not symmetric, as in effect argued by Rohrbaugh and deRosset (2004, pp. 718ff).[16,17]

Both $\Box\, Fa$ and $\Box\, Origin\, \langle a, b\rangle$ are supposed to be conclusions of the argument schema that Kripke introduces (1971, p. 153):

$$\begin{array}{ll} (1) & P \supset \Box\, P \\ (2) & P \\ \hline (3) & \Box\, P. \end{array}$$

Thus, let P be $(a = b)$, 'a', 'b' Kripkean proper names:

$$\begin{array}{ll} (1.1) & (a = b) \supset \Box\, (a = b) \\ (2.1) & a = b \\ \hline (3.1) & \Box\, (a = b) \end{array}$$

Kripke argues that the necessity in the consequent of (1.1) and in (3.1) is metaphysical necessity a posteriori. If $a = b$, then necessarily so, since everything is necessarily self-identical. Or phrased in the idiom of rigid designation: if two rigid designators co-designate at one world they do so at all worlds (with provisos for inexistence); i.e., '$a = b$' will express a necessary truth.

But the necessitation of $a = b$ can be argued for on strictly logical grounds. If true, (3.1) is just the logical triviality that a is self-identical. Only the triviality of the argument is masked by the notation. When unmasked, the argument is

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[16] If a table, T_1, has its origin in a hunk of wood, H_1, at one world then T_1 must have its origin in H_1 at all other worlds as well, except that there are worlds where H_1 fails to exist and T_1, therefore, also fails to exist. However, there are still other worlds at which H_1 exists without T_1 existing; the existence of H_1 is a necessary but not sufficient condition for T_1 to exist. (Hence, the set of worlds at which T_1 exists is a proper subset of the set of worlds at which H_1 exists.) Rohrbaugh and deRosset allow that at worlds lacking T_1, H_1 may be the origin of wooden objects different from T_1 or of no artefacts at all. However, their principle of *origin uniqueness* (ibid., p. 715) is not immune to the infinite-regress objection. The principle grounds the necessary distinctness of T_1, T_2 in the distinctness of their origins H_1, H_2. But the necessary distinctness of H_1, H_2 must in turn be grounded in the distinctness of *their* origins; and so on, with no end in sight.

[17] Cameron (2005, p. 264) says, 'Given a block of wood I could make a table that was four-legged or three-legged, tall or short, round or square, thin or wide. Am I to believe that it would be the same table I was making in each case?' Cameron thinks not, citing a lack of essentialist intuitions. But the obvious answer is Yes—for being four-legged and all the rest are all *accidental* properties of one and the same table (entailing that the table might be many different *kinds* of table).

$$\begin{array}{ll} (1.1.1) & (a = a) \supset \Box\, (a = a) \\ (2.1.1) & a = a \\ \hline (3.1.1) & \Box\, (a = a). \end{array}$$

Further, there is nothing a posteriori in the premises or the conclusion of either of the notational variants of the argument. It only requires linguistic competence to know whether '*a*', '*b*' co-denote, and linguistic competence is acquired a priori.[18]

The necessitation of *Fa* and *Origin*⟨*a*, *b*⟩, by contrast, must be another, since neither is a logical truth. Once we start casting about for an alternative sort of necessity, the only reasonable candidate would be nomological necessity, since the necessity of *Fa* and *Origin*⟨*a*, *b*⟩ is supposed to be a posteriori. If 'metaphysical' necessity reduces to nomological necessity, then the former is redundant and can be done away with. It is not obvious to us what the added value of the category of 'metaphysical' modality might be. In TIL, at least, there is neither need nor room for it. What we suggest instead are analytic and nomological necessity, but the former seems to be too strong, and the latter too weak, to match the intended modal profile of metaphysical necessity.[19]

Alleged cases of necessity a posteriori are what Kripke terms 'theoretical identifications' (1980, pp. 99ff).[20] One famous example is that water is H_2O.[21] TIL makes available two ways of construing this 'identification'. The first is

$$[{}^0Req_1\ {}^0F\ {}^0G]$$

F, $G/(o\iota)_{\tau\omega}$. If F, G are co-intensional, then $F = G$, which is trivial; so *being water* and *having the molecular structure* H_2O would need to be two different properties. But if they are different, which should be a requisite of the other? The choice is

[18] See Sections 3.3, 4.3.

[19] Though there is no way to find out, since no advocate of metaphysical modality that we are aware of has ever bothered to actually define the notion. This is not a satisfactory situation, considering the frequency and abandon with which the notion is being bandied about.

[20] Kripke famously claims that, 'One might very well discover essence empirically.' (1980, p. 110.) We agree with Nortmann's qualification of this claim. A chemist, he says, may very well discover the essence (e.g., the molecular structure) of some liquid; but he can hardly be said to have discovered *that* this molecular structure (or whatever) is the essence of the liquid in question. Discovering what the essence of some stuff is, is not a purely empirical matter ('keine allein in der Natur vorfindbare Tatsache'), as it also contains conventional components, (ibid., p. 10.) *Perhaps* Kripke makes a similar qualification in 1971 (p. 153) when claiming that one knows by philosophical analysis a priori that if some table is made of wood then it is necessarily not made of ice, while knowing a posteriori whether some particular table is wooden.

[21] The discussion appears to fall within a larger discussion of identity sentences, yet two examples of theoretical identifications Kripke gives are 'light is *a* stream of photons' and 'lightning is *an* electrical discharge' (ibid., p. 116, emphasis ours); so theoretical identifications need not be phrased as identity sentences; so it is not certain that the identification of water as H_2O should be, either.

obvious: *Having the molecular structure H_2O* should be the requisite in order to define liquids as water. The resulting modality is a priori, *necessary* and *analytic*. It leaves open the possibilities that G (i.e., *being water*) should have more requisites than just F (i.e., *having the molecular structure H_2O*) and, if so, that something should have the molecular structure H_2O without being water. The second construal is

$$\lambda w\ [\forall t\ [[{}^0Exist_{wt}\ {}^0G] \supset [\forall x\ [{}^0True_{wt}\ \lambda w\lambda t\ [{}^0G_{wt}\ x] \supset [{}^0F_{wt}\ x]]]]].$$

The resulting modality is a posteriori, logically *contingent* and at least *quasi-nomological*. It leaves open the possibility that there be possible worlds outside the set of worlds so constructed at which it does not follow that if x has G then x also has F. It falls to chemistry to ascertain whether the actual world is an element of the set of worlds just constructed. Neither construal, however, can be the full story about whether water is H_2O. For H_2O will in turn have to be 'identified' (we would prefer: defined), which can happen only relative to a body of chemical propositions; i.e., a chemical theory. A venture into philosophy of science would take us too far afield; here we intended merely to point out the possibilities of (partially) defining G in terms of F and of offering a construal on which Kripke's theoretical identification comes out both a posteriori and necessary, albeit not 'metaphysically' but physically so.

The thesis of the necessity of origin claims that it is 'metaphysically' necessary for a given individual a to have its material origin in some other particular individual b. If a is a wooden table then if a is a wooden table and b is a chunk of wood then if b is a's origin then this is so as a matter of 'metaphysical' necessity. Or if a is a person (better: a human body) and b another person (better: another human body) then if b is a's origin then this is so as a matter of 'metaphysical' necessity. TIL offers two ways of construing

Necessarily, a is an offspring of b.

The first makes the property of being an offspring of b a requisite of the office A:

$$[{}^0Req_3\ \lambda w\lambda t\ [\lambda x\ [{}^0Offspring_{wt}\ x\ {}^0b]]\ {}^0A].$$

Types: $x \rightarrow \iota$; $Offspring/(o\iota\iota)_{\tau\omega}$; b/ι; $A/\iota_{\tau\omega}$.

Alternatively, *Offspring* may take an office B as its argument.[22] Whether b or B, this construal is not in Kripke's spirit, since *Offspring* is a relation-in-*intension* between a particular individual b and whatever individual (if any) is the value of A at $\langle w, t\rangle$ or between two such values (i.e., B_{wt}, A_{wt}). Kripke seems to envision a as 'growing out of' b and being related in-extension to b. Lacking a notion akin to

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[22] Some sort of 'pedigree essentialism' construed in terms of requisites may be relevant to inheritance in monarchies, clan-based Stalinist regimes and suchlike. By the way, Kripke's origin essentialism comes with a tacit physicalist premise pertaining to personal identity that we see no cogent reason for adopting.

requisite, Kripke is in no position to define a relation-in-extension between two intensions, but only relations-in-intension or relations- in-extension between two individuals. The former will make it contingent that *a* is related to *b* as its offspring; only the latter will make their relation necessary. But care must be taken not to turn the Kripkean offspring relation into a logical or mathematical relation. The means Kripke has available to him are his *accessibility* relations between worlds. The worlds at which the offspring relation between two individuals obtains need to be restricted to a proper subset of logical space. Some worlds will need to be inaccessible from the actual world, such that there will be worlds at which it is not true that *a* is the offspring of *b*.[23] The accessibility relations can be modelled as functions from a world *w* of evaluation to a set of worlds accessible from *w*, making their type ((οω)ω). As is seen, the accessibility relation is an intension. In any sub-S5 system the resulting set of worlds will be a proper subset of logical space. Hence, the accessibility relation characterizing the given system will be a non-trivial intension. Hence, it will be a posteriori whether *b* is the origin of *a*. In S5 the equivalence relation will still be an intension, but non-triviality will instead have to be obtained by means of varying domains, making it a posteriori whether *a* and *b* exist.

The recourse to accessibility relations is a viable alternative to our requisite proposal. But there is a philosophical problem. Which is that the accessibility proposal fails to indicate which particular modal system (equipped with a particular accessibility relation defined over its frames) models metaphysical necessity best. This failure is symptomatic of what we see as a lacuna in Kripke's oeuvre. There is the mathematical logic of the accessibility relations. And there is the intuitively argued philosophy of metaphysical modality. But there is no philosophical logic to bridge between the two by privileging one particular modal system. (On the other hand, it is widely agreed that S5 best models logical modality, that S4 best models intuitionistic logic and epistemic modality, that S4.3 best models temporal modality, etc.)

Our second construal is

$$\lambda w\ [\forall t\ [{}^0Occ_{wt}\ {}^0A] \supset [{}^0Offspring_{wt}\ {}^0A_{wt}\ \ {}^0b]].$$

(Again, *b* may be replaced by B_{wt}.) This makes it at least quasi-nomologically necessary that whenever *A* is occupied its occupant originates from b/B_{wt}. The modal profile of this proposition is necessity a posteriori. The construal fails to exclude that there be a possible world at which *A* is occupied and A_{wt} is not an offspring of b/B_{wt}. This is fine, since any such world will be inaccessible from the world of evaluation. But again, it is not clear what the formal properties of the accessibility

[23] Thus, Cocchiarella says, '[N]ot only need not all the worlds in a given logical space be in the model structure..., even the worlds in the model structure need not all be possible alternatives to one another... Clearly, such a restriction...only deepens the sense in which the necessity in question is no longer a logical but a material or metaphysical modality.' (1984, p. 323).

relation would have to be. Hence, it is not clear, either, which modal system is best suited to model such a possibility.

We wish to push the point that metaphysical necessity is best identified with (or 'reduced to') physical/nomological necessity.[24] The connection between physical and metaphysical modality is inspired by a remark made by Graeme Forbes:

> We need a theory according to which our conception of the thisness of an individual is formed in the temporal case and then projected to transworld identity, to fix the boundaries of significance on de re hypotheses about the individual (1985, p. 147, n. 11).

It seems fair enough that once individual *a* is a wooden table, *a* could not, 'metaphysically' speaking, have been an elephant nor ever become one. The physical building-blocks making up a wooden table are not the right stuff for making an elephant(!), or *vice versa*. As for the traffic up and down the temporal axis we have no quarrel with metaphysical modality thus construed. The construction following below constructs a set of worlds V such that for each individual x which, at any $w \in V$, is a table there is no moment t at which x is an elephant. In more natural English, if something is a table in V then it is never an elephant in V.

$$\lambda w\ [\forall t\ [\forall x\ [[{}^0Table_{wt}\ x] \supset \neg[\exists t'\ [{}^0Elephant_{wt'}\ x]]]]].$$

The laws of physics, biology, chemistry, etc., that rule within V rule out the physical possibility that a table turn into an elephant. Even cutting-edge physical, biochemical, etc., engineering, no matter its stage of development, will bump up against the laws of nature that hold sway in V. However, there must be other classes of worlds where a table can indeed turn into an elephant. The laws of nature obtaining at those worlds are well likely to defy human comprehension. What is more, it is even conceivable, and logically possible, that there should be worlds devoid of laws of nature.[25] Such mind-boggling worlds must be capable of existing,

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[24] This is argued by, e.g., Cocchiarella (1984, p. 325), Berkovski (2005) , Farrell (1981), whereas Kripke appears to be suggesting that metaphysical modality is identical, or else very close, to logical modality; see, e.g., (1980, pp. 99, 125). Rohrbaugh and DeRosset (2004) assume an independent category of metaphysical necessity, yet fail in our view to differentiate it from either nomological or logical necessity. On the one hand, when talking about the production processes from hunks of wood to wooden tables, the possibilities and impossibilities they consider are in effect nomological (ibid., pp. 711ff). On the other hand, all four formulae in the formal argument in ibid. (p. 715, fn. 18) contain strings like '…□…⇒…'. But ⇒ is entailment, which is a *logical* relation between (hyper-) propositions and too strong for metaphysical necessity, provided it is to be something other than logical necessity. Furthermore, '…□…⇒…' looks like overkill. Entailment is defined as the necessitation of implication, $\Box(p\supset q)$; so what is the point and sense of necessitating entailment?

[25] We do not consider the—admittedly interesting—question of whether possible worlds devoid of laws of nature could possibly have elephants and tables in them. A reasoned answer to this question would presuppose a discussion of what the nomological prerequisites are for a given intension to be instantiated.

since otherwise the laws of logic and mathematics would coincide extensionally with the laws of the natural sciences.[26] For instance, Hanson says,

> No one has ever succeeded in building [a *perpetuum mobile*]. And, given *our* physical world, no one ever will. … But it need not be self-contradictory to suppose [this circumstance] to obtain; it would just be false. [Both "A *perpetuum mobile* is impossible" and "Nothing travels father than light" are] not conceivably false and yet not tautologically true (1967, p. 88).

We propose the term 'temporal essentialism' to stand for the doctrine above in terms of which we interpret metaphysical modality. The temporal essentialist now makes the further claim that no individual that *exists* within V exists without V. If, *per impossibile*, this were the case then we might indeed have an example of an individual that was a table in one world and an elephant in another. But the notion of metaphysical modality was launched exactly to narrow the modal span of an object down to what is physically, or temporally, possible within some subset of *all* the logically possible worlds. A thought-provoking passage in Forbes reads:

> It is presumably true that more or less anything can develop into more or less anything, given sufficiently sophisticated engineering, so taking the acorn c which grows into a certain oak tree in the actual world, we can consider a world where c is treated in such a way that it develops into a small vegetable. Then (PI) entails that that oak tree could have been, e.g., a cabbage, and therefore that there are entities which can be oak trees in some world and cabbages in others (Ibid., p. 146).

(PI) says: if x at world u has the same propagules as y at world v then $x = y$. Forbes rejects (PI) on the ground that the principle points toward bare particulars by allowing what he calls 'ungrounded identity'. However, our bare particular anti-essentialism is not predicated on applying engineering, whether sophisticated or pedestrian, to acorns, zygotes, or whatnot. Introducing cunning engineering into the story gives the wrong idea about what counterfactual scenarios involving essences are all about.

Metaphysical modality depends on fixing some set of worlds within which one member plays the role of the 'home world' from which all the other worlds are targeted as 'merely possible or non-actual'. But such a set of worlds would, *ex hypothesī*, not exhaust all of *logical* space. We suspect that the notion of metaphysical modality is fuelled by the illusion that philosophical investigations can somehow fix the modal span of at least some kinds of object. For instance, an acorn, genetically or otherwise tampered with, may turn into a cabbage rather than an oak, but surely not into an elephant or a wooden table. Or so the intuition goes.

But why not? It is hardly acceptable that the laws of nature of some particular set of worlds, for instance, those of the set of worlds containing the actual world as a member, should play any role in analytic philosophising, which is concerned with conceptual analysis. Yet this is exactly what happens when the empirical

[26] Worlds whose laws of nature deviate from the actual ones are what G. Priest calls 'nomologically impossible worlds' (1992, p. 292).

laws defining *V* are allowed to determine which properties *b* might possibly have had and which not. The kind of engineering that could possibly be applied to Forbes' *c* in *V* will be hedged in by the laws of *V*. It follows then that *c* may at most exhibit its full *physical*, or 'metaphysical', potential within *V*, but not its full *logical* potential. We are, therefore, in flat opposition to the second half of the quote by Forbes:

> In the time of a single world, the same individual can undergo a change of sex, but it is less clear that an individual of one sex could have been, from the outset, an individual of another [.] (Ibid., p. 148).

If 'from the outset' means from the beginning of time within *V* then the truth of the claim presupposes metaphysical necessity. If 'from the outset' means from the beginning of time within logical space *in totō* then bare particular anti-essentialism is only happy to embrace that possibility. The way we look at it, the question should not be whether anything can become anything else thanks to engineering, which is something drawing upon the notion of natural laws. Instead the question ought to be whether anything could turn into anything else thanks to logic. In the case of intensions, the answer is a resounding No. In the case of individuals, the answer is a no less resounding Yes. In logical space the sky is the limit (Which is not to say that TIL spills over into the space of logical impossibilities).

Our quarrel with temporal essentialism is not only to do with its stealing empirical laws into questions of essence. A narrower objection concerns *existence*. Consider this Closure:

$$\lambda w\ [\forall t\ [[{}^0 Wooden_{wt}\ {}^0 a] \equiv [{}^0 Exist'_{wt}\ {}^0 a]]].$$

Types: *Wooden*, $Exist'/(o\iota)_{\tau\omega}$. The point is this: individual *a* exists′ wherever and whenever *a* is wooden and is wooden whenever and wherever it exists; so *a* is essentially wooden.

Our objection concerns existence as an $(o\iota)_{\tau\omega}$-entity.[27] Within an intensional system the tendency would be to conceive of existence as something along the lines of an $(o(\alpha_{\tau\omega}))_{\tau\omega}$-entity: an empirical property of intensions. By contrast, *Exist′* above would come out a trivial intension, returning as it would for every $\langle w, t\rangle$ the set of those objects that are the elements of the universe of discourse. Existence, on our theory, is the property an intension *Int* exemplifies at those $\langle w, t\rangle$ pairs at which *Int* is occupied/instantiated. What is fundamentally at play is probably that when we speak of *individuals*, intending ι-entities (i.e., bare particulars), those who construe existence as a non-trivial property of what they call 'individuals' intend what we would take to be something like *persons*, typing personhood as $(o\iota)_{\tau\omega}$. For now it will suffice to observe that conceptualising *Person* as an intension turns it into the right sort of thing to come into and go out of existence non-trivially. Thus, rather than operating with varying domains we operate with

[27] See also Section 2.3.

modally and temporally varying extensions of *Person*. It is along these lines we would make sense of the claims that there might have been more or fewer persons, or that there might have been other persons than those who actually exist.

Tichý's strongest argument against varying domains is this:

> Suppose that an unactualized world *W* featuring a unique winged horse has been successfully specified. Will the winged horse of *W* constitute an example of an individual absent from the actual world? Not necessarily. Having wings is surely a contingent matter. Hence the horse which is winged in *W* will presumably be wingless in some other worlds. The actual world, where wingless horses are legion, may well be one of these worlds. Should this turn out to be the case, the individual in question would not be missing from the actual world after all. Thus in order to furnish an example of an individual which is actually missing, *W* would have to be specified as a world in which the [office] of the winged horse is filled by an individual numerically distinct from all individuals existing in the actual world. But how can this be done? If there are non-existent individuals, there will presumably be more than one. Clearly any world in which one of them is the winged horse is distinct from any world in which another one is. *W* won't be specified until it is specified *which* non-existent individual is its winged horse. The task of giving an example of a non-existent individual is thus hardly facilitated by appeal to the [office] *the winged horse*. To be able to exploit the [office] in pinpointing such an individual, one has to have an epistemic handle on the individual's numerical identity in the first place (1988, p. 181).

The argument, in a nutshell, is the following. A non-actual individual cannot be identified by ostension but only by description. So one might attempt to identify some numerically specific individual as the unique F at $\langle w, t\rangle$. But the individual office of the unique F will not be powerful enough to identify, or pinpoint, some numerically specific individual, for the occupant of the F-office at $\langle w, t\rangle$ will just be whoever or whatever is the unique F at $\langle w, t\rangle$. (Worse, the F-office may even fail to take a value at $\langle w, t\rangle$.) The specification of *which* (non-actual) individual is the unique F at $\langle w, t\rangle$ will thus be circular. This incapacity to pinpoint a numerically specific individual is shared by all offices. What is required is identification of an individual independently of its satisfying some condition at some $\langle w, t\rangle$. This brings us back to ostension; but again, ostension is inapplicable to non-actuals.[28]

If existence is no longer a property non-trivially applicable to individuals, but is instead a property of intensions, the construction $\lambda w\ [\forall t\ [[{}^0Wooden_{wt}\ {}^0a] \equiv [{}^0Exist_{wt}\ {}^0a]]]$, $Exist/(o(\iota_{\tau\omega}))_{\tau\omega}$, will simply involve a type-theoretic category mistake. It would be impossible, for this reason, to define non-trivial essential properties in terms of the (non-) existence of individuals. For instance, one among countless ways of defining equivalence classes of worlds is in terms of the existence of some particular individual *a*. The essential properties of *a* will be just those that *a*

[28] A recent discussion of a cluster of arguments whose conclusion is that Aristotle exists necessarily is a good example of what we have in mind (see Stephanou, 2000). Stephanou assumes that people would find the conclusion counter-intuitive because they would find it unacceptable that Aristotle should exist of necessity. But nobody is in a position to know whom Stephanou is talking about for want of a description of the intended individual. An individual office would have come in handy.

exemplifies in all worlds within that class. However, since existence applies only trivially to individuals, none of *a*'s properties exemplified anywhere will be both essential and non-trivial.

By adhering to a fixed domain of discourse, TIL adheres to ontological actualism. The only individuals we acknowledge are actual, eschewing merely possible individuals. Simchen also espouses ontological actualism in 2004, 2006,[29] but proposes a rival solution to how actualism can maintain

> [T]hat there are no merely possible things in the face of properties that are both actually uninstantiated and cannot be had contingently (2006, p. 9).

Simchen's solution centres around a notion of innate *potentiality* suggesting some brand of ('Aristotelian') realism *in rē*:

> An oak seed is no possible oak and a fertilized human egg is no possible human. But an oak seed and a fertilized human egg are *potentially* an oak and a human, respectively. Potentiality is a matter pertaining to what the seed and the egg might become. Potentiality is possible becoming (2006, p. 21).

The intuitive idea seems to be that an individual with a given property (e.g., the property of being a fertilized human egg) is a thing with a potential that circumscribes the modal variability of this thing and anything like it. For instance, though the properties of being a donkey and being a talker may be co-instantiated consistently, no donkey is potentially a talking donkey (nor is any talker potentially a talking donkey): 'So it is impossible that there be a talking donkey.' (Ibid., p. 8.)

But Simchen pays little attention to the fact that anchoring modal variability to potentiality goes via *nomological* modality. We may grant that, given the actual laws of nature, no donkey will have the potential, or make-up, to be able to talk. But, if a donkey lacking the actual potentiality to talk is transplanted to a world obeying relevantly different laws of nature (or perhaps none at all?), its potentiality thus embedded may (logically speaking) well manifest itself differently in such a way as to enable it to talk. Or the other way around with a talker being a donkey at such a world.[30] So the properties of being a donkey and a talker will have intersecting extensions at at least one $\langle w, t\rangle$. Whether the actual world and the present moment is such a pair can be established only empirically.

Our solution to the problem Simchen wishes ontological actualism to take on is this. If G (e.g., *being a mammal*) is actually uninstantiated,

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[29] Simchen claims, 'To be an [ontological] actualist requires dealing with the metaphysics directly and letting the logic track the metaphysics rather than the other way around.' (2006, p. 18.) We beg to differ, since this conceptual order of priority is tantamount to rejecting analytic philosophy as we know and love it. Analytic philosophy starts out with a logical analysis of expressions and concepts pertaining to a particular discourse on (say) metaphysics and only then enters the sphere of (say) metaphysics proper.

[30] Similarly, though less interestingly, it is logically possible that there be nomologically more restrictive worlds at which the donkeys' (the talkers') potential is a fraction of what it is at the actual world.

$$\lambda w \lambda t \neg[{}^0Exist_{wt}\ {}^0G]$$

and if G is a requisite of F (e.g., *being a whale*),

$$[{}^0Req\ {}^0G\ {}^0F]$$

then necessarily, if there had been F's then there would also have been G's. In case G is instantiated at any $\langle w, t\rangle$ only in its capacity as requisite, then G cannot be had contingently. That is, no individual is ever a mammal, pure and simple. Something is a mammal only in virtue of being some particular kind of animal (be it a zebra, a human being, a wombat, or whatnot).

Now, having rejected individual essentialism, our particulars are 'bare', not in the sense of lacking properties at any world and time, but in the sense of possessing no purely contingent properties of necessity.[31] The introduction of bare particulars is the only way to preserve the non-triviality of the predication that a is an F. Imagine now that there is an object before you that you wish to take a closer look at. After turning it inside out and upside down, you make the observations that it is a table, is wooden, is two metres long, and dark-brown. Could these four pieces of knowledge have been obtained a priori? Surely not. Only empirical inquiry can decide what is actually and presently true of the individual you are taking apart. At the beginning of the inquiry the individual can rationally be checked for any property whatsoever: is it a planet, a table, an elephant, a speck of dust, etc.? At this initial stage logic is no guide to any of its actual properties. As the results start coming in, logic will become useful, however. For instance, if the object before you is a Roman Catholic cardinal, you may infer, thanks to the requisites of cardinalhood, that the individual is also a human being, a man of faith, fluent in Latin, and a host of others. Also an infinite string of properties can be ruled out. Since no cardinal is inanimate, it follows that he is not inanimate, and since only inanimate objects can be planets, he cannot be a planet. The point is that the empirical investigation must begin from absolute scratch. If some purely contingent properties were true of the individual a priori, the empirical tests would already have something to begin from. But then it would not be informative to get to know that the object before you was a table, say, rather than a cardinal; it would be just as exciting as getting to know that the individual was self-identical. Yet it seems incontrovertible that by correctly ranking a among the tables and not among the cardinals you have made a discovery about the actual world: you have established that the actual world belongs to that set of worlds where it is true that a is a table. Had a radically different world been actual instead, a would not have been a table, but a cardinal, a banknote, a drop of water, or whatever, and your ranking a among the tables would have been a miss instead of a hit.

David Lewis (crediting Tichý with making him think less unfavourably of bare particulars) would call bare particular anti-essentialism *extreme haecceitism* (1986, pp. 293ff). A haecceitist is someone who thinks that above and beyond its

[31] For purely contingent properties, see Section 1.4.2.1. See Bergmann (1967, pp. 24ff) for the term 'bare particular'.

qualities an individual has a non-qualitative core. A haecceitist is extreme if no qualities are privileged in the sense of forming a protective belt around the core. Lewis, needless to say, has little time for haecceitism, but basically argues that if somebody wants to be a haecceitist then they would be much better off being an extreme haecceitist. The reason is that the latter discharges themselves of a burden that the former will have to lift. The burden is how to lay down the qualitative constraints which would constitute the protective belt of some individual (or species or natural kind as well, presumably). Certain choices of qualities might intuitively have something going for them, but justifying those intuitions is hard. We would add that it is hard also to formulate such a protective belt of qualities if those qualities are to be drawn from among purely contingent intensions without infringing their non-triviality. The situation is somewhat simpler for the extreme haecceitist. In Lewis' words,

> A moderate haecceitist says that there are qualitative constraints on haecceitistic difference; there is no world at all, however inaccessible, where you are a poached egg. Why not? He owes us some sort of answer, and it may be no easy thing to find a good one. Once you start it's hard to stop—those theories that allow haecceitistic differences at all do not provide any very good way to limit them. The extreme haecceitist needn't explain the limits—because he says there aren't any (Ibid., p. 241).

We draw from this the morale that since we are trafficking in bare particulars, we ought to make sure that they really are bare and not clad, however scantily, in a few select intrinsic non-trivial qualities. Otherwise we end up with individual essentialism. In Tichý's words,

> [T]he notion of object and that of an [intension] of an object are conflated and the result is presented as the doctrine of individual essentialism. According to this doctrine, the properties instantiated by an individual divide into two kinds: accidental and essential. Accidental properties are those that the individual might conceivably lack. Essential properties are those which the individual could not possibly lack. It is beyond dispute that every individual instantiates properties which are essential in this sense. Self-identity, and membership of any class to which the individual belongs, are examples of such. Elizabeth II, for example, could not possibly fail to be identical with herself, or fail to be a member of a class consisting of herself and Prince Philip, and so on. But the thesis of individual essentialism is to the effect that not all essential properties are of this trivial sort; some of them, it maintains, are substantive and their possession by an individual can be established only empirically (1988, p. 185).

That is, also TIL admits of a kind of individual essentialism, but of a hollow kind, since the necessity of $a = a$ or $a \in \{\ldots, a, \ldots\}$ is logical, not 'metaphysical'. There is nothing about those two necessities that could furnish a with a qualitative core.

It might seem as if we had flung the door open to anarchy at this point. As Stalnaker rightly observes, any individual might have had the properties of any other:

> [I]f [Babe Ruth] does have the logical potential to be a billiard ball, it is of no interest that he does since on the bare particular theory this does not distinguish him from anything else (Ibid., p. 349).

True, individuals are indistinguishable as far as their logical potential goes. What is possibly true of one individual is also possibly true of any other individual. Still, no two individuals can be (in) the extension of the same intensions at all the same possible worlds at the same time. For instance, no two individuals can be the extension of the intension *the King of France* at the same $\langle w, t\rangle$. Since individuals are ground types in TIL, they are logical atoms and, therefore, pairwise disjoint, their identity and difference being a matter of bare, or numerical, identity and difference. On the other hand, each and every individual gets to be King of France at some $\langle w, t\rangle$ or other. Individuals, which are the same for all worlds and times, are in and by themselves nothing but numerical individuators that exemplify any empirical property only contingently. As Ruth Barcan Marcus says in so many words, what we want is the 'description-neutral peg on which to hang descriptions across possible worlds' (1993, p. 61). Individuals as such are of little logical importance, since they are not themselves functions, but only functional arguments or values.

So while anarchy, if you like, does rule in the extensional basement, order reigns on the intensional ground floor in virtue of the requisites hosted there. The state space of any individual is bounded only by type-theoretic constraints. For instance, it is impossible for any individual to be a prime number, since functions of type $(o\tau)$ do not apply to ι-objects. But intensions are bounded not only by type theory but also by other intensions.[32] What we are interested in is studying the conceptual interplay between intensions, and not the interplay between intensions and extensions, which is roughly the question of which individuals have which properties at which worlds and times.[33]

For a final illustration of the distinction between intensional and extensional, or individual, essentialism, consider the difference between two construals *de dicto* and a construal *de re* of the sentence

'Wooden tables are necessarily wooden'.

The intensional, or *de dicto*, construals make it a necessary truth that wooden tables are wooden. The first construal expresses the Composition

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[32] Along similar lines, Jaakko Hintikka says, '[I]n the question, Who administers the oath to a new President?, the relevant alternatives might be the different officers (offices) (Secretary of State, Chief Justice, Speaker of the House, etc.) rather than persons holding them. Then my criterion of answerhood will require that the questioner knows what office it is that an answer refers to, not that he knows who the person is who holds it' (Hintikka, 1962, p. 45). Similarly, Fred Dretske says, 'Once an object occupies such an office, its activities are constrained by the set of relations connecting that office to other offices…; it must do some things, and it cannot do other things' (1977, pp. 264ff). To be sure, Dretske is concerned to make an analogy between legal and nomological modalities, but his discussion of what he himself dubs 'offices' is kindred to ours, particularly 'by talking about the relevant properties rather than the sets of things that have these properties' (ibid., p. 266).

[33] For examples of such interplay, see Sections 3.3.1, 4.3.

$$[{}^0Req_1\ {}^0Wooden\ \lambda w\lambda t\ \lambda x\ [[{}^0Wooden_{wt}\ x] \wedge [{}^0Table_{wt}\ x]]]$$

Type: *Wooden*, *Table*/$(o\iota)_{\tau\omega}$.

The second construal *de dicto* mixes *Wooden*/$(o\iota)_{\tau\omega}$ with the property modifier *Wooden′*/$((o\iota)_{\tau\omega}\,(o\iota)_{\tau\omega})$:[34]

$$[{}^0Req_1\ {}^0Wooden\ [{}^0Wooden'\ {}^0Table]].$$

The extensional, or *de re*, construal makes the sentence denote a falsehood, for now it expresses the Closure

$$\lambda w\lambda t\ \forall x\ [[[{}^0Wooden_{wt}\ x] \wedge [{}^0Table_{wt}\ x]] \supset \forall w'\ \forall t'\ [{}^0Wooden_{w't'}\ x]].$$

The so constructed proposition returns **T** at those $\langle w, t\rangle$ at which it holds that, for all x, if x exemplifies *Table* and *Wooden* then it is necessary that x exemplifies *Wooden*. Does any $\langle w, t\rangle$ satisfy this truth-condition? Yes; thanks to the truth-table for $\supset$, the condition is satisfied by all and only $\langle w, t\rangle$ pairs that falsify the antecedent. The proposition returns **F** for all the remaining $\langle w, t\rangle$-pairs.

By way of summary, it is rigid what the requisites of an intension are, and it is flexible who or what instantiates or occupies a given intension. This general point can be rephrased thus. Any instance of the instantiation or occupation relation between an individual a and a purely contingent intension *Int* is accidental; and: some instances of the co-instantiation or co-occupation relation between any two intensions Int_i, Int_j are analytically necessary, such that every instance of necessary co-instantiation or co-occupation of intensions is an instance of a *requisite* relation.

4.2.1 Quine's mathematical cyclist

Quine put forward his by now famous biking-mathematician example, in 1960, to create the paradox that it is both necessary and not necessary of the same individual that it be rational and bipedal. However, the argument rides on flat tyres, as pointed out in Plantinga (1974, Chapter 2), Marcus (1993, Chapter 1). At the same time, we agree with the purpose for which Quine put forward his argument; he wished to show that individual essentialism is incoherent. In the previous Section 4.2 we also argued against individual essentialism—but in favour of an essentialism of a different ilk, which we called *intensional essentialism*.

We can make explicit the fallacy of Quine's argument using the notion of requisite (see Section 4.1). Quine's argument goes as follows.

[34] See Section 4.4 on property modification.

1. *Mathematicians are necessarily rational but not necessarily bipedal.*
2. *Cyclists are necessarily bipedal but not necessarily rational.*
3. *Charles is both a cyclist and a mathematician.*
4. *∴Charles is necessarily rational but not necessarily bipedal.*
5. *∴Charles is necessarily bipedal but not necessarily rational.*

Contradiction, for conclusion (4) contradicts conclusion (5).

Let $M, R, C, B/(o\iota)_{\tau\omega}$); Ch/ι. Then

1′. $[{}^0Req\ {}^0R\ {}^0M], [{}^0\neg[{}^0Req\ {}^0B\ {}^0M]$

2′. $[{}^0Req\ {}^0B\ {}^0C],\ [{}^0\neg[{}^0Req\ {}^0R\ {}^0C]$

3′. $\lambda w\lambda t\ [{}^0C_{wt}\ {}^0Ch],\ \lambda w\lambda t\ [{}^0M_{wt}\ {}^0Ch]$.

The definition of the requisite relation between individual properties yields:

1″. $\forall w\forall t\ [\forall x\ [[{}^0M_{wt}x] \supset [{}^0R_{wt}x]]], \exists w\exists t\ [\exists x\ [[{}^0M_{wt}x] \wedge \neg[{}^0B_{wt}x]]]$

2″. $\forall w\forall t\ [\forall x\ [[{}^0C_{wt}x] \supset [{}^0B_{wt}x]]],\ \exists w\exists t\ [\exists x\ [[{}^0C_{wt}x] \wedge \neg[{}^0R_{wt}x]]]$.

From 1″ and 3′ we get

$$\lambda w\lambda t\ [{}^0R_{wt}\ {}^0Ch], \text{ but not } \forall w\forall t\ [{}^0R_{wt}\ {}^0Ch],$$

and from 2″ and 3′

$$\lambda w\lambda t\ [{}^0B_{wt}\ {}^0Ch], \text{ but not } \forall w\forall t\ [{}^0B_{wt}\ {}^0Ch].$$

It is not possible to derive $\lambda w\lambda t\ [\neg[{}^0B_{wt}\ {}^0Ch]]$ or $\lambda w\lambda t\ [\neg[{}^0R_{wt}\ {}^0Ch]]$. The point is that (3) is not *Charles is necessarily both a cyclist and a mathematician.*

Here explicit intensionalization has shown what also Marcus showed when pointing out that from $\Box(A \supset B)$ it follows that $(\Box A \supset \Box B)$, while $(A \supset \Box B)$ does not. The fallacy thrives on confusing the necessitation of the consequence with the necessitation of the consequent.

4.3 Requisites and substitution in simple sentences

The discussion of the semantics, pragmatics and logic of so-called simple sentences like ‘It is raining’ has received renewed attention over the last 10 years in the form of a substitution puzzle involving ‘Superman’ and ‘Clark Kent’.[35] The discussion is due to Saul (1997). According to Saul’s (negative) characterisation, simple sentences are ‘sentences which contain no attitude, modal, or quotational constructions’ (1997, p. 102, n. 1).

[35] See, for instance, Barber (2000), Forbes (1997, 1999), Moore (1999), Pitt (2001), Predelli (2004), and Spencer (2006). For further critique of Saul’s puzzles, see Jespersen (2008b).

In Section 2.7 we discuss the principle of substitution, claiming that the substitution of co-referential expressions is valid when the expressions occur in *de re* supposition. The 'Superman'/'Clark Kent' puzzle appears to throw doubts on the principle. In this section we show that the substitution principle *is* valid in the *de re* case. The 'puzzle' can be easily explained away by showing (a) that there are two readings (one *de dicto*, the other *de re*) of the sentence 'Superman is Clark Kent', such that on its *de re* reading 'Superman' and 'Clark Kent' are not necessarily co-referential, (b) that substitution is invalid due to a shift in time, and (c) that the *de dicto* reading of 'Superman is Clark Kent' is analysed as expressing that an antisymmetric requisite relation obtains between the two individual offices of Superman and Clark Kent.[36]

First, we examine whether a certain argument whose validity Saul has drawn into doubt is valid. We conclude, uncontroversially, that the argument is obviously valid, provided Leibniz's Law applies. Then we offer an alternative analysis of the premises and the conclusion based on the notion of requisite (see Section 4.1) and along the lines of the analysis of 'Hesperus is Phosphorus' (see Section 3.3.1). The TIL analysis is intended to bring out the rational core of the anti-substitution sentiments that the 'Millian' analysis is unable to bring out. On the 'Millian' analysis of 'Superman is Clark Kent' the sentence just means that an individual bearing two different names is self-identical. Our aim is to demonstrate that a purely semantic explanation of the anti-substitution intuitions rivalling the prevalent pragmatics-based ones is available.[37]

The puzzle we investigate here substitutes 'Clark Kent' for 'Superman' in 'Clark Kent enters the phone booth and Superman emerges' (see Saul, ibid., p. 102). Call it (*).[38] If the anti-substitution intuitions are correct, then (*) will at at least one world/time have this distribution of truth-values:

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[36] If being Superman is a *sufficient* condition for being Clark Kent, whereas being Clark Kent is a *necessary* condition for being Superman, it follows that the Superman office is *more exclusive* than the Clark Kent office, in the sense specified in Section 4.1, Definition 4.2, and the following Remark.

[37] Another attempt at a purely semantic approach is Forbes (1999), which introduces (so-called!) logophors that receive no mention in the sentences under analysis. What speaks against Forbes' proposal is, as Predelli observes (2004, p. 112), that Forbes' allegedly simple sentences are not simple, logophors being a quotational device.

[38] Saul'z puzzle bears some resemblance to the Partee puzzle from around 1970 (see Section 2.6):

The temperature is 90°F
The temperature is rising
———————————
90°F is rising.

Saul also wishes to come up with a flawed argument in order to make a point, but none of Saul's arguments in 1997 is invalid, provided Saul's semantic stipulations are accepted.

(*)

(1)	Clark Kent enters and Superman emerges	**T**
(2)	Superman = Clark Kent	**T**
(3)	Clark Kent enters and Clark Kent emerges	**F**

But then (*) would have to be an *invalid* argument. Yet, if (as Saul assumes) 'Superman' and 'Clark Kent' are 'Millian' names of individuals and if Leibniz's Law is valid, then the substitution of 'Clark Kent' for 'Superman' in (3) does go through. Hence, Saul's puzzle thrives on the collision between a valid argument and an intuition to the effect that the argument is, or ought to be, invalid.

Below we consider whether one-way and two-way substitution are valid. These are the general forms of one-way and two-way substitution, respectively:

$$\frac{\ldots a \ldots b \ldots}{\ldots a \ldots a \ldots}\ (\textit{one-way}_1) \qquad \frac{\ldots a \ldots b \ldots}{\ldots b \ldots b \ldots}\ (\textit{one-way}_2)$$

$$\frac{\ldots a \ldots b \ldots}{\ldots b \ldots a \ldots}\ (\textit{two-way})$$

Two-way substitution is trivially valid due to self-implication, if the expressions are co-denoting semantic proper names, the respective conclusions being but a rephrasing of a premise in the respective arguments. But substitution can be rendered non-trivial. The solution we offer below is an extensive elaboration of one of the several candidate solutions that Saul herself considers and rejects. The solution goes a long way toward accommodating her anti-substitution intuitions by validating only one-way substitution. At the same time, it also contains the extra means to validate two-way substitution in those cases when this ought to be validated, and to block it when it should not be validated. Not so with the 'Millian' approach to 'Superman', 'Clark Kent', which validates two-way substitution *tout court*.

The semantic solution we are proposing is capable of proving that (*) is valid, whereas another argument is invalid:

(**)

(1) Clark Kent enters and Superman emerges
(2) Superman = Clark Kent

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(3) Superman enters and Superman emerges

But then the semantics of 'Superman' and 'Clark Kent' must be different from the naïve one that 'Millianism' embodies.

Our solution to the phone booth puzzle is pivoted on, first, making both 'Superman' and 'Clark Kent' denote individual offices rather than individuals and, second, conjoining these two offices by an antisymmetric requisite relation (see Section 4.1). While anything remotely like requisites and the requisite relation seem to play no role in the extensive literature on the Superman puzzle, Saul gives individual offices (what she calls 'ordinary senses' of singular terms) short shrift by claiming that they cannot have any of the properties that apply to individuals, such as entering and emerging from phone booths. Of course, they cannot. But Saul overlooks the fact that if an individual office is extensionalized then an individual fully capable of entering and exiting from phone booths will emerge.

More specifically, we argue that a non-trivial semantic analysis of the example should take account of the *diachronicity* of Clark Kent's entrance and Superman's exit while preserving the internal link between being Superman and being Clark Kent. We suggest the following. If 'Superman' and 'Clark Kent' denote two different individual offices, then 'Superman is Clark Kent' no longer expresses the self-identity of an individual bearing two names, but the fact that two named offices are held together by the requisite relation: wherever and whenever someone occupies the office of Superman the same individual also occupies the Clark Kent office, whereas there are exceptions to the converse. This link is preserved by arranging the two offices in a requisite relation, such that the occupant of the Superman office co-occupies the Clark Kent office, while the converse is not always true, since the Clark Kent office may be occupied without the Superman office being occupied. The semantic analysis always validates the substitution of 'Clark Kent' for 'Superman', but validates the substitution of 'Superman' for 'Clark Kent' only if the additional condition is met that somebody should occupy the Superman office.

The *rule of substitution* that Saul tacitly assumes is Leibniz's Law of substitution of identicals for identicals. The general formulation of the rule is as follows, 'Φ' an n-ary predicate and 'μ', 'ν' singular terms:

(*Leibniz's Law*)

$$\frac{\Phi\langle\mu_1, \ldots, \mu_n\rangle \qquad \mu_i = \nu_i}{\Phi\langle\mu_1, \ldots, \nu_i, \ldots, \mu_n\rangle, \text{ for any } i \in (1, \ldots, n).}$$

First-order predicate logic with identity suffices throughout to spell out the relevant measure of logical structure in (*). It is obvious what the logical structure of the respective arguments is, once we assume this logical framework and accept Saul's assumption that the terms involved denote individuals.

$$\begin{array}{ll} (1) & Fa \wedge Gb \\ (2) & a = b \\ \hline (3) & Fa \wedge Ga \end{array}$$

Leibniz's Law assumes the following form in the case of (*):

$$\frac{\begin{array}{c}\Phi(\mu_1)\\ \mu_1 = \nu_1\end{array}}{\Phi(\nu_1).}$$

Conclusion (3) follows uncontroversially from {(1), (2)} via Leibniz's Law. So does (3′), as well as (3″):

(3′) $Fb \wedge Gb$
(3″) $Fb \wedge Ga$

But those who harbour anti-substitution sentiments may object that, although we *may* validly substitute both one-way and two-way, we *ought* not to do so at least in certain simple sentences. There are pragmatic constraints on the uses of '*a*', '*b*' that are not reflected in their semantics. Pragmatically speaking, '*a*', '*b*' are not interchangeable. Consonantly with this, Saul says, when outlining a similar response,

> We accompany our favourite standard semantic account with the explanatory claim that such truth-preserving substitutions may well yield sentences which are quite misleading, due to false pragmatic implicatures (1997, p. 106).

This suggests a two-tiered policy combining valid substitution with false implicatures.[39] The perhaps most important consequence of this policy is that it locates the origin of Saul's puzzle in pragmatics and not in semantics or logic. This sort of cohabitation between pragmatics and semantics has something to be said for it, *if* 'Superman' and 'Clark Kent' are names of individuals. It pretty much allows us to have our cake and eat it. One may admit that the conclusion is contrived or baffling while at the same time leaving the validity of Leibniz's Law unscathed by (*). Analogously, the 'paradoxes of material implication' are both almost universally deemed unnatural and are at the same time classically valid.

If 'Superman' and 'Clark Kent' denote individuals then, from a logical point of view, (*) is no puzzle at all.[40] Yet both one-way and two-way substitution do leave one with a sense of dodgy reasoning. In our view this uneasy feeling can be put down to the fact that the diachronicity of Clark Kent's entrance and Superman's

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[39] However, see Spencer (2006) for a convincing case that 'Russellian' philosophers of language in effect overstretch Grice's concept of implicature.

[40] Note that if *Clark Kent*, *Superman* are of type ι then both ^{0}*Kent* and ^{0}*Superman* occur with ι-intensional supposition in the premises. Thus two-way substitution is valid in intensional contexts; see Section 2.7 for the intensional rule of substitution.

exit is wholly absent from the (shallow) logical analysis considered so far.[41] It does seem to matter that it is *Clark Kent* who enters and, later, *Superman* who exits. An easy fix would be to construe Clark Kent and Superman as two different individuals, such that one guy enters and another guy later exits. But this 'fix' would falsify the second premise, that Superman is Clark Kent. So we need to preserve some internal link between being Superman and being Clark Kent. Only that link should not be self-identity.

The sentence 'Superman is Clark Kent' lends itself to two readings. On its *de dicto* reading it denotes the necessarily true proposition *TRUE* that the Clark Kent office is a requisite of the Superman office:

$$\lambda w\lambda t\ [{}^0Req_2\ {}^0Kent\ {}^0Superman].$$

That is,

$$\lambda w\lambda t\ [\forall w\forall t\ [[{}^0Occ_{wt}\ {}^0Superman] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [{}^0Superman_{wt} = {}^0Kent_{wt}]]]].$$

Types: $Req_2/(\text{o}\iota_{\tau\omega}\iota_{\tau\omega})$; $Occ/(\text{o}\iota_{\tau\omega})_{\tau\omega}$; $Superman$, $Kent/\iota_{\tau\omega}$; $True/((\text{oo}_{\tau\omega})_{\tau\omega})$.

On its *de re* reading the sentence denotes the properly partial proposition *P* constructed by the Closure

$$\lambda w\lambda t\ [{}^0Superman_{wt} = {}^0Kent_{wt}].$$

Though *P* comes close to being necessarily true, it is not equal to *TRUE.* There are worlds/times at which *P* lacks a truth-value; namely, those worlds/times at which either Superman or Clark Kent (or both) fails to exist.

We need to operate with two distinct instants of time, for Clark Kent's entering the phone booth cannot be simultaneous with Superman's exiting it without rendering 'Superman is Clark Kent' false—nobody, including superhuman aliens, can enter and exit in one go. So Clark Kent's entrance must precede Superman's exit. To bring out the temporal profile of 'Clark Kent went into the phone booth, and Superman came out' in the logical syntax, the truth-conditions spelt out below come with an explicit time indication to capture temporal variability. Let T_0, T_1 be moments of time, such that T_0 precedes T_1. These two times are those of Clark's entrance and Superman's exit, respectively. Further, let *W* be some possible world the scenario is set at.

Our semantic analysis validates two-way substitution only if the additional condition that somebody occupy the Superman office when Clark Kent enters is met, while the substitution of 'Clark Kent' for 'Superman' in '…exits…' follows unconditionally. So we always have one-way substitution, but two-way substitution only conditionally. The lack of symmetry is due to two factors. One is the

[41] The only other commentator that we know of to point out the *non-symmetry* between being Superman and being Clark Kent caused by the *diachronicity* between Clark Kent's entrance and Superman's exit is Zimmermann (2005, p. 55, pp. 68ff).

diachronicity between Clark Kent's entrance and Superman's exit. The other is the construal both of Superman and Clark Kent as individual offices, arranged in the following (antisymmetric) relation: necessarily, whoever occupies the Superman office co-occupies the Clark Kent office, though not always *vice versa*. In plain English, if you are Superman then you are also Clark Kent, while if you are Clark Kent then you may, or may not, be Superman. Consequently, in virtue of Leibniz's Law, whatever is true of the occupant of the Superman office is true of the occupant of the Clark Kent office, while again the converse is not always true.[42] The Clark Kent office being a requisite of the Superman office, if the occupant of the Superman office exits at WT_1 then so does the occupant of the Clark Kent office at WT_1, hence 'Clark Kent' may be substituted for 'Superman' in the second conjunct. But the occupant of the Clark Kent office enters at WT_0 without the occupant of the Superman office entering at WT_0, in case the Superman office is vacant at WT_0. As it stands, the argument does not allow us to infer that the Superman office is occupied at WT_0. Hence, 'Superman' may not be substituted for 'Clark Kent' in the first conjunct. The key to two-way substitution, then, consists in adding the premise that the Superman office is occupied at WT_0.

The intuition that (*) is invalid might be fuelled by the fact that {(1), (2), (3)} also lends itself to the following interpretation:

(1*) Clark Kent (who is not Superman yet) enters and Superman (hence also Clark) emerges
(2*) Superman = Clark Kent (at the moment of emerging)

(3*) Clark Kent enters and Clark Kent (but *not* Superman) emerges

This argument is obviously *invalid*. The situation depicted in the argument is a possible one; the occupant of the office of Clark Kent, distinct from the occupant of the office of Superman (as this office is vacant when Clark Kent enters) enters, and while in the booth he obtains the additional properties required to make him occupy the Superman office. Hence, the occupant of the Clark Kent office, when emerging, co-occupies the Superman office; from this it can be inferred that Superman exists. Therefore, it is impossible that anyone who would emerge and be Clark Kent would not also be Superman. Such a reading is borne out by pragmatic considerations: in order to pick out an individual, it makes good sense to denote the more exclusive office, if possible. When using a less exclusive office, we want to express the fact that some individual lacks some properties that are requisites of the more exclusive office. For instance, when referring to Johannes Ratzinger, we would typically use the term 'the Pope' and not 'the Head of State of the Vatican', the office of Head of State being a requisite of the office of Pope. If somebody

[42] The Superman office might just as well have been a requisite of the Clark Kent office, but since the sentence to be analysed is 'Superman is Clark Kent' and not 'Clark Kent is Superman' (cf. Saul, ibid., p. 104, display [11]), the antisymmetry is in this particular direction.

would use the latter term without mentioning Ratzinger's papacy, the hearer would typically suppose that Ratzinger had resigned as Pope.[43]

The interplay between occupancy and vacancy can be phrased in terms of rules. Let X, Y be variables ranging over individual offices and let Occ be the property of being occupied. Then:

$$\frac{\begin{array}{c}[{}^0Req_2\ Y\ X]\\ [{}^0Occ_{wt}\ X]\end{array}}{[{}^0Occ_{wt}\ Y]} \qquad \frac{\begin{array}{c}[{}^0Req_2\ Y\ X]\\ \neg[{}^0Occ_{wt}\ Y]\end{array}}{\neg[{}^0Occ_{wt}\ X].}$$

On the other hand, if $\neg[{}^0Occ_{wt}\ X]$ then neither $\neg[{}^0Occ_{wt}\ Y]$ nor $[{}^0Occ_{wt}\ Y]$ follows. And if $[{}^0Occ_{wt}\ Y]$ then neither $[{}^0Occ_{wt}\ X]$ nor $\neg[{}^0Occ_{wt}\ X]$ follows.

