

Manuel Vogel

Particle Confinement in Penning Traps

An Introduction



Springer

Springer Series on Atomic, Optical, and Plasma Physics

Volume 100

Editor-in-chief

Gordon W. F. Drake, Department of Physics, University of Windsor, Windsor,
ON, Canada

Series Editors

James Babb, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA, USA

Andre D. Bandrauk, Faculté des Sciences, Université de Sherbrooke,
Sherbrooke, QC, Canada

Klaus Bartschat, Department of Physics and Astronomy, Drake University,
Des Moines, IA, USA

Philip George Burke, School of Maths and Physics, Queen's University,
Belfast, UK

Robert N. Compton, Knoxville, TN, USA

Tom Gallagher, University of Virginia, Charlottesville, VA, USA

Charles J. Joachain, Faculty of Sciences, Université Libre Bruxelles, Bruxelles,
Belgium

Peter Lambropoulos, FORTH, IESL, University of Crete, Iraklion, Crete, Greece

Gerd Leuchs, Institute of Physics, Universität Erlangen-Nürnberg, Erlangen,
Germany

Pierre Meystre, Optical Sciences Center, University of Arizona, Tucson, AZ, USA

The Springer Series on Atomic, Optical, and Plasma Physics covers in a comprehensive manner theory and experiment in the entire field of atoms and molecules and their interaction with electromagnetic radiation. Books in the series provide a rich source of new ideas and techniques with wide applications in fields such as chemistry, materials science, astrophysics, surface science, plasma technology, advanced optics, aeronomy, and engineering. Laser physics is a particular connecting theme that has provided much of the continuing impetus for new developments in the field, such as quantum computation and Bose-Einstein condensation. The purpose of the series is to cover the gap between standard undergraduate textbooks and the research literature with emphasis on the fundamental ideas, methods, techniques, and results in the field.

More information about this series at <http://www.springer.com/series/411>

Manuel Vogel

Particle Confinement in Penning Traps

An Introduction



Springer

Manuel Vogel
GSI Helmholtz Centre for Heavy
Ion Research
Darmstadt, Hessen
Germany

ISSN 1615-5653 ISSN 2197-6791 (electronic)
Springer Series on Atomic, Optical, and Plasma Physics
ISBN 978-3-319-76263-0 ISBN 978-3-319-76264-7 (eBook)
<https://doi.org/10.1007/978-3-319-76264-7>

Library of Congress Control Number: 2018934436

© Springer International Publishing AG, part of Springer Nature 2018

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG
part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

Penning traps have established themselves as premier tools for high-precision spectroscopy in the radio frequency, optical and microwave domains. They facilitate mass measurements of elements and their isotopes, measurements of magnetic moments of the electron/positron, of the proton/antiproton and of the bound electron. In that, they yield access to fundamental constants and enable stringent tests of fundamental symmetries. Frequency standards and antimatter studies have been realised with the help of traps, and due to their versatility, they enable us to perform a broad variety of measurements beyond these few examples, based on the confinement and numerous interrogation techniques available.

This book is intended to provide an introduction to the field of Penning traps and to the experimental techniques related with them. It should serve as a primer for those entering the field, and in part as a quick reference for those working in it. Its writing has been motivated by the observation that often a vast number of different sources have to be explored to gain a good overview of Penning trap principles. This is especially true for students who experience additional difficulty from the different styles of presentation and notation. Here, I will try to give a broad introductory overview in unified notation.

We will see how Penning traps work, how they are built and have a close look at the confinement properties and the techniques to produce, capture, confine, select, cool, detect and generally manipulate charged particles in them. We will rarely look at derivations that root back to the underlying Hamiltonians, but rather at an account of the most important results and aspects from the standpoint of an experimenter.

One focus will be on cryogenic Penning traps and the implementation of radio frequency, optical and microwave spectroscopy of confined particles. This includes methods for mass spectrometry and spectroscopy, and in particular for the determination of magnetic moments both by application of the continuous Stern–Gerlach effect and by optical and microwave spectroscopy of highly charged ions. We will mostly stay on the classical and macroscopic side of Penning trap physics; hence,

we will not speak about confined particles close to the ground state of their motion, non-classical states of trapped particles or about the field of quantum information processing, as this is a separate topic on its own.

Darmstadt, Germany

Manuel Vogel

Contents

1	A Bit of History and Context	1
1.1	Namesake and Developmental History	1
1.2	Fields of Operation	4
1.3	The Notion of Confinement	5
1.4	Terms and Notation	6
	References	7
2	Penning Trap Concept and Implementation	9
2.1	Penning Trap Confinement	9
2.2	Field Configuration	9
2.3	Penning Traps Compared to Other Types of Traps	12
2.4	Composition and Materials of Penning Traps	14
2.5	The Embedding of Penning Traps	16
2.5.1	Trap Chamber and Vacuum	16
2.5.2	Superconducting Magnets	17
	References	19
3	Hyperbolic and Cylindrical Penning Traps	21
3.1	Hyperbolic Penning Trap	21
3.2	Cylindrical Closed-Endcap Penning Trap	22
3.2.1	Mechanical Compensation	22
3.2.2	Electric Compensation	23
3.3	Cylindrical Open-Endcap Penning Trap	24
3.4	Cylindrical Half-Open Penning Trap	26
3.4.1	Light Collection Efficiency	26
3.4.2	Conducting Window Endcap	28
	References	30
4	Variations on the Penning Trap Concept	31
4.1	Planar Penning Traps	31
4.2	Combined Trap	33

4.3	Nested Traps	35
4.4	Elliptical Penning Traps	36
4.5	Unitary Penning Traps	36
4.6	Penning-Ioffe Traps	37
4.7	Penning-Malmberg Traps	38
4.8	Conical-Endcap Penning Trap	39
4.9	Wire Traps, Plate Traps and Pad Traps	39
4.9.1	Wire Traps	40
4.9.2	Plate Traps	41
4.9.3	Pad Traps	42
	References	43
5	Motion of a Single Particle in an Idealised Penning Trap	45
5.1	Particle Trajectory	45
5.2	Particle Oscillation Frequencies	47
5.3	The Different Nature of the Three Motions	50
5.3.1	Axial Motion	50
5.3.2	Radial Motions	51
5.3.3	A Closer Look at the Magnetron Motion	54
5.4	Brillouin Frame of Reference	56
	References	59
6	Motion of a Single Particle in a Real Penning Trap	61
6.1	Spatial Imperfections of the Electrostatic Field	61
6.1.1	Spherical Expansion of the Electrostatic Potential	61
6.1.2	Cylindrical Expansion of the Electrostatic Potential	64
6.2	Effect of Electrostatic Imperfections	66
6.3	Spatial Magnetic Field Imperfections	70
6.3.1	Spherical Expansion of the Magnetostatic Field	70
6.3.2	Cylindrical Expansion of the Magnetostatic Field	72
6.4	Effect of Magnetostatic Imperfections	75
6.5	Temporal Field Imperfections	77
6.5.1	Electrostatic Field	77
6.5.2	Magnetostatic Field	78
6.6	Field Misalignment and Ellipticity	80
6.6.1	Effect of Ellipticity	80
6.6.2	Combined Effect of Ellipticity and Tilt	82
6.7	The Effect of Conical Endcaps	84
6.8	Motional Spectrum in Presence of Field Imperfections	86
	References	86

7	Inherent Effects of Single-Particle Confinement	89
7.1	Image Charge Effects	89
7.2	Image Current Effect	91
7.3	Particle Polarisability	92
7.4	Radiative Damping	93
7.5	Cavity Effects	94
7.6	Relativistic Frequency Shifts	95
7.7	Sokolov-Ternov Effect	96
7.8	Spontaneous Decay of the Spin State	97
7.9	Charge-Changing Reactions	97
7.10	The Presence of a Second Particle	98
	References	101
8	Particle Ensembles in Penning Traps	105
8.1	Space Charge as a Confinement Limitation	105
8.2	Space Charge as a Source of Intrinsic Frequency Widths	106
8.3	Space Charge and Resonant Particle Loss	108
8.4	Confined Particle Ensembles as Non-neutral Plasmas	110
8.5	Ion Crystals	112
8.5.1	Small Ion Crystals	113
8.5.2	Mesoscopic Ion Crystals	114
8.5.3	Large Ion Crystals	116
8.6	Multi-species Ion Crystals	117
8.7	Centrifugal Separation	118
8.8	Plasma Modes and Resonant Particle Loss	120
8.9	Calculation of Particle Trajectories and Simulations	123
	References	126
9	Loading a Trap with Particles	131
9.1	In-trap Ion Production	131
9.1.1	Electron Impact Ionisation	131
9.1.2	Secondary Electron Creation	134
9.1.3	Cryogenic Gas Source	135
9.2	Dynamic Particle Capture	135
9.2.1	Method	136
9.2.2	Accumulation Timing and Efficiency	137
9.3	Intra-trap Particle Transport	138
9.4	Potential Asymmetry	139
9.5	Particle Deceleration	140
9.5.1	Potential Cages and Decelerators	141
9.5.2	Pulsed Drift Tube	141
9.6	Magnetic Mirror Effect	143
	References	145

10	Excitation of Particle Motions	147
10.1	Excitation Geometries	147
10.1.1	Dipole Excitation	149
10.1.2	Quadrupole Excitation	151
10.2	Resonant Ejection of Unwanted Particles	151
10.3	Resonant Collisional Excitation of Particles	154
10.4	Resonant Excitation: Penning-Trap Mass Spectrometry	155
10.5	Non-resonant Excitation: Modulation of the Trap Potential	156
10.5.1	Slow Modulation	156
10.5.2	Fast Modulation: Parametric Excitation	156
10.6	Non-resonant Excitation: Kick and Sidekick	158
	References	158
11	Resonant Motional Coupling	161
11.1	General Discussion of Resonant Coupling	161
11.2	Simultaneous Determination of Oscillation Frequencies	163
11.3	Cooling Via Motional Coupling	164
11.4	Magnetron Centring	164
	References	167
12	Motional Cooling in Penning Traps	169
12.1	The Meaning of Particle Cooling in a Penning Trap	169
12.2	The Notion of Particle Temperature	169
12.3	Particle Temperature Measurement	171
12.3.1	Electronic Noise Temperature Measurement	171
12.3.2	Fluorescence Lineshape Temperature Measurement	172
12.3.3	Magnetic Bottle Temperature Measurement	173
12.3.4	Active Motional Coupling Temperature Measurement	173
12.3.5	Passive Motional Coupling Temperature Measurement	174
12.4	Laser Doppler Cooling	176
12.5	Laser Cooling with Resolved Sidebands	180
12.6	Laser Cooling in the Presence of Residual Gas	182
12.7	More Involved Laser Cooling Techniques	184
12.8	Buffer Gas Cooling	185
12.9	Evaporative Cooling	187
12.10	Resistive Cooling: General Discussion	188
12.11	Effective Electrode Distance D	190
12.12	Resistive Cooling of a Single Particle	191
12.12.1	Cooling Time Constant and Final Temperature	191
12.12.2	The Equivalence of Cooling and Detection	194

12.13	Resistive Cooling of Particle Ensembles	197
12.13.1	Generalisation to N Particles	197
12.13.2	Energy Flow Model	199
12.13.3	Observability	200
12.13.4	Thermal Width of the Oscillation Frequency Distribution	202
12.13.5	Spectral Overlap Between Particle Oscillation and Circuit	203
12.14	Continuous Feedback Cooling	205
12.15	Non-continuous Feedback Cooling: Stochastic Cooling	207
12.16	Sympathetic Cooling	209
12.16.1	A Special Case: Electron Cooling	211
12.16.2	Another Special Case: Mediated Sympathetic Cooling	212
12.17	Suppression of Unwanted Heating: Filters	214
12.17.1	Types of Noise and Sources	214
12.17.2	Filters	215
12.18	Adiabatic Cooling	218
	References	218
13	Particle Ensemble Density: Rotating Wall	225
13.1	The Concept and Implementation	225
13.2	The Brillouin Limit	229
13.3	Amplitude and Sense of Rotation of the Drive	232
13.4	Examples	233
	References	235
14	The Concept of Image Charges	237
14.1	Method	237
14.2	Charge Between Two Parallel Conducting Plates	237
14.3	Charge Between Two Hollow Conducting Spheres	240
14.4	Charge Inside a Hollow Cylinder	242
14.5	Charge Inside a Hollow Cylinder Segment	243
14.6	Frequency Shifts Revisited	245
14.6.1	Image-Charge Shift	245
14.6.2	Space-Charge Shift	246
	References	248
15	Electronic Detection and Frequency Measurement	249
15.1	Method	249
15.2	General Statements About Superconductors	250
15.3	Particle-Resonator Combined Circuit	253
15.4	Resonator and Detection Design Principles	255
15.5	Resonance Frequency Shift by Varactor Diodes	258

15.6	Superconducting Resonators in a Magnetic Field	259
15.7	Detection of Small Frequency Differences	262
	References	263
16	Radio-Frequency Spectroscopy: Penning-Trap Mass Spectrometry	265
16.1	Penning Traps as Mass Spectrometers	265
16.2	Transient Recording: FT-ICR	266
16.3	Narrow-Band Scan: ARES	267
16.4	Energy Dependence of Spectral Features	269
16.5	Destructive Detection Techniques	270
	References	271
17	Radio-Frequency Spectroscopy: Outreach	275
17.1	Expressions for Frequency Shifts	275
17.2	Explicit Expressions: Quantum Picture	276
17.3	Magnitudes of the Involved Parameters	279
17.4	Classification of Field Effects	280
17.5	Group-1-Terms: Optical Spectroscopy	281
	17.5.1 Transition Energies	281
	17.5.2 Transition Rates	282
17.6	Group-2 Terms: Microwave Spectroscopy	284
17.7	Group-3 Terms: ‘Weighing’ Photons	284
17.8	Spectroscopic Use of Trap Properties: Overview	286
	References	288
18	Optical Spectroscopy	289
18.1	Applications of Optical Spectroscopy in Penning Traps	289
18.2	Light Source and Fluorescence Detection	289
18.3	Photon Detection Rate and Counting Statistics	290
18.4	Transition Line Broadening Effects	291
	18.4.1 Natural Broadening	291
	18.4.2 Collisional Broadening	292
	18.4.3 Power Broadening	292
	18.4.4 Doppler Broadening	293
18.5	Light Emission Characteristics	295
	References	295
19	Microwave Spectroscopy	297
19.1	Applications of In-Trap Microwave Spectroscopy	297
19.2	Microwave Source and Saturation Intensity	297
	19.2.1 Microwave Source and Guide	297
	19.2.2 Saturation Intensity	298

19.3	Microwave Zeeman Transitions in Highly Charged Ions	300
19.4	Microwave Larmor Frequency Measurement	301
19.4.1	Measurement of Spin Flips	301
19.4.2	Saturation Behaviour	302
19.4.3	Power Shift of the Larmor Frequency	303
19.5	Stern-Gerlach-Type g-Factor Measurements	304
	References	305
20	Laser-Microwave Double-Resonance Spectroscopy	307
20.1	Application of Laser-Microwave Double-Resonance Spectroscopy	307
20.2	Optical Transitions in Highly Charged Ions	308
20.2.1	Hyperfine Transition Energies	309
20.2.2	Fine Structure Transition Energies	311
20.2.3	Optical Transition Lifetimes	311
20.3	Microwave Zeeman Transitions and Magnetic Moments	313
20.4	Double-Resonance and Higher-Order Zeeman Effect	315
	References	317
21	Magnetic Bottles as Implemented in Penning Traps	319
21.1	Applications of Specific Magnetic Field Geometries	319
21.2	Magnetic Bottles	320
21.3	Magnetic Bottle and the Invariance Theorem	321
21.4	Magnetic Bottles and Continuous Stern-Gerlach Effect	322
21.5	Implementation of a Magnetic Bottle	324
21.6	Measurement of the Magnetic Bottle Strength	325
21.7	Superconducting Magnetic Bottle	326
21.7.1	Field Before Superconductivity	327
21.7.2	Superconductivity Before Field	327
21.7.3	Magnetic Field Removal from a Superconducting Bottle	328
21.8	Residual Field of a Magnetic Bottle	329
	References	332
22	Application of the Continuous Stern Gerlach Effect: Magnetic Moments	335
22.1	Magnetic Moment Measurements	335
22.2	Electron and Positron Magnetic Moments	336
22.3	Proton and Anti-proton Magnetic Moments	337
22.4	Bound Electron Magnetic Moment	339
22.4.1	Applicability of Stern-Gerlach Experiments	339
22.4.2	Difference Between the Unbound and the Bound Electron	339
	References	343

Appendix A: Early History of Charged-Particle Traps	347
Appendix B: Penning Trap Confinement: The Brief Version	365
Appendix C: Magnetic Field Creation and Control	377
Appendix D: Equivalence of Results for Magnetic Bottle Strength	391
Appendix E: Cryogenics	395
Appendix F: Collisional Effects in Penning Traps	399
Appendix G: General Design Considerations	415
Appendix H: A More Detailed Look at the Potential	419
Appendix I: A Special Case of Mediated Cooling	425
Index	445

Chapter 1

A Bit of History and Context



Abstract This chapter gives a brief account of the history of the Penning trap, the central characters involved in its development, and presents the main fields of operation of such traps together with a discussion of the sense in which the word ‘confinement’ needs to be understood in this context. It also clarifies some of the most important terminology and introduces the main ingredients of a quantitative description of confinement properties.

1.1 Namesake and Developmental History

The name ‘Penning trap’ goes back to Dutch physicist Frans Michel Penning (1894–1953), who in 1936 implemented his idea of improving the operation of a vacuum gauge by adding a magnetic field which forces the electrons on a radial path about the magnetic field lines (on account of the Lorentz force), hence increasing their time spent within the volume relevant for the measurement [1, 2].

‘[In the presence of an axial magnetic field], if there is a sufficient number of gas molecules in the chamber, an electron can collide with these molecules. If it loses energy in these collisions the return to the cathode is impossible and the electron will describe a significantly longer path before eventually impinging onto the anode...’

The first concept of what would later become a Penning trap is due to J.R. Pierce (1910–2002). He pointed out that it is possible to obtain a sinusoidal motion of electrons trapped in a combination of a quadrupole electric and axial magnetic field. **‘There is one particular field which perhaps merits individual attention [...]. This field can be produced by hyperbolic electrodes [...]. Suppose we have also a uniform magnetic field of strength B in the z direction [...]. We see that it is possible to obtain a pure sinusoidal motion of electrons trapped in this combination of electric and magnetic fields.’**

By this part of his book ‘Theory and Design of Electron Beams’ (1949) [3] he inspired H.G. Dehmelt (1922–2017) to build and first use such a kind of device for electron storage in 1959. Dehmelt coined this kind of trap a ‘Penning trap’ after Frans Michel Penning (Fig. 1.1).



Fig. 1.1 Artist's sketches of Penning, Dehmelt, Paul and Pierce

In the biographical of his 1989 Nobel contribution, Dehmelt stated: ‘I was not satisfied with the plasma trapping scheme used for the electrons and asked my student, Keith Jefferts, to study ion trapping in an electron beam traversing a field free vacuum space between two grids. Also, I began to focus on the magnetron/Penning discharge geometry, which, in the Penning ion gauge, had caught my interest already at Göttingen and at Duke. In their 1955 cyclotron resonance work on photoelectrons in vacuum Franken and Liebes had reported undesirable frequency shifts caused by accidental electron trapping. Their analysis made me realize that in a pure electric quadrupole field the shift would not depend on the location of the electron in the trap. This is an important advantage over many other traps that I decided to exploit. A magnetron trap of this type had been briefly discussed in J.R. Pierce’s 1949 book, and I developed a simple description of the axial, magnetron, and cyclotron motions of an electron in it. With the help of the expert glassblower of the Department, Jake Jonson, I built my first high vacuum magnetron trap in 1959 and was soon able to trap electrons for about 10 sec and to detect axial, magnetron and cyclotron resonances.’

In the following years, he and his co-workers performed a number of spectroscopic measurements with stored ions and protons [4–8]. In 1973, Dehmelt and Eckstrom proposed to use the force of an inhomogeneous magnetic field on the spin of an electron confined in a Penning trap [9]. This way, a precise measurement of the oscillation frequency yields information on the spin direction. They used this effect for the detection of induced changes of the spin direction of an electron by observing the corresponding changes in the electron’s oscillation frequency in a Penning trap. Since the confined particle’s spin direction is monitored continuously, Dehmelt called this the ‘continuous Stern-Gerlach effect’ [10]. It has since been successfully applied to a number of magnetic moment studies on both free and bound electrons as we will see below. Dehmelt was awarded the 1989 Nobel Prize in physics together with Wolfgang Paul (1913–1993) for ‘the development of the ion trap technique’. The prize was shared with Norman F. Ramsey ‘for the invention of the separated oscillatory fields method and its use in the hydrogen maser and other atomic clocks’. See also the Nobel lectures of Paul and Dehmelt [11, 12].

The theory of single charged particles confined in a Penning trap was called ‘Geonium theory’ by Dehmelt, to point out the notion of the confined particle and the rest of the world constituting a ‘Geonium atom’. This spirit is also captured

in the title ‘Single Elementary Particles at Rest in Free Space’ of Dehmelt’s 1979 publication [13].

Historically, the Penning trap was preceded by the Kingdon trap [14], sometimes called ‘orbitron trap’ [15], which is an electrostatic device (1923), and by the Paul trap [16], which is an electric radio-frequency device (1953) (Fig. 1.2). The Kingdon trap is named after Kenneth Hay Kingdon (1894–1982), who worked on the topic of electric discharges and the like for the General Electric company in Schenectady, New York. The Paul trap is named after Wolfgang Paul (1913–1993), who worked on improvements in mass spectrometry at the University of Bonn, Germany. A further treatment of early historical aspects of traps can be found in [17] and in appendix A.

In more recent times, the field of Penning trap techniques and physics was advanced by the work of many people, in our context notably by the work of Gerald Gabrielse (*1951), David Wineland (*1944) and their respective co-workers. The former is known for his work with anti-particles and anti-matter formation in traps, and for precise measurements of the electron magnetic moment. David Wineland

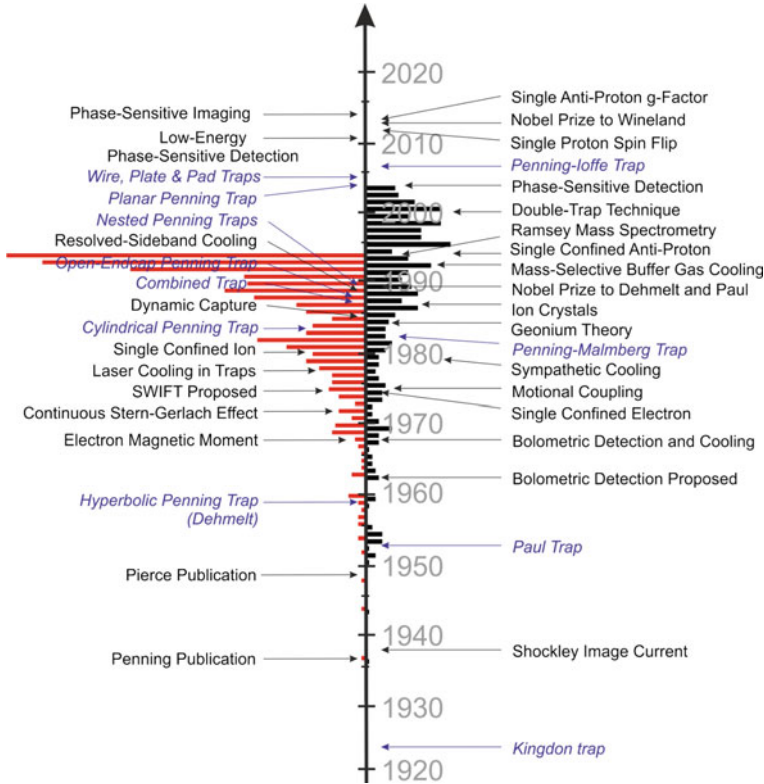


Fig. 1.2 Timeline of important events in connection with Penning traps (types of traps shown in italic). Bars in the background show the unscaled numbers of publications cited in [18] (right hand side, published 2005) and in [19] (left hand side, published 1995) as a function of the year

was awarded the 2012 Nobel Prize in Physics, together with Serge Haroche, for ‘ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems’ [20], which include laser cooling methods and quantum information processing with confined ions.

1.2 Fields of Operation

Figure 1.3 gives a rough overview of the main areas of research in which Penning traps and their derivatives play an important role. This list is not intended to be exhaustive, but to give an idea of the most typical situations in which one will encounter a trap of the type we are going to discuss. The three boxes on the bottom indicate the main capabilities of Penning traps that are exploited in that research, namely the possibility to localise, confine and cool particles for extended periods up to months, to select and isolate specific species, and to apply a wide variety of non-destructive methods for manipulation, detection and analysis.

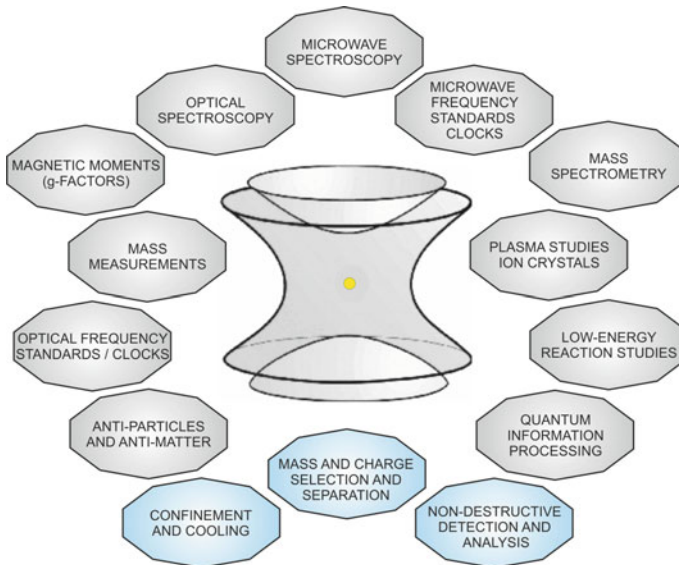


Fig. 1.3 Main areas of research in which Penning traps play an important role. Lower boxes: main capabilities of Penning traps that are exploited in that research. They may be thought of as the foundation on which the research builds

1.3 The Notion of Confinement

We will speak of ‘confinement’ as a stable configuration in which particles are localised in all three dimensions to a defined volume for long periods of time. ‘Stable’ means that the particle trajectory may be closed or unclosed, but is in any case limited to a small volume not too different from a sphere, and that at least in principle the confinement period can be infinite. The typical volumes to which particles are confined in the traps we discuss is usually between cm^3 and mm^3 or below. ‘Long periods’ commonly means times on the scale of seconds to months, depending on the application. Although, as we will see, in Penning traps there is one component of the motion that requires attention with regard to its stability, experiments have achieved confinement times of several months [21–23].

Historically, the first term used for some localisation of charged particles was ‘imprisonment’, brought forward by Kingdon [14]. The Kingdon trap, being an electrostatic device, is not capable of confinement in the sense above, and initial ‘imprisonment’ times were severely limited. They could later be prolonged in designs like the dynamic Kingdon trap [24] or the so-called ‘orbitrap’ [25, 26], but like in any other electrostatic device, the achievable times cannot compare with Paul or Penning traps. Purely electrostatic devices do not provide stable confinement but can create complicated escape paths for charged particles in them, and are often used for mass spectrometry [27–30].

‘Storage’ and ‘trapping’ are common synonyms for ‘confinement’, although ‘storage’ is not unique to the situation (since there are for example devices like storage rings which have nothing in common with the traps we will discuss), and ‘trapping’ is sometimes used in the same sense as ‘dynamic capture’ or ‘in-flight capture’ [31], namely for externally produced particles becoming confined in the trap. There are some traps which are designed solely for particle storage [23], but commonly, the trap is a device for experimental investigations of confined particles, and thus one of the most evident implementations of Carl Friedrich von Weizsäcker’s ‘interrogation of nature’ [32].

The unifying concept behind Penning traps is the combination of a static magnetic field with an electrostatic potential, leading to both radial and axial confinement of charged particles with respect to the direction of the magnetic field axis. In many cases, the magnetic field is homogeneous and of considerable strength of several Tesla. The electrostatic potential is commonly of quadrupolar shape such that charged particles perform the motion of a harmonic oscillator. We will discuss the details and implications below.

A three-dimensional potential that creates harmonic motions has the Cartesian form $U \propto ax^2 + by^2 + cz^2$. The Laplace equation in free space $\Delta U = 0$ requires us to have $a + b + c = 0$ [33]. Out of the possible solutions, the choice $a = +1$, $b = -1$, $c = 0$ is a linear quadrupole configuration, and the solution $a = -1$, $b = -1$, $c = +2$ is a three-dimensional quadrupole. In the following, this is the solution we are interested in for the Penning trap.

1.4 Terms and Notation

A word on the involved quantities and their notation: we will generally use B_0 as the field strength of the homogeneous part of the magnetic field used for confinement, in units of Tesla (T), and higher-order components B_k of the field given in units of T/m^k . The expansion coefficients $C_k < 1$ of the electrostatic confining potential U are scaled by the creating voltage U_0 and are hence of unit 1 ('dimension-less'). Spatial coordinates are most often Cartesian (x, y, z) or cylindrical (ρ, z, ϕ) , sometimes spherical (r, θ, ϕ) . The azimuthal angle ϕ , however, rarely occurs since Penning traps are desired to have radial symmetry such that azimuthal dependences only represent imperfections. Oscillation frequencies of confined particles are denoted by $\omega = 2\pi\nu$, frequency shifts by $\Delta\omega$, and widths of frequency distributions by $\delta\omega$. q is the signed electric charge of a particle, and m is its mass. Dotted quantities are time derivatives. With a few exceptions of traditional notation that are especially indicated, we use S.I. units throughout. Furthermore we have as most important quantities:

- the 'reduced cyclotron frequency' ω_+ of the perturbed cyclotron motion
- the 'magnetron frequency' ω_- of the drift motion in the crossed confining fields
- the 'axial frequency' ω_z of the oscillation along the magnetic field axis
- the cyclotron frequency ω_c , sometimes called 'free cyclotron frequency' (in absence of the electric field E created by the potential U), it is given by the modulus of qB_0/m
- the frequency ω_0 , the signed value of qB_0/m , i.e. we have $|\omega_0| = \omega_c$
- the frequency ω_1 , it has no name but appears in the context of the combined radial motion of a particle. Care is in place, since it is sometimes defined as $\omega_1^2 = \omega_c^2 - 2\omega_z^2$, and sometimes as $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$, we adapt the latter for the discussions to come
- the 'trapping parameter' $\beta = U_0/U_{\text{MAX}}$ given by the ratio of the actual trap voltage and the maximum value that still allows confinement according to the stability criterion
- the 'stability parameter' given by $\gamma = \omega_c/(\sqrt{2}\omega_z)$, such that for $\gamma > 1$ the stability criterion is fulfilled
- the particle number density $n = N/V$, not to be confused with a charge density
- the 'plasma frequency' ω_p , a characteristic frequency of a confined non-neutral plasma connected with the oscillation of the charge distribution
- the 'plasma parameter' Γ_p which measures the Coulomb potential energy of an arrangement of charged particles with respect to their kinetic energy
- the 'Brillouin frequency' $\omega_c/2$ which plays a special role in the description of confined particle plasmas ('Brillouin density'), and in a specific frame of reference ('Brillouin frame of reference').

References

1. F.M. Penning, Glow discharge between coaxial cylinders at low pressures in an axial magnetic field. *Physica (Utrecht)* **3**, 873 (1936)
2. F.M. Penning, Ein neues Manometer für niedrige Gasdrücke, insbesondere zwischen 10^{-3} und 10^{-5} mm. *Physica* **4**, 71 (1937)
3. J.R. Pierce, *Theory and Design of Electron Beams* (Van Nostrand, Princeton, 1949)
4. E.N. Fortson, F.G. Major, H.G. Dehmelt, Ultrahigh resolution $F=0,1$ $^3\text{He}^+$ Hfs spectra by an ion-storage collision technique. *Phys. Rev. Lett.* **16**, 221 (1966)
5. F.G. Major, H.G. Dehmelt, Exchange-collision technique for the rf spectroscopy of stored ions. *Phys. Rev.* **170**, 91 (1968)
6. H.G. Dehmelt, F.L. Walls, "Bolometric" technique for the rf spectroscopy of stored ions. *Phys. Rev. Lett.* **21**, 127 (1968)
7. D.A. Church, H.G. Dehmelt, Radiative cooling of an electrostatically contained proton gas. *J. Appl. Phys.* **40**, 3421 (1969)
8. H.A. Schuessler, E.N. Fortson, H.G. Dehmelt, Hyperfine structure of the ground state of $^3\text{He}^+$ by the ion-storage exchange-collision technique. *Phys. Rev.* **187**, 5 (1969)
9. H. Dehmelt, P. Ekstrom, Proposed $g - 2$ experiment on stored single electron or positron. *Bull. Am. Phys. Soc.* **18**, 727 (1973)
10. H. Dehmelt, Continuous Stern-Gerlach effect: principle and idealized apparatus. *Proc. Natl. Acad. Sci. USA* **83**, 2291 (1986)
11. H.G. Dehmelt, Experiments with an isolated subatomic particle at rest (Nobel Lecture), July 1990, <https://doi.org/10.1002/anie.199007341>
12. W. Paul, Electromagnetic traps for charged and neutral particles (Nobel Lecture), July 1990, <https://doi.org/10.1002/anie.199007391>
13. H. Dehmelt, R.S. Van Dyck, P.B. Schwinberg, G. Gabrielse, Single elementary particles at rest in free space. *Bull. Am. Phys. Soc.* **24**, 757 (1979)
14. K.H. Kingdon, A method for the neutralization of electron space charge by positive ionization at very low gas pressures. *Phys. Rev.* **21**, 408 (1923)
15. D.J. Wineland, Spectroscopy of stored ions, in: *Precision Measurement and Fundamental Constants II*, ed. by B.N. Taylor, W.D. Phillips, Natl. Bur. Stand. (U.S.), Spec. Publ. **617** (1984)
16. W. Paul, H. Steinwedel, Ein neues Massenspektrometer ohne Magnetfeld. *Z. Naturforsch. A* **8**, 448 (1953)
17. M.H. Holzschteier, A brief history in time of ion traps and their achievements in science. *Phys. Scr.* **T59**, 69 (1995)
18. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
19. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)
20. D.J. Wineland, Nobel lecture: superposition, entanglement, and raising Schrödinger's cat. *Rev. Mod. Phys.* **85**, 1103 (2013)
21. G. Gabrielse et al., Thousandfold improvement of the measured antiproton mass. *Phys. Rev. Lett.* **65**, 1317 (1990)
22. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
23. C. Smorra et al., A reservoir trap for antiprotons. *Int. J. Mass Spectrom.* **389**, 10 (2015)
24. H. Blümel, Dynamic Kingdon trap. *Phys. Rev. A* **51**, R30 (1995)
25. Q. Hu et al., The Orbitrap: a new mass spectrometer. *J. Mass Spectrom.* **40**, 430 (2005)
26. R.H. Perry, R.G. Cooks, R.J. Noll, Orbitrap mass spectrometry: instrumentation, ion motion and applications. *Mass. Spec. Rev.* **27**, 661 (2008)
27. H.T. Schmidt, H. Cederquist, J. Jensen, A. Fardi, Conetrapp: a compact electrostatic ion trap. *Nucl. Instrum. Methods Phys. Res. B* **173**, 523 (2001)
28. M. Dahan et al., A new type of electrostatic ion trap for storage of fast ion beams. *Rev. Sci. Instr.* **69**, 76 (1998)

29. M. Wang, A.G. Marshall, A “screened” electrostatic ion trap for enhanced mass resolution, mass accuracy, reproducibility, and upper mass limit in Fourier-transform ion cyclotron resonance mass spectrometry. *Anal. Chem.* **61**, 1288 (1989)
30. A. Makarov, Electrostatic axially harmonic orbital trapping: a high-performance technique of mass analysis. *Anal. Chem.* **72**, 1156 (2000)
31. H. Schnatz et al., Inflight capture of ions into a Penning trap. *Nucl. Inst. Meth. A* **251**, 17 (1986)
32. C.F. von Weizsäcker, *The Unity of Nature* (Farrar, Straus, and Giroux, New York, 1980)
33. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)

Chapter 2

Penning Trap Concept and Implementation



Abstract In this chapter, we have a look at the concepts involved in Penning-trap confinement, and the way in which the concepts can be implemented. It compares the confinement situation in Penning traps to those of other types of confinement such as in optical and acoustic traps, and shows the most important features of experimental implementations of Penning traps and their embedding.

2.1 Penning Trap Confinement

Penning traps can be found in a variety of geometries, sizes, and manufacturing detail. From the multitude of other types of traps (see for example the monographs by Ghosh [1] and Werth et al. [2, 3]) we call a trap ‘Penning trap’ when particle confinement is due to a superposition of a (homogeneous) magnetostatic field B_0 with a (harmonic) electrostatic potential U (Fig. 2.1).

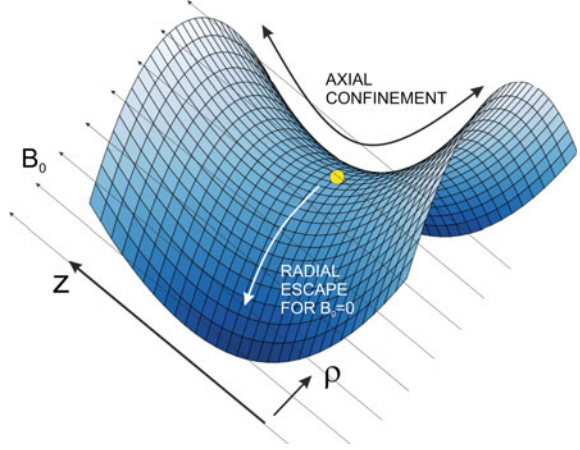
2.2 Field Configuration

As stated by Samuel Earnshaw (1805–1888) [4] in his famous theorem [5, 6], it is not possible to create a static electromagnetic field that confines a charged particle in all three dimensions. The electrostatic field of the Penning trap $\mathbf{E} = -\nabla U$ created by a three-dimensional quadrupolar potential U of the shape $(2z^2 - x^2 - y^2)$ represents a harmonic potential well along the axial direction z for a charged particle with $qU > 0$, but at the same time a repulsive potential in the radial direction ρ . This situation is shown in Fig. 2.2. A particle would hence escape radially within a short period. Radio-frequency (Paul) traps avoid this by rotating the saddle-shaped potential fast enough to hinder the escape, as often demonstrated by the mechanical analogue, a ball on a rotating saddle (‘pringle’), although the analogy is somewhat limited [7]. In a Penning trap, this is provided for by the strong magnetic field $\mathbf{B} = (0, 0, B_0)$ in the axial direction. A particle trying to escape radially must hence move perpendicular to the magnetic field which forces it on a radial orbit on account of the Lorentz force



Fig. 2.1 Cartoon of a Penning trap being a cage for a charged particle by a combination of electric and magnetic fields

Fig. 2.2 Electrostatic potential saddle created in a Penning trap. It becomes obvious that the charged particle's motion is stable in the axial direction z , but unstable in the radial direction ρ when the magnetic field B_0 is absent



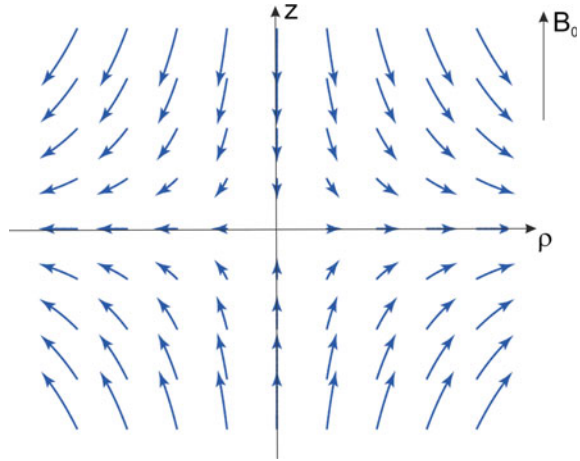
$\mathbf{F} = -q\nabla U + q(\mathbf{v} \times \mathbf{B}_0)$. Since the magnetic field and the repulsive electrostatic field are crossed, the particle is forced to perform a drift motion ('magnetron motion'), trying to slide down the electrostatic potential hill while thereby being forced on a perpendicular motion.

The electric field resulting from the potential shown in Fig. 2.2 is displayed in Fig. 2.3. Obviously, the field (and hence the forces) along the z -axis point towards the trap centre at $(z, \rho) = 0$, assuring axial confinement. At the same time, the field along the radial direction points away from the trap centre, which would lead to radial particle loss if the magnetic field were absent.

Note, that the energy required to move a particle of mass m and charge q radially in a uniform magnetic field from zero to R as given by

$$E_B = \frac{1}{2m} R^2 q^2 B_0^2, \quad (2.1)$$

Fig. 2.3 Electrostatic field configuration created in a Penning trap. It becomes obvious that a charged particle is forced towards the trap centre in the axial direction z , but away from the trap centre radially along ρ



and the energy that is required to move the same particle axially from zero to z as given by

$$E_E = qU \frac{z^2}{d^2} \quad (2.2)$$

for the same travelled distance ($z = R$) have a ratio of

$$\frac{E_B}{E_E} = \frac{1}{2m} d^2 q \frac{B_0^2}{U}, \quad (2.3)$$

which for typical parameters is roughly in the range of 10^2 – 10^3 . This means that the magnetic confinement is usually stronger than the electric confinement by that ratio. We will discuss this in more detail in Chap. 5.

Figure 2.4 illustrates a variety of realisations of the underlying Penning trap concept. The ‘traditional’ hyperbolic Penning trap is shown in the centre. This is the geometry initially used by Dehmelt and Paul for Penning and Paul traps. In the case of the Penning trap, there is an axial (here: vertical) magnetic field and an electrostatic potential created by a voltage between the central ring and the two endcaps. In a Paul-trap application, the magnetic field would be replaced by an additional radio-frequency voltage added to the static potential. Around the hyperbolic trap there are some examples for variations on the concept, all of which work with a magnetic field and an electrostatic potential. Many of these developments find their roots in specialised needs, mainly for improved optical access to the confined particles.

In the following, we will discuss these and some more Penning trap realisations in varying breadth, depending on their role in experimental applications. Note that this overview is not extensive in that it leaves out the multitude of possible combinations of such traps like double traps, triple traps and so forth. It also leaves out the micro-trap

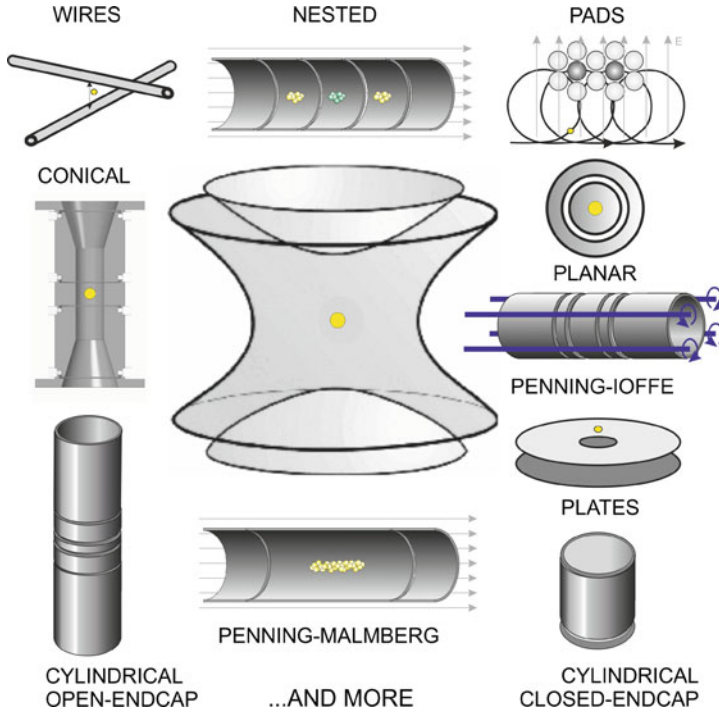


Fig. 2.4 Illustration of a variety of realisations of the Penning trap concept, with the ‘traditional’ hyperbolic Penning trap shown in the centre

arrays and chips designed for quantum-information processing as this is a different subject.

2.3 Penning Traps Compared to Other Types of Traps

Penning traps have the general advantage of being able to confine a broad range of charge-to-mass ratios simultaneously. Also, in contrast to traps that employ non-static fields, there is no inherent heating mechanism. The absence of a radio-frequency (rf) field of considerable strength as for example present in Paul traps facilitates the use of rf-based manipulation techniques as well as spectroscopy in that domain. The confinement is much tighter than in optical traps, allowing particles at much higher energies to remain confined. One disadvantage is the ever-presence of a strong

magnetic field, which typically necessitates a non-compact apparatus and brings along the feature that there is always a radial or rotary motion due to the magnetic field. To obtain high magnetic field strengths and a magnetic field that is constant both in time and space, most often a superconducting magnet surrounding the Penning trap is used. There are, however, also a number of Penning traps which make use of electromagnets or permanent magnets [8–11], including portable traps with battery-based confinement [12].

Let us briefly compare the confinement in a Penning trap to other types. For stable confinement, we require a potential energy minimum of some sort, i.e. an equilibrium position in space such that for any displacement there is a restoring force. This can be achieved by different choices of the medium that exerts the force and in various geometric arrangements, see also Table 2.1. Note that certain electro- and magneto-static devices [13–15] are sometimes counted as a trap, hence we mention them here, although confinement in them is not stable.

For the **Penning trap** and likewise for the **Paul trap** and the **combined trap**, it is the electromagnetic force of static and/or dynamic electric and magnetic fields that provides confinement. In optical traps, it is the electromagnetic radiation typically in the form of laser beams, sometimes in combination with magnetic or electric fields. They are typically used for neutrals, but traps such as the optical dipole trap can also confine charged particles.

In acoustic traps, it is a standing wave in a gas (and in principle also a fluid) that creates a local density minimum and thus the restoring force. Just to get some idea of the quantities involved in the description, we try to compare these situations by their potentials. For simplicity, we just look in the axial direction z .

As we will discuss in detail below, the axial potential created by a Penning trap (apart from eventual geometry coefficients of order unity) is given by

$$U(z) = \frac{U_0}{2d^2} (2z^2) \quad (2.4)$$

Table 2.1 Different families of traps that can be used with charged particles, the ‘grip’ i.e. the property by which the particle is influenced, and the media used for confinement

Trap family	Grip	Medium
Penning traps	Charge	Static E-field + static B-field
Paul traps	Charge	Static E-field + rf E-field
Combined traps	Charge	All of the above
Electrostats	Charge/moments	Static E-field (inhomogeneous)
Magnetostats	Charge/moments	Static B-field (inhomogeneous)
Optical traps	Level scheme	Electromagnetic radiation (+ fields)
Acoustic traps	Volume	Waves in gas/fluid

where d represents the geometric size of the trap, and U_0 is the voltage applied between the trap electrodes. In principle, there is almost no limit to the absolute values of the fields used, and this is likewise true for Paul and combined traps. In reality, one is typically limited to magnetic fields of several Tesla and to voltages of several kV. The confinement is specific to the electric charge of the particle under confinement, and most confinement properties depend mainly on the ratio of particle mass to electric charge.

An **optical dipole trap** [16] creates a potential well of depth

$$U(z) = \frac{3\pi c^2}{2\omega_0^3} \frac{\Gamma}{\Delta} I(z) \quad \text{with} \quad \Gamma = \frac{e^2 \omega^2}{6\pi \epsilon_0 m_e c^3}, \quad (2.5)$$

where ω_0 is the resonance frequency of the confined particle's optical transition in use, Δ is the detuning of the actual laser frequency ω of the confining beam from ω_0 , Γ is the classical damping rate corresponding to the spontaneous decay rate of the excited level and I is the laser field intensity. A more thorough discussion of this can be found in [16]. For Gaussian beam profiles the potential well is harmonic around its centre. The achievable well depths depend mainly on the optical properties (level scheme) of the particle and on the laser power. Note, that optical traps are mostly used for neutral particles like atoms, but can similarly be used for confinement of charged particles. The well depths then are much lower than in Penning and Paul traps. For an overview and a comparison of neutral-particle and charged-particle traps see [17].

An **acoustic trap** creates a potential well given by [18, 19]

$$U(z) = \frac{5}{16} \rho_0 v^2 V f(kd_s) \cos(2kz) \quad \text{with} \quad f(kd_s) = \frac{1}{kd_s^2} \left(\frac{\sin kd_s}{kd_s} - \cos kd_s \right), \quad (2.6)$$

where ρ_0 is the density of the medium used for acoustic wave propagation, v is the wave velocity in the medium, V is the volume of the (spherical) confined particle and kd_s is the relative particle diameter with the wave number k and the particle diameter d_s . From the equation it follows that the potential is deepest for a relative particle size of $kd_s = 2\pi/3$. The achievable well depths depend mainly on the medium used for confinement, denser media allow stronger confinement. The trap works with neutral and charged particles, however usually macroscopic ones.

2.4 Composition and Materials of Penning Traps

Macroscopic Penning traps most often are built from metal electrodes separated by insulator elements, with the electrodes kept in place by some exo-frame. The electrode geometry is usually chosen such that from the trap centre (the confinement volume) only metallic surfaces are visible, in order to avoid charging-up of insulator

elements which would distort the electrostatic potential close to the trap centre. The electrode material is normally desired to have high electric conductivity (both in the dc and rf domains) and in the case of cryogenic set-ups also high thermal conductivity. Hence, the most common electrode material is copper (sometimes silver), typically of high purity and oxygen-free (such as Oxygen-Free High thermal Conductivity copper, OFHC copper). Often, there is a surface coating of gold on a diffusion barrier [20] of silver, to avoid oxidation and to keep the electric patch effects [21–23] on the surface small, see also Sect. 6.5.

Figure 2.5 shows a generic example of the composition of a Penning trap in a sectional view. It is a stack of high-purity copper electrodes with gold plating. The electrodes have lids to shield the separating insulators from the trap's interior. An exoframe holds the electrode stack together. This example is a cylindrical Penning trap with a ring that is radially split into 8 segments, with one endcap on either side, and an additional electrode for particle capture on either side. We will discuss the function of such an arrangement below.

Apart from special applications (like for example magnetic bottles as will be discussed in Chap. 21), ferromagnetic materials such as iron, nickel, cobalt and samarium are avoided as they deform the magnetic field. Insulators are commonly made of ceramics such as Macor or PEEK (PolyEther-Ether-Keton), aluminium-nitrides such as Shapal, sometimes also Nylon and the like are used. The decay of the price of sapphire components has led to their increased use, mainly because of their excellent insulator properties combined with high thermal conductivity, see right in Fig. (2.6).

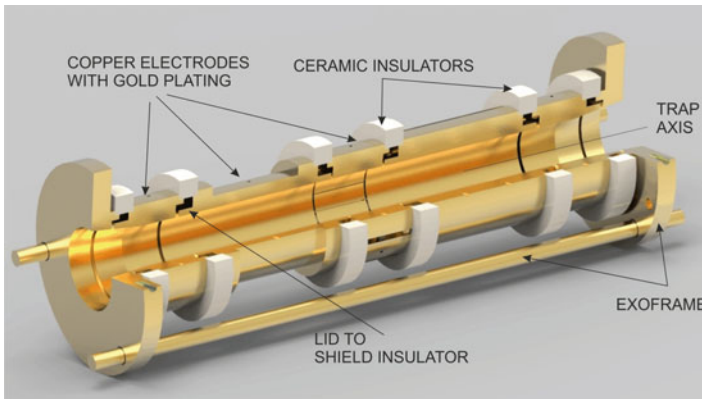


Fig. 2.5 Photo-realistic sectional view of a generic example of the composition of a Penning trap

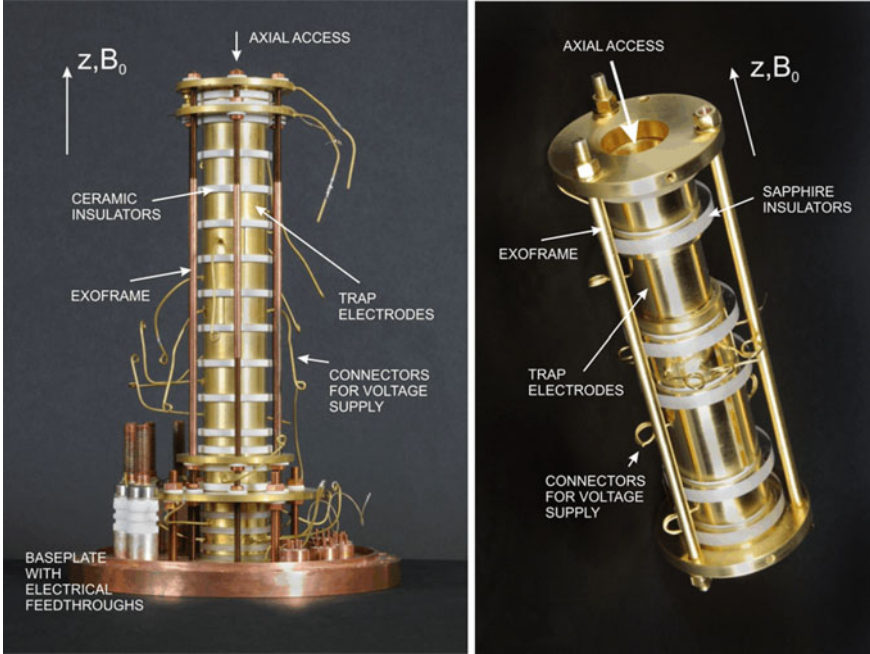


Fig. 2.6 Two examples of cylindrical Penning traps. They are mainly comprised of hollow (gold-plated) copper cylinders insulated by ceramics

2.5 The Embedding of Penning Traps

2.5.1 Trap Chamber and Vacuum

Confinement and localisation of charged particles for extended periods is one of the key motivations for the use of Penning traps. This most often makes it necessary to create good vacuum conditions inside the trap. It is usually undesirable to have any reactions of the confined species with residual gas (either chemical or physical processes like charge exchange), and it is typically also undesirable to have a high rate of collisions between the species of interest and residual gas, as this may eventually lead to particle loss from the trap, as will be discussed below. Hence, most Penning traps are placed in UHV chambers which are pumped to the vacuum region of 10^{-9} mbar and below, often they are located in a cryogenic surrounding where the trap electrodes and other components like electronics attached to the electrodes are cooled to some low temperature that ensures efficient cryo-pumping (below about 20 K, see Sect. D.3) down to residual gas pressures below 10^{-16} mbar. In many such cases, liquid-helium temperature (around 4 K) is used since that also enables the

use of superconducting electronic equipment. With the use of dilution refrigerators, temperatures of the order of 100 mK and below can be achieved, see for example [24, 25], which facilitates electronic cooling to the motional ground state of particles in the trap.

2.5.2 Superconducting Magnets

The magnetic field is usually required to be strong (of the order of several Tesla) and homogeneous as well as constant, so often superconducting magnets are used for its production. Since the centre of the trap is typically positioned at the centre of the magnetic field, the trap is in most cases located inside the superconducting magnet, see Fig. 2.7. Either this is a ‘warm-bore’ magnet which has a room-temperature opening for positioning of the trap (the manufacturers call it ‘the probe’), or a ‘cold-bore’ magnet in which the trap is integrated into the magnet’s cryo- and vacuum system. The prior case is more common due to its greater flexibility and accessibility, but requires a separate trap vacuum and in case of a cryogenic trap also a separate cryostat to be inserted into the magnet bore.

Magnet geometries depend on the designated use. Most common are solenoids around the field centre, see the left and lower right examples in Fig. 2.7. When radial access to the centre is desired, for example for optical beams, a Helmholtz-arrangement is usually chosen [26], for an example see upper right of Fig. 2.7. Depending on the design homogeneity of the central field, additional coils may be used to correct the field close to the centre (‘shim coils’ [27]). Often, these are supplied with electrical current in series with the main coils, for highly homogeneous fields, however, they need to be tuned separately. The desired field homogeneity is typically defined by the maximum allowed relative field difference within a specified volume like a central cm^3 or so, which usually is the designated confinement volume. A value of 10^{-4} (100 ppm) is normally achieved without significant efforts, and values of 10^{-9} (1 ppb) and better are possible by active (independent) shimming. This applies to bore diameters which are typically between about 50 and 200 mm. In the examples in Fig. 2.7, the bore diameters are 160 mm (left), 80 mm (top right), and 150 mm (lower right), respectively. The coils usually have several tens of thousands of windings of thin insulated wire. Most commonly, niobium-titanium (NbTi) is used as the superconductor, which has a zero-field critical temperature of 9.2 K and can be used up to magnetic field strengths of about 15 T [28]. For higher field strengths of up to around 25–30 T niobium-tin (Nb_3Sn) is often used, which maintains superconductivity at much higher current densities, but is more difficult to manufacture [29]. In this high-field region, also high-temperature superconductors such as yttrium barium copper oxide (YBCO) have been used for fields of the order of 30 T [30, 31].

Typical currents are around 50–200 A and above, the central field strength created is commonly between 0.01 and 0.1 T/A. The current is normally run in series with a current supply through a switch which is non-superconducting when heated for insertion of the current, and becomes superconducting once the heating is removed.

