STUDIES IN *FUZZINESS*
AND *SOFT COMPUTING*

Elisabeth Rakus-Andersson

Fuzzy and Rough Techniques in Medical Diagnosis and Medication

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Fuzzy and Rough Techniques in Medical Diagnosis and Medication

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To Irena and Christer

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1 Introduction

In the late eighties of the twentieth century I encountered a paper referring to mathematics dealing with imprecision applied to medical diagnosis. I was working in the area of medical statistics at that time, and obviously I was interested in the contents of the paper. After I had read it I became fascinated by new possibilities of medical data interpretation and processing proposed by the author. The new world of fuzziness, originated by Professor Lotfi Zadeh, seemed to open up to me and I started reading all accessible material about fuzzy set theory.

Many years have passed since then. We are now living in the information society and we do not experience troubles in reaching scientific material. Lately I have read many papers and books about treating medical tasks solved by using fuzzy ideas. I am still keen on tracing applications in medicine, and by myself I have been contributing to some concepts in this subject that has motivated me to prepare my own book. The objective of writing such a book has been a little particular, namely, I have intended to present the subject of fuzzy tools and techniques in medicine for eventual users. These, maybe representatives of medical or pharmacological staffs, are not expected to possess a large amount of mathematical knowledge. On the contrary, we have a feeling that mathematics is a subject making non-professionally educated specialists almost afraid of meeting comprehensive difficulties. To build a bridge between theoretical, mathematical excerpts and practical applications, I have formed my material as a sequence of occurrences in which a patient appears to be recovered from his or her illness. The patient needs to be diagnosed and treated by effective means to become healthy again. In this way the reader should have an impression that he or she follows the patient and his or her problems.

Therefore I have decided to avoid inserting a large number of mathematical definitions and theorems that have not much meaning in practice. I have only selected formal fuzzy concepts that are needed for medical models. To facilitate the understanding process for a reader, I have added many examples in which even simple operations are thoroughly analyzed. The user can follow the steps of examples without implementing a computer program. As we can expect, the book is not a survey of all theoretical concepts typical of fuzzy sets; such monographs already exist. Nevertheless, some beginners can use it to learn the basics included in fuzzy set theory. In spite of limiting the mathematical dimension, the work should convince the reader of the richness of applicable fuzzy models in natural sciences, especially medicine.

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Incidentally, I also discovered the usefulness of rough set theory for the classification of objects. Thus, some rough classification methods are considered as well.

The first part of the book – Chapter 2 – introduces some necessary elements of fuzzy set theory to enable the reader to repeat the processes and interpret imprecise information in further studies. A classical fuzzy diagnostic model with my own extensions is discussed in Chapter 3. Since a diagnostic decision is sometimes not quite clear, I have added Chapter 4 filled with my own contributions to improve informative validity of collected data. The model, presenting the essence of clinical examinations extended throughout time, constitutes one of the supplements.

After stating the right diagnosis the patient needs to be cured. Methods of drug effectiveness measurements are compared in Chapter 5, while a choice of the optimal medicine is made in Chapter 6. The last models are totally based on my own research results. Lastly, the solution of an approximation problem that is of considerable importance for our discussion is suggested in Chapter 7. Many times we obtain very irregular two-dimensional point sets that cannot be approximated by applying them to standard numerical methods. Some fuzzy membership functions have been adopted to provide the sets with smooth curves acting for them as tight envelopes. Even though the method is tested without medical examples, we are able to notice its usefulness in practical tests.

It should be emphasized that all models are also applicable to other fields, especially to technical domains after necessary adaptations.

This book could not have filled its role without professional medical support. I would like to thank Professors in Medicine – Alicja Kurnatowska and Anna Jegier – for a piece of medical advice and simple but illustrative examples.

I am very grateful to the representatives of Springer Verlag and to the series editor Professor Janusz Kacprzyk for giving me a chance to publish this material concerning applicable fuzzy medical models presented from my point of view. I hope that this work can bridge a gap between scientific reports, strictly treating separate domains, and, in this way, interdisciplinary groups of researchers can surely notice that there are prospects of linking theoretical fuzzy tools to practical medical exercises.

Karlskrona – Sweden
July 2006

Elisabeth Rakus-Andersson

2 Fundamental Items

2.1 Introduction

We are still accustomed to our traditional tools of reasoning being strict and precise. In conventional binary logic a statement can be true or false, and there is no place for even a little uncertainty in this judgement. By looking at sets, we can state that an element either belongs to a set or does not. We call these kinds of sets *crisp sets*. In practice we often experience those real situations that are represented by crisp sets, as impossible to describe accurately. If we assign a truth-value of one to the element that is included in the set, and a truth-value comparable to zero to such an element that lies outside the set, we create the range of two-valued logic. This sort of logic assumes that precise symbols must be employed, and it is therefore not applicable to the real existence but only to an imagined existence.

The creator of fuzzy set theory, Lotfi A. Zadeh, referred to the last hypothesis when he wrote: "As the complexity of a system increases, our ability to make precise and yet significant statements about its behaviour diminishes until the threshold is reached beyond which precision and significance become almost mutually exclusive characteristics" [88, 95].

If we consider the characteristic features of real world systems, we will conclude that real situations are very often uncertain or vague in a number of ways. If the information demanded by a system is lacking, the future state of such a system may not be known completely. This type of uncertainty has been handled by probability theories and statistics, and it is called stochastic uncertainty. The vagueness, concerning the description of the semantic meaning of the events, phenomena, or statements themselves, is called fuzziness [95].

Fuzziness can be found in many areas of daily life, especially in medicine. We look for the methods that help us to express the borders of such sets as "*young*", "*middle-aged*", "*old*", "*seldom*", "*rarely*", "*often*", "*high temperature*", "*low sugar level*" and the like. By using the traditional methods we are not able to express exactly the range of a set, e.g., "*young*" when defining the upper limit as, say, 26. Someone can ask "Why not 27?"

Almost every human being cannot be classified as "*totally healthy*" or "*totally ill*" in accordance with the two states of truth assumed by binary logic. It is more so since the conditions of health should be characterized by grades of a scale that propose the many nuances between "*total health*" and "*total sickness*". Thus we introduce the fuzzy apparatus to extend a notion of the set under the circumstances of vagueness.

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2.2 Fuzzy Sets

If we use the expression “a set” we will interpret it as a given collection of objects that are listed exactly, or that have the same property. Define $A = \{4,5,6,7\}$. The set A contains a determined number of elements and it thus is called a finite set. If we cannot count the number of elements in a set, e.g., $B = \{1,2,3,4,5,6,7,\ldots\}$ we will call B an infinite set.

Let us introduce a function μ_C as a characteristic function of a crisp set C. The crisp set C of universe $X = \{x\}$ is represented by its characteristic function μ_C as follows.

Definition 2.1

The function $\mu_C : X \to \{0,1\}$ is the characteristic function of the set C if and only if for all $x \in X$

$$y = \mu_C(x) = \begin{cases} 1 & \text{for} \quad x \in C, \\ 0 & \text{for} \quad x \notin C. \end{cases} \tag{2.1}$$

Example 2.1

Figure 2.1 shows the characteristic function of the crisp set $C = [4, 8]$.

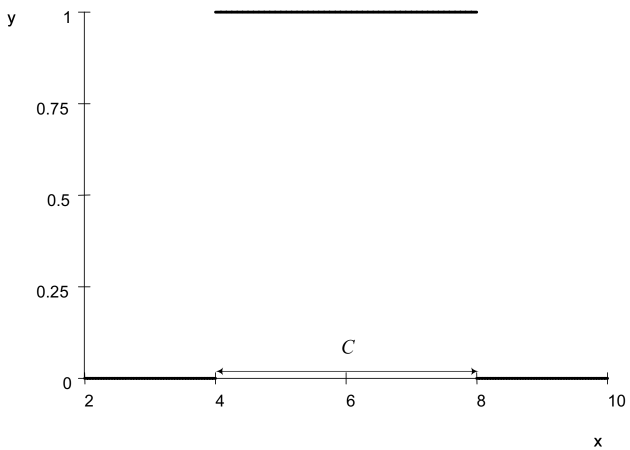

Figure 2.1: The characteristic function of the crisp set $C = [4, 8]$

In *fuzzy set theory* classical sets are called *crisp sets* in order to distinguish them from *fuzzy sets*. Let C be a crisp set defined with the domain named X. Then for any element $x \in X$ (the sign "$\in$" denotes that an element x belongs to the set X) we have that $x \in C$ or $x \notin C$ if the sign "$\notin$" stands for "does not belong". In fuzzy set theory this property is generalized. Therefore, in fuzzy set A, it is not necessary that either $x \in A$ or $x \notin A$ [10, 12, 14, 40, 41, 44, 48, 54, 55, 69, 88, 89, 95].

The generalization is assumed as follows. According to the above statement for any crisp set C it is possible to define a characteristic function $\mu_C : X \to \{0, 1\}$. In fuzzy set theory, the above characteristic function is generalized to *a membership function* that assigns every $x \in X$ a value from the unit interval [0, 1] instead of being assigned from the two-element set {0, 1}. The set A, defined on the basis of a generalized characteristic function, is called a fuzzy set.

Definition 2.2

The membership function μ_A of fuzzy set A is a function defined as $\mu_A : X \to [0, 1]$. Every element $x \in X$ has thus *a membership degree* $y = \mu_A(x) \in [0, 1]$. The fuzzy set A is finally completely determined by the set of pairs

$$A = \{(x, y) = (x, \mu_A(x))\} \ , \ x \in X. \tag{2.2}$$

Definition 2.3

The important part of fuzzy set A is *a support* denoted by supp(A) and defined as a non-fuzzy set (the sign ":" denotes "for which") [40, 88, 95]

$$\mathrm{supp}(A) = \{x \in X : \mu_A(x) > 0\}. \tag{2.3}$$

Example 2.2

Suppose that fuzzy set A has the support $\mathrm{supp}(A) = \{x : 0 \le x \le 10\}$, and its membership function is given by

$$y = \mu_A(x) = \begin{cases} \frac{1}{3}x & \text{for} \quad 0 \le x < 3, \\ 1 & \text{for} \quad 3 \le x < 8, \\ -\frac{1}{2}x + 5 & \text{for} \quad 8 \le x \le 10\,. \end{cases}$$

By analyzing the last formula we can sketch the membership function of A as a graph placed in Fig. 2.2.

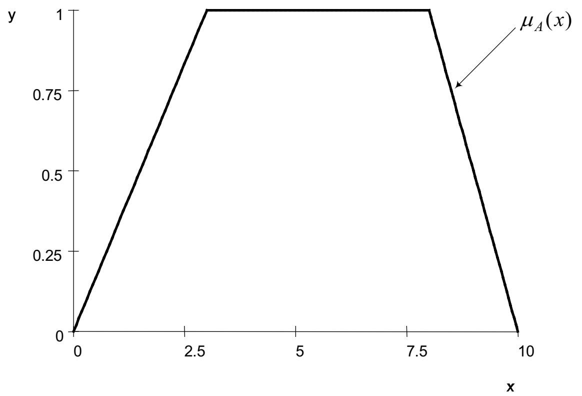

Figure 2.2: Fuzzy set $A = \{(x, \mu_A(x))\}, x \in [0, 10]$

Fuzzy sets are extremely useful when we want to formalize mathematically the descriptions of some uncertain occurrences. How do we define such sets as, for example, "*young*", "*old*" or "*cold*"? We may not decide that the maximal value in the set "*young*" will be 25. The age "26" should be included in "*young*" as well but the association of 26 with "*young*" will be weaker, for instance, 0.9 instead of 1.

Example 2.3

Let us state the non-fuzzy finite set "*young*" = {18, 20, 25, 30, 35, 40, 45, 50}, and let us intuitively decide strength of the relationship between the set and each value belonging to its support.

$$"young" = \{(18,1), (20,1), (25,0.9), (30,0.7), (35,0.5), (40,0.3), (45,0.1), (50,0)\}$$

that now becomes the fuzzy set that can be listed in another shape as

$$"young" = 1/18 + 1/20 + 0.9/25 + 0.7/30 + 0.5/35 + 0.3/40 + 0.1/45 + 0/50 \, .$$

We explain that the number over the slash constitutes a value of the membership degree of an element x. The element x is placed under the slash. The sign "+" links members of the set and has only a symbolic meaning.

If we design the membership function of the continuous set "*young*" as the figure with boundary lines

$$y = \mu_{"young"}(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < 25, \\ -\dfrac{1}{25}x + 2 & \text{for} \quad 25 \le x \le 50, \end{cases}$$

then we will obtain the membership degree of an arbitrary age that belongs to [0, 50] in conformity with a graph displayed in Fig. 2.3.

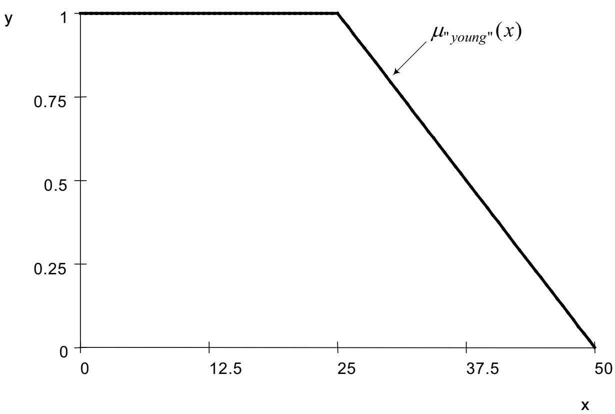

Figure 2.3: The continuous fuzzy set "*young*"

By examining the membership function sketched in the graph we are able to establish the connection between a value of age, e.g., $x = 33$ and the set "*young*" as nearly 0.7. The number indicates the strength of the relationship between the age of 33 and the notion of "*young*".

When the domain X of fuzzy set A is continuous, then the membership function μ_A will be regarded as a continuous membership function. We will designate A as a symbolic sum

$$A = \sum_{x \in X} \mu_A(x) / x = \{(x, y) = (x, \mu_A(x))\}, x \in X, \tag{2.4}$$

for which the Σ-sign denotes an infinite enumeration of its elements.

The membership function of a fuzzy set, which has the graph designed as a collection of straight lines linked piece by piece, is the easiest one to apply. However, the split linear function does not yield the only possibility of expressing an association between the set and its elements. We can – as a prognosis of the mentioned relationship – introduce other functions that are continuous and map the support of a fuzzy set onto the interval [0, 1]. As an alternative membership function of the fuzzy set A we demonstrate the s-class function $s(x, \alpha, \beta, \gamma)$ with the parameters α, β and γ that are included in the formula [2, 3, 12, 40, 41, 49, 50, 67, 70, 91]

$$y = \mu_A(x) = s(x,\alpha,\beta,\gamma) = \begin{cases} 0 & \text{for } x \le \alpha, \\ 2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 & \text{for } \alpha < x \le \beta, \\ 1-2\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 & \text{for } \beta < x \le \gamma, \\ 1 & \text{for } x > \gamma, \end{cases} \tag{2.5}$$

where $\beta = \dfrac{\alpha+\gamma}{2}$.

Example 2.4

The function $s(x, 25, 37.5, 50)$ is plotted in Fig. 2.4.

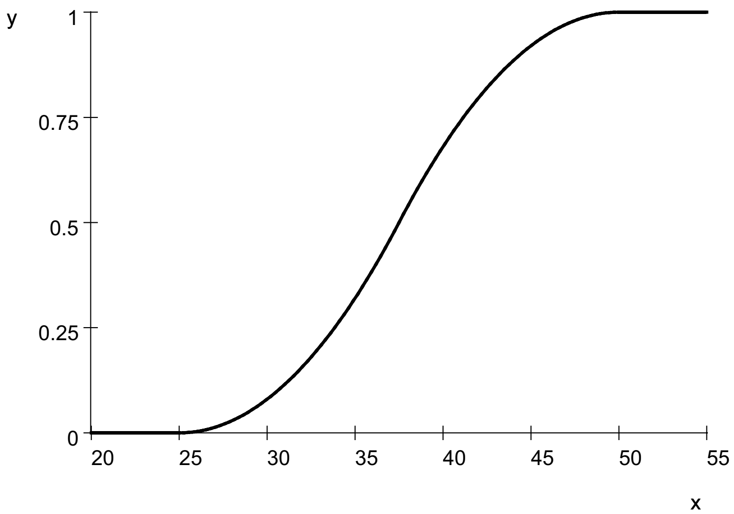

Figure 2.4: The function $s(x, 25, 37.5, 50)$

The s-function holds a number of properties that stand out as very advantageous. As a continuous polynomial of the second degree the s-function can assist further analytical operations, such as differentiation or integration without making them very complicated. It is also evident from the formula (2.5) that the range of s covers the interval [0, 1] that is a desirable feature of the membership function.

Example 2.5

By proposing the membership function of the set "*young*" as a formula $\mu_{"young"}(x) = 1 - s(x, 25, 37.5, 50)$ we create another relationship between the concept of "*young*" and its elements when comparing to Ex. 2.3. Figure 2.5 interprets the alternative set "*young*".

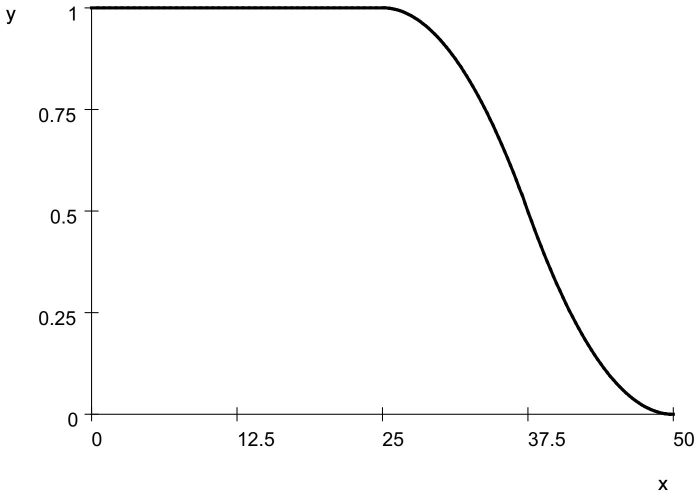

Figure 2.5: The membership function of the set "*young*" as the *s*-class function

The same value of age can be a member of many sets with different membership degrees. Let us introduce three sets named "*young*", "*middle-aged*" and "*old*". To suggest a membership function of the set "*middle-aged*" we test a function of the π-class constructed as

$$y = \pi(x, \alpha, \beta) = \begin{cases} s\left(x, \beta - \alpha, \beta - \dfrac{\alpha}{2}, \beta\right) & \text{for} \quad x \le \beta, \\ 1 - s\left(x, \beta, \beta + \dfrac{\alpha}{2}, \beta + \alpha\right) & \text{for} \quad x > \beta. \end{cases} \tag{2.6}$$

Example 2.6

When we decide $\alpha = 20$ and $\beta = 45$, we will accommodate (2.6) to the formula

$$y = \pi(x,20,45) = \begin{cases} s(x,25,35,45) & \text{for} \quad x \le 45, \\ 1 - s(x,45,55,65) & \text{for} \quad x > 45. \end{cases}$$

The graph of $\pi(x, 20, 45)$ is drawn in Fig. 2.6.

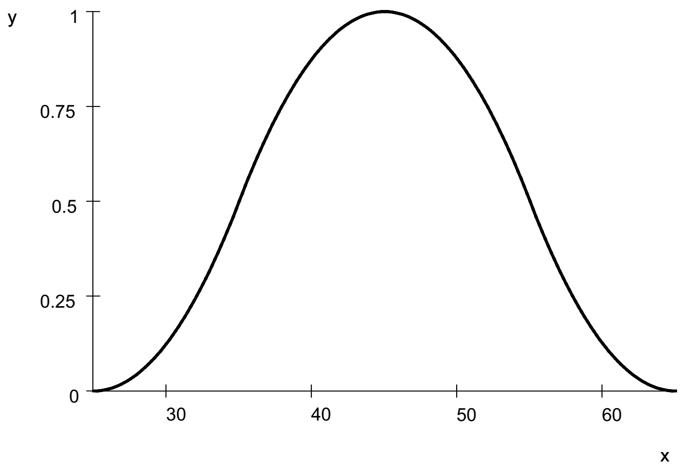

Figure 2.6: The function $\pi(x, 20, 45)$

By constructing three fuzzy sets with supports that overlap each other we demonstrate again the idea of imprecision in the following task.

Example 2.7

Three fuzzy sets of the universe $X = [0, 100]$, made for different groups of age, are now put forward as "*young*" with $y = \mu_{\text{"young"}}(x) = 1 - s(x, 25, 37.5, 50)$, "*middle-aged*" with $y = \mu_{\text{"middle-aged"}}(x) = \pi(x, 20, 45)$ and, finally, "*old*" characterized by the membership function $y = \mu_{\text{"old"}}(x) = s(x, 40, 52.5, 65)$. The sets have no sharp borders; on the contrary, some non-empty intersections of all supports are built in the sets' domains. Figure 2.7 visually explains the concept of vagueness even better since it helps us to understand the effects of fuzziness when studying the sets that are not disjoint in spite of the different classifiers of age.

We compute that $x = 42$ belongs to "*young*" with the membership degree equal 0.2048 because of $y = 1 - s(42, 25, 37.5, 50) = 1 - \left(1 - 2\left(\frac{42-50}{50-25}\right)^2\right), 42 \in (37.5, 50]$.

The connection between $x = 42$ and "*middle-aged*" is evaluated as 0.955 in relation to $y(42) = s(42, 25, 35, 45)$. The value of $x = 42$ is not a member of the set "*old*".

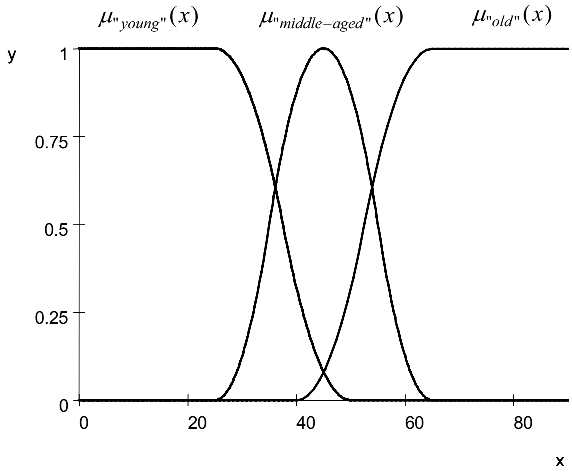

Figure 2.7: The fuzzy sets "*young*", "*middle-aged*" and "*old*" in $X = [0, 100]$

Measurable, medical parameters can be interpreted as fuzzy sets to prepare data to models described in the next parts of this work. Clinical symptoms involved in mathematical decision patterns will be represented by values of membership degrees. The degrees, as numbers, express the intensities of symptom presence or symptom importance without engaging in complicated verbal descriptions. The possibility of computing with words, which are replaced by real numbers assigned to them, facilitates the communication among researchers who represent different scientific fields.

Example 2.8

The existence of fuzzy sets enables an introduction of medical models that operate with clinical symptoms constituting the basis for a decision. To evaluate the importance of increasing body temperature via a corresponding membership degree, we propose adopting the fuzzy set "*body temperature*" with a membership function expanded by

$$y = \mu_{\text{"body temperature"}}(x) = \begin{cases} 1 - s(x, 35°, 35.8°, 36.6°) & \text{for} \quad 35° \le x \le 36.6°, \\ s(x, 36.6°, 39.3°, 42°) & \text{for} \quad 36.6° < x \le 42°. \end{cases}$$

The membership function of the set "*body temperature*" is sketched in Fig. 2.8. The values of membership degrees provide us with an estimation of the temperature importance for states in which body temperature is too high or too low in comparison to its normal value.

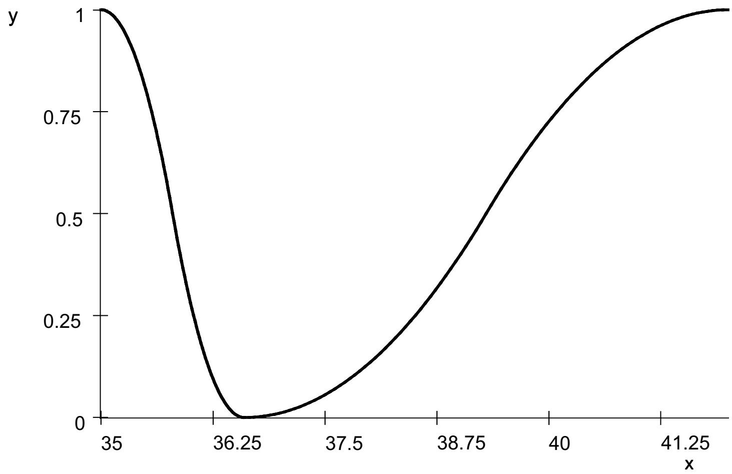

Figure 2.8: The membership function of the set "*body temperature*"

2.3 Basic Operations on Fuzzy Sets

The definition of a fuzzy set that differs from the concept of a crisp set has entailed new approaches to the operations on fuzzy sets. In order to connect two fuzzy sets let us study their union and intersection.

At first, we recall the classical operation of the union performed on two non-fuzzy sets.

Definition 2.4

The union of two crisp sets $A = \{x : x \in A\}$ and $B = \{x : x \in B\}$, $A, B \subset X = \{\mathrm{x}\}$ (denoted by "$\cup$") is a set $A \cup B = \{x : x \in A$ or $x \in B\}$. We remember that the sign ":" is read as "for which".

In practice, the set $A \cup B$ is a collection of all elements that belong to either A or B provided that identical elements are counted only once.

Example 2.9

For $A = [2, 6]$ and $B = [4, 9]$ the union is determined as $A \cup B = [2, 9]$.

A union of two fuzzy sets should also be a fuzzy set. By accepting the operation of the classical union for the supports of fuzzy sets, we establish a common support of the union of fuzzy sets while the connective union operation for membership degrees can be suggested in the following definition.

Definition 2.5

Let $A = \{(x, \mu_A(x))\}$ and $B = \{(x, \mu_B(x))\}$, $x \in X$, denote two fuzzy sets. The union of A and B is a fuzzy set $A \cup B = \{(x, \mu_{A \cup B}(x))\}$, $x \in (\text{supp}(A) \cup \text{supp}(B))$, for which the membership function is given by [12, 40, 88, 95]

$$\mu_{A \cup B}(x) = \max_{x \in (\text{supp}(A) \cup \text{supp}(B))} (\mu_A(x), \mu_B(x)). \tag{2.7}$$

Example 2.10

If A has supp(A) = [2, 6] and $\mu_A(x) = 1 - s(x, 2, 4, 6)$, while B is fixed precisely by supp(B) = [4, 9] and $\mu_B(x) = s(x, 4, 6.5, 9)$, then the union of A and B will consist of supp($A \cup B$) = [2, 9] and $\mu_{A \cup B}(x) = \max(1 - s(x, 2, 4, 6),\ s(x, 4, 6.5, 9))$ for every $x \in [2, 9]$ in accordance with a dotted curve placed in Fig. 2.9.

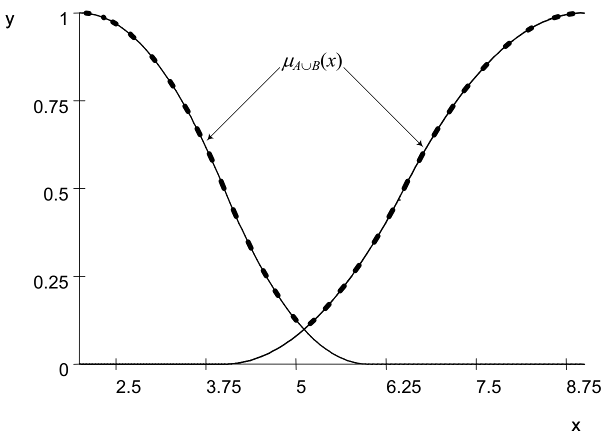

Figure 2.9: The union of fuzzy sets A and B from Ex. 2.10

In order to estimate the membership degree of, e.g., $x = 4.5$ in the union $A \cup B$ we will follow the expression $\mu_{A \cup B}(4.5) = \max(1 - s(4.5, 2, 4, 6), s(4.5, 4, 6.5, 9))$

$$= \max\left\{1 - \left[1 - 2\left(\frac{4.5-6}{4}\right)^2\right], 2\left(\frac{4.5-4}{5}\right)^2\right\} = \max(0.28125, 0.02) = 0.28125.$$

The replacement of the operation "max" in the membership function of the union of the two fuzzy sets A and B by the dual operation "min", generates an intersection of A and B. We state that a support of the intersection between A and B is composed of these elements that take place in the union of the sets' supports. Since some elements of the intersection get the membership degrees equal to zero, then we should reduce the common support to its essential part in which is $\mu_{A\cap B}(x) > 0$.

Definition 2.6

For $A = \{(x, \mu_A(x))\}$ and $B = \{(x, \mu_B(x))\}$, $x \in X$, that are two fuzzy sets, the intersection of A and B, marked by "$\cap$", is a fuzzy set $A \cap B = \{(x, \mu_{A\cap B}(x))\}$ for all $x \in (\text{supp}(A) \cup \text{supp}(B))$. Its membership function is shaped by [12, 40, 88, 95]

$$\mu_{A\cap B}(x) = \min_{x\in(\text{supp}(A)\cup\text{supp}(B))} (\mu_A(x), \mu_B(x)). \tag{2.8}$$

Example 2.11

For the sets A and B from Ex. 2.10 the membership function of the intersection is indicated by the dotted line drawn in Fig. 2.10.

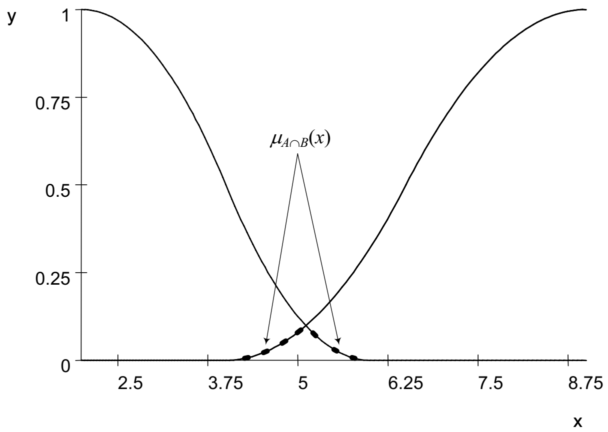

Figure 2.10: The intersection of A and B from Ex. 2.10

There also exists the complement of fuzzy set A introduced by the following definition [12, 40, 88, 95].

Definition 2.7

Let $A = \{(x, \mu_A(x))\}$, $x \in X$. The complement of A, denoted A\`, is a fuzzy set with the membership function $\mu_{A`}$ given by the formula

$$\mu_{A'}(x) = 1 - \mu_A(x), \text{ for } x \in X. \qquad (2.9)$$

Example 2.12

Suppose that B – the fuzzy set examined in Ex. 2.10 – is determined by supp(B) = [4, 9] and $\mu_B(x) = s(x, 4, 6.5, 9)$. We thus establish a membership function of the complement B` as the curve presented by Fig. 2.11 for these x that belong to supp(B).

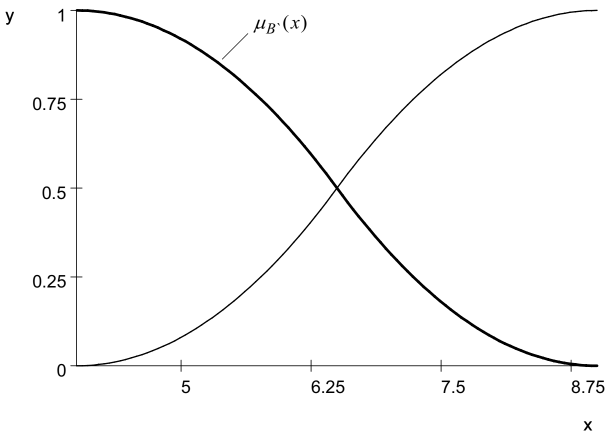

Figure 2.11: The complement of the set B from Ex. 2.10

Let us discuss another example that inserts a simple and practical aspect of the union and the intersection of two fuzzy sets.

Example 2.13

We consider two fuzzy sets "*young*" and "*experienced*" in the space of ages X = [0, 100]. The set "*young*" is still determined by the membership function

$$\mu_{\text{"young"}}(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < 25, \\ -\frac{1}{25}x + 2 & \text{for} \quad 25 \le x < 50, \\ 0 & \text{for} \quad 50 \le x \le 100, \end{cases}$$

in accordance with Ex. 2.3. The rule "*more experienced with growing age*" suggests a creation of the set "*experienced*" as a fuzzy set related to the membership function

$$\mu_{\text{"}experienced\text{"}}(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < 15, \\ \dfrac{1}{45}x - \dfrac{1}{3} & \text{for} \quad 15 \le x < 60, \\ 1 & \text{for} \quad 60 \le x \le 100. \end{cases}$$

The adaptation of (2.7) to a union of the sets "*young*" and "*experienced*" causes the existence of a fuzzy set "*young or experienced*" with the membership function

$$\mu_{\text{"}young\ or\ experienced\text{"}}(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < 25, \\ -\dfrac{1}{25}x + 2 & \text{for} \quad 25 \le x < 37.63, \\ \dfrac{1}{45}x - \dfrac{1}{3} & \text{for} \quad 37.63 \le x < 60, \\ 1 & \text{for} \quad 60 \le x \le 100, \end{cases}$$

that is depicted in Fig. 2.12.

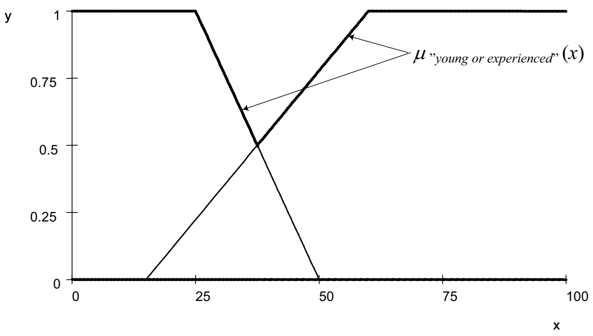

Figure 2.12: The membership function of the set "*young or experienced*"

Equation (2.8) is a tool of deciding an intersection "*young and experienced*" of the sets "*young*" and "*experienced*". This intersection, being a fuzzy set, is modelled by the function

$$\mu_{"young\ and\ experienced"}(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < 15, \\ \frac{1}{45}x - \frac{1}{3} & \text{for} \quad 15 \le x < 37.63, \\ -\frac{1}{25}x + 2 & \text{for} \quad 37.63 \le x < 50, \\ 0 & \text{for} \quad 50 \le x \le 100. \end{cases}$$

Figure 2.13 proposes the graph of "*young and experienced*".

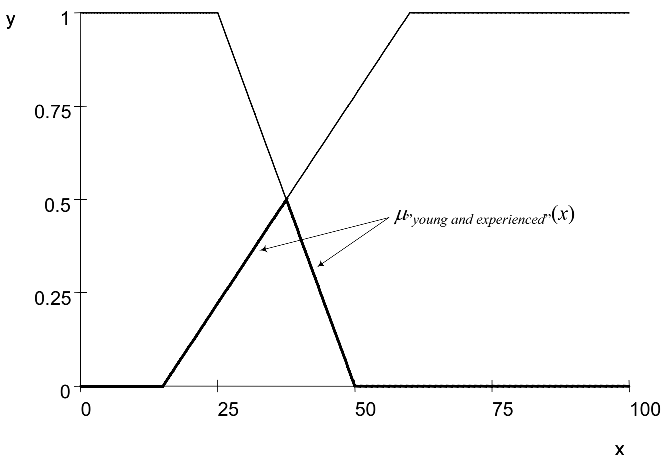

Figure 2.13: The membership function of the set "*young and experienced*"

It is easy to verify that the membership function of the intersection operation (2.8) then rewritten as

$$\mu_{A \cap B}(x) = \min_{x \in X}(\mu_A(x), \mu_B(x)) = t(\mu_A(x), \mu_B(x)) \tag{2.10}$$

is a function from [0, 1] × [0, 1] into [0, 1]. The sign "×", used symbolically as a notion of the Cartesian product, informs us that we should take a pair of two numbers from [0, 1] to map this pair of quantities in another number belonging to the interval [0, 1] as well. The minimum function satisfies some pre-defined properties. If other functions behave themselves like the minimum, then we should generate for them a class of functions named *t*-norms. The *t*-norms have the following features

$$\begin{aligned}&1.\ t(0,0)=0;\ t(\mu_A(x),1)=t(1,\mu_A(x))=\mu_A(x), x\in X,\\&2.\ t(\mu_A(x),\mu_B(x))\le t(\mu_C(x),\mu_D(x)) \text{ if } \mu_A(x)\le\mu_C(x) \text{ and } \mu_{\ldots}\\&3.\ t(\mu_A(x),\mu_B(x))=t(\mu_B(x),\mu_A(x)),\\&4.\ t(\mu_A(x),t(\mu_B(x),\mu_C(x)))=t(t(\mu_A(x),\mu_B(x)),\mu_C(x)).\end{aligned}\tag{2.11}$$

For minimum the first condition is fulfilled because $t(0, 0) = \min(0, 0) = 0$ and $t(\mu_A(x), 1) = \min(\mu_A(x), 1) = \mu_A(x)$.

The minimum operator is a *t-norm* [12, 41, 44, 95]. If we define any function $t(a, b) = c$ for $a, b, c \in [0, 1]$, $t : [0,1] \times [0,1] \to [0,1]$, due to conditions 1-4 in (2.11) we will form another intersection operation for two fuzzy sets, e.g., the bounded product, the algebraic product, the Einstein product, the Yager product and the like [44, 95]. To watch the difference between the minimum norm, called also the largest norm, and the bounded norm defined as

$$t(\mu_A(x),\mu_B(x)) = \max_{x\in X}(\mu_A(x)+\mu_B(x)-1,0) \tag{2.12}$$

we compare a value of the t-minimum norm for a pair, e.g., (0.8, 0,5) equal to $\min(0.8, 0.5) = 0.5$ to the t-bounded norm value calculated as $\max(0.8 + 0.5 - 1, 0) = 0.3$.

The membership function (2.7) of the union of two fuzzy sets that is interpreted as the function

$$\mu_{A\cup B}(x) = \max_{x\in X}(\mu_A(x),\mu_B(x)) = s(\mu_A(x),\mu_B(x)), \tag{2.13}$$

also is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ with the following properties

$$\begin{aligned}&1.\ s(1,1)=1;\ s(\mu_A(x),0)=s(0,\mu_A(x))=\mu_A(x),\ x\in X,\\&2.\ s(\mu_A(x),\mu_B(x))\le s(\mu_C(x),\mu_D(x))\\&\quad \text{if } \mu_A(x)\le\mu_C(x) \text{ and } \mu_B(x)\le\mu_D(x),\\&3.\ s(\mu_A(x),\mu_B(x))=s(\mu_B(x),\mu_A(x)),\\&4.\ s(\mu_A(x),s(\mu_B(x),\mu_C(x)))=s(s(\mu_A(x),\mu_B(x)),\mu_C(x)).\end{aligned}\tag{2.14}$$

When checking for the maximum, the reliability of the first condition from (2.14), we state that $s(1, 1) = \max(1, 1) = 1$ and $s(\mu_A(x), 0) = \max(\mu_A(x), 0) = \mu_A(x)$.

The function $s(a, b) = c$, $a, b, c \in [0, 1]$, is called an *s-norm* (coincidentally the same notion like the s-function $s(x, \alpha, \beta, \gamma)$). If we propose another definition satisfying the features 1-4 in (2.14) we will create a class of union operations on fuzzy sets.

The operations concatenating two fuzzy sets will constitute an important basis for solutions to medical projects.

2.4 Linguistic Variables

The conception of a variable in classical mathematics makes us think of joining a number to a name of a certain symbol, usually denoted by the letters x, y, z and the like. Nobody doubts that the classical variable takes values associated with it for a hundred percent certainty. In fuzzy set theory we join values to the name of a variable even if the values reveal a weaker relationship with the variable than a "sure connection".

The assigned values to a variable need not be numbers. It is possible to use some words or some structures to express a connection between the name of the variable and its range if the connection is imprecise and cannot be described exactly.

If we think about the idea of a variable then we can imagine constructing an equation

$$x = a \tag{2.15}$$

that assigns a value of a to the name x.

Generally, let us accept the name of the variable as X and let us recognize A as a set of all values taken by X. The set A is called the range of X. If we further restrict the values of the range A by imposing a constraint $R(X) \subseteq A$ on the values of X, it will mean that X takes only the values that belong to $R(X)$. A new equation that assigns the value of a to X is derived as

$$x = a,\ a \in R(X)\,. \tag{2.16}$$

Suppose that the constraint $R(X)$ is a fuzzy set. The variable X, which takes its values in the range A and possesses the fuzzy constraint $R(X)$, is now renamed as *a fuzzy variable* [28, 40, 90, 95]. The values assigned to X are the elements of a fuzzy set (a fuzzy constraint) and thus they are equipped with a corresponding membership degree.

By using the equation (2.16) we make the next trial of associating the variable X and its values provided that (2.16) is supplemented by a so-called *compatibility degree* $c(a) \in [0, 1]$. The mentioned equation (2.16) is now rewritten in the form of

$$x = a,\ c(a) = \mu_{R(X)}(a),\ a \in \mathrm{supp}(R(x)),\ c(a) \in [0,1]\,. \tag{2.17}$$

The compatibility degree $c(a)$ is computed for the value of a by adopting a formula of the membership function defined for $R(X)$.

Example 2.14

Let us examine the relation between a certain diagnosis D and one of clinical symptoms typical of D that is named S. The relationship between S and D can illustrate a concept of the fuzzy variable. We pose a question referring to the frequency of the symptom present in the selected diagnosis. We expect – as an answer – such a frequency description like, e.g., "*often*", "*seldom*", "*never*", "*almost always*" and other formulations related to intensity of the symptom present. We use words as the answers but we wish to convert them into numbers that intend to represent the linguistic structures in future computations.

Suppose that a basic reference to set $A = \{0, 1, 2, \ldots, 100\}$ includes one hundred patients. We determine the name of a fuzzy variable X as "*often*". Let a fuzzy set R("*often*") be the fuzzy constraint laid on the set A by the variable X. The membership function of R("*often*") is given by the formula

$$\mu_{R(\text{"}often\text{"})}(a) = s(a,50,60,70),$$

that is interpreted in the explicit form, in accordance with (2.5), as

$$y = \mu_{R(\text{"}often\text{"})}(a) = s(a,50,60,70) = \begin{cases} 0 & \text{for} \quad a \le 50, \\ 2\left(\dfrac{a-50}{70-50}\right)^2 & \text{for} \quad \text{for } 50 < a \le 60, \\ 1-2\left(\dfrac{a-70}{70-50}\right)^2 & \text{for} \quad 60 < a \le 70, \\ 1 & \text{for} \quad a > 70. \end{cases}$$

The appearance of a curve plotted in Fig. 2.14 is an instance of influence of the constraint R("*often*") on the range of the reference set A. It is no doubt that a status of the variable name gives rise to the selection of the parameters α, β and γ placed in the membership function of R("*often*").

Figure 2.14: The constraint R("*often*") over the reference set A = [0, 100]

In the equation "*often*" = 40, $c(40) = 0$, the compatibility between the value of 40 and the frequency notion "*often*" is equal to zero in the space of one hundred patients. If "*often*" = 58 then $c(58) = 2\left(\dfrac{58-50}{70-50}\right)^2 = 0.32$. The grade indicates that the strength of the connection between the name "*often*" and a sample of fifty-eight patients in comparison with one hundred patients is appreciated as 0.32. Another equation – "*often*" = 72, $c(72) = 1$ – confirms the total compatibility between the name of the variable "*often*" and its value 72.

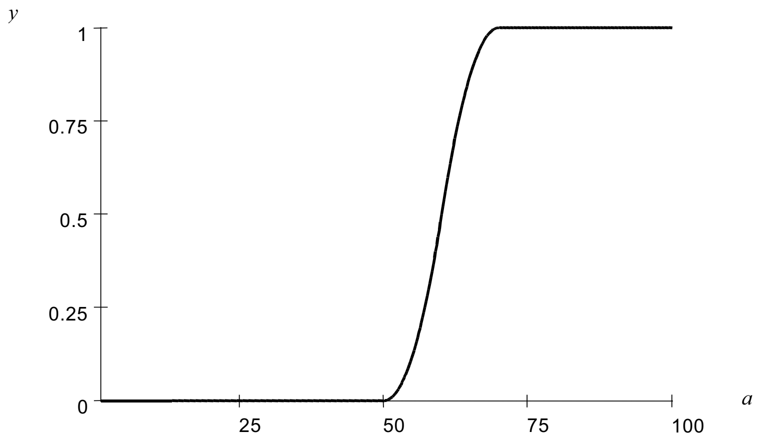

Figure 2.14: The constraint R(“*often*”) over the reference set $A = [0, 100]$

The introduction of the fuzzy variable “*often*” allows us to replace the word *often* by the set of numbers that are associated with this linguistic item. However, we realize that we should operate with many more words or structures to describe a certain occurrence like, e.g., *intensity of presence* introduced in Ex. 2.14. In other words, we want to have at our disposal a full list of words, and moreover, we would like to be able to express every word in the form of a number set in the common space. This idea contributes to the evolution of a variable range by giving access as verbal expressions as its members. A *linguistic variable* that offers commonly used words as its range characters is interpreted as such.

The linguistic variable is a variable taking values expressed by words. These are names of fuzzy variables defined in a space such as $A = \{0, 1, 2, \ldots, 100\}$. Let us denote the linguistic variable by L and a set of its terms by $T = \{T_1, T_2, \ldots, T_n\}$, where each T_i, $i = 1, 2, \ldots, n$, is the name of a fuzzy variable restricted by a fuzzy constraint $R(T_i)$ definable in the space A. An equation

$$L = T_i, i = 1, \ldots, n \tag{2.18}$$

that links a “value” T_i to the linguistic variable L, reveals a relation between a general name of the variable and one of its semantic terms.

Example 2.15

Suppose that we are able to upgrade the frequency of a symptom in the associated diagnosis by employing average verbal expressions that emphasize the importance of symptom presence. We can thus utilize a list of words to make a conversation with a cooperating physician much more comfortable. We should realize that the physician must experience some difficulties in telling us his opinion as a strict mathematical number that describes the grade of symptom presence. Let us decide a content T of the list "*presence of symptom S in diagnosis D*" as $T = \{T_1 =$ "*never*", $T_2 =$ "*almost never*", $T_3 =$ "*very seldom*", $T_4 =$ "*seldom*", $T_5 =$ "*moderately*", $T_6 =$ "*often*", $T_7 =$ "*very often*", $T_8 =$ "*almost always*", $T_9 =$ "*always*"$\}$. Each of the terms from the list constitutes the name of a fuzzy variable placed in the reference set $A = \{0, 1, 2, \ldots, 100\}$. In order to characterize the intervals in the set of one hundred patients that are typical of the names from the list, the following restrictions are put forward by their membership functions [2, 3, 56]

$$\mu_{R(T_1=\text{"}never\text{"})}(a) = 1 - s(a,0,5,10),$$

$$\mu_{R(T_2=\text{"}almost\ never\text{"})}(a) = 1 - s(a,6,10,14),$$

$$\mu_{R(T_3=\text{"}very\ seldom\text{"})}(a) = 1 - s(a,15,20,25),$$

$$\mu_{R(T_4=\text{"}seldom\text{"})}(a) = 1 - s(a,30,40,50),$$

$$\mu_{R(T_5=\text{"}moderately\text{"})}(a) = \pi(a,20,50),$$

$$\mu_{R(T_6=\text{"}often\text{"})}(a) = s(a,50,60,70),$$

$$\mu_{R(T_7=\text{"}very\ often\text{"})}(a) = s(a,75,80,85),$$

$$\mu_{R(T_8=\text{"}almost\ always\text{"})}(a) = s(a,86,90,94),$$

$$\mu_{R(T_9=\text{"}always\text{"})}(a) = s(a,90,95,100).$$

Figure 2.15 gives an impression of dividing space A into subintervals that selects the terms assigned to the linguistic variable "*presence of symptom S in diagnosis D*". It is remarkable to notice that the intervals build non-disjointed intersections. The occurrence of overlapping the supports of fuzzy sets emphasizes fuzzy performance of sets once again.

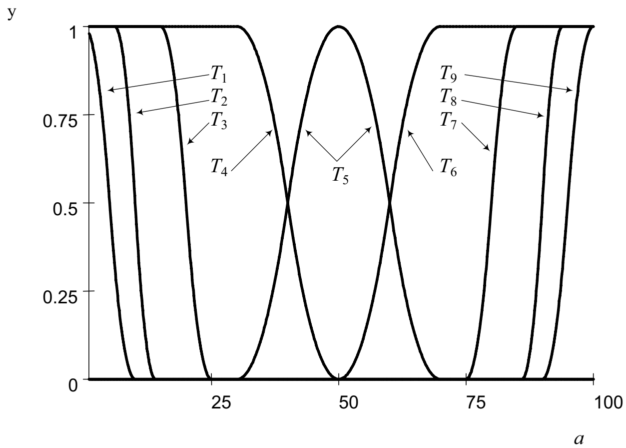

Figure 2.15: The fuzzy constraints of terms representing "*presence of S in D*"

The boundary values of fuzzy constraints from Fig. 2.15 have been decided in compliance with the physician's advice. In one of the next chapters we intend to introduce formal, mathematical evidence that explains how to construct fuzzy restrictions for terms belonging to the list of a linguistic variable. The concept of atomic words and hedges will be involved in the evidence as conclusive factors of the solutions.

An equation, in which a term is attributed to the variable name, determines the relationship between the linguistic variable, e.g., "*presence of symptom S in diagnosis D*" and one of its terms such as "*almost never*". In this way we can state the connection

$$\text{“}\textit{presence of symptom S in diagnosis D}\text{”} = \text{"}\textit{almost never}\text{"}\ .$$

The notions of fuzzy variables and linguistic variables constitute very important tools in medical models that are discussed in the next chapters. The linguistic variable makes it possible to convert words or other verbal structures into numbers, and this possibility then opens up an understandable dialogue between physicians and mathematicians working together.

2.5 Fuzzy Relations

Fuzzy relations join two-non fuzzy sets in the common set of pairs on condition that each pair has the membership degree assigned.

To understand better the idea of a fuzzy relation let us first study the basic definition of a Cartesian product of two non-fuzzy sets.

Definition 2.8

Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two finite sets. The Cartesian product of X and Y, denoted by $X \times Y$, is a set of ordered pairs (x_i, y_j), $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, for $x_i \in X$ and $y_j \in Y$.

Example 2.16

If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$ then $X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.

If each pair (x_i, y_j) included in $X \times Y$ is equipped with a membership degree then the Cartesian product that is mapped in the interval [0, 1] by a membership function, will be called a fuzzy relation. Formally, let us introduce the fuzzy relation $\widetilde{R}$ in the following way [12, 40, 88, 95].

Definition 2.9

Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two finite sets; then

$$\widetilde{R} = \{((x_i, y_j), \mu_{\widetilde{R}}(x_i, y_j))\}, (x_i, y_j) \in X \times Y, \mu_{\widetilde{R}} : X \times Y \to [0,1] \tag{2.19}$$

for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Fuzzy relations are often presented in the form of two-dimensional tables. The rows of such a table are all marked by x while the columns are all indicated by y. An entry of the table corresponds to this membership degree of the pair (x, y) that belongs to the intersection of row x and column y.

An $m \times n$ table-matrix constitutes a comfortable way of entering the fuzzy relation $\widetilde{R}$ that presents a format suggested below

$$\widetilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \\ \begin{bmatrix} \mu_{\widetilde{R}}(x_1, y_1) & \mu_{\widetilde{R}}(x_1, y_2) & \cdots & \mu_{\widetilde{R}}(x_1, y_n) \\ \mu_{\widetilde{R}}(x_2, y_1) & \mu_{\widetilde{R}}(x_2, y_2) & \cdots & \mu_{\widetilde{R}}(x_2, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{\widetilde{R}}(x_m, y_1) & \mu_{\widetilde{R}}(x_m, y_2) & \cdots & \mu_{\widetilde{R}}(x_m, y_n) \end{bmatrix} \end{array}. \tag{2.20}$$

Example 2.17

Let X = "*body height*" = [160, 190] and Y = "*body weight*" = [60, 90] be two sets containing the measurements of two parameters x = "*height*" and y = "*weight*" that are characteristic of the man's silhouette. We design a relation "*a strong and proportional construction of the man's body*"$(x, y) = \widetilde{R}(x, y)$, in which the membership degree values of the pairs computed in accordance with the function [28]

$$z = \mu_{\text{"a strong and proportional construction of the man's body"}}(x, y)$$

$$= \mu_{\widetilde{R}}(x, y) = \begin{cases} 0 & \text{for } x < 160,\ y < 60, \\ \dfrac{x - 160}{60} + \dfrac{y - 60}{60} & \text{for } 160 \le x \le 190,\ 60 \le y \le 90, \\ 1 & \text{for } x > 190,\ y > 90, \end{cases}$$

confirm a grade of adaptation of the physical features to the definition of the relation. For instance, the pair (170, 65) has $\mu_{\tilde{R}}(170,65) = \dfrac{170-160}{60} + \dfrac{65-60}{60} =$ 0.25, while another pair (182, 89) shows the compatibility degree with the definition of the relation estimated as equal to $\mu_{\tilde{R}}(182,89) = \dfrac{182-160}{60} + \dfrac{89-60}{60} =$ 0.85.