Let us generalise how the anti-symmetry between X and Y is decisive for which predications *de re* are true. If at $\langle w, t\rangle$ X_{wt} (i.e., the occupant of X) has the property H then at $\langle w, t\rangle$ Y_{wt} is also an H. But if at $\langle w, t\rangle$ Y_{wt} is an H then either X is occupied and its occupant is an H or X is vacant and it is not true that X_{wt} is an H. In terms of rules:[44]

First Rule of Predication de re (*P1*)

$$\frac{\begin{array}{c}[{}^0Req_2\ Y\ X]\\ [{}^0H_{wt}\ X_{wt}]\end{array}}{[{}^0H_{wt}\ Y_{wt}].}$$

Second Rule of Predication de re (*P2*)

$$\frac{\begin{array}{c}[{}^0Req_2\ Y\ X]\\ [{}^0Occ_{wt}\ X]\\ [{}^0H_{wt}\ Y_{wt}]\end{array}}{[{}^0H_{wt}\ X_{wt}].}$$

Since the requisite relation between any two offices holds for all worlds and times, 'Superman is Clark Kent' expresses (on the *de dicto* reading we are championing) a necessary truth. But it is not a requisite either of the Superman or the

[43] To the best of our knowledge, the offices of Pope and Head of State of the Vatican are distinct, and can be ordered in the requisite relation, such that the latter is a requisite of the former. This relation obtains on condition that it be conceptually possible that the office of Head of State is occupied while the papacy goes vacant. This scenario might obtain if, for instance, somebody is the political leader of the Vatican while nobody is its religious leader.

[44] What underlies both rules is the principle of predication *de re* explained in Section 2.6.

Clark office that whoever is its occupant at some given instant must be identical to whoever is the occupant either of the Superman or Clark office at some earlier or later moment. In other words, it is not necessary that there be diachronic co-occupation of both offices by the same individual 'throughout' the conjunction 'Clark Kent enters and Superman exits'. Thus, for instance, it is possible that whoever occupies the Clark office at WT_0 not be identical to whoever occupies the Superman office at WT_1. Consequently, if this possibility is realised, the one who is Clark at WT_0 is not the one who is Clark at WT_1.[45] Odd it may be; impossible not. There is no logically compelling reason why, for instance, the following scenario should not obtain: the occupant of the Clark office enters the phone booth at WT_0 and ceases occupying the office upon entering, whereas someone else already waiting inside exits at WT_1 either as the occupant of the Superman office (hence, also of the Clark office) or as the occupant of the Clark office (though not necessarily as the occupant of the Superman office).

Such a scenario cannot be articulated in a language that construes 'Superman' and 'Clark Kent' as 'Millian' names. Any such language obliterates the differences between (being) Superman and (being) Clark Kent and also renders the diachronicity between Clark's entrance and Superman's exit irrelevant, since the one who enters must be identical to the one who exits.

We are now able to specify our take on two-way substitution in Saul's phone booth argument when interpreted in terms of offices and requisites. Here is the argument in (slightly stilted) prose first.

(i) The Clark Kent office is a requisite of the Superman office
(ii) At WT_0, the Superman office is occupied
(iii) At WT_0, the occupant of the Clark Kent office enters, and at WT_1, the occupant of the Superman office exits

---

(iv) At WT_0, the occupant of the Superman office enters, and at WT_1, the occupant of the Clark Kent office exits.

Let $t_0, t_1 \to \tau$ such that $t_0 < t_1$ (t_0 preceding t_1). Let X, Y be variables ranging over individual offices ($X, Y \to \iota_{\tau\omega}$) and F, G variables ranging over properties ($F, G \to (o\iota)_{\tau\omega}$). Then the logical form (see Section 1.5.1, Definition 1.11) of the argument underlying this three-premise analysis of the inference of 'Superman enters the phone booth and Clark Kent exits' is

(1)	$[F_{wt0}\ Y_{wt0}] \wedge [G_{wt1}\ X_{wt1}]$	Assumption
(2)	$[{}^0Req\ Y X]$	Assumption
(3)	$[{}^0Occ_{wt0}\ X]$	Assumption
(4)	$[F_{wt0}\ Y_{wt0}]$	1, $\wedge$E

[45] Similarly, when in a monarchy the previous king is dead and the new king is proclaimed—'The king is dead. Long live the king!'—the old king and the new king are two different individuals.

(5)	$[F_{wt0}\ X_{wt0}]$	2, 3, 4, *P2*
(6)	$[G_{wt1}\ X_{wt1}]$	1, $\wedge$E
(7)	$[G_{wt1}\ Y_{wt1}]$	2, 6, *P1*
(8)	$[F_{wt0}\ X_{wt0}] \wedge [G_{wt1}\ Y_{wt1}]$	5, 7, $\wedge$I.

The two intermediate conclusions are (5) and (7). The main conclusion, (8), then follows by adjoining them by means of conjunction introduction. One-way substitution (invalidating two-way substitution) is obtained by leaving out (3), so that (5) cannot be inferred.

We finish by briefly addressing two other puzzles.[46]

Superman is more successful with women than Clark Kent	*Rab*
Superman is Clark Kent	$a = b$
Superman is more successful with women than Superman	*Raa*.

Superman leaps tall buildings, and Clark Kent does not	$Fb \wedge \neg Fa$
Superman is Clark Kent	$a = b$
Superman leaps tall buildings, and Superman does not	$Fa \wedge \neg Fa$.

As long as 'Superman', 'Clark Kent' are names of individuals, these two arguments are valid (though unsound). Consequently, Superman cannot outdo Clark Kent in anything. But since the plotline of the *Superman* comics drives us to accept that Superman *does* outdo Clark Kent (in courtship, in leaping tall buildings, etc.), the arguments seem to be puzzles. However, if 'Superman' and 'Clark Kent' are names of offices, then both sets of premises are *true* on their *de dicto* reading. The arguments then come out *invalid*, because the conclusions are false. The conclusion of the first argument is false, because the relation of being more successful is irreflexive; the conclusion of the second argument is a contradiction.

If Superman and Clark Kent are re-construed as offices, our take on the first puzzle is this. An individual occupying the office of Superman (thereby co-occupying the office of Clark Kent) is more successful with women than an individual occupying only the office of Clark Kent, because the office of Superman is (in some unspecified way) greater than the office of Clark Kent.

The puzzle of Superman, but not Clark Kent, leaping tall buildings can be solved in the same manner. The first premise cannot be true on its *de re* reading, for the first conjunct entails that Clark Kent leaps tall buildings, which contradicts the second conjunct. However, on its *de dicto* reading the first premise is true. It expresses again a relation between the offices of Superman and Clark Kent that

[46] The first is culled from Saul (1997, p. 103), while the second is adapted from 'Superman leaps tall buildings more frequently than Clark Kent', originally occurring in Joseph G. Moore (1999, p. 92, n. 1).

makes the former greater than the latter (assuming that greatness is exemplified by, for instance, leaping tall buildings). No occupant of the Clark Kent office ever leaps tall buildings, when the Superman office is vacant. When the Superman office is occupied, some of its occupants leap tall buildings. This is to say, due to the requisite relation between the two offices, that whenever Superman leaps tall buildings then Clark Kent also does so.

Formally, both puzzles are unravelled thus:

$$\frac{[{}^0Greater_{wt}\,{}^0Superman\,{}^0Kent]}{\begin{array}{c}\forall w\forall t\,[[{}^0True_{wt}\,\lambda w\lambda t\,[{}^0Suc_{wt}\,{}^0Kent_{wt}]] \supset [{}^0True_{wt}\,\lambda w\lambda t\,[{}^0Suc_{wt}\,{}^0Superman_{wt}]]] \wedge \\ \exists w^*\exists t^*\,[\neg[{}^0Suc_{w^*t^*}\,{}^0Kent_{w^*t^*}] \wedge [{}^0Occ_{w^*t^*}\,{}^0Kent] \wedge \neg[{}^0Occ_{w^*t^*}\,{}^0Superman]]\end{array}}$$

Gloss: 'Being successful as Superman is a necessary condition for Clark Kent to be successful (at whatever). When the Superman office is vacant, Clark Kent is not successful.'

4.4 Property modification and pseudo-detachment

Gamut (the Dutch equivalent of Bourbaki) claims that if Jumbo is a small elephant, then it does not follow that Jumbo is small (1991, §6.3.11). We are going to show that the conclusion does follow. To this end we define *the rule of pseudo-detachment* (PD). The rule validates a certain inference schema, which on first approximation is formalized as follows:

$$\frac{a \text{ is an } AB}{a \text{ is an } A}$$

where 'a' names an appropriate subject of predication (e.g., an individual or a property), while 'A' is an adjective and 'B' a noun phrase compatible with a.

The reason why we need the rule of pseudo-detachment is that A as it occurs in AB is a *modifier* and, therefore, cannot be transferred to the conclusion to figure as a *property*. If a is an individual and B a function of type $(o\iota)_{\tau\omega}$, whereas A is a function of type $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ then no actual detachment of A from AB is possible, and Gamut is insofar right. But PD makes it possible to replace the modifier A by the property A^* compatible with a to obtain the conclusion that a is an A^*. PD introduces a new property A^* 'from the outside' rather than by obtaining A 'from the inside', by extracting a part from a compound already introduced. The applicability of PD presupposes the validity of existential generalisation over properties and of substituting identical properties, something we are not going to doubt.

It might be objected, however, that the rule of pseudo-detachment is far too liberal. Apparently it is nonsensical for A to stand on its own, except when A is an intersective modifier. It seems indisputable that somebody can be happy, full stop, whence follows we may factor out 'happy' from 'happy B'. But the objection applies to two particular kinds of modifier. One kind are the non-subsective ones, which divide into *privative* like *forged* and *former*, and *modal* such as *alleged* and *apparent*. The other kind of modifier are *scalar* and other relative ones, e.g., *small* as in *small elephant* and *good* as in *good flutist*. Our claim that PD is logically valid entails that forged banknotes are forged and small elephants are small (though definitely not that forged banknotes are banknotes). For instance, if you factor out *small* from *small elephant*, say, the conclusion says that Jumbo is small, period. Yet this would seem a strange thing to say, for something appears to be missing: Jumbo is a small *what*? Nothing or nobody can be said to be small—or forged, temporary, larger than, the best, good, notorious, or whatnot, without any sort of qualification. A complement providing some sort of qualification to provide an answer to the question, 'a … *what*?' is required. Or so it appears. We are going to show why we, nonetheless, find the conclusion reasonable whatever the property and how to dismantle the objection that PD is an invalid rule.

First case. We consider the following argument valid:

$$\frac{a \text{ is a forged banknote and } b \text{ is a forged passport}}{a \text{ and } b \text{ are forged.}}$$

For instance, if the customs officers seize a forged banknote and a forged passport, they may want to lump together all the forged things they have seized that day, abstracting from the particular nature of the forged objects. This lumping together is feasible only if it is logically possible to, as it were, abstract *forged* from *a being a forged B* and *b being a forged C* to form the new predications that *a is forged* and that *b is forged*, which are subsequently telescoped into a conjunction.

Second case. We consider this argument valid, too:

$$\frac{c \text{ is a small elephant}}{c \text{ is small.}}$$

For instance, somebody may insist that c is a large mammal, a large land-living animal, and certainly much larger than any mouse around. But it ought to be possible to counter that c is also small, most other elephants dwarfing c.

Properties are scalar when requiring a scale for their application. Without a scale to differentiate small elephants from average-sized and large elephants, it becomes nonsensical to predicate smallness of an elephant. The reason is because nothing is absolutely small (or average-sized or large), but only small relative to something else. This 'something else' is other elephants, when we say about some elephant that it is small. Exactly how many, or which, elephants it takes to constitute a norm is another matter. What is important, from a logical and linguistic point of view, is that some scale or other be already in place. So it would seem a non-starter to argue that we may, nonetheless, predicate smallness of *c* without specifying a scale. As indeed it would be, as the rule stands. The claim that *c* is small invites the standard rejoinder, 'A small *what*?' But one thing is to indicate a specific scale; another thing is to indicate an unspecified scale. The essence of our solution is to claim that *c* is small with respect to some scale without stating which one. The obvious way to introduce an unspecified scale is to existentially quantify over scales. We model scales as properties: if Jumbo is a small elephant, it is with respect to the property of being an elephant that Jumbo is small. But now, if we introduce quantification over properties, does it not, then, become trivial to say that there is some property or other relative to which *c*, or any other individual, is small (or large, or whatnot)? Yes, it does. It is to say very little, virtually nothing, that there is some property with respect to which *c* is small. But this very triviality explains why we do not hesitate to embrace the rule of pseudo-detachment.

The temporary rule above is incomplete as it stands, for two related reasons. First, on our interpretation the two occurrences of *A* denote two different functions that are type-theoretically distinct. The first occurrence is as a modifier; the second, as a property: a distinction the rule above glosses over.[47] Secondly, therefore, the full pseudo-detachment rule must contain more premises to bridge between the original premise and the conclusion.

Here is the full pseudo-detachment rule, SI being substitution of identicals.[48]

(1)	*a* is an *AB*	Assumption
(2)	*a* is an (*A* something)	1, EG
(3)	*A** is the property (*A* something)	Definition
(4)	*a* is an *A**	2, 3, SI

[47] In *some* natural languages the two types seem to be flagged grammatically. For instance, 'Jumbo ist ein *kleiner* Elefant', but 'Jumbo ist *klein'*. Strictly speaking, however, we are imposing a particular interpretation on German grammar by claiming that the form '*kleiner*' as it occurs in 'ein *kleiner* Elefant' signals that the adjective denotes a modifier here. It could be objected that '*glückliches*' as it occurs in 'Karl ist ein *glückliches* Kind' is an intersective adjective and that the sentence has been generated by telescoping the conjunction 'Karl ist *glücklich*, und Karl ist ein Kind'. Though 'ein kleiner Elefant' and 'ein glückliches Kind' are grammatically on a par, 'klein' denotes a modifier and 'glücklich' a property. But even if we grant this point, we are still able to claim that the morphology of German grammar displays a grammatical link between 'kleiner' and 'Elefant' (which is absent in the corresponding English phrase 'small elephant') that shows that 'kleiner' calls for complementation, as is indeed characteristic of modifying predicates. ('Jumbo ist kleiner' is actually well-formed, but means that Jumbo is *smaller*, not small, and demands complementation.)

[48] More precisely, substitution of identical properties according to the intensional rule of substitution; see Section 2.7.

Let $[AB]$ be the property resulting from applying A to B, and let $[AB]_{wt}$ be the result of applying the property $[AB]$ to the world and time variables w, t to obtain a set, in the form of a characteristic function, applicable to a. Further, let = be the identity relation between properties, and let p range over properties, x over individuals. Then the *proof* of the rule is this:

1. $[[AB]_{wt}\, a]$	assumption
2. $\exists p\, [[Ap]_{wt}\, a]$	1, EG
3. $[\lambda x\, \exists p\, [[Ap]_{wt} x]\, a]$	2, β-expansion
4. $[\lambda w'\lambda t'\, [\lambda x\, \exists p\, [[Ap]_{w't'} x]]_{wt}\, a]$	3, β-expansion
5. $A^* = \lambda w'\lambda t'\, [\lambda x\, \exists p\, [[Ap]_{w't'} x]]$	definition
6. $[A^*_{wt}\, a]$	4, 5, Leibniz's Law

Any valuation of the free occurrences of the variables w, t that makes the first premise true will also make the second, third and fourth steps true. The fifth premise is introduced as valid by definition. Hence, any valuation of w, t that makes the first premise true will, together with step five, make the conclusion true. Therefore, the following argument is valid:

$$\frac{\lambda w\lambda t\, [[AB]_{wt}\, a];\;\; A^* = \lambda w'\lambda t'\, [\lambda x\, \exists p\, [[Ap]_{w't'} x]]}{\lambda w\lambda t\, [A^*_{wt}\, a]}$$

Here is an instance of the rule.

(1′) *a* is a forged banknote
(2′) *a* is a forged something
(3′) Forged* is the property of being a forged something
(4′) *a* is forged*.

If it is to be a logically valid rule, PD must apply indiscriminately to intersective, subsective, modal/intensional and privative modifiers (We do not consider the so-called modal modifiers, which appear to be well-nigh logically lawless[49]).

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[49] 'Modal modifier' is the term used by, e.g., Partee; see (2001, p. 7). Such modifiers are also known as 'intensional'. E.g., Cresswell (1978, p. 17) suggests that 'Arabella walked across the park for fifteen minutes' fails to entail, 'Arabella walked across the park', making *for* an intensional modifier. Nor does 'Arabella walked across the park for fifteen minutes' exclude that Arabella did walk across the park. Rotstein and Winter (2004, p. 276, n. 14) point out that, 'Many modifiers, especially intensional ones, are neither restrictive nor co-restrictive. For instance, the sentence *John is hopefully a good student* does not entail that John is a good student, and it does not entail that John is not a good student. Hence, the (sentential or predicational) modifier *hopefully* is neither restrictive nor co-restrictive.' Similarly, the intensional property modifier *alleged* allows that some alleged assassins are assassins while others are not. It is interesting to note a strong similarity between modal modifiers and *non-factive attitudes*. If, for instance, *a believes* that *b* is an assassin then it does not follow that *b* is an assassin, but nor does its negation. We suppose that a deeper study of modal/intensional modifiers (a hitherto marginalized kind of modifiers in formal semantics and linguistics) will reveal that many of them are attitudinal in nature, as exemplified by *a hoped-for result* or *being presumed innocent*.

Here is a taxonomy of the three kinds of modifier, $\|$ forming sets from properties.[50] (In Section 4.1 *requisite* was defined as a relation-in-extension of type $(o\alpha_{\tau\omega}\beta_{\tau\omega})$ that inputs an ordered pair of intensions and yields **T** iff the first element is a *requisite* of the second element.)

Intersective. 'If a is a *happy* child, then a is happy and a is a child'.

$$AB(a) \therefore A^*(a) \wedge B(a).$$
$$\text{Necessarily, } \|AB\| = \|A^*\| \cap \|B\|.$$
$$[{}^0Req\ \lambda w\lambda t\ [[A^*_{wt}\ x] \wedge [B_{wt}\ x]]\ [AB]].$$

Types: $A \to ((o\alpha)_{\tau\omega}(o\alpha)_{\tau\omega})$; $A^*, B \to (o\alpha)_{\tau\omega}$; $x \to \alpha$; $Req/(o(o\alpha)_{\tau\omega}(o\alpha)_{\tau\omega})$.

Thus the class of modifiers which are intersective with respect to a property F is defined as

$$\lambda g\ [{}^0Req\ [\lambda w\lambda t\ \lambda x\ [[g^*_{wt}\ x] \wedge [{}^0F_{wt} x]]]\ [g\ {}^0F]].$$

Types: $g \to ((o\alpha)_{\tau\omega}(o\alpha)_{\tau\omega})$; $F/(o\alpha)_{\tau\omega}$; $g^* \to (o\alpha)_{\tau\omega}$; $x \to \alpha$.

Intersectivity is the least interesting form of modification, since antecedent and consequent, or premise and conclusion, are equivalent. Still, even in the case of the apparently logically trivial intersectives we cannot transfer A from the premise to the conclusion. The reason, again, is that a modifier cannot also occur as a property. Hence A^* instead of just A.

Subsective. 'If a is a *skilful* surgeon, then a is a surgeon.'

$$AB(a) \therefore B(a).$$
$$\text{Necessarily, } \|AB\| \subseteq \|B\|.$$
$$[{}^0Req\ B\ [AB]].$$

Thus the class of modifiers which are subsective with respect to a property F is defined as

$$\lambda g\ [{}^0Req\ {}^0F\ [g\ {}^0F]].$$

The major difference between subsective and intersective modification is that subsectivity bans this sort of argument: $AB(a)$, $C(a) \therefore AC(a)$. Charles may be a skilful surgeon, and he may be a drummer too, but this does not make him a skilful drummer. Scalar properties are subsective modifiers. Again, Jumbo may be a small elephant, as well as a mammal, but this does not make Jumbo a small mammal.

[50] Hence $\|$ is an operation of extensionalising properties, which corresponds in TIL to B_{wt}.

Privative. 'If a is a *forged* banknote, then a is not a banknote', 'If b is an *ex*-Stalinist, then b is not a Stalinist.'[51,52]

$AB(a) \therefore \neg B(a)$.
Necessarily, $\|AB\| \cap \|B\| = \varnothing$

or equivalently,

Necessarily, $\|AB\| \subset \|\text{non-}B\|$.
$[{}^0Req\ \lambda w\lambda t\ [\neg[B_{wt}\ x]]\ [AB]]$.

Thus the class of modifiers which are privative with respect to a property F is defined as

$$\lambda g\ [{}^0Req\ \lambda w\lambda t\ \lambda x\ \neg[{}^0F_{wt}\ x]\ [g\ {}^0F]].$$

The pseudo-detachment schema is immune to whether A in AB is an intersective, subsective or privative modifier. Happy children are happy, skilful surgeons are skilful, fake Ming vases are fake. The reason is the same for all three. The existential generalisation in the pseudo-detachment schema quantifies B away, replacing AB by $\exists p(\mathrm{A}p)$. Since we impose no restrictions on which of the three kinds of modifier A in AB may be, it follows that anything that is capable of being an AB-object is *ipso facto* capable of being an Ap-object. Via the identification of Ap with A^*, the indifference to the particular nature of the modifier A is transferred to A^*.

Up until now we have been arguing intuitively why the pseudo-detachment schema ought to be valid and shown in prose how to assign a semantics to it that is capable of validating it. Now we commit ourselves to a full-fledged semantics in accordance with TIL. The functional application of A to B is the logical operation underlying the formation of a compound predicate 'AB' containing a modifying

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[51] In the second case, however, it can be inferred that b once was a Stalinist.

[52] Partee (2001) attempts to reduce privative modifiers to subsective modifiers so that 'the [linguistic] data become much more orderly' (ibid.). In her case guns would divide into fake guns and real guns, and fur into fake fur and real fur. Her argument is that only this reduction can do justice to the meaningfulness of asking the following sort of question: 'Is this gun real or fake?' At first blush, however, it would seem the question pre-empts the answer: if some individual is correctly identified as a gun, then surely it is a real gun, something being a gun if, and only if, it is a real gun. However, if we go along with the example, we think the argument is easily rebutted by putting scare quotes around 'gun' so that the question becomes, 'Is this 'gun' fake or real?' The scare quotes indicate that 'gun' is something like 'gun-like', including toy guns, which are not guns. If the answer is that the gun-like object is a fake gun (hence not a gun), the scare quotes stay on. If the answer is that it is a real gun (i.e., a gun), the scare quotes are lifted. Similarly with 'Is this 'fur' fake or real?' A more direct way of phrasing the question would be, 'Is this fur?', which does not pre-empt the answer and which does not presuppose that there be two kinds of fur, fake and real. For an intuitive test, ask yourself what the sum is of a fake 10-Euro bill and a 10-Euro bill. For a comparison between the respective kinds of procedural semantics of TIL and Martin-Löf's constructive type theory, see Jespersen and Primiero (forthcoming).

adjective '*A*' and a modified noun '*B*' and denoting the property *AB*. This is why *A* needs to be of type $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$.[53] Let *A*, *B*, *A** be constructions, typed as above. Then the Composition [*AB*] of *A* and *B* *v*-constructs the property of being an *AB*. The predication of this property of *a* proceeds as explained in Section 2.4.2:

$$\lambda w \lambda t\ [[AB]_{wt}\ {}^0a].$$

$[AB]_{wt}$ *v*-constructs the set of *AB*-things at $\langle w, t\rangle$. $\lambda w \lambda t\ [[AB]_{wt}\ {}^0a]$ constructs a proposition that is true at all and only those worlds and times at which *a* is in the extension of the property constructed by [*AB*]. Notice that the *w* and *t* parameters, for intensional descent, must be appended to [*AB*] and not to either of *A*, *B* in isolation. Wrong typing aside, the very point of employing 'modified' properties would be lost if $[AB]_{wt}$ were replaced by either $[A_{wt} B_{wt}]$ or $[AB_{wt}]$.

PD, dressed up in full TIL notation, is this:

$$\frac{\begin{array}{c}[[AB]_{wt}\ {}^0a] \\ [A^* = \lambda w \lambda t\ \lambda x\ {}^0\exists p\ [[Ap]_{wt} x]]\end{array}}{[A^*_{wt}\ {}^0a].}$$

Types: $\pi = (o\iota)_{\tau\omega}$; $\exists/(o(o\pi))$; $p/{*_1} \to_v \pi$; $A \to (\pi\pi)$; $A^*, B \to \pi$; $=/(o\pi\pi)$.

The whole argument stipulates that there is a logically necessary connection between two properties, *AB* and *A**. The stipulation is to the effect that whenever something is an *AB*-thing it is an *A**-thing. But the connection is established only via worlds, times and individuals, which accounts for the use of intensional descent. However, since we are already operating within an intensional system, why not link the two intensions directly? This can be done using the requisite relation. Here is how. The specific type of the relation we need here is $Req_1/(o\pi\pi)$: $[{}^0Req_1\ A^*\ AB]$.[54] When employing the schema of pseudo-detachment below, we shall condense it into this one-premise rule:

$$\frac{[[AB]_{wt}\ {}^0a]}{[A^*_{wt}\ {}^0a].}$$

The schema extends to all (appropriately typed) simple-type objects. For instance, let the inference be, 'Spelunking is an exciting hobby; therefore, spelunking is exciting'. Then *a* is of type π, $B \to (o\pi)_{\tau\omega}$, $A \to ((o\pi)_{\tau\omega}\ (o\pi)_{\tau\omega})$, and $A^* \to (o\pi)_{\tau\omega}$.

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[53] For the sake of simplicity we now consider only individual properties. Generalization to any type of property is straightforward.

[54] See Section 4.1, Definition 4.1.

Let us return to the first of the two examples set out above. We can now easily show why this argument must be valid:

$$\frac{\text{Charles has a forged banknote and a forged passport}}{\text{Charles has (at least) two forged things.}}$$

$$\frac{\begin{array}{c}\lambda w\lambda t\ \exists xy\ [[{}^0Have_{wt}\ {}^0Charles\ x] \wedge [{}^0Have_{wt}\ {}^0Charles\ y] \wedge \\ [[{}^0Forged\ {}^0Bank]_{wt}\ x] \wedge [[{}^0Forged\ {}^0Pass]_{wt}\ y] \wedge [{}^0\neq x\ y]]\end{array}}{\begin{array}{c}\lambda w\lambda t\ \exists xy\ [[{}^0Have_{wt}\ {}^0Charles\ x] \wedge [{}^0Have_{wt}\ {}^0Charles\ y] \wedge \\ [{}^0Forged^*{}_{wt}\ x] \wedge [{}^0Forged^*{}_{wt}\ y] \wedge [{}^0\neq x\ y]]\end{array}}$$

$$\overline{\lambda w\lambda t\ [{}^0Card\ \lambda x\ [[{}^0Have_{wt}\ {}^0Charles\ x] \wedge [{}^0Forged^*{}_{wt}\ x]] >= {}^02].}$$

Types: *Card*(*inality of a set of individuals*)/$(\tau(o\iota))$; *Bank*(*note*), *Pass*(*port*), *Forged**/$(o\iota)_{\tau\omega}$; *Have*/$(o\iota\iota)_{\tau\omega}$; *Forged*/$((o\iota)_{\tau\omega}\,(o\iota)_{\tau\omega})$.

Since *Forged* is privative, a forged banknote is *not* a banknote that is forged, such that there would be two kinds of banknotes: those that are genuine and those that are forged. The sum of four genuine banknotes and one forged banknote is four banknotes and not five (though five pieces of paper).[55] This is also to say that *Genuine* is an idle modifier: anything is a genuine *F* iff it is an *F*.[56] This is not to say that the same material object may not be genuine in one respect and fail to be genuine in another. For instance, an artefact being passed off as a paper banknote may fail to be a banknote (being a forged banknote), while being indeed made of paper (rather than polymer, say), thereby being a paper artefact. ('The "banknote" is fake, the paper is real').

Now we are going to tackle four conceivable objections to the validity of PD.

First objection. If Jumbo is a small elephant and if Jumbo is a big mammal, then Jumbo is not a small mammal; hence Jumbo is small and Jumbo is not small. Contradiction!

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[55] In colloquial speech we may ask, 'Is this a genuine banknote or a *Monopoly* banknote?', where it would be sufficient to ask, 'Is this a banknote or a *Monopoly* banknote?', *Monopoly* having the effect of a privative modifier not unlike *toy* in *being a toy gun*.

[56] There is a viable alternative to construing *Genuine*, *True* and suchlike as trivial modifiers, though not along the lines suggested by Partee (2001). Consider the sentence, 'True men of the desert know no fear'. It would be tempting to construe it as having only rhetoric import, as 'true' does in, 'True beer lovers prefer Czech Budweiser to American Budweiser'. But the property *True man of the desert* may also partition a set of men of the desert into those who are male desert dwellers and those who are male desert dwellers plus something more. The latter would be the natural-born male desert dwellers. They know no fear; the former may, and some of them no doubt will. One might, though need not, go one step further and construe *Knowing_no_fear* as a requisite of *True_man_of_the_desert*: $[{}^0Req\ Know\_no\_fear\ [{}^0True\ {}^0Man\_of\_the\_desert]]$.

The contradiction is only apparent, however. To show that there is no contradiction, we apply PD:

$$\frac{\lambda w \lambda t\, [[{}^0 Small\ {}^0 Elephant]_{wt}\ {}^0 Jumbo]}{\lambda w \lambda t\, \exists p\, [[{}^0 Small\ p]_{wt}\ {}^0 Jumbo]}$$

$$\frac{\lambda w \lambda t\, [[{}^0 Big\ {}^0 Mammal]_{wt}\ {}^0 Jumbo]}{\lambda w \lambda t\, \exists q\, [[{}^0 Big\ q]_{wt}\ {}^0 Jumbo].}$$

Additional types: *Small*, *Big*/$(\pi\pi)$; *Mammal*, *Elephant*/π; *Jumbo*/ι; $p, q/*_1 \rightarrow \pi$.

Now the only conclusion we can draw is that Jumbo is small (i.e., a small something) and big (i.e., a big something else). To obtain a contradiction, we would need an additional premise; namely, that, necessarily, any individual that is large (i.e., a large something) is not small (the same 'something'). Symbolically,

$$\forall w \forall t\, \forall x\, \forall p\, [[[{}^0 Big\, p]_{wt}\ x] \supset \neg[[{}^0 Small\ p]_{wt}\ x]].$$

Applying this fact to Jumbo, we have:

$$\forall w \forall t\, \forall p\, [[[{}^0 Big\, p]_{wt}\ {}^0 Jumbo] \supset \neg[[{}^0 Small\ p]_{wt}\ {}^0 Jumbo]].$$

This construction is equivalent to

$$\forall w \forall t\, \neg\exists p\, [[[{}^0 Big\, p]_{wt}\ {}^0 Jumbo] \wedge [[{}^0 Small\ p]_{wt}\ {}^0 Jumbo]].$$

But the only conclusion we obtained by applying PD expresses the construction:

$$\lambda w \lambda t\, [\exists p\, [[{}^0 Small\ p]_{wt}\ {}^0 Jumbo] \wedge \exists q\, [[{}^0 Big\ q]_{wt}\ {}^0 Jumbo]],$$

which obviously does not entail that

$$\lambda w \lambda t\, \exists p\, [[[{}^0 Small\ p]_{wt}\ {}^0 Jumbo] \wedge [[{}^0 Big\ p]_{wt}\ {}^0 Jumbo]].$$

Hence, no contradiction.

The conclusion ought to strike us as being trivial. If we grant, as we should, that nobody is absolutely good or absolutely bad, then everybody has something they do well and something they do poorly. And if we grant, as we should, that nobody and nothing is absolutely small or absolutely large, then everybody is made small by something and made large by something else. That is, everybody is both good and bad, which here just means being good at something and being bad at something else, without generating paradox:

$$\lambda w \lambda t\ \forall x\ [\exists p\ [[{}^0Good\ p]_{wt}\ x] \wedge \exists q\ [[{}^0Bad\ q]_{wt}\ x]].$$

But nobody can be good at something and bad *at the same thing* simultaneously:

$$\forall w \forall t\ \forall x\ \neg\exists p\ [[[{}^0Good p]_{wt}\ x] \wedge [[{}^0Bad\ p]_{wt}\ x]].$$

Additional type: *Good*, *Bad*/($\pi\pi$).

Second objection. It would appear that too liberal a use of pseudo-detachment, together with an innocuous-sounding premise, enables the following argument:

$$\frac{\text{Jumbo is a small elephant} \wedge \text{Mickey is a big mouse}}{\text{Jumbo is small} \wedge \text{Mickey is big}}$$

$$\frac{\text{If } x \text{ is big and } y \text{ is small, then } x \text{ is bigger than } y}{\text{Mickey is bigger than Jumbo.}}$$

Similarly, if Jumbo is a small elephant and Mickey a big mouse, we cannot deduce that Mickey is bigger than Jumbo. We can only infer the necessary truth that *if* x is a small something and y is a big object of the same kind, *then* y is a bigger object of that kind than x:

$$\forall w \forall t\ \forall x\ \forall y\ \forall p\ [[[[{}^0Small\ p]_{wt} x] \wedge [[{}^0Big\ p]_{wt}\ y]] \supset [{}^0Bigger_{wt}\ y\ x]].$$

Type: *Bigger*/$(o\iota\iota)_{\tau\omega}$. This cannot be used to generate a contradiction from these constructions as premises:

$$\exists p\ [[{}^0Small\ p]_{wt}\ {}^0a];\ \exists q\ [[{}^0Big\ q]_{wt}\ {}^0b]].$$

Geach (1956) launches an argument similar to the one we just dismantled to argue against a rule of inference has the same effect of PD of pseudo-detaching a property. He claims that that rule would license an invalid argument. And indeed, the following argument *is* invalid:

> *a* is a big flea, so *a* is a flea and *a* is big; *b* is a small elephant, so *b* is an elephant and *b* is small; so *a* is a big animal and *b* is a small animal (Ibid., p. 33).

But pseudo-detachment licenses no such argument. Geach's illegitimate move is to steal the property *being an animal* into the conclusion, thereby making *a* and *b* commensurate. Indeed, both fleas and elephants are animals, but *a*'s being big and *b*'s being small follow from *a*'s being a *flea* and *b*'s being an *elephant*, so pseudo-detachment only licenses the following two inferences, $p \neq q$:

$$\exists p\ [[{}^0Big\ p]_{wt}\ {}^0a];\ \exists q\ [[{}^0Small\ q]_{wt}\ {}^0b].$$

And a big p may well be smaller than a small q, depending on the values assigned to p, q.

Third objection. If we do not hesitate to use ‘small’ not only as a modifier expression but also as a predicate, then it would seem we could not possibly block the following fallacy:

Jumbo is small
Jumbo is an elephant

Jumbo is a small elephant.

$$\lambda w \lambda t\ \exists p\ [[{}^0Small\ p]_{wt}\ a]$$
$$\lambda w \lambda t\ [{}^0Elephant_{wt}\ {}^0a]$$

$$\lambda w \lambda t\ [[{}^0Small\ {}^0Elephant]_{wt}\ {}^0a].$$

But we can block this obviously invalid argument. The premises do not guarantee that the property p with respect to which Jumbo is small is identical to the property *Elephant*. As was already pointed out, one cannot start out with a *premise* that says that Jumbo is small (is a small something) and conclude that Jumbo is a small B.

Fourth objection. If it is valid to infer that Charles is happy from Charles’ being a happy child, then it would seem that if the premise is that a and b are French fries then a and b are French; which should not follow, of course. And if some piece of paper is a forged banknote then it appears to follow that the piece of paper is forged; which should not follow, of course.

The morale is that we must be careful not to mechanically apply the rule of pseudo-detachment without conducting a prior semantic analysis of the terms and the grammar of the premises. The first example concerns illegitimate substituends for ‘AB’; the second, illegitimate substituends for ‘a’.

First example.

a, b are French fries

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a, b are French.

Valid? No. ‘French fries’ is not a compound descriptive name of a property, describing things that are both French and fries, though the surface grammar of English would make it appear to be such a name. From a logical and semantic point of view, ‘French fries’ is a non-composite expression (an idiom) denoting the property of being some particular kind of sliced and fried potatoes. Not surprisingly, some other languages use grammatically simple expressions for this property, such as ‘patat’ in Dutch and ‘hranolky’ in Czech, with no ‘Franse’ or ‘francouzské’ appended.

Second example.

$$\frac{\text{This piece of paper is a forged banknote}}{\text{This piece of paper is forged}}$$

The conclusion might mean that a certain piece of paper is a forged piece of paper. But then we generate the contradiction that something is a piece of paper and also a forged piece of paper (i.e., a piece of non-paper). The contradiction comes about because nothing seems to block ‘forged’, which qualifies ‘banknote’ in the premise, from qualifying ‘This piece of paper’ in the conclusion. Such a reading of the argument is fallacious, however, because ‘This piece of paper is forged’ can be read either as above or as ‘This piece of paper is a forged *something’*. Obviously the latter is the correct reading that does follow from the premise and that does not generate paradox.

4.4.1 Malfunction: subsective vs. privative modification

Property modification is also indispensable when analysing properties like *being a malfunctioning F*, since something malfunctions only with respect to a property and not absolutely. The modifier *Malfunctioning* is also susceptible to pseudo-detachment. If a is a malfunctioning F, it follows that a is malfunctioning*; namely, with respect to the ∃-bound property p. Let *Malf* (*Malfunctioning*)/$((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; = $(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; $p \rightarrow (o\iota)_{\tau\omega}$; $x \rightarrow \iota$. Then this argument is valid:

$$\frac{\begin{array}{c}[[{}^0Malf\ {}^0F]_{wt}\ {}^0a] \\ \exists p\ [[{}^0Malf\ p]_{wt}\ {}^0a] \\ {}^0Malf^* = \lambda w\lambda t\ \lambda x\ \exists p\ [[{}^0Malf\ p]_{wt}\ x]\end{array}}{[{}^0Malf^*_{wt}\ {}^0a].}$$

A logically interesting question is whether the converse holds; i.e., whether a malfunctioning F (be it an organism or a device) is no less of an F for that.[57] For instance, is a malfunctioning heart a heart? Is a malfunctioning piston a piston?

[57] Proper-function theory holds that a malfunctions as an F iff a falls short of fulfilling its proper function, and systemic-function theory holds that a malfunctions as an F iff a lacks the current capacity to function as an F. See Kroes and Meijers (2006) and Jespersen and Carrara (ms).

Logically, this is the question whether *Malf* is subsective or privative. Here we do not take a stand either way, because this would require a thorough philosophical discussion that would stray too much from modification proper. Instead we show how the notion of requisite (see Section 4.1) may be useful in defining *Malf*, whether subsective or privative. Therefore, we distinguish between $Malf_s$ and $Malf_p$, where $Malf_s$ is subsective:

$$(Malf_s\ F)a \therefore Fa,$$

and $Malf_p$, privative:

$$(Malf_p\ F)a \therefore \neg Fa.$$

Once the definitions of $Malf_s$, $Malf_p$ in terms of requisites are in place, it is straightforward how to infer that a malfunctioning$_s$ F is an F and a malfunctioning$_p$ F is not an F.

The tricky bit consists rather in the bit of homework needed to be done before setting out the definitions of $Malf_s$, $Malf_p$. Technically, we need the modifier *Func_as* (for '*Functioning_as*'), as well as two additional mappings. *Func_as* forms the property *functioning_as_an_F* from the property F. One of the elements of the essence of F is a property specifying what F-objects are for. Thus, if F is *being a gun*, then let its *what-for* property be *firing_bullets*. So to function as a gun is to be used to fire bullets (without necessarily being *designed* as a gun, to leave room for improper, or unintended, use of an artefact).

As a notational convention, let 'π' abbreviate '$(o\iota)_{\tau\omega}$'. The first mapping we need is of type $(\pi(o\pi))$: given the essence of F as argument, the mapping extracts the property that specifies what F's are for; so this mapping is the '*what-for*' mapping. We dub it '*Extract*'. The second mapping we need is one we encountered in Section 4.1; namely, $Essence_2$, here specifically of type $((o\pi)\pi)$. Given F as argument, $Essence_2$ returns the essence of F; so *Essence* is the 'essence-extracting' mapping. Finally, let F' be the *what-for* property of some property F. Given F' as argument, *Func_as*, of type $(\pi\pi)$, yields as value the property of functioning as an F. The property *functioning_as_an_F* is then formed thus:

$$[{}^0Func\_as\ [{}^0Extract\ [{}^0Essence_2\ {}^0F]]].$$

If F is *being a gun*, as above, then $[{}^0Essence_2\ {}^0F]$ is the essence of *being a gun*, $[{}^0Extract\ [{}^0Essence_2\ {}^0F]]$ is the property *firing_bullets*, and $[{}^0Func\_as\ [{}^0Extract\ [{}^0Essence_2\ {}^0F]]]$ is the property *functioning as a gun*. The predication of that property of a then looks like this:

$$\lambda w \lambda t\ [[{}^0Func\_as\ [{}^0Extract\ [{}^0Essence_2\ {}^0F]]]_{wt}\ {}^0a].$$

What we have done here is merely spell out the bare logical bones of how to form the property *functioning_as_an_F*. This is not an ad hoc solution, however, but more of a schema of what any logical analysis of that property would have to look like, provided properties have essences and a what-for property. With $[{}^0Func\_as\ [{}^0Extract\ [{}^0Essence_2\ {}^0F]]]$ in place, we now show how to define the modifiers $Malf_s$, $Malf_p$ of type $(\pi\pi)$. We must define two mappings from π to π. The property figuring as functional argument is a property F. But what is going to be the property figuring as functional value? The property *being an F and not functioning as an F* in case of $Malf_s$ and the property *not being an F and not functioning as an F* in case of $Malf_p$. The definitions are as follows.

Definition 4.5 (***subsective malfunctioning: Malf$_s$***). Let $p, q \rightarrow \pi$; $Req/(o\pi\pi)$. Then the *subsective modifier* $Malf_s/(\pi\pi)$ is the mapping

$$\lambda p\ \iota q\ [[{}^0Req\ p\ q] \wedge$$
$$[{}^0Req\ \lambda w\lambda t\ \lambda x\ \neg[[{}^0Func\_as\ [{}^0Extract\ [{}^0Essence_2\ p]]]_{wt}\ x]\ q]]. \qquad \square$$

Corollary 1. The set $\{[{}^0Malf_s\ {}^0F], {}^0F\}$ is a subset of the *essence* of the property $[{}^0Malf_s\ {}^0F]$.

Definition 4.6 (***privative malfunctioning: Malf$_p$***). Let $p, q \rightarrow \pi$; $Req/(o\pi\pi)$. Then the *privative modifier* $Malf_p/(\pi\pi)$ is the mapping

$$\lambda p\ \iota q\ [[{}^0Req\ \lambda w\lambda t\ \lambda x\ \neg[p_{wt}\ x]\ q] \wedge$$
$$[{}^0Req\ \lambda w\lambda t\ \lambda x\ \neg[[{}^0Func\_as\ [{}^0Extract\ [{}^0Essence_2\ p]]]_{wt}\ x]]]\ q]]. \qquad \square$$

Corollary 2. The set $\{[{}^0Malf_p\ {}^0F], \lambda w\lambda t\ \lambda x\ \neg[{}^0F_{wt}\ x]\}$ is a subset of the *essence* of the property $[{}^0Malf_P\ {}^0F]$.

With these two definitions in place, the following two derivations are straightforward. Both derivations invoke the propositional property $True/(oo_{\tau\omega})_{\tau\omega}$ introduced in Section 1.5.2.1. Where $P \rightarrow o_{\tau\omega}$, the definition of *True* is:

$$[{}^0True_{wt}\ P]\ v\text{-constructs } \mathbf{T} \text{ iff } P_{wt}\ v\text{-constructs } \mathbf{T},$$
$$\text{otherwise } [{}^0True_{wt}\ P]\ v\text{-constructs } \mathbf{F}.$$

Thus, the rule of *True* introduction is

$$P_{wt} \vdash [{}^0True_{wt}\ P],$$

and the rule of *True* elimination,

$$[{}^0True_{wt}\ P] \vdash P_{wt}.$$

First, the derivation that a malfunctioning$_s$ F is still an F. We are to prove that the argument

$$\frac{\lambda w \lambda t\ [[\mathit{Malf}_s\ F]_{wt}\ a]}{\lambda w \lambda t\ [F_{wt}\ a]}$$

is valid. From Definition 1.13 of *valid argument*, it follows that we are to prove that for any $\langle w, t\rangle$ at which $[[\mathit{Malf}_s\ F]_{wt}\ a]$ v-constructs **T**, $[F_{wt}\ a]$ v-constructs **T** as well.

(1)	$[[\mathit{Malf}_s\ F]_{wt}\ a]$	Assumption
(2)	$[\mathit{True}_{wt}\ \lambda w \lambda t\ [[\mathit{Malf}_s\ F]_{wt}\ a]]$	1′, *True* I
(3)	$[\mathit{Req}\ F\ [\mathit{Malf}_s\ F]]$	Corollary 1
(4)	$\forall w' \forall t'\ \forall x\ [[\mathit{True}_{w't'}\ \lambda w \lambda t\ [[\mathit{Malf}_s\ F]_{wt} x]] \supset [\mathit{True}_{w't'}\ \lambda w \lambda t\ [F_{wt}\ x]]]$	Definition 4.1
(5)	$[[\mathit{True}_{wt}\ \lambda w \lambda t\ [[\mathit{Malf}_s\ F]_{wt}\ a]] \supset [\mathit{True}_{wt}\ \lambda w \lambda t\ [F_{wt}\ a]]]$	4, ∀E, *a*/*x*, *w*/*w*′
(6)	$[\mathit{True}_{wt}\ \lambda w \lambda t\ [F_{wt}\ a]]$	2, 5, MPP
(7)	$[F_{wt}\ a]$	6, *True* E.

Second, the derivation that a malfunctioning$_p$ F is not an F:

(1′)	$[[\mathit{Malf}_p\ F]_{wt}\ a]$	Assumption
(2′)	$[\mathit{True}_{wt}\ \lambda w \lambda t\ [[\mathit{Malf}_s\ F]_{wt}\ a]]$	1′, *True* I
(3′)	$[\mathit{Req}\ [\lambda w \lambda t\ [\lambda x\ \neg[F_{wt}\ x]]]\ [\mathit{Malf}_p\ F]]$	Corollary 2
(4′)	$\forall w' \forall t'\ \forall x\ [[\mathit{True}_{w't'}\ \lambda w \lambda t\ [[\mathit{Malf}_p\ F]_{wt} x]] \supset [\mathit{True}_{w't'}\ \lambda w \lambda t\ \neg[F_{wt}\ x]]]$	Definition 4.1
(5′)	$[[\mathit{True}_{wt}\ \lambda w \lambda t\ [[\mathit{Malf}_s\ F]_{wt} a]] \supset [\mathit{True}_{wt}\ \lambda w \lambda t\ \neg[F_{wt}\ a]]]$	4′, ∀E, *a*/*x*, *w*/*w*′
(6′)	$[\mathit{True}_{wt}\ \lambda w \lambda t\ \neg[F_{wt}\ a]]]$	2′, 5′, MPP
(7′)	$\neg[F_{wt}\ a]$	6′, *True* E.

These two definitions and derivations apply across the board to any modifier that comes in both a subsective and a privative variant, so there is insofar no particular reason to exemplify the distinction between subsective and privative modification by means of *Malf* rather than any other modifier also susceptible to this bifurcation. But it is philosophically interesting that both variants apply to *Malf*, since the respective associated notions of (biological or technical) *function* are going to be somewhat different.[58]

Having presented both variants of *Malf*, we will put them to good philosophical use by showing how to solve the following puzzle. Consider multiple-function artefacts. Jesse Hughes (2009, n. 12, §4) brings up the example of a claw hammer, saying that a claw hammer with a broken claw functions properly with respect to pounding nails, but not with respect to prying nails. Hughes is right about this. But what general lessons can be extracted? Assume that to function as a claw is to pry nails and to function as a hammer is to pound nails. Then there are two options. The first option is this: something is a functioning claw hammer if, and only if, it functions as a claw *and* it functions as a hammer. The second option is this: something is a functioning claw hammer if, and only if, it functions as a claw *or* it functions as a hammer (*or*, inclusive disjunction). The first option entails this: something is a malfunctioning claw hammer if, and only if, it fails to function as a claw or it fails to function as a hammer. That is, it is possible that something be a malfunctioning claw hammer and still function as a claw or as a hammer, depending on whether its clawing capacity or its hammering capacity is compromised. The second option entails this: something is a malfunctioning claw hammer if, and only if, it fails to function as a claw and it fails to function as a hammer. That is, a malfunctioning claw hammer functions neither as a claw nor as a hammer. As is seen, the two respective entailments are instances of De Morgan's laws: the negation of a conjunction is equivalent to the disjunction of the negations of its conjuncts; the negation of a disjunction is equivalent to the conjunction of the negations of its disjuncts.

The difference between the subsective and the privative view of malfunction can be schematised in the following manner. Let *Claw* be a property modifier and *Claw** a property. Then the subsective view validates the inference

$$\frac{\dfrac{[[^0 Malf_s\ [^0 Claw\ ^0 Hammer]]_{wt}\ ^0 a]}{[[^0 Claw\ ^0 Hammer]_{wt}\ ^0 a]}}{[^0 Hammer_{wt}\ ^0 a]}$$

That a claw hammer is a claw is in turn inferable via the rule of pseudo-detachment. The privative view validates the inference

[58] For details, see Jespersen and Carrara (ms.).

$$\frac{\dfrac{[[{}^0Malf_p\ [{}^0Claw\ {}^0Hammer]]_{wt}\ {}^0a]}{\neg[[{}^0Claw\ {}^0Hammer]_{wt}\ {}^0a]}}{[\neg[{}^0Claw*_{wt}\ {}^0a] \vee \neg[{}^0Hammer_{wt}\ {}^0a]]}$$

That is, a malfunctioning claw hammer is a non-(claw hammer) that is either not a claw or not a hammer, or neither a claw nor a hammer.

So what is the essence of a multiple-function tool like a claw hammer? It is the union of the sets of essential properties defining each of its functions. Where *being a claw* and *being a hammer* each comes with a set of essential properties, *being a claw hammer* has as its essence the union of these two sets.

4.5 Nomological necessity

By 'nomological or nomic necessity' we understand the sort of necessity that pertains to laws of nature. We are not attempting to analyze causality, which we expect to possess a somewhat more elaborate modal profile than the one we suggest for nomic necessity.

Nomic necessity is logically contingent, so the source of the universality that any kind of necessity requires is another. We obtain universality by suspending temporal variability, which means that it is true (false) at all instants of time that if so-and-so then so-and-so, or that a certain equality between magnitudes obtains. For instance, for all times t, for all individuals x, if x is hot air at t, then x rises at t. This seems to correspond to the sound intuition that the laws of nature are always the same, yet might logically speaking have been different.[59] Nomic necessity is a law-bound correlation between two empirical properties or two propositions (states-of-affairs, events), such that if one is exemplified or obtains then so must the other, or between two magnitudes such that they are equal at all t.

Our position on laws of nature is kindred to the *universalism* of Armstrong, Tooley and Dretske, especially because we are also after contingent ('soft') necessitation and apply a top-down approach starting out with universals. But ours is importantly different in at least one regard. The universalist contention that physical necessity is a relation between universals—$N(F, G)$, in universalist notation—can be expressed by means of *Req*. Thus, if F, G are properties of type $(o\iota)_{\tau\omega}$ then we have $[{}^0Req\ {}^0G\ {}^0F]$. Similarly, if P, Q are propositions then we have $[{}^0Req'\ {}^0Q\ {}^0P]$, meaning that, for all $\langle w, t\rangle$, if P is true at $\langle w, t\rangle$ then so is Q. But the fact that this can be done goes to show that universalism is too strong, as soon as the physical necessities that apply to some logically possible worlds are supposed not to extend

[59] Cf. Mitchell (2000, p. 247): 'Laws are about our world for all time.' However, we bracket the question of whether theoretical physics will eventually bear out this assumption.

to all other logically possible worlds as well.[60] Thus, $[^0Req\ ^0Q\ ^0P]$ is definable as *P entails Q*; yet this is surely too strong a relation to capture nomic necessity.

On the other hand, this is too weak:

$$\lambda w \lambda t\ [\forall x\ [^0F_{wt}\ x] \supset [^0G_{wt}\ x]].$$

This amounts to an empirical generalisation holding for a set of times relative to a set of worlds. It is incapable of guaranteeing that the implication from being an F to being a G may not previously have failed to hold or may not later fail to hold. If true, it just reports the fact that at the given time of evaluation it is true that all F's are G's.

This would seem to steer the right course, though:

$$\lambda w\ [\forall t\ \forall x\ [^0F_{wt}\ x] \supset [^0G_{wt}\ x]].$$

What is constructed is a set of worlds; namely, the set of worlds at which it holds for all times and all individuals that F's are G's. This does not exclude the logical possibility of counterlegals, only not within this set of worlds. So the Closure arguably succeeds in pinning down at least one essential feature of nomic necessity.