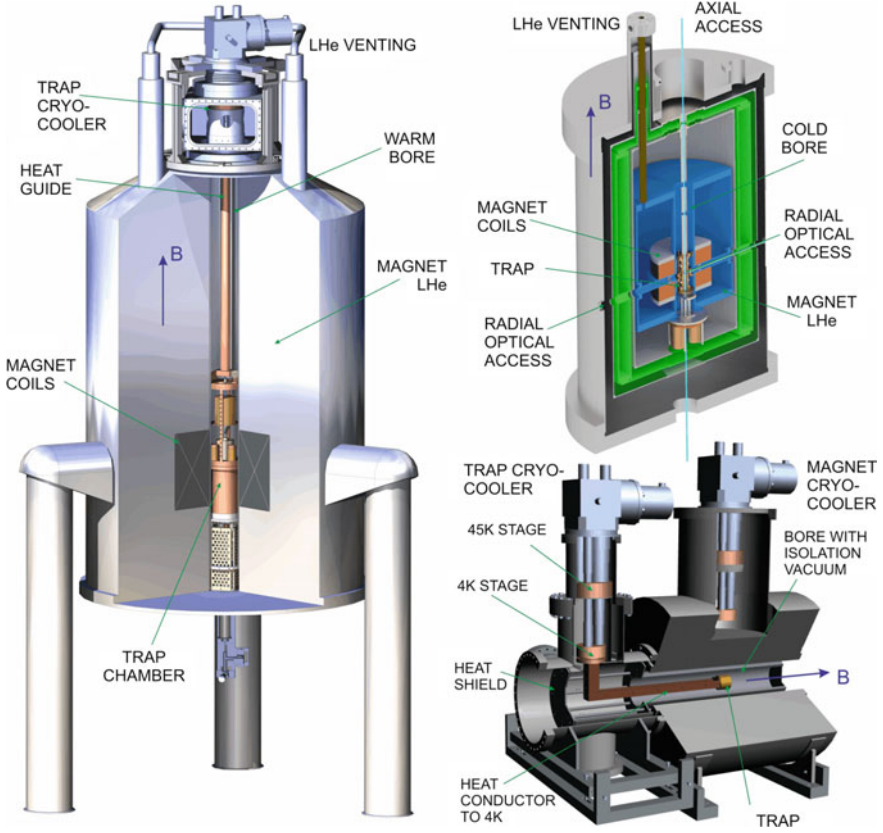


Fig. 2.7 Three examples of Penning trap setups with superconducting magnets: Left: vertical warm-bore magnet with liquid-He cooling. Top right: Vertical cold-bore magnet with Helmholtz configuration and liquid-He cooling. Lower right: Horizontal warm-bore magnet with cryo-cycle-cooling. Roughly to scale

The current then runs in the so-called ‘persistent mode’ [32] and the current supply can be removed. In many applications, the magnetic field is established once and then left untouched for long periods up to many years. Lower limits for the decay half-life of the superconducting current are of the order of 10^5 years [33]. The superconducting state can be lost on account of undesired magnetic flux fluctuations which is called a ‘magnet quench’ [32]. Most magnets have special safety installations to protect the magnet from the effects and dump the energy stored in the current and the magnetic field when superconductivity is lost. This energy can have considerable values of the order kJ to MJ. To reduce the likelihood of a quench to occur during operation, magnets are usually ‘trained’ by the first few introductions of the current to allow the coil windings to settle to stable positions, thus minimising the sources of flux jumps [34].

Cryostats for cooling of the superconducting magnet coils can either be dewars that evaporate liquid helium for cooling (left and upper right examples in Fig. 2.7), or they can be closed-cycle systems with ‘cold-heads’ such as the magnet in the lower right of Fig. 2.7. The latter have the advantage of not using up liquid gases, but produce cost for electricity and maintenance. They also may cause undesired vibrations, which is why they are uncommon in high-precision experiments.

References

1. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)
2. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
3. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
4. W.T. Scott, Who was Earnshaw? *Am. J. Phys.* **27**, 418 (1959)
5. S. Earnshaw, On the nature of the molecular forces which regulate the constitution of the luminiferous ether. *Trans. Camb. Phil. Soc.* **7**, 97 (1842)
6. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)
7. R.I. Thompson, T.J. Harmon, M.G. Ball, The rotating-saddle trap: a mechanical analogy to RF-electric-quadrupole ion trapping? *Can. J. Phys.* **80**, 1433 (2002)
8. V. Gomer et al., Magnetostatic traps for charged and neutral particles. *Hyp. Int.* **109**, 281 (1997)
9. J.N. Tan, S.M. Brewer, N.D. Guise, Penning traps with unitary architecture for storage of highly charged ions. *Rev. Sci. Instr.* **83**, 023103 (2012)
10. S.M. Brewer, N.D. Guise, J.N. Tan, Capture and isolation of highly-charged ions in a unitary Penning trap. *Phys. Rev. A* **88**, 063403 (2013)
11. L. Suess, C.D. Finch, R. Parthasarathy, S.B. Hill, F.B. Dunning, Permanent magnet Penning trap for heavy ion storage. *Rev. Sci. Instr.* **73**, 2861 (2002)
12. C. Tseng, G. Gabrielse, Portable trap carries particles 5000 kilometers. *Hyp. Int.* **76**, 381 (1993)
13. D. Zajfman et al., High resolution mass spectrometry using a linear electrostatic ion beam trap. *Int. J. Mass Spectrom.* **229**, 55 (2003)
14. O. Kelly et al., Femtosecond lasers for mass spectrometry: proposed application to catalytic hydrogenation of butadiene. *The Analyst* **137**, 64 (2011)
15. J. Perez-Riosa, How does a magnetic trap work? *Am. J. Phys.* **81**, 836 (2013)
16. R. Grimm, M. Weidemüller, Y.B. Ovchinnikov, Optical dipole traps for neutral atoms. *Adv. At. Mol. Opt. Phys.* **42**, 95 (2000)
17. D.E. Pritchard, M.P. Bradley, Atom traps compared with ion traps. *Phys. Scr.* **T59**, 131 (1995)
18. L.V. King, On the acoustic radiation pressure on spheres. *Proc. Roy. Soc.* **147**, 212 (1934)
19. E.G. Lierke, Akustische Positionierung. *Acoust. Acta Acoust.* **82**, 220 (1996)
20. M.R. Pinnel, J.E. Bennett, Mass diffusion in polycrystalline copper/electroplated gold planar couples. *Metall. Trans.* **3**, 1989 (1972)
21. J. Labaziewicz et al., Temperature dependence of electric field noise above gold surfaces. *Phys. Rev. Lett.* **101**, 180602 (2008)
22. M. Brownnutt, M. Kumph, P. Rabl, R. Blatt, Ion-trap measurements of electric-field noise near surfaces. *Rev. Mod. Phys.* **87**, 1419 (2015)
23. A. Safavi-Naini, P. Rabl, P.F. Weck, H.R. Sadeghpour, Microscopic model of electric-field-noise heating in ion traps. *Phys. Rev. A* **84**, 023412 (2011)
24. G. Gabrielse et al., Precise Matter and Antimatter Tests of the Standard Model, in *Fundamental Physics in Particle Traps, Springer Tracts in Modern Physics*, 256 (Springer, Heidelberg, 2014)
25. P. Bushev et al., Electrons in a cryogenic planar Penning trap and experimental challenges for quantum processing. *Eur. Phys. J. D* **50**, 97 (2008)
26. L. Gruber, J.P. Holder, D. Schneider, Formation of strongly coupled plasmas from multi-component ions in a Penning trap. *Phys. Scr.* **71**, 60 (2005)

27. W.A. Anderson, Electrical current shims for correcting magnetic fields. *Rev. Sci. Inst.* **32**, 241 (1961)
28. G.B. Yntema, Superconducting winding for electromagnets. *Phys. Rev.* **98**, 1197 (1955)
29. J.E. Kunzler, E. Buehler, F.S.L. Hsu, J.H. Wernick, Superconductivity in Nb_3Sn at high current density in a magnetic field of 88 kilogauss. *Phys. Rev. Lett.* **6**, 890 (1961)
30. M.D. Bird, I.R. Dixon, J. Toth, Large. High-field magnet projects at the NHMFL. *IEEE Trans. Appl. Supercond.* **25**, 4300606 (2015)
31. New mag lab record promises more to come, News Release of the National High Magnetic Field Laboratory, USA (2007)
32. M.N. Wilson, *Superconducting Magnets* (Clarendon Press, Oxford, 1983)
33. J.C. Gallop, *SQUIDS, the Josephson Effects and Superconducting Electronics* (CRC Press, Boca Raton, 1990). ISBN 0-7503-0051-5
34. G. Pasztor, C. Schmidt, Dynamic stress effects in technical superconductors and the ‘training’ problem of superconducting magnets. *J. Appl. Phys.* **49**, 886 (1978)

Chapter 3

Hyperbolic and Cylindrical Penning Traps



Abstract The oldest Penning trap geometry in use is the hyperbolic shape, which provides good confinement properties by design, however was difficult to machine to high precisions at the time of its introduction, and offers limited access for particles and laser beams. Thus, cylindrical designs were brought forward, including more open structures. Here, we briefly review the hyperbolic and the most important forms of cylindrical Penning traps. Following that, we will have a look at variations on the concept and at the confinement properties.

3.1 Hyperbolic Penning Trap

A hyperbolic Penning trap (sometimes called ‘quadrupole Penning trap’ [1]) consists of a ring and two endcaps of which the ring obeys the geometry (radial coordinate ρ , axial coordinate z)

$$\left(\frac{\rho}{\rho_0}\right)^2 - 2\left(\frac{z}{\rho_0}\right)^2 = 1 \quad (3.1)$$

and the endcaps follow the form

$$\frac{1}{2}\left(\frac{\rho}{z_0}\right)^2 - 2\left(\frac{z}{z_0}\right)^2 = -1, \quad (3.2)$$

where z_0 is the distance of each of the endcaps to the trap centre and ρ_0 is the central ring inner radius (Fig. 3.1). When a voltage $U_0 = U_e - U_r$ is applied between the ring and the combined endcaps, this arrangement creates an electrostatic potential of the form

$$U = \frac{U_0}{2d^2} (2z^2 - x^2 - y^2) \quad (3.3)$$

in which d is the so-called ‘characteristic trap size’ given by

$$d^2 = \frac{1}{2} \left(z_0^2 + \frac{\rho_0^2}{2} \right). \quad (3.4)$$

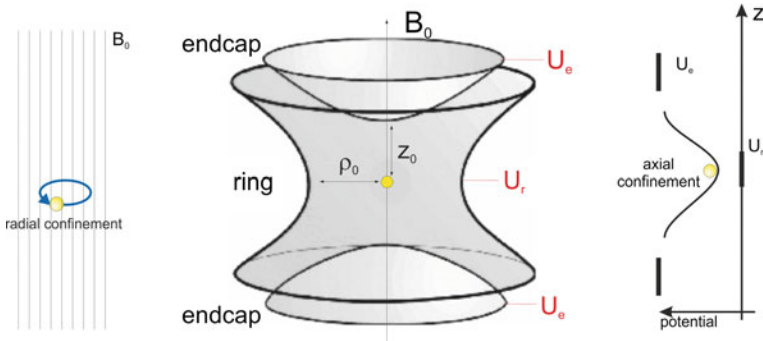


Fig. 3.1 Schematic of a hyperbolic Penning trap. The confining electric and magnetic fields are indicated separately aside the trap

When the ratio ρ_0/z_0 is chosen to equal $\sqrt{2}$, the potential on the ring and endcaps is equal and opposite with respect to the centre of the trap. Any choice of the ratio ρ_0/z_0 will lead to a quadrupolar potential [2], but in this special case, we obtain the largest effective confinement space. This trap geometry has the advantage that it *by design* creates a quadrupolar potential which leads to a harmonic axial oscillation. It has, however, the disadvantage of being almost completely closed both mechanically and optically.

A common solution is the introduction of holes or slits in the electrodes. The approximate effects of such openings on the trap potential have been discussed analytically in [3]. For an axial hole of diameter h , one finds an octupolar field contribution $C_4 \propto (h/d)^3$ and for an azimuthal slit of length l a corresponding contribution $C_4 \propto (l/d)^2$. We will discuss the consequences of such contributions in detail below.

Also, the precise machining of such shapes was a demanding ('painstaking') task before the upcoming of computerised manufacturing tools [4]. The effects of a truncation of the hyperboloidal electrodes to open up the trap structure for better access have been discussed in detail in [5].

Sometimes, one encounters the same arrangement in cubic form (like cubic ion-cyclotron-resonance (ICR) cells for mass spectrometry [6]) with side length $2a$ [7]. Near the centre, this produces the same potential as in (3.3), with $d = a$ and an additional factor of 0.693432 [8].

3.2 Cylindrical Closed-Endcap Penning Trap

3.2.1 Mechanical Compensation

One possibility to simplify the geometry is the use of cylindrical electrodes. The most simple configuration is a three-pole cylindrical trap with closed endcaps, i.e. a

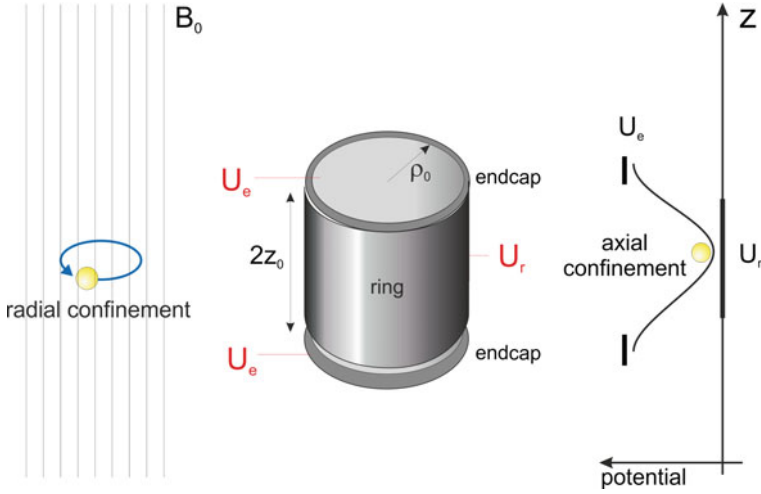


Fig. 3.2 Schematic of a mechanically compensated closed-endcap cylindrical Penning trap. The confining electric and magnetic fields are indicated separately aside the trap

hollow cylinder of inner radius ρ_0 as ring electrode and flat closed endcap electrodes (Fig. 3.2).

Such a trap has cylindrical symmetry along the z -axis and mirror symmetry with respect to the (x, y) -plane though $z = 0$. The deviation from the hyperbolic shape causes the electrostatic potential U as created by the applied voltage difference $U_0 = U_e - U_r$ inside the trap not to be quadrupolar, which leads to non-harmonic axial confinement. By appropriate choice of the endcap distance $2z_0$ with respect to the inner diameter ρ_0 of the electrodes.

$$\frac{\rho_0}{z_0} = 1.203, \quad (3.5)$$

it is possible to obtain a compensation of the confining potential, such that close to the trap centre it becomes quadrupolar. This is called ‘mechanical compensation’, sometimes also called ‘geometric compensation’, for details see the discussion in [9].

3.2.2 Electric Compensation

One alternative to mechanical compensation is to add correction electrodes, also called ‘compensation electrodes’ on either side of the ring, i.e. between the ring and either endcap, see Fig. 3.3. When we supply these electrodes with a voltage U_c , we have another degree of freedom to tune the trap potential close to the centre. Compensation electrodes allow one to electrically tune the potential in order to achieve a harmonic trapping potential in spite of the electrodes having non-hyperbolic

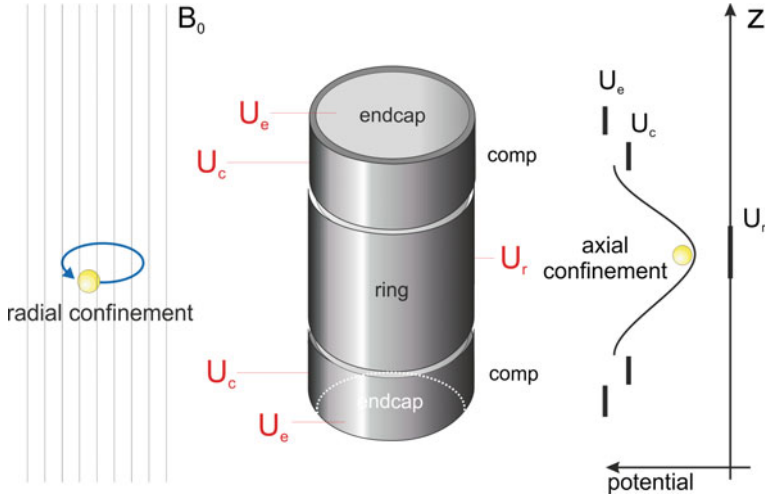


Fig. 3.3 Schematic of an electrically compensated closed-endcap cylindrical Penning trap. The confining electric and magnetic fields are indicated separately aside the trap

geometry and not being mechanically compensated. The details of the proper choice of the applied voltages U_c , U_e and U_r are given in [9].

3.3 Cylindrical Open-Endcap Penning Trap

The issue of inaccessibility of a closed-endcap trap is to some extent solved by removing the caps, and in turn elongating the endcaps to provide harmonic confinement close to the trap centre.

Such a trap has been described in detail in [4, 10]. Axially, the particle is confined by a constant voltage $U_0 = U_r - U_e$ applied between the central ring electrode and the endcap electrodes. Additionally, compensation electrodes with voltage U_c are placed on either side of the ring between the ring and the endcaps to create a harmonic potential around the trap centre. A schematic of such a trap is given in Fig. 3.4. Since this trap geometry finds wide application, we discuss its properties in a bit more detail. As we will see later, we can write the electrostatic potential near the trap centre by the expansion [10]

$$U = \frac{1}{2}U_0 \sum_{k=0}^{\infty} C_k \left(\frac{r}{d}\right)^k P_k(\cos \theta); \quad (k \text{ even}) \quad (3.6)$$

where the summation is over even k only, $r = (\rho^2 + z^2)^{1/2}$ is the distance to the trap centre and $P_k(\cos \theta)$ are Legendre polynomials of the k -th degree with the argument

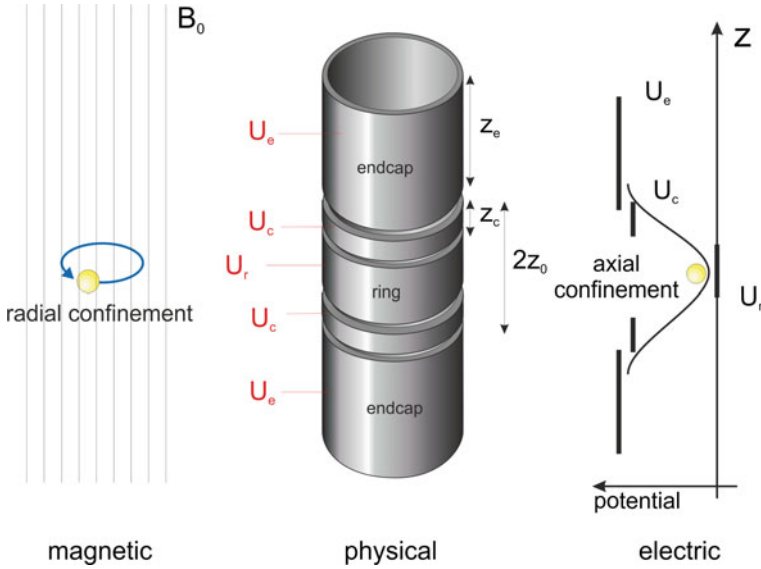


Fig. 3.4 Schematic of a five-pole open-endcap cylindrical Penning trap. The confining electric and magnetic fields are indicated separately aside the trap

$$\cos(\theta) = \frac{z}{\sqrt{\rho^2 + z^2}} = \frac{z}{r}, \quad (3.7)$$

z being the axial distance to the trap centre. d again is the ‘characteristic trap size’ given by $d^2 = (z_0^2 + \rho_0^2/2)/2$. Note, that this is not the only way to represent the potential, but it is the one historically chosen first, so we refer to it here. We will discuss alternatives below. Since the trap has electric symmetry with respect to the trap centre, i.e. cylindrical symmetry with respect to the central trap axis and mirror symmetry perpendicular to that axis at the position of the trap centre, only even expansion coefficients have non-zero values. The coefficient C_0 is an overall potential offset and hence irrelevant for the particle motion. C_2 represents the desired quadrupole term relevant for electrostatic confinement and reflects the ‘efficiency’ of the trap in creating a potential well from the applied voltages. Its value chiefly determines the exact values of the oscillation frequencies of particles in the trap, as we will see in Chap. 5. A hyperbolic Penning trap has $C_2 = 1$ by design. For cylindrical Penning traps C_2 is typically about 0.5. In cylindrical Penning traps, the dominant electric imperfection is characterized by the term C_4 , the octupole component of the electrostatic potential. The next term C_6 (the dodecapole component) is suppressed with respect to the term in C_4 by a factor of $(r/d)^2$, which is often of the order of 10^{-4} or smaller. Usually, terms above C_6 may be neglected as for particles close to the trap centre their contribution is below the experimental resolution, but we will discuss this in more detail below.

Importantly, the coefficients, in particular also C_2 and C_4 , depend on the applied voltages and can be written as [10]

$$C_2 = C_2^{(0)} + D_2 \frac{U_c}{U_0} \quad \text{and} \quad C_4 = C_4^{(0)} + D_4 \frac{U_c}{U_0} \quad (3.8)$$

where the $C_k^{(0)}$ and D_k are given by the trap geometry [10]. It is hence possible to choose the value of U_c/U_0 , the so-called ‘tuning ratio’, such that $C_4 = 0$. In general, due to a non-zero D_2 , this however also changes the value of C_2 such that the trap potential near the centre and hence all the oscillation frequencies are changed. Luckily, for any choice of z_c/z_0 there is a unique choice of ρ_0/z_0 such that $D_2 = 0$. This special geometry is called an ‘orthogonalised trap’, in which C_4 can be tuned out by choice of the proper tuning ratio U_c/U_0 without changing the well depth $C_2 U_0$. For one specific value of z_c/z_0 , the tuning ratio U_c/U_0 does not only tune out C_4 , but it simultaneously tunes out C_6 as well. This special point has the geometry

$$\frac{z_c}{z_0} = 0.8351 \quad \text{and} \quad \frac{\rho_0}{z_0} = 1.0239. \quad (3.9)$$

For the replacement of closed endcaps by open cylinders to work, the endcaps need to be sufficiently elongated, [10] states that for $z_e/\rho_0 > 3$ the coefficients are expected to be within 1% of their true values. We will have a closer look at field imperfections and their consequences below.

3.4 Cylindrical Half-Open Penning Trap

This Penning trap design is a variation on a closed cylindrical Penning trap in which one closed endcap is replaced by open cylindrical electrodes (‘anti-compensator’ and ‘anti-ring’) to allow injection and ejection of particles, while the other endcap is electrically closed but optically transparent [11], which is realized by a window with conductive coating that we will discuss below. This dedicated arrangement of electrodes brings the electric trap centre and hence the confinement region in close proximity to the optically open endcap, and thus leads to a significantly larger solid angle of detection as one would have in the standard open-endcap design (Fig. 3.5).

3.4.1 Light Collection Efficiency

To quantify this statement, we look at the typical dimensions of such traps. The inner trap radius is ρ_0 and the lengths of the ring, compensation and endcap electrodes are z_r , z_c and z_e , respectively. We ignore the small gaps between electrodes, hence the distance z of the trap centre from the end surface of the trap endcap is given

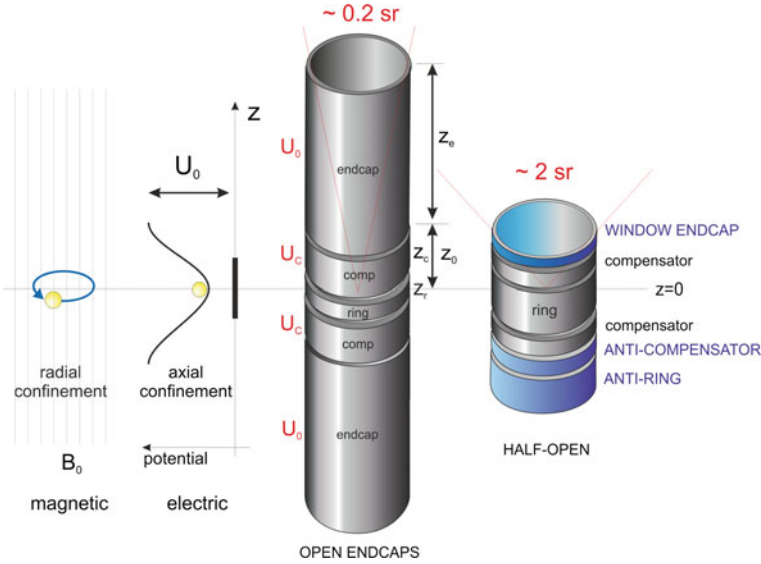


Fig. 3.5 Comparison of a five-pole open-endcap cylindrical Penning trap (left) and a half-open cylindrical Penning trap (right). The main differences to the classic open-endcap design are highlighted and marked with capital text. The confining electric and magnetic fields are indicated separately aside the trap

by $z = z_r/2 + z_c + z_e$. In the ‘half-open’ trap where $z_e = 0$, this distance is about $z \approx 1.5\rho_0$, while in standard open-endcap cylindrical traps the endcaps need to be very long to achieve comparable harmonicity (z_e at least about $4z_0$ while $z_0 \approx \rho_0$, see [10]), such that this distance is about $z \approx 5\rho_0$. The full opening angle Θ of the light cone from the trap centre is given by

$$\Theta = 2 \tan^{-1} \left(\frac{\rho_0}{z} \right), \quad (3.10)$$

and the corresponding solid angle of detection Ω is given by

$$\Omega = 4\pi \sin^2 \left(\frac{\Theta}{4} \right). \quad (3.11)$$

Hence, the ratio of opening angles Θ between the traps is about 10/3 and the corresponding ratio of solid angles is about 11 in favour of the half-open design.

The advantage of light detection along the trap axis as possible in an optically open endcap trap as compared to radial light detection may be even more pronounced in situations with favourable angular distribution of radiation emitted from confined particles. For example, magnetic dipole radiation in the presence of the confining field has an angular distribution given by [12]

$$I(\theta)d\theta \propto (1 + \cos^2 \theta)d\theta, \quad (3.12)$$

where θ is the spherical polar angle with respect to the central trap axis. This directional characteristic enhances emission along the axis of the magnetic field (i.e. the trap axis) by a factor of 2 with respect to radial emission [12], which further increases the ratio of light collected through the window to the total light emitted, see also the discussion in Chap. 18.

3.4.2 Conducting Window Endcap

To retain the electrostatic potential of the trap, the inner surface of the endcap window can be coated with a thin conductive layer of indium tin oxide (ITO) or a similar optically transparent coating. ITO is a doped n-type semiconductor with a band gap of around 4 eV and is therefore close to completely transparent in the visible part of the spectrum. It usually consists of about 75% indium (In), 17% oxygen (O₂) and 8% tin (Sn) in a homogeneous layer on a substrate. Besides ITO (In₂O₃:Sn), a number of other coatings are available, for example fluorine-tin-oxide FTO (SnO₂:F) and antimony-tin-oxide ATO (SnO₂:Sb) which have roughly similar properties, for details see [13]. ITO has a bulk resistance of about $10^{-5} \Omega\text{m}$ (three orders of magnitude higher than copper), which typically corresponds to a sheet resistance of around $80 \Omega/\text{sq}$ at a layer thickness of 125 nm [13]. The optical transmission in the visible regime is around 80%. Importantly, the conducting surface is geometrically better defined by several orders of magnitude (when compared to machined bulk metal surfaces) which facilitates highly harmonic confining potentials. This is an important feature since in the half-open trap design, the particles are confined in close proximity to the closed endcap such that the potential at their position is sensitive to imperfections of the electrode's conducting surface (Fig. 3.6).

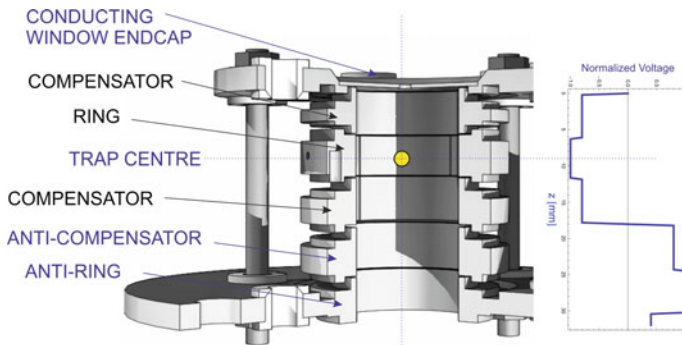


Fig. 3.6 Left: sectional view of a half-open cylindrical Penning trap with transparent endcap. Right: applied voltages

Such a coating has been shown to be suited for the operating conditions of cryogenic Penning traps, including the use as a Faraday cup for destructive particle detection [14]. At cryogenic temperatures, the overall shape of the resistance curve follows the Bloch-Grüneisen law (after Felix Bloch (1905–1983) and Eduard Grüneisen, 1877–1949), as expected for metallic layers like the present one [15, 16]. Note that for much smaller layer thicknesses below 40 nm, significantly different behaviour has been reported [17]. The observed increase of resistance below about 100 K [14] is in accordance with measurements of similar ITO films [15, 16] and is attributed to effects of weak localisation and electron-electron interaction [15]. The temperature dependence of the resistance is expected to be different when an external magnetic field is applied, this however is a minor effect [16]. Measurements of the electrical conductivity have been reported [18] up to a frequency of 20 GHz and show that it remains roughly at the dc value of about 0.5×10^6 S/m. The skin depth of ITO up to 20 GHz is above $1 \mu\text{m}$, and hence much larger than the layer thickness [18]. From this, ITO layers can be expected to behave similarly throughout the whole radio-frequency domain.

Figure 3.7 shows an artist's view of what is deemed possible with ITO or other transparent conductive layers on bent substrates like glass. Note, that in situations where high thermal conductivity is required, a sapphire substrate can be used. The common coating techniques include sputtering, spray coating and vapour deposition, all of which allow one to fully coat or to produce two-dimensional structures on a substrate surface, which is also true for bent substrates. A cylindrical Penning trap with transparent ring and endcap electrodes would lend itself to flexible access with laser beams and provide large solid angles of optical detection.

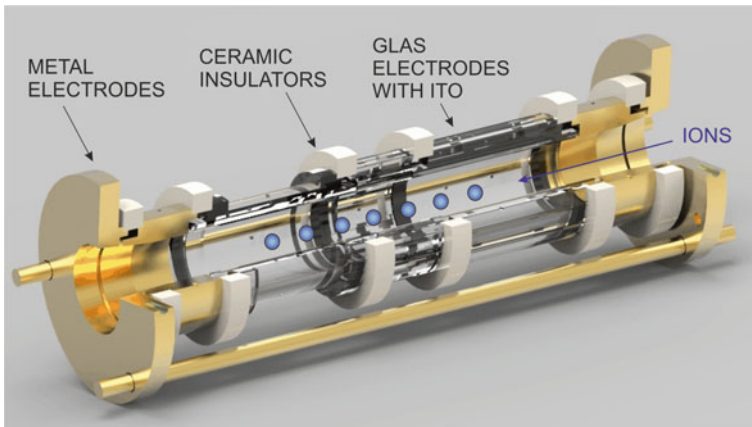


Fig. 3.7 Sectional view of a possible application of a cylindrical Penning trap with transparent central electrodes that feature a conductive coating.eps

References

1. B.M. Dyavappa, D. Datar, S. Ananthamurthy, Dependence of the confinement time of an electron plasma on the magnetic field in a quadrupole Penning trap. *EPJ Techn. Instrum.* **4**, 4 (2017)
2. R.D. Knight, The general form of the quadrupole ion trap potential. *Int. J. Mass Spectrom. Ion Proc.* **51**, 127 (1983)
3. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
4. J. Tan, G. Gabrielse, One electron in an orthogonalized cylindrical Penning trap. *Appl. Phys. Lett.* **55**, 2144 (1989)
5. E.C. Beatty, Calculated electrostatic properties of ion traps. *Phys. Rev. A* **33**, 3645 (1986)
6. D.W. Mitchell, Theory of trapped ion motion in the non-quadrupolar electrostatic potential of a cubic ion cyclotron resonance cell. *Int. J. Mass Spectrom. Ion Proc.* **142**, 1 (1995)
7. V.L. Campbell, Z. Guan, V.H. Vartanian, D.A. Laude, Cell geometry considerations for the Fourier transform ion cyclotron resonance mass spectrometry remeasurement experiment. *Anal. Chem.* **67**, 420 (1995)
8. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)
9. G. Gabrielse, F.C. Macintosh, Cylindrical Penning traps with orthogonalized anharmonicity compensation. *Int. J. Mass. Spectrom. Ion Proc.* **57**, 1 (1984)
10. G. Gabrielse, L. Haarsma, S.L. Rolston, Open-endcap Penning traps for high precision experiments. *Int. J. Mass Spectrom. Ion Proc.* **88**, 319 (1989)
11. D. von Lindenfels et al., Half-open Penning trap with efficient light collection for precision laser spectroscopy of highly charged ions. *Hyp. Int.* **227**, 197 (2014)
12. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)
13. H. Bisht, H.-T. Eun, A. Mehrtens, M.A. Aegerter, Comparison of spray pyrolyzed FTO, ATO and ITO coatings for flat and bent glass substrates. *Thin Solid Films* **351**, 109 (1999)
14. M. Wiesel et al., Optically transparent solid electrodes for precision Penning traps. *Rev. Sci. Instr.* **88**, 123101 (2017)
15. Z.Q. Li, J.J. Lin, Electrical resistivities and thermopowers of transparent Sn-doped indium oxide films. *J. Appl. Phys.* **96**, 5918 (2004)
16. B.-T. Lin, Y.-F. Chen, J.-J. Lin, C.-Y. Wu, Temperature dependence of resistance and thermopower of thin indium tin oxide films. *Thin Solid Films* **518**, 6997 (2010)
17. G. Mei-Zhen, R. Job, X. De-Sheng, W. Fahrner, Thickness dependence of resistivity and optical reflectance of ITO films. *Chin. Phys. Lett.* **25**, 1380 (2008)
18. E.A. Alwan, A. Kiourti, J.L. Volakis, Indium tin oxide film characterization at 0.120 GHz. *IEEE Access* **3**, 648 (2015)

Chapter 4

Variations on the Penning Trap Concept



Abstract In this chapter, we briefly have a look at a number of variations on the idea of a Penning trap, i.e. at specific deviations from the combination of a homogeneous magnetostatic field with an aligned quadrupolar electrostatic potential. We discuss only those designs that use magnetic *and* electric fields, i.e. we are not concerned with Paul traps, Kingdon traps and so forth, which are a huge topic on their own.

4.1 Planar Penning Traps

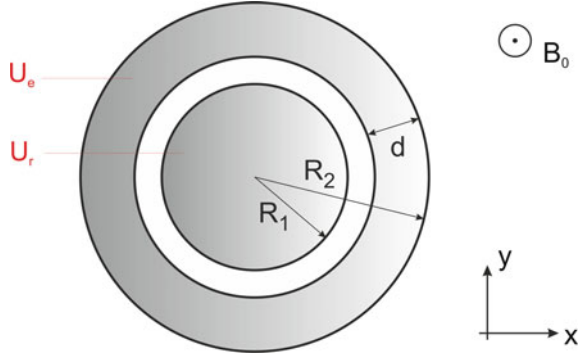
Converting the three-dimensional electrode array of a cylindrical Penning trap as above into a two-dimensional design is achieved by removing the upper endcap and embedding the ring and lower endcap in the (x, y) -plane. The result is displayed in Fig. 4.1. This is the simplest two-dimensional configuration possible. Further ring electrodes may be added if necessary to enhance the properties of the trap.

The electrodes of a planar trap may be implemented by use of well-established thin-film or thick-film technology on a ceramic substrate with a metallic material such as gold. To avoid charging-up of the substrate, an additional surface layer of low-conductivity material can be used. Thin layers of graphite or gold can avoid a charging-up, yet isolate the electrodes sufficiently to avoid short-circuits. The electrodes may be connected to the respective voltage supplies by contacts through the substrate.

In contrast to conventional Penning traps, the planar design represents a more open geometry. This allows access to the trapped particles with various kinds of radiation or beams. The position of the confined particles as well as the depth of the potential can be varied to a certain extent by appropriate choice of the applied voltages.

The electrostatic potential for a fixed set of parameters exhibits a minimum at a well-defined distance above the trap's plane. This minimum provides axial confinement of a charged particle, while radial confinement is assured by the magnetic field along the z -direction, as in conventional Penning traps. The electrostatic potential of a cylindrically symmetric configuration with electrodes filling the entire $z = 0$ plane and open ends in both $z \rightarrow \infty$ and $\rho \rightarrow \infty$ can be written in the form [1]

Fig. 4.1 Simplest possible planar Penning trap structure. The magnetic field is perpendicular to the plane in which the electrodes (grey areas) are embedded. The confinement properties can be improved by addition of further concentric electrodes



$$U(z, \rho) = \int_0^\infty dk A(k) e^{-kz} J_0(k\rho), \quad (4.1)$$

where $J_0(k\rho)$ are Bessel functions [2]. In the plane $z = 0$, the potential $U(\rho)$ can be written as

$$U(\rho) \equiv U(0, \rho) = \int_0^\infty dk A(k) J_0(k\rho) \quad (4.2)$$

which, by use of the Hankel transform [2], yields [1]

$$A(k) = k \int_0^\infty d\rho \rho U(\rho) J_0(k\rho). \quad (4.3)$$

As $U(\rho)$ is constant within each electrode, the coefficients $A(k)$ are a sum of the contributions from each electrode:

$$A(k) = \sum_i U_i [R_i J_1(kR_i) - (R_i - d_i) J_1(k(R_i - d_i))], \quad (4.4)$$

where R_i denotes the distance at which electrode i ends. Thus, the central electrode ends at R_1 , the inner ring electrode at R_2 and so forth. Note, that for the potential $U(\rho)$ to be expressed in terms of the coefficients in (4.3), the planar trap needs to be surrounded by an infinite grounded electrode so that the configuration fills the $z = 0$ plane completely. The deviation from a realistic case with appropriate spacing between the electrodes and a finite grounded surrounding has been calculated by comparison with numerical codes. In the region of the potential minimum, i.e. in the region that determines the trapping behaviour, the deviations are found to be negligible [1]. Based on the coefficients $A(k)$ of the expansion (4.2) one can integrate the potential in the whole space. As we are interested in confinement along the z -axis the desired solution is obtained by [1]

$$U(z, 0) \equiv U(z) = \int_0^\infty dk A(k) e^{-kz}, \quad (4.5)$$

which can be represented by a sum of contributions from each electrode of the type

$$U(z) = \sum_i U_i(z) = \sum_i U_i \left(\frac{1}{\sqrt{1 + \frac{(R_i - d_i)^2}{z^2}}} - \frac{1}{\sqrt{1 + \frac{R_i^2}{z^2}}} \right). \quad (4.6)$$

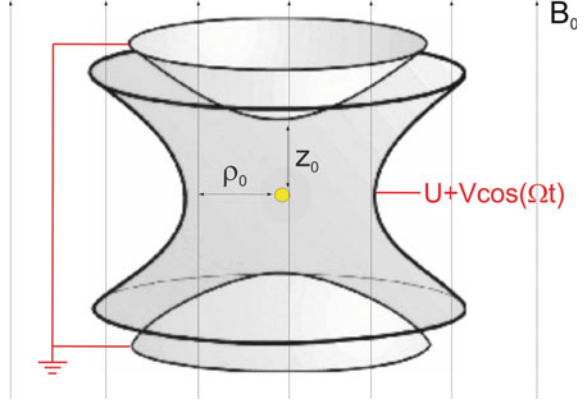
A more detailed account on this can be found in [1]. The motional frequencies resulting from these potentials have been given in [3]. The planar structures can be extended to more electrodes for better compensation of anharmonic effects [4], typically by addition of outer concentric ring electrodes, and to designs that include wires and pads in planes [5, 6]. There is also a number of similar planar Penning trap structures such as the ‘onion trap’ and the ‘pixel trap’ which consists of hexagonal pads manifolded in the plane [7]. Let us note that such planar structures need not be Penning traps, there are also planar traps with alternating trap voltages, including low-frequency planar traps that confine microscopic charged particles in air [8]. There is quite some interest in miniature or even microscopic planar structures on chips because of the high scalability and possible applications in quantum information processing [1, 9–12]. The same is true for extending application from the radio-frequency into the microwave domain, such as the coplanar-waveguide Penning trap [13].

4.2 Combined Trap

The so-called ‘combined trap’ is a hybrid form of a Penning trap and a radio-frequency (Paul) trap that uses a strong magnetostatic field *and* a radio-frequency electric field superimposed on a Paul trap structure [14–17]. It has thus the disadvantages of both designs, i.e. the need for a homogeneous magnetic field and the heating process due to the radio-frequency micro-motion of confined particles, but the advantage that it allows to simultaneously confine particles of both signs of charge. As compared to a pure radio-frequency (Paul) trap, it has also a larger area of stability, i.e. it allows a broader range of operation parameters [17, 18]. In a combined trap, the confining electric potential is a superposition of the static quadrupolar component U and the radio-frequency component $V \cos(\Omega t)$ where V is the radio-frequency amplitude and Ω is the radio-frequency (Fig. 4.2). The combined potential U' thus has the form

$$U' = q \frac{U - V \cos(\Omega t)}{2d^2} (2z^2 - x^2 - y^2), \quad (4.7)$$

Fig. 4.2 Example of a combined trap: the geometry here is identical to the hyperbolic Penning trap, but the voltage between the ring electrode and the endcaps has a dc component U and a rf component with amplitude V and frequency Ω



which for $V = 0$ is identical to the potential of the Penning trap as given in (3.3). This potential leads to equations of motion

$$\ddot{x} - \frac{q}{md^2} (U + V \cos(\Omega t)) x - \omega_c \dot{y} = 0 \quad (4.8)$$

$$\ddot{y} - \frac{q}{md^2} (U + V \cos(\Omega t)) y - \omega_c \dot{x} = 0 \quad (4.9)$$

$$\ddot{z} + \frac{2q}{md^2} (U + V \cos(\Omega t)) z = 0 \quad (4.10)$$

which again for vanishing V take the form of the equations of motion in a Penning trap, as we will discuss in detail below. The typical way to solve these equations is by transformation into a reference frame rotating at frequency $\omega_c/2 = q B_0/(2m)$ ('Brillouin frame of reference'), we will do this in Sect. 5.4. A detailed discussion of the oscillation frequencies in a combined trap can be found in [19, 20]. The effects of field imperfections in a combined trap have been discussed in detail in [21]. The properties of a linear combined trap, the hybrid of a linear Paul trap and a Penning trap, have been presented in [22, 23].

A noteworthy feature of the combined trap is the possibility of a magnetron-free operation, which is not possible in a Penning trap. As we will see in (5.47), the magnetron frequency in a combined trap is given by

$$\omega_- = \frac{\omega_c}{2} - \frac{\kappa_\rho \Omega}{2} \quad (4.11)$$

where

$$\kappa_\rho^2 = \frac{\omega_c^2}{\Omega^2} - \frac{4qU}{md^2\Omega^2} + \frac{2q^2V^2}{m^2d^4\Omega^4} \quad (4.12)$$

such that for the choice $\kappa_\rho \Omega = \omega_c$ (which can be achieved by appropriate choice of the voltages U and V) the magnetron frequency becomes zero. When we compare

the motions of a particle in Penning and Paul traps, we can obtain a physical picture of the magnetron-free operation. As we have seen, the electrostatic field of a Penning trap confines the particle in the axial direction but leads to a radially outward force. In contrast, the radio-frequency field of a Paul trap leads to confining forces in all directions and an inward radial force. The presence of the crossed electric and magnetic fields causes a magnetron drift around the trap axis in opposite directions for the two cases. When the two effects cancel each other, the magnetron motion vanishes. This has been studied in detail in [24].

4.3 Nested Traps

The idea of a nested trap structure [25] is to use more electrodes than required for a single cylindrical Penning trap, such that in the same homogeneous magnetostatic field one can have more than one electrostatic minimum (Fig. 4.3). The relative depths of these several potential wells can be chosen to accommodate different ensembles of particles separated from each other at a distance given by the electrode structure and voltage settings used. In particular, one can establish a relative potential minimum for positive charges that appears between two potential minima for negative charges (or inverted). This way, when the particles of different charge signs have sufficiently small kinetic energy, they can be spatially very close while still confined to different wells, which for example facilitates electron cooling of ions [26] that would undergo a capture process if not separated, like highly charged ions. The same is true for confinement of positrons and anti-protons and the formation of cold anti-hydrogen [27, 28] when two minima are merged at low energy.

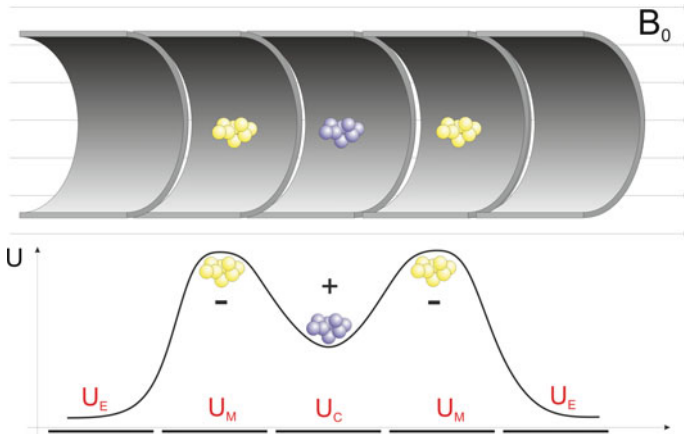


Fig. 4.3 Sectional view of a simple nested Penning trap structure for confinement of particles with different charge

4.4 Elliptical Penning Traps

In an elliptical Penning trap, the radial field symmetry is broken by an additional azimuthal quadrupolar electrostatic field proportional to $(x^2 - y^2)$. In principle, this can be achieved by a modification of the shape of the electrode surfaces as given in (3.1) and (3.2) of the kind

$$2z^2 - (1 - \varepsilon)x^2 - (1 + \varepsilon)y^2 = -z_0^2 \quad (4.13)$$

for the endcap and

$$2z^2 - (1 - \varepsilon)x^2 - (1 + \varepsilon)y^2 = \rho_0^2 \quad (4.14)$$

for the ring, where z_0 is the distance of each of the endcaps to the trap centre and ρ_0 is the central ring inner radius. The ellipticity is parameterised by the ellipticity parameter ε . When the ellipticity of the trap is due to the shape of the electrodes, the value of the ellipticity parameter cannot be changed during an experiment. Thus, it is more practical to introduce an ellipticity by application of an additional voltage between opposing segments of the trap's ring electrode. A configuration like this has been investigated theoretically [29] and experimentally [30]. A non-zero ellipticity parameter ε adds a term

$$U_\varepsilon = -\frac{U_0}{4d^2}\varepsilon(x^2 - y^2) \quad (4.15)$$

to the harmonic trap potential such that the total electrostatic potential is given by

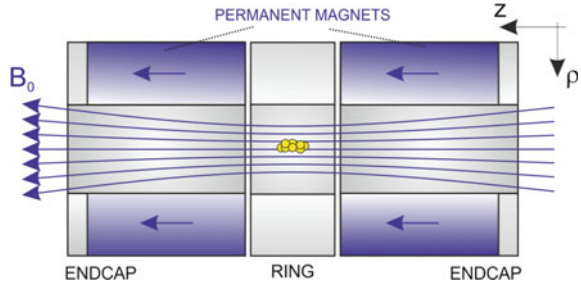
$$U = \frac{U_0}{2d^2} \left(z^2 - \frac{1}{2}(x^2 + y^2) - \frac{1}{2}\varepsilon(x^2 - y^2) \right). \quad (4.16)$$

The orientation of the ellipse is determined by the sign of ε . Stable confinement is possible only for $-1 < \varepsilon < 1$, as has been carefully discussed in [29]. The additional field modifies the equations of motion of confined particles, see Sect. 6.6.1, which may be used for specific applications [31–33]. There are also planar Penning trap designs that are elliptical by nature, such as the coplanar-waveguide Penning trap (CPW-trap), the ‘Geonium chip’ [13]. We will discuss the effects of non-vanishing ellipticity in Sect. 6.6.

4.5 Unitary Penning Traps

A unitary Penning trap is not a specific variation of the standard Penning trap, hence many forms are possible. The unifying idea is the integration of the magnets that create the confining magnetic field into the electrode structure. This is done by use

Fig. 4.4 Schematic of the idea of a unitary Penning trap, where the magnetic field is created by permanent magnets integrated into the electrode structure



of permanent magnets in a compact arrangement as shown exemplarily in Fig. 4.4. It is hence one form of a permanent-magnet Penning trap [34] with the feature of being very compact. Since the functions of creating the electrostatic potential U and the confining magnetic field B_0 are united in the same component, the name ‘unitary’ was coined for it. With permanent magnets, field strengths of around one tesla and slightly above can be readily achieved, see also the discussion in Sect. 21.5. Penning traps of this kind have been used for confinement of highly charged ions in [9, 10, 35].

4.6 Penning-Ioffe Traps

In a Penning-Ioffe trap, the field configurations of a Penning trap and a Ioffe trap are superimposed. The Ioffe trap features a magnetostatic radial quadrupole field that is usually created by a four-rod racetrack configuration (‘Ioffe bars’ [36]) of electrical currents parallel to the trap axis (Fig. 4.5). Some traps use an octupole configuration. Note, that it is named after M. S. Ioffe, not after the more famous A. F. Ioffe. It is, however, the same Ioffe as in ‘Ioffe-Pritchard trap’ [37], which is named also after D. E. Pritchard (*1941). Its magnetic field has the form

$$B = B_0 + b_2 z^2 + \rho^2 \left(\frac{b_1^2}{2B_0} - \frac{b_2}{2} \right) \quad (4.17)$$

with coefficients b_1 and b_2 usually chosen such that $b_2 \ll b_1/B_0$ to form a potential minimum along the trap axis z . The corresponding potential V experienced by a neutral species with dipole moment μ is given by

$$V = \mu B = \frac{1}{2} m \omega_z z^2 + \frac{1}{2} \omega_\rho \rho^2, \quad (4.18)$$

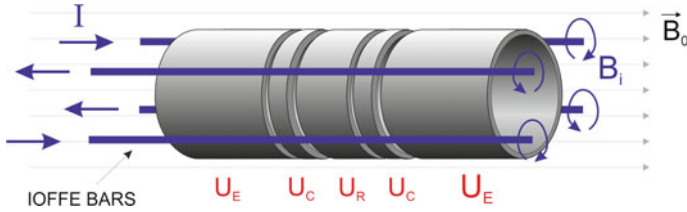


Fig. 4.5 Schematic of a Penning-Ioffe trap. It is similar to a cylindrical Penning trap, with an added radial magnetic quadrupole field B_r created by currents I through the ‘Ioffe bars’ as indicated

where the axial and radial frequencies are given by

$$\omega_z^2 = \frac{2\mu b_2}{m} \quad \text{and} \quad \omega_\rho^2 = \frac{\mu}{m} \left(\frac{b_1^2}{B_0} - \frac{b_2}{2} \right) \quad (4.19)$$

Ioffe traps confine neutral species on account of the forces of magnetic field gradients on dipole moments. The combination of both field configurations in a Penning-Ioffe trap is used for simultaneous confinement of charged and neutral species. The stability conditions for this are somewhat delicate [38], but such traps have successfully been operated [39, 40]. Note that the Ioffe part of the confining potential is commonly weaker than the Penning part by several orders of magnitude.

4.7 Penning-Malmberg Traps

A Penning-Malmberg trap [41] is very closely related to a cylindrical Penning trap, the main difference being that the Penning-Malmberg trap trades a high harmonicity of the confining potential for an extended confinement volume along the trap axis. Such traps are often found in ion plasma confinement, where the plasmas are desired to have a substantial extension, mainly in the axial direction (prolate ellipsoids with aspect ratios of axial extension to radial extension that are typically $\alpha < 1$), but where the exact values of the oscillation frequencies are of lesser importance, see for example [42]. Usually, a Penning-Malmberg trap looks like a cylindrical Penning trap with a long central ring electrode neighbored by comparatively short endcap electrodes. There are also concepts to combine a Penning-Malmberg trap with a minimum-B trap for neutral particles [43], and modifications with a central conducting wire [44]. Also, there are annular Penning-Malmberg traps of various field configurations applied in plasma research [45–47] (Fig. 4.6).

The Penning-Malmberg trap is named after F. M. Penning (1894–1953) and J. H. Malmberg (1927–1992), sometimes it is also called ‘Malmberg-Penning trap’.

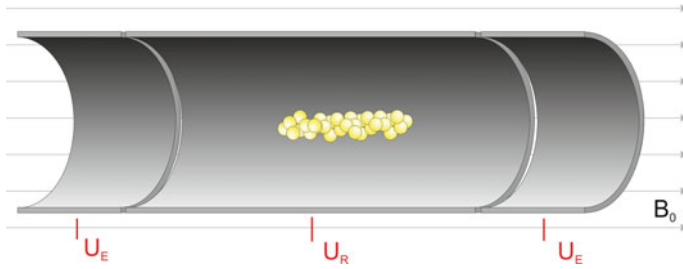


Fig. 4.6 Sectional view of a Penning-Malmberg trap. It is similar to a three-pole open-endcap cylindrical Penning trap, but with an elongated ring and short endcaps

4.8 Conical-Endcap Penning Trap

As we have seen before, the optical access to a standard 3-pole or 5-pole cylindrical Penning trap is limited by geometry. A possible way around this is the ‘half-open’ design as discussed in Sect. 3.4. That concept, however, is limited to a large solid angle of access on one side. In situations where the solid angle of access needs to be maximised on both sides, we need another geometry. Access from both sides is for example relevant when strongly focussed light is to enter and leave the trap, with the focal point in the trap centre where the particles are confined. In this situation, a possible solution is to break the cylindrical symmetry of the endcaps and/or outer electrodes and give them a conical shape with an opening angle θ . This trap still has the symmetries of the cylindrical trap (i.e. radial symmetry with respect to the trap axis $(x, y) = 0$ and the xy -plane through $z = 0$), and we will see in Sect. 6.7 that the field distortions from the cones still make this a workable trap in practice. Such a trap is displayed in Fig. 4.7. One possible application of such a trap is the study of confined particle interaction with highly focused high-intensity laser light. This application has been discussed in [48–50].

4.9 Wire Traps, Plate Traps and Pad Traps

In principle, any electrode configuration that is able to produce a sufficiently harmonic and extended electrostatic minimum in a magnetic field can be chosen. These electrodes need not be hyperbolic or cylindrical or cubic configurations around the confinement region, nor do they need to be extended flat surfaces. A number of such configurations have been proposed and demonstrated [51], mainly motivated by increased optical access from various directions.

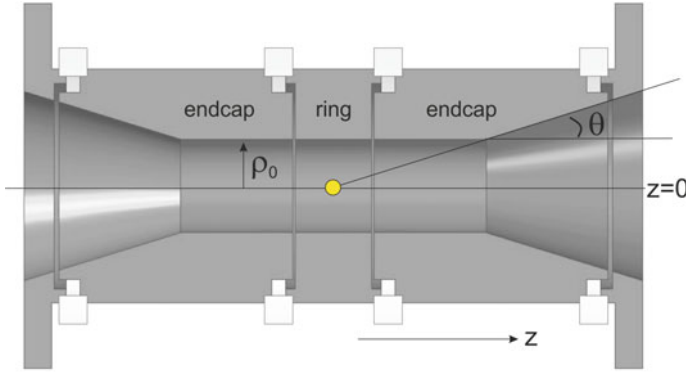


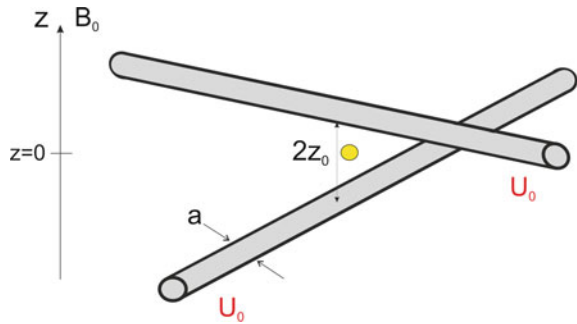
Fig. 4.7 Schematic sectional view of a mechanically compensated cylindrical Penning trap with conical endcap openings for increased axial access

4.9.1 Wire Traps

The most simple configuration is a trap that consists of two long perpendicular wires separated axially (in the magnetic field direction) by $2z_0$ with symmetry about $z = 0$. In the projection onto the (x, y) -plane, the wires cross at $(x, y) = 0$. The wires have diameter $a \ll 2z_0$ and are located in a grounded surrounding that is at distance R (Fig. 4.8). When the wires are both on a voltage U_0 with respect to the ground at R , then the potential U they create is given by

$$U = \frac{U_0}{\ln \frac{R^2}{az_0}} \left(\ln \frac{R^2}{y^2 + (z - z_0)^2} + \ln \frac{R^2}{x^2 + (z + z_0)^2} \right). \quad (4.20)$$

Fig. 4.8 Schematic of the most simple wire trap: two thin wires of diameter a separated by $2z_0$ around $z = 0$ creating a potential minimum at $(x, y, z) = 0$



This potential has a minimum around $(x, y, z) = 0$ and close to this trap centre can be approximated by

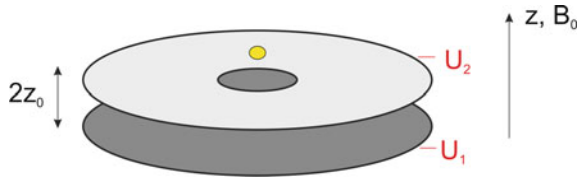
$$U = \frac{2U_0}{\ln \frac{R^2}{az_0}} \left(2 \ln \frac{R}{z_0} - \frac{1}{z_0^2} (x^2 + y^2 - 2z^2) \right). \quad (4.21)$$

such that the potential has a position-dependent contribution of the same geometry (3.3) as the cases discussed above, and will consequently lead to similar equations of motion, as has been detailed in [51]. The properties of the potential can be improved by employment of more wires, for example of two perpendicular wire triplets. This creates a deeper potential, allows tuning, and additionally creates minima outside the region between wire triplets [51].