The general dependence of the membership degrees $\mu_{\tilde{R}}(x,y)$ on the growth of both biological parameters can be observed in Fig. 2.16.

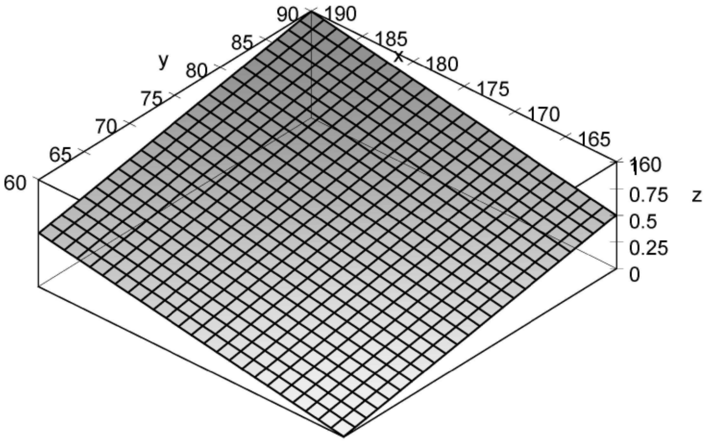

Figure 2.16: "*A strong and proportional construction of the man's body*"

Fuzzy relations can be combined with each other by the operation of "composition". In order to understand better the aggregation of two fuzzy relations, let us first recall the definition of a matrix multiplication for two matrices (dot product).

Definition 2.10

If $R = (r_{ij})_{i=1,\ldots,m,\, j=1,\ldots,n}$ and $Q = (q_{jk})_{j=1,\ldots,n,\, k=1,\ldots,p}$ then $S = R$ Q is a product matrix with elements $s_{ik} = \sum_{j=1}^{n} r_{ij} \cdot q_{jk}$, $i = 1, \ldots, m$, $k = 1, \ldots, p$. The number of columns in R must be the same as the row number in Q.

The matrices R and Q in table forms are used to clarify the multiplication. Let

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix} \text{ and } Q = \begin{bmatrix} q_{11} & \cdots & q_{1p} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{np} \end{bmatrix}. \text{ Hence, } S = \begin{bmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & \ddots & \vdots \\ s_{m1} & \cdots & s_{mp} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} \cdot q_{11} + \cdots + r_{1n} \cdot q_{n1} & \cdots & r_{11} \cdot q_{1p} + \cdots + r_{1n} \cdot q_{np} \\ \vdots & \ddots & \vdots \\ r_{m1} \cdot q_{11} + \cdots + r_{mn} \cdot q_{n1} & \cdots & r_{m1} \cdot q_{1p} + \cdots + r_{mn} \cdot q_{np} \end{bmatrix}.$$

Example 2.18

We design two matrices $R_{2\times 3}$ and $Q_{3\times 2}$.

Let us multiply the matrices $R = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 6 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & 2 \\ 2 & 3 \\ 5 & 7 \end{bmatrix}$.

The product matrix S is obtained in conformity with Def. 2.10 as the matrix

$$S = \begin{bmatrix} 2\cdot 4 + 1\cdot 2 + 5\cdot 5 & 2\cdot 2 + 1\cdot 3 + 5\cdot 7 \\ \\ 3\cdot 4 + 6\cdot 2 + 2\cdot 5 & 3\cdot 2 + 6\cdot 3 + 2\cdot 7 \end{bmatrix} = \begin{bmatrix} 35 & 42 \\ \\ 34 & 38 \end{bmatrix}.$$

By returning to fuzzy relations, filled with membership degrees, we will treat the operation of "product" as "composition". To find some adequate operations that replace the external sum and the internal multiplication in the product of two matrices, the different suggestions have been made. The most popular composition is a version with the maximum applied instead of the outer sum, and the minimum replaces the inner multiplication.

Definition 2.11 (max-min composition) [12, 40, 88, 95]

Let $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_p\}$. We introduce $\tilde{R}$ with $\mu_{\tilde{R}}(x_i, y_j)$, $(x_i, y_j) \in X \times Y$, and $\tilde{Q}$ with $\mu_{\tilde{Q}}(y_j, z_k)$, $(y_j, z_k) \in Y \times Z$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, p$, as two fuzzy relations. The max-min composition of $\tilde{R}$ with $\tilde{Q}$, denoted by $\tilde{R} \circ \tilde{Q}$, will be a fuzzy relation [12, 40, 95]

$$\tilde{S} = \tilde{R} \circ \tilde{Q} = \left\{ \left((x_i, z_k), \mu_{\tilde{R}\circ\tilde{Q}}(x_i, z_k) = \max_{y_j \in Y} \left\{ \min \left\{ \mu_{\tilde{R}}(x_i, y_j), \mu_{\tilde{Q}}(y_j, z_k) \right\} \right\} \right) \right\}. \quad (2.21)$$

The next example throws more light on the last definition by explaining the meaning of performed operations. In order to introduce the entry data we propose using two rectangular matrices that correspond to two fuzzy relations.

Example 2.19

The relation $\widetilde{R}$ reveals via values of membership degrees $\mu_{\widetilde{R}}(x_i, y_j)$, $i = 1, 2$, $j = 1, 2, 3$, the connective dependence between X = "*the intensity of sun radiation*" = {x_1 – "*low radiation*", x_2 – "*high radiation*"} and Y = "*the daily temperature*" = {y_1 – "*the temperature in the morning*", y_2 – "*the temperature at noon*", y_3 – "*the temperature in the evening*"}. After inserting the membership degrees to each pair of $\widetilde{R}$ we state its content by the matrix

$$\widetilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.5 & 0.7 \\ 0.4 & 0.9 & 0.5 \end{bmatrix} \end{array}.$$

The next relation $\widetilde{Q}$ settles the relationship between Y = "*the daily temperature*" represented by y_1, y_2, y_3, and two states of Z = "*the moisture of soil*" = {z_1 – "*low moisture*", z_2 – "*high moisture*"}. The membership degrees $\mu_{\widetilde{Q}}(y_j, z_k)$, $j = 1, 2, 3$, $k = 1, 2$, express the truthfulness of a connection between Y and Z as

$$\widetilde{Q} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.3 & 0.9 \\ 0.8 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \end{array}.$$

By composing the relations $\widetilde{R}$ and $\widetilde{Q}$ we should obtain the result that reveals the association between X and Z described by the values of membership degrees as follows

$$\widetilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} \max(\min(0.8,0.3), \min(0.5,0.8), \min(0.7,0.5)) & \max(\min(0.8,0.9), \min(0.5,0.3), \min(0.7,0.7)) \\ \max(\min(0.4,0.3), \min(0.9,0.8), \min(0.5,0.5)) & \max(\min(0.4,0.9), \min(0.9,0.3), \min(0.5,0.7)) \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} \max(0.3,0.5,0.5) & \max(0.8,0.3,0.7) \\ \max(0.3,0.8,0.5) & \max(0.4,0.3,0.5) \end{bmatrix} \end{array} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.8 \\ 0.8 & 0.5 \end{bmatrix} \end{array}.$$

The value 0.8 assigned to the pairs (low radiation, high moisture) and (high radiation, low moisture) seems to confirm a truthful association between the examined parameters "*sun radiation*" and "*soil moisture*". It is also acceptable to admit that the connection between low radiation and low moisture or high radiation and high moisture is appreciated as true in the grade 0.5.

Even this simple example convinces us about the importance of a decisive character of the max-min composition. In spite of the greatest popularity, the results of the operation sometimes are interpreted as too "sharp" because of the action of the inner operation min that causes an essential consideration of the smallest values. To smooth this inconvenient effect, it is suggested to introduce another general definition of the composition of two fuzzy relations.

Definition 2.12

Let $\tilde{R}$ and $\tilde{Q}$ be the relations from Def. 2.11. The max-∗ composition of $\tilde{R}$ and $\tilde{Q}$ is now defined as

$$\tilde{R} \underset{*}{\circ} \tilde{Q} = \left\{ \left((x_i, z_k), \mu_{\tilde{R} \underset{*}{\circ} \tilde{Q}}(x_i, z_k) = \max_{y_j \in Y} \left\{ \mu_{\tilde{R}}(x_i, y_j) * \mu_{\tilde{Q}}(y_j, z_k) \right\} \right) \right\} \tag{2.22}$$

for $x_i \in X, y_j \in Y, z_k \in Z$, where "∗" is an associative operation that is monotonically non-decreasing in each argument [95].

An arbitrary *t*-norm satisfies the conditions of monotonicity and associativity (properties 2. and 4. in (2.11)); therefore it can be utilized in Eq. 2.22. Two special cases of the operation max-∗ are taken into consideration in the following definition.

Definition 2.13

Let $\tilde{R}$ and $\tilde{Q}$ be fuzzy relations of the same shape as in Defs 2.11 and 2.12. Hence, the max-*prod* composition $\tilde{R} \underset{\cdot}{\circ} \tilde{Q}$ and the max-*av* composition $\tilde{R} \underset{av}{\circ} \tilde{Q}$ are proposed as fuzzy relations

$$\tilde{R} \underset{\cdot}{\circ} \tilde{Q} = \left\{ \left((x_i, z_k), \mu_{\tilde{R} \underset{\cdot}{\circ} \tilde{Q}}(x_i, z_k) = \max_{y_j \in Y} \left\{ \mu_{\tilde{R}}(x_i, y_j) \cdot \mu_{\tilde{Q}}(y_j, z_k) \right\} \right) \right\} \tag{2.23}$$

and

$$\tilde{R} \underset{av}{\circ} \tilde{Q} = \left\{ \left((x_i, z_k), \mu_{\tilde{R} \underset{av}{\circ} \tilde{Q}}(x_i, z_k) = \frac{1}{2} \cdot \max_{y_j \in Y} \left\{ \mu_{\tilde{R}}(x_i, y_j) + \mu_{\tilde{Q}}(y_j, z_k) \right\} \right) \right\} \quad (2.24)$$

for $x_i \in X, y_j \in Y, z_k \in Z$.

The operation $\mu_{\tilde{R}} + \mu_{\tilde{Q}}$ used in (2.24) has not all properties of a *t*-norm but it fulfils the conditions 2. and 4. composed in (2.11). This classifies the sum of membership degrees as the appropriate operation "∗", accepted in the composition of fuzzy relations [95].

We return to the basic relations $\tilde{R}$ and $\tilde{Q}$ from Ex. 2.19 to test results of the formulas (2.23), (2.24) and to compare these results to the effects of the max-min composition already obtained in Ex. 2.19.

Example 2.20

Let us compute the relations $\tilde{R} \underset{\cdot}{\circ} \tilde{Q}$ and $\tilde{R} \underset{av}{\circ} \tilde{Q}$ for $\tilde{R}$ and $\tilde{Q}$ from Ex. 2.19. By applying Eq. (2.23) we obtain

$$\tilde{R} \underset{\cdot}{\circ} \tilde{Q} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} \max(0.8 \cdot 0.3, 0.5 \cdot 0.8, 0.7 \cdot 0.5) & \max(0.8 \cdot 0.9, 0.5 \cdot 0.3, 0.7 \cdot 0.7) \\ \max(0.4 \cdot 0.3, 0.9 \cdot 0.8, 0.5 \cdot 0.5) & \max(0.4 \cdot 0.9, 0.9 \cdot 0.3, 0.5 \cdot 0.7) \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} \max(0.24, 0.4, 0.35) & \max(0.72, 0.15, 0.49) \\ \max(0.12, 0.72, 0.25) & \max(0.36, 0.27, 0.35) \end{bmatrix} \end{array} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.4 & 0.72 \\ 0.72 & 0.36 \end{bmatrix} \end{array}$$

while the formula (2.24) used to the max-*av* composition yields a matrix

$$\tilde{R} \underset{av}{\circ} \tilde{Q} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} \frac{1}{2} \cdot \max(0.8 + 0.3, 0.5 + 0.8, 0.7 + 0.5) & \frac{1}{2} \cdot \max(0.8 + 0.9, 0.5 + 0.3, 0.7 + 0.7) \\ \frac{1}{2} \cdot \max(0.4 + 0.3, 0.9 + 0.8, 0.5 + 0.5) & \frac{1}{2} \cdot \max(0.4 + 0.9, 0.9 + 0.3, 0.5 + 0.7) \end{bmatrix} \end{array}$$

$$= \begin{matrix} \\ x_1 \\ x_2 \end{matrix} \begin{matrix} z_1 & z_2 \\ \left[\begin{matrix} \frac{1}{2} \cdot \max(1.1, 1.3, 1.2) \\ \frac{1}{2} \cdot \max(0.7, 1.7, 1.0) \end{matrix} \right. & \left. \begin{matrix} \frac{1}{2} \cdot \max(1.7, 0.8, 1.4) \\ \frac{1}{2} \cdot \max(1.3, 1.2, 1.2) \end{matrix} \right] \end{matrix} = \begin{matrix} \\ x_1 \\ x_2 \end{matrix} \begin{matrix} z_1 \quad z_2 \\ \begin{bmatrix} 0.65 & 0.85 \\ 0.85 & 0.65 \end{bmatrix} \end{matrix}.$$

By analyzing the results of three tested operations $\tilde{R} \circ \tilde{Q}$, $\tilde{R} \underset{\cdot}{\circ} \tilde{Q}$ and $\tilde{R} \underset{av}{\circ} \tilde{Q}$, we conclude that it would be the most reliable to choose the second composition $\tilde{R} \underset{\cdot}{\circ} \tilde{Q}$ in further medical investigations. The choice is justified by the fact that the difference between two essential stages (*high radiation*, *high moisture*) and (*high radiation*, *low moisture*) is emphasized in the remarkable way in the composition $\tilde{R} \underset{\cdot}{\circ} \tilde{Q}$.

In some special cases, when $X = \{x_1\}$, the relation $\tilde{R}$ becomes a one row-relation. Definition 2.13 thus takes the following modified version.

Definition 2.14

Let $X = \{x_1\}$, $Y = \{y_1, \ldots, y_n\}$, $Z = \{z_1, \ldots, z_p\}$. We set fuzzy relations $\tilde{R}$ with $\mu_{\tilde{R}}(x_1, y_j)$, $(x_1, y_j) \in X \times Y$ and $\tilde{Q}$ characterized by $\mu_{\tilde{Q}}(y_j, z_k)$, $(y_j, z_k) \in Y \times Z$. The max-*prod* composition of $\tilde{R}$ with $\tilde{Q}$, denoted by $\tilde{R} \underset{\cdot}{\circ} \tilde{Q}$, is a one row fuzzy set

$$\tilde{R} \underset{\cdot}{\circ} \tilde{Q} = \left\{ \left((x_1, z_k), \mu_{\tilde{R} \underset{\cdot}{\circ} \tilde{Q}}(x_1, z_k) = \max_{y_j \in Y} \left\{ \mu_{\tilde{R}}(x_1, y_j) \cdot \mu_{\tilde{Q}}(y_j, z_k) \right\} \right) \right\} \tag{2.25}$$

for $x_1 \in X, y_j \in Y, z_k \in Z$.

The short and simple introduction that has been accomplished in this chapter is necessary to lead to a further discussion about different mathematical fuzzy models. The models are fitted for the same objective, namely, how to extract different concepts and properties developed by fuzzy set theory in order to transform them to facilitate solutions to medical problems.

3 Medical Diagnosis

3.1 Introduction

The creators of fuzzy set theory, who develop mathematical models applied to different technical domains, have also made representative contributions in medical investigations. One of the earliest models created by Sanchez [72, 73] and discussed by other scientists [6, 8, 11, 17, 27, 45, 48, 56, 69, 70, 74, 75, 76, 77, 87] has given some answers to questions concerning a choice of diagnosis. The choice should only be made on the basis of clinical symptoms when assuming that the symptoms are typical of all considered diagnoses.

To decide an appropriate diagnosis in one patient we introduce three non-fuzzy sets:

the set of symptoms $S = \{S_1, S_2, ..., S_n\}$,

the set of diagnoses $D = \{D_1, D_2, ..., D_p\}$,

the set of patients $P = \{P_1\}$.

The symptoms occurring in set S are associated with the diagnoses from set D. Moreover, we assume that information about all symptoms belonging to S is complete in the patient's case. By using his medical experience as a foundation a physician then establishes connections between the symptoms and the diagnoses.

3.2 The Modus Ponens Law in Medical Diagnosis

The symptoms S_1, S_2, …, S_n, that are stated in set S, are included in the pairs (P_1, S_1), (P_1, S_2), …, (P_1, S_n). These constitute the relation PS ("*patient – symptom*"). Let us write down the fuzzy relation PS as a one-row matrix

$$PS = P_1 \begin{array}{c} \\ \left[\right. \end{array} \begin{array}{cccc} S_1 & S_2 & \cdots & S_n \\ \mu_{PS}(P_1,S_1) & \mu_{PS}(P_1,S_2) & \cdots & \mu_{PS}(P_1,S_n) \end{array} \begin{array}{c} \\ \left. \right] \end{array}, \tag{3.1}$$

where $\mu_{PS}(P_1, S_j)$, $j = 1, …, n$, is a value of the membership degree providing us with evaluation of the intensity of S_j in P_1.

The next relation consists of the pairs (S_1, D_1), (S_1, D_2), …, (S_1, D_p), …, (S_n, D_p). The fuzzy relation, in which each value of the membership degree tied to the pair (S_j, D_k), $j = 1, 2, …, n$, $k = 1, 2, …, p$, expresses strength of the relationship

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between the symptom and the associated diagnosis, is called "*symptom – diagnosis*". The relation has the name *SD* and is projected as a matrix

$$SD = \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & \cdots & D_p \end{array} \\ \begin{bmatrix} \mu_{SD}(S_1,D_1) & \mu_{SD}(S_1,D_2) & \cdots & \mu_{SD}(S_1,D_p) \\ \mu_{SD}(S_2,D_1) & \mu_{SD}(S_2,D_2) & \cdots & \mu_{SD}(S_2,D_p) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{SD}(S_n,D_1) & \mu_{SD}(S_n,D_2) & \cdots & \mu_{SD}(S_n,D_p) \end{bmatrix} \end{array}. \tag{3.2}$$

In the further discussion we suppose that the relation *PS* is generally interpreted as a statement

"p" = "the symptoms S_j, $j = 1, 2, \ldots, n$, are found in patient P_1".

The relation *SD* stands for an implication

"p' IMPLIES q" = "the symptoms S_j confirm presence of the diagnoses D_k (if S_j then D_k), $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$".

The statement p' is nearly the same as p, but these two sentences p and p' do not need to be worded identically.

By quoting the rule *modus ponens*

IF "p" AND "p' IMPLIES q" THEN q'"

we expect getting the conclusion

"q'"= "the diagnoses D_k, $k = 1, 2, \ldots, p$, are assigned to P_1".

As the result of the last sentence q' a relation *PD* has been produced. The *PD* ("*patient – diagnosis*") relation is a matrix

$$PD = P_1 \overset{\begin{array}{cccc} D_1 & D_2 & \cdots & D_p \end{array}}{[\mu_{PD}(P_1,D_1) \;\; \mu_{PD}(P_1,D_2) \;\; \cdots \;\; \mu_{PD}(P_1,D_p)]}, \tag{3.3}$$

in which the membership degrees reveal associations between the patient and the diagnoses.

To make a proper choice of the diagnosis that is the most applicable for the examined patient when regarding his or her symptoms, we employ the *modus ponens* rule as follows.

Definition 3.1

If the premise “the symptoms S_j, $j = 1, 2, \ldots, n$, are found in patient P_1” is given by the relation *PS*, and the hypothesis ”the symptoms S_j imply the diagnoses D_k, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$” is represented by the relation *SD*, then the relation $PD = PS \circ SD$, formed as a result of the thesis “P_1 suffers from D_k”, will be composed of membership degrees allowing us to estimate associations between the patient and the considered diagnoses.

We will study some modifications of Def. 3.1 in the next subsections.

3.3 The *Patient –Symptom* Relation

Each symptom belonging to the set *S* will be represented as a fuzzy set.

We adopt three basic types of biological parameters [29, 30, 32, 56]:

1. Simple qualitative features,
2. Compound qualitative features,
3. Quantitative (measurable) features.

Example 3.1

Suppose that symptoms from set *S*, characteristic of P_1, are listed as: S_1 – “*hereditary inclination*”, S_2 – “*ECG changes in resting position*”, S_3 – “*smoking*”, S_4 – “*lack of physical activity*”, S_5 – “*pain in chest*”, S_6 –“*breathlessness*”, S_7 – “*feeling of sickness*”, S_8 – “*hypertension*”, S_9 – “*increased level of LDL-cholesterol*”, S_{10} – “*obesity*”.

By studying the nature of the symptoms we can divide them into three following groups:

1. S_1 is interpreted as the simple qualitative feature that is present or lacking;
2. S_2, S_3, S_4, S_5, S_6, S_7 are investigated with the help of a questionnaire as compound qualitative symptoms;
3. S_8, S_9, S_{10} are typical measurable parameters that are described by values obtained in examinations carried out.

Since every symptom is considered as a fuzzy set then we should decide the set’s support and values of membership degrees assigned to the members of the support. The ways of membership degree constructions should reflect intensities of symptoms acting as important indications of the patient’s health.

We now intend to concentrate this discussion on three types of symptoms that will be designed as fuzzy sets. Since they represent different features then we will be obliged to invent varying methods of designing membership functions for them.

3.3.1 Simple Qualitative Biological Parameters

The fuzzy set characterizing this symptom is defined in a space {0, 1}. If the symptom S_j does not occur in the patient, the number 0 will symbolize its lack. It means that element x of the fuzzy set S_j takes the value 0, and the membership degree of this element is fixed to be 0, too.

If the symptom S_j is found in the patient, we agree to note this fact down as the number 1. Thus, element x takes the value 1 and the membership degree of x is also determined as 1.

The fuzzy set S_j corresponding to the symptom S_j is often written down symbolically as

$$S_j = 0/0 + 1/1 . \qquad (3.4)$$

In Eq. (3.4) both the symptom and the fuzzy set representing it have the same denotation in order not to introduce too many symbols.

Example 3.2

Suppose that patient P_1 has had relatives who have suffered from cardiovascular system diseases. Hence, his tendency to inherit these diseases is evaluated as $x = 1$ and $S_1(1) = 1$. Consequently, the value of $S_1(1) = 1$ takes also its place in the relation PS that is actualized as

$$\begin{array}{rccccccccccc} & & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \\ PS = P_1 & [& 1 & & & & & & & & &] . \end{array}$$

The qualitative attribute wearing the simple complexity is the simplest possible. It is frequently required to have a wider approach to the ascertainment of qualitative features, for example, by applying a questionnaire with questions and alternative answers to the questions being posed. This is designed in the case of compound qualitative features for which the construction of fuzzy set S_j corresponding to the symptom S_j is thoroughly explained below.

3.3.2 Compound Qualitative Biological Features

We assume now that the symptom S_j is no longer determined by one of the alternatives "*present – absent*", but it needs a certain verbal description. Let us introduce a new variable supporting S_j, the linguistic variable $Q^{S(j)}$, with the set of terms formed by the questions (asked by a physician or by a questionnaire) that are denoted here by $q_p^{S(j)}$, $p = 1, 2, ..., Q^{S(j)-last\,question}$ [29, 30, 32, 56, 70]. The symbol $Q^{S(j)\text{-}last\ question}$ stands for the quantity of questions associated with the symbol S_j.

To each of the questions $q_p^{S(j)}$ posed to the person being examined in connection with the symptom S_j found, he/she has a possibility of choosing one of several answers that are usually furnished with numbers-codes $s_{p,t}^{S(j)} = 0,1,2, ..., N$, $p = 1,2, ..., Q^{S(j)-last\,question}$, $t = 1,2, ..., t^{(p)}$. The symbol $t^{(p)}$ designates the number of alternative answers to the question $q_p^{S(j)}$, and we understand that the value $t = 1$ is connected to the choice of the code "0", while $t = t^{(p)}$ must be associated with pointing to the code "N". The codes 0, 1, …, N are hierarchical replacements of the answers from the most negative (denying the presence of S_j) to the most positive (confirming the existence of the symptom).

In the further procedure, one should assign weights $w_{p,t}^{S(j)}$ to the encoded (proposed) answers $s_{p,t}^{S(j)}$ [70]. These weights are suggested to be numbers belonging to the interval [–1, 1]. It is assumed that negative numbers correspond to negative answers to the question posed, i.e., ones that do not confirm the occurrence of the symptom (never, rarely, very rarely and the like), with that –1 defines the most negative answer. The positive value of the weight gives a suitable positive character, certifying the presence of the symptom in the patient, and the value +1 confirms the entire presence of the symptom. The weight equal to zero (or close to zero) is reserved for the case of the lack of the answer or a statement that does not bring any (or almost any) information.

The set of questions in $Q^{S(j)}$ is treated as a list of terms representing the linguistic variable $Q^{S(j)}$. Each question $q_p^{S(j)}$, $p = 1,2, ..., Q^{S(j)-last\,question}$, is interpreted as a fuzzy variable with the support, established via weights, equal to [–1, 1]. This conception helps to build a fuzzy set S_j reflecting the compound qualitative symptom S_j with the same name.

The value of a weight lying in the subintervals [–1, 0] and [0, 1], respectively, depends on the number of a code 1, 2, …, N, chosen by the patient who answers a corresponding question. The above-mentioned intervals are, in general, decomposed evenly according to the number of alternative answers. The endpoints of the subintervals thus formed constitute the values of the weights for all the alternative answers. The number of the weights is finite and equal, as a rule, to a slight number of answers. To represent the symptom S_j by one value being a result of an operation concatenating all weights assigned to the questions $q_p^{S(j)}$, p = 1, 2, …, $Q^{S(j)-last\ question}$, it seems to be purposeful to bring into use a continuous aggregation function.

To sum up, let S_j be the fuzzy set for a compound qualitative feature. This set is fully defined when both its support and its membership function is given. The function should be, in turn, considered in a real space X and built as a set of weight aggregation results. Therefore it is essential to transpose the qualitative feature to measurable values, i.e., to values x belonging to X, where X is regarded as a support of S_j. To concatenate the weights of all answers as a common value x describing S_j, we suggest the mapping

$$x = \sum_{p=1}^{Q^{S(j)-last\,question}} w_{p,t}^{S(j)} . \tag{3.5}$$

The operation means the addition of the weights determined for the answers to the questions that collect information about S_j (every question is represented by only one weight joined to the alternative answer $t = 1, \ldots, t^{(p)}$).

The values of x form an interval bounded below by some number α that is calculated from the formula

$$\alpha = \sum_{p=1}^{Q^{S(j)-last\,question}} \min_{1 \le t \le t^{(p)}} (w_{p,t}^{S(j)}) , \tag{3.6}$$

i.e., is assumed to be the sum of the weights, smallest as far as their relative values are concerned, assigned to the most negative answers to each question giving a picture of S_j.

A number γ bounds this interval from above and is fixed in accordance with the equation

$$\gamma = \sum_{p=1}^{Q^{S(j)-last\,question}} \max_{1 \le t \le t^{(p)}} (w_{p,t}^{S(j)}) \tag{3.7}$$

that means that one should calculate the sum of the weights greatest as far as their relative values are concerned. They are quantities representing the extreme positive answers to the questions concerning the symptom S_j.

The conclusions, implied above, have led to the construction of the support of the compound qualitative attribute S_j defined as the interval $[\alpha, \gamma]$. The membership function over $[\alpha, \gamma]$ is put forward for consideration as [29, 30, 32, 56]

$$y = \mu_{S_j}(x) = s(x, \alpha, \beta, \gamma) , \tag{3.8}$$

in which $s(x, \alpha, \beta, \gamma)$ is given by Eq. (2.5).

Example 3.3

If a value of the weight assigned to the extreme negative answer is fixed at -1 and the weight of its most positive variant is equal to $+1$, and the rule is preserved for each question, then the graph of the function (3.8), designed for five questions characteristic of S_j and formed as the equation $y = \mu_{S_j}(x) = s(x, -5, 0, 5)$, will take the symmetrical form as given in Fig. 3.1.

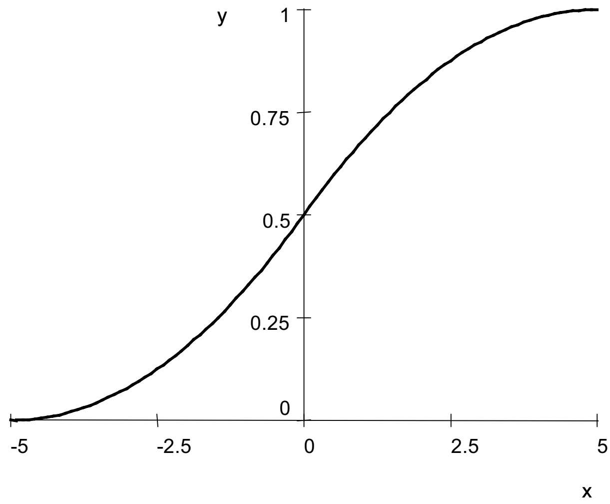

Figure 3.1: The membership function of the compound qualitative symptom S_j

By analysing the graph we can notice there the number $\alpha = -5$. That is the sum of the minimal weights representing the five answers negating the symptom presence, it has the assigned membership degree equal to zero. This is a sign of "health". The number $\gamma = 5$ is a sum of maximally positive weights confirming the revealed symptom and, in consequence, the membership degree associated with it takes the value 1. An ambiguous piece of information or its lack, expressed by β, has its mapping in the form of the membership degree of the value 0.5.

To illustrate the action of computing the membership degrees for compound qualitative symptoms, let us state the values of degrees for some of the symptoms of this character found in patient P_1, with whom we have already made an acquaintance in Ex. 3.1.

Example 3.4

By composing a questionnaire, which yields the information of intensity accompanying the symptoms S_3 and S_4, we assign the membership degree to each of them.

An inquiry of the first considered symptom S_3 – "*smoking*", is accomplished by answering questions, e.g.:

$q_1^{S(3)}$ ="*How often do you smoke cigarettes?*"

$q_2^{S(3)}$ ="*How long have you smoked?*"

The alternative answers to these questions may be formulated, for instance, as follows:

– to the first question $q_1^{S(3)}$

1. *I do not smoke* $s_{1,1}^{S(3)} = 0$ $w_{1,1}^{S(3)} = -1$
2. *I smoke seldom* $s_{1,2}^{S(3)} = 1$ $w_{1,2}^{S(3)} = -\frac{1}{2}$
3. *I smoke moderately* $s_{1,3}^{S(3)} = 2$ $w_{1,3}^{S(3)} = 0$
4. *I smoke often* $s_{1,4}^{S(3)} = 3$ $w_{1,4}^{S(3)} = \frac{1}{2}$
5. *I smoke constantly* $s_{1,5}^{S(3)} = 4$ $w_{1,5}^{S(3)} = 1$

– to the second question $q_2^{S(3)}$

1. *I have never smoked* $s_{2,1}^{S(3)} = 0$ $w_{2,1}^{S(3)} = -1$
2. *A few months* $s_{2,2}^{S(3)} = 1$ $w_{2,2}^{S(3)} = -\frac{2}{3}$
3. *1–2 years* $s_{2,3}^{S(3)} = 2$ $w_{2,3}^{S(3)} = -\frac{1}{3}$
4. *I cannot determine* $s_{2,4}^{S(3)} = 3$ $w_{2,4}^{S(3)} = 0$
5. *3–4 years* $s_{2,5}^{S(3)} = 4$ $w_{2,5}^{S(3)} = \frac{1}{3}$
6. *More than 5 years* $s_{2,6}^{S(3)} = 5$ $w_{2,6}^{S(3)} = \frac{2}{3}$
7. *I have smoked since my teens* $s_{2,7}^{S(3)} = 6$ $w_{2,7}^{S(3)} = 1$.

We assume that patient P_1 has chosen answer *4.* to $q_1^{S(3)}$ and answer *6.* to $q_2^{S(3)}$; hence, $x = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$, $\alpha = (-1) + (-1) = -2$, $\gamma = 1 + 1 = 2$, $\beta = 0$, and

$\mu_{S_3}(x) = s(x, -2, 0, 2) = 1 - 2\left(\frac{x-2}{2-(-2)}\right)^2$ in view of $x > 0$, that is $\mu_{S_3}\left(\frac{7}{6}\right) \approx 0.913$.

The pair (P_1, S_3) of the relation *PS* has therefore the membership degree equal to 0.913. This degree characterizes a feature such as "*smoking*" efficiently, since it reflects every subtle distinction in the patient's report when comparing the extreme information of the type "*I smoke*" contrary to "*I do not smoke*".

The other parameter S_4 – "*lack of physical activity*" can be described, as an instance, by three questions:

$q_1^{S(4)}$ = "*Do you sometimes exercise?*"

$q_2^{S(4)}$ = "*How intensively do you exercise?*"

$q_3^{S(4)}$ = "*How long have you exercised?*"

The answers proposed in the questionnaire with their codes and weights are established as:

– to the first question $q_1^{S(4)}$

1. *I exercise every day* $w_{1,1}^{S(4)} = -1$
2. *I exercise often* $w_{1,2}^{S(4)} = -\frac{1}{2}$
3. *I cannot say* $w_{1,3}^{S(4)} = 0$
4. *I exercise little* $w_{1,4}^{S(4)} = \frac{1}{2}$
5. *I do not exercise* $w_{1,5}^{S(4)} = 1$

– to the second question $q_2^{S(4)}$

1. *I exercise very hard* $w_{2,1}^{S(4)} = -1$
2. *I exercise rather hard* $w_{2,2}^{S(4)} = -\frac{1}{2}$
3. *It is difficult to say* $w_{2,3}^{S(4)} = 0$
4. *I do light exercises* $w_{2,4}^{S(4)} = \frac{1}{2}$
5. *I do not do any exercises* $w_{2,5}^{S(4)} = 1$

– to the third question $q_3^{S(4)}$

1. *I have exercised since my teens* $w_{3,1}^{S(4)} = -1$
2. *More than 5 years* $w_{3,2}^{S(4)} = -\frac{2}{3}$
3. *3–4 years* $w_{3,3}^{S(4)} = -\frac{1}{3}$
4. *I cannot determine* $w_{3,4}^{S(4)} = 0$
5. *1–2 years* $w_{3,5}^{S(4)} = \frac{1}{3}$
6. *A few months* $w_{3,6}^{S(4)} = \frac{2}{3}$
7. *I have never exercised* $w_{3,7}^{S(4)} = 1$.

Let us notice that the lack of *exercising* is regarded as a harmful symptom for some diseases. This is expressed by assigning the positive values of weights to the answers that confirm the bad physical condition. On the contrary, a well-trained person does not run a great risk of falling ill. Exercising can increase the health condition and therefore is added to prior physical factors. The answers pointing to a good physical form have thus the negative weights attached.

If patient P_1 marks the answers *3. 4.* and *4.*, respectively, we should find the value of the element x for S_4 as $x = 0 + \frac{1}{2} + 0 = \frac{1}{2}$, $\alpha = (-1) + (-1) + (-1) = -3$, $\gamma = 1 + 1 + 1 = 3$, $\beta = 0$, and $\mu_{S_4}(x) = s(x, -3, 0, 3) = 1 - 2\left(\frac{x-3}{3-(-3)}\right)^2$ because of the positive value of x that gives $\mu_{S_4}\left(\frac{1}{2}\right) \approx 0.653$.

The compound qualitative attribute S_4 has been assessed by the value of the membership degree 0.653. We act in the same way as discussed above to state the membership degrees of the rest of the compound qualitative attributes, calculating them as, e.g.,

$\mu_{S_2 - \text{"}\mathit{ECG\ changes\ in\ resting\ position}\text{"}}(x) = 0.515$,

$\mu_{S_5 - \text{"}\mathit{pain\ in\ chest}\text{"}}(x) = 0.345$,

$\mu_{S_6 - \text{"}\mathit{breathnessless}\text{"}}(x) = 0.632$,

$\mu_{S_7 - \text{"}\mathit{feeling\ of\ sickness}\text{"}}(x) = 0.720$.

The values of x via weights characterize the questionnaire answers of patient P_1. The membership degrees of x fill up some remaining empty positions in the relation *PS*. After completing the missing values the relation *PS* has its content presented as

$$PS = \begin{array}{c} \\ P_1 \end{array} \begin{array}{cccccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \\ [\ 1 & 0.515 & 0.913 & 0.653 & 0.345 & 0.632 & 0.720 & & & \ ] \end{array}.$$

There still exist three symptoms S_8, S_9 and S_{10} that do not have their membership degrees determined. The mentioned parameters belong to the third symptom class created for quantitative features. To find their membership values we refer to "arguing", which is portrayed in the next subsection.

3.3.3 Compound Quantitative Biological Features

A quantitative (measurable) feature is the last type of a biological parameter for which a model of defining the membership degree in the relation *PS* has been produced. These features take values continuously from a known interval determined by a physician. The measurable symptom S_j can be represented as the fuzzy set S_j with values from the interval containing all possible quantities taken by this feature. We denote this interval by [*VMIN*, *VMAX*] with the notations; *VMIN* is the minimal value taken by the parameter and, respectively, *VMAX* is its maximal value. Let us give the symbols VN_1, VN_2 to the limits of the interval in which there occur numbers characteristic of healthy man. It should be noticed that, outside the interval (VN_1, VN_2), both the deficiency and the excess of a biological indicator is a disease sign, most frequently connected with different diagnoses. Therefore one should divide the interval [*VMIN*, *VMAX*] adopted originally and write it down as [29, 30, 32, 56]

$$[VMIN, VMAX] = [VMIN, VN_1] \cup (VN_1, VN_2) \cup [VN_2, VMAX]. \tag{3.9}$$

The membership function of the set S_j, defined on the interval (3.9) ought to express reliably all disease states examined on the basis on the symptom S_j. For symptoms whose uniform growth of values is connected with a uniformly progressing disease, one proposes the membership function

$$y = \mu_{S_j}(x) = \begin{cases} s\left(x, VN_2, \frac{VN_2 + VMAX}{2}, VMAX\right) & \text{for} \quad VN_2 \le x \le VMAX, \\ 0 & \text{for} \quad VN_1 < x < VN_2, \\ 1 - s\left(x, VMIN, \frac{VMIN + VN_1}{2}, VN_1\right) & \text{for} \quad VMIN \le x \le VN_1, \end{cases} \tag{3.10}$$

where x stands for the value taken by S_j.

Example 3.5

The quantitative symptom S_8 – "*hypertension*" from Ex. 3.1 is a consequence of increased values of systolic blood pressure. This parameter has an interval of normal values, typical of a healthy man, established as (VN_1, VN_2) = (90 *mmHg*, 120 *mmHg*) while *VMIN*, *VMAX* are appreciated as 50 *mmHg*, 250 *mmHg*, respectively. It is worth mentioning that high values of systolic blood pressure are assigned – as warning signals – to cardiovascular diseases. Opposite to it, very low values of systolic blood pressure are typical of, e.g., acute bleeding. On the condition that the membership function of S_8 reflects a uniformly progressive sickly state, we propose to explore it by the equation

$$y = \mu_{S_8}(x) = \begin{cases} s\left(x, 120, \frac{120 + 250}{2}, 250\right) & \text{for} \quad 120 \le x \le 250, \\ 0 & \text{for} \quad 90 < x < 120, \\ 1 - s\left(x, 50, \frac{50 + 90}{2}, 90\right) & \text{for} \quad 50 \le x \le 90, \end{cases}$$

represented by the graph displayed in Fig. 3.2.

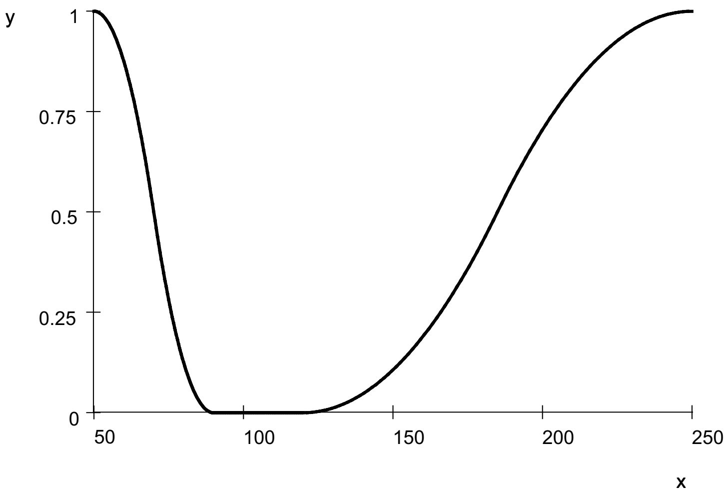

Figure 3.2: The membership function of the symptom S_8

More complicated medical phenomena can be divided into two following groups. The physician often finds that the growth of a value characterizing the symptom and belonging to the interval [VN_2, $VMAX$] (or to [$VMIN$, VN_1] when the value is lowered instead) does not matter essentially till some moment, and only a high level of the indicator is connected with a violent deterioration of the health condition.

Then it would be purposeful to apply a concentration operation CON for the membership function of the fuzzy set S_j over the interval [VN_2, $VMAX$] (or [$VMIN$, VN_1]), that is, to use the membership function of the type

$$y = \mu_{CON(S_j)}(x) = \left(\mu_{(S_j)}(x)\right)^2 . \tag{3.11}$$

Conversely, if the physician describes that what constitutes the greatest danger for one's health is the growth of the symptom value at the first stage, then it is advisable to introduce a dilution operation DIL for the symptom S_j that changes the membership function $\mu_{S_j}(x)$ in the following manner

$$y = \mu_{DIL(S_j)}(x) = \sqrt{\mu_{S_j}(x)} . \tag{3.12}$$

The membership function of S_j modified by (3.11) or (3.12) better reflects the patient's physical state when an irregular development of the symptom S_j has an importance in finding the appropriate diagnosis.

Example 3.6

The graphs of the function (3.10) as well as the above-described tendencies towards a reliable adoption of the membership function for a measurable feature S_8, discussed in Ex. 3.5, are shown in Fig. 3.3.

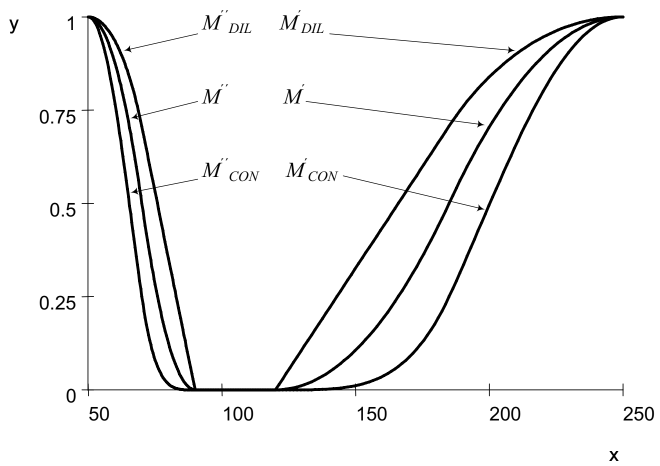

Figure 3.3: The membership function of the symptom S_8 with modifications

In the description of the figure we have adopted the notations

$$M' = s\left(x, 120, \frac{120+250}{2}, 250\right) \quad \text{for} \quad 120 \le x \le 250,$$

$$M'' = 1 - s\left(x, 50, \frac{50+90}{2}, 90\right) \quad \text{for} \quad 50 \le x \le 90.$$

The results of Eqs (3.11) and (3.12) obtained in the form of modified membership degrees, in comparison to the effects of (3.10), are intended to be treated as reliable indicators of the symptom's decisive effect on a diagnosis choice.

Example 3.7

The physician decides that the value of S_8, found in P_1, is equal to 210 *mmHg* and points to a severe state of the patient's health. We expect that an appropriate membership degree will be associated with the value of S_8 in order to indicate the essential impendence of P_1's health. To start the computations, we choose the first part of the function $\mu_{S_8}(x)$ from Ex. 3.5 because we notice that the value of $x = 210$ belongs to the interval [120, 250].

Hence,

$$\mu_{S_8}(210) = s\left(210,120,\frac{120+250}{2},250\right) = s(210,120,185,250) =$$

$$1-2\left(\frac{210-250}{250-120}\right)^2 \approx 0.81.$$

To focus on the essentially heightened blood pressure value of $x = 210$, which convinces us that the patient's health is in danger, we should increase $\mu_{S_8}(210)$ by adopting the dilution operation as $\mu_{DIL(S_8)}(210) = \sqrt{\mu_{S_8}(210)} \approx \sqrt{0.81} = 0.9$.

It is worth noticing that we should use the third part of the formula derived in Ex. 3.5, i.e., $\mu_{S_j}(x) = 1 - s\left(x,50,\frac{50+90}{2},90\right)$ if the value of systolic blood pressure is less than 90 in any patient. It can happen when the patient meets with violent bleeding. The lower values of systolic blood pressure than the quantities placed in [90, 120] characterize another symptom that differs from S_8.

The growth of *LDL*-cholesterol also informs a physician about a worse condition of the examined patient. To accentuate the importance of the increased level of S_9 – "*increased level of LDL-cholesterol*" we try to generate a reliable value of its membership degree in *PS*.

Example 3.8

By carrying out *LDL-cholesterol* level examinations in P_1, the physician has established the value of S_9 as 145 *mg*/*dl*. If he also decides the values of *VMIN* = 50 *mg*/*dl*, *VMAX* = 250 *mg*/*dl* and states the interval (VN_1, VN_2) as (100 *mg*/*dl*, 135 *mg*/*dl*), then we are capable of assigning the membership degree of $x = 145$ in accordance with the first "branch" of the membership function given by (3.10) as $\mu_{S_9}(145) = s(145, 135, 192.5, 250) = 2\left(\frac{145-135}{250-135}\right)^2 \approx 0.02$. Since the value of $x =$ 145 does not lay emphasis on any greater threat for P_1's health, then we should lower the value of the membership degree for the x-number. We will reduce it if we apply the concentration operation (3.11) in the form of $\mu_{CON(S_9)}(145)$ $= (\mu_{S_9}(145))^2 \approx 0.0004$.

By using these similar techniques on the symptom S_{10}, we assign to its missing representative, in the relation *PS*, the value of the membership degree decided as $\mu_{S_{10}-\text{"obesity"}}(x) = 0.353$, when measuring "*weight*" = 115 *kg* in relation to "*height*" = 1.80 *m*. The grade of obesity x is determined by applying the body mass index x = *BMI* computed as a quotient "*body weight*/(*body height in meters*)^2" in units equal to kg/m^2. By taking $x = 35.5\ kg/m^2$, $VMIN = 12\ kg/m^2$, $VMAX = 50\ kg/m^2$, $VN_1 = 18\ kg/m^2$ and $VN_2 = 25\ kg/m^2$ as the parameters' values in

(3.10), we confirm the result 0.353 as the membership degree of S_{10}, provided that the growth of body weight has the uniform meaning in the diagnostic decision.

Possessing all the values of the membership degrees for the symptoms listed in Ex. 3.1, we finally complete the matrix *PS* and save it to further computations as

$$\begin{array}{cccccccccccc} & & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \\ PS = & P_1 [& 1 & 0.515 & 0.913 & 0.653 & 0.345 & 0.632 & 0.720 & 0.9 & 0.0004 & 0.353\ ]. \end{array}$$

The next stage in our investigations is to prepare the matrix *SD* introduced by (3.2) that constitutes the other factor (besides *PS*) in the decision equation $PD = PS \circ SD$.

3.4 The *Symptom – Diagnosis* Relation

A mathematical process of developing the forms of associations among the symptoms $S_j, j = 1, 2, \ldots, n,$ and diagnoses D_k, $k = 1, 2, \ldots, p$, constitutes another important task to be fulfilled in the diagnostic problem posed. The connection between S_j and D_k in each pair (S_j, D_k) is rendered as a value of the membership degree accompanied by this pair. The membership degree, in turn, expresses the signification of symptom S_j for diagnosis D_k. To determine the intensity of the symptom influence on the diagnosis decisive character, a physician asks two essential questions, namely [2, 3]:

1. How often is S_j found in D_k?
2. How often is S_j decisive for D_k?

The physician uses his experience to answer the questions by selecting verbal expressions that are included in a certain list. We, by following his advice, concentrate on replacing the chosen word by an appropriate number. To begin with, let us first decide the terms of a common list identically constructed for "*presence*" and "*decisive character*". We decide "*presence*" = "*decisive character*" ={"*never*", "*almost never*", "*very seldom*", "*seldom*", "*rather seldom*", "*moderately*", "*rather often*", "*often*", "*very often*", "*almost always*", "*always*"}. By employing the membership functions (the constraints) of fuzzy variables that correspond to the expressions collected above, we will try to find these membership degrees of the variables that represent them in the most adequate way.

Some suggestions referring to the constraints imposed on fuzzy variables have already been made in Ex. 2.15, but we should honestly admit that the borders of the restrictions proposed there have been chosen empirically rather than by involving any computation technique. There exists a domain in fuzzy set theory that deals with computing with words. Within this field we can mathematically modify some sophisticated linguistic expressions coming from basic verbal items. The supports of fuzzy variables and the membership functions laid over them thus will be elaborated without guessing their values, which seems to improve a correctness of further computations.

3.4.1 Numerical Representations of Linguistic Variables

One of the most important features of fuzzy set theory, which makes it very attractive for applications, is its potential for the modelling of natural language expressions. Most works done on this topic focus on some parts of natural language, mostly those that correspond to the so-called "*evaluating linguistic expressions*", i.e., the dissertations show how to build fuzzy constraints for the expressions that mark characteristic limits on an ordered scale [40, 49, 90, 91, 92, 93, 94, 95].

Keeping in mind the premises from Subsection 2.4, let us preserve a reference set as the range $A = [0, A_l]$. This contains supports of all fuzzy variables corresponding to the terms placed in the lists "*presence*" and "*decisive character*". We first define three atomic expressions in A, i.e., "*the leftmost*" = "*seldom*", "*in the middle*" = "*moderately*" and "*the rightmost*" = "*often*". We thus propose the following constrains for "*the leftmost*" and "*the rightmost*" variables, provided that the abbreviation "*se*" points at the parameters of "*seldom*" [49]:

$$y = \mu_{\text{"}seldom\text{"}}(x) = \begin{cases} 1 & \text{for } x \le \alpha_{se}, \\ \dfrac{2(\beta_{se} - \alpha_{se})^2 - (x - \alpha_{se})^2}{2(\beta_{se} - \alpha_{se})^2} & \text{for } \alpha_{se} < x \le \beta_{se}, \\ \dfrac{(\gamma_{se} - x)^2}{2(\gamma_{se} - \beta_{se})^2} & \text{for } \beta_{se} < x < \gamma_{se}, \\ 0 & \text{for } x \ge \gamma_{se}, \end{cases} \tag{3.13}$$

while "*often*" is dependent on the parameters of "*seldom*" in the way

$$y = \mu_{\text{"}often\text{"}}(x) = \\ \begin{cases} 0 & \text{for } x \le A_l - \gamma_{se}, \\ \dfrac{(x - (A_l - \gamma_{se}))^2}{2((A_l - \beta_{se}) - (A_l - \gamma_{se}))^2} & \text{for } A_l - \gamma_{se} < x \le A_l - \beta_{se}, \\ \dfrac{2((A_l - \alpha_{se}) - (A_l - \beta_{se}))^2 - (x - (A_l - \alpha_{se}))^2}{2((A_l - \alpha_{se}) - (A_l - \beta_{se}))^2} & \text{for } A_l - \beta_{se} < x < A_l - \alpha_{se}, \\ 1 & \text{for } x \ge A_l - \alpha_{se}. \end{cases} \tag{3.14}$$

After transforming Eq. (3.13) we realize that it is identical with $\mu_{\text{"}seldom\text{"}}(x) = 1 - s(x, \alpha_{se}, \beta_{se}, \gamma_{se})$ while Eq. (3.14) is identified with $\mu_{\text{"}often\text{"}}(x) = s(x, \alpha_{of}, \beta_{of}, \gamma_{of})$, where $\alpha_{of} = A_l - \gamma_{se}$, $\beta_{of} = A_l - \beta_{se}$, $\gamma_{of} = A_l - \alpha_{se}$, and the abbreviation "*of*" is intended for the variable "*often*". Let us also suppose that $\gamma_{se} = \alpha_{of} = \dfrac{A_l}{2}$.