The type $((o o_{\tau\omega})\omega)$ is the type of propositional properties—given a world, we are given a set of propositions; to wit, those eternally true at the given world. One such empirical property is the property of being a nomically necessary proposition. Thus, the Closure

$$\lambda w\ [\lambda p\ \forall t\ p_{wt}]$$

constructs a function from worlds to sets of propositions. The set is the set of propositions p that are eternal at a given world. Nomically necessary propositions constitute a subset of this set.[61]

Some laws are phrased as *generalizations*: 'As a matter of nomic necessity, all F's are G's'. Others are phrased as *equations*: 'As a matter of nomic necessity, the magnitude M is proportional to the magnitude N'. The best-known example of the latter is probably Einstein's 1905 equation of mass with energy,

$$E = mc^2.$$

It bears witness to the thoroughgoing mathematization of physics that the syntactic form of the formula does not reveal that the proportion between energy and mass times the speed of light squared is empirical. Assuming that the special theory of

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[60] As pointed out in Materna (2005, n. 1, p. 62), the source of the problem is that N is a relation-in-*extension*, according to Dretske (1977, p. 263), aligning N with mathematical and logical relations.

[61] See Materna (2005).

relativity is true, Einstein's discovery of this equivalence was an empirical one. What he discovered was the physical law that two particular magnitudes coincide or are proportional to one another. A unit of energy will be equal to the result of multiplying a unit of mass by the square of the constant of the speed of light. So his equation will be an instance of the following logical form:

$$\lambda w\,[\forall t\,[M_{wt} = N_{wt}]].$$

Types: $M, N \rightarrow \tau_{\tau\omega}$ (i.e., constructions of magnitudes); $=/(o\tau\tau)$.

When making explicit the empirical character of $E = mc^2$, it is obvious that E, m must be modally and temporally parameterized. But so must c. Though a constant value, the value is constant only relative to a proper subset of the space of all logically possible worlds. It is a logical possibility that in at least some nomologically deviant universe light will have a different velocity.[62] Einstein's equation is constructible thus:

$$\lambda w\,[\forall t\,[[{}^0\mathit{Mult}\ m_{wt}\ [{}^0\mathit{Square}\ {}^0c_{wt}]] = E_{wt}]].$$

Types: *Mult(iplication)*/($\tau\tau\tau$); *Square*/($\tau\tau$); $=/(o\tau\tau)$; $E, m \rightarrow \tau_{\tau\omega}$; $c/\tau_{\tau\omega}$.

What is constructed is the set of worlds at which it is eternally true that E_{wt} is identical to the result of multiplying m_{wt} with the square of c_{wt}.

The crux of our conception of nomic necessity is as modally flexible and temporally rigid. But 'freezing' the temporal parameter is not uncontroversial. For would it not be analytically possible for something both to be a law of nature and either change or be replaced over time? And if there are laws of nature now, were they always in place? In particular, to put it naïvely, if there was a Big Bang at the dawn of time, were the (presumed) laws and the values of the (presumed) constants settled simultaneously with the inception of the physical universe or did a fraction of whatever time unit pass before they were? It would be tempting to answer Yes and No, respectively, backing the answer up by a definition of law of nature to the effect that such laws apply to *all* times within a set of worlds. However, we are open to the possibility that something more deserving of the predicate 'law of nature' may have a somewhat more complicated modal profile. Thus, it is probably a non-negotiable constraint on any viable notion of nomic necessity that it be logically and analytically contingent.[63] But this constraint will most probably

[62] Though we acknowledge that essentialists about the velocity of light will claim that c is the same value for all logically possible physical universes. This is not to say that light will travel at the speed of c in all logically possible universes; for at some of them light will not travel at all or light will be missing altogether. So it still constitutes a non-trivial, empirical discovery that the speed of light is c and not any other numerical value.

[63] So-called *necessitarians* flatly deny, of course, that logical contingency is a constraint at all. Instead (strong) necessitarianism holds that the laws of the actual world are identical to the laws of all other logically possible worlds. For a clear statement of (strong) necessitarianism, see Bird (2004). Bird's theory is based on the highly problematic premise of dispositional essentialism,

turn out not to be sufficient. It thus remains a partly open issue how to exactly capture nomic necessity in TIL.

4.6 Counterfactuals

It is a well-established constraint on laws of nature that they must sustain counterfactuals in contrast to cosmic coincidences, which do not. The idea is this: *x* is not hot air; but if it had been, then *x* would have risen, as a matter of nomic necessity. By contrast, as a matter of cosmic coincidence, each *x* that is hot air also rises. But this generalization is not guaranteed to extend to any or every new *x* that is hot air, so it may happen, as a matter of fact, that *x* is hot air and *x* fails to rise.

We are sympathetic to this constraint, for it ought to be a defining feature of laws of nature that they are capable of issuing just this sort of guarantee for any universe in which they hold sway. It seems obvious, however, that the relationship between laws and counterfactuals needs to be another than so-called scientific essentialism claims, as convincingly argued in Lange (2004). The root of the problem is that scientific essentialism (e.g., Ellis' dispositional essentialism) does not keep natural kinds and laws sufficiently separate. According to essentialism, a natural kind such as electrons comes with an essence, e.g., in the form of an electric charge, that determines which lawful proportions electrons enter into. Therefore,

> [T]he laws in which […] a natural kind figures must be laws in any world in which that […] natural kind exists. In a world with different […] natural kinds, the law is again true—vacuously—but it is not a law in that world (Lange, 2004, p. 227).

That is, if you have electrons then you thereby also have certain laws; so without those laws you do not have electrons. But this imposes excessively narrow constraints on counterfactuals:

> If a counterlegal automatically posits an entirely new population of natural kinds, then essentialism cannot readily account for the preservation of certain laws […] under that counterlegal supposition (Ibid., p. 233).

Essentialism appears in fact to make counterlegals impossible, for a counterfactual involving electrons cannot also be a counterlegal on pain of either not admitting electrons into that counterfactual scenario or redefining the notion of electron by means of that contrary-to-fact law. Yet

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which is a variant of extensional essentialism as applied to natural kinds. Given that properties are defined (and individuated) in terms of their dispositional essences, it is little wonder that if (as Bird argues) all logically possible worlds share the same properties then all worlds must share the same laws. But it does leave one wondering how the strong necessitarians avoid making it cognizable a priori what the laws of nature are.

> Essentialism was supposed to *explain why* the laws and natural kinds would have been no different under various counterfactual perturbations. (Ibid., p. 234.)

In our view the logical root of the predicament of scientific essentialism is that it is a version of *extensional* essentialism (as discussed in Section 4.2). One obvious way out of the problem that Lange has raised would be to define natural kinds and laws of nature independently of one another, so that electrons (and not just near-identical replicas) may exist in a universe where some or all of those laws deviate from the actual ones. Natural kinds could be defined in terms of requisites (see Section 4.1) and laws of nature at least partially as sketched in Section 4.5.

For want of a fully-fledged notion of nomic necessity, we are not yet in a position to demonstrate how laws of nature exactly sustain counterfactuals. We do have, however, a full theory of counterfactuals. The problem of counterfactuals can be illustrated by the following example.

(Cond) '*If* Charles had owned something, then *he* would have taken care of *it*.'

The sentence indicates what would be the case if its antecedent *were* true. This is to be contrasted with an indicative conditional, which indicates what the case is if its antecedent *is* true. The latter would be expressed by the sentence

'If Charles owns something, then he takes care of it'

and analysed simply by means of material implication.

The problem of how to analyse counterfactual sentences like (Cond) is a wide-ranging one. Here we just outline the gist of the solution based on Tichý's tacit-premise theory.[64] Tichý proposes an amendment of the Mill-Ramsey-Chisholm theory, which is the following. A counterfactual sentence expresses a construction of the form

$$\lambda w \lambda t \, [A \angle B]$$

where w, t can occur free in the construction A and/or in the construction B (A, $B/*_1 \rightarrow_v o_{\tau\omega}$) and the implication function $\angle$ is of type $(o \; o_{\tau\omega} \, o_{\tau\omega})$, taking a pair of propositions to a truth-value. This function takes the value **T** if in all the world-time couples in which the proposition v-constructed by A is true it holds that also the consequent proposition v-constructed by B is true. Moreover, parts of the construction A are often tacitly understood rather than explicitly spelled out in the antecedent of the conditional sentence. The reason for using the implication function $\angle$ (instead of the common material implication $\supset$) is the fact that arguments of this function can often be v-constructed by open propositional constructions or picked out by a propositional office of type $(o_{\tau\omega})_{\tau\omega}$, as for instance in the sentence,

[64] For details, see Tichý (1984, 2004, pp. 543–75).

'If John's most favourite proposition had been true
then he would have weighed a ton.'

Hence, a conditional sentence is not an analytical sentence, as it does not express a construction of the proposition *TRUE*. Rather it is an *empirical* sentence. Informally, the explicitly stated antecedent proposition is itself too weak to logically imply the consequent proposition. However, the proposition denoted by the conditional sentence is nevertheless true, if in all those worlds and times $\langle w^*, t^*\rangle$ that differ from the actual $\langle w, t\rangle$ only in some obvious aspects, the antecedent proposition implies the consequent proposition. The antecedent proposition is then v-constructed in such a way that in $\langle w^*, t^*\rangle$ that tacit assumption is true (hence 'counterfactual').

In (Cond) the tacit premise is the proposition that Charles does not own anything. Hence, the sentence can be explained as expressing the proposition that Charles does not own anything, but in all the worlds and times that are the same as the actual, except that Charles does own something, it is true that Charles takes care of his belongings.

Let us analyse first a simpler case of the conditional statement without taking into account the anaphoric reference of 'he':

(Cond1) 'Charles does not own anything,
but if he had owned something then he would have taken care of it.'

Types: *Charles*/ι; *Own* (something)/$(o\iota\iota)_{\tau\omega}$; (take) *Care* (of something)/$(o\iota\iota)_{\tau\omega}$; x, $y \rightarrow_v \iota$.

(Cond1′)
$$\lambda w \lambda t\, [\lambda w^* \lambda t^*\, [\neg\exists x\, [{}^0Own_{wt}\, {}^0Charles\ x] \wedge \exists x\, [{}^0Own_{w^*t^*}\, {}^0Charles\ x]] \angle \lambda w^* \lambda t^*\, \forall y\, [[{}^0Own_{w^*t^*}\, {}^0Charles\ y] \supset [{}^0Care_{w^*t^*}\, {}^0Charles\ y]]].$$

Now we have to take into account the fact that the meaning of the consequent clause 'if *he* had owned something then *he* would have taken care of it' is the open construction (the variable $he \rightarrow_v \iota$)

$$\lambda w^* \lambda t^*\, \forall y\, [[{}^0Own_{w^*t^*}\ he\ y] \supset [{}^0Care_{w^*t^*}\ he\ y]]$$

that is to be completed by substituting the Trivialization of *Charles* for the variable *he*. To this end we use the substitution function Sub_1:[65]

$$^2[{}^0Sub_1\ {}^{00}Charles\ {}^0he\ {}^0[\lambda w^* \lambda t^*\, \forall y\, [[{}^0Own_{w^*t^*}\ he\ y] \supset [{}^0Care_{w^*t^*}\ he\ y]]]].$$

[65] See Section 3.5.

Double Execution is indispensable here, because the result of the substitution is a propositional *construction*, whereas the second argument of $\angle$ is a proposition. The analysis of (Cond1) is thus as follows:

(Cond1″) $$\lambda w \lambda t\, [\lambda w^* \lambda t^*\, [\neg \exists x\, [{}^0 Own_{wt}\, {}^0 Charles\ x] \wedge \exists x\, [{}^0 Own_{w^*t^*}\, {}^0 Charles\ x]] \angle\, {}^2[{}^0 Sub_1\, {}^{00} Charles\, {}^0 he\, {}^0[\lambda w^* \lambda t^*\, \forall y\, [[{}^0 Own_{w^*t^*}\ he\ y] \supset [{}^0 Care_{w^*t^*}\ he\ y]]]]].$$

Counterfactuals do not always have the form of conditionals. Thus, another common kind of counterfactual modality is expressed by sentences of the form

‘The *F* might not have been an *F’*

e.g., ‘Smith’s murderer might not have murdered Smith’.

We construe ‘the *F*’ as expressing the following Closure and denoting the individual office so constructed ($F/(o\iota)_{\tau\omega}$):

$$\lambda w \lambda t\, [\iota x\, [{}^0 F_{wt}\, x]].$$

The sentence ‘The *F* might not have been an *F’* is standardly considered ambiguous between these two readings:

Possibly, the *F* is not an *F*.

The *F* is such that it possibly is not an *F*.

The first construal is *de dicto*, the sentence expressing the Closure

$$\lambda w \lambda t\, [\exists w'\, \exists t'\, \neg [{}^0 F_{w't'}\, [\iota x\, [{}^0 F_{w't'}\, x]]]].$$

Gloss: ‘The *F* at $<w', t'>$ is a member of the set of those *x* that are not an *F* at $<w', t'>$.’

What is constructed? Very simple: a proposition that yields **T** for *no* world/time pair; i.e., an impossible proposition.[66] No *F* can be a non-*F* at the same world/time pair.

The second construal is *de re*. On this reading the sentence comes close to being necessarily true: it denotes a proposition that takes **T** at all the worlds/times at which the *F* exists. The modality is ascribed to a *res*, *in casu* the individual that is picked out as the *F* at the first of two world/time pairs. The point is that the predication of possibly not being an *F* demands, on the construal *de re*, that ‘is not an *F*’ be evaluated at a world/time different from the one at which the unique *F* was identified. So in the first world/time the set of *F*-objects is singled out and the

[66] There are many impossible propositions, which differ only by being false or undefined at different $\langle w, t\rangle$ pairs.

unique F is identified. Next, in the second world/time the individual so identified is predicated to be a member of the set of those individuals who, at the second world-time, are not F's (i.e., the set that is the complement of the F-set at the second world/time pair).

Our analysis of the sentence, 'The F might not have been an F' when understood *de re* is

$$\lambda w \lambda t\ [\lambda x\ [\exists w'\ \exists t'\ \neg[{}^0F_{w't'}\ x]]\ [\iota y\ [{}^0F_{wt}\ y]]].$$

Gloss: 'The F at $\langle w, t\rangle$ is a member of the set of those x that are not an F at $\langle w', t'\rangle$.'

Juggling (at least) two world/time pairs simultaneously is what underpins counterfactual modality, and if restricted to just worlds is what has become known as 'two-dimensional', or 'multi-dimensional', modal logic.[67] The semantic actualist argues that without reference, implicitly or explicitly, to the actual world, locutions in the vein of 'The F might not have been an F' read *de re* possess an expressive power that can be accommodated only by an actualist semantics. We wish to show that the amount of expressive power required can be obtained otherwise, without recourse to the actual world as a semantic component.[68]

For illustration, consider the following two examples. First, H.T. Hodes's 'There could be something which doesn't actually exist' (1984, p. 28). His formula is this ('@' being explained ibid., p. 27):

$$\Diamond(\exists x)@\neg \mathrm{E}x.$$

We interpret the sentence as, 'Of all the things that do not exist, some might have' or 'Something which does not exist might have existed', and stating insofar a necessary truth about the contingency of existence. For example, although no unicorns exist, unicorns might well have existed, and although zebras do exist, zebras might not have existed. Following Frege, and differing from Hodes, TIL treats existence as a second-degree property of intensions of type $(o\alpha_{\tau\omega})_{\tau\omega}$: unicorns exist at $\langle w, t\rangle$ iff the intension *Unicorn* returns a non-empty set at $\langle w, t\rangle$; the Queen of Belgium exists at $\langle w, t\rangle$ iff the intension *The Queen of Belgium* returns exactly one individual at $\langle w, t\rangle$; etc.[69] The formalization in TIL, x ranging over a given intension (i.e., $x \rightarrow \alpha_{\tau\omega}$, $Exist/(o\ \alpha_{\tau\omega})_{\tau\omega}$), is:

$$\lambda w \lambda t\ [\exists x\ [\exists w^*\ \exists t^*\ [[{}^0Exist_{w^*t^*}\ x] \wedge \neg[{}^0Exist_{wt}\ x]]]].$$

[67] Cf. Davies and Humberstone (1980), as well as Segerberg's seminal (1973): '[I]n "two-dimensional" modal logic one wants to evaluate formulas at two points: at a point x, with respect to a point y.' (Ibid., p. 79.)

[68] See Section 2.4.1 for objections to semantic actualism.

[69] see Section 2.3.2.

Gloss: ‘There is an intension x and a world/time couple $\langle w^*, t^*\rangle$ such that x exists at $\langle w^*, t^*\rangle$ and x does not exist at $\langle w, t\rangle$.’

Next up is M. Davies’s ‘It is possible that everything which is actually red should have been shiny’ (1981, pp. 220–1). His formula is (‘A’ being explained ibid., p. 221)

$$\Diamond(\forall x)\ (A(x \text{ is red}) \rightarrow x \text{ is shiny}).$$

We read the sentence as, ‘It is possible that everything which is red should have been shiny’. It is easy to make sense of the idea that, for instance, all the individuals that are red at one world/time pair are shiny at another. We use a set to single out some individuals that we then insert into another set. This idea forms the foundation of this formalization:

$$\lambda w \lambda t\ [\exists w^*\ \exists t^*\ [\forall x\ [[{}^0Red_{wt}\ x] \supset [{}^0Shiny_{w^*t^*}\ x]]]].$$

Gloss: ‘Possibly, every x that is a member of the set of red things at $\langle w, t\rangle$ is a member of the set of shiny things at $\langle w^*, t^*\rangle$.’

5
Attitudes and information

Contexts involving attitudes are notorious for challenging the principle of compositionality and for occasioning reference shift from extensions to intensions.[1] It is a widespread prejudice that the logic and semantics of attitudes will have to be different than the logic and semantics of so-called extensional contexts. It is commonly held, in particular, that Leibniz's Law is invalid in attitude contexts and that its invalidity may even define intensional contexts, whereby 'intensional' is simply synonymous with 'non-extensional'. In Section 1.5.2 we presented Tichý's objection to identifying intensionality with non-extensionality, as well as his circularity argument to the effect that the definitions of extensional context and the validity of the substitution of co-referential singular terms and existential generalization presuppose one another.

One reaction to the existence of allegedly non-extensional contexts is to set up a logic and semantics to emulate the 'non-extensional' behaviour of attitude contexts (Montague). Another reaction is to see the apparent collapse of extensional logic vis-à-vis attitudes as a challenge to come up with an attitude logic that is, after all, extensional (Davidson). A third reaction, however, is to devise a logic and semantics that flouts none of the principles of extensional logic and is, at the same time, intensional. Ours is such a reaction. To get an extensional intensional logic, if you like, off the ground the first step is not to construe intensionality as non-extensionality. Instead what is wanted is an extensional logic in which to manipulate intensions. As Bealer says concerning Frege and Church,

> [T]here is no genuinely intensional language; when *prima facie* intensional language is properly analysed, it turns out to be extensional language concerning intensional entities (1982, p. 148).

The TIL 'language of constructions' is such an extensional language. It contains the semantic means to talk about intensions and their properties. And not only that; it also contains the semantic means to talk about hyperintensions and their properties.[2] Both are needed for a two-tiered attitude logic capable of manipulating and controlling attitudes of two different degrees of granularity. One

---

[1] Concerning the problem of propositional attitudes, see also Richard (1990).

[2] Sierszulska correctly says that, 'Tichy does not need a special treatment of intensional contexts because his entire approach is hyper-intensional … For Tichy, the question of mediating objects is central, the entire conception is built around it.' (2006, p. 498.)

M. Duží et al., *Procedural Semantics for Hyperintensional Logic*, Logic, Epistemology, and the Unity of Science 17, DOI 10.1007/978-90-481-8812-3_5,

granularity is the one known from possible-world semantics. The principle of individuation of the intensions of possible-world semantics is logical equivalence. Another granularity is hyperintensional, which is any individuation finer than logical equivalence. In popular terms, hyperintensional logic is able to distinguish between a half-full and a half-empty glass. This distinction presupposes the possibility of operating with two or more different (yet equivalent) modes of presentation, or conceptualisations, of the same property. In the case of attitudes, we need to be able to operate with two or more different (yet equivalent) hyperintensional modes of presentation of the same proposition. The relevance to epistemic logic is that even though *a* knows, hyperintensionally, that the glass is half-full, it does not follow that *a* knows that the glass is half-empty (or the other way around). The same proposition is presented or conceptualised—or *constructed*—in two different manners; first, in terms of the glass being half-*full*, then in terms of it being half-*empty*. Something analogous holds for mathematical knowledge, except that now we are no longer looking for two or more constructions of the same proposition, but of the same truth-value (namely, **T**). The analogy is that even though *b* knows, hyperintensionally, that the figure before her is tri*angular*, it does not follow that *b* knows that the figure is tri*lateral* (or the other way around).

For background, Gamut points out that *if* ‘Hesperus’, ‘Phosphorus’ are co-designating rigid designators *then* if *a* believes that Hesperus is Hesperus it follows that *a* believes that Hesperus is Phosphorus—‘which is clearly unrealistic’:

> So intensional semantics clearly runs into complications when applied to the verb *believe*. No consensus has been reached on how to get around this. It has been proposed that the solution lies in a more refined intensional semantics. The above examples indicate that more than just logical equivalence [co-intensionality] is required for interchangeability *salva veritate* in hyperintensional contexts. Apparently expressions need to have more semantic properties in common than just the property of having the same reference in all possible worlds. Perhaps the ways in which the intensions of expressions are built up from the intensions of their composite parts should also be taken into account. It has also been proposed that the hyperintensional contexts lie beyond the limits of (intensional) semantics and that a satisfactory solution will mean getting beyond these limits. It is argued that semantics must join forces with pragmatics in order to give an adequate treatment of hyperintensional contexts like that created by the verb *believe*. The relations between language and language users can to a large extent be abstracted away in semantics, but not entirely, and the analysis of belief contexts is thought to be one area in which the semantic interpretation must take language users into account. Be that as it may, if hyperintensional contexts should lie at or beyond the limits of intensional semantics, that would in no way diminish its utility in research into the semantics of natural language. Moreover, even with the proposed refinements or extensions added, intensional semantics would still have an essential part to play (1991, pp. 73–4).

We disagree that pragmatics would be the place to seek a solution. This holds in particular if this would involve embracing some form of sententialism (see below), as Gamut is perhaps gesticulating toward. What is desirable is instead a fully-fledged semantics ‘big enough’ to encompass hyperintensional contexts. But Gamut makes an important point that tends to be neglected in hyperintensional quarters. Even though intensional semantics is demonstrably insufficient, it is nonetheless indispensable. We still need an intensional level between extensions and hyperintensions.[3] For instance, as we argue in Section 5.2, the default interpretation of notional attitudes is as intensional attitudes. There are also times when an ascriber does more justice to an ascribee's attitude by ascribing a propositional (as opposed to hyperpropositional) attitude. This is so when the attitude concerns an empirical state-of-affairs where it is irrelevant how the state-of-affairs is presented to, or conceptualized by, the ascribee.

TIL offers two variants of hyperintensional individuation, *constructional* and *conceptual*, the latter being slightly coarser than the former. Constructions are slightly finer than procedures, which means that they draw differences where sometimes none are needed for logical or semantic purposes. Concepts, on the other hand, are procedures (see Section 2.2). Whenever an agent entertains a hyperintensional attitude, we wish to relate the agent to a procedure. So we construe hyperintensional attitudes as *conceptual* attitudes. However, since concepts are themselves (normalized closed) constructions, we shall most often talk of constructional attitudes instead, since any procedurally isomorphic (closed) construction can serve as a representative of the concept to which an agent is related.

Possible-world propositions, for their part, are an overworked notion, serving as they do in at least five different capacities:

- truth-bearer (*P is true*)
- truth-condition (*P is satisfied*)
- state-of-affairs (*P obtains*)
- object of attitude (*P is known; P is believed; etc.*)
- argument of intensional connectives (*$\{P_1,\ldots,P_n\}$ entails P*; *if P had been true then P′ would have been true; etc.*).

It is a thrice-told tale that, and why, mere possible-world propositions are inherently insufficient for most purposes of attitude logic. In particular, they are unsuitable for modelling mathematical attitudes, since the object of attitude will in each and every case be the same, to wit, the necessary proposition *TRUE*. Nor can they distinguish between inverse relations; if *a* knows (believes, hopes, etc.) that *b* is taller than *c* then it follows that *a* knows (etc.) that *c* is shorter than *b*. And they also over-generate knowledge (belief, hope, etc.) when various closure principles are applied; e.g., if *a* knows (etc.) that *P*, and if (*a* knows that) *P* entails *P′* then *a* knows (etc.) that *P′*.[4] For instance, if *a* knows that Prague is the capital of the Czech Republic then *a* knows that Prague is the capital of the Czech Republic and all whales are mammals. Much effort, therefore, has gone into imposing restrictions to logically block undesired conclusions. Throughout the rest of Chapter 5 we show which form our efforts take.

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[3] To mention but two, Almog (1986) and King (2001) skip the intensional level entirely. An unwelcome consequence is that hyperintensions will lead directly to their actual-world values; this amounts to semantic actualism, a stance which we objected to in Section 2.4.1.

[4] See Hales (1995) for a catalogue of closure principles.

The following schema applies equally to 'propositional' and 'notional' attitudes; e.g., *knowing that the sun is shining* and *seeking the fountain of youth*:

$$\lambda w \lambda t\ [Att_{wt}\ a\ X]$$

Att is a construction of an attitude; *a*, of an ascribee; and *X*, of an object of attitude.

Each and every attitude we consider in this book is a relation-in-intension between a solitary attitude agent (the ascribee *a*) and an attitude object *X*. This holds whether the object is 'propositional' or 'notional'. What varies are the granularity of the objects (intensional vs. hyperintensional) and the type of the particular hyperintensions and intensions (first-order and higher-order hyperintensions *vs.* propositions and other intensions).[5] So the type of *X* will vary a good deal, and the type of *Att* will vary with it. As is the case with fellow hyperintensional attitude logics, we operate with basically four different kinds of attitude:

- intensional *de dicto*
- intensional *de re*
- hyperintensional *de dicto*
- hyperintensional *de re*.

In addition, when dealing with *knowing whether*, we consider two forms of intensional attitudes *de re* and two forms of hyperintensional attitudes *de re*.[6]

Let *X* construct intensions, *Att* relations-in-intension between an individual and an intension and *Att** relations-in-intension between an individual and a construction: $X/*_n \rightarrow \alpha_{\tau\omega}$; $Att/*_n \rightarrow (o\iota\alpha_{\tau\omega})_{\tau\omega}$; $Att^*/*_n \rightarrow (o\iota*_m)_{\tau\omega}$.

Then the schema above is the schema of intensional attitudes, while this is the schema of hyperintensional attitudes (notice the Trivialization of *X*):

$$\lambda w \lambda t\ [Att^*_{wt}\ a\ {}^0X].$$

Throughout this chapter we adopt the notational convention that '*Att*' denotes constructions of relations to intensions, and '*Att**' constructions of relations to constructions. Thus the same schema applies also to mathematical attitudes, where $X/*_n \rightarrow \alpha$; $Att^*/(o\iota*_n)_{\tau\omega}$. The Trivialization of *X* is what makes the difference between a *used* and a *mentioned* occurrence of a construction.[7] The difference, in attitude logic, is the difference between relating an ascribee to what a construction constructs and to the construction itself.

In order to make the schematic analyses that follow easier to read, we also adopt the following conventions: a, $b/*_n \rightarrow \iota$ are constructions of individuals; $F/*_n \rightarrow (o\iota)_{\tau\omega}$ is a construction of a property; P, $Q/*_n \rightarrow o_{\tau\omega}$ of a proposition. Thus in a

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[5] By 'hyperintension' we henceforth mean *construction*.

[6] See Section 5.1.2.

[7] See Section 2.6.

schematic analysis of, e.g., 'b is an F' we do not use Trivialisations of b and F—$\lambda w \lambda t\ [F_{wt}\ b]$—because '$F$' and '$b$' are here metalinguistic signs for constructions. Whenever needed, we will apply the substitution of a particular construction for these signs. Thus, for example, a construction complying with the schema '$\lambda w \lambda t\ [F_{wt}\ b]$' is the analysis of 'The Pope is wise', *viz.* $\lambda w \lambda t\ [{}^0Wise_{wt}\ {}^0Pope_{wt}]$; $Wise/(o\iota)_{\tau\omega}$; $Pope/\iota_{\tau\omega}$.

5.1 Propositional attitudes

For illustration, consider the (formulaic) sentence,

'a believes that b is an F'.

Its standard possible-world-semantic analysis relates a to the proposition that b is an F:

$$\lambda w \lambda t\ [{}^0Believe_{wt}\ a\ [\lambda w \lambda t\ [F_{wt}\ b]]].$$

Its hyperintensional analysis relates a to the Trivialization of the proposition that b is an F:

$$\lambda w \lambda t\ [{}^0Believe^*_{wt}\ a\ {}^0[\lambda w \lambda t\ [F_{wt}\ b]]].$$

The bifurcation of attitudes into relations to intensions and to constructions has as a consequence that there can be no such thing as knowing, believing, hoping, seeking, etc., *simpliciter*.[8] These attitudes are always typed, so there can only be propositional knowing, constructional knowing*, intensional seeking, etc.[9] For this reason, nor can there be unrestricted universal quantification over *all* objects of knowledge (etc.), as in the standard argument

Everything x knows, x believes
a knows that P

---

a believes that P.

[8] See also Duží and Materna (2000).

[9] The difference between *knowing* and *knowledge* (in a world w at time t) is the difference between a *relation* (to a proposition or a construction of a proposition) and a *set* (of known propositions or known* constructions). However, when no confusion can arise, we may not highlight the difference linguistically.

Once we type the domain of objects of knowledge and belief, quantification proceeds smoothly. Thus, if quantifying over propositions, the argument goes into TIL notation thus:

$$\frac{\begin{array}{c}\lambda w\lambda t\ [\forall x\ [\forall p\ [{}^0Know_{wt}\ x\ p] \supset [{}^0Believe_{wt}\ x\ p]]]\\ \lambda w\lambda t\ [{}^0Know_{wt}\ a\ q]\end{array}}{\lambda w\lambda t\ [{}^0Believe_{wt}\ a\ q].}$$

Types: *Know*, *Believe*/$(o\iota o_{\tau\omega})_{\tau\omega}$; variables $p, q/*_n \rightarrow o_{\tau\omega}$.

The underlying argument schema will have to be duplicated to cover also the case of constructional knowledge and belief:

$$\frac{\begin{array}{c}\lambda w\lambda t\ [\forall x\ [\forall c\ [{}^0Know^*_{wt}\ x\ c] \supset [{}^0Believe^*_{wt}\ x\ c]]]\\ \lambda w\lambda t\ [{}^0Know^*_{wt}\ a\ d]\end{array}}{\lambda w\lambda t\ [{}^0Believe^*_{wt}\ a\ d].}$$

Types: *Know**, *Believe**/$(o\iota *_n)_{\tau\omega}$; variables $c, d \rightarrow *_n$; ${}^2c, {}^2d \rightarrow o_{\tau\omega}$.

The restriction is not to say, however, that propositional and constructional knowing (etc.) may not occur within the same context. Here is an example (without assessing its validity):

$$\frac{\begin{array}{c}\text{Everything } x \text{ knows*}, x \text{ knows}\\ a \text{ knows* that } d\end{array}}{a \text{ knows that } {}^2d.}$$

$$\frac{\begin{array}{c}\lambda w\lambda t\ [\forall x\ [\forall c\ [{}^0Know^*_{wt}\ x\ c] \supset [{}^0Know_{wt}\ x\ {}^2c]]]\\ \lambda w\lambda t\ [{}^0Know^*_{wt}\ a\ d]\end{array}}{\lambda w\lambda t\ [{}^0Know_{wt}\ a\ {}^2d].}$$

The argument says that, for every propositional construction, if you know* it then you also know that the proposition it constructs is true.[10]

[10] *Iteration* of attitudes does not pose a technical problem. Any attitude, whether intensional or hyperintensional or mixed, can always be iterated. In particular, different variants of *KK*, disregarding the issue of validity, can be technically obtained in TIL. The basic idea is this. Any construction of any attitude will always construct a *proposition*, which may in turn serve as an input of an attitude. This holds even in the case of constructional attitudes, because the input proposition may be Trivialized.

5.1.1 Three puzzles and a non-puzzle

Frege's second puzzle. Frege's Hesperus (Evening Star)/Phosphorus (Morning Star) puzzle can be dissolved already in terms of intensional attitudes. Gamut *is* right that it is 'clearly unrealistic' that if *a* believes that Hesperus is Hesperus then *a* believes that Hesperus is Phosphorus: for otherwise the identity of Hesperus with Phosphorus would constitute no contingent, astronomical discovery. Only this undesirable conclusion does follow if 'Hesperus', 'Phosphorus' rigidly co-designate (be it an individual or an individual office). In Section 3.3.1 we showed how to reconcile rigid designation with the non-triviality that Hesperus is Phosphorus. Let 'Hesperus', 'Phosphorus' rigidly denote two different individual offices. If Hesperus and Phosphorus are co-occupied at some $\langle w, t\rangle$ then it can be known only a posteriori at $\langle w, t\rangle$ that Hesperus is Phosphorus. Co-occupancy is non-trivial because there is a $\langle w', t'\rangle$ at which Hesperus and Phosphorus are not co-occupied.

To show why the intuition Gamut voices is on the right track, let *a* believe (or know, for that matter) that Hesperus is Hesperus. Then it does not follow that *a* believes (knows) that Hesperus is Phosphorus.[11]

(1) $\lambda w\lambda t\ [{}^0Believe_{wt}\ a\ \lambda w\lambda t\ [{}^0{=}\ [{}^0H_{wt}\ {}^0H_{wt}]]]$
(2) $\lambda w\lambda t\ [{}^0{=}\ [{}^0H_{wt}\ {}^0P_{wt}]]$

---

(3) $\lambda w\lambda t\ [{}^0Believe_{wt}\ a\ \lambda w\lambda t\ [{}^0{=}\ [{}^0H_{wt}\ {}^0P_{wt}]]]$.

Types: $Believe/(o\iota o_{\tau\omega})_{\tau\omega}$; $H, P/\iota_{\tau\omega}$; $=/(o\iota\iota)$.

Premise (1) constructs the proposition that *a* believes that the occupant of Hesperus is self-identical.[12] (2) constructs the proposition that Hesperus and Phosphorus are co-occupied. (3) constructs the proposition that *a* believes that Hesperus and Phosphorus are co-occupied. The argument is invalid, because the offices *P* and *H* are not identical. Thus the propositions that Hesperus is Hesperus and Hesperus is Phosphorus are two *different* propositions. The former is almost necessarily true, because it is true in all worlds at all times at which Hesperus is occupied. The latter is true much more rarely. So ${}^0P$ may not be validly substituted for ${}^0H$ in $[{}^0H_{wt}\ {}^0H_{wt}]$ to construct $[{}^0H_{wt}\ {}^0P_{wt}]$.[13]

---

[11] Nor is the reversal of premise and conclusion valid; *a* most probably believes that Hesperus is Hesperus as soon as *a* has any beliefs at all about Hesperus, but this does not follow from believing that Hesperus is Phosphorus.

[12] Or, the sentence 'Hesperus is Hesperus' might mean that the *office* of Hesperus is self-identical, though only on pain of construing 'is identical to' differently than in 'Hesperus is Phosphorus'. The self-identity of an office would be constructed thus: $[{}^0{=}'\ {}^0H\ {}^0H]$, $='/(o\ \iota_{\tau\omega}\iota_{\tau\omega})$.

[13] Since ${}^0H$, ${}^0P$ are *v*-congruent but not equivalent constructions, they can be validly substituted only in extensional contexts, which is not the case here. See Section 2.7 for valid rules of substitution.

Half-empty vs. half-full. Attitudes involving inverse relations, like *smaller-than* and *larger-than*, are typical cases where both coarse-grained intensional and fine-grained hyperintensional attitudes naturally fit in. Coarse attitudes, if the agent is related merely to an empirical state-of-affairs, where it is immaterial whether a glass is conceptualized as half-full or half-empty, since the amount of liquid is the same, anyway. Fine attitudes, if the agent is related to one among several ways of conceptualizing a state-of-affairs, and where the agent's own way of conceptualizing the state-of-affairs matters. The trite saying that optimists think of the glass as half-full and pessimists as half-empty presupposes that hyperintensional attitudes are available.[14] Let *a* think that the glass before him is half-full. Does it follow that *a* thinks that the glass is half-empty? You can have it either way.[15] If *thinking* is a coarse-grained relation, then the argument needs to look like this.

(It is an innocuous simplification here to construe 'the glass before him' as the constant '*b*'.)

$$\frac{\begin{array}{c}\lambda w\lambda t\ [{}^0Think_{wt}\ a\ \lambda w\lambda t\ [[{}^0Half\ {}^0Full]_{wt}\ b]] \\ \forall w\forall t\ [\forall x\ [[[{}^0Half\ {}^0Empty]_{wt} x] \equiv [[{}^0Half\ {}^0Full]_{wt} x]]]\end{array}}{\lambda w\lambda t\ [{}^0Think_{wt}\ a\ \lambda w\lambda t\ [[{}^0Half\ {}^0Empty]_{wt}\ b]].}$$

Types: $Think/(o\iota o_{\tau\omega})_{\tau\omega}$; $Empty$, $Full/(o\iota)_{\tau\omega}$; $Half/((o\iota)_{\tau\omega}\,(o\iota)_{\tau\omega})$.[16]

Premise (2) is a meaning postulate regulating the intrinsic relation between the properties denoted by the predicates 'is half-empty' and 'is half-full'. The premise means that, necessarily, everything is half-empty if and only if it is half-full. Consequently, the properties of being half-empty and half-full are co-intensional, hence identical. Hence, $\lambda w\lambda t\ [[{}^0Half\,{}^0Full]_{wt}\ b]$ is one and the same proposition as $\lambda w\lambda t\ [[{}^0Half\ {}^0Empty]_{wt}\ b]$. So we may substitute the latter for the former in (1) to obtain (3), since *a*'s attitude is merely intensional.

This substitution is only quite natural, since the states-of-affairs that *a* is related to in both (1) and (3) are identical. The brute fact is that *a*'s pint of beer is down to half a pint, whichever way *a* looks at it—though not whichever way *a conceptualizes* this (all-too) brute fact. If what was hitherto a pint of beer has been decimated to half a pint, then the glass is better described as being 'half-empty'. But, if what was hitherto an empty glass has had half a pint poured into it, then the glass is better described as being 'half-full'. Or, if *a* is an optimistic beer drinker then he will conceptualise his half-finished glass as being half-full. To accommodate these differences

[14] The following, hopefully less-trite joke, does not call for hyperintensional-attitude treatment. *Q*: 'What's the difference between an optimist and a pessimist?' *A*: 'An optimist *believes* that the actual world is the best of all possible worlds; a pessimist *knows* that it is.'

[15] See Sullivan (1998) for a discussion of coarse-grained 'Russellian singular propositions' and fine-grained 'Fregean singular thoughts', the former applying to states-of-affairs and the latter to conceptualizations of states-of-affairs.

[16] See Section 4.4 on modifiers.

in optimistic and pessimistic conceptualisation, *thinking* needs now to be a relation to a propositional construction. The resulting argument is

$$
\begin{array}{ll}
(1.1) & \lambda w\lambda t\,[{}^0\mathit{Think}^*_{wt}\ a\ {}^0[\lambda w\lambda t\,[[{}^0\mathit{Half}\ {}^0\mathit{Full}]_{wt}\ b]]] \\
(2) & \forall w\ \forall t\,[\forall x\,[[[{}^0\mathit{Half}\ {}^0\mathit{Empty}]_{wt}\,x] \equiv [[{}^0\mathit{Half}\ {}^0\mathit{Full}]_{wt}\,x]]] \\
\hline
(3.1) & \lambda w\lambda t\,[{}^0\mathit{Think}^*_{wt}\ a\ {}^0[\lambda w\lambda t\,[[{}^0\mathit{Half}\ {}^0\mathit{Empty}]_{wt}\ b]]].
\end{array}
$$

Type: $\mathit{Think}^*/(o\iota{*_1})_{\tau\omega}$.

Has *a* transmogrified into a pessimist? No, for the argument is invalid. There is only one property, to be sure, but $[{}^0\mathit{Half}\ {}^0\mathit{Empty}]$ and $[{}^0\mathit{Half}\ {}^0\mathit{Full}]$ are now mentioned* in the hyperintensional context of (1.1), (3.1), respectively. In such a context only *procedurally isomorphic* constructions can be substituted.[17] Though $[{}^0\mathit{Half}\ {}^0\mathit{Empty}]$ and $[{}^0\mathit{Half}\ {}^0\mathit{Full}]$ are equivalent (constructing as they do one and the same property), they are not procedurally isomorphic, because ${}^0\mathit{Full}$ and ${}^0\mathit{Empty}$ are not identical.[18] Hence in the hyperintensional context of (1.1) ${}^0\mathit{Empty}$ cannot be substituted for ${}^0\mathit{Full}$ in $[{}^0\mathit{Half}\ {}^0\mathit{Full}]$ to construct $[{}^0\mathit{Half}\ {}^0\mathit{Empty}]$.

An arithmetic example. Let *a* know* that seven plus five makes twelve. Does it follow that *a* knows* that five plus seven makes twelve? No, for it is not logically necessary that *a* be able to invert the order of the numbers.

The logical way to block the conclusion is via the argument

$$
\begin{array}{ll}
(1) & \lambda w\lambda t\,[{}^0\mathit{Know}^*_{wt}\ a\ {}^0[{}^0{=}\ [{}^0{+}\ {}^07\ {}^05]\ {}^012]] \\
(2) & [{}^0{=}'\ [{}^0{=}\ [{}^0{+}\ {}^07\ {}^05]\ {}^012]\ [{}^0{=}\ [{}^0{+}\ {}^05\ {}^07]\ {}^012]] \\
\hline
(3) & \lambda w\lambda t\,[{}^0\mathit{Know}^*_{wt}\ a\ {}^0[{}^0{=}\ [{}^0{+}\ {}^05\ {}^07]\ {}^012]].
\end{array}
$$

Types: $=/(o\nu\nu)$; $='/(ooo)$; $\mathit{Know}^*/(o\iota{*_1})_{\tau\omega}$.

It is *invalid*, because in (2) the arguments of =’ are truth-values and not constructions of truth-values. The attitude relatum in (1) is a Composition of a truth-value, so the right premise would be instead

$$
(2') \qquad [{}^0{=}''\ {}^0[{}^0{=}\ [{}^0{+}\ {}^07\ {}^05]\ {}^012]\ {}^0[{}^0{=}\ [{}^0{+}\ {}^05\ {}^07]\ {}^012]].
$$

Type: $=''/(o{*_1}{*_1})$.

The argument {(1), (2′), (3)} is *valid*. But also *unsound*, since (2′) is false: the two Compositions are just that—two and not one. But, it might be objected, the Composition $[{}^0{=}\ [{}^0{+}\ {}^07\ {}^05]\ {}^012]$ is equivalent to $[{}^0{=}\ [{}^0{+}\ {}^05\ {}^07]\ {}^012]$; so why cannot a Trivialization of the latter be substituted for a Trivialization of the former? The answer is that, Yes, the Compositions are, of course, equivalent, as (2) says; but it is precisely constructions of constructions that need to be substituted inside a hyperintensional

[17] See the hyperintensional rule of substitution in Section 2.7.1, and the following Remark.

[18] Obviously, these constructions are not even equivalent.

context, and not just constructions of non-constructions. The difference is here between Trivializations of constructions and constructions of truth-values.[19]

So *a* does not get a second attitude for free. But what *a* could do is *infer* that five plus seven makes twelve. To validate the conclusion that *a* knows* that five plus seven makes twelve, the following premises are needed. First, (1) as an assumption to get something to work with. Second, the premise that *a* knows* that the two Compositions above are equivalent. Third, a premise to the effect that if *x* knows* that the Compositions C, C' are equivalent then *x* infers C'. The conclusion, then, is that *a* knows* that C'. However, we are not going to pursue this solution, so we are not going to formalise in TIL what the third premise would look like. The reason is the philosophical one that the premise would have to embody the principle that every agent infers every Composition known* to be equivalent to an already known* Composition. This bloats the agents' base of inferential knowledge, taxing the inferential and other resources of resource-bounded agents, whether people or machines. As an alternative to *inferential* knowledge, we have worked out a somewhat more restrictive notion of *inferable* knowledge, set out in Section 5.1.5.

Mates' puzzle. This is a staple in epistemic logic which we have not attempted to solve, for the simple reason that we consider it a non-puzzle. Roughly, the idea of the 'puzzle' is that even if '*F*' and '*F'*' are stipulated to be synonymous predicates, *a* may know that *b* is an *F* and still not know that *b* is an *F'*. Standard pairs are {'is a woodchuck'/'is a groundhog'} and {'is a fortnight'/'is a period of 14 days'}.[20]

As soon as '*F*' and '*F'*' are introduced as synonymous predicates we fail to see how there might be any room for a distinction on the hyperintensional level of linguistic sense. And surely hyperintensional logic should not make distinctions where there is no difference. If two terms are synonymous then there is only one sense (concept);[21] and since in hyperintensional attitude contexts TIL relates agents to senses, there is one thing and not two to relate the agents to. Thus, for instance, $^{0}Woodchuck$ and $^{0}Groundhog$ are not two constructions, but one and the same: $^{00}Woodchuck = {}^{00}Groundhog$, $=/(o*_1*_1)$. Syntactic differences between synonymous expressions and other merely notational variations are logically irrelevant.[22] Mates-style 'puzzles' do not arise for non-sententialist theories of attitude

[19] The solution to Frege's first puzzle in 1892a, about the intersection of two medians, may be gleaned from this example. 'The intersection of two medians is identical to the intersection of the other two medians of a triangle' provides non-trivial analytical information, unlike 'The median point of a triangle is identical to itself'. Thus one can easily know the latter without knowing the former. See also Section 5.4 on analytical information.

[20] See Mates (1950).

[21] See Definition 2.10 in Section 2.2.1.

[22] Penco points out that, 'Mates' puzzle could...be resolved by the application of a Fregean (conditional) Transparency Principle: *if* we grasp the senses of two expressions, we always recognize whether these expressions have the same sense. It is apparent that failures in sense recognition bring about failures in substitutability; but we may not be interested in these kinds of failures.'

ascription, because (to use own Mates' example) '…chew…' and '…masticate…' are *synonymous*, since 'to chew', 'to masticate' are assumed to be synonymous. Thus, non-linguistic substitution would be of a proposition/hyperproposition for —itself. If, to complicate the case, we construe knowing hyperintensionally, then the argument

$$\frac{\begin{array}{c} a \text{ knows* that } b \text{ is an } F \\ \text{the sense of } F \text{ is the sense of } F' \end{array}}{a \text{ knows* that } b \text{ is an } F'}$$

is analysed as

$$\frac{\begin{array}{c} \lambda w \lambda t\, [{}^0Know^*_{wt}\ a\ {}^0[\lambda w \lambda t\, [F_{wt}\ b]]] \\ [{}^0{=}\ {}^0F\ {}^0F'] \end{array}}{\lambda w \lambda t\, [{}^0Know^*_{wt}\ a\ {}^0[\lambda w \lambda t\, [F'_{wt}\ b]]].}$$

The argument is valid, for a construction is being substituted for itself.[23] If F is the sense of 'is a woodchuck' and F' the sense of 'is a groundhog', then if a knows* that b is a woodchuck then a thereby knows* that b is a groundhog. The attributer simply elects, for whatever reason, to deploy not one predicate but two when reporting a's knowledge* that b is some particular kind of rodent.

Individual language-users may, of course, fail to know that 'F', 'F'' are synonymous. But inflicting linguistic incompetence on attributees means changing the topic. One topic is *what* somebody knows. The other topic is *how* to attribute and report this knowledge; i.e., which words to use. If the two topics are run together, the result is *sententialism*. Whatever the particular variations, sententialism holds that 'propositional', or 'that'-clause, attitudes are really sentential attitudes, which

(Penco 2003, p. 109). This sums up our take on Mates' puzzle. Failure of substitutability rooted in linguistic incompetence is logically and semantically irrelevant, as it is a pragmatic problem. Cf. Sullivan (1998, pp. 119–23): 'How can a reasonable agent both believe and not believe the same thing?' (They cannot, as Sullivan agrees.) But Sullivan's set-up of the discussion is flawed. He has the Italian speaker Paolo, who knows 'Monte Bianco' but not 'Mont Blanc', assert, 'Monte Bianco is more than 4,000 m high', but pass judgment on, 'Mont Blanc is more than 4,000 m high'. Sullivan, inadvertently, turns an issue to do with belief content into a matter of linguistic competence. Palo's attitude is rightly to do with Mont Blanc or a presentation of Mont Blanc, not 'Mont Blanc'. So Sullivan's question, 'How …?' does not arise from the preceding scenario.

[23] See the hyperintensional rule of substitution in Section 2.7.1.

are relations to types or tokens of sentences, inscriptions, or whatever other linguistic entities.[24]

Once we jettison sententialism, we can expose as false the dilemma between (a) and (b) (Mates', near enough):

(a) $word_1$, $word_2$ of language *L* are synonymous if and only if they are validly intersubstitutable in each sentence in each kind of sentence (bar quotational contexts); this logical equivalence backs up a compositional truth theory

(b) any definition of synonymy is adequate if and only if it validates intersubstitutivity as described *ad* (a).

The pair {'to chew', 'to masticate'} is then launched as an example of a pair of synonyms that are not universally substitutable. Allegedly, it is *true* that nobody doubts that whoever believes that *b* chews, believes that *b* chews, but *false* that nobody doubts that whoever believes that *b* chews, believes that *b* masticates. For one, Moffett (2002, p. 162) holds that one of (a), (b) must be given up in consequence. But why cannot we have both (a) and (b) in the same package? All it takes is arguing to the effect that it is false that it is false that nobody doubts that whoever believes that *b* chews, believes that *b* masticates. It can be true that it is false only if believing, doubting (etc.) are made sensitive to a particular choice of words (or notation). We agree with Moffett that,

> [I]t appears consistent to assume that *x* is not so confused or irrational as to doubt that [the mathematical proposition that *f* is recursive is the proposition that *f* is recursive]
> (Ibid., p. 164).

But then Moffett, in keeping with Mates, goes on to argue that *x* may well doubt that the mathematical proposition that *f* is recursive is the proposition that *f* is computable, even though 'is recursive' and 'is computable' are stipulated to be synonymous (ibid., p. 161).[25] However, from the viewpoint of a hyperintensional attitude logic based on senses and not sentences, *if* 'is recursive' and 'is computable' are synonymous and if *x* is acquainted with the mathematical language used, *then x* is no less confused and irrational when doubting this as when doubting that

[24] See Tichý (1986a, pp. 265–67, 2004, pp. 663–65) for a (contrived) account of what he calls 'linguistic attitudes'.

[25] However, Moffett's pair {'is recursive'/'is computable'} is ill-chosen as an example of synonymous predicates. The *definition* of recursive function is usually taken not to be identical to the definition of computable function. The former is a recursive definition, whereas the latter standardly means λ-computable. Thus it does not suffice to 'suppose' that the respective predicates are synonymous, as this usually (on the ordinary *intensional* approach) means that the defined *set* of functions is identical, which would be similar to 'is an equilateral triangle' and 'is an equiangular triangle'. On the hyperintensional approach the two are *not* synonymous, so that one can easily believe that *f* is recursive without believing that *f* is computable, and vice versa. The *construction* of the set of recursive functions is different from the *construction* of (the same) set of computable functions.

the mathematical proposition (in TIL: mathematical construction) that *f* is recursive is the proposition that *f* is recursive.[26]

We will deal with hyperintensional propositional attitudes in the next section 5.1.2, where we provide a basic insight into the difference between intensional/hyperintensional attitudes *de dicto* and *de re.*

5.1.2 Propositional attitudes de dicto vs. de re

Let *a* believe that the Pope is not the Pope. The ascription of an attitude *de dicto* makes a confused or irrational fool of the attributee; for then *a*'s belief is to the effect that whoever is the occupant of the papacy is not self-identical. The ascription of an attitude *de re*, by contrast, makes *a* fully lucid and rational. Now the attributer uses the papacy to pick out an individual, of whom the attributee believes that he is not the Pope, without the attributee necessarily making the connection between this individual and the papacy. It is the attributer who is responsible for creating an air of paradox by employing the individual office of Pope twice over, for any office co-occupied by the Pope would have served equally well to pick out the individual occupying the office. For example, the attributer might have said instead that the German with the highest rank in the Roman-Catholic Church is such that he is believed by *a* not to be the Pope; equivalently, that *a* believes of the German with the highest rank in the Roman-Catholic Church that he is not the Pope.

But apart from the fact that they are obviously different from attitudes *de dicto*, what are attitudes *de re* actually? They are at the heart both of natural language and natural-language semantics, yet there is little consensus on their proper analysis. Quine takes a predictably dim view of their very viability:

> Spelling dissolves the syntax and lexicon of the content clause and blends it with that of the ascriber's language. So long as we rest with the unanalyzed quotational form, on the other hand, the inverted commas mark an opaque interface between two ontologies, two worlds: that of the man in the attitude, however benighted, and that of our responsible ascriber of the attitude (1992, pp. 69–70).
>
> I conclude that the propositional attitudes *de re* resist annexation to scientific language, as propositional attitudes *de dicto* do not. At best the ascriptions *de re* are signals pointing at a direction in which to look for informative ascriptions *de dicto* (Ibid. p. 71).