4.9.2 Plate Traps

The model for this type of trap is an arrangement of two parallel disks separated by an axial distance z_0 , on two different voltages and in an axial magnetic field (Fig. 4.9). One disk has a central hole, above which an electrostatic potential minimum can be created by appropriate choice of the voltages U_1 and U_2 [51]. Like the wire traps, this concept has good scaling properties, is straightforward to miniaturise and yields good optical access to the particle confined above the disk surface. It may be seen as a variation on the planar Penning trap design with two planes instead of one, which may improve the geometric options for supply with voltages and hence the scalability. For a closer discussion see [51].

Fig. 4.9 Schematic of a simple plate trap: two thin plates separated by $2z_0$ around $z = 0$, with a hole in the upper plate creating the potential minimum above its surface



4.9.3 Pad Traps

Pad traps are a variation on the plate trap design and feature central circular pads surrounded by a hexagonal arrangement of six similar circular pads, forming a honeycomb structure across the (x, y) -plane [51]. The whole setup is a sandwich of two such parallel honeycombs separated along the z direction (Fig. 4.10). The magnetic field is again applied in the z direction. The design is optimized for a rapid transport of confined particles between adjacent electrostatic minima and across the magnetic field. This is possible in the following fashion: assume a homogeneous electric field E is applied in the y direction, the trajectory of a particle initially at rest at the origin is given by the parametric equations

$$x = -\frac{1}{\omega_c} v \sin(\omega_c t + vt) \quad \text{and} \quad y = \frac{1}{\omega_c} v (1 - \cos \omega_c t) \quad (4.22)$$

where $v = E/B_0$ is the drift velocity in the crossed electric and magnetic fields and $\omega_c = qB_0/m$ is the cyclotron frequency. The (x, y) -motion is therefore a series of loops in the (x, y) -plane such that the particle drifts in the x direction and periodically comes to rest in the y direction. Consider a pair of pad traps whose axes are along the z -direction but whose centres are displaced in the x -direction. If the trap potential is switched off and replaced by a linear electric field as described above it would be possible for the particle to hop from the centre of one trap to the centre of the adjacent trap in a single cycloid loop. A discussion of possible applications can be found in [51].

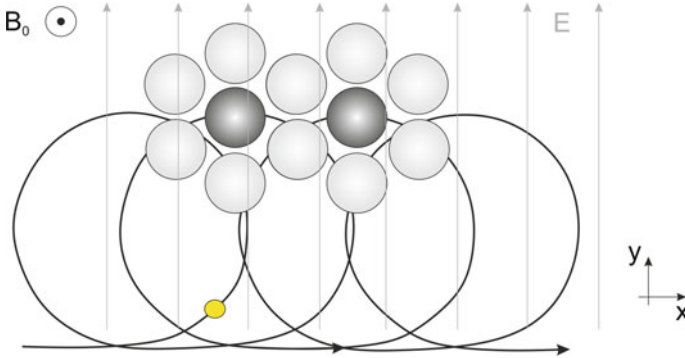


Fig. 4.10 Schematic of a pad trap arrangement. The same pads are located below the ones shown, and the particle trajectory lies in the plane between them. The particle motion is due to a homogeneous electric field in the (x, y) -plane crossed with the magnetic field along the z direction

References

1. S. Stahl et al., A planar Penning trap. *Eur. Phys. J. D* **32**, 139 (2005)
2. A.D. Poularikas, *Transforms and Applications Handbook*, 3rd edn. (CRC Press, Boca Raton, 2010). ISBN 9781420066524
3. F. Galve, G. Werth, Motional frequencies in a planar Penning trap. *Hyp. Int.* **174**, 41 (2007)
4. J. Pinder, J. Verdu, A planar Penning trap with tunable dimensionality of the trapping potential. *Int. J. Mass Spectrom.* **356**, 49 (2013)
5. J.R. Castrejon-Pita, R.C. Thompson, Proposal for a planar Penning ion trap. *Phys. Rev. A* **72**, 013405 (2005)
6. J.R. Castrejon-Pita, *Design, Development and Operation of Novel Ion Trap Designs* (VDM Publishing, Saarbrücken, 2009)
7. M. Hellwig, A. Bautista-Salvador, K. Singer, G. Werth, F. Schmidt-Kaler, Fabrication of a planar micro Penning trap and numerical investigations of versatile ion positioning protocols. *New J. Phys.* **12**, 065019 (2010)
8. L. Jiang, W.B. Whitten, S. Pau, A planar ion trapping microdevice with integrated waveguides for optical detection. *Opt. Express* **19**, 3037 (2011)
9. P. Bushev et al., Electrons in a cryogenic planar Penning trap and experimental challenges for quantum processing. *Eur. Phys. J. D* **50**, 97 (2008)
10. F. Galve, P. Fernandez, G. Werth, Operation of a planar Penning trap. *Eur. Phys. J. D* **40**, 201 (2006)
11. J. Goldman, G. Gabrielse, Optimized planar Penning traps for quantum-information studies. *Phys. Rev. A* **81**, 052335 (2010)
12. J. Goldman, G. Gabrielse, Optimized planar Penning traps for quantum information studies. *Hyp. Int.* **199**, 279 (2011)
13. J. Verdu, Theory of the coplanar-waveguide Penning trap. *New J. Phys.* **13**, 113029 (2011)
14. E. Fischer, Three-dimensional stabilisation of charged particles in a quadrupole field. *Z. Phys.* **156**, 1 (1959)
15. G.Z. Li, A quantum particle in a combined trap. *Z. Phys. D* **10**, 451 (1988)
16. W.M. Itano, Atomic ion frequency standards. *Proc. IEEE* **79**, 936 (1991)
17. R. Blümel, Comment on 'regular and chaotic motions in ion traps: a nonlinear analysis of trap equations'. *Phys. Rev. A* **48**, 854 (1992)
18. G.-Z. Li, G. Werth, The combined trap and some possible applications. *Phys. Scr.* **46**, 587 (1992)
19. D.J. Bate, K. Dholakia, R.C. Thompson, D.C. Wilson, Ion oscillation frequencies in a combined trap. *J. Mod. Opt.* **39**, 305 (1992)
20. K. Dholakia, G. Horvath, D.M. Segal, R.C. Thompson, Photon correlation measurement of ion oscillation frequencies in a combined trap. *J. Mod. Opt.* **39**, 2179 (1992)
21. G.Z.K. Horvath, J.-L. Hernandez-Pozos, K. Dholakia, J. Rink, D.M. Segal, R.C. Thompson, Ion dynamics in perturbed quadrupole ion traps. *Phys. Rev. A* **57**, 1944 (1998)
22. Y. Huang, G.-Z. Li, S. Guan, A.G. Marshall, A combined linear ion trap for mass spectrometry. *J. Am. Soc. Mass Spectrom.* **8**, 962 (1997)
23. M.A. van Eijkelenborg, M.E.M. Storkey, D.M. Segal, R.C. Thompson, Ion dynamics in a novel linear combined trap. *Int. J. Mass Spectrom.* **188**, 155 (1999)
24. M. Yan, X. Luo, X. Zhu, Potential usage of the magnetron-motion-free mode of one ion confined in a combined trap. *Appl. Phys.* **67**, 235 (1998)
25. G. Gabrielse, S.L. Rolston, L. Haarsma, W. Kells, Antihydrogen production using trapped plasmas. *Phys. Lett. A* **129**, 38 (1988)
26. D.S. Hall, G. Gabrielse, Electron cooling of protons in a nested Penning trap. *Phys. Rev. Lett.* **77**, 1962 (1996)
27. G. Gabrielse et al., Driven production of cold antihydrogen and the first measured distribution of antihydrogen states. *Phys. Rev. Lett.* **89**, 233401 (2002)
28. M. Amoretti et al., Production and detection of cold antihydrogen atoms. *Nature* **419**, 456 (2002)

29. M. Kretschmar, Theory of the elliptical Penning trap. *Int. J. Mass Spectrom.* **275**, 21 (2008)
30. M. Breitenfeldt et al., The elliptical Penning trap: experimental investigations and simulations. *Int. J. Mass Spectrom.* **275**, 34 (2008)
31. L.S. Brown, G. Gabrielse, Precision spectroscopy of a charged particle in an imperfect Penning trap. *Phys. Rev. A* **25**, 2423 (1982)
32. G. Gabrielse, Why is sideband mass spectrometry possible with ions in a Penning trap? *Phys. Rev. Lett.* **102**, 172501 (2009)
33. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
34. L. Suess, C.D. Finch, R. Parthasarathy, S.B. Hill, F.B. Dunning, Permanent magnet Penning trap for heavy ion storage. *Rev. Sci. Instr.* **73**, 2861 (2002)
35. S.F. Hoogerheide, Experiments with highly-ionized atoms in unitary Penning traps. *Atoms* **3**, 367 (2015)
36. Y.V. Gott, M.S. Ioffe, V.G. Telkowskii, Some results on confinement in magnetic trapping, *Nucl. Fusion, Suppl.* **2**, Pt. 3, 1045 (1962)
37. D.E. Pritchard, Cooling neutral atoms in a magnetic trap for precision spectroscopy. *Phys. Rev. Lett.* **51**, 1336 (1983)
38. T.M. Squires, P. Yesley, G. Gabrielse, Stability of a combined Penning-Ioffe trap. *Phys. Rev. Lett.* **86**, 5266 (2001)
39. G. Gabrielse et al., Antiproton confinement in a Penning-Ioffe trap for antihydrogen. *Phys. Rev. Lett.* **98**, 113002 (2007)
40. G. Gabrielse et al., Antihydrogen production within a Penning-Ioffe trap. *Phys. Rev. Lett.* **100**, 113001 (2008)
41. J.H. Malmberg, C.F. Driscoll, Long-time containment of a pure electron plasma. *Phys. Rev. Lett.* **44**, 654 (1980)
42. D.H.E. Dubin, Plasmas in Penning traps, in *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson (World Scientific, 2016)
43. J. Fajans, A. Schmidt, Malmberg-Penning and Minimum-B trap compatibility: the advantages of higher-order multipole traps. *Nucl. Instr. Meth. A* **521**, 318 (2004)
44. D.L. Eggleston, Confinement of test particles in a Malmberg-Penning trap with a biased axial wire. *Phys. Plasmas* **4**, 1196 (1997)
45. S. Robertson, Annular Malmberg-Penning trap for studies of plasma confinement. *Rev. Sci. Instr.* **70**, 2993 (1999)
46. S. Robertson, Electron confinement in an annular Penning trap. *Phys. Plasmas* **7**, 2340 (2000)
47. J. Espejo, Q. Quraishi, S. Robertson, Experimental measurement of neoclassical mobility in an annular Malmberg-Penning trap. *Phys. Rev. Lett.* **84**, 5520 (2000)
48. M. Vogel et al., A Penning trap for advanced studies with particles in extreme laser fields. *Nucl. Instr. Meth. B* **285**, 65 (2012)
49. S. Kumar et al., Design of a mechanically compensated Penning trap for the study of ions in extreme laser field. *J. Phys. Conf. Ser.* **635**, 092070 (2015)
50. S. Ringleb et al., HILITE - ions in intense photon fields. *J. Phys. Conf. Ser.* **635**, 092124 (2015)
51. J.R. Castrejon-Pita et al., Novel designs for Penning Ion traps. *J. Mod. Opt.* **54**, 1581 (2007)

Chapter 5

Motion of a Single Particle in an Idealised Penning Trap



Abstract This chapter is concerned with the motion that is performed by a single confined test particle in a Penning trap. For now we disregard any realisation of a Penning trap and see it as the abstract combination of an ideally homogeneous static magnetic field B_0 perfectly aligned with a quadrupolar electrostatic potential U that creates a harmonic well across a ‘characteristic trap size’ d . We are concerned only with the motion of a single test particle of mass m and electric charge q in such an arrangement. Later we will discuss the effects that arise when each of these idealisations are dropped.

5.1 Particle Trajectory

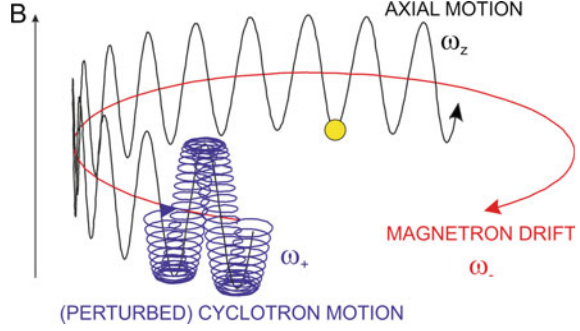
The trajectory of a single particle in an ideal Penning trap can be calculated analytically [1]. For two and three particles, this is in part possible [2], see also Sect. 7.10. For more than three particles, no analytical solutions exist. In an experiment, the precise trajectory of any given particle is usually not of interest, and the observables are rather quantities such as oscillation amplitudes and oscillation frequencies. As causation of the specific motion we have the homogeneous magnetic field B_0 along the axial z -direction and the quadrupolar electrostatic potential of the form $U \propto (2z^2 - x^2 - y^2)$ as discussed in detail in Sect. 2.2. For this configuration we obtain two coupled equations of motion in the radial (x, y) -plane and one independent equation of motion for the axial degree of freedom (Fig. 5.1).

For a single particle we can write equations of motion in Cartesian coordinates in the form

$$\ddot{x} = \omega_c \dot{y} + \frac{\omega_z^2}{2} x \quad ; \quad \ddot{y} = -\omega_c \dot{x} + \frac{\omega_z^2}{2} y \quad ; \quad \ddot{z} = -\omega_z^2 z, \quad (5.1)$$

from which we obtain a trajectory $(x, y, z)(t)$ with

Fig. 5.1 Depiction of the motion of a confined particle in a Penning trap. The motion has an axial and two radial components at the frequencies ω_z and $\omega_{\pm}$. While these frequencies are related, the amplitudes of the three motional components are independent. In this example we have $\rho_+ : a_z : \rho_- = 1.5 : 5 : 20$



$$x(t) = \rho_+ \sin(\omega_+ t + \phi_+) + \rho_- \sin(\omega_- t + \phi_-) \quad (5.2)$$

$$y(t) = \rho_+ \cos(\omega_+ t + \phi_+) + \rho_- \cos(\omega_- t + \phi_-) \quad (5.3)$$

$$z(t) = a_z \sin(\omega_z t + \phi_z), \quad (5.4)$$

where ω_+ , ω_- and ω_z are characteristic oscillation frequencies that we will discuss in detail below, and ϕ_+ , ϕ_- and ϕ_z are the corresponding phases of these oscillations, which depend on initial conditions. The motional amplitudes ρ_+ , ρ_- and a_z are independent from one another. They are related to the corresponding kinetic energies E_+ , E_- and E_z by (Fig. 5.1)

$$\rho_+^2 = \frac{2E_+}{m(\omega_+^2 - \omega_z^2/2)} d^2 \quad (5.5)$$

$$\rho_-^2 = \frac{2E_-}{m(\omega_-^2 - \omega_z^2/2)} d^2 \quad (5.6)$$

$$a_z^2 = \frac{E_z}{qU} d^2. \quad (5.7)$$

Note that when the magnetron radius ρ_- is calculated from (5.6) by taking the square root, the argument of the square root on the right hand side appears to be negative due to $\omega_z^2/2 > \omega_-^2$, but as we will discuss below, the magnetron energy E_- is always negative as well, so that the amplitude comes out positive.

We can obtain the particle trajectory in the radial coordinate $\rho^2 = x^2 + y^2$ from the Cartesian coordinates (x, y) given by (5.2) and (5.3) by [3]

$$\rho^2(t) = \rho_+^2 + \rho_-^2 + 2\rho_+\rho_- \cos(2\omega_1 t + \phi_+ - \phi_-) \quad (5.8)$$

in which the frequency ω_1 is given by $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$. The radius $\rho(t)$ is hence bounded between the values $|\rho_+ - \rho_-|$ and $|\rho_+ + \rho_-|$, as can be seen in Fig. 5.2. The instantaneous kinetic energy (i.e. at any point in time) of a particle is given by

cies in the eigenvalues of the Hamiltonian of the confined motion [4]. For any given mass-to-charge ratio m/q of a particle and the effective trap size d , the frequencies are uniquely fixed by the values of the confining potentials, i.e. by choice of U and B_0 . Other than the corresponding motional amplitudes, which are completely unrelated in a perfect trap, the frequencies are related to one another by the potentials.

The single-particle oscillation frequencies are given by

$$\omega_z^2 = \frac{qU}{md^2} \quad \text{and} \quad \omega_{\pm} = \frac{\omega_c}{2} \pm \left(\frac{\omega_c^2}{4} - \frac{\omega_z^2}{2} \right)^{1/2}, \quad (5.13)$$

where ω_c is the so-called ‘free-particle’ cyclotron frequency

$$\omega_c = \frac{qB_0}{m}, \quad (5.14)$$

which is the value of ω_+ for $\omega_z = 0$ (i.e. for $U = 0$).

Sometimes, ω_c is called the ‘true’ cyclotron frequency. The eigenfrequencies fulfill the identities

$$\omega_+ + \omega_- = \omega_c \quad \text{and} \quad \omega_+ \omega_- = \frac{\omega_z^2}{2}. \quad (5.15)$$

Sometimes, (5.13) are written in the form

$$\omega_z^2 = \frac{qU}{md^2} \quad \text{and} \quad \omega_{\pm} = \frac{1}{2}\omega_c \pm \omega_1, \quad (5.16)$$

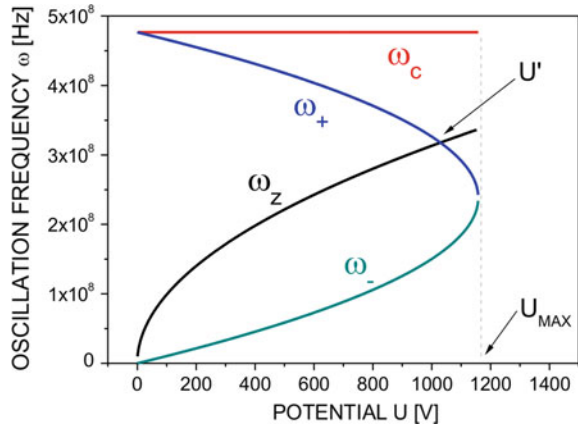
where again $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$. Commonly, the oscillation frequencies obey the hierarchy

$$\omega_c > \omega_+ \gg \omega_z \gg \omega_-, \quad (5.17)$$

although, as we will see, while $\omega_c > \omega_+$ and $\omega_z > \omega_-$ are strictly true throughout the whole allowed parameter space, the relation $\omega_+ > \omega_z$ is only usually true.

Figure 5.3 shows all the oscillation frequencies for the example of a proton in a given trap with $d = 1$ cm and $B_0 = 5$ T as a function of the trap potential U . Obviously, the cyclotron frequency ω_c does not depend on the electrostatic potential, as expected from (5.14). The axial oscillation frequency ω_z depends to first order on the potential and increases like the square root of U , as given by (5.13). The reduced cyclotron frequency and the magnetron frequency depend on the potential via the value of the axial oscillation frequency and their values become identical at the critical potential U_{MAX} , beyond which confinement is lost. Note, that there is a range of values of U for which the axial frequency ω_z is *larger* than the reduced cyclotron frequency ω_+ . This range is close to the critical value and is usually not

Fig. 5.3 Oscillation frequencies of a particle (a proton as an example) in a Penning trap as a function of the trap potential U



a range in which a trap is operated, as we will also discuss in relation to resonant particle loss in Sect. 8.3. The voltage U' at which $\omega_z = \omega_+$ follows from $3\omega_z = 2\omega_c$ [5] and is given by

$$U' = \frac{4}{9} \frac{q d^2 B_0^2}{m} = \frac{8}{9} U_{\text{MAX}}. \quad (5.18)$$

At this voltage, we additionally have $\omega_z = 2\omega_-$. The necessity for the argument of the square root in (5.13) to be larger than zero leads to the so-called ‘stability criterion’

$$2\omega_z^2 < \omega_c^2. \quad (5.19)$$

Sometimes $\gamma = \omega_c/(\sqrt{2}\omega_z)$ is called ‘stability parameter’ and the stability criterion then is expressed as $\gamma > 1$. This defines an upper limit for the allowed electrostatic potential U at any given magnetic field strength B_0 . This maximum voltage U_{MAX} follows from setting $\omega_z = \omega_c/\sqrt{2}$ in (5.13) and reads

$$U_{\text{MAX}} = \frac{1}{2} q d^2 \frac{B_0^2}{m}. \quad (5.20)$$

In the physical picture, the electrostatic radial outward force at this trap potential is strong enough to overcome the magnetic confinement, such that the particles are lost radially.

Figure 5.4 shows the maximum voltage U_{MAX} as a function of magnetic field strength B_0 in the same trap with $d = 1$ cm for different particles (left), and the axial oscillation frequencies of these particles as a function of the actual trap potential U for $B_0 = 5$ T (right). Note, that for the heavy particles the scale extends beyond the particles’ U_{MAX} at the chosen field strength of 5 T. For example, at this field strength, the maximum voltage for confinement of $^{40}\text{Ca}^+$ is about 1000 V, for C_{60}^+ it is about 100 V, and for a singly charged hemoglobin molecule ($m \approx 64500$ u) it is about 1 V.

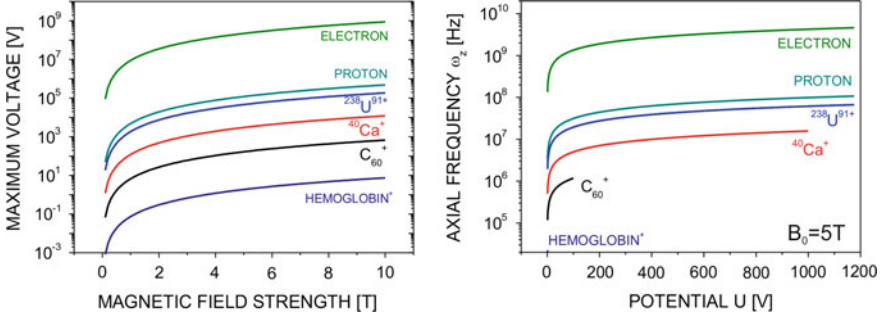


Fig. 5.4 Left: maximum voltage U_{MAX} as a function of magnetic field strength B_0 in a trap with $d = 1$ cm for different particles. Right: axial oscillation frequencies of these particles as a function of the actual trap potential in the same trap. Note, that for the heavy particles this scale extends beyond the particles' U_{MAX} at the chosen field strength of 5 T

As expected, the frequencies relate to each other like the inverse square roots of the respective particle masses, with the lightest particle having the highest oscillation frequency.

The requirement that the arguments of the square roots in (5.13) need to be positive also leads to a maximum particle mass that can stably be confined for a given set of trap parameters. We find the maximum particle mass given by

$$m_{\text{MAX}} = \frac{qd^2}{2} \frac{B_0^2}{U} \quad \text{or} \quad \left(\frac{m}{q} \right)_{\text{MAX}} = \frac{d^2}{2} \frac{B_0^2}{U}. \quad (5.21)$$

Note that $m/m_{\text{MAX}} = U/U_{\text{MAX}}$. By appropriate choice of U , the mass limitation can be used to remove undesired heavy particles from the trap. We will discuss other limitations such as confinement time limitation and number or density limitations in Sect. 8.1. It is one major advantage of a Penning trap to be able to simultaneously confine all particle masses between basically zero and m_{MAX} which is a considerable span.

5.3 The Different Nature of the Three Motions

For the following discussions, it is useful to first have a look at the natures of the three individual motions.

5.3.1 Axial Motion

Other than the two radial motions, the axial motion is alone in its spatial degree of freedom and merely a harmonic oscillation parallel to the z -axis. It is thus the

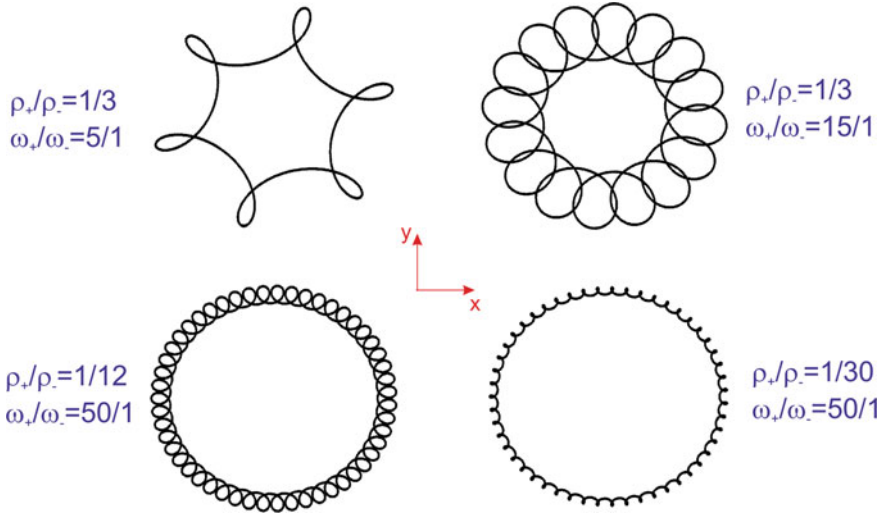


Fig. 5.5 Motion of a single confined particle in the radial plane for four different combinations of the ratios ρ_+/ρ_- and ω_+/ω_- leading to closed orbits

motion that often is easiest to manipulate and detect, also since it is the only motion to depend on the value of the trapping voltage to first order. The axial projection of the trajectory is a straight line centred around $z = 0$ and extended from $-a_z$ to $+a_z$.

5.3.2 Radial Motions

The radial motion of a confined particle is a composition of the magnetron drift motion in the crossed electric and magnetic fields, and the perturbed cyclotron motion. It is called ‘perturbed cyclotron motion’, because the presence of the electric field created by U perturbs the pure (‘free’) cyclotron motion of a charged particle around the magnetic field B_0 and reduces the oscillation frequency from ω_c to the lower value $\omega_+ = \omega_c - \omega_-$, the ‘reduced cyclotron frequency’. To illustrate this, Fig. 5.5 shows the motion of a single confined particle in the radial plane, i.e. a projection of the motion displayed in Fig. 5.1 on the (x, y) -plane, for four different combinations of the ratios ρ_+/ρ_- and ω_+/ω_- . These examples have closed trajectories on account of the chosen integer ratios ω_+/ω_- of the radial frequencies, and constant motional amplitudes ρ_+ and ρ_- . We will discuss these requirements in more detail below.

The shape of this projection is an epitrochoid, a specific form of a roulette. The epitrochoid is created by the trace of a point co-rotating with a circle of radius

$$s \equiv \frac{\omega_-}{\omega_+} \rho_-, \quad (5.22)$$

which rolls on the outside of a circle with radius

$$l \equiv \frac{2\omega_1}{\omega_+} \rho_- = \frac{\sqrt{\omega_c^2 - 2\omega_z^2}}{\omega_+} \rho_-, \quad (5.23)$$

such that the distance from the point to the centre of the rolling circle is equal to the cyclotron radius ρ_+ . One can distinguish three cases for the radius s , namely to be larger, smaller, or equal to the radius ρ_+ . In the first case, we obtain a ‘shortened’ epitrochoid, in the second case an ‘elongated’ one, and the special case of $s = \rho_+$ creates a so-called ‘epicycloid’. The shortened epitrochoid, by the way, is exactly the situation we encounter in the ‘Spirograph’ toys [6]. The different cases are depicted in Fig. 5.6, where the relevant quantities in the three cases are shown. The trajectory traced out by the radial coordinate $\rho(t)$ is given by (5.8) and is bounded between the values $|\rho_+ - \rho_-|$ and $|\rho_+ + \rho_-|$. For $l = s$, the resulting specific curve is a so-called ‘limaçon’ or ‘limaçon of Pascal’, named after Etienne Pascal (1588–1651), not after his more famous son Blaise Pascal (1623–1662). The corresponding epicycloid, i.e. when additionally we have $\rho_+ = s$, is a so-called ‘cardioid’ which features a cusp at which the motion has the most abrupt change of direction possible. Using whole numbers m and n to parameterise the circulations, we can write an expression for $\rho(t)$ by

$$\rho(t) = (l + s) \begin{pmatrix} \cos(\omega_- t) \\ \sin(\omega_- t) \end{pmatrix} - \rho_+ \begin{pmatrix} \cos\left(\frac{l+s}{s} \omega_- t\right) \\ \sin\left(\frac{l+s}{s} \omega_- t\right) \end{pmatrix}, \quad (5.24)$$

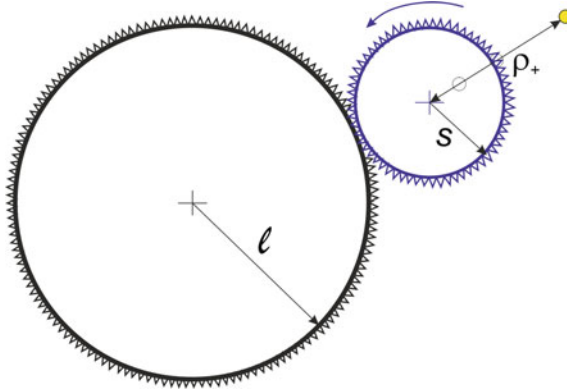


Fig. 5.6 Depiction of the generation of an epitrochoid: the circle of radius s rolls on the outside of the circle with radius l , and the point at distance ρ_+ from the centre on a rigid axis through the small circle traces out the epitrochoid. This point can be outside ($\rho_+ > s$), inside ($\rho_+ < s$) or on the small circle ($\rho_+ = s$). The gears are drawn to point out the resemblance to the Spirograph [6]

where the relation between the radius s and the radius l is given by

$$\frac{s}{l} = \frac{1}{\frac{m}{n} - 1}. \quad (5.25)$$

If m and n are relatively prime, we obtain a closed radial orbit after n circulations of the small circle around the large one, at a time $2\pi n/\omega_-$ [3].

If additionally the frequency ratio ω_z/ω_- is rational, we can write it as $\omega_z/\omega_- = m/n$ with m and n being relatively prime. Then we can express certain frequency ratios by these coprimes [3]

$$\frac{\omega_+}{\omega_z} = \frac{m}{2n} \quad \text{and} \quad \frac{\omega_c}{\omega_z} = \frac{m^2 + 2n^2}{2nm} \quad \text{and} \quad \frac{2\omega_1}{\omega_z} = \frac{m^2 - 2n^2}{2nm}, \quad (5.26)$$

and the condition for a closed three-dimensional orbit is given by $m > \sqrt{2}n$. Then, the period of a periodic orbit is given by $2\pi mn/\omega_z$ for even m , and $4\pi mn/\omega_z$ for odd m [3]. Examples for closed orbits are given in Table 5.1. These are the closed three-dimensional orbits (m, n) with periods T up to $12 T_z = 12 \times 2\pi/\omega_z$. A closer discussion and a number of further examples can be found in the appendix of [3]. The combination of a (comparatively fast) cyclotron motion around the magnetic field with a (comparatively slow) drift due to an electric field is often described in terms of the so-called ‘guiding centre approximation’ where the guiding centre is the centre point of the cyclotron motion that drifts slowly under the electric influence [7]. This description is not restricted to Penning trap situations, but is sometimes used in that context [8].

Note, that the cyclotron frequency ω_c can be determined from the observable oscillation frequencies. In principle, this can be done by use of the relation $\omega_c = \omega_+ + \omega_-$, which follows directly from (5.13). This, however, is only true in the ideal case. Often, it is favourable to use the so-called ‘invariance theorem’ [4].

Table 5.1 Frequency ratios of periodic orbits up to $T = 12 T_z$ [3] as given by parameters m and n

m	n	T/T_z	ω_z/ω_-	ω_+/ω_z	ω_c/ω_z
2	1	2	2	1	2
4	1	4	4	2	4
3	1	6	3	3/2	6
6	1	6	6	3	6
8	1	8	8	4	8
5	1	10	5	5/2	10
10	1	10	10	5	10
12	1	12	12	6	12
3	2	12	3	3/4	12

The invariance theorem relates the three single-particle oscillation frequencies in a Penning trap to the cyclotron frequency in absence of an electric potential by

$$\omega_c^2 = \omega_-^2 + \omega_+^2 + \omega_z^2. \quad (5.27)$$

The name ‘invariance theorem’ comes from the fact that a number of small imperfections do affect the individual frequencies, but leave ω_c invariant, as will be discussed in detail in Sect. 6.6. In an experiment, it may not be possible to measure all three oscillation frequencies or both ω_+ and ω_- directly. In this case, ω_c may be determined as the upper sideband $\omega_c = \omega_+ + \omega_-$ in magnetron centring, see the discussion in Sect. 11.4. Alternatively, if only ω_+ can be measured, it is possible to perform measurements of ω_+ as a function of the trap voltage U and use

$$\lim_{U \rightarrow 0} \omega_+ = \omega_c \quad (5.28)$$

to extrapolate to the unperturbed cyclotron frequency.

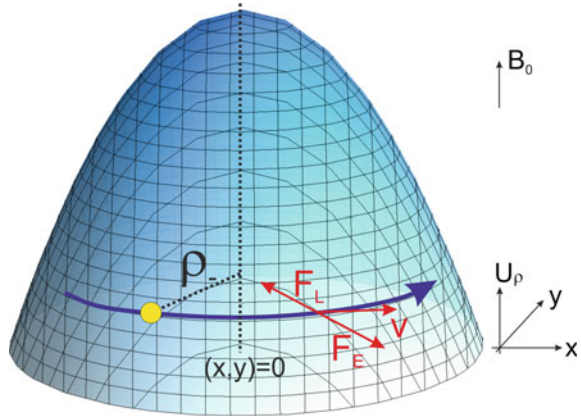
5.3.3 A Closer Look at the Magnetron Motion

The magnetron motion deserves a more thorough discussion, because it is different from the perturbed cyclotron and axial motions in some ways. It has been described as a ‘slow rotation of the centre of cyclotron motion around the central axis of a Penning trap that results from a drift of the cyclotron centre along the electric equipotential lines perpendicular to the magnetic field direction’. This description motivates the term ‘guiding-centre approximation’ as mentioned above. Often, the motion is called ‘unstable’ or ‘non-thermal’ [4], but we need to have a closer look at what it means. The field configuration is clear: we have a homogeneous axial magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_z$ and an electrostatic quadrupole potential that is a harmonic well in the axial direction and the inverse situation in the radial direction, see Fig. 2.2. In an ideal trap, the motions are uncoupled and we now only look at the magnetron motion. The potential hill with its downward slope from the trap centre in the radial direction corresponds to an electric field $\mathbf{E}$ which exerts an outward radial force of $\mathbf{F}_E = q\mathbf{E}$, while the magnetic field is source of an inward Lorentz force of size $\mathbf{F}_L = q\mathbf{v} \times \mathbf{B}_0$ (Fig. 5.7). The equation of motion for the radial coordinate $\mathbf{r}$ is then

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}_0) = q(\mathbf{E} + \mathbf{v}_- \times \mathbf{B}_0) = q\left(\frac{U}{2d^2}\rho_- + \omega_- \rho_- B_0\right)\mathbf{e}_\rho \quad (5.29)$$

in which the electric field at radius ρ_- is given by

Fig. 5.7 Depiction of the radial potential hill of a particle in a Penning trap in Cartesian coordinates ($x^2 + y^2 = \rho^2$) with the magnetron drift motion indicated. The lowest-energy state for the magnetron motion is the one with the largest magnetron radius ρ_-



$$\mathbf{E} = \frac{U}{2d^2} \rho_- \mathbf{e}_\rho. \quad (5.30)$$

Since the electric and magnetic fields are perpendicular in this arrangement, the situation is similar to a Wien filter, which in our case means that there is no net force on the particle when the outward radial force balances the inward Lorentz force. This also means that we have a fixed relation between the velocity of the particle on its circular magnetron trajectory about the top of the potential hill and the radius of it by $v_- = \omega_- \rho_-$, in which the magnetron frequency ω_- is fixed by the values of the electric and magnetic fields. The magnetic field determines the cyclotron frequency ω_c while the electric field determines the axial frequency ω_z which lead to the magnetron frequency

$$\omega_-(U, B_0) = \frac{\omega_c(B_0)}{2} - \left(\frac{\omega_c^2(B_0)}{4} - \frac{\omega_z^2(U)}{2} \right)^{1/2}. \quad (5.31)$$

It is often described as a ‘drift’ motion in the crossed electric and magnetic fields. It is also often called ‘unstable’ which becomes evident when we look at the energies involved. Corresponding to the velocity v_- of the magnetron motion we have a (positive) kinetic energy of

$$E_k = \frac{1}{2} m v_-^2 = \frac{1}{2} m \rho_-^2 \omega_-^2. \quad (5.32)$$

On the other hand, as we have the repulsive force from the potential hill, the corresponding (negative) potential energy is given by

$$E_p = -\frac{1}{4} m \rho_-^2 \omega_z^2, \quad (5.33)$$

which for typical choices of the confining fields dominates the kinetic energy by far since then $\omega_z \gg \omega_-$, such that the total magnetron energy is always negative. We hence have the particle orbiting the potential hill at the fixed magnetron frequency on a radius depending on its velocity (in an experiment given by initial conditions). As long as there is no energy dissipation, this situation will not change.

There are, however, several possible sources of energy dissipation.

- Even in a perfect trap, there is energy dissipation through synchrotron radiation from the magnetron motion. This reduces the motional energy and thus increases the radius, however, for a frequency as low as the magnetron frequency this is in most cases irrelevant, as we will see in Sect. 7.4.
- When (cold) residual gas is present, collisions on average reduce the magnetron energy and thereby increase the orbit, as discussed in Sect. 12.8.
- A resonant dipole excitation at the magnetron frequency (with arbitrary phase and duration) increases the orbital velocity and through $v_- = \omega_- \rho_-$ leads to an increase of the radius, see Sect. 10.1.1.

It may be instructive to look at the problem also in the quantum picture. There, we have an equally spaced ladder of energy levels from zero to negative infinity with spacings of $\hbar\omega_-$. The zero point corresponds to the particle sitting on the top of the hill (at radius zero). From there, it may in principle reach negative infinity at infinite radius (in reality of course cut off by the trap radius). Any of the above-mentioned interactions will lead to a random walk on the ladder. In the first two cases with step sizes of a certain distribution, in case of resonant excitation with step sizes of one. Since the probability distribution is asymmetric [9], and also because l is capped at zero but infinite to positive numbers, the random walk will eventually lead to a higher quantum number, which means down the ladder, which means to an increase of magnetron radius. This is often referred to by the sentence ‘no dissipative process which removes energy can reduce the magnetron radius’. The only well-defined way to reduce a finite magnetron orbit is by motional interconversion with another degree of freedom that is cooled, as has been described in detail by Brown and Gabrielse, and which is called ‘magnetron cooling’ or sometimes ‘magnetron centring’ and we will have a closer look at it in Sect. 11.4.

5.4 Brillouin Frame of Reference

We have seen that from the perspective of the laboratory, the two radial motions in the (x, y) -plane (perturbed cyclotron motion and magnetron motion) cannot be regarded independently. To give a common description, it is possible to introduce the complex variable $u = x + iy$ and write the two radial equations of motion

$$\ddot{x} = \omega_c \dot{y} + \frac{\omega_z^2}{2} x \quad \text{and} \quad \ddot{y} = -\omega_c \dot{x} + \frac{\omega_z^2}{2} y \quad (5.34)$$

in one equation such as

$$\ddot{u} + i \omega_0 \dot{u} - \frac{1}{2} \omega_z^2 u = 0, \quad (5.35)$$

which can be solved by the characteristic equation [3]

$$\omega^2 - \omega_0 \omega + \frac{1}{2} \omega_z^2 = 0, \quad (5.36)$$

and gives solutions for the radial frequencies which read

$$\omega_+ = \frac{1}{2} \omega_c + \omega_1 \quad \text{and} \quad \omega_- = \frac{1}{2} \omega_c - \omega_1. \quad (5.37)$$

Note that ω_0 is the signed value of $q B_0 / m$ such that $|\omega_0| = \omega_c$. The frequency ω_1 is given by

$$4\omega_1^2 = \omega_c^2 - 2\omega_z^2. \quad (5.38)$$

If we now look at the situation from the standpoint of an observer who rotates at the frequency $\omega_c/2$, then we see the outward radial force due to the rotation and the inward Lorentz force due to the magnetic field exactly cancel each other, such that the two radial motions become independent. We then have radial equations of motion

$$\ddot{x} = -\omega_1^2 x \quad \text{and} \quad \ddot{y} = -\omega_1^2 y. \quad (5.39)$$

In fact, they now both appear as simple harmonic motions at the same frequency ω_1 , in the way that the perturbed cyclotron motion appears as an orbital motion at $+\omega_1$ while the magnetron motion appears as an orbital motion at $-\omega_1$ [2]. The trajectory is depicted in Fig. 5.8 for the case of a single particle with a perturbed cyclotron motion at frequency ω_+ and radius ρ_+ , and a magnetron motion at frequency ω_- and radius $\rho_- > \rho_+$. The right hand side shows the radial trajectory in the laboratory frame as we have seen it before. The left hand side shows the trajectory seen in the frame co-rotating at the frequency $\omega_c/2$: the trajectory becomes a simple ellipse with semi-axes $\rho_+ + \rho_-$ and $\rho_+ - \rho_-$, respectively [2].

Under the idealised conditions in this section, the ellipse is fixed in orientation, however when relativistic effects are considered, it starts to rotate at a rate mainly given by the particle kinetic energy [10, 11].

We will see the special role of the frequency $\omega_c/2$ in connection with the rotation of ion plasmas, where the density assumes a maximum for this value of the global rotation frequency. This density limit is called ‘Brillouin limit’, after French-American physicist Leon Nicolas Brillouin (1889–1969) [12], hence we may call the presently discussed frame of reference the ‘Brillouin frame of reference’. See also the discussions in [2, 13].

This specific frame of reference is helpful in solving the equations of motion of a combined trap. The potential of the combined trap as given in (4.7) leads to equations of motion

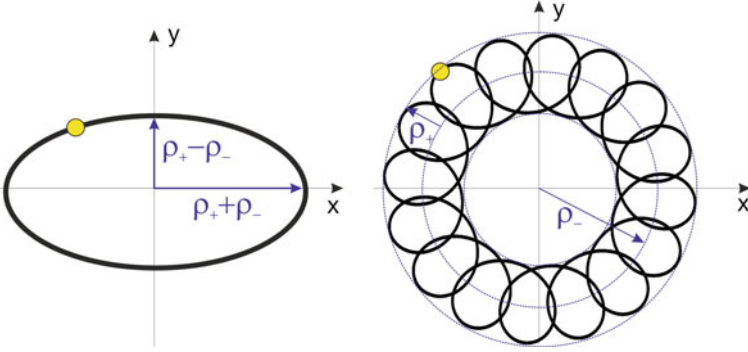


Fig. 5.8 Single-particle radial trajectories as seen in the different frames of reference: The right hand side shows the radial trajectory in the laboratory frame. The left hand side shows the trajectory seen in the frame co-rotating at the frequency $\omega_c/2$: the trajectory becomes an ellipse with semi-axes $\rho_+ + \rho_-$ and $\rho_+ - \rho_-$, respectively

$$\ddot{x} = \omega_c \dot{y} + \frac{q}{md^2}(U - V \cos(\Omega t))x \quad (5.40)$$

$$\ddot{y} = -\omega_c \dot{x} + \frac{q}{md^2}(U - V \cos(\Omega t))y \quad (5.41)$$

$$\ddot{z} = -\frac{2q}{md^2}(U - V \cos(\Omega t))z, \quad (5.42)$$

which for $V = 0$ simplify to give the Penning trap equations of motion (5.1). These coupled equations are difficult to solve. However, when we apply a transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \cos\left(\frac{\omega_c}{2}t\right) + y' \sin\left(\frac{\omega_c}{2}t\right) \\ -x' \sin\left(\frac{\omega_c}{2}t\right) + y' \cos\left(\frac{\omega_c}{2}t\right) \end{pmatrix}, \quad (5.43)$$

this situation changes. The transformation is equivalent to looking at the problem in a frame of reference rotating at frequency $\omega_c/2$, the Brillouin frame of reference. In this frame, with the shorthand $\eta = \Omega t/2$, Equations (5.40)–(5.42) become

$$\frac{\partial^2 x'}{\partial \eta} = -\left(\frac{\omega_c^2}{\Omega^2} - \frac{4qU}{md^2\Omega^2} + \frac{4qV}{md^2\Omega^2} \cos(2\eta)\right)x' \quad (5.44)$$

$$\frac{\partial^2 y'}{\partial \eta} = -\left(\frac{\omega_c^2}{\Omega^2} - \frac{4qU}{md^2\Omega^2} + \frac{4qV}{md^2\Omega^2} \cos(2\eta)\right)y' \quad (5.45)$$

$$\frac{\partial^2 z}{\partial \eta} = -\left(\frac{8qU}{md^2\Omega^2} - \frac{8qV}{md^2\Omega^2} \cos(2\eta)\right)z, \quad (5.46)$$

which are identical with the Mathieu equation of the pure radio-frequency trap. We have thus again transformed into a frame in which the magnetic field is absent from the equations of motion. These equations can now be solved to give the well-known

solutions for the radio-frequency trap [14]. Transforming back into the laboratory frame, we obtain solutions for the three characteristic frequencies ω_+ , ω_- and ω_z by

$$\omega_+ = \frac{\omega_c}{2} + \frac{\kappa_\rho \Omega}{2} \quad (5.47)$$

$$\omega_- = \frac{\omega_c}{2} - \frac{\kappa_\rho \Omega}{2} \quad (5.48)$$

$$\omega_z = \frac{\kappa_z \Omega}{2}, \quad (5.49)$$

where the radial and axial contributions κ_ρ and κ_z are given by

$$\kappa_\rho^2 = \frac{\omega_c^2}{\Omega^2} - \frac{4qU}{md^2\Omega^2} + \frac{2q^2V^2}{m^2d^4\Omega^4} \quad (5.50)$$

$$\kappa_z^2 = \frac{8qU}{md^2\Omega^2} + \frac{8q^2V^2}{m^2d^4\Omega^4}. \quad (5.51)$$

References

1. M. Kretschmar, Particle motion in a Penning trap. *Eur. J. Phys.* **12**, 240 (1991)
2. R.C. Thompson, D.C. Wilson, The motion of small numbers of ions in a Penning trap. *Z. Phys. D* **42**, 271 (1997)
3. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
4. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
5. G.Z.K. Horvath, J.-L. Hernandez-Pozos, K. Dholakia, J. Rink, D.M. Segal, R.C. Thompson, Ion dynamics in perturbed quadrupole ion traps. *Phys. Rev. A* **57**, 1944 (1998)
6. R.J. Whitaker, Mathematics of the Spirograph. *Sch. Sci. Math.* **88**, 554 (2010)
7. T.G. Northrop, The guiding center approximation to charged particle motion. *Ann. Phys.* **15**, 79 (1961)
8. T.S. Pedersen, The guiding center approximation, in *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson (World Scientific, 2016)
9. C.J. Foot, *Atomic Physics*, Oxford Master Series in Atomic, Optical and Laser Physics (Oxford University Press, Oxford, 2005, reprint 2009)
10. Y. Yaremko, Relativistic shifts of eigenfrequencies in an ideal Penning trap. *Int. J. Mass Spectrom.* **405**, 64 (2016)
11. Y. Yaremko, M. Przybylska, A.J. Maciejewski, Dynamics of a relativistic charge in the Penning trap. *Chaos* **25**, 053102 (2015)
12. Remy Mosseri, *Leon Brillouin (1889–1969). A la croisee des ondes*, Belin, Paris, ISBN 2-7011-2299-6 (1999)
13. R.C. Thompson, Penning traps, in *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson (World Scientific, 2016)
14. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)

Chapter 6

Motion of a Single Particle in a Real Penning Trap



Abstract Deviations from the idealised situation described in Chap. 5 as they are present in real Penning traps give rise to a number of effects. Some of these can be minimized by efforts to approach the ideal situation as closely as possible. Others are inherent to the situation and cannot be avoided or circumvented, and we will discuss them in the next chapter. In particular in precision measurements, a number of imperfections may be of relevance. The most prominent imperfections in experimental setups are non-ideal electric and magnetic fields, misalignments of the trap axis with respect to the axis of the magnetic field and mechanical imperfections of trap electrodes. Such imperfections can in principle be avoided by appropriate choice of experimental parameters, in reality however, they are present and need to be taken into account in the analysis of the data. Let us have a look at the effects and their magnitudes.

6.1 Spatial Imperfections of the Electrostatic Field

For Penning traps that deviate from the ideal hyperbolic geometry, we have seen in Sect. 3.3 that in order to obtain a sufficiently harmonic potential close to the trap centre, it is necessary to choose the proper geometry of trap electrodes and the proper set of applied voltages. Let us quantify this statement and look in particular at the most important case, namely the cylindrical one.

6.1.1 *Spherical Expansion of the Electrostatic Potential*

The potential created by the set of electrodes of a real trap is imperfect for a number of reasons. For the cylindrical trap, it is mainly the choice of geometry itself in its deviation from a hyperbolic form that gives rise to radially symmetric field imperfections which can be quantified in an expansion of the potential.

The electrostatic potential of a trap can be expanded in Legendre polynomials, which introduce expansion coefficients $C_k \leq 1$ in the form

$$U = U_0 \sum_{k=0}^{\infty} C_k \left(\frac{r}{d}\right)^k P_k(\cos \theta). \quad (6.1)$$

For mirror symmetry about the (x, y) -plane through $z = 0$, terms with odd k are absent. In a cylindrical trap not all coefficients with $k > 2$ vanish.

This expansion is in spherical coordinates $[r, \theta, \phi]$, however the azimuthal angle ϕ does not appear because of the rotational trap symmetry. We can use the relations $\cos(\theta) = z/(\rho^2 + z^2)^{1/2} = z/r$ and $r^2 = \rho^2 + z^2$ to make the transition to cylindrical coordinates, but the form (6.1) is the one we find in many foundational publications [1–4], so we stay with this form here. Later, we will detail an approach that is in cylindrical coordinates from the beginning. The general expression for the Legendre polynomials $P_k(x)$ is given by [5]

$$P_k(x) = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j}^2 (x-1)^{k-j} (x+1)^j. \quad (6.2)$$

Inserting into (6.1) and using the fact that $\cos \theta = z/r$, the leading terms in the expansion have the geometries

$$\begin{aligned} k=0 & : 1 \\ k=1 & : \frac{1}{d} \frac{r}{r} \left(\frac{z}{r}\right) \\ k=2 & : \frac{1}{2} \frac{r^2}{d^2} \left(\frac{3z^2}{r^2} - 1\right) \\ k=3 & : \frac{1}{2} \frac{r^3}{d^3} \left(\frac{5z^3}{r^3} - \frac{3z}{r}\right) \\ k=4 & : \frac{1}{8} \frac{r^4}{d^4} \left(\frac{35z^4}{r^4} - \frac{30z^2}{r^2} + 3\right) \\ k=5 & : \frac{1}{8} \frac{r^5}{d^5} \left(\frac{63z^5}{r^5} - \frac{70z^3}{r^3} + \frac{15z}{r}\right) \\ k=6 & : \frac{1}{16} \frac{r^6}{d^6} \left(\frac{231z^6}{r^6} - \frac{315z^4}{r^4} + \frac{105z^2}{r^2} - 5\right) \\ k=7 & : \frac{1}{16} \frac{r^7}{d^7} \left(\frac{429z^7}{r^7} - \frac{693z^5}{r^5} + \frac{315z^3}{r^3} - \frac{35z}{r}\right) \\ k=8 & : \frac{1}{128} \frac{r^8}{d^8} \left(\frac{6435z^8}{r^8} - \frac{12012z^6}{r^6} + \frac{6930z^4}{r^4} - \frac{1260z^2}{r^2} + 35\right). \end{aligned} \quad (6.3)$$

For the cylindrical trap in Sect. 3.3 with additional contributions from the compensation electrodes, the potential can be written as

$$U = \frac{1}{2}U_0 \sum_{k=0}^{\infty} C_k \left(\frac{r}{d}\right)^k P_k(\cos \theta), \quad (6.4)$$

in which the coefficients are given by [2]

$$C_k = C_k^{(0)} + D_k \frac{U_c}{U_0}, \quad (6.5)$$

where we have a part $C_k^{(0)}$ in absence of the compensation potential U_c given by [2]

$$C_k^{(0)} = \frac{(-1)^{\frac{k}{2}}}{k!} \frac{\pi^{k-1}}{2^{k-3}} \left(\frac{d}{z_0 + z_e}\right)^k \sum_{n=0}^{\infty} (2n+1)^{k-1} \frac{A_n}{J_0(ik_n \rho_0)} \quad (6.6)$$

with

$$A_n = \frac{1}{2} \left((-1)^n - \sin(k_n z_0) - \sin(k_n(z_0 - z_c)) \right) \quad (6.7)$$

and

$$k_n = \frac{(n + \frac{1}{2})\pi}{z_0 + z_e}, \quad (6.8)$$

plus a contribution $D_k \cdot U_c / U_0$ from the compensation potential with the coefficient [2]

$$D_k = \frac{(-1)^{\frac{k}{2}}}{k!} \frac{\pi^{k-1}}{2^{k-3}} \left(\frac{d}{z_0 + z_e}\right)^k \sum_{n=0}^{\infty} (2n+1)^{k-1} \frac{A'_n}{J_0(ik_n \rho_0)} \quad (6.9)$$

with

$$A'_n = \sin(k_n z_0) - \sin(k_n(z_0 - z_c)) \quad (6.10)$$

and

$$k_n = \frac{(n + \frac{1}{2})\pi}{z_0 + z_e}. \quad (6.11)$$

In these equations, J_0 is the Bessel function of order zero that can be denoted by the sum

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}. \quad (6.12)$$

Note, that the expression $J_0(ik\rho_0)$ in (6.6) and (6.9) is sometimes written as $I_0(k\rho_0)$ on account of the identity

$$I_n(x) = i^{-n} J_n(ix), \quad (6.13)$$

which links the Bessel functions J_n to the modified Bessel functions I_n .

The coefficient C_0 is an overall potential offset and hence irrelevant to the particle motion. C_2 represents the desired quadrupole term relevant for electrostatic confinement and reflects the ‘efficiency’ of the trap in creating a potential well from the applied voltages. Its value chiefly determines the actual oscillation frequencies of particles in the trap, as discussed in Chap. 5. A hyperbolic Penning trap has $C_2 = 1$ by design. For cylindrical Penning traps C_2 is typically about 0.5, and the dominant electric imperfection is characterized by the term C_4 (the octupole component of the electrostatic potential).

6.1.2 Cylindrical Expansion of the Electrostatic Potential

The expansion in Legendre polynomials as given by (6.1) is not the only possibility. As stated in [6], this treatment is likely rooted in [7], but not perforce.

Alternatively, one may write the potential in terms of Bessel functions and then perform a Taylor expansion. In essence, one uses the expansion (6.1) with $\cos(\theta) = z/(\rho^2 + z^2)^{1/2}$ and the identities

$$\rho^n [\cos(\theta)]^{n-2j} = z^{n-2j} \rho^{2j} \quad (6.14)$$

and the sum representation of the Legendre polynomials in the form [5]

$$P_n(x) = \frac{1}{2^n} \sum_{j=0}^{n'} (-1)^j \binom{n}{j} \binom{2n-2j}{n} x^{n-2j}, \quad (6.15)$$

which leads to the expression [8]

$$U = U_0 \sum_{n=1}^{\infty} \frac{C_n}{d^n} \sum_{j=0}^{n'} A_n^{(j)} z^{n-2j} \rho^{2j}. \quad (6.16)$$

Here, n' is the floor of $n/2$ (the next integer smaller than $n/2$ if $n/2$ is not an integer), and the coefficients $A_n^{(j)}$ follow from Poisson’s equation for the given symmetry (omitting the azimuthal term due to the rotational trap symmetry)

$$\Delta U = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) + \frac{\partial^2 U}{\partial z^2} = 0, \quad (6.17)$$

and can therefore be written in the form [8]

$$A_n^{(j)} = \frac{(-1)^{2n-j} \Gamma(n+1)}{2^{2j} \Gamma^2(j+1) \Gamma(n-2j+1)} = \frac{(-1)^{2n-j} n!}{2^{2j} (j!)^2 (n-2j)!} \quad (6.18)$$

where $\Gamma(x)$ is the Gamma function with $\Gamma(x+1) = x!$ for integer x . So the first terms of the inner sum in (6.16) have the geometries $A_n^{(j)} z^{n-2j} \rho^{2j}$:

$$\begin{aligned} n=0 & : 1 \\ n=1 & : z \\ n=2 & : z^2 - \frac{1}{2} \rho^2 \\ n=3 & : z^3 - \frac{3}{2} z \rho^2 \\ n=4 & : z^4 - \frac{6}{2} z^2 \rho^2 + \frac{3}{8} \rho^4 \\ n=5 & : z^5 - \frac{10}{2} z^3 \rho^2 + \frac{15}{8} z \rho^4 \\ n=6 & : z^6 - \frac{15}{2} z^4 \rho^2 + \frac{45}{8} z^2 \rho^4 - \frac{5}{16} \rho^6 \\ n=7 & : z^7 - \frac{21}{2} z^5 \rho^2 + \frac{105}{8} z^3 \rho^4 - \frac{35}{16} z \rho^6 \\ n=8 & : z^8 - \frac{28}{2} z^6 \rho^2 + \frac{210}{8} z^4 \rho^4 - \frac{140}{16} z^2 \rho^6 + \frac{35}{128} \rho^8, \end{aligned} \quad (6.19)$$

where each expression has $n' + 1$ terms since we sum from $j = 0$ to $j = n'$. The sign of the terms alternates because of the contribution $(-1)^{2n-j}$ in (6.18). The coefficient of the first terms ($j = 0$) is always one. Written in the given fractions (1/2 for $j = 1$, and 1/8 for $j = 2$, 1/16 for $j = 3$ etc.), the numerators of the second coefficients ($j = 1$) are triangular numbers

$$1, 3, 6, 10, 15, 21, 28, 36, 45, \dots \quad (6.20)$$

of the sequence A000217 in the OEIS [9], and the numerators of the third coefficients ($j = 2$) are tri-triangular numbers

$$3, 15, 45, 105, 210, 378, 630, 990, \dots \quad (6.21)$$

of the sequence A050534 in the OEIS [9], given by

$$\binom{n+1}{2} \quad \text{and} \quad \binom{\binom{n+2}{2}}{2}, \quad (6.22)$$

respectively, and so forth, in close relation to the coefficients in the probabilists' Hermite polynomials [5]. Each of the denominators

$$2, 8, 16, 128, 256, 1024, 2048 \dots \quad (6.23)$$

of the terms in (6.19) is the corresponding denominator of the Legendre polynomials (see also 6.3), which are given by the sequence A060818 in the OEIS [9].