Example 3.9

We accept as a common range for all the variables the reference set A = [0, 1, 2, …, 100] by following the results obtained in Ex. 2.15. Then A_l takes the value of 100 as the largest value in the range A. If we also state the values of α_{se}, β_{se}, γ_{se} as 30, 40, 50, respectively, in "*seldom*" and "*often*" we then will implement the membership functions of the "*leftmost*" and "*rightmost*" atomic words derived as the split definitions

$$y = \mu_{\text{"}seldom\text{"}}(x) = \begin{cases} 1 & \text{for} \quad x \le 30, \\ \dfrac{2(40-30)^2 - (x-30)^2}{2(40-30)^2} & \text{for} \quad 30 < x \le 40, \\ \dfrac{(x-50)^2}{2(50-40)^2} & \text{for} \quad 40 < x < 50, \\ 0 & \text{for} \quad x \ge 50, \end{cases} \tag{3.15}$$

and

$$y = \mu_{\text{"}often\text{"}}(x) = \\ \begin{cases} 0 & \text{for } x \le 100-50, \\ \dfrac{(x-(100-50))^2}{2((100-40)-(100-50))^2} & \text{for } 100-50 < x \le 100-40, \\ \dfrac{2((100-30)-(100-40))^2 - (x-(100-30))^2}{2((100-30)-(100-40))^2} & \text{for } 100-40 < x < 100-30, \\ 1 & \text{for } x \ge 100-30. \end{cases} \tag{3.16}$$

We emphasize that we only need to define "*the leftmost*" description to implement both membership functions for "*seldom*" and "*often*".

The membership function of "*moderately*" still remains equal to the function introduced by (2.6), and is adopted here as an atomic expression formulated as "*in the middle*". This takes a form of

$$\pi\left(x, \frac{A_l}{2} - \alpha_{se}, \frac{A_l}{2}\right) = \begin{cases} s\left(x, \alpha_{se}, \dfrac{A_l + 2\alpha_{se}}{4}, \dfrac{A_l}{2}\right) & \text{for} \quad x \le \dfrac{A_l}{2}, \\ 1 - s\left(x, \dfrac{A_l}{2}, \dfrac{3A_l - 2\alpha_{se}}{4}, A_l - \alpha_{se}\right) & \text{for} \quad \text{x} > \dfrac{A_l}{2}. \end{cases} \tag{3.17}$$

Example 3.10

For A_l = 100 and α_{se} = 30, which generate the borders in (3.17), the last formula becomes

$$y = \pi(x,20,50) = \begin{cases} s(x,30,40,50) & \text{for} \quad x \le 50, \\ 1 - s(x,50,60,70) & \text{for} \quad \mathrm{x} > 50. \end{cases}$$

We often need to widen the list of expressions coming into existence from the atomic words with the membership functions established in Ex. 3.9. If we want to use the descriptions "*very seldom*" or "*rather seldom*", then we should adjust the membership functions of new fuzzy variables that possess names consisting of both the atomic words and hedges. The hedges are interpreted as additional descriptions (usually adjectives) added to atomic words. In the word compositions "*very seldom*" or "*rather seldom*", the hedges are found as "*very*" and "*rather*". To generate membership functions of sophisticated linguistic formulations, including such adjectives as "*very*", "*rather*", "*almost*" and the like, we add a parameter δ to the parameters α_{se}, β_{se}, γ_{se}, already existing in (3.13) and (3.14). The action of the parameter δ introduces either a narrowing or a widening effect in membership functions of these fuzzy variables that are derived from atomic expressions.

Example 3.11

We suggest the formulas of membership functions with hedges for two groups of fuzzy variables. We modify Eq. (3.15) as

$$y = \mu_{\text{"}hedge\ seldom\text{"}}(x) = \begin{cases} 1 & \text{for } x \le 30\delta, \\ \dfrac{2\delta^2(40-30)^2 - (x-30\delta)^2}{2\delta^2(40-30)^2} & \text{for } 30\delta < x \le 40\delta, \\ \dfrac{(x-50\delta)^2}{2\delta^2(50-40)^2} & \text{for } 40\delta < x < 50\delta, \\ 0 & \text{for } x \ge 50\delta, \end{cases} \tag{3.18}$$

to produce membership functions of the variables originating from "*seldom*".

The changes in (3.16) made as

$$y = \mu_{\text{"hedge often"}}(x) = \begin{cases} 0 & \text{for } x \le 100-50\delta, \\ \dfrac{(x-(100-50\delta))^2}{2\delta^2((100-40)-(100-50))^2} & \text{for } 100-50\delta < x \le 100-40\delta, \\ \dfrac{2\delta^2((100-30)-(100-40))^2-(x-(100-30\delta))^2}{2\delta^2((100-30)-(100-40))^2} & \text{for } 100-40\delta < x < 100-30\delta, \\ 1 & \text{for } x \le 100-30\delta, \end{cases} \tag{3.19}$$

give a class of functions possessing "*often*" in their names.

The parameter δ works in accordance with the following criteria [49]:

1. $\delta = 1$ where no hedge in (3.18) and (3.19) is needed (*empty hedge*);
2. $0 < \delta < 1$ is applied for hedges with narrowing effects;
3. $\delta > 1$ is introduced for hedges with widening effects.

Example 3.12

We have tested different values of a parameter δ to finally decide that the most appropriate values of δ in the case of "*seldom*" can be stated as $\delta = 0.75$ for "*very seldom*", $\delta = 0.5$ for "*very, very seldom*" = "*almost never*", $\delta = 0.25$ for "*very, very, very seldom*" = "*never*" and $\delta = 1.25$ for "*rather seldom*". The values of $\delta <$ 1 will narrow supports of "*hedge seldom*" variables but $\delta > 1$, on the contrary, widens an outlook of "*rather seldom*".

The graphs of the membership functions generated by "*seldom*" when taking into account the parameters designed above are depicted in Fig. 3.4.

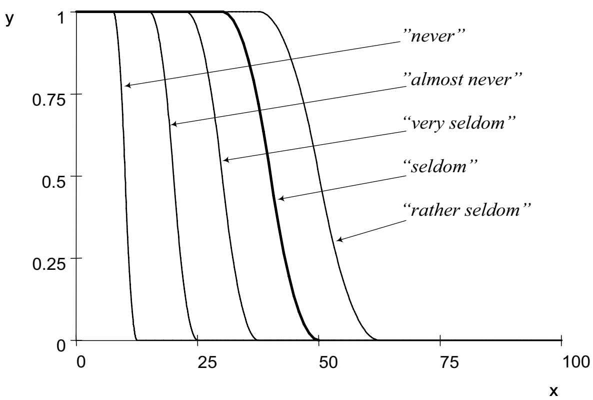

Figure 3.4: The membership functions of fuzzy variables generated by "*seldom*"

The formula (3.19) yields further composed structures proceeding from the other atomic word "*often*". By determining $\delta = 0.75$ we generate the membership function of "*very often*", $\delta = 0.5$ gives "*very, very often*" = "*almost always*" and we get "*very, very, very often*" = "*always*" for $\delta = 0.25$. To create the membership function of "*rather often*" we exploit the widening effect of δ and decide its value as $\delta = 1.25$.

The common result of employing the parameter δ as a factor changing the membership function of "*often*" for the sophisticated expressions containing this word is seen in Fig. 3.5.

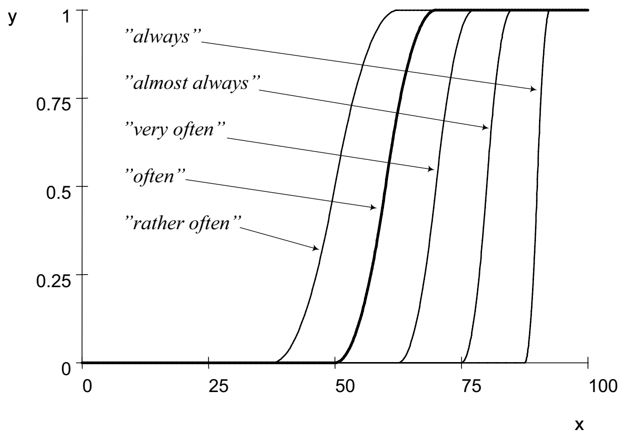

Figure 3.5: The membership functions of fuzzy variables generated by "*often*"

We now sample the results of all latest investigations that have led to the construction of new membership functions. These, as fuzzy variables shown in Fig. 3.6, represent the terms of "*presence*" and "*decisive character*".

In the further step of our efforts leading to a creation of the *SD* matrix, we desire to extract only one value of the support that represents each fuzzy variable belonging to "*presence*" and "*decisive character*". It seems to be reliable to accept, as a representative, this element of the variable support that is treated as a certain border of the variable's membership function. We can establish the boundary value, x, as the x-coordinate of an intersection point between the line $\mu_{variable}(x) = 1$ and a part of the membership function in which $\mu_{variable}(x) < 1$. The expressions coming from "*seldom*" have thus the borders determined as $x = 30\delta$ while the descriptions created by "*often*" form the group with representatives equal to $x = (A_l = 100) - 30\delta$.

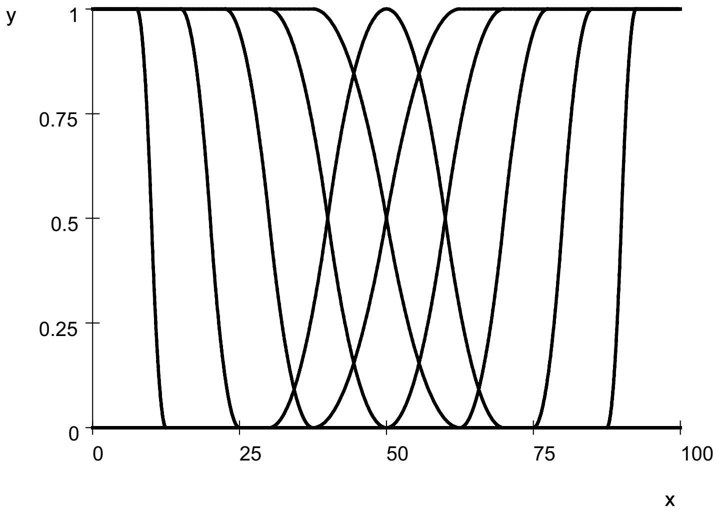

Figure 3.6: The terms from the lists "*presence*" and "*decisive character*"

Example 3.13

The representatives of the variables "*never*",…, "*always*" are sampled in Table 3.1.

Table 3.1: The representatives of the variables "*never*", …, "*always*"

Fuzzy variables	δ	The representatives for the fuzzy variables in the reference set [0, 100]
"*never*"	0.25	$x = 30 \cdot 0.25 = 7.5$
"*almost never*"	0.5	$x = 30 \cdot 0.5 = 15$
"*very seldom*"	0.75	$x = 30 \cdot 0.75 = 22.5$
"*seldom*"	1	$x = 30 \cdot 1 = 30$
"*rather seldom*"	1.25	$x = 30 \cdot 1.25 = 37.5$
"*moderately*"	–	$x = 50$
"*rather often*"	1.25	$x = 100 – 30 \cdot 1.25 = 62.5$
"*often*"	1	$x = 100 – 30 \cdot 1 = 70$
"*very often*"	0.75	$x = 100 – 30 \cdot 0.75 = 77.5$
"*almost always*"	0.5	$x = 100 – 30 \cdot 0.5 = 85$
"*always*"	0.25	$x = 100 – 30 \cdot 0.25 = 92.5$

To give expression to the verbal descriptions of presence and decisive character via values of the membership degrees, we finally plan a common restriction

$$\mu_{\text{"}common\text{"}}(x) = s(x, \alpha_{common}, \beta_{common}, \gamma_{common}) \tag{3.20}$$

that completely covers the space formed for the representative values from Table 3.1.

Example 3.14

Let us accept $\alpha_{common} = 7.5$, $\beta_{common} = 50$ and $\gamma_{common} = 92.5$ in (3.20). It should be mentioned that the borders of the space for "*common*" are decided to be equal to 7.5 respectively 92.5 to obtain the value of zero as the membership degree standing for "*never*", and the value of one assigned to "*always*" in accordance with the physicians' ability to interpret "*never*" and "*always*". By setting the numbers from the last column of Table 3.1 in (3.20) as x-values, we decide the association between the names of variables and the corresponding membership degrees assigned to them. We state the results of appropriate computations in Table 3.2.

Table 3.2: The numerical description of fuzzy variables in "*presence*"

Fuzzy variables	x	$\mu_{\text{"}common\text{"}}(x)$
"*never*"	7.5	0
"*almost never*"	15	0.016
"*very seldom*"	22.5	0.062
"*seldom*"	30	0.14
"*rather seldom*"	37.5	0.25
"*moderately*"	50	0.5
"*rather often*"	62.5	0.75
"*often*"	70	0.86
"*very often*"	77.5	0.938
"*almost always*"	85	0.984
"*always*"	92.5	1

Table 3.2 provides us with the information on how to tie the words taking place in the lists, "*presence*" and "*decisive character*", to real numbers that replace them in the fuzzy relations "*symptom – diagnosis*", which we will generate in the next subsection.

3.4.2 Relations of "*Presence*" and "*Decisive Character*"

When a physician is asked to decide, e.g., the presence of a symptom in the corresponding diagnosis, then he should only choose a word from the list containing the items that determine "*presence*". In computations assisting a mathematical model a number replaces the verbal expression approved by the physician.

Example 3.15

We consider three diagnoses D_1 = "*high risk of cardiovascular diseases*", D_2 = "*coronary heart disease*" and D_3 = "*myocardial infarct*". These are associated with the symptoms S_1, …, S_{10} already discussed in Ex. 3.1. To answer the questions: "*How often is S_j found in D_k?*" and "*How often is S_j decisive for D_k?*", $j = 1, …, 10$, $k = 1, 2, 3$, the physician selects a word from the list defining "*presence*" and "*decisive character*". His answers are collected in Table 3.3.

This table inserts the information in the mathematical model of diagnosing sometimes called "medical knowledge" because of its expressing a correlation between clinical symptoms and diagnoses. The relations *PS*, made for individual patients vary a lot from each other, but "the medical knowledge" remains invariant when looking for the most reliable diagnosis with regards to the same symptoms.

Table 3.3: Linguistic frequency and importance of S_1, …, S_{10} in D_1, D_2, D_3

Symptoms	Presence			Decisive character		
	D_1	D_2	D_3	D_1	D_2	D_3
S_1	*often*	*often*	*often*	*often*	*often*	*almost always*
S_2	*almost never*	*rather seldom*	*very often*	*very seldom*	*moderately*	*often*
S_3	*rather often*	*often*	*often*	*almost always*	*often*	*often*
S_4	*often*	*very often*	*very often*	*often*	*often*	*often*
S_5	*almost never*	*often*	*very often*	*almost never*	*almost always*	*always*
S_6	*almost never*	*seldom*	*rather often*	*almost never*	*seldom*	*often*
S_7	*almost never*	*very seldom*	*moderately*	*almost never*	*seldom*	*very often*
S_8	*very often*	*often*	*often*	*very often*	*often*	*often*
S_9	*very often*	*very often*	*very often*	*very often*	*very often*	*often*
S_{10}	*often*	*rather often*	*rather often*	*often*	*often*	*moderately*

The results obtained in Table 3.2 are used to the expressions put in Table 3.3 to replace them by numbers shown in Table 3.4.

For example, we ask a physician about the association of symptom S_{10} = "*obesity*" and diagnosis D_3 = "*myocardial infarct*" in the context of "*presence*". As an answer we get a piece of information stated as "*rather often*". By applying Eq. (3.20) we have computed $\mu_{\text{"}common\text{"}}(62.5) = 1 - 2\left(\frac{62.5 - 92.5}{92.5 - 7.5}\right)^2 \approx 0.75$. It means that the physician's statement will be utilized as 0.75 in a mathematical diagnostic model.

Table 3.4: Numerical frequency and importance of $S_1, \ldots, S_{10}$ in D_1, D_2, D_3

Symptoms	Presence			Decisive character		
	D_1	D_2	D_3	D_1	D_2	D_3
S_1	0.86	0.86	0.86	0.86	0.86	0.984
S_2	0.016	0.25	0.938	0.062	0.5	0.86
S_3	0.75	0.86	0.86	0.984	0.86	0.86
S_4	0.86	0.938	0.938	0.86	0.86	0.86
S_5	0.016	0.86	0.938	0.016	0.984	1
S_6	0.016	0.14	0.75	0.016	0.14	0.86
S_7	0.016	0.062	0.5	0.016	0.14	0.938
S_8	0.938	0.86	0.86	0.938	0.86	0.86
S_9	0,938	0.938	0.938	0.938	0.938	0.86
S_{10}	0.86	0.75	0.75	0.86	0.86	0.5

The contents of Table 3.4 can be rewritten as two matrices named "*symptom – presence*" and "*symptom – decisive character*". The first matrix forms a fuzzy relation SD_P that informs us about the presence of symptoms in the considered diagnoses. The other matrix creates a relation SD_D containing the knowledge about importance of the symptoms for the diagnoses. We introduce the relations as the matrices

$$SD_P = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.86 & 0.86 & 0.86 \\ 0.016 & 0.25 & 0.938 \\ 0.75 & 0.86 & 0.86 \\ 0.86 & 0.938 & 0.938 \\ 0.016 & 0.86 & 0.938 \\ 0.016 & 0.14 & 0.75 \\ 0.016 & 0.062 & 0.5 \\ 0.938 & 0.86 & 0.86 \\ 0.938 & 0.938 & 0.938 \\ 0.86 & 0.75 & 0.75 \end{bmatrix} \end{array} \qquad SD_D = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.86 & 0.86 & 0.984 \\ 0.062 & 0.5 & 0.86 \\ 0.984 & 0.86 & 0.86 \\ 0.86 & 0.86 & 0.86 \\ 0.016 & 0.984 & 1 \\ 0.016 & 0.14 & 0.86 \\ 0.016 & 0.14 & 0.938 \\ 0.938 & 0.86 & 0.86 \\ 0.938 & 0.938 & 0.86 \\ 0.86 & 0.86 & 0.5 \end{bmatrix} \end{array}.$$

The relations: PS discussed in Section 3.3, SD_P and SD_D constitute important components in the set of equations being the mathematical formalizations of the compositional rule of inference *modus ponens*, already mentioned in Section 3.2. Even another mathematical law, *modus tollens*, is utilized to improve the decision making process regarding the most reliable choice of a diagnosis based on clinical symptoms.

3.5 The *Patient – Diagnosis* Relation

Fuzzy relations PS, SD_P and SD_D are elements of fuzzy relation equations that bring solutions regarded as fuzzy relations of the type PD = "*patient – diagnosis*". These new relations contain pairs (P_1, D_k), k = 1, 2,…, p. By superposing the fuzzy relations in the way suggested in Eq. (2.21), we obtain equations with the operation of max-min type.

We return to Def. 3.1, which develops the meaning of the compositional rule of inference *modus ponens*, already cited in Subsection 3.2, to recall its diagnostic interpretation:

"If symptom S_j appears in patient P_1 with the membership degree $\mu_{PS}(P_1, S_j)$"

and

"If the presence of S_j results in D_k with the membership degree $\mu_{SD_P}(S_j, D_k)$ or $\mu_{SD_D}(S_j, D_k)$"

then

"Diagnosis D_k occurs in patient P_1 with the membership degree $\mu_{PD}(P_1, D_k)$".

Fuzzy relations, replacing the statements of the rule formulated above, are components of a fuzzy relation equation [2, 3]

$$PD_1 = PS \circ SD_P, \tag{3.21}$$

in which the relation PD_1 has the membership function (2.21) customized as

$$\mu_{PD_1}(P_1, D_k) = \max_{S_j \in S}(\min(\mu_{PS}(P_1, S_j), \mu_{SD_P}(S_j, D_k))) . \tag{3.22}$$

The relation SD_D has found its place in the next relation equation

$$PD_2 = PS \circ SD_D . \tag{3.23}$$

The membership function of PD_2 is derived as

$$\mu_{PD_2}(P_1, D_k) = \max_{S_j \in S}(\min(\mu_{PS}(P_1, S_j), \mu_{SD_D}(S_j, D_k))) \tag{3.24}$$

for j = 1, 2, …, n, k = 1, 2, …, p.

The relations PD_1 and PD_2, discussed in Eqs (3.21) and (3.23), are involved in an equation

$$PD_3 = \min(PD_1, PD_2) , \tag{3.25}$$

where the membership function of PD_3 is constructed as

$$\mu_{PD_3}(P_1, D_k) = \min(\mu_{PD_1}(P_1, D_k), \mu_{PD_2}(P_1, D_k)). \tag{3.26}$$

The relation PD_3 decides about an acceptance of the diagnosis in patient P_1 by means of comparing the membership degrees in PD_3. The higher the membership degree of the diagnosis D_k in P_1 is, the more certain the approval of D_k will be.

The membership degrees appearing in the row of P_1 in PD_3 sometimes do not differ essentially to indicate an optimal diagnosis as a clear-cut decision. They may sometimes differ in minutely; therefore it is also recommended that one analyzes the possibility of rejecting the diagnoses.

Another rule of inference, known as *modus tollens*, is a logical law of the shape

IF "NOT q" AND "p IMPLIES q" THEN "NOT p".

If we interpret the premises of the law as

"NOT q" = "Symptom S_j does not appear in patient P_1 with the membership degree $1 - \mu_{PS}(P_1, S_j)$"

and

"p IMPLIES q" = "D_k requires the presence of S_j with the membership degree $\mu_{SD_P}(S_j, D_k)$"

then we draw the conclusion

"NOT p" = "Diagnosis D_k is rejected in patient P_1 with the membership degree $\mu_{PD}(P_1, D_k)$".

We interpret the mathematical meaning of the *modus tollens* law as an equation

$$PD_4 = (1 - PS) \circ SD_P \cdot \tag{3.27}$$

The membership function of PD_4 is presented in the form of

$$\mu_{PD_4}(P_1, D_k) = \max_{S_j \in S}(\min(1 - \mu_{PS}(P_1, S_j), \mu_{SD_P}(S_j, D_k))), \tag{3.28}$$

for each diagnosis from the set D.

By applying the *double negation* law "NOT(NOT q)) = q" we modify *modus tollens* as a statement

IF "q" AND "p IMPLIES (NOT q)" THEN "NOT p."

A translation of the premises in the last version of the *modus tollens* law into an understandable medical sentence can be formulated as follows.

If

"q" = "Symptom S_j appears in patient P_1 with the degree $\mu_{PS}(P_1, S_j)$"

and
"q IMPLIES (NOT p)" = "D_k does not need the presence of S_j with the membership degree $1-\mu_{SD_P}(S_j,D_k)$"
then we will come to the thesis
"NOT p" = "Diagnosis D_k is rejected for patient P_1 with the membership degree $\mu_{PD}(P_1, D_k)$".

The last adaptation of *modus tollens* has given rise to a fuzzy relation involved in the diagnostic process in the way of a composition

$$PD_5 = PS \circ (1 - SD_P), \tag{3.29}$$

where the relation PD_5 is characterized by the membership function

$$\mu_{PD_5}(P_1,D_k) = \max_{S_j \in S}(\min(\mu_{PS}(P_1,S_j),1-\mu_{SD_P}(S_j,D_k))) . \tag{3.30}$$

The fuzzy relations PD_4 and PD_5 resolve of the rejection of a diagnosis assisting patient P_1. The higher the value of the membership degree associated with the diagnosis in PD_4 and PD_5 is, the greater the certainty of the diagnosis rejection will be.

The formulas (3.22), (3.24), (3.26), (3.28) and (3.30) are valid for $j = 1, 2,\dots, n$ and $k = 1, 2,\dots, p$.

The final decision concerning the acceptance of a proper diagnosis assumes the simultaneous and thorough comparison of the membership degrees originating from the relations PD_3, PD_4 and PD_5.

Example 3.16

We can already provide patient P_1 with the relations PS, SD_P and SD_D that have been determined for his sake in Ex. 3.8 and 3.15 respectively. A computing process of the relations PD_1–PD_5 is carried through by involving Eqs (3.21)–(3.30), in turn, as it is executed below.

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If

$$\begin{array}{cccccccccccc} & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \\ PS = P_1[& 1 & 0.515 & 0.913 & 0.653 & 0.345 & 0.632 & 0.720 & 0.9 & 0.0004 & 0.353 \;] \end{array}$$

then we would like to have access to the relation SD_P as the second component in Eq. (3.21) to appreciate PD_1.

Hence,

$$PD_1 = PS \circ SD_P$$

or

$$PD_1 = P_1 \overset{\begin{matrix} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \end{matrix}}{[\begin{matrix} 1 & 0.515 & 0.913 & 0.653 & 0.345 & 0.632 & 0.720 & 0.9 & 0.0004 & 0.353 \end{matrix}]} \circ$$

$$\begin{matrix} & D_1 & D_2 & D_3 \\ S_1 & 0.86 & 0.86 & 0.86 \\ S_2 & 0.016 & 0.25 & 0.938 \\ S_3 & 0.75 & 0.86 & 0.86 \\ S_4 & 0.86 & 0.938 & 0.938 \\ S_5 & 0.016 & 0.86 & 0.938 \\ S_6 & 0.016 & 0.14 & 0.75 \\ S_7 & 0.016 & 0.062 & 0.5 \\ S_8 & 0.938 & 0.86 & 0.86 \\ S_9 & 0.938 & 0.938 & 0.938 \\ S_{10} & 0.86 & 0.75 & 0.75 \end{matrix} = P_1 \overset{\begin{matrix} D_1 & D_2 & D_3 \end{matrix}}{[\begin{matrix} 0.9 & 0.86 & 0.86 \end{matrix}]} .$$

To calculate a membership degree for the pair (P_1, D_1) we follow the operations recommended by (3.22). Hence, $\mu_{PD_1}(P_1, D_1)$ = max(min(1, 0.86), min(0.515, 0.016), min(0.913, 0.75), min(0.653, 0.86), min(0.345, 0.016), min(0.632, 0.016), min(0.720, 0.016), min(0.9, 0.938), min(0.0004, 0.938), min(0.353, 0.86)) = max(0.86, 0.016, 0.75, 0.653, 0.016, 0.016, 0.016, 0.9, 0.0004, 0.353) = 0.9.

In accordance with (3.23) and (3.24) we evaluate

$$PD_2 = PS \circ SD_D = P_1 \overset{\begin{matrix} D_1 & D_2 & D_3 \end{matrix}}{[\begin{matrix} 0.913 & 0.86 & 0.984 \end{matrix}]}$$

and by returning to (3.25) and (3.26), we obtain

$$PD_3 = \min(PD_1, PD_2) = P_1 \overset{\begin{matrix} D_1 & D_2 & D_3 \end{matrix}}{[\begin{matrix} 0.9 & 0.86 & 0.86 \end{matrix}]}$$

as the final decision of accepting the diagnosis.

Since the membership degrees in PD_3 are almost equal, we should examine the possibility of rejecting the diagnoses as a supplementary decisive factor. In further computations we exploit two complements to matrices already introduced. Let us consequently find $1 - PS$ as a table

$$1 - PS = P_1 \overset{\begin{matrix} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \end{matrix}}{[\begin{matrix} 0 & 0.485 & 0.087 & 0.347 & 0.655 & 0.368 & 0.280 & 0.1 & 0.9996 & 0.647 \end{matrix}]}$$

filled with membership degrees that are calculated by subtracting $\mu_{PS}(P_1, D_k)$ from one, $k = 1, \ldots, p$. The other matrix $1 - SD_P$, is an algebraic complement to one and has a form

$$1 - SD_P = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.14 & 0.14 & 0.14 \\ 0.984 & 0.75 & 0.062 \\ 0.25 & 0.14 & 0.14 \\ 0.14 & 0.062 & 0.062 \\ 0.984 & 0.14 & 0.062 \\ 0.984 & 0.86 & 0.25 \\ 0.984 & 0.938 & 0.5 \\ 0.062 & 0.14 & 0.14 \\ 0.062 & 0.062 & 0.062 \\ 0.14 & 0.25 & 0.25 \end{bmatrix} \end{array}.$$

We now use (3.27) and (3.28) to state the contents of PD_4 as the table

$$PD_4 = (1 - PS) \circ SD_P = P_1 \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.938 & 0.938 & 0.938 \end{bmatrix} \end{array}.$$

PD_4 is the first matrix that provides us with a decision about excluding the diagnoses. By adopting (3.29) and (3.30) we calculate the entries of the matrix PD_5 that closes the series of matrices participating in the decision making process with respect to the optimal diagnosis. PD_5, which is founded on the complement of the fuzzy relation SD_P, possesses the following membership degrees

$$PD_5 = PS \circ (1 - SD_P) = P_1 \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.720 & 0.720 & 0.500 \end{bmatrix} \end{array}.$$

To make the final decision let us sum up the obtained data in Table 3.5

Table 3.5: P_1's diagnostic decision based on PD_3, PD_4 and PD_5

Patient	PD_3			PD_4			PD_5			Decision
	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3	
P_1	0.90	0.86	0.86	0.938	0.938	0.938	0.72	0.72	0.5	D_1 or D_3

After carrying out a thorough analysis of all membership degrees, we agree with the decision of accepting D_1 or D_3 as the disease that patient P_1 suffers from.

Unfortunately, we cannot decide which disease is right since the obtained information is not clear. We motivate making our choice in the way described below:

1. The membership degree of D_1 in PD_3 is the largest value that convinces us to approve D_1 as the most plausible disease;
2. The matrix PD_4 has all the membership degrees equal, which means that the decision is lacking;
3. Finally, by rejecting the diagnoses D_1 and D_2 in P_1, since they have the highest membership degrees in PD_5, we leave D_3 as the most probable diagnosis in the considered patient.

The physician has sampled the data about the patient's state and confirmed that P_1's health state is very severe. He risks a myocardial infarct in the substantial grade, which justifies our doubtful decision between D_1 and D_3.

Let us also provide a bit of information concerning another patient P_2.

Example 3.17

The data sample for P_2 is placed in the matrix PS in the symptom order as

$$\begin{array}{cccccccccccc} & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & \\ PS = P_2[& 0 & 0.2 & 0.4 & 0.5 & 0.1 & 0.3 & 0.8 & 0.9 & 0.8 & 0.6 &]. \end{array}$$

Table 3.6 contains membership degrees that describe P_2's diagnostic conditions to decide his most credible diagnosis.

Table 3.6: P_2's diagnostic decision based on PD_3, PD_4 and PD_5

Patient	PD_3			PD_4			PD_5			Decision
	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3	
P_1	0.90	0.86	0.86	0.86	0.86	0.9	0.8	0.8	0.5	D_3

Once again we meet the patient's case that is not easy to diagnose. The differences among the membership degrees representing PD_3 are not substantial enough to indicate the diagnosis accepted for the patient. Even the degrees in the rejection matrix PD_4, as close to each other, do not convince us completely about the choice of a proper diagnosis assigned to P_2. Only the informative character of PD_5 can be considered as reliable because of essential differences among the membership degrees placed in this relation. Since the value of D_3 is smallest of all in PD_5 then we will admit that the recognized diagnosis is identified as D_3. The patient's medical reports certify this choice as well.

We can observe some harmful effects of the maximum and minimum operations included in compositions of relations. These operations deprive many membership degrees of their power in the final decisions. Therefore we need to use "softer" calculations that take into considerations all values presented by the matrices "*patient – symptom*" and "*symptom – diagnosis*".

Many patient cases that do not deliver diagnostic specifications in order to make a clear choice among diagnoses should be supported by complementary solutions. We intend to discuss the supplementary details of diagnostic models in the next part of the dissertation.

4 Complementary Solutions in Diagnostic Models

4.1 Introduction

We should admit that the case of patient P_1 in Ex. 3.16 has not been very easy to solve especially when you consider the proper interpretation of PD_3. By equipping us with equal values of the membership degrees it has not made it easy enough to make the proper choice of an unknown diagnosis.

Moreover, the patient has been examined only once according to existing reports about his health. If the patient visits the doctor's office more than one time, then we can notice some changes in values of biological parameters under consideration. The increasing or decreasing values of clinical symptoms, when observing them many times, can absolutely exclude this diagnosis that has been approved after the first examination. The analysis of a diagnostic model extended in time could provide us with more accurate information that assists in making a better choice of a disease.

To limit some doubtful diagnostic decisions made by means of mathematical tools, we propose the supplementary solutions handled in this chapter.

4.2 OWA Operators in Decision Relations

The results from Ex. 3.16, obtained as the membership degrees taking place in five decision matrices, are possible to interpret even if the values in PD_3 have become equal or the values in PD_4 have not varied much from each other. Nevertheless, we desire to obtain clearer information that is accessible in the model's final relations and to be capable of conveying a right conclusion.

The almost equal membership degrees calculated on account of (3.22), (3.24), (3.26), (3.28) and (3.30) have been influenced by an action of the maximum operation. By taking the maximum values in the sets of minimum compounds consisting of μ_{PS} and μ_{SD} we have lost a large part of the useful information since only the largest values of compounds have affected output data. Even the application of (2.23) or (2.24) does not eliminate an unfavourable impact of the maximum operator.

To prevent a loss of valuable explanations in the future let us suggest another concatenation operation in a composition of two matrices, i.e., such a one which takes into consideration all membership degrees included in *PS* and *SD*.

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A new definition considering weights is now proven to make diagnostic results clearly interpretable. To accomplish new computations that should lead to an easier analysis of final results in the diagnostic model, we suggest making an acquaintance with OWA aggregating operations. We thus cite a general definition of an OWA operator [46, 82, 84, 86].

Definition 4.1

If x_1, x_2, …, x_n are some estimates of the same quantity x, then an aggregation operation called Ordered Weighted Averaging (OWA) has a type

$$f(x_1, x_2, ..., x_n) = a_0 + a_1 \cdot x_{(1)} + a_2 \cdot x_{(2)} + \cdots + a_n \cdot x_{(n)}, \tag{4.1}$$

where a_0, a_1, …, a_n are constants.

The values $x_{(1)}$, $x_{(2)}$, …, $x_{(n)}$ are described in the terms of minimum and maximum as

- $x_{(1)} = \min(x_1, x_2, ..., x_n)$
- $x_{(2)} = \max(x(1), ..., x(n))$, where $x(i)$ is the minimum of all the values except the i^{th}, i.e.,
  - $x(1) = \min(x_2, x_3, x_4, ..., x_n)$,
  - $x(2) = \min(x_1, x_3, x_4, ..., x_n)$,
  - …
  - $x(n) = \min(x_1, x_2, x_3, ..., x_{n-1})$;
- $x_{(3)} = \max(x(1,2), x(1,3), ..., x(n-1, n))$, where $x(i, j)$ is the minimum of all the values except the i^{th} and the j^{th};
- etc.

Example 4.1

The mean $\frac{x_1 + x_2}{2}$ is the OWA operation for $a_0 = 0$, $a_1 = a_2 = \frac{1}{2}$. It can be expanded in the series (4.1) as $\frac{1}{2}\min(x_1, x_2) + \frac{1}{2}\max(\min(x_2), \min(x_1))$. If we set $x_1 = 40$ and $x_2 = 30$ then $\frac{1}{2}\min(40,30) + \frac{1}{2}\max(\min(30), \min(40)) = \frac{30}{2} + \frac{40}{2} = 35$ that is the exact value of the arithmetic mean for 30 and 40.

Both Def. 4.1 and Ex. 4.1 should convince us about the classification of an arithmetic mean as a modern OWA operator.

Definition 4.2

We recall the general equation $PD = PS \circ SD$ showed in Section 3.2 that constitutes the most important part in the diagnostic model. We define an operation, denoted symbolically by "$\underset{+}{\circ}$" in order to compose two fuzzy relations *PS* and *SD* introduced by (3.1) and (3.2) respectively. The membership function of the relation *PD* is proposed as

$$\mu_{PD}(P_1, D_k) = \frac{\sum_{j=1}^{n} \mu_{PS}(P_1, S_j) \cdot \mu_{SD}(S_j, D_k)}{\sum_{j=1}^{n} \mu_{SD}(S_j, D_k)} = \\ \frac{\mu_{SD}(S_1, D_k)}{\sum_{j=1}^{n} \mu_{SD}(S_j, D_k)} \cdot \mu_{PS}(P_1, S_1) + \cdots + \frac{\mu_{SD}(S_n, D_k)}{\sum_{j=1}^{n} \mu_{SD}(S_j, D_k)} \cdot \mu_{PS}(P_1, S_n). \tag{4.2}$$

The value of the quotient $\mu_{PD}(P_1, D_k)$ is a number belonging to the interval [0, 1]. To explain it we first notice that $\mu_{PS}(P_1, S_j) \cdot \mu_{SD}(S_j, D_k) \le \mu_{SD}(S_j, D_k)$ since both $\mu_{PS}(P_1, S_j)$ and $\mu_{SD}(S_j, D_k)$ are less than one for all j and k, $j = 1, \ldots, n$, $k = 1, \ldots, p$. This causes the value of a product to be less than the values of both factors. We thus conclude that the numerator is less than or equal to the denominator, which guarantees that the entire value of the quotient is a member from [0, 1]; therefore it can be approved as a membership degree of the pair (P_1, D_k).

We also notice that the sum placed in the denominator of the quotient never becomes equal to zero, since almost one of the examined symptoms must express any presence or decisive character for diagnoses included in the designed model. This assumption is very important for truthfulness of the diagnostic model that cannot provide operations on undefined structures.

Let us accommodate (4.2) to the assumptions of (4.1). The value of a sum $\mu_{SD}(S_1, D_k) + \cdots + \mu_{SD}(S_n, D_k)$ and even the quantities $\mu_{SD}(S_j, D_k)$ play roles of invariants for different patients, i.e., they become unchangeable for varying collections of $\mu_{PS}(P_1, S_j)$, $j = 1, \ldots, n$. The mentioned invariants can be reasonably regarded as constants in coefficients

$$a_j = \frac{\mu_{SD}(S_j, D_k)}{\mu_{SD}(S_1, D_k) + \cdots + \mu_{SD}(S_n, D_k)} \tag{4.3}$$

used in the sum (4.1). Further,

$$\mu_{PD}(P_1, D_k) = f(\mu_{PS}(P_1, S_1), ..., \mu_{PS}(P_1, S_n)) = a_{j_1}\mu_{PS}(P_1, S_1) + \cdots + a_{j_n}\mu_{PS}(P_1, S_n) \quad (4.4)$$

for $k = 1, \ldots, p$. The order of coefficients $a_{j_1}, \ldots, a_{j_n}$ constitutes a new rearrangement of the sequence $a_1, \ldots, a_n$ made in order to fulfil the assumptions of Def. 4.1. After explaining the meaning of (4.3) and (4.4) we can claim that the proposed operation (4.2) is certified to be assigned to the class of OWA operators.

In the suggested formula (4.2) all membership degrees from the relations *PS* and *SD* are equally valuable for computations. This means that each of the values affects a result. The membership degrees of the k^{th} column belonging to *SD* act as weights that balance the signification of tested symptoms. To summarize, we come to a conclusion that a proposed value of the membership degree for the pair (P_1, D_k), computed by (4.2), has shown itself to be more intermediary when comparing it to an effect of the sharp value of maximum.

Let us accomplish necessary changes in (3.22), (3.24), (3.26), (3.28) and (3.30) to accommodate them to new circumstances forced by (4.2).

We begin with the composition $PD_1 = PS \underset{+}{\circ} SD_P$ to change its membership function as

$$\mu_{PD_1}(P_1, D_k) = \frac{\sum_{j=1}^{n} \mu_{PS}(P_1, S_j) \cdot \mu_{SD_P}(S_j, D_k)}{\sum_{j=1}^{n} \mu_{SD_P}(S_j, D_k)} \quad (4.5)$$

while the relation $PD_2 = PS \underset{+}{\circ} SD_D$ has a membership function derived by the replacement of the relation SD_P by SD_D in (4.5). This yields a result

$$\mu_{PD_2}(P_1, D_k) = \frac{\sum_{j=1}^{n} \mu_{PS}(P_1, S_j) \cdot \mu_{SD_D}(S_j, D_k)}{\sum_{j=1}^{n} \mu_{SD_D}(S_j, D_k)} \quad (4.6)$$

for $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$.

The relations calculated by applying of (4.5) and (4.6) are involved in the equation $PD_3 = \text{mean}(PD_1, PD_2)$ in which the relation PD_3 is characterized by a membership function

$$\mu_{PD_3}(P_1, D_k) = \frac{\mu_{PD_1}(P_1, D_k) + \mu_{PD_2}(P_1, D_k)}{2}. \tag{4.7}$$

The membership degrees of PD_3 decide, as before, the approval of the most possible diagnosis in a diagnostic hierarchy.

We upgrade diagnoses in another hierarchical order when we try to reject them. To exclude diagnoses which a patient cannot suffer from, we prepare a membership function of $PD_4 = (1 - PS) \underset{+}{\circ} SD_P$ as

$$\mu_{PD_4}(P_1, D_k) = \frac{\sum_{j=1}^{n} (1 - \mu_{PS}(P_1, S_j)) \cdot \mu_{SD_P}(S_j, D_k)}{\sum_{j=1}^{n} \mu_{SD_P}(S_j, D_k)} \tag{4.8}$$

for each diagnosis from the set D.

The last equation $PD_5 = PS \underset{+}{\circ} (1 - SD_P)$ completes the conclusive material that helps us to exclude doubtful diagnoses in the examined patient. After adapting the operation (4.2) to (3.30) we get

$$\mu_{PD_5}(P_1, D_k) = \frac{\sum_{j=1}^{n} \mu_{PS}(P_1, S_j)) \cdot (1 - \mu_{SD_P}(S_j, D_k))}{\sum_{j=1}^{n} (1 - \mu_{SD_P}(S_j, D_k))} \tag{4.9}$$

for $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$.

Let us confirm the validity of newly suggested operations (4.5)–(4.9) by reconsidering the well-known case that concerns the input data of patient P_1 from Ex. 3.16.

Example 4.2

We use the entries of the same matrices PS, SD_P and SD_D that have already been tested in Ex. 3.16, but now we intend to perform the operations on their membership degrees by executing the operations related to (4.5)–(4.9).

The relation $PD_1 = PS \underset{+}{\circ} SD_P$, has a component PS, accepted according with Ex. 3.8 as

$$PS = P_1 \begin{array}{c} \\ [\end{array} \begin{array}{cccccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \\ 1 & 0.515 & 0.913 & 0.653 & 0.345 & 0.632 & 0.720 & 0.9 & 0.0004 & 0.353 \end{array} \begin{array}{c} \\ ] \end{array}.$$

After the composition of PS with SD_P, as recommended by (4.5), we find PD_1 in the form of

$$PD_1 = P_1 \begin{array}{c} \\ [\end{array} \begin{array}{cccccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \\ 1 & 0.515 & 0.913 & 0.653 & 0.345 & 0.632 & 0.720 & 0.9 & 0.0004 & 0.353 \end{array} \begin{array}{c} \\ ] \end{array} \underset{+}{\circ}$$

$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \end{array} \begin{array}{ccc} D_1 & D_2 & D_3 \\ \hline 0.86 & 0.86 & 0.86 \\ 0.016 & 0.25 & 0.938 \\ 0.75 & 0.86 & 0.86 \\ 0.86 & 0.938 & 0.938 \\ 0.016 & 0.86 & 0.938 \\ 0.016 & 0.14 & 0.75 \\ 0.016 & 0.062 & 0.5 \\ 0.938 & 0.86 & 0.86 \\ 0.938 & 0.938 & 0.938 \\ 0.86 & 0.75 & 0.75 \end{array} = \begin{array}{c} \\ P_1 \end{array} \begin{array}{ccc} D_1 & D_2 & D_3 \\ [0.624 & 0.591 & 0.593] \end{array}.$$

The membership degree of (P_1, D_1) has been induced by computations:
$\mu_{PD_1}(P_1, D_1) = (1\cdot0.86+0.515\cdot0.016+0.913\cdot0.75+0.653\cdot0.86+0.345\cdot0.016+0.632\cdot0.016+0.720\cdot0.016+0.9\cdot0.938+0.0004\cdot0.938+0.353\cdot0.86)/(0.86+0.016+0.75+0.86+0.016+0.016+0.016+0.938+0.938+0.86) = 3.2899/5.27 = 0.624$.

After employing (4.6) the relation PD_2 is decided to be a matrix

$$PD_2 = PS \underset{+}{\circ} SD_D = \begin{array}{c} \\ P_1 \end{array} \begin{array}{ccc} D_1 & D_2 & D_3 \\ [0.635 & 0.581 & 0.616] \end{array}.$$

By utilizing (4.7) we determine the elements of PD_3 as a one-row table

$$PD_3 = \text{mean}(PD_1, PD_2) = \begin{array}{c} \\ P_1 \end{array} \begin{array}{ccc} D_1 & D_2 & D_3 \\ [0.629 & 0.586 & 0.604] \end{array},$$

in which the membership degrees distinctly appear as the indicators of a sequence of possible diagnoses taken in the order D_1, D_3, D_2.

In order to confirm the decision made above, let us also consider results of the operations excluding diagnoses. The matrix PD_4, calculated by applying of (4.8), is stated as

$$PD_4 = (1 - PS) \underset{+}{\circ} SD_P = \begin{matrix} & D_1 & D_2 & D_3 \\ P_1 & [0.375 & 0.408 & 0.407] \end{matrix}.$$

PD_4 clearly provides us with the order of rejected diagnoses. By taking into account the value order among membership degrees in the last relation, we reject diagnoses in P_1 in the sequence D_2, D_3, D_1. This still testifies the fact that D_1 is the most probable illness typical of these symptoms that have been found and reported by a doctor for P_1's sake.

The entries of PD_5 are effects of performed operations in accordance with (4.9). They are written down in the matrix

$$PD_5 = PS \underset{+}{\circ} (1 - SD_P) = \begin{matrix} & D_1 & D_2 & D_3 \\ P_1 & [0.579 & 0.624 & 0.655] \end{matrix}.$$

The numbers still assure that D_1 should be assigned to P_1 as the most probable diagnosis because of the least value of the membership degree accompanying D_1 in the last "rejection" matrix.

The final decision is now submitted in Table 4.1.

Table 4.1: Weighted relation compositions in the diagnostic decision

Patient	PD_3			PD_4			PD_5			Decision
	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3	
P_1	0.629	0.586	0.604	0.375	0.408	0.407	0.579	0.624	0.655	D_1

There is no doubt that D_1 satisfies all conditions that the optimal diagnosis should fulfil. The membership degree of D_1 in the matrix of acceptance PD_3 is the largest of all observed values. The membership degrees of D_1 in the matrices of rejection PD_4 and PD_5 are the smallest that confirm the rules already discussed in Section 3.4.

The next example explains how to use the OWA definition to compute the membership value of one entry belonging to the relation PD_1.

Example 4.3

The membership degree of (P_1, D_1), already evaluated in Ex. 4.2, also constitutes a result of the OWA definition expansion (4.1).

If $x_1, x_2, \ldots, x_{10}$ are equal to $\mu_{PS}(P_1, S_1)$, $\mu_{PS}(P_1, S_2)$, …, $\mu_{PS}(P_1, S_{10})$ respectively, then:

- $a_0 = 0$;
- $x_{(1)} = \min(1, 0.515, 0.913, 0.653, 0.345, 0.632, 0.720, 0.9, 0.0004, 0.353) = 0.0004$, which suggests accepting a_1 as
$$a_1 = \frac{\mu_{SD_P}(S_9, D_1)}{\mu_{SD_P}(S_1, D_1) + \cdots + \mu_{SD_P}(S_{10}, D_1)} = \frac{0.938}{5.27} \approx 0.178;$$
- $x_{(2)}$ = max(min(0.515, 0.913, 0.653, 0.345, 0.632, 0.720, 0.9, 0.0004, 0.353), min(1, 0.913, 0.653, 0.345, 0.632, 0.720, 0.9, 0.0004, 0.353), min(1, 0.515, 0.653, 0.345, 0.632, 0.720, 0.9, 0.0004, 0.353), min(1, 0.515, 0.913, 0.345, 0.632, 0.720, 0.9, 0.0004, 0.353), min(1, 0.515, 0.913, 0.653, 0.632, 0.720, 0.9, 0.0004, 0.353), min(1, 0.515, 0.913, 0.653, 0.345, 0.720, 0.9, 0.0004, 0.353), min(1, 0.515, 0.913, 0.653, 0.345, 0.632, 0.9, 0.0004, 0.353), min(1, 0.515, 0.913, 0.653, 0.345, 0.632, 0.720, 0.0004, 0.353), min(1, 0.515, 0.913, 0.653, 0.345, 0.632, 0.720, 0.9, 0.353), min(1, 0.515, 0.913, 0.653, 0.345, 0.632, 0.720, 0.9, 0.0004)) = max(0.0004, 0.0004, 0.0004, 0.0004, 0.0004, 0.0004, 0.0004, 0.0004, 0.345, 0.0004) = 0.345, which generates
$$a_2 = \frac{\mu_{SD_P}(S_5, D_1)}{\mu_{SD_P}(S_1, D_1) + \cdots + \mu_{SD_P}(S_{10}, D_1)} = \frac{0.016}{5.27} \approx 0.003;$$

…

- $x_{(10)}$ = max(min(1), min(0.515), min(0.913), min(0.653), min(0.345), min(0.632), min(0.720), min(0.9), min(0.0004), min(0.353)) = 1. The corresponding coefficient a_{10} is a result of the computation
$$a_{10} = \frac{\mu_{SD_P}(S_1, D_1)}{\mu_{SD_P}(S_1, D_1) + \cdots + \mu_{SD_P}(S_{10}, D_1)} = \frac{0.86}{5.27} \approx 0.163.$$

Equation (4.1) is used as a basis of the evaluation of the membership degree $\mu_{PD_1}(P_1, D_1) = f(x_1, x_2, ..., x_{10}) = a_0 + a_1 \cdot x_{(1)} + a_2 \cdot x_{(2)} + \cdots + a_{10} \cdot x_{(10)} =$ $0.178 \cdot 0.0004 + 0.003 \cdot 0.345 + \cdots + 0.86 \cdot 1 = 0.624$.

The performed operations in Ex. 4.3 are not recommended to apply in practical cases. We only want to convince a reader that the proposed formula (4.2) is logically correct as a kind of the OWA operation introduced by (4.1).

The use of weighed operations in decision equations (4.5)–(4.9) instead of earlier suggested max-min compositions makes the diagnostic process clearer and more reliable. The differences among membership degrees in decision matrices are large enough to recognize the appropriate diagnosis without making a mistake.

We can conclude that the common influence of all membership degrees, computed in *PS* and *SD*, on values placed in the decision matrices PD_1–PD_5 improve the quality of a final decision.

Some researchers, who deal with theoretical assumptions of fuzzy set theory, sometimes criticize operations containing computations of mean values. To defend this sort of calculations involved in the diagnostic model we should emphasize that each little change in the symptom index brings valuable information and cannot be lost in the decisive process. The results of operations that resemble mean estimates of some parameters assure that important input data will not disappear.

4.3 Fuzzy Set Distances in Diagnostic Decisions

The new operations introduced by the previous section have substantially elucidated the changes made in the diagnostic model to get clear decisions. However, they cannot help when biological parameters measured in a patient indicate a tendency to agree with more diseases than one. We thus observe little differences among membership degree values, or contradictory membership values in matrices PD_3, PD_4 and PD_5 that make the obtained specifications of the patient's data almost unreadable.

We can note that a conception of the metrics is a rather popular tool of investigations in most interdisciplinary fields developed by researchers dealing with fuzzy set theory.

Let us recall the formula for computing a distance between two fuzzy sets [40, 95].

Definition 4.3

For two fuzzy sets $A = \{(x_i, \mu_A(x_i))\}$ and $B = \{(x_i, \mu_B(x_i))\}$, determined in the universe $X = \{x_i\}$, $i = 1, \ldots, n$, the Euclidean distance $d(A, B)$ between them is approximated by a formula [40, 95]

$$d(A,B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2} \;. \tag{4.10}$$

The Euclidean distance $d(A, B)$ maps $[0, 1] \times [0, 1]$ into $R \cup \{0\}$ (the set of non-negative values) and fulfils the following conditions:

1. $d(A, B) \geq 0$,
2. If $A = B$ then $d(A, B) = 0$,
3. $d(A, B) = d(B, A)$,
4. $d(A, C) \leq d(A, B)$" $*$ "$d(B, C)$,

where " $*$ " is a certain operation, e.g., addition.

To comprehend an action of the formula (4.10) we go through a simple example that explains the order of performed operations.