We will show, however, that attitudes *de re* can be logically fully accommodated while heeding the compositionality principle. We analyse *de re* attitudes in a rigorous way by means of explicit intensionalization, which enables us to keep separate the two 'worlds' that Quine alludes to; namely, the perspective of the *attributee* (the believer, the knower, etc.) and that of the *attributer*. The philosophical difference between attitudes *de dicto* and *de re* is pivoted on an *inversion of*

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[26] Church advocated a tack similar to ours; see Anderson (1998, pp. 144–46ff). For a hyperintensional logician who sets great store by Mates' puzzle, see Bealer (1982, pp. 69ff).

perspective: an attribution *de dicto* reproduces the attributee's perspective; an attribution *de re*, the attributer's.

The disambiguation of a's papal attitude induces two different truth-conditions and has him believe one of two different propositions. Therefore, the readings are also associated with two different propositional constructions. The difference between the respective constructions hinges on whether the respective constructions of the office of Pope occur with supposition *de dicto* or *de re*. Choosing unambiguous wordings for these two readings is difficult at least for the reading *de dicto*, since the only candidate is identical to the original ambiguous sentence (unless one opts for a paraphrase in stilted, semi-formal logician's English):

(*de dicto*) 'a believes that the F is an F.'

On the other hand, the analysis *de re* affords two unambiguous readings:

(*de re act*) 'a believes of the F that he/she is an F'

(*de re pas*) 'The F is believed by a to be an F.'

Remark. (*de re act*) is the *active* variant involving the anaphora 'she'/'he'; (*de re pas*), the *passive* variant.

To put our approach into a wider context, in the prevalent notation of doxastic logic the *de dicto*/*de re* distinction is, according to Hintikka and Sandu (1989), characterised as the contrast between

(*de dicto*) $B_a F[f]$

(*de re*) $(\exists x)\ (x = f \wedge B_a F[x])$.

But there are worrisome questions concerning the *de re* analysis in particular. Thus Hintikka and Sandu wonder where the existential quantifier in the *de re* case comes from, as there is no trace of it in the original sentence; how can two such similar sentences have such different logical forms? Hintikka and Sandu propose a remedy by means of Independence Friendly logic:

> [I]ndependence of the sort IF first-order logic deals with is a frequent and important feature of natural language semantics. Without the notion of independence, we cannot fully understand the logic of such concepts as belief, knowledge, questions and answers, or the *de dicto* vs. *de re* contrast (1996, p. 173).
>
> [T]he [*de dicto/de re*] distinction does not involve any difference whatsoever between different kinds of knowledge. Both kinds of knowledge have precisely the same objects, in the sense that the same entities are involved in the models of either kind of knowledge statement. In general, the regularities governing the *de dicto* vs. *de re* distinction are consequences of the more general regularities governing informational independence (1997, pp. 399–400).

Their analysis of attitudes *de re* makes use of the independence indicator '/':

(*de re**) $B_a F[f/B_a]$.

This is certainly a more plausible analysis, closer as it is to the syntactic form of the original sentence. Moreover, the independence indicator zooms in on the core of the matter, which is that there are two *independent* questions involved: (i) 'Who is the F?', (ii) 'What does a think of that individual?'. Still, the semantics of '$/B_a$' is not transparent, as we pointed out in Section 1.2.1. What the slash does is clarify rather than solve the problem of attitudes *de re*. We shall show that informational independence can be precisely captured by means of explicit intensionalization without using any new non-standard operators.

For philosophical background, Richard Foley distinguishes between two accounts of belief *de re*:

> [T]hose accounts that make it relatively difficult to believe *de re* of an object that it has some characteristic, because they require believers to have a special, intimate relation of some sort with objects about which they have *de re* beliefs; and those accounts that do not require there to be such a relation and thus make it relatively easy to have *de re* beliefs (1986, pp. 332–33).

'A special, intimate relation' amounts to something like the believer being intellectually or perceptually related to a numerically specific individual in a manner that circumvents pretty much anything true of the individual. The individual is, as it were, given immediately. Ours is not an account that requires this level or kind of intimacy. What it requires is that the attributee have some kind of 'intimate relation' to the individual that is singled out by the attributer via an office. But the attributee need not connect this individual with the respective office. Nor does the attribution reveal how the attributee is related to the occupant. The attributer uses the office as a pointer to the *res*, as the attributee does not. Hence our account is rather of Foley's second kind. Indeed, the agent of an attitude *de re* need do absolutely nothing to entertain such an attitude. What he must do, however, is having already adopted some other attitude, whether *de dicto* or *de re*, since attitudes *de re* on our construal are parasitic on prior attitudes. So attitudes *de dicto* are conceptually prior to attitudes *de re*. We, in effect, generalise what Foley claims to hold for only some attitudes *de re*:

> [C]ases involving *de re* beliefs about epistemically remote objects are cases where it is plausible to think that the person has these beliefs in virtue of having other beliefs [.] (Ibid., p. 341).

The inversion of perspective adumbrated above consists in the perspective shifting from attributee to attributer in the case of attitudes *de re*, and from attributer to attributee in the case of attitudes *de dicto*. However, attitudes *de dicto* and *de re* are logically independent, as neither entails the other, so there is no smooth logical traffic between the two. Adding a certain premise, though, validates the inference of an attitude *de re* from an attitude *de dicto*, and *vice versa*.

A prerequisite for attributing an attitude *de re* to a is that the attributer must know that the relevant office is occupied at $\langle w, t\rangle$. If the attributer wishes to substitute $Office_{wt}$ for $Office'_{wt}$, he must also know that $Office_{wt} = Office'_{wt}$. The attributee need know neither. Substituting a construction of one co-occupied office for another in the case of attitudes that are already *de re* is one way of generating another attitude *de re*. Another way is to take as premises an attitude *de dicto* involving

Office and the attributee's knowledge of which individual is $Office_{wt}$, and deduce the conclusion that the attributee has an attitude *de re* toward $Office_{wt}$. The reason why an attitude *de re* cannot come into being *ex nihilo* is because the attributee needs first to believe or know that somebody occupies some office or exemplifies some property. Only then can the attributer introduce a second office *Office′* and begin to query whether the office is occupied so that the attributee's attitude content (i.e., a construction) can be approached from the attributer's own vantage point. So if *Office* and *Office′* are co-occupied at $\langle w, t\rangle$ then the attributer can swap freely between using constructions of *Office* and *Office′* with supposition *de re.*

The ascription of any attitude is, of course, wrapped within the attributer's inescapably idiosyncratic perspective. But the attribution must be such that the attributee not only would, but must endorse it as their own. This requirement is typically cashed out in the demand that the attributee, if confronted with a sentence describing the attitude attributed to them, would, and rationally must, assent to it.

5.1.2.1 Intensional propositional attitudes *de dicto* and *de re*

Here is the solution to the papal example. It illustrates how intensional attitudes *de dicto* and *de re* may be constructed:

(*de dicto*) $\lambda w\lambda t\ [{}^0B_{wt}\ a\ \lambda w^*\lambda t^*\ \neg[{}^0Pope_{w^*t^*} = {}^0Pope_{w^*t^*}]]$

(*de re act-preliminary*) $\lambda w\lambda t\ [{}^0B\_of_{wt}\ a\ {}^0Pope_{wt}\ \lambda w^*\lambda t^*\ \neg[{}^0Pope_{w^*t^*} = he]]$

(*de re pas*) $\lambda w\lambda t\ [\lambda x\ [{}^0B_{wt}\ a\ \lambda w^*\lambda t^*\ \neg[x = {}^0Pope_{w^*t^*}]]\ {}^0Pope_{wt}]$.

Types: *B(elieve)*/$(o\iota o_{\tau\omega})_{\tau\omega}$; *B_of*/$(o\iota\iota o_{\tau\omega})_{\tau\omega}$; *Pope*/$\iota_{\tau\omega}$; =/$(o\iota\iota)$; x, *he*/$*_1 \rightarrow \iota$.

Remark. *-superscripted letters for w, t variables represent the attributee's perspective, while those without superscript represent the attributer's.

Remark. *B_of* is, as its type indicates, a relation-in-intension between an individual (the attributee), an individual (the one of whom something is believed), and a proposition. If $Office/*_n \rightarrow \iota_{\tau\omega}$ and $p \rightarrow *_n$, ${}^2p \rightarrow o_{\tau\omega}$, this constructional schema is the logical form of an attitude (*de re act*):

$$\lambda w\lambda t\ [{}^0B\_of_{wt}\ a\ Office_{wt}\ {}^2p].$$

Gloss: '*a* believes of the occupant of *Office* that he/she is such that ${}^2p$ is true.'

The above (*de re act-preliminary*) is just a schema of the analysis, because there is a free variable *he* occurring as a constituent. To obtain an adequate analysis, the relation-in-intension *B_of* of believing of somebody that they are thus-and-so is constructed as follows:

$${}^0B\_of =' \lambda w\lambda t\ \lambda xyp\ [{}^0B_{wt}\ x\ {}^2[{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ \ p]].$$

Additional types: $x, y/*_1 \rightarrow_v \iota$; $p/*_2 \rightarrow_v *_1$; ${}^2p \rightarrow_v o_{\tau\omega}$; $Tr/(*_1\iota)$; $Sub/(*_1*_1*_1*_1)$; $='/((o(o\iota\iota o_{\tau\omega})_{\tau\omega}(o\iota\iota o_{\tau\omega})_{\tau\omega})_{\tau\omega})$.

In our case *B_of* is predicated of *a* and the occupant of the office of Pope. Hence we substitute *a* for *x*, ${}^0Pope_{wt}$ for *y,* and ${}^0[\lambda w^*\lambda t^* \neg[he = {}^0Pope_{w^*t^*}]]$ for *p*, thus obtaining the adequate analysis

(*de re act*) $\lambda w\lambda t\ [{}^0B_{wt}\ a\ {}^2[{}^0Sub\ [{}^0Tr\ {}^0Pope_{wt}]\ {}^0he$
${}^0[\lambda w^*\lambda t^* \neg[he = {}^0Pope_{w^*t^*}]]]]$.

The attitudes *de re* contain occurrences of anaphoric reference that must be semantically pre-processed by substitution.[27] Note that (*de re pas*) and (*de re act*) are equivalent, since (*de re act*) is the result of executing β-conversion 'by value'.[28,29] One might wonder, however, whether β-conversion renders beliefs (*de dicto*) and (*de re act/pas*) β-equivalent. It does not, for the following reasons. In the former, ${}^0Pope$ occurs in the generic intensional context of $\lambda w^*\lambda t^* \ldots$, so ${}^0Pope$ occurs *de dicto*. *Pope* undergoes intensional descent, but within the attributee's perspective. By contrast, in the latter, ${}^0Pope$ occurs in the extensional context of both $[{}^0B\_of_{wt}\ a\ {}^0Pope_{wt}\ \lambda w^*\lambda t^* \neg[{}^0Pope_{w^*t^*} = he]]$ and $[\lambda x\ [{}^0B_{wt}\ a\ \lambda w^*\lambda t^* \neg[x = {}^0Pope_{w^*t^*}]]\ {}^0Pope_{wt}]$, so ${}^0Pope$ occurs *de re*. *Pope* undergoes intensional descent within the attributer's perspective. Unlike *de dicto*, belief *de re* comes with the existential presupposition that *Pope* must be occupied at $\langle w, t\rangle$. There must be an individual of whom to have a belief. Hence belief *de re* does not follow from belief *de dicto.* One might wonder, however, whether belief *de dicto* follows from belief *de re*. It does not, for the following reason.

If conversion 'by name' is carried out on (*de re pas*), the result is

(*contractum*) $\lambda w\lambda t\ [{}^0B_{wt}\ a\ \lambda w^*\lambda t^* \neg[{}^0Pope_{wt} = {}^0Pope_{w^*t^*}]]$.

Since *Pope* is a properly partial function, the Composition $[{}^0Pope_{wt}]$ will be *v*-improper for any $\langle w, t\rangle$ pair at which *Pope* goes vacant. In virtue of the compositionality constraint, for any such pair, the Composition

$$[\lambda x\ [{}^0B_{wt}\ a\ [\lambda w^*\lambda t^* \neg[x = {}^0Pope_{w^*t^*}]]]\ {}^0Pope_{wt}]$$

will not *v*-construct a truth-value, but nothing at all. Hence (*de re pas*) will construct a proposition undefined at $\langle w, t\rangle$. But (*contractum*) may well construct a

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[27] See Section 3.5.

[28] See Claim 2.6 in Section 2.7.

[29] The passive form is a way to indicate the *topic-focus* articulation of a sentence. Thus in the sentence 'The Pope is believed by *a* not to be the Pope' the topic is the Pope and the focus ascribed to the topic is the property of being believed by *a* not to be the Pope. The topic is connected with a presupposition, in this case that the Pope should exist. See Duží (2009) for details.

proposition that takes the value **T** at any such $\langle w, t\rangle$. Since $\lambda w^*\lambda t^* \neg[{}^0Pope_{w^*t^*} = {}^0Pope_{wt}]$ is never *v*-improper but *v*-constructs the degenerate proposition undefined at all $\langle w^*, t^*\rangle$ pairs in case $[{}^0Pope_{wt}]$ is *v*-improper, *redex* and *contractum* are not β-equivalent.[30]

Moreover, (*contractum*) is obviously also not equivalent to (*de dicto*). The latter will construct the proposition that *a* believes that the Pope is not the Pope, which would be a strange and stupid thing to believe. But regardless of the occupancy of the papal office, the proposition takes a truth-value dependently on whether *a* does, or does not, believe this impossible proposition. Belief not being a factive attitude, the ability to believe this or that comes with no existential presupposition. On the other hand, (*contractum*) does not relate *a* to an impossible proposition, but to a proposition that may well be true at those $\langle w, t\rangle$ where the Pope exists.

Partiality, which models the $\langle w, t\rangle$-relative vacancy of, e.g., individual offices, blocks unrestricted β-conversion between the constructions of attitudes *de re* and *de dicto*. Hence attitudes *de dicto* and *de re* are logically independent. However, we said above that the addition of an extra premise would make attitudes *de dicto* and *de re* mutually inferable. To set the stage, consider this argument (A, A' individual offices).

(1) *a* believes that the occupant of A is an F
(2) *a* knows that A and A' are co-occupied

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(3) *a* believes that the occupant of A' is an F.

It is valid, to be sure, provided *believe* is intensional, but it does not allow us to transform these two beliefs *de dicto* into beliefs *de re*. It can, of course, be inferred that both A and A' are occupied, thanks to the factivity of (2). But *a* may not know *who* is the shared occupant of A, A'. If we add the premise that *a* also *knows* this then the attitudes *de dicto* (1) and (3) can be transformed into attitudes *de re*. The premise we need to add is that *a* knows that A is occupied by *b*. This fresh premise kills two birds with one stone. First, due to the factivity of knowledge, it is true that *b* is A and that A is occupied. Second, *a*'s knowledge of the occupation of A by *b* entails that *b* belongs to the extension of the property of being believed by *a* to be an F. Since *b* is A, the respective *de dicto* and *de re* attitudes are mutually transferable via *b*. However, in order to perform the transformation, we need the additional assumption that knowing entails believing. This is uncontroversial, though, provided it is granted that knowledge is true belief (plus whatever else in terms of justification, warrant, or whatnot).[31] Thus the additional assumption and the ensuing consequence are as follows.

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[30] See Claim 2.5 in Section 2.7.

[31] By 'believing' we do not mean something like *feeling convinced*, but *holding to be true*, which is an intellectual and not an emotional stance.

(i) *a* knows that *A* is occupied by *b*

(i′) $\lambda w \lambda t\ [{}^0Know_{wt}\ a\ \lambda w \lambda t\ [A_{wt} = b]]$.

In virtue of factivity, (i) entails

(ii) *A* is occupied by *b*;

(ii′) $\lambda w \lambda t\ [A_{wt} = b]$.

Knowledge entailing belief, (i) entails

(iii) *a* believes that *A* is occupied by *b*

(iii′) $\lambda w \lambda t\ [{}^0B_{wt}\ a\ \lambda w \lambda t\ [A_{wt} = b]]$.

Types: *Know, B(elieve)*/$(o\iota o_{\tau\omega})_{\tau\omega}$; $a, b/*_n \rightarrow \iota$; $A/*_n \rightarrow \iota_{\tau\omega}$.

Inversion of perspective then comes about in the following manner.

I. From *de dicto* to *de re*:

(iv) *a* believes that the occupant of *A* is an *F*

(iv′) $\lambda w \lambda t\ [{}^0B_{wt}\ a\ \lambda w \lambda t\ [F_{wt}\, A_{wt}]]$.

Additional type: $F/*_n \rightarrow (o\iota)_{\tau\omega}$.

Conjunction introduction on (iii) and (iv) yields:

(v) *a* believes that the occupant of *A* is an *F* and also that this occupant is *b*

(v′) $\lambda w \lambda t\ [[{}^0B_{wt}\, a\ \lambda w \lambda t\ [F_{wt}\, A_{wt}]] \wedge [{}^0B_{wt}\, a\ \lambda w \lambda t\ [A_{wt} = b]]]$.

At any $\langle w, t\rangle$ at which *a* evaluates the propositions constructed by

$$\lambda w \lambda t\ [F_{wt}\, A_{wt}]$$

and

$$\lambda w \lambda t\ [A_{wt} = b]$$

as being true, *a* also holds that $[[F_{wt}\, A_{wt}] \wedge [A_{wt} = b]]$ and $[F_{wt}\, b]$ both construct **T**. In general, as far as intensional attitudes are concerned, any particular $\langle w, t\rangle$ at which both propositions are true is indistinguishable from any $\langle w', t'\rangle$ at which the proposition constructed by $[\lambda w \lambda t\ [F_{wt}\, b]]$ is true.[32] Hence, *a* is logically committed to believing that *b* is an *F*:

[32] That *a* believes, knows, doubts, etc., a proposition *P* does not mean that *a* is able to grasp the entire intension $P/o_{\tau\omega}$, which is an instance of actual uncountable infinity. Nobody (with the possible

(vi) a believes that b is an F

(vi′) $\lambda w \lambda t\ [{}^0B_{wt}\, a\ \lambda w \lambda t\ [F_{wt}\, b]]$.

Since b is not v-improper, an equivalent abstraction over b yields

(vii) b is believed by a to be an F

(vii′) $\lambda w \lambda t\ [\lambda x\ [{}^0B_{wt}\, a\ \lambda w \lambda t\ [F_{wt} x]]\ b]$.

Now, for *any* $\langle w, t\rangle$ pair at which (ii) and (vii) come out true, the Compositions $[A_{wt} = b]$ and $[\lambda x\ [{}^0B_{wt}\ a\ [\lambda w \lambda t\ [F_{wt}\ x]]]\ b]$ v-construct **T**. Thus we can apply the rule of substitution of identicals,[33] yielding $[\lambda x\ [{}^0B_{wt}\ a\ [\lambda w \lambda t\ [F_{wt}\ x]]]\ A_{wt}]$. Abstracting over w and t, we get the *de re* attitude

(viii) The occupant of A is believed by a to be an F

(viii′) $\lambda w \lambda t\ [\lambda x\ [{}^0B_{wt}\, a\ \lambda w \lambda t\ [F_{wt} x]]\ A_{wt}]$.

II. From *de re* to *de dicto*.

Since in any $\langle w, t\rangle$ pair at which (ii) comes out true, the occupant of A is b, (ii) and (viii) yield, *via* the substitution of identicals,

(ix) b is believed by a to be an F

(ix′) $\lambda w \lambda t\ [\lambda x\ [{}^0B_{wt}\, a\ \lambda w \lambda t\ [F_{wt}\ x]]\ b]$.

Since b is not v-improper, β-reduction applied to (ix) entails[34]

(x) a believes that b is an F

(x′) $\lambda w \lambda t\ [{}^0B_{wt}\, a\ \lambda w \lambda t\ [F_{wt}\ b]]$.

Via conjunction introduction on (iii) and (x), we get

(xi) a believes that b is an F and also that the occupant of A is b

(xi′) $\lambda w \lambda t\ [[{}^0B_{wt}\, a\ \lambda w \lambda t\ [F_{wt}\, b\,]] \wedge [{}^0B_{wt}\, a\ \lambda w \lambda t\ [A_{wt} = b]]]$.

Now, similarly as above, (xi) entails (xii):

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exception of someone omniscient) is able to do so. But a is able to evaluate the instructions yielding P in *any* w at *any* t; in other words, a has access to potential infinity.

[33] The constituents b and A_{wt} are v-congruent and occur in extensional contexts of the respective constructions: the substitution is valid according to the extensional rule of substitution. See Section 2.7.

[34] See Claim 2.5, Section 2.7.

(xii) *a* believes that the occupant of *A* is an *F*

(xii′) $\lambda w\lambda t\ [{}^0B_{wt}\, a\ \lambda w\lambda t\ [F_{wt} A_{wt}]]$.

Remark. It is essential that *a* should know that *A* is occupied by some particular individual *b*. This individual enables us to perform two-way transformation *via* λ-conversion 'by name', because *b* is not *v*- improper.
Remark. The proofs rely critically on intensional attitudes being closed under entailment. The mutual transformation of *de dicto* and *de re* beliefs is not valid for hyperintensional attitudes, which we are going to deal with next.

5.1.2.2 Hyperintensional propositional attitudes *de dicto* and *de re*

Intensional propositional attitudes are ascribed to an atributee *a* from the outside, from the attributer's perspective. Once *a* knows (believes, doubts, etc.) that *P* then *a* knows (etc.) all the equivalent transformations of *P*. Therefore, intensional propositional attitudes inevitably lead to some variant or other of the paradox of logical/mathematical omniscience.[35] The tightest restriction that can be obtained by the intensional approach is up to equivalence, because equivalent propositions cannot be distinguished on this approach, since the attitude complements are one and the same proposition.[36]

Already Carnap (1947) recognized the need for fine-grained, hyperintensional analysis of propositional-attitude reports, characterizing 'belief sentences' as being neither extensional nor intensional. In TIL lingo, hyperpropositional attitudes are constructional attitudes. Agents are not related to propositions, but to *propositional constructions*, or in the case of mathematics, to *truth-value constructions*. In the case of mathematical sentences the hyperintensional character of belief sentences is obvious. All true mathematical sentences are equivalent in virtue of denoting the truth-value **T**. Yet if *a* knows/believes that 1+1=2, it does not follow that *a* knows/believes any further mathematical truth, unless *a* is mathematically omniscient. The same phenomenon crops up in the case of empirical sentences involving mathematical expressions. For instance, if *a* believes that the number of inhabitants of Prague equals decimal number 1048576, it does not follow that *a* believes that the number of inhabitants of Prague equals hex number 100000. Though the complement clauses 'The number of inhabitants of Prague equals decimal number 1048576' and 'The number of inhabitants of Prague equals hex number 100000' are equivalent due to denoting one and the same proposition, *a* need not master the transition from the decimal to the hexadecimal number system. This transition is only superficially to do with shifting between notational

[35] See also Stalnaker (1999).

[36] See Definition 2.11, Section 2.2.1.

systems. What is at stake is, at heart, a shift from one *calculation* to another. Relations to calculations are relations to *constructions*, not formulae belonging to some system of mathematical notation, even though the attitude must be reported by means of a particular such system.

We encounter the same problem even with purely empirical sentences not involving any mathematics. For instance, a student can easily believe that it is not true that if he/she studies hard they will pass the exam, without believing that they will study hard and yet will not pass the exam. Every student not blessed with logical omniscience needs to be taught that sentences of the form '$\neg(P \supset Q)$' and '$(P \wedge \neg Q)$' are equivalent.

Similarly, *a* can believe that the Pope is wise without believing that the Pope is wise and no bachelor is married. Yet the sentences 'The Pope is wise', 'The Pope is wise and no bachelor is married' are equivalent in virtue of denoting one and the same proposition. As has been shown in Section 1.5.1, the sentence 'No bachelor is married' is analytically true, denoting the proposition *TRUE*, which is true for all $\langle w, t\rangle$ pairs. The sentence 'The Pope is wise' is an empirical one, thus denoting a proposition true for some but not all $\langle w, t\rangle$ pairs. Therefore, the set of $\langle w, t\rangle$ pairs in which the proposition denoted by 'The Pope is wise' is true is identical to the set of $\langle w, t\rangle$ pairs in which the proposition denoted by 'The Pope is wise and no bachelor is married' is true. In other words, the two sentences denote one and the same proposition.

At the outset of this chapter, we put forward this general schema of hyperintensional attitudes:

$$\lambda w \lambda t\, [Att^*_{wt}\ a\ {}^0X],$$

where in the case of attitudes to propositional constructions the types are:
$Att^*/*_m \rightarrow (o\iota *_n)_{\tau\omega}$; $a \rightarrow \iota$; $X/*_n \rightarrow o_{\tau\omega}$.

To show how hyperintensional attitudes *de dicto* and *de re* are constructed, we will analyse a similar schema of *de dicto*/*de re* attitudes as they occurred above.

(*de dicto*)	'*a* believes* that the *F* is a *P*'
(*de re act*)	'*a* believes* of the *F* that he/she is a *P*'
(*de re pas*)	'The *F* is believed* by *a* to be a *P*'

The analysis of *de dicto* case is straightforward:

(*de dicto*)* $\lambda w \lambda t\, [{}^0Believe^*_{wt}\ a\ {}^0[\lambda w \lambda t\, [P_{wt}\, F_{wt}]]]$.

Types: $Believe^*/(o\iota *_1)_{\tau\omega}$; $P/*_n \rightarrow (o\iota)_{\tau\omega}$; $F/*_n \rightarrow \iota_{\tau\omega}$.

In order to analyse *de re* attitudes, we apply a similar method as was applied to intensional attitudes:

(*de re act-preliminary*)* $\lambda w\lambda t\ [{}^0B\_of^*_{wt}\ a\ F_{wt}\ {}^0[\lambda w'\lambda t'\ [P_{w't'}\ he]]]$.

Additional types: $B\_of^*/(o\iota *_1)_{\tau\omega}$; $he/*_1 \rightarrow_v \iota$.

The hyperintensional relation-in-intension B_of^* is now defined as follows:

$${}^0B\_of^* =' \lambda w\lambda t\ \lambda xyp\ [{}^0Believe^*_{wt}\ x\ [{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p]].$$

Additional types: $x,\ y/*_1 \rightarrow_v \iota$; $p/*_2 \rightarrow_v *_1$; ${}^2p \rightarrow_v o_{\tau\omega}$; $Tr/(*_1\iota)$; $Sub/(*_1 *_1 *_1 *_1)$; $='/((o(o\iota *_1)_{\tau\omega}(o\iota *_1)_{\tau\omega})_{\tau\omega})$.

As with intensional attitudes, hyperintensional attitudes *de re* contain occurrences of anaphoric reference that must be semantically pre-processed by substitution.[37] Note that unlike the intensional case Double Execution is not applied to the Composition $[{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p]$. The reason is that $Believe^*/(o\iota *_n)_{\tau\omega}$ is a relation-in-intension between an individual and a propositional construction and the Composition v-constructs a propositional construction. Thus we have

$[{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p] \rightarrow_v *_1$, whereas ${}^2[{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p] \rightarrow_v o_{\tau\omega}$.

Now we can refine the Composition $[{}^0B\_of^*_{wt}\ a\ F_{wt}\ {}^0[\lambda w'\lambda t'\ [P_{w't'}\ he]]]$ by substituting the right-hand side of the above definition of ${}^0B\_of^*$ by

$$[\lambda w\lambda t\ \lambda xyp\ [{}^0Believe^*_{wt}\ x\ [{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p]]_{wt}\ a\ F_{wt}\ {}^0[\lambda w'\lambda t'\ [P_{w't'}\ he]]].$$

By performing the respective β-reductions 'by name' we simplify the Composition.[38] We begin with innocuous reduction of the $\lambda w\lambda t$-Closure:

$$[\lambda xyp\ [{}^0Believe^*_{wt}\ x\ [{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p]]\ a\ F_{wt}\ {}^0[\lambda w'\lambda t'\ [P_{w't'}\ he]]].$$

In order to reduce further, we now substitute a for x, F_{wt} for y and ${}^0[\lambda w'\lambda t'\ [P_{w't'}\ he]]$ for p into $[{}^0Believe^*_{wt}\ x\ [{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p]]$. Since the variables x, y, p occur in the extensional context of $[{}^0Believe^*_{wt}\ x\ [{}^0Sub\ [{}^0Tr\ y]\ {}^0he\ p]]$, such a substitution is admissible: F is to occur with supposition *de re*. Thus, if F_{wt} is v-improper the whole Composition must be v-improper as well. The result of the second reduction is

$$[{}^0Believe^*_{wt}\ a\ [{}^0Sub\ [{}^0Tr\ F_{wt}]\ {}^0he\ {}^0[\lambda w'\lambda t'\ [P_{w't'}\ he]]]].$$

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[37] See Section 3.5.

[38] For β-reduction 'by name', see Section 2.7.

Abstraction over w, t yields the final analysis of the active form of *de re* attitudes:

$$(de\ re\ act)^* \qquad \lambda w\lambda t\ [^0Believe^*_{wt}\ a\ [^0Sub\ [^0Tr\ F_{wt}]\ ^0he\ ^0[\lambda w'\lambda t'\ [P_{w't'}\ he]]]].$$

It might seem as though the passive form could be analysed analogously to the intensional case, along the lines of

$$\lambda w\lambda t\ [\lambda x\ [^0Believe^*_{wt}\ a\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]\ F_{wt}].$$

However, this Closure is *not* an admissible analysis of (*de re pas*). The problem is that the variable x is now ^{0}bound and so not free for substitution. The property of being believed* by a to be a P has to be constructed using the substitution method in the following manner:

$$\lambda w\lambda t\ [\lambda x\ [^0Believe^*_{wt}\ a\ [^0Sub\ [^0Tr\ \ x]\ ^0x\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]]].$$

Now the first occurrence of x in $[^0Sub\ [^0Tr\ \ x]\ ^0x\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]$ is free, unlike the second and third occurrence. Application of the so constructed property to the occupant of the F-office yields

$$(de\ re\ pas)^* \qquad \lambda w\lambda t\ [\lambda x\ [^0Believe^*_{wt}\ a\ [^0Sub\ [^0Tr\ \ x]\ ^0x\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]]\ F_{wt}].$$

Since the first occurrence of x in $[^0Believe^*_{wt}\ a\ [^0Sub\ [^0Tr\ \ x]\ ^0x\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]]$ is extensional, it is admissible to perform β-reduction 'by name' to obtain the reduced Closure

$$\lambda w\lambda t\ [^0Believe^*_{wt}\ a\ [^0Sub\ [^0Tr\ \ F_{wt}]\ ^0x\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]].$$

Thus (*de re pas*)* is equivalent to (*de re act*)*.

Indeed, the Compositions

(i) $\qquad [\lambda x\ [^0Believe^*_{wt}\ a\ [^0Sub\ [^0Tr\ \ x]\ ^0x\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]]\ F_{wt}]$

and

(ii) $\qquad [^0Believe^*_{wt}\ a\ [^0Sub\ [^0Tr\ \ F_{wt}]\ ^0x\ ^0[\lambda w'\lambda t'\ [P_{w't'}x]]]]$

are v-congruent for all valuations v, because F occurs extensionally in both Compositions. If F_{wt} v-constructs an individual b then the hyperproposition believed by a is in both cases the Closure $[\lambda w'\lambda t'\ [P_{w't'}\ b]]$. If F_{wt} is v-improper then both (i) and (ii) come out v-improper.

For similar reasons as in the intensional case, (*de dicto*)* and (*de re act/pas*)* constructional attitudes are logically independent.

That a *de re* belief* does not follow from a *de dicto* belief* is obvious; a *de re* belief* comes with the existential presupposition that the office F be occupied, unlike a *de dicto* belief*. There must be an individual of whom or which a has a belief* *de re*.

We are now going to show that belief* *de dicto* does not follow from belief* *de re*. Since we have just showed that (*de re act*)* and (*de re pas*)* are equivalent, it is sufficient to prove that

(*de re act*)* $\lambda w \lambda t\, [{}^0 Believe^*_{wt}\; a\, [{}^0 Sub\, [{}^0 Tr\, F_{wt}]\; {}^0 he\; {}^0[\lambda w' \lambda t'\, [P_{w't'}\; he]]]]$

cannot be equivalently reduced to

(*de dicto*)* $\lambda w \lambda t\, [Believe^*_{wt}\; a\; {}^0[\lambda w \lambda t\, [P_{wt}\, F_{wt}]]]$.

To achieve this, we have to prove that for at least one valuation v the Compositions

(ii) $[{}^0 Believe^*_{wt}\; a\, [{}^0 Sub\, [{}^0 Tr\, F_{wt}]\; {}^0 he\; {}^0[\lambda w' \lambda t'\, [P_{w't'}\; he]]]]$

and

(iii) $[Believe^*_{wt}\; a\; {}^0[\lambda w \lambda t\, [P_{wt}\, F_{wt}]]]$

are not v-congruent. That they are not is intuitively obvious, because F_{wt} occurs extensionally in (ii) and hyperintensionally in (iii). Yet for a detailed proof, we consider two cases again:

(a) If F_{wt} is v-improper, then (ii) is v-improper, unlike (iii).
(b) Let F_{wt} v-construct b. Then $[{}^0 Tr\, F_{wt}]$ v-constructs ${}^0 b$. Thus the Composition $[{}^0 Sub\, [{}^0 Tr\, F_{wt}]\; {}^0 he\; {}^0[\lambda w' \lambda t\, [P_{w't'}\; he]]]$ v-constructs the *Closure* $[\lambda w' \lambda t'\, [P_{w't'}\; {}^0 b]]$. This Closure is not procedurally isomorphic to $[\lambda w \lambda t\, [P_{wt}\; F_{wt}]]$, because F_{wt} and ${}^0 b$ are neither α- nor η-convertible. These constructions are not even equivalent. Therefore, the Closure $[\lambda w' \lambda t'\, [P_{w't'}\, {}^0 b]]$ cannot be equivalently substituted for $[\lambda w \lambda t\, [P_{wt}\; F_{wt}]]$ into the hyperintensional context of (iii).[39]

This time around, in the hyperintensional case not only partiality but also the notion of procedural isomorphism prevent constructions of attitudes* *de re* and *de dicto* from being equivalent, as illustrated by part (b) of the above proof.

On the other hand, intensional attitudes *implicitly* relate an individual a to any construction equivalent to the literal meaning of the embedded clause in virtue of relating a to the proposition constructed by all these mutually equivalent propositional constructions. This is also the reason why we were able to prove that the extra premise that a knows that the F is b plus the additional assumption that knowing

[39] See the hyperintensional rule of substitution in Section 2.7.1.

entails believing make intensional attitudes *de dicto* and *de re* mutually transferable via *b*.

However, even though the assumption that knowing* entails believing* is presumably not too controversial, the substitution of ${}^0b$ for F_{wt}, and *vice versa*, inside hyperintensional attitudes is anything but uncontroversial. The reason is this. The key step of the proof transforming a *de dicto* attitude into a *de re* one is that

(v) *a* believes that the *F* is a *P* and also that the *F* is *b*.

From this it follows that

(vi) *a* believes that *b* is a *P*.

But if belief is hyperintensional, we cannot deduce from (v) that *a* is logically committed to believing that *b* is a *P*. So this argument step is invalid. To show why, let us analyse the step.

(v′) $\lambda w \lambda t\, [[{}^0Believe^*_{wt}\, a\, {}^0[\lambda w \lambda t\, [P_{wt} F_{wt}]]] \wedge$
$[{}^0Believe^*_{wt}\, a\, {}^0[\lambda w \lambda t\, [F_{wt} = {}^0b]]]];$

(vi′) $\lambda w \lambda t\, [{}^0Believe^*_{wt}\, a\, {}^0[\lambda w \lambda t\, [P_{wt}\, {}^0b]]].$

At any $\langle w, t\rangle$ at which *a* believes* that the propositional constructions

$$\lambda w \lambda t\, [P_{wt} F_{wt}]$$

and

$$\lambda w \lambda t\, [F_{wt} = {}^0b]$$

construct propositions true at $\langle w, t\rangle$, *a* also assents to $[P_{wt} F_{wt}]$ and $[F_{wt} = {}^0b]$ both constructing **T**, provided *a* is logically rational. However, unless *a* masters the rule of substitution of *v*-congruent constructions occurring in extensional contexts, *a* need not assent to $[P_{wt}\, {}^0b]$ constructing **T**.

In general, as far as hyperintensional attitudes go, only procedurally isomorphic constructions are indistinguishable and thus mutually substitutable. This is so because, when in the realm of hyperintensional attitudes, we cannot presume any inferential abilities in an agent *a* that would parallel the logical closure characteristic of intensional attitudes. Hyperintensional attitudes are ascribed to an atributee *a* from the attributee's inner perspective. Formally, this is modelled by closing the meaning of the believed clause by Trivialisation. In a Composition of the form $[{}^0Believe^*_{wt}\, a\, {}^0[\lambda w \lambda t\, [P_{wt}\ F_{wt}]]]$ the Closure $[\lambda w \lambda t\, [P_{wt}\ F_{wt}]]$ is only *mentioned* (rather than *used* to construct a proposition), and thus not accessible to direct logical manipulation.

Similarly, the argument

a believes* that the occupant of *A* is an *F*
a knows* that *A* and *B* are co-occupied

---

a believes* that the occupant of *B* is an *F*.

which is valid in case of intensional belief, is in general *not valid* hyperintensionally.

We have seen that the hyperintensional analysis of propositional attitudes blocks undesirable consequences that might yield paradox. This is a point in favour of the hyperintensional approach. Yet it might seem that it is occasionally too restrictive. It portrays the attributee *a* as being little more than a logical moron, incapable as he is even of such a simple inference rule as the substitution of identicals. However, in Section 5.1.5 we are going to introduce the notion of inferable knowledge to accommodate various ways of calibrating *a*'s inferential abilities. The notion of inferable knowledge makes it possible to tune *a* in such a way that *a* comes out neither a moron nor a logically/mathematically omniscient genius, but a much more realistic agent.

5.1.2.3 Summary of attitudes

This section provides a taxonomy and a schematic analysis of propositional and hyperpropositional attitude reports.

Types: $a/*_n \rightarrow \iota$ (attitude agent); $B/*_n \rightarrow \iota_{\tau\omega}$ (subject of the attitude); $F/*_n \rightarrow (o\iota)_{\tau\omega}$ (the property ascribed to *B*).

I. Implicit (propositional) attitudes: $Att \rightarrow (o\iota o_{\tau\omega})_{\tau\omega}$
   (a) *De dicto*: *a Att*-s that *B* is an *F*.
   (b) *De re*:
      (i) *B* is *Att*-ed by *a* to be an *F*. (passive variant)
      (ii) *a Att*-s of *B* that *he* [*B*] is an *F*. (active variant with anaphora 'he')

II. *Explicit (constructional) attitudes*: $Att^* \rightarrow (o\iota *_n)_{\tau\omega}$
   (a) *De dicto*: *a Att**s that *B* is an *F*.
   (b) *De re*:
      (i) *B* is *Att**-ed by *a* to be an *F*. (passive variant)
      (ii) *a Att**-s of *B* that *he* [*B*] is an *F*. (active variant with anaphora 'he')

Analytic schemas

***Ad* I.** *Implicit (propositional) attitudes*

I. (a) *de dicto*: $\lambda w\lambda t\ [Att_{wt}\ a\ \lambda w\lambda t\ [F_{wt}\ B_{wt}]]$

I. (b) (i) *de re passive variant*.

First, a coarse-grained form:

$$\lambda w \lambda t\ [Att\_a\_F_{wt}\, B_{wt}],$$

$Att\_a\_F/*_n \rightarrow (o\iota)_{\tau\omega}$ the property of being *Att*-ed by a to be an F.
Second, we define the property Att_a_F ($x \rightarrow \iota$):

$$Att\_a\_F = \lambda w \lambda t\ [\lambda x\ [Att_{wt}\ a\ \lambda w \lambda t\ [F_{wt}\, x]]].$$

Third, the analysis of **I. (**b) (i) is obtained by replacing the left-hand side construction by the right-hand side definition of the property:

$$\lambda w \lambda t\ [[\lambda w \lambda t\ [\lambda x\ [Att_{wt}\ a\ \lambda w \lambda t\ [F_{wt}\, x]]]]_{wt}\ B_{wt}],$$

β-reducible to

$$\lambda w \lambda t\ [\lambda x\ [Att_{wt}\ a\ \lambda w \lambda t\ [F_{wt}\, x]]\ B_{wt}].$$

Further β-reduction 'by name' would not be valid, because we would then be substituting B_{wt} for x into the generic intensional context of $\lambda w \lambda t\ [F_{wt}\, x]$, which is not an equivalent transformation due to partiality, even in the case of a substitution that prevents the collision of variables by renaming. We cannot equivalently draw the extensional *de re* occurrence of B into the intensional context of $\lambda w \lambda t\ [F_{wt}\, x]$.[40]

I. (b) (ii) *de re active variant*
First, a coarse-grained form:

$$\lambda w \lambda t\ [Att\_of_{wt}\, a\ B_{wt}\ {}^2p].$$

Second, we define $Att\_of/*_n \rightarrow (o\iota\iota o_{\tau\omega})_{\tau\omega}$. It is a construction of an intension relating an individual x to another individual y and a proposition ${}^2p \rightarrow o_{\tau\omega}$. Schematically, x is the individual *who* believes of the individual y that *he* (*i.e., y*) is such that ${}^2p$:

$$Att\_of = \lambda w \lambda t\ \lambda x y\, p\ [Att_{wt}\, x\ {}^2[{}^0Sub_n\ [{}^0Tr_\iota\ y]\ {}^0he\ \ p]].$$

Types: $x, y, he \rightarrow \iota$; $Sub_n/(*_n *_n *_n *_n)$; $Tr_\iota/(*_1 \iota)$.

Double Execution is necessary here in order to descend from the hyperintensional context of the propositional construction (the result of applying the Sub_n function) to the intensional context of the proposition to which the individual *v*-constructed by y is related.

Third, the analysis of **I. (**b) (ii) is obtained by (a) extensionalizing *Att_of*, (b) applying Att_of_{wt} to a, B_{wt} and ${}^0[\lambda w \lambda t\ [F_{wt}\ he]]$, and (c) abstracting over w, t. The result is the schema of the *de re* active-variant analysis:

$$\lambda w \lambda t\ [{}^0Att_{wt}\ \ a\ {}^2[{}^0Sub_n\ [{}^0Tr_\iota\ B_{wt}]\ {}^0he\ {}^0[\lambda w \lambda t\ [F_{wt}\ he]]]].$$

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[40] See Section 2.7, Claim 2.5.

Notice that the substitution of B_{wt} for y is valid here, because the variable y occurs in the extensional context of the Composition $[{}^0Att_{wt}\, x\, {}^2[{}^0Sub_n\, [{}^0Tr_\iota\, y]\, {}^0he\;\; p]]$.

(*Ad* II) *Explicit (constructional) attitudes*

II. (a) *de dicto*: $\lambda w\lambda t\, [Att^*_{wt}\, a\, {}^0[\lambda w\lambda t\, [F_{wt}\, B_{wt}]]]$

II. (b) (i) *de re passive variant*

First, a coarse-grained analysis rendering the logical form is

$$\lambda w\lambda t\, [{}^0Att^*\_a\_F_{wt}\, B_{wt}],$$

$Att^*\_a\_F/*_n \rightarrow (o\iota)_{\tau\omega}$: the property of being Att^*-ed by a to be an F. Next, we refine the analysis by defining the property $(x \rightarrow \iota)$:

$$Att^*\_a\_F = \lambda w\lambda t\, [\lambda x\, [Att^*_{wt}\, a\, [{}^0Sub_n\, [{}^0Tr_\iota\, x]\, {}^0x\, {}^0[\lambda w\lambda t\, [F_{wt}\, x]]]]].$$

Now we have to use Sub_n and Tr_ι, because x occurs in the hyperintensional context of ${}^0[\lambda w\lambda t\, [F_{wt}\, x]]$, and so is not free for λ-binding. However, Double Execution of the result of applying Sub_n is not needed, because a is related to the hyperproposition and not what it constructs.

Second, by substituting the above definition of the property, we obtain a fine-grained analysis schema:

$$\lambda w\lambda t\, [\lambda w\lambda t\, [\lambda x\, [Att^*_{wt}\, a\, [{}^0Sub_n\, [{}^0Tr_\iota\, x]\, {}^0x\, {}^0[\lambda w\lambda t\, [F_{wt}\, x]]]]]_{wt}\, B_{wt}],$$

β-reducible to:

$$\lambda w\lambda t\, [\lambda x\, [Att^*_{wt}\, a\, [{}^0Sub_n\, [{}^0Tr_\iota\, x]\, {}^0x\, {}^0[\lambda w\lambda t\, [F_{wt}\, x]]]]\, B_{wt}].$$

Further β-reduction 'by name' *is* now an equivalent transformation. However, if executed, the result is identical to the analysis of the active variant *ad* II.b. (ii):

$$\lambda w\lambda t\, [Att^*_{wt}\, a\, [{}^0Sub_n\, [{}^0Tr_\iota\, B_{wt}]\, {}^0x\, {}^0[\lambda w\lambda t\, [F_{wt}\, x]]]].$$

II. (b) (ii) *de re active variant*

First, a coarse-grained analysis:

$$\lambda w\lambda t\, [Att^*of_{wt}\, a\, B_{wt}\;\; p]$$

$Att^*\_of/*_n \rightarrow (o\iota *_n)_{\tau\omega}$; $p \rightarrow *_n$; ${}^2p \rightarrow o_{\tau\omega}$.

Second, we define Att^*_of (*x-who*, *y-of whom*, *that-he* (i.e., y) *is an F*):

$${}^0Att^*\_of = \lambda w\lambda t\, \lambda xy\, p\, [{}^0Att^*_{wt}\, x\, [{}^0Sub_n\, [{}^0Tr_\iota\, y]\, {}^0he\;\; p]].$$

Third, the analysis of **II.** (b) (ii) is obtained by substituting a for x, B_{wt} for y, and ${}^0[\lambda w\lambda t\, [F_{wt}\, he]]$ for p, which is correct even if B_{wt} is v-improper, because y occurs in the extensional context of the Composition $[{}^0Att_{wt}\, x\, [{}^0Sub_n\, [{}^0Tr_\iota\, y]\, {}^0he\;\; p]]$:

$$\lambda w \lambda t\ [Att^*_{wt}\ a\ [{}^0Sub_n\ [{}^0Tr_\iota\ B_{wt}]\ {}^0he\ {}^0[\lambda w \lambda t\ [F_{wt}\ he]]]].$$

*Remark***.** It is easy to prove that *de re* and *de dicto* attitudes are logically independent, as neither kind of attitude entails the other.

However, if 'b' is a *rigid designator* of an individual, $b/*_n \to \iota$, making b *v-proper* for any v, then in case I. the *de dicto* and *de re* attitudes are logically equivalent, whereas in case II. they are not. In the second case only the active and passive variants of the *de re* attitude are logically equivalent.

Case I. *Implicit propositional attitudes*:

$$\lambda w \lambda t\ [Att_{wt}\ a\ \lambda w \lambda t\ [F_{wt}\ b]] = \qquad (de\ dicto)$$

$$\lambda w \lambda t\ [\lambda x\ [Att_{wt}\ a\ \lambda w \lambda t\ [F_{wt}\ x]]\ b] = \qquad (de\ re\ \text{passive})$$
$$\lambda w \lambda t\ [Att_{wt}\ a\ {}^2[{}^0Sub_n\ [{}^0Tr_\iota\ b]\ {}^0x\ {}^0[\lambda w \lambda t\ [F_{wt}\ x]]]] \qquad (de\ re\ \text{active}).$$

Case II. *Explicit hyperpropositional attitudes*:

$$\lambda w \lambda t\ [Att^*_{wt}\ a\ {}^0[\lambda w \lambda t\ [F_{wt}\ b]]] \neq \qquad (de\ dicto)$$

$$\lambda w \lambda t\ [\lambda x\ [Att^*_{wt}\ a\ [{}^0Sub_n\ [{}^0Tr_\iota\ x]\ {}^0x\ {}^0[\lambda w \lambda t\ [F_{wt}\ x]]]]\ b] =$$
$$(de\ re\ \text{passive})$$
$$\lambda w \lambda t\ [Att^*_{wt}\ a\ [{}^0Sub_n\ [{}^0Tr_\iota\ b]\ {}^0x\ {}^0[\lambda w \lambda t\ [F_{wt}\ x]]]] \qquad (de\ re\ \text{active}).$$

The non-equivalence is due to the fact that the hyperpropositions to which a is related may not be *procedurally isomorphic*; e.g.,

$${}^0[\lambda w \lambda t\ [F_{wt}\ b]]] \neq [{}^0Sub_n\ [{}^0Tr_\iota\ b]\ {}^0x\ {}^0[\lambda w \lambda t\ [F_{wt}\ x]]].$$

They are only *equivalent* (on the assumption just made of b being proper):

$$[\lambda w \lambda t\ [F_{wt}\ b]]] = {}^2[{}^0Sub_n\ [{}^0Tr_\iota\ b]\ {}^0x\ {}^0[\lambda w \lambda t\ [F_{wt}\ x]]].$$

The reason for this is the fact that while $[{}^0Tr_\iota\ b]$ v-constructs the Trivialization of the individual v-constructed by b, b itself may be a Composed construction v-constructing the same individual. For instance, though ${}^0Charles = [\iota x\ [x = {}^0Charles]]$, the left-hand side Trivialization is neither identical nor procedurally isomorphic to the right-hand side Composition. Thus one could believe* that Charles is an F without believing* that the only individual who is equal to Charles is an F. Constructional attitudes are very restrictive and exacting: the attributer must reproduce the interior agent's attitude in a way that literally and exactly reproduces the agent's own perspective and procedure.

This completes our summary of attitudes as these have been dealt with up to now.

5.1.3 Inconsistent belief

A hyperintensional attitude logic is indispensable for analyzing inconsistent beliefs.[41] In particular, in order to safeguard the rationality of anyone entertaining an inconsistent belief, it is important that the agent may be related to a hyperproposition (procedure) without thereby being related to the proposition it presents (its product). Intuitively, what happens when entertaining an inconsistent belief is that one is mistaken about the nature of the product. Had one realized that the procedure engenders paradox one would not have embraced it.[42]

Inconsistent attitudes are extreme attitudes for at least three reasons. First, no instance of $A \wedge \neg A$ can be true; therefore, none can figure as an object of knowledge. Second, they cannot be relations to propositions (i.e., sets of logically *possible* worlds) on pain of all inconsistent attitudes converging in the class of attitudes to impossible propositions[43]:

$$^{0}Untrue = \lambda p\ [\forall w \forall t\ [[^{0}False_{wt}\,p] \vee [^{0}Undef_{wt}\,p]]],\ p \rightarrow o_{\tau\omega}.$$

Third, their attitude *relata* are inherently logically complex, since constructions of the connectives $\wedge$, $\neg$ are needed to generate the complex hyperproposition $A \wedge \neg A$.[44]

The notion of constructional (conceptual) attitude makes possible a simple and natural answer to the question of what an agent entertaining an inconsistent attitude is related to. We relate the agent (in the empirical case) to a construction of a proposition belonging to the class *Untrue* or (in the mathematical case) either to a o-construction of **F** or an improper construction.

So ours is a hyperintensional solution. By contrast, an extensional ('Meinongian') account will move 'downwards' by introducing an ontology of round squares, even primes distinct from 2, Russellian barbers, etc., as impossible extensional entities. And an intensional account will move 'sideways' by introducing a twin logical space of impossible worlds at which contradictions are true, as in 'impossible worlds semantics'.[45] A

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[41] More drastically, and much less realistically, one could stipulate that agents never entertain inconsistent beliefs. This stipulation is found in, e.g., the widespread system *KD45* used to modelling belief, whose axiom D is $\neg(\Diamond p \wedge \Diamond\neg p)$.

[42] Save paraconsistent attitude agents or agents whose attitudes are being modelled by paraconsistent logicians. See Priest (2000).

[43] Traditionally, a world w' is doxastically accessible to agent a from world w just in case w' is compatible with a's information state in w. If at w a knows/believes that P then in all possible worlds accessible to a from w it is the case that P. Thus, if at w a knows/believes that P, and P is impossible, then no world is doxastically accessible to a from w, or from any other logically possible world, for that matter.

[44] *Inconsistent* beliefs must not be confused with *incompatible* beliefs. There is a difference in the scope of $\wedge$: $B_a(A \wedge \neg A)$ *vs.* $(B_a A) \wedge (B_a \neg A)$, resp. Besides, an inconsistent belief is one attitude, whereas incompatible beliefs are at least two in number.

[45] See Hintikka (1975) and Priest (2000); for criticism, see MacPherson (1993) or Thrush (2001).

hyperintensional account, on the other hand, moves 'upwards' by introducing a sphere of hyperintensions that present contradictions without themselves being contradictory.

The reason why it is possible for a rational agent to entertain an inconsistent attitude is because what is constructed by a construction is not a constituent of the attitude *relatum*. The agent fully understands the procedure encoded by the construction, yet fails to appreciate the true nature of the product so produced. In the empirical case, the agent believes that the construction constructs a true proposition (whereas it constructs a proposition that is either false or undefined). In the mathematical case, the agent believes that the construction constructs **T** (whereas it either constructs **F** or is improper). Famously and tragically, Frege believed that his system was consistent, until Russell's polite letter pointed out the paradox to him.