For a trap with mirror symmetry about the (x, y) -plane through $z = 0$, terms with odd n do not appear in the potential U . It then has the explicit form

$$\begin{aligned} \frac{U}{U_0} = & \frac{C_2}{d^2} \left(z^2 - \frac{1}{2} \rho^2 \right) + \frac{C_4}{d^4} \left(z^4 - 3z^2 \rho^2 + \frac{3}{8} \rho^4 \right) \\ & + \frac{C_6}{d^6} \left(z^6 - \frac{15}{2} z^4 \rho^2 + \frac{45}{8} z^2 \rho^4 - \frac{5}{16} \rho^6 \right) \\ & + \frac{C_8}{d^8} \left(z^8 - \frac{21}{2} z^6 \rho^2 + \frac{210}{8} z^4 \rho^4 - \frac{140}{16} z^2 \rho^6 + \frac{35}{128} \rho^8 \right) + \dots \end{aligned} \quad (6.24)$$

This convention has been used for example in [2, 6, 8, 10–12]. Note, that the terms in equation (6.24) individually fulfill Poisson's equation.

6.2 Effect of Electrostatic Imperfections

As initially stated, we are mainly concerned with potential contributions of even order due to the trap symmetry. The zero-order term C_0 is irrelevant for the particle motion since it is a potential offset. The term C_2 is the desired quadrupolar component of the potential, and in the ideal trap, it is the only contribution to the total potential. In this case, the oscillation frequencies are given by (5.13) with $U = U_0 C_2$ and are independent of motional energies, i.e. independent of motional amplitudes. The effect of the next higher contribution C_4 is an energy-dependence of the oscillation frequencies which close to the trap centre can be described by [1]

$$\begin{pmatrix} \Delta\omega_+/\omega_+ \\ \Delta\omega_z/\omega_z \\ \Delta\omega_-/\omega_- \\ \Delta\omega_L/\omega_L \end{pmatrix} = \frac{6C_4}{C_2} \frac{1}{qU} \begin{pmatrix} \eta^4/4 & -\eta^2/2 & -\eta^2 \\ -\eta^2/2 & 1/4 & 1 \\ -\eta^2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_+ \\ E_z \\ E_- \end{pmatrix}, \quad (6.25)$$

where the matrix contains the dependences of all oscillation frequencies on all oscillation energies and $\eta \equiv \omega_z/\omega_+$ (which is typically of order 10^{-2}). The bottom line of the matrix contains only zeros since the Larmor frequency (of the particle's spin precession in the magnetic field of the trap) is a purely magnetic property and thus not

affected by electric anharmonicities. Note that the original equation in [1] describes a hyperbolic trap with $C_2 = 1$, so we have explicitly introduced the coefficient C_2 . We can then write down these equations in a more explicit form like

$$\frac{\Delta\omega_+}{\omega_+} = \frac{C_4}{C_2} \frac{1}{qU} \left(\frac{\omega_z}{\omega_+} \right)^2 \left[-3E_z + \frac{3}{2} \left(\frac{\omega_z}{\omega_+} \right)^2 E_+ - 6E_- \right] \quad (6.26)$$

$$\frac{\Delta\omega_z}{\omega_z} = \frac{C_4}{C_2} \frac{1}{qU} \left[\frac{3}{2} E_z - 3 \left(\frac{\omega_z}{\omega_+} \right)^2 E_+ + 6E_- \right] \quad (6.27)$$

$$\frac{\Delta\omega_-}{\omega_-} = \frac{C_4}{C_2} \frac{1}{qU} \left[6E_z - 6 \left(\frac{\omega_z}{\omega_+} \right)^2 E_+ + 6E_- \right]. \quad (6.28)$$

Let us now have a closer look at the dependence of the axial frequency on the axial energy, as this is a most often used relation. While the octupole term C_4 describes a relative frequency dependence of ω_z on E_z that is given by

$$\frac{\Delta\omega_z}{\omega_z} = \frac{3}{2} \frac{C_4}{C_2} \frac{E_z}{m\omega_z^2 d^2} \quad (6.29)$$

and that is linear in the axial amplitude, the dodecapole term C_6 accounts for a quadratic dependence, and hence this term contributes less by a factor of about $(z/d)^2$. For the axial motion, it may be regarded as an amplitude-dependent C_4 -term C'_4 which is given by [2]

$$C'_4 = C_4 + \frac{5}{2} \frac{E_z}{qU} C_6, \quad (6.30)$$

and which leads to a relative shift of the axial frequency of

$$\frac{\Delta\omega_z}{\omega_z} = \frac{3}{2} \frac{C'_4}{C_2} \frac{E_z}{m\omega_z^2 d^2} = \frac{3}{2} \frac{C_4}{C_2} \frac{E_z}{m\omega_z^2 d^2} + \frac{15}{4} \frac{C_6}{C_2} \left(\frac{E_z}{m\omega_z^2 d^2} \right)^2. \quad (6.31)$$

For well-cooled particles the effect due to C_6 can be smaller than the one due to C_4 by several orders of magnitude, but in special cases it can be the dominating contribution [13]. The general effects of an octupolar perturbation of the trap potential have also been studied in [14, 15].

Let us use the representation (6.24) of the potential to look explicitly at the situation away from the trap axis, i.e. for finite radius ρ of the motion. We focus on the mixed terms of the potential (6.24) in both axial and radial coordinates. We hence find the axial oscillation frequency ω_z to depend on the axial and radial energies since all terms of the form $z^2 \rho^{2k}$ create a force that is linear in the axial coordinate z . The frequency ω_z thus becomes explicitly dependent on ρ , independent of the axial energy. For this, we find

$$\begin{aligned}\omega_z^2(\rho) &= \omega_z^2 + \Delta\omega_z^2(\rho) \\ &= \frac{2qU_0C_2}{md^2} + \left(-\frac{6\rho^2qU_0C_4}{md^4} + \frac{45\rho^4qU_0C_6}{md^6} + \dots \right),\end{aligned}\quad (6.32)$$

where the first term represents the ideal axial oscillation, while the further terms describe the radial dependence of the axial frequency at zero axial energy. When we add the terms that come additionally due to finite axial energy, we find

$$\begin{aligned}\omega_z^2(E, \rho) &= \omega_z^2(\rho) + \Delta\omega_z^2(E, \rho) \\ &= \omega_z^2(\rho) + \left(\frac{C_4}{C_2} \frac{3}{md^2} - \frac{45C_6}{C_2} \frac{\rho^2}{md^4} \right) E + \left(\frac{15}{4md^2} \frac{C_6}{C_2^2} \right) E^2 + \dots\end{aligned}\quad (6.33)$$

where the first term summarizes the radial dependence at zero axial energy, and the further terms explicitly state the dependence on the axial energy.

By linearisation of this expression, we can find a combined equation for the relative shift of the axial frequency

$$\frac{\Delta\omega_z}{\omega_z} = \frac{\omega_z(\rho) - \omega_z}{\omega_z} + \left(\frac{3C_4}{2C_2} \frac{1}{m\omega_z^2d^2} - \frac{45C_6}{2C_2} \frac{\rho^2}{m\omega_z^2d^4} \right) E + \frac{15}{4} \frac{C_6}{C_2} \left(\frac{E}{m\omega_z^2d^2} \right)^2 + \dots\quad (6.34)$$

Not surprisingly, for $\rho = 0$ this simplifies to the well-known relation [2, 3, 16] we have written down by equation (6.31)

$$\frac{\Delta\omega_z}{\omega_z} = \frac{3}{2} \frac{C_4}{C_2} \frac{E}{m\omega_z^2d^2} + \frac{15}{4} \frac{C_6}{C_2} \left(\frac{E}{m\omega_z^2d^2} \right)^2. \quad (6.35)$$

From (6.25) and the invariance theorem, one can find an expression for the relative shift of the free-particle cyclotron frequency which is often of particular importance

$$\frac{\Delta\omega_c}{\omega_c} = \frac{3}{4} \frac{C_4}{C_2} \frac{\omega_z^2}{\omega_c^3} (\rho_+^2 - \rho_-^2) + \frac{15}{4} \frac{C_6}{C_2} \frac{1}{\omega_c d^2} [\rho_z^2 (\rho_-^2 - \rho_+^2) - (\rho_-^4 - \rho_+^4)] + \dots\quad (6.36)$$

Here, we use the relations between kinetic energies E and amplitudes of motion as given in (5.5)

$$\begin{aligned}\rho_+^2 &= \frac{2E_+}{m(\omega_+^2 - \omega_z^2/2)} d^2 \\ \rho_-^2 &= \frac{2E_-}{m(\omega_-^2 - \omega_z^2/2)} d^2 \\ a_z^2 &= \frac{E_z}{qU} d^2.\end{aligned}\quad (6.37)$$

We can similarly write the absolute frequency shifts as a function of the amplitudes of motion following the formalism in [17] and find for the shifts due to $C_4 \neq 0$ (octupole term)

$$\Delta\omega_+ = \frac{3}{4d^2} \frac{C_4}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_+^2 + 2\rho_-^2 - 2a_z^2) \quad (6.38)$$

$$\Delta\omega_z = \frac{3\omega_z}{4d^2} \frac{C_4}{C_2} (a_z^2 - 2\rho_-^2 - 2\rho_+^2) \quad (6.39)$$

$$\Delta\omega_- = -\frac{3}{4d^2} \frac{C_4}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_+^2 + 2\rho_-^2 - 2a_z^2), \quad (6.40)$$

and for the shifts due to $C_6 \neq 0$ (dodecapole term)

$$\Delta\omega_+ = -\frac{15}{16d^4} \frac{C_6}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_+^4 + 6\rho_+^2\rho_-^2 + 3\rho_-^4 - 6a_z^2\rho_+^2 - 12a_z^2\rho_-^2 + 3a_z^4)$$

$$\Delta\omega_z = \frac{15\omega_z}{16d^4} \frac{C_6}{C_2} (3\rho_+^4 + 12\rho_+^2\rho_-^2 + \rho_-^4 - 6a_z^2\rho_+^2 - 6a_z^2\rho_-^2 + a_z^4)$$

$$\Delta\omega_- = \frac{15}{16d^4} \frac{C_6}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_-^4 + 6\rho_+^2\rho_-^2 + 3\rho_+^4 - 6a_z^2\rho_-^2 - 12a_z^2\rho_+^2 + 3a_z^4).$$

Note, that for contributions in C_6 and all higher contributions we have mixed terms in the axial and radial coordinate, as we have already seen in (6.32) to (6.36). For the shifts due to $C_8 \neq 0$ (hexadecapole term) we have

$$\begin{aligned} \Delta\omega_+ &= \frac{35}{32d^6} \frac{C_8}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_+^6 + 18\rho_+^4\rho_-^2 + 18\rho_+^2\rho_-^4 + 4\rho_-^6 - 4a_z^6 \\ &\quad + 18a_z^4\rho_+^2 + 36a_z^4\rho_-^2 + 12a_z^2\rho_+^4 + 72a_z^2\rho_+^2\rho_-^2 + 36a_z^2\rho_-^4) \\ \Delta\omega_z &= \frac{35\omega_z}{32d^6} \frac{C_8}{C_2} (4\rho_+^6 + 36\rho_+^4\rho_-^2 + 36\rho_+^2\rho_-^4 + 4\rho_-^6 + 12a_z^4\rho_+^2 + 12a_z^4\rho_-^2 \\ &\quad - 18a_z^2\rho_+^4 - 72a_z^2\rho_+^2\rho_-^2 - 18a_z^2\rho_-^4 - a_z^6) \\ \Delta\omega_- &= -\frac{35}{32d^6} \frac{C_8}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_-^6 + 18\rho_-^4\rho_+^2 + 18\rho_-^2\rho_+^4 + 4\rho_+^6 - 4a_z^6 \\ &\quad + 18a_z^4\rho_-^2 + 36a_z^4\rho_+^2 + 12a_z^2\rho_-^4 + 72a_z^2\rho_+^2\rho_-^2 + 36a_z^2\rho_+^4). \end{aligned}$$

From these three equations and the invariance theorem, we find an expression for the shift of the free cyclotron frequency by

$$\begin{aligned} \Delta\omega_c &= \frac{3}{4d^2} \frac{C_4}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_-^2 - \rho_+^2) \\ &\quad + \frac{15}{8d^4} \frac{C_6}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_+^2 + \rho_+^2 - 3a_z^2)(\rho_-^2 - \rho_+^2) \\ &\quad + \frac{105}{32d^6} \frac{C_8}{C_2} \frac{\omega_+\omega_-}{\omega_+ - \omega_-} (\rho_+^4 + 3\rho_+^2\rho_-^2 + \rho_-^4 - 8a_z^2\rho_+^2 - 8a_z^2\rho_-^2 + 6a_z^4)(\rho_-^2 - \rho_+^2). \end{aligned} \quad (6.41)$$

In general, the shifts of the oscillation frequencies in all orders C_n are given by [12]

$$\begin{aligned}\frac{\Delta\omega_+}{\omega_+} &= \frac{\omega_-}{\omega_+ - \omega_-} \frac{(2n)!}{2^{2n-1}d^{2n-2}} \frac{C_{2n}}{C_2} \sum_{k=1}^n \sum_{p=0}^{k-1} \frac{(-1)^k (p+1) \rho_+^{2p} \rho_-^{2(k-1-p)} z^{2(n-k)}}{((n-k)!(p+1)(k-p-1)!)^2} \\ \frac{\Delta\omega_-}{\omega_-} &= -\frac{\omega_+}{\omega_+ - \omega_-} \frac{(2n)!}{2^{2n-1}d^{2n-2}} \frac{C_{2n}}{C_2} \sum_{k=1}^n \sum_{p=0}^{k-1} \frac{(-1)^k (p+1) \rho_+^{2p} \rho_-^{2(k-1-p)} z^{2(n-k)}}{((n-k)!(p+1)(k-p-1)!)^2} \\ \frac{\Delta\omega_z}{\omega_z} &= \frac{(2n)!}{2^{2n}d^{2n-2}} \frac{C_{2n}}{C_2} \sum_{k=1}^{n-1} \sum_{p=0}^k \frac{(-1)^k (n-k) \rho_+^{2p} \rho_-^{2(k-p)} z^{2(n-k-1)}}{((n-k)!p!(k-p)!)^2},\end{aligned}$$

such that for the free cyclotron frequency we have

$$\begin{aligned}\frac{\Delta\omega_c}{\omega_c} &= \frac{\omega_+ \omega_-}{\omega_c(\omega_+ - \omega_-)} \frac{(2n)!}{2^{2n-1}d^{2n-2}} \\ &\times \frac{C_{2n}}{C_2} \sum_{k=1}^n \frac{(-1)^k z^{2(n-k)}}{((n-k)!)^2} \sum_{p=0}^{k-1} \frac{\rho_+^{2p} \rho_-^{2(k-1-p)}}{(p!(k-p-1)!)^2} \left(\frac{1}{p+1} - \frac{1}{k-p} \right)\end{aligned}\tag{6.42}$$

The presence of shifts of the cyclotron frequency ω_c is of particular importance for all measurements that rely on a calibration of the magnetic field strength by that quantity, or that want to perform absolute mass measurements via the cyclotron frequency in a known magnetic field. We will discuss these cases in the context of the application of the continuous Stern-Gerlach effect and mass measurements in Penning traps in Sects. 16 and 19.5.

6.3 Spatial Magnetic Field Imperfections

Let us look at a quantification of geometric imperfections of the magnetic field. Since we speak about the field rather than about a potential, we now have an axial and a radial component. Also, for convenience, we do not scale the radial coordinate ρ by the trap size d and the magnetic field components by B_0 .

6.3.1 Spherical Expansion of the Magnetostatic Field

We can expand the magnetic field in analogy to the electric potential as done in (6.1), however the field components require separate treatments.

In analogy to the expansion of the electrostatic potential (6.1), the axial and radial components of the magnetic field can be written as [1]

$$B_z = \sum_{k=0}^{\infty} B_k r^k P_k(\cos \theta) \quad (6.43)$$

$$B_\rho = - \sum_{k=0}^{\infty} B_k r^k \frac{1}{k+1} P_k^1(\cos \theta), \quad (6.44)$$

where the P_k are Legendre polynomials and the P_k^1 are the associated Legendre polynomials [5] of order 1.

Thus, B_0 is the homogeneous field part in Tesla, B_1 a linear gradient in T/m, B_2 measures the strength of a magnetic bottle in T/m² and so forth. Again here, we stay with spherical coordinates to be consistent with the discussion of expansion (6.1). Below, we will detail an alternative expansion of the axial and radial magnetic field components that is in cylindrical coordinates. The Legendre polynomials in (6.43) are identical to those in the expansion (6.1) of the electrostatic potential, the explicit expressions are given by (6.3). The associated Legendre polynomials of order 1 in the expansion of the radial field component (6.44) follow from the relation [1]

$$P_k^1(\cos \theta) = \sin \theta \frac{dP_k(\cos \theta)}{d \cos \theta}. \quad (6.45)$$

We define the shorthand

$$s^2 \equiv 1 - \cos^2 \theta = 1 - \frac{z^2}{r^2}. \quad (6.46)$$

Then, the geometries of the radial magnetic field terms in (6.44) are given by the expressions

$$\begin{aligned} k=0 & : 0 \\ k=1 & : -\frac{1}{2}sr \\ k=2 & : -\frac{3}{3}sr^2\frac{z}{r} \\ k=3 & : -\frac{3}{8}sr^3\left(\frac{5z^2}{r^2} + 1\right) \\ k=4 & : -\frac{1}{2}sr^4\left(\frac{7z^3}{r^3} + \frac{3z}{r}\right) \\ k=5 & : -\frac{5}{16}sr^5\left(\frac{21z^4}{r^4} + \frac{14z^2}{r^2} - 1\right) \end{aligned} \quad (6.47)$$

$$\begin{aligned}
k = 6 & : -\frac{3}{8}sr^6 \left(\frac{33z^5}{r^5} + \frac{30z^3}{r^3} - \frac{5z}{r} \right) \\
k = 7 & : -\frac{7}{128}sr^7 \left(\frac{429z^6}{r^6} + \frac{495z^4}{r^4} - \frac{135z^2}{r^2} + 5 \right) \\
k = 8 & : -\frac{1}{16}sr^8 \left(\frac{715z^7}{r^7} + \frac{1001z^5}{r^5} - \frac{385z^3}{r^3} + \frac{35z}{r} \right).
\end{aligned}$$

We see from a comparison of (6.47) with (6.3) that while the zero-order contribution to the axial magnetic field is 1, the corresponding radial contribution is zero. Also, while the first-order axial magnetic field component has no radial dependence, the first-order radial magnetic field component has a non-zero radial dependence.

6.3.2 Cylindrical Expansion of the Magnetostatic Field

In similarity to the cylindrical expansion of the electrostatic potential, we can treat the magnetostatic field within the same formalism. We should keep in mind that while we use the electrostatic potential U to describe the electric field in the Penning trap, it is more practical to use the magnetic field directly instead of the magnetostatic potential, and to treat its axial and radial components separately.

In analogy to the expansion of the electrostatic potential (6.16) in Sect. 6.1.2, we can write the axial component of the magnetic field in the form

$$B_z(\rho, z) = \sum_{n=0}^{\infty} B_n \sum_{j=0}^{n'} A_n^{(j)} z^{n-2j} \rho^{2j} \quad (6.48)$$

where as before we have n' defined as the floor of $n/2$ (the next integer smaller than $n/2$ if $n/2$ is not an integer) and the geometry coefficients $A_n^{(j)}$ are given by

$$A_n^{(j)} = \frac{(-1)^{2n-j} \Gamma(n+1)}{2^{2j} \Gamma^2(j+1) \Gamma(n-2j+1)} = \frac{(-1)^{2n-j} n!}{2^{2j} (j!)^2 (n-2j)!}, \quad (6.49)$$

which are the same geometry functions as we have used in the cylindrical expansion of the electrostatic potential, see (6.19). Remember that in contrast to (6.16) we have not scaled the radial coordinate ρ by the trap size d and the magnetic field components by B_0 . This means that B_0 is the homogeneous field part in Tesla, B_1 a linear gradient in T/m, B_2 measures the strength of a magnetic bottle in T/m² and so forth.

The radial component of the magnetic field can be expanded by

$$B_\rho(\rho, z) = \sum_{n=0}^{\infty} B_n \sum_{j=1}^{n''} A_n^{(j)} z^{n-2j+1} \rho^{2j-1} \quad (6.50)$$

where n'' is the floor of $(n+1)/2$ and the geometry coefficients of the radial magnetic field components are given by [12]

$$A_n^{(j)} = \frac{(-1)^j j n!}{2^{2j-1} (n-2j+1)! (j!)^2}. \quad (6.51)$$

Obviously, the expansion of the radial magnetic field component has the same structure as the expansion of the axial field component, note however, that the inner sum in equation (6.48) starts at $j = 0$, while in (6.50) it starts at $j = 1$. We will discuss the consequences below. So the first terms of the inner sum of the expansion in (6.50) have the geometries $A_n^{(j)} z^{n-2j} \rho^{2j}$:

$$\begin{aligned} n=0 & : 0 \\ n=1 & : -\frac{1}{2}\rho \\ n=2 & : -\frac{2}{2}z\rho \\ n=3 & : -\frac{3}{2}z^2\rho + \frac{3}{8}\rho^3 \\ n=4 & : -\frac{4}{2}z^3\rho + \frac{12}{8}z\rho^3 \\ n=5 & : -\frac{5}{2}z^4\rho + \frac{30}{8}z^2\rho^3 - \frac{5}{16}\rho^5 \\ n=6 & : -\frac{6}{2}z^5\rho + \frac{60}{8}z^3\rho^3 - \frac{30}{16}z\rho^5 \\ n=7 & : -\frac{7}{2}z^6\rho + \frac{105}{8}z^4\rho^3 - \frac{105}{16}z^2\rho^5 + \frac{35}{128}\rho^7 \\ n=8 & : -\frac{8}{2}z^7\rho + \frac{168}{8}z^5\rho^3 - \frac{280}{16}z^3\rho^5 + \frac{280}{128}z\rho^7. \end{aligned} \quad (6.52)$$

There is a particularly straight-forward way to obtain the coefficients of the terms in these geometry functions. When written in the given fractions ($1/2$ for $j = 1$, and $1/8$ for $j = 2$, and $1/16$ for $j = 3$ etc.) then we look at the difference triangle of the harmonic unit fractions:

$$\begin{array}{cccccccccc}
1/1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & \dots \\
-1/2 & -1/6 & -1/12 & -1/20 & -1/30 & -1/42 & -1/56 & -1/72 & -1/90 & \dots & \\
1/3 & 1/12 & 1/30 & 1/60 & 1/105 & 1/168 & 1/252 & 1/360 & \dots & & \\
-1/4 & -1/20 & -1/60 & -1/140 & -1/280 & -1/504 & -1/840 & \dots & & & \\
1/5 & 1/30 & 1/105 & 1/280 & 1/630 & 1/1260 & \dots & & & &
\end{array}$$

where the denominators in the odd-numbered rows $2j - 1$ are the numerators of the coefficients in terms j . They are also the denominators of the $(2j - 1)$ th diagonal in the Leibniz harmonic triangle (sequence A003506 in the OEIS [9]), which can be given by

$$n \binom{n-1}{2j-2}. \quad (6.53)$$

Comparing the radial geometry functions (6.52) to the axial ones (which are the same as in 6.19), we realize that all powers of ρ are odd, and all powers of z have the opposite parity of n . Thus, the radial magnetic field is symmetric with respect to z for n odd, and antisymmetric for n even. The radial component $B_\rho(\rho, z)$ has been calculated for a real magnetic geometry in [18] and for the typical field of high-homogeneity superconducting magnets we have B_ρ/B_z of the order of 10^{-12} .

Looking at the total magnetic field, for symmetry about the (x, y) -plane through $z = 0$ we have a geometry with even terms only, as in the electrostatic case. Hence, the magnetic field has the form

$$\begin{aligned}
\mathbf{B}(\rho, z) = & B_0 \mathbf{e}_z + B_2 \left[\left(z^2 - \frac{1}{2} \rho^2 \right) \mathbf{e}_z + (-z\rho) \mathbf{e}_\rho \right] \\
& + B_4 \left[\left(z^4 - 3z^2 \rho^2 + \frac{3}{8} \rho^4 \right) \mathbf{e}_z + \left(-2z^3 \rho + \frac{3}{2} z \rho^3 \right) \mathbf{e}_\rho \right] + \dots
\end{aligned} \quad (6.54)$$

In this representation, the first term is the homogeneous part of the magnetic field, which only has an axial component along $\mathbf{e}_z$ of strength B_0 . The second term characterises a magnetic bottle of strength B_2 that is of relevance to the application of the continuous Stern-Gerlach effect in Penning traps, as we will detail in Chap. 21. The third term is the first of the higher-order terms that go along with non-linear dependences of oscillation frequencies on the respective oscillation energies. We will discuss this in some more detail in the connection with residual fields from a magnetic bottle in Sect. 21.8.

6.4 Effect of Magnetostatic Imperfections

The term B_1 of the axial field represents a linear field gradient that to first order does not affect the oscillation about the trap centre $(\rho, z) = (0, 0)$. The presence of $B_2 \neq 0$ results in a dependence of the oscillation frequencies on the motional energies. Hence, the frequencies shift with the energies (amplitudes) of the motions. Making use of the hierarchy $\omega_- \ll \omega_z \ll \omega_+$ of the oscillation frequencies [2], close to the trap centre this dependence can be expressed by [1]

$$\begin{pmatrix} \Delta\omega_+/\omega_+ \\ \Delta\omega_z/\omega_z \\ \Delta\omega_-/\omega_- \\ \Delta\omega_L/\omega_L \end{pmatrix} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} \begin{pmatrix} -\eta^2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & -2 \\ -\eta^2 & 1 & 2 \end{pmatrix} \begin{pmatrix} E_+ \\ E_z \\ E_- \end{pmatrix}, \quad (6.55)$$

where again the matrix contains the dependences of all oscillation frequencies on all oscillation energies and $\eta \equiv \omega_z/\omega_+$. It expresses that all frequencies depend linearly on all energies except for the axial frequency which does not depend on the axial energy since the corresponding matrix element is zero (due to the purely electrostatic nature of the motion). Here, ω_L is the Larmor frequency (spin precession frequency) of the particle which is not a classical oscillation but can be described within the same formalism. These expressions can be written explicitly by

$$\frac{\Delta\omega_+}{\omega_+} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} \left[-\left(\frac{\omega_z}{\omega_+}\right)^2 E_+ + E_z - 2E_- \right] \quad (6.56)$$

$$\frac{\Delta\omega_z}{\omega_z} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} [E_+ - E_-] \quad (6.57)$$

$$\frac{\Delta\omega_-}{\omega_-} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} [2E_+ - E_z - 2E_-] \quad (6.58)$$

$$\frac{\Delta\omega_L}{\omega_L} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} \left[-\left(\frac{\omega_z}{\omega_+}\right)^2 E_+ + E_z + 2E_- \right]. \quad (6.59)$$

We will detail the effects of terms in B_2 and higher-order terms in Chap. 21 about magnetic bottles.

Again using the amplitude dependences (6.37), we can write the frequency shifts in B_2 as a function of motional amplitudes by [12]

$$\frac{\Delta\omega_+}{\omega_+} = \frac{\omega_+ + \omega_-}{2(\omega_+ - \omega_-)} \frac{B_2}{B_0} \left(a_z^2 - \rho_+^2 - \left(1 + \frac{\omega_-}{\omega_+}\right) \rho_-^2 \right) \quad (6.60)$$

$$\frac{\Delta\omega_z}{\omega_z} = \frac{\omega_+ + \omega_-}{4(\omega_+ \omega_-)} \frac{B_2}{B_0} (\omega_+ \rho_+^2 + \omega_- \rho_-^2) \quad (6.61)$$

$$\frac{\Delta\omega_-}{\omega_-} = -\frac{\omega_+ + \omega_-}{2(\omega_+ - \omega_-)} \frac{B_2}{B_0} \left(a_z^2 - \left(1 + \frac{\omega_+}{\omega_-}\right) \rho_+^2 - \rho_-^2 \right), \quad (6.62)$$

and correspondingly the terms in B_4 by

$$\frac{\Delta\omega_+}{\omega_+} = \frac{3(\omega_+ + \omega_-)}{8(\omega_+ - \omega_-)} \frac{B_4}{B_0} \left(a_z^4 + \rho_+^4 + \left(1 + 2\frac{\omega_-}{\omega_+} \right) \rho_-^4 \right. \quad (6.63)$$

$$\left. - 4\rho_+^2 a_z^2 - 4 \left(1 + \frac{\omega_-}{\omega_+} \right) \rho_-^2 a_z^2 + 4 \left(1 + \frac{\omega_-}{2\omega_+} \right) \rho_+^2 \rho_-^2 \right)$$

$$\frac{\Delta\omega_z}{\omega_z} = \frac{3(\omega_+ + \omega_-)}{8\omega_+ \omega_-} \frac{B_4}{B_0} \quad (6.64)$$

$$\times \left(\omega_- (-\rho_-^4 + \rho_-^2 a_z^2 - 2\rho_+^2 \rho_-^2) + \omega_+ (-\rho_+^4 + \rho_+^2 a_z^2 - 2\rho_+^2 \rho_-^2) \right)$$

$$\frac{\Delta\omega_-}{\omega_-} = -\frac{3(\omega_+ + \omega_-)}{8(\omega_+ - \omega_-)} \frac{B_4}{B_0} \left(a_z^4 + \rho_-^4 + \left(1 + 2\frac{\omega_+}{\omega_-} \right) \rho_+^4 \right. \quad (6.65)$$

$$\left. - 4\rho_-^2 a_z^2 - 4 \left(1 + \frac{\omega_+}{\omega_-} \right) \rho_+^2 a_z^2 + 4 \left(1 + \frac{\omega_+}{2\omega_-} \right) \rho_+^2 \rho_-^2 \right),$$

such that for the combined shift of the free cyclotron frequency we find

$$\frac{\Delta\omega_c}{\omega_c} = \frac{B_2}{2B_0} \left[a_z^2 - \rho_-^2 \frac{\omega_+}{\omega_+ - \omega_-} + \rho_+^2 \frac{\omega_-}{\omega_+ - \omega_-} \right] \quad (6.66)$$

$$+ \frac{3B_4}{8B_0} \left[a_z^4 + 2\rho_+^2 \rho_-^2 + \frac{\omega_c}{\omega_+ - \omega_-} (\rho_-^4 - \rho_+^4) + \frac{4a_z^2}{\omega_+ - \omega_-} (\rho_+^2 \omega_- - \rho_-^2 \omega_+) \right].$$

Equations for the first-order frequency shifts in all even orders of B_k are given by [12]

$$\frac{\Delta\omega_+}{\omega_+} = \frac{\omega_c}{\omega_+(\omega_+ - \omega_-)} \frac{(2n)!}{2^{2n}} \frac{B_{2n}}{B_0} \sum_{k=0}^n \frac{(-1)^k z^{2(n-k)}}{(k!(n-k)!)^2} \quad (6.67)$$

$$\times \sum_{p=0}^k \binom{k}{p} \left[\omega_+ \binom{k}{p} + \omega_- \binom{k}{p+1} \right] \rho_+^{2p} \rho_-^{2(k-p)}$$

and

$$\frac{\Delta\omega_-}{\omega_-} = -\frac{\omega_c}{\omega_-(\omega_+ - \omega_-)} \frac{(2n)!}{2^{2n}} \frac{B_{2n}}{B_0} \sum_{k=0}^n \frac{(-1)^k z^{2(n-k)}}{(k!(n-k)!)^2} \quad (6.68)$$

$$\times \sum_{p=0}^k \binom{k}{p} \left[\omega_- \binom{k}{p} + \omega_+ \binom{k}{p+1} \right] \rho_-^{2p} \rho_+^{2(k-p)}$$

and

$$\begin{aligned} \frac{\Delta\omega_z}{\omega_z} = & -\frac{\omega_c}{\omega_+\omega_-} \frac{(2n)!}{2^{2n+1}} \frac{B_{2n}}{B_0} \sum_{k=1}^n \frac{(-1)^k k z^{2(n-k)}}{(k!(n-k)!)^2 (n-k+1)} \\ & \times \sum_{p=0}^k \binom{k}{p} \left[\omega_+ \binom{k-1}{p-1} + \omega_- \binom{k-1}{p} \right] \rho_+^{2p} \rho_-^{2(k-p)}, \end{aligned} \quad (6.69)$$

such that for the free cyclotron frequency shift we have

$$\begin{aligned} \frac{\Delta\omega_c}{\omega_c} = & \frac{1}{\omega_+ - \omega_-} \frac{(2n)!}{2^{2n}} \frac{B_{2n}}{B_0} \sum_{k=0}^n \frac{(-1)^k z^{2(n-k)}}{(k!(n-k)!)^2} \\ & \times \sum_{p=0}^k \binom{k}{p} \left[\omega_+ \left(\binom{k}{p} - \binom{k}{p-1} \right) + \omega_- \left(\binom{k}{p+1} - \binom{k}{p} \right) \right] \rho_+^{2p} \rho_-^{2(k-p)}. \end{aligned} \quad (6.70)$$

What we have said in the discussion of (6.42) holds true here as well, with the exception that magnetic imperfections cannot be tuned out as simply as electric imperfections. Regions with purposeful distortions $B_2 \neq 0$ such as magnetic bottles for the application of the continuous Stern-Gerlach effect are commonly located away from regions of precision frequency measurements, as we will discuss in the context of the double-trap technique in Sect. 21.7.3. Note, that the present treatment of imperfections is within the frame of a first-order expansion of the potentials and fields. So far, it has apparently not been necessary to include higher-order expansions to explain or predict effects that are of experimental relevance. A second-order treatment of field imperfections has been given in [12], but we will not detail it here.

6.5 Temporal Field Imperfections

As the Penning trap relies on electrostatic and magnetostatic fields for confinement, any temporal gradient of these fields may be seen as an imperfection. Unless used for manipulation of the particle motion or other experimental techniques, field drifts and fluctuations are commonly undesired. This is true in particular for precision experiments that suffer from field changes during a measurement which may span many hours.

6.5.1 Electrostatic Field

Once in thermal equilibrium, the electrode geometry and thus the effective trap size d will not change on an observable level. Hence the electrostatic field may

only change when the voltages applied to the trap electrodes change. High-precision voltage supplies reach about 10 ppm (1 in 10^5) and better relative voltage accuracy on the level of several tens of volts, temperature dependences of the order of 1 ppm (1 in 10^6) per kelvin or better, and relative fluctuations of the order of 10 ppb (1 in 10^8) within minutes [19]. This is facilitated by specific designs [19, 20], including battery-based sources [21]. The influence of external electric fields can usually be neglected due to shielding by the trap and its metallic surrounding.

Note that the radial frequencies depend on the actual value of the electric field strength to a much lesser degree than the axial motion, such that the influence of fluctuations on them is much smaller. When we look at the derivatives of the frequencies with respect to the trap voltage

$$\frac{\partial \omega_z}{\partial U} = \frac{q}{2md^2\omega_z} \quad \text{and} \quad \frac{\partial \omega_{\pm}}{\partial U} = \mp \frac{q}{4md^2 \left(\frac{\omega_z^2}{4} - \frac{\omega_z^2}{2} \right)^{1/2}} \quad (6.71)$$

then for typical confinement conditions we will have $\partial \omega_z / \partial U \gg \partial \omega_{\pm} / \partial U$ because of the frequency relation $\omega_c^2 \gg \omega_z^2$.

In particular for small traps, fluctuations of the electrostatic surface potentials represent a source of temporal field imperfections [22–24]. A metal surface is commonly thought of as an electrical equipotential, but in reality shows significant variations of the potential of up to hundreds of millivolts over distances of micrometres [22]. These so-called ‘patch potentials’ generate local electric fields that have a static component due to the different values of the work function of the different patches, which may further be modified by adsorbates on the surface when cryo-adsorption is present, see also Sect. D.3. Thermal fluctuations of the patches create a non-static component that if the source of $1/f$ -noise in the radio-frequency domain [22] and thus an issue in particular for the small traps common to quantum-information processing with trapped ions [23] and precision experiments in general. It has been shown that the noise depends significantly on the surface temperature [22], so the effects are to some extent suppressed in cryogenic traps.

6.5.2 Magnetostatic Field

The magnetic field has several aspects to it. For one, external magnetic fields may not be negligible as they usually are shielded to a much smaller extent than electric fields. To avoid their influence, self-shielding superconducting solenoid systems have been brought forward [25–28], which largely cancel the effect of an external field at the position of the trap.

The intrinsic magnetic field of a superconducting magnet that is cooled by liquid helium is subject to changes of the internal temperature by changes of the helium evaporation rate [30]. These can be caused either by changes of the exterior atmospheric pressure or temperature [29, 31], which basically means that they are weather-

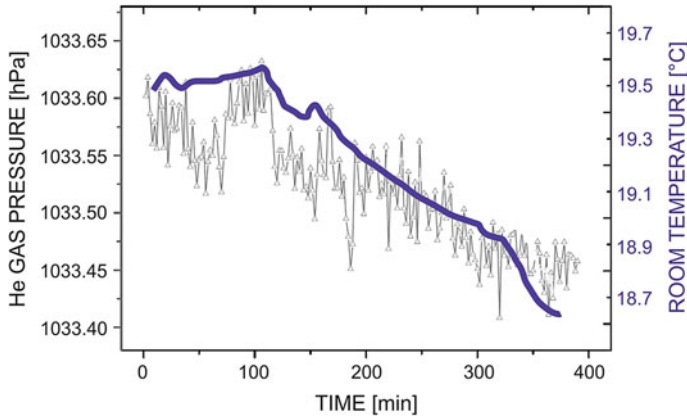


Fig. 6.1 Measurement of the helium pressure in the magnet (triangles) and the room temperature (curve) over the course of about six hours. Data from [29]

dependent. Such changes occur on the time scale of minutes to hours or above, and a number of experimental setups have been actively and / or passively stabilized against such influences [32]. A measurement example of fluctuations is shown in Fig. 6.1: the helium pressure in the magnet dewar shows short-term fluctuations and a long-term correlation with the room temperature. Free-running superconducting magnets commonly show magnetic field drifts on the scale of ppb per hour, see for example [33].

Several techniques have been successfully applied and have helped to reach relative temporal stabilities of the order of 10 ppt (1 in 10^{11}) per hour and better [31, 34, 35]. One other possible source of temporal magnetic field changes is the temperature-dependent magnetic susceptibility of in-bore materials [36–38]. These, however, vanish once the system has reached thermal equilibrium.

When the magnetic field is created by a superconducting magnet, the field is subject to short-term fluctuations on the ppb scale caused by flux jumps in the superconductor [39]. Additionally, the current in the superconducting coils of the magnet decreases on long terms due to the so-called ‘flux creep’ phenomenon [39, 40] which occurs when flux lines that are pinned to inhomogeneities of the superconducting material jump between pinning sites. This phenomenon obviously depends critically on manufacturing detail and for modern superconducting magnets the field decay can be seen as linear over the course of many years.

Finally, one may need to account for acoustic influences, i.e. vibration-induced fluctuations, see for example [41]. A number of experiments hence feature acoustic dampening and vibration isolation, see for example [42, 43].

6.6 Field Misalignment and Ellipticity

The most prominent field imperfections apart from non-vanishing higher-order field components are a tilt of the trap axis with respect to the magnetic field axis, and a non-vanishing ellipticity ε of the electric field which breaks its rotational symmetry. Figure 6.2 gives a schematic of a Penning trap with a polar angle θ between the trap's main axis z and the magnetic field B_0 , as well as an azimuthal angle ϕ to describe the orientation of the elliptic electric potential in the plane normal to z . Let us briefly discuss the effect of a non-zero ellipticity alone, and then combine its effect with a non-zero tilt of the trap axis with respect to the magnetic field.

6.6.1 Effect of Ellipticity

A non-vanishing ellipticity ε adds a term proportional to $-\varepsilon(x^2 - y^2)/2$ to the confining electrostatic potential such that the total electrostatic potential reads

$$U_\varepsilon = \frac{U_0}{2d^2} \left(z^2 - \frac{1}{2}(x^2 + y^2) - \frac{1}{2}\varepsilon(x^2 - y^2) \right). \quad (6.72)$$

Correspondingly, a single particle has axial and radial potential energies [44]

$$E_z(z) = \frac{1}{2}m\omega_z^2 z^2, \quad (6.73)$$

$$E_\rho(x, y) = -\frac{1}{4}m\omega_z^2 ((1 - \varepsilon)x^2 + (1 + \varepsilon)y^2). \quad (6.74)$$

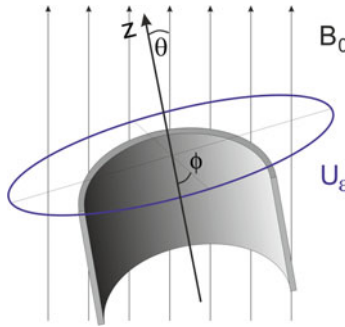


Fig. 6.2 Schematic of a Penning trap with polar angle θ between the trap's main axis z and the magnetic field B_0 , and azimuthal angle ϕ to describe the orientation of the ellipticity of the electric potential U_ε in the plane normal to z . It is caused by the presence of an additional azimuthal component of the electrostatic field, which breaks rotational symmetry and changes oscillation frequencies and particle trajectories

The ellipticity is caused by the presence of the additional quadrupolar azimuthal component of the electrostatic field, which changes oscillation frequencies and particle trajectories. To this end, an additional voltage U' can be applied to opposing segments of the azimuthally split ring electrode of the Penning trap. The ellipticity parameter is then given by

$$\varepsilon = \chi \frac{U'}{U_0}, \quad (6.75)$$

where χ is a geometry parameter that mainly depends on the arrangement of the electrodes used to apply the voltage U' , a method for its determination has been given in [45]. In the presence of an ellipticity, the axial motion is still decoupled from the radial motions, like in the ideal Penning trap, as becomes obvious from the single-particle equations of motion in the presence of $\varepsilon \neq 0$ [46]

$$\ddot{x} = \frac{1}{2}(1 - \varepsilon)\omega_z^2 x + \omega_c \dot{y} \quad (6.76)$$

$$\ddot{y} = \frac{1}{2}(1 + \varepsilon)\omega_z^2 y - \omega_c \dot{x} \quad (6.77)$$

$$\ddot{z} = -\omega_z^2 z. \quad (6.78)$$

The values of the frequencies, however, change and we find generalized reduced cyclotron and magnetron frequencies. While the reduced cyclotron frequency depends only weakly on the value of ε , the magnetron frequency decreases rapidly with increasing ellipticity, and vanishes at a maximum value of ε which represents the stability limit for the magnetron motion [44]. The shape of the magnetron orbit depends significantly on the value of ε and is elliptical. Its numerical eccentricity η approaches 1 for the maximum value of the ellipticity. Numerical eccentricity is defined as

$$\eta = \frac{c}{a}, \quad (6.79)$$

where c is the distance from the centre to a focus of the ellipse, and a is the distance from that focus to a vertex, see Fig. 6.3. This means that a circle has $\eta = 0$ and becomes ‘squeezed’ to a line for $\eta = 1$. In general, due to the different magnitude of the effects on ω_+ and ω_- , the ‘free’ cyclotron frequency ω_c becomes dependent on ε . A noteworthy feature of such a trap is the extended range of mass-to-charge ratios that can be confined. The presence of an ellipticity modifies the critical value of mass to charge as given by (5.21) in the way [45]

$$\left(\frac{m}{q}\right)'_{\text{MAX}} = \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \left(\frac{m}{q}\right)_{\text{MAX}}, \quad (6.80)$$

which theoretically allows one to double the mass-to-charge limit with respect to the standard Penning trap that has $\varepsilon = 0$.

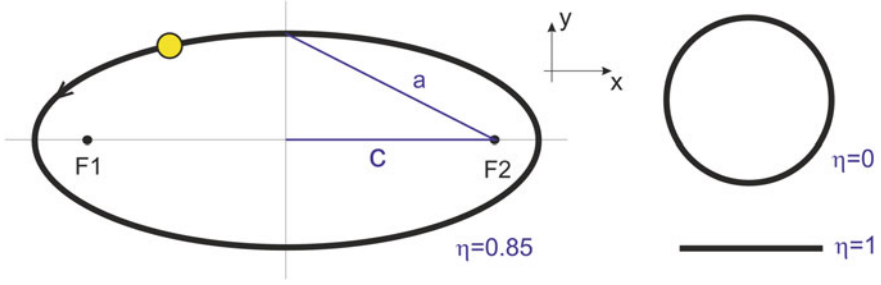


Fig. 6.3 Definition of numerical eccentricity $\eta = c/a$ (left) in an ellipse with focal points $F1$ and $F2$, and the extreme cases of a circle ($\eta = 0$) and a straight line ($\eta = 1$) (right)

In the absence of a tilt of the trap axis with respect to the magnetic field, the ellipticity ε shifts the radial oscillation frequencies to [44]

$$\omega_{\pm}^2 = \frac{1}{2}(\omega_c^2 - \omega_z^2) \pm \frac{1}{2}\sqrt{4\omega_1^2\omega_c^2 + \varepsilon^2\omega_z^4}, \quad (6.81)$$

while the axial oscillation frequency remains unaffected. It fulfills the useful identity

$$\omega_z^2 = \frac{2\omega_+^2\omega_-^2}{\sqrt{1 - \varepsilon^2}}. \quad (6.82)$$

This situation has been carefully discussed in [44]. Remember that the frequency ω_1 is given by $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$. Let us now look at the situation when we additionally introduce a tilt between the central trap axis and the magnetic field axis.

6.6.2 Combined Effect of Ellipticity and Tilt

When we first look at the effect of a tilt of the quadrupolar potential with respect to the magnetic field axis, we can describe the situation by introduction of the polar angle θ between the respective axes. Choosing for example the tilt to be in the xz -plane, we find the equations of motion of a single particle by

$$\ddot{x} = \frac{1}{2}(\cos^2 \theta - 2 \sin^2 \theta)\omega_z^2 x - \frac{3}{2}(\sin \theta \cos \theta)\omega_z^2 z + \omega_c \dot{y} \quad (6.83)$$

$$\ddot{y} = \frac{1}{2}\omega_z^2 y - \omega_c \dot{x} \quad (6.84)$$

$$\ddot{z} = -\frac{1}{2}(2 \cos^2 \theta - \sin^2 \theta)\omega_z^2 z - \frac{3}{2}(\sin \theta \cos \theta)\omega_z^2 x. \quad (6.85)$$

This situation and the resulting motion have been carefully discussed in [46].

When we combine such a tilt with a non-vanishing field ellipticity, we additionally need to take into account the relative azimuthal orientation of the tilt axis and the ellipticity axis by the angle ϕ . In this case, the magnetic field as seen from the trap aligned with the ellipticity is no longer $\mathbf{B} = (0, 0, B_0)$ but takes the form

$$\mathbf{B} = B_0(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (6.86)$$

in which θ and ϕ are the angles characterizing the relative orientation of the axes. The equations of motion in the presence of an ellipticity ε and tilt are given by [17]

$$\ddot{x} - \frac{1}{2}\omega_z^2(1 + \varepsilon)x - \omega'_0(\dot{y} \cos \theta - \dot{z} \sin \theta \sin \phi) = 0 \quad (6.87)$$

$$\ddot{y} - \frac{1}{2}\omega_z^2(1 - \varepsilon)y - \omega'_0(-\dot{x} \cos \theta + \dot{z} \sin \theta \cos \phi) = 0 \quad (6.88)$$

$$\ddot{z} + \omega_z^2 z - \omega'_0(\dot{x} \sin \theta \sin \phi - \dot{y} \sin \theta \cos \phi) = 0. \quad (6.89)$$

where $|\omega'_0| = \omega'_+ + \omega'_-$. The solutions for the oscillation frequencies can be found from the characteristic equation of (6.87)–(6.89) which is given by [17]

$$\omega^6 - \omega_c^2 \omega^4 + (1 - \eta)\omega_z^2 \omega_c^2 \omega^2 - \frac{1}{4}(3 + \varepsilon^2)\omega_z^4 \omega - \frac{1}{4}(1 - \varepsilon^2)\omega_z^6 = 0, \quad (6.90)$$

where the quantity η is the so-called misalignment parameter given by

$$\eta = \frac{1}{2} \sin^2 \theta (3 + \varepsilon \cos 2\phi). \quad (6.91)$$

For small values of the ellipticity and tilt, one finds modified axial and radial oscillation frequencies given by [47]

$$\omega'_\pm \approx \omega_\pm + \eta \omega_- \quad \text{and} \quad \omega'_z \approx \omega_z - \frac{\eta}{2} \omega_z. \quad (6.92)$$

The oscillation frequencies in the presence of a small ellipticity ε and a small tilt angle θ can be written as

$$\omega'_\pm \approx \omega_\pm + \frac{1}{2} \omega_- \sin^2 \theta (3 + \varepsilon \cos 2\phi) \quad (6.93)$$

$$\omega'_z \approx \omega_z \left(1 - \frac{1}{4} \sin^2 \theta (3 + \varepsilon \cos 2\phi) \right). \quad (6.94)$$

When inserting these shifted frequencies into the invariance theorem, one finds

$$\omega_+^2 + \omega_z^2 + \omega_-^2 \approx \omega_c^2, \quad (6.95)$$

which means that the invariance theorem is valid despite the presence of small tilts and ellipticities [47, 48], which is the origin of its name.

The invariance theorem together with (6.93) yield a possibility to give respective bounds on the ellipticity or tilt. When we look at the frequency $\omega'_c = \omega'_+ + \omega'_-$, that is the frequency that resonantly couples the actual shifted perturbed cyclotron and magnetron motions, and write its deviation from the true cyclotron frequency by $\Delta\omega_c = \omega'_c - \omega_c$, the resulting value is [47]

$$\Delta\omega_c = \omega'_- \left(\frac{9}{4}\theta^2 - \frac{1}{2}\varepsilon^2 \right). \quad (6.96)$$

Now assuming that we have $\varepsilon = 0$, the maximum tilt is found from

$$\theta = \sqrt{\frac{4}{9} \frac{\omega'_+ + \omega'_- - \omega_c}{\omega'_-}}. \quad (6.97)$$

Turning the argument around, when we assume $\theta = 0$, we can determine the maximum ellipticity by

$$\varepsilon = \sqrt{-2 \frac{\omega'_+ + \omega'_- - \omega_c}{\omega'_-}}. \quad (6.98)$$

Special properties of an artificially elliptical trap may be used for special applications, see for example [45].

6.7 The Effect of Conical Endcaps

Let us have a look at the effect of conical endcap openings such as in the trap shown in Sect. 4.8. We compare a numerical simulation of such a trap with an analytic calculation that ignores the conical shape and treats the trap like a cylindrical one. For small cone opening angles Θ this results in only small deviations of the coefficients when the electrostatic potential near the trap centre is expanded like in (3.6). Beyond a certain critical angle, the potential in the trap centre becomes influenced by the fringe fields of the increasing endcap openings. A similar effect occurs when the endcap length is reduced below a certain critical value, as discussed in Sect. 3.3. Figure 6.4 shows the potential expansion coefficients C_2 , C_4 and C_6 as a function of the cone opening angle Θ . One can identify a critical value for the opening angle, in the current example about 35° , beyond which the coefficients begin to change significantly.

Fig. 6.5 shows a comparison of the expansion coefficient C_2 between a cylindrical and a conical endcap trap of otherwise same geometry as a function of the endcap

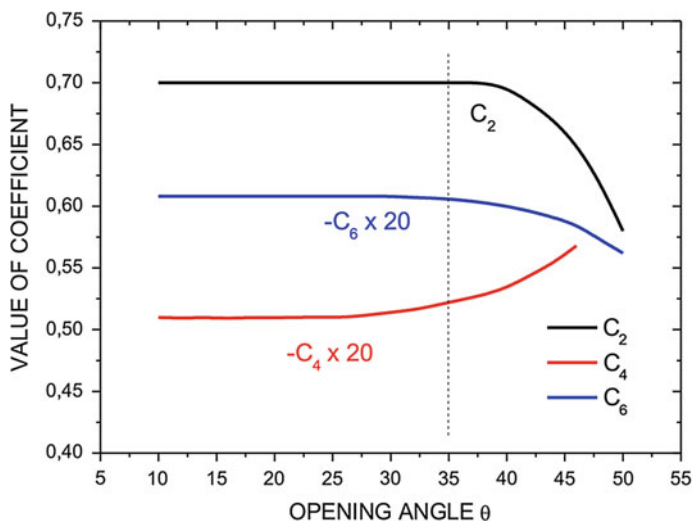


Fig. 6.4 Expansion coefficients C_2 , C_4 and C_6 of the electrostatic potential U as a function of the cone opening angle. The values of C_4 and C_6 have been scaled by a factor of 20 for better display

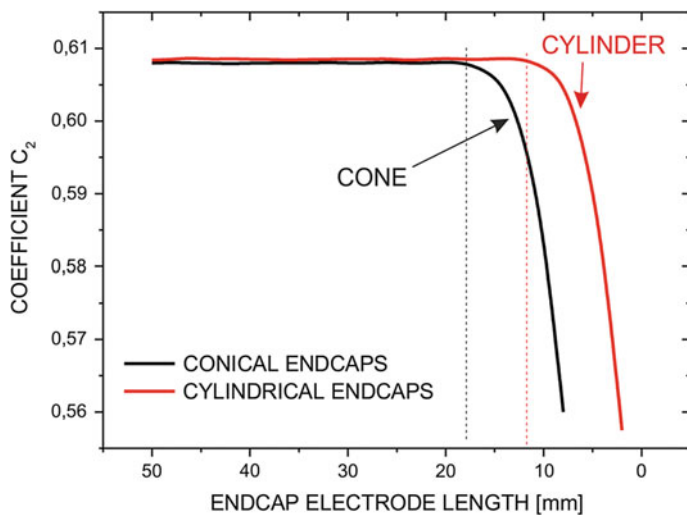


Fig. 6.5 Comparison of the potential expansion coefficient C_2 between cylindrical and conical endcap trap as a function of the endcap length

length. Apparently, as the endcaps are shortened successively, the potential begins to change significantly beyond a certain value. This value is smaller, however, for the cylindrical trap because of its more closed geometry in the axial direction.

6.8 Motional Spectrum in Presence of Field Imperfections

In a perfect trap, the spectrum of motional frequencies contains only the three eigen-frequencies ω_+ , ω_z and ω_- . The axial motion can only be excited by a resonant signal at ω_z that has an axial component, while the radial motions can only be excited by a resonant signal at ω_+ or ω_- , respectively, that has a radial component. Usually, for axial excitation, the endcaps of the Penning trap are used, while for radial excitation one typically uses segments of an azimuthally split electrode, for example a split ring. For a closer discussion of excitation geometries, see Sect. 10.1.

Hence, when we apply a signal that has all field geometry components (for example by use of an unspecific radio-frequency antenna) and perform a sweep over the whole frequency band, we expect to see particle excitation (possibly observed as a loss from the trap) only for these three eigen-frequencies.

In the presence of field imperfections, this is no longer true, since now the confining fields are a superposition of many multipole components that lead to motional coupling, and make accessible a large number of combinations of the eigen-frequencies. The effect of different motional combinations is differently strong, such that the observed spectrum depends on the power of the excitation signal. Such observations have been made for example for electrons in a Penning trap, where apart from the eigen-frequencies also combinations like $\omega_z + \omega_z$, $\omega_+ - \omega_-$, $\omega_+ + \omega_z - \omega_-$, $\omega_+ + \omega_-$ and others were identified [49].

The presence of field imperfections also leads to confinement instabilities in the absence of external excitation, mainly by resonant energy transfer between the motions, as we will discuss in more detail in Sect. 8.3.