Example 4.4

We define two fuzzy sets A and B in the common universe $X = [1, 10]$, where $A = {}^{0.2}/_{3} + {}^{0.5}/_{4} + {}^{0.6}/_{5} + {}^{1}/_{7}$ and $B = {}^{0.2}/_{4} + {}^{0.4}/_{6} + {}^{0.6}/_{7} + {}^{1}/_{8} + {}^{0.8}/_{9} + {}^{0.5}/_{10}$. At first the distance $d(A, B)$ is pre-evaluated by a number $(d(A,B))^2 = (0.2-0)^2 + (0.5-0.2)^2 + (0.6-0)^2 + (0-0.4)^2 + (1-0.6)^2 + (0-1)^2 + (0-0.8)^2 + (0-0.5)^2 = 2.65$ and afterwards measured by $d(A,B) = \sqrt{2.65} \approx 1.63$.

The concept of a distance between fuzzy sets will be utilized in a diagnostic model in order to improve some decision criteria in doubtful cases. We count on the helpful role of a complementary distance method when analyzing almost equal membership degrees, or opposite values in decision matrices that do not provide us with clear conclusions.

Let us restrict the set of diagnoses D to three diagnoses D_1, D_2 and D_3 to make the following discussion comprehensive in details. The mentioned diagnoses can be found in patient P.

Suppose that the fuzzy set [56]

$$AP(PD_3) = {}^{1}\!/_{(P,D_1)} + {}^{1}\!/_{(P,D_2)} + {}^{1}\!/_{(P,D_3)} \tag{4.11}$$

is associated with the state of a total acceptance of each diagnosis. Each value of the membership degree in a one-row "ideal" acceptance relation-matrix, PD_3, should be compared to one. This matrix in reality, has other values of membership degrees computed for the symptoms evaluated for any patient P. The true set PD_3 is generally stated as a fuzzy set (a one-row matrix)

$$PD_3 = {}^{\mu_{PD_3}(P,D_1)}\!/_{(P,D_1)} + {}^{\mu_{PD_3}(P,D_2)}\!/_{(P,D_2)} + {}^{\mu_{PD_3}(P,D_3)}\!/_{(P,D_3)}. \tag{4.12}$$

In Fig. 4.1 we draw ellipses to mark membership degrees of the pairs (P, D_1), (P, D_2), (P, D_3) coming from the set $AP(PD_3)$, and we use squares as symbols of genuine membership values found in PD_3. Figure 4.1, built artificially for the purpose of making concluding remarks, gives us some hints how to rank diagnoses. Due to an impression given by Fig 4.1 we should place them in order D_1, D_2, D_3.

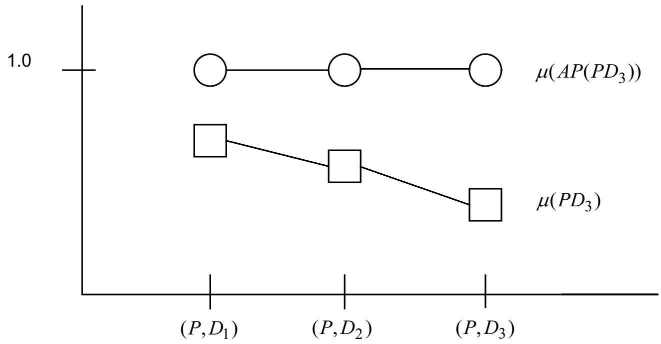

Figure 4.1: The comparison of sets PD_3 and $AP(PD_3)$ for patient P

Moreover, Fig. 4.1 has another role to fill in; it ought to, via a general image about the distances between real and extreme decision values constructed for PD_3, provide us with conclusions that confirm the diagnostic order stated above.

Let us now suppose that we theoretically choose diagnosis D_1 with the total security, which introduces the membership degree equal to one in the place of $\mu_{PD_3}(P,D_1)$ in the new set

$$AP(PD_3)_{D_1} = 1\Big/(P,D_1) + \mu_{PD_3}(P,D_2)\Big/(P,D_2) + \mu_{PD_3}(P,D_3)\Big/(P,D_3). \tag{4.13}$$

Analogously, we introduce a set

$$AP(PD_3)_{D_2} = \mu_{PD_3}(P,D_1)\Big/(P,D_1) + 1\Big/(P,D_2) + \mu_{PD_3}(P,D_3)\Big/(P,D_3) \tag{4.14}$$

if we fully adopt D_2 in the theoretical way in spite of its real value $\mu_{PD_3}(P,D_2)$.

We also construct a set

$$AP(PD_3)_{D_3} = \mu_{PD_3}(P,D_1)\Big/(P,D_1) + \mu_{PD_3}(P,D_2)\Big/(P,D_2) + 1\Big/(P,D_3) \tag{4.15}$$

that corresponds to the acceptation of diagnosis D_3 as a totally true decision.

Let us first visually estimate a distance of $AP(PD_3)_{D_1}$ from $AP(PD_3)$ by regarding the location of theoretically designed membership degrees of both sets in Fig. 4.2. All membership degrees are placed in the same manner as in Fig. 4.1

except for the degree of D_1. If D_1 is theoretically accepted without any doubts, then the sign marking its membership degree $\mu_{PD_3}(P,D_1)$ in Fig. 4.1 should be removed to such a position in Fig. 4.2, that shows how the membership degree of $AP(PD_3)_{D_1}$ covers the membership degree of $AP(PD_3)$ for the mutual diagnosis D_1.

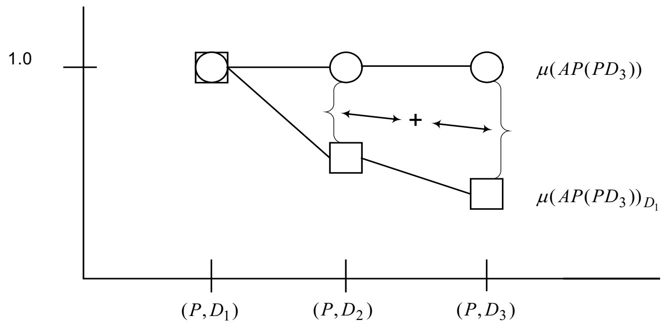

Figure 4.2: The distance between $AP(PD_3)_{D_1}$ and $AP(PD_3)$

To compare some measurements between other characteristic fuzzy sets, we theoretically accept D_3 as an absolute diagnosis in P. This entails the following changes in Fig. 4.2: we move the membership degree of D_3 to the position of one, and return with the membership degree of D_1 to the previous location as calculated in PD_3. By making the recommended corrections in Fig 4.2, we obtain Fig. 4.3 to evaluate the distance between the sets $AP(PD_3)_{D_3}$ and $AP(PD_3)$.

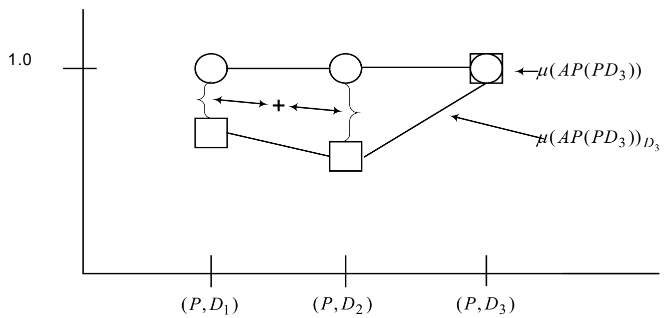

Figure 4.3: The distance between $AP(PD_3)_{D_3}$ and $AP(PD_3)$

The closer analysis of Fig. 4.1, combined with visual estimations of distances revealed by Figs 4.2 and 4.3, provides us with the following conclusion: the larger value of the distance between $AP(PD_3)_{D_k}$, $k = 1, 2, 3$, and $AP(PD_3)$ points to this D_k that possesses the value of one in the set $AP(PD_3)_{D_k}$ as the more truthful diagnosis in P.

It is obvious that D_1, which has the largest value of all the membership degrees in the set PD_3 in accordance with Fig 4.1, is the approved diagnosis in P. At the same time the distance of the set $AP(PD_3)_{D_1}$, which is associated with D_1, measured from $AP(PD_3)$ is the largest in comparison to other distances with respect to D_2 and D_3. This finally confirms that D_1 should be approved as a recognized illness for P.

Let us formulate a conclusion by summing up the premises expressed above.

Conclusion 4.1

If the membership degrees in the acceptance matrix PD_3 are almost equal or they differ a little from each other, then it will be rather impossible to find a proper diagnosis. We thus recommend an additional method based on distances between fuzzy sets.

We will successively calculate the distances $d_k = d(AP(PD_3), AP(PD_3)_{D_k})$, $k = 1,2,3$, in the case of three diagnoses belonging to the set $D = \{D_1, D_2, D_3\}$. The set D can be extended to as many diagnoses as we can assign to the considered symptoms. We finally approve this D_k that has contributed in the largest value of the distance d_k, $k = 1, 2, 3$.

We apply (4.10) to find that

$$d_1 = d(AP(PD_3), AP(PD_3)_{D_1}) = \sqrt{(1-\mu_{PD_3}(P,D_2))^2 + (1-\mu_{PD_3}(P,D_3))^2}\ , \tag{4.16}$$

$$d_2 = d(AP(PD_3), AP(PD_3)_{D_2}) = \sqrt{(1-\mu_{PD_3}(P,D_1))^2 + (1-\mu_{PD_3}(P,D_3))^2} \tag{4.17}$$

and

$$d_3 = d(AP(PD_3), AP(PD_3)_{D_3}) = \sqrt{(1-\mu_{PD_3}(P,D_1))^2 + (1-\mu_{PD_3}(P,D_2))^2}\ . \tag{4.18}$$

To reach a higher grade of accuracy in medical diagnosis, we can also investigate a possibility of rejecting the diagnoses when we still face some unclear data in PD_3. If membership degrees in the matrices PD_4 and PD_5 do not differ essentially from each other, then they will concern us enough to make the best possible decision. To omit this obstacle we propose another diagnostic method based on distances. We preserve $D = \{D_1, D_2, D_3\}$ as a set of diagnoses. Let us introduce two sets

$$JP(PD_4) = {}^{1}\!/\!_{(P,D_1)} + {}^{1}\!/\!_{(P,D_2)} + {}^{1}\!/\!_{(P,D_3)} \tag{4.19}$$

and

$$JP(PD_5) = {}^{1}\!/\!_{(P,D_1)} + {}^{1}\!/\!_{(P,D_2)} + {}^{1}\!/\!_{(P,D_3)}\,. \tag{4.20}$$

The sets (4.19) and (4.20) correspond to the states of total rejections of all diagnoses in P while the sets PD_4 and PD_5, constructed for patient P, are generally denoted as fuzzy sets (one row-matrices)

$$PD_4 = \mu_{PD_4}(P,D_1)/(P,D_1) + \mu_{PD_4}(P,D_2)/(P,D_2) + \mu_{PD_4}(P,D_3)/(P,D_3) \tag{4.21}$$

and

$$PD_5 = \mu_{PD_5}(P,D_1)/(P,D_1) + \mu_{PD_5}(P,D_2)/(P,D_2) + \mu_{PD_5}(P,D_3)/(P,D_3)\,. \tag{4.22}$$

Figure 4.4 lets us perceive the range of the distance between true rejection sets and total rejection sets if we preserve the indications (ellipses and squares) introduced in Fig. 4.1.

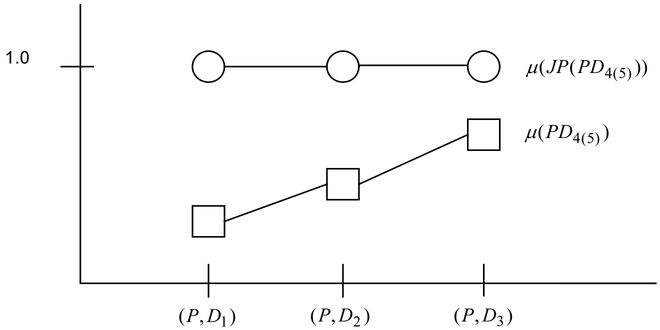

Figure 4.4: The comparison between PD_4 (or PD_5) and $JP(PD_4)$ (or $JP(PD_5)$)

By following the same way of reasoning as in the case of accepted diagnoses, we consider a secure, theoretical choice of D_1. This implies the set $JP(PD_4)_{D_1}$ in which the membership degree of D_1 takes the value of one. The set has an appearance as a fuzzy set

$$JP(PD_4)_{D_1} = 1/(P,D_1) + \mu_{PD_4}(P,D_2)/(P,D_2) + \mu_{PD_4}(P,D_3)/(P,D_3). \tag{4.23}$$

Since we are furnished with two matrices of rejection we should also introduce the set $JP(PD_5)_{D_1}$, as a counterpart of $JP(PD_4)_{D_1}$, in the form of

$$JP(PD_5)_{D_1} = 1/(P,D_1) + \mu_{PD_5}(P,D_2)/(P,D_2) + \mu_{PD_5}(P,D_3)/(P,D_3). \tag{4.24}$$

The sets $JP(PD_4)_{D_2}$ and $JP(PD_5)_{D_2}$ correspond to an unquestionable exclusion of D_2 and are arranged as

$$JP(PD_4)_{D_2} = \mu_{PD_4}(P,D_1)/(P,D_1) + 1/(P,D_2) + \mu_{PD_4}(P,D_3)/(P,D_3) \tag{4.25}$$

and

$$JP(PD_5)_{D_2} = \mu_{PD_5}(P,D_1)/(P,D_1) + 1/(P,D_2) + \mu_{PD_5}(P,D_3)/(P,D_3). \tag{4.26}$$

Finally, to refuse entirely an existence of D_3 we place the value of one as the membership degree of D_3 in sets

$$JP(PD_4)_{D_3} = \mu_{PD_4}(P,D_1)/(P,D_1) + \mu_{PD_4}(P,D_2)/(P,D_2) + 1/(P,D_3) \tag{4.27}$$

and

$$JP(PD_5)_{D_3} = \mu_{PD_5}(P,D_1)/(P,D_1) + \mu_{PD_5}(P,D_2)/(P,D_2) + 1/(P,D_3). \tag{4.28}$$

Looking at Fig. 4.4 we experience that the larger value of a distance between $JP(PD_4)_{D_k}$, k = 1, 2, 3, and $JP(PD_4)$ (or $JP(PD_5)_{D_k}$ and $JP(PD_5)$) is related to the more sensible rejection of this D_k that is recognized by the value of one in $JP(PD_4)_{D_k}$ or $JP(PD_5)_{D_k}$.

D_1, which now is represented by the smallest value in the rejection matrices PD_4 and PD_5 according to Fig. 4.4, is still the approved diagnosis for P. This holds true because the distance of the set $JP(PD_4)_{D_1}$ from $JP(PD_4)$ (and probably the distance of $JP(PD_5)_{D_1}$ from $JP(PD_5)$) is smallest of all distances computed for D_2 and D_3 with respect to $JP(PD_{4(5)})_{D_2(D_3)}$ and $JP(PD_{4(5)})$.

We go through the observations that have been made lately and write them down as the following outline.

Conclusion 4.2

If the membership degrees in the rejection matrices PD_4 and PD_5 differ a little from each other, or they induce a contraposition in the diagnostic exclusion, then we will experience difficulties in pointing out some rejected diagnoses. In spite of this inconvenience we supply the next trial of the model improvement still based on distances between fuzzy sets.

We estimate a sequence of distances $d_k^{'} = d(JP(PD_4), JP(PD_4)_{D_k})$, $k = 1,2,3$, (or $d_k^{''} = d(JP(PD_5), JP(PD_5)_{D_k})$) for three diagnoses belonging to set $D = \{D_1, D_2, D_3\}$ (the number of D's members can be definitely enlarged). We reject this D_k, which has the largest value of the distance $d_k^{'}$ (or $d_k^{''}$), $k = 1, 2, 3$.

Let us derive formulas for making calculations of distances. We introduce quantities of

$$d_1^{'} = d(JP(PD_4), JP(PD_4)_{D_1}) = \sqrt{(1-\mu_{PD_4}(P,D_2))^2 + (1-\mu_{PD_4}(P,D_3))^2}\,, \quad (4.29)$$

$$d_2^{'} = d(JP(PD_4), JP(PD_4)_{D_2}) = \sqrt{(1-\mu_{PD_4}(P,D_1))^2 + (1-\mu_{PD_4}(P,D_3))^2}\,, \quad (4.30)$$

$$d_3^{'} = d(JP(PD_4), JP(PD_4)_{D_3}) = \sqrt{(1-\mu_{PD_4}(P,D_1))^2 + (1-\mu_{PD_4}(P,D_2))^2} \quad (4.31)$$

as well as

$$d_1^{''} = d(JP(PD_5), JP(PD_5)_{D_1}) = \sqrt{(1-\mu_{PD_5}(P,D_2))^2 + (1-\mu_{PD_5}(P,D_3))^2}\,, \quad (4.32)$$

$$d_2'' = d(JP(PD_5), JP(PD_5)_{D_2}) = \sqrt{(1-\mu_{PD_5}(P,D_1))^2 + (1-\mu_{PD_5}(P,D_3))^2} \quad (4.33)$$

and

$$d_3'' = d(JP(PD_5), JP(PD_5)_{D_3}) = \sqrt{(1-\mu_{PD_5}(P,D_1))^2 + (1-\mu_{PD_5}(P,D_2))^2} \,. \quad (4.34)$$

To make a final decision regarding a choice of the most probable diagnosis in vague decision circumstances we should elaborate the analysis of all obtained distances in accordance with the following criteria:

1. We agree to this diagnosis D_k, for which the distance d_k, $k = 1, 2, 3$ (generally $k = 1, 2, \ldots, p$) is largest;
2. We neglect this diagnosis D_k, that influences the distance d_k' or d_k'' to be the largest value for $k = 1, 2, 3$ (generally $k = 1, 2, \ldots, p$).

The method based on distances assists the diagnostic model projected for the next patient P_3, whose indices fit for all adequate diagnoses that have been associated with a collection of chosen symptoms.

Example 4.5

We assume that patient P_3 suffers from one of the diagnoses that have already been investigated in Ex. 3.16. The examinations of ten symptoms, listed in Ex. 3.1, have been converted to the values of membership degrees that constitute the contents of the one-row matrix *PS.* We exploit the formulas (4.5)–(4.9) to make the necessary computations collected in Table 4.2.

Table 4.2: The diagnostic decision concerning patient P_3

Patient	PD_3			PD_4			PD_5			Decision
	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3	
P_3	0.755	0.795	0.62	0.3	0.62	0.755	0.82	0.41	0.41	*unknown*

By taking into consideration the membership degrees in PD_3, we are able to assign to P_3 either a D_1 or D_2 since they have the largest membership degrees. With respect to PD_4 we should exclude D_3 and D_2 because they show the largest in magnitude degrees. However, this contradicts the results obtained in PD_5 where D_1 ought to be rejected since its membership degree is largest of all. We have come to contradictory conclusions that make P_3's diagnostic problem unsolvable.

In order to improve the data's decisive character we estimate the distances d_k, $d_k^{'}$ and $d_k^{''}$, k = 1, 2, 3. Table 4.3 now consists of the revised specification of P's health conditions.

Table 4.3: The distances d_k, $d_k^{'}$ and $d_k^{''}$, k = 1, 2, 3, evaluated for patient P_3

Patient	D_1			D_2			D_3			Decision
	d_1	$d_1^{'}$	$d_1^{''}$	d_2	$d_2^{'}$	$d_2^{''}$	d_3	$d_3^{'}$	$d_3^{''}$	
P_3	0.43	0.45	0.83	0.45	0.74	0.61	0.32	0.8	0.61	D_1

Having compared d_1, d_2 and d_3 we can admit to a placement of D_1 or D_2 at the top of a hierarchy ladder of the considered diagnoses. The revision of $d_1^{'}$, $d_2^{'}$ and $d_3^{'}$ gives us a tool for deciding that D_2 and D_3, as the diagnoses with the largest rejection distances $d_2^{'}$ and $d_3^{'}$, are not taken into consideration anymore as possible diagnoses in P_3. The numbers $d_1^{''}$, $d_2^{''}$ and $d_3^{''}$ do not vary from each other in the substantial grade anymore, which allows us to omit their influence on the final decision. Since D_1 is characterized by the essential low value of $d_1^{'}$, and by the substantial high value of d_1 then we can take a risk of choosing this diagnosis as a primary diagnosis in the patient. The choice is confirmed by the experienced physician who has examined P_3.

The distance method of diagnosing can be helpful in cases that contain hardly interpretable or vague decision data, but we can imagine that a physician should obtain better diagnostic results after more than one examination of a patient. Some wider and richer reports of symptom observations can prevent a diagnostician from making a mistake when the clinical state of a patient shows a tendency to some changes.

4.4 Diagnostic Processes Extended in Time Intervals

This section refers to earlier results obtained in Chapter 3 and Section 4.2 and constitutes their essential complement and extension. A new assumption aims at the introduction of repeated medical examinations in which measurements of symptoms are regularly made. In this way we can render all essential changes in symptom values resulting in making an appropriate diagnostic decision. The model offered below concerns the observations of symptoms in an individual patient at a time interval.

The behaviour of the symptoms over a period of time, conduces to the access of some additional information. This sometimes is very important in a diagnostic process in which several clinical pictures of a patient, obtained during a certain time interval, differ from each other and point to different diagnoses. It may occur that the change in the intensity of a symptom decides an acceptance of another diagnosis, when, after some time, the patient does not feel better.

The objective now is to fix an optimal diagnosis on the basis of clinical symptoms typical of several diagnoses with respect to the changes of these symptoms throughout time. Both the intensity of some symptoms, and the retreat of another group of them observed during a certain specific period of time, constitutes an additional factor that supports the selection of a diagnosis.

In order to solve a diagnostic model extended in time, we again modify fuzzy relation equations as discussed in Subsections 3.4 and 4.2. Moreover, in the final decision concerning the choice of an adequate diagnosis, the adoption of a normalized Euclidean distance is suggested as a measure between an objective decision and an "ideal" decision. As usual we check the relevance of the model by testing some sampled clinical data.

We consider three non-fuzzy sets representing only one patient [56, 58, 61]:

1. A set of *"stages of observations"* $T = \{T_1, T_2,\ldots, T_m\}$, where each symbol T_i, $i = 1, 2, \ldots, m$, stands for a new phase of the examination;
2. A set of symptoms $S = \{S_1, S_2,\ldots, S_n\}$ in which each biological symptom-parameter S_j, $j = 1, 2, \ldots, n$, has been described or measured in the successive examination T_i;
3. A set of diagnoses $D = \{D_1, D_2,\ldots, D_p\}$, where to each diagnosis D_k, $k = 1, 2, \ldots, p$, one may assign the symptoms occurring in the set S.

Each of the symptoms $S_j \in S$, $j = 1, 2, \ldots, n$, is a fuzzy set with the membership function being modelled according to a kind of symptom (see Section 3.3) and allowing one to assign the membership degree to a fix value of this symptom.

The "*stage – symptom*" fuzzy relation formed as a collection of membership degrees of the pairs (T_i, S_j), $i = 1, 2,\ldots, m$, $j = 1, 2,\ldots, n$, is written down as a matrix

$$TPS = \begin{array}{c} \\ T_1 \\ T_2 \\ \vdots \\ T_m \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & \cdots & S_n \end{array} \\ \begin{bmatrix} \mu_{TPS}(T_1,S_1) & \mu_{TPS}(T_1,S_2) & \cdots & \mu_{TPS}(T_1,S_n) \\ \mu_{TPS}(T_2,S_1) & \mu_{TPS}(T_2,S_2) & \cdots & \mu_{TPS}(T_2,S_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{TPS}(T_m,S_1) & \mu_{TPS}(T_m,S_2) & \cdots & \mu_{TPS}(T_m,S_n) \end{bmatrix} \end{array} . \tag{4.35}$$

The fuzzy relation $1-TPS$ introduced by a membership function

$$\mu_{1-TPS}(T_i,S_j) = 1 - \mu_{TPS}(T_i,S_j) \tag{4.36}$$

is one of components taking part in decision equations.

We now focus on the presence of symptom S_j in diagnosis D_k on one hand, and on the other hand, the decisive character of S_j for D_k so as to allow one to judge the intensity of a relationship between the symptoms S_j and the diagnoses D_k. To each pair (S_j, D_k) we assign two membership degrees fixed for the linguistic variables "*presence*" and "*decisive character*" (see Subsection 3.4.1). By employing the technique already described in Subsection 3.4.2, we insert two fuzzy relations of "*medical knowledge*" listed as

$$SD_P = \begin{matrix} & D_1 & \cdots & D_p \\ S_1 & & & \\ \vdots & & \mu_{\text{"}presence\text{"}}(S_j, D_k) & \\ S_n & & & \end{matrix} \tag{4.37}$$

and

$$SD_D = \begin{matrix} & D_1 & \cdots & D_p \\ S_1 & & & \\ \vdots & & \mu_{\text{"}decisive\ character\text{"}}(S_j, D_k) & \\ S_n & & & \end{matrix}. \tag{4.38}$$

The fuzzy relations (4.35), (4.37) and (4.38) are elements of equations yielding relations *TPD* standing for connections "*stage – diagnosis*". These consist of pairs (T_i, D_k), $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, p$. By superposing the fuzzy relations *TPS* with SD_P or SD_D with respect to the operation "$\underset{+}{\circ}$" we reproduce the membership functions of relations *TPD* in accordance with the general formula

$$\mu_{TPD}(T_i, D_k) = \frac{\sum_{j=1}^{n} \mu_{TPS}(T_i, S_j) \cdot \mu_{SD}(S_j, D_k)}{\sum_{j=1}^{n} \mu_{SD}(S_j, D_k)} \tag{4.39}$$

for $i = 1, \ldots, m, j = 1, \ldots, n$ and $k = 1, \ldots, p$.

The inference rule *modus ponens* (cited in Subsections 3.2 and 3.4) induces an interpretation:

"If the symptom S_j emerges in stage T_i with the membership degree $\mu_{TPS}(T_i, S_j)$"

and

"If the appearance of S_j results in D_k with the membership degree $\mu_{SD_P}(S_j, D_k)$ or $\mu_{SD_D}(S_j, D_k)$"

then
"The diagnosis D_k occurs in the stage T_i with the general membership degree $\mu_{TPD}(T_i, D_k)$".

On the basis of the rule above we will consider a fuzzy relation equation

$$TPD_1 = TPS \underset{+}{\circ} SD_P \tag{4.40}$$

as well as

$$TPD_2 = TPS \underset{+}{\circ} SD_D \,. \tag{4.41}$$

The relations TPD_1 and TPD_2 are parts of a mean rule leading to

$$TPD_3 = mean(TPD_1, TPD_2)\,, \tag{4.42}$$

provided that the relation TPD_3 is a crucial factor deciding the final acceptance of an optimal diagnosis after each examination $T_1, \ldots, T_m$.

An acceptance criterion for the diagnosis S_j at the stage T_i is the same as the conclusion stated in Section 3.4, i.e., the higher membership degree of the diagnosis D_k at the stage T_i corresponds to the more certain approval of D_k.

It can happen that the membership degrees in the row T_i of the relation TPD_3 (i.e., at the stage T_i) differ a little and do not indicate the optimal diagnosis as a clear-cut decision. Therefore it is also recommended to inspect an opportunity of rejecting the diagnosis.

Another rule of inference *modus tollens*, already familiar to us, creates a foundation for the statement:
"If the symptom S_j does not appear in stage T_i with the membership degree $1 - \mu_{TPS}(T_i, S_j)$"
and
"It is true that D_k requires presence of S_j with the membership degree $\mu_{SD_P}(S_j, D_k)$"
then
"The diagnosis D_k is rejected in the patient at the stage T_i with the membership degree $\mu_{TPD}(T_i, D_k)$".

The above interpretation of the *modus tollens* law involved in the diagnosis exclusion gives rise to setting the next fuzzy relation equation

$$TPD_4 = (1 - TPS) \underset{+}{\circ} SD_P \,. \tag{4.43}$$

By proving a modification of the same logical *modus tollens* law, we formulate the next equation as the compound operation involving the relations *TPS* and 1 – SD_P in a relation

$$TPD_5 = TPS \underset{+}{\circ} (1 - SD_P) \,. \tag{4.44}$$

The fuzzy relations TPD_4 and TPD_5 play an essential role in the rejecting of inadequate diagnoses at the successive stages T_i, $i = 1, \ldots, m$. The higher the membership degree value of D_k at the T_i stage in TPD_4 and TPD_5, the greater the certainty that the D_k-diagnosis will be rejection. All the conclusions are valid for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$.

The final decision concerning the acceptance of the proper diagnosis assumes a thorough analysis of the entire period of observations at the stages $T_1, T_2, \ldots, T_m$. The hierarchy of diagnoses during this period of time in a considered patient is established by the estimation of the Euclidean distances between fuzzy sets.

Each diagnosis D_k, $k = 1, 2, \ldots, p$ occurring as the k^{th} column in the relations TPD_t ("*stage – diagnosis*"), $t = 1, 2, 3, 4, 5$, is interpreted as a fuzzy set

$$D_k = \mu_{TPD_t}(T_1, D_k)/T_1 + \mu_{TPD_t}(T_2, D_k)/T_2 + \cdots + \mu_{TPD_t}(T_m, D_k)/T_m \,. \tag{4.45}$$

Let us demonstrate an uncomplicated example to make an interpretation of the set D_k a bit easier.

Example 4.6

We consider three diagnoses D_1, D_2, D_3 in a relation TPD_1, computed for four stages of symptom observations. The relation TPD_1, written down as the matrix

$$TPD_1 = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.3 & 0.6 \\ 0.6 & 0.5 & 0.5 \\ 0.8 & 0.4 & 0.3 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \end{array},$$

introduces diagnosis D_1 as the fuzzy set

$$D_1 = \mu_{TPD_1}(T_1, D_1)/T_1 + \mu_{TPD_1}(T_2, D_1)/T_2 + \mu_{TPD_1}(T_3, D_1)/T_3 + \mu_{TPD_1}(T_4, D_1)/T_4$$

or, more specifically, as the set $D_1 = 0.7/T_1 + 0.6/T_2 + 0.8/T_3 + 0.5/T_4$.

A set of "total acceptance" or "total rejection" of diagnosis D_k on the basis of observed symptoms at stages $T_1, \ldots, T_m$, is assumed to be the set given by

$$D = \frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_m} \tag{4.46}$$

because the membership degree of the fully accepted diagnosis D_k in the relation TPD_3 and the totally rejected diagnosis in TPD_4 and TPD_5, when considering each stage, should be equal to the rarely achieved value "*one*".

The distance of the set D_k from the set D is estimated by applying the Euclidean normalized distance

$$e_k = e(D_k, D) = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(\mu_{TPD_t}(T_i, D_k) - 1)^2}\ , \tag{4.47}$$

for $t = 3, 4, 5$ and $k = 1, \ldots, p$.

It is easy to conclude that there exist some relationships between magnitudes of the distances e_k estimated for D_k and the decisions of its acceptance or rejection from a set of diagnoses possible in a patient. The smaller the distance from the set D to the fuzzy set D_k, created in TPD_3, the stronger the acceptance of the diagnosis D_k will be assumed in the considered patient. A similar conclusion concerns the diagnoses forming the columns of the relations TPD_4 and TPD_5, i.e., the small distance of D_k from set D that indicates the excluded illness in a patient.

It ought to appear a theoretical connection, e.g., for two diagnoses D_1 and D_2:

a) If D_1 is the accepted diagnosis, then D_1 tends to have the smaller distance $e(D_1, D)$ than $e(D_2, D)$, for D_1, D_2 taken as the columns in TPD_3.

b) If D_2 is the rejected diagnosis, then D_2 shows the smaller distance $e(D_2, D)$ than $e(D_1, D)$, for D_1, D_2 appearing as the columns in TPD_4 and TPD_5.

The next medical sample of data is tested to prove the successful action of observations throughout time.

Example 4.7

The relation *TPS,* built for patient P_1, collects the membership degrees assigned to his symptom values that have been estimated three times during separate visits at the hospital. We introduce *TPS* as the matrix

$$PS = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} \end{array} \\ \begin{bmatrix} 1 & 0.515 & 0.913 & 0.653 & 0.345 & 0.632 & 0.720 & 0.9 & 0.0004 & 0.353 \\ 1 & 0.875 & 0.523 & 0.569 & 0.543 & 0.576 & 0.641 & 0.7 & 0.0008 & 0.342 \\ 1 & 0.712 & 0.320 & 0.436 & 0.634 & 0.543 & 0.436 & 0.6 & 0.0005 & 0.326 \end{bmatrix} \end{array}.$$

We note that the patient has improved the values of some unfavourable parameters like smoking, hypertension and lack of physical activity after his first consultation with a doctor. We can imagine that P_1 has taken the doctor's advice into serious consideration.

By using Eqs (4.40)–(4.42) we compute membership degrees of the relation TPD_3 whereas Eqs (4.43) and (4.44) give rise to TPD_4 and TPD_5. All relations are demonstrated in Table 4.4.

Table 4.4: The relations TPD_3, TPD_4 and TPD_5 made for P_1

Stage	TPD_3			TPD_4			TPD_5			Decision
	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3	
T_1	0.629	0.586	0.604	0.376	0.408	0.407	0.579	0.625	0.655	D_1
T_2	0.520	0.539	0.583	0.481	0.462	0.426	0.641	0.649	0.590	D_3
T_3	0.445	0.485	0.510	0.553	0.519	0.497	0.560	0.539	0.488	D_3

The final decision, pointing out a right diagnosis, is rather clear on the basis of clinical symptoms observed during each distinct visit at the doctor's. Let us prove the distance method according to (4.47) that helps us to weigh intensities of the examined symptoms in time. To carry out the comparison of all involved distances we place the obtained results in Table 4.5.

Table 4.5: The final acceptance of diagnosis by means of distances

Patient	TPD_3			TPD_4			TPD_5			Decision
	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3	
P_1	0.474	0.465	0.436	0.534	0.539	0.558	0.408	0.398	0.428	D_3

To explain the procedure of computing the membership degrees that are the results of (4.47), we go through a basic example of executing the necessary operations to get e_1 in TPD_3 as $e_1 = \sqrt{\frac{1}{3}\left((0.629-1)^2 + (0.520-1)^2 + (0.445-1)^2\right)} =$ 0.474.

We should reject D_1 and D_2 on the basis of the relations TPD_4 and TPD_5 for which e_1 and e_2 are the smallest values. The value e_3 computed for TPD_3 as the smallest of all confirms that D_3 should be recognized for P_1's sake. We thus decide that P_1, who has suffered from D_1 at the first examination stage, actually runs the high risk of going down with D_3 (infarct) if he is not careful enough and does not improve risk factors in his parameters.

The proposed method for establishing the correct diagnosis on the basis of clinical symptoms *observed in time*, constitutes an essential improvement of the diagnostic process because it optimizes the diagnosing by correction and verification of decisions with respect being paid to the variability of symptoms in time. The changes in intensities of symptom presence at some time influence not only a choice of the most appropriate diagnosis, but also affect a rejection of less accurate diseases.

4.5 Rough Set Theory in the Classification of Diagnoses

Rough set theory is a new mathematical approach to intelligent data analysis and data mining [50, 51, 52, 53].

Rough set philosophy is founded on the assumption that some information is associated with every object of the considered universe set. The objects characterized by the same information are indiscernible (similar) in view of the available information about them. The indiscernibility relation generated for similar objects is the mathematical basis of rough set theory. Any set of similar objects, being the equivalence class of the similarity relation, is called an elementary set. Any union of some elementary sets (equivalence classes) is a crisp set (a precise set). Such union of elementary sets, which has boundary-line cases, i.e., objects that cannot be classified with certainty, constitutes a rough set (an imprecise, vague set).

With any rough set, a pair of precise sets – called a lower and an upper approximation of the rough set – is associated. The lower approximation consists of all objects that surely belong to the set, and the upper approximation contains all objects that possibly belong to the set. A difference between the upper and the lower approximation constitutes the boundary region of the rough set. Approximations are two basic operations in the rough set theory.

Let us first introduce the theoretical background of rough sets and afterwards let us prove their usefulness via presenting a practical problem concerning medical diagnosing. All conceptions and annotations will be accommodated to a medical model to make it easier at the stage of practical interpretation.

We start with an information system constructed as a data table whose columns are labelled by attributes. Objects of interest label the table rows, and entries of the table are attribute values. In a new scenario of the diagnostic discussion, interpreted now as a classification of diagnoses, we adopt the set of patients $P = \{P_1, \ldots, P_m\}$ with objects P_i, $i = 1, \ldots, m$, as a *universe set P*. The set of *condition attributes S* is established as a set of symptoms $S = \{S_1, \ldots, S_n\}$. With every attribute $S_j \in S$, $j = 1, \ldots, n$, we associate a set $V_{S_j} = \{x_{S_j}^1, x_{S_j}^2, \ldots, x_{S_j}^{t(S_j)}\}$ of its values, called the *domain* of S_j. In the diagnostic problem the set V_{S_j} will contain some linguistic terms or values of the membership degrees of S_j expressed by codes that correspond to the intensity grades of S_j. Any subset B of S determines a binary relation $I(B)$ on B, which will be called an *indiscernibility* relation. The relation $I(B)$ is defined by an inclusion operation

$$(P_i, P_l) \in I(B) \text{ if } S_j(P_i) = S_j(P_l)\,, \tag{4.48}$$

for each $S_j \in B \subseteq S$, i, l = 1, …, m, j = 1, …, n, where $S_j(P_i)$ denotes the value $x^c_{S_j}$, c = 1, …, $t(S_j)$, of attribute S_j for the element P_i.

The relation $I(B)$ is *reflexive* because $(P_i, P_i) \in I(B) \leftrightarrow S_j(P_i) = S_j(P_i)$ for each $P_i \in P$.

Since $(P_i, P_l) \in I(B) \leftrightarrow S_j(P_i) = S_j(P_l) \leftrightarrow S_j(P_l) = S_j(P_i) \leftrightarrow (P_l, P_i) \in I(B)$ for P_i, $P_l \in I(B)$, then $I(B)$ will be a *symmetric* relation, too.

Finally, the assumptions $(P_i, P_l) \in I(B)$ and $(P_l, P_r) \in I(B)$ for P_i, P_l, $P_r \in P$ imply $S_j(P_i) = S_j(P_r) \leftrightarrow (P_i, P_r) \in I(B)$. $I(B)$ thus is a *transitive* relation.

The sign "↔" is interpreted as "which is equivalent to".

For the reason of such properties as reflexivity, symmetry and transitivity $I(B)$ is recognized as an equivalence relation.

It is possible to make a partition of the set P, with respect to B, by means of the relation $I(B)$ to obtain equivalence classes $IB(P_i)$ defined by

$$IB(P_i) = \{P_l : (P_i, P_l) \in I(B)\}\,, \tag{4.49}$$

for each i, l = 1, …, m. The classes $IB(P_i)$ are additionally called elementary sets. We realize that these sets contain the objects P_i, which are identical, i.e., in the considered case, they gather patients who suffer from a presence of the same symptoms characterized by the same intensity.

The symptoms S_j constitute the condition attributes in the diagnostic model. Besides these, we also consider a decision attribute – the diagnosis D_1. D_1 has a set of values determined as "*yes*" if it is found in the patient, "*no*" if the patient is free from it and "*unknown*" when a decision about the presence of the diagnosis cannot be clearly formulated.

By resuming the assumptions made so far we can come to a conclusion that the contents of the classification table, giving rise to the indiscernibility relation $I(B)$, corresponds to a *triple* (P, S, D_1) in the model of diagnoses. The patients P_i are placed in the first column of the table, the three values of D_1 appear in the last column while the rest of the table positions are filled with the values of condition attributes.

The aim of the classification, accomplished by $I(B)$ or rather its equivalence classes, is to divide the patients belonging to P in three groups. These three groups are; a group of patients who surely are ill with D_1, a sample of patients who may suffer from D_1 and a collection of patients who do not have diagnosis D_1.

Let us create a set $P_{yes} \subseteq P$ in accordance with the following definition

$$P_{yes} = \{P_i : D_1 \text{ has decision "}yes\text{" assigned}\}\,, \tag{4.50}$$

for i = 1, …, m.

We now state two sets surrounded $P_{yes} \subseteq P$ that are treated as its lower and upper approximations.

The lower approximation $B_*(P_{yes})$ of P_{yes} is built by an inclusion operator as

$$B_*(P_{yes}) = \{P_i : IB(P_i) \subseteq P_{yes}\} \tag{4.51}$$

and rendered as a set of these P_i that have D_1 assigned with a full security.

Another set, the upper approximation $B^*(P_{yes})$ of P_{yes} is designed by

$$B^*(P_{yes}) = \{P_i : IB(P_i) \cap P_{yes} \neq 0\} \tag{4.52}$$

and accepted as a sampling of those objects P_i that possibly are members of the class D_1 possessing the attribute "*yes*" (D_1 = "*yes*").

The set P_{yes} is thus bounded by two sets in compliance with the inclusion $B_*(P_{yes}) \subseteq P_{yes} \subseteq B^*(P_{yes})$ and referred to the approximation sets as rough or inexact with respect to B.

Even a boundary set

$$B_{border}(P_{yes}) = B^*(P_{yes}) - B_*(P_{yes}) \tag{4.53}$$

contains some useful information about the objects that are uncertain members of the class D_1 = "*yes*".

To measure a grade of membership uncertainty in D_1 = "*yes*" for each P_i, we recommend applying the formula for computing membership degrees

$$\mu_{D_1="yes"}(P_i) = \frac{|P_{yes} \cap IB(P_i)|}{|IB(P_i)|}, \tag{4.54}$$

in which the symbol "$|\ |$" denotes the cardinality of a set (the number of elements belonging to a set).

A selection of the B-subset of S should be made with the special care to assure good classification results. We can measure a coefficient α_B called the accuracy of approximation in conformity with

$$\alpha_B(P_{yes}) = \frac{|B_*(P_{yes})|}{|B^*(P_{yes})|} \tag{4.55}$$

to measure an adaptation grade of B to the decision table (P, S, D_1).

We demonstrate the utility of rough sets in the diagnosis classification process by studying steps of the following example.

Example 4.8

In Ex. 3.1 we have already listed 10 symptoms that are the elements of the set of symptoms S. Let us select set $B \subseteq S$ as $B = \{S_3, S_4, S_8, S_9, S_{10}\}$. Set B contains the most significant symptoms that are characteristic of diagnosis D_1.

We now prepare sets of values corresponding to the selected symptoms.

Since S_3 and S_4 are compound qualitative parameters measured by means of a questionnaire, then we can place their membership degrees in the continuous interval [0, 1]. The quantitative indicators S_8, S_9 and S_{10} possess the same property. In order to vary some intensity grades of the symptoms' appearance as discrete characteristic quantities, we construct the following codes associated with the membership values $\mu_{S_j}(P_i)$, $j = 3, 4, 8, 9, 10$, belonging to subintervals of [0, 1]. We assign the code 0 to $\mu_{S_j}(P_i) \in [0, 0.25)$, 1 – to $\mu_{S_j}(P_i) \in [0.25, 0.5)$, 2 – to $\mu_{S_j}(P_i) \in [0.5, 0.75)$ and, finally, 3 – to $\mu_{S_j}(P_i) \in [0.75, 1]$. The codes generate sets $V_{S_j} = \{0,1,2,3\}$, $j = 3, 4, 8, 9, 10$.

Suppose that $P = \{P_1, P_2, P_3, P_4, P_5, P_6\}$. The patients P_1, P_2, P_5 suffer from D_1, P_3, P_6 have D_2 assigned and the diagnosis concerning D_4 is unknown. We decide the members of set $P_{yes} = \{P_1, P_2, P_5\}$. To regard P_{yes} as rough, we should find its lower and upper approximation. In this way we count on classifying the unknown object P_4.

By assuming that the knowledge of clinical symptoms is absolutely correct, we fill in Table 4.6 known as (P, S, D_1) that constitutes a basis for establishing an indiscernibility relation $I(B)$.

Table 4.6: The table (P, S, D_1) in diagnosis classification

Patients	Codes characteristic of symptoms					Decision about D_1
	S_3	S_4	S_8	S_9	S_{10}	
P_1	1	1	2	1	2	*yes*
P_2	2	3	1	2	3	*yes*
P_3	2	2	2	3	1	*no*
P_4	2	3	1	2	3	*unknown*
P_5	1	2	2	2	2	*yes*
P_6	1	1	3	2	1	*no*

The relation $I(B)$ consists of the pairs of patients (P_i, P_l), $i, l = 1, \ldots, 6$, which when comparing rows i and l, all have equal codes.

We list $I(B)$ as $I(B) = \{(P_1, \mathrm{P}_1), (P_2, \mathrm{P}_2), (P_3, \mathrm{P}_3), (P_4, \mathrm{P}_4), (P_5, \mathrm{P}_5), (P_6, \mathrm{P}_6), (P_2, \mathrm{P}_4), (P_4, \mathrm{P}_2)\}$. The elementary sets of $I(B)$ or its equivalent classes are given as the sets $IB(P_1) = \{P_1\}$, $IB(P_2) = \{P_2, \mathrm{P}_4\}$, $IB(P_3) = \{P_3\}$, $IB(P_4) = \{P_2, \mathrm{P}_4\}$, $IB(P_5) = \{P_5\}$, $IB(P_6) = \{P_6\}$.

The lower approximation of P_{yes} is established as $B_*(P_{yes}) = \{P_1, P_5\}$ while its upper approximation is obtained as $B^*(P_{yes}) = \{P_1, P_2, P_4, P_5\}$.

The boundary set $B_{border}(P_{yes}) = \{P_2, P_4\}$.

The membership degrees, whose sizes confirm the patients' membership in the D_1 = "*yes*" class, have been evaluated as

$$\mu_{D_1="yes"}(P_1) = 1, \quad \mu_{D_1="yes"}(P_2) = \frac{1}{2}, \quad \mu_{D_1="yes"}(P_3) = 0, \;\; \mu_{D_1="yes"}(P_4) = \frac{1}{2},$$
$$\mu_{D_1="yes"}(P_5) = 1, \quad \mu_{D_1="yes"}(P_6) = 0.$$

We can assume that P_1 and P_5 have D_1 with a one hundred percent confidence, while P_2 and P_4 may suffer from D_1 to a certain grade. We can also notice that P_4 affects a status of P_2 negatively, and to the contrary, we can see that P_2 upgrades an importance of P_4 as a member in the D_1 = "*yes*"-class.

The accuracy approximation coefficient $\alpha_B(P_{yes}) = \frac{1}{2}$ does not give us a feeling of absolute trust in the choice of set B as a reliable source of information in the finished classification. This configuration of symptoms is not sufficient for a reliable classification since the accuracy coefficient has a low value of 0.5.

The supplementary solutions, proposed in Chapter 4, may improve the basic diagnosis model discussed in Chapter 3. We may apply them in the patients' cases that provide us with fuzzy data difficult to interpret in order to make a reliable decision. We thus should realize that a combination of different mathematical methods could support and improve an appropriate solution that is founded on clinical symptoms.

5 Evaluation of Medicine Action Levels

5.1 Introduction

In the previous chapters, we have discussed some ways of determining the most credible diagnosis in a patient who could be identified by his set of clinical symptoms. The same symptoms are usually found in several illnesses. Therefore, it is often difficult to recognize the value of each of their deterministic yet individual characteristics all at once. After improving the diagnostic model by adding complementary solutions we are at last aware of a diagnosis of the patient. The next step would be to prescribe him medication that will lead to a cure. It is seldom possible to give the patient only one remedy to remove completely all unfavourable symptoms. In order to broaden a list of medicines that complement each other, we usually want to evaluate levels of one medicine and its impact on all of the symptoms. Preferably, we want to estimate the lowest and the highest levels of effectiveness of the medicines tested, one by one, when considering their curative powers.

We make a simple attempt of eigen fuzzy set theory applications to respond to the question concerning the possibility of deciding the degree of effectiveness on a drug that is expected to affect some of the determined symptoms. It can be concluded that a final minimal and maximal level of the drug's action, found theoretically, does not change even if the patient takes the medicine for a long time. Such a conclusion is the result of adopting the eigen fuzzy set associated with a given fuzzy relation.

The existence of the greatest eigen fuzzy set of a fuzzy relation was confirmed in the 1980's [34, 72, 73, 78]. In the latest investigations, the scientists have proved that even the least eigen fuzzy set can be generated for the given relation [4, 24, 37]. The eigen fuzzy sets have already been applied to the evaluationf of a medicine's action levels when considering the medicine influence on clinical symptoms [27, 31, 64].

The basic operation, performed in the eigen problem, is the max-min composition already introduced in Chapter 2.

5.2 Theoretical Assumptions of Eigen Fuzzy Problem

By studying the contents of Def. 2.11 we have approached the conception of a composition of two fuzzy relations.

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If we suppose that one of these relations is a fuzzy set, we can make the max-min composition between the relation and the set, see Section 2.5. If the result of the composition is known in advance, then we will be prepared for demonstrating a particular case of the relation composition known as the eigen fuzzy problem.

Definition 5.1

Assume that $X = \{x\}$ is a set of real numbers. The eigen fuzzy set of a fuzzy relation $R \subseteq X \times X$ is a set $A \subseteq X$ that satisfies the condition $A \circ R = A$.

R is the fuzzy relation determined as $R \subseteq X \times X$ with the membership function $\mu_R : X \times X \to [0,1]$, $\mu_R(x,x') \in [0,1]$, x, $x' \in X$. We prove that the eigen fuzzy set $A \subseteq X$, $\mu_A : X \to [0,1]$, $\mu_A(x) \in [0,1]$, $x \in X$, satisfying $A \circ R = A$, should exist.

Some theoretical considerations that confirm the existence of set *A* are based on the papers of Sanchez [73, 74].

We define set A_0 with $\mu_{A_0}(x) = a_0$ for all $x \in X$, where

$a_0 = \min_{x' \in X}(\max_{x \in X} \mu_R(x,x'))$.

The fuzzy connection $A_0 \circ R = A_0$ is a true equality because of

$\mu_{A_0 \circ R}(x') = \max_x(\min(\mu_{A_0}(x), \mu_R(x,x'))) = \max_x(\min(a_0, \mu_R(x,x')))$

$= \min(a_0, \max_x \mu_R(x,x')) = a_0 = \mu_{A_0}(x'),\ x,\ x' \in X.$

Hence, A_0 is an eigen fuzzy set of *R*.

We have shown that at least one eigen fuzzy set can be found because the equation $A_0 \circ R = A_0$ is a true statement.

The next set A_1 is identified by its membership function given by

$$\mu_{A_1}(x') = \max_{x \in X} \mu_R(x,x') \tag{5.1}$$

for all $x' \in X$.

The fuzzy sets, which are members of the sequence $(A_n)_n$

$$A_2 = A_1 \circ R = A_1 \circ R^1,\ A_3 = A_2 \circ R = A_1 \circ R^2,\ \cdots, A_{n+1} = A_n \circ R = A_1 \circ R^n, \tag{5.2}$$

exist for all integers $n > 0$.

The sets satisfy inclusions

$$A_0 \subseteq \cdots \subseteq A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_2 \subseteq A_1. \tag{5.3}$$

Before starting to prove (5.3) we will insert the definition of an inclusion $A \subseteq B$ for two fuzzy sets *A* and *B* [12, 40, 88, 95].

Definition 5.2

Let $A = \{(x, \mu_A(x))\}$ and $B = \{(x, \mu_B(x))\}$ be two finite fuzzy sets in X. We say that A is a fuzzy subset of B ($A \subseteq B$) if $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$.

Example 5.1

We define $X = [0, 100]$. Let A be a fuzzy set given by $\mu_A(x) = s(x, 30, 50, 70)$ and let B be another fuzzy set introduced by $\mu_B(x) = s(x, 10, 50, 90)$. Since $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$, as Fig. 5.1 reveals, then $A \subseteq B$.

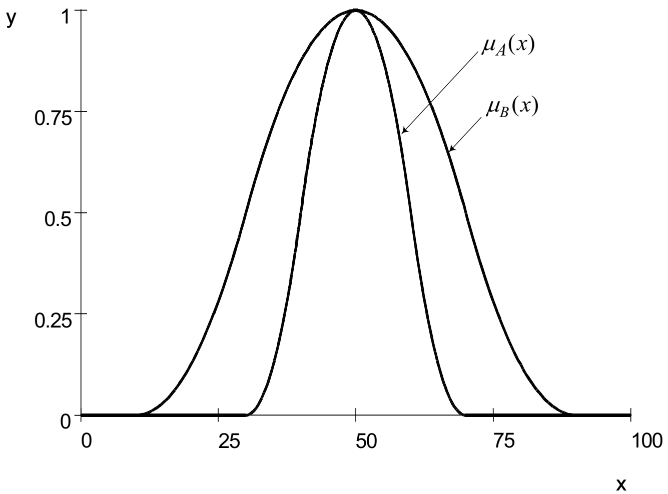

Figure 5.1: $A \subseteq B$ for $\mu_A(x) = s(x, 30, 50, 70)$ and $\mu_B(x) = s(x, 10, 50, 90)$

To confirm (5.3) we apply the mathematical induction. This method helps to prove the validity of a formula that contains a non-negative integer n. In order to accomplish the induction proof we follow three steps:

1. The formula should be true for $n = 0$ (or other low values of n).
2. We assume that the formula is valid for n.
3. We prove the reliability of the formula for $n + 1$ by applying the induction assumption in the proof.