For an empirical example, suppose the believer believes that it is both raining and not raining. Then, sketchily, the believer is related to a construction whose subconstructions are these:

(a) a construction of *Raining*
(b) a construction of *non-Raining*
(c) a construction of $\wedge$
(d) an application of $\wedge$ to *Raining*, *non-Raining*.

The Closure of the impossible proposition that it is raining and not raining is

$$\lambda w \lambda t\ [[{}^0\mathit{Raining}_{wt}] \wedge \neg[{}^0\mathit{Raining}_{wt}]].$$

The Closure of the proposition that *a* believes* that it is raining and not raining is

$$\lambda w \lambda t\ [{}^0B^*_{wt}\ a\ {}^0[\lambda w \lambda t\ [[{}^0\mathit{Raining}_{wt}] \wedge \neg[{}^0\mathit{Raining}_{wt}]]]].$$

Types: $B^*/(o\iota*_1)_{\tau\omega}$; $a/*_n \rightarrow \iota$; $\mathit{Raining}/o_{\tau\omega}$.

The template of the logical form of an inconsistent belief is

$$\lambda w \lambda t\ [{}^0B^*_{wt}\ a\ {}^0[[\ldots] \wedge \neg[\ldots]]].$$

Yet some forms are more complicated and not so obviously contradictory. For a more elaborate example, let *a* believe* that all pigs fly and that there is some pig that does not fly:

$$\lambda w \lambda t\ [{}^0B^*_{wt}\ a\ {}^0[\lambda w \lambda t\ [[{}^0\mathit{All}\ {}^0\mathit{Pig}_{wt}]\ {}^0\mathit{Fly}_{wt}] \wedge [\exists y\ [[{}^0\mathit{Pig}_{wt}\ y] \wedge \neg[{}^0\mathit{Fly}_{wt}\ y]]]]].$$

Types: $\mathit{All}/((o(o\iota))(o\iota))$; $\mathit{Pig}, \mathit{Fly}/(o\iota)_{\tau\omega}$; $\exists/(o(o\iota))$; $y \rightarrow \iota$.

Zalta's abstract-object approach is kindred to ours. Some (hyperintensional) *abstract* objects *encode* properties that cannot be *exemplified* (i.e., some abstract objects cannot be matched by a concrete object). As for worlds, an impossible world is an abstract object encoding properties that no world would be able to exemplify.[46] As in TIL, Zalta locates his impossibilities neither on the extensional nor on the intensional level, but on the hyperintensional level. Zalta is able to offer a hyperintensional solution to inconsistent attitudes. Thus, the sentence

‘*a* believes that there is a barber who shaves all and only those who do not shave themselves’

is analysed as[47]

$$Bel\ (a, [\exists x\ (Barber\ x\ \&\ \forall y\ (Shave\ x\ y \leftrightarrow \neg Shave\ y\ y))]).$$

a’s belief relatum is an abstract object ‘presenting’ the empty set of possible worlds, to which *a* is *not* related. Unlike TIL, however, possible worlds (and times) are not mentioned in the analysis itself. This makes it impossible to distinguish between empirical and mathematical attitudes. For comparison, our analysis of the sentence is

$$\lambda w\lambda t\ [{}^0B^*_{wt}\ a\ {}^0[\lambda w\lambda t\ \exists x\ [[{}^0Barber_{wt}\ x] \wedge [\forall y\ [{}^0Shave_{wt}\ x\ y] \equiv \neg[{}^0Shave_{wt}\ y\ y]]]]].$$

An example of an inconsistent mathematical attitude would be

‘a believes* that two plus two makes four and that two plus five makes eight’.

Obviously *a* does not believe the truth-value **F**. He is related to the construction of **F** and believes that the respective construction constructs **T**.

The analysis is (=/(ονν))

$$\lambda w\lambda t\ [{}^0B^*_{wt}\ a\ {}^0[[{}^0{=}\ [\ {}^0{+}\ {}^02\ {}^02]\ {}^04] \wedge [{}^0{=}\ [{}^0{+}\ {}^02\ {}^05]\ {}^08]]].$$

5.1.4 Knowing whether

We have shown in Sections 5.1.1 and 5.1.2 how to analyse ‘*a* knows that *P*’, where *P* is a proposition, and ‘*a* knows* that *C*’, where *C* is a construction of a proposition. And we have shown in Section 5.1.2 how to analyse attitudes *de dicto* and *de re*. In this section we show how to analyse propositional and constructional attitudes of the form, ‘*a* knows whether *P*’ and ‘*a* knows* whether *C*’ in their *de dicto* and *de re* variants.[48]

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[46] See Zalta (1997, §7).

[47] Kindly provided in personal communication (see Jespersen, 2002, p. 135, n. 21).

[48] Rescher calls it an ‘[E]pistemic resolution regarding a proposition [*A*] when the knower [*a*] knows whether *A* is true or not: $KaA \vee Ka\neg A$’ (2005, p. 24). See also Hintikka (1975) and Lewis (1998). $KaA \vee Ka\neg A$ is not a tautology, for *a* may know neither *A* nor $\neg A$, and should not be confused with the classical tautology $KaA \vee \neg KaA$ (See Genesereth and Nilsson, 1987, p. 227).

The most important difference between *knowing that A* and *knowing whether A* is that the latter is *not factive*: *knowing whether A* is logically compatible with $\neg A$ (i.e., the negation of A being true) or with A lacking a truth-value. Where A is an arbitrary object of knowledge, we construe *knowing whether A* as a special case of a general case. The general case is

knowing which disjunct (if any) of A or B is true.

If you know whether $A \vee B$ it is because any one of the following four options obtains:

- *knowing that A*
- *knowing that B*
- *knowing that both A and B*
- *knowing that neither A nor B*.

The reason why *knowing whether A* is a special case is because B is here $\neg A$, with the proviso that the third option cannot obtain, unless the background logic is a paraconsistent one.

In general, an ascription of *knowledge whether* does not reveal which of the four (three) options obtains. Nor need the ascriber know which obtains in order to make a true ascription. But the ascriber must know that the ascribee knows which it is. For illustration, imagine that you know that Fermat had a proof of whether his Last Theorem is, indeed, a theorem, but do not know which way the proof went. Then you know that Fermat knew whether the Theorem is a theorem, while you may not know what Fermat knew. What you do know is that Fermat would have been the one to turn to for a conclusive answer.

The third option presupposes that $B \neq \neg A$, as we do not allow that knowledge may be inconsistent.[49] The fourth option presupposes that if A, B are propositions then they must be properly partial functions; and that if A, B are constructions then they must construct propositions with this property, or must be v-improper for some valuation v.

For an example of the third option, suppose you know whether the sun is shining or it is raining. If it is both raining and the sun is shining, then what you know when knowing *whether* the sun is shining or it is raining is *that* the sun is shining and that it is raining.

For an example of the fourth option, suppose you know whether Jupiter's only moon is larger than the Earth's moon. Since Jupiter does not have only one moon (but four major and several minor ones), Jupiter's only moon is neither larger nor not larger than ours. There is no fact of the matter as to whether Jupiter's only

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Hart et al. argue that *knowing whether* and *knowing that* are interdefinable, such that a knows that A iff A and a knows whether A (1996, p. 254). They also point out that *knowing whether* is 'invariant under complementation'; a knows whether A iff a knows whether $\neg A$. This is due to the non-factivity of *knowing whether*, and is symptomatic of its paucity of information.

[49] However, see Section 5.1.3 for inconsistent beliefs.

moon is larger than ours, since there is no unique moon of Jupiter's of which it would be either true or false that it was larger than our moon.[50] So what you know when knowing *whether* Jupiter's only moon is larger than ours is *that* Jupiter's only moon is neither larger nor not larger than ours. This is still to know an empirical fact, though the fact is nothing to do with any relation between Jupiter's unique moon and ours. Instead what you know is that the proposition that Jupiter's only moon is larger than the Earth's lacks a truth-value. This you know because you know that the individual office of Jupiter's unique moon goes vacant at the actual world at the present moment. Whenever an existential presupposition goes unsatisfied, an ascription of *knowledge whether* must employ the fourth option.

To express the fourth option in logical notation, we need to make use of the property of being undefined (*Undef*). Let $True, False, Undef/(o o_{\tau\omega})_{\tau\omega}$;[51] let C be a propositional construction. Then

$$\forall w \forall t\, [[{}^0Undef_{wt}\, C] = [\neg[{}^0True_{wt}\, C] \wedge \neg[{}^0False_{wt}\, C]]].$$

Knowing whether requires two definitions, because in the empirical case *knowing* may be either a relation (-in-intension) to a proposition or a relation (-in-intension) to a propositional construction. We use the notation and typing introduced in Section 5.1.1:

$K/(o\iota o_{\tau\omega})_{\tau\omega}$ ('to know that a proposition is true')

$K^*/(o\iota *_n)_{\tau\omega}$ ('to know that a construction constructs a true proposition').

Let P, $Q/o_{\tau\omega}$; C, $D/*_1 \rightarrow o_{\tau\omega}$; $p/*_1 \rightarrow o_{\tau\omega}$; c, $d/*_2 \rightarrow *_1$; ${}^2c$, ${}^2d \rightarrow o_{\tau\omega}$; $=_1/(o o_{\tau\omega} o_{\tau\omega})$; $=_2/(o *_1 *_1)$; $\iota/(o_{\tau\omega}(o o_{\tau\omega}))$; $\iota^*/(*_1(o *_1))$. Here C, D are propositional constructions, and c, d variables ranging over propositional constructions. We only define the cases in which ${}^0Q =_1 \lambda w \lambda t\, [\neg{}^0P_{wt}]$, ${}^0D =_2 {}^0[\lambda w \lambda t\, [\neg C_{wt}]]$ and C, D are constructions of order 1 to keep the definitions as economic as possible. The respective general cases may be readily reconstructed from the definitions.

Definition 5.1 (*knowing whether P*) Let C construct P.
Then *a knows whether P* iff

$$\lambda w \lambda t\, [{}^0K_{wt}\, a\, [\iota p\, [p_{wt} \wedge [[p =_1 C] \vee [p =_1 \lambda w \lambda t\, \neg[C_{wt}]] \vee [p =_1 \lambda w \lambda t\, [{}^0Undef_{wt}\, C]]]]]]. \qquad \square$$

Remark. When P is a total function, what is known is

$$[\iota p\, [p_{wt} \wedge [[p =_1 C] \vee [p =_1 \lambda w \lambda t\, \neg[C_{wt}]]]]].$$

[50] See Section 1.5.2 for partial functions and existential presuppositions.

[51] See also Section 1.5.2.1.

Definition 5.2 (*knowing* whether C*) Let C construct P.
Then a *knows* whether C* iff

$$\lambda w \lambda t\, [{}^0K_{wt}\, a\, [\iota^*c\, [[{}^2c]_{wt} \wedge [[c =_2 {}^0C] \vee [c =_2 {}^0[\lambda w \lambda t\, [\neg C_{wt}]]] \vee [c =_2 {}^0[\lambda w \lambda t\, [{}^0Undef_{wt}\, C]]]]]]]. \qquad \square$$

Having set out the foundations of our logic of *knowing whether*, we now apply this logic to a six-fold disambiguation of the ascription

'*a* knows whether Scott is the author of *Waverley*'

in order to express *knowing whether* and *knowing* whether* in their *de dicto* and *de re* variants. We need to employ the partial option defined above, since the individual office of author of *Waverley* is a properly partial function and, hence, so is the proposition that Scott is the author of *Waverley*.

Notice that the propositional and constructional attitudes *de re* will both have two variants, as soon as we allow that the ascribed sentence may also be read as, '*a* knows whether the author of *Waverley* is Scott'.[52] The disambiguations of the ascribed sentence are the following paraphrases:

- (propositional, *de dicto*)
  '*a* knows whether the proposition that Scott is the author is true, false or undefined';
- (*propositional*, *de re*)
  (i) '*a* knows of Scott whether the proposition that he is the author is true, false or undefined';
  (ii) '*a* knows of the author whether the proposition that he/she is Scott is true or not';
- (*constructional*, *de dicto*)
  '*a* knows* whether the construction constructing the proposition that Scott is the author constructs a true, false or undefined proposition';
- (*constructional*, *de re*)
  (i) '*a* knows* of Scott whether the construction constructing the proposition that he is the author constructs a true, false or undefined proposition';
  (ii) '*a* knows* of the author whether the construction constructing the proposition that he/she is Scott constructs a true, false or undefined proposition'.

Let s/ι (Scott); $AW/\iota_{\tau\omega}$ (the individual office of author of *Waverley*); $=/(o\iota\iota)$; $Sub/(*_1*_1*_1*_1)$; $Tr/(*_1\iota)$; $y/*_1 \rightarrow \iota$. Then:

[52] In order to disambiguate the topic-focus articulation of the sentence we use two different word orders; viz., 'The author of *Waverley* is Scott' *vs*. 'Scott is the author of *Waverley*'. The topic of the former is the author of *Waverley*, whereas the topic of the latter is Scott. Thus, the former comes with the presupposition that the author of *Waverley* should exist. See Duží (2009) for details on topic and focus, as explained in terms of TIL.

(*propositional*, *de dicto*)

$$\lambda w\lambda t\ [{}^0K_{wt}\ a\ [\iota p\ [p_{wt} \wedge [[p =_1 \lambda w\lambda t\ [{}^0AW_{wt} = {}^0\mathrm{s}]] \vee [p =_1 \lambda w\lambda t\ \neg[{}^0AW_{wt} = {}^0\mathrm{s}]] \vee [p =_1 \lambda w\lambda t\ [{}^0Undef_{wt}\ \lambda w\lambda t\ [{}^0AW_{wt} = {}^0\mathrm{s}]]]]]]]$$

(*propositional*, *de re*)

(i) $\lambda w\lambda t\ [{}^0K_{wt}\ a\ [\iota p\ [p_{wt} \wedge {}^2[{}^0Sub\ {}^0\mathrm{s}\ {}^0y\ {}^0[[p =_1 \lambda w\lambda t\ [y = {}^0AW_{wt}]] \vee [p =_1 \lambda w\lambda t\ \neg[y = {}^0AW_{wt}]] \vee [p =_1 \lambda w\lambda t\ [{}^0Undef_{wt}\ \lambda w\lambda t\ [y = {}^0AW_{wt}]]]]]]]]$

(ii) $\lambda w\lambda t\ [{}^0K_{wt}\ a\ [\iota p\ [p_{wt} \wedge {}^2[{}^0Sub\ [{}^0Tr\ {}^0AW_{wt}]\ {}^0y\ {}^0[[p =_1 \lambda w\lambda t\ [y = {}^0\mathrm{s}]] \vee [p =_1 \lambda w\lambda t\ \neg[y = {}^0\mathrm{s}]]]]]]]$

(*constructional*, *de dicto*)

$$\lambda w\lambda t\ [{}^0K^*_{wt}\ a\ [\iota^*c\ [[{}^2c]_{wt} \wedge [[c =_2 {}^0[\lambda w\lambda t\ [{}^0AW_{wt} = {}^0\mathrm{s}]]] \vee [c =_2 {}^0[\lambda w\lambda t\ \neg[{}^0AW_{wt} = {}^0\mathrm{s}]]] \vee [c =_2 {}^0[\lambda w\lambda t\ [{}^0Undef_{wt}\ \lambda w\lambda t\ [{}^0AW_{wt} = {}^0\mathrm{s}]]]]]]]]$$

(*constructional*, *de re*)

(i) $\lambda w\lambda t\ [{}^0K^*_{wt}\ a\ [\iota^*c\ [[{}^2c]_{wt} \wedge {}^2[{}^0Sub\ {}^0\mathrm{s}\ {}^0y\ {}^0[[c =_2 {}^0[\lambda w\lambda t\ [y = {}^0AW_{wt}]]] \vee [c =_2 {}^0[\lambda w\lambda t\ \neg[y = {}^0AW_{wt}]]] \vee [c =_2 {}^0[\lambda w\lambda t\ [{}^0Undef_{wt}\ \lambda w\lambda t\ [y = {}^0AW_{wt}]]]]]]]]]$

(ii) $\lambda w\lambda t\ [{}^0K^*_{wt}\ a\ [\iota^*c\ [[{}^2c]_{wt} \wedge {}^2[{}^0Sub\ [{}^0Tr\ {}^0AW_{wt}]\ {}^0y\ {}^0[[c =_2 {}^0[\lambda w\lambda t\ [y = {}^0\mathrm{s}]]] \vee [c =_2 {}^0[\lambda w\lambda t\ \neg[y = {}^0\mathrm{s}]]]]]]]]$.

Remark. The attitudes *de re* contain occurrences of anaphoric reference.[53] These *de re* cases help elucidate the notion of attitude *de re* in general. On the one hand, the attributer is responsible for replacing the occurrences of y (the sense of 'he/she') by (i) the Trivialization of s, or (ii) the Trivialization of the individual playing the role of author of *Waverley*, whereas the possible identity between the author and Scott may well elude attributee a altogether. On the other hand, the attributee knows whether the proposition is true or false (or neither, in case (i)), whereas the attributer may not.

Remark. On its readings *de re ad* (ii), 'a knows whether the author is Scott' denotes a proposition that is undefined for any $\langle w, t\rangle$ at which the author fails to exist. If the author does exist and the office is occupied by individual X, then the known proposition p is either the trivial proposition that $X = s$ or the likewise trivial proposition that $X \neq s$. On the other hand, on its *de re* readings *ad* (i), 'a knows whether Scott is the author' denotes a proposition that is never undefined, whereas the known proposition p may be undefined, or the known* propositional construction constructs proposition p that may be undefined.

Mathematical attitudes invariably demand constructional treatment. Knowing* whether Fermat's Last Theorem is true is to know* which of two constructions constructs **T**. The *analysandum* is the sentence (disregarding tense)

[53] See Section 3.5.

'Fermat knows whether there are positive integers a, b, c, n ($n > 2$) such that $a^n + b^n = c^n$.'

Let ν be the type of natural numbers. Let a, b, c, n, $x/*_1 \rightarrow \nu$; *Pos(itive integers)*/($o\nu$); $2/\nu$; *Fermat*/ι; $\forall$, $\exists/(o(o\nu))$; $d/*_2 \rightarrow *_1$; ${}^2d \rightarrow o$. We write 'x^n' for '$[{}^0Exp\ n\ x]$', *Exp*/($\nu\nu\nu$) the power function taking x to its n^{th} power. The analysis is then the Closure

$$\begin{aligned}&\lambda w\lambda t\ [{}^0K^*_{wt}\ {}^0Fermat\ [\iota^*d\ [[{}^2d] \wedge \\ &[d =_2 {}^0[\exists abcn\ [[{}^0Pos\ a] \wedge [{}^0Pos\ b] \wedge [{}^0Pos\ c] \wedge [{}^0{>}\ n\ {}^02] \wedge \\ &\qquad [{}^0{=}\ [{}^0{+}\ a^n\ b^n]\ c^n]]] \vee \\ &d =_2 {}^0[\forall abcn\ [[{}^0Pos\ a] \wedge [{}^0Pos\ b] \wedge [{}^0Pos\ c] \wedge [{}^0{>}\ n\ {}^02] \supset \\ &\qquad \neg[{}^0{=}\ [{}^0{+}\ a^n\ b^n]\ c^n]]]]]]].\end{aligned}$$

5.1.5 Epistemic closure and inferable knowledge

In Section 5.1.1 we briefly mentioned the notion of inferable knowledge as a weaker alternative to the notion of inferential knowledge. The rationale for the new notion is that, given an agent endowed with a stock of knowledge and command of some rule(s) of inference, we as exterior agents infer what the interior agent *might* validly infer from his or her existing knowledge by means of the rule(s) he or she masters. We do not ascribe any new *inferential* knowledge to him or her, as this would presuppose that the agent had actually gone through the process of inferring new knowledge and adding it to his or her stock of existing knowledge. Going through this process may (as we also briefly mentioned in Section 5.1.1) be too much of a good thing, as it were, since the act of drawing inferences may strain the agent's restricted resources and, moreover, lead to overloading his or her knowledge base with irrelevant knowledge. These are considerations to do with pragmatic rationality; a consideration to do with logical rationality is that swapping inferential knowledge for inferable knowledge seems promising, as far as blocking logical omniscience goes.[54]

We introduce the notion of inferable knowledge by showing how it tackles epistemic closure. The particular variant we wish to consider is

$$[K_a\,\varphi\ \wedge\ K_a\,(\varphi \supset \psi)] \supset K_a\,\psi \qquad\qquad (\textit{Epistemic Closure})$$

[54] The notion of inferable knowledge introduced in this section is mainly applicable to theories of how *computational* agents acting in a multi-agent system reason. In such a system the problem of modeling agents' knowledge adequately in order to prevent over-inferring (and under-inferring) is particularly important, for the system must not engender inconsistencies. See also Duží et al. (2005).

Epistemic Closure (EC) is epistemic logic's counterpart of *modus ponendo ponens*, with the epistemic operator K_a (*know*) preceding both the premises and the conclusion: *a* knows that φ; *a* knows that φ implies ψ; therefore, *a* knows that ψ. This principle is formulated, in epistemic logic, as the *K axiom* (of *logical rationality*).

The question we wish to raise is, Is EC valid? The K axiom is not a logical truth, so the answer is not a straightforward Yes. However, we are going to argue that EC is valid—at least if restricted in certain respects that we are going to set out below. The claim is not that our way is the only way to secure validity (nor are we going to canvass a long parade of other attempts to do so), but that ours is one provably efficient way of doing so. These restrictions give rise to our notion of *inferable knowledge*.

Though the axiom is not valid for just any interpretation of K_a, it is trivially valid if K_a models *implicit* knowledge. The implicit knowledge of an agent *a* is any set of propositions that are logically compatible with *a*'s initial stock of knowledge. When combined with the rule of necessitation, $\varphi \vdash K_a \varphi$, *logical omniscience* ensues. Logical omniscience is innocuous, as long as we are modelling only implicit knowledge, since the K axiom simply traces all the logical consequences of a given stock of knowledge that obtain whether the agent is aware of them or not. The axiom does not entail that the agent should know *explicitly* what he implicitly knows. A way of putting the relationship between implicit and explicit knowledge is that for each piece of explicit knowledge there is a set of implicit knowledge, namely the set of logical consequences of the former.

However, the situation is somewhat different in the case of explicit knowledge. Explicit-knowledge closure is conceptually incompatible with resource-bounded epistemic agents. EC thus understood would mean that every agent explicitly knew every logical consequence of every proposition in his initial knowledge base. Resource-bounded agents cannot possess such a magnitude of explicit knowledge. Therefore, although EC may be a principle of *logical* rationality, it would not be possible for resource-bounded agents to adhere to explicit-knowledge EC as a principle guiding their policy of drawing inferences. Nor would it be pragmatically rational for them to (attempt to) infer each and every conclusion following from their supply of explicit knowledge. They (we) would be inundated with irrelevant and useless knowledge taxing their (our) resources. And even if the agents were not required to infer *every* conclusion and could thus avoid deriving useless knowledge, they might not be able to infer *any* conclusion that is needed, in case they lack the capabilities required for such inferences.[55]

All the same, there is a philosophical interpretation of EC on which it is valid and at the same time compatible with resource-bounded agents. The basic idea is to calculate the stock of *inferable* knowledge of a given agent, in the following manner. Given an agent *a*, a possible world *w* and an instant of time *t*, the inferable

[55] For more on resource-bounded agents, see, e.g., Wassermann (1999) or Pollock (2006, Ch. 1).

knowledge of *a* at $\langle w, t\rangle$ functionally depends on *a*'s stock of explicit knowledge at $\langle w, t\rangle$ together with a set of inference rules that *a* masters at $\langle w, t\rangle$. By 'mastering a rule of inference' we mean that the agent would be able to apply the rule to all suitable elements of his stock of knowledge at $\langle w, t\rangle$. We calculate neither the agent's inferred knowledge nor the set of logical consequences of the set of propositions the agent does know (This does not rule out the theoretical possibility of the set of inferable knowledge coinciding with the set of conclusions drawn by the agent). Thus, our EC contains much less idealisation than both explicit- and implicit-knowledge EC. There is still a residual element of idealisation, though, due to the fact that we cannot know, and must instead stipulate, which rules the agent masters at $\langle w, t\rangle$. The idealisation is that we allow ourselves to assign a set of rules of inference to *a.* This explains why the logic of our EC is in essence a logic that validates the implication that *if a* masters this or that rule *then a*'s set of inferable knowledge is thus and so.[56]

Categorical ascription of particular rules of inference realistically applies only to designed agents, since in this case we are in charge of the package of inferential capacities being fed into the agent. Real-world examples of such designed agents would be computer software. The ascription of rules is only subject to the constraint that the agent must be known not to malfunction, but instead to draw inferences in strict accordance with the rules assigned to it by its designer(s). When applied to non-designed, whimsical human agents, the ascription of rules does not proceed by saying that oftentimes, every Wednesday, or in fair weather, the agent gets it right, as this would amount to a logically inoperative empirical generalisation. Instead, if proceeding conditionally—*if* the agent possesses rules $R_1, \ldots, R_n$ *then* the agent will have this or that stock of inferable knowledge—the ascription of one or more rules to an agent becomes logically operative. It also enables us to make fallible, but reasonable, empirical hypotheses about particular individuals. On the basis of their past inferential practices exhibiting a certain pattern, we hypothesise that *if* the agent sticks to the same pattern on future occasions *then* the agent's inferable knowledge will be such-and-such.

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[56] Our strategy of imposing limitations on the inferential power of our epistemic agents is kindred to the strategy pursued in Thijsse (1993). Furthermore, the so-called step-logic of Elgot-Drapkin shares some similarities with our approach, especially by being quite restrictive as for the number of conclusions that may be drawn at a particular step. We basically agree with Elgot-Drapkin that, 'Intuitively, we view an agent as an inference mechanism' subject to various constraints such that 'for real-time effectiveness and cognitive plausibility, at each step [of drawing inferences] we want only a finite number of conclusions to be drawn.' (1991, p. 413). Only we are more restrictive, in that on our theory of infer*able* knowledge no inferences may be actually drawn at each step. Our theory only catalogues the inferences it would be valid for a given agent to draw. In a wider perspective, the heightened attention epistemic logicians are paying to the *acts* of inference by means of which epistemic agents acquire inferential knowledge (always a staple tenet of intuitionism/constructivism, however) is a step in the right direction. The connection between finite agents and inferable conclusions is a porous one and the place to look for real-world limitations on what real-world agents may and may not acquire of inferential knowledge.

Before turning to inferable knowledge, we sketch two standard intensional approaches to knowledge modelling, namely Kripke and Montague-Scott structures. Afterwards we recursively define *inferable knowledge* and explain the mechanisms of computing it.

In a Kripke semantics each formula's truth-value is determined not only by a valuation but also by the state of a *possible world*.[57] A Kripke model M is a tuple $\langle W, I, R_1, \ldots, R_n \rangle$, where W is a set of possible worlds, R_i binary relations of accessibility over W, and I a function that assigns subsets of W to formulae. Formula φ is true in $w \in W$ iff $w \in I(\varphi)$, denoted '$(M, w) \models \varphi$'. The knowledge of an agent a_i is defined as *truth in all worlds* accessible to a_i according to the relation R_i: $(M, w) \models K_{ai}\varphi$ iff for all w' ($w\ R_i\ w'$) it holds that $(M, w') \models \varphi$. In these systems an overly strong version of EC holds, dubbed the *Inferential Accessibility Principle* by Rescher (2002, p. 479):

$$\text{if } (M, w) \models K_a \varphi \text{ and } (\varphi \models \psi) \text{ then } (M, w) \models K_a \psi.$$

If a knows any empirical proposition, then a also knows everything logically implied by it. And a immediately knows all analytical truths as well, because they follow from the empty set of assumptions; or semantically put, they are true in every possible world.

Such a system can be characterised as a 'view from the outside': ψ is known by a, whenever ψ cannot be falsified by a. An agent's knowledge is something ascribed to a by, say, a system designer. Agents are regarded neither as gaining their knowledge by themselves, nor as being explicitly aware of their knowledge. Such agents are in effect nothing other than inert points of evaluation.

Montague-Scott neighbourhood semantics is another version of possible-world semantics.[58] The truth-value of a formula φ is (as in Kripke structures) determined by the particular state of a world. A *Montague-Scott structure* is a tuple $\langle W, I, C_1, \ldots, C_n \rangle$, where $C_i(w)$ is a set of subsets of W; namely, the set of propositions that the agent a_i knows in w. The formula $K_{ai}\,\varphi$ is true in w, $(M, w) \models K_{ai}\,\varphi$, iff the set of worlds w' in which φ is true is a member of $C_i(w)$: $\{w' \mid (M, w') \models \varphi\} \in C_i(w)$. Montague-Scott structures can be used to model more realistic situations. For instance, agent$_1$ may not know whether φ or $\neg\varphi$, but he may know that agent$_2$ knows whether φ is true. Knowledge in Montague-Scott semantics is not closed under entailment, because the set $C_i(w)$ of propositions that the agent a_i knows in w is imposed on a_i; it does not have to contain all consequences of propositions explicitly known by a_i. Yet it is closed under equivalence ('$\Leftrightarrow$' standing for logical equivalence):

$$\text{if } K_a \varphi \text{ and } (\varphi \Leftrightarrow \psi) \text{ then } K_a \psi.$$

[57] See Kripke (1963).

[58] For details, see Fagin et al. (2003, pp. 316–20), Montague (1970) and Scott (1970).

Thus, omniscience is restricted to equivalence. This is the tightest restriction possible in any modal intensional semantics whose intensions are extensionally individuated, since equivalent formulas are semantically indistinguishable.

In general, the possible-world-semantics approach to epistemic closure perpetuates (various manifestations of) the paradox of omniscience. Knowers are construed as logical or mathematical geniuses: a wholly unrealistic scenario irrelevant to human or machine-aided reasoning.

We need to reconsider our logical foundations if we are to bar our agents' knowledge from proliferating too rapidly. One consideration concerns a differentiation between different kinds of knowledge. Another concerns the nature of the objects of knowledge. Concerning the former, we mentioned in Section 5.1.2 *implicit* and *explicit knowledge.* Levesque (1984) distinguishes between *implicit* and *explicit belief.*[59] Transposing *belief* to *knowledge*, the two kinds of knowledge may be characterized thus:

> *Implicit knowledge*, which is ascribed to an agent from the outside. Implicit knowledge is closed under entailment or under equivalence: *a implicitly* knows anything that follows from (or, in the restricted Montague-Scott version, is equivalent to) propositions *a* already knows. Rescher's terms for this is 'accessible knowledge' (2002, p. 478.)
>
> *Explicit knowledge,* which is knowledge that *a* is aware of and is able to use, e.g., when drawing inferences or as a map by means of which to steer through one's environment (as Ramsey put it). Rescher dubs this 'occurrent knowledge' (ibid.).

Intensional approaches such as Kripke's or Montague's are apt for modelling implicit knowledge. When it comes to applications, in particular when modelling computational aspects of machine-aided reasoning, explicit knowledge is what is wanted.

The other consideration is what sort of object knowers are related to. We reject formulae and the propositions of possible-world semantics and embrace hyperpropositions in the form of *propositional constructions*.

A bit of background to set the stage. Highly desirable features of a theory of actively reasoning agents are the theory's ability both to *quantify* over and/or *talk about* the objects of propositional attitudes and express self-referential statements. While this is beyond the expressive power of classical 1st-order and modal logics, quantification can be expressed in a higher-order intensional Montague-like typed logic. However, as we explained above, a variant of logical-mathematical omniscience is inevitable here. The only alternative seems to be a *syntactical theory* of explicit knowledge. On a syntactic approach propositions are considered syntactic

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[59] Fitting (2005) is not content with what he calls 'potential knowledge' (our inferable knowledge, roughly) and pushes for what he calls 'actual knowledge', which is explicit knowledge that a knower has for reasons known to him. Thus, where 't' is a term and 'X' a formula, the formula '$t{:}X$' means that X is known for reason t (Hence, if $t \neq t'$ then knowing X for reason t and knowing X for reason t' are going to be two different things). However, the relation between reasons for knowing and pieces of knowledge is black-boxed. Interestingly, a parallel notion of knowing something for a reason has been developed within constructivist type theory, in which '$a{:}A$' means that a is a proof of A and is, therefore, a (cogent) reason for A. See Martin-Löf (1984).

objects and identified with sentences in some 1st-order formal language of representation. This view seems to provide a fine-grained notion of attitude object as required for explicit-knowledge representation. However, syntactic approaches are prone to inconsistency when self-reference enters. The difference between syntactic theories and modal intensional theories may be characterised as being analogous to the difference between direct quotation ('John believes "Bill walks"') and indirect quotation ('John believes that Bill walks'), respectively. Hence, a syntax-based logic of propositional attitudes requires terms denoting sentences (such as 'walks(Bill)') and a way of 'disquoting' them. 'Disquotation' is achieved by the introduction of a truth predicate which says of any of its arguments that the sentence it denotes is true: True("walks(Bill)") $\Leftrightarrow$ walks(Bill). It is easy to show that a 1st-order theory involving the truth predicate is inconsistent. Naming in a language L is included by allowing each statement φ of L to have a name, $<\varphi>$. The above bivalence axiom for Truth—True($<\varphi>$) $\Leftrightarrow$ φ, for all φ—together with the Diagonalization lemma that guarantees that there is a ψ such that $\psi \Leftrightarrow \neg$True($<\psi>$) lead to inconsistency. Montague (1974a) proves some negative results regarding the consistency of 1st-order syntactic theories of knowledge, and Thomason (1980) shows how to extend these results to the weaker notion of belief. In order to avoid inconsistency, one needs to restrict either the syntax of the language or the logic involved. The former is realised by a hierarchy of (meta-) languages each of which establishes its 'grounded truths'; the latter by theories of truth and syntactic modalities in which a 'stable truth' is defined.[60]

Despite their technical viability, syntactic approaches face a major philosophical objection. In brief, the attitude objects ought not to be the sentences themselves, but rather their semantic content. When an agent knows something, it is not the symbols of a formal language that he or she is related to. Rather the agent is related to its *meaning*, which is the abstract object that the sentence expresses. Our foremost reason for rejecting sententialism is that it ties knowledge to a particular notation. This leads to the notorious, and unacceptable, problem of translatability across notational systems.[61]

We do not restrict by decree, i.e., arbitrarily and artificially, the set of propositional constructions the agent is said to know. Instead we *compute* the *inferable* knowledge relative to the inference *rule*(*s*) that have been assigned to the agent and which he/she/it is *able* to use.[62] By assigning rules of inference to an agent, we are assigning an *intelligence* with a very specific calibration. The resulting logic enables an exterior

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[60] See Turner (1990) and Fasli (2003) for results.

[61] See Schiffer (1987, Chapter 5). On a polemic note, we would say that sententialism is hyperintensionality on the cheap.

[62] The *justification* component of the tripartite definition of knowledge as *justified true belief* has proved to be the most troublesome by far. But in the limiting case of *deductive inferable* knowledge, the internal agent a has available a robust answer to the question of how he knows that ψ is true: 'I know that ψ is true, because I know that ψ is the conclusion of a sound argument: I know that my premises are true, and that my rule of inference is valid, so I know that my conclusion has got to be true, and ψ is that conclusion.' Whether a actually avails himself of this justification when challenged is extraneous to our non-constructivist logic of knowledge representation.

agent b to draw valid inferences about the inferable knowledge of an interior agent a, who receives mention in the formula $K_a \psi$. This interplay between interior and exterior agents can be extended. Thus, by mentioning agent b in the syntax, we can set up a logic that enables the exterior agent c to draw valid inferences about the knowledge that the interior agent b validly infers about the inferable knowledge of the likewise interior agent a; and so on.[63]

To specify a logical system describing the behaviour of autonomous, resource-bounded agents, we propose distinguishing between *three* ways in which an agent may possess knowledge.

$K^{exp}(a)/(o{*_n})_{\tau\omega}$ — *explicit knowledge*: a's knowledge is a set of propositional constructions that a actively possesses (for instance, as built into a's memory or knowledge base).

$K^{imp}(a)/(o{*_n})_{\tau\omega}$ — *implicit knowledge*: a's knowledge is a set of propositional constructions analytically compatible with a's base of explicit knowledge.

$K^{inf}(a)/(o{*_n})_{\tau\omega}$ — *inferable knowledge*: in every state $\langle w, t\rangle$ a's inferable knowledge is the set of constructions that a is able to infer from a's base of explicit knowledge.[64]

An exterior agent c attempting to draw valid inferences about the interior agents a's and b's information bases needs a closure principle to validate his inferences. But he must take into account a's and b's inferential capabilities. To specify a principle guiding c's inferential policy, we introduce the functions $Inf(R)/((o{*_n})(o{*_n}))$ associating an input set Γ of constructions with the set of constructions derivable from Γ using a set of rules R.

Henceforth, let $c, c' \rightarrow {*_n}$ v-construct a specific inferable piece of knowledge, let $d \rightarrow (o{*_n})$ v-construct a stock of knowledge, and let $R/(o({*_n}\ (o{*_n})))$ be a set of rules of inference, $r \rightarrow ({*_n}\ (o{*_n}))$ v-constructing a particular element of R. Then the following constructional schema specifies the function $Inf(R)$:

$$\lambda d\, \lambda c\, [[d\ c] \vee \exists r\, [[{}^0R\ r] \wedge (d \vdash_r c)]]$$

'$(d \vdash_r c)$' denoting derivation by means of r, i.e., the Composition $[[r\ d] = c]$. The schema can be glossed as, 'From any set d of constructions (λd) a construction c is inferable (λc), if c belongs to d ($[d\ c]$), or c is derivable from d using rule r.'

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[63] We are using the meanings of 'a', 'b', 'c' as constructions of individual agents.

[64] Inferable knowledge is a case where, as Hintikka puts it, 'the source of information is…the Inquirer's own brain' (Hintikka, J., and M.B. Hintikka 1989, pp. 29–30). If inferable knowledge is realized as inferred knowledge, the latter sort of knowledge (knowing) becomes what Hintikka calls 'completely active knowledge'; i.e., knowledge 'put forward by the Inquirer as outcomes of…deductive moves…' (ibid., p. 31) (In essay 2 of his 1989, however, Hintikka is not considering inferable knowledge per se, but erotetic logic as an instance of epistemic logic).

Example. Let R contain the rule of disjunctive syllogism, the substitution rule, the β-reduction rule, and the rule ${}^{20}\mathrm{C} \mid\!- \mathrm{C}$. Then $Inf(R)$ is defined as follows:

$$ {}^{0}Inf(R) =_{df} \lambda d\, \lambda c\, \big[[d\, c] \vee \exists c' \big[[d\, c'] \wedge [d\ {}^{c,c'}[\lambda w \lambda t\, [\neg({}^{2}c')_{wt} \vee ({}^{2}c)_{wt}]]]\big]\big]. $$

Gloss: Given a stock of knowledge d a piece of knowledge c is inferable iff either c belongs to d ($[d\ c]$) or there is a piece of knowledge c' such that c' belongs to d ($[d\ c']$) and non-c' or c belongs to d as well ($[d\ {}^{c,c'}\ [\lambda w \lambda t\, [\neg({}^{2}c')_{wt} \vee ({}^{2}c)_{wt}]]]$).

Technical complications abound, however. For one thing, the stock of knowledge constructed by d is usually a set of empirical concepts, viz. closed constructions of propositions. But secondly, since we are talking about the very objects of a's epistemic attitudes, we need to *mention* the constructions by Trivializing them (which corresponds to *calling* a subprocedure with formal parameters c, c'). To release the variables c, c' bound by Trivialization, we must use the *Sub* function to bring about the substitution of the actual values for the formal parameters. The Double-Executing variables ranging over propositional constructions ${}^{2}c$, ${}^{2}c'$ return the respective propositions, which are then subjected to intensional descent in order to obtain a truth-value. And thirdly, β-reductions and the rule transforming ${}^{20}\mathrm{C}$ into C (${}^{20}\mathrm{C} \mid\!- \mathrm{C}$) are to be performed. The upper indices c, c' are a notational abbreviation of these devices.

Thus,

$$ {}^{c,c\prime}\, [\lambda w \lambda t\, [\neg({}^{2}c')_{wt} \vee ({}^{2}c)_{wt}]] $$

is to be unpacked as

$$ [{}^{0}Sub\ [{}^{0}Tr\ c]\ {}^{0}c\ [{}^{0}Sub\ [{}^{0}Tr\ c']\ {}^{0}c'\ {}^{0}[\lambda w \lambda t\, [\neg({}^{2}c')_{wt} \vee ({}^{2}c)_{wt}]]]]. $$

In this case Tr is of type $(*_{n+1}\ *_n)$, and its argument is an $*_n$–construction. That is, Tr applied to a construction C of type $*_n$ returns ${}^{0}C$ of type $*_{n+1}$.

With a equipped with a finite set of propositional constructions $K^{exp}(a)_{wt}$ (i.e., a's current knowledge) and some intelligence (i.e., the set R of rules of inference that a masters), an exterior agent is now in a position to compute the derivable pieces of a's knowledge.

To adduce a very simple example, let a's knowledge base contain (i) that Charles is bald and (ii) that Charles is not bald or Charles is a king. Then, if a masters the rules R, a is able to deduce that Charles is a king ($Bald$, $King/(o\iota)_{\tau\omega}$; $Charles/\iota$):

$$ d \rightarrow_v \{\ldots\ [\lambda w \lambda t\, [{}^{0}Bald_{wt}\, {}^{0}Charles]], \ldots, $$
$$ [\lambda w \lambda t\, [\neg[{}^{0}Bald_{wt}\ {}^{0}Charles] \vee [{}^{0}King_{wt}\ {}^{0}Charles]]], \ldots\} $$

$$ c' \rightarrow_v [\lambda w \lambda t\, [{}^{0}Bald_{wt}\ {}^{0}Charles]] $$

$$ c \rightarrow_v [\lambda w \lambda t\, [{}^{0}King_{wt}\ {}^{0}Charles]] $$

$$ [{}^{0}Sub\ [{}^{0}Tr\ c]\ {}^{0}c\ [{}^{0}Sub\ [{}^{0}Tr\ c']\ {}^{0}c'\ {}^{0}[\lambda w \lambda t\, [\neg({}^{2}c')_{wt} \vee ({}^{2}c)_{wt}]]]] \rightarrow_v $$

$$[\lambda w\lambda t\ [\neg{}^{20}[\lambda w\lambda t\ [{}^{0}Bald_{wt}\ {}^{0}Charles]]_{wt} \vee {}^{20}[\lambda w\lambda t\ [{}^{0}King_{wt}\ {}^{0}Charles]]_{wt}]] = ({}^{20}\mathrm{C} \vdash \mathrm{C})$$

$$[\lambda w\lambda t\ [\neg[\lambda w\lambda t\ [{}^{0}Bald_{wt}\ {}^{0}Charles]]_{wt} \vee [\lambda w\lambda t\ [{}^{0}King_{wt}\ {}^{0}Charles]]_{wt}]] =$$

$$[\lambda w\lambda t\ [\neg[{}^{0}Bald_{wt}\,{}^{0}Charles] \vee [{}^{0}King_{wt}\,{}^{0}Charles]]]\ (\textit{via}\ \beta\text{-reduction}).$$

This last construction is contained in *a'*s knowledge base, which means that actually the construction *v*-constructed by *c* (i.e., $[\lambda w\lambda t\ [{}^{0}King_{wt}\,{}^{0}Charles]]$) is derivable by means of the rules *R*.

Some, however, may still be left with the feeling that our construction-based approach offers too much of a good thing by being *too* fine-grained. Consider the case where *a*, *b* start out with equivalent knowledge bases (i.e., two sets of not procedurally isomorphic constructions constructing the same set of possible-world propositions) and master the same set of rules *R*. Then *a*, *b* are going to arrive at two equivalent pools of inferable knowledge that are, nonetheless, *different* from one another. Or consider the case where *a*, *b* start out with an *identical* knowledge base (i.e., the elements of *a*'s set of propositional constructions are either identical or procedurally isomorphic to the elements of *b*'s set). Then if *a*, *b* use different, albeit equivalent, rules of inferences they will arrive at two different, albeit equivalent, pools of inferable knowledge. For instance, *a* may use natural-deduction rules, and *b* general resolution and unification principles. Are we not operating with a distinction between pieces of knowledge where there should be none? One could certainly make the case that pieces of knowledge ought in many cases to be cut more coarsely. Only not so with inferable knowledge. Otherwise we will be letting back in stocks of knowledge that swell too much too fast and the introduction of the notion of inferable knowledge would lose much of its relevance.[65] Besides, since inferable knowledge is constructional, and constructions are procedures, the very *procedure* that a particular agent deploys when extracting a new piece of knowledge from an old piece of knowledge is relevant. To demonstrate the relevancy, consider software agents *a* and *b* starting with an identical knowledge base of empirical concepts and equipped with equivalent inference rules. Yet, while *a* has a built-in library of sophisticated mathematical procedures, *b* can use only basic arithmetic. Then obviously the inferable knowledge of *a* is much greater than the inferable knowledge of *b*.

[65] From the point of view of an external agent *c* who may be in the position of a dispatcher controlling both *a* and *b*, it would be useful to create (i) classes of equivalent sets of rules of inference, (ii) classes of agents according to equivalent inferential capabilities, and (iii) to partially order such agent classes according to their inferential capacities and primordial supply of knowledge. This would make it possible to apply a method of formal conceptual lattices to facilitate *dynamic* aspects of the system. This is a topic for further research.

Above we introduced the notion of a rule as a function of type $(*_n\ (o*_n))$. We assumed that the rules R assigned to a are valid rules of inference and that the function $Inf(R)$ meets the following conditions for any agent a.[66]

- $Inf(R)$ is *sub-classical*: if φ is derived from a stock of knowledge Γ, then φ is entailed by Γ; i.e., if C_n is the function associating Γ with the set of its logical consequences then $[Inf(R)\,\Gamma] \subseteq [C_n\,\Gamma]$.
- $Inf(R)$ is *reflexive*: $\Gamma \subseteq [Inf(R)\,\Gamma]$ (‘a does not forget what a already knows’).

These two assumptions express the basic conditions that inference about knowledge should satisfy. $Inf(R)$ has to be logically sound, because knowledge has to be true (factive); and since we suppose that agents are resource-bounded, the inferable stock of knowledge needs to be a subset of the set of logical entailments.

A noteworthy consequence of the assumptions of sub-classicality and reflexivity is monotonicity:

- If $Inf(R)$ is sub-classical and reflexive, then it is *monotonic*: if $\Gamma \subseteq \Gamma'$ then $[Inf(R)\,\Gamma] \subseteq [Inf(R)\,\Gamma']$.

The function $Inf(R)$ ‘computes’ only one step of inference, so we do *not* assume it to be idempotent:

- $Inf(R)$ is not *idempotent:* $[Inf(R)\ [Inf(R)\ \Gamma']]$ is not a subset of $[Inf(R)\ \Gamma']$ for some Γ'.

At this point we are able to recursively define *the inferable knowledge of a* mastering R at $\langle w, t\rangle$ by using the fixed-point technique.

The knowledge of a at $\langle w, t\rangle$, whether implicit, explicit or inferable, is a set of propositional constructions. The drawing of valid inferences about a’s inferable knowledge is, for any $\langle w, t\rangle$, executed step-wise. At step 0 we take a’s explicit knowledge as the base of the induction $K_0(a)_{wt} = K^{exp}(a)_{wt}$. Step 1 consists in applying the function $Inf(R)$ to this knowledge, thus obtaining a new set of derived constructions $K_1(a)_{wt} = [Inf(R)\ K^{exp}(a)_{wt}]$. The new set is a superset of the initial knowledge. But it is not necessarily equal to a’s inferable knowledge yet: there may be more inferences to be drawn. Step 2 consists in applying $Inf(R)$ to the result of Step 1 to obtain a new set: $K_2(a)_{wt} = [Inf(R)\ K_1(a)_{wt}]$. By iteration, an increasing sequence of sets of constructions $K_1(a)_{wt} \subseteq K_2(a)_{wt} \subseteq K_3(a)_{wt} \ldots$ is obtained, such that each set $K_{n+1}(a)_{wt}$ depends only on the preceding set $K_n(a)_{wt}$. But at which step will the iteration stop? There are two possibilities. Either there is a step m such that no more constructions can be inferred: $K_{m+1}(a)_{wt} = K_m(a)_{wt}$, and $K_m(a)_{wt}$ is the *supremum* of the sequence $K_i(a)_{wt}$. Or else there is no such finite m,

[66] For the sake of simplicity, we now use the standard infix notation for set-theoretical inclusion $\subseteq$, union $\cup$, and omit Trivialization when no confusion can arise. In particular, instead of ‘${}^0Inf(R)$’ we write simply ‘$Inf(R)$’.

the sequence increasing *ad infinitum* for want of a maximum element. Still, even in the latter case there exists a least upper bound of the sequence:

$$K_\infty(a)_{wt} = \bigcup_{k=1}^{\infty} K_k(a)_{wt}$$

This potentially infinite set is well-defined: it is the result of a potentially infinite number of finite computational steps, and $K_\infty(a)_{wt} = [Inf(R)\ K_\infty(a)_{wt}]$ holds.

If the initial set of explicit knowledge is a finite set of constructions, and if the function $Inf(R)$ is algorithmically computable (i.e., a partial recursive function), then $K_\infty(a)_{wt}$ is recursively enumerable.

In any case, the $Inf(R)$ function is not decreasing (monotonic) and has a *supremum*. According to Tarski's fixed-point theorem, there is a least fixed point of $Inf(R)$ containing $K^{exp}(a)_{wt}$, and since no more inferences can be drawn, this fixed-point set is the entire inferable knowledge of a at some particular $\langle w, t\rangle$.

Definition 5.3 (*inferable knowledge*)

- $K_0(a)_{wt} = K^{exp}(a)_{wt}$
- $K_{n+1}(a)_{wt} = [Inf(R)\ K_n(a)_{wt}]$
- Nothing other is a set of inferable knowledge, unless it so follows from the two preceding clauses. □

The entire set of constructions validly inferable by a—a's *inferable* knowledge—is a fixed point of $Inf(R)$:

$$K^{inf}(a)_{wt} = [Inf(R)\ K^{inf}(a)_{wt}]$$

and it is the least fixed point of $Inf(R)$ containing a's *explicit* knowledge:

$$K^{inf}(a)_{wt} = \mu\ \lambda x\ [Inf(R)\ [x \cup K^{exp}(a)_{wt}]].$$

Now, necessarily (i.e., at every $\langle w, t\rangle$ for every a) it holds that

$$K^{exp}(a)_{wt} \subseteq K^{inf}(a)_{wt} \subseteq K^{imp}(a)_{wt}.$$

The introduction of the concept of inferable knowledge allows us to tread a subtle path between two unrealistic extremes; either the explicit knowledge of an 'idiot', deprived of any inferential capabilities, or the implicit knowledge of a logical/mathematical genius. The logic set out so far has been proposed as the logic that accommodates the philosophical desiderata to be met by the notion of the inferable knowledge had by an autonomous, intelligent agent who masters *some* but not all the valid rules of inference. Given an agent a furnished with a stock of recursively enumerable explicit knowledge and a flawless command of only some rules of inference R, there is an upper limit to the knowledge it would be logically possible for the agent to derive from his existing knowledge. This limit is the closure of *a*'s explicit knowledge. Thus, the manner in which we make good on our promise to provide a conception of EC on which it is valid consists in restricting EC to inferable knowledge.[67,68]

5.1.6 Factivity and epistemic shift

As mentioned above, knowing differs from believing due to the rule of factivity: whatever is known is true, while some of what is believed may not be true. A certain technical problem arises for any system of hyperintensional logic within which hyperpropositions are capable of figuring as objects of knowledge but not also as truth-bearers.[69] Call this the problem of *epistemic shift*.[70] In TIL the problem, in the empirical case, is how to descend from a known* propositional construction to the proposition it constructs, and in the mathematical case, how to descend from a known* construction to **T**.

In the case of propositional knowledge, the rule of factivity is straightforward, since $p/*_1 \to o_{\tau\omega}$ is both known and true:

$$\frac{[{}^0K_{wt}\, a\, p]}{p_{wt}}$$

The implication corresponding to this rule of inference is

$$\forall w \forall t\, \forall p\, [[{}^0K_{wt}\, a\, p] \supset [p_{wt}]].$$

In the case of hyperpropositional knowledge, we are forced to qualify the principle that what is known is true in such a way that what is known* constructs either

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[67] However, our theory still comes with the unrealistic assumptions that *a* has boundless storage capacity and unlimited time available to him. Future research will be devoted to relaxing this assumption by differentiating between various degrees of *computational complexity* with a view to developing a theory of *computationally tractable* inferable knowledge.

[68] Without the assumption that every rule the agent uses is *valid*, we would be able to consider *non-monotonic* reasoning as well. However, as long as we are modelling *knowledge*, which we regard as factive and incapable of giving rise to inconsistent information bases, all the reasoning must be monotonic. On the other hand, if we wish to model *belief*, we are forced to allow the agent to use invalid rules of non-monotonic reasoning that are liable to occasion inconsistency.

[69] Put differently, our stance marks a deviation from the *unity thesis* that one sort of entity will fulfill all the roles pre-theoretically assigned to propositions, such as functioning as truth-bearer, as object of knowledge, etc. See Carrara and Sacchi (2006) for an elaborate discussion of the functional roles of propositions.

[70] See Moffett (2003) for a discussion of doxastic shift.

a true truth-bearer or else **T**. For K^* we provide two kinds of rules of factivity; one for propositional constructions and another for mathematical constructions.