References

1. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
2. G. Gabrielse, L. Haarsma, S.L. Rolston, Open-endcap Penning traps for high precision experiments. *Int. J. Mass Spectr. Ion Proc.* **88**, 319 (1989)
3. G. Gabrielse, F.C. Macintosh, Cylindrical Penning traps with orthogonalized anharmonicity compensation. *Int. J. Mass. Spec. Ion Proc.* **57**, 1 (1984)
4. G. Gabrielse, Relaxation calculation of the electrostatic properties of compensated Penning traps with hyperbolic electrodes. *Phys. Rev. A* **27**, 2277 (1983)
5. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover (1972)
6. G.J. Ketter, Theoretical treatment of miscellaneous frequency-shifts in penning traps with classical perturbation theory, Ph.D. thesis, University of Heidelberg (2015)
7. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)
8. J. Steinmann, Modellierung und Simulation der Widerstandskuehlung von hochgeladenen Ionen, Ph. D. thesis, University of Erlangen-Nürnberg, 2015
9. N.J.A. Sloane, ed., The On-Line Encyclopedia of Integer Sequences, published electronically at <https://oeis.org> (2017)
10. T. Murböck, Preparation and cooling of magnesium ion crystals for sympathetic cooling of highly charged ions in a Penning trap, Ph.D. thesis, TU Darmstadt (2017)

11. J. Steinmann, J. Groß, F. Herfurth, G. Zwicknagel, MD simulations of resistive cooling in HITRAP using GPUs. *AIP Conf. Proc.* **1521**, 240 (2013)
12. J. Ketter et al., First order perturbative calculation of the frequency-shifts caused by static cylindrically symmetric electric and magnetic imperfections of a penning trap. *Int. J. Mass. Spectrom.* **358**, 1 (2014)
13. J. Verdu et al., Penning trap measurement of the magnetic moment of the antiproton. *AIP Conference Proc.* **796**, 260 (2005)
14. M. Lara, J.P. Salas, Octupolar perturbation of a single ion in a Penning trap. *Phys. Rev. E* **67**, 027401 (2003)
15. M. Lara, J.P. Salas, Octupolar perturbation of a single ion in a Penning trap, chaos: an interdisciplinary. *J. Nonlinear Sci.* **14**, 763 (2004)
16. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
17. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
18. J. Verdu, Ultrapräzise Messung des elektronischen g-Faktors in wasserstoffähnlichem Sauerstoff, Ph.D. thesis, University of Mainz (2003)
19. Ch. Böhm et al., An ultra-stable voltage source for precision Penning-trap experiments. *Nucl. Inst. Meth. A* **828**, 125 (2016)
20. D.B. Pinegar, K. Blaum, T.P. Biesiadzinski, S.L. Zafonte, R.S. Van Dyck Jr., Stable voltage source for Penning trap experiments. *Rev. Sci. Instr.* **80**, 064701 (2009)
21. A. Wagner et al., A battery-based, low-noise voltage source. *Rev. Sci. Instr.* **81**, 064706 (2010)
22. J. Labaziewicz et al., Temperature dependence of electric field noise above gold surfaces. *Phys. Rev. Lett.* **101**, 180602 (2008)
23. M. Brownnutt, M. Kumph, P. Rabl, R. Blatt, Ion-trap measurements of electric-field noise near surfaces. *Rev. Mod. Phys.* **87**, 1419 (2015)
24. A. Safavi-Naini, P. Rabl, P.F. Weck, H.R. Sadeghpour, Microscopic model of electric-field-noise heating in ion traps. *Phys. Rev. A* **84**, 023412 (2011)
25. G. Gabrielse, J. Tan, Self-shielding superconducting solenoid systems. *J. Appl. Phys.* **63**, 5143 (1988)
26. G. Gabrielse et al., A superconducting solenoid system which cancels fluctuations in the ambient magnetic field. *J. Mag. Res.* **91**, 564 (1991)
27. W. Jhe, D. Phillips, L. Haarsma, J. Tan, G. Gabrielse, Cylindrical penning traps and self-shielding superconducting solenoids for high precision experiments. *Phys. Scr.* **46**, 264 (1992)
28. G. Gabrielse, J.N. Tan, L.A. Orozco, S.L. Rolston, C.H. Tseng, R.L. Tjoelker, A superconducting solenoid system which cancels fluctuations in the ambient magnetic field. *J. Mag. Res.* **91**, 564 (1991)
29. S. Djekic, Implementation of new techniques for high precision g factor measurements, Ph.D. thesis, University of Mainz, 2004
30. K. Blaum et al., ISOLTRAP pins down masses of exotic nuclides. *J. Phy. G: Nucl. Part. Phys.* **31**, 1775 (2005)
31. R.S. Van Dyck Jr., D.L. Farnham, S.L. Zafonte, P.B. Schwinberg, Ultrastable superconducting magnet system for a Penning trap mass spectrometer. *Rev. Sci. Instr.* **70**, 1665 (1999)
32. M. Marie-Jeanne et al., Magnetic field stabilization for high-accuracy mass measurements on exotic nuclides. *Nucl. Instrum. Meth. A* **587**, 464 (2008)
33. Ch. Gerz, D. Wilsdorf, G. Werth, A high precision Penning trap mass spectrometer. *Nucl. Inst. Meth. B* **47**, 453 (1990)
34. E.G. Myers, The most precise atomic mass measurements in Penning traps. *Int. J. Mass Spectrom.* **349–350**, 107 (2013)
35. R.S. Van Dyck Jr., S. Zafonte, S. Van Liew, D.B. Pinegar, P.B. Schwinberg, Ultraprecise atomic mass measurement of the α particle and ^4He . *Phys. Rev. Lett.* **92**, 220802 (2004)
36. G.L. Salinger, J.C. Wheatley, Magnetic susceptibility of materials commonly used in the construction of cryogenic apparatus. *Rev. Sci. Instr.* **32**, 872 (1961)
37. Z. Xia et al., Magnetization of materials used in cryostats. *J. Low Temp. Phys.* **126**, 655 (2002)
38. D.M. Ginsberg, Magnetic susceptibilities of some materials which may be used in cryogenic apparatus. *Rev. Sci. Instr.* **41**, 1661 (1970)

- 39. P.W. Anderson, Y.B. Kim, Hard superconductivity: theory of the motion of Abrikosov flux lines. *Rev. Mod. Phys.* **36**, 39 (1964)
- 40. P.W. Anderson, Theory of flux creep in hard superconductors. *Phys. Rev. Lett.* **9**, 309 (1962)
- 41. J.W. Britton et al., Vibration-induced field fluctuations in a superconducting magnet. *Phys. Rev. A* **93**, 062511 (2016)
- 42. H.J. Schneider-Muntau, High-field NMR magnets. *Solid State Nucl. Magn. Reson.* **9**, 61 (1997)
- 43. T. Kiyoshi et al., HTS-NMR: present status and future plan. *IEEE Trans. Appl. Supercond.* **20**, 714 (2010)
- 44. M. Kretzschmar, Theory of the elliptical Penning trap. *Int. J. Mass Spectrom.* **275**, 21 (2008)
- 45. M. Breitenfeldt et al., The elliptical Penning trap: experimental investigations and simulations. *Int. J. Mass Spectrom.* **275**, 34 (2008)
- 46. G.Zs.K. Horvath, J.-L. Hernandez-Pozos, K. Dholakia, J. Rink, D.M. Segal, R.C. Thompson, Ion dynamics in perturbed quadrupole ion traps, *Phys. Rev. A* **57**, 1944 (1998)
- 47. G. Gabrielse, Why is sideband mass spectrometry possible with ions in a Penning trap? *Phys. Rev. Lett.* **102**, 172501 (2009)
- 48. G. Gabrielse, The true cyclotron frequency for particles and ions in a Penning trap. *Int. J. Mass Spectrom.* **279**, 107 (2009)
- 49. P. Paasche et al., Individual and center-of-mass resonances in the motional spectrum of an electron cloud in a Penning trap. *Eur. Phys. J. D* **18**, 295 (2002)

Chapter 7

Inherent Effects of Single-Particle Confinement



Abstract In the previous chapter, we have discussed the effects of field imperfections on the motion of a single confined particle in a Penning trap. These effects can in principle be avoided, and can in reality be minimised by careful choice of parameters and of the trap manufacturing details. Here, we will discuss effects that are inherent to the confinement situation and can thus not be avoided even in principle.

7.1 Image Charge Effects

The interaction of charged particles with the image charges they create in conductors was first described by E.M. Purcell (1912–1997) [1]. The effects of image charges on the confining potential and the motion of confined particles have been studied theoretically by [2–5] and systematic experimental observations have been made by [6]. The effect has to be taken into account in precision experiments, for example in the experimental determination of stored ions' magnetic moments [7] or precision mass measurements [8], and has to be considered in specific experimental designs [9].

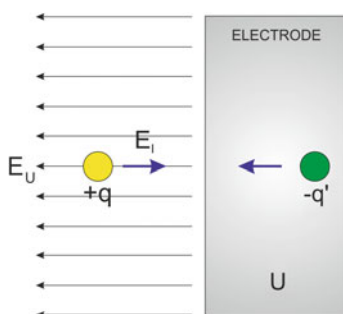


Fig. 7.1 Depiction of the image-charge effect. A particle with real charge $+q$ in proximity to an (in this example endcap) electrode creates an image charge $-q'$ that is of opposite sign, to which it is attracted. This effectively reduces the repulsive potential from the endcap due to the voltage U , and thus reduces the axial oscillation frequency

The presence of the induced image charges in the conducting electrodes of the trap effectively changes the confining potential at the particle position and thus alters its oscillation frequencies [2, 3]. A depiction of the effect can be seen in Fig. 7.1: a particle with real charge q induces an image charge $-q'$ in the (in this example the endcap) electrode at potential U . The charges have opposite sign, hence attracting each other, which reduces the total repulsive electric field E_U of the electrode by an amount E_I . Thus, the effective potential and hence the axial oscillation frequency is reduced. A more detailed discussion of the method of image charges is given in Chap. 14.

Measurements of the image-charge shift of confined particles have been performed for $^1\text{H}^+$, $^2\text{H}^+$, $^3\text{He}^+$, $^3\text{He}^{2+}$, and $^{12}\text{C}^{4+}$ ions [6]. The result for that trap was an observed absolute shift of the reduced cyclotron and magnetron frequencies of about 20 mHz per confined charge, and a relative shift of these frequencies by about 0.25 ppb per atomic mass unit of the confined ion species, in good agreement with theoretical expectations [6]. Detailed calculations of the influence of image charges on the oscillation frequencies of confined particles have also been performed by [3, 4] and [10], including more general trap geometries.

In the general situation we have frequency shifts due to the image charge given by

$$\Delta\omega_z = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{m} \frac{P_z}{2\omega_z} \quad (7.1)$$

$$\Delta\omega_{\pm} = \mp \frac{1}{4\pi\epsilon_0} \frac{q^2}{m} \frac{P_{\rho}}{\omega_{+} - \omega_{-}} \approx \mp \frac{q}{e} \frac{P_{\rho}}{B_0}, \quad (7.2)$$

in which the P_z and P_{ρ} are geometry-dependent electric field gradients that scale roughly like $1/d^3$ [3].

The geometry of the electric fields as reflected in the gradients P_z and P_{ρ} can be described analytically for a number of trap geometries [11, 12], in particular for the spherical and cylindrical case. For the hyperbolic Penning trap used in [6], the effect of the image charge is calculated in a sphere of radius ρ_0 that approximates the trap geometry. This leads to frequency shifts of [6, 13]

$$\Delta\omega_{\pm} \approx \mp \frac{1}{4\pi\epsilon_0} \frac{2q^2}{m\rho_0^3\omega_c} \quad \text{and} \quad \Delta\omega_z \approx -\frac{1}{4\pi\epsilon_0} \frac{q^2}{2m\rho_0^3\omega_z}, \quad (7.3)$$

such that the free-particle cyclotron frequency ω_c remains unaffected to first order:

$$\frac{\Delta\omega_c}{\omega_c} \approx 0. \quad (7.4)$$

In the general case, the situation is different. In arbitrary geometry, we expect a relative shift of the cyclotron frequency given by [12]

$$\frac{\Delta\omega_c}{\omega_c} = -\frac{m}{e} \frac{2P_\rho + P_z}{2B_0^2}. \quad (7.5)$$

which vanishes only for the special case $2P_\rho + P_z = 0$. In a cylindrical geometry, this is not fulfilled. To describe the effect of image charges in a cylindrical Penning trap of inner radius ρ_0 , it is necessary to separate the transversal from the longitudinal component of the electric field of the image charge at the position of the particle. The longitudinal image-field effect is significant and causes frequency shifts such that for the free cyclotron frequency we have [14]

$$\frac{\Delta\omega_c}{\omega_c} = \left(\frac{\omega_-}{\omega_c} - \frac{\omega_+}{\omega_c} \right) \frac{1}{4\pi\epsilon_0} \frac{q^2}{m\rho_0^3\omega_c^2} \propto \frac{m}{\rho_0^3}. \quad (7.6)$$

This shift may be relevant for precision oscillation frequency measurements in small traps, particularly for particles with large mass m . For a discussion of the concept of image charge in a broader sense see Chap. 14.

7.2 Image Current Effect

Apart from the image-charge effect which is due to the presence of conducting electrodes, there is also an image-current effect which is due to the presence of an impedance (of the resonant circuit) connecting these electrodes. A resonant RLC-circuit used for resistive cooling or particle detection has an impedance $Z(\omega)$ which consists of a real and an imaginary part. In resonance, the imaginary part $\text{Im}(Z)$ vanishes and the real part $\text{Re}(Z)$ gives rise to an exponential damping of the particle motion as discussed in Sect. 12.12.1. Away from resonance, the imaginary part gives rise to an effective electrostatic potential that is phase-shifted with respect to the particle motion. This shift in phase effectively reduces the respective frequency. The relative shifts are given by [15]

$$\frac{\Delta\omega_+}{\omega_+} \approx -\frac{q}{m} \frac{1}{2\omega_+ D_+^2} \text{Im}(Z_+(\omega_+)) \quad (7.7)$$

$$\frac{\Delta\omega_z}{\omega_z} \approx -\frac{q}{m} \frac{1}{2\omega_z D_z^2} \text{Im}(Z_z(\omega_z)) \quad (7.8)$$

where again D is the respective effective trap size for the arrangement of electrodes used for the cooling, see Sect. 12.11. Depending on the actual detuning from the resonance, i.e. depending on the value of $\text{Im}(Z)$, this effect may be within achievable experimental resolutions. See also the discussion in [15].

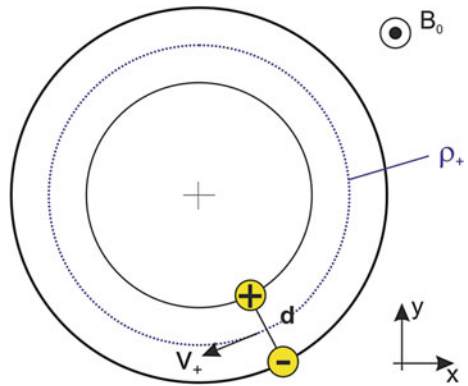
7.3 Particle Polarisability

When we look at a particle that has a non-vanishing electric dipole polarisability α , then the dipole moment in its rotation with the cyclotron motion in the trap leads to a shift of that frequency. As the particle moves on its cyclotron orbit at a velocity v_+ perpendicular to the magnetic field B_0 , the Lorentz force seen in its rest frame at any time is equivalent to an electric field $\mathbf{E}_+ = \mathbf{v}_+ \times \mathbf{B}_0$. If the particle is polarisable, this leads to a dipole moment given by $\mathbf{d} = \alpha \mathbf{E}_+$, which always points towards the centre of the cyclotron orbit. This dipole orientation adiabatically follows the electric field, which leads to its two ends moving at different velocities. The velocity difference gives rise to an effective additional Lorentz force on the particle that is directed along the radial direction as well. The induced dipole moment $\mathbf{d}$ is proportional to the particle velocity $v_+ = \omega_+ \rho_+$, which itself is proportional to the cyclotron radius ρ_+ . This situation results in a constant frequency shift [16]

$$\frac{\Delta\omega_+}{\omega_+} = -\alpha \frac{B_0^2}{m} \quad (7.9)$$

since the additional Lorentz force scales linearly with the cyclotron radius, as does the original Lorentz force. Effectively, the induced dipole moves the centre of charge of the particle to a cyclotron radius which is different from that of its centre of mass. The fractional frequency shift is then given by the ratio of the distance between the two centres and the cyclotron radius ρ_+ . This effect has been experimentally observed for confined CO^+ ions [16], and can be relevant on the scale of relative precisions of 10^{-9} and may, for high magnetic fields and strong polarisabilities, enter the 10^{-7} region. It can in turn be used for polarisability measurements in a Penning trap, as has been shown in [17].

Fig. 7.2 Depiction of a polarisable particle on a cyclotron orbit as two separated charges on account of the dipole moment $\mathbf{d}$ induced in the field $\mathbf{E}_+ = \mathbf{v}_+ \times \mathbf{B}_0$. The orientation of the dipole rotates such that it is always oriented towards the centre of the cyclotron motion



7.4 Radiative Damping

Radiative damping of the particle motion, i.e. kinetic energy loss due to radiation of the accelerated charge, is necessarily connected with the confined particle's oscillation. It leads to exponential energy loss according to

$$E(t) = E_0 \cdot \exp(-\gamma_R t), \quad (7.10)$$

where γ_R is the radiative damping rate. It is significant for light particles such as electrons and positrons but usually negligible for ions since the damping constants γ_R of motions are given by [18]

$$\gamma_{R,+} = \frac{4q^2\omega_+^2}{3mc^3} \approx \frac{4q^4B_0^2}{3m^3c^3} \quad (7.11)$$

$$\gamma_{R,z} = \frac{2q^2\omega_z^2}{3mc^3} = \frac{2q^3U_0^2}{3m^3c^3} \quad (7.12)$$

$$\gamma_{R,-} = \frac{4q^2\omega_-^2}{3mc^3}, \quad (7.13)$$

respectively, and are hence even for highly charged ions suppressed by a factor of about 10^9 with respect to electrons. Figure 7.3 shows the corresponding cooling rates (inverse of the cooling time constants) of an electron/positron due to synchrotron radiation as a function of the magnetic field strength. The data points are electron measurements from [19]. Radiative cooling of protons has been reported in [20]. The

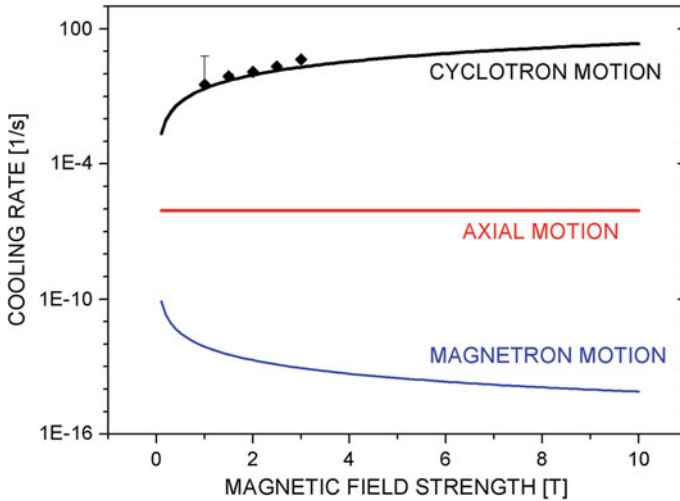


Fig. 7.3 Electron cooling rates due to synchrotron radiation as a function of the magnetic field strength for the three motions. Data points: measurements from [19]

‘self-cooling’ of electrons and positrons on the time scale of fractions of a second via their cyclotron oscillation may be an experimental advantage (see for example the case discussed in section H), or a disadvantage because of potential magnetron loss from the trap, as has been discussed in [21]. It is, however, possible to inhibit this spontaneous emission when the trap acts like a cavity which is detuned from the emitted radiation [22]. The cyclotron radiation of an electron or positron in a magnetic field of a few Tesla is at a microwave frequency and the effective application of this method hence requires a small trap in which the cavity modes can be predicted sufficiently well. This way, suppression of spontaneous emission by factors of 100 and above has been achieved [21].

7.5 Cavity Effects

The conducting surfaces of the trap electrodes surrounding the confinement volume effectively form a cavity for electromagnetic radiation, with a spectrum of electromagnetic modes depending on the details of the geometry [23, 24]. The prediction of modes is facilitated in a cylindrical geometry with respect to hyperboloidal geometries [25, 26]. For traps of sizes of the order of mm to cm as presently discussed, the relevant cavity modes are in the microwave domain, and hence only of interest when particle oscillation frequencies are in that domain. This is true in particular for the (perturbed) cyclotron motion of electrons and positrons at frequency ω_+ . The trap cavity modifies the density of states of the radiation modes of free space, and the oscillation frequency of a confined particle is affected when a mode is nearby in frequency. This is depicted in Fig. 7.4 which shows a generic example of modes in a cavity, with three example frequencies indicated at which spontaneous emission is enhanced with respect to average (left), inhibited because the mode strength is small (middle), and the ‘frequency pulling’ towards a strong mode (right)

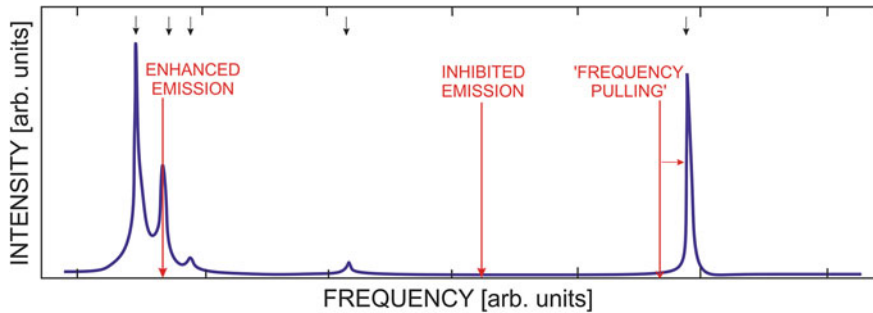


Fig. 7.4 Generic example of modes in a cavity, with three example frequencies indicated at which spontaneous emission is enhanced with respect to average (left), inhibited because the mode strength is small (middle), and the ‘frequency pulling’ towards a strong mode (right)

(middle), and the ‘frequency pulling’ towards a strong mode (right). The corresponding shifts have been calculated in [25, 26], and are of relevance in high-precision measurements such as for example in [21].

7.6 Relativistic Frequency Shifts

The relativistic shifts of the frequencies in the external degrees of freedom (oscillation frequencies) can be understood in terms of the relativistic mass effect due to the motional energy of the particle. Given the high precisions possible in Penning trap mass measurements, these effects are visible already at low particle velocities (energies) [27]. These shifts can be written in analogy to the magnetic and electric field effects in (6.25) and (6.55) as [18]

$$\begin{pmatrix} \Delta\omega_+/\omega_+ \\ \Delta\omega_z/\omega_z \\ \Delta\omega_-/\omega_- \\ \Delta\omega_L/\omega_L \end{pmatrix} = -\frac{1}{mc^2} \begin{pmatrix} 1 & 1/2 & -\eta^2 \\ 1/2 & 3/8 & -\eta^2/4 \\ -\eta^2 & -\eta^2/4 & -\eta^4/4 \\ 2/9 & 1/2 & -\eta^2 \end{pmatrix} \begin{pmatrix} E_+ \\ E_z \\ E_- \end{pmatrix}. \quad (7.14)$$

A detailed treatment of relativistic effects on the oscillation frequencies and the particle trajectories has been given in [28] and [29]. We can write down the equations in (7.14) explicitly, and for the oscillation frequencies we find

$$\frac{\Delta\omega_+}{\omega_+} = -\frac{1}{mc^2} \left[E_+ + \frac{1}{2}E_z - \left(\frac{\omega_z}{\omega_+}\right)^2 E_- \right] \quad (7.15)$$

$$\frac{\Delta\omega_z}{\omega_z} = -\frac{1}{mc^2} \left[\frac{1}{2}E_+ + \frac{3}{8}E_z - \frac{1}{4}\left(\frac{\omega_z}{\omega_+}\right)^2 E_- \right] \quad (7.16)$$

$$\frac{\Delta\omega_-}{\omega_-} = -\frac{1}{mc^2} \left[-\left(\frac{\omega_z}{\omega_+}\right)^2 E_+ - \frac{1}{4}\left(\frac{\omega_z}{\omega_+}\right)^2 E_z - \frac{1}{4}\left(\frac{\omega_z}{\omega_+}\right)^4 E_- \right], \quad (7.17)$$

while for the Larmor frequency we have

$$\frac{\Delta\omega_L}{\omega_L} = -\frac{1}{mc^2} \left[\frac{2}{9}E_+ + \frac{1}{2}E_z - \left(\frac{\omega_z}{\omega_+}\right)^2 E_- \right]. \quad (7.18)$$

For the Larmor frequency of a bound electron, a more thorough discussion is in place. Following the statements in [30–32], for the relativistic effect on the Larmor precession frequency, two cases need to be distinguished, depending on the spin orientation relative to the magnetic field. We use $\beta = v/c$ where v is the ion velocity, and the corresponding relativistic gamma-factor is $\gamma = (1 - \beta^2)^{-1/2}$. Then, the free ion cyclotron frequency is given by $\omega_c = qB_0/(\gamma m)$. Denoting the bound electron’s

magnetic moment anomaly by $a = g/2 - 1$ one finds the Larmor frequency to be $\omega_L = (1+a)\omega_c^e$ for an orientation parallel to the magnetic field, and $\omega_L = (1+\gamma a)\omega_c^e$ for perpendicular orientation, where ω_c^e is the true free electron cyclotron frequency. For ions at motional temperatures of about 4 K and for an anomaly of about 10^{-3} the factor γ is of order 10^{-15} , and hence outside current experimental resolutions. Even if ions need to be excited to motional energies of several eV in order to be properly detected, the magnitude of this effect is below 10^{-12} [14].

Note, however, that the above-said is true only for the relativistic effect of typical *kinetic* energies of confined ions. When ions change their internal (electronic excitation) energy, e.g. by absorption or emission of photons, the relativistic mass effect may be significant, as we will discuss in detail in Sect. 17.7.

7.7 Sokolov-Ternov Effect

The Sokolov-Ternov effect describes the self-polarization of charged particles moving in a magnetic field. The self-polarization occurs through the emission of spin-flip synchrotron radiation. It was predicted in 1963 by Igor Ternov and then justified by Arsenij Sokolov [33] using exact solutions of the Dirac equation. Potentially, it obstructs measurements of magnetic moments which use the continuous Stern-Gerlach effect, because the spin orientation of the confined particle in a Penning trap is affected during the measurement time by the magnetic field used for radial confinement. For an ensemble of trapped ions one would observe an increasing orientation ξ of the spins anti-parallel to the magnetic field with an exponential time behaviour given by

$$\xi(t) = A \left(1 - e^{-t/\tau_{ST}} \right), \quad (7.19)$$

where the saturation $A = 8\sqrt{3}/15 \approx 0.924$ is slightly below 1 and the orientation time constant τ_{ST} is given by

$$\tau_{ST} = A \frac{\hbar^2}{mcq^2} \left(\frac{mc^2}{E_k} \right)^2 \left(\frac{B_S}{B_0} \right)^3. \quad (7.20)$$

Here, $B_S = m_e^2 c^2 / (e\hbar) \approx 4.41 \times 10^9$ T is the Schwinger field [34] (after Julian Schwinger (1918–1994), Nobel Prize in physics 1965), B_0 is the magnetic field strength in the trap and E_k is the particle kinetic energy. While this effect of radiative polarization provides a unique capability for creating polarized beams of high-energy particles in storage ring experiments, in typical trap experiments it can be ignored as for any particle and any realistic set of trapping parameters the time constant τ_{ST} exceeds the lifetime of the experiment. A thorough discussion of this effect and its close relation to the Unruh effect is given in [35].

7.8 Spontaneous Decay of the Spin State

The measurement of magnetic moments by the continuous Stern-Gerlach effect relies on manipulation and detection of a particles's spin state in the external magnetic field, and a spontaneous radiative decay of a higher-energy to a lower-energy spin state during the measurement would obstruct the applicability of the continuous Stern-Gerlach effect and of Zeeman spectroscopy. Let us see whether this is a concern. A decay from a state $|M_J + 1\rangle$ to $|M_J\rangle$ is energetically possible via a magnetic dipole (M1) transition at a rate given by

$$\gamma_s = \frac{4}{3} \frac{\omega^3 \mu_B^2 \mu_0}{\hbar c^3} \Pi, \quad (7.21)$$

in which the term

$$\Pi = \frac{1}{2}(J - M_J)(J + M_J + 1) \left(\frac{3J(J + 1) - L(L + 1) + \frac{3}{4}}{2J(J + 1)} \right)^2 \quad (7.22)$$

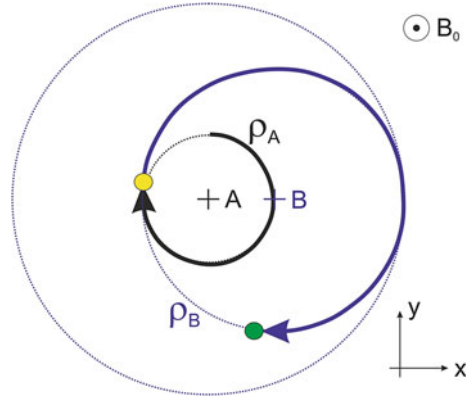
accounts for the spin statistics. Here, ω is the microwave transition frequency, and for the s-state we have $\Pi = 2$. Note, that equation (7.21) is a non-relativistic approximation excluding QED effects. According to this equation, the natural lifetime of the bound electron's upper spin state in highly charged ions such as in [36–38] is of the order of years and can safely be ignored.

The same formula applies to the bound electron in excited Zeeman sub-states of the fine structure and hyperfine structure as employed in measurements by use of laser-microwave double-resonance spectroscopy of highly charged ions [39]. Here, $\Pi = 2/9$ for $p_{1/2}$ -states, $\Pi = 32/9$ for $p_{3/2}$ -states ($M_J = +1/2$ to $M_J = -1/2$) and $\Pi = 24/9$ for $p_{3/2}$ -states ($M_J = +3/2$ to $M_J = +1/2$). For transitions at frequencies of the order of several tens of GHz, the corresponding lifetimes of the upper Zeeman sub-states are also of the order of years and thus outside of experimental time scales as well.

7.9 Charge-Changing Reactions

A confined particle may undergo a process that changes its charge state, for examples anions or highly charged ions that perform a charge exchange with neutral gas or other species that make a transition from a higher absolute charge state to a lower one. This obviously changes all of the particle's oscillation frequencies, but more importantly it changes the particle trajectory and is a possible mechanism for particle loss, both in the axial and radial directions. Figure 7.5 shows a schematic of the radial motion of a particle that undergoes a change of its charge from $q_1 = 2e$ to $q_2 = 1e$. In general, assuming an initial cyclotron radius $\rho_+^{(1)}$ at zero magnetron radius, the

Fig. 7.5 Schematic of the radial motion of a particle that undergoes a change of its charge from $q_1 = 2e$ to $q_2 = 1e$. This changes the cyclotron radius from ρ_A to $\rho_B = 2\rho_A$. Assuming zero initial magnetron radius, the centre of the motion is shifted from point A to point B, giving rise to a magnetron radius of ρ_A , hence extending the overall motion to within a radius of $3\rho_A$



change of the charge from q_1 to $q_2 < q_1$ changes the cyclotron frequency from $\omega_+^{(1)}$ to

$$\omega_+^{(2)} \approx \frac{q_2}{q_1} \omega_+^{(1)}, \quad (7.23)$$

while the reaction leaves the particle velocity nearly unchanged (apart from a small recoil). This leads to an increase of the cyclotron radius to

$$\rho_+^{(2)} \approx \frac{q_1}{q_2} \rho_+^{(1)}, \quad (7.24)$$

while the centre of the motion is shifted, giving rise to a finite magnetron radius of

$$\rho_-^{(2)} \approx \left(\frac{q_1}{q_2} - 1 \right) \rho_+^{(1)}. \quad (7.25)$$

In particular for small numbers q_1 and q_2 , the relative changes are large. As depicted in Fig. 7.5, for $q_1 = 2$ and $q_2 = 1$, the cyclotron radius is doubled, and due to the shift of the motional centre and the corresponding magnetron radius of about $\rho_+^{(1)}$, the overall extension of the motion in the radial plane is tripled, which may lead to radial particle loss, as has been investigated in [40].

7.10 The Presence of a Second Particle

As a step towards the following chapter, we look at situations with two and three confined particles. When two or more particles are confined, the mutual Coulomb interaction modifies the equations of motion. For identical particles ‘1’ and ‘2’ we can write equations of motion for particle ‘1’ given by [41]

$$\ddot{x}_1 + \omega_c \dot{y}_1 - \frac{\omega_z^2}{2} x_1 + \frac{1}{4\pi\epsilon_0} \frac{(x_1 - x_2)q^2}{mr^3} = 0 \quad (7.26)$$

$$\ddot{y}_1 + \omega_c \dot{x}_1 - \frac{\omega_z^2}{2} y_1 + \frac{1}{4\pi\epsilon_0} \frac{(y_1 - y_2)q^2}{mr^3} = 0 \quad (7.27)$$

$$\ddot{z}_1 + \omega_z^2 z_1 + \frac{1}{4\pi\epsilon_0} \frac{(z_1 - z_2)q^2}{mr^3} = 0, \quad (7.28)$$

and symmetric equations for particle ‘2’ with the indices exchanged, so we have 6 coupled equations. Here r is the particle-particle distance given by

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2. \quad (7.29)$$

We can find a solution by separating the centre-of-mass motion from the relative motions. This gives again 6 equations, this time 3 for the centre of mass and 3 for the relative motion. The motion of the centre of mass is given by (bold symbols as centre-of-mass coordinates) [41]

$$\ddot{\mathbf{x}} + \omega_c \dot{\mathbf{y}} - \frac{\omega_z^2}{2} \mathbf{x} = 0 \quad (7.30)$$

$$\ddot{\mathbf{y}} + \omega_c \dot{\mathbf{x}} - \frac{\omega_z^2}{2} \mathbf{y} = 0 \quad (7.31)$$

$$\ddot{\mathbf{z}} - \omega_z^2 \mathbf{z} = 0, \quad (7.32)$$

which leads to the same oscillation frequencies for the centre of mass as for the single particle.

The centre of mass of an ensemble of identical particles has the same equations of motion like a single particle, irrespective of the particle number, since the mutual Coulomb interactions do not contribute to its motion.

The relative motion is then described by the equations

$$\ddot{x} + \omega_c \dot{y} - \frac{\omega_z^2}{2} x + \frac{1}{4\pi\epsilon_0} \frac{xq^2}{mr^3} = 0 \quad (7.33)$$

$$\ddot{y} + \omega_c \dot{x} - \frac{\omega_z^2}{2} y + \frac{1}{4\pi\epsilon_0} \frac{yq^2}{mr^3} = 0 \quad (7.34)$$

$$\ddot{z} - \omega_z^2 z + \frac{1}{4\pi\epsilon_0} \frac{zq^2}{mr^3} = 0, \quad (7.35)$$

and there are two cases for small motional energies with particularly simple solutions, one where the two particles are aligned along the z -axis of the trap, and the other when they are aligned in the central radial plane. When they are aligned along the

z -axis, their equilibrium positions are symmetric around the trap centre at $z = 0$ and the distance s between these positions is given by [42]

$$s^3 = \frac{2}{4\pi\epsilon_0} \frac{q^2}{m\omega_z^2}. \quad (7.36)$$

The frequency of the vibration against each other along the axis is given by $\sqrt{3}\omega_z$. When the two particles are aligned in the radial plane, the situation is more complicated. Looking only at the radial motion, it depends on the particle separation in the way that for close proximity the orbital frequency tends towards the Brillouin frequency $\omega_c/2$ at which the particle separation is smallest and has the value [42]

$$s^3 = \frac{2}{4\pi\epsilon_0} \frac{q^2}{m\omega_1^2} = \frac{2}{4\pi\epsilon_0} \frac{4q^2}{m(\omega_c^2 - 2\omega_z^2)}. \quad (7.37)$$

As we will see later in Sect. 13.2, this behaviour is also reflected in larger ensembles of particles at low energy that behave like non-neutral plasmas.

In experiments with two simultaneously confined particles [43], there is the possibility to resonantly interconvert the radial motions of the particles by application of a π -pulse, see also Chap. 11. This can be used to ‘park’ one particle in a large radial orbit, perform a cyclotron frequency measurement on the other one close to the trap centre, then swap their positions and perform the same measurement on that one [44]. This finds applications for example in precision mass measurements where the systematic effects can hence be minimized [45]. The central particle is subjected to much smaller frequency shifts caused by field imperfections due to its much smaller energy and amplitude of motion. It is, however, influenced by the presence of the second particle. The non-resonant particle-particle interaction can be modelled by treating the outer particle as a continuous ring of charge. This charge distribution modifies the potential near the centre of the trap and causes frequency shifts of the inner particle’s motion in the form [46]

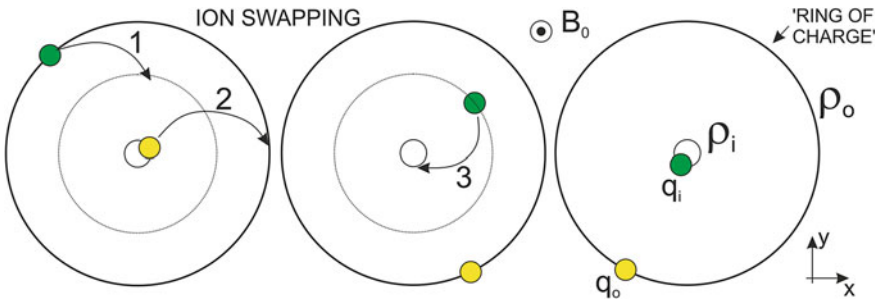


Fig. 7.6 Schematic of ion swapping by resonant coupling pulses (left and middle), and the measurement situation (right), where the cyclotron frequency of the inner particle is influenced by the outer particle, modelled as a ‘ring of charge’

$$\Delta\omega_+ = \frac{\omega_z}{\omega_+} \Omega_z \left[\frac{9}{16} \left(\frac{\rho_i}{\rho_o} \right)^2 + \frac{75}{128} \left(\frac{\rho_i}{\rho_o} \right)^4 + \frac{1225}{2048} \left(\frac{\rho_i}{\rho_o} \right)^6 + \dots \right] \quad (7.38)$$

$$\Delta\omega_z = \Omega_z \left[\frac{9}{16} \left(\frac{a_i}{\rho_o} \right)^2 - \frac{75}{128} \left(\frac{a_i}{\rho_o} \right)^4 + \frac{1225}{2048} \left(\frac{a_i}{\rho_o} \right)^6 + \dots \right], \quad (7.39)$$

where the index ‘i’ refers to the inner particle, the index ‘o’ to the outer particle, and the frequencies are the frequencies of the inner particle. The frequency Ω is given by

$$\Omega = \frac{1}{\omega_z} \frac{q_i q_o}{4\pi \epsilon_0 m_i \rho_o}, \quad (7.40)$$

where again ω_z refers to the inner particle.

When we look at the situation with three identical particles in a plane, similar to our initial discussion of two particles, we can make statements about their motions in the Brillouin frame of reference we have introduced in Sect. 5.4. We find four modes involving motion in the plane, a breathing mode where all three ions move in and out together at a frequency $\omega = \sqrt{3} \omega_1$, a rotation at $\omega = 0$ for low amplitude, and two modes where one particle moves inwards while the other two move outwards at a frequency $\omega = \sqrt{3}/2 \omega_1$. Also, there are two modes for a rocking motion in the axial direction at a frequency of $\omega = (\omega_z^2 - \omega_1^2)^{1/2}$, as has been discussed in detail in [41]. For more particles, this approach becomes tedious and there seem to be no examples in literature. In the following chapter, we will discuss the behaviour of many simultaneously confined particles, their mutual interactions and collective effects. In such a treatment, the individual particles lose their identity and the quantities involved refer to the ensemble as a whole.

References

1. E.M. Purcell, Spontaneous emission probabilities at radio frequencies. *Phys. Rev.* **69**, 681 (1946)
2. D.F.A. Winters, M. Vogel, D. Segal, R.C. Thompson, Electronic detection of charged particle effects in a Penning trap. *J. Phys. B* **39**, 3131 (2006)
3. J.V. Porto, Series solution for the image charge fields in arbitrary cylindrically symmetric Penning traps. *Phys. Rev. A* **64**, 023403 (2001)
4. E. Fischbach, N. Nakagawa, Apparatus-dependent contributions to $g-2$ and other phenomena. *Phys. Rev. D* **30**, 2356 (1984)
5. D.G. Boulware, L.S. Brown, T. Lee, Apparatus-dependent contributions to $g-2$? *Phys. Rev. D* **32**, 729 (1985)
6. R.S. Van Dyck Jr., F.L. Moore, D.L. Farnham, P.B. Schwinberg, Number dependency in the compensated Penning trap. *Phys. Rev. A* **40**, 6308 (1989)
7. M. Vogel, W. Quint, Magnetic Moment of the bound electron, in: *Fundamental Physics in Particle Traps, Springer Tracts in Modern Physics*, vol. 256 (Springer, Heidelberg 2014)

8. R.S. Van Dyck Jr., S.L. Zafonte, S. Van Liew, D.B. Pinegar, P.B. Schwinberg, Ultraprecise atomic mass measurement of the α particle and ^4He . *Phys. Rev. Lett.* **92**, 220802 (2004)
9. C. Roux et al., The trap design of PENTATRAP. *Appl. Phys. B* **107**, 997 (2012)
10. D.G. Boulware, L.S. Brown, T. Lee, Apparatus-dependent contributions to $g - 2$? *Phys. Rev. D* **32**, 729 (1985)
11. M.D. Tinkle, S.E. Barlow, Image charge forces inside conducting boundaries. *J. Appl. Phys.* **90**, 1612 (2001)
12. G.J. Ketter, Theoretical treatment of miscellaneous frequency-shifts in Penning traps with classical perturbation theory, Ph.D. thesis, University of Heidelberg, 2015
13. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
14. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
15. F. Köhler et al., The electron mass from g -factor measurements on hydrogen-like carbon $^{12}\text{C}^{5+}$, *J. Phys. B: At. Mol. Opt. Phys.* **48** (2015)
16. J.K. Thompson, S. Rainville, D.E. Pritchard, Cyclotron frequency shifts arising from polarization forces. *Nature* **430**, 58 (2004)
17. M. Redshaw et al., Precision mass spectrometry and polarizability shifts with one and two ions in a Penning trap, in *TCP 2006*, ed. by J. Dilling, M. Comyn, J. Thompson, G. Gwinner (Springer, Berlin, Heidelberg, 2006)
18. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
19. Z. Andelkovic et al., Status of deceleration and laser spectroscopy of highly charged ions at HITRAP. *Hyp. Int.* **235**, 37 (2015)
20. D.A. Church, H.G. Dehmelt, Radiative cooling of an electrostatically contained proton gas. *J. Appl. Phys.* **40**, 3421 (1969)
21. G. Gabrielse et al., Precise Matter and Antimatter Tests of the Standard Model, in: *Fundamental Physics in Particle Traps, Springer Tracts in Modern Physics*, vol. 256 (Springer, Heidelberg 2014)
22. G. Gabrielse, H. Dehmelt, Observation of inhibited spontaneous emission. *Phys. Rev. Lett.* **55**, 67 (1985)
23. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)
24. L.S. Brown, K. Helmerson, J. Tan, Cyclotron motion in a spherical microwave cavity. *Phys. Rev. A* **34**, 2638 (1986)
25. L.S. Brown, G. Gabrielse, J.N. Tan, K.C.D. Chan, Cyclotron motion in a penning trap microwave cavity. *Phys. Rev. A* **37**, 4163 (1988)
26. L.S. Brown, G. Gabrielse, K. Helmerson, J. Tan, Cyclotron motion in a microwave cavity: lifetime and frequency shifts. *Phys. Rev. A* **32**, 3204 (1985)
27. G. Gabrielse, Relativistic mass increase at slow speeds. *Am. J. Phys.* **63**, 568 (1995)
28. Y. Yaremko, Relativistic shifts of eigenfrequencies in an ideal Penning trap. *Int. J. Mass Spectrom.* **405**, 64 (2016)
29. Y. Yaremko, M. Przybylska, A.J. Maciejewski, Dynamics of a relativistic charge in the Penning trap. *Chaos* **25**, 053102 (2015)
30. H. Mendlowitz, Double scattering of electrons with magnetic interaction. *Phys. Rev.* **97**, 33 (1955)
31. V. Bargmann, L. Michel, V.L. Telegdi, Precession of the polarization of particles moving in a homogeneous electromagnetic field. *Phys. Rev. Lett.* **2**, 435 (1958)
32. F. Combley, F.J.M. Ferley, J.H. Field, E. Picasso, $g - 2$ Experiments as a test of special relativity. *Phys. Rev. Lett.* **42**, 1383 (1979)
33. A.A. Sokolov, I.M. Ternov, *Dokl. Akad. Nauk SSSR* **153**, 1052 (1963)
34. J. Schwinger, On gauge invariance and vacuum polarization. *Phys. Rev.* **82**, 664 (1951)
35. E.T. Akhmedov, D. Singleton, On the relation between Unruh and Sokolov-Ternov effects. *Int. J. Mod. Phys. A* **22**, 4797 (2007)
36. H. Häffner et al., High-accuracy measurement of the magnetic moment anomaly of the electron bound in hydrogen-like carbon. *Phys. Rev. Lett.* **85**, 5308 (2000)

37. J. Verdú et al., Electronic g factor of hydrogen-like oxygen $^{16}\text{O}^{7+}$. Phys. Rev. Lett. **92**, 093002 (2004)
38. S. Sturm et al., g factor of hydrogen-like $^{28}\text{Si}^{13+}$. Phys. Rev. Lett. **107**, 023002 (2011)
39. D. von Lindenfels et al., Experimental access to higher-order Zeeman effects by precision spectroscopy of highly charged ions in a Penning trap. Phys. Rev. A **87**, 023412 (2013)
40. A. Herlert, L. Schweikhard, M. Vogel, Charge-changing reactions and their influence on the ion motion in a Penning trap. AIP Conference Proceedings **606**, 652 (2002)
41. R.C. Thompson, D.C. Wilson, The motion of small numbers of ions in a Penning trap. Z. Phys. D **42**, 271 (1997)
42. R.C. Thompson, Penning traps, in: *Trapped Charged Particles*, M. Knoop, N. Madsen, R.C. Thompson, eds., World Scientific (2016)
43. E.A. Cornell, K.R. Boyce, D.L.K. Fyngenson, D.E. Pritchard, Two ions in a Penning trap: implications for precision mass spectroscopy. Phys. Rev. A **45**, 3049 (1992)
44. M. Redshaw, J. McDaniel, W. Shi, E.G. Myers, Mass ratio of two ions in a Penning trap by alternating between the trap center and a large cyclotron orbit. Int. J. Mass Spectrom. **251**, 125 (2006)
45. M. Redshaw, J. McDaniel, E.G. Myers, Dipole moment of PH^+ and the atomic masses of ^{28}Si , ^{31}P by comparing cyclotron frequencies of two ions simultaneously trapped in a Penning trap. Phys. Rev. Lett. **100**, 093002 (2008)
46. M.J. Höcker, *Precision Mass Measurements at THE-Trap and the FSU trap*, Ph.D. thesis, University of Heidelberg, 2016

Chapter 8

Particle Ensembles in Penning Traps



Abstract In contrast to the situation with a single particle in a trap, the presence of many charged particles gives rise to long-range particle-particle interactions, collective effects, and partial shielding of the trap potential that leads to shifts of the oscillation frequencies. Here, we briefly discuss the most important collective effects and review the treatment of ion ensembles as non-neutral plasmas that show phase transitions between phases including a crystal-like state.

8.1 Space Charge as a Confinement Limitation

When many particles are confined, the corresponding charge distribution locally lifts the trapping potential and changes the oscillation frequencies [1]. This also sets a limit to the amount of charge that can be confined in a given trap. Assuming cold particles in a nearly spherical ensemble, the space-charge potential has the same quadratic shape as the external confining potential.

The presence of space charge results in a shift of the oscillation frequencies given by [2]

$$\omega'_z = \omega_z \sqrt{1 - \frac{\omega_p^2}{3\omega_z^2}} \quad \text{and} \quad \omega'_\pm = \frac{\omega_c}{2} \left(1 \pm \sqrt{1 - \left(1 + \frac{2\omega_p^2}{3\omega_z^2} \right) \frac{2\omega_z^2}{\omega_c^2}} \right), \quad (8.1)$$

where ω_p is the plasma frequency (characteristic frequency of charge oscillation in the plasma) given by

$$\omega_p^2 = \frac{q^2 n}{\epsilon_0 m}, \quad (8.2)$$

in which n is the particle number density and $\epsilon_0 = 8.854 \times 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 \text{ A}^2$ is the permittivity of free space. The square roots in (8.1) create two conditions, which are

$$\frac{\omega_p^2}{\omega_z^2} \leq 3 \quad \text{and} \quad \frac{\omega_p^2}{\omega_z^2} \leq \frac{3}{2} \left(\frac{\omega_c^2}{2\omega_z^2} - 1 \right). \quad (8.3)$$

The first condition contains electrostatic quantities only, but the second one links electrostatic and magnetostatic properties. When these conditions are written in terms of experimental parameters and solved for the particle number density n they yield

$$n \leq \frac{3\varepsilon_0 U_0 C_2}{qd^2} \quad \text{and} \quad n \leq \frac{3}{4} \frac{\varepsilon_0 B_0^2}{m} - \frac{3}{2} \frac{\varepsilon_0 U_0 C_2}{qd^2}. \quad (8.4)$$

For a trap potential of

$$U_0 C_2 = \frac{1}{6} \frac{q B_0^2 d^2}{m} \quad (8.5)$$

the right hand sides of both (8.4) become equal to the Brillouin limit

$$n_{\text{MAX}} = \frac{\varepsilon_0 B_0^2}{2m}, \quad (8.6)$$

which is the maximum achievable particle density, typically of the order of 10^9 charges per cm^3 .

Dropping the condition of a spherical ensemble, the Brillouin limit can be reached for any value of the trapping potential that fulfills the stability criterion $2\omega_z^2 < \omega_c^2$. For higher trapping potentials, the terms on the right hand sides of (8.4) become identical and confinement is lost. The potential well becomes shallower with increasing number of particles N until confinement is lost when N surpasses the critical number of particles N_{MAX} which is proportional to n_{MAX} . The effects of space charge on the confinement and the particle oscillation frequencies have been treated for Penning traps in [1–4], and for cubic Penning traps (ICR cells) in [1, 5]. See also the discussions in [6, 7] and our treatment in Sect. 14.6.2.

8.2 Space Charge as a Source of Intrinsic Frequency Widths

Let us assume a confined particle ensemble with all particles initially oscillating with the same frequency. Then, the spatial distribution of charge will lead to partial shielding of the trap potential for particles closer to the trap centre by particles around them. Looking at the axial degree of freedom, this results in a distribution of axial oscillation frequencies roughly between the value given by (8.1) and the single-particle value ω_z . For simplicity, we assume a roughly spherical ensemble,

as the typical deviations from this are usually small and do not affect the further arguments. Then, the axial oscillation frequency spectrum of a single species has a relative width of about

$$\frac{\delta\omega_z}{\omega_z} \approx 1 - \sqrt{1 - \frac{q^2 n}{3\omega_z^2 \varepsilon_0 m}}, \quad (8.7)$$

which is zero for $n = 0$. For example, assuming C^{5+} ions with a density of $n = 10^4/\text{cm}^3$ at $\omega_z = 2\pi \times 780 \text{ kHz}$, this relative frequency width has a value of 2.5×10^{-4} .

One alternative way to approach this problem is to assume a spherical ensemble with a Boltzmann-type density profile of the kind

$$n(z) = n_0 \exp\left(-\frac{qU_0 C_2}{k_B T}\right), \quad (8.8)$$

as commonly observed in similar situations [8]. Application of Poisson's equation leads to a modified axial potential of the kind [6]

$$V(z) \approx \left(\frac{U_0 C_2}{d^2} - \frac{n_0 q}{\varepsilon_0}\right) \frac{z^2}{2} \left[1 + \frac{1}{3} \left(\frac{z}{\lambda_D}\right)^2 + \dots\right], \quad (8.9)$$

where λ_D is the Debye length [after Peter Debye (1884–1966), Nobel Prize in chemistry 1936] given by

$$\lambda_D^2 = \frac{\varepsilon_0 k_B T}{2n_0 q^2}. \quad (8.10)$$

Solving the axial equation of motion

$$- \frac{q}{m} \frac{dV(z)}{dz} + \omega_z^2 z = 0 \quad (8.11)$$

with this modified potential and saving only terms of first and second order in z^2 leads to a relative frequency width given by

$$\frac{\delta\omega_z}{\omega_z} \approx \frac{n_0 q}{\varepsilon_0} \left(\frac{U_0 C_2}{d^2} - \frac{n_0 q}{\varepsilon_0}\right)^{-1}, \quad (8.12)$$

which for the above conditions is about 2.7×10^{-4} , hence in good agreement with the result from (8.7).

This frequency width is intrinsic to the particle ensemble, such that according to Newton's third law, the common centre-of-charge frequency remains unaffected. This situation is different from frequency widths due to imperfections of the confining fields as described in Sect. 6.1. Hence, the space-charge-broadened frequency spectrum is not necessarily resolved in a resonant electronic pick-up of the axial signal. The observed power density may be different due to the fact that the current

induced by a particle depends on details of its motion such as position. An optical detection, however, would observe the Doppler shift and broadening connected with the particles oscillating at different frequencies and with different amplitudes, see also the discussion in [9].

8.3 Space Charge and Resonant Particle Loss

The confinement in a Penning trap can become resonantly unstable when the potentials are chosen such that the oscillation frequencies have certain relations [10]. This effect has been studied experimentally with confined electrons [11] and ions [2, 12, 13]. For example, instabilities have been found for $2\omega_z + \omega_- = \omega_+$, $2\omega_z = \omega_+$, $\omega_z + 3\omega_- = \omega_+$, $\omega_z + 2\omega_- = \omega_+$, $3\omega_z = 2\omega_+$, $\omega_z + \omega_- = \omega_+$ and so forth [11].

When the oscillation frequencies fulfill the condition

$$N_+\omega_+ + N_-\omega_- + N_z\omega_z = 0, \quad (8.13)$$

with integer coefficients N_+ , N_- and N_z , confinement can become unstable ('non-linear resonances') since energy can be resonantly transferred between the degrees of freedom [10].

The instabilities arising from different combinations of N_+ , N_- and N_z are different in the magnitude of the effect of particle loss, as can be seen from stability diagrams of the trap, see for example [7, 11, 14].

Coming back to the discussion of orbits in Sect. 5.3.2, we can relate integer combinations of the numbers $N_+ \geq 0$, N_- and N_z to the frequency ratios of non-linear resonances with non-periodic orbits [7]. The quantity $|N_+| + |N_z| + |N_-| > 0$ is called the order of the resonance, and Table 8.1 shows the non-linear resonances up to order 4. This relation has been discussed in detail in [7].

In the presence of field imperfections such as higher-order multipoles adding to the quadrupolar electrostatic field, particle confinement can become unstable due to the frequency shifts connected with space charge [2]. When the confined particle number is varied (thus shifting the oscillation frequencies due to the space charge effect), resonant particle loss is observed for the above conditions [2]. Hence, in the presence of field imperfections, a process such as gradual particle loss (or gain) may lead to resonant (abrupt) loss of confinement for specific particle numbers.

The dominant mechanism of such loss is a resonant transfer of energy between the axial motion and the magnetron motion, increasing the magnetron orbit until the spatial confinement limits are reached. Following the discussion in [2], we define an order k of the resonance as

$$k = \frac{\omega'_z}{\omega'_-}, \quad (8.14)$$

Table 8.1 Frequency ratios of non-linear resonances with non-periodic orbits [7]

N_+	N_-	N_z	ω_+/ω_z	ω_+/ω_-	ω_z/ω_-
1	0	-1	1	2	2
2	0	-1	2	8	4
1	1	-1	$(1 + \sqrt{3})/2$	$2 + \sqrt{3}$	$1 + \sqrt{3}$
3	0	-1	3	18	6
2	-1	-1	$(1 + \sqrt{5})/4$	$(3 + \sqrt{5})/4$	$(1 + \sqrt{5})/2$
1	-2	-1	$(1 + \sqrt{5})/2$	$3 + \sqrt{5}$	$1 + \sqrt{5}$
1	1	-2	$(2 + \sqrt{2})/2$	$3 + 2\sqrt{2}$	$2 + \sqrt{2}$
1	-1	-2	$(2 + \sqrt{6})/2$	$5 + 2\sqrt{6}$	$2 + \sqrt{6}$

where ω'_z and ω'_- are the space-charge-shifted frequencies according to (8.1). Note that they depend on the particle number density n . We also come back to the definition of the ‘stability parameter’

$$\gamma = \frac{\omega_c}{\sqrt{2}\omega_z} \quad (8.15)$$

and for convenience define the quantity

$$\alpha = \frac{\omega_p^2}{\omega_z^2} = \frac{q^2 n}{\varepsilon_0 m \omega_z^2}, \quad (8.16)$$

in which ω_p is the plasma frequency corresponding to the particle number density n . Inserting the expressions (8.1) into the definition of the instability resonance order k , we find [2]

$$k = \frac{(2 - 2\alpha/3)^{1/2}}{\gamma - (\gamma^2 - (1 + 2\alpha/3))^{1/2}}. \quad (8.17)$$

For any given order k , this equation connects α , which is proportional to the particle number density n , to a certain stability parameter γ , which is proportional to the magnetic field B_0 . Defining short hands

$$X = \frac{k^2 \gamma^2 + (k^2/2 + 1)(k^2 - 1)}{(k^2 - 1)^2} \quad (8.18)$$

$$Y = \frac{2k^2 \gamma^2 - (k^2/2 + 1)^2}{(k^2 - 1)^2}, \quad (8.19)$$

we can write down the solutions of (8.17) for critical density parameters by

$$\alpha_k(\gamma) = 3(X^2 + Y)^{1/2} - 3X. \quad (8.20)$$

This equation yields a certain critical particle density expressed by the value $\alpha_k(\gamma)$ for each value of γ for which the k th resonance occurs. Instabilities and corresponding resonant particle loss have been observed for orders up to $k = 6$ [2].