Let us take $n = 0$. On the basis of the definition of A_0 we conclude that $A_0 \subseteq A_1$ since $\mu_{A_0}(x') = \min_{x'}(\max_{x} \mu_R(x,x')) \leq \max_{x} \mu_R(x,x') = \mu_{A_1}(x')$. We will deduce that an inclusion $A_2 \subseteq A_1$ is also true. By conveying, that $\mu_{A_2}(x') = \mu_{A_1 \circ R}(x') =$

$\max_{x \in X}(\min(\mu_{A_1}(x), \mu_R(x,x'))) \leq \max_{x \in X} \mu_R(x,x') = \mu_{A_1}(x')$, for every $x' \in X$, we use Def. 5.2 to state that $A_2 \subseteq A_1$.

We should now prove that an assumption $A_n \subseteq A_{n-1}$ induces the conclusion $A_{n+1} \subseteq A_n$, $n \in N \cup \{0\}$. We start with the assumption to get an implication $A_n \subseteq A_{n-1} \rightarrow A_n \circ R \subseteq A_{n-1} \circ R \leftrightarrow A_{n+1} \subseteq A_n$, whose thesis $A_{n+1} \subseteq A_n$ is the true statement (the sign "$\leftrightarrow$" still stands for "is equivalent to").

The set A_0 is the eigen set of R. A_1, the other introduced set, rarely is a solution of the restriction $A_1 \circ R = A_1$. If $A_n \circ R = A_n$, for A_n being a member of the sequence of sets given by (5.2), we will allege that A_n is the expected greatest eigen set of the relation R that differs from A_0. The set A_0 is the least set in the chain of sets in (5.2), and all sets included between A_1 and A_n are not eigen.

Suppose that $A_0 \neq A_{n+k} = \cdots = A_{n+1} = A_n \neq \cdots \neq A_2 \neq A_1$; then the composition $A_n \circ R$ leads to $A_n \circ R = A_1 \circ R^{n-1} \circ R = A_1 \circ R^n = A_{n+1} = A_n$.

A_n thus is the greatest eigen fuzzy set of R provided that $A_n = A_{n+1}$. The inclusion (5.3) ensures that $A_n \subseteq A_1$. Moreover, the inclusion confirms the existence of at least one A_n.

The introduction of the eigen set item is sufficient for medical applications proposed in the further part of this chapter. For more mathematical details; we refer to works by Sanchez and other authors who have developed this topic [34, 72, 73, 78].

There exist three fundamental algorithms of determining GEFS (the Greatest Eigen Fuzzy Set). We prefer adopting the procedure that consists of the commands listed below.

Algorithm 5.1

A relation $R \subseteq X \times X$ with the membership function $\mu_R(x,x')$ is given.

1. Find the set A_1 identified by $\mu_{A_1}(x') = \max_{x \in X} \mu_R(x,x')$ for all $x' \in X$.
2. Set the index $n = 1$.
3. Calculate $A_{n+1} = A_n \circ R$.
4. $A_{n+1} \overset{?}{=} A_n \genfrac{}{}{0pt}{}{\rightarrow No \rightarrow n=n+1 \rightarrow \text{Go to step 3}}{\rightarrow Yes \rightarrow A = A_{n+1}.}$

We recall that membership degrees of A_{n+1} are calculated as

$$\mu_{A_{n+1}}(x') = \mu_{A_n \circ R}(x') = \max_{x \in X}(\min(\mu_{A_n}(x), \mu_R(x,x'))) \tag{5.4}$$

for each $x' \in X$.

We demonstrate an action of the algorithm by selecting the greatest fuzzy set of a matrix introduced in the next example.

Example 5.2

We wish to find the greatest fuzzy set of a matrix

$$R = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 \\ x_2 \\ x_3 \end{matrix} \begin{bmatrix} 0.7 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.3 \\ 0.9 & 0.8 & 0.2 \end{bmatrix}$$

defined on $X \times X$, for $X = \{x_1, x_2, x_3\}$. The set A_1 has the membership degrees of x_j found as the largest values in columns j, $j = 1, 2, 3$, and is thus determined as

$$A_1 = \begin{bmatrix} 0.9 & 0.8 & 0.3 \end{bmatrix} \quad (x_1, x_2, x_3).$$

For $n = 1$ we obtain

$$A_2 = A_1 \circ R = \begin{bmatrix} 0.9 & 0.8 & 0.3 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.3 \\ 0.9 & 0.8 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 & 0.3 \end{bmatrix}.$$

We estimate $\mu_{A_2}(x_1)$, according to the max-min composition, as the quantity $\mu_{A_2}(x_1) = \max(\min((0.9,0.7),(0.8,0.4),(0.3,0.9))) = 0.7$.

Since $A_2 \neq A_1$, we set $n = 2$ in Step 4. of Algorithm 5.1 in order to compute A_3 as a set

$$A_3 = A_2 \circ R = \begin{bmatrix} 0.7 & 0.6 & 0.3 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.3 \\ 0.9 & 0.8 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 & 0.3 \end{bmatrix}$$

that satisfies the equality $A_3 = A_2$. The set A_3 is accepted as the greatest eigen fuzzy set of the relation R and we notice that A_3 holds $A_3 \subseteq A_2 \subseteq A_1$.

It can be desirable to find the smallest eigen fuzzy set of a given fuzzy relation as well. In spite of some accomplished investigations [4, 24] let us propose our own contribution as the following proof of the least eigen fuzzy set existence.

We define a new set A_0 with $\mu_{A_0}(x) = a_0$ for all $x \in X$, where $a_0 = \max_{x' \in X}(\min_{x \in X} \mu_R(x, x'))$. A_0 is the eigen set of R as it has been proved before.

The set A_1 also gets new membership degrees determined by

$$\mu_{A_1}(x') = \min_{x \in X} \mu_R(x, x') \tag{5.5}$$

for all $x' \in X$.

We preserve the same sequence of fuzzy sets $(A_n)_n$, $A_2 = A_1 \circ R = A_1 \circ R^1$, $A_3 = A_2 \circ R = A_1 \circ R^2, \cdots, A_{n+1} = A_n \circ R = A_1 \circ R^n$ that satisfy inclusions

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots \subseteq A_0. \tag{5.6}$$

The boundary inclusion $A_1 \subseteq A_0$ in the chain is true because of $\mu_{A_0}(x') = \max_{x'}(\min_x \mu_R(x, x')) \geq \min_x \mu_R(x, x') = \mu_{A_1}(x')$.

To prove other inclusions in (5.6) we once again recall the assumptions of mathematical induction. To check if $A_1 \subseteq A_2$ we evaluate relationships among the membership degrees of both investigated sets. We make a comparison $\mu_{A_2}(x') = \mu_{A_1 \circ R}(x') = \max_{x \in X}(\min(\mu_{A_1}(x), \mu_R(x, x'))) \geq \min_{x \in X} \mu_R(x, x') = \mu_{A_1}(x')$, with respect to $x' \in X$, to get $A_2 \supseteq A_1$ according to Def. 5.2.

The last connection certainly confirms that $A_1 \subseteq A_2$ since $\mu_{A_1}(x') \leq \mu_{A_2}(x')$.

The induction assumption $A_{n-1} \subseteq A_n$ is utilized in the proof to get the conclusion $A_n \subseteq A_{n+1}$. We begin with $A_{n-1} \subseteq A_n$ to compose both sides of the inclusion with R in the way: $A_{n-1} \circ R \subseteq A_n \circ R$. The last inclusion is equivalent to $A_n \subseteq A_{n+1}$.

As in the previous case the set A_0 is the eigen set of R, while A_1 seldom is regarded as eigen. Let us assume that A_n is one of the sets listed in (5.6) and fulfils $A_n \circ R = A_n$ for $A_1 \neq A_2 \neq \cdots \neq A_n = A_{n+1} = \cdots = A_{n+k} \neq A_0$. Then A_n will be the least eigen fuzzy set (LEFS) of the relation R that is different from A_0. We notice that A_0 is the greatest set in the collection of sets in (5.6) and eigen as well, and we cannot find other eigen sets between A_1 and A_n, thus A_n must be the least eigen set.

To evaluate the least eigen fuzzy set of R we make an important change in Algorithm 5.1, namely, we state A_1 accordingly to the definition proposed by (5.5).

Algorithm 5.2

A relation $R \subseteq X \times X$ with the membership function $\mu_R(x, x')$ is given.

1. Find the set A_1 defined by $\mu_{A_1}(x') = \min_{x \in X} \mu_R(x, x')$ for all $x' \in X$.
2. Set the index $n = 1$.
3. Calculate $A_{n+1} = A_n \circ R$.
4. $A_{n+1} \stackrel{?}{=} A_n$ $\rightarrow No \rightarrow n = n+1 \rightarrow$ Go to step 3; $\rightarrow Yes \rightarrow A = A_{n+1}$.

Example 5.3

By returning to Ex. 5.2 we repeat the matrix

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.3 \\ 0.9 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

defined on $X \times X$, $X = \{x_1, x_2, x_3\}$. We then intend to calculate the entries of the least eigen fuzzy set. The membership degrees of x_j in the set A_1 are the smallest values in columns $j, j = 1, 2, 3$, when applying (5.5). Hence,

$$A_1 = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix} \end{array}.$$

For $n = 1$ we create A_2 as a fuzzy set

$$A_2 = A_1 \circ R = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.3 \\ 0.9 & 0.8 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 & 0.3 \end{bmatrix}.$$

that satisfies $A_2 \neq A_1$. We thus put $n = 2$ in Algorithm 5.2 to find A_3

$$A_3 = A_2 \circ R = \begin{bmatrix} 0.4 & 0.5 & 0.3 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.3 \\ 0.9 & 0.8 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 & 0.3 \end{bmatrix}.$$

A_3 is the component of the equality $A_3 = A_2$. The set A_3 will be treated as the least eigen fuzzy set of the relation R. By the way, we check that $A_1 \subseteq A_2 \subseteq A_3$, which confirms the proper choice of the least set.

The relation R keeps the given fuzzy set invariant. An occurrence in which the system (the matrix) does not produce any effect on the given input (the eigen set), apparently fits to the medical appearance when a medicine has no more effect in the curative process. If the relation is stated as "pharmacological knowledge" about some configurations of drug effectiveness created for pairs of symptoms, then an eigen set of the relation estimates, via its membership degrees, the medicine effectiveness level related to each symptom.

In the next subsection, we will suggest two definitions of fuzzy relations of the "pharmacological knowledge" type. The relations contribute in deciding the membership degrees of eigen fuzzy sets associated with them. These in turn give us a tool of determining the lowest and the highest threshold of the drug action on a collection of selected medical symptoms.

5.3 Eigen Sets in Medicine Effectiveness Levels

By possessing the results of examinations carried out on a group of patients with some symptoms, we can then estimate a theoretical level of effectiveness concerning medicine that is recommended to the patients belonging to the considered group. We involve the eigen problem technique for fuzzy sets to make a trial of finding the minimal and the maximal level of recovery. Although the range is stated theoretically, it ought not to change in practice during an extended period of treatment.

Let us assume that some characteristic symptoms are found in a sample of *m* observed patients. All patients have the same symptoms. These should disappear entirely after the treatment if the drug is highly effective. Nevertheless, the symptoms can persist when the drug is not efficient enough.

Let us denote a set of symptoms by $S = \{S_1, \ldots, S_n\}$. S_j is the j^{th} symptom, $j = 1, \ldots, n$, and S is non-fuzzy.

The estimation of the maximal level is possible by employing a fuzzy relation, created due course to the definition formulated by a sentence: "The action of the drug on the j^{th} symptom is equal or stronger than on the k^{th} symptom in patient, $j, k = 1, \ldots, n$". We call the relation $R_{\max}$ and we realize that $R_{\max}$ is a set of pairs (S_j, S_k). The membership degree $\mu_{R_{\max}}(S_j, S_k)$, as a number from the range [0, 1], indicates the grade to which the statement defining $R_{\max}$ is true for the j^{th} and the k^{th} symptom [27, 31].

A comparison of the drug's influence on the considered symptoms has to be executed for each pair of the relation $R_{\max}$. Suppose that *m* denotes a number of patients having been examined (the sample cardinality). If *b* stands for a number of patients for whom the description of $R_{\max}$ constitutes a true sentence, then we will compute the membership degrees $\mu_{R_{\max}}(S_j, S_k)$ as

$$\mu_{R_{\max}}(S_j, S_k) = \frac{b}{m} \tag{5.7}$$

for $j, k = 1, \ldots, n$.

Example 5.4

As a simple example that explains a way of estimating the membership degree for two symptoms included in the same pair, we consider a group consisted of seven patients. Suppose that "–" is assigned as the lack of a symptom after the treatment and "+" designates its presence in the patient after the medication [27, 31].

To count b, we should consider two configuration patterns of these signs, i.e.,
"–" "–" that is interpreted as "The drug acts as strongly on S_j as on S_k",
"–" "+" that corresponds to "The drug acts more strongly on S_j than on S_k".

The first combination of signs signals that the examined patient is now rid of both symptoms tied to each other by the considered pair order. The second arrangement of signs explains that the patient is recovered from the first symptom while the second symptom is still prevailing after the treatment.

These configurations can be arranged as the contents of Table 5.1.

Table 5.1: Sign configurations for symptoms S_j, S_k in P_1–P_7

Patient	S_j	S_k
P_1	–	–
P_2	–	+
P_3	–	+
P_4	+	+
P_5	–	–
P_6	+	–
P_7	+	+

The membership degree of the pair (S_j, S_k) is evaluated as 0.571 (4/7) and for (S_k, S_j) as 0.429 (3/7).

The fuzzy relation $R_{\max}$ can be written down as a matrix

$$R_{\max} = \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & \cdots & S_n \end{array} \\ \begin{bmatrix} \mu_{R_{\max}}(S_1,S_1) & \mu_{R_{\max}}(S_1,S_2) & \cdots & \mu_{R_{\max}}(S_1,S_n) \\ \mu_{R_{\max}}(S_2,S_1) & \mu_{R_{\max}}(S_2,S_2) & \cdots & \mu_{R_{\max}}(S_2,S_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{R_{\max}}(S_n,S_1) & \mu_{R_{\max}}(S_n,S_2) & \cdots & \mu_{R_{\max}}(S_n,S_n) \end{bmatrix} \end{array} . \tag{5.8}$$

We should emphasize that a diagonal entry $\mu_{R_{\max}}(S_j,S_j)$ of $R_{\max}$ has b estimated as a number of "–" signs counted for S_j.

Next, we utilize the conception of the greatest eigen fuzzy set associated with the relation $R_{\max}$. Due to Def. 5.1 there exists the greatest fuzzy set in the universe S that is associated with $R_{\max}$. The set is called $A_{\max}$, and we include it, as a crucial part, in an equation

$$A_{\max} \circ R_{\max} = A_{\max} . \tag{5.9}$$

$A_{\max}$ is a result of computations due to Algorithm 5.1, in which A_1 is defined by (5.1). The relation, designed in accordance with the statement: "The drug acts equally or more strongly on the j^{th} symptom than on the k^{th} symptom, j, k = 1, …, n", has its eigen set $A_{\max}$. The set $A_{\max}$ does not change in spite of many compositions with the relation. This lack of variations leads to a conclusion that the membership degrees of A_{max} show a level: "the drug action on the considered symptoms is not stronger". A_{max} can thus be an indicator of how to estimate the maximal level to which the medicine can be effective since, evidently, A_{max} is the greatest solution of Eq. (5.9) in the sense of the largest membership degree values.

Estimation of the minimal medicine effect aims at stating another fuzzy relation R_{min} proposed as a clue: "The action of the drug on the j^{th} symptom is equal or weaker than on the k^{th} symptom in patient, j, $k = 1, \ldots, n$." The suggested formula of calculating membership degrees of R_{min} is similar to Eq. (5.7), but arrangements of the signs used before should be reconsidered to find b. We take into account the configurations:

"–" "–" that is valid if: "The drug acts as strongly on S_j as on S_k",

"+" "–" that is approved if: "The drug acts more weakly on S_j than on S_k".

The relation R_{min} also generates its own, this time the least, eigen fuzzy set A_{min} that constitutes the compound of an equation

$$A_{min} \circ R_{min} = A_{min} . \tag{5.10}$$

To decide A_{min} we perform the steps of Algorithm 5.2 in which the membership function of A_1 is yielded by (5.5).

Since A_{min} does not change its membership degrees after the next composition with R_{min} = "The drug works equally or more weakly for the j^{th} symptom in comparison to the k^{th} symptom", then the membership degrees of the least eigen set, corresponding to symptoms $S_1, \ldots, S_n$, should point out the minimal effectiveness level. A_{min} is the least eigen set of R_{min}, and its existential meaning can be adequate to the thesis "the action of the medicine on the considered symptoms cannot be weaker".

We treat the values of $\mu_{A_{min}}(S_j)$ and $\mu_{A_{max}}(S_j)$, $j = 1, \ldots, n$, as borders of this interval that limits the range of medicine effectiveness for each symptom S_j. Even if we obtain ranges for single symptoms, we should realize that these results are effects of the simultaneous appreciation of the medicine strength measured for pairs of symptoms when following the relations' definitions.

Example 5.5

The diagnosis D known as a throat inflammation is recognized by the set of symptoms $S = \{S_1$ = "*sore throat* (*pain*)", S_2 = "*temperature*", S_3 = "*inflammation state*"$\}$. The physician has prescribed *Bayer's aspirin* as a remedy that should improve the health conditions in the group of 30 patients suffering from throat inflammation.

By applying Eq. (5.7) and the sign pattern "–" "–" and "–" "+" we compute A_{max} as

$$R_{\max} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.8 & 0.8 & 0.8 \\ 0.6 & 0.6 & 0.6 \end{bmatrix} \end{array}$$

with the corresponding greatest eigen fuzzy set decided as

$$A_{\max} = \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}.$$

Equation (5.7), in which the quantity of the associations "–" "–" and "+" "–" constitutes a basis of the b value computations, results in the relation $A_{\min}$ yielded as the matrix

$$R_{\min} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.8 & 0.6 \\ 0.5 & 0.8 & 0.6 \\ 0.5 & 0.8 & 0.6 \end{bmatrix} \end{array}$$

possessing the least eigen set

$$A_{\min} = \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.8 & 0.6 \end{bmatrix} \end{array}.$$

By interpreting the membership degrees of $A_{\min}$ and $A_{\max}$ in the percentage scale we conclude that *Bayer's aspirin* removes S_1 in 50%–80% and S_2 – in 80%. S_3 disappears in 60%–80% in the sample of examined patients.

By constructing the relations we consider the drug influence on pairs of symptoms to learn about the symptoms' interactions in the process of reacting on the treatment. Even if we appreciate effectiveness levels for individual symptoms, we will be aware of the complex dependency among symptoms that affects single ranges. This aspect of complexity is an advantage of fuzzy research when comparing fuzzy results to computations of statistical ranges that do not consider interactions among the examined objects.

We have engaged two different eigen sets to estimate effectiveness ranges. In spite of this, we sometimes could not find two boundaries of the range as in the case of S_2. In the next subsection, we will extend the procedure of deciding eigen sets by adding to them fuzzy numbers.

5.4 Order Operations on Fuzzy Numbers

Our next attempt to appreciate a drug level is accomplished by using another approach to eigen fuzzy sets. In contrast to the previous method presented in Section 5.3, we introduce fuzzy numbers as the membership degrees of a relation R instead of real numbers belonging to the interval [0, 1].

The fuzzy number is also a fuzzy set that fulfils some particular conditions. Since we do not intend to apply fuzzy numbers defined on finite sets, we would like to quote the definition of the fuzzy number that has a continuous support [19, 20, 22, 23, 25, 36, 40, 42, 47, 95].

Definition 5.3

The fuzzy number N is a fuzzy set of L-R type in the real universe Z if there exist two continuous reference functions L, R and scalars $\alpha > 0$, $\beta > 0$ included in the membership function of N as follows

$$\mu_N(z) = \begin{cases} L\left(\dfrac{m-z}{\alpha}\right) & \text{for} \quad m-\alpha \le z \le m, \\ R\left(\dfrac{z-m}{\beta}\right) & \text{for} \quad m \le z \le m+\beta, \end{cases} \tag{5.11}$$

$z \in Z$, where m, called the mean value of N, is a real number, and α and β are called the left and right spreads, respectively. The functions L and R map R^+ in $[0,1]$. L should satisfy $L(0) = 1$, $L(z) < 1$ for every $z > 0$; $L(z) > 0$ for every $z < 1$; $L(1) = 0$. The same conditions refer to R. Symbolically, N is denoted by $N = (m_N, \alpha_N, \beta_N)$. We state a notion of the space of fuzzy numbers in the L-R representation as $FN(LR)$. For the purpose of medical applications, we suppose that only fuzzy numbers satisfying the condition $m_N \ge 0$ belong to the space.

To be able to construct a practical version of the membership function we propose the $L(z)$ and $R(z)$ functions as [25, 63, 64]

$$L(z) = R(z) = -z + 1 \tag{5.12}$$

for $z \in Z$.

By studying (5.11) and (5.12) we realize that N is such a fuzzy set whose membership function forms a triangle with the peak at the point $(m_N, 1)$. The left side of the triangle has increasing values of the membership degrees from zero to one, while the right side is as a slope that goes down to zero.

The triangles constitute the most popular shapes of membership functions assigned to fuzzy numbers.

Example 5.6

We perform the operations defined by (5.11) and (5.12) to find a membership function of the fuzzy number $N = (40, 2, 3)$.

We accommodate (5.11) to $m = 40$, $\alpha = 2$, $\beta = 3$. The argument z in $L(z)$ is replaced by $\frac{40-z}{2}$ and in $R(z)$ – by $\frac{z-40}{3}$. We obtain a function

$$\mu_N(z) = \begin{cases} L\left(\frac{40-z}{2}\right) = 1 - \frac{40-z}{2} = \frac{z-38}{2} & \text{for} \quad 38 \le z \le 40, \\ R\left(\frac{z-40}{3}\right) = 1 - \frac{z-40}{3} = \frac{43-z}{3} & \text{for} \quad 40 \le z \le 43, \end{cases}$$

that is plotted in Fig. 5.2.

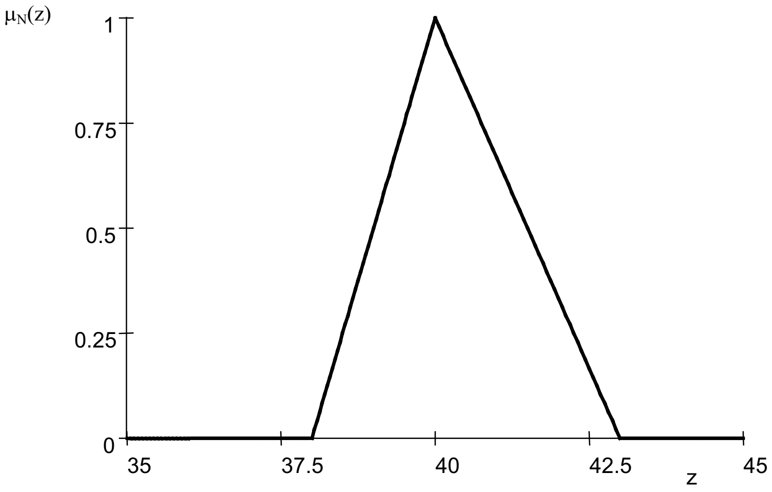

Figure 5.2: The fuzzy number $N = (40, 2, 3)$

The arithmetic over the space $FN(LR)$ is discussed in many research reports [19, 20, 22, 25, 26, 36, 40, 42, 47, 95]. In particular, order operations in $FN(LR)$ attract the scientists' attention [16, 18, 19, 20, 23, 36, 42, 47, 79, 95]. Below we discuss two sorts of order operations on fuzzy numbers in the L-R form to select the most applicable operations and to reconsider the eigen fuzzy model.

Let us first study an approach to minimum and maximum operations proposed by Dubois and Prade [19, 20].

Definition 5.4

Minimum for two fuzzy numbers $N_1 = (m_{N_1}, \alpha_{N_1}, \beta_{N_1})$, $N_2 = (m_{N_2}, \alpha_{N_2}, \beta_{N_2})$ is decided to be a fuzzy number

$$\begin{aligned} &\min(N_1, N_2) = (m_{N_1}, \alpha_{N_1}, \beta_{N_1}) \\ &\text{if } m_{N_1} < m_{N_2} \text{ and } \operatorname{supp}(N_1) \cap \operatorname{supp}(N_2) = 0 \end{aligned} \tag{5.13}$$

or

$$\begin{aligned} &\min(N_1, N_2) = (\min(m_{N_1}, m_{N_2}), \max(\alpha_{N_1}, \alpha_{N_2}), \min(\beta_{N_1}, \beta_{N_2})) \\ &\text{if } m_{N_1} \neq m_{N_2} \text{ or } m_{N_1} = m_{N_2} \text{ and } \operatorname{supp}(N_1) \cap \operatorname{supp}(N_2) \neq 0\,. \end{aligned} \tag{5.14}$$

Example 5.7

We set N_1 = (25, 2, 3) and N_2 = (40, 1, 5). Since the sets supp(N_1) = [23, 28] and supp(N_2) = [39, 45] have no common elements we immediately decide that min(N_1, N_2) = N_1 as pointed out in Fig. 5.3.

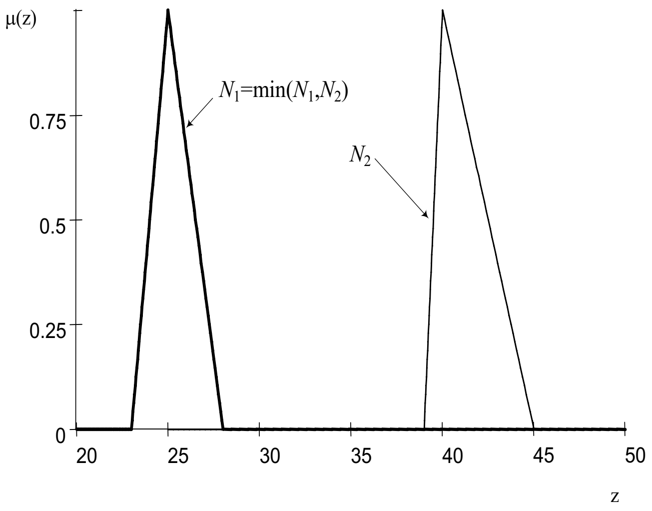

Figure 5.3: Minimum for N_1 = (25, 2, 3) and N_2 = (40, 1, 5) according to (5.13)

Let us explain the meaning of (5.14) by investigating data involved in the next example.

Example 5.8

We let N_1 = (40, 2, 3) and N_2 = (42, 1, 5). In accordance with (5.14), when the intersection of supp(N_1) = [38, 43] and supp(N_2) = [41, 47] is not the empty set, we decide min(N_1, N_2) = (min(40, 42), max(2, 1), min(3, 5)) = (40, 2, 3).

The membership function of the minimal fuzzy number (40, 2, 3) has a formula

$$\mu_{(40,2,3)}(z) = \begin{cases} \dfrac{z-38}{2} & \text{for} \quad 38 \le z \le 40, \\ \dfrac{43-z}{3} & \text{for} \quad 40 \le z \le 43, \end{cases}$$

illustrated by the graph drawn in Fig. 5.4, see Ex. 5.6.

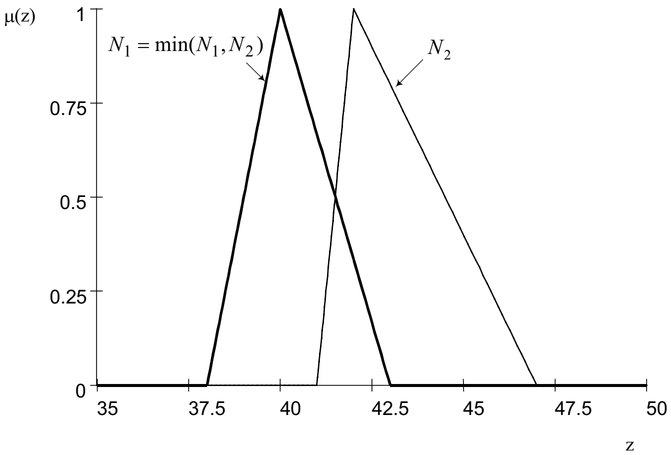

Figure 5.4: Minimum for N_1 = (40, 2, 3) and N_2 = (42, 1, 5) made by (5.14)

Another definition lets us determine the largest fuzzy number chosen for two numbers from the pair (N_1, N_2).

Definition 5.5

Maximum for $N_1 = (m_{N_1}, \alpha_{N_1}, \beta_{N_1})$ and $N_2 = (m_{N_2}, \alpha_{N_2}, \beta_{N_2})$ is a fuzzy number

$$\begin{aligned} &\max(N_1, N_2) = (m_{N_1}, \alpha_{N_1}, \beta_{N_1}) \\ &\text{if } m_{N_1} > m_{N_2} \text{ and } \operatorname{supp}(N_1) \cap \operatorname{supp}(N_2) = 0 \end{aligned} \tag{5.15}$$

alternatively

$$\max(N_1, N_2) = (\max(m_{N_1}, m_{N_2}), \min(\alpha_{N_1}, \alpha_{N_2}), \max(\beta_{N_1}, \beta_{N_2}))$$
$$\text{if } m_{N_1} \neq m_{N_2} \text{ or } m_{N_1} = m_{N_2} \text{ and } \operatorname{supp}(N_1) \cap \operatorname{supp}(N_2) \neq 0. \tag{5.16}$$

Example 5.9

For two numbers from Ex. 5.7 N_2 is found as the rightmost element and the maximal fuzzy number in the pair (N_1, N_2) in compliance with Fig. 5.5.

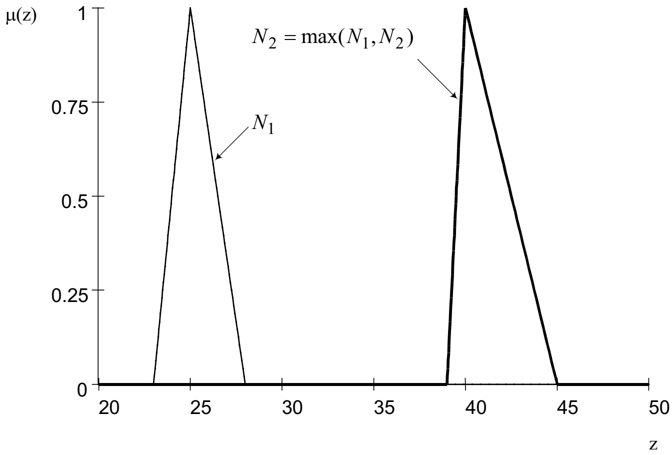

Figure 5.5: Maximum for N_1 = (25, 2, 3) and N_2 = (40, 1, 5) due to (5.15)

Example 5.10

Again we set N_1 = (40, 2, 3) and N_2 = (42, 1, 5). By applying (5.16) we choose max(N_1, N_2) = (max(40, 42), min(2, 1), max(3, 5)) = (42, 1, 5).

The maximal fuzzy number is given by a membership function

$$\mu_{(42,1,5)}(z) = \begin{cases} L\left(\dfrac{42-z}{1}\right) = 1 - \dfrac{42-z}{1} = \dfrac{z-41}{1} & \text{for} \quad 41 \le z \le 42, \\ R\left(\dfrac{z-42}{5}\right) = 1 - \dfrac{z-42}{5} = \dfrac{47-z}{5} & \text{for} \quad 42 \le z \le 47, \end{cases}$$

sketched in Fig. 5.6.

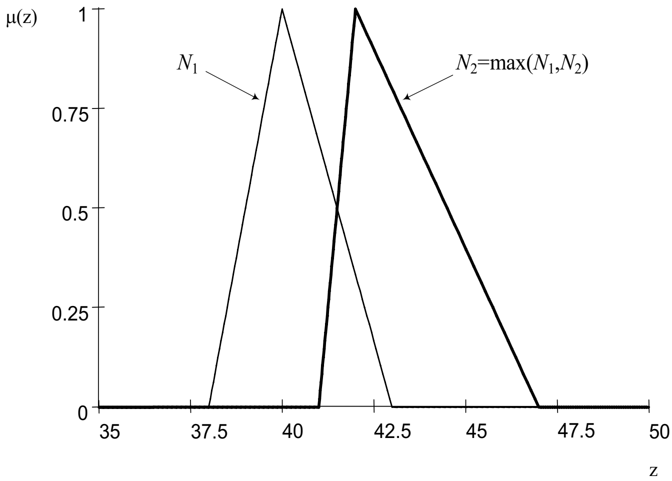

Figure 5.6: Maximum for N_1 = (40, 2, 3) and N_2 = (42, 1, 5) computed by (5.16)

We emphasize that the results of (5.13)–(5.14) and (5.15)–(5.16) are fuzzy numbers preserving the *L*-*R* representation and a triangular shape. To make some conclusions about the usability of different approaches to the concept of order among *L*-*R* fuzzy numbers, let us also study another attempt of defining the order operations proposed by Chih-Hui Chiu and Wen-June Wang [16].

We start with the minimum notion.

Definition 5.6

1) Suppose that two fuzzy numbers N_1 and N_2 given in the *L*-*R* representation satisfy the condition $\mathrm{supp}(N_1) \cap \mathrm{supp}(N_2) \neq 0$ (the supports of numbers are not disjoint). If continuous membership functions of N_1 and N_2 have one intersection point possessing the *z*-coordinate equal to z_m that lies between the mean values m_{N_1} and m_{N_2}, then

$$\mu_{\min(N_1,N_2)}(z) = \begin{cases} \max(\mu_{N_1}(z), \mu_{N_2}(z)) & \text{for} \quad z < z_m, \\ \min(\mu_{N_1}(z), \mu_{N_2}(z)) & \text{for} \quad z \geq z_m. \end{cases} \tag{5.17}$$

2) If $\mathrm{supp}(N_1) \cap \mathrm{supp}(N_2) = 0$ (the supports of fuzzy numbers are disjoint) then we will exploit (5.17) for any value of z_m that fulfils the restriction $(z_{N_1} - z_m)(z_{N_2} - z_m) < 0$ for all $z_{N_1} \in \mathrm{supp}(N_1)$ and all $z_{N_2} \in \mathrm{supp}(N_2)$.

Let us first concentrate on the first part of Def. 5.6 that entails some comments.

Example 5.11

Once again we consider $N_1 = (40, 2, 3)$ and $N_2 = (42, 1, 5)$.

N_1 has the membership function expanded as (see Ex. 5.6 and Ex. 5.8)

$$\mu_{N_1}(z) = \begin{cases} \dfrac{z-38}{2} & \text{for} \quad 38 \le z < 40, \\ \dfrac{43-z}{3} & \text{for} \quad 40 \le z \le 43, \end{cases}$$

while the function of N_2 is expressed by (see Ex. 5.10)

$$\mu_{N_2}(z) = \begin{cases} \dfrac{z-41}{1} & \text{for} \quad 41 \le z < 42, \\ \dfrac{47-z}{5} & \text{for} \quad 42 \le z \le 47. \end{cases}$$

Both functions have an intersection point between the lines $\mu_{N_1}(z) = \dfrac{43-z}{3}$ and $\mu_{N_2}(z) = \dfrac{z-41}{1}$ that provides us with $z_m = 41.5$. Hence, we use (5.17) to get

$$\mu_{\min(N_1,N_2)}(z) = \begin{cases} \max\limits_{38 \le z \le 41.5} (\mu_{N_1}(z), \mu_{N_2}(z)) = \begin{cases} \dfrac{z-38}{2} & \text{for} \quad 38 \le z < 40, \\ \dfrac{43-z}{3} & \text{for} \quad 40 \le z \le 41.5, \end{cases} \\ \min\limits_{41.5 \le z \le 47} (\mu_{N_1}(z), \mu_{N_2}(z)) = \begin{cases} \dfrac{43-z}{3} & \text{for} \quad 41.5 \le z < 43, \\ 0 & \text{for} \quad 43 \le z \le 47, \end{cases} \end{cases}$$

$$= \mu_{N_1}(z).$$

The result of applying (5.17) is exactly the same as the effect of adopting (5.14).

To study better the action of case 2) in Def. 5.6 we should go through the next example.

Example 5.12

We test (5.17) related to case 2) of Def. 5.6 on $N_1 = (25, 2, 3)$ and $N_2 = (40, 1, 5)$. By returning to (5.11) and (5.12) we develop the membership function of N_1 as

$$\mu_{N_1}(z) = \begin{cases} L\left(\dfrac{25-z}{2}\right) = 1 - \dfrac{25-z}{2} = \dfrac{z-23}{2} & \text{for} \quad 23 \le z < 25, \\ R\left(\dfrac{z-25}{3}\right) = 1 - \dfrac{z-25}{3} = \dfrac{28-z}{3} & \text{for} \quad 25 \le z \le 28, \end{cases}$$

while the membership function of N_2 is equal to

$$\mu_{N_2}(z) = \begin{cases} L\left(\dfrac{40-z}{1}\right) = 1 - \dfrac{40-z}{1} = \dfrac{z-39}{1} & \text{for} \quad 39 \le z < 40, \\ R\left(\dfrac{z-40}{5}\right) = 1 - \dfrac{z-40}{5} = \dfrac{45-z}{5} & \text{for} \quad 40 \le z \le 45. \end{cases}$$

The supports [23, 28] and [39, 45] are disjoint sets. We can thus choose the value of z_m, say, 35 because of the condition $(z_{N_1} - 35)(z_{N_2} - 35) < 0$ that is satisfied for all $z_{N_1} \in [23,28]$ and all $z_{N_2} \in [39,45]$. By returning to (5.17) we decide minimum for N_1 and N_2 as

$$\mu_{\min(N_1,N_2)}(z) = \begin{cases} \max\limits_{23 \le z \le 35} (\mu_{N_1}(z), \mu_{N_2}(z)) = \begin{cases} \dfrac{z-23}{2} & \text{for} \quad 23 \le z < 25, \\ \dfrac{28-z}{3} & \text{for} \quad 25 \le z \le 28, \\ 0 & \text{for} \quad 28 \le z \le 35, \end{cases} \\ \min\limits_{35 \le z \le 45} (\mu_{N_1}(z), \mu_{N_2}(z)) = \{0 & \text{for} \quad 35 < z \le 45, \end{cases}$$

$$= \mu_{N_1}(z).$$

Maximum for two fuzzy numbers, from the point of view presented by [16], is outlined in the following statement.

Definition 5.7

1) If two fuzzy numbers N_1 and N_2 obtained in the *L*-*R* representation have a non-empty intersection between supp(N_1) and supp(N_2), and if there exists one intersection point for continuous membership functions of N_1 and N_2 that has the *z*-coordinate equal to z_m placed between the mean values m_{N_1} and m_{N_2}, then

$$\mu_{\max(N_1,N_2)}(z) = \begin{cases} \min(\mu_{N_1}(z), \mu_{N_2}(z)) & \text{for} \quad z < z_m, \\ \max(\mu_{N_1}(z), \mu_{N_2}(z)) & \text{for} \quad z \ge z_m. \end{cases} \tag{5.18}$$

2) For disjoint fuzzy numbers N_1 and N_2 (supp(N_1) $\cap$ supp(N_2) = 0) we apply (5.18) for any value of z_m, provided that $(z_{N_1} - z_m)(z_{N_2} - z_m) < 0$ for all $z_{N_1} \in \text{supp}(N_1)$ and all $z_{N_2} \in \text{supp}(N_2)$.

Example 5.13

We recall $N_1 = (40, 2, 3)$ and $N_2 = (42, 1, 5)$ from Ex. 5.11. For $z_m = 41.5$ we decide $\max(N_1, N_2)$ by means of its function

$$\mu_{\max(N_1,N_2)}(z) = \begin{cases} \min\limits_{38 \le z \le 41.5} (\mu_{N_1}(z), \mu_{N_2}(z)) = \begin{cases} 0 & \text{for} \quad 38 \le z < 41, \\ \dfrac{z-41}{1} & \text{for} \quad 41 \le z \le 41.5, \end{cases} \\ \max\limits_{41.5 \le z \le 47} (\mu_{N_1}(z), \mu_{N_2}(z)) = \begin{cases} \dfrac{z-41}{1} & \text{for} \quad 41.5 \le z < 42, \\ \dfrac{47-z}{5} & \text{for} \quad 42 \le z \le 47, \end{cases} \end{cases}$$

$$= \mu_{N_2}(z).$$

For two non-disjoint fuzzy numbers with one intersection point between their membership functions the results of Defs 5.4, 5.6 and Defs 5.5, 5.7 are coincident. We now discuss the case of searching the minimal number for a pair (N_1, N_2) that consists of fuzzy numbers intersecting each other in at least two points.

Example 5.14

Let $N_1 = (40, 7, 8)$ and $N_2 = (42, 3, 2)$. Let us set the numbers in (5.14) to establish $\min(N_1, N_2) = (\min(40, 42), \max(7, 3), \min(8, 2)) = (40, 7, 2)$ that preserves the triangular shape in the *L-R* form, see Fig. 5.7.

To watch effects of (5.17) we find the membership function of (40, 7, 8) as

$$\mu_{(40,7,8)}(z) = \begin{cases} \dfrac{z-33}{7} & \text{for} \quad 33 \le z < 40, \\ \dfrac{48-z}{8} & \text{for} \quad 40 \le z \le 48. \end{cases}$$

The membership function of (42, 3, 2) is computed as

$$\mu_{(42,3,2)}(z) = \begin{cases} \dfrac{z-39}{3} & \text{for} \quad 39 \le z < 42, \\ \dfrac{44-z}{2} & \text{for} \quad 42 \le z \le 44. \end{cases}$$

As z_m we accept the z-value solving the equation $\dfrac{48-z}{8} = \dfrac{z-39}{3}$. We find $z_m =$ 41.45 and we check that it satisfies the inequality $m_{N_1} < z_m < m_{N_2}$.

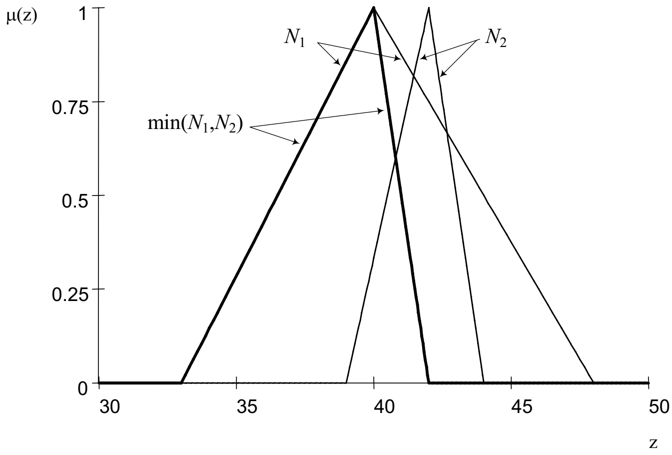

Figure 5.7: Minimum for $N_1 = (40, 7, 8)$ and $N_2 = (42, 3, 2)$ as the result of (5.14)

By expanding (5.17) we construct a function presented below and depicted in Fig. 5.8.

$$\mu_{\min(N_1,N_2)}(z) = \begin{cases} \max\limits_{33 \le z \le 41.45} (\mu_{N_1}(z), \mu_{N_2}(z)) = \begin{cases} \dfrac{z-33}{7} & \text{for} \quad 33 \le z < 40, \\ \dfrac{48-z}{8} & \text{for} \quad 40 \le z \le 41.45, \end{cases} \\ \min\limits_{41.45 \le z \le 48} (\mu_{N_1}(z), \mu_{N_2}(z)) = \begin{cases} \dfrac{48-z}{8} & \text{for} \quad 41.45 \le z < 42.67, \\ \dfrac{44-z}{2} & \text{for} \quad 42.67 \le z < 44, \\ 0 & \text{for} \quad 44 \le z \le 48, \end{cases} \end{cases}$$

The result of (5.17), found as a minimal fuzzy number for $N_1 = (40, 7, 8)$ and $N_2 = (42, 3, 2)$, does not emerge as a fuzzy number in *L-R* representation, and moreover, this minimum does not maintain the shape of a regular triangle.

The thorough analysis of properties, typical of the most popular approaches to order operations on fuzzy numbers, provides us with important hints as to the use of one alternative in further medical application. We wish to utilize these definitions of order operations on fuzzy numbers that are easy to apply, and preferably, we expect the operations to yield triangular numbers in the *L-R* form.

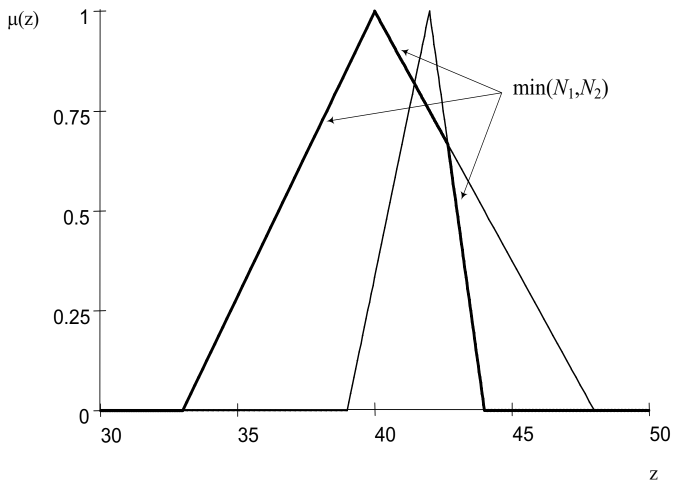

Figure 5.8: Minimum for N_1 = (40, 7, 8) and N_2 = (42, 3, 2) due to (5.17)

Hence, we choose operations (5.13)–(5.14) to look for the minimum and (5.15)–(5.16) to find the maximum for two fuzzy numbers in the *L-R* form. The formulas, selected above, are supposed to replace the order operations performed in algorithms that are developed to reveal eigen fuzzy sets.

5.5 Eigen Fuzzy Sets with Fuzzy Numbers

We use the operations on fuzzy numbers suggested by (5.13)–(5.14) and (5.15)–(5.16) to propose a new conception of the composition of a relation with a set. This time both the relation and the set have membership degrees formed as fuzzy numbers in the *L-R* form.

Definition 5.8

We recall that $FN(LR) = \{N : N = (m_N, \alpha_N, \beta_N)\}$.

Let R be a fuzzy relation $R \subseteq X \times Y$, $X = \{x\}$, $Y = \{y\}$, with the membership function $\mu_R : X \times Y \to FN(LR)$, $\mu_R(x,y) \in FN(LR)$, $(x,y) \in X \times Y$, and let A be a fuzzy set $A \subseteq X$ given by the membership function $\mu_A : X \to FN(LR)$, $\mu_A(x)$ $\in FN(LR)$, $x \in X$.

If the membership degrees $\mu_A(x) = (m_{\mu_A(x)}, \alpha_{\mu_A(x)}, \beta_{\mu_A(x)})$ of set A and the degrees $\mu_R(x,y) = (m_{\mu_R(x,y)}, \alpha_{\mu_R(x,y)}, \beta_{\mu_R(x,y)})$ of the relation R are expressed by fuzzy numbers belonging to *FN*(*LR*), then we will identify the set as $B = A \circ R$ by its membership function [34, 64]

$$\begin{aligned}\mu_B(y) &= (m_{\mu_B(y)}, \alpha_{\mu_B(y)}, \beta_{\mu_B(y)}) = \max_{x \in X}(\min(\mu_A(x), \mu_R(x,y))) \\ &= \max_{x \in X}(\min((m_{\mu_A(x)}, \alpha_{\mu_A(x)}, \beta_{\mu_A(x)}), (m_{\mu_R(x,y)}, \alpha_{\mu_R(x,y)}, \beta_{\mu_R(x,y)})))\end{aligned} \tag{5.19}$$

for all $y \in Y$. We keep in mind that the maximum and minimum operations are performed on fuzzy numbers from *FN*(*LR*) due to (5.13)–(5.16).

Example 5.15

Let $X = \{10, 20, 30\}$. Set $A \subseteq X$ is a fuzzy set whose membership degrees are stated as the *L*-*R* fuzzy numbers, e.g., in set $Z = [0, 50]$. For instance, A can be proposed as

$$A = {(30,5,7)}/{10} + {(20,3,2)}/{20} + {(15,3,4)}/{30} = \begin{matrix} 10 & 20 & 30 \\ [(30,5,7) & (20,3,2) & (15,3,4)] \end{matrix}.$$

The fuzzy relation $R \subseteq X \times X$ ($Y = X$) is constructed in the form of a 3×3 matrix. Each entry of the matrix R is approved as a fuzzy number with the support constituting a subset of Z. We can suggest the matrix R as a table

$$R = \begin{matrix} & 10 & 20 & 30 \\ 10 & (3,3,7) & (42,4,4) & (11,6,5) \\ 20 & (25,2,8) & (5,4,6) & (13,3,2) \\ 30 & (9,5,4) & (18,3,5) & (24,1,4) \end{matrix}.$$

We execute the operations suggested by (5.19), via (5.14)–(5.16) (see Ex. (5.7)–(5.10)), to get a subset of X denoted by B and computed as the fuzzy set

$$B = [(30,5,7) \quad (20,3,2) \quad (15,3,4)] \circ \begin{bmatrix} (3,3,7) & (42,4,4) & (11,6.5) \\ (25,2,8) & (5,4,6) & (13,3,2) \\ (9,5,4) & (18,3,5) & (24,1,4) \end{bmatrix}$$

$$= [(20,3,2) \quad (30,5,7) \quad (15,3,4)] = {(20,3,2)}/{10} + {(30,5,7)}/{20} + {(15,3,4)}/{30}$$

B possesses the fuzzy numbers as its membership degrees.

Example 5.16

We intend to explain the difference between a type-1 (simple) fuzzy set B_1 and a type-2 (compound) fuzzy set B_2. In B_2 the membership degrees are given as fuzzy sets. Let us accept $B_1 = 0.2/10 + 0.3/20 + 0.15/30$ while B_2 is considered as, say, the result of Ex. 5.15. We thus take $B_2 = (20,3,2)/10 + (30,5,7)/20 + (15,3,4)/30$.

The membership degrees of B_1 are real values from [0, 1]. Hence, set B_1 can be sketched in the x-$\mu(x)$ coordinate plane as the set of points (x, $\mu(x)$) marked by ellipses in Fig. 5.9.

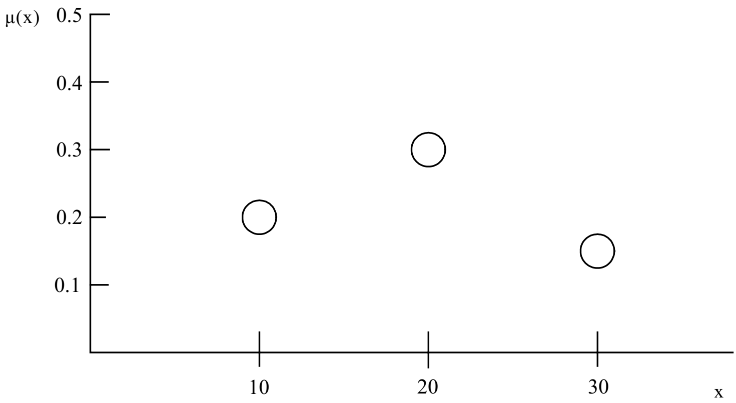

Figure 5.9: The type-1 fuzzy set B_1 = {(10, 0.2), (20, 0.3), (30, 0.15)}

The picture of B_2, see Fig. 5.10, is more sophisticated. We first assign supports of fuzzy numbers to the x-elements 10, 20, 30 belonging to X, and then we design the membership function for each support. The way from x to $\mu(z)$ via z is now three-dimensional and can be described as a path from the x-value to a segment along the z-axis and finally up along the $\mu(z)$-axis for $z \in [0, 50]$.

When set B remains equal to A after the max-min composition with the relation R (see Def. 5.8) then we will regard A as the eigen fuzzy set of relation R. We still assume that all membership degrees of the set and the relation appear as L-R fuzzy numbers.