Let $C/*_n \rightarrow o_{\tau\omega}$ be a propositional construction, such that C is known* by a. Then we wish to express that the proposition that C constructs is true:

$$\forall w \forall t\, [[{}^0K^*_{wt}\ a\ {}^0C] \supset [C_{wt}]].$$

The respective rule of inference is

$$\frac{[{}^0K^*_{wt}\ a\ {}^0C]}{[C_{wt}]}$$

Let $c/*_2 \rightarrow *_1$, ${}^2c \rightarrow o_{\tau\omega}$. Then we wish to express that, necessarily and for all c, if c is known*, then the proposition that is v-constructed by what is v-constructed by c is true:

$$\forall w \forall t\, \forall c\, [[{}^0K^*_{wt}\ a\ c] \supset [[{}^2c]_{wt}]].$$

Remark. Variable c v-constructs a propositional construction; ${}^2c$ v-constructs a proposition; $[{}^2c]_{wt}$ v-constructs a truth-value.

The respective rule of inference

$$\frac{[{}^0K^*_{wt}\ a\ c]}{[[{}^2c]_{wt}]}$$

is thus valid.

If you know that 5 + 7 makes 12 then the mathematical proposition that 5 + 7 makes 12 is true, but the problem is no longer how to descend from a known* propositional construction to the proposition (of type $o_{\tau\omega}$) that it constructs. Instead the problem is how to descend from a known* construction to the truth-value that it constructs. Let the known* piece of knowledge be the Composition

$$[[{}^0{+}\ {}^05\ {}^07] = {}^012].$$

This construction is incapable of *being* true; but it *constructs* **T**. Thus it holds that

$$\forall d\, [[{}^0K^*_{wt}\ a\ d] \supset {}^2d]$$

and the relevant rule of factivity

$$\frac{[{}^0K^*_{wt}\ a\ d]}{{}^2d}$$

is valid. Types: $d \rightarrow *_n$; ${}^2d \rightarrow_v o$.

The philosophical rationale for the rules is that if C, or what c, d range over, respectively, is known* at some $\langle w, t\rangle$ then the proposition that C, or ${}^2c$, respectively, constructs must be true at any such pair of parameters, and the given value v-constructed by ${}^2d$ must be the truth-value **T**. Otherwise the link of factivity extending from knowledge* to truth would be severed. And factivity is one feature of knowledge/knowledge* that we are not prepared to give up, for it ensures that only truths are knowable/knowable*. Factivity is the *externalization* constraint on knowledge/knowledge* that anchors any piece of knowledge/knowledge* that any epistemic agent has internalized to a portion of reality outside the agent's knowledge/knowledge* base.[71]

5.2 Notional attitudes

The survey of 'propositional' attitudes showed that attitudes divide into relations (-in-intension) to propositions/$o_{\tau\omega}$ and to constructions/$*_n$. The same bifurcation applies to so-called *notional* attitudes, which also divide into relations-in-intension to intensions and to constructions. Examples of notional attitudes would be *seeking the fountain of youth*, *calculating the* n*th decimal in the decimal expansion of* π, *finding one's keys*, *worshipping God* and *regretting ever being a Platonist*.

A solid philosophical reason for studying them is that notional-attitude attributions deal with central parts of our psychological life, like hoping, remembering, and wishing. Besides, notional-attitude verbs are part and parcel of both everyday language and more technical and scientific vocabularies (like 'to design', 'to prove', 'to experimentally test'), so a semantic analysis of them should not be missing from any theory designed to analyse a wide array of expressions. Surprisingly, though, notional attitudes have received much less attention than 'that'-clause attitudes in analytic philosophy of language. Since notional attitudes are no less important than their 'propositional' cousins, making up half the sphere of attitudes, TIL contributes to rectifying the situation by treating notional attitudes at length.

When analysing notional attitudes, a key question to ask is,

Which kind of object is the agent related to?

[71] One may then look upon the *belief constraint*—'if you *know* that *P*/*know** that *C* then you *believe* that *P*/*believe** that *C*'—as one of *internalization*.

Whatever the answer, substitution tests should always justify our answer. But in the case of notional attitudes there is a preliminary, more fundamental question; namely,

Which attitudes are notional?

At first blush, the answer would seem to be simple: notional attitudes are attitudes to some *notion*, i.e., a *concept*, a construction, though not to a propositional construction. In this section we show that the answer cannot be that simple. The main result is that the default interpretation of empirical notional attitudes is as *intensional* attitudes to an α-intension of any type, usually an ι-property of type $(o\iota)_{\tau\omega}$, a proposition of type $o_{\tau\omega}$, or an ι-office of type $\iota_{\tau\omega}$. In the mathematical case, however, notional attitudes are invariably of one kind, namely constructional.

Similarly, attitudes that must meet with failure must be constructional. This is the notional counterpart of inconsistent beliefs, discussed in Section 5.1.3. An example, which we shall analyze in more detail below, has *a* wishing to become the Pope and also wishing not to become head of state of the Vatican: *a* has strong religious, but no political, aspirations. Assume now that the office of Pope and the office of head of state of the Vatican are one and the same office, and thus defined in terms of one and the same requisites. Then *a*'s wish is an inconsistent one, for *a* simultaneously wants to occupy and also not to occupy a particular office, and so for strictly logical reasons his wish cannot be true. If *a*'s wish were reported in intensional terms, no good sense could be made of his wish, just as little as a propositional, as opposed to hyperpropositional, account of an inconsistent belief could make good sense of such a belief. The report or account must be hyperintensional. Thus, what the account ought to point out is that *a* is intentionally related to one construction of the office and not so related to another construction of the same office.

(a) *Attitudes to mathematical notions*

Consider the sentence

(1) 'Charles calculates 2 + 5.'

To which object is Charles related? In Section 1.2 we showed that it cannot be the denotation of '2 + 5', for Charles does not calculate 7. It cannot be the respective expression '2 + 5' either, because Charles can calculate 2 + 5 without knowing any notation, instead playing with an abacus or doing mental arithmetic.[72]

[72] Of course, as the procedure of calculating gets more and more complicated, *executing* such a procedure unaided by proper notation is hardly imaginable. The importance of symbolic notation, images, etc., in mathematics is stressed in Brown (1999, pp. 92–93), where he talks about the 'computational role' and 'computational power' of (good) mathematical notation. Still, the sentence does not say anything about *how* the act of calculating the sum is being *executed*. Reference to the notational system that Charles is employing on some occasion of making a calculation

He is related to a *construction*, trying as he is to perform a particular procedure in order to find out which number is its product. Hence *Calc*(ulating) is an object of type $(o\iota*_1)_{\tau\omega}$, and the analysis of (1) is:

(1′) $\lambda w \lambda t\ [{}^0Calc_{wt}\ {}^0Charles\ {}^0[{}^0{+}\ {}^02\ {}^05]]$.

The sentence

(2) ‘Charles is seeking the greatest prime’

is analysed along similar lines. Since there is no greatest prime, Charles cannot be related to the denotation of ‘the greatest prime’, simply because there is none. He is related to the sense of ‘the greatest prime’, for he is trying to find out which particular number is identified by the concept of the greatest prime. This attempt is bound to fail because there is no such number to be found. Only this fact does not detract from his activity of seeking such a number. Besides, relating Charles to a number would misconstrue the activity of seeking, turning a seeker into a finder right away. Hence *Seek*(ing) is here again an object of type $(o\iota*_1)_{\tau\omega}$.

Let ν be the type of natural numbers; $Prime/(o\nu)$; $Greatest/(\nu(o\nu))$: the function associating a set of natural numbers with its greatest element; $x, y \to \nu$. Then the literal analysis of (2) is

(2′) $\lambda w \lambda t\ [{}^0Seek_{wt}\ {}^0Charles\ {}^0[{}^0Greatest\ {}^0Prime]]$.

Of course, this analysis can be refined[73] by replacing the Composition $[{}^0Greatest\ {}^0Prime]$ by a more fine-grained equivalent construction, which would correspond to analysing the sentence

(2r) ‘Charles is seeking the only number such that it is a prime and is greater than any other prime’.

The resulting finer analysis would be

(2r′) $\lambda w \lambda t\ [{}^0Seek_{wt}\ {}^0Charles\ {}^0[\iota x\ [[{}^0Prime\ x] \wedge \forall y\ [[{}^0Prime\ y] \supset [x \geq y]]]]]$.

Another refinement would be to use a definition[74] of *prime number*:

Prime number $=_{\text{df}}$ the class of natural numbers that have exactly two factors.

Thus the respective ontological definition of the class *Prime* becomes as follows:

$$\lambda x\ [[{}^0Nat\ x] \wedge [{}^0Card\ \lambda y\ [[{}^0Nat\ y] \wedge [{}^0Div\ x\ y]] = {}^02]].$$

Types: Nat(ural)/$(o\tau)$; Div(isible by)/$(o\tau\tau)$; $Card/\tau(o\tau)$; $x, y \to \tau$.

can be made, but then the *analysandum* becomes another, along the lines of ‘Charles is calculating the square root of 525 *by means of X*’, where *X* is Charles’ notational system.

[73] For the definition of *refinement*, see Section 5.4, Definition 5.5.

[74] See Section 2.2.2.

Replacing the Trivialization ${}^{0}Prime$ by this Closure, we can obtain a still finer analysis. No matter the degree of refinement, however, Charles' attitude is to a *construction* of a number. This is in line with the main tenet of the philosophy of mathematics of TIL, that the subject-matter of mathematics is constructions.[75]

In general, *notional attitudes* to *mathematical* objects are invariably of type $(o\iota*_n)_{\tau\omega}$, n mostly equal to 1.

(b) *Attitudes to empirical notions*

In this case the situation is rather more complicated, and we will show that, though empirical attitudes come in both intensional and hyperintensional variants, their default interpretation is as intensional.

First, simple relations-in-intension of type $(o\iota\iota)_{\tau\omega}$ of one individual to another, like *kicking*, *loving*, *touching*, *talking to*, do not qualify as attitudes. For instance, in

(3) 'Charles is talking to the Mayor of Dunedin'

'talking to' ($Talk\_to/(o\iota\iota)_{\tau\omega}$) denotes a relation-in-intension between two individuals, though one and not two particular individuals are mentioned in (3). The construction of the office of Mayor of Dunedin (Composed of the constructions of $Mayor\_of/(\iota\iota)_{\tau\omega}$ and $Dunedin/\iota$) occurs with *de re* supposition. The office serves only as a pointer to an unspecified individual. The only alternative would have been the absurdity of Charles talking to the *office* and not its incumbent. The correct analysis *de re* is

(3′) $\lambda w\lambda t\ [{}^{0}Talk\_to_{wt}\ {}^{0}Charles\ [\lambda w\lambda t\ [{}^{0}Mayor\_of_{wt}\ {}^{0}Dunedin]]_{\boldsymbol{wt}}]$.

The two *de re* principles—existential presupposition and substitution of co-referential expressions[76]—are valid. The existential presupposition is that the Mayor of Dunedin exist in order that the proposition have a truth-value. Thus the arguments (E_1) and (E_2) are valid:[77]

(E_1) Charles is talking to the Mayor of Dunedin,
$\lambda w\lambda t\ [{}^{0}Talk\_to_{wt}\ {}^{0}Charles\ [\lambda w\lambda t\ [{}^{0}Mayor\_of_{wt}\ {}^{0}Dunedin]]_{wt}]$

The Mayor of Dunedin exists,
$\lambda w\lambda t\ [{}^{0}Exist_{wt}\ \lambda w\lambda t\ [{}^{0}Mayor\_of_{wt}\ {}^{0}Dunedin]]$

[75] See Tichý (1995). Interestingly, constructivists/intuitionists would agree verbally. But there is a major substantial difference. For TIL mathematical constructions are modes of presentation of pre-existing mathematical entities. For constructivists/intuitionists constructions are proofs.

[76] See Section 1.5.2.1.

[77] The argument (E_2) is valid provided the Mayor of Dunedin is contained in the topic of the assumption. For details on the topic-focus distinction, see Duží (2009).

(E$_2$) Charles does not talk to the Mayor of Dunedin,
$\lambda w \lambda t\, \neg[{}^0 Talk\_to_{wt}\, {}^0 Charles\, [\lambda w \lambda t\, [{}^0 Mayor\_of_{wt}\, {}^0 Dunedin]]_{wt}]$

The Mayor of Dunedin exists,
$\lambda w \lambda t\, [{}^0 Exist_{wt}\, \lambda w \lambda t\, [{}^0 Mayor\_of_{wt}\, {}^0 Dunedin]]$

The *de re* principle of substitution is that if 'the Mayor of Dunedin' and 'Mr *X*' are co-referential, then Charles is talking to Mr *X*:

(S) $\lambda w \lambda t\, [{}^0 Talk\_to_{wt}\, {}^0 Charles\, [\lambda w \lambda t\, [{}^0 Mayor\_of_{wt}\, {}^0 Dunedin]]_{wt}]$
$\lambda w \lambda t\, [[\lambda w \lambda t\, [{}^0 Mayor\_of_{wt}\, {}^0 Dunedin]]_{wt} = {}^0 X]$

$\lambda w \lambda t\, [{}^0 Talk\_to_{wt}\, {}^0 Charles\, {}^0 X]$

Charles may be talking to Mr *X* without knowing that this individual is the Mayor of Dunedin, but even then the attributer may truthfully use (3) to report the situation. Hence, in describing such a situation, the *notion* of the respective office is dispensable and (3) should not be considered an example of a *notional attitude*, for Charles is involved in the *relation* of talking to Mr *X* the individual.

If, however, Charles wants to become Mayor of Dunedin, the situation is different. Obviously, Charles does not want to, nor could he, become another individual, so Charles does not want (absurdly) to become the individual who is already occupying the office of Mayor. Instead, what is going on is that Charles would like to hold the office of Mayor. Charles is thus related to the office, and the sentence reports a *notional attitude.*

In what follows we are going to analyse three types of this attitude, namely the attitudes of wishing/wanting to, seeking and finding.

5.2.1 Wishing and wanting to

Since the sentence

(4) 'Charles wants to become Mayor of Dunedin.'

has no reasonable *de re* reading, we proceed immediately to its analysis *de dicto*. First, a coarse-grained analysis ($WB/(o\iota\iota_{\tau\omega})_{\tau\omega}$: the relation-in-intension of an individual to the ι-office $MD/\iota_{\tau\omega}$ of wanting to occupy it):

(4′) $\lambda w \lambda t\, [{}^0 WB_{wt}\, {}^0 Charles\, {}^0 MD]$.

In order to refine (4′) to obtain a finer literal analysis of (4), we should take into account the meaning of the expressions 'to want to', 'to become', 'Mayor of' and 'Dunedin'. In the previous section the office *MD* was constructed by

$$\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin].$$

Concerning the former, ‘to become’ denotes a relation-in-intension between an individual and an office: $Become/(o\iota\iota_{\tau\omega})_{\tau\omega}$. There are, however, two analyses of ‘to want to’, one reducible to the other. Either we conceive of wanting to and wishing (something) as relations-in-intension between an individual and a *property* which the individual would like to possess, or as relations-in-intension between an individual and a *proposition* which the individual would like to be true. Thus we have W^1(ant_to)/$(o\iota(o\iota)_{\tau\omega})_{\tau\omega}$ and W^2(ant_to)/$(o\iota o_{\tau\omega})_{\tau\omega}$ defined as follows. Let $a/*_n \rightarrow \iota$ be a construction of an individual wanting to be a $P/*_n \rightarrow (o\iota)_{\tau\omega}$. Then $(y \rightarrow_v \iota)$

(i) $\lambda w \lambda t\ [{}^0W^1{}_{wt}\ a\ [\lambda w \lambda t\ \lambda y\ [P_{wt}\,y]]]$

(ii) $\lambda w \lambda t\ [{}^0W^2{}_{wt}\ a\ [\lambda w \lambda t\ [P_{wt}\,a]]]$.

The equivalence of W^1 and W^2 is anchored to the fact that, necessarily, whenever a wants to obtain the property P he wants that the proposition that he is a P be true, and *vice versa*:[78]

$$\forall w \forall t\ [[{}^0W^1{}_{wt}\ a\ [\lambda w \lambda t\ \lambda y\ [{}^0P_{wt}\,y]]] = [{}^0W^2{}_{wt}\ a\ [\lambda w \lambda t\ [{}^0P_{wt}\,a]]]].$$

In our case the property P is to become Mayor of Dunedin $(x \rightarrow_v \iota)$, constructed by

$$\lambda w \lambda t\ \lambda x\ [{}^0Become_{wt}\,x\ \lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]],$$

and we have two equivalent analyses of (4):

(4″) $\lambda w \lambda t\ [{}^0W^1{}_{wt}\ {}^0Charles$
$\lambda w \lambda t\ [\lambda x\ [{}^0Become_{wt}\,x\ \lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]]]$

(4‴) $\lambda w \lambda t\ [{}^0W^2{}_{wt}\ {}^0Charles$
$\lambda w \lambda t\ [{}^0Become_{wt}\,{}^0Charles\ \lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]]$.

Gloss: ‘Charles wants that he should become Mayor of Dunedin.’

This ambivalence does not matter here, because the analyses are equivalent, being reducible to one another. In this case we do not have any criterion to determine which analysis to prefer, the sentence (4) obviously being ambivalent in this innocuous way.

Consider, however, a variant of (4):

(5) ‘Charles wants that Peter should become Mayor of Dunedin.’

Using a variant of the schema (ii), namely

[78] Properly speaking, a wants *himself* (cf. Castañeda’s *he**) to become a P; cf. Section 3.4.1. Thus we ought to analyse the anaphoric reference to a in terms of ‘*he*’ and use the substitution method, which yields $\lambda w \lambda t\ [{}^0W^2{}_{wt}\ a\ {}^2[{}^0Sub\ [{}^0Tr\ a]\ {}^0he\ {}^0[\lambda w \lambda t\ [{}^0P_{wt}\,he]]]$. Yet, since this distinction is irrelevant here, we decided to disregard *he**.

(ii′) $\lambda w \lambda t\ [{}^0W^2_{wt}\ a\ [\lambda w \lambda t\ [P_{wt}\ b]]]$,

the analysis is easy:

(5′) $\lambda w \lambda t\ [{}^0W^2_{wt}\ {}^0Charles$
$\lambda w \lambda t\ [{}^0Become_{wt}\ {}^0Peter\ \lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]]$.

The schema (i) seems not to be applicable in this case.[79]

Though the ambivalence of the sentence (4) is innocuous, the ambivalence of sentences expressing notional attitudes is a general problem, which is often not reducible to the possibility of choosing between two different, but equivalent, analyses.

It turns out that disambiguation of (4) and (5) is possible *via* topic-focus articulation:

(5a) 'Charles wants that the wisest citizen should become Mayor of Dunedin.'

(5b) 'Charles wants the wisest citizen to become Mayor of Dunedin.'

The corresponding interrogative sentences indicate the topic. When inquiring about the state-of-affairs expressed by (5a), one would use the form

'What does Charles want?'

whereas an inquiry into (5b) would go via

'What does Charles want the wisest citizen to do?'.

As is seen, the topic of (5a) is Charles' wanting (something) and the topic of (5b) is Charles' wanting the wisest citizen to (do something). Therefore, 'the wisest citizen' occurs *de dicto* in (5a) and *de re* in (5b); (5b) comes with the existential presupposition that the wisest citizen of Dunedin should exist, unlike (5a).

The sentence (5a) thus expresses as its meaning the Closure

(5a′) $\lambda w \lambda t\ [{}^0W^2_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Become_{wt}\ \lambda w \lambda t\ [{}^0Wisest_{wt}\ {}^0Citizen_{wt}]_{wt}$
$\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]]$.

Additional types: $Wisest/(\iota(o\iota))_{\tau\omega}$; $Citizen(\_of\_Dunedin)/(o\iota)_{\tau\omega}$.

The Closure $\lambda w \lambda t\ [{}^0Wisest_{wt}\ {}^0Citizen_{wt}]$ occurs with extensional supposition in (5a′), because it is Composed with w and t. Yet it occurs intensionally in (5a′), because it occurs in the generic intensional context of the Composition $[{}^0W^2_{wt}\ {}^0Charles$ $\lambda w \lambda t\ [{}^0Become_{wt} \ldots]]$. Thus due to the dominancy of intensional context over extensional one it occurs *de dicto* in (5a′).[80] There is no existential presupposition;

[79] The only way would be to construct a derivative constant property $[\lambda w \lambda t\ \lambda y\ [P_{wt}\ b]]$, which is trivially possessed by all individuals whenever b is a P, i.e., whenever the proposition constructed by $[\lambda w \lambda t\ [P_{wt}\ b]]$ is true. This solution has been applied by Kuchyňka (2005), where *wishing* is analysed primarily as a relation to a proposition and derivatively as a relation to the property involved.

[80] See Definitions 2.19, 2.20 and 2.21.

even if the office of the wisest citizen of Dunedin is vacant, Charles may nonetheless want that the political situation in Dunedin were such that the wisest man of Dunedin would become Mayor.

The scenario of (5b) is different. Now Charles knows a particular person who happens to be the wisest citizen of Dunedin, and Charles wants that this person become Mayor. Thus the analysis of (5b) must respect the presupposition of the existence of the wisest citizen. The analysis invokes the substitution method:

(5b′) $\lambda w \lambda t\, [{}^0W^2{}_{wt}\, {}^0Charles\, {}^2[{}^0Sub\, [{}^0Tr\, [{}^0Wisest_{wt}\, {}^0Citizen_{wt}]]\, {}^0he$
${}^0[\lambda w \lambda t\, [{}^0Become_{wt}\ he\ \lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0Dunedin]]]]].$

Another example of the *de dicto*/*de re* ambiguity of sentences expressing notional attitudes is

(6) ‘Charles would like to talk to the Mayor of Dunedin.’

(a) *De re* reading: Charles would like to talk to Mr *X*, because Charles wants to talk to some numerically specific individual and not whomever occupies the office of Mayor, so the reporter just uses this office to point to Mr *X*. Hence ‘would like to talk to’ denotes the object WLT^r of type $(o\iota\iota)_{\tau\omega}$. We get, on a first approximation,

$$\lambda w \lambda t\, [{}^0WLT^r{}_{wt}\, {}^0Charles\, [\lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0Dunedin]]_{wt}].$$

The constituent $[\lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0Dunedin]]$ occurs with *de re* supposition here.

(b) *De dicto* reading: Charles would like to talk to the Mayor of the city, perhaps because he wishes to discuss a political matter with the political leader of Dunedin, and may and need not know who, if indeed any, is the Mayor. Hence ‘would like to talk to’ denotes the object WLT^d of type $(o\iota\iota_{\tau\omega})_{\tau\omega}$, and we get, on a first approximation,

$$\lambda w \lambda t\, [{}^0WLT^d{}_{wt}\, {}^0Charles\, [\lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0Dunedin]]].$$

The constituent $[\lambda w \lambda t\, [{}^0Mayor\_of_{wt}\, {}^0Dunedin]]$ occurs with *de dicto* supposition.

Obviously, in the case *ad* (a), the two *de re* principles are satisfied. However, does this reading of the sentence express a notional attitude, in case WLT^r is a relation (-in-intension) of an individual to an individual? Yes, it does. *Talking to* is extensional, for sure, but *wanting to talk to* is intensional. This becomes obvious, as soon as we refine the analysis. Again, we have to do it by replacing the Trivialization ${}^0WLT^r$ by a Composed construction, which is the ontological definition of the relation. To this end we can use either the schema (i) or (ii). In both cases the refined analysis will apply a relation (-in-intension) of *wishing* to an *intension*, which is how we obtain a notional attitude.

There is again a technical complication stemming from the dominancy of supposition *de dicto*, however. The property P is now the property of talking to the Mayor of Dunedin, which is constructed by ($x \to \iota$; $Talk\_to/(o\iota\iota)_{\tau\omega}$)

$$\lambda w \lambda t\ [\lambda x\ [{}^0Talk\_to_{wt}\ x\ [\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]_{\boldsymbol{wt}}]].$$

Though the construction of the office of Mayor occurs *de re* here, the whole Closure itself occurs *de dicto*. A direct application of one of the schemas, together with the respective β-reductions, leads to

(6′) $\lambda w \lambda t\ [{}^0W^2_{wt}\ {}^0Charles$
$[\lambda w \lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ [\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]_{\boldsymbol{wt}}]]]$,

which is *not* an admissible analysis of the *de re* reading of (6). The reason is that the Closure $[\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]$ occurs in the generic intensional context of the Composition

$$[{}^0W^2_{wt}\ {}^0Charles\ [\lambda w \lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ [\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]_{\boldsymbol{wt}}]]],$$

so it occurs *de dicto* in (6′).[81] Note, however, that (6′) is an admissible analysis of the *de dicto* reading of (6); or, of its rather free but less ambiguity-riddled reformulation,

‘Charles wants that he [Charles] would be talking to the Mayor of Dunedin.’[82]

Charles is related to the *proposition* denoted by the embedded clause and thus also to the office itself, but not to the unspecified occupant of the office (if any). In particular, the principle of existential presupposition does not apply. If the Mayor of Dunedin does not exist at a given $\langle w, t\rangle$-pair of evaluation, Charles may still want to talk to the Mayor. When reporting on such a situation, the respective *notion* of the office is indispensable, and we have an example of a *notional attitude.*

To wrap up the example, we are to find an admissible analysis of the *de re* reading of (6). We have seen that (6′) cannot serve this goal. Had the case been *de re*, the construction $[\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]_{wt}$ would have had to occur in a non-generic extensional context of the respective Composition. There are two ways out. Similarly as in the case of propositional attitudes *de re*, there are again two readings, one active and the other passive. The readings, in stilted but unambiguous English, are:

(*de re act*) ‘Charles wants that *he* should talk to the individual *who* occupies the office of Mayor of Dunedin.’

[81] More precisely, the Closure occurs with extensional supposition but in the $\omega\tau$-generic context of this Composition, due to the first Closure [$\lambda w \lambda t$ ...], which constructs the proposition that Charles talks to the Mayor of Dunedin. See Section 2.7, Definitions 2.18, 2.19 and 2.20.

[82] Again, for the sake of simplicity, (6′) disregards the anaphoric reference ‘he’.

(*de re pas*) 'The Mayor of Dunedin is the one whom Charles would like to talk to.'

When analysing (*de re act*), we again bump into the problem of anaphoric reference; this time 'he' refers to 'Charles' and 'who' to the *de re* occurrence of Mayor of Dunedin. The embedded (incomplete) clause 'he talks to whom' expresses the open construction:

(Em) $\lambda w \lambda t\ [^0 Talk\_to_{wt}\ he\ who]$

Types: $Talk\_to/(o\iota\iota)_{\tau\omega}$, $he, who/*_1 \to \iota$.

In order to obtain an analysis of (*de re act*), we have to substitute the meanings of the antecedents for *he* and *who*. Inserting the Trivialization of Charles for *he* into (Em) is not a problem, since $^0 Charles$ cannot be *v*-improper. But inserting a construction of the individual, if any, who occupies the office of Mayor of Dunedin at $\langle w, t\rangle$ is a problem. We need to insert a construction of an extension—the value of the office at $\langle w, t\rangle$—into the ($\omega\tau$-generic) intensional context of (Em). But the direct replacement of *who* by $[\lambda w \lambda t\ [^0 Mayor\_of_{wt}\ ^0 Dunedin]]_{wt}$ is *invalid*, for we would be drawing a construction occurring extensionally into an intensional context. We invoke once again the functions *Tr* and *Sub*.

(6-*de re act*) $\lambda w \lambda t\ [^0 W^2{}_{wt}\ ^0 Charles\ ^2[^0 Sub\ ^{00} Charles\ ^0 he$
$[^0 Sub\ [^0 Tr\ [\lambda w \lambda t\ [^0 Mayor\_of_{wt}\ ^0 D]]_{\boldsymbol{wt}}]\ ^0 who$
$^0[\lambda w \lambda t\ [^0 Talk\_to_{wt}\ he\ who]]]]]$.

It is easily verified that this analysis is adequate: for any such state-of-affairs $\langle w, t\rangle$ at which the Mayor of Dunedin fails to exist, the intensional descent $[\lambda w \lambda t\ [^0 Mayor\_of_{wt}\ ^0 Dunedin]]_{wt}$ is *v*-improper, as is the Composition $[^0 Tr\ [\lambda w \lambda t\ [^0 Mayor\_of_{wt}\ ^0 Dunedin]]_{wt}]$, which is why *Sub* receives no argument to operate on. So the whole Composition $[^0 W^2{}_{wt}\ ^0 Charles\ \ldots]$ is *v*-improper, and the proposition it constructs has no truth-value. This is precisely as it should be. At those $\langle w, t\rangle$ where the Mayor does exist and the office is occupied by *b*, say, the result of the substitutions is the *construction*

(T) $[\lambda w \lambda t\ [^0 Talk\_to_{wt}\ ^0 Charles\ ^0 b]]$,

the Execution of which will construct the proposition that Charles is related to. Thus Double Execution is required. The first Execution is the instruction to pre-process the open construction $\lambda w \lambda t\ [^0 Talk\_to_{wt}\ he\ who]$ in order to obtain the closed construction (T), and the second Execution of (T) constructs the respective proposition which Charles wants to be true.

Note that the analysis of (*de re act*) is *not* equivalent to the construction obtained *after* the semantic pre-processing of the anaphoric references. This is due to the partiality of the office of Mayor. Provided the pre-processing does not fail, the result is the Closure

$$\lambda w \lambda t\ [{}^0W^2{}_{wt}\ {}^0Charles\ [\lambda w \lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ {}^0B]]].$$

Let this Closure construct a proposition Q. What is constructed by (6-*de re act*) is a proposition P that is undefined at those $\langle w, t\rangle$ at which the office of Mayor goes vacant. However, at these $\langle w, t\rangle$ pairs Q is *not* undefined, because the above Closure does not have a constituent constructing this office. Instead, it contains as a constituent the Trivialisation ${}^0b$ of the occupant of the office of Mayor. But Trivialisation is not v-improper for any valuation v. At the other $\langle w, t\rangle$ pairs both P and Q share the same truth-value.

To complete the example of the active variant, here is the solution to (*de re act*) using the relation W^1:

(6-*de re act*)′ $\lambda w \lambda t\ [{}^0W^1{}_{wt}\ {}^0Charles$
${}^2[{}^0Sub\ [{}^0Tr\ [\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]_{wt}]\ {}^0who$
${}^0[\lambda w \lambda t\ \lambda y\ [{}^0Talk\_to_{wt}\ y\ who]]]]$.

Now we are going to analyse the passive variant of (6). First, a coarse-grained analysis of (*de re pas*) to get the ball rolling:

$$\lambda w \lambda t\ [{}^0ChWT_{wt}\ [\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]]_{wt}].$$

Gloss. ‘The Mayor of Dunedin has the property $ChWT/(o\iota)_{\tau\omega}$.’

In order to refine the analysis, we deploy an ontological definition of the property *ChWT*. To this end we construct the property *ChWT* by Composing constructions of the objects that receive mention here; namely, $Charles/\iota$; $Talk\_to/(o\iota\iota)_{\tau\omega}$, and one of the two possible variants of *Want_to*, either $W^1(ant\_to)/(o\iota(o\iota)_{\tau\omega})_{\tau\omega}$ or $W^2(ant\_to)/(o\iota o_{\tau\omega})_{\tau\omega}$. Here is how this works. Let $x, y \rightarrow_v \iota$; then we get either

$$\lambda w \lambda t\ \lambda x\ [{}^0W^1{}_{wt}\ {}^0Charles\ \lambda w \lambda t\ \lambda y\ [{}^0Talk\_to_{wt}\ y\ x]],$$

or

$$\lambda w \lambda t\ \lambda x\ [{}^0W^2{}_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ x]].$$

If using the latter, the analysis of (*de re pas*) is

(6-*de re pas*) $\lambda w \lambda t\ [\lambda w \lambda t\ [\lambda x\ [{}^0W^2{}_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ x]]]_{wt}$
$\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]_{wt}]$.

This analysis can be simplified by means of β-reduction to

$$\lambda w \lambda t\ [\lambda x\ [{}^0W^2{}_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Talk\_to_{wt}\ {}^0Charles\ x]]\ [{}^0Mayor\_of_{wt}\ {}^0Dunedin]].$$

Consider now a sentence similar to (6):

(7) 'Charles would like to marry a princess.'

Again, two readings. The distinction is well-known and traditionally gets characterised as the difference between the existential quantifier taking narrow or wide scope.

(a) The wide-scope reading may be glossed as,
'There is a particular princess that Charles wishes to marry'.
If *Princess*/$(o\iota)_{\tau\omega}$; *Marry*/$(o\iota\iota)_{\tau\omega}$; $x, y \to \iota$, then we get

$(7_w')$ $\lambda w \lambda t\, \exists x\, [[{}^0Princess_{wt}\, x] \wedge [{}^0W^1{}_{wt}\, {}^0Charles\, \lambda w \lambda t\, \lambda y\, [{}^0Marry_{wt}\, y\, x]]]$.

Gloss: 'There is a princess such that Charles wishes to obtain the property of being married to her'. The implication is that it is irrelevant to Charles whether this particular girl is a princess; she just happens to be one (it is the girl he is after, not her title).

Or alternatively,

$(7_w'')$ $\lambda w \lambda t\, \exists x\, [[{}^0Princess_{wt}\, x] \wedge [{}^0W^2{}_{wt}\, {}^0Charles\, \lambda w \lambda t\, [{}^0Marry_{wt}\, {}^0Charles\, x]]]$.

Gloss: 'There is a princess such that Charles wishes *he* were married to her.'

(b) The narrow-scope reading can be freely glossed as,
'Charles' nuptial intentions revolve around the property of being a princess, rather than a particular princess, so any female will do as wife, as long as she is a princess':

$(7_n')$ $\lambda w \lambda t\, [{}^0W^1{}_{wt}\, {}^0Charles\, [\lambda w \lambda t\, \lambda y\, \exists x\, [[{}^0Princess_{wt}\, x] \wedge [{}^0Marry_{wt}\, y\, x]]]]$.

Gloss: 'Charles wishes to obtain the property of being married to some princess or other'.

Or alternatively,

$(7_n'')$ $\lambda w \lambda t\, [{}^0W^2{}_{wt}\, {}^0Charles\, [\lambda w \lambda t\, \exists x\, [[{}^0Princess_{wt}\, x] \wedge [{}^0Marry_{wt}\, {}^0Charles\, x]]]]$.

Gloss: 'Charles wishes that he were married to some princess or other'.[83]

The morale so far is that sentences expressing somebody's wishing this or that are ambiguous. The alternatives are readings *de dicto* and *de re*, wide-scope and narrow-scope. In this case, at least, we identify the *de dicto* reading with the narrow-scope reading and the *de re* reading with the wide-scope reading. Moreover, another ambiguity arises from two possible ways of analysing the meaning of 'to want to', 'would like to', etc.; namely, either as denoting a relation-in-intension of

[83] The anaphoric reference of 'he*' is disregarded again.

an individual to a property or else to a proposition. We illustrated particular kinds of admissible analysis, showing that in all the cases an agent is related to an *intension.* We also characterised attitudes of this kind, relating an agent to an intension, as notional, as opposed to the relations obtaining between two individuals.

However, the question is whether such attitudes ought rather to be analysed in a way analogous to propositional attitudes; that is, as relations of an agent to *constructions* of the respective intensions. What speaks against it is that, unlike knowing, believing, doubting, etc., which concern primarily *modes* of presenting intensions to the attributee (whose deductive abilities are strongly dependent on the respective constructions), the *intentional* activities of agents are primarily related to the intensions themselves, regardless of the way in which the intensions are presented to them. The reason for this is that the attribution of notional attitudes is set within the attributee's third-person perspective, and does not attempt to reproduce the attributee's first-person perspective. Only when particular agent-relative perspectives need to be reproduced are constructions called for.

To put it another way, some wishes must indeed be analysed hyperintensionally, as we said at the outset of this subsection. People have many impossible wishes. For instance, many mathematicians wish that arithmetic were recursively axiomatizable, despite Gödel's incompleteness proof. Many children wish to become a world champion in athletics, say, without wishing to put in the effort required to get there. However, one cannot wish and at the same time not wish one and the same thing. Thus if P is a proposition, then one cannot consistently want that P be true and at the same time not wish P be true. Or the stronger variant: one cannot simultaneously both wish and not P to be true. And if Q is a property, then one cannot consistently both wish and not wish to obtain Q. However, if C_1 and C_2 are two different constructions of the proposition P, an agent a can want that C_1 would construct P, without wanting that C_2 would construct P. Similarly, if D_1 and D_2 are two different constructions of the property Q, a may want to obtain the property constructed by D_1 without wanting to obtain the property constructed by D_2.

An example that we broached above had a wishing to become the Pope and also wishing not to become head of state of the Vatican. Assume that the office of Pope and the office of head of the state of Vatican City are one and the same office. Assume also that if a wishes not to become head of state of the Vatican then a does not wish to become head of state of the Vatican (though not vice versa). Then a truthful report on such a situation must be hyperintensional. Here is why.

Let $[{}^0Pope =_o \lambda w\lambda t\, [{}^0Head\_of_{wt}\, {}^0Vatican]]$.

Types: $Pope/\iota_{\tau\omega}$; $Head\_of/(\iota\iota)_{\tau\omega}$; $Vatican/\iota$; $\lambda w\lambda t\, [{}^0Head\_of_{wt}\, {}^0Vatican] \rightarrow \iota_{\tau\omega}$; $=_o/(o\iota_{\tau\omega}\iota_{\tau\omega})$.

First we show that an intensional analysis yields inconsistency:

$$\frac{\begin{array}{c}\lambda w\lambda t\,[[{}^0W^1{}_{wt}\,{}^0a\;\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\,{}^0Pope]]]\wedge\\ \neg[{}^0W^1{}_{wt}\,{}^0a\;\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\;\lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]]]]\end{array}}{\lambda w\lambda t\,\exists p\,[[{}^0W^1{}_{wt}\,{}^0a\,p]\wedge\neg[{}^0W^1{}_{wt}\,{}^0a\,p]]}$$

Additional types: $W^1(\text{ant\_to})/(o\iota(o\iota)_{\tau\omega})_{\tau\omega}$; $p \rightarrow (o\iota)_{\tau\omega}$.

Proof. Since by assumption $[{}^0Pope =_o \lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]]$, and both ${}^0Pope$ and $\lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]$ occur intensionally in

$$\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\,{}^0Pope]]$$

and

$$\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\;\lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]]],$$

respectively, the intensional rule of substitution yields:[84]

$$\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\,{}^0Pope]] =_p$$
$$\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\;\lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]]].$$

Additional type: $=_p/(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$.

The consequence now trivially follows by existential generalization.

However, hyperintensional wishes do not yield inconsistency. Let W^* be hyperintensional wishing of type $(o\iota*_n)_{\tau\omega}$. Then from

$$\begin{array}{c}\lambda w\lambda t\,[[{}^0W^*{}_{wt}\,{}^0a\,{}^0[\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\,{}^0Pope]]]]\wedge\\ \neg[{}^0W^*{}_{wt}\,{}^0a\,{}^0[\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\;\lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]]]]]\end{array}$$

one can derive neither

$$\lambda w\lambda t\,\exists p\,[[{}^0W^*{}_{wt}\,{}^0a\,{}^0p]\wedge\neg[{}^0W^*{}_{wt}\,{}^0a\,{}^0p]]$$

nor

$$\lambda w\lambda t\,\exists c\,[[{}^0W^*{}_{wt}\,{}^0a\,c]\wedge\neg[{}^0W^*{}_{wt}\,{}^0a\,c]].$$

Additional type: $c \rightarrow *_n$.

Since both ${}^0Pope$ and $\lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]$ occur hyperintensionally in

$${}^0[\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\,{}^0Pope]]]$$

and

$${}^0[\lambda w\lambda t\,[\lambda x\,[{}^0Become_{wt}\,x\,{}^0Head\_of_{wt}\_{}^0Vatican]]],$$

respectively, the intensional rule of substitution is not applicable. The hyperintensional rule of substitution is not applicable either, because ${}^0Pope$ and $\lambda w\lambda t\,[{}^0Head\_of_{wt}\,{}^0Vatican]$ are equivalent but *not* procedurally isomorphic constructions.

[84] See Section 2.7.1 for the rules of substitutions.

5.2.2 Seeking, finding and looking for

Attitudes of seeking and finding have famously been dealt with by Montague who turned seeking unicorns into a favourite pastime amongst Montagovians in the 1970s and after.[85] These attitudes have also been tackled within the TIL framework in, e.g., Jespersen (1999), Gahér (2003) and Duží (2003b). Since some points made there are in need of correction and much has since been done on *seeking* and *finding*, we present here an up-dated, comprehensive survey of the main results so far.

We do not use 'to look for' or 'to seek' to talk about going to get something of which we know what it is and where it is (for which we use 'to fetch', 'to pick up' and suchlike). In other words, we cannot seek something whose identity and whereabouts are known to us. Charles can be looking for the author of *Waverly*, and the detectives seeking the murderer, only if they do not know who the author, or the murderer, is (And if they do know it, they do not know where he or she is; see below). They investigate who occupies the respective *office*. The seeker is related to an office, and we have another example of a typical *notional attitude*:

(8) 'Charles is seeking the murderer of *X*.'

Types: *Charles*/ι; *Seek*/$(o\iota\iota_{\tau\omega})_{\tau\omega}$; *Murderer_*(of)/$(\iota\iota)_{\tau\omega}$; *X*/$\iota$; *MX*/$\iota_{\tau\omega}$ (the murderer of *X*).

Analysis:

(8′) $\lambda w \lambda t\ [{}^0 Seek_{wt}\ {}^0 Charles\ \lambda w \lambda t\ [{}^0 Murderer_{wt}\ {}^0 X]]$.

The subconstruction $\lambda w \lambda t\ [{}^0 Murderer_{wt}\ {}^0 X]$ occurs *de dicto* in (8′); the office of the murderer is not used as a pointer to its occupant at the $\langle w, t\rangle$ of evaluation. There is no existential presupposition: Charles may conduct his search even if Mr *X* has not been murdered or has more than one murderer (Recall Agatha Christie's *The Murder on the Orient Express*). The substitution *salva veritate* of co-referential expressions is not valid, either. If the gardener is the murderer of *X*, it does not follow from (8) that Charles is seeking the gardener.

Still, 'looking for' and 'seeking' are again homonymous expressions. It makes perfect sense to say,

(9) 'Václav Havel is looking for Dagmar Havlová.'

Does it mean that this search is an object of type $(o\iota\iota)_{\tau\omega}$, a relation of an individual to an *individual*? Yes, it does. Types: *VH* (Václav Havel), *DH* (Dagmar Havlová)/ι; *Look_for*/$(o\iota\iota)_{\tau\omega}$. A coarse-grained analysis of (9) would be[86]

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[85] See Montague (1974a) and Gamut (1991, pp. 165–70, 197). Some results in this section were inspired by discussions with Jiří Raclavský.

[86] In this section we use the expression 'look_for' as a name for a relation-in-intension of an individual to an individual, and 'seek' as a name for a relation-in-intension of an individual to an

(9′) $\lambda w \lambda t\ [{}^0\mathit{Look\text{-}for}_{wt}\ {}^0VH\ {}^0DH]$.

What does Václav then find when successfully conducting such a search? This kind of search is different from the activity of seeking as stipulated above, for the existential question never arises for individuals, and Václav certainly knows exactly which individual Dagmar is. But Václav does not know *where* Dagmar is; he is trying to *locate* her. As we explained in Section 3.5.1, the object of such a search is a location office, i.e., an my-office.

Let *Loc*(ation) be an empirical function assigning to an individual *a* the place on the face of the Earth where *a* is, and let the type of the place be μ;[87] thus $Loc/(\mu\iota)_{\tau\omega}$. Our task is now to define *Look_for* in terms of $Seek^{L}$, where $Seek^{L}$ is this time of type $(o\iota\mu_{\tau\omega})_{\tau\omega}$: a relation-in-intension of an individual to a μ-office; $x, y \rightarrow \iota$. Obviously, the explication of *looking for* is

(D) ${}^0\mathit{Look\_for} = \lambda w \lambda t\ \lambda xy\ [{}^0Seek^{L}{}_{wt}\ x\ \lambda w \lambda t\ [{}^0Loc_{wt}\ y]]$.

Gloss: 'An individual *x* is looking for an individual *y* iff *x* is seeking the location of *y*.'

When conducting his search, Václav is now related to the μ-office *Dagmar's location*. To obtain a more fine-grained analysis of (9), we have to substitute the right-hand side of the above equation for ${}^0\mathit{Look\_for}$ in (9′):

$$\lambda w \lambda t\ [\lambda w \lambda t\ \lambda xy\ [{}^0Seek^{L}{}_{wt}\ x\ \lambda w \lambda t\ [{}^0Loc_{wt}\ y]]_{wt}\ {}^0VH\ {}^0DH].$$

By performing the respective β-reductions, we obtain:[88]

(9″) $\lambda w \lambda t\ [{}^0Seek^{L}{}_{wt}\ {}^0VH\ \lambda w \lambda t\ [{}^0\mathrm{Loc}_{wt}\ {}^0DH]]$.

We see that this search is again a notional attitude, this time directed to the μ-office constructed by

$$\lambda w \lambda t\ [{}^0Loc_{wt}\ {}^0DH] \rightarrow \mu_{\tau\omega}.$$

It could be objected that (8) is also ambiguous, as it might be read *de re* as well. As indeed it may, in case the police were to announce, 'The murderer *Y* is wanted for questioning', and Charles the detective was looking for *Y*'s whereabouts. A coarse-grained analysis of the *de re* reading of (8) is

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office. Of course, Charles can be looking for the author of *Waverley*, being thus related to the *office*. But, to distinguish the two different types of relations, we will use the above convention.

87 The place can be determined, for instance, by a (continuous) set of 3D co-ordinates with respect to the centre of the Earth; the type μ is then equal to $(o\tau\tau\tau)$.

88 These β-reductions consist in substituting variables *w*, *t* for variables *w*, *t*, and Trivialisations ${}^0VH$, ${}^0DH$ for *x* and *y*, respectively. Since neither a Trivialisation nor a variable can be *v*-improper, such a transformation is guaranteed to be an equivalent one.

(8^{r}) $\lambda w \lambda t\ [{}^0Look\_for_{wt}\ {}^0Charles\ [\lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]]_{wt}]$.

Hence, when analysing the *de re* reading of (8), we must use the construction of the murderer of *X* with *de re* supposition, as in (8^{r}). Of course, Charles is not locating an ι-office (an abstract functional object) but the individual (if any) that occupies the ι-office. The fact that Charles happens to know who the murderer is, is irrelevant, however. In those states of affairs $\langle w, t\rangle$ where the murderer of *X* does not exist, the sentence (8) has (on its *de re* reading) *no* truth-value. The murderer of *X* is the topic of the sentence on its *de re* reading, and the non-vacancy of the office is presupposed. Obviously, Charles cannot be locating the occupant of a vacant office. And if Charles is locating the murderer of *X* and the murderer of *X* is *Y*, then Charles is locating *Y*. The two *de re* principles are satisfied.

We are at this point confronted once again with technical complications. According to (D), ${}^0Look\_for_{wt}$ is equivalent to the Closure

$$\lambda xy\ [{}^0Seek^{\mathrm{L}}{}_{wt}\ x\ \lambda w \lambda t\ [{}^0Loc_{wt}\ y]],$$

and we need to Compose the latter with ${}^0Charles$ and the constituent

$$[\lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]]_{wt}$$

v-constructing the occupant (if any) of the office of the murderer:

($8^{\mathrm{r}\prime}$) $\lambda w \lambda t\ [\lambda xy\ [{}^0Seek^{\mathrm{L}}{}_{wt}\ x\ \lambda w \lambda t\ [{}^0Loc_{wt}\ y]]\ {}^0Charles$
$[\lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]]_{wt}]$.

As we have proved in Section 2.7 and illustrated by means of numerous examples, the substitution of a construction such as $[\lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]]_{wt}$ for a variable such as *y* into a context such as ($8^{\mathrm{r}\prime}$) results in a non-equivalent construction due to the partiality of the office of the murderer, because *y* occurs in a ωτ-generic intensional context of ($8^{\mathrm{r}\prime}$).[89] Thus, though ($8^{\mathrm{r}\prime}$) is equivalent to

($8^{\mathrm{r}\prime\prime}$) $\lambda w \lambda t\ [\lambda y\ [{}^0Seek^{\mathrm{L}}{}_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Loc_{wt}\ y]]\ [\lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]]_{wt}]$,

it is *not* equivalent to

$$\lambda w \lambda t\ [{}^0Seek^{\mathrm{L}}{}_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Loc_{wt}\ [\lambda w' \lambda t'\ [{}^0Murderer_{w't'}\ {}^0X]]_{wt}]].$$

Even renaming the variables *w*, *t*, to prevent collision of variables, is to no avail here. The construction $\lambda w \lambda t\ [{}^0Loc_{wt}\ y]$ is never *v*-improper, even if *v*-constructing a degenerate function. And this construction has to occur with *de dicto* supposition, because Charles is related just to the μ-office (not to its occupant, which is a

[89] See Claim 2.5 concerning β-reduction ‘by name’.

particular place) when finding the location of the murderer. Even if Charles found himself on the very spot, he would fail to make the connection between the murderer and the location, unless he knew it to be the place where the murderer was. However, $[\lambda w\lambda t\ [^0Murderer_{wt}\ ^0X]]$ must occur with *de re* supposition, which conflicts with the former.

There are two possible ways of solving the problem: either not to perform further β-reduction and accept ($8^{r''}$) as an admissible analysis of (8), or to make use of the valid rule of β-reduction 'by value', as introduced in Section 2.7 by Claim 2.6. The former, i.e. ($8^{r''}$), corresponds to the passive variant of (8),

'The murderer of X is sought by Charles'

whose reading *de re* can be rephrased as,

(8^{r}passive) 'The murderer of X is such that Charles is seeking his/her location.'

To obtain an analysis of the active *de re* variant of (8), we apply the *Sub* and *Tr* functions in order to obtain equivalence between the following two constructions:

$$\lambda w\lambda t\ [^0Look\_for_{wt}\ ^0Charles\ [\lambda w\lambda t\ [^0Murderer_{wt}\ ^0X]]_{wt}] =$$

($8^{r'''}$) $\lambda w\lambda t\ [^0Seek^{L}{}_{wt}\ ^0Charles\ ^2[^0Sub\ [^0Tr\ [\lambda w\lambda t\ [^0Murderer_{wt}\ ^0X]]_{wt}]\ ^0y\ ^0[\lambda w\lambda t\ [^0Loc_{wt}\, y]]]]$.

We can easily check that this analysis is adequate: in any such state-of-affairs $\langle w, t\rangle$ in which the murderer of X fails to exist, the intensional descent $[\lambda w\lambda t\ [^0Murder_{wt}\ ^0X]]_{wt}$ fails (i.e., is v-improper), as does the Composition $[^0Tr\ [\lambda w\lambda t\ [^0Murder_{wt}\ ^0X]]_{wt}]$, which is why *Sub* receives no argument to operate on. Hence, the substitution

$$[^0Sub\ [^0Tr\ [\lambda w\lambda t\ [^0Murder_{wt}\ ^0X]]_{wt}]\ ^0y\ ^0[\lambda w\lambda t\ [^0Loc_{wt}\, y]]]$$

is v-improper, and so is the Double Execution of it, and also the whole Composition

$$[^0Seek^{L}{}_{wt}\ ^0Charles\ ^2[^0Sub\ [^0Tr\ [\lambda w\lambda t\ [^0Murder_{wt}\ ^0X]]_{wt}]\ ^0y\ ^0[\lambda w\lambda t\ [^0Loc_{wt}\, y]]]].$$

The constructed proposition has no truth-value. This is only as it should be, though, because otherwise, at those $\langle w, t\rangle$ where the murderer of X does exist and the office is occupied (say, by Y), the result of the substitution is the *construction* $[\lambda w\lambda t\ [^0Loc_{wt}\ ^0Y]]$, the execution of which constructs the respective μ-office to which Charles is related.

Note that *Double* Execution is required here, because the first Execution is the instruction to perform the substitution that operates on the construction of the μ-office, and only the second Execution is the instruction to construct the μ-office.

Consider next Church's classic[90]

[90] In 1956, (p. 8, n. 20).

(10) 'Schliemann sought the site of Troy.'

When Schliemann embarked upon his quest, he did not know whether Troy existed, though he may have felt pretty sure of its existence. Hence Troy cannot be analysed as an individual, because individuals trivially exist, as argued in Section 2.3. Instead, Troy is an ι-office, $Troy/\iota_{\tau\omega}$. Evidently, Schliemann did not only want to know whether Troy existed; he also wanted to find its site. Therefore, he performed the activity of seeking that we denoted by '$Seek^L$'. But now the construction of the ι-office *Troy* does not occur *de re*: there is no existential presupposition associated with (10), so Schliemann could perform his search even if *Troy* had not existed. Hence both the construction of the μ-office (the site of *Troy*) and the construction of the respective ι-office (*Troy*) need to occur *de dicto*. The adequate analysis of (10) is thus easily found ($Loc/(\mu\iota)_{\tau\omega}$)[91]:

(10′) $\lambda w\lambda t\ [{}^0Seek^L{}_{wt}\ {}^0Schliemann\ [\lambda w\lambda t\ [{}^0Loc_{wt}\ {}^0Troy_{wt}]]]$.