Similar observations have been made for confined ions when the trap voltage has been varied. For voltages U_0 above the critical value (see also 5.20)

$$U_{\text{MAX}} = \frac{1}{2} q d^2 \frac{B_0^2}{m} \quad (8.21)$$

confinement was lost, as expected. In addition, an instability with corresponding ion loss was observed below this critical value [15]. This is explained by a resonance of the kind stated in (8.13), that occurs for a trapping parameter $\beta = U_0/U_{\text{MAX}} = 8/9$, which corresponds to the smallest possible integer ratio of the frequencies when $\omega_+ = \omega_z = 2\omega_-$ and $\omega_- = \omega_z/2 = \omega_c/3$, see also our discussion in Sect. 5.2.

Experimental results further show that in real traps, stable confinement conditions are not easily obtained when the trap voltage exceeds about half the value allowed by the stability criterion $2\omega_z^2 < \omega_c^2$ [7]. The confinement properties of a Penning trap close to the stability boundary ('marginally stable Penning trap') have also been studied in [16]. It is important to realise that any additional fields (not only those due to the space charge) alter the stability conditions of the trap, as has for example been investigated for a rotating electric quadrupole field [17] and for magnetic fields in a Ioffe arrangement [18].

8.4 Confined Particle Ensembles as Non-neutral Plasmas

A confined ensemble of charged particles may need to be regarded as a non-neutral plasma under conditions we will discuss below. If residual neutral species need not be considered, it may be called a fully ionised (or fully de-neutralised) plasma, if only one particle species is involved, it is a single-component non-neutral plasma. We will consider this to be the most common case in Penning trap experiments. Such plasmas have been created from electrons [19], from ions [20], from mixtures of ions with positrons [21], from mixtures of electrons with anti-protons [22], and also from anions [23]. The consequences of plasma excitation away from equilibrium ('plasma modes') have been initially discussed by Trivelpiece and Gould [24], and more specifically by Dubin [25, 26]. The complex behaviour of confined plasmas opens numerous experimental possibilities and requires theoretical modelling [25–40]. A general discussion of non-neutral plasmas can be found in [41].

When a particle ensemble confined in a Penning trap is dense enough such that the Debye length

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{2nq^2}} \quad (8.22)$$

is much smaller than any of the ensemble dimensions, the particles are coupled by their mutual Coulomb forces such that their dynamics cannot be described by a single-particle approach (or independent-particle approach [6]), but instead by terms of a single-component plasma. Assuming an approximately spherical ensemble with radius

$$\left(\frac{3N}{4\pi n} \right)^{1/3}, \quad (8.23)$$

this is the case for densities much larger than

$$n = \frac{2}{9} \pi^2 \frac{1}{N^2} \left(\frac{\varepsilon_0 k_B T}{q^2} \right)^3, \quad (8.24)$$

where N is the total particle number in the trap.

A useful quantity for the characterization of ion plasmas is the plasma parameter Γ_p [42] which measures the Coulomb energy between ions relative to their thermal energy. It is defined by

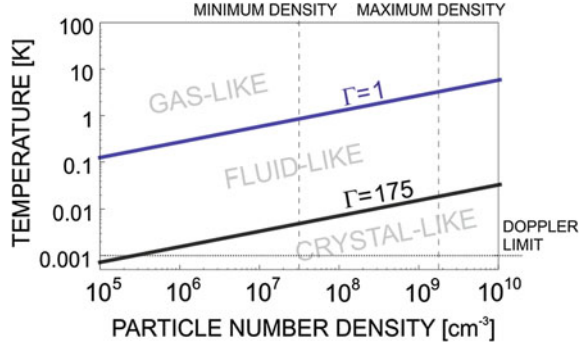
$$\Gamma_p \equiv \frac{q^2}{4\pi \varepsilon_0 a_{\text{WS}} k_B T} \quad (8.25)$$

where q is the ion charge, T is the ensemble temperature and $a_{\text{WS}} = (4/3 \pi n)^{1/3}$ is the Wigner-Seitz radius [43] measuring the effective ion-ion distance at a given ion number density n . Ion cooling increases the value of Γ_p :

Commonly, one speaks of a weakly correlated plasma (a gas-like state) for $\Gamma_p \ll 1$, and of a strongly correlated plasma for $\Gamma_p > 1$. Theoretical studies predict a fluid-like behaviour for $174 > \Gamma_p > 2$ [44] and a crystal-like behaviour for $\Gamma_p > 174$ [30], which has been corroborated experimentally [45].

As an example, for magnesium ions at a density of $5 \times 10^7/\text{cm}^3$, a value of $\Gamma \approx 175$ is reached for $T \approx 5$ mK. In a Penning trap, a strongly correlated plasma in equilibrium is forced to rotate globally at a frequency $\omega_- < \omega_r < \omega_+$, we will discuss this in more detail in Chap. 13. Figure 8.1 shows the phase diagram of a confined ensemble of Mg^+ ions in a Penning trap. The two coloured lines indicate values of the plasma parameter Γ_p of 1 and 175, respectively, thereby separating the three distinguishable phases of a gas-like plasma (weakly coupled), a fluid-like plasma (intermediate coupling) and a crystal-like plasma (strong coupling). The vertical lines indicate the minimum and maximum allowed ion number densities for the working parameters of the trap. The minimum density corresponds to a global plasma rotation at ω_- while the maximum

Fig. 8.1 Phase diagram of a confined ensemble of Mg^+ ions in a Penning trap. The solid lines at $\Gamma = 1$ and $\Gamma = 175$ separate three regimes in which the confined ensemble may be described as gas-like, liquid-like and crystal-like, respectively



density is the Brillouin density for a global rotation at $\omega_c/2$. Also indicated is the Doppler cooling limit of about 1 mK for these ions.

From the standpoint of particle dynamics, at zero internal energy, the system is frozen in a crystalline state. Apart from the global rotation, at nearly zero energy, the particles have a quasi-harmonic motion around their respective equilibrium positions. This is the regime of normal modes of vibration. At slightly higher energies, the system may change from the regime of quasi-harmonic normal modes to the regime of local modes. At energies high enough for exchanges of ions between the equilibrium positions, their motion becomes erratic and the crystal melts, leaving a fluid-like state. At still higher energies, the ions form a thermal ensemble in which the mean Coulomb potential energy becomes negligible with respect to the mean kinetic energy and the mean harmonic potential energy. This is the thermal regime which can be described by statistical mechanics, with a well-defined ensemble temperature and entropy which is given by [46]

$$S(T) \approx N k_B \ln \frac{\exp^4(1) (k_B T)^3}{N \hbar^3 \omega_- \omega_+ \omega_z}. \quad (8.26)$$

The different dynamical regimes and the order-chaos transition can be characterised by their Lyapunov exponents in a model of the ensemble as a confined non-neutral microplasma [46].

8.5 Ion Crystals

At sufficiently low kinetic energies, i.e. for sufficiently large plasma parameters $\Gamma_p \geq 175$, see (8.25), confined particles ‘crystallise’ into well-defined structures given by their mutual Coulomb repulsion in the presence of the confining trap potential. In this case, the Debye length becomes smaller than the mean inter-particle separation, and the plasma oscillations (Langmuir oscillations [47], after Irving Langmuir (1881–1957), Nobel Prize for chemistry in 1932) become modes of vibration of the crystal

lattice. This notion of a ‘crystal’ is of course different from the common one, since here we do not have attractive forces between particles, but the repulsive Coulomb force that is balanced with the confining forces of the trap’s potential well and the centripetal forces of the plasma rotation.

The observation of ion crystals in a Penning trap is complicated by the presence of the magnetic field which forces the crystal into that global rotation. In order to resolve individual particles in imaging of the crystal, it is either necessary to have extremely short exposure times, which is limited by the amount of scattered light from the particles, or to use a stroboscopic (phase-locked) technique in which several exposures paused by one global rotation period are added.

Typical ion-ion distances in such crystals are of the order of $10\text{ }\mu\text{m}$. Note, that the treatment of such states in one-component plasmas goes back to well before ion crystals in Penning traps. The value of the plasma parameter Γ_p at the transition to a crystal-like state had previously been predicted to be $\Gamma_p \approx 125$ [48], $\Gamma_p \approx 155$ [49], $\Gamma_p \approx 168$ [50] and then $\Gamma_p \approx 172 - 174$ [51].

The geometric properties of ion plasmas and the formation of ion Coulomb crystals have been described in detail for example in [52–60]. An overview of ion Coulomb crystals in Penning traps is given in [61]. Commonly, three regimes are distinguished.

8.5.1 Small Ion Crystals

We may speak about an ion crystal to be small when ‘each ion counts’ in the sense that the overall appearance is distinct from ion number to ion number, here we speak about, say, up to 100 ions or so. The geometry assumed by ion crystals depends on the confinement parameters, specifically on the depth of the electrostatic potential well. We will see this in detail in the discussion of the aspect ratio of a confined plasma. For small ion crystals, this dependence manifests itself in a visible discrete change between different orders since the total number of particles is so small. This is shown in Fig. 8.2 where a crystal of 15 ions is shown for six different values of the trap depth, which increases from left to right. Obviously, in a shallow

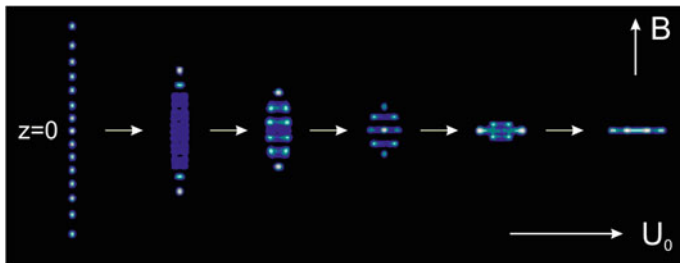


Fig. 8.2 Ion Coulomb crystal consisting of 15 ions as a function of the trap voltage U_0 which increases from left to right. Data from [57]

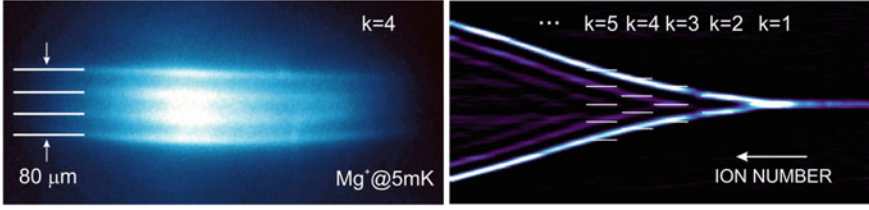


Fig. 8.3 Left: planar shell ion crystal with $k = 4$ shells. Right: number of shells as a function of ion number in a comparison of data with theory. Data from [62]

trap, the ions form a linear and nearly equidistant chain along the trap axis, while with increasing trap depth more ions are forced to assume positions with larger radii. For a more detailed discussion and some nice examples see [57]. The observed structures can be very well explained by a simulation of the total potential including ion-ion interaction, as has been detailed in [57].

8.5.2 Mesoscopic Ion Crystals

For ion numbers roughly in the range of $10^3 - 10^5$, and aspect ratios (axial extension to radial extension) of typically $\alpha \ll 1$, the so-called ‘planar-shell model’ is a good approximation of the geometry assumed by a confined plasma (Fig. 8.3). It applies to the case of a spheroidal plasma with a radius sufficiently large such that the curvature of the shells can be neglected close to the trap axis. While in a real plasma, the number of shells depends on the radial position in the crystal and decreases towards the edges, this model describes the plasma throughout as a series of parallel shells (‘stack of pancakes’).

For the sake of simplicity, it does not explicitly account for correlations between shells, in which case one would also expect lateral offsets in the ion positions between neighbouring shells [56]. We assume the area charge density $q\sigma_i$ of each shell is identical and that the shells are spaced by a uniform distance D [30]. The total area number density σ is the sum of all σ_i , and the spacing D is linked to σ and the number of shells k via the relation

$$D = \frac{\sigma}{nk}. \quad (8.27)$$

This situation is schematically depicted in Fig. 8.4. The contributions to the energy per particle are the self-energy of the set of k shells, the energy due to the external potential, and the negative correlation energy associated with each shell. The total energy per particle is given by [30]

$$\frac{E}{N} = \pi e^2 \left[L\sigma - \frac{1}{6} \frac{\sigma^2}{n} \right] + \frac{E_c}{N}, \quad (8.28)$$

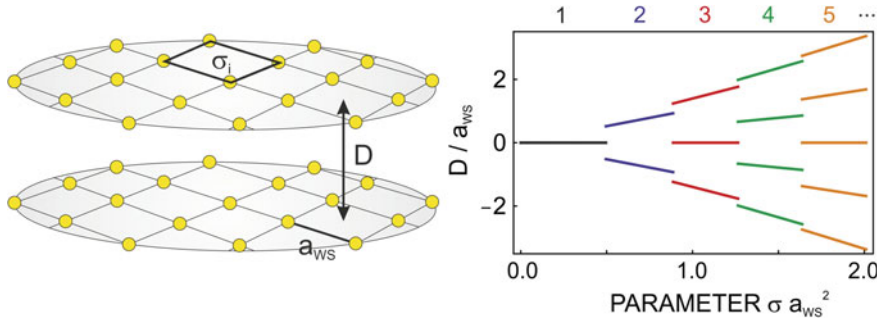


Fig. 8.4 Planar shell model geometry (left), and planar-shell model prediction of the crystal shell structure (number of shells k) as a function of $\bar{\sigma} = \sigma a_{ws}^2$ (right)

where $2L$ is the axial extension of the crystal and E_c is the ion-ion correlation energy given by [30]

$$\frac{E_c}{N} = \frac{e^2}{a_{ws}} \left[\frac{2\pi^2}{9} \left(\frac{\bar{\sigma}^2}{k} \right) - \frac{\eta}{2} \left(\frac{\bar{\sigma}}{k} \right)^{1/2} \right], \quad (8.29)$$

where $\bar{\sigma} = \sigma a_{ws}^2$. In this equation, η accounts for the Madelung energy [43] of the two-dimensional lattice. We use the value $\eta = 3.921$ of the hexagonal lattice, which has the lowest Madelung energy in two dimensions [43].

A structure of k parallel shells has a higher energy than a uniformly spread charge, which is reflected in the fact that the first term of (8.29) is positive. The second term accounts for the ion-ion correlations within each plane. It is negative, which facilitates the formation of a finite set of ordered planes. Hence, the number of shells that form results from the competition between these two terms.

The correlation energy per particle E_c/N takes a minimum value with respect to the shell number k given by

$$k_{\min} = \left(\frac{16\pi^2}{9\eta} \right)^{2/3} \bar{\sigma}, \quad (8.30)$$

in which case the distance D between two shells is given by

$$\frac{D}{a_{ws}} = \left(\frac{3\eta^2}{4\pi} \right)^{1/3} = 1.54. \quad (8.31)$$

Since the number k must be an integer, $k = k_{\min}$ can only be fulfilled for certain values of $\bar{\sigma}$, and the actual value of k will be an integer close to $k_{\min}$. Figure 8.5 shows the correlation energy per particle E_c/N according to (8.29) for shell numbers $k = 1$ to $k = 8$. For the given range of $\bar{\sigma}$, the shell number k that gives the minimum correlation energy was chosen to calculate the inter-shell distance D with (8.27).

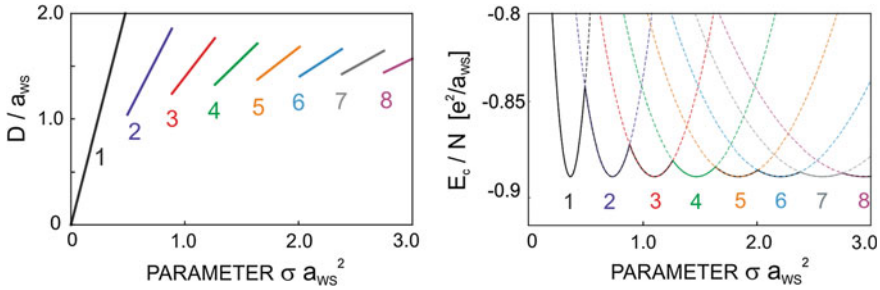


Fig. 8.5 Distance D for which the energy per particle has a minimum as a function of the normalized area (left), and correlation energy per particle as a function of $\bar{\sigma} = \sigma a_{ws}^2$ given for various numbers of shells k (right)

One can observe that for larger $\bar{\sigma}$, the variation of the minimum correlation energy gets smaller, hence the actual correlation energy is for large shell numbers k close to the minimum value. Likewise, the variation of the shell distance D decreases for large shell numbers and approaches $D \approx 1.54 a_{ws}$. The corresponding axial plane positions as a function of the normalized area charge density are shown in Fig. 8.4. It illustrates the stepwise increase of the number of shells for increasing charge density, i.e. for increasing ion number. For a closer discussion and some examples see [30, 62].

8.5.3 Large Ion Crystals

Larger crystals approach a bulk-like lattice with long-range ordering [56], as has been shown in crystallographic Bragg scattering experiments, see for example [63]. Typical sizes in experiments are in the range between several 10^4 and 10^6 ions. An infinitely extended plasma at zero energy would assume a bcc (body-centred cubic) lattice, as this has the lowest energy [64], although the relative energy differences to fcc (face-centred cubic) and hcp (hexagonal close-packed) structures are small [55] (Fig. 8.6).

The question as to the minimum size of such a crystal to exhibit a bcc lattice structure has been answered in studies which found that a plasma size of more than 60 inter-particle spacings is required for bulk behaviour to be exhibited, and that more than about 10^5 particles are needed in a spherical plasma to form a bcc lattice [55]. A somewhat lower bound was found in simulations which require about 1.5×10^4 particles for such a lattice in a spherical plasma [64]. Experimentally, bcc structures have been observed for spherical plasmas above 2×10^5 ions. Other crystal types such as fcc and hcp have been observed for spherical plasmas in the range between 2×10^4 and 2×10^5 ions [56]. It should be noted, that also for large ion crystals there is a non-negligible influence of the boundary geometry in the given confinement

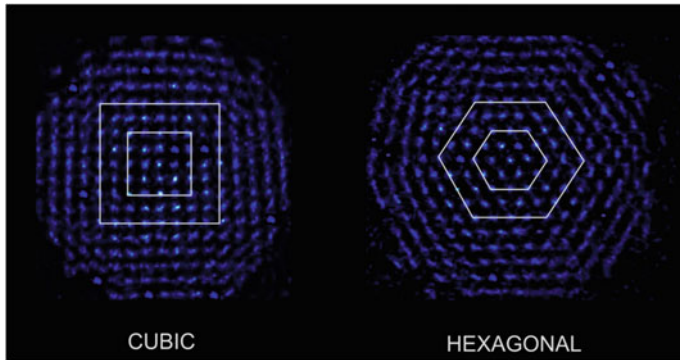


Fig. 8.6 Visualisation of large ion Coulomb crystals with cubic structure (left) and hexagonal structure (right)

situation that leads to deviations from the infinitely extended crystal case or from naive theory. Simulations suggest that in large confined plasmas, concentric shells with nearly equal spacing are formed [65, 66], that have a surface structure of plane hexagonal symmetry. For sufficiently large plasmas, the interior of these crystals evolves to exhibit bcc-like structure.

8.6 Multi-species Ion Crystals

In similarity to single-component ion plasmas forming single-species ion Coulomb crystals at low temperatures, multi-species ion Coulomb crystals can be formed from multi-component plasmas, or by injection of admixture ions into existing crystals. In general, one ion species is laser-cooled while the admixture ions are not. Depending on the details of the formation process, admixture ions can be embedded into forming or existing crystals, which may be seen as a special form of cryo-trapping (Fig. 8.7). Sometimes, this is referred to as a form of ‘sympathetic crystallisation’ [7]. The ions that form the crystal are subjected to laser cooling, and to heating by interaction with the admixture ions. At particle energies of few eV and below, laser cooling is an efficient cooling mechanism, hence the rapid crystallisation previously observed for pure Mg^+ crystals [62]. The admixture ions are subjected to sympathetic cooling via the laser-cooled ions. As sympathetic cooling depends on more parameters than laser cooling which vary during the course of observation, one can expect the competition between these two processes to display more features.

Injection of a number of different ion species into existing crystals has been discussed in [67]. In this case, ions both with smaller and larger charge-to-mass ratios than the species forming the crystal have been injected from an external source. The injection of highly charged ions into existing crystals has been demonstrated in [68].

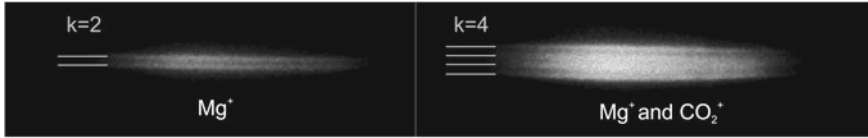


Fig. 8.7 Mg^+ ion crystal without (left) and with admixture of CO_2^+ ions (right). After admixture, the geometric structure is preserved and more layers are formed. The heavier admixture ions concentrate more Mg^+ ions close to the centre, hence increasing the observed fluorescence

Ion crystals have become a powerful tool for sympathetic cooling of admixture ions, and the terms are sometimes even used in situations with single ions of different species, where one ion of interest is sympathetically cooled by one laser-cooled ion [69]. This finds application to precision spectroscopy and quantum information processing [70]. One important mechanism in strongly coupled multi-component plasmas and hence during the formation of multi-species ion crystals is centrifugal separation.

8.7 Centrifugal Separation

When different particle species are confined within the same trap volume, one observable process is centrifugal separation, in which species with different m/q assume different radial equilibrium distributions about the trap centre. At very low energies, this may lead to complete separation of species into concentric rings. Generally, it leads to separation with the ions of smaller m/q on average assuming smaller radii. This separation is not driven by cooling and leads to different ion species probing different values of the trapping field imperfections which leads to different shifts of their axial frequencies independent of the cooling.

Generally, for a multi-species plasma at low temperature, its rotation tends to produce centrifugal separation of its components. As long as the plasma is not yet in thermal equilibrium, the different species rotate with different frequencies, and collisions between the two species will cause a net momentum transfer. This creates an inward radial flux of lighter species and an outward radial flux of heavier species [71]. The plasma evolves until it reaches thermal equilibrium, which is characterized by a global rigid-body rotation. In this case, both species rotate with the same frequency ω_r , but experience different centrifugal potentials due to their different ratios of mass to charge m/q . The degree of separation depends on the plasma temperature and on the difference of the involved mass-to-charge ratios. If the separation is complete, the species are arranged in separate concentric rings with the ion with smaller mass-to-charge ratio in the centre of the trap. This has been observed for example for two-species ensembles consisting of cold beryllium and mercury ions [72] as well as for combinations of singly charged and highly charged ions [29]. Similar observations have been made also for ensembles of beryllium and cadmium ions

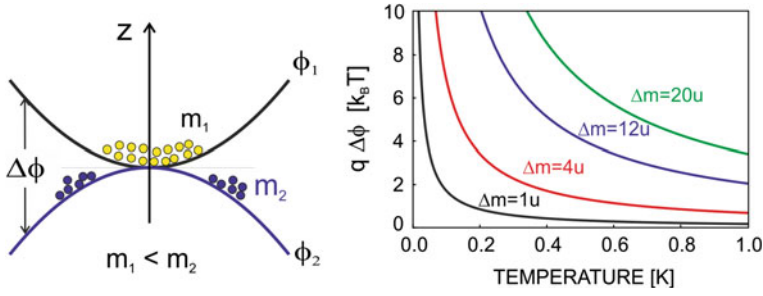


Fig. 8.8 Schematic of centrifugal separation of two particle species with different masses (left), and the expected potential difference experienced by the different species as a function of temperature for four different values of the mass difference (right)

[73], mixed-isotope ensembles containing ^{24}Mg and ^{26}Mg [74], and for ensembles containing anti-particles, namely beryllium ions and positrons, and electrons and anti-protons [22, 75, 76]. Partial separation has been observed in two-species ion crystals of magnesium ions and several lighter and heavier species [67].

The criterion for centrifugal separation between particle species ‘1’ and ‘2’ is

$$e \left| \frac{m_1}{q_1} - \frac{m_2}{q_2} \right| \omega_r^2 R_p^2 > k_B T, \quad (8.32)$$

where ω_r is the global rotation frequency of the ensemble and R_p is the radial extension of the plasma.

Figure 8.8 (left) shows a schematic of centrifugal separation of two particle species with different masses on account of the difference $\Delta\phi$ of the effective potentials experienced by the different species. Figure 8.8 (right) shows the expected effective potential difference experienced by the different species as a function of the ensemble temperature for four different values of the mass difference. The potential difference and hence the separation is most pronounced for particles with a large difference of the mass-to-charge ratios and for low values of the ensemble temperature.

Centrifugal separation of ions is usually expected to occur at the μeV scale. A useful quantity is the so-called ‘separation length’ given by

$$l = \frac{k_B T}{|m_1 - m_2| \omega_r^2 R_p}, \quad (8.33)$$

in which R_p again is the radial extension of the plasma. When the separation length is smaller than R_p , separation occurs. If the separation length is smaller than even the Debye length

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{2nq^2}} \quad (8.34)$$

in the plasma ($l < \lambda_D$), then the separation is complete and distinct concentric rings are observed. In general, the relative radial density profiles of two equally charged species takes the form

$$\frac{n_1}{n_2} = K_{12} \exp\left(\frac{1}{2k_B T}(m_1 - m_2)\omega_r^2 r^2\right), \quad (8.35)$$

in which K_{12} depends on the relative numbers of the species involved. For a more detailed discussion, see for example [77].

8.8 Plasma Modes and Resonant Particle Loss

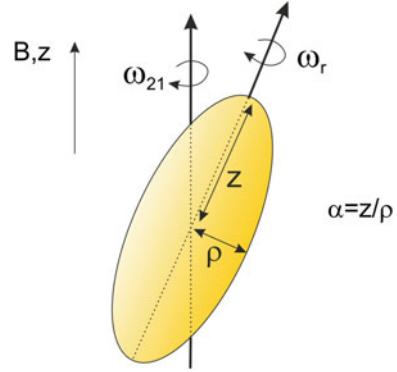
A particle ensemble that may be regarded as a plasma possesses many possible modes of internal oscillation, the so-called ‘plasma modes’. These can be resonantly excited by irradiation of electromagnetic radiation of the proper geometry.

The geometry of plasma modes is described in terms of Legendre polynomials of the first and second kind P_l^m and Q_l^m [78] with positive integer numbers l and m with $l \geq 1$ and $m \geq 0$ where $m \leq l$ [20]. The number of possible modes depends on l and m : there are $2l$ modes for $m = 0$, for non-zero m and even $l - m$ there are $2(l - m) + 2$ modes, while for non-zero m and odd $l - m$ there are $2(l - m) + 1$ modes. Modes with $l = 1$ maintain the plasma shape and have the plasma’s centre of mass perform one of the three single-particle motions in a Penning trap, ($l = 1, m = 0$) representing the axial motion and the two ($l = 1, m = 1$)-modes representing the perturbed cyclotron and magnetron motions. Modes with $l = 2$ are quadrupolar deformations of the plasma shape. The value of m describes the azimuthal dependence of the potential of the plasma mode by a factor $\exp(im\phi)$.

A more detailed discussion can be found in [20]. Note that in the frame of reference rotating with the plasma, static field imperfections of the trap appear with the periodicity of the global plasma rotation frequency. Hence, when the plasma rotation frequency equals a plasma mode frequency, the static field imperfections may resonantly drive a plasma mode. In this case, energy from the field is dissipated via the plasma mode into internal degrees of freedom of the particle ensemble and thus heats it up [79]. This leads to a sudden decrease of the particle number density at certain rotation frequencies that equal a plasma mode frequency [80].

At least one of the three possible ($l = 2, m = 1$) plasma modes is a prominent source of heating and particle loss in [20, 80]. This mode is a precession motion of

Fig. 8.9 Depiction of a confined plasma in a Penning trap performing a global rotation at a frequency ω_r with the $(l = 2, m = 1)$ -plasma mode at frequency ω_{21} indicated. This mode is equivalent to a precession of the plasma's main axis about the magnetic field axis z



the plasma's main axis about the magnetic field lines, as indicated in Fig. 8.9. It is hence sensitive to misalignments of the magnetic field axis with respect to the trap axis, in particular for strong magnetic fields.

The mode spectrum depends on the applied confining potential and by appropriate sudden switching, a plasma mode may jump over a given rotation frequency [20]. To be able to make use of this, the plasma-mode frequencies need to be known to sufficient accuracy. For the $(l = 2, m = 1)$ plasma mode the three mode frequencies ω_1 , ω_2 and ω_3 result from the three solutions η_1 , η_2 and η_3 of the cubic equation [20]

$$a_3\eta^3 + a_2\eta^2 + a_1\eta + a_0 = 0 \quad (8.36)$$

and are given by

$$\omega_{21}(k) = (-1)^{k+1} \sqrt{-\omega_p^2 f_\alpha \frac{\alpha^2/\eta_k + 1}{3}}, \quad (8.37)$$

for $k = 1, 2, 3$. Here, ω_p and f_α are defined as

$$\omega_p = \sqrt{\frac{q^2 n}{\varepsilon_0 m}} \quad \text{and} \quad f_\alpha = \frac{3}{2} \left(\frac{1 - 3\omega_z^2/\omega_p^2}{1 - \alpha^2} \right), \quad (8.38)$$

where ω_p is the plasma frequency. The coefficients a_0 , a_1 , a_2 and a_3 are given by

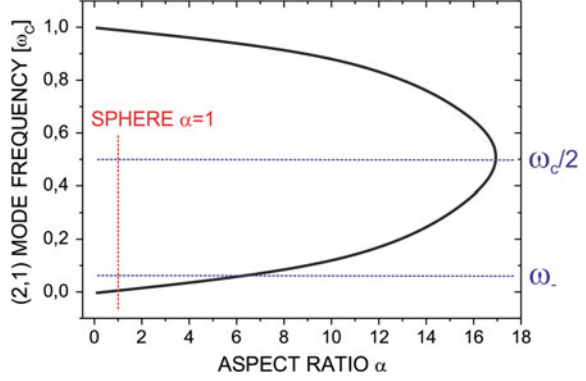
$$a_0 = a_3 \alpha^2 \quad (8.39)$$

$$a_1 = a_3(1 - 2\alpha^2) + \omega_v^2 \quad (8.40)$$

$$a_2 = a_3(\alpha^2 - 2) \quad (8.41)$$

$$a_3 = \frac{1}{3} \omega_p^2 f_\alpha, \quad (8.42)$$

Fig. 8.10 (2,1)-plasma mode frequency ($k = 1$) as a function of the aspect ratio α (for $q/m = 3e/u$ at $T=10\text{ K}$ and $B_0 = 1\text{ T}$). For $\alpha \approx 1$ there is no (2,1) mode between ω_- and $\omega_c/2$



where ω_v is the plasma vortex frequency given by $\omega_v = \omega_c - 2\omega_r$. The vortex frequency is the cyclotron frequency as seen from a co-rotating frame of reference. It vanishes at the Brillouin limit $\omega_c/2$, see the discussion in [28].

In Fig. 8.10 the ($l = 2, m = 1$) plasma mode frequency ($k = 1$) is shown as a function of the aspect ratio α for ions with $q/m = 3e/u$ at $T = 10\text{ K}$ and $B_0 = 1\text{ T}$. The plasma mode frequency for $k = 2$ has opposite sign and the frequency for $k = 3$ is smaller than ω_- , hence they play no role in our discussion. Under these conditions, for an aspect ratio of $\alpha \approx 1$ (i.e. a nearly spherical plasma) there is no ($l = 2, m = 1$) plasma mode between the minimum (ω_m) and maximum ($\omega_c/2$) possible global rotation frequencies. This means we are not in danger of exciting a mode when the rotating wall is applied. However, for more prolate plasma shapes, this is a potential issue. One example is shown in Fig. 8.11. It shows the calculated ($l = 2, m = 1$) plasma mode frequencies with $k = 1 \dots 3$ for the conditions detailed in [80]. The crossing of the dotted line with the ω_{21} -line for $k = 1$ indicates that the ensemble rotation frequency equals that mode frequency and heating of the ensemble

Fig. 8.11 Calculated frequencies of the ($l = 2, m = 1$)-plasma modes for $k = 1 \dots 3$. The crossing of the dotted line with the ω_{21} -line for $k = 1$ indicates that the ensemble rotation frequency equals that mode frequency and heating of the ensemble is expected

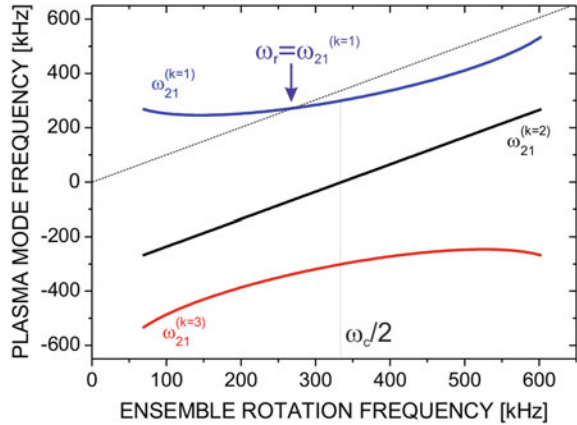
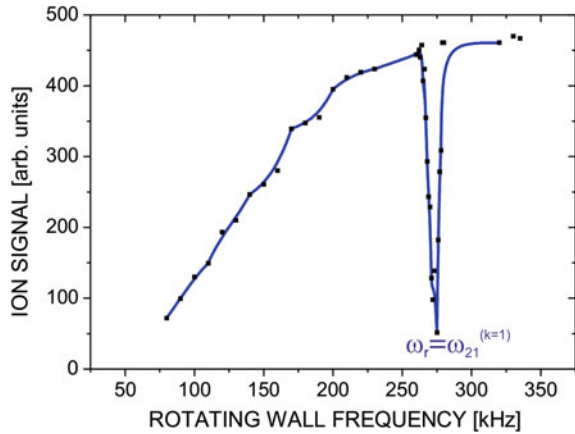


Fig. 8.12 Ion signal as a function of rotating wall frequency ω_r , indicating a resonant heating by a loss of ion signal for $\omega_r = \omega_{21}(k = 1)$. Data from [80]



is expected. In Fig. 8.12, the observed ion fluorescence signal is plotted as a function of rotating-wall frequency ω_r . It increases with increasing rotating-wall frequency, as expected from the corresponding increase of the particle density. As Fig. 8.11 predicts, for $\omega_r = \omega_{21}(k = 1)$, there is a significant loss of signal due to resonant heating which dilutes the ensemble and reduces the observed fluorescence.

8.9 Calculation of Particle Trajectories and Simulations

Let us briefly discuss the issue of calculating or simulating the trajectories of particles in confined ensembles as compared to the single-particle case.

As we have seen in (5.2)–(5.4), the trajectory of a single particle can be calculated analytically from three equations of motion. When we regard the centre of mass of an ensemble of identical particles, the same is true as long as external forces such as friction are absent. The centre of mass in that case is identical to the centre of charge and the centre of momentum, and the situation is the same whether we take particle-particle interaction into account or not, on account of Newton's third law.

In general, the trajectories of N particles follow from the solution of $3N$ equations of motion, for which case there is no analytical solution. Particle-particle interaction adds a computational complication which scales roughly like N^2 since in general the effect of each particle on each other particle needs to be accounted for. For special cases such as highly correlated motions of confined particles, molecular-dynamics methods may be employed which reduce the effective number of parameters [65, 81]. It is possible to solve the equations of motion numerically or to perform simulations based on the method of finite elements. Some particle-in-cell (PIC) codes exist that treat confined plasmas in similarity to a fluid with charge and volume [82, 83]. Commercial software solutions such as SimIon or ComSol can produce meaningful

results in nearly arbitrary geometry, both with and without particle-particle interaction, so long as the particle number is small enough for the given computational power.

In cases that allow one to give the confining potentials in an analytical form, like for the traps we have discussed, it is not required to rely on the method of finite elements, and the equations of motion can be solved by comparatively simple commercial or non-commercial software, both with and without the consideration of field imperfections like we will discuss below.

The situation is significantly complicated when external forces such as a frictional force due to resistive cooling is to be considered. In principle, the additional friction term only adds to the required computational power, however, to obtain realistic results for large numbers of particles also requires to account for the mutual interaction of the particle ensemble with the cooling circuit, which is not readily obtained [84–87].

When we look at an axially symmetric situation and the voltage difference u between the electrodes used for cooling following from the induced current i through the circuit, we find the differential equation [85]

$$\frac{d^2 u}{dt^2} + \frac{R}{L} \frac{du}{dt} + \omega_0^2 u = R\omega_0 \left(i + \frac{L}{R} \frac{di}{dt} \right), \quad (8.43)$$

where $R = \text{Re}(Z)$ is the real part of the impedance of the circuit, L is its inductance, and the current i is the total induced current through the impedance and depends on all phase space coordinates of the particles, that is on all positions and velocities. For small amplitudes, we have [85]

$$\frac{di}{dt} \approx 2qC_1 \ddot{z}, \quad (8.44)$$

where C_1 is the geometry parameter for the given electrode arrangement. The induced current thus leads to a voltage drop across the impedance that in turn reduces the effective trap voltage and hence couples back to the axial oscillation. The radial motions to first order remain unaffected by this, however the axial equation of motion $\ddot{z} = -\omega_z^2 z$ becomes [85]

$$\ddot{z} = -\omega_z^2 z + \frac{q}{m} \frac{C_1}{2} u, \quad (8.45)$$

in which the voltage u obeys the equation [85]

$$\ddot{u} + \frac{1}{RC} \dot{u} + \frac{1}{LC} u = \frac{2qC_1}{C} \ddot{z}, \quad (8.46)$$

such that we have a non-trivial set of coupled equations for the axial motion.

Let us now look at the modification of all equations of motion when we additionally take into account particle-particle interaction. We obtain the $3N$ coupled

equations in absence of friction plus N expressions that describe the frictional coupling to the axial motions [85]

$$\ddot{x}_j = \omega_c \dot{x}_j + \frac{\omega_z^2}{2} x_j + \frac{q^2}{4\pi\epsilon_0} \sum_{j \neq k}^N \frac{x_j - x_k}{|\mathbf{r}_j - \mathbf{r}_k|^3} \quad (8.47)$$

$$\ddot{y}_j = -\omega_c \dot{y}_j + \frac{\omega_z^2}{2} y_j + \frac{q^2}{4\pi\epsilon_0} \sum_{j \neq k}^N \frac{y_j - y_k}{|\mathbf{r}_j - \mathbf{r}_k|^3} \quad (8.48)$$

$$\ddot{z}_j = -\omega_z^2 z_j + \frac{q}{m} \frac{C_1}{2} u_j + \frac{q^2}{4\pi\epsilon_0} \sum_{j \neq k}^N \frac{z_j - z_k}{|\mathbf{r}_j - \mathbf{r}_k|^3} \quad (8.49)$$

$$\ddot{u}_j = \frac{2qC_1}{C} \ddot{z} - \frac{1}{RC} \dot{u}_j - \frac{1}{LC} u_j. \quad (8.50)$$

For radial cooling, the set of equations would be similar, just with the voltage term in the axial equations of motion then appearing in the radial ones, and the geometry coefficient then referring to the electrode geometry for the radial cooling.

In a perfect trap, as described by these equations of motion, axial cooling will not cool the radial motions. This only changes when the motions are coupled either by field imperfections or active coupling. Field imperfections complicate the situation significantly, since their presence makes all particle frequencies dependent on the particle positions. Looking for example only at the leading electrostatic anharmonicity and its influence on the axial motion, we would have to replace the first term in (8.49) in the way

$$-\omega_z^2 z_j \rightarrow -\omega_z^2 \left(1 + \frac{2C_4}{C_2} z_j^2 \right) z_j, \quad (8.51)$$

such that the axial frequency becomes dependent on the axial position z_j , and looking at the expansion of the electrostatic potential in for example the form (6.24) it becomes obvious that also higher-order terms and particularly mixed terms will appear and couple the equations of motion in a non-trivial way.

An additional complication comes from the fact that for the simulation to be a realistic representation of all interactions, the time steps in which the temporal coordinate is subdivided usually need to be much smaller than any of the involved time scales of oscillations or interactions [88], so to gain insight into the behavioural trends over many oscillation cycles, a huge number of time steps is required, which affords correspondingly huge computational powers. Similar statements are true about the spatial coordinates.

This kind of multi-particle simulation is in any case facilitated by the use of graphics processing units (GPUs), which by design allow for a highly parallel computation which is suitable for a case where the same kind of calculation needs to be performed for a large number of particles [84, 85]. Computational requirements of numerical simulations of particle ensemble dynamics have also been discussed in [88].

In all cases, however, the comparison of experimental results with the outcome of calculations or simulations generally suffers from a lack of knowledge with regard to the initial conditions of the experiment, such as the exact initial velocities and phases of all particles involved. Also, in general, the external forces in the experiment may not be well-known, like for example the full information about the cooling impedance $Z(\omega)$ and its interaction with the confined particles.

References

1. J.B. Jeffries, S.E. Barlow, G.H. Dunn, Theory of space charge shift of ion cyclotron resonances. *Int. J. Mass Spectrom. Ion Process.* **54**, 169 (1983)
2. J. Yu, M. Desaintfuscien, F. Plumelle, Ion density limitation in a Penning trap due to the combined effect of asymmetry and space charge. *Appl. Phys. B* **48**, 51 (1989)
3. D.F.A. Winters, M. Vogel, D.M. Segal, R.C. Thompson, Electronic detection of charged particle effects in a Penning trap. *J. Phys. B* **39**, 3131 (2006)
4. R.S. Van Dyck Jr., F.L. Moore, D.L. Farnham, P.B. Schwinberg, Number dependency in the compensated Penning trap. *Phys. Rev. A* **40**, 6308 (1989)
5. T.J. Franci et al., Experimental determination of the effects of space charge on ion cyclotron resonance frequencies. *Int. J. Mass Spectrom. Ion Proc.* **54**, 189 (1983)
6. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)
7. G. Werth, V.N. Gheorghe, F.G. Major, *Charged particle traps* (Springer, Heidelberg, 2005)
8. G.-Z. Li, S. Guan, A.G. Marshall, Comparison of equilibrium ion density distribution and trapping force in penning, paul, and combined ion traps. *J. Am. Soc. Mass. Spectrom.* **9**, 473 (1998)
9. K. Dholakia, G.Z.K. Horvath, D.M. Segal, R.C. Thompson, D.M. Warrington, D.C. Wilson, Photon-correlation detection of ion-oscillation frequencies in quadrupole ion traps. *Phys. Rev. A* **47**, 441 (1993)
10. M. Kretschmar, Ideal gas approximation for an ion cloud in a penning trap. *Z. Naturf.* **45a**, 965 (1990)
11. P. Paasche et al., Instabilities of an electron cloud in a Penning trap. *Eur. Phys. J. D* **22**, 183 (2003)
12. H. Hübner et al., Instabilities of ion confinement in a Penning trap. *Europhys. Lett.* **37**, 459 (1997)
13. G. Tommaseo, P. Paasche, C. Angelescu, G. Werth, Subharmonic excitation of the eigenmodes of charged particles in a Penning trap. *Eur. Phys. J. D* **28**, 39 (2004)
14. G. Werth, V.N. Gheorghe, F.G. Major, *Charged particle traps II* (Springer, Heidelberg, 2009)
15. L. Schweikhard, J. Ziegler, H. Bopp, K. Lützenkirchen, The trapping condition and a new instability of the ion motion in the ion cyclotron resonance trap. *Int. J. Mass Spectrom. Ion Proc.* **141**, 77 (1995)
16. H.M. Holzschneider, Ion confinement in a marginally stable Penning trap. *Nucl. Inst. Meth. A* **240**, 457 (1985)
17. T. Hasegawa, M.J. Jensen, J.J. Bollinger, Stability of a Penning trap with a quadrupole rotating electric field. *Phys. Rev. A* **71**, 023406 (2005)
18. T.M. Squires, P. Yesley, G. Gabrielse, Stability of a combined Penning-Ioffe Trap. *Phys. Rev. Lett.* **86**, 5266 (2001)
19. J.H. Malmberg, J.S. deGrassie, Properties of nonneutral plasma. *Phys. Rev. Lett.* **35**, 577 (1975)
20. J.J. Bollinger et al., Electrostatic modes of ion-trap plasmas. *Phys. Rev. A* **48**, 525 (1993)
21. B.M. Jelenkovic et al., Sympathetically laser-cooled positrons. *Nucl. Inst. Meth. B* **192**, 117127 (2002)

22. G.B. Andresen et al., Centrifugal separation and equilibration dynamics in an electron-antiproton plasma. *Phys. Rev. Lett.* **106**, 145001 (2011)
23. D.H.E. Dubin, Plasmas in Penning traps, in *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson (World Scientific, 2016)
24. A.W. Trivelpiece, R.W. Gould, Space charge waves in cylindrical plasma columns. *J. Appl. Phys.* **30**, 1784 (1959)
25. D.H.E. Dubin, Theory of electrostatic fluid modes in a cold spheroidal non-neutral plasma. *Phys. Rev. Lett.* **66**, 2076 (1991)
26. D.H.E. Dubin, Plasmas Modes, in *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson (World Scientific, 2016)
27. R.J. Hendricks, E.S. Phillips, D.M. Segal, R.C. Thompson, Laser cooling in the Penning trap: an analytical model for cooling rates in the presence of an axializing field. *J. Phys. B* **41**, 035301 (2008)
28. J.J. Bollinger et al., Nonneutral ion plasmas and crystals in Penning traps. *Phys. Scr.* **T59**, 352 (1995)
29. L. Gruber et al., Evidence for highly charged ion coulomb crystallization in multicomponent strongly coupled Plasmas. *Phys. Rev. Lett.* **86**, 636 (2001)
30. D.H.E. Dubin, T.M. O'Neil, Trapped nonneutral plasmas, liquids, and crystals (the thermal equilibrium states). *Rev. Mod. Phys.* **71**, 87 (1999)
31. T.M. O'Neil, D.H.E. Dubin, Thermal equilibria and thermodynamics of trapped plasmas with a single sign of charge. *Phys. Plasmas* **5**, 2163 (1998)
32. L.R. Brewer et al., Static properties of a non-neutral ${}^9\text{Be}^+$ ion plasma. *Phys. Rev. A* **38**, 859 (1988)
33. X.P. Huang et al., Steady-State Confinement of Non-neutral Plasmas by Rotating Electric Fields. *Phys. Rev. Lett.* **78**, 875 (1997)
34. X.P. Huang, J.J. Bollinger, T.B. Mitchell, W.M. Itano, Phase-locked rotation of crystallized non-neutral Plasmas by rotating electric fields. *Phys. Rev. Lett.* **80**, 73 (1998)
35. X.P. Huang et al., Precise control of the global rotation of strongly coupled ion plasmas in a Penning trap. *Phys. Plasmas* **5**, 1656 (1998)
36. F. Anderegg, E.M. Hollmann, C.F. Driscoll, Rotating field confinement of pure electron plasmas using trivelpiece-gould modes. *Phys. Rev. Lett.* **81**, 4875 (1998)
37. E.M. Hollmann, F. Anderegg, C.F. Driscoll, Confinement and manipulation of non-neutral plasmas using rotating wall electric fields. *Phys. Plasmas* **7**, 2776 (2000)
38. D.J. Heinzen et al., Rotational equilibria and low-order modes of a non-neutral ion plasma. *Phys. Rev. Lett.* **66**, 2080 (1991)
39. M.D. Tinkle, R.G. Greaves, C.M. Surko, Modes of spheroidal ion plasmas at the Brillouin limit. *Phys. Plasmas* **3**, 749 (1996)
40. D.H.E. Dubin, J.P. Schiffer, Normal modes of cold confined one-component plasmas. *Phys. Rev. E* **53**, 5249 (1996)
41. P.M. Bellan, *Fundamentals of Plasma Physics* (Cambridge University Press, Cambridge, 2006)
42. J.H. Malmberg, T.M. O'Neil, Pure electron plasma, liquid, and crystal. *Phys. Rev. Lett.* **39**, 1333 (1977)
43. C. Kittel, *Introduction to solid state physics*, 8th edn. (Wiley, New York, 2005)
44. S.L. Gilbert, J.J. Bollinger, D.J. Wineland, Shell-structure phase of magnetically confined strongly coupled plasmas. *Phys. Rev. Lett.* **60**, 2022 (1988)
45. M.J. Jensen, T. Hasegawa, J.J. Bollinger, D.H.E. Dubin, Rapid heating of a strongly coupled plasma near the solid-liquid phase transition. *Phys. Rev. Lett.* **94**, 025001 (2005)
46. P. Gaspard, Lyapunov exponent of ion motion in microplasmas. *Phys. Rev. E* **68**, 056209 (2003)
47. L. Tonks, I. Langmuir, Oscillations in ionized gases. *Phys. Rev.* **33**, 195 (1929)
48. S.G. Brush, H.L. Sahlin, E. Teller, Monte carlo study of a one-component plasma. *J. Chem. Phys.* **45**, 2102 (1966)
49. E.L. Pollock, J.P. Hansen, Statistical mechanics of dense ionized matter. II. Equilibrium properties and melting transition of the crystallized one-component plasma. *Phys. Rev. A* **8**, 3110 (1973)

50. W.L. Slattery, G.D. Doolen, H.E. DeWitt, Improved equation of state for the classical one-component plasma. *Phys. Rev. A* **21**, 2087 (1980)
51. S. Ichimaru (ed.), *Strongly Coupled Plasma Physics* (North-Holland Publ. Co., Amsterdam, 1990)
52. F. Diedrich et al., Observation of a phase transition of stored laser-cooled ions. *Phys. Rev. Lett.* **59**, 2931 (1987)
53. G. Birkel, S. Kassner, H. Walther, Multiple-shell structures of laser-cooled $^{24}\text{Mg}^+$ ions in a quadrupole storage ring. *Nature* **357**, 310 (1992)
54. D.H.E. Dubin, Effect of correlations on the thermal equilibrium and normal modes of a non-neutral plasma. *Phys. Rev. E* **53**, 5268 (1996)
55. D.H.E. Dubin, Correlation energies of simple bounded Coulomb lattices. *Phys. Rev. A* **40**, 1140 (1989)
56. T.B. Mitchell et al., Direct observations of structural phase transitions in planar crystallized ion plasmas. *Science* **282**, 1290 (1998)
57. S. Mavadia et al., Control of the conformations of ion Coulomb crystals in a Penning trap. *Nat. Comm.* **4**, 2571 (2013)
58. L. Hornekaer, M. Drewsen, Formation process of large ion Coulomb crystals in linear Paul traps. *Phys. Rev. A* **66**, 013412 (2002)
59. M. Drewsen, Ion Coulomb crystals. *Physica B: Condensed Matter* **460**, 105 (2015)
60. J.J. Bollinger et al., Crystalline order in laser-cooled, non-neutral ion plasmas. *Phys. Plasmas* **7**, 7 (2000)
61. R.C. Thompson, Ion Coulomb crystals. *Contemporary Physics* **56**, 63 (2015)
62. T. Murböck et al., Rapid crystallization of externally produced ions in a Penning trap. *Phys. Rev. A* **94**, 043410 (2016)
63. J. Tan et al., Long-range order in laser-cooled, atomic-ion Wigner crystals observed by Bragg-scattering. *Phys. Rev. Lett.* **75**, 4198 (1995)
64. H. Totsuji, T. Kishimoto, C. Totsuji, K. Tsuruta, Competition between two forms of ordering in finite Coulomb clusters. *Phys. Rev. Lett.* **88**, 125002 (2002)
65. D.H.E. Dubin, T.M. O'Neil, Computer simulation of ion clouds in a Penning trap. *Phys. Rev. Lett.* **60**, 511 (1988)
66. R.W. Hasse, V.V. Avilov, Structure and madelung energy of spherical Coulomb crystals. *Phys. Rev. A* **44**, 4506 (1991)
67. S. Schmidt et al., Sympathetic cooling in two-species ion crystals in a Penning trap, *J. Mod. Opt.*, <https://doi.org/10.1080/09500340.2017.1342877> (2017)
68. L. Schmöger et al., Coulomb crystallization of highly charged ions. *Science* **347**, 1233 (2015)
69. V.A. Alekseev, D.D. Krylova, V.S. Letokhov, Sympathetic cooling of two trapped ions. *Physica Scripta* **51**, 368 (1995)
70. J.B. Wübbena, S. Amairi, O. Mandel, P.O. Schmidt, Sympathetic cooling of mixed-species two-ion crystals for precision spectroscopy. *Phys. Rev. A* **85**, 043412 (2012)
71. T.M. O'Neil, Centrifugal separation of a multispecies pure ion plasma. *Phys. Fluids* **24**, 1447 (1981)
72. D.J. Larson, J.C. Bergquist, J.J. Bollinger, W.M. Itano, D.J. Wineland, Sympathetic cooling of trapped ions: a laser-cooled two-species nonneutral ion plasma. *Phys. Rev. Lett.* **57**, 70 (1986)
73. H. Imajo et al., Spatial separation of ion clouds between sympathetically laser-cooled Cd^+ -ion isotopes in a Penning trap. *Phys. Rev. A* **55**, 1276 (1997)
74. M. Affolter, F. Anderegg, D.H.E. Dubin, C.F. Driscoll, Cyclotron mode frequencies and resonant absorption in multi-species ion plasmas. *Phys. Plasmas* **22**, 055701 (2015)
75. B.M. Jelenkovic, A.S. Newbury, J.J. Bollinger, W.M. Itano, T.B. Mitchell, Sympathetically cooled and compressed positron plasma. *Phys. Rev. A* **67**, 063406 (2003)
76. G. Gabrielse et al., Centrifugal separation of antiprotons and electrons. *Phys. Rev. Lett.* **105**, 213002 (2010)
77. F. Anderegg, Rotating Wall Technique and Centrifugal Separation, in: *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen and R.C. Thompson (World Scientific, 2016)
78. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover (1972)

79. L.R. Brewer et al., Static properties of a non-neutral ${}^9\text{Be}^+$ -ion plasma. *Phys. Rev. A* **38**, 859 (1988)
80. S. Bharadia, M. Vogel, D.M. Segal, R.C. Thompson, Dynamics of laser-cooled Ca^+ ions in a Penning trap with a rotating wall. *Appl. Phys. B* **107**, 1105 (2012)
81. D.H.E. Dubin, T.M. O'Neil, Trapped nonneutral plasmas, liquids, and crystals. *Rev. Mod. Phys.* **71**, 87 (1999)
82. G.L. Delzanno, G. Lapenta, J.M. Finn, KANDINSKY: a PIC code for fluid simulations of Penning traps. *IEEE Transactions on Plasma Science* **30**, 34 (2002)
83. G. Lapenta, G.L. Delzanno, J.M. Finn, Nonlinear PIC simulation in a Penning trap. *AIP Conf. Proc.* **606**, 486 (2002)
84. J. Steinmann, J. Groß, F. Herfurth, G. Zwicknagel, MD simulations of resistive cooling in HITRAP using GPUs. *AIP Conf. Proc.* **1521**, 240 (2013)
85. J. Steinmann, Modellierung und Simulation der Widerstandskuehlung von hochgeladenen Ionen, Ph.D thesis University of Erlangen-Nürnberg, 2015
86. G. Maero et al., Numerical investigations on resistive cooling of trapped highly charged ions. *Appl. Phys. B* **107**, 1087 (2012)
87. G. Maero, Cooling of highly charged ions in a Penning trap for HITRAP, Ph.D. thesis, University of Heidelberg 2008
88. N. Sillitoe and L. Hilico, Numerical Simulations of Ion Cloud Dynamics, in *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson, (World Scientific 2016)

Chapter 9

Loading a Trap with Particles



Abstract Obviously, there are two options to arrive at a situation in which particles can be confined, namely to produce them inside the trap where they remain confined, or to capture and confine particles that have been produced externally. We will see that internal production is somewhat limited in terms of the available particle species that may be produced under the circumstances inside a trap, while for external production, options are nearly unlimited, however a defined capture process is required prior to confinement.

9.1 In-trap Ion Production

There are a number of ways to produce ions inside or close to the trapping volume of a trap, such as for example laser ablation / ionisation [1, 2], thermionic emission / field ionisation (from ovens or filaments), and electron-beam sputtering and ionisation [3, 4]. While thermionic ionisation and electron-impact ionisation is generally unselective, more specific ionisation into desired states can be achieved by use of narrow-band lasers [5–7]. In-trap field ionisation into high Rydberg states has been demonstrated in [8].

9.1.1 *Electron Impact Ionisation*

One particularly simple form of ion production from residual or injected gas in a Penning trap is by electron-impact ionisation with electrons from a heated filament. Such thermionic emission from heated metals is easy to control, and the emitted electrons are guided by the magnetic field of the trap to the trap centre, such that conveniently ion production occurs predominantly in the confinement region. The filament may be located at some distance on the magnetic field axis and at a potential difference with respect to the trap such that the voltage between filament and trap determines the electron kinetic energy. This energy is usually chosen to be about two to three times the ionisation potential of the desired particle, since then the electron

impact ionisation cross section is highest. The emitted electron beam current density j is given by the Richardson-Dushman equation [9]

$$j = \lambda AT^2 \exp\left(-\frac{W}{k_B T}\right), \quad (9.1)$$

where T is the filament temperature and W is the work function of the metal. The quantity $\lambda \approx 0.5$ is a material-specific constant that modifies the Richardson-constant A which is given by

$$A = \frac{4\pi me k_B^2}{h^3} \quad (9.2)$$

and has the value of $A = 1.2017 \times 10^6 \text{ Am}^{-2}\text{K}^{-2}$. Efficient electron emission usually requires the proper choice of the filament material (often tungsten or rhenium annealed with small amounts of thorium or such) and heating to temperatures of several hundreds of Kelvin [10].