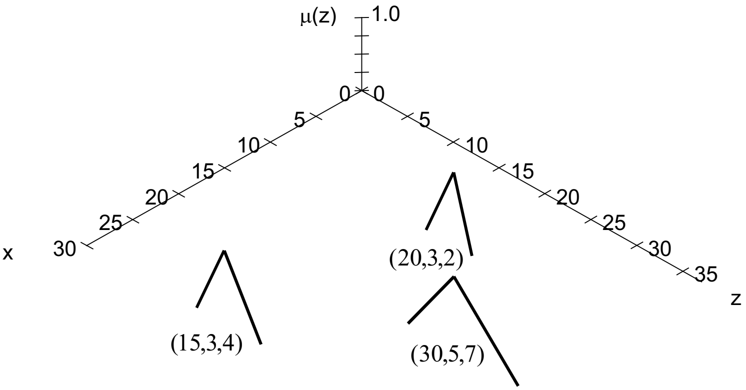

Figure 5.10: The type-2 set B_2 = {(10, (20, 3, 2)), (20, (30, 5, 7)), (30, (15, 3, 4))}

Definition 5.9

The eigen fuzzy set of a fuzzy relation $R \subseteq X \times X$ is a set $A \subseteq X$, $X = \{x\}$, that satisfies the condition $A \circ R = A$.

R is the fuzzy relation with the membership function $\mu_R : X \times X \to FN(LR)$, $\mu_R(x,x') \in FN(LR)$, x, $x' \in X$. We should prove that the greatest eigen fuzzy set $A \subseteq X$ of relation R, $\mu_A : X \to FN(LR)$, $\mu_A(x) \in FN(LR)$, $x \in X$, which constitutes a crucial part of the equation $A \circ R = A$, exists [64].

Some theoretical considerations that warrant the existence of the eigen fuzzy set, possessing fuzzy numbers as the contents, are similar to the conclusions included in Subsection (5.2) and the papers of Sanchez [72, 73].

Let us first verify that the set A_0 with its membership function defined by $\mu_{A_0}(x) = (m_{\mu_{A_0}(x)}, \alpha_{\mu_{A_0}(x)}, \beta_{\mu_{A_0}(x)}) = (m_{N_0}, \alpha_{N_0}, \beta_{N_0})$ for all $x \in X$, where

$$(m_{N_0}, \alpha_{N_0}, \beta_{N_0}) = \min_{x' \in X} \max_{x \in X} (m_{\mu_R(x,x')}, \alpha_{\mu_R(x,x')}, \beta_{\mu_R(x,x')})),$$

is an eigen set of R.

We notice that

$$\mu_{A_0 \circ R}(x') = \max_x (\min((m_{N_0}, \alpha_{N_0}, \beta_{N_0}), (m_{\mu_R(x,x')}, \alpha_{\mu_R(x,x')}, \beta_{\mu_R(x,x')})))$$
$$= \min((m_{N_0}, \alpha_{N_0}, \beta_{N_0}), \max_x (m_{\mu_R(x,x')}, \alpha_{\mu_R(x,x')}, \beta_{\mu_R(x,x')}))$$
$$= (m_{N_0}, \alpha_{N_0}, \beta_{N_0}) = \mu_{A_0}(x'),$$

since $(m_{N_0}, \alpha_{N_0}, \beta_{N_0})$ is a constant fuzzy number.

We further define the set A_1 by its membership function

$$\mu_{A_1}(x') = (m_{\mu_{A_1}(x')}, \alpha_{\mu_{A_1}(x')}, \beta_{\mu_{A_1}(x')}) = \max_{x \in X} \mu_R(x, x')$$
$$= \max_{x \in X}(m_{\mu_R(x,x')}, \alpha_{\mu_R(x,x')}, \beta_{\mu_R(x,x')}) \quad (5.20)$$

for all $x' \in X$, and we introduce the sequence $(A_n)_n$ of the type-2 fuzzy sets (the sets whose membership degrees are determined by other fuzzy sets – in this case – fuzzy numbers)

$$A_2 = A_1 \circ R = A_1 \circ R^1,\ A_3 = A_2 \circ R = A_1 \circ R^2,\ \cdots,\ A_{n+1} = A_n \circ R = A_1 \circ R^n \quad (5.21)$$

for all integers $n > 1$.

We conclude that

$$A_0 \subseteq \cdots \subseteq A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_2 \subseteq A_1, \quad (5.22)$$

that can be compared to (5.3).

We prove the inclusions (5.22) in the same manner as the inclusions (5.3) but all the operations assisting the proofs are performed with respect to fuzzy numbers from *FN*(*LR*).

The set A_0 always is the eigen set of R, while A_1 sometimes is regarded as an eigen set of the considered relation. When $A_n \circ R = A_n$ for A_n from the collection (5.21) then A_n will be the greatest eigen set of the relation R, and A_n seldom equals A_0. To put this assertion to the test we repeat this method of concluding that was already accomplished in section 5.2, but we remember that fuzzy numbers replace membership degrees in all developments involving fuzzy sets and fuzzy relations.

To find the greatest eigen set $A = A_n$ of the fuzzy relation R we access Algorithm 5.1 in which we execute the operations on fuzzy numbers in accord with (5.19). As the minimum and maximum operations, we exploit (5.13)–(5.16), which warrant that the results will be obtained as numbers in the *L-R* forms. The *L-R* forms in turn facilitate the interpretation of number membership functions.

The eigen value model producing eigen sets with fuzzy numbers as the membership degrees is essential in the fitness procedure when appreciating an upper threshold of the drug action. The supports of fuzzy numbers are supposed to indicate effectiveness levels of the medicine that should bring some relief to a patient.

5.6 The *L-R* Fuzzy Numbers as Drug Efficiency Intervals

To obtain levels of drug efficiency that are bounded by a lower and an upper limit, we make a new attempt of solving the problem of appreciating the effectiveness level of a medicine when using it against the symptoms typical of an illness. We introduce a fuzzy relation with elements equal to fuzzy numbers in the *L-R* representation to describe some connections among the symptoms. The fuzzy numbers that appear in the relation replace the verbal expressions decided by physicians in accordance with the definition of the relation.

The fuzzy relation discussed in Subsection 2.5 is a counterpart of the fuzzy relation filled with fuzzy numbers. The latter produces an eigen fuzzy set whose membership degrees also are structured as fuzzy numbers. These, via their supports, appreciate the levels of drug influence on clinical symptoms [64].

Assume that a certain disease is characterized by some typical symptoms placed within the set of symptoms $S = \{S_1, \cdots, S_n\}$. We try to appreciate the influence level of the drug on each symptom by researching an eigen fuzzy set associated with the fuzzy relation $R \subseteq S \circ S$. We believe in the physicians' experience and therefore we believe that the essential relief concerning one symptom, e.g., sharp ache, improves the patient's mood and physical condition even if other symptoms still are present. We agree with this observation of the patients' reaction on the treatment, and as an expression of our belief, we define the relation R by R = "the cumulated effectiveness of the drug action for S_i and S_j, $i, j = 1, \ldots, n$".

Let us state a content of the list with verbal descriptions of the effectiveness in accordance with the physician's advice. The list containing the grades of growing effectiveness is proposed to be $L = \{N_0$ = "*none*", N_1 = "*almost none*", N_2 = "*very little*", N_3 = "*little*", N_4 = "*rather little*", N_5 = "*medium*", N_6 = "*rather large*", N_7 = "*large*", N_8 = "*very large*", N_9 = "*almost complete*", N_{10} = "*complete*"$\}$.

In order to replace words by fuzzy numbers in the *L-R* form we utilize (5.12) and construct the membership functions of N_k, $k = 0, 1, 2, \ldots, 10$, as

$$\mu_{N_k}(z) = \begin{cases} L\left(\dfrac{10\cdot k - z}{10}\right) = \dfrac{z - 10\cdot k + 10}{10} & \text{for} \quad 10\cdot k - 10 \le z < 10\cdot k, \\[2ex] R\left(\dfrac{z - 10\cdot k}{10}\right) = \dfrac{10\cdot k + 10 - z}{10} & \text{for} \quad 10\cdot k \le z \le 10\cdot k + 10. \end{cases} \tag{5.23}$$

Hence

$$N_k = (10\cdot k, 10, 10)\,. \tag{5.24}$$

We are enabled to derive Eqs (5.23) in another manner as well. In similarity with Examples 3.9–3.11 we can arrange three atomic effectiveness descriptions such as "*seldom*", "*medium*" and "*large*" to get the rest of formulations by adding hedges. Later, the specified values of the parameter δ (see Subsection 3.4.1) let us model shapes of all membership functions assigned to the effectiveness terms.

Example 5.17

The overview of different effectiveness designs is available in Fig. 5.11. We assume that an adequate theoretical reference set for all terms describing effectiveness is chosen as $Z = [0, 100]$. This even corresponds to the percentage scale 0%–100%. Let us extend the interval to [–10, 110] to make space for all supports of the fuzzy numbers associated with the effectiveness. The numbers are placed along the z-axis that constitutes their common domain.

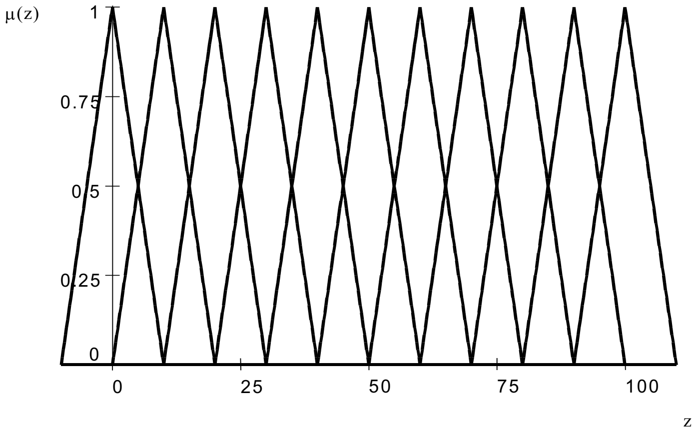

Figure 5.11: Terms of drug effectiveness expressed as fuzzy numbers

For instance, we set the value of $k = 3$ in (5.23) and involve (5.12) when we want to evaluate the membership function of "*little*" = N_3. The structure "*little*" = $N_3 = (30,10,10)$ has a membership function derived as

$$\mu_{N_3}(z) = \begin{cases} L\left(\dfrac{10\cdot 3 - z}{10}\right) = -\dfrac{10\cdot 3 - z}{10} + 1 = \dfrac{z-20}{10} & \text{for} \quad 20 \le z < 30, \\ R\left(\dfrac{z - 10\cdot 3}{10}\right) = -\dfrac{z-10\cdot 3}{10} + 1 = \dfrac{40-z}{10} & \text{for} \quad 30 \le z \le 40. \end{cases}$$

Denote the set of $N_0, \ldots, N_{10}$ by $E=\{N_k\}$, $k = 0, 1, \ldots, 10$. The supports of the fuzzy numbers N_k are the subsets of the set $[-10,110]$ (the technical extension of [0, 100] that is the best reference set used in medicine). The spreads $\alpha = \beta = 10$ are modelled after the consultation with the physicians who have advised these numbers as proper for the considered problem.

After determining some relationship between the verbal expressions graduating the effectiveness and the fuzzy numbers suggested by (5.23), we create the contents of the relation R. The physician assigns the semantic structure describing the effectiveness of a tested medicine to the single symptom S_i, $i = 1, \ldots, n$. The fuzzy number N_k, $k = 0, \ldots, 10$ substitutes this word afterwards. Nevertheless, we realize that we should propose a connective operator for fuzzy numbers representing the pairs of symptoms with regard to the definition of R.

The values of "effectiveness $(S_i) = N_{i_k}$" and "effectiveness $(S_j) = N_{j_k}$", describing the medicine curative power in the case of the symptoms S_i, S_j, $k = 0, 1, \ldots, 10$, $N_{i_k}, N_{j_k} \in E$, constitute crucial factors of the membership degree $\mu_R(S_i, S_j)$ fixed by a formula [64]

$$\begin{aligned}\mu_R(S_i,S_j) &= \frac{N_{i_k}+N_{j_k}}{2} = \frac{(m_{N_{ik}},\alpha_{N_{ik}},\beta_{N_{ik}})+(m_{N_{jk}},\alpha_{N_{jk}},\beta_{N_{jk}})}{2} \\ &= \left(\frac{m_{N_{ik}}+m_{N_{jk}}}{2},\frac{\alpha_{N_{ik}}+\alpha_{N_{jk}}}{2},\frac{\beta_{N_{ik}}+\beta_{N_{jk}}}{2}\right)\end{aligned} \tag{5.25}$$

for $i, j = 1, \ldots, n$.

The aggregation operation for two fuzzy numbers suggested in (5.25) is based on the mean values that sometimes are the items of critical remarks. In spite of them, experienced physicians recommend the mean operations in medicine to avoid accepting too sharp results being effects of maximum and minimum operators. We even know that OWA aggregation operator techniques based on mean values [46, 81, 83, 85], already discussed in Subsection 4.2, have acquired a high preference in different applications.

Example 5.18

The mean $\dfrac{m_{N_{ik}}+m_{N_{jk}}}{2}$ is the OWA operation for $a_0 = 0$, $a_1 = a_2 = \dfrac{1}{2}$ and can be expanded in the sum (5.25) as $\dfrac{1}{2}\min\left(m_{N_{ik}}, m_{N_{jk}}\right) + \dfrac{1}{2}\max\left(\min\left(m_{N_{jk}}\right), \min\left(m_{N_{ik}}\right)\right)$ accordingly to Def. 4.1.

If $m_{N_{ik}} = 40$ and $m_{N_{jk}} = 30$ then $\dfrac{1}{2}\min(40,30) + \dfrac{1}{2}\max(\min(30), \min(40)) = \dfrac{30}{2} + \dfrac{40}{2} = 35$.

The functional significance of Def. 4.1 is authorized by the results obtained in Ex. 5.18. We should now be totally convinced that the operation of computing an arithmetic mean is classified as a modern OWA operator and there is nothing wrong in its adaptation.

If some symptoms are more important than others in the clinical picture of the disease, then a weighted aggregation operation ought to be used. We define a simple and intuitive scale of weights as the set, e.g., $H=\{h_1,h_2,h_3,h_4\}=\{1,2,3,4\}$, where the value of 4 is reserved for symptom S_i revealing the greatest importance in the observed disease. By *importance* we mean the harmful impact of a symptom on the patient's state. Let us estimate a membership grade $\mu_R(S_i,S_j)$ of (S_i, S_j) as

$$\begin{aligned}\mu_R(S_i,S_j)&=\frac{h_{i_l}N_{i_k}+h_{j_l}N_{j_k}}{h_{i_l}+h_{j_l}}=\frac{h_{i_l}(m_{N_{ik}},\alpha_{N_{ik}},\beta_{N_{ik}})+h_{j_l}(m_{N_{jk}},\alpha_{N_{jk}},\beta_{N_{jk}})}{h_{i_l}+h_{j_l}}\\&=\left(\frac{h_{i_l}m_{N_{ik}}+h_{j_l}m_{N_{jk}}}{h_{i_l}+h_{j_l}},\frac{h_{i_l}\alpha_{N_{ik}}+h_{j_l}\alpha_{N_{jk}}}{h_{i_l}+h_{j_l}},\frac{h_{i_l}\beta_{N_{ik}}+h_{j_l}\beta_{N_{jk}}}{h_{i_l}+h_{j_l}}\right)\end{aligned}\tag{5.26}$$

for $h_{i_l},h_{j_l}\in H, N_{i_k},N_{j_k}\in E$, $i, j=1, \ldots, n$, $k=0,1,\ldots,10$, $l=1,2,3,4$. The proposed weights are related to the symptoms when the importance of the symptoms should be emphasized.

If all weights are equal to 1 it will mean no important difference among the symptoms, and consequently we return to (5.25).

Example 5.19

Even the operations defined by (5.26) belong to the OWA category operators. For the mean value of the fuzzy number (5.26) we consider two cases:

1) If $m_{N_{ik}}<m_{N_{jk}}$ then, for $a_0=0$, $a_1=\dfrac{h_{i_l}}{h_{i_l}+h_{j_l}}$ and $a_2=\dfrac{h_{j_l}}{h_{i_l}+h_{j_l}}$ we create

$$\frac{h_{i_l}m_{N_{ik}}+h_{j_l}m_{N_{jk}}}{h_{i_l}+h_{j_l}}=\frac{h_{i_l}}{h_{i_l}+h_{j_l}}\min(m_{N_{ik}},m_{N_{jk}})+\frac{h_{j_l}}{h_{i_l}+h_{j_l}}\max(m_{N_{ik}},m_{N_{jk}}).$$

When we set, e.g., $m_{N_{ik}}=30$, $m_{N_{jk}}=50$, $h_{i_l}=3, h_{j_l}=4$ then we will make a calculus $\dfrac{3\cdot30+4\cdot50}{3+4}=\dfrac{3}{3+4}\min(30,50)+\dfrac{4}{3+4}\max(30,50)$, see (5.26).

2) For $m_{N_{ik}}>m_{N_{jk}}$ we put $a_0=0$, $a_1=\dfrac{h_{j_l}}{h_{i_l}+h_{j_l}}$ and $a_2=\dfrac{h_{i_l}}{h_{i_l}+h_{j_l}}$ in the mean value of (5.26). We compute

$$\frac{h_{i_l}m_{N_{ik}}+h_{j_l}m_{N_{jk}}}{h_{i_l}+h_{j_l}}=\frac{h_{j_l}}{h_{i_l}+h_{j_l}}\min(m_{N_{ik}},m_{N_{jk}})+\frac{h_{i_l}}{h_{i_l}+h_{j_l}}\max(m_{N_{ik}},m_{N_{jk}}).$$

If , for example, $m_{N_{ik}} = 50$, $m_{N_{jk}} = 30$, $h_{i_l} = 3, h_{j_l} = 4$, then we will prove

$$\frac{3 \cdot 50 + 4 \cdot 30}{3+4} = \frac{4}{3+4}\min(30,50) + \frac{3}{3+4}\max(30,50) .$$

The relation R expresses the synchronized effectiveness of the drug action for every pair of symptoms. R has its eigen fuzzy set A that does not change after the next composition with it. The support of A consists of the elements of S, and the membership grades $\mu_A(S_j)$ of S_j, $j = 1, \ldots, n$, belonging to A are fuzzy numbers in the L-R representation. Furthermore, the supports of the numbers can be interpreted as the levels of medicinal action for the examined symptoms one by one.

Set A, fulfilling $A \circ R = A$, is represented by the following connection derived for the membership functions, see (5.19)

$$\mu_{A\circ R}(S_j) = \max_{S_i \in S}(\min(\mu_A(S_i), \mu_R(S_i, S_j))) = \mu_A(S_j) = (m_{\mu_A(S_j)}, \alpha_{\mu_A(S_j)}, \beta_{\mu_A(S_j)}),$$
$$\mu_A(S_i),\ \mu_A(S_j), \mu_R(S_i, S_j) \in FN(LR), \tag{5.27}$$

in which the maximum and minimum operations are performed due to (5.13)–(5.16) for $i, j = 1, \ldots, n$.

Let us designate $\mu_A(S_j) = (m_{\mu_A(S_j)}, \alpha_{\mu_A(S_j)}, \beta_{\mu_A(S_j)})$ as $A(S_j)$ in the next step of investigations.

Even if the levels concern single symptoms, we shall remember that the positive reaction after the treatment, assigned to one symptom, also affects the ranges of other symptoms. The levels, established by means of the eigen set, do not change in spite of the extended curative period.

Example 5.20

We intend to test the designed model on clinical data from Ex. 5.5 that we have already been acquainted with.

By giving the patients *Bayer's aspirin* we cure the inflammation of the throat. Patients who suffer from it are often troubled with selected symptoms; S_1 = "*sore throat* (*pain*)", S_2 = "*temperature*", S_3 = "*inflammation state*". Due to the physician's opinion the approximate effectiveness of the drug has been decided as "m" = "*medium*" for S_1, "vlg" = "*very large*" for S_2 and "lg" = "*large*" in the case of S_3. The fuzzy relation R = "the cumulated effectiveness of the drug action for S_i and S_j, $i, j = 1, 2, 3$" is then expressed by the table

$$R = \begin{matrix} & S_1 & S_2 & S_3 \\ \begin{matrix} S_1 \\ S_2 \\ S_3 \end{matrix} & \left[\begin{matrix} \text{"m"} \\ \text{"vlg" and "m"} \\ \text{"lg" and "m"} \end{matrix}\right. & \begin{matrix} \text{"m" and "vlg"} \\ \text{"vlg"} \\ \text{"lg" and vlg"} \end{matrix} & \left.\begin{matrix} \text{"m" and "lg"} \\ \text{"vlg" and "lg"} \\ \text{"lg"} \end{matrix}\right] \end{matrix}.$$

After assigning the weights 4, 2 and 3 to S_1, S_2 and S_3 respectively, we form in accordance with (5.24) and (5.26) the table

$$R = \begin{bmatrix} \frac{4(50,10,10)}{4} & \frac{4(50,10,10)+2(80,10,10)}{4+2} & \frac{4(50,10,10)+3(70,10,10)}{4+3} \\ \frac{2(80,10,10)+4(50,10,10)}{2+4} & \frac{2(80,10,10)}{2} & \frac{2(80,10,10)+3(70,10,10)}{2+3} \\ \frac{3(70,10,10)+4(50,10,10)}{3+4} & \frac{3(70,10,10)+2(80,10,10)}{3+2} & \frac{3(70,10,10)}{3} \end{bmatrix}$$

as a counterpart of the linguistically defined relation R. We compute the entries of R to get it in the final version

$$R = \begin{bmatrix} (50,10,10) & (60,10,10) & (59,10,10) \\ (60,10,10) & (80,10,10) & (74,10,10) \\ (59,10,10) & (74,10,10) & (70,10,10) \end{bmatrix}.$$

To determine the corresponding greatest eigen fuzzy set A we exploit the steps of Algorithm 5.1 as follows.

1. A_1 is decided as a type-2 fuzzy set

$$A_1 = \begin{matrix} S_1 & S_2 & S_3 \\ [(60,10,10) & (80,10,10) & (74,10,10)] \end{matrix}.$$

2) The action of (5.13)–(5.16) leads to the first composition of A_1 with R and results in

$$A_2 = [(60,10,10) \quad (80,10,10) \quad (74,10,10)] \circ \begin{bmatrix} (50,10,10) & (60,10,10) & (59,10,10) \\ (60,10,10) & (80,10,10) & (74,10,10) \\ (59,10,10) & (74,10,10) & (70,10,10) \end{bmatrix}$$
$$= [(60,10,10) \quad (80,10,10) \quad (74,10,10)].$$

3) Since $A_2 = A_1$ we accept $A = A_2$.

By coming back to the formula $L(z) = R(z) = -z + 1$ we finally determine the levels of *Bayer's aspirin* curative effect as supports of the fuzzy numbers $A(S_1)$, $A(S_2)$ and $A(S_3)$, specified by membership functions

$$\mu_{A(S_1)}(z) = \begin{cases} \dfrac{z-50}{10} & \text{for} \quad 50 \le z < 60, \\ \dfrac{70-z}{10} & \text{for} \quad 60 \le z \le 70, \end{cases}$$

$$\mu_{A(S_2)}(z) = \begin{cases} \dfrac{z-70}{10} & \text{for} \quad 70 \le z < 80, \\ \dfrac{90-z}{10} & \text{for} \quad 80 \le z \le 90, \end{cases}$$

and

$$\mu_{A(S_3)}(z) = \begin{cases} \dfrac{z-64}{10} & \text{for} \quad 64 \le z < 74, \\ \dfrac{84-z}{10} & \text{for} \quad 74 \le z \le 84.. \end{cases}$$

The membership functions of $A(S_1)$, $A(S_2)$ and $A(S_3)$ are drawn in Fig. 5.12.

The results can be interpretable in the percentage scale, for the supports of fuzzy numbers are subsets of the standard population [0, 100]. We take into consideration the most important parts of the number supports associated with the membership grades greater than, say, 0.5. Summing up, we make a trial of approximating the curative effect of *Bayer's aspirin* in 55%–65% for S_1, 75%85% for S_2 and 69%–79% with respect to S_3.

Let us emphasize some advantages of the application of the eigen fuzzy model with fuzzy numbers to an appreciation of the drug level as follows.

Even if the fuzzy relation has the elements equal to fuzzy numbers, the eigen fuzzy set associated with the relation will exist. The supports of fuzzy numbers obtained in the eigen fuzzy set yield the expected levels of drug action for every distinct symptom.

The levels are the most optimistic prognoses of drug action since the given eigen fuzzy set is greatest. Moreover, the effects of common medicine action in regard to pairs of symptoms positively influence the range of single levels.

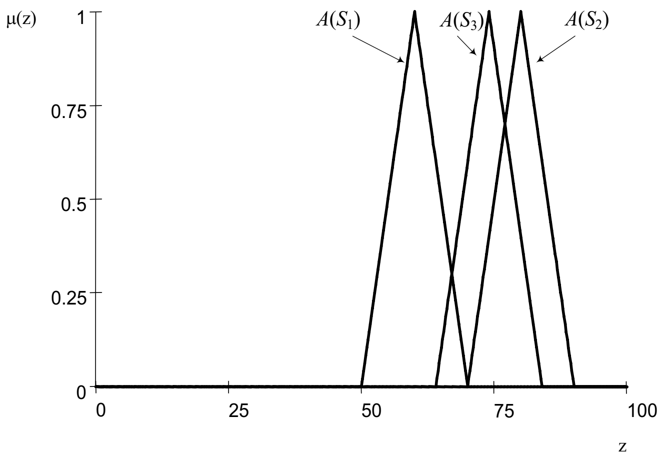

Figure 5.12: Fuzzy numbers $A(S_1)$, $A(S_2)$, $A(S_3)$ of the eigen set A of the relation R

Even the importance of symptoms in a disease also affects the appreciation of levels. The results do not depend on any sample size that can be unstable, but they are based on the stabile expressions formulated by experienced physicians.

At last, the results should satisfy the expectations of medicine manufacturers who wish to recommend the most efficacious, curative remedy.

We have been concentrating our attention on the estimation of one-drug levels concerning several symptoms. In the next chapter, we maintain the same collection of symptoms but we make some different approaches. We will attempt the selection of the best medicine from a sample of remedies recommended in a certain disease.

6 The Choice of Optimal Medicines

6.1 Introduction

We have already used many auxiliary methods coming from fuzzy set theory to make attempts at solutions in such medical tasks as diagnosing or appreciation of drug efficiency. In Chapter 5 we have tested eigen fuzzy set techniques to appreciate the optimal levels of one drug action in the case of several symptoms characteristic of a disease.

We often experience that there can occur such a pathologic process in which the symptoms do not disappear after the treatment when using only one medicine. The medication can improve too high or too low of a level of the quantitative symptom, but the symptom still indicates the presence of the disease. We sometimes have some problems in making a choice of this medicine, which acts best; because it can happen that most drugs influence the same symptoms while they do not improve the others.

By employing different fuzzy decision-making models, we try to make it easier to find such a drug that affects most of the symptoms in the highest degree. We also want to discuss the task of selecting the best possible medicine within the circumstances provided when some decision-makers have different opinions about the priority of tested drugs.

In fuzzy decision making models, often applied to technical solutions like in [33], we also use non-fuzzy sets. It is remarkable to observe how the assumptions of fuzzy set theory link crisp sets to imprecise collections of elements to obtain a harmonic mixture of decisive information.

Some readers may still experience the advantage of translating average words originating from "spoken" languages into numbers (we have already discussed the problem in Chapter 3). This emphasizes the richness of applications offered by processes of numerical fuzzifying of some appearances that cannot be strictly defined.

6.2 Fuzzy Utilities in Decision-Making Models

Let us introduce the notions of a space of states-results $X = \{x_1, x_2, ..., x_m\}$ and a decision space $A = \{a_1, a_2, ..., a_n\}$. The mentioned universes of discourse constitute the main data basis in Jain's decision-making model.

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6.2.1 Jain's Utility Matrix as the *Drug – Symptom* Table

If a rational decision maker makes a decision $a_i \in A$, $i = 1, 2, \ldots, n$, concerning states-results $x_j \in X$, $j = 1, 2, \ldots, m$, then the problem is reduced to the consideration of the ordered triplet (X, A, U), where X is a set of states-results, A – a set of decisions and U – the utility matrix [38, 39, 40]

$$U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1m} \\ U_{21} & U_{22} & \cdots & U_{2m} \\ \vdots & & \ddots & \vdots \\ U_{n1} & U_{n2} & \cdots & U_{nm} \end{bmatrix} \tag{6.1}$$

in which each element U_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, is a fuzzy set defining the fuzzy utility following from the decision a_i with the result x_j.

Assume now that the state-result is not exactly known, but as a fuzzy set $S \subseteq X$ given in the form

$$S = \mu_S(x_1)/x_1 + \mu_S(x_2)/x_2 + \cdots + \mu_S(x_m)/x_m \,. \tag{6.2}$$

A decision method thus concerns such fuzzy decision situation in which both the knowledge about the state and the utilities are fuzzy. To solve the decision problem under circumstances that are given above, means to find the best decision a_i influenced by all constraints.

The theoretical model with the triplet (X, A, U) and the fuzzy set of states S, thus very shortly sketched, can find its practical application in the processes of choosing an optimal drug. If a given disease is recognized by the symptoms accompanying it, then we, by giving a medicine, try to liquidate these symptoms or at least try to reduce their unfavorable influence upon the patient's health. Not all symptoms retreat after the cure has been carried out. Sometimes, one can only soothe their negative effects by, for example, the lowering of an excessive level of the indicator, the relief of pain, and the like, but cannot ascertain that the patient is fully free from them. The problem of choosing an optimal drug (decision), which soothes the symptoms or has some power to eliminate them in full, corresponds to the theoretical assumptions presented above [59, 60, 62].

In order to show the algorithm for finding such a decision let us consider a model with n drugs $a_1, a_2, \ldots, a_n \in A$. On the basis of the physician's decision, the drugs are prescribed to the patient (thus may be treated as decisions $a_1, a_2, \ldots, a_n$) with a view to have an effect on m symptoms $x_1, x_2, \ldots, x_m \in X$ representing certain states characteristic of the given disease. We actually rename the symptoms as x_j instead for S_j as we used in Chapters 3–5 to agree with some symbolic terms assigned to states-results being parts of fuzzy decision-making models. To simplify the symbols let us further assume that each symptom $x_j \in X$, where X is a space of symptoms (states), is understood as the result of the treatment of the symptom after the cure with the drugs $a_1, a_2, ..., a_n$ has been carried out. On the basis of earlier experiments the physician knows how to define in words the curative drug efficiency in the case of considered symptoms. In accordance with his advice, we suggest a list of terms, already known from Section 5.6 that introduces a linguistic variable named "the curative drug effectiveness regarding a symptom" = {R_1 ="*none*", R_2 = "*almost none*", R_3 = "*very little*", R_4 = "*little*", R_5 = "*rather little*", R_6 = "*medium*", R_7 = "*rather large*", R_8 = "*large*", R_9 = "*very large*", R_{10} = "*almost complete*", R_{11} = "*complete*"}. Each notion from this list of terms is the name of a fuzzy set. Assume that all sets are defined in the space $Z = [0,100]$, see Ex. 5.17, which is suitable as a reference set to measure a number of patients who have been affected by a medicine in the grade corresponding to each name. We use this technique of building a list of expressions for the third time. Each time we change the forms of constraints to demonstrate how many options in the creation of membership functions are allowable.

To avoid further complicated computations we suggest membership functions of the fuzzy sets from the list, called "the curative drug effectiveness regarding a symptom", as simple linear functions [2, 59, 60, 62]

$$L(z,\alpha,\beta) = \begin{cases} 0 & \text{for} \quad z \le \alpha, \\ \dfrac{z-\alpha}{\beta-\alpha} & \text{for} \quad \alpha < z \le \beta, \\ 1 & \text{for} \quad z > \beta, \end{cases} \tag{6.3}$$

and

$$\pi(z,\alpha,\gamma,\beta) = \begin{cases} 0 & \text{for} \quad z \le \alpha, \\ L(z,\alpha,\gamma) & \text{for} \quad \alpha < z \le \gamma, \\ 1 - L(z,\gamma,\beta) & \text{for} \quad \gamma < z \le \beta, \\ 0 & \text{for} \quad z > \beta, \end{cases} \tag{6.4}$$

where z is an independent variable belonging to [0, 100] and α, β, γ are some parameters.

Let us define

$$\mu_{R_k}(z) = \begin{cases} 1 - L(z,\alpha_k,\beta_k) & \text{for} \quad k = 1,2,3,4,5, \\ L(z,\alpha_k,\beta_k) & \text{for} \quad k = 7,8,9,10,11, \end{cases} \tag{6.5}$$

and

$$\mu_{R_6}(z) = \pi(z, \alpha_6, \gamma, \beta_6) \tag{6.6}$$

in which $z \in Z = [0, 100]$, while α_k, β_k, γ are borders for the fuzzy supports and they also constitute some numbers from the interval [0, 100].

Let us further decide the values of the boundary parameters α_k, β_k, γ in order to construct constrains for the fuzzy sets that represent the terms of the mentioned list "the curative drug effectiveness regarding a symptom".

Example 6.1

We suggest the following linear functions that can be approved as the membership functions of terms constituting the contents of the effectiveness list

$$\begin{aligned}
\mu_{R_1}(z) &= \mu_{\text{"}none\text{"}}(z) = 1 - L(z,0,20)\,,\\
\mu_{R_2}(z) &= \mu_{\text{"}almost\,none\text{"}}(z) = 1 - L(z,10,30)\,,\\
\mu_{R_3}(z) &= \mu_{\text{"}very\ little\text{"}}(z) = 1 - L(z,20,40)\,,\\
\mu_{R_4}(z) &= \mu_{\text{"}little\text{"}}(z) = 1 - L(z,30,50)\,,\\
\mu_{R_5}(z) &= \mu_{\text{"}rather\ little\text{"}}(z) = 1 - L(z,40,60)\,,\\
\mu_{R_6}(z) &= \mu_{\text{"}medium\text{"}}(z) = \pi(z,30,50,70)\,,\\
\mu_{R_7}(z) &= \mu_{\text{"}rather\ large\text{"}}(z) = L(z,40,60)\,,\\
\mu_{R_8}(z) &= \mu_{\text{"}large\text{"}}(z) = L(z,50,70)\,,\\
\mu_{R_9}(z) &= \mu_{\text{"}very\ large\text{"}}(z) = L(z,60,80)\,,\\
\mu_{R_{10}}(z) &= \mu_{\text{"}almost\ complete\text{"}}(z) = L(z,70,90)\,,\\
\mu_{R_{11}}(z) &= \mu_{\text{"}complete\text{"}}(z) = L(z,80,100)
\end{aligned}$$

for $z \in [0, 100]$.

The parameters α_k, β_k and γ in equations above have been proposed in conformity with the physician's suggestion. In order to give an image of the restrictions' appearance we sketch them in Fig. 6.1.

The membership functions presented in Ex. 6.1 can be adopted as a foundation of other fuzzy sets, this time finite sets corresponding to R_1–R_{11}. To accomplish a process of generating a class of discrete sets, replacing R_1–R_{11}, we take only the essential parts of sets into consideration, i.e., the elements of their supports that possess membership degrees greater than 0.5. The continuous membership functions serve as a tool for calculating the membership degrees of some of the chosen elements coming from the set supports.

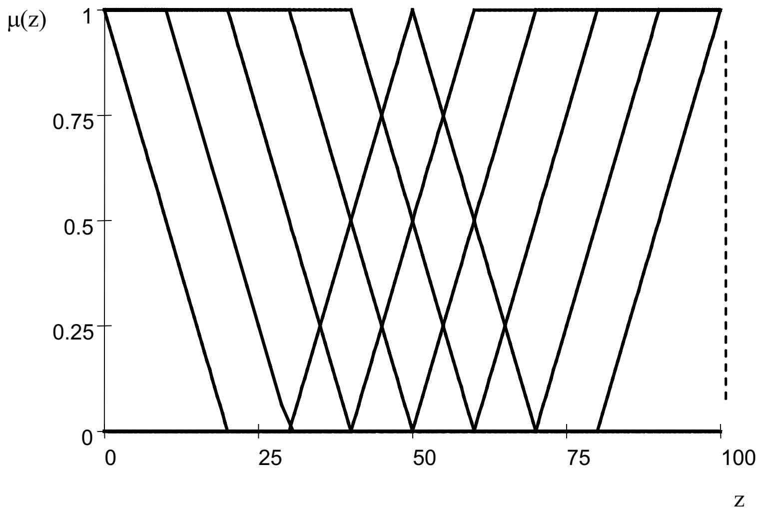

Figure 6.1: The fuzzy constraints R_1–R_{11}

Example 6.2

By involving the patterns from Ex. 6.1, we form the following discrete fuzzy sets, which act on behalf of R_1–R_{11}

$$R_1 = \text{"}none\text{"} = 1/0 + 0.9/2 + 0.8/4 + 0.7/6 + 0.6/8\,,$$

$$R_2 = \text{"}almost\ none\text{"} = 1/10 + 0.9/12 + 0.8/14 + 0.7/16 + 0.6/18\,,$$

$$R_3 = \text{"}very\ little\text{"} = 1/20 + 0.9/22 + 0.8/24 + 0.7/26 + 0.6/28\,,$$

$$R_4 = \text{"}little\text{"} = 1/30 + 0.9/32 + 0.8/34 + 0.7/36 + 0.6/38\,,$$

$$R_5 = \text{"}rather\ little\text{"} = 1/40 + 0.9/42 + 0.8/44 + 0.7/46 + 0.6/48\,,$$

$$R_6 = \text{"}medium\text{"} = 0.6/42 + 0.7/44 + 0.8/46 + 0.9/48 + 1/50 + 0.9/52 + 0.8/54 + 0.7/56 + 0.6/58\,,$$

$$R_7 = \text{"}rather\ large\text{"} = 0.6/52 + 0.7/54 + 0.8/56 + 0.9/58 + 1/60\,,$$

$$R_8 = \text{"}large\text{"} = 0.6/62 + 0.7/64 + 0.8/66 + 0.9/68 + 1/70\,,$$

$$R_9 = \text{"}very\ large = 0.6/72 + 0.7/74 + 0.8/76 + 0.9/78 + 1/80\,,$$

$R_{10} = \text{"}almost\ complete\text{"} = {}^{0.6}\!/_{82} + {}^{0.7}\!/_{84} + {}^{0.8}\!/_{86} + {}^{0.9}\!/_{88} + {}^{1}\!/_{90}$,

$R_{11} = \text{"}complete\text{"} = {}^{0.6}\!/_{92} + {}^{0.7}\!/_{94} + {}^{0.8}\!/_{96} + {}^{0.9}\!/_{98} + {}^{1}\!/_{100}$.

The fuzzy sets U_{ij} from the utility matrix U can be now replaced by the built fuzzy sets R_1–R_{11}. To state a connection between a_i (medicine) and the effectiveness of the retreat of x_j (symptom) the physician uses the word from the list "the curative drug effectiveness regarding a symptom" and this word is "translated" into the fuzzy set R_k, $k = 1, 2, \ldots, 11$.

Let us also admit that the physician possesses a general experience as to the "difficulties" in the remission of the symptoms x_j, $j = 1, 2, \ldots, m$. His medical knowledge, based on observations, can contribute in a classification of symptoms that are harder to treat, and those symptoms that recede more readily during the treatment process. Via the words from the list, "the curative drug effectiveness regarding a symptom", one may assign to each symptom a general ability to retreat, fixed, for instance, by observing the cure of many patients with different drugs. For instance, it is commonly known that a fever disappears quicker than some changes in tissues after inflammation. Such an average classification of symptoms found its place in the fuzzy set S defined theoretically by (6.2), in which the membership degrees $\mu_S(x_j)$, $j = 1, 2, ..., m$, correspond now to the fuzzy sets R_k, $k = 1, 2, \ldots, 11$. These express the mean effectiveness of treatment independently of a prescribed medicine. By the "cure", one can mean the level of the retreating symptom, the decrease of the heightened index, and the like.

In accordance with Jain's theory of decision-making, the fuzzy utility [38, 39, 40, 59, 60, 62] for each decision-drug a_i, $i = 1, 2, \ldots, n$, with the fuzzy state $S \subseteq X$ characterized by means of the membership degrees $\mu_S(x_j)$ is defined to be the set

$$U_i = {}^{\mu_S(x_1)}\!\Big/_{U_{i1}} + {}^{\mu_S(x_2)}\!\Big/_{U_{i2}} + \cdots + {}^{\mu_S(x_m)}\!\Big/_{U_{im}} \tag{6.7}$$

for $i = 1, 2, \ldots, n$. The set allows observing the relationship between the general ability to soothe, and this effect in soothing which the drug a_i causes for each symptom x_j. Both the membership degrees $\mu_S(x_j)$ and the elements U_{ij} in the support of the set U_i are the fuzzy sets of the discrete type R_1–R_{11}.

It is not possible to make further calculations on such sets that have fuzzy sets as the elements of supports and the membership degrees. We thus need an operation that reduces this family of fuzzy sets to one fuzzy set. This is grounded on the single support with clearly determined membership degrees. We test a concatenation operator [59, 62]

$$U_i = \sum_{j=1}^{m} \mu_S(x_j) \Big/ U_{ij} = \sum_{x_j \in X} \sum_{t=1}^{r} \mu_{R_{k_\phi}}(z_t) \Big/ z_t \Bigg/ \sum_{c=1}^{q} \mu_{R_{k_\varphi}}(z_c) \Big/ z_c =$$

$$= \sum_{\substack{(z_t, z_c) \in \\ \{z_1, z_2, \ldots, z_r\} \\ \times \\ \{z_1, z_2, \ldots, z_q\}}} \min(\mu_{R_{k_\phi}}(z_t), \mu_{R_{k_\varphi}}(z_c)) \Bigg/ \frac{z_t + z_c}{2} = \sum_{z \in Z} \mu_{U_i}(z) \Big/ z \tag{6.8}$$

in which $\mu_S(x_j)$ is the equal of a fuzzy set $\sum_{t=1}^{r} \mu_{R_{k_\phi}}(z_t) \Big/ z_t$ while U_{ij} is another fuzzy set given as $\sum_{c=1}^{q} \mu_{R_{k_\varphi}}(z_c) \Big/ z_c$ for R_{k_ϕ} and R_{k_φ} belonging to the class of the sets R_1–R_{11}. Each U_{ij} expresses the fuzzy utility following the decision a_i with the result x_j. However, by $\mu_S(x_j)$ we judge if x_j is a symptom having a tendency to disappear.

When the same element z in the support of the fuzzy set appears with different membership degrees $\mu_1(z)$ and $\mu_2(z)$, we will aggregate their values by adopting the Jain operation

$$\mu_1(z) \oplus \mu_2(z) = \mu_1(z) + \mu_2(z) - \mu_1(z) \cdot \mu_2(z) . \tag{6.9}$$

The sign "⊕" denotes a symbolic addition of two different membership degrees assigned to the same support element.

Example 6.3

Suppose that $A = 0.8/2 + 0.4/3 + 0.5/2 + 0.2/2 + 1/3$. Since the support members x = 2, 3 appear more than once then we should rearrange A due to the following operations

$$A = 0.8/2 + 0.4/3 + 0.5/2 + 0.2/2 + 1/3 = 0.8 + 0.5 - 0.8 \cdot 0.5/2 + 0.2/2$$
$$+ 0.4 + 1 - 0.4 \cdot 1/3 = 0.9/2 + 0.2/2 + 1/3 = 0.9 + 0.2 - 0.9 \cdot 0.2/2 + 1/3$$
$$= 0.92/2 + 1/3 .$$

We notice that this sort of a concatenation operation, recommended by (6.9), raises the aggregated values of membership degrees computed for the same elements in the support of a fuzzy set when comparing them to primary values. We will employ the operations of the (6.9) type to induce the most optimistic prognosis in further investigations.

6.2.2 The Solution of Jain's Decision Case

The problem of choosing an optimal decision is solved according to the algorithm developed by Jain [38, 39]. The steps of the action line are listed in the following order.

Algorithm 6.1

1. We form a non-fuzzy set Y as the union of supports characteristic of U_i, $i = 1, 2, \ldots, n$. This set contains the elements $z \in Z$, which appear in all sets U_i. Hence, we have access to the range of the common utility expressed as $Y = \bigcup_{i=1}^{n} \text{supp}(U_i)$.
2. We select the maximal element of the set Y, so-called $z_{\max}$.
3. We define the fuzzy sets U_i' as

$$U_i' = \sum_{z \in Z} \mu_{U_i'}(z) / z \tag{6.10}$$

   for $z \in \text{supp}(U_i)$. This means that the supports of U_i' and U_i are the same sets. The membership degrees of U_i' are computed by means of the formula

$$\mu_{U_i'}(z) = \frac{z_{U_i}}{z_{\max}}, \tag{6.11}$$

   where z_{U_i} stands for an element belonging to the support of the set U_i. U_i''s membership degrees evaluate the "deviation" between the elements of U_i, and the maximal z found in the union of all U_i.
4. The next introduced fuzzy set has the form of

$$U_{i0} = \sum_{z \in Z} \mu_{U_{i0}}(z) / z, \tag{6.12}$$

provided that the membership degree $\mu_{U_{i0}}(z)$ is calculated according to the rule

$$\mu_{U_{i0}}(z) = \min(\mu_{U_i}(z), \mu_{U_i'}(z)) . \tag{6.13}$$

The fuzzy utility U_{i0}, constructed for each medicine a_i gathers all possible factors that can affect appreciation of the soothing power of a_i. The minimum operation is used in (6.13) in order to reduce too large values of final results, which as we remember, are effects of the operation $\oplus$ induced by (6.9).

5. We slowly close the action of Algorithm 6.1 by the adoption of a new fuzzy set A^* composed of elements $a_1, a_2, \ldots, a_n$ ($a_i \in A$, $i = 1, 2, \ldots, n$) and formalized by

$$A^* = \sum_{i=1}^{n} \mu_{A^*}(a_i) \big/ a_i \, . \tag{6.14}$$

The membership degree for each a_i is generated by

$$\mu_{A^*}(a_i) = \underset{z \in \mathrm{supp}(U_{i0})}{\text{mean value}}(\mu_{U_{i0}}(z)) . \tag{6.15}$$

In practice we compute the arithmetic mean for a sample of membership degrees appearing in each set U_{i0}. This value expresses the decisive character of every a_i in accordance with a rule: the higher the value of the membership degree assigned to a_i is, the better the influence of a_i on the patient's health is to be expected.

6. To terminate the choice of an optimal decision a^* we accept as a^* this a_i whose membership degree satisfies the equation

$$\mu_{A^*}(a^*) = \max_{1 \le i \le n}(\mu_{A^*}(a_i)) , \tag{6.16}$$

and we ascertain that the application of the drug a^* should yield the best effects in the retreating process of the symptoms x_j, $j = 1, 2, \ldots, m$.

Example 6.4

The Jain model is tested on the clinical data coming from the investigation carried out among patients who suffer from D = "*coronary heart disease*". We consider the most typical symptoms accompanying the illness, i.e., x_1 = "*pain in chest*", x_2 = "*changes in ECG*", and x_3 = "*increased level of LDL-cholesterol*". A physician has recommended a_1 = *nitroglycerin*, a_2 = *beta-adrenergic blockade*, a_3 = *acetylsalicylic acid* (*aspirin*) and a_4 = *statine LDL-reductor* as the medicines expected to improve the patient's state. The physician has also decided that the set S and the matrix U should have the following descriptions

$$S = {large}/{x_1} + {medium}/{x_2} + {rather\ large}/{x_3}$$

and

$$U = \begin{matrix} & x_1 & x_2 & x_3 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{matrix} & \multicolumn{3}{l}{\begin{bmatrix} complete & very\ large & almost\ none \\ medium & medium & little \\ little & little & very\ little \\ little & little & very\ large \end{bmatrix}} \end{matrix}.$$

We begin the computations with determining supports of the sets U_i. For instance, the set U_1 is decided as

$$U_1 = {large}/{complete} + {medium}/{very\ large} + {rather\ large}/{almost\ none} =$$

$$\frac{0.6/62 + \cdots + 1/70}{0.6/92 + \cdots + 1/100} + \frac{0.6/42 + \cdots + 1/50 + \cdots + 0.6/58}{0.6/72 + \cdots + 1/80} +$$

$$\frac{0.6/52 + \cdots + 1/60}{1/10 + \cdots + 0.6/18} =$$

$$\frac{\min(0.6,0.6)}{\frac{62+92}{2}} + \cdots + \frac{\min(1,1)}{\frac{70+100}{2}} + \frac{\min(0.6,0.6)}{\frac{42+72}{2}} + \cdots + \frac{\min(0.6,1)}{\frac{58+80}{2}} +$$

$$\frac{\min(0.6,1)}{\frac{52+10}{2}} + \cdots + \frac{\min(1,0.6)}{\frac{60+18}{2}} =$$

$$0.6/31 + 0.88/32 + 0.938/33 + 0.998/34 + 0.964/35 + 0.998/36 + 0.976/37 + 0.88/38 +$$
$$0.6/39 + 0.6/57 + 0.84/58 + 0.952/59 + 0.986/60 + 0.997/61 + 0.9986/62 + 0.997/63 +$$
$$0.999/64 + 0.1/65 + 0.998/66 + 0.9761/67 + 0.88/68 + 0.6/69 + 0.6/77 + 0.84/78 +$$
$$0.952/79 + 0.974/80 + 0.998/81 + 0.999/82 + 0.988/83 + 0.999/84 + 1/85$$

when applying (6.8) and (6.9).

By repeating the procedure developed above we obtain the sets

$$U_2 = 0.6/41 + 0.952/42 + 0.976/43 + 0.998/44 + 1/45 + 0.998/46 + 0.97/47 + 0.988/48 + 0.6/49 + 1/50 + 0.96/52 + 0.84/53 + 0.99/54 + 0.986/55 + 0.999/56 + 0.998/57 + 0.998/58 + 0.999/59 + 1/60 + 0.992/61 + 0.976/62 + 0.88/63 + 0.6/64,$$

$$U_3 = 0.84/36 + 0.986/37 + 0.999/38 + 0.999/39 + 1/40 + 0.999/41 + 0.999/42 + 0.999/43 + 0.999/44 + 0.986/45 + 0.98/46 + 0.994/47 + 0.99/48 + 0.992/49 + 1/50 + 0.992/51 + 0.976/52 + 0.88/53 + 0.6/54$$

and

$$U_4 = 0.6/36 + 0.88/37 + 0.976/38 + 0.99/39 + 1/40 + 0.999/41 + 0.999/42 + 0.998/43 + 0.998/44 + 0.986/45 + 0.98/46 + 0.98/47 + 0.984/48 + 0.992/49 + 1/50 + 0.992/51 + 0.976/52 + 0.88/53 + 0.6/54 + 0.6/62 + 0.84/63 + 0.952/64 + 0.986/65 + 0.998/66 + 0.998/67 + 0.976/68 + 0.99/69 + 1/70.$$

The non-fuzzy sum of all supports emerges as a set

$$\bigcup_{i=1}^{4} \mathrm{supp}(U_i) =$$
{31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 77, 78, 79, 80, 81, 82, 83, 84, 85}

in which the largest element is found as $z_{\max} = 85$.