Note that the construction ${}^0Troy$ occurs with *de dicto* supposition in (10'). The office is extensionalized with respect to Schliemann's empirical context $\langle w, t\rangle$, rather than the external attributer's empirical context; α-renaming the respective variables makes things clearer:

$$\lambda w\lambda t\ [{}^0Seek^L{}_{wt}\ {}^0Schliemann\ [\lambda w'\lambda t'\ [{}^0Loc_{w't'}\ {}^0Troy_{w't'}]]].$$

Due to the dominancy of the generic intensional context, both the constructions of the office *Troy* and its location are merely *mentioned* in (10′) and the respective offices are not used to point to their occupants. If Burbank had been the site of Troy then, as Kaplan rightly observes, Schliemann would not have been seeking Burbank.[92] And even if he happened to find himself at the mouth of the river Hissarlik and stumbled upon the ruins of Troy, he would ignore the place until and unless realizing the connection between it and the offices. On the other hand, he might have successfully sought the site of Troy without ever actually being on the relevant location. He might have had access to sources he knew to be truthful and simply put two and two together, thus figuring out which place on the face of the Earth is the site of Troy.

To sum up, the activities of seeking relate an individual to an office (a $\iota_{\tau\omega}$-object or a $\mu_{\tau\omega}$-object),[93] so they are *notional* attitudes, of the types $(o\iota\mu_{\tau\omega})_{\tau\omega}$ or

[91] We continue to disregard tenses. See, however, Section 2.5.2.

[92] Kaplan (1975, p. 718).

[93] Or, generally, to an α-intension. One can, for instance, seek a unicorn. The object of such a search is an individual property, of type $(o\iota)_{\tau\omega}$. The seeker wants to find an instance of the property.

$(o\iota_{\tau\omega})_{\tau\omega}$. When we seek something or look for something, we are trying to find the occupant of the office (who or what may not exist, since the office may be vacant).[94]

Now, the search may meet with success, in which case the seeker becomes a finder, as the agent finds what he was seeking or looking for. Or the search may fail (hence neither 'to seek' or 'to look for' are so-called success verbs.) If Charles was seeking (or looking for) the murderer of *X*, then one of the two following propositions has to be true and the other false:

(11) Charles found the murderer of *X*.

(12) Charles did not find the murderer of *X*.

(12) is true in either of two situations: either Charles was not good enough at finding the murderer, or the murderer did not exist at all, for *X* was not murdered or had more than one murderer. If there was an existential presupposition and the murderer did not exist, the propositions denoted by (11) and (12) would both be without truth-value, which violates our intuition that in such a situation (12) ought to be false. Thus there is no existential presupposition concerning the occupant of the respective office, and *finding* does not relate an individual to an individual but to an office. These are again *notional* attitudes of the same types as those corresponding to seeking or looking for, i.e., $Find/(o\iota_{\tau\omega})_{\tau\omega}$ and $Find^{L}/(o\iota\mu_{\tau\omega})_{\tau\omega}$, respectively.

Of course, if the search is successful, then the murderer must exist; thus (11) implies, but does not presuppose, that the murderer must exist. We say that *finding* comes with an existential *commitment*. To elucidate the difference between existential presupposition and commitment, recall Definition 1.14 in Section 1.5.2. If *Q* is a presupposition of *P*, then *Q* is entailed (i.e., necessarily implied) both by *P* and non-*P*:

$$\forall w \forall t\ [[{}^0True^{\mathrm{P}}{}_{wt}\,P] \vee [{}^0False^{\mathrm{P}}{}_{wt}\,P]] \supset [{}^0True^{\mathrm{P}}{}_{wt}\,Q]].$$

Thus if the existence of the murderer was a presupposition of him or her being found, then both (11) and (12) would entail the existence of the murderer; which is not desirable. (11), but not (12), entails existence. So the difference between presupposition and commitment is that if *Q* is a commitment of *P*, then *Q* is entailed by *P* but not by non-*P*.

What is more, if (11) is true then Charles has to know *who* the murderer is, or *where* he is, dependently on the type of the foregoing search.

Let us consider first the type of the foregoing search as $Seek/(o\iota_{\tau\omega})_{\tau\omega}$. Then the corresponding *Find* is also of type $(o\iota_{\tau\omega})_{\tau\omega}$, and the analyses of (11) and (12) are:

(11′) $\lambda w \lambda t\ [{}^0Find_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]]$

(12′) $\lambda w \lambda t\ \neg[{}^0Find_{wt}\ {}^0Charles\ \lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]]$.

[94] 'Looking for' is, however, used in English only in case the existence of the sought object is guaranteed; otherwise we use 'seeking'. This fact motivates our convention of distinguishing between *Seek* (*Seek*^{L}) and *Look_for*.

Note that both in (11′) and (12′) the subconstruction $\lambda w \lambda t\ [{}^0Murderer_{wt}\ {}^0X]$ occurs *de dicto*. The two *de re* principles do not hold. In particular, if *Y* is the murderer of Mr *X*, it does not follow that Charles found/did not find *Y*.

As pointed out above, having successfully sought the murderer entails finding out *who* the murderer is, which however does not logically follow from (11′).[95] To obtain that entailment, we need to specify some *requisites* of *finding*. It seems fair to say that in any state-of-affairs ⟨*w*, *t*⟩ in which somebody performs a successful search of an ι-office they also identify the respective occupant of the office. But what does it mean *to identify the respective occupant of the office*? If Charles found the murderer of *X* and if the murderer is *Y* then Charles identified *Y*. So it might seem that the type of *Identify* ought to be $(o\iota\iota)_{\tau\omega}$. But if *Y* is not only the murderer of *X* but also, say, the President of the Silesian Spelunking Society, then it would not be correct to deduce that Charles had identified the President of the Silesian Spelunking Society. How could he have identified the President? He was not even looking for the President. Yet it seems fair to say that Charles did identify the President *as the murderer of X*. Even if Charles does not know that the President is the murderer, the *attributer* may correctly report the situation using *the President of the Silesian Spelunking Society* as a pointer to the individual *Y* who has been identified by Charles as the murderer of *X*. The underlying principle is that the attributer may use as a means to pinpoint the murderer any office that the murderer also occupies at the same ⟨*w*, *t*⟩.

Therefore, the type of *Identify* is $(o\iota\iota_{\tau\omega})_{\tau\omega}$: a relation-in-intension between an individual (*who* identifies), an individual (*whom*) and an office (as *what*). This yields a fifth instance of a requisite relation, this time between two relations-in-intension: $Req_5/(o(o\iota\iota_{\tau\omega})_{\tau\omega}(o\iota_{\tau\omega})_{\tau\omega})$, defined as follows:[96]

$$[{}^0Req_5\ {}^0Identify\ {}^0Find] =$$
$$\forall w \forall t\ [\forall xu\ [[{}^0Find_{wt}\ x\ u] \supset [{}^0True_{wt}\ \lambda w \lambda t\ [{}^0Identify_{wt}\ x\ u_{wt}\ u]]]].$$

Types: $Identify/(o\iota\iota_{\tau\omega})_{\tau\omega}$; $Find/(o\iota_{\tau\omega})_{\tau\omega}$; $x \to \iota$; $u, c \to \iota_{\tau\omega}$; $Exist/(o\iota_{\tau\omega})_{\tau\omega}$; $=/(ooo)$.

Hence, necessarily, if somebody finds *u*, then they identify the holder of *u*, so we can specify ***Rules for finding after a prior search***:

$$\text{(I)} \qquad \frac{[{}^0Find_{wt}\ x\ u]}{[{}^0Identify_{wt}\ x\ u_{wt}\ u]}$$

Since the first occurrence of *u* in the consequence of (I) is extensional, the rule of substitution of *v*-congruent constructions is valid as well:[97]

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[95] For the definition of logical entailment, see Section 1.5.1, Definition 1.13.

[96] For definitions of the four other kinds of requisite relation, see Section 4.1.

[97] See Section 1.5.2.1.

$$\text{(II)} \qquad \frac{[{}^0Find_{wt}\ x\ u],\ [u_{wt} = c_{wt}]}{[{}^0Identify_{wt}\ x\ c_{wt}\ u].}$$

If somebody identifies the occupant of *u*, the office *u* must be occupied, and so we have the auxiliary rule

$$\text{(III)} \qquad \frac{[{}^0Identify_{wt}\ x\ u_{wt}\ u]}{[{}^0Exist_{wt}\ u]}$$

From (I) and (III) we get, by transitivity, the rule

$$\text{(IV)} \qquad \frac{[{}^0Find_{wt}\ x\ u]}{[{}^0Exist_{wt}\ u]}$$

Note that none of these rules are valid in case of an unsuccessful search. Indeed, *finding after a prior search* does not come with an existential presupposition, only with an existential commitment, since existence is implied, but not presupposed, by this kind of finding. Hence, another requisite of *finding* is defined as follows:

$$[{}^0Req_6\ {}^0Exist\ {}^0Find] = \forall w \forall t\ [\forall xu\ [[{}^0Find_{wt}\ x\ u] \supset [{}^0Exist_{wt}\ u]]].$$

$Req_6/(o(o\iota_{\tau\omega})_{\tau\omega}(o\iota\iota_{\tau\omega})_{\tau\omega})$.

Next up is *finding a location*. It is true that Schliemann found the site of Troy, hence the following construction constructs the proposition that is true in, among other, the actual world at the present time:

$$\lambda w \lambda t\ [{}^0Find^L{}_{wt}\ {}^0Schliemann\ [\lambda w \lambda t\ [{}^0Loc_{wt}\ {}^0Troy_{wt}]]],$$

where $Find^L/(o\iota\mu_{\tau\omega})_{\tau\omega}$.

But another scenario is thinkable: if Troy does not exist or Schliemann is less lucky, then it will be true that Schliemann's search was unsuccessful:

'Schliemann did not find the site of Troy',

which is analysed as

$$\lambda w \lambda t\ \neg[{}^0Find^L{}_{wt}\ {}^0Schliemann\ [\lambda w \lambda t\ [{}^0Loc_{wt}\ {}^0Troy_{wt}]]].$$ [98]

However, if the search turns out successful, then Troy exists and Schliemann identifies its site. Since Schliemann's search did meet with success, he singled out the mouth of Hissarlik. And since he identified the place, the site of Troy exists.

[98] Again, for the sake of simplicity, we disregard tenses.

Thus similar requisite relations and the corresponding rules apply also in case of finding a location after a prior search.

Additional types: $Exist^L/(o\mu_{\tau\omega})_{\tau\omega}$; $x \rightarrow \iota$; l, $m \rightarrow \mu_{\tau\omega}$, $Identify^L/(o\iota\mu\mu_{\tau\omega})_{\tau\omega}$; $Req_7/(o(o\iota\mu\mu_{\tau\omega})_{\tau\omega}(o\iota\mu_{\tau\omega})_{\tau\omega})$; $Req_8/(o(o\mu_{\tau\omega})_{\tau\omega}(o\iota\mu_{\tau\omega})_{\tau\omega})$;

$$[{}^0Req_7\ {}^0Identify^L\ {}^0Find^L] =$$
$$\forall w \forall t\ [\forall x\ l\ [[{}^0Find^L{}_{wt}\ x\ l] \supset [{}^0True_{wt}\ \lambda w \lambda t\ [{}^0Identify^L{}_{wt}\ x\ l_{wt}\ l]]]].$$

$$[{}^0Req_8\ {}^0Exist^L\ {}^0Find^L] = \forall w \forall t\ [\forall x\ l\ [[{}^0Find^L{}_{wt}\ x\ l] \supset [{}^0Exist^L{}_{wt}\ l]]].$$

Here are the corresponding ***Rules for successfully searching the location***:

(V) $$\frac{[{}^0Find^L{}_{wt}\ x\ l]}{[{}^0Identify^L{}_{wt}\ x\ l_{wt}\ l]}$$

(VI) $$\frac{[{}^0Find^L{}_{wt}\ x\ l],\ [l_{wt} = m_{wt}]}{[{}^0Identify^L{}_{wt}\ x\ m_{wt}\ l]}$$

(VII) $$\frac{[{}^0Find^L{}_{wt}\ x\ l]}{[{}^0Exist^L{}_{wt}\ l]}$$

One may wonder whether the reverse relations and rules are valid as well; i.e., the question is whether we can *define* the notional attitudes $Find/(o\iota\iota_{\tau\omega})_{\tau\omega}$, $Find^L/(o\iota\mu_{\tau\omega})_{\tau\omega}$ in terms of *Identify*, $Identify^L$, respectively. We cannot; *Find*, $Find^L$ concern the offices to which the seeker was related in the foregoing search. But you may identify Mr *Y* as the murderer of *X* without having sought the murderer of *X*. Or, you may arrive at the mouth of Hissarlik and identify this place as the site of Troy without having sought the site.

There is, however, a second kind of finding. Charles, on his way home, may stumble upon a piece of metal, pick it up, and only upon returning home come to realize that it is the single most valuable coin in the entire history of numismatics. The situation can be reported as,

(13) 'Charles found the single most valuable coin.'

This time Charles is not primarily related to the office of the most valuable coin, for he was not even looking for its occupant. We have here a case of finding something by chance, without a preceding search. Thus, $Find^C/(o\iota\iota)_{\tau\omega}$. The analysis of (13) is:

Types: $The\_most/(\iota(o\iota))_{\tau\omega}$; $Valuable/((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$: a property modifier;[99] $Coin/(o\iota)_{\tau\omega}$.

(13′) $\lambda w\lambda t\ [{}^0Find^C_{wt}\ {}^0Charles\ \lambda w\lambda t\ [{}^0The\_most_{wt}\ [{}^0Valuable\ {}^0Coin]_{wt}]_{wt}]$.

The constituent $\lambda w\lambda t\ [{}^0The\_most_{wt}\ [{}^0Valuable\ {}^0Coin]_{wt}]$ occurs *de re*; i.e., the two *de re* principles hold. In particular, if the most valuable coin is the only 1933 US Mint Gold Double Eagle still around (suppose only one survived the melt-down), then Charles found the only 1933 US Mint Gold Double Eagle still around. Or, if the most valuable coin is Charles' favourite coin, then Charles found his favourite coin.

The difference between $Find^L$ and $Find^C$ is the difference between seeking-and-purposefully-finding and accidentally finding something by happening upon it. Since seeking is guided by an intension, the successful seeker is always able to conceptualize what they have found. Not so with accidentally finding something, which is unaided by an intension. Therefore, a finder^C may well fail to conceptualize what they have found.

To sum up, sentences reporting on seeking and finding are systematically ambiguous, for 'seeking', 'looking for' and 'finding' are homonymous. The first may denote $Seek/(o\iota\iota_{\tau\omega})_{\tau\omega}$ in case the seeker is trying to find out *who* occupies the respective ι-office, or $Seek^L/(o\iota\mu_{\tau\omega})_{\tau\omega}$ in case the seeker is trying to find out *where* the respective individual is. In general, $Seek/Seek^L$ is an object of type $(o\iota\alpha_{\tau\omega})_{\tau\omega}$; the seeker is trying to find an instance of an α-intension. Both *Seek* and $Seek^L$ are to be characterised as *notional attitudes.* An agent is intentionally related to an α-intension. On the other hand, 'finding' may simply express incidental finding,[100] in which case it denotes a relation of type $Find^C/(o\iota\iota)_{\tau\omega}$, which is not an attitude at all. In case finding (or possibly not finding) was preceded by a search,[101] $Seek/(o\iota\iota_{\tau\omega})_{\tau\omega}$ or $Seek^L/(o\iota\mu_{\tau\omega})_{\tau\omega}$, or in general *Seek* of type $(o\iota\alpha_{\tau\omega})_{\tau\omega}$, it is the notional attitude $Find/(o\iota\iota_{\tau\omega})_{\tau\omega}$, $Find^L/(o\iota\mu_{\tau\omega})_{\tau\omega}$, or *Find* of type $(o\iota\alpha_{\tau\omega})_{\tau\omega}$, respectively.

There may be still another kind of ambiguity involved, though. In Duží (1993) this ambiguity was characterized as *strong polymorphism* in distinction to *weak polymorphism*. The latter holds for functions like cardinality, equality, quantifiers, etc. When using a quantifier, we should always specify its type, for instance $\forall^\iota/(o(o\iota))$, $\forall^\tau/(o(o\tau))$, $\exists^\iota/(o(o\iota))$, $\exists^\tau/(o(o\tau))$. Though we sometimes write '$\forall^\alpha$', '$\exists^\alpha$', or simply, '$\forall$', '$\exists$', we are always able to specify the proper unique type α. Not so with strongly polymorphic functions. Consider, for instance, the sentences

(14) 'Charles is contemplating something.'
(15) 'Charles is seeking something.'

[99] See Section 4.4.

[100] 'Találni' in Hungarian; see Jespersen (1999).

[101] 'Megtalálni' in Hungarian; see Jespersen (1999).

(16) ‘What is Charles thinking about?’

If we try to find the respective construction representing the meaning of these sentences, we find ourselves in trouble. The first attempt to analyse, e.g., (14), might be:

$$\lambda w \lambda t\ [{}^0\exists^\alpha\ \lambda x\ [{}^0Contemplate_{wt}\ {}^0Charles\ x]].$$

But what is the type of *Contemplate*? Which proper type does α stand for and which type does the variable *x* range over? Since Charles can be contemplating an entity of *any* type—or even a type, a type of a type, simply *anything*—we would be entangled in an absurdity; there would be uncountably infinitely many constructions corresponding to this simple sentence. In Duží (1993) the solution proposed consisted in expanding the ramified hierarchy of types into a three-dimensional hierarchy. Entities of types defined by the ramified hierarchy were entities of the 1st kind. Entities of the 2nd kind were constructions involving variables ranging over the collection of all types of the first kind and constructions containing such variables; and so on up *ad infinitum*. However, even such a super-ramified hierarchy of types would allow Charles to contemplate only entities up to kind *n*, for arbitrary *n.* This renders the typing overly restrictive. To put it another way, a formal type theory is bound to be outstripped by the resources of natural language. An absolute solution to the problem of strong polymorphism is in principle impossible. But, the solution is perhaps just this very insight. For the insight is that there is no such thing as contemplating *simpliciter*, but only stratified, type-relative contemplation, which means that contemplation comes in many different flavours and colours, depending on what sort of thing is being contemplated. This insight (if we are agreed that that is what it is) will be lost on any theory of type-free contemplation.

5.3 Quantifying in

In Section 1.4.3 we mentioned existentially quantifying into modal and attitude contexts as one of the stumbling blocks upholding the progress of analytic philosophy of language.[102] Here we demonstrate how to technically overcome this problem.

The logical possibility of quantifying-in demonstrates, as we argued in Section 1.5.2, that (hyper-) intensional logic is not defined by failure to validate quantifying-in. Nor is the difference between attitudes *de dicto* and *de re* anything to do with only the latter validating quantifying-in, for both kinds of attitude do. Rather, since our project is to develop and implement an extensional logic of intensions

[102] The two classics are, of course, Quine (1956) and Kaplan (1968).

and hyperintensions, the ability to quantify-in should not be missing from our repertory.

Moreover, quantifying-in is rational on independent philosophical grounds. It seems evident that if Charles believes that Smith's murderer is left-handed, then there is something (somebody) which (whom) Charles believes to be left-handed. Or if Charles calculates seven plus five, then there is something that Charles is calculating. The key to quantifying-in is the ability to assign intensional as well as constructional ranges to variables. In the former example it must range over $\iota_{\tau\omega}$; in the latter, over $*_1$. So, not surprisingly, we need at least $*_2$-constructions in our ontology to get quantifying-in off the ground.

Needless to say, the ability to quantify into empirical, not to mention mathematical, attitudes is way beyond the pale for any logic whose domain of quantification is limited to extensional entities. If we were to quantify over such entities, the respective analyses would say that there is an individual whom Charles believes to be left-handed and that there is a number that Charles is calculating. However, calculating is not a relation to numbers, but to numerical constructions. So the conclusion should not quantify away a number and replace it by a variable ranging over numbers. And to believe that the *F* is a *G* is not to believe that some numerically specific individual is a *G*. So the conclusion should not quantify away an individual and replace it by a variable ranging over individuals. Furthermore, an extensionalist analysis of quantifying-in will be left groping in vain for something to quantify over when, e.g., Charles believes, wrongly, that Smith's murderer is left-handed (because Smith has no unique murderer) or Charles contemplates, in vain, the largest natural number (because no one natural number is larger than the rest). The extensionalist conclusions would magically create a unique murderer of Smith and a largest natural number, the existence of either is neither presupposed nor entailed by the premises.

So how does TIL go about the business of quantifying-in? The derivation, in stilted prose to begin, that there is an individual office whose occupant Charles believes to be left-handed is

$$\frac{\text{At } \langle w, t\rangle\ a \text{ believes that the } F \text{ is a } G}{\text{At } \langle w, t\rangle \text{ there is an } x \text{ such that } a \text{ believes that } x_{wt} \text{ is a } G}$$

Type: $x/*_1 \rightarrow \iota_{\tau\omega}$.

Remark. x_{wt} is not the intensional descent of a variable (which would be nonsensical), but of its value, *in casu* an individual office, to obtain an individual.

The derivation that there is a construction which Charles is calculating is

$$\frac{\text{At } \langle w, t\rangle\ a \text{ calculates seven plus five}}{\text{At } \langle w, t\rangle \text{ there is a } c \text{ such that } a \text{ calculates } c}$$

Type: $c/*_2 \rightarrow *_1$.

Intensional 'propositional' attitudes *de dicto* and constructional 'notional' attitudes *de dicto* are only two out of altogether eight different kinds of attitude. The full list is

(1) intensional 'propositional' *de dicto*
(2) intensional 'propositional' *de re*
(3) intensional 'notional' *de dicto*
(4) intensional 'notional' *de re*
(5) constructional 'propositional' *de dicto*
(6) constructional 'propositional' *de re*
(7) constructional 'notional' *de dicto*
(8) constructional 'notional' *de re*.

Remark. By (5)–(8) we mean attitudes of an agent to a *construction* $C(X/y)$, where the subject of the attitude *v*-constructed by X occurs (a) mentioned* in the constructional context of (5) or (7), or (b) used* *de re*; i.e., with an extensional supposition in the non-generic extensional context of (6) or (8).[103]

The first step toward a solution is to avail oneself of variables ranging over intensions and constructions. Once these are granted, the second step is to display the exact logical analyses of quantifying into the attitudes (1)–(8). Our analyses and rules are as follows.

Rules ad (1) and (3) Let X/*n $\to \alpha_{\tau\omega}$ be a constituent occurring in C *intensionally*,[104] and let X be *v*-proper. Then:

$$\frac{[Att_{wt}\ a\ C(X/x)]}{[\exists x\ [Att_{wt}\ a\ C(x)]].}$$

Types: $x/*_1 \to \alpha_{\tau\omega}$; $a/*_n \to \iota$; $\exists/(o(o\alpha_{\tau\omega}))$; $C(X/x)/*_n \to \beta_{\tau\omega}$; $Att/*_n \to (o\iota\alpha_{\tau\omega})_{\tau\omega}$: an intensional 'propositional' or 'notional' attitude relation;

Proof. Follows directly from the intensional rule of substitution[105] by applying existential generalisation (EG): if the premise of the rule $[Att_{wt}\ a\ C(X/x)]$ *v*-constructs **T**, then it is *v*-proper, and so is $C(X/x)$. Since X is *v*-proper by assumption, the class *v*-constructed by $\lambda x\ [^0Att_{wt}\ a\ C(x)]$ is non-empty.

Examples.

Ad (1): '*a* believes that *b*'s murderer is left-handed; hence, there is an individual office such that *a* believes that its occupant is left-handed.'

[103] '$C(X/y)$' stands for a construction arising from C by a correct substitution of X for y (see Definition 2.22) and '$C(x)$' stands for a construction with a free variable x.

[104] See Section 2.7.

[105] See Section 2.7.

$$\frac{\lambda w\lambda t\ [{}^0B_{wt}\ a\ \lambda w\lambda t\ [{}^0LH_{wt}\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]]}{\lambda w\lambda t\ [\exists x\ [{}^0B_{wt}\ a\ \lambda w\lambda t\ [{}^0LH_{wt}\ x_{wt}]]].}$$

Types: $a, b/*_n \rightarrow \iota$; $\exists/(o(o\iota_{\tau\omega}))$; $LH/(o\iota)_{\tau\omega}$; $M\_of/(\iota\iota)_{\tau\omega}$; $x/*_1 \rightarrow \iota_{\tau\omega}$; $B/(o\iota o_{\tau\omega})_{\tau\omega}$.
Remark. It may seem logically untoward that *M_of* is an *attribute*, while *x* ranges over *offices*. But it should be borne in mind that $\lambda w\lambda t\ [{}^0M\_of_{wt}\ b]$ constructs an office. The particular way this office happens to be constructed is irrelevant to the validity of quantifying in.

Ad (3): ‘*a* is seeking *b*’s murderer; hence, there is something that *a* is seeking.’

$$\frac{\lambda w\lambda t\ [{}^0Seek_{wt}\ a\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]]}{\lambda w\lambda t\ [\exists x\ [{}^0Seek_{wt}\ a\ x]]}$$

Types: $Seek/(o\iota\iota_{\tau\omega})_{\tau\omega}$; $x/*_n \rightarrow \iota_{\tau\omega}$; $\exists/(o(o\iota_{\tau\omega}))$.

Rules ad (2) and (4):

Active variant:

$$\frac{[Att_{wt}\ a\ {}^2[{}^0Sub\ [{}^0Tr\ X_{wt}]\ {}^0y\ {}^0C(y)]]}{[\exists z\ [Att_{wt}\ a\ C(z/y)]]}$$

Passive variant:

$$\frac{[\lambda y\ [Att_{wt}\ a\ C(y)]\ X_{wt}]}{[\exists z\ [Att_{wt}\ a\ C(z/y)]]}$$

Types: $X/*_n \rightarrow \alpha_{\tau\omega}$; $Att/*_n \rightarrow (o\iota\alpha_{\tau\omega})_{\tau\omega}$; $a/*_n \rightarrow \iota$; $y, z/*_1 \rightarrow \alpha$; $\exists/(o(o\alpha))$; y free in $C(y) \rightarrow \beta_{\tau\omega}$.
Remark. We omit indexing the type of the polymorphous functions Sub_n and Tr^α.

Proof. Follows directly from the rules of existence (see Section 2.7) by applying EG; if the assumption v-constructs **T**, then it is v-proper, and so is $X_{wt} \rightarrow \alpha$. Hence there is an α-object occupying X.

Examples.
Ad (2): ‘*a* believes of *b*’s murderer that he is left-handed; hence, there is an individual such that *a* believes that he/she is left-handed.’

Active variant:

$$\frac{\lambda w\lambda t\ [{}^0B_{wt}\ a\ {}^2[{}^0Sub\ [{}^0Tr\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ he]]]]}{\lambda w\lambda t\ [\exists z\ [[{}^0B_{wt}\ a\ \lambda w\lambda t\ [{}^0LH_{wt}\ z]]]}$$

Passive variant:

$$\frac{\lambda w\lambda t\ [\lambda z\ [{}^0B_{wt}\ a\ \lambda w\lambda t\ [{}^0LH_{wt}\ z]]\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]}{\lambda w\lambda t\ [\exists z\ [[{}^0B_{wt}\ a\ \lambda w\lambda t\ [{}^0LH_{wt}\ z]]]}$$

Types: a, $b/*_n \rightarrow \iota$; z, $he/*_1 \rightarrow \iota$; $\exists/(o(o\iota))$.
Remark. The conclusion may be equivalently rephrased as, 'There is an individual who is believed by *a* to be left-handed.' Indeed, if *a* believes of *b*'s murderer that he is left-handed, then the class of those who are believed by *a* to be left-handed is non-empty, which means that there is such an individual.

Ad (4): '*a* is looking for (the location of) *b*'s murderer; hence, there is someone whose location is sought by *a*.'

Active variant:

$$\frac{\lambda w\lambda t\, [{}^0Seek^L_{wt}\ a\ {}^2[{}^0Sub\ [{}^0Tr\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]\ {}^0whom\ {}^0[\lambda w\lambda t\ [{}^0Loc\_of_{wt}\ whom]]]}{\lambda w\lambda t\ [\exists z\ [{}^0Seek^L_{wt}\ a\ \lambda w\lambda t\ [{}^0Loc\_of_{wt}\ z]]].}$$

Passive variant:

$$\frac{\lambda w\lambda t\ [\lambda z\ [{}^0Seek^L_{wt}\ a\ \lambda w\lambda t\ [{}^0Loc\_of_{wt}\ z]]\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]}{\lambda w\lambda t\ [\exists z\ [{}^0Seek^L_{wt}\ a\ \lambda w\lambda t\ [{}^0Loc\_of_{wt}\ z]]].}$$

Types: $Seek^L/(o\iota\mu_{\tau\omega})_{\tau\omega}$; z, $whom/*_1 \rightarrow \iota$; $Loc(ation)\_of/(\mu\iota)_{\tau\omega}$; $\exists/(o(o\iota))$.
Remark. In both cases it also follows that *a* is looking for somebody:

$$\lambda w\lambda t\ [\exists z\ [{}^0Look\_for_{wt}\ az]],$$

where $Look\_for/(o\iota\iota)_{\tau\omega}$ is defined as: $\lambda w\lambda t\ \lambda xz\ [{}^0Seek^L_{wt}\ x\ \lambda w\lambda t\ [{}^0Loc\_of_{wt}\ z]]$.

Notional attitudes *de re* were analysed in Section 5.2. We showed that when seeking an individual, as in 'Václav is seeking Dagmar', the seeker is related to an individual whose identity he/she knows. Yet he/she is seeking something unknown to him/her, namely the location (of type μ) of the sought individual. Thus we proposed two analyses, a coarse-grained one (by using *Look_for*) and a fine-grained one (by using $Seek^L$), the latter obtained by equivalently refining the former.

So much for intensional attitudes. A *caveat* is in place before proceeding, though, since quantifying into constructional contexts brings with it a major technical complication. To appreciate which, here is an example of reckless deriving (Let $B^*/(o\iota^*_1)_{\tau\omega}$ and $u/*_1 \rightarrow \iota_{\tau\omega}$).

$$\frac{\lambda w\lambda t\ [{}^0B^*_{wt}\ a\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]]]}{\lambda w\lambda t\ [\exists u\ [{}^0B^*_{wt}\ a\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ u_{wt}]]]].}$$

To bring out the contrast between the respective conclusions of (1) and (5), the latter may be recast as, 'There is a construction of an individual office such that *a* believes* that the construction of the proposition that its occupant is left-handed constructs a true proposition.'

Why is the conclusion no good? The occurrence of u in ${}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ u_{wt}]]$ is ^{0}bound, so u is *mentioned* inside a hyperintensional context, hence not available for manipulation. It is, as it were, shielded from $\exists$ by the first ^{0} in the Trivialization ${}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ u_{wt}]]$. A linguistic parallel would be to attempt to quantify into a quotational context, where the quotation marks would have an analogous shielding effect.

Yet, intuitively, it seems that from the premise it does follow that there is an office such that a believes* that its occupant is left-handed, namely the office of b's murderer. And it does follow. It is just that we have to use a *construction* of the office in order to exactly reproduce a's explicit belief. The solution is to untie the occurrence of a variable from its binding. To do so, we again use the substitution technique.[106] But this time we must use a variable ranging over *constructions* of individual offices. If we used $u \to \iota_{\tau\omega}$, the result would be:

(Flawed) $\lambda w\lambda t\ [\exists u\ [{}^0B^*_{wt}\ a\ [{}^0Sub\ [{}^0Tr\ u]\ {}^0u\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ u_{wt}]]]]]$.

Type-theoretically, this construction is correct, because the result of the substitution is a propositional construction. However, (Flawed) does not follow from the premise. To see why, imagine that the variable u v-constructs the office X of b's murderer. Then the result of the substitution is the Closure $[\lambda w\lambda t\ [{}^0LH_{wt}\ {}^0X_{wt}]]$, which may *not* be explicitly believed* by a on the assumption that a has the explicit attitude B^* to the construction $[\lambda w\lambda t\ [{}^0LH_{wt}\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]]$. A valid conclusion is obtained by using a variable $c/*_2 \to *_1;\ {}^2c/*_3 \to \iota_{\tau\omega}$:

(Correct) $\lambda w\lambda t\ [\exists c\ [{}^0B^*_{wt}\ a\ [{}^0Sub_2\ c\ {}^0c\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ c_{wt}]]]]]$.

Note that the Composition $[{}^0LH_{wt}\ c_{wt}]$ is improper, as it does not construct anything, due to wrong typing. The variable c ranges over constructions of order 1, so c cannot be Composed with $\langle w, t\rangle$. This is the other kind of improperness mentioned in Section 1.3.2. However, the whole Closure figuring in (Correct) is proper and entailed by the assumption. To see why, assume that at $\langle w, t\rangle$ the Composition $[{}^0B^*_{wt}\ a\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]]]$ v-constructs **T** and consider the particular execution steps specified by (Correct):

(i) ${}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ c_{wt}]]$ constructs $[\lambda w\lambda t\ [{}^0LH_{wt}\ c_{wt}]]$
(ii) $[{}^0Sub_2\ c\ {}^0c\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ c_{wt}]]]$ v-constructs a Closure dependently on the valuation of c; the first occurrence of c is free here and

[106] Materna (1997) considers only constructional attitudes, and the analysis put forward there is slightly more complicated. The reason for the complication is that the *Sub*/*Tr* segment is placed *before* the construction of the attitude relation. Since *Sub* returns a construction (here of type $*_1$), Double Execution of $[{}^0Sub\ [{}^0Tr\ldots]]$ would be needed in the respective conclusions to provide $\exists$ with the right type of argument, namely a truth-value. Double Execution is missing from (1997), but will not be needed, anyway, if the construction of the attitude relation *precedes* the *Sub*/*Tr* segment. The approach employed above was first presented in Duží (1999). However, in the latter, an invalid argument in the vein of (Flawed) is put forward in one case.

- at $\langle w, t\rangle$ the Composition $[{}^0Sub_2\ c\ {}^0c\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ c_{wt}]]]\ v(\lambda w\lambda t\ [{}^0M\_of_{wt}\ b]/c)$-constructs the Closure $[\lambda w\lambda t\ [{}^0LH_{wt}\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]]$
- at $\langle w, t\rangle$ the Composition $[{}^0B^*_{wt}\ a\ [{}^0Sub_2\ c\ {}^0c\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ c_{wt}]]]]\ v(\lambda w\lambda t\ [{}^0M\_of_{wt}\ b]/c)$-constructs **T**
- at $\langle w, t\rangle$ the Composition $[\exists\lambda c\ [{}^0B^*_{wt}\ a\ [{}^0Sub_2\ c\ {}^0c\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ c_{wt}]]]]]$ v-constructs **T** as well, because the class of constructions v-constructed by $\lambda c\ [{}^0B^*_{wt}\ ...]$ is non-empty.

Remark. If we want to avoid the type-theoretical improperness of the mentioned* subconstruction on which the substitution operates, we will have to use Double Execution as follows:

$$\lambda w\lambda t\ [\exists c\ [{}^0B^*_{wt}\ a\ [{}^0Sub\ [{}^0Tr\ c]\ {}^0c\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ {}^2c_{wt}]]]]].$$

But this would be entailed only if the additional assumption were added that the agent a knew* that ${}^{20}C = C$ for any construction C.

Now we are ready to specify the rules for constructional attitudes. Let $X/*_n$ be a construction.

Rules ad (5) and (7):

$$\frac{[Att^*_{wt}\ a\ {}^0C(X/c)]}{[\exists c\ [Att^*_{wt}\ a\ [{}^0Sub\ c\ {}^0c\ {}^0C(c)]]].}$$

Types: $c/*_{(n+1)} \to *_n$; $a/*_n \to \iota$; $\exists/(o(o*_n))$; $Att^*/(o\iota*_n)_{\tau\omega}$: a hyperpropositional or hypernotional attitude relation.

Proof. Follows from the hyperintensional rule of substitution[107] by applying EG.

Example ad (7): ‘*a* is proving* Fermat’s Last Theorem; hence, there is a numerical construction being proved by *a*.’

$$\frac{\lambda w\lambda t\ [{}^0Prove^*_{wt}\ a\ {}^0[\forall x\ y\ z\ n\ [[n > {}^02] \supset \neg[x^n+y^n = z^n]]]]}{\lambda w\lambda t\ [\exists c\ [{}^0Prove^*_{wt}\ a\ [{}^0Sub\ c\ {}^0c\ {}^0c]]].}$$

Types: $\exists/(o(o*_1))$; $c/*_2 \to *_1$; $Prove^*/(o\iota*_1)_{\tau\omega}$; the other types are obvious.

Remark. The conclusion is equivalent to $\lambda w\lambda t\ [\exists c\ [{}^0Prove^*_{wt}\ a\ c]]$.

Remark. The premise is analogous to *a* contemplating π, which does not have *a* contemplating some one particular real number, but instead a *construction* of whatever number is the ratio of the circumference of a circle to its diameter (where there is a plethora of equivalent constructions of this ratio). That is, *a* is

[107] See Section 2.7.

related to a *procedure* rather than its *product*.[108] The corresponding conclusion, then, would likewise be that there is some construction *d* such that *a* is contemplating *d*.

Remark. In the case of constructional attitudes, the assumption that *X* is a proper construction is not needed, unlike in the case of the intensional attitudes *ad* (1) and (3). To illustrate the need of this assumption in case of intensional beliefs, consider the example

‘*a* believes that the occupant of *b*’s (most) favourite individual office is left-handed’.

$$\lambda w \lambda t\, [{}^0B_{wt}\, a\, \lambda w \lambda t\, [{}^0LH_{wt}\ [{}^0Fav\_of_{wt}\, b]_{wt}]].$$

Types: $Fav\_of/(\iota_{\tau\omega}\iota)_{\tau\omega}$; $a,b/*_n \to \iota$; $LH/(o\iota)_{\tau\omega}$; $B/(o\iota o_{\tau\omega})_{\tau\omega}$.

We cannot derive that there is an office whose occupant is believed by *a* to be left-handed, because *a* may believe that the occupant of *b*’s favourite individual office is left-handed, even if *b* actually does not have any one favourite office. For such a $\langle w, t\rangle$,

$$[{}^0B_{wt}\, a\, \lambda w \lambda t\, [{}^0LH_{wt}\ [{}^0Fav\_of_{wt}\, b]_{wt}]]$$

v-constructs **T**, but

$$[\exists x\, [{}^0B_{wt}\ a\ \lambda w \lambda t\, [{}^0LH_{wt}\ x_{wt}]]]$$

v-constructs **F**, because $[{}^0Fav\_of_{wt}\, b]$ is *v*-improper.

Types: $x \to \iota_{\tau\omega}$; $\exists/(o(o\iota_{\tau\omega}))$; $B/(o\iota o_{\tau\omega})_{\tau\omega}$.

However, if *a* believes*, for instance, that the greatest prime is even (or exists, or whatever else) we can derive that there is a construction of a number such that the number is believed* by *a* to be even (or whatever):

$$\frac{\lambda w \lambda t\, [{}^0B^*_{wt}\ a\ {}^0[{}^0Even\ \iota x\, [[{}^0Prime\, x] \wedge \forall y\, [[{}^0Prime\, y] \supset [x \geq y]]]]]]}{\lambda w \lambda t\, [\exists c\, [{}^0B^*_{wt}\ a\ [{}^0Sub\ c\ {}^0c\ {}^0[{}^0Even\ c]]]]].}$$

Types: $\exists/(o(o*_1))$; $c/*_2 \to *_1$; $B^*/(o\iota*_1)_{\tau\omega}$; the other types are obvious.

Rules ad (6) and (8):

Active variant:

$$\frac{[Att^*_{wt}\, a\, [{}^0Sub\, [{}^0Tr\, X]\ {}^0y\ {}^0\mathrm{C}(y)]]}{[\exists z\, [Att_{wt}\, a\ C(z/y)]]}$$

Passive variant:

$$\frac{[\lambda y\, [Att^*_{wt}\, a\, [{}^0Sub\, [{}^0Tr\, X]\ {}^0y\ \ {}^0\mathrm{C}(y)]\ X]]}{[\exists z\, [Att_{wt}\, a\ C(z/y)]]}$$

[108] See Section 3.2.1.

Types: $X/*_n \rightarrow \alpha$; $Att/*_n \rightarrow (o\iota*_n)_{\tau\omega}$; $a/*_n \rightarrow \iota$; $y, z\ /*_1 \rightarrow \alpha$; $\exists/(o(o\alpha))$; y free in $C(y)/*_n$.

Remark. We again omit indexing the respective types of the polymorphous functions Sub_n and Tr_α.

Proof. Follows directly from the rules of existence[109] by applying EG: if the premise v-constructs **T**, then it is v-proper, and so is X.

Examples.

Ad (6): '*a* believes* of *b*'s murderer that he is left-handed; hence, there is an individual such that *a* believes that he is left-handed.'

Active variant:

$$\frac{\lambda w\lambda t\ [{}^0B^*_{wt}\, a\ [{}^0Sub\ [{}^0Tr\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt}\ he]]]]}{\lambda w\lambda t\ [\exists z\ [[{}^0B^*_{wt}\, a\ \lambda w\lambda t\ [{}^0LH_{wt}\ z]]]]}$$

Passive variant:

$$\frac{\lambda w\lambda t\ [\lambda z\ [{}^0B^*_{wt}\, a\ [{}^0Sub\ [{}^0Tr\ z]\ {}^0z\ {}^0[\lambda w\lambda t\ [{}^0LH_{wt} z]]]\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]]}{\lambda w\lambda t\ [\exists z\ [[{}^0B^*_{wt}\ a\ \lambda w\lambda t\ [{}^0LH_{wt}\ z]]]]}$$

Types: z, $he/*_1 \rightarrow \iota$; $\exists/(o(o\iota))$.

Remark. It is difficult to imagine a notional attitude that would be both constructional and *de re*. The following argument does not exactly correspond to the above schema, but it illustrates the extensional reading of constructional attitudes:

'*b*'s favourite number is the result of *a*'s calculation;
hence, there is a number that is the result of *a*'s calculation.'

$$\frac{\lambda w\lambda t\ [{}^2[\iota c\ [{}^0Calc_{wt}\ a\ c]] = [{}^0Fav\_num_{wt}\ b]]}{\lambda w\lambda t\ [\exists x\ [{}^2[\iota c\ [{}^0Calc_{wt}\ a\ c] = x]]].}$$

Types: $c/*_2 \rightarrow *_1$; ${}^2c \rightarrow \tau$; $x/*_1 \rightarrow \tau$; $\iota/(*_1(o*_1))$; $=/(o\tau\tau)$; a, $b/*_n \rightarrow \iota$; $\exists/(o(o\tau))$; $Calc/(o\iota*_1)_{\tau\omega}$; $Fav\_num/(\tau\iota)_{\tau\omega}$.

The argument is obviously valid. Both ${}^2[\iota c\ [{}^0Calc_{wt}\ a\ c]]$ and $[{}^0Fav\_num_{wt}\ b]$ occur extensionally in the Composition $[{}^2[\iota c\ [{}^0Calc_{wt}\ a\ c]] = [{}^0Fav\_num_{wt}\ b]]$. Therefore, the substitution of a v-congruent construction and EG are both valid.

Another example of an extensional *de re* reading of a notional attitude would be

[109] See Section 2.7.

$$\frac{\text{The largest natural number is being contemplated by } a}{\text{There is a number a construction of which is being contemplated by } a.}$$

On its extensional reading the premise can be reformulated as follows:

'The largest natural number is such that
a construction of *it* is being contemplated by *a*.'

On this reading the denoted proposition has no truth-value, because there is no largest natural number, and the proposition denoted by the conclusion takes the value **F**. The argument is valid, but unsound. *If* there were a largest number and the premise were true *then* the conclusion would be bound to be true as well. To illustrate the situation, let us analyse another premise:

'The least natural number is such that
a construction of *it* is being contemplated by *a*'.

First, the anaphoric clause '*a* contemplates a construction *c* and *c* constructs *it*' expresses:

$$\lambda w \lambda t\, [[{}^0Cont_{wt}\, a\, c] \wedge [{}^2c = it]].$$

The analysis of the premise is obtained by substituting the construction of the least number for *c* and a Trivialization of the so constructed number for *it*:

$$\frac{\begin{gathered}{}^2[{}^0Sub\ {}^{00}[\iota x\, \forall y\, [x \leq y]]\ {}^0c\ [{}^0Sub\ [{}^0Tr\, [\iota x\, \forall y\, [x \leq y]]]\ {}^0it \\ {}^0[\lambda w \lambda t\, [[{}^0Cont_{wt}\, a\, c] \wedge [{}^2c = it]]]]]\end{gathered}}{\lambda w \lambda t\, [\exists z\, [\exists^* \mathrm{c}\ \lambda w \lambda t\, [[{}^0Cont_{wt}\, a\, c] \wedge [{}^2c = z]]]].}$$

Types: x, y, z, $it/*_1 \rightarrow \nu$; $\leq/(o\nu\nu)$; $c/*_2{\rightarrow}*_1$; ${}^2c{\rightarrow}\nu$; $\iota/(\nu(o\nu))$; $Cont/(o\iota*_1)_{\tau\omega}$; $\exists/(o(o\nu))$; $\exists^*(o(o*_1))$.

Gloss: 'There is a number and a construction such that the latter v-constructs the former and *a* contemplates the latter.'

The result of the substitution is the Closure

$$\lambda w \lambda t\, [[{}^0Cont_{wt}\, a\ {}^0[\iota x\, \forall y\, [x \leq y]]] \wedge [{}^{20}[\iota x\, \forall y\, [x \leq y]] = {}^00]].$$

The Execution of this Closure constructs the proposition that *a* contemplates the construction of the least number and this construction constructs the number 0.

Since in the premise the second occurrence of the Composition $[\iota x\, \forall y\, [x \leq y]]$ is used* as a constituent, and the Singularizer ${}^0Sing$ (abbreviated 'ι') occurs with

($o\nu$)-extensional supposition in the non-generic extensional context of the premise, the conclusion follows by EG according to the rule of existence[110]:

$$\frac{Proper\ ([\iota x\ \forall y\ [x \le y]])}{[[{}^0Exist\ \lambda x\ \forall y\ [x \le y]\ {}^0Sing].}$$

Since $[[{}^0Exist\ \lambda x\ \forall y\ [x \le y]]\ {}^0Sing] = \exists z\ [[{}^0Sing\ \lambda x\ \forall y\ [x \le y]] = z]$, and both constructions construct **T**, the conclusion is true on the assumption that the premise is true. The argument is valid and possibly sound.

Type: $Exist/((o(\nu(o\nu)))(o\nu))$.

If read extensionally, the following is a similar valid argument:

$$\frac{\text{The ratio of the circumference of a circle to its diameter is contemplated by } a}{\text{There is a number a construction of which is being contemplated by } a.}$$

a is contemplating a construction of the number π, even though 'π' does not express as its meaning a Trivialization of a particular real number, but instead a Composed construction of it. If the assumption is read extensionally, then any equivalent construction can be substituted *salva veritate* for the meaning of 'the ratio of the circumference of a circle to its diameter'. The argument is valid and possibly sound for the same reasons as above.

We only considered *attitudes* up to now. However, once attitudes have been sorted out, *modalities* are easily handled. Here we analyse a case of nomic necessity, a case of requisites, and a case of propositional properties.[111]

$$\frac{\text{Necessarily, hot air rises}}{\text{There is a property such that, necessarily, everything with this property rises.}}$$

$$\frac{\lambda w\ [\forall t\ [\forall x\ [[{}^0Hot\ {}^0Air]_{wt}\ x] \supset [{}^0Rise_{wt}\ x]]]}{\lambda w\ [\exists y\ [\forall t\ [\forall x\ [y_{wt}\ x] \supset [{}^0Rise_{wt}\ x]]]].}$$

Types: $Hot/((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; Air, $Rise/(o\iota)_{\tau\omega}$; $x/*_1 \rightarrow \iota$; $y/*_1 \rightarrow (o\iota)_{\tau\omega}$.

F is a requisite of *G*

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110 For particular rules and definitions, see Section 2.7, Definitions 2.15, 2.18 and 2.19. Similarly as in Section 2.7.1, '*Proper*(*A*)' stands for the construction *A* being *v*-proper for all valuations *v*;

111 For details on nomic necessity and requisites, see Sections 4.5, 4.1 and 4.4, respectively.

There is a property such that it is a requisite of *G*.

$$[{}^0Req\ F\ G]$$

$$[\exists x\ [{}^0Req\ x\ G]].$$

Types: $Req/(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; $F, G/*_n \rightarrow (o\iota)_{\tau\omega})$; $\exists/(o(o(o\iota)_{\tau\omega}))$; $x/*_1 \rightarrow (o\iota)_{\tau\omega}$.

Allegedly, *b*'s murderer is left-handed

There is an individual office such that, allegedly, its occupant is left-handed.

$$\lambda w\lambda t\ [{}^0Allegedly_{wt}\ \lambda w\lambda t\ [{}^0LH_{wt}\ \lambda w\lambda t\ [{}^0M\_of_{wt}\ b]_{wt}]]$$

$$\lambda w\lambda t\ [\exists x\ [{}^0Allegedly_{wt}\ \lambda w\lambda t\ [{}^0LH_{wt}\ x_{wt}]]].$$

Types: $Allegedly/(oo_{\tau\omega})_{\tau\omega}$: a property of propositions; $\exists/(o(o\iota_{\tau\omega}))$; $x/*_1 \rightarrow \iota_{\tau\omega}$.

Allegedly is construed as a propositional *property* rather than a propositional *modifier*, of type $(o_{\tau\omega}\ o_{\tau\omega})$, to allow alleged propositions to be merely contingently alleged to be true. The reason is that it ought to be contingent that, allegedly, *b*'s murderer is left-handed, leaving room for alternative worlds and times at which the proposition that *b*'s murderer is left-handed does not have the property of being allegedly true. If *Allegedly* were instead a modifier taking a proposition *P* to the proposition *Allegedly, P* then any proposition so modified would be necessarily so. The difference is due to propositional properties of type $(oo_{\tau\omega})_{\tau\omega}$ being intensions and modifiers, of any type, not being intensions, but functions from intensions to intensions that are not themselves intensional entities.

5.4 Information and inference

Cohen and Nagel (1934) put forward the *paradox of inference*:

> If in an inference the conclusion is not contained in the premises, it cannot be valid; and if the conclusion is not different from the premises, it is useless; but the conclusion cannot be contained in the premises and also possess novelty; hence inferences cannot be both valid and useful (Cohen and Nagel, 1934, p. 173).

This paradox arises because of the tension between (a) the *validity* (legitimacy) of an inference, and (b) the *utility* of an inference. One can reformulate the question posed by the paradox thus: how can (deductive) logic function as a useful epistemological tool?[112] For an inference to be legitimate, the recognition of the premises as true must already have accomplished what is needed for the recognition of the truth of the conclusion; but if the conclusion is to be useful the recognition of its truth should not already take place when the truth of the premises is ascertained.

For illustration, consider this argument:

Everybody is at home or has gone shopping
If Charles has gone shopping, then he is buying milk
Charles is not buying milk

Charles is at home.

If we recognise the premises as being true thanks to empirical investigation, then we need not empirically investigate the state of the world in order to get to know that Charles is at home. Pure reasoning is sufficient to establish that he is.

For a mathematical example, consider the following argument:

All numbers divisible by 4 or 6 are even
Some numbers which are divisible by 6 are also divisible by 4

It is not true that no even number is divisible by 4.

This time we do not examine the state of the world in order to establish whether the premises are true. However, no matter how we establish their truth-value (be it by running a proof or consulting a textbook on mathematics), once they have been established, no further investigation is called for. Pure reasoning suffices to determine that is not true that no even number is divisible by 4.

As many fellow logicians and mathematicians will no doubt agree, the conclusion of a valid argument is often very useful, and can often be surprising too. It seems evident that there is *something* that we *learn* when deducing the conclusion of a sound argument. So it won't do to claim that we learn nothing. We show in this section what it is that we learn when drawing valid inferences from premises known to be true. Deductive logic is as useful an epistemic tool as we always knew it to be; but the nature of what a sound argument teaches us is liable to come as a surprise.

The paradox of inference is an instance of the broader problem of the usefulness of analytically true sentences. Since we adopt the explication that an analytically true sentence is true solely in virtue of its meaning, we defined analytically true sentences in Section 1.5.1.1 as sentences expressing a construction that constructs

[112] Historically, this 'paradox' seems to have been discovered as early as in the seventeenth century, by Francis Bacon, who raised the problem in order to criticize deduction. See http://www.blupete.com/Literature/Biographies/Philosophy/Bacon.htm#Writings.

T in all worlds and times, or independently of worlds and times in case of mathematical sentences.[113]

Every deductively valid argument with premises $P_1,\ldots, P_n$ and conclusion P corresponds to an analytically true conditional sentence of the form, 'If P_1 and … and P_n, then P'. Of course, if the argument is deductively valid, then it cannot be the case that all the premises $P_1,\ldots,P_n$ are true and the conclusion P is not true. So the conditional sentence must be analytically true.

We emphasize that by 'argument' we do not mean 'inference', if by 'inference' is meant the *act* of inferring conclusion from premises. Rather, by 'argument' we understand a set of declarative sentences (premises) along with another declarative sentence (a conclusion). The act of inferring involves a *judgement*, which is the execution of one or more cognitive procedures. These two concepts of argument are discussed in Sundholm (1997), where he warns against conflating the notions of logical consequence and valid inference:

> The relata in logical consequence are propositions, whereas an inference affects a passage from known judgements to a novel judgement that becomes known in virtue of the inference in question (1997, p. 27).