In cryogenic set-ups, methods that use heat in any form are usually excluded. In such situations, ions of interest, in particular highly charged ions, may also be produced in a miniature electron beam ion source ('mini-EBIS' [3, 4]) which is part of the trap set-up. To that end, a cold electron source (field emission point) produces an electron beam that sputters material from a target or hits present gas which is subsequently ionized by electron-impact ionisation in the same beam [3]. The field emission characteristics of metals near zero temperature are modified by the presence of the magnetic field [11, 12] and in general, it is difficult to produce field emitters like field-emission points or arrays with predictable and well-defined characteristics of the emitted electron current as a function of the voltage [13], such that initial gauging measurements may be needed. The beam energy is chosen for optimized production of the charge state of interest.

Fig. 9.1 Cartoon of charge breeding inside a Penning trap by an electron beam emitted from a field emission point. The electron beam is reflected between the outer electrodes and create ions confined in the centre of the arrangement

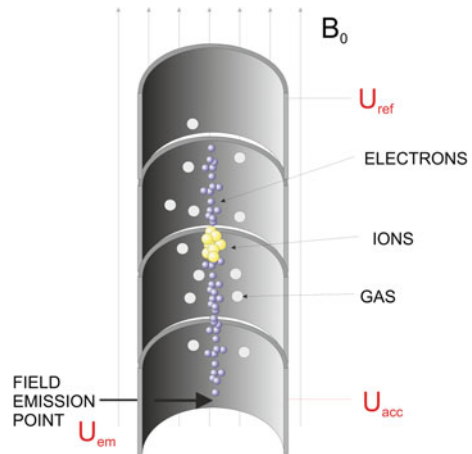
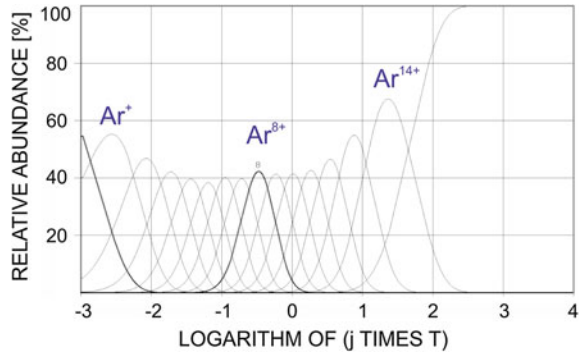


Fig. 9.2 Charge breeding simulation for argon ions by use of the CBSIM software. Relative abundance of argon charge states as a function of production parameters jT (electron current density j times charge breeding time T), at an electron beam energy of $E = 900\text{ eV}$



Each ion species has an energy-dependent cross section σ for further ionisation by electron impact which can be described by an empirical formula like the Lotz formula [14, 15]

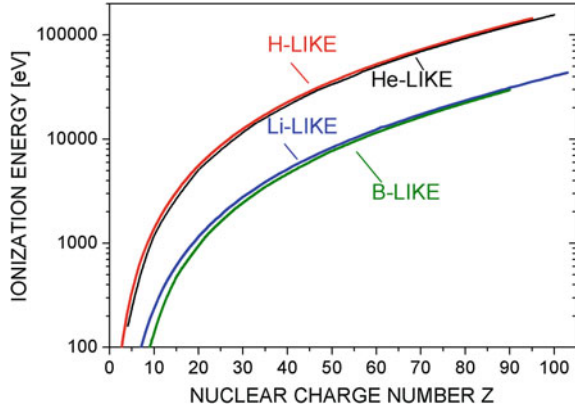
$$\sigma[\text{cm}^2] \approx 4.49 \times 10^{-14} \frac{k}{I^2[\text{eV}^2]} \frac{\ln(u+1)}{u+1}, \quad (9.3)$$

with $u = E_b/I - 1$ basically being the electron beam energy E_b divided by the ionisation potential I of the ion, and k being the number of equivalent electrons in the same shell. Commonly, this cross section has a maximum at two to three times the ionisation potential of the ion under consideration. For any given species, the ionisation rate is given by the product of that cross section and the electron beam current density. Alternative to sputtering from a target, small amounts of gas can be injected into the production section of the trap and then subsequently ionized by the electron beam, as is depicted in Fig. 9.1.

Note, that this kind of electron-beam ion source is different from an electron-beam ion trap (EBIT) [16] due to the lack of radial confinement by the electron beam, which in this application is used only for electron-impact ionisation. Often, the EBIS is operated in a ‘reflectron mode’ such that the electron beam is reflected many times between suitable HV electrodes until the electrons are lost radially due to their mutual repulsion. In particular for high charge states, there are competing processes like (radiative) recombination, loss by Coulomb collisions, and charge exchange which narrow down the production parameters to specific combinations of electron current density, electron energy and breeding time [17, 18].

It is then favourable to employ simulation software such as CBSIM [19] or others to find the optimum parameter set prior to production. Figure 9.2 shows the simulated charge breeding evolution for the example of argon ions and an electron beam of $E = 900\text{ eV}$ kinetic energy. It becomes obvious that the relative abundance of the desired ion (in the present example Ar^{14+}) can be expected to be significant only for about $1 < \log(jT) < 1.5$, where jT is the product of the electron current density and the charge breeding time. By adapting E to the experimentally available electron current density j , the necessary breeding time of a specific charge state can be minimized. If j is not well known or timing is problematic, E can be set above I of the desired

Fig. 9.3 Ionisation energies required to produce hydrogen-like, helium-like, lithium-like and boron-like ions as a function of the nuclear charge number Z . Data from the NIST Ionisation Energy Database



charge state (but below I of the subsequent charge state), such that it is produced in equilibrium, i.e. after long times T .

The ionisation potential for the hydrogen-like state can be approximated by the equation

$$I \approx m_e c^2 \left[1 - (1 - \hat{q}^2 \alpha^2)^{1/2} \right], \quad (9.4)$$

where $\hat{q} = q/e = Z - 1$ is the charge state of the desired ion and $\alpha \approx 1/137$ is the fine structure constant. Figure 9.3 shows the ionisation energies required to produce hydrogen-like ions and lithium-like ions as a function of the nuclear charge number Z . The red line represents the Dirac equation (9.4).

9.1.2 Secondary Electron Creation

For studies of electron attachment or electron recombination, low-energy electrons within the confinement volume are required. In many situations, it is non-trivial to inject such electrons from sources outside of the trap, and a possible solution is the production of secondary electrons from neutral species in the confinement volume that are subjected to an electron beam through the trap. Corresponding studies have been performed and indicate that in the case of electron attachment to anions, the rate is increased when secondary electrons are produced in presence of the species foreseen for electron attachment, as compared to a situation in which the trap is filled with secondary electrons before dynamic capture of that species [20, 21]. A possible explanation is the efficient ‘self-cooling’ of electrons in the strong magnetic field used for confinement on account of the synchrotron radiation from the perturbed cyclotron motion, which apparently leads to electrons below the energy region of efficient attachment because the low-energy electrons cannot cross the Coulomb barrier of the anion [21].

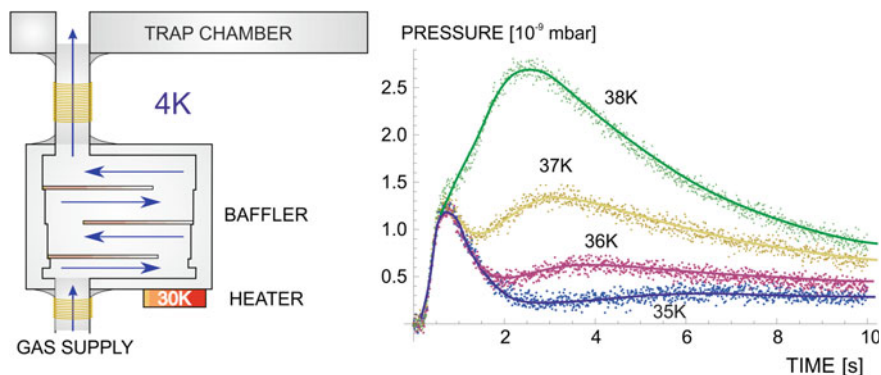


Fig. 9.4 Schematic of a baffler as a cryogenic gas source (left), and example measurements of the resulting gas pressure inside a trap chamber as a function of time for different values of the heater temperature

9.1.3 Cryogenic Gas Source

In setups that operate above the condensation temperature of a gas to be injected for subsequent ionisation (or other purposes like buffer gas cooling, reaction studies etc.), it is straight forward to use some kind of mechanical or piezo-electric valve. At or close to liquid-helium temperature, this is not feasible and normally requires artificial heating. When the amount of gas to be injected is small, as is often the case, then one can use a cryogenic gas source. It is basically a baffler positioned in proximity to the confinement volume and cooled to cryogenic temperature, filled with gas that freezes on the interior surfaces. A heater element like a resistor can then be used to temporarily heat the baffler, evaporating some of the gas and injecting it into the trap. Due to the low heat capacities of metals at low temperatures, only a small amount of energy is required, and the source thermalises fast with the ambience after heating is ended. Such a device is schematically depicted in Fig. 9.4. It shows a baffler as a cryogenic gas source, with a heater element, and measurements of the resulting gas pressure inside a trap chamber as a function of time for different values of the heater temperatures. Obviously, the resulting gas pulse depends critically on the temperature, as is also reflected in the vapour pressure curves of gases, see figure D.1.

9.2 Dynamic Particle Capture

In cases where in-trap production of the species of interest is not possible, particles may be produced externally and then captured in the trap. There is an enormous number of different types of ion sources, for an overview see [22]. Dynamic capture

is usually achieved by fast lowering of a confining potential to allow particles in and fast restoring of the potential before the particles have time to leave the trap again. This process is called ‘dynamic capture’ or ‘in-flight capture’ [23]. It is in some sense the time-reverse of evaporative cooling as discussed in Sect. 12.9.

9.2.1 Method

Figure 9.5 shows a cartoon of the procedure: the trap voltage U_0 is temporarily lowered to U_D on the entrance side when the particles enter the trap and then immediately restored, thus capturing the ions. Sometimes, there is no well-defined entrance side, and this asymmetric loading is replaced by symmetric loading where the total potential is lowered and raised again. This routine can be applied repeatedly (called ‘stacking’ or ‘accumulation’) [24]. The efficiency of this process decreases as a function of the number of repetitions, because with increasing number of particles in the trap, the occupied volume increases, and the likelihood for escape during the next cycle increases as well. Hence, one usually observes a saturation behaviour of the number of particles $N(k)$ as a function of cycle number k by

$$N(k) = N_\infty \left(1 - \exp \left(-\frac{k}{k_0} \right) \right), \quad (9.5)$$

where N_∞ is the equilibrium particle number after infinite time, and the ‘time’ constant k_0 is given by gain-to-loss rates during an individual stacking cycle. The time constant of this saturation is shorter when there is an efficient cooling process for the already confined ions [25]. Figure 9.6 schematically shows the dynamic capture of an ion bunch in a trap. The bunch length is shorter than the endcap distance $2z_0$, which ensures efficient capture for proper timing. When the upper endcap is equipped with a charge counter, the incoming ion bunch produces a signal at the time of the first pass and then after every cycle. The top right hand graph of the figure shows these initial two signals. The bottom right hand graph shows the bunch reflection observed over many cycles, indicating that the bunch structure may be preserved even after many reflections.

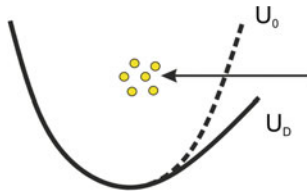


Fig. 9.5 Cartoon of dynamic capture (‘in-flight capture’): the trap potential U_0 is temporarily lowered to U_D when the particles enter the trap and then immediately restored, thus capturing the ions

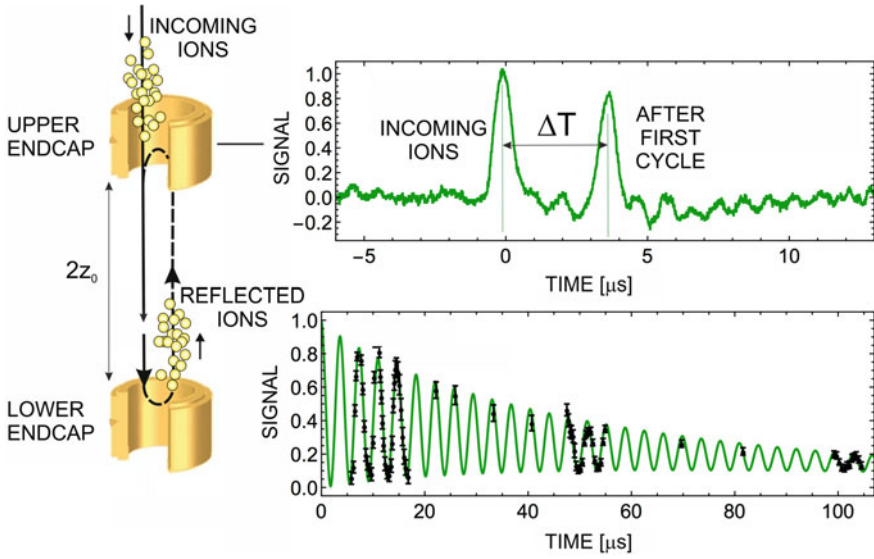


Fig. 9.6 Schematic of the dynamic capture of an ion bunch in a trap (left) and corresponding signals induced in the upper electrode as a function of time. Data from [26]

9.2.2 Accumulation Timing and Efficiency

We can distinguish different cases for the accumulation of particles in the trap, from which different requirements for timing and choice of voltages follow.

9.2.2.1 Bunch Capture

One possible situation in particle accumulation by repeated dynamic capture is the accumulation of well-defined particle bunches that maintain an oscillation with fixed phase before the next bunch is captured, as shown in Fig. 9.6. In this case, the exact value of U_D with respect to U_0 is of much smaller importance than the proper timing of switching the potentials. When the timing is such that the next bunch is captured in the same phase as the confined particles, accumulation can be highly efficient.

9.2.2.2 Thermal Particles

When the motion of the confined particles has de-phased or is thermal to begin with, then if $k_B T$ is not much less than qU_D and for opening times T_o longer than the corresponding oscillation period $2\pi/\omega$, the fraction of confined particles with kinetic energies E exceeding the value of qU_D will leave the trap during capture. Since

for thermal particle ensembles, the kinetic energies follow a Maxwell-Boltzmann distribution, there is always a certain fraction of particles with $E > qU_D$ such that U_D should not be chosen much smaller than U_0 if possible. Already confined particles may experience a non-negligible acceleration a during capture. For a cylindrical Penning trap it is given by

$$a = \frac{q(U_0 - U_D)}{2mz_0}, \quad (9.6)$$

such that the particles are translated by

$$\frac{1}{2} \frac{q(U_0 - U_D)}{2mz_0} T_o^2 \quad (9.7)$$

during the opening time T_o . The endcap distance from the trap centre z_0 limits the allowed opening time at given value of U_D (or vice versa) according to

$$z_0 > \frac{1}{2} \frac{q(U_0 - U_D)}{2mz_0} T_o^2 \quad \leftrightarrow \quad T_o < \sqrt{\frac{4mz_0^2}{q(U_0 - U_D)}}. \quad (9.8)$$

This limit on T_o is usually weaker than the above $T_o < 2\pi/\omega$.

9.3 Intra-trap Particle Transport

Many cylindrical Penning traps feature a number of electrodes such that multiple potential wells can be created. Often, this is motivated by the need to have different confinement conditions or different regions with contradicting experimental conditions. One example is the so-called ‘double-trap technique’ used in magnetic moment measurements by use of the continuous Stern-Gerlach effect, where one trap needs to feature a strong magnetic bottle while another trap needs to have a highly homogeneous magnetic field [27]. Another example is the separation of a trap for ion creation from a trap for precision confinement and spectroscopy [28]. This usually requires the particles to be transported from one trap to another, sometimes also back, sometimes for a number of cycles. This is achieved by subsequent switching of the individual electrode voltages in order to create a moving well, as depicted in Fig. 9.7. The details of this process such as the precise timing and the exact values of all involved voltages can significantly differ from case to case, and need to be adapted to each situation. Often, we wish to transport a particle or a particle ensemble adiabatically, which means without excitation of any internal degrees of freedom. For a single particle, this means that there are no structural and electromagnetic transitions, including changes of the spin.

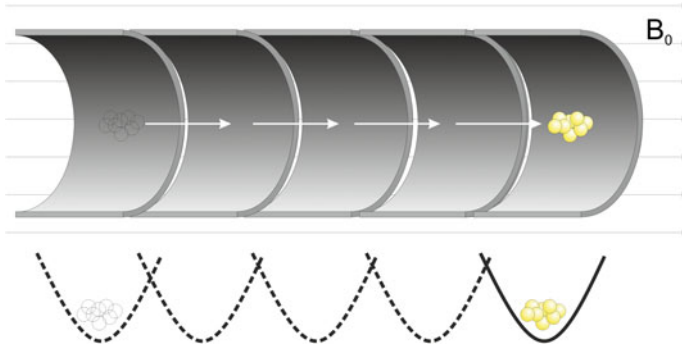


Fig. 9.7 Cartoon of particle transport between different electrodes of the same Penning trap

If we want to transport a particle through a magnetic field gradient without a change of the spin state, we need to fulfill the condition [29, 30]

$$\frac{1}{B} \left| \frac{\partial B}{\partial t} \right| = \frac{v}{B} \left| \frac{\partial B}{\partial z} \right| \ll \frac{\mu B}{\hbar} = \omega_L, \quad (9.9)$$

where v is the velocity of transport and the Larmor frequency ω_L is the spin precession frequency in the magnetic field B .

For an ensemble, the same may be true, additionally in this case, also relative motions are internal degrees of freedom, and adiabatically then means that there is no change of the ensemble temperature. For a detailed understanding we need to study the adiabatic invariants involved in the specific situation [31]. For particle ensembles, the transport time T usually needs to be much longer than any of the involved periods of individual or collective oscillations in order to avoid heating. In the presence of magnetic field gradients, the first adiabatic invariant, the magnetic moment, is of additional relevance [31]. This is an important criterion in the application of the double-trap technique (see Sect. 21.8) where an ion needs to be transported between adjacent Penning traps without a change of the spin state.

9.4 Potential Asymmetry

The trap potential can be made asymmetric along the axial direction by adding a voltage U_A to one of the endcap voltages U_0 . If $U_A \ll U_0$, to first order, the frequencies remain unchanged (see 21.8), but the electrostatic centre of the trap (the

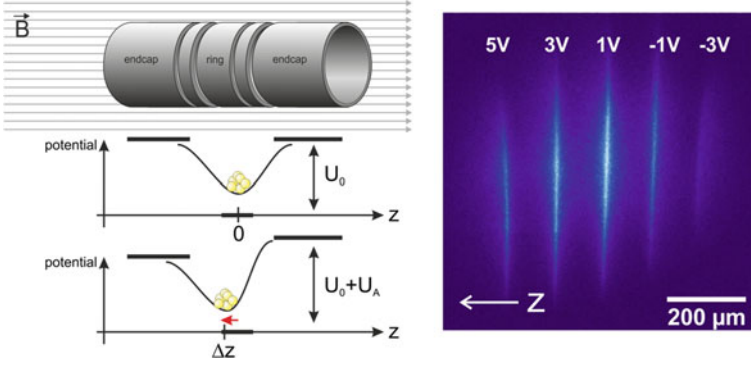


Fig. 9.8 Left: Schematic of a potential asymmetry shifting the trap centre. Right: Images of confined particle ensembles shifted along the z -axis with corresponding asymmetry voltages indicated. The slight tilt is due to optical distortions. Data from [32]

position of the potential minimum) is shifted with respect to its geometric centre by an amount

$$\Delta z = \frac{1}{2} \frac{d^2}{z_0} \frac{C_1}{C_2} \frac{U_A}{U_0}. \quad (9.10)$$

This yields good control over the position of the electric trap centre and may hence be used for axial positioning of confined particles [32, 33]. The same is possible in the radial direction if the electrodes are segmented accordingly. Note, however, that accelerating a charged particle across the strong magnetic field B_0 requires significantly more energy than a similar shift along the the electric field. Figure 9.8 gives a schematic of the voltage difference applied to trap electrodes (left) and a CCD image of oblate ion ensembles (thin disks rotating about the trap axis z) in the SpecTrap experiment when shifted by the indicated voltage asymmetries [32].

9.5 Particle Deceleration

In-flight capture of charged particles into a Penning trap is unfortunately limited by the maximum voltages that can be applied to electrodes, usually these may not exceed values of several hundreds of volts up to a few kilo-Volts. For particles that are transported to the trap at somewhat higher energies, such as from an EBIS or from a low-energy beam line, direct dynamic capture is normally not feasible. In this situation, there are several concepts for deceleration prior to capture. We will not go into detail about gas stopping cells, funnels, radio-frequency quadrupoles and the like, since these are usually not applied in the low-energy domain we focus on.

9.5.1 *Potential Cages and Decelerators*

One solution that is conceptionally simple but demanding to implement is to have the complete experiment at a higher potential than the particle source, such that incoming particles lose kinetic energy while going up the potential hill. For this, commonly the whole experiment has to be located in a high-voltage (HV) cage. This brings a lot of complications, but has the advantage that the height of the potential hill can be of the order of many kilo-Volts.

Other than that, concepts from accelerator physics can be adapted for deceleration, like radio-frequency quadrupole decelerators and such. Also in this case, the set-up is demanding to implement and operate, but also here, initial energies can exceed 100 kV per charge. Examples for trap experiments that feature such devices are given in [34–36].

9.5.2 *Pulsed Drift Tube*

A comparatively simple device is a pulsed drift tube. It is an elongated conducting tube that the incoming particles pass through. This of course works best with particle bunches that are much shorter than the tube. During their transit, the potential of the tube is pulsed down such that the exit kinetic energy is lower than the entrance kinetic energy. This is a fairly simple device that mainly requires a fast HV switch. The details of tube geometry and initial and final energies are best obtained from a simulation.

It is usually advisable to perform the deceleration in the presence of a coaxial magnetic field, which usually means that the pulsed drift tube (PDT) is located adjacent to the trap, in the strong magnetic field used also for confinement. Since this form of deceleration conserves the phase space of the particle bunch, deceleration in the axial direction leads to a radial acceleration, meaning that the bunch extends radially and may be lost on the tube walls. The presence of the magnetic field counteracts this expansion by forcing radially accelerated particles on a cyclotron orbit. This has two aspects to it, the absolute value of the magnetic field B and the magnetic field gradient $\partial B/\partial z$ experienced by the particles entering the magnetic field along the trap axis.

- As far as axial deceleration is concerned, we may wish to efficiently transfer axial energy into radial energy. This requires the process of entering the magnetic field of the trap to be adiabatic, which requires incoming particles to be slow enough for the given value of the axial magnetic field gradient.
- As far as the increase of radial energy (cyclotron radius) connected with axial deceleration is concerned, the set-up geometry sets bounds beyond which the particles are lost, hence the absolute value of the magnetic field must be high enough to keep cyclotron radii below the bounds.

Let us look at the involved quantities: In the present case, a particle enters the magnetic field gradient adiabatically if the experienced relative change of the magnetic field strength is slower than the cyclotron oscillation at the position of the particle. Let $B(z)$ be the axial component of the magnetic field strength at axial position z , then this can be expressed as

$$\frac{1}{B(z)} \frac{\partial B(z)}{\partial t} \ll \frac{\omega_c}{2\pi}, \quad (9.11)$$

and we can substitute

$$\frac{\partial B(z)}{\partial t} = \frac{\partial B(z)}{\partial z} \frac{\partial z}{\partial t} = \frac{\partial B(z)}{\partial z} v \quad (9.12)$$

and use the definition of the cyclotron frequency $\omega_c = qB/m$.

The adiabaticity condition is then given by

$$2\pi \frac{1}{B^2(z)} \frac{m}{q} \frac{\partial B(z)}{\partial z} v \ll 1. \quad (9.13)$$

Hence, for any magnetic field gradient there is a maximum particle velocity v such that the particle enters the field adiabatically.

A closer discussion of this topic can for example be found in [37, 38]. Figure 9.9 shows the axial magnetic field strength of a superconducting magnet (here: the magnet of the ARTEMIS experiment) as a function of axial distance from the field centre (trap centre). The field close to the centre is highly homogeneous, and displays a maximum field gradient (inset of the figure) at an axial position around $z \approx 20$ cm. This kind of field plot is required for an optimal positioning of a pulsed drift tube.

The second aspect is the absolute value of the magnetic field in the region of particle deceleration. The kinetic energy of a charged particle on a cyclotron orbit of radius R is given by

$$E = \frac{1}{2} m R^2 \omega_c^2 = \frac{1}{2m} R^2 q^2 B_0^2. \quad (9.14)$$

The voltage $U = E/q$ required to move a particle from radius 0 to R would hence amount to

$$U = \frac{1}{2m} R^2 q B_0^2. \quad (9.15)$$

Assuming a radius R of 10 mm in a magnetic field of 6 T, for a singly charged ion of mass number 40, the voltage U is 4500 V, for a charge state of 10, it is 45 kV. This is the maximum allowed acceleration voltage perpendicular to the deceleration direction for the particle still to perform its cyclotron orbit within the bounds given by the geometry. In other words, this is the maximum allowed deceleration voltage

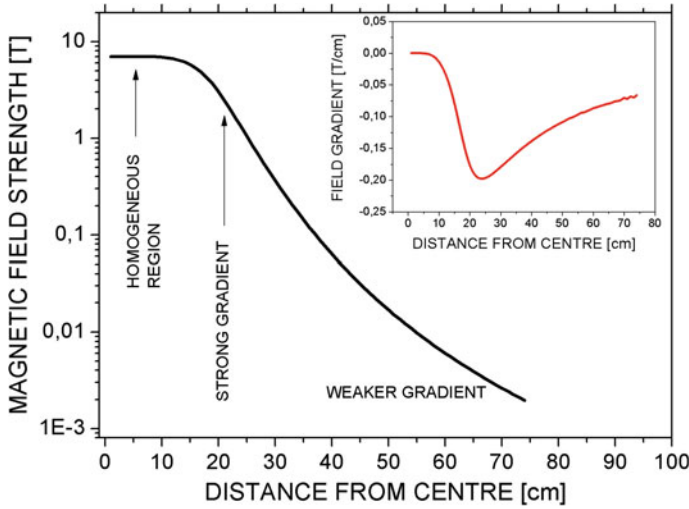


Fig. 9.9 Axial magnetic field strength of a typical superconducting magnet as a function of axial distance from the field centre. Inset: corresponding field gradient

in the axial direction, given that all axial energy is converted to radial energy, see also the adiabaticity condition above.

Such pulsed-tube deceleration devices have successfully been operated to decelerate ion bunches from energies of keV per charge to below 100 V per charge [37, 39].

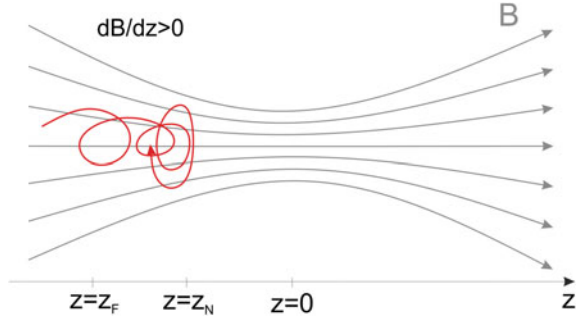
Note, that such tubes also find application as detecting devices when the signal induced in the tube by particles is picked up by a single-pass non-destructive charge counter as described in detail in [40]. Such detection may be seen as the non-resonant counterpart to Schottky mass spectrometry.

9.6 Magnetic Mirror Effect

Connected with the question about adiabaticity of particles entering a magnetic field gradient (when for example approaching the trap along the magnetic field axis of the field B_0 used for confinement) is the question about magnetic mirror forces (Fig. 9.10). While adiabaticity is a question of comparing the field gradient with the particle velocity along the field lines, the magnetic mirror effect is relevant when there is a particle velocity component perpendicular to the field, i.e. when the field is entered at an angle. Let us have a look at the mirror force $\mu \nabla B$ and apply cylindrical coordinates to the problem. We then have the non-existence of magnetic charges ($\nabla B = 0$) in the form

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{\partial}{\partial z} B_z = 0, \quad (9.16)$$

Fig. 9.10 Schematic of a the magnetic mirror effect when a particle enters a magnetic field gradient



where we have neglected B_θ since the field of magnets as used in Penning trap experiments have no volumetric currents. When we assume B_z to be roughly constant over the radial particle motion, then by integration this leads to

$$B_\rho = -\frac{\rho}{2} \frac{\partial}{\partial z} B_z, \quad (9.17)$$

which means that there is a component of the Lorentz force pointing in the negative z -direction. The force is given by

$$F_z = -\frac{m}{2} v_p^2 \frac{1}{B} \frac{\partial B_z}{\partial z}, \quad (9.18)$$

where v_p is the velocity component perpendicular to the magnetic field and is the source of a magnetic moment

$$\mu = \frac{mv_p^2}{2B} \quad (9.19)$$

associated with the particle. When the energy

$$E_m = \int F_z(z) dz \quad (9.20)$$

is larger than the kinetic energy E of the incoming particle, it will be reflected by the magnetic field gradient. Looking at a roughly constant magnetic field gradient between two axial positions z_F (far) and z_N (near) then we have axial magnetic fields B_F and $B_N > B_F$ at these positions which are related by

$$B_N - B_F = \frac{\partial B_z}{\partial z} (z_N - z_F). \quad (9.21)$$

The condition for magnetic mirror particle reflection when travelling between z_F and z_N can then be stated as

$$\frac{v_p^2}{v_0^2} > \frac{B_F}{B_N}. \quad (9.22)$$

This means that if the velocity v_p perpendicular to the field (when compared to the total velocity v_0) exceeds a certain value proportional to the field gradient, the induced dipole force is sufficient to counteract the initial kinetic energy over the travelled distance. We can state this last equation also in the form

$$\sqrt{\frac{B_F}{B_N}} < \sin \Theta, \quad (9.23)$$

where Θ is the so-called ‘pitch angle’ at which the particle enters the field.

References

1. D.A. Davies et al., A high-power laser ablation ion source for Penning trap studies of nuclear reaction products. *J. Phys.: Conf. Ser.* **59**, 136 (2007)
2. C.G. Gill, A.W. Garrett, P.H. Hemberger, N.S. Nogar, Selective laser ablation/ionization for ion trap mass spectrometry: resonant laser ablation. *Spectrochim. Acta Part B: At. Spectrosc.* **51**, 851 (1996)
3. J. Alonso et al., A miniature electron beam ion source for in-trap creation of highly charged ions. *Rev. Sci. Instr.* **77**, 03A901 (2006)
4. B. Schabinger et al., Creation of highly-charged calcium ions for the g-factor determination of the bound electron. *J. Phys. Conf. Ser.* **163**, 012108 (2009)
5. N. Kjaergaard et al., Isotope selective loading of an ion trap using resonance-enhanced two-photon ionization. *Appl. Phys. B* **71**, 207 (2000)
6. S. Removille et al., Photoionisation loading of large Sr^+ ion clouds with ultrafast pulses. *Appl. Phys. B* **97**, 47 (2009)
7. S. Gulde et al., Simple and efficient photo-ionization loading of ions for precision ion-trapping experiments. *Appl. Phys. B* **73**, 861 (2001)
8. J. Estrada, T. Roach, J.N. Tan, P. Yesley, G. Gabrielse, Field ionization of strongly magnetized Rydberg positronium: a new physical mechanism for positron accumulation. *Phys. Rev. Lett.* **84**, 859 (2000)
9. C.R. Crowell, The Richardson constant for thermionic emission in Schottky barrier diodes. *Solid-State Electron.* **8**, 395 (1965)
10. T. Murböck, S. Schmidt, Z. Andelkovic, G. Birkel, W. Nörtershäuser, M. Vogel, A compact source for bunches of singly charged atomic ions. *Rev. Sci. Instr.* **87**, 043302 (2016)
11. F.J. Blatt, Field emission in a magnetic field. *Phys. Rev.* **131**, 166 (1963)
12. R.F. Waitest, H.A. Schwettman, Field emission from bismuth and tungsten in a magnetic field. *Phys. Rev. B* **8**, 2420 (1973)
13. D. Temple, Recent progress in field emitter array development for high performance applications. *Mater. Sci. Eng.* **R24**, 185 (1999)

14. W. Lotz, An empirical formula for the electron-impact ionization cross-section. *Z. Phys.* **206**, 205 (1967)
15. V.A. Bernshtam, YuV Ralchenko, Y. Maron, Empirical formula for cross section of direct electron-impact ionization of ions. *J. Phys. B* **33**, 5025 (2000)
16. M.A. Levine et al., The electron beam ion trap: a new instrument for atomic physics measurements. *Physica Scripta* **T22**, 157 (1988)
17. E.D. Donets, in *The Physics and Technology of Ion Sources*, ed. by I.G. Brown (Wiley, New York, 1989)
18. R. Becker, in *Handbook of Ion Sources*, ed. by B. Wolf (CRC Press, Boca Raton, NY, London, Tokyo, 1995)
19. R. Becker, O. Kester, Th. Stöhlker, Simulation of charge breeding for trapped ions. *J. Phys.: Conf. Ser.* **58**, 443 (2007)
20. A. Herlert, K. Hansen, L. Schweikhard, M. Vogel, Multiply charged titanium cluster anions: production and photodetachment. *Hyp. Int.* **127**, 529 (2000)
21. A. Herlert et al., First observation of doubly charged negative gold cluster ions. *Physica Scripta* **T80**, 200 (1999)
22. B. Wolf, in *Handbook of Ion Sources*, (CRC Press, Boca Raton, ISBN 978-0-8493-2502-1, 1995)
23. H. Schnatz et al., Inflight capture of ions into a Penning trap. *Nucl. Inst. Meth. A* **251**, 17 (1986)
24. M. Rosenbusch et al., Ion bunch stacking in a Penning trap after purification in an electrostatic mirror trap. *Appl. Phys. B* **114**, 147 (2014)
25. H.-U. Hasse et al., External-ion accumulation in a Penning trap with quadrupole excitation assisted buffer gas cooling. *Int. J. Mass Spectrom. Ion Proc.* **132**, 181 (1994)
26. S. Schmidt et al., Sympathetic cooling in two-species ion crystals in a Penning trap. *J. Mod. Opt.* (2017). <https://doi.org/10.1080/09500340.2017.1342877>
27. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
28. D. von Lindenfels et al., Experimental access to higher-order Zeeman effects by precision spectroscopy of highly charged ions in a Penning trap. *Phys. Rev. A* **87**, 023412 (2013)
29. E. Majorana, Atomî orientati in campo magnetico variabile. *Nuovo Cimento* **9**, 43 (1932)
30. V. Gomer et al., Magnetostatic traps for charged and neutral particles. *Hyp. Int.* **109**, 281 (1997)
31. D. Boccaletti, G. Pucacco, The theory of adiabatic invariants, in *Theory of Orbits*, Astronomy and astrophysics library (Springer, Berlin, Heidelberg, 1999)
32. T. Murböck, Preparation and cooling of magnesium ion crystals for sympathetic cooling of highly charged ions in a Penning trap, Ph.D. thesis (Technische Universität Darmstadt, 2016)
33. G. Gabrielse, Detection, damping, and translating the center of the axial oscillation of a charged particle in a Penning trap with hyperbolic electrodes. *Phys. Rev. A* **29**, 462 (1984)
34. M. Mukherjee et al., ISOLTRAP: an on-line Penning trap for mass spectrometry on short-lived nuclides. *Eur. Phys. J. A* **35**, 1 (2008)
35. F. Herfurth et al., A linear radiofrequency ion trap for accumulation, bunching, and emittance improvement of radioactive ion beams. *Nucl. Inst. Meth. A* **469**, 254 (2001)
36. H.J. Kluge et al., HITRAP: a facility at GSI for highly charged ions. *Adv. Quantum Chem.* **53**, 83 (2007)
37. L. Gruber, J.P. Holder, D. Schneider, Formation of strongly coupled plasmas from multi-component ions in a Penning trap. *Physica Scripta* **71**, 60 (2005)
38. D. Schneider et al., Confinement in a cryogenic penning trap of highest charge state ions from EBIT. *Rev. Sci. Inst.* **65**, 3472 (1994)
39. T. Murböck et al., Rapid crystallization of externally produced ions in a Penning trap. *Phys. Rev. A* **94**, 043410 (2016)
40. S. Schmidt et al., Non-destructive single-pass electronic detection of ions in a beamline. *Rev. Sci. Inst.* **86**, 113302 (2015)

Chapter 10

Excitation of Particle Motions



Abstract Oscillatory motions of confined particles can be resonantly excited by a suited electromagnetic excitation at the frequency of the respective motion. This refers to individual axial and radial oscillation frequencies of single particles or particle ensembles, excitation of sidebands for motional coupling, and also to collective frequencies of ensembles such as plasma modes. Non-resonant excitation can be used to force particular motions upon particles, such as in rotating wall applications.

10.1 Excitation Geometries

The most common types of motional excitation in Penning traps are

- Resonant electric dipole excitation of the axial oscillation (as used for example for resonant axial ejection of unwanted particles)
- Resonant electric dipole excitation of motional frequencies in the radial plane (as used for example for resonant radial ejection of unwanted particles)
- Resonant electric quadrupole excitation in the radial plane (as used for example for magnetron centring)
- Non-resonant electric dipole or quadrupole excitation in the radial plane (as used for example in a ‘rotating wall’).

Figure 10.1 shows generic examples of the most common radial excitation geometries: dipole excitation (left), quadrupole excitation (middle), and rotating dipole excitation (right). We will discuss these excitations and their applications in more detail. For an account on single-frequency dipole and quadrupole excitation geometries and their effect, including quadrature excitation on cubic Penning traps, see also [1]. Higher-order excitations are comparatively rare. For a rotating quadrupole or a static octupole, eight segments can be used like depicted in Fig. 10.2. Sometimes, six segments are used for rotating quadrupole excitation, see for example [2]. In practice, it is often not important to be efficient in creating a field of the right geometry, so long as the created field contains a component of the desired geometry. This is largely owed to the fact that in many cases, small excitation amplitudes are sufficient to obtain the sought effect. Octupolar excitation finds application for example in Penning trap mass spectrometry [3, 4], where azimuthal octupolar excitation

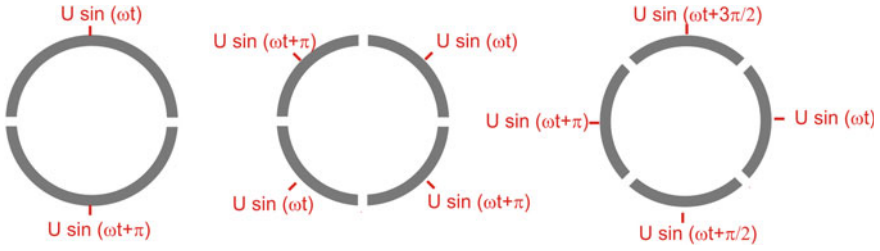


Fig. 10.1 Generic cases of radial excitation geometries. From left to right: dipole excitation, quadrupole excitation, and rotating dipole excitation

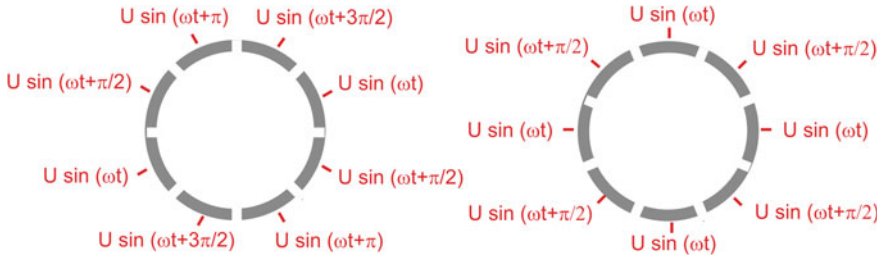


Fig. 10.2 Generic cases of radial excitation geometries: rotating quadrupole (left) and non-rotating octupole (right)

at $2\omega_c$ appears to have advantages over quadrupolar excitation at ω_c [5]. Rotating quadrupoles are employed for example in confined non-neutral plasma experiments [2, 6]. It is understood that all of these excitations are applied for times much longer than the oscillation periods.

The general situation for rotating multipoles can be expressed by

$$U_k(t) = U \cos(m(\phi_k - \omega t)), \quad (10.1)$$

where the voltage $U_k(t)$ is applied to the k th electrode (along the sense of rotation), and where the phase ϕ_k is given by

$$\phi_k = 2\pi \frac{k}{n}, \quad (10.2)$$

in which n is the number of segments in use, and the quantity m represents dipole excitation ($m = 1$), quadrupole excitation ($m = 2$) and so forth.

10.1.1 Dipole Excitation

Dipolar excitation is most efficient when two symmetrically opposing electrodes (with respect to the trap centre) are used to create a dipole field at the desired frequency. These are often the trap endcaps in case of axial dipole excitation, or two opposing segments of the ring electrode in case of radial dipole excitation. In any case, one electrode is supplied with a voltage $U(t) = U \sin(\omega t)$ while the other electrode is supplied with $U(t) = U \sin(\omega t + \pi)$, i.e. the same amplitude and frequency but with a phase shift of a half cycle.

The resulting field geometry is shown in Fig. 10.3 for two opposing sets of electrodes that each span half the inner circumference of the trap. We have chosen the example of an eight-fold segmented ring electrode in a cylindrical Penning trap, so electrodes (2,3,4,5) form one half and (1,8,7,6) form the other half. The figure shows a false-colour representation of the electric field components (left: x -component, right: y -component) created by a voltage of +1 V at electrode (2,3,4,5) and a voltage of -1 V at electrode (1,8,7,6). Note that in these plots, all other electrodes are on ground potential, including the not visible cylindrical endcaps of same radius above and below the ring. This makes the shown true field configuration in such cases more complicated than in the ideal case of infinite axial extension of the used electrodes.

When the opposing electrodes used for dipolar excitation each span less than half the inner circumference of the trap, the resulting field geometry is shown in Fig. 10.4. It shows again a false-colour representation of the electric field components (left:

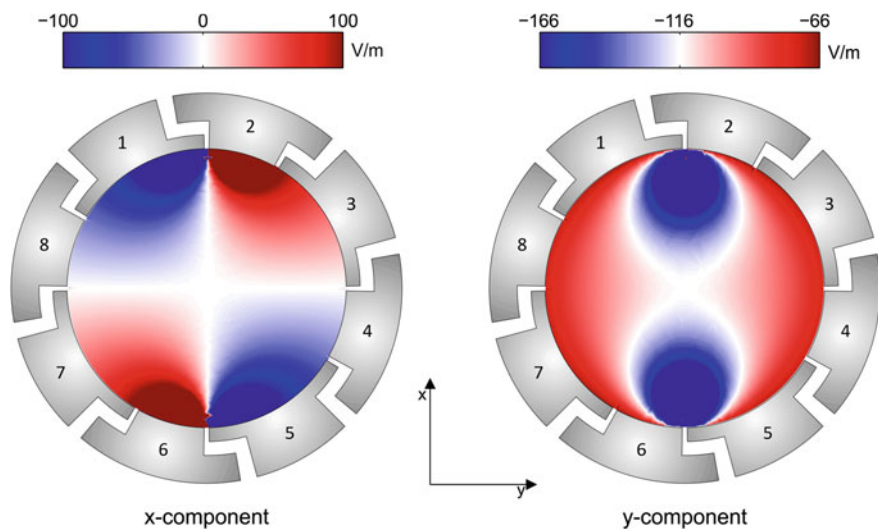


Fig. 10.3 Dipole excitation: False-colour representation of the electric field components (left: x -component, right: y -component) created by a voltage of +1 V at ring electrode (2,3,4,5) and a voltage of -1 V at ring electrode (1,8,7,6). The invisible endcap electrodes above and below the ring are grounded

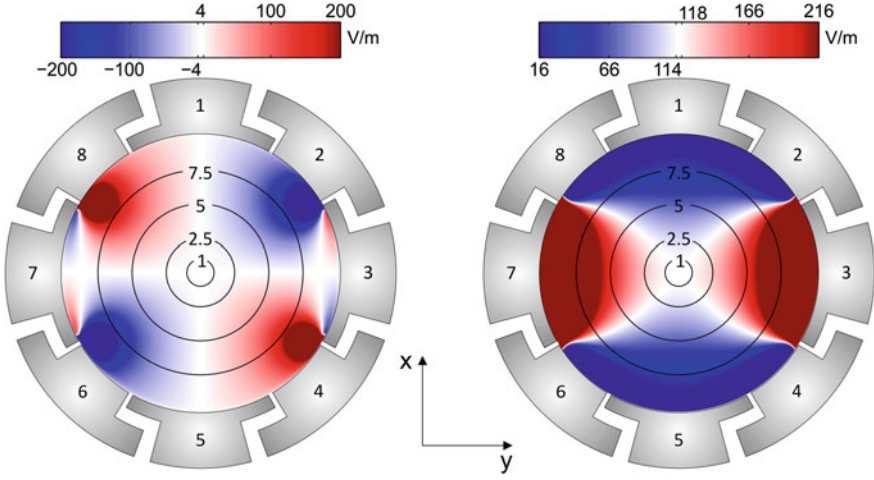


Fig. 10.4 Dipole excitation: False-colour representation of the electric field components (left: x -component, right: y -component) created by a voltage of +1 V at ring electrode number 3 and a voltage of -1 V at ring electrode number 7. The invisible endcap electrodes above and below the ring are grounded

x -component, right: y -component) created by a voltage of +1 V at electrode number 3 and a voltage of -1 V at electrode number 7.

Dipole excitation at the oscillation frequency leads to an increase of the kinetic particle energy in the excited motion. The motional amplitudes (assumed to be zero initially) are excited to

$$\rho_+ = \frac{C_{1,\rho} U T}{d B_0} \quad , \quad a_z = \frac{C_{1,z} U T}{\sqrt{\frac{2mU_0}{q}}} \quad , \quad (10.3)$$

where T is the duration of the excitation and C_1 is the geometry parameter of the respective electrode arrangement used for excitation. Note that the radius ρ_+ is independent of the mass-to-charge ratio m/q of the particle. The kinetic energy of the particle after excitation (again assumed to be zero initially) is given by

$$E_\rho = \frac{q^2 C_{1,\rho}^2 U^2 T^2}{2md^2} \quad , \quad E_z = \frac{q^2 C_{1,z}^2 U^2 T^2}{2md^2} \quad , \quad (10.4)$$

which of course is identical for radial and axial excitation except for the different geometry parameter. Using the phase information, short dipolar excitations can also be used to reduce motional amplitudes, including the magnetron orbit [7]. We will discuss applications of dipolar excitation in the following sections, and the special case of a non-resonant rotating dipole field (‘rotating wall’) in Chap. 13.

10.1.2 Quadrupole Excitation

Quadrupole excitation is most often applied to the radial plane by supplying two opposing electrodes (again with respect to the trap centre) with a voltage $U(t) = U \sin(\omega t)$ while a second pair of opposing electrodes (with the opposition axes perpendicular to each other) is supplied with $U(t) = U \sin(\omega t + \pi)$, i.e. the same amplitude and frequency but with a phase shift of a half cycle.

The resulting field geometry is shown in Fig. 10.5. It shows a false-colour representation of the electric field components (left: x -component, right: y -component) created by a voltage of +1 V at electrodes number 2 and 6, and a voltage of –1 V at electrodes number 4 and 8. We will discuss applications of quadrupolar excitation in connection with motional coupling in Chap. 11. The confinement stability conditions of Penning traps in the presence of a rotating electric quadrupole field has been discussed in [8].

10.2 Resonant Ejection of Unwanted Particles

In many cases, in-trap production or dynamic capture of particles results in a spectrum of confined particle species. Often, only one of these is desired for the experiment. The trap can be ‘cleaned’ by (resonant) ejection of unwanted particles. To that end,

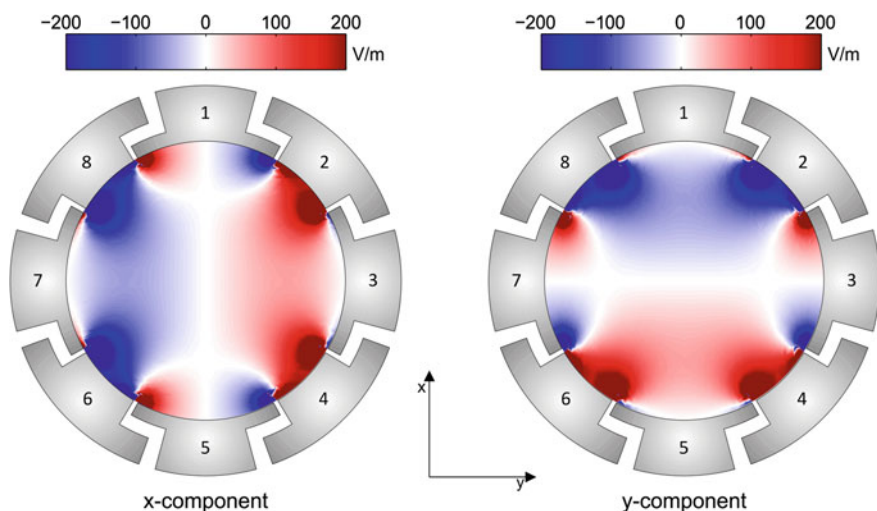


Fig. 10.5 Quadrupole excitation: False-colour representation of the electric field components (left: x -component, right: y -component) created by a voltage of +1 V at ring electrodes number 2 and 6, and a voltage of –1 V at ring electrode numbers 4 and 8. The invisible endcap electrodes above and below the ring are grounded

their axial and/or radial motion can be excited until the motional amplitude exceeds the trap confinement region and the unwanted particles are lost. Four main techniques are common:

- narrow-band irradiation of individual oscillation frequencies
- a frequency sweep over the bands to be ejected
- broadband irradiation of white noise with a gap at the frequency of the desired particle ('notch filter'), see also Sect. 12.17
- irradiation of a signal transient, the Fourier transform of which corresponds to the frequency bands of all unwanted particles ('SWIFT').

The first technique is useful when few unwanted species with well-defined frequencies are present and confinement times are sufficiently long to allow subsequent resonant ejection of all unwanted particle species individually. The notch filter approach addresses all unwanted particle at the same time, however it requires to cover a much broader frequency band. The 'SWIFT' technique ('Stored Waveform Inverse Fourier Transform' [9, 10]) is in some sense a compromise between the former two ideas. When the spectrum of unwanted particles is known sufficiently well, frequency bands around each unwanted species are defined. A Fourier transform of that combined frequency domain into the time domain yields a transient signal which is irradiated into the trap and excites just those unwanted particle frequencies (Fig. 10.6). Efficient application of this technique requires the proper shaping of the excitation transient in use [11].

Sidenote: a non-resonant ejection ('cleaning') technique is for example the ejection of heavy particles beyond a certain mass by change of the trap potential, as we have mentioned at the end of Sect. 5.2. It is one example of ejection by a selective instability, in analogy to a Finnegan scan as performed in Paul trap mass spectrometry [12, 13]. This method is selective but not resonant.

All of the resonant ejection techniques can in principle be applied to the axial motion or to the perturbed cyclotron motion, with different advantages and disadvantages. The energy necessary to excite the axial motion of a particle to beyond the confinement region is of the order of the electrostatic well depth, i.e. $E_z \approx qU_0$, while for the radial motion it is

$$E_R \approx m\omega_+^2 R^2 \approx \frac{1}{m} q^2 B_0^2 R^2, \quad (10.5)$$

when R is the radius at which the particles lose confinement, usually the inner trap radius ρ_0 . Under common trapping conditions with U_0 of the order of few Volts to few tens of Volts and B_0 of the order of several Tesla, the cyclotron motion is much harder to excite to the radius R at which the particles will be lost from the trap than the axial motion to leave the well (commonly by about two orders of magnitude).

There are, however, more aspects to this: Firstly, the rf excitation needs to be resonant with the particle motion until the particle has gained enough energy to leave the trap ('escape energy'). In the presence of trapping field imperfections this broadens the required frequency bands. The oscillation frequencies are prone mainly

to anharmonicities as described by electrostatic potential coefficients C_4 and C_6 and by magnetic field inhomogeneities as given by terms with B_2 , see Sects. 6.1 and 21.2. Additionally, the axial frequency undergoes significant space charge shifts with the confined charge density (see also Sect. 8.1), which are commonly negligible for the cyclotron frequency.

Secondly, the overall excitation time is limited by the thermalisation (mainly through Coulomb collisions) amongst particle species: when the time required for excitation to escape energy is of the order of the thermalisation time constant or longer, then the desired particles are indirectly excited as well. Hence, the pun with the name of the procedure ‘SWIFT’. In a given experiment, this requires the rf power to be sufficiently high such that the particles in unwanted frequency bands are excited to escape energy before the desired particles are heated significantly. Assuming a thermalised ensemble of particles of different charge states of the same species which represent a uniform density n at a common temperature T , the thermalisation time constant may be estimated roughly from the Spitzer self-collision time [14]

$$\tau \approx (4\pi\epsilon_0)^2 \frac{3\sqrt{m} (k_B T)^{3/2}}{4\sqrt{\pi} n q^4 \ln \Lambda}, \quad (10.6)$$

in which $\ln \Lambda \approx 10$, see the discussion in Sect. E.2.1, and we may take q to be the average charge of the charge state distribution. The two crucial quantities are the density n and the temperature T at the time of excitation. Let us discuss one example, in which we have confined argon ions of which Ar^{10+} is the desired ion species. We

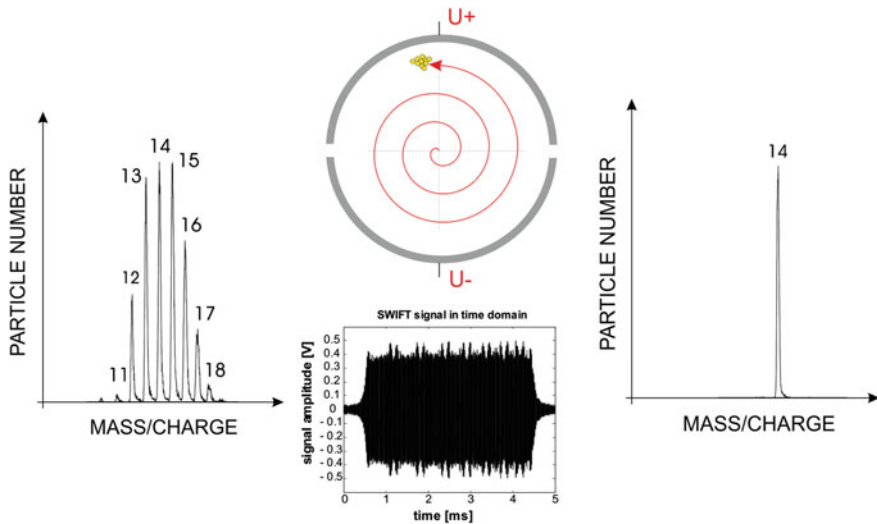


Fig. 10.6 Schematic representation of the SWIFT technique, using measured spectra. The initial particle distribution (left) is dipole-excited with the SWIFT transient signal U which resonantly ejects all but the remaining species (left)

assume an overall ion number density of, say, $n = 10^5/\text{cm}^3$ at a temperature of, say, $T = 10^5$ K. Then τ is of the order of some 10^{-4} s. Let us take the perspective of one ion that is to be excited to escape energy. This is reached once its cyclotron radius is R , at which point it has velocity $\omega_+ R$. Assuming it was at rest initially, it has been accelerated by $\omega_+ R/\tau$, for which a force $m\omega_+ R/\tau$ was necessary. This force is due to the electric field U_{rf}/D acting on the ion charge q . The field is created by the excitation voltage U_{rf} across the electrodes used for radial excitation, let us assume them at distance $D \approx 2R$.

The minimum excitation voltage for efficient radial ejection by SWIFT excitation follows from

$$U_{\text{rf}} \approx m \frac{\omega_+ R D}{q \tau} \approx m \frac{2R^2 \omega_+}{q \tau} \approx \frac{2R^2 B_0}{\tau}, \quad (10.7)$$

where R is the trap radius, D is the effective electrode distance and τ is the thermalisation time constant of the particle ensemble.

For our example with $R \approx 10$ mm, it amounts to several Volts. At a 50Ω impedance, this corresponds to an rf power of around one Watt, to be irradiated for just under one milli-second. This power significantly increases for colder and denser ion ensembles, due to the shorter admissible excitation time. Hence, a compromise must be made (with respect to the ion temperature) between the well-defined oscillation frequencies (and thus narrow excitation bands) of cold ions, and the increased necessary rf power in that case.

10.3 Resonant Collisional Excitation of Particles

The procedures described above may also be used for a specific excitation of confined particles other than resonant ejection from the trap. This enables for example reaction [15] and dissociation [16] studies of confined atomic and molecular ions with a neutral species in the confinement volume, or alternative ways for electron impact ionisation and the like.

ECR ion sources (ECRIS)

Electron Cyclotron Resonance Ions Sources (ECRIS) [17] employ microwave heating of a confined plasma to ionize gas or volatile compounds [18] to the end of producing highly charged ions. Similar resonant electron excitation is possible in Penning traps.

Collision-Induced Excitation / Dissociation (CIE / CID)

When for example confined atomic or molecular ions are resonantly excited to specific kinetic energies, they gain internal excitation in collisions with neutral gas (CIE). This may be used for excitation of internal states of molecular ions or for dissociation of polyatomic systems (CID) [16, 19].