Equations (6.10) and (6.11) give rise to the creation of new sets U_i', $i = 1, 2, 3, 4$. U_1' – the first set in the sequence – appears as the following fuzzy collection of elements

$$U_1' = \frac{31}{85}\Big/31 + \frac{32}{85}\Big/32 + \frac{33}{85}\Big/33 + \cdots + \frac{84}{85}\Big/84 + \frac{85}{85}\Big/85 =$$
$$0.36/31 + 0.376/32 + 0.388/33 + 0.4/34 + 0.41/35 + 0.42/36 + 0.435/37 + 0.447/38 +$$
$$0.459/39 + 0.67/57 + 0.682/58 + 0.694/59 + 0.706/60 + 0.718/61 + 0.729/62 +$$
$$0.741/63 + 0.753/64 + 0.765/65 + 0.776/66 + 0.788/67 + 0.8/68 + 0.811/69 +$$
$$0.9/77 + 0.917/78 + 0.929/79 + 0.94/80 + 0.953/82 + 0.965/82 + 0.976/83 +$$
$$0.988/84 + 1/85 .$$

The other sets of U_1' 's type, $i = 2, 3, 4$, are expanded as

$$U_2' = 0.48/41 + 0.494/42 + 0.506/43 + 0.517/44 + 0.529/45 + 0.54/46 + 0.553/47 +$$
$$0.565/48 + 0.576/49 + 0.588/50 + 0.612/52 + 0.623/53 + 0.635/54 + 0.647/55 +$$
$$0.659/56 + 0.67/57 + 0.682/58 + 0.694/59 + 0.706/60 + 0.718/61 + 0.729/62 +$$
$$0.741/63 + 0.753/64,$$

$$U_3' = 0.42/36 + 0.435/37 + 0.447/38 + 0.459/39 + 0.470/40 + 0.482/41 + 0.494/42 +$$
$$0.506/43 + 0.518/44 + 0.529/45 + 0.541/46 + 0.553/47 + 0.564/48 + 0.576/49 +$$
$$0.588/50 + 0.6/51 + 0.612/52 + 0.623/53 + 0.635/54$$

and

$$U_4' = 0.42/36 + 0.435/37 + 0.447/38 + 0.459/39 + 0.470/40 + 0.482/41 + 0.494/42 +$$
$$0.506/43 + 0.518/44 + 0.529/45 + 0.541/46 + 0.553/47 + 0.564/48 + 0.576/49 +$$
$$0.588/50 + 0.6/51 + 0.612/52 + 0.623/53 + 0.635/54 + 0.729/62 + 0.741/63 +$$
$$0.753/64 + 0.765/65 + 0.776/66 + 0.788/67 + 0.8/68 + 0.811/69 + 0.823/70 .$$

We follow the next step of Algorithm 6.1 to arrange the sets U_{i0}, $i = 1, 2, 3, 4$ as

$$U_{10} = \frac{\min(0.6,0.36)}{31} + \frac{\min(0.88,0.376)}{32} + \cdots + \frac{\min(1,1)}{85} = \frac{0.36}{31} + \frac{0.376}{32} + \frac{0.388}{33} + \frac{0.4}{34} + \frac{0.41}{35} + \frac{0.42}{36} + \frac{0.435}{37} + \frac{0.447}{38} + \frac{0.459}{39} + \frac{0.6}{57} + \frac{0.682}{58} + \frac{0.694}{59} + \frac{0.706}{60} + \frac{0.718}{61} + \frac{0.729}{62} + \frac{0.741}{63} + \frac{0.753}{64} + \frac{0.765}{65} + \frac{0.776}{66} + \frac{0.788}{67} + \frac{0.8}{68} + \frac{0.6}{69} + \frac{0.6}{77} + \frac{0.84}{78} + \frac{0.929}{79} + \frac{0.94}{80} + \frac{0.953}{81} + \frac{0.965}{82} + \frac{0.976}{83} + \frac{0.988}{84} + \frac{1}{85},$$

$$U_{20} = \frac{0.48}{41} + \frac{0.494}{42} + \frac{0.506}{43} + \frac{0.517}{44} + \frac{0.529}{45} + \frac{0.541}{46} + \frac{0.553}{47} + \frac{0.565}{48} + \frac{0.576}{49} + \frac{0.588}{50} + \frac{0.612}{52} + \frac{0.623}{53} + \frac{0.635}{54} + \frac{0.647}{55} + \frac{0.659}{56} + \frac{0.67}{57} + \frac{0.682}{58} + \frac{0.694}{59} + \frac{0.706}{60} + \frac{0.718}{61} + \frac{0.729}{62} + \frac{0.741}{63} + \frac{0.6}{64},$$

$$U_{30} = \frac{0.42}{36} + \frac{0.435}{37} + \frac{0.447}{38} + \frac{0.459}{39} + \frac{0.470}{40} + \frac{0.482}{41} + \frac{0.494}{42} + \frac{0.506}{43} + \frac{0.518}{44} + \frac{0.529}{45} + \frac{0.541}{46} + \frac{0.553}{47} + \frac{0.564}{48} + \frac{0.576}{49} + \frac{0.588}{50} + \frac{0.6}{51} + \frac{0.612}{52} + \frac{0.623}{53} + \frac{0.6}{54}$$

and

$$U_{40} = \frac{0.42}{36} + \frac{0.435}{37} + \frac{0.447}{38} + \frac{0.459}{39} + \frac{0.470}{40} + \frac{0.482}{41} + \frac{0.494}{42} + \frac{0.506}{43} + \frac{0.518}{44} + \frac{0.529}{45} + \frac{0.541}{46} + \frac{0.553}{47} + \frac{0.564}{48} + \frac{0.576}{49} + \frac{0.588}{50} + \frac{0.6}{51} + \frac{0.612}{52} + \frac{0.623}{53} + \frac{0.6}{54} + \frac{0.6}{62} + \frac{0.741}{63} + \frac{0.753}{64} + \frac{0.765}{65} + \frac{0.776}{66} + \frac{0.788}{67} + \frac{0.8}{68} + \frac{0.811}{69} + \frac{0.823}{70}.$$

The decision set A^* has been decided as

$$A^* = \frac{\text{mean}(0.36,0,376, ...,1)}{a_1} + \frac{\text{mean}(0.48,0.494, ...,0.6)}{a_2} + \frac{\text{mean}(0.42,0.435, ...,0.6)}{a_3} + \frac{\text{mean}(0.42,0.435, ...,0.823)}{a_4} = \frac{0.685}{a_1} + \frac{0.611}{a_2} + \frac{0.527}{a_3} + \frac{0.603}{a_4}.$$

The magnitudes of the membership degrees give us a hint about priorities of drugs, i.e., a_1 should have the strongest soothing power when regarding the considered symptoms, and it should be accepted as the optimal decision-drug. Moreover, we can state the hierarchy of drugs in the following order: $a_1 \succ a_2 \succ a_4 \succ a_3$. The notion $a_i \succ a_j$ indicates that a_i acts better than a_j, $i, j = 1, 2, 3, 4$.

6.3 Group Decision-Making in the Selection of Drugs

We have followed the procedure of comparing the healing effect of medicines on the condition that some descriptions, which concern the decisive character of the pairs "*drug – symptom*", are made by one physician. Nevertheless, everyone knows that such opinions are often shared. If we involve several physicians in a discussion about the drug priority, then we can experience that they will hold different views about the curative power of considered medicines. In this section we will give a piece of information about a new technique contributing in the choice of an optimal medicine in spite of contradictory judgements.

Each physician treated as a decision-maker would like to create the matrix U and the set S according to his own experience and judgement. As a result, we obtain different priorities in the set A^*. How shall we choose the best medicine in the case when the sets A^* differ a greatly from each other? The question is answered by means of the algorithm based on graphs [40]. We thus adapt the theoretical graph model to a medical task that is sketched below [62].

Let us state a set of physicians $P = \{P_1, P_2, ..., P_t\}$ who appreciate the drugs belonging to set $A = \{a_1, a_2, ..., a_n\}$. A fuzzy relation $\Re \subset A \times A$ with the membership function $\mu_\Re : A \times A \to [0,1]$, called the group order, is represented by membership degrees $\mu_\Re(a_i, a_j)$. These, in turn, appreciate the intensity grade of preference concerning decision a_i in comparison with a_j. If we define

$$\delta_{ij} = \left\{P_s : P_s \quad \textit{tells} \quad \textit{that} \quad a_i \succ a_j\right\}, \tag{6.17}$$

$s = 1, 2, \ldots, t$, $i, j = 1, 2, \ldots, n$, then we will generate membership degrees in relation $\Re$ as

$$\mu_\Re(a_i, a_j) = \frac{|\delta_{ij}|}{t}. \tag{6.18}$$

We now need a definition for the α-level of a fuzzy set and apply it in the further part of the discussed many-decision-making-model [12, 40, 95].

Definition 6.1

For a fuzzy set $A = \{(x, \mu_A(x))\}, x \in X$, we determine a non-fuzzy set

$$A_\alpha = \{x : \mu_A(x) \geq \alpha\}, \tag{6.19}$$

called the α-level of A.

Example 6.5

Suppose that $A = 0.1/1 + 0.2/2 + 0.4/3 + 0.5/4 + 0.7/5 + 0.8/7 + 0.9/8 + 1/9 + 0.6/10$ in $X = \{1, \ldots, 10\}$. If $\alpha = 0.4$ then $A_{0.4} = \{3, 4, 5, 7, 8, 9, 10\}$.

For A given by the membership function $\mu_A(x) = \pi(x, 20, 50)$ we state, e.g., $A_{0.5} = [40, 60]$ in accordance with Fig. 6.2.

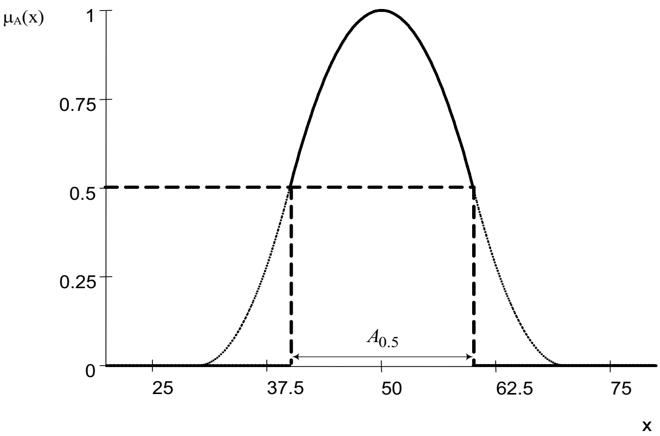

Figure 6.2: The 0.5-level of A characterized by $\pi(x, 20, 50)$

Analogously, the α-level R_α of the relation $\mathfrak{R}$ is decided as the set

$$R_\alpha = \left\{\left(a_i, a_j\right) : \mu_{\mathfrak{R}}\left(a_i, a_j\right) \geq \alpha\right\}. \tag{6.20}$$

For the greatest α we seek a set R_α that contains all decisions a_i and is totally ordered. We treat a_i values as vertices in a directed graph. The order in the pair (a_i, a_j) indicates the direction of the arrow that ties a_i and a_j together. We should explain that the order in the graph is interpreted as total if each pair of nodes has a connection formed by the arrow. The vertex, that concentrates the most endpoints of the arrows in accordance with R_α, is determined as a group decision.

The steps that follow the determination procedure of selecting an optimal medicine are collected in the algorithm developed below.

Algorithm 6.2

1. Find "α values"= $\{\mu_{\Re}(a_i,a_j) : \mu_{\Re}(a_i,a_j)$ are different in $\Re\} = \{\alpha_1, \alpha_2, ..., \alpha_p\}$ as a set of values arranged in the descending order.
2. Set $k = 1$.
3. Find R_{α_k}, $k = 1, \ldots, p$, and sketch a directed graph for R_{α_k} due to the pair order. The notation $a_i \succ a_j$ corresponds to the ordered pair (a_i, a_j) generating the direction $a_i \leftarrow a_j$.
4. Is the order in R_{α_k} total? →No. Set $k=k+1$. Go To Step 3
   →Yes. Choose the vertex with the largest number of arrow endpoints as an optimal decision

Example 6.6

In order to test the group decision model of choosing the best medicine among the four drugs already introduced by Ex. 6.4, we have asked six physicians P_1, P_2, …, P_6 for evaluating the curative effects of the drugs: a_1, a_2, a_3, a_4. The medicines, as we remember, show a healing power in "*coronary heart disease*". The priority levels are listed in the following schedule:

$$P_1 = P_3 = P_4 = (a_1 \succ a_2 \succ a_4 \succ a_3),$$
$$P_2 = (a_1 \succ a_4 \succ a_2 \succ a_3),$$
$$P_5 = (a_2 \succ a_1 \succ a_4 \succ a_3),$$
$$P_6 = (a_2 \succ a_4 \succ a_1 \succ a_3).$$

We adopt (6.18) to decide the contents of the matrix $\Re$ as

$$\Re = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0 & 0.67 & 1 & 0.83 \\ 0.33 & 0 & 1 & 0.83 \\ 0 & 0 & 0 & 0 \\ 0.17 & 0.17 & 1 & 0 \end{bmatrix} \end{array}.$$

The "α-values" set is sorted as "α-values" = {1, 0.83, 0.67, 0.33, 0.17}.

For $k = 1$ we find $R_1 = \{(a_1,a_3),(a_2,a_3),(a_4,a_3)\}$ .The associated graph R_1, plotted in Fig. 6.3, does not reveal the total order.

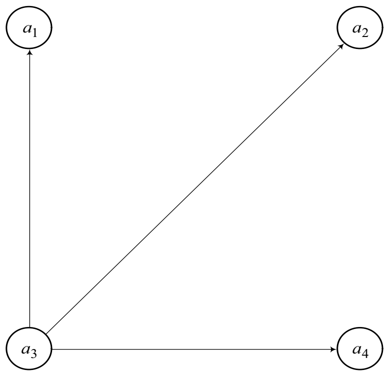

Figure 6.3: The set R_1 as the directed graph

If $k = 2$ we set $R_{0.83} = \{(a_1,a_3),(a_2,a_3),(a_4,a_3),(a_1,a_4),(a_2,a_4)\}$ that constitutes a basis of the graph $R_{0.83}$ presented in Fig. 6.4.

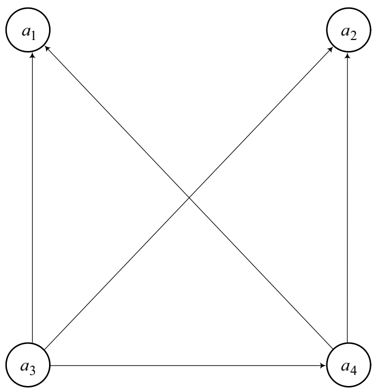

Figure 6.4: The set $R_{0.83}$ as the directed graph

The graph $R_{0.83}$ has not the total order either, since the pair (a_1, a_2) is lacking a connection.

We thus prove if the set $R_{0.67} = \{(a_1,a_3),(a_2,a_3),(a_4,a_3),(a_1,a_4),(a_2,a_4),$ $(a_1,a_2)\}$, generating the graph $R_{0.67}$ drawn in Fig. 6.5, will be totally ordered.

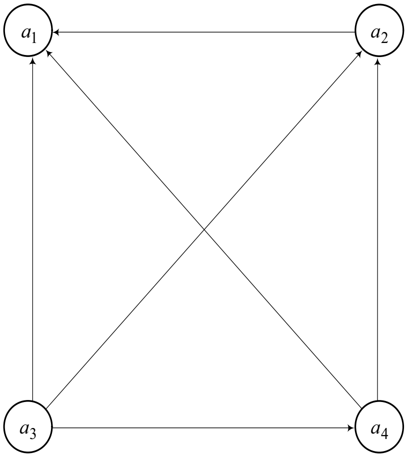

Figure 6.5: The set $R_{0.67}$ as the directed graph

Finally, we have determined the set of pairs that form the total order in the associated graph. By counting the number of arrow endpoints, tended towards each vertex, we should state the group decision concerning a hierarchy of drugs as $a_1 \succ a_2 \succ a_4 \succ a_3$.

The models discussed in Sections 6.2 and 6.3 should be simple for an eventual user since they do not require deep knowledge in making calculations.

In the last models the physicians' data reports have been tested very thoroughly because of introducing the finite fuzzy sets of effectiveness. We want to emphasize that even one value representing the effectiveness – we have numerically stated presence in the diagnostic model in such a way – should yield satisfactory results. However, the occurrence of expressing the effectiveness terms as fuzzy sets, eliminates a risk of casual results and should be admitted by users as a safe decision step. The decisions obtained lately seem to be very reliable in spite of different interactions between the drugs and their influence on the symptoms. The hierarchy of drugs is even debatable when involving many decision-makers in the process of their evaluation. Physicians often share different opinions concerning medicine priorities, but it is still mathematically possible to sum up all conclusions as a final selection of the most efficacious remedy. The obtained results have also been confirmed by experienced pharmacologists.

6.4 Unequal Objectives in the Choice of Medicines

The purpose of this section is to present some ideas on the applications of fuzzy sets to multi-objective decision making, with particular emphasis on a means of including differing degrees of importance to different objectives. Different approaches to aggregation of weighted decision criteria have constituted a subject of lively discussions during last decades [5, 13, 21, 57, 80, 83, 85].

The primary reasons for the usefulness of fuzzy sets in handling multi-objectives are: ability to represent objectives, convenient forms for combining objectives and means of including differing degrees of importance to the objectives [81].

6.4.1 The Design of Objectives-Constraints

We still consider a decision model in which *n* drugs $a_1, ..., a_n \in A$ act as decisions. These affect *m* symptoms $x_1, ..., x_m \in X$ that are typical of a morbid unit under consideration. The drugs-decisions constitute *n* elements in supports of fuzzy sets K_t, $t = 1, \ldots, m, m+1, m+2$, determined as some criteria-objectives restricting the set *A*. Thus, we can recognize each set K_t as a fuzzy subset of *A*, i.e., $\mu_{K_t} : A \to [0,1]$, $t = 1, ..., m+2$. In the model of accepting the most optimal medicine A_i, $i = 1, \ldots, n$, we assume that some of restriction sets K_j, $j = 1, \ldots, m$, are defined by

$$K_j = \text{"influence of } a_1, ..., a_n \text{ on symptom } x_j\text{"} = a_1\text{'s } \textit{effectiveness} \text{ regarding } x_j \Big/ a_1 + \cdots + a_n\text{'s } \textit{effectiveness} \text{ regarding } x_j \Big/ a_n . \tag{6.21}$$

In spite of drug effectiveness, which definitely is the most important factor in the appreciation of drug action, we can introduce other important factors assisting drug decision-making like side effects of medicines or their prices. We thus form the following fuzzy sets

$$K_{m+1} = \text{"estimation of } \textit{side effects} \text{ of } a_1, ..., a_n \text{ supporting the decision positively"} = 1 - \textit{side effects} \text{ of } a_1 \Big/ a_1 + \cdots + 1 - \textit{side effects} \text{ of } a_n \Big/ a_n \tag{6.22}$$

and

$$K_{m+2} = \text{"estimation of } \textit{price availability} \text{ for } a_1, ..., a_n\text{"} = \textit{price availability} \text{ of } a_1 \Big/ a_1 + \cdots + \textit{price availability} \text{ of } a_n \Big/ a_n \tag{6.23}$$

in order to enlarge a number of decisive indications.

Example 6.7

We return to the clinical data from Ex. 6.4, which concerns D = "*coronary heart disease*". We still consider the symptoms x_1 = "*pain in chest*", x_2 = "*changes in ECG*" and x_3 = "*increased level of LDL-cholesterol*". Even the medicines are unchanged and we list them as a_1 = *nitroglycerin*, a_2 = *beta-adrenergic blockade*, a_3 = *acetylsalicylic acid* (*aspirin*) and a_4 = *statine LDL-reductor*.

The procedure of stating effectiveness has been based on fuzzy sets in Ex. 6.2. Nevertheless, even if the mathematical presentation of each effectiveness as a distinct fuzzy set has been very efficient and thorough, we probably do not need such accuracy in determining the sets K_j, $j = 1, \ldots, m$, because of concentration on their importance instead. We assign only one value to every effectiveness term that is an approved procedure as shown in Chapter 3. To decide adequate representatives $z \in [0, 100]$ of the effectiveness descriptions from Ex. 6.2, we take, when we return to (6.5), $z = \alpha_k$ for $k = 1, 2, 3, 4, 5$, and $z = \beta_k$ for $k = 7, 8, 9, 10, 11$, respectively $z = \gamma$ for $k = 6$ due to (6.6). On the basis of Ex. 6.1, we select z-values, which stand for the exponents of the following expressions: $z_{\text{"none"}} = 0$, $z_{\text{"almost none"}} = 10$, $z_{\text{"very little"}} = 20$, $z_{\text{"little"}} = 30$, $z_{\text{"rather little"}} = 40$, $z_{\text{"medium"}} = 50$, $z_{\text{"rather large"}} = 60$, $z_{\text{"large"}} = 70$, $z_{\text{"very large"}} = 80$, $z_{\text{"almost complete"}} = 90$, $z_{\text{"complete"}} = 100$. If we fit a membership function $\mu_{\text{"effectiveness"}}(z) = L(z, 0, 100)$ over [0, 100] as recommended by (6.3), we will obtain the final membership values $\mu(z)$ for z sorted above. These replace the terms of effectiveness according to the pattern shown in Table 6.1.

Table 6.1: The representatives of linearly modeled effectiveness terms

Effectiveness	Representing z-value	$\mu(z)$
"*none*"	0	0
"*almost none*"	10	0.1
"*very little*"	20	0.2
"*little*"	30	0.3
"*rather little*"	40	0.4
"*medium*"	50	0.5
"*rather large*"	60	0.6
"*large*"	70	0.7
"*very large*"	80	0.8
"*almost complete*"	90	0.9
"*complete*"	100	1

We reconstruct the sets K_j, $j = 1, 2, 3$, due to (6.21), by applying the columns from the matrix U introduced by Ex. 6.4. Hence

$$K_1 = \text{"influence of } a_1, a_2, a_3, a_4 \text{ on } x_1\text{"} =$$

$$complete\big/a_1 + medium\big/a_2 + little\big/a_3 + little\big/a_4 =$$

$$1\big/a_1 + 0.5\big/a_2 + 0.3\big/a_3 + 0.3\big/a_4 ,$$

$$K_2 = \text{"influence of } a_1, a_2, a_3, a_4 \text{ on } x_2\text{"} = \\ {very\ large}/a_1 + {medium}/a_2 + {little}/a_3 + {little}/a_4 = \\ 0.8/a_1 + 0.5/a_2 + 0.3/a_3 + 0.3/a_4$$

and

$$K_3 = \text{"influence of } a_1, a_2, a_3, a_4 \text{ on } x_3\text{"} = \\ {almost\ none}/a_1 + {little}/a_2 + {very\ little}/a_3 + {very\ large}/a_4 = \\ 0.1/a_1 + 0.3/a_2 + 0.2/a_3 + 0.8/a_4 \ .$$

The physician has estimated side effects of the drugs in the set K_4 by assimilating the words from the first columns of Table 6.4. The side effects of a_i, $i = 1, \ldots, n$, are rather unfavorable occurrences; therefore their lack in a_i, e.g., "*side effects* of a_i" = "*almost none*", should be emphasized by the larger membership value assigned to a_i as an indication of safe medicine consumption. For the purpose of enlarging membership values of these medicines that do not have side effects, we use the complement operation 1 – estimation of *side effects*. Set K_4 is established in accordance with (6.22) as

$$K_4 = \text{"estimation of } \textit{side effects} \text{ of } a_1, a_2, a_3, a_4 \text{ supporting the decision positively"} = \\ 1 - {very\ little}/a_1 + 1 - {little}/a_2 + 1 - {rather\ large}/a_3 + 1 - {very\ little}/a_4 = \\ 1 - 0.2/a_1 + 1 - 0.3/a_2 + 1 - 0.6/a_3 + 1 - 0.2/a_4 = 0.8/a_1 + 0.7/a_2 + 0.4/a_3 + 0.8/a_4 \ .$$

The prices of all medicines are not in the least inconvenient for patients to purchase them. Thus, if we note that the large value of a membership degree corresponds to a rather cheap and available medicine we can state the set K_5 by adopting (6.23) as

$$K_5 = \text{"estimation of } \textit{price availability} \text{ for } a_1, a_2, a_3, a_4\text{"} = \\ 0.8/a_1 + 0.8/a_2 + 0.9/a_3 + 0.8/a_4 \ .$$

After preparing the criteria-objectives we are ready to make a fuzzy decision, which is affected by all of them.

The fuzzy decision D, which takes into account K_1 and K_2 and … and K_{m+2} is made in accordance with the minimum decision rule [9, 40]

$$D = K_1 \cap K_2 \cap \cdots \cap K_m \cap \cdots \cap K_{m+2}\,. \tag{6.24}$$

This provides us with the membership function

$$\mu_D(a_i) = \min(\mu_{K_1}(a_i), \mu_{K_2}(a_i), ..., \mu_{K_m}(a_i), ..., \mu_{K_{m+2}}(a_i)) \tag{6.25}$$

for each $a_i \in A$.

The optimal drug-decision is accepted as this a_i, $i = 1, \ldots, n$, which has the maximal value of the membership degree in D as defined by (6.16).

6.4.2 The Power-Importance of Objectives

If we can associate with each fuzzy objective K_t, $t = 1, \ldots, m, m+1, m+2$, a non negative number that indicates its power or importance in the decision according to the rule: the higher the number the more important criterion K_t, then we could raise each fuzzy criterion set to this power before combining to form D. We regard $w_1, w_2, \ldots, w_m, \ldots, w_{m+2}$ as powers-weights of $K_1, K_2, \ldots, K_m, \ldots, K_{m+2}$ to modify (6.24) as a richer and more extended decision

$$D = K_1^{w_1} \cap K_2^{w_2} \cap \cdots \cap K_m^{w_m} \cap \cdots \cap K_{m+2}^{w_{m+2}} \tag{6.26}$$

in which the membership degree of each $a_i \in A$ is determined as

$$\mu_D(a_i) = \min((\mu_{K_1}(a_i))^{w_1}, (\mu_{K_2}(a_i))^{w_2}, ..., (\mu_{K_m}(a_i))^{w_m}, ..., (\mu_{K_{m+2}}(a_i))^{w_{m+2}}). \tag{6.27}$$

We note that each K_t always takes the values of membership degrees from [0, 1]. If w_t gets bigger then $(\mu_{K_t}(a_i))^{w_t}$, $t = 1, \ldots, m, \ldots, m + 2$, $i = 1, \ldots, n$, will get smaller, closer to zero. On the contrary, $w_t \rightarrow 0$ implies $(\mu_{K_t}(a_i))^{w_t} \rightarrow 1$. This behaviour of K_t's membership degrees emphasizes that the choice of the minimum operation in (6.27) is proper. The membership grade in all objectives having little importance ($w_t < 1$) becomes larger, and while those in objectives having more importance ($w_t > 1$) become smaller. Since we use the minimum operation to the membership degrees in the decision set D, we will exclude the larger values that are rather unimportant. This has the effect of making the membership function of the decision set D as useful in the decision making process as possible when taking care of all decisive factors.

A procedure for obtaining a ratio scale of importance for a group of $m + 2$ elements (like in the drug-decision model) was developed by Saaty [68].

Assume that we have $m + 2$ objectives and we want to construct a scale, rating these objectives as to their importance with respect to the decision. We ask a decision-maker to compare the objectives in paired comparison. If we are comparing objective t with objective l, we assign the values b_{tl} and b_{lt} as follows

(1) $b_{lt} = \dfrac{1}{b_{tl}}$.

(2) If objective t is more important than objective l then b_{tl} gets assigned a number according to the following scheme:

Intensity of importance expressed by the value of b_{tl}	*Definition*
1	Equal importance of K_t and K_l
3	Weak importance of K_t over K_l
5	Strong importance of K_t over K_l
7	Demonstrated importance of K_t over K_l
9	Absolute importance of K_t over K_l
2, 4, 6, 8	Intermediate values

If objective l is more important than objective t, we assign the value of b_{lt}.

Having obtained the above judgments an $(m + 2) \times (m + 2)$ importance matrix B is constructed in the drug decision problem sketched above.

Example 6.8

By involving the computation technique suggested in the description of matrix B we try to find the weights for objectives K_t, $t = 1, \ldots, 5$, already stated in Ex. 6.7.

The physical status of a patient is subjectively better if the pain disappears that means that a physician tries to release the patient from symptom x_1 = "*pain in chest*". The next priority is assigned to x_2 = "*changes in ECG*" and finally, we concentrate our attention on getting rid of x_3 = "*increased level of LDL-cholesterol*". The last symptom does not disappear very quickly and the patient must be treated for some time to be free from it.

These remarks are helpful when constructing a content of the matrix B as

$$B = \begin{array}{c} \\ K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \end{array} \begin{array}{c} \begin{array}{ccccc} K_1 & K_2 & K_3 & K_4 & K_5 \end{array} \\ \begin{bmatrix} 1 & 3 & 5 & 7 & 7 \\ \frac{1}{3} & 1 & 3 & 7 & 7 \\ \frac{1}{5} & \frac{1}{3} & 1 & 7 & 7 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 1 & 3 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{3} & 1 \end{bmatrix} \end{array}.$$

Matrix B constitutes a crucial part in the procedure of determining the degrees of importance $w_1, \ldots, w_m, \ldots, w_{m+2}$ that affect the decision set D in a substantial way (in accord with (6.27)). The weights are decided as components of this eigen vector which corresponds to the largest in magnitude eigen value of the matrix B.

Definition 6.2

The value of λ and the vector V are called "the eigen value of the matrix B" respectively "the eigen vector of the matrix B" if they satisfy the equation

$$BV = \lambda V . \tag{6.28}$$

B has type $(m + 2) \times (m + 2)$. We can find $m + 2$ eigen values of B by solving a characteristic equation

$$\det(B - \lambda I) = 0 \tag{6.29}$$

where I is a unit matrix of the same type $(m + 2) \times (m + 2)$.

Among $m + 2$ roots of (6.29) there exists the largest one. By returning to Eq. (6.28) we determine the coordinates of a corresponding eigen vector V. These constitute weights of the objectives taking place in the decision set D.

Example 6.9

Equation (6.29) results in a determinant equation

$$\det\begin{bmatrix} 1-\lambda & 3 & 5 & 7 & 7 \\ \frac{1}{3} & 1-\lambda & 3 & 7 & 7 \\ \frac{1}{5} & \frac{1}{3} & 1-\lambda & 7 & 7 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 1-\lambda & 3 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{3} & 1-\lambda \end{bmatrix} = 135\lambda^5 - 675\lambda^4 - 2172\lambda^2 - 1440\lambda - 320 = 0$$

which has only one real root $\lambda = 5.5805$. The associated eigen vector $V = (0.83215, 0.46393, 0.26609, 0.08575, 0.055586)$ is composed of components that are interpreted as the weights sought for K_t, $t = 1, \ldots, 5$.

The sets K_t, $t = 1, \ldots, 5$, already found in Ex. 6.7, are now completed by introducing their grades of importance.

Thus,

$$K_1 = 1^{0.83215}\big/ a_1 + 0.5^{0.83215}\big/ a_2 + 0.3^{0.83215}\big/ a_3 + 0.3^{0.83215}\big/ a_4 ,$$

$$K_2 = 0.8^{0.46393}\big/ a_1 + 0.5^{0.46393}\big/ a_2 + 0.3^{0.46393}\big/ a_3 + 0.3^{0.46393}\big/ a_4 ,$$

$$K_3 = 0.1^{0.26609}/a_1 + 0.3^{0.26609}/a_2 + 0.2^{0.26609}/a_3 + 0.8^{0.26609}/a_4 \;,$$

$$K_4 = 0.8^{0.08575}/a_1 + 0.7^{0.08575}/a_2 + 0.4^{0.08575}/a_3 + 0.8^{0.08575}/a_4$$

and

$$K_5 = 0.8^{0.055586}/a_1 + 0.8^{0.055586}/a_2 + 0.9^{0.055586}/a_3 + 0.8^{0.055586}/a_4 \;.$$

The final decision D is obtained as a fuzzy set due to the recommended Eqs (6.26) and (6.27)

$$\begin{aligned} D = {} & \min(1^{0.83215}, 0.8^{0.46393}, 0.1^{0.26609}, 0.8^{0.08575}, 0.8^{0.055586})/a_1 + \\ & \min(0.5^{0.83215}, 0.5^{0.46393}, 0.3^{0.26609}, 0.7^{0.08575}, 0.8^{0.055586})/a_2 + \\ & \min(0.3^{0.83215}, 0.3^{0.46393}, 0.2^{0.26609}, 0.4^{0.08575}, 0.9^{0.055586})/a_3 + \\ & \min(0.3^{0.83215}, 0.3^{0.46393}, 0.8^{0.26609}, 0.8^{0.08575}, 0.8^{0.055586})/a_4 = \\ & \min(1, 0.902, 0.542, 0.981, 0.987)/a_1 + \min(0.562, 0.725, 0.726, 0.969, 0.987)/a_2 + \\ & \min(0.367, 0.572, 0.652, 0.924, 0.994)/a_3 + \\ & \min(0.367, 0.572, 0.942, 0.981, 0.988)/a_4 = \\ & 0.5418/a_1 + 0.5616/a_2 + 0.3671/a_3 + 0.3671/a_4 \;. \end{aligned}$$

We conclude that the curative power of considered medicines is ranked in the order $a_2 \approx a_1 \succ a_4 = a_3$. We have not only considered the effectiveness of drugs regarding their action on symptoms, but also the priority of symptoms. The importance order among the symptoms points out that the ones that should disappear first, for the reason of their harm, mostly influence the patient's mental and psychical condition.

6.4.3 Minimization of Regret

The action of the minimum operation in the final decision formula has provided us with a very cautious prognosis referring to the drug hierarchy. Some high values of degrees that reflect a positive effect of medicine, impact on considered symptoms and have no chance of influencing finally computed decision values. We can even say that the minimum operation acts as a filter for high values by depriving them of their power.

We try to obtain clearer results by applying another fuzzy decision-making technique known as a minimization of regret [84]. Let us prepare a new medical apparatus by reorganizing the sets previously introduced. We preserve a decision space (a space of alternatives) $A=\{a_1, ..., a_n\}$ but we complement a space of states as $X=\{x_1, x_2, ..., x_m, x_{m+1}, x_{m+2}\}$. In X there are symbols possessing the following meanings: x_1 – the 1st symptom, …, x_m – the m^{th} symptom, x_{m+1} – medicine side effects, x_{m+2} – medicine price availability. We form a basic payoff matrix

$$C = \begin{array}{c} \\ a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{array} \begin{array}{c} x_1 \quad \dots \quad x_t \quad \dots \quad x_{m+2} \\ \left[\begin{array}{ccc} & & \\ & & \\ & c_{it} & \\ & & \\ & & \end{array} \right] \end{array}, \tag{6.30}$$

where c_{it} is the payoff to a decision-maker if he connects a_i to x_t, $i = 1, …, n$, $t = 1, …, m+2$.

In a continuation of the proposed approach to the choice of an optimal medicine, we first obtain the regret matrix R. Its components r_{it} indicate the decision-maker's regret in selecting alternative a_i when the state of X is x_t. We then calculate the maximal regret for each alternative.

A procedure of selecting an optimal a_i should follow some steps listed below:

1. For each x_t calculate $C_t = \max_{1 \le i \le n} c_{it}$.
2. For each pair a_i and x_t calculate $r_{it} = C_t - c_{it}$.
3. Suppose that matrix B from Subsection 6.4.2 consists of b_{tl}, which now describe the importance scale when comparing states x_t and x_l, $t, l = 1, …, m+2$. The coordinates of this eigen vector that assists the largest in magnitude eigen value of B still constitute weights $w_1, …, w_{m+2}$ assigned to states $x_1, …, x_{m+2}$ stated in X. The weights are involved in the computations of estimates $RT_i = w_1 r_{i1} + \dots + w_{m+2} r_{i,m+2}$ for each a_i. It can be proved that the formulas derived for calculations of RT_i satisfy the conditions of OWA operators [82, 86].
4. Select a_{i^*}, such that $RT_{i^*} = \min_{1 \le i \le n} RT_i$.

The values r_{it} constitute the entries of the matrix R called the regret matrix. We shall refer to C_t as the horizon under x_t.

Example 6.10

The sets K_1–K_5 found in Ex. 6.7 are now utilized as columns of the matrix C, determined by a table

$$C = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1^* & 0.8^* & 0.1 & 0.8^* & 0.8 \\ 0.5 & 0.5 & 0.3 & 0.7 & 0.8 \\ 0.3 & 0.3 & 0.2 & 0.4 & 0.9^* \\ 0.3 & 0.3 & 0.8^* & 0.8^* & 0.8 \end{bmatrix} \end{array}$$

in which "*" points to the largest element in each column due to Step 1.

The regret matrix R is computed as the next table

$$R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.7 & 0 & 0.1 \\ 0.5 & 0.3 & 0.5 & 0.1 & 0.1 \\ 0.7 & 0.5 & 0.6 & 0.4 & 0 \\ 0.7 & 0.5 & 0 & 0 & 0.1 \end{bmatrix} \end{array}.$$

For $w_1 \approx 0.83$, $w_2 \approx 0.46$, $w_3 \approx 0.27$, $w_4 \approx 0.09$, $w_5 \approx 0.05$ (Ex. 6.9) the values of RT_i, $i = 1, \ldots, 4$, are appreciated as

$RT_1 = 0.83 \cdot 0 + 0.46 \cdot 0 + 0.27 \cdot 0.7 + 0.09 \cdot 0 + 0.05 \cdot 0.1 = 0.194$,
$RT_2 = 0.702$, $RT_3 = 1.009$, $RT_4 = 0.816$.

Finally, we decide the hierarchical order of drugs with respect to their curative abilities. We state them in sequence $a_1 \succ a_2 \succ a_4 \succ a_3$ that totally confirms the results obtained by the technique of unequal objectives. Moreover, we notice that the last decision is very clearly interpretable and easy to understand without special doubts. This emphasizes an advantage of applying the OWA weighted operations that prevent a loss of substantial information. The OWA operations have resulted in the simultaneous engagement of all effectiveness quantities in mean decision-making values involved in the regret model.

In Section 6.4 we have adapted Yager's theoretical fuzzy decision models in the process of extracting the best medicine from the collection of proposed remedies. The basis of the investigations has been mostly restricted to a judgment of medicinal influence on clinical symptoms that accompany the disease. By employing the factors of importance associated with decisive objectives we could strengthen their crucial power as well.

We have shown some useful fuzzy decision making models in the process of selecting the most efficacious medicine. The decision patterns should be particularly helpful in doubtful cases when we observe unequal, curative abilities of different medicines in the case of the same symptoms, or, when some specialists who make a trial of prioritizing the medicines have shared opinions in their judgments.

7 Approximation of Clock-like Point Sets

7.1 Introduction

This chapter has a theoretical character and can be studied by some medical staff researchers that seek methods of approximation of very irregular point sets. When the shape of an obtained polygon based on the point set is similar to a chain of bells, then it will be difficult to find a continuous standard curve that should approximate the polygon without making a large approximation error. The studies of some medical data give rise to the creation of polygons consisting of finite numbers of points tied together. Since the polygons are not formalized by some mathematical expressions, we suggest creating continuous functions that approximate them thoroughly in spite of their irregular shapes. To warrant a high accuracy of approximation, otherwise impossible to obtain when using standard curves, we test a continuous function composed of joined truncated π-functions or joined truncated s-functions.

By operating with the functions representing polygons that have unusual shapes, we attempt a classification of medical data. We adopt rough sets to assign the members to an investigated medical class even if their origin sometimes is unknown.

Since we do not possess medical data that comes from solidly accomplished investigations, we will discuss the matter of approximation and classification theoretically. Nevertheless, we hope that some scientists can find patterns of points in their research work that resemble the shapes assumed below. In this way they can find the proposed approximation method useful in possible research investigations of medical results.

7.2 Fitting of π-functions to Clock-like Polygons

Some examinations of the behaviour of the two variables named X and Y, provide us with strings of values x and y, which can be included in the pairs (x, y), and treated further as the coordinates of points in the two-dimensional system. We suppose that the finite set A consists of the points (x, y), thus it can be illustrated as a polygon with its points joined together by segments of straight lines.

Certain experiments, in which $y \in [0, 1]$, deliver the polygon (set A) composed of parts looking like bells (or hills), e.g., like A, sketched in Fig. 7.1. The polygon,

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which ties a lot of straight-line bits, cannot constitute a piecewise interpolation of the points. There can be too many first-degree equations to make the further analysis of set A efficient, and moreover, the linear interpolation is not smooth enough.

The most popular classical method of approximating applied to a set of points is known as the least-square regression with modern variants [15]. Other algorithms of approximating that we can mention, adopt such technical tools as cubic polynomials based on four points [43], tangent curves [1], free algebras [35] or weighted approximations [71].

As the counterpart of the listed procedures, we consider an approximation of multi-shapes from Fig. 7.1 by π-truncated functions used piecewise [65, 66]. The y-values of π functions and the y-coordinates of the points constituting the elements of A belong to the interval [0, 1]. The procedure forms the approximation of A by truncated π-functions, and tied by pieces of straight lines if needed.

Example 7.1

In experimental domains of science like some medical investigations, we encounter polygons as results of accomplished observations in which the variable Y is dependent on the variable X. We observe the behaviour of two variables X and Y, to determine a finite set of pairs $A = \{(x, y)\}$, $x \in [0, 50]$, $y \in [0, 1]$. Suppose that an experiment delivers set A resembling the polygon from Fig. 7.1

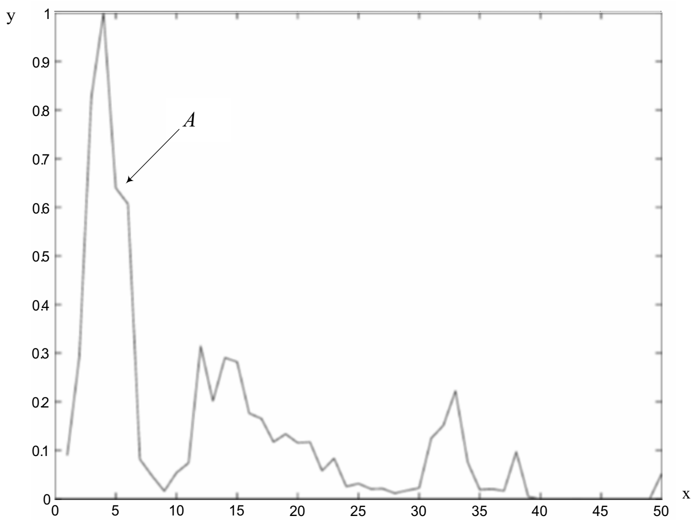

Figure 7.1: The polygon reflecting $A = \{(x, y)\}$

.

We will introduce "the sampled truncated π" as a split curve that approximates the polygon 7.1. This will consist of first and second degree-polynomials. The curve should follow the polygon's shape very closely to cumulate a very low error referring to deviations between the approximating curve and the polygon. We assume that a continuous function, which provides us with y-values corresponding to regularly chosen x (not always appearing in the set of points), is more useful in the further analysis of polygons, e.g., their comparison.

We now intend to explain how to adjust a π-function to the shape of a polygon. Let us first suppose that one part A_1 of the obtained polygon A, whose shape resembles a bell, is determined by a set of pairs (x, y) that represent the finite set of pairs $A_1 \subseteq A$.

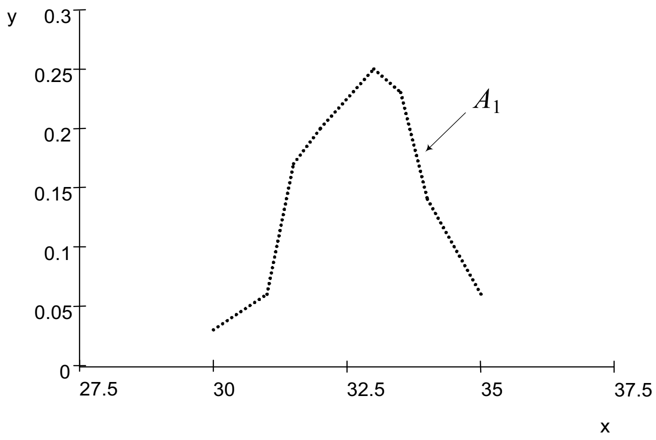

Figure 7.2: The polygon representing A_1

Example 7.2

We examine the values of pairs included in set A_1, which constitutes a part of A presented by Fig. 7.1, over the interval (30, 35). We find that A_1 is determined by A_1 = {(30, 0.03), (31, 0.06), (31.5, 0.17), (32, 0.20), (33, 0.25), (33.5, 0.23), (34, 0.14), (35, 0.06)}. The points corresponding to the pairs given above are tied together to build the polygon A_1 drawn in Fig. 7.2.

To an approximation of the pattern of points from Fig. 7.2, the π-function, already inserted by (2.6) best fits because of its clock-like shape and its range constituting the interval [0, 1].

We quote the formula of π in the fully developed form as [65, 66]

$$y=\begin{cases}(1)\ 0 & \text{for} \quad x<\alpha_1,\\ (2)\ 2\varepsilon\left(\dfrac{x-\alpha_1}{\gamma_1-\alpha_1}\right)^2 & \text{for} \quad \alpha_1\le x<\beta_1,\\ (3)\ \varepsilon\left(1-2\left(\dfrac{x-\gamma_1}{\gamma_1-\alpha_1}\right)^2\right) & \text{for} \quad \beta_1\le x<\gamma_1,\\ (4)\ \varepsilon & \text{for} \quad x=\gamma_1=\alpha_2,\\ (5)\ \varepsilon\left(1-2\left(\dfrac{x-\alpha_2}{\gamma_2-\alpha_2}\right)^2\right) & \text{for} \quad \alpha_2<x<\beta_2,\\ (6)\ 2\varepsilon\left(\dfrac{x-\gamma_2}{\gamma_2-\alpha_2}\right)^2 & \text{for} \quad \beta_2\le x\le\gamma_2,\\ (7)\ 0 & \text{for} \quad x>\gamma_2.\end{cases} \tag{7.1}$$

The function possesses six standard parameters α_1, β_1, γ_1, α_2, β_2, γ_2, and it has the additional parameter ε, added in (7.1), which accommodates the height of the function to the real data existing in set A_1. The parameters β_1 and β_2 are estimated by

$$\beta_1=\frac{\alpha_1+\gamma_1}{2},\ \beta_2=\frac{\alpha_2+\gamma_2}{2}. \tag{7.2}$$

Example 7.3

Once again we intend to recall what the π-function given by (7.1) and (7.2) looks like. If we suppose that, e.g., $\alpha_1 = 30$, $\gamma_1 = \alpha_2 = 32.5$, $\gamma_2 = 35$, and $\varepsilon = 0.25$ then $\beta_1=\dfrac{30+32.5}{2}=31.25$, $\beta_2=\dfrac{32.5+35}{2}=33.75$ and the function will have the graph depicted in Fig. 7.3.

The pairs in set A_1 from Ex. 7.2 have no y-coordinates equal to zero and that means that the values of α_1, and γ_2 in the π-function, which is expected to approximate A_1, are unknown. By accepting the value of ε as the largest y-coordinate in A_1, corresponding to the x-coordinate taken as $\gamma_1 = \alpha_2$, we reconstruct the values of remaining parameters α_1, γ_2 according to the following patterns:

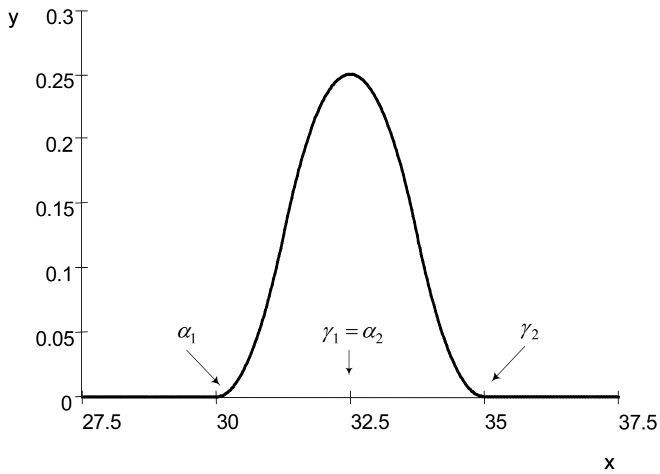

Figure 7.3: The π-function for $\alpha_1 = 30$, $\gamma_1 = \alpha_2 = 32.5$, $\gamma_2 = 35$ and $\varepsilon = 0.25$

Case of α_1

Denote by $A_1(X)$ all x-values that belong to n points from A_1. If the pair $(x_{\min}, y(x_{\min}))$ begins set A_1, which means that $x_{\min} = \min_{1 \le k \le n} x_k, x_k \in A_1(X)$ and $y(x_{\min})$ is the corresponding y-value to $x_{\min}$, then

a) $\alpha_1 = \dfrac{x_{\min} - \gamma_1 \sqrt{\frac{y(x_{\min})}{2\varepsilon}}}{1 - \sqrt{\frac{y(x_{\min})}{2\varepsilon}}}$ for $y(x_{\min}) < \dfrac{\varepsilon}{2}$. The value of α_1 is computed from the equality $2\varepsilon \left(\dfrac{x_{\min} - \alpha_1}{\gamma_1 - \alpha_1} \right)^2 = y(x_{\min})$. This case entails the changes in (7.1) in accordance with

$$y = \begin{cases} (1) & 0 & \text{for} \quad x < x_{\min}, \\ (2) & 2\varepsilon \left(\dfrac{x - \alpha_1}{\gamma_1 - \alpha_1} \right)^2 & \text{for} \quad x_{\min} \le x < \beta_1, \\ (3)-(7) & \text{without changes.} & \end{cases} \tag{7.3}$$

b) $\alpha_1 = \gamma_1 - \dfrac{\gamma_1 - x_{\min}}{\sqrt{\frac{\varepsilon - y(x_{\min})}{2\varepsilon}}}$ for $y(x_{\min}) \ge \dfrac{\varepsilon}{2}$. The result is obtained from a connection

$\varepsilon\left(1-2\left(\frac{x_{\min}-\gamma_1}{\gamma_1-\alpha_1}\right)^2\right)=y(x_{\min})$. Then the $\pi(x)$ formula appears as

$$y=\begin{cases}(1)-(2) & 0 & \text{for} & x<x_{\min}, \\ (3) & \varepsilon\left(1-2\left(\frac{x-\gamma_1}{\gamma_1-\alpha_1}\right)^2\right) & \text{for} & x_{\min}\le x<\gamma_1, \\ (4)-(7) & \text{without changes.} & & \end{cases} \tag{7.4}$$

Case of γ_2

The pair $(x_{\max}, y(x_{\max}))$ is the last pair in set A_1, which is associated with the formula $x_{\max}=\max\limits_{1\le k\le n} x_k, x_k\in A_1(X)$. Hence

c) $\gamma_2=\dfrac{x_{\max}-\alpha_2\sqrt{\frac{y(x_{\max})}{2\varepsilon}}}{1-\sqrt{\frac{y(x_{\max})}{2\varepsilon}}}$ is evaluated from part $2\varepsilon\left(\frac{x_{\max}-\gamma_2}{\gamma_2-\alpha_2}\right)^2=y(x_{\max})$ for $y(x_{\max})<\frac{\varepsilon}{2}$. We thus suggest the following changes in (7.1) to adapt it to the new assumptions

$$y=\begin{cases}(1)-(5) & \text{without changes} & & \\ (6) & 2\varepsilon\left(\frac{x-\gamma_2}{\gamma_2-\alpha_2}\right)^2 & \text{for} & \beta_2\le x<x_{\max}, \\ (7) & 0 & \text{for} & x\ge x_{\max}.\end{cases} \tag{7.5}$$

d) $\gamma_2=\alpha_2+\dfrac{x_{\max}-\alpha_2}{\sqrt{\frac{\varepsilon-y(x_{\max})}{2\varepsilon}}}$ for $y(x_{\max})\ge\frac{\varepsilon}{2}$ when taking into consideration the association $\varepsilon\left(1-2\left(\frac{x_{\max}-\alpha_2}{\gamma_2-\alpha_2}\right)^2\right)=y(x_{\max})$. We adjust the $\pi(x)$ formula as

$$y=\begin{cases}(1)-(4) & \text{without changes} & & \\ (5) & \varepsilon\left(1-2\left(\frac{x-\alpha_2}{\gamma_2-\alpha_2}\right)^2\right) & \text{for} & \alpha_2\le x<x_{\max}, \\ (6)-(7) & 0 & \text{for} & x\ge x_{\max}.\end{cases} \tag{7.6}$$

The modified π constitutes a segment of the classical π-function, therefore we will name it a truncated π-function.

We select the minimal and the maximal x-values as well as the maximal y-value existing in set A_1 by examining the slope of A_1. Equations (7.3)–(7.6) are applied to computations of unknown parameters α_1 and γ_2. The point in which the y-coordinate takes the ε-value and the x-coordinate – the $\alpha_2 = \gamma_1$ value, belongs both to the polygon and the function π. In spite of reconstructing the values of α_1 and γ_2, the approximating function is not intersected by the x-axis. The domain of π begins with $A_1(X)$'s minimal x-value and is ended by the maximal x value in $A_1(X)$. This warrants that the polygon and the curve lie very close to each other.