In this quotation Sundholm uses the term 'proposition' in the sense of Bolzano's *Satz an sich* as the content of a judgement. If translated into TIL, it corresponds to a *propositional construction* rather than a proposition in the modern sense understood as a mapping from worlds to truth-values. According to the procedural semantics we have set out in this book, an argument encodes a *procedure*, which may or may not be executed by an epistemic agent, rather than the *process* of actually executing the procedure. Also recall that the procedure encoded by an analytically true sentence does not have to be *effectively* executable. Accordingly, a passage from known premises to the novel conclusion of a valid argument does not have to be effectively executable. For this reason we imposed the distinction between analytically and logically true sentences, and between analytically and logically valid arguments. A logically true sentence is a sentence whose literal logical form v-constructs **T** in the mathematical case and the proposition *TRUE* in the empirical case for every valuation v. If a sentence is logically true then it is also analytically true, but not vice versa. This is because analytical validity conceived as truth in all possible worlds at all times is independent of the logical system in which its meaning happens to be regimented. A logically true sentence is provable in some sufficiently expressive logical calculus, though discovering such a calculus is obviously a non-trivial task. Similarly, an analytically valid argument need not be logically valid. If an argument is logically valid then it is analytically valid, but not vice versa.[114]

Our point of departure is the characterisation of information as being objective and of a semantic nature, as found in Floridi (2004, 2005). Moreover, we are

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[113] See Definition 1.9.

[114] See Section 1.5.1.1, Definitions 1.10, 1.11, 1.12 and 1.13.

going to investigate information that is independent of any informee. Thus we adopt a *realist* view of meaning and information as defended in Floridi (2005).

If we put things in terms of possible-world semantics (PWS), then sentences are informative due to the PWS propositions they are associated with by denoting them. Empirical sentences are informative because the propositions they denote are *contingently* true. Sentences have *empirical content*, which is the set of world/time pairs excluded by the propositions denoted by them. The more world/time pairs a sentence excludes, the greater empirical content it has, and thus the more informative it is. And the more we know, the more 'powerful' propositions we are in a position to assert, where a proposition is more powerful the fewer possible worlds and times it is true at. If we were empirically omniscient, we would be in a position to assert the most informative proposition true in the singleton set consisting only of the actual world and the present moment.

It is, however, readily seen that PWS does not solve the problem of the informativeness of *analytically* true sentences. Analytically true sentences denote the necessary proposition *TRUE*, which takes the truth-value **T** in all possible worlds at all times, and so these sentences come out uninformative from the point of view of PWS.[115]

The conclusion of a valid argument is true in a superset of the set of world/time pairs at which all the premises are true. Equivalently, the set of world/time pairs excluded by the conclusion is a subset of the set of world/time pairs excluded by the premises. In this sense it is true that the empirical semantic content of the conclusion of a valid argument *is contained* in the premises, which explains why we do not gain any *novel* piece of *empirical information* by validly inferring the conclusion of a sound argument.

Moreover, mathematical sentences are true or false in all possible worlds and at all times, so possible worlds and times are, strictly speaking, out of place here. Rather, mathematical sentences are true or false *independently* of worlds and times. Frege touched upon the topic of the informational value of mathematical sentences in 1892a when he considered the case of a triangle's medians: the mathematical sentence that the point of intersection of two pairs of a triangle's medians is one and the same point is informative, unlike the sentence that the point is self-identical.[116] Yet both sentences are equivalent by denoting the truth-

[115] The other extreme would be analytically untrue sentences, which exclude all worlds/times. They denote impossible propositions or, in the mathematical case, they either do not denote anything or denote the truth-value **F**. This is the counterpart of the 'scandal of deduction' put forward in Hintikka (1970), which is the Bar-Hillel/Carnap paradox of contradictory sentences. Since a contradictory sentence denotes a proposition that excludes *all* possible worlds (and all times), it should be the most informative one possible. Yet, we are not going to deal with this problem; since knowledge presupposes truth, we presume that the sentence has to be true in order to provide useful information.

[116] This example is the first Frege gives in (1892a), prior to setting out his heralded Hesperus/Phosphorus example. While the latter may lend itself to a solution within PWS, the former does not. This is already reason enough not to explicate Frege's intuitive notion of *Sinn* in terms of PWS intensions across the board.

value **T**. From this point of view, mathematics turns out to be just about the so-called *great fact*.[117] For another example, we readily grant that

(i) ‘12 = 12’

is not at all informative, whereas we no less readily grant that

(ii) ‘7 + 5 = 9 + 3’

is informative. But why?

A similar problem arising along the same line of reasoning is the question whether analytically equivalent sentences yield equal semantic information. Analytically equivalent sentences denote the same PWS proposition and thus exclude the same set of world/time pairs. Does this mean that they are equally informative? PWS says they are. However, wouldn’t we say that, for instance, the sentence ‘Prague is the capital of the Czech Republic’ is less informative than the sentence ‘Prague is the capital of the Czech Republic and no bachelor is married’? Surely we would (and should). Yet, since the proposition denoted by ‘no bachelor is married’ is the proposition *TRUE*, which excludes no world/time pairs, the two sentences exclude exactly the same set of such pairs. Thus PWS predicts them to be equally informative. So much the worse for PWS as a general theory of information, as many others have pointed out; see, e.g., Allo (2007) and Sequoiah-Grayson (2006) for recent statements of this objection to the crude individuation of informational content.

In this section we begin and end with the paradox of inference, since the paradox is what provoked us to probe into the principles underlying semantic information. Thus our narrow aim is to offer a principled solution to this paradox. The challenge is to explain how the validity and the utility of a deductive argument do not cancel one another out. The main goal is to present a solution based on a distinction between two kinds of information: *empirical (factual)* and *analytical*. The broader aim, however, is to offer a no less principled account of why analytic information is far from being trivial. This is to say that the semantic framework within which the paradox of inference is solved is not tailored to that paradox only: the framework is of much wider scope than that. The scope extends to explaining how and why analytically true sentences are informative, and why analytically equivalent sentences do not have to convey equal analytic information.

In the next subsection we first briefly recapitulate, for background, some classical theories of semantic information.

[117] See, e.g., Wagner (1986).

5.4.1 Empirical semantic information and 'the scandal of deduction'

The questions concerning the amount of informational yield of sentences are squarely *semantic* in kind. The classical attempt at answering them is Bar-Hillel and Carnap (1952), in which a theory of (*empirical*) *semantic information (ESI)* is presented. The guiding intuition is the same as outlined above: the more possibilities a proposition excludes, the more informative it is.[118] For instance, the conjunction $P \wedge Q$ excludes more possibilities than does the disjunction $P \vee Q$; therefore, a conjunction is more informative than a disjunction. Bar-Hillel and Carnap built *ESI* around a monadic predicate language. They conceived of possibilities as a set of so-called state descriptions, which is the set of atomic sentences that can be formulated in the language by applying primitive monadic predicates to individual constants.

Let W be a set of state descriptions, S a sentence. Then the semantic *content* of S is defined as the set of $w \in W$ that make S false: $Cont(S) =_{df} \{w \in W: w \models \neg S\}$. Obviously, for any logically/analytically true sentence T it holds that $Cont(T) = \varnothing$.

ESI is concerned not only with the individuation of empirical information content (*Cont*), but also with its *measure.* Since the set of individual constants and primitive monadic predicates of their language is finite, the set of state descriptions is finite as well. Thus the measure of the informativeness of sentences can be based on the probability of the states it describes as being the case. *ESI* defines two distinct methods of measuring, a *content measure (cont)* and an *information measure (inf).* Yet both the content measure and the information measure of an analytically true sentence amount to zero. Moreover, *ESI* predicts that analytically equivalent sentences convey the same information, and that the informational content of the conclusion of a valid argument is contained in the informational content of the premises. In sum, these are all the wrong answers to the questions posed by the paradox of inference.

Hintikka (1970) characterises the failure of *ESI* to provide an account of the information yield of deductive inferences as the *scandal of deduction.* He makes an attempt to obtain a measure of the information yield by distinguishing between what he calls *depth information* and *surface information.* However, Sequoiah-Grayson shows in 2008 that Hintikka's attempt fails. It fails primarily because his method, based on the concept of distributive normal form, applies only to a restricted set of deductions in the polyadic predicate calculus, and so fails to apply at all to the deductions in the monadic predicate calculus and the propositional calculus. Sequoiah-Grayson says:

> The consequence is that the problem of obtaining a measure of the information yield of deductive inferences remains an open one. The failure of Hintikka's proposal will suggest

[118] For a summary of Bar-Hillel and Carnap's theory, see also Duží (1992) and Sequoiah-Grayson (2006).

> that a purely syntactic approach to the problem be abandoned in favour of an intrinsically semantic one (2008, p. 67).

Another attempt to solve the problem is Sequoiah-Grayson (2006). There he investigates the measure of information along the lines of *ESI*, where particular doxastic states of agents (or, *situations*) rather than state descriptions or possible worlds are deployed. He specifies the basis of a theory of *psychological information* (*PI*) in which the frame semantics of substructural logic is invoked. By weakening the axioms of linear and relevant logics he specifies the theory of *PI* and provides it with a Kripke-style frame semantics. *Situations* serve the same purpose as do possible worlds, with the augmentation that they may be incomplete and/or inconsistent, because resource-bounded rational agents may entertain inconsistent beliefs. Impossible situations are logically impossible situations and correspond to confused epistemic states of agents. In such a model each agent is allowed to be confused in each their own way, without all such confused states being identified.

The result is a theory with a relevance semantics specified in terms of the information flow between the doxastic states of agents. The idea of plugging the basis of a semantic theory into doxastic states is not new, as Sequoiah-Grayson admits. For instance, Gärdenfors (1988) comes with belief sets, where these are sets of sentences closed under logical consequence. Gärdenfors thus presents a theory of idealized, logically omniscient agents. What is novel about Sequoiah-Grayson's theory is that it takes into account resource-bounded and fallible agents. The theory is still work in progress, though. On this point Sequoiah-Grayson says:

> [So] there exists a tension between the frame conditions it would appear that we would like, given a putative theory of *PI*, and those that we may get, given the constraints imposed by a theory of information flow. This remains an open problem (2006, p. 394).

However, in our opinion, such a theory will meet with similar problems as Bar-Hillel and Carnap's did. In principle, Sequoiah-Grayson's theory is set-theoretical. Thus within the theory we can hardly explain *what* an agent *a* learns when inferring a conclusion of a sound argument. True, having executed a valid inference *a* finds himself or herself in a new state, which may even be an 'impossible situation'. If *P* and *Q* are equivalent, *a* may believe that *P* without believing that *Q*. In this sense *PI* can distinguish equivalent sentences. However, unless we admit that what the agent learns is just a new piece of syntax, which is not a satisfactory solution, such a set-theoretical theory does not provide an answer to the question *what* the agent learns, and thus does not provide a satisfactory solution of the paradox of inference. Since we are going to show that equivalent sentences often convey different information, we are not going to pursue the tack outlined above. Our goal is to investigate the objective notion of analytic information conveyed by sentences, independently of whether a sentence is contained in the doxastic state of any agent.

As we have seen above, the basic principle of informational content adopted by Bar-Hillel and Carnap, in its most general formulation, is given by the inverse relationship principle:

> Whenever there is an increase in available information, there is a decrease in possibilities, and *vice versa* (Allo, 2007, p. 662).

This formulation gives rise to questions concerning the character of the excluded 'possibilities'. If we limit ourselves to Tarskian models, world-like entities or situations, we won't be in a position to answer the basic question posed by the paradox of inference, *What* do we learn when inferring the conclusion of a sound argument? We are going to show that we learn a new *procedure* the product of which is the proposition (or truth-value, in the case of mathematics) denoted by the conclusion. To put it in another way, we must primarily investigate the *procedure* yielding a proposition/truth-value as its *product*, and only secondarily the product itself. For this reason we need a *procedural* semantic framework such as TIL.

To adduce an example, the expressions '$3^2 - 2^2$' and '$(3 + 2) \times (3 - 2)$' do not have the same sense. They encode two different ways of constructing the number 5 in terms of two other numbers. The sense of '$3^2 - 2^2$' is the procedure that consists in the application of the square function to the number 3, application of the square function to the number 2, and subtraction of the result of the latter from the result of the former. The sense of '$(3 + 2) \times (3 - 2)$' is a different procedure. It is the procedure that consists in adding 3 and 2, subtracting 2 from 3, and multiplying the former by the latter. Thus, though the sentences '$3^2 - 2^2 = 5$' and '$(3 + 2) \times (3 - 2) = 5$' do not exclude any possible world or time and thus do not convey any factual information, they convey two distinct pools of analytical information by selecting two distinct constructions of the truth-value **T** from among the infinitely many possible ones, viz. $[^0\mathit{Minus}\ [^0\mathit{Square}\ ^03]\ [^0\mathit{Square}\ ^02]]$ and $[^0\mathit{Multiply}\ [^0\mathit{Plus}\ ^03\ ^02]\ [^0\mathit{Minus}\ ^03\ ^02]]$. Types. *Minus*, *Plus*/(τττ); *Square*/(ττ); 3, 2/τ.

We now turn to solving the paradox of inference and the related problem of the information value of analytically true sentences. The paradox of inference can be summarized thus:

(1.1) Valid arguments are uninformative,
because their conclusion is contained in the premises.
(1.2) Uninformative arguments are epistemically useless.

(1.3) Valid arguments are epistemically useless.

Since we intuitively reject the conclusion as false, the truth of at least one of the premises seems doubtful. While the second premise is trivially true, we are going to show that the first premise is false. The procedural semantics of TIL allows us to distinguish between *empirical* and *analytical* information. By the *empirical information* conveyed by a sentence *S* we mean the factual content of *S*, which is the set of world/time pairs excluded by *S*. By the *analytic content* of *S* we mean the set of procedural constituents of the meaning (construction) expressed by *S*. So we can reformulate the above argument thus:

(2.1) Valid arguments are *factually uninformative*, because the empirical information conveyed by the conclusion is contained in the empirical information conveyed by the premises.
(2.2) Factually *and analytically* uninformative arguments are epistemically useless.

(2.3) Valid arguments are epistemically useless.

This argument is invalid, thus blocking the derivation of the conclusion that valid arguments are epistemically useless. In what follows we show that some valid arguments are analytically informative and, therefore, epistemically useful.

A related problem that we are going to address as well is the problem of the *great fact* broached above. Since all true mathematical sentences converge in the truth-value **T**, mathematics is only about **T**, 'the great fact', which is obviously an unhappy outcome. Moreover, all analytically true sentences, even those containing empirical expressions, are factually uninformative. So an argument parallel to the one above summarizing the paradox of inference would be:

(3.1) Analytically true sentences are uninformative, because they do not exclude any possible world or time.
(3.2) Uninformative sentences are epistemically useless.

(3.3) Analytically true sentences are epistemically useless.

Again, we can reformulate the argument to bring out its invalidity:

(4.1) Analytically true sentences are *factually* uninformative, because they do not exclude any possible world or time.
(4.2) Factually *and analytically* uninformative sentences are useless.

(4.3) Analytically true sentences are useless.

In what follows we are going to show that analytically true sentences are useful because they have analytical content. In general, a (mathematical, analytical or empirical) sentence *S* reveals analytic information about the constituent steps to be executed in order to arrive at the entity (if any) that the meaning of *S*, i.e., a particular construction, constructs. It might seem, naïvely, that the analytical informativeness of a sentence would depend on the number of steps involved in the respective construction. However, since a shorter procedure may still be more informative than a longer one, we need some qualitatively better criteria. We are not going to propose an *absolute measure* of analytic information. Instead, we are going to examine some criteria for *comparing* the information yield of equivalent sentences. Thus in the next subsections we are going to apply the TIL apparatus to an investigation of the information conveyed by sentences.

5.4.2 Empirical content of sentences

As stated above, some sentences are factually informative thanks to their empirical content. We characterized the empirical content of a sentence S as the set of possible worlds and times excluded by S. Now we can put this characterisation on a more solid ground. Let $P/o_{\tau\omega}$ be the proposition denoted by a sentence S. Then the *empirical content of* S, $EC(S)$, is the set of $\langle w, t\rangle$-pairs at which P does not take the truth-value **T**, because ${}^0P_{wt}$ either v-constructs **F** or is v-improper.

Let T be an analytically true sentence. Then T denotes the proposition *TRUE*, which takes the value **T** at all $\langle w, t\rangle$-pairs, and $EC(T) = \varnothing$. Let T' be a contradictory sentence denoting an impossible proposition, i.e. a proposition that does not take **T** in any possible world or at any time.[119] Then $EC(T')$ is the set W of all $\langle w, t\rangle$-pairs. Thus neither T nor T' conveys any epistemically useful empirical information about the state of the world.

On the other hand, an empirical sentence S conveys non-trivial empirical information, because $EC(S)$ is non-empty and $EC(S) \subset W$. When we combine analytical and empirical sentences, we can compare the empirical content of the resulting sentences. For example, consider the following atomic sentences:

S: ‘Charles is a logician’,

M_1: ‘1+1 = 2’,

M_2: ‘1+1 = 3’

and the four compound sentences $S_1 - S_4$:

S_1: ‘Charles is a logician and 1+1 = 2’,

S_2: ‘Charles is a logician and 1+1 = 3’,

S_3: ‘Charles is a logician or 1+1 = 2’,

S_4: ‘Charles is a logician or 1+1 = 3’.

The sentences M_1 and M_2 are not empirically informative, because their empirical content is an empty set and the whole logical space, respectively. The empirical content of the compound sentences S_1 through S_4 is obtained by applying the

[119] We say ‘*a* proposition’, not ‘*the* proposition’, because due to partiality there are many impossible propositions, which differ by taking **F** or no value at different $\langle w, t\rangle$-pairs.

set-theoretic operators of union (∪) and intersection (∩) to the *EC* of the atomic sentences:

$$EC(S_1) = EC(S) \cup EC(M_1) = EC(S) \cup \varnothing = EC(S)$$

$$EC(S_2) = EC(S) \cup EC(M_2) = EC(S) \cup W = W$$

$$EC(S_3) = EC(S) \cap EC(M_1) = EC(S) \cap \varnothing = \varnothing$$

$$EC(S_4) = EC(S) \cap EC(M_2) = EC(S) \cap W = EC(S).$$

The conjunction of an analytically true sentence with an empirical sentence *S* does not increase the empirical content of *S* (the case of S_1). The conjunction of an analytically false sentence with an empirical sentence *S* voids the empirical content of *S* by yielding the whole logical space (the case of S_2). The disjunction of an empirical sentence *S* and an analytically true sentence also voids the empirical content of *S* by yielding an empty content (the case of S_3). Finally, the disjunction of an empirical sentence *S* and an analytically false sentence does not change the empirical content of *S* (the case of S_4).[120]

Nonetheless, we are going to explain below that even analytically true sentences convey information, because they have non-empty *analytical*, as opposed to *empirical*, content.

5.4.3 Analytical content of sentences

Analytically true sentences are true exclusively in virtue of their meaning, independently of states-of-affairs.[121] For example, the sentence 'Whales are not dolphins' contains the empirical predicates 'is a whale' and 'is a dolphin', yet the sentence is analytical. At no world/time are the properties *being a whale* and *being a dolphin* co-instantiated by the same individual. Mathematical sentences are analytical in the above sense: when evaluating their truth-values, possible worlds and times do not matter as points of evaluation.

Compare the compound Closure $\lambda x\ [^0{+}\ x\ ^01]$ and the atomic Trivialization ^{0}*Successor*. Though they are equivalent by constructing the same function *Successor*, of type (ττ), the former is more informative than the latter. ^{0}*Successor* is the instruction to select the successor function amongst all the other (ττ)-typed functions, and qualifies insofar as a fully-fledged instruction, though one with a serious

[120] A similar conception was advocated by Wittgenstein (1922, §4.46).

[121] Recall that we are not analysing a formal, non-interpreted language. Instead, we presuppose full understanding and full linguistic competence. Thus from our point of view, 'is analytically true' is not synonymous with 'is true on all interpretations'; rather, it is synonymous with 'is true with respect to all possible worlds and times' or 'is true independently of possible worlds and times'.

flaw: it fails to specify *how* to select that function.[122] The rival Closure $\lambda x\ [{}^0{+}\ x\ {}^0 1]$ offers a *how*. It conveys an interesting piece of information on how to obtain the mapping.[123] The Closure $\lambda x\ [{}^0{+}\ x\ {}^0 1]$ consists of the constituents ${}^0{+}$, x, ${}^0 1$, $[{}^0{+}\ x\ {}^0 1]$ and itself. ^{0}*Successor*, in contrast, consists only of itself. So there are no further procedures to decompose ^{0}*Successor* into.

To examine analytically true sentences involving empirical expressions, consider the sentence

(*W*) 'No whale is a dolphin'.

Its literal meaning is the Closure

(*) $\lambda w \lambda t\ [[{}^0 No\ {}^0 Whale_{wt}]\ {}^0 Dolphin_{wt}]$.

Types: *Whale*, *Dolphin*/$(o\iota)_{\tau\omega}$; *No*/$((o(o\iota))(o\iota))$: the restricted quantifier which, when applied to a set *S*, returns the set of all those sets that have an empty intersection with *S*.

The proposition constructed by (*) is the necessary proposition *TRUE*. However, the above analysis does not make it possible to *prove* it. We need to refine the analysis. To this end we make use of the fact that the property of being a whale can be *defined* as the property of being a marine mammal of the order *Cetacea* that is neither a dolphin nor a porpoise.[124] Thus the ontological definition of the property of being a whale is

$$\lambda w \lambda t\ \lambda x\ [[{}^0 Mammal_{wt}\ x] \wedge [{}^0 Marine_{wt}\ x] \wedge [{}^0 Cetacea_{wt}\ x] \wedge \neg[{}^0 Dolphin_{wt}\ x] \wedge \neg[{}^0 Porpoise_{wt}\ x]]$$

Types: $x \to \iota$; *Cetacea*, *Mammal*, *Marine*, *Dolphin*, *Porpoise*/$(o\iota)_{\tau\omega}$.

Substituting this definition for ^{0}*Whale* into (*) we get:

(**) $\lambda w \lambda t\ [{}^0 No\ \lambda x\ [[{}^0 Mammal_{wt}\ x] \wedge [{}^0 Marine_{wt}\ x] \wedge [{}^0 Cetacea_{wt}\ x] \wedge \neg[{}^0 Dolphin_{wt}\ x] \wedge \neg[{}^0 Porpoise_{wt}\ x]]\ {}^0 Dolphin_{wt}]$.

Gloss: 'No individual *x* such that *x* is a marine mammal of the order *Cetacea* and *x* is neither a dolphin nor a porpoise is a dolphin'.

Remark. Properly speaking, the literal meaning of the predicate 'is a marine mammal which is neither a dolphin nor a porpoise' is the Closure

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[122] Moreover, the atomic construction ^{0}*Successor* is a non-executable, ineffective procedure. The execution of this procedure amounts to taking the infinite mapping *Successor* and delivering it as a result. Cf. the discussion of ${}^0\pi$ in Section 3.2.1.

[123] It is an easily executable procedure, though one that may potentially be executed infinitely many times. For details, see Duží and Materna (2004).

[124] See, for instance, http://mmc.gov/species/speciesglobal.html#cetaceans or http://www.crru.org.uk/education/factfiles/taxonomy.htm

$$\lambda w \lambda t\ [\lambda x\ [[{}^0Marine^m\ {}^0Mammal]_{wt}\, x] \wedge [{}^0Cetacea_{wt}\, x] \wedge \neg[{}^0Dolphin_{wt}\, x] \wedge \neg[{}^0Porpoise_{wt}\, x]]$$

where $Marine^m/((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ is a subsective modifier: necessarily, if a is a marine mammal then a is a mammal. In (**) we applied the rule of pseudo-detachment.[125]

The sentence encoding (**) is obviously analytically, *ex definitione*, true. It is also logically true, because the corresponding logical form

$$\lambda w \lambda t\ [{}^0No\ \lambda x\ [[M_{wt}\, x] \wedge [Mar_{wt}\, x] \wedge [C_{wt}\, x] \wedge \neg[D_{wt}\, x] \wedge \neg[P_{wt}\, x]]\ D_{wt}]$$

v-constructs *TRUE* for any valuation v; the set v-constructed by D_{wt} is disjoint with the set v-constructed by $\lambda x\ [[M_{wt}\, x] \wedge [Mar_{wt}\, x] \wedge [C_{wt}\, x] \wedge \neg[D_{wt}\, x] \wedge \neg[P_{wt}\, x]]$ for any valuation v of the variables $M, Mar, C, D, P \rightarrow (o\iota)_{\tau\omega}$.

Still, to prove it we need a finer analysis that makes use of the definition of the restricted quantifier *No*. It is a function that operates on sets of individuals defined as follows. Let variables $m, n/*_1 \rightarrow_v (o\iota)$; then the definition of *No* is this:

$${}^0No = \lambda m\, n\ [\neg\exists x\ [[m\, x] \wedge [n\, x]]].$$

Now by using this definition we obtain:

$$[[{}^0No\ {}^0Whale_{wt}]\ {}^0Dolphin_{wt}] =$$

$$[{}^0No\ \lambda x\ [[{}^0Mammal_{wt}\, x] \wedge [{}^0Marine_{wt}\, x] \wedge [{}^0Cetacea_{wt}\, x] \wedge \neg[{}^0Dolphin_{wt}\, x] \wedge \neg[{}^0Porpoise_{wt}\, x]]\ {}^0Dolphin_{wt}] =$$

$$\neg\exists x\ [[[{}^0Mammal_{wt}\, x] \wedge [{}^0Marine_{wt}\, x] \wedge [{}^0Cetacea_{wt}\, x] \wedge \neg[{}^0Dolphin_{wt}\, x] \wedge \neg[{}^0Porpoise_{wt}\, x]] \wedge [{}^0Dolphin_{wt}\, x]].$$

Since this last construction obviously and provably v-constructs **T** for any valuation v, we can generalize to

$$\forall w \forall t\ \neg\exists x\ [[[{}^0Mammal_{wt}\, x] \wedge [{}^0Marine_{wt}\, x] \wedge [{}^0Cetacea_{wt}\, x] \wedge \neg[{}^0Dolphin_{wt}\, x] \wedge \neg[{}^0Porpoise_{wt}\, x]] \wedge [{}^0Mammal_{wt}\, x]].$$

Substituting this construction for the Composition $[[{}^0No\ {}^0Whale_{wt}]\ {}^0Dolphin_{wt}]$ into (*), we obtain

$$(***)\qquad \lambda w \lambda t\ [\forall w \forall t\ \neg\exists x\ [[[{}^0Mammal_{wt}\, x] \wedge [{}^0Marine_{wt}\, x] \wedge [{}^0Cetacea_{wt}\, x] \wedge \neg[{}^0Dolphin_{wt}\, x] \wedge \neg[{}^0Porpoise_{wt}\, x]] \wedge [{}^0Mammal_{wt}\, x]]].$$

We have proven that the sentence 'No whale is a dolphin' denotes the proposition *TRUE.*

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[125] See Section 4.4.

This example illustrates the method of refining a construction by replacing a constituent of it by an equivalent one so that the resulting construction reveals a provable way to the product. But now, *which* of the equivalent constructions (*), (**), (***) should be assigned to the sentence (*W*) as its meaning? The method of semantic analysis introduced in Section 2.1 heeds the Parmenides principle. It yields a construction *C* (assigned to *E* as its meaning) such that every closed subconstruction of *C* constructs an object mentioned by *E*; i.e., an object denoted by a subexpression of *E*. Hence the construction $\lambda w \lambda t\ [[^0 No\ ^0 Whale_{wt}]\ ^0 Dolphin_{wt}]$ is the literal meaning of the sentence 'No whale is a dolphin', while this Closure is not:

$$\lambda w \lambda t\ [[[^0 No\ ^0 Whale_{wt}]\ ^0 Dolphin_{wt}] \wedge [^0{<}\ ^0 2\ ^0 5]].$$

Though both constructions construct one and the same proposition *True*, the latter contains subconstructions constructing objects that do not receive mention in the sentence (viz., the numbers 2, 5 and the relation <). This is also why we adhere to the Parmenides principle. An adequate analysis of an expression *E* contains constructions of all and only objects that receive explicit mention in *E*.

The construction (**) might be an adequate analysis of the sentence (*W*), provided we assigned the above ontological definition of the property of being a whale to 'is a whale'. Then the meaning of the syntactically simple expression 'is a whale' would not be $^0 Whale$ but the compound Closure $\lambda w \lambda t\ \lambda x\ [[^0 Mammal_{wt}\ x] \wedge [^0 Marine_{wt}\ x] \wedge \neg[^0 Dolphin_{wt}\ x] \wedge \neg[^0 Porpoise_{wt}\ x]]$. Yet one might object that (*W*) does not explicitly mention the properties of being a marine mammal, a dolphin and a porpoise. Thus on its literal reading, which we prefer, (**) is the analysis of another sentence; namely, this one:

(*W′*) 'No individual *x* such that *x* is a marine mammal of the order *Cetacea* and *x* is neither a dolphin nor a porpoise, is a dolphin.'

Similarly, (***) is the analysis of this sentence:

(*W″*) 'Necessarily, there is no individual *x* such that *x* is a marine mammal of the order *Cetacea* that is neither a dolphin nor a porpoise and *x* is a dolphin.'

It is readily seen that (*W′*) is more informative than (*W*), and (*W″*) is more informative than (*W′*). Due to the more detailed specification of the property of being a whale, the logical validity of (*W′*) and (*W″*) is obvious. We shall say that (*W″*) has greater analytical content than (*W′*), and that (*W′*) has greater analytical content than (*W*).

Recalling our problem of the analytical information conveyed by sentences, the question arises, What do we learn when knowing that (*W*), (*W′*) and (*W″*), respectively? We do not learn anything about the state of the world, for sure. It is not the proposition *True* that we get to know, because nobody with finite, human-like capacities can survey all the elements of an uncountably infinite set such as the *True* mapping. It would amount to knowing an actual infinity, which is beyond our finite capacities.

Yet we have the capacity to *potentially* know infinities. In *any* world at *any* time, following the instructions presented by (W), (W') or (W'') amounts to arriving at the truth-value **T**. Understanding these sentences amounts to knowing three different constructions detailing how to arrive, in any given world at any given time, at **T**.

For an analogy, imagine you need to go from place A to place B. You consult a map and figure out an itinerary from A to B via X. Did you learn something? Surely you did. You obtained a piece of information to the effect that the mapping $\{A\} \rightarrow \{B\}$ can be realised by composing the mappings $\{A\} \rightarrow \{X\}$ and $\{X\} \rightarrow \{B\}$. Now you start from A moving toward X but, alas, the stretch around X is closed due to the area being flooded. You conduct a search on traffic web sites and discover a new itinerary from A to B via Y and Z. As a result, you follow the instruction to go from A to B via Y and Z, realising the mapping $\{A\} \rightarrow \{B\}$ by composing the mappings $\{A\} \rightarrow \{Y\}$, $\{Y\} \rightarrow \{Z\}$ and $\{Z\} \rightarrow \{B\}$. Wasn't it again a useful piece of new information you learnt? Surely it was. Yet from the set-theoretical point of view, in all three cases the result comes out the same, *viz*. the mapping $\{A\} \rightarrow \{B\}$. TIL constructions are such itineraries, specifying a route to an output given some input entities.[126] In a broader sense, not only a declarative sentence but any meaningful expression conveys some itinerary.

Thus we define:

Definition 5.4 (*analytical information, analytical content*) The *analytical information conveyed by an expression E* is the literal meaning of *E. The analytical content of an expression E* is the set of constituents of the literal meaning of *E*. □

Remark. Given a construction, the set of its *content* can be constructed. This set is emphatically not to be confused with the very construction itself. A construction is a procedure, and as such structured and complex, whereas a set (also when construed as a characteristic function, as in TIL) lacks structure and is therefore simple. A set cannot figure as a procedure, but at most as input or output of a procedure. For instance, the analytical content of the sentence 'All whales are mammals' is the set $\{\lambda w \lambda t\ [[{}^0All\ {}^0Whale_{wt}]\ {}^0Mammal_{wt}],\ [[{}^0All\ {}^0Whale_{wt}]\ {}^0Mammal_{wt}],\ [{}^0All\ {}^0Whale_{wt}],\ {}^0Whale_{wt},\ [{}^0Whale\ w],\ {}^0Whale,\ {}^0Mammal_{wt},\ [{}^0Mammal\ w],\ {}^0Mammal,\ {}^0All,\ w,\ t\}$. Each of the elements of this set is a procedure that must be executed in order to execute the first constituent. However, this very set cannot be executed. It is not a procedure.

As stated in Section 1.5.1.1 and as the whale example illustrates, any logically true sentence is analytically true, whereas the converse does not hold. Moreover, it seems at first blush that a logically true sentence would be more informative than

[126] Some of these routes are 'roads to nowhere' in the form of improper constructions, which are procedures that fail to deliver a product but are no less procedures for that.

the corresponding equivalent, analytically true sentence in virtue of its greater analytical content. Yet the situation is not that simple. Definition 5.4, together with the definitions found in Section 1.5.1.1, in particular Definition 1.10 of the literal meaning of an expression, provides us with the logical machinery required to compare the degree of informational value of analytically/logically true sentences, or in general of analytically equivalent sentences. This we are going to do below.

5.4.4 Information content of analytically equivalent sentences

In Section 5.4.1 we showed that classical theories of empirical semantic information fail to distinguish between the information content of analytically equivalent sentences. This is so because analytically equivalent sentences share the same empirical content. Yet, as we demonstrated above, analytically equivalent sentences do not necessarily share the same analytical content.

It might seem that sentences A, B are analytically equivalent if A entails B and *vice versa.* However, co-entailment is only a necessary but not sufficient condition for equivalency. Here is why. Trivially, a valid argument is *truth-preserving from premises to conclusion*. However, due to partiality, the entailment relation may fail to be *falsity-preserving from conclusion to premises*. As a consequence, even if $A \models B$ and $B \models A$ it may still happen that A, B are not analytically equivalent. If A, B are propositional constructions then the propositions they construct may not be identical, though they take the truth-value **T** at exactly the same worlds/times.

For an example of co-entailing constructions that are not equivalent, consider Russell's elimination of definite descriptions discussed in Section 3.1. The Russellian rephrasing of

S_1: 'The President of the Czech Republic is an economist'

would be S_2:

S_2: 'There is a unique individual such that (s)he is the President of the Czech Republic and (s)he is an economist.'[127]

Literal analyses of these sentences come down to

S_1': $\lambda w \lambda t\ [{}^0Economist_{wt}\ {}^0PresCR_{wt}]$

and

S_2': $\lambda w \lambda t\ [\exists x\ [[{}^0PresCR_{wt} = x] \wedge [{}^0Economist_{wt}\ x]]]$.

[127] For present purposes, it is not necessary to explicitly specify the uniqueness of the Czech Presidency, as it is given by the meaning of the expression 'The President of the Czech Republic', and thus by the type of the denoted entity. Native Czech speakers (such as ⅔ of the authors of this book) know that 'President České republiky' means that at most one person at a time gets to be President of the Czech Republic.

It holds that $S_1' \models S_2'$ and $S_2' \models S_1'$, but the two are *not equivalent*. If S_1 is true then S_2 is true, and vice versa. For instance, in the actual world before 1992, S_1 had *no* truth-value (because there was no such thing as the presidency of the Czech Republic) whereas S_2 was just *false*. Moreover, from S_1, as well as from its negation, that the President of the Czech Republic is *not* an economist, we can validly infer that the President of the Czech Republic exists. Not so with S_2.[128]

Therefore, when comparing the informational value of sentences, we cannot rely on co-entailment in order to determine whether the sentences are analytically equivalent. Instead what we need to do is examine their meanings, and compare their analytical content.

Now we are going to put forward some criteria for comparing the informative value of analytically equivalent sentences. First, we want to define exactly how fine-grained our individuation of analytical information is. So we need to lay down when two sentences are informationally indistinguishable. Let the analytical content of a sentence S be denoted by '$AC(S)$'. Obviously, the following condition is valid:

If $AC(S_1) = AC(S_2)$
then the sentences S_1 and S_2 are *equally analytically informative*.

Actually, if S_1 and S_2 have the same analytical content, then S_1, S_2 have the same meaning. For example, the sentences '5 is a prime number', 'Five is a prime number', and 'Fünf ist eine Primzahl' are synonymous. They have the same meaning and thus the same analytical content; hence, they are informationally indistinguishable. The reason is because the literal meaning of all these sentences is $[{}^0Prime\ {}^05]$. The Trivialization ${}^0Prime$ is identical with the Trivialization ${}^0Primzahl$, because what is Trivialized is the *set* of primes, independently of the various (English or German or whatever other) name we use to denote this set.

However, having identical analytical content is a sufficient, though not necessary, condition for two expressions to be synonymous. As we showed in Section 2.2, the definition of synonymy has to be slightly weakened: expressions E_1, E_2 are *synonymous* iff their literal meanings are *procedurally isomorphic*. Procedural isomorphism was defined in Definition 2.3 as the transitive closure of α- and η-equivalence. For instance, these constructions are procedurally isomorphic
$(x, y, z \rightarrow \iota)$:

$$
{}^0Whale,\ \lambda w\lambda t\ \lambda x\ [{}^0Whale_{wt}\ x],\ \lambda w\lambda t\ \lambda y\ [{}^0Whale_{wt}\ y],\\
\lambda w\lambda t\ \lambda z\ [\lambda x\ [{}^0Whale_{wt}\ x]\ z],\ \dots,
$$

while

[128] In other words, S_1 comes attached with the existential presupposition that the President of the Czech Republic exist. This is due to the fact that ${}^0PresCR$ occurs with supposition *de re* in S_1'. This is not true of S_2': ${}^0PresCR$ occurs here intensionally, in the ι-generic intensional context of the Composition $[{}^0\exists\ \lambda x\ [[{}^0PresCR_{wt} = x] \wedge [{}^0Economist_{wt}\ x]]]$. See Section 2.6 for details.

$$\lambda w \lambda t\, \lambda x\, [[{}^0 Mammal_{wt}\, x] \wedge [{}^0 Marine_{wt}\, x] \wedge [{}^0 Cetacea_{wt}\, x] \wedge \neg[{}^0 Dolphin_{wt}\, x] \wedge \neg[{}^0 Porpoise_{wt}\, x]]$$

is only equivalent to all of them.

Procedural isomorphism gives rise to this principle:

> The sentences S_1, S_2 are *synonymous*, and thus *convey the same analytical information*, iff their literal meanings are procedurally isomorphic.

Since we want to compare the analytical informational value of sentences that are equivalent but not synonymous (which would be trivial), we might consider comparing the *number* of constituents contained in the respective analytic contents. If the number of constituents of the literal meaning of a sentence S_1 is greater than the number of constituents of a sentence S_2, we will write '$|AC(S_1)| > |AC(S_2)|$'. However, a simple criterion based on the number of steps is impossible. The mere number of steps is insufficient to define the (relative) degree of information, since some steps do not contribute to the informational value of sentences, while others are incomparable. For one thing, how would we compare the empirical content of sentences with their analytical content?

It might seem that if $AC(S_1) \subset AC(S_2)$ then S_1 would be less analytically informative than S_2. Again, a moment's reflection reveals that this is not so. This is due to the fact that the analytical content of a sentence is construed as the *set* of its meaning constituents. As pointed out above, while constructions are procedures and as such structured and complex, sets lack structure. Consider, for instance, sentences of the form 'p' and 'p or q'. Though the analytical content of the former is a (proper) subset of the latter, 'p or q' is less informative than 'p'. In general, if we know that p then the degree of our uncertainty decreases to a greater extent than when knowing that p or q. A criterion given by simple set-theoretic inclusion is *not* plausible, either.

The first criterion (AC), which we are going to formulate now, is based on the fact that a conjunctive extension of the analytical content of a sentence by an analytically true subclause results in an equivalent, but more analytically informative, sentence. For instance, 'Whales are mammals' is less analytically informative than 'Whales are mammals belonging to the order *Cetacea*', or, 'Whales are mammals *and* the problem of logical validity is not decidable within first-order predicate logic'. Thus we formulate the criterion (AC):

(AC) Let sentences S_1, S_2 be analytically equivalent, and let $AC(S_1) \subset AC(S_2)$. If S_2 is of a form equivalent to 'S_1 and T', for some non-contradictory sentence T, then S_2 is analytically more informative than S_1.

This simple criterion is, however, not applicable in the case of equivalencies obtained by refining the meaning, like the above case of 'No whale is a dolphin'. The analytical contents of (W), (W') and (W'') are not comparable by set-theoretical inclusion. We need a structurally qualitative criterion. Intuitively, the analytical information increases from (W) to (W''). This is because one constituent of (W) or (W') has been replaced by a more complex, but still equivalent, constituent.[129]

Recall the example of sentences T_1, T_2 and T_3 from Section 1.5.1.1 and their respective literal analyses:

T_1 'If 2 < 5 and 5 < 11 then 2 < 11'

T_1' $[[[^0\!< {}^02\ {}^05] \wedge [^0\!< {}^05\ {}^011]] \supset [^0\!< {}^02\ {}^011]]$;

T_2 'If 2 < 5, 5 < 11 and the relation < is transitive, then 2 < 11'

T_2' $[[[^0\!< {}^02\ {}^05] \wedge [^0\!< {}^05\ {}^011] \wedge [^0\mathit{Transitive}\ {}^0\!<]] \supset [^0\!< {}^02\ {}^011]]$;

T_3 'If 2 < 5 and 5 < 11 and if $\forall x \forall y \forall z\ (x < y \supset (y < z \supset x < z))$ then 2 < 11'

T_3' $[[[^0\!< {}^02\ {}^05] \wedge [^0\!< {}^05\ {}^011] \wedge [\lambda r\ \forall x \forall y \forall z\ [[r\ x\ y] \supset [[r\ y\ z] \supset [r\ x\ z]]]\ {}^0\!<]] \supset [^0\!< {}^02\ {}^011]]$.

Types: $x, y, z \to \tau$; $r \to (o\tau\tau)$; the other types are obvious.

The analytical information conveyed by these sentences increases from T_1 to T_3. The logically true sentence T_3 provides, thanks to its meaning, such detailed instructions on how to construct a truth-value that its truth is easily provable; and it is readily seen that it provides the greatest analytical information of the three. Sentences T_1 and T_2 are comparable *via* the criterion (AC). The analytical content of T_1 is a proper subset of the analytical content of T_2, and the extension of $AC(T_1)$ to $AC(T_2)$ does not consist in the irrelevant addition of an analytically true disjunct. Yet (AC) is not applicable to T_2 and T_3. Their analytical contents differ only in that the simple concept $^0\mathit{Transitive}$ occurring as a constituent of T_2 has been replaced by a complex definition of the set of transitive binary relations; viz., by the Closure $\lambda r\ \forall x \forall y \forall z\ [[r\ x\ y] \supset [[r\ y\ z] \supset [r\ x\ z]]]$. We say that the meaning of T_3 is a *refinement* of the meaning of T_2.

In order to render the increase in information yield such as that from T_2 to T_3, or from (W) to (W''), we are now going to define a qualitative criterion based on the *refinement of the meaning* of sentences.

Definition 5.5 (*refinement of a construction*) Let C_1, C_2, C_3 be constructions. Let ^0X be a simple concept of X, and let ^0X occur as a constituent of C_1. If C_2 differs from C_1 only by containing in lieu of ^0X an ontological definition of X, then C_2 is a *refinement of* C_1. If C_3 is a *refinement* of C_2 and C_2 is a *refinement* of C_1, then C_3 is a *refinement of* C_1.

Corollaries. If C_2 is a refinement of C_1, then

[129] For this reason (W), (W') and (W'') are equivalent. See the intensional rule of substitution in Section 2.7.1.

(1) C_1, C_2 are equivalent ($C_1 \approx C_2$) but *not* procedurally isomorphic
(2) $AC(C_1)$ is not a subset of $AC(C_2)$
(3) $|AC(C_2)| > |AC(C_1)|$.

Recall that, according to Definition 1.5, C_1, C_2 are equivalent if either C_1, C_2 v-construct the same entity for all valuations v, or C_1, C_2 are v-improper for all valuations v. Obviously, if C_1, C_2 are procedurally isomorphic, then they are equivalent, but not vice versa. For example, the Closure $\lambda x\ [{}^0Card\ \lambda y\ [{}^0Divide\ y\ x] = {}^02]$ is a refinement of the atomic construction ${}^0Prime$. However, the Closure $\lambda x\ [{}^0Prime\ x]$ is not a refinement of ${}^0Prime$, these two constructions being procedurally isomorphic.

The involved types are: ν, the type of natural numbers; $Card/(\nu(o\nu))$: the cardinality of a set of natural numbers; $Divide/(o\nu\nu)$: the relation of x being divisible by y; the other types are obvious.

There can be more than one refinement of a construction C. As was already pointed out, the Trivialization ${}^0Prime$ is in fact the least informative procedure for producing the set of primes. Using particular definitions of the set of primes, we can refine ${}^0Prime$ in many ways, including:

$$\lambda x\ [{}^0Card\ \lambda y\ [{}^0Divide\ y\ x] = {}^02],$$

$$\lambda x\ [[x \neq {}^01] \wedge \forall y\ [[{}^0Divide\ y\ x] \supset [[y = {}^01] \vee [y = x]]]],$$

$$\lambda x\ [[x > {}^01] \wedge \neg\exists y\ [[y > {}^01] \wedge [y < x] \wedge [{}^0Divide\ y\ x]].$$

When having two analytically equivalent sentences, if the meaning of one is a refinement of the meaning of the other, then the former is more informative than the latter. Thus we formulate the following criterion based on *analytical refinement*, (*AR*):

(*AR*) Let S_1 and S_2 be sentences with literal meanings CS_1, CS_2, respectively, such that CS_2 is a refinement of CS_1. Then CS_2 *is more analytically informative than* CS_1 ($CS_2 >_{an} CS_1$). If $CS_2 >_{an} CS_1$ then we also say that sentence S_2 *is more informative than* sentence S_1.

If applying (*AR*) to sentences T_2, T_3, it is easily seen that $T_3' >_{an} T_2'$, hence T_3 is more informative than T_2.

By refining the meaning C_S of a sentence S we uncover a more fine-grained construction C_S' such that C_S and C_S' are equivalent, yet not procedurally isomorphic, and such that the latter is more informative than the former. The relation $>_{an}$ defined by (*AR*) is transitive. But in principle, we could keep refining one and the same construction *ad infinitum*, possibly criss-crossing between various conceptual systems.[130] For instance, we could still refine the definitions above of the set of primes by refining the Trivialization ${}^0Divide$:

[130] See Section 2.2.3.

$$^0Divide = \lambda yx\ [\exists z\ [x = [^0Mult\ yz]]].$$

Types: $x, y, z \to \nu$; $Mult/(\nu\nu\nu)$: the function of multiplication defined over the domain of natural numbers ν.

Substituting the Closure for the Trivialization yields a more informative refinement:

$$^0Prime <_{an} [\lambda x\ [^0Card\ \lambda y\ [^0Divide\ y\ x] = {}^02]] <_{an}$$
$$[\lambda x\ [^0Card\ \lambda y\ [\exists z\ [x = [^0Mult\ yz]]] = {}^02]] <_{an} \ldots$$

The uppermost level of refinement depends on the *conceptual system* in use. Recall Definition 2.14: a conceptual system is determined by the set of its *primitive* concepts, which is the set of Trivializations of entities of a type of order 1 (non-constructions). The *derived* concepts of a conceptual system are then the compound closed constructions that can be obtained by using the primitive concepts and variables of appropriate types. It has been proven that the set of refinements of a construction C obtainable within a given conceptual system forms a complete lattice with respect to the partial order defined as follows:
$C_1 \leq C_2$ iff $C_1 <_{an} C_2$ or C_1, C_2 are procedurally isomorphic.[131]

The criteria (*AC*) and (*AR*) enable us to *compare* the analytical informational value of sentences such as

(S_i) ‘5 is a prime’

(S_{ii}) ‘5 is a number with exactly two factors’

(S_{iii}) ‘5 is a number with exactly two factors and 1 < 2’

(S_{iv}) ‘5 is a number with exactly two factors or the problem of logical validity is decidable in first-order predicate logic’

(S_v) ‘5 is a number not equal to the number 1 divisible only by 1 and itself’

in the following manner:

- (S_i) is less informative than (S_{ii}) according to criterion (*AR*).
- (S_{ii}) is less informative than (S_{iii}) according to criterion (*AC*).
- (S_{ii}) and (S_{iv}) are not comparable according to criteria (*AR*), (*AC*).
- (S_{iii}) and (S_{iv}) are not comparable according to (*AR*), (*AC*) for the same reasons as above.
- (S_i) is less informative than (S_v) according to (*AR*).
- None of (S_{ii}), (S_{iii}), (S_{iv}) are comparable with (S_v) *via* (*AR*), (*AC*).

[131] See Materna and Duží (2005).

5.4.5 The information value of a valid argument

Now we are going to apply the above method of exploring analytical informational value to the case of valid arguments as well. As explained at the outset of this section, every argument can be transformed into a corresponding conditional sentence (via the rule of implication introduction). A deductively valid argument gets transformed into an analytically true sentence. In the case of a mathematical argument, the resulting implicative sentence denotes the truth-value **T**. In the case of an empirical argument, the implicative sentence denotes the proposition *TRUE*.

The notions of analytically and logically valid argument, respectively, were defined in Section 1.5.1.1, Definition 1.13. An argument is analytically/logically valid if the corresponding implicative sentence is analytically/logically true. In both cases the implicative sentence lacks empirical content; however, it does have non-empty analytical content, and thus conveys some analytical information.

5.4.5.1 The paradox of inference

At the outset of this section we raised the question posed by the paradox of inference concerning how deductive logic can function as a useful epistemological tool. Here is an outline of how the paradox is engendered and how we solve it. [132]

If an argument A is valid then, necessarily, the set W_P of states-of-affairs $\langle w, t\rangle$ at which all the premises are true is a subset of the set W_C of states-of-affairs at which the conclusion is true ($W_P \subseteq W_C$). Hence the conclusion excludes a subset of the set of states-of-affairs $\langle w, t\rangle$ excluded by the set of premises. In other words, the conclusion of a valid argument is *redundant* with respect to the *empirical information* conveyed by the set of premises. It is in this sense that the conclusion may be said to be *contained* in the premises. Thus, by inferring S from $S_1,\ldots,S_n$ one cannot acquire any new empirical information.

One could barely hope to offer a solution to this paradox if one's method were truth-conditional, which is to say essentially set-theoretical. As soon as one accepts a procedural semantics such as the one found in TIL, however, the informational value of a valid argument is preserved. The pivotal point is the procedure/product bifurcation. A *construction* of the conclusion may not occur in the premises; if it does not we have to discover it, and the construction we discover is new to us, hence epistemically useful and non-trivial. What happens is that we acquire a novel piece of *analytic information* about a particular *way of constructing* the proposition/truth-value denoted by the conclusion.

Thus we may paraphrase Cohen and Nagel's paradox along these lines:

If in an argument the *empirical content* of the conclusion
is not contained in the *empirical content* of the premises, it cannot be valid.

[132] This subsection draws in part on material appearing in Duží (2010).

If the conclusion is not different from any of the premises,
it is epistemically useless.
The conclusion cannot be such that its *empirical content* is contained in the *empirical content* of the premises and it also possesses novelty.

Arguments cannot be both valid and epistemically useful.

This argument is valid but not sound. The third premise is not true. To restate the point, though the empirical content of the conclusion of a valid argument is contained in the empirical content of its premises, the *analytical content* of the conclusion need not be so contained. And if it is not, then the literal meaning of the conclusion is a novelty.

On the other hand, if the literal meaning of the conclusion does occur among the premises—i.e., if the literal meaning of the conclusion is procedurally isomorphic with the literal meaning of some premise(s)—then the argument is trivial in the sense of offering no new analytic information. Such an argument is circular.

To adduce two simple examples of what we have in mind, the argument

Anna wears a sky-blue blouse

Anna wears an azure blouse

is trivial. The literal meaning of the premise and the conclusion is the same, provided 'is sky-blue' and 'is azure' are synonymous predicates.

On the other hand, the argument

Charles is a bachelor

Charles is an unmarried man

is not trivial. The literal meaning of the conclusion is a refinement of the literal meaning of the premise, and so the conclusion is more informative than the premise. Thus, though the conclusion denotes the same proposition as does the premise, it provides a new (and more informative) construction of this proposition.

By way of conclusion, the powerful framework provided by the procedural semantics of TIL is suitable for dealing with the problems posed by the paradox of inference. Whereas traditional set-theoretic theories of semantics cannot give a satisfactory account of these problems, we showed *what* one learns when validly inferring a conclusion from true premises. While the *product* of the procedure assigned to the conclusion as its meaning—namely, the proposition (in the empirical case) or the truth-value (in the mathematical case) denoted by the conclusion—*is* informationally contained in the premises, the *procedure* itself need *not* be (namely, whenever the argument is non-circular). Provided it is not contained, then what we have learnt is a new procedure producing the relevant proposition/truth-value.

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M. Duží et al., *Procedural Semantics for Hyperintensional Logic*, Logic, Epistemology, and the Unity of Science 17, DOI 10.1007/978-90-481-8812-3,

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