Example 7.4

The adjustments, accomplished for the data describing A_1 from Ex. 7.2, should be made by applying both (7.3) and (7.5). We determine $x_{min} = 30$, $x_{max} = 35$ and $\varepsilon = 0.25$. The x-coordinate associated with the largest y-value in A_1 accepted as ε is equal to $\gamma_1 = \alpha_2 = 33$. Since $y(x_{\min}) = 0.03$ satisfies the condition $y(x_{\min}) < \frac{\varepsilon}{2}$, then we will compute the lacking value of the function parameter α_1 as

$$\alpha_1 = \frac{x_{\min} - \gamma_1\sqrt{\frac{y(x_{\min})}{2\varepsilon}}}{1-\sqrt{\frac{y(x_{\min})}{2\varepsilon}}} = \frac{30-33\sqrt{\frac{0.03}{2\cdot 0.25}}}{1-\sqrt{\frac{0.03}{2\cdot 0.25}}} = 29.0278\,.$$

The value of $y(x_{max}) = 0.06$ fulfils $y(x_{\max}) < \frac{\varepsilon}{2}$ and we will generate

$$\gamma_2 = \frac{x_{\max} - \alpha_2\sqrt{\frac{y(x_{\max})}{2\varepsilon}}}{1-\sqrt{\frac{y(x_{\max})}{2\varepsilon}}} = \frac{35-33\sqrt{\frac{0.06}{2\cdot 0.25}}}{1-\sqrt{\frac{0.06}{2\cdot 0.25}}} = 36.0918\,.$$

With $\beta_1 = 31.0139$ and $\beta_2 = 34.5459$ as the complementary parameters of the truncated π_1-function accommodated to the set A_1, now has a full expansion as

$$y = \begin{cases} 0.25\cdot 2\left(\dfrac{x-29.0278}{33-29.0278}\right)^2 & \text{for} \quad 30 \le x < 31.0139, \\ 0.25\left(1-2\left(\dfrac{x-33}{33-29.0278}\right)^2\right) & \text{for} \quad 31.0139 \le x < 33, \\ 0.25 & \text{for} \quad x = 33, \\ 0.25\left(1-2\left(\dfrac{x-33}{36.0918-33}\right)^2\right) & \text{for} \quad 33 < x < 34.5459, \\ 0.25\cdot 2\left(\dfrac{x-36.0918}{36.0918-33}\right)^2 & \text{for} \quad 34.5459 \le x \le 35. \end{cases}$$

Figure 7.4 shows the total effects of evaluating the finite point set A_1 by a continuous function $\pi_1(x)$ possessing the reconstructed parameters $\alpha_1 = 29.0278$, $\gamma_2 = 36.0918$ and $\varepsilon = 0.25$.

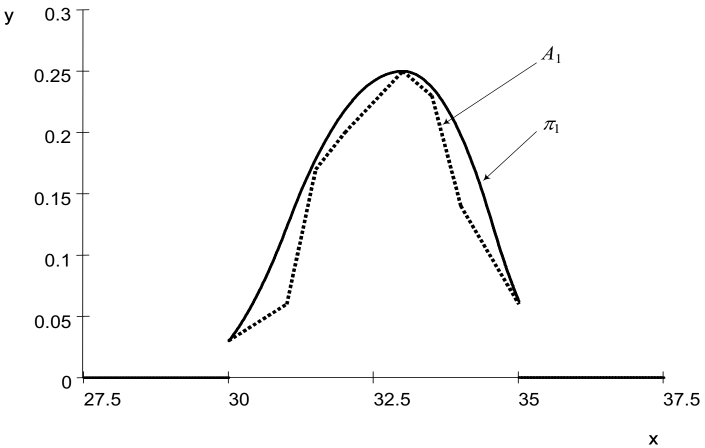

Figure 7.4: The approximation of A_1 by the truncated π-function

To get the collection of split definitions covering the total x-interval of A, we divide A's x-interval in subintervals according to the magnitude of y-values. If the points included in A create structures that resemble bells, then the y-coordinates of the points should be arranged in ascending order until they reach the maximal value. Afterwards the y-values ought to be placed in a sequence that is characterized by their descending order. If the order is ascending again, we should design a new interval for another truncated π function. A straight line will tie the borders of two adjacent π curves.

Example 7.5

Let us add the next set of points to A_1, known from Ex. 7.2, over the interval [35, 39] to introduce $A^* = \{(30, 0.03), (31, 0.06), (31.5, 0.17), (32, 0.20), (33, 0.25), (33.5, 0.23), (34, 0.14), (35, 0.06), (36.5, 0.07), (36.9, 0.08), (37.2, 0.09), (38, 0.1), (38.2, 0.08), (38.5, 0.06), (38.7, 0.04), (39, 0.02)\}$. We study the slopes of the polygon A^* by examining the inequality relations among the y-values as $0.03 < 0.06 < 0.17 < 0.20 < 0.25 > 0.23 > 0.14 > 0.06 < 0.07 < 0.08 < 0.09 < 0.1 > 0.08 > 0.06 > 0.04 > 0.02$. The y-values form two clock shapes over [30, 35] and [36.5, 39]. We thus recognize two point sets $A_1 = \{(30, 0.03), (31, 0.06), (31.5, 0.17), (32, 0.20), (33, 0.25), (33.5, 0.23), (34, 0.14), (35, 0.06)\}$ and $A_2 = \{(36.5, 0.07), (36.9, 0.08), (37.2, 0.09), (38, 0.1), (38.2, 0.08), (38.5, 0.06), (38.7, 0.04), (39, 0.02)\}$ in A^*. The approximation of A_1 has already been accomplished in Ex. 7.4. By repeating the steps of the procedure from Ex. 7.4, we decide the unknown parameters of π_2 that intends to approximate A_2. We find $x_{min} = 36.5$, $x_{max} = 39$ and ε

= 0.1. The x-coordinate corresponding to ε has a value of $\gamma_1 = \alpha_2 = 38$. We check that $y(x_{\min}) = 0.07$ fits for $y(x_{\min}) > \frac{\varepsilon}{2}$, which generates the parameter α_1 as $\alpha_1 = \gamma_1 - \frac{\gamma_1 - x_{\min}}{\sqrt{\frac{\varepsilon - y(x_{\min})}{2\varepsilon}}} = 38 - \frac{38 - 36.5}{\sqrt{\frac{0.1-0.07}{2\cdot 0.1}}} = 34.124$. The value of $y(x_{\max}) = 0.02$, however, satisfies the constraint $y(x_{\max}) < \frac{\varepsilon}{2}$; therefore we will calculate the value of $\gamma_2 = \frac{x_{\max} - \alpha_2\sqrt{\frac{y(x_{\max})}{2\varepsilon}}}{1 - \sqrt{\frac{y(x_{\max})}{2\varepsilon}}} = \frac{39 - 38\sqrt{\frac{0.02}{2\cdot 0.1}}}{1 - \sqrt{\frac{0.02}{2\cdot 0.1}}} = 39.449$. The additional parameters β_1 = 36.062 and β_2 = 38.724 are also included in the truncated π_2-function that matches set A_2 in accordance with (7.4) and (7.5).

For two points $(x_1, y_1) = (35, 0.06)$ that ends A_1 and $(x_2, y_2) = (36.5, 0.07)$ which begins A_2, we apply the equation of a straight line $y = kx + l$ in order to tie them together.

The coefficients k and l are computed by the formulas

$$k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.07 - 0.06}{36.5 - 35} = 0.0067 \text{ and } l = y_2 - x_2\frac{y_2 - y_1}{x_2 - x_1} =$$

$$0.07 - 36.5\frac{0.07 - 0.06}{36.5 - 35} = -0.1746.$$

Figure 7.5 presents the graphs of A_1 and A_2 as well as the approximating curves π_1 and π_2 joined by $y = 0.0067x - 0.1746$.

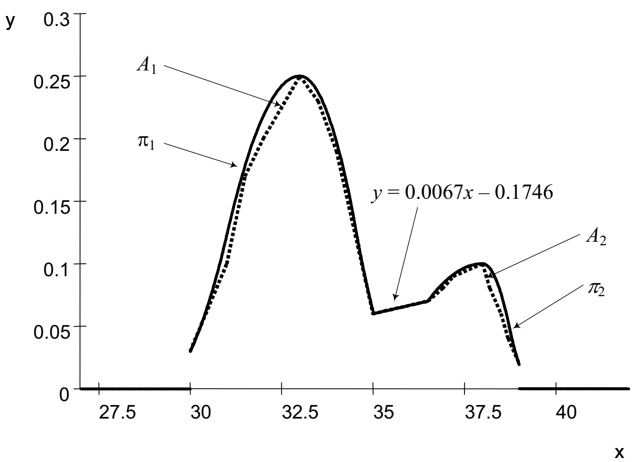

Figure 7.5: The approximation of A^* by truncated π functions

If we add the function created for A_1 in Ex. 7.4 to

$$y = \begin{cases} 0.0067x - 0.1746 & \text{for} \quad 35 \le x < 36.5, \\ 0.1\left(1 - 2\left(\dfrac{x-38}{38-34.124}\right)^2\right) & \text{for} \quad 36.5 \le x < 38, \\ 0.1\left(1 - 2\left(\dfrac{x-38}{39.449-38}\right)^2\right) & \text{for} \quad 38 \le x < 38.724, \\ 0.1\left(2\left(\dfrac{x-39.449}{39.449-38}\right)^2\right) & \text{for} \quad 38.724 \le x \le 39, \end{cases}$$

then we will obtain the total approximating function for A^*.

By using the same procedure to all “bells” visible in A in Fig. 7.1, we obtain other functions of the π type. We join the functions by inserting equations of straight lines to plot a full, continuous curve $\pi(x)$ approximating A entirely.

Since we adapt several π functions to truncated forms, then we will call a sampled approximation “sampled, truncated π”.

Example 7.6

Figure 7.1 is an example of the point set, which delivers an irregular polygon $A = \{(x, y)\}$. The polygon is composed of segments of straight lines that tie (x, y) together. Figure 7.6 gives the approximation of the shape's image from Fig. 7.1, by a collection of truncated π functions joined together by pieces of lines to guarantee continuity of the approximating function.

It is worth noticing that the collective error that measures the deviations of π from A is not large and that is very important for the approximation of a composed polygon consisting of many “bells”.

A number of split functions that are included in the sampled definition of $\pi(x)$ is substantially less than a number of linear functions that define short line pieces placed among the nodes (x, y) in the polygon A. One π segment can surround a great many pairs (x, y). This reduces the number of piecewise definitions and the number of subintervals in the “sampled truncated π”. By introducing π we simplify a collective definition of the function approximating A when comparing it to the linear parts taking place in the interpolation of A. This property of π should be regarded as its advantage.

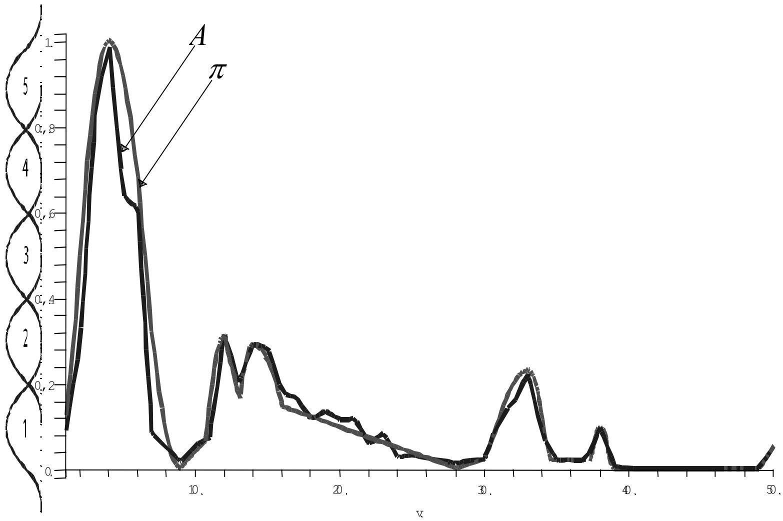

Figure 7.6: The sampled π in the approximation of A

7.3 Rough Sets in Classifying of Clock-like Polygons

In order to include the unknown sets of the A type within classes already possessing the declared members, we apply some elements of rough set theory [50, 51, 52, 53] that have already proven useful in the process of a disease classification.

The y-axis in Fig. 7.6 is divided in five regions. We would like to assign codes associated with the subintervals of the same length created for the y-values. Some scientists have a custom of applying fuzzy sets with their membership functions to accomplish the determination of interval borders [50]. Anyhow, we do not want to engage new elements of fuzzy set theory in this chapter, we only want to announce another possibility of finding the boundaries for the named intervals by drawing five membership functions along the y-axis in Fig. 7.6. Independently of the method, we list intervals of the y-variable and associated with them codes in Table 7.1.

Each considered point set has an envelope created by a continuous function that approximates it. When regarding any value placed on the x-axis we are capable of establishing the association between the x-value and the code. To achieve this we should first compute the $\pi(x)$ value and then place it in the appropriate interval from Table 7.1. We thus accept the set $A = \{(x, y)\} \approx \{(x, \pi(x))\} = \{(x, code(x)\}$, where $code(x)$ is the code of the $\pi(x)$ interval.

Table 7.1: The relationship between y-values and codes

Interval of y-values	Code
(0.0, 0.2)	1
(0.2, 0.4)	2
(0.4, 0.6)	3
(0.6, 0.8)	4
(0.8, 1.0)	5

Let us introduce a universe set $U=\{A_1, \ldots, A_n\}$ composed of clock-like polygons. Assume that some of them are members of class "Class 1", while the others have an unknown membership or belong to a different class other than "Class 1". Our purpose is to assign membership degrees to all polygons from U in order to classify them within "Class 1".

The objects of U are determined by two groups of attributes, so called condition and decision attributes, presented by the sets B and D respectively. We assume that set B consists of m chosen x-sizes x_j, mapped into a set of values $code_{A_i}(x_j)$, $i = 1, \ldots, n, j = 1, \ldots, m$. The codes are equal to the integers 1, 2, 3, 4, 5. Set D has an attribute stated as "the membership of a polygon in "Class 1"", where the membership is expressed as "*yes*", "*no*" and "*unknown*".

The triple $I = (U, B, D)$ forms the decision table that constitutes a data basis for an equivalence relation $I(B)$ called the indiscernibility relation and defined by the relationship

$$I(B) = \left\{(A_i, A_k) : code_{A_i}(x_j) = code_{A_k}(x_j)\right\} \quad \textit{for each size } x_j, \tag{7.7}$$

where $j = 1, 2, \ldots, m,\ i,\ k = 1, 2, \ldots, n.$

We find the equivalence classes of the relation $I(B)$, i.e., the blocks $IB(A_i)$ as the sets

$$IB(A_i) = \{A_k : (A_i, A_k) \in I(B)\}. \tag{7.8}$$

By following a general rough set procedure we create a set $X = \{A_i$: membership "*yes*" to "Class 1" is assigned$\}$.

The first decision set (the lower approximation of X)

$$B_*(X) = \{A_i : IB(A_i) \subseteq X\} \tag{7.9}$$

reveals the polygons which surely match "Class 1".

The other decision set (the upper approximation of X)

$$B^*(X) = \{A_i : IB(A_i) \cap X \neq 0\} \tag{7.10}$$

contains these members of U that may belong to "Class 1".

The elements of a boundary set

$$B_{border}(X) = B^*(X) - B_*(X) \tag{7.11}$$

are interpreted as members of “Class 1” in a certain grade.

The membership degree of A_i, interpreted as a degree of being a member in “Class 1”, is computed as

$$\mu_{\text{"Class 1"}}(A_i) = \frac{|X \cap IB(A_i)|}{|IB(A_i)|}. \tag{7.12}$$

Example 7.7

We collect the data concerning six point sets A_1–A_6 and approximate the obtained polygons by “sampled truncated π” π_1–π_6 as shown in Fig. 7.7. The continuous and smooth curves replace sharp polygons to give access to every pair (x, y) over the common x-interval under consideration.

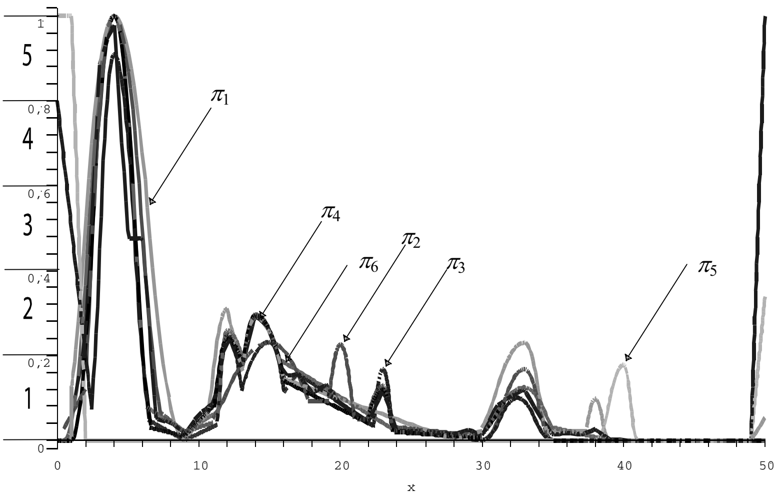

Figure 7.7: π_1–π_6 in approximation of polygons A_1–A_6

We state $U = \{A_1, A_2, A_3, A_4, A_5, A_6\}$.

The decision table $I = (U, B, D)$, made for the condition (codes 1, 2, 3, 4, 5) and decision (*yes*, *no*, *unknown*) attributes, shows the properties of members of U expanded in Table 7.2.

Table 7.2: The decision table $I = (U, B, D)$ of clock-like polygons A_1–A_6

$A_i \backslash x_j$	0	4	8	12	16	20	24	28	"Class 1"
A_1	1	5	1	2	1	1	1	1	*yes*
A_2	1	5	1	1	1	2	1	1	*yes*
A_3	5	5	1	2	1	1	1	1	*yes*
A_4	1	5	1	2	1	1	1	1	*yes*
A_5	5	1	1	1	1	1	1	1	*no*
A_6	1	5	1	2	1	1	1	1	*unknown*

The equivalence relation $I(B)$ is formed by a set of pairs

$$I(B) = \{(A_1, A_1), (A_2, A_2), (A_3, A_3), (A_4, A_4), (A_5, A_5), (A_6, A_6), (A_4, A_6), (A_6, A_4)\}.$$

The equivalence classes of $I(B)$ are created as the sets

$$IB(A_1) = \{A_1\}, IB(A_2) = \{A_2\}, IB(A_3) = \{A_3\}, IB(A_4) = \{A_4, A_6\}, IB(A_5) = \{A_5\},$$
$$IB(A_6) = \{A_6, A_4\}$$

according to (7.8).

The semantic value of the decision attribute "Class 1" = "*yes*" generates set $X = \{A_1, A_2, A_3, A_4\}$ that in turn is an essential factor implementing the sets $B_*(X) = \{A_1, A_2, A_3\}$, $B^*(X) = \{A_1, A_2, A_3, A_4, A_6\}$ and $B_{border}(X) = \{A_4, A_6\}$.

The polygon membership degrees whose sizes confirm the membership in "Class 1" are obtained as

$$\mu_{\text{"}Class\ 1\text{"}}(A_1) = 1,\ \mu_{\text{"}Class\ 1\text{"}}(A_2) = 1,\ \mu_{\text{"}Class\ 1\text{"}}(A_3) = 1,\ \mu_{\text{"}Class\ 1\text{"}}(A_4) = \frac{1}{2},$$

$$\mu_{\text{"}Class\ 1\text{"}}(A_5) = 0,\ \mu_{\text{"}Class\ 1\text{"}}(A_6) = \frac{1}{2}.$$

We can assume that A_1, A_2 and A_3 are the true members of "Class 1" in U while A_4 and A_6 may belong to the investigated class to certain degrees. We can also notice that A_6 affects a status of A_4 negatively, and on the contrary, we can see that A_4 upgrades the importance of A_6 in the considered class "Class 1".

7.4 *s*-functions in Fitting to Letter-shaped Polygons

As effects of some experiments, in which $y \in [-1, 1]$, we obtain polygons (sets A) composed of parts looking like bells or even half-bells that lie over and under the x-axis.

Example 7.8

Consider the polygon A sketched in Fig. 7.8.

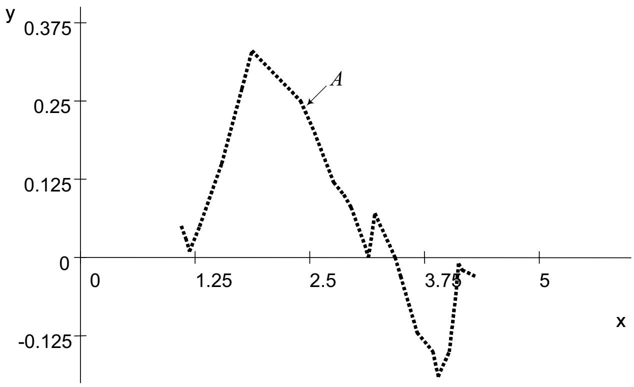

Figure 7.8: The example of a letter-shaped polygon reflecting $A = \{(x, y)\}$

We assume an approximation of multi-shapes from Fig. 7.8 by *s*-truncated functions used piecewise as another approach to the numerical problem of a smooth curve fitting to point sets. Since we recognize half-bells as dominant shapes in A, then we should prefer adopting the appearance of the *s*-functions as approximating segments. "The sampled truncated *s*", as we call an entire approximation curve, consists of first and second degree-polynomials. This would follow the polygon's shape very closely and results in cumulating very low error, measuring deviations between the approximating curve and the polygon.

Let us suppose that the *y*-values of curves, that are similar in shape to the set depicted in Fig. 7.8, are important indicators in the further classification process of the curves. These can resemble some letters, e.g., *N*, *W*, or *M*, and can occur in different places along the *x*-axis. In order to assign the curves to proper classes denoted by *N*, *W* or *M*, we should compare their *y*-coordinates. It is not possible if the curves are scattered in different segments of the *x*-axis. To make the curves comparable, we should move them over the interval [0, 1].

The approach to approximation of irregular polygons presented below constitutes a solution [65, 66, 67] that differs from other modern procedures of seeking approximation curves [1, 15, 35, 43, 71].

We discover that the *x*-values of pairs included in A belong to interval $\left[x_{\min}(A), x_{\max}(A)\right]$, in which $x_{\min}(A)$ is the smallest and $x_{\max}(A)$ is the largest *x*-value in A. In the next step we divide the whole *x*-interval into subintervals $\left[x_{\min(A_j)}, x_{\max(A_j)}\right]$, where A_j, $j = 1, 2, \ldots, Q$, are parts of A. In parts A_j we can experience either the growth or the decrease of the *y*-values corresponding to these perience either the growth or the decrease of the y-values corresponding to these x that are placed between the borders $x_{\min(A_j)}$ and $x_{\max(A_j)}$ functioning as the smallest, and respectively, the largest value of x in A_j. S-functions or segments of straight lines attached to two adjacent s-curves approximate the A_j components.

Example 7.9

The pairs, which create the polygon depicted in Fig. 7.8, are the members of A = {(1.1, 0.05), (1.15, 0.03), (1.19, 0.01), (1.3, 0.05), (1.54, 0.15), (1.76, 0.27), (1.87, 0.33), (2.4, 0.25), (2.55, 0.2), (2.76, 0.12), (2.87, 0.1), (2.96, 0.08), (3.1, 0.02), (3.14, 0), (3.21, 0.07), (3.48, 0), (3.49, –0.03), (3.67, –0.12), (3.84, –0.15), (3.9, –0.19), (4.02, –0.15), (4.09, –0.06), (4.12, –0.01), (4.16, –0.02), (4.3, –0.03)}. By measuring the direction of changes in the y-values, which point out extreme nodes in A's shape, we consider the subintervals [1.1, 1.19], [1.19, 1.87], [1.87, 3,14], [3.14, 3.21], [3.21, 3.43], [3.43, 3.9], [3.9, 4.12], [4.12, 4.3]. Over the intervals either s-functions or straight lines will be applied as approximation tools.

The s-function with the standard parameters α, β, γ and an additional parameter ε, introduced by (2.5) and modified by the equation

$$y = s(x,\alpha,\beta,\gamma,\varepsilon) = \begin{cases} (1) \quad \varepsilon\left(2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2\right) & \text{for} \quad \alpha \le x < \beta, \\ (2) \quad \varepsilon\left(1-2\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2\right) & \text{for} \quad \beta \le x \le \gamma, \end{cases} \tag{7.13}$$

where $\beta = \dfrac{\alpha+\gamma}{2}$, is suitable for the occurrences of "half-bells" A_j. Since the y-values of the classical s belong to the interval [0, 1] (–s has its y-values in [–1, 0]) then we should insert an additional parameter ε in (7.13) to accommodate a height of the function to the data existing in the set A_j, j = 1, 2, …, Q. We have already introduced the partition of A by means of subsets A_j, looking like "half-bells", then we should denote each s-function that approximates A_j by $s_{A_j}(x,\alpha_{A_j},\beta_{A_j},\gamma_{A_j},\varepsilon_{A_j})$.

We now discuss different cases of A_j's approximation that is dependent on the sizes of y-coordinates in the set A_j.

Let us assume that the values of the y-coordinates in A_j associated with the x-values belonging to $\lfloor x_{\min(A_j)}, x_{\max(A_j)} \rfloor$ appear in the ascending order, and let us notice that no y-coordinate is equal to zero. The pair $(x_{\min(A_j)}, y(x_{\min(A_j)}))$ ($(y(x_{\min(A_j)})$ corresponds to $x_{\min(A_j)}$) begins the set A_j but we cannot identify $x_{\min(A_j)}$ as α_{A_j}. Thus, the value of α_{A_j} in the s_{A_j}-function, expected to approximate A_j, is unknown. To find α_{A_j} we, at first, accept the value of ε_{A_j} as the largest y-coordinate in A_j associated with the x-coordinate γ_{A_j}. We can now reconstruct the value of the remaining parameter α_{A_j} according to patterns that are almost identical with "Case of α_1" already discussed for π-functions:

a) $\alpha_{A_j} = \dfrac{x_{\min(A_j)} - \gamma_{A_j}\sqrt{\frac{y(x_{\min(A_j)})}{2\varepsilon_{A_j}}}}{1-\sqrt{\frac{y(x_{\min(A_j)})}{2\varepsilon_{A_j}}}}$ for $y(x_{\min(A_j)}) < \dfrac{\varepsilon_{A_j}}{2}$. It changes (7.13) as

$$y = \begin{cases} (1) \quad \varepsilon_{A_i}\left(2\left(\dfrac{x-\alpha_{A_j}}{\gamma_{A_j}-\alpha_{A_j}}\right)^2\right) & \text{for} \quad x_{\min(A_j)} \le x < \beta_{A_j}, \\ (2) \quad \varepsilon_{A_i}\left(1-2\left(\dfrac{x-\gamma_{A_j}}{\gamma_{A_j}-\alpha_{A_j}}\right)^2\right) & \text{for} \quad \beta_{A_j} \le x < \gamma_{A_j}. \end{cases} \tag{7.14}$$

b) $\alpha_{A_j} = \gamma_{A_j} - \dfrac{\gamma_{A_j} - x_{\min(A_j)}}{\sqrt{\frac{\varepsilon_{A_j}-y(x_{\min(A_j)})}{2\varepsilon_{A_j}}}}$ for $y(x_{\min(A_j)}) \ge \dfrac{\varepsilon_{A_j}}{2}$. The $s_{A_j}(x)$ formula appears as

$$y = \begin{cases} (1) \quad 0 & \text{for} \quad x < x_{\min(A_j)}, \\ (2) \quad \varepsilon_{A_j}\left(1-2\left(\dfrac{x-\gamma_{A_j}}{\gamma_{A_j}-\alpha_{A_j}}\right)^2\right) & \text{for} \quad x_{\min(A_j)} \le x \le \gamma_{A_j}. \end{cases} \tag{7.15}$$

It happens that the position of pairs in the set A_j introduces the descending order among points with respect to the y-coordinate values. We assume that none of them is equal to zero. The pair $(x_{\max(A_j)}, y(x_{\max(A_j)}))$ will end the set A_j, but $x_{\max(A_j)} \neq \gamma_{A_j}$. Let us assign the largest value of y in A_j, regarded as ε_{A_j}, to the x-coordinate $x_{\min(A_j)} = \alpha_{A_j}$. Then it is possible to restore the missing value of γ_{A_j}, which is one of the parameters included in function $1 - s_{A_j}(x, \alpha_{A_j}, \beta_{A_j}, \gamma_{A_j}, \varepsilon_{A_j})$ applied to approximate A_j.

We make the following distinction between two different cases of adjusting the parameter γ_{A_j} to the data set A_j:

c) $\gamma_{A_j} = \dfrac{x_{\max(A_j)} - \alpha_{A_j}\sqrt{\dfrac{y(x_{\max(A_j)})}{2\varepsilon_{A_j}}}}{1-\sqrt{\dfrac{y(x_{\max(A_j)})}{2\varepsilon_{A_j}}}}$ for $y(x_{\max(A_j)}) < \dfrac{\varepsilon_{A_j}}{2}$. We suggest the following changes in (7.13) to adapt it to the new assumptions

$$y = \begin{cases} (1) \quad \varepsilon_{A_j}\left(1-2\left(\dfrac{x-\alpha_{A_j}}{\gamma_{A_j}-\alpha_{A_j}}\right)^2\right) & \text{for} \quad \alpha_{A_j} \le x < \beta_{A_j}, \\ (2) \quad \varepsilon_{A_j}\left(2\left(\dfrac{x-\gamma_{A_j}}{\gamma_{A_j}-\alpha_{A_j}}\right)^2\right) & \text{for} \quad \beta_{A_j} \le x \le x_{\max(A_j)}. \end{cases} \tag{7.16}$$

d) $\gamma_{A_j} = \alpha_{A_j} + \dfrac{x_{\max(A_j)} - \alpha_{A_j}}{\sqrt{\dfrac{\varepsilon_{A_j} - y(x_{\max(A_j)})}{2\varepsilon_{A_j}}}}$ for $y(x_{\max(A_j)}) \ge \dfrac{\varepsilon_{A_j}}{2}$. We adjust the $s_{A_j}(x)$ formula as

$$y = \begin{cases} (1) \quad \varepsilon_{A_j}\left(1-2\left(\dfrac{x-\alpha_{A_j}}{\gamma_{A_j}-\alpha_{A_j}}\right)^2\right) & \text{for} \quad \alpha_{A_j} \le x \le x_{\max(A_j)}, \\ (2) \quad 0 & \text{for} \quad x > x_{\max(A_j)}. \end{cases} \tag{7.17}$$

The s_{A_j} function is a section of the classical *s*-function and therefore we will name it a truncated *s*-function. By selecting the minimal and the maximal *x*-value and the maximal *y*-value, which exist in the set A_j, we prepare (7.14)–(7.17) for computing the unknown parameters α_{A_j} or γ_{A_j}. The point, in which the *y*-coordinate takes the ε_{A_j}-value and the *x*-coordinate is the equal of the γ_{A_j} value for the function $s_{A_j}(x, \alpha_{A_j}, \beta_{A_j}, \gamma_{A_j}, \varepsilon_{A_j})$, and respectively the α_{A_j} value for the complement $1 - s_{A_j}(x, \alpha_{A_j}, \beta_{A_j}, \gamma_{A_j}, \varepsilon_{A_j})$, is one of the vertices in A and it constitutes the common element of A_j and the function s_{A_j}, $j = 1, \ldots, Q$. The total approximation s_A of A is called "sampled truncated *s*".

To preserve the right shape of the approximating curve, it is advisable to tie two adjacent functions s_{A_j}, $s_{A_{j+1}}$ between the points $(x_{\max(A_j)}, y(x_{\max(A_j)}))$, $(x_{\min(A_{j+1})}, y(x_{\min(A_{j+1})}))$ by the segment of a straight line having an equation

$$y = line_{A_j}(x) = k_{A_j}x + l_{A_j} \quad \text{for} \quad x_{\max(A_j)} \le x < x_{\min(A_{j+1})}. \tag{7.18}$$

Example 7.10

The "sampled truncated *s*", made for the data from Ex. 7.8, is shown in Fig. 7.9.

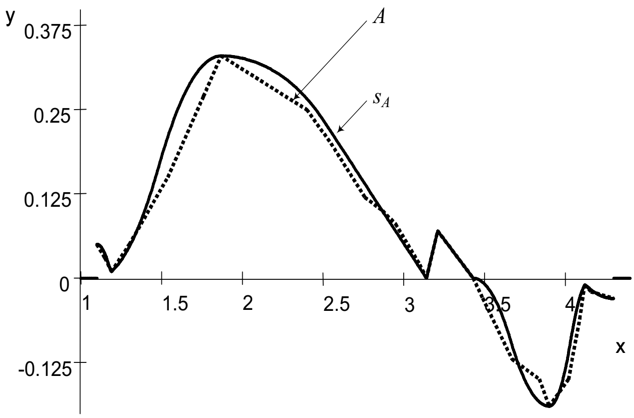

Figure 7.9: The approximation of *A* by "sampled truncated *s*"

The first set of points $A_1 \subset A$, in which the *y*-coordinates form the descending order, is placed over [1.1, 1.19] as decided in Ex. 7.9. Since no *y*-value is equal to zero we will re-construct the value of a parameter $\gamma_{A_1} = \dfrac{1.19 - 1.1\sqrt{\frac{0.01}{2\cdot 0.05}}}{1-\sqrt{\frac{0.01}{2\cdot 0.05}}} = 1.2316$ for $\varepsilon_{A_1} = 0.05$, $\alpha_{A_1} = 1.1$, $x_{\max(A_1)} = 1.19$ and $y(x_{\max(A_1)}) = 0.01$ in accordance with c).

In the next interval A_2 = [1.19, 1.87] the value of α_{A_2} should be estimated. If we re-quest the values of $\varepsilon_{A_2} = 0.33$, $\gamma_{A_2} = 1.87$, $x_{\min(A_2)} = 1.19$ and $y(x_{\min(A_2)}) = 0.01$ then $\alpha_{A_2} = \dfrac{1.19 - 1.87\sqrt{\frac{0.01}{2\cdot 0.33}}}{1-\sqrt{\frac{0.01}{2\cdot 0.33}}}$ =1.0945 due to a).

The formula of s_A for *A* is expanded as the following split definition

$$y = s_A(x) = \begin{cases} 0.05\left(1-2\left(\dfrac{x-1.1}{1.2316-1.1}\right)^2\right) & \text{for} \quad 1.1 \le x < 1.1658, \\ 0.05\left(2\left(\dfrac{x-1.2316}{1.2316-1.1}\right)^2\right) & \text{for} \quad 1.1658 \le x < 1.19, \\ \vdots & \vdots \quad \vdots \\ (-0.31818)x + 1.0914 & \text{for} \quad 3.21 \le x < 3.43, \\ \vdots & \vdots \quad \vdots \\ -0.03\left(1-2\left(\dfrac{x-4.3}{4.3-3.9958}\right)^2\right) & \text{for} \quad 4.1479 \le x < 4.3. \end{cases}$$

We can prove some additional operations on the s-function values, e.g., $y = (s(x))^2$ or $y = (s(x))^{\frac{1}{2}}$ to match a shape of the function to the given polygon in the best way.

It is worth noticing that the total error that collects the deviations of $s_A(x)$ from A is very small.

The curve created for A has a particular pattern since it resembles the letter N. In some medical or technical problems we obtain sets of points that will be approximated by some shapes of letters, e.g., N, M or W. The shapes of mentioned letters can be disturbed or vague, which makes difficult to classify them properly, i.e., we do not know exactly if we should include the curves in classes determined by N, M and W. In order to ensure if a vague or unknown object can belong to the considered class or not, we accomplish a classification according to the rules of rough set theory.

If we are given several polygons then we, at the first stage, want to collect all approximated objects over a common interval [0, 1] to measure their deviations in y-values with respect to the same x values.

Example 7.11

Suppose that we have obtained different shapes of the curves originating from point sets A^1–A^5. Each of them is approximated by a continuous function that consists of s-sections and pieces of straight lines that link the parts of s-functions if it is necessary. Figure 7.10 provides the polygons and the approximating functions over their original intervals along the x-axis. We assume the following polygon membership: A^1, A^3 and A^5 belong to the "N" class, A^4 is a member of the "W" class, while the origin of A^2 is unknown.

In further analysis we use only the continuous curves, also named A^1–A^5.

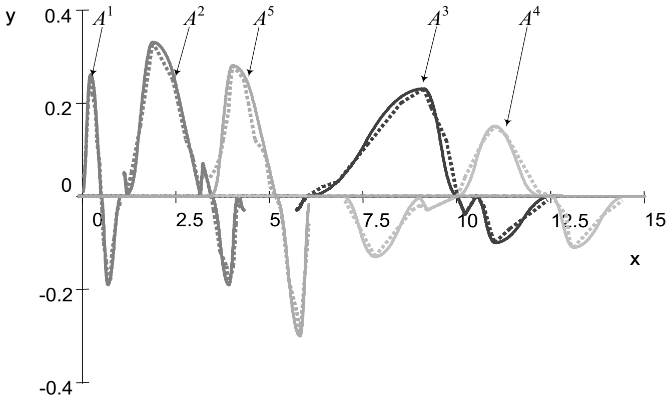

Figure 7.10: The approximated polygons A^1–A^5

To move all curves to the same starting point settled as the origin of the *x*-*y* coordinate system, we suggest the following transformations.

Suppose that the A^i-curve, $i = 1, \ldots, n$, is placed in the *x*-subinterval $\left[x_{\min}(A^i), x_{\max}(A^i)\right]$. We move the j^{th} segment $s_{A^i_j}$, approximating the subset A^i_j of A^i, $i = 1, \ldots, n$, $j = 1, \ldots, Q$, to a position close to the origin by introducing the formula

$$y = \begin{cases} (1)\ \varepsilon_{A^i_j}\left(2\left(\dfrac{x-(\alpha_{A^i_j}-x_{\min}(A^i))}{\gamma_{A^i_j}-\alpha_{A^i_j}}\right)^2\right) & \text{for } x_{\min(A^i_j)} - x_{\min}(A^i) \le x < \beta_{A^i_j} - x_{\min}(A^i), \\ (2)\ \varepsilon_{A^i_j}\left(1-2\left(\dfrac{x-(\gamma_{A^i_j}-x_{\min}(A^i))}{\gamma_{A^i_j}-\alpha_{A^i_j}}\right)^2\right) & \text{for } \beta_{A^i_j} - x_{\min}(A^i) \le x \le \gamma_{A^i_j} - x_{\min}(A^i). \end{cases} \tag{7.19}$$

The straight line (7.18) is transferred nearby the origin by the action of an equation

$$y = line_{A^i_j}(x) = k_{A^i_j}x + l_{A^i_j} + k_{A^i_j} \cdot x_{\min}(A^i) = K_{A^i_j}x + L_{A^i_j}$$
$$\text{for } x_{\max(A^i_j)} - x_{\min}(A^i) \le x < x_{\min(A^i_{j+1})} - x_{\min}(A^i) \tag{7.20}$$

Example 7.12

Figure 7.11 shows A^1–A^5 attached to the origin after performing (7.19) and (7.20).

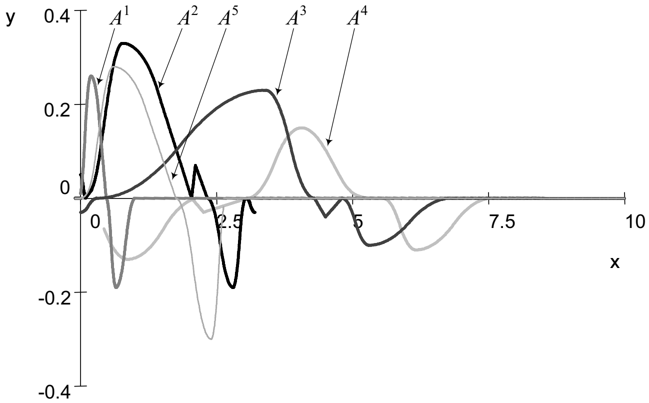

Figure 7.11: The curves A^1–A^5 with their start points at the origin

In Fig. 7.11 we recognize A^2 as A from Ex. 7.8. We decide $x_{\min}(A^2)=1.1$ and modify "sampled truncated s" for $A_2 = A$, as a function

$$y = s_{A^2}(x) = \begin{cases} 0.05\left(1-2\left(\dfrac{x-(1.1-1.1)}{1.2316-1.1}\right)^2\right) & \text{for} \quad 1.1-1.1 \le x < 1.1658-1.1, \\ 0.05\left(2\left(\dfrac{x-(1.2316-1.1)}{1.2316-1.1}\right)^2\right) & \text{for} \quad 1.1658-1.1 \le x < 1.19-1.1, \\ \vdots & \vdots \quad \vdots \\ (-0.31818)x+1.0914 & \\ +(-0.31818)\cdot 1.1 = & \text{for} \quad 3.21-1.1 \le x < 3.43-1.1, \\ (-0.31818)x+0.7414 & \\ \vdots & \vdots \quad \vdots \\ -0.03\left(1-2\left(\dfrac{x-(4.3-1.1)}{4.3-3.9958}\right)^2\right) & \text{for} \quad 4.1479-1.1 \le x < 4.3-1.1, \end{cases}$$

which displaces A_2's start point to the origin.

The comparison of all curves will be successful if we can observe them at a common interval. Let us determine the interval [0, 1] as a new domain for all split-functions A^1–A^n. Each piece $s_{A^i_j}$ or $line_{A^i_j}$, $i = 1, \ldots, n$, $j = 1, \ldots, Q$, should be shrunk or enlarged proportionally to fit it for the interval [0, 1] together with other pieces.

In order to achieve the required movements of s_j^i over [0, 1], we initiate the parameter $\delta_{A^i} = \dfrac{1}{x_{\max}(A^i) - x_{\min}(A^i)}$ in (7.19), which generates a new formula [49, 67]

$$y = \begin{cases} (1) \quad \varepsilon_{A_j^i}\left(2\left(\dfrac{x-(\alpha_{A_j^i} - x_{\min}(A^i))\delta_{A^i}}{(\gamma_{A_j^i} - \alpha_{A_j^i})\delta_{A^i}}\right)^2\right) \\ \text{for} \quad (x_{\min(A_j^i)} - x_{\min}(A^i))\delta_{A^i} \le x < (\beta_{A_j^i} - x_{\min}(A^i))\delta_{A^i}, \\ (2) \quad \varepsilon_{A_j^i}\left(1-2\left(\dfrac{x-(\gamma_{A_j^i} - x_{\min}(A^i))\delta_{A^i}}{(\gamma_{A_j^i} - \alpha_{A_j^i})\delta_{A^i}}\right)^2\right) \\ \text{for} \quad (\beta_{A_j^i} - x_{\min}(A^i))\delta_{A^i} \le x \le (\gamma_{A_j^i} - x_{\min}(A^i))\delta_{A^i}. \end{cases} \tag{7.21}$$

Before equipping (7.20) with the parameter δ_{A^i} we should find another form of (7.20) adapted to the range [0, 1] as [7, 67]

$$y = K_{A_j^i} x + L_{A_j^i} = \frac{x + \dfrac{L_{A_j^i}}{K_{A_j^i}}}{\dfrac{1}{K_{A_j^i}}} \quad \text{for} \quad x_{\max(A_j^i)} - x_{\min}(A^i) \le x < x_{\min(A_{j+1}^i)} - x_{\min}(A^i). \tag{7.22}$$

We can now place δ_{A^i} in (7.22) according to a pattern

$$y = \frac{x + \dfrac{L_{A_j^i}\delta_{A^i}}{K_{A_j^i}}}{\dfrac{\delta_{A^i}}{K_{A_j^i}}} \quad \text{for} \quad (x_{\max(A_j^i)} - x_{\min}(A^i))\delta_{A^i} \le x < (x_{\min(A_{j+1}^i)} - x_{\min}(A^i))\delta_{A^i}. \tag{7.23}$$

Example 7.13

The applications of (7.21) and (7.23) to every *s*-section and every line segment that takes place in A^1–A^5, yields the effect of collecting all curves over the *x*-domain [0, 1]. The curves A^1–A^5, lying in [0, 1], are plotted in Fig. 7.12.

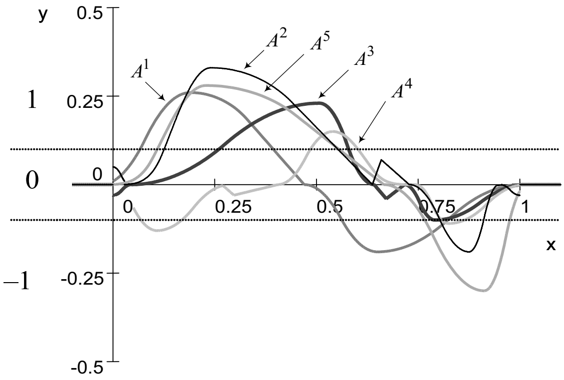

Figure 7.12: The curves A^1–A^5 over the common interval [0, 1]

After the executed transformations (7.21) and (7.23) the y-coordinates of A^2, for which $\delta_{A^2} = \dfrac{1}{4.3-1.1} \approx 0.31$, are computed as values

$$y = s_{A^2}(x) = \begin{cases} 0.05\left(1-2\left(\dfrac{x-(1.1-1.1)0.31}{(1.2316-1.1)0.31}\right)^2\right) \\ \text{for } \ (1.1-1.1)0.31 \le x < (1.1658-1.1)0.31, \\ 0.05\left(2\left(\dfrac{x-(1.2316-1.1)0.31}{(1.2316-1.1)0.31}\right)^2\right) \\ \text{for } \ (1.1658-1.1)0.31 \le x < (1.19-1.1)0.31, \\ \vdots \\ \dfrac{x+\frac{0.7414\cdot 0.31}{-0.31818}}{\frac{0.31}{-0.31818}} \\ \text{for } \ (3.21-1.1)0.31 \le x < (3.43-1.1)0.31, \\ \vdots \\ -0.03\left(1-2\left(\dfrac{x-(4.3-1.1)0.31}{(4.3-3.995)0.31}\right)^2\right) \\ \text{for } \ (4.1479-1.1)0.31 \le x < (4.3-1.1)0.31. \end{cases}$$

The mathematical tools used for polygons result in the creation of a common collection of curves representing the polygons over [0, 1]. Next, the selected elements of rough set theory will constitute a foundation for the classification of the curves.

7.5 The Classification of Letter-shaped Polygons

We return to the curves presented by Ex. 7.11 in order to accomplish their classification provided that we would like to determine their membership in the "*N*" class. The *y*-axis in Fig. 7.12 is divided in three regions. After analyzing the importance of the *y*-values we consider three intervals for them. The values of *y* belong to the interval (–0.3, 0.35) in the recognized case. Suppose that the *y*-values, occurring from –0.1 to 0.1 cannot provide us with essential information about the curve character and they are ignored. As a consequence, the code assigned to the *y*-value belonging to [–0.1, 0.1] is equal to 0. For decisive, positive *y*-values, the code of 1 is reserved while the negative *y*-values of a deterministic character obtain the code stated as –1.

Let us introduce the universe set $U = \{A^1, A^2, \ldots, A^n\}$ composed of continuous curves A^1, A^2, …, A^n approximating the polygons bearing the same names and representing different shapes of letters. The objects of *U* are determined by condition and decision attributes defined by the sets *B* and *D* respectively. We assume that the set *B* consists of sizes $x_k \in [0, 1]$, $k = 1, \ldots, m$, associated with values $code_{A^i}(x_k)$, $i = 1, \ldots, n$ that are equal to the integers –1, 0 and 1.

Since we want to assign some members to the "*N*" class, then set *D* obtains an attribute stated as "the membership of a polygon in "*N*"", where the membership is expressed as "*yes*", "*no*", "*unknown*".

The triple $I = (U, B, D)$ forms the decision table whose analysis generates the equivalence relation already introduced by (7.7), its classes given by (7.8) and two approximation sets of *X*, as recommended by (7.9) and (7.10). The relationship between class "*N*" and each member of *U* is estimated by (7.12).

Example 7.14

We consider the data concerning A^1–A^5 and sampled in Fig. 7.10 as pictures of different letter-shaped cases. We decide $U = \{A^1, A^2, A^3, A^4, A^5\}$. The decision triple $I = (U, B, D)$ is expanded in Table 7.3. The objective of investigations is to revise the hypothesis formulated earlier in Ex. 7.11. As we remember, we have supposed that A^1, A^3 and A^5 fit the "*N*" class, A^4 resembles the letter "*W*", and we cannot strictly decide about the origin of A^2. Let us accomplish the appropriate rough classification by, at first, filling in some entry data in the mentioned decision table 7.3.

The table contains the values of condition and decision attributes.

Table 7.3: The decision table $I = (U, B, D)$ for letter-like curves A^1–A^5

$A^i \backslash x_k$	0.125	0.250	0.375	0.500	0.625	0.750	0.875	Class"*N*"
A^1	1	1	1	0	–1	–1	0	*yes*
A^2	1	1	1	1	0	0	–1	*unknown*
A^3	0	1	1	1	0	0	0	*yes*
A^4	–1	0	0	1	0	0	0	*no*
A^5	1	1	1	1	0	0	–1	*yes*

The equivalence relation $I(B)$, provided in accordance with (7.7), is a set of pairs $I(B) = \{(A^1, A^1), (A^2, A^2), (A^3, A^3), (A^4, A^4), (A^5, A^5), (A^2, A^5), (A^5, A^2)\}$.

The equivalence classes of $I(B)$ are decided as the sets $IB(A^1) = \{A^1\}$, $IB(A^2) = \{A^2, A^5\}$, $IB(A^3) = \{A^3\}$, $IB(A^4) = \{A^4\}$, $IB(A^5) = \{A^2, A^5\}$ ·

The value of decision attribute "*N*" = "*yes*", generates the set $X = \{A^1, A^3, A^5\}$ that in turn is the most essential factor implementing sets $B_*(X) = \{A^1, A^3\}$, $B^*(X) = \{A^1, A^2, A^3, A^5\}$ and $B_{border}(X) = \{A^2, A^5\}$.

The polygon membership degrees, whose sizes confirm the membership in the "*N*" class, are obtained as:

$$\mu_{"N"}(A^1) = 1,\ \mu_{"N"}(A^2) = \frac{1}{2},\ \mu_{"N"}(A^3) = 1,\ \mu_{"N"}(A^4) = 0, \mu_{"N"}(A^5) = \frac{1}{2}.$$

By looking at the results of the accomplished analysis we can conclude that A^1 and A^3 are the true members of the "*N*" class in *U*, while A^2 and A^5 may belong to the investigated class to certain grades. The recognition of the curve nature aims at the special treatment of all sure and possible objects belonging to "*N*". We often know how to handle a class on the condition that its members are recognized.

Some finite sets of pairs are often interpolated by polygons that seldom have convenient equations mathematically expanded. Although there exists a large number of approximation methods applied to point sets, especially the different variations of least square regressions, we suggest applying a new procedure of approximation. This originates from the standard π or *s*-functions in truncated forms that approximate the irregular parts of the polygons very smoothly.

The functions, called by us "the sampled, truncated π (*s*)" are composed of the first and second degree-polynomials in the form of split definitions. The low degrees of approximating functions make further operations on them rather easy, which is an essential advantage of the method. One truncated π or *s*-segment can approximate many nodes belonging to the point set that reduces a number of piecewise functions involved in the general definition of an approximating collection. But most of all we notice that "the sampled, truncated π (*s*)" follows the changes of the polygon's pattern very sensitively, which guarantees the high thoroughness of approximation results.

A new process of approximation sometimes is invented in mathematics as an interesting theoretical item without greater practical validity. To prove the empirical aspect of "sampled truncated π or s" we want to consider a classification praxis.

The accomplishment of a successful classification of unknown objects, possessing only some features typical of the considered class, is not an easy task. By applying rough set theory combined with earlier achievements in approximation, we could classify polygons within the same class even if they have an unknown origin. Two introduced sets, B_* and B^*, act as a lower and an upper approximation of the investigated class. This makes it possible to assign its sure members and such ones that have most of the properties characteristic of the class. Moreover, we can easily exclude the polygons that do not satisfy the class's attributes.

If we need a pattern for another classification of objects obtained as some point sets, we can return to the discussed model and adapt it to other assumptions.

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Index

Elisabeth Rakus-Andersson
Fuzzy and Rough Techniques in Medical Models and Medication
The back page

This volume provides readers with selected fuzzy and rough tools used to medical tasks, especially diagnosing and medication. To build a link between theoretical, mathematical excerpts and practical medical applications, the contents is formed as a sequence of occurrences in which a patient appears to be diagnosed and cured. The fuzzy and rough elements are inserted in the book in the order required by the presentation of medical substance to maintain the logical unity of the book's essence.

In conformity with this pattern the essay presents in turn some necessary elements of fuzzy set theory, the classical fuzzy diagnostic model with extensions, the fuzzy diagnostic model with clinical examinations extended throughout time based on distance theory, methods of drug effectiveness measurements and algorithms selecting the optimal medicine. As the complement, the solution of an approximation problem is suggested to find a curve that surrounds two-dimensional clock-like point sets with the little approximation error.

A lot of appealing examples are added to facilitate comprehension of theoretical principles for a reader, so that even a beginner in fuzzy set theory can follow calculation steps without implementing computer programs. It should be emphasized that all models are also applicable to other fields, especially to technical domains after necessary adaptations. This confirms the existence of the large spectrum of applicable fuzzy and rough methods not only in medicine but also in natural sciences.