Apart from this, Penning traps open the way to a variety of reaction and collisional excitation studies that benefit from motional cooling, excitation to well-defined energies, potentially long confinement times, and the broad range of charge-to-mass ratios that can be confined simultaneously. These studies comprise ion-molecule reactions [15], including low-temperature measurements with astrophysical relevance [20], ion-atom and ion-molecule charge-exchange reactions [12], including polarisation-dependent studies [21], ion-electron recombination studies [22, 23], time-resolved photo-dissociation and electron-detachment investigations [24, 25], ionisation cross-section measurements [26], studies of collisional activation [27], infrared multiple-photon dissociation spectroscopy of ions in Penning traps [28], and many more.

10.4 Resonant Excitation: Penning-Trap Mass Spectrometry

Let us briefly mention that other than the axial motion of particle ensembles, the radial motions typically have much smaller frequency widths, mainly due to the absence of a first-order space-charge effect. This allows for a high selectivity in the mass-to-charge ratio when the radial (perturbed cyclotron) motion is excited. This and the conversion from radial to axial energy in a magnetic field gradient when the particles are ejected from the trap is exploited for a technique called ‘time-of-flight ion-cyclotron-resonance technique’ (TOF-ICR). It makes use of the fact that resonantly excited particles gain higher radial energy that is converted to axial energy during ejection, which shortens the flight time in the subsequent time-of-flight mass spectrometer. Of the total kinetic particle energy E_0 , the radial particle motion in the trap at energy E_ρ represents a magnetic moment

$$\mu = \frac{E_\rho}{B_0}, \quad (10.8)$$

which affects the time of flight T from the trap at z_t to the detector at z_d according to

$$T = \int_{z_t}^{z_d} \left(\frac{m}{2} \frac{1}{E_0 - qU(z) - \mu(\omega)B(z)} \right) dz, \quad (10.9)$$

which therefore become minimal for $\omega = \omega_+$, such that the minimum T in a scan of ω reveals the value of ω_+ from which the mass can be determined. This technique is faster than non-destructive methods, which is why it is particularly well-suited to mass determination of short-lived rare isotopes. For a closer look at the concept and the applications, see for example [3, 29–32].

10.5 Non-resonant Excitation: Modulation of the Trap Potential

Instead of a constant trap voltage U_0 , we may apply a modulated trap voltage. This finds a number of applications in motional excitation and detection we will discuss here, and also in particle cooling, see Sect. 12.18.

10.5.1 Slow Modulation

If the voltage modulation is much smaller than the time-averaged voltage, and the frequency of the modulation ω is much smaller than the axial frequency ω_z , this leads to a time-dependent axial frequency given by [33]

$$\omega_z^2(t) = \omega_z^2 (1 + \varepsilon \cos \omega t), \quad (10.10)$$

in which ε is a measure of the relative amplitude of the modulation. Under the above conditions, we have $\varepsilon \ll 1$ and $\omega \ll \omega_z$. The presence of the modulation gives rise to a second-order shift of the fundamental (average) axial frequency given by [33]

$$\omega'_z \approx \omega_z \left(1 + \frac{\varepsilon^2}{16} \right), \quad (10.11)$$

which is mainly due to the non-linearity of the square root in $\omega_z \propto \sqrt{U_0}$. Likewise, we obtain shifted radial frequencies given by [34]

$$\omega'_\pm \approx \omega_\pm \mp \frac{\omega_z^4}{(\omega_+ - \omega_-)^3} \frac{\varepsilon^2}{8}. \quad (10.12)$$

Most importantly, the modulation creates equidistant side bands around the axial frequency with spacing $\omega \ll \omega_z$. We can hence excite the axial motion by irradiation of a side-band frequency. This is of possible advantage in situations when a sensitive detection of the axial frequency must not be obstructed by the excitation signal. The thus required frequency difference can be chosen by choice of the modulation frequency ω . This situation has been discussed in detail in [33]. The technique has first been applied to the excitation and detection of the axial oscillation of a single electron in a Penning trap [35]. It is sometimes called ‘sideband detection’ and also finds application in Penning-trap mass spectrometry [36, 37].

10.5.2 Fast Modulation: Parametric Excitation

A second example of the potential modulation is the so-called ‘parametric oscillator’ as described in [38]. Here, the axial motion of a confined single particle (electron) is ‘parametrically’ excited by a modulation of the trap potential at a frequency

$\omega = 2(\omega_z + \varepsilon)$ which drives a particle response at frequency $\omega_z + \varepsilon$, where $\varepsilon \ll \omega_z$ may be a small detuning. When we take into account the lowest-order field imperfections of octupolar and dodecapolar form (given by expansion parameters C_4 and C_6), then such a driven particle follows an axial equation of motion given by [38]

$$\ddot{z} + \gamma \dot{z} + ((1 + \eta \cos(\omega t)) z + \lambda_4 z^3 + \lambda_6 z^5) \omega_z^2 = 0, \quad (10.13)$$

where $\gamma \ll \omega_z$ is the damping of the axial motion due to a detection / cooling circuit, η is a measure of the drive amplitude, ω again is the drive frequency, and λ_4 and λ_6 are anharmonicity terms given by

$$\lambda_4 = \frac{2C_4}{1 + C_2} \quad \text{and} \quad \lambda_6 = \frac{3C_6}{1 + C_2}. \quad (10.14)$$

The driven change of the axial oscillation amplitude is small unless the strength of the drive reaches a value of $\eta' = 2\gamma/\omega_z$. Then, there is a significant excitation within the frequency band with $\varepsilon_- < \varepsilon < \varepsilon_+$ around ω_z where

$$\varepsilon_{\pm} = \pm \frac{1}{4} \omega_z \sqrt{\eta'^2 - \eta'^2}. \quad (10.15)$$

Parametric excitation has been shown to be highly efficient in creating a driven response in single-particle oscillators, as discussed in detail in [38, 39]. This again is of advantage in situations where particle motions need to be efficiently excited at frequencies that do not coincide with resonant detection frequencies. It has been shown that the resonances of parametrically driven particles can be very narrow, even for large ensembles [40]. This facilitates application to mass spectrometry [41, 42]. The same situation has been studied with particle ensembles in Penning traps for a number of manipulation and observational techniques [43, 44]. There have also been theoretical investigations of parametric excitation by modulation of the magnetic field [45], this however is technically much more demanding than a modulation of the voltages. In general, parametric excitation changes the values of the oscillation frequencies due to the voltage modulation.

A modulation of the trap voltage U_0 by an additional voltage $V \cos \omega t$ between ring and endcaps leads to perturbed oscillation frequencies given by [46]

$$\omega'_{\pm} = \omega_{\pm} \mp \frac{1}{2} \left(\frac{qC_2}{2m\omega_1} \right)^2 \frac{2\omega_1}{4\omega_1^2 + \omega^2} \quad (10.16)$$

$$\omega'_z = \omega_z - \left(\frac{qC_2}{m\omega_z} \right)^2 \frac{\omega_z}{\omega_z^2 + \omega^2}, \quad (10.17)$$

where as before $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$, and only the lowest-order contribution to the frequency shift has been taken into account.

To first order, the shifts of the radial frequencies are symmetric with opposite sign, such that the cyclotron frequency also remains unchanged to first order. This has also been discussed in the frame of a more general perturbation treatment in [12].

10.6 Non-resonant Excitation: Kick and Sidekick

Axial as well as radial motions can be non-resonantly excited by pulses of short duration, often of dipolar or predominantly dipolar geometry. In the axial direction, a pulse of duration $t \ll 1/\omega_z$ applied to one endcap electrode in phase with the motion will increase the axial amplitude and act like a ‘kick’. This can be used for axial ejection from the trap with well-defined particle energy and starting position and can hence be applied in time-of-flight mass spectrometry with Penning traps. Pulses of opposite phase can be used for non-thermal cooling such as in stochastic cooling, see Sect. 12.15.

It has been shown that also the magnetron radius of confined particles and particle ensembles can be increased in a well-controlled way by dipolar pulses of short duration that are applied azimuthally, in similarity to a ‘sidekick’. The obtained orbit has been shown to remain stable and the method can be used in combination with an axial ‘kick’ for off-axis ejection of confined particles [47]. Radial kicks also find application in FT-ICR mass spectrometry, where they can be used in combination with ‘tickle pulses’ to specifically excite certain species to large magnetron orbits where they can be subjected for example to off-axis laser irradiation [48]. Similarly, electric dipole fields of time-varying strength and orientation in the presence of the magnetic field (‘Lorentz steerer’) can be used to prepare desired magnetron orbits for specific applications in ion-cyclotron resonance mass spectrometry [49].

The spectral widths of such ‘kicks’ is of course much broader than the widths achievable in resonant excitation (which are applied for times much longer than one oscillation period) on account of the Fourier limit. This however, is generally not of concern unless filters between the signal source and the trap electrodes block relevant spectral regions.

References

1. L. Schweikhard, A.G. Marshall, Excitation modes for Fourier transform-ion cyclotron resonance mass spectrometry. *J. Am. Soc. Mass Spectrom.* **4**, 433 (1993)
2. X.P. Huang et al., Precise control of the global rotation of strongly coupled ion plasmas in a Penning trap. *Phys. Plasmas* **5**, 1656 (1998)
3. S. Eliseev et al., Octupolar-excitation Penning-trap mass spectrometry for Q-value measurement of double-electron capture in ^{164}Er . *Phys. Rev. Lett.* **107**, 152501 (2011)
4. M. Rosenbusch et al., A study of octupolar excitation for mass-selective centering in Penning traps. *Int. J. Mass. Spectrom.* **314**, 6 (2012)
5. R. Ringle, G. Bollen, P. Schury, S. Schwarz, T. Sun, Octupolar excitation of ion motion in a Penning trap—a study performed at LEBIT. *Int. J. Mass Spectrom.* **262**, 33 (2007)

6. X.P. Huang, J.J. Bollinger, T.B. Mitchell, W.M. Itano, Phase-locked rotation of crystallized non-neutral plasmas by rotating electric fields. *Phys. Rev. Lett.* **80**, 73 (1998)
7. K. Blaum et al., Recent developments at ISOLTRAP: towards a relative mass accuracy of exotic nuclei below 10^{-8} . *J. Phys. B* **36**, 921 (2003)
8. T. Hasegawa, M.J. Jensen, J.J. Bollinger, Stability of a Penning trap with a quadrupole rotating electric field. *Phys. Rev. A* **71**, 023406 (2005)
9. A.G. Marshall, C.L. Hendrickson, G.S. Jackson, Fourier transform ion cyclotron resonance mass spectrometry: a primer. *Mass Spectrom. Rev.* **17**, 1 (1998)
10. S. Guan, A.G. Marshall, Stored waveform inverse Fourier transform (SWIFT) ion excitation in trapped-ion mass spectrometry: theory and applications. *Int. J. Mass Spectrom. Ion Proc.* **157–158**, 5 (1996)
11. S. Guan, R.T. McIver Jr., Optimal phase modulation in stored wave form inverse Fourier transform excitation for Fourier transform mass spectrometry. I. Basic algorithm. *J. Chem. Phys.* **92**, 5841 (1990)
12. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)
13. G.C. Stafford, P.E. Kelley, J.E.P. Syka, W.E. Reynolds, J.F.J. Todd, Recent improvements in and analytical applications of advanced ion trap technology. *Int. J. Mass Spectrom. Ion Proc.* **60**, 85 (1984)
14. L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers Inc, New York, 1956)
15. U. Rieth et al., Ion-molecule reactions of Ru⁺ and Os⁺ with oxygen in a Penning trap. *Radiochim. Acta* **90**, 337 (2002)
16. S. Krückeberg et al., Multiple-collision induced dissociation of trapped silver clusters. *J. Chem. Phys.* **110**, 7216 (1999)
17. B. Wolf, *Handbook of Ion Sources* (CRC Press, Boca Raton, ISBN 978-0-8493-2502-1, 1995)
18. H. Koivisto, M. Nurmia, Metal ion beams from an ECR ion source using volatile compounds. *Nucl. Inst. Meth. B* **94**, 291 (1994)
19. J.M. Wells, S.A. McLuckey, Collision-induced dissociation (CID) of peptides and proteins. *Methods Enzymol.* **402**, 148 (2005)
20. J.A. Luine, G.H. Dunne, Ion-molecule reaction probabilities near 10 K. *Astrophys. J.* **299**, L67 (1985)
21. R. Loch, R. Stengler, G. Werth, Th Zenker, Spin-dependence of low-energy charge exchange between H₂⁺ and Na⁺. *Z. Phys. D* **7**, 189 (1987)
22. F.L. Walls, G.H. Dunn, Storing ions for collision studies. *Phys. Today* **27**, 30 (1974)
23. F.L. Walls, G.H. Dunn, Measurement of total cross section for electron recombination with NO⁺ and O₂⁺ using ion storage techniques. *J. Geophys. Res.* **79**, 1911 (1974)
24. M. Vogel, K. Hansen, A. Herlert, L. Schweikhard, Model-free determination of dissociation energies of polyatomic systems. *Phys. Rev. Lett.* **87**, 013401 (2001)
25. A. Herlert, L. Schweikhard, M. Vogel, Photoinduced dissociation of anionic and electron detachment of dianionic gold clusters by use of a laser pointer. *Int. J. Mass Spectr.* **213**, 157 (2002)
26. C. Walther, A. Herlert, J.I. Kim, F.J. Scherbaum, L. Schweikhard, M. Vogel, Absolute cross sections for the non-resonant multiphoton ionization of toluene in the gas phase. *Chem. Phys.* **265**, 243 (2001)
27. H. Weidele, M. Vogel, A. Herlert, S. Krückeberg, P. Lievens, R.E. Liverans, L. Schweikhard, C. Walther, Decay pathways of stored metal cluster anions after collisional activation. *Eur. Phys. J. D* **9**, 173 (1999)
28. J.R. Eyler, Infrared multiple photon dissociation spectroscopy of ions in Penning traps. *Mass Spectrom. Rev.* **28**, 448 (2009)
29. G. Bollen et al., The accuracy of heavy-ion mass measurements using time of flight-ion cyclotron-resonance in a Penning trap. *J. Appl. Phys.* **68**, 4355 (1990)
30. M. König et al., Quadrupole excitation of stored ion motion at the true cyclotron frequency. *Int. J. Mass Spectr.* **142**, 95 (1995)
31. G. Gräff, H. Kalinowsky, J. Traut, A direct determination of the proton-electron mass ratio. *J. Appl. Phys.* **68**, 4355 (1990)

32. S. Eliseev et al., Phase-imaging ion-cyclotron-resonance measurements for short-lived nuclides. *Phys. Rev. Lett.* **110**, 082501 (2013)
33. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
34. G.J. Ketter, Theoretical Treatment of Miscellaneous Frequency-Shifts in Penning Traps with Classical Perturbation Theory, Ph.D. thesis (University of Heidelberg, 2015)
35. D. Wineland, P. Ekstrom, H. Dehmelt, Monoelectron oscillator. *Phys. Rev. Lett.* **31**, 1279 (1973)
36. E.G. Myers, The most precise atomic mass measurements in Penning traps. *Int. J. Mass Spectrom.* **349–350**, 107 (2013)
37. R.S. Van Dyck, S.L. Zafonte, P.B. Schwinberg, in *Atomic Physics at Accelerators: Mass Spectrometry*, ed. by D. Lunney, G. Audi, H.J. Kluge. Ultra-precise mass measurements using the UW-PTMS (Springer, Dordrecht, 2001)
38. C.H. Tseng, D. Enzer, G. Gabrielse, F.L. Walls, 1-bit memory using one electron: parametric oscillations in a Penning trap. *Phys. Rev. A* **59**, 2094 (1999)
39. L.J. Lapidus, D. Enzer, G. Gabrielse, Stochastic phase-switching of a parametrically-driven electron in a Penning trap. *Phys. Rev. Lett.* **83**, 899 (1999)
40. P. Bushev et al., Electrons in a cryogenic planar Penning trap and experimental challenges for quantum processing. *Eur. Phys. J. D* **50**, 97 (2008)
41. L. Schweikhard, M. Lindinger, H.-J. Kluge, Parametric mode excitation/dipole mode detection Fourier transform ion cyclotron resonance spectrometry. *Rev. Sci. Instr.* **61**, 1055 (1990)
42. D.L. Rempel, E.B. Ledford, S.K. Huang, M.L. Gross, Parametric mode operation of a hyperbolic Penning trap for Fourier transform mass spectrometry. *Anal. Chem.* **59**, 2527 (1987)
43. J. Tan, G. Gabrielse, Synchronization of parametrically pumped electron oscillators with phase bistability. *Phys. Rev. Lett.* **67**, 3090 (1991)
44. J. Tan, G. Gabrielse, Parametrically pumped electron oscillators. *Phys. Rev. A* **48**, 3105 (1993)
45. N.D. Naumov, YuG Pavlenko, Parametric excitation of oscillations of charged particle bunches in a Penning trap. *Tech. Phys.* **42**, 22 (1997)
46. M. Kretzschmar, Single particle motion in a Penning trap: description in the classical canonical formalism. *Physica Scripta* **46**, 544 (1992)
47. T. Mortensen, A. Deller, A. Isaac, P. van der Werf, M. Charlton, J.R. Machacek, Manipulation of the magnetron orbit of a positron cloud in a Penning trap. *Phys. Plasmas* **20**, 012124 (2013)
48. S.A. Hofstadler, K.A. Sannes-Lowery, H.R. Griffey, *m/z*-selective infrared multiphoton dissociation in a Penning trap using sidekick trapping and an rf-tickle pulse. *Rapid Commun. Mass Spectrom.* **15**, 945 (2001)
49. R. Ringle et al., A “Lorentz” steerer for ion injection into a Penning trap, *Int. J. Mass Spectrom.* **263**, 38 (2007)

Chapter 11

Resonant Motional Coupling



Abstract Particle oscillations in different motional degrees of freedom in the Penning trap can purposefully be resonantly coupled by suitable electromagnetic radiation. Here, we briefly discuss the phenomenology and the main applications.

11.1 General Discussion of Resonant Coupling

When two of the independent oscillations of a particle in a Penning trap are coupled by irradiation of a suited electric radio-frequency field, the corresponding motions are amplitude-modulated by the interaction with the external field [1]. The equations of motion of those coupled oscillations can formally be described in analogy to a driven quantum-mechanical two-level system [2]. If the external radio frequency is identical to a sideband frequency (e.g. the difference or sum frequency) of the two oscillations, the coupling is resonant. It is based on the fact that an oscillator that simultaneously performs two motions at frequencies ω_A and ω_B can resonantly be excited at frequencies ω_A , ω_B and $\omega_A \pm \omega_B$. In a Penning trap, this is sometimes called ‘parametric coupling’ because it is achieved by a radio-frequency modulation upon the confining field.

Let us look at the active coupling of the reduced cyclotron and axial motions by irradiation of an electrical field at frequency $\omega_D = \omega_+ - \omega_z$, which has the form

$$\mathbf{E}(\mathbf{r}, t) = \text{Re} \left(E_D \exp(i\omega_D t) (z\mathbf{e}_\rho - \rho\mathbf{e}_z) \right), \quad (11.1)$$

where E_D is the amplitude of the irradiated field. This field geometry is obtained by use of a radially segmented electrode around $z = 0$. On resonance, the coupling leads to an energy transfer between the two motions which we express as an amplitude modulation of the axial motion of the form

$$\begin{aligned} z(t) &= z_0 \cos\left(\frac{\Omega}{2}t\right) \sin(\omega_z t) \\ &= z_0 \left[\sin\left(\left(\omega_z + \frac{\Omega}{2}\right)t\right) + \sin\left(\left(\omega_z - \frac{\Omega}{2}\right)t\right) \right]. \end{aligned} \quad (11.2)$$

Here, Ω is the on-resonance Rabi frequency (after I.I. Rabi, 1898–1988) which depends on the field amplitude E_D . The frequency spectrum of the coupled motion has two components at frequencies

$$\omega_L = \omega_z - \frac{\Omega}{2} \quad \text{and} \quad \omega_R = \omega_z + \frac{\Omega}{2}, \quad (11.3)$$

the so-called ‘classical dressed states’. This is the minimum splitting achievable for active sideband coupling.

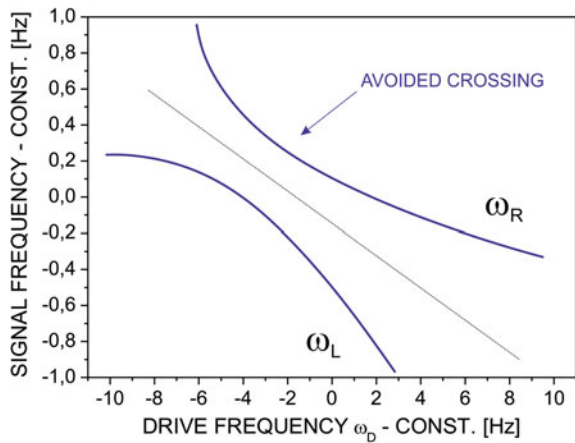
If, however, the radio-frequency drive frequency ω_D is detuned by $\delta_- = \omega_D - (\omega_z + \omega_-)$ or $\delta_+ = \omega_D - (\omega_+ - \omega_z)$, respectively, the resulting frequencies are given by

$$\omega_L = \omega_z - \frac{\delta_{\pm}}{2} - \frac{\Omega_{\pm}}{2} \quad \text{and} \quad \omega_R = \omega_z - \frac{\delta_{\pm}}{2} + \frac{\Omega_{\pm}}{2}, \quad (11.4)$$

where $\Omega_{\pm} = \sqrt{\Omega^2 + \delta_{\pm}^2}$ is the off-resonant Rabi frequency. By measuring ω_L and ω_R as a function of the drive frequency ω_D , a ‘classical avoided crossing’ is observed [3], as schematically depicted in Fig. 11.1.

When the amplitude of the drive and its duration are chosen such that exactly one complete cycle from axial to cyclotron motion or vice versa is induced, this is called a ‘ π -pulse’. This usually requires fine tuning and precise knowledge of the frequencies involved in the coupling. An imprecise knowledge of the coupling frequency can be overcome by a sweep of the drive across the coupling resonance. It has been found that over a wide range of coupling amplitudes E_D and sweep rates, a frequency sweep leads to a complete transfer between the coupled modes [4]. It hence effectively acts on the motions like a π -pulse, without the need for careful tuning. This behaviour is the classical analogue of rapid adiabatic passage [5].

Fig. 11.1 Illustration of avoided crossing of the two signals at ω_L and ω_R as a function of the coupling frequency ω_D



11.2 Simultaneous Determination of Oscillation Frequencies

Note that the above equations include three unknown quantities: the Rabi frequency $\Omega_{\pm}$, the frequency of the radial mode $\omega_{\pm}$, and the axial frequency ω_z . By recording such a spectrum with two dips and a measurement of the axial frequency ω_z , the respective radial frequency can be extracted by solving this system of equations.

For zero detuning, i.e. when the coupling frequency is identical to the sideband frequency, the frequency splitting is minimal. In case of symmetric splitting the widths of the two resonances at ω_L and ω_R are identical.

By use of a time-dependent or modulated coupling frequency, the resulting spectrum contains not only the two split components of a certain frequency, but also the frequency itself. This is shown in Fig. 11.2 as the additional appearance of the original dip at ω_z .

The split components can again be coupled to the third motion such that five signals appear in the spectrum, see Fig. 11.2 (right). From these, all three frequencies ω_+ , ω_z and ω_- can be determined simultaneously, as has been demonstrated for a single confined proton [3, 6] and for a single confined highly charged ion [7].

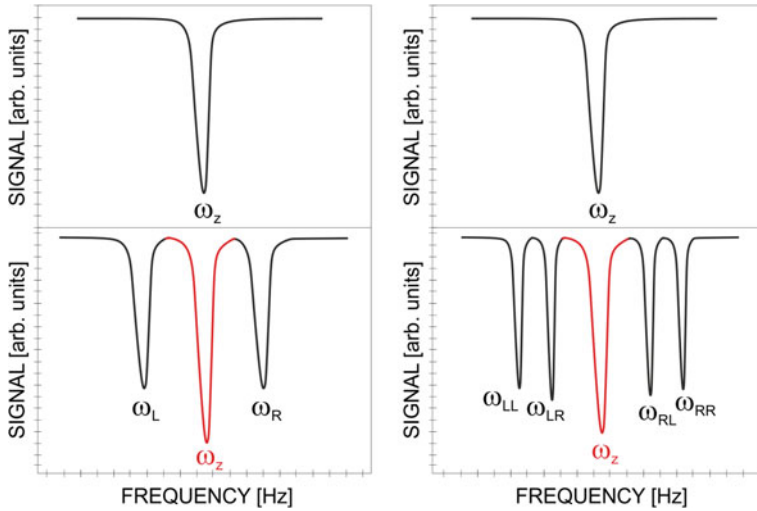


Fig. 11.2 Illustration of the spectral result of active motional coupling. Left: the dip at frequency ω_z (top) splits into two components at ω_L and ω_R (bottom). The original dip at ω_z also appears in the spectrum if the coupling is modulated. Right: same situation when two coupling frequencies are used simultaneously

11.3 Cooling Via Motional Coupling

Another feature of sideband coupling can be used for cooling of a lower-frequency degree of freedom below the ambient temperature when it is coupled to a cooled higher-frequency degree of freedom. Typically, in the case of a coupling between the perturbed cyclotron motion and the axial motion, this is achieved by irradiation of the sum frequency $\omega_+ + \omega_z$. Such coupling leads to an equalization of the quantum numbers of the two oscillations [8]. Let the higher-frequency motion at ω_+ be cooled for example to liquid-helium temperature of $T_+ \approx 4$ K. The sideband-coupling cooling limit for the axial temperature T_z is then given by

$$T_z = \frac{\omega_z}{\omega_+} T_+ \approx \frac{1}{B_0 d} \sqrt{\frac{m U_0}{q}} T_+. \quad (11.5)$$

For heavy few-electron ions in typical traps, the frequency ratio ω_z/ω_+ is of order 10^{-2} – 10^{-1} , such that for the axial temperature values of the order of 100 mK are achieved. Not only is the final temperature low, but also the cooling time constant is small even for weak coupling. Sideband coupling leads to an exponential energy loss in full analogy to the resistive cooling discussed in Sect. 12.12.1, however with a time constant τ_c which in case of weak resonant coupling is given by

$$\tau_c^{-1} = 4\pi\epsilon_0 \frac{3ma^2 c^3 \omega_z^3}{16q^2 \omega_c^2 (\omega_+ - \omega_-)} \approx 4\pi\epsilon_0 \frac{3a^2 c^3}{16B_0^3} \frac{U_0^{3/2}}{d^3} \sqrt{\frac{m^5}{q^7}}, \quad (11.6)$$

where $a = \eta U_d / U_0$ is a dimensionless constant of order 10^{-4} given by a trap geometry factor η and the amplitude U_d of the coupling drive [8]. For heavy few-electron ions confined at typical trap parameters, the cooling time constant τ_c is commonly below seconds. Thus, if the coupling is sufficiently effective, the cooling of the axial motion at ω_z is limited by the resistive cooling time constant τ_+ of the radial motion (as given by 12.66) rather than by the time constant τ_c itself.

11.4 Magnetron Centring

As we have discussed in Sect. 5.3.3, the amplitude (radius) of the magnetron drift motion cannot be reduced by direct energy dissipation, but rather needs to be resonantly coupled to a second motion that is cooled in some way. This second motion can either be the axial motion or the perturbed cyclotron motion. Here we focus on the latter case as an example. This coupling is achieved by a radio-frequency quadrupolar field of amplitude U irradiated at the upper magnetron sideband frequency of the reduced cyclotron motion, i.e. at a frequency $\omega = \omega_+ + \omega_- = \omega_c$. The coupling represents a motional interconversion such that the cooling of the reduced cyclotron

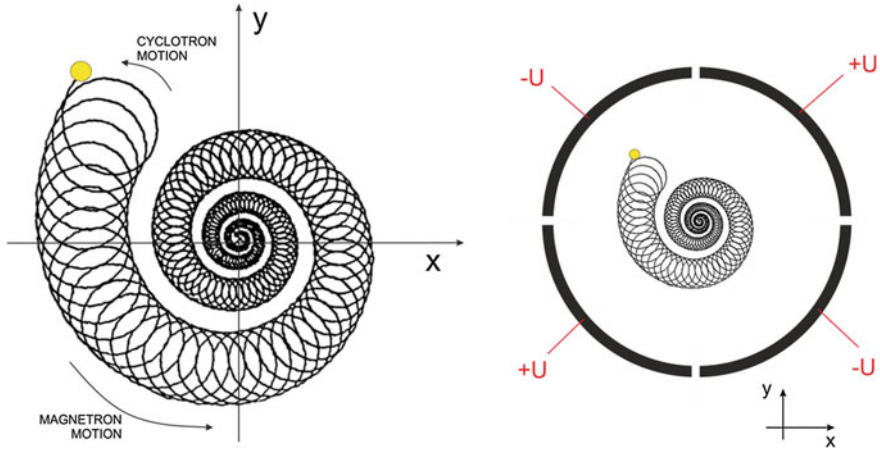


Fig. 11.3 Phenomenology of magnetron centring by resonant coupling to the cooled cyclotron motion

energy (and hence decrease of its radius) is translated into a decrease of the magnetron radius as well, forcing the particle uphill in the electrostatic trap potential (Fig. 11.3). This means that the (negative) potential energy of the magnetron motion is increased in magnitude, while the magnetron kinetic energy is decreased. At the same time, both kinetic and potential energy of the reduced cyclotron motion are decreased by the outer cooling mechanism. Since the radial particle motion is decreased in radius and thus the particle radially closes in on the trap axis, the technique is sometimes called ‘axialisation’. This has been shown for example with buffer gas cooling of the reduced cyclotron motion [9, 10], with resistive cooling of the reduced cyclotron motion [11], in combination with radial laser cooling [12, 13], and with coupling to the cooled axial motion [14].

In the absence of a cooling mechanism, the coupling would lead to an ongoing energy conversion from one motion to the other and back. Let us remind ourselves of the total instantaneous radial kinetic energy given by

$$E_{\rho}(t) = \frac{1}{2}m (\rho_{+}(t)\omega_{+} + \rho_{-}(t)\omega_{-}) . \quad (11.7)$$

Assuming the initial perturbed cyclotron and magnetron radii to be $\rho_{+} = 0$ and $\rho_{-} > 0$, respectively, the initial energy is given by

$$E_{\rho}(t = 0) = \frac{1}{2}m (\rho_{-}\omega_{-}) . \quad (11.8)$$

The time T_C for one complete conversion to $\rho_{-}(T_C) = 0$ is given by

$$T_C = \frac{1}{U}(\omega_+ - \omega_-) \frac{m\rho_-^2}{q} \approx \frac{B_0\rho_-^2}{U}. \quad (11.9)$$

As an example, for an initial magnetron radius ρ_- of 1 mm in a magnetic field B_0 of 6 T, an amplitude of $U = 1$ mV leads to a time T_C of about 1/4 s. Note, that the conversion time is independent of the mass-to-charge ratio of the particle.

When the coupling is not resonant, i.e. when we have $\omega \neq \omega_c$, the motions will not be completely converted after one conversion time T_C as predicted from (11.9). This means that the energy difference between the motions will not be the total initial energy as given by

$$\Delta E_\rho(T_C) = \frac{1}{2}m(\rho_-(t=0)\omega_-), \quad (11.10)$$

but will instead be given by

$$\Delta E_\rho(T_C) = \frac{1}{2}m(\rho_+(T_C)\omega_+ - \rho_-(T_C)\omega_-), \quad (11.11)$$

which can be expressed as

$$\Delta E_\rho(T_C) = \frac{q^2 U^2}{2m\rho_-^4} \frac{\omega_+^2}{(\omega_+ - \omega_-)^2} \frac{\sin^2(\omega T_C)}{\omega_c^2}, \quad (11.12)$$

in which ω_c is given by

$$\omega_c = \frac{1}{2}\sqrt{(\omega - \omega_c)^2 + k^2} \quad \text{with} \quad k = \frac{2qU}{md^2} \frac{1}{\omega_+ - \omega_-}. \quad (11.13)$$

When the perturbed cyclotron motion is cooled at a rate $\gamma < 1/T_C$, one observes a reduction of both the radial amplitudes at a time constant given by the cooling. Hence, we have a reduction of the magnetron radius according to

$$\rho_\pm(t) = \rho_\pm(t=0) \exp\left(-\frac{\omega_\pm}{\omega_+ - \omega_-} \gamma t\right). \quad (11.14)$$

Note that this technique is frequency-selective, such that out of a mixed ensemble of particles with different m/q , a specific particle species can be magnetron-centred. This has applications in Penning-trap mass spectrometry, see for example [15, 16].

References

1. E.A. Cornell, R.M. Weisskoff, K.R. Boyce, D.E. Pritchard, Mode coupling in a Penning trap: π pulses and a classical avoided crossing. *Phys. Rev. A* **41**, 312 (1990)
2. M. Kretzschmar, in *Trapped Charged Particles and Fundamental Physics*, ed. by D.H.E. Dubin, D. Schneider, (AIP Press, 1999)
3. S. Ulmer, C. Smorra, The magnetic moments of the proton and the antiproton, in *Fundamental Physics in Particle Traps*, Springer tracts in modern physics **256** (Springer, 2014)
4. M.J. Höcker, Precision Mass Measurements at THE-Trap and the FSU trap, Ph.D. thesis (University of Heidelberg, 2016)
5. B.W. Shore, M.V. Gromovyy, L.P. Yatsenko, V.I. Romanenko, Simple mechanical analogs of rapid adiabatic passage in atomic physics. *Am. J. Phys.* **77**, 1183 (2009)
6. S. Ulmer et al., Direct measurement of the free cyclotron frequency of a single particle in a Penning trap. *Phys. Rev. Lett.* **107**, 103002 (2011)
7. J. Verdú et al., Determination of the g -factor of single hydrogen-like ions by mode coupling in a Penning trap. *Physica Scripta* **T112**, 68 (2004)
8. D.J. Wineland, H.G. Dehmelt, Principles of the stored ion calorimeter. *J. Appl. Phys.* **46**, 919 (1975)
9. G. Savard et al., A new cooling technique for heavy ions in a Penning trap. *Phys. Lett. A* **158**, 247 (1991)
10. Ch. Lichtenberg, G. Marx, G. Tommaseo, P.N. Ghosh, G. Werth, Axialisation, cooling and quenching of Ba^+ ions in a Penning trap. *Eur. Phys. J. D* **2**, 29 (1998)
11. K. Blaum et al., Penning traps as a versatile tool for precise experiments in fundamental physics. *Contemp. Phys.* **51**, 149 (2010)
12. H.F. Powell, D.M. Segal, R.C. Thompson, Axialization of laser cooled magnesium ions in a Penning trap. *Phys. Rev. Lett.* **89**, 093003 (2002)
13. H.F. Powell, S.R. de Echaniz, E.S. Phillips, D.M. Segal, R.C. Thompson, Improvement of laser cooling of ions in a Penning trap by use of the axialization technique. *J. Phys. B.* **36**, 1 (2003)
14. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
15. A.G. Marshall, S. Guan, Observation, manipulation, and uses for magnetron motion in ion cyclotron resonance mass spectrometry. *Physica Scripta* **T59**, 155 (1995)
16. N.K. Kaiser, J.E. Bruce, Reduction of ion magnetron motion and space charge using radial electric field modulation. *Int. J. Mass Spectrom.* **265**, 271 (2007)

Chapter 12

Motional Cooling in Penning Traps



Abstract The possibility to cool the motions of confined particles is one key motivation for the use of Penning traps, particularly in precision spectroscopy in any frequency domain. Here, we discuss the notion of a particle temperature, its measurement in different experimental situations, and review the most important cooling techniques applied in Penning traps.

12.1 The Meaning of Particle Cooling in a Penning Trap

Motional cooling means a reduction of the particle's kinetic energy and hence a reduction of the amplitude of oscillation. This is true for the axial and perturbed cyclotron motions. For the magnetron motion, a more careful discussion is in place, see Sect. 5.3.3.

A multitude of cooling techniques for confined particles have been developed, one overview is given in [1]. The most common techniques used in Penning traps are resistive cooling [1–3] and several forms of laser cooling [4–6], which either require conducting surfaces close to the ions, good optical access, or both.

In an ideal Penning trap, i.e. for vanishing inhomogeneities and anharmonicities, there is no energy transfer between the motional degrees of freedom of a single particle. Hence, the oscillation energies (amplitudes) are independent and different temperatures can be assigned to the different motional degrees of the particle [7]. This also means that motional degrees of freedom can be cooled or excited individually. In the absence of imperfections, the motions are harmonic and thus the oscillation frequencies are independent of the motional energies.

12.2 The Notion of Particle Temperature

The concept of temperature is strictly defined only for infinite ensembles, usually it is applied also to sufficiently large ensembles of particles and one needs to examine the meaning of 'sufficiently' from case to case. For a single isolated particle it is,

in fact, meaningless. Under certain conditions, the concept of temperature can be given a meaning also with individual particles, when they are in equilibrium with a heat bath of sufficient size. It has, for example, been successfully applied to a single electron confined in a Penning trap [8], to a single highly charged ion in a Penning trap [7], and to a single proton in a Penning trap [9]. Generally, in the case of a thermal equilibrium with the environment at a temperature T , if one measures the motional energy E of an individual particle repeatedly, the probability of obtaining a certain value for the energy follows an exponential $\exp(-E/k_B T)$, where k_B is the Boltzmann constant. It thus represents a Boltzmann distribution which is characterized by a temperature. However, this is possible only in ergodic systems, where the result of a measurement with an ensemble of particles, if averaged, yields the same result as given by several consecutive measurements of the same observable with a single particle, time-averaged [10]. This is commonly the case for particles in a trap in thermal equilibrium with a macroscopic heat bath. An example is shown in Fig. 12.1. The particle energy has been repeatedly measured and the number of occurrences of a certain value has been plotted as a function of that energy. A Boltzmann fit reveals a corresponding temperature of about 5 K. In a situation where the particle is in resonance with a RLC-circuit that couples to it with a cooling rate γ , we expect the instantaneous energy of the particle to fluctuate at that rate between all values allowed by the Boltzmann distribution that corresponds to the charge carrier temperature of the circuit and the trap electrodes.

The amplitude of a particle oscillation is a direct measure for the corresponding kinetic energy. In an ideal Penning trap, the amplitudes of the three oscillations are independent and therefore every degree of freedom of the particle can independently

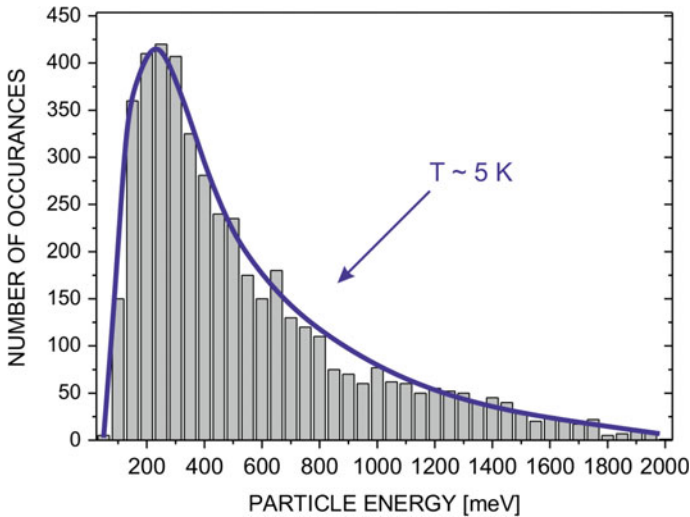


Fig. 12.1 Binned plot of repeated particle energy measurements leading to a Boltzmann distribution corresponding to a particle temperature of about 5 K. Data from [7]

be attributed with a different temperature. Field imperfections lift this degeneracy to a certain degree, however not necessarily strong enough to compensate for the different energy dissipations into the system, e.g., due to different external noise levels in the respective resonance circuits. Therefore, even in an imperfect trap, different thermal equilibria can develop in different motional degrees of freedom of the particle. Note, that it is the charge carrier temperature of the electrodes and the electronics that is predominantly relevant for the equilibrium temperature of confined particles. The association of a temperature to the magnetron degree of freedom is possible only when the particle can be in a thermal equilibrium with its surrounding and still be stored. This will commonly not be the case, since the radius of the magnetron motion increases with decreasing energy and in a cryogenic surrounding, the corresponding radius can be much larger than the trap, leading to particle loss.

12.3 Particle Temperature Measurement

Let us have a brief look at possibilities to measure the temperature of a confined particle or particle ensemble.

12.3.1 Electronic Noise Temperature Measurement

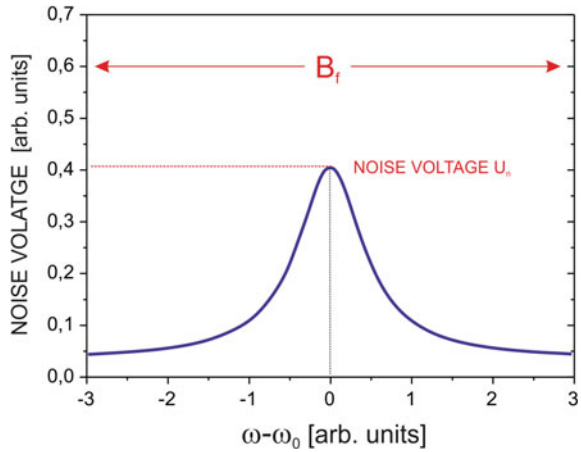
Assuming an equilibrium between the noise temperature of a resonance circuit attached to the trap and the particles confined in the trap, a value for the particle or particle ensemble temperature can be obtained by a measurement of the noise temperature of the electronics, as has been shown in [7]. To that end, the Johnson noise level voltage U_n (after J. B. Johnson, 1887–1970) is measured by use of a spectrum analyser. The relevant quantities are shown in Fig. 12.2: the on-resonance noise voltage is measured within a bandwidth B_f . An absolute value of the noise level can be calculated by use of the corresponding attenuation factor between the resonance circuit and the spectrum analyser. This number (the ‘transfer function’) needs to be determined by a combination of measured attenuation factors and calculations based on the known electronic properties of the relevant components.

The expression that relates the measured Johnson noise level U_n to the temperature T can be solved to give

$$T = \frac{U_n^2}{4k_B B_f R} \quad (12.1)$$

where B_f is the observation bandwidth, and R is the resonance resistance of the circuit.

Fig. 12.2 Quantities used in a direct particle temperature measurement by the on-resonance noise voltage U_n when the spectral noise voltage across a resonant detection circuit is recorded within a bandwidth B_f



12.3.2 Fluorescence Lineshape Temperature Measurement

Confined particles that are irradiated by a laser which resonantly drives an observable electromagnetic transition yield information about the particle temperature encoded in the line shape of the observed fluorescence. This is possible in a straight-forward fashion in the Doppler regime, where the particle velocity distribution is still large enough to produce a Doppler-broadened line. In the Lamb-Dicke regime, where side bands are resolved, this method does not apply.

The line shape of the observed transition contains the natural lifetime, and broadening contributions from collisional broadening, the laser linewidth itself, but in particular a temperature-dependent Doppler broadening contribution. The observed line shape then is a convolution of the natural Lorentz shape of the general form

$$L(x, \gamma) = \frac{x^2}{\pi (x^2 + \Gamma^2)}, \quad (12.2)$$

where Γ is the characteristic width of the distribution, and the Gaussian Doppler contribution of the general form

$$G(x, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right). \quad (12.3)$$

This convolution is called ‘Voigt line shape’, named after W. Voigt (1850–1919). It is commonly expressed by

$$I(\omega - \omega_0) = I_0 \frac{1}{4\pi \sqrt{\ln 2}} \Gamma \delta\omega \int_{-\infty}^{\infty} \frac{1}{(\omega - \omega_0 - kv\eta)^2 + \Gamma^2/4} \exp(-\eta) d\eta, \quad (12.4)$$

in which the velocity v is the temperature-sensitive part. ω_0 and Γ are the resonance frequency and the line width of the transition, respectively, and $\delta\omega$ is the full Doppler width. The temperature T is encoded in the velocity v by

$$v^2 = \frac{2k_B T}{m}, \quad (12.5)$$

such that it can be determined from a fit of the measured line shape, as for example shown in [11, 12].

12.3.3 Magnetic Bottle Temperature Measurement

In particular for a single particle, the kinetic oscillation energy can be measured from the frequency shift in a magnetic bottle of known strength. The value of the expected shift as a function of energy is given by the equations discussed in Sect. 6.3. As a straight-forward example, we can have a look at the shift $\Delta\omega_z$ of the axial frequency of a $^{12}\text{C}^{5+}$ ion in a magnetic field of $B_0 = 3.8 \text{ T}$ in the presence of a magnetic bottle of strength $B_2 = 8.2 \text{ mT/mm}^2$. It reveals the energy of the perturbed cyclotron motion by

$$E_+ = m\omega_z \frac{B_0}{B_2} \Delta\omega_z, \quad (12.6)$$

which for the present case is 5 Hz per meV [7]. In principle, all the dependences in Sect. 6.3 can be used for this kind of measurement, in practice, however, we will usually choose the ones with the highest coupling coefficient as given by the matrix in (6.55). The temperature then follows from the distribution of E_+ as determined from a repeated measurement of $\Delta\omega_z$

$$T_+ = \frac{\langle E_+ \rangle}{k_B} = \frac{1}{k_B} m\omega_z \frac{B_0}{B_2} \langle \Delta\omega_z \rangle. \quad (12.7)$$

Details of the method and its application are given in [7].

12.3.4 Active Motional Coupling Temperature Measurement

There are cases where a measurement as described in the previous section is not possible, for example when the axial energy is to be measured, and we can only measure the axial frequency. From (6.55) we learn that the coupling coefficient in that case is zero, i.e. we have $\Delta\omega_z(E_z) = 0$ and a measurement as described fails. However, by active coupling of axial and radial motions as described in Chap. 11, this problem can be circumvented. We look at a coupling of the cyclotron motion to

the axial motion by a quadrupolar radio-frequency field at the frequency $\omega_+ - \omega_z$ applied in the radial plane of the trap. This coupling makes the quantum numbers of both oscillations identical. For a detailed description of this effect, see [13]. Then we have the relation

$$\frac{E_+}{\omega_+} = \frac{E_z}{\omega_z} \quad (12.8)$$

between the cyclotron and axial energies. This can be used to determine the axial energy from a measurement of the cyclotron energy. When the cooling time constant of the axial motion is much smaller than the cyclotron cooling time constant, the axial energy fixes the cyclotron energy. The axial motion is first cooled to equilibrium with the resistive cooling circuit. To ensure the equivalence of the quantum numbers the sideband coupling is performed during a time much longer than the energy exchange time between the motions. The cyclotron energy is measured as described in the previous section by measuring the axial frequency in a magnetic bottle. For a repeated measurement of E_+ we obtain the axial temperature by

$$T_z = \frac{\langle E_z \rangle}{k_B} = \frac{\langle E_+ \rangle}{k_B} \frac{\omega_z}{\omega_+}. \quad (12.9)$$

This method has been demonstrated for example in [7, 14].

12.3.5 *Passive Motional Coupling Temperature Measurement*

A passive technique for an axial temperature measurement makes use of the fact that the electrostatic potential can be purposefully chosen not to be harmonic. Then, in contrast to the above case, the axial frequency will depend on the axial energy and the line shape of the particle's resonance signal will be asymmetric.

In such a case, the electric potential U along the symmetry axis of the trap has a non-vanishing octupole term C_4 in the expansion of the potential

$$U(z, 0) = \sum_{j=0}^{\infty} C_j z^j; \quad C_j = \frac{1}{j!} \left. \frac{\partial^j U}{\partial z^j} \right|_{(0,0)} \quad (12.10)$$

The harmonicity of the potential U can be influenced by the choice of the voltages applied to the correction electrodes of the Penning trap. This is expressed in terms of the so-called ‘tuning ratio’ U_c/U_0 , which is the ratio of the voltages applied to the correction electrodes and the ring electrode. The term C_4 is related to the difference between the optimal tuning ratio (the one corresponding to a harmonic trap potential) and the one actually applied. For a trap geometry as described in Sect. 3.3, this relation is given by [7, 15]

$$C_4 = \frac{1}{4!} \cdot \sum_{m=1,3,5}^{\infty} \left(\frac{m\pi}{L} \right)^4 \left(A_m + B_m \frac{U_c}{U_0} \right), \quad (12.11)$$

where L is the length of the cylindrical trap and A_m and B_m are tuning ratio parameters which are calculated by use of the given trap geometry [7, 15]. The dependence of the axial frequency on the axial energy can be written as

$$\omega_z = \omega_{z,E_z=0} + \beta E_z \quad (12.12)$$

where $\omega_{z,E_z=0}$ is the axial frequency for vanishing axial energy. β is given by

$$\beta = \frac{3\omega_z}{8\pi q U} \frac{C_4}{C_2^2} \quad (12.13)$$

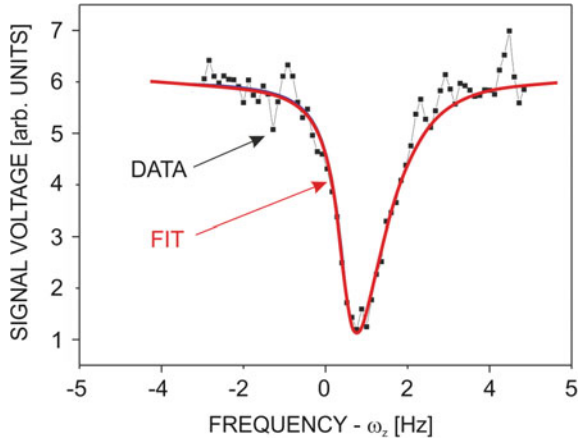
and describes the dependence of the axial frequency ω_z on the axial energy E_z in terms of the determined C_4 .

By choice of a non-optimum tuning ratio and measurement of the axial resonance signal of the particle, the motional temperature can be extracted from the line shape of the resonance signal. To that end, the line shape for a fixed axial energy E_z of the ion is convoluted with a Boltzmann distribution corresponding to that energy. Thus, we start with a Lorentzian line shape as given by

$$U(\omega_z) \propto \sqrt{\frac{(\omega_{z,r} - \omega_z)^2}{\gamma^2/4 + (\omega_{z,r} - \omega_z)^2}} \quad (12.14)$$

where $U(\omega_z)$ is the measured signal voltage and $\omega_{z,r}$ is the resonance frequency of the circuit. γ is the frequency-dependent damping factor due to the impedance $Z(\omega)$ of the detection/cooling system given by

Fig. 12.3 Axial particle temperature determination from a fit of (12.16) to passive motional coupling data [7]



$$\gamma = \frac{q^2}{mD^2} Z(\omega). \quad (12.15)$$

The convolution of equation (12.14) with a Boltzmann distribution yields

$$U(\omega_z) \propto \int_0^\infty \exp\left(\frac{-E_z}{k_B T}\right) \sqrt{\frac{(\omega_{z,r} - (\omega_z + \beta E_z))^2}{\gamma^2/4 + (\omega_{z,r} - (\omega_z + \beta E_z))^2}} dE_z. \quad (12.16)$$

Thus, by numerical inversion of equation (12.16), a value for the axial energy and hence for the axial temperature can be found from the measured axial resonance curve at a fixed trap detuning (Fig. 12.3).

12.4 Laser Doppler Cooling

Laser cooling of confined ions has been demonstrated in the late 1970s and early 1980s for ion clouds in Paul (radio-frequency) traps [16, 17] and in Penning traps [18]. Laser cooling of single confined ions has been demonstrated successfully by [19, 20]. In those measurements, a single ion has been laser-cooled even to the quantum-mechanical ground state of the ion motion in the trap [21]. There is a broad variety of laser cooling techniques for confined particles, see for example [6, 22]. Here, we briefly review laser Doppler cooling, have a look at the situation with resolved sidebands, and discuss its combination with buffer gas cooling. For more involved techniques that reach into lower-energy domains such as polarisation gradient cooling (Sisyphus cooling), stimulated Raman cooling, or EIT-cooling we give only brief indications and refer to more thorough treatments.

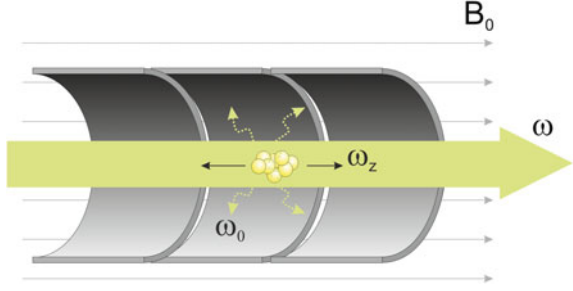
Laser Doppler cooling is an effective tool to reduce the temperature of confined ions, particularly from temperatures of up to several thousands of kelvin down to the Doppler limit, which is commonly in the mK range [1, 6, 23, 24]. For ions such as for example magnesium and calcium, this has been demonstrated under various confinement conditions [16, 19, 25–27].

Let a two-level ion absorb photons from a laser beam along the oscillation axis, see Fig. 12.4. Each absorbed photon gives the ion a momentum in the direction of the laser, while the spontaneously emitted photons on average have no preferred direction. When the incident laser is red-detuned from the transition in use, on average, this results in a scattering force which slows the ion down, proportional to the rate of photon absorption and detuning. This force is given by [28]

$$F = \hbar k \frac{\Gamma}{2} \frac{I/I_s}{1 + I/I_s + 4\delta^2/\Gamma^2}, \quad (12.17)$$

where Γ is the excited-level decay rate (line width), and I_s is the saturation intensity given from the transition wavelength λ_0 by

Fig. 12.4 Cartoon of axial laser cooling in a Penning trap. Light at a frequency ω red-detuned from the laser-cooling transition at frequency ω_0 is irradiated along the trap axis



$$I_s = \frac{2\pi^2 \hbar \Gamma c}{3\lambda_0^3}. \quad (12.18)$$

The detuning of the laser is given by $\delta = \omega_0 - \omega - kv$ at the particle wave number k and velocity (in the direction of the photon) v . Here, ω_0 is the transition frequency, ω is the actual laser frequency and kv is the Doppler shift. For $I \rightarrow \infty$, the scattering force converges to the limit

$$F = \hbar k \frac{\Gamma}{2}, \quad (12.19)$$

where the factor 1/2 reflects the fact that for a two-level ion at large intensities the upper and the lower levels of the cooling transition are equally populated. The final thermal energy of the particle is given by [28]

$$k_B T = \frac{\hbar \Gamma}{4} \frac{1 + \frac{4\delta^2}{\Gamma^2}}{-\frac{2\delta}{\Gamma}}. \quad (12.20)$$

The final temperature assumes a minimum for a laser detuning of $\delta = \Gamma/2$, the so-called ‘Doppler cooling limit’

$$k_B T = \frac{\hbar \Gamma}{2}, \quad (12.21)$$

which for ions is typically in the single-mK range as the line widths in use are typically on the scale of many MHz. The particular influence of laser cooling on the motions of particles confined in a Penning trap has been discussed in detail in [29, 30].

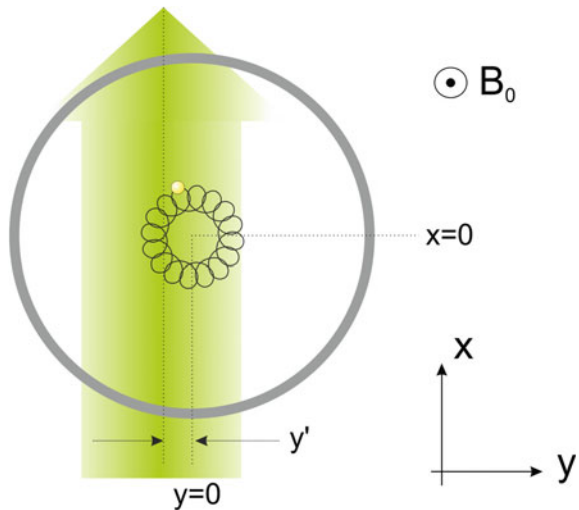
At the limit of cooling, the particle velocity is dominated by the cumulative effect of the momenta of the absorbed and re-emitted photons. The particle may be seen as executing a random walk in momentum space in units of $\hbar k$. Hence the lowest temperature achieved is a balance between absorbing red-detuned photons that cool the particle, and recoil heating from spontaneous emission. These two processes are bal-

anced for a laser detuning of $\Gamma/2$ from the line centre, and under these circumstances the temperature is minimised to the value given in (12.21).

Laser cooling of the radial motions is not as straight-forward as in the axial case, since the radial motion of a particle is not a simple harmonic when seen in the laboratory frame, and the two radial motions cannot be geometrically decoupled, see also our discussion in Sect. 5.3.2. Even cooling of only the perturbed cyclotron motion by use of a laser perpendicular to the trap axis [31] is not straight-forward and does not reach the final temperatures expected from naive theory, as has been detailed in [31]. A cooling of the magnetron motion (i.e. a reduction of the magnetron radius ρ_-) requires an increase of its energy E_- (for which the laser needs to be blue-detuned), while at the same time a cooling of the perturbed cyclotron motion requires a decrease of its radius ρ_+ and energy E_+ (for which the laser needs to be red-detuned like in the axial case). This, however, is not possible in conventional Doppler laser cooling, such that we may cool only one of the motions at a time. One solution is the so-called ‘intensity gradient cooling’, which makes use of a radial cooling beam that is inhomogeneous at the location of the ion. When a Gaussian beam is offset from the ion, it provides an intensity gradient parameterised by the offset y' , see Fig. 12.5. The laser is red-detuned so that frequency selection over the cyclotron orbit causes this motion to be cooled, and the beam is chosen such that the intensity is higher on the side of the magnetron orbit where the ion is moving in the same direction as the laser wave vector, thus increasing the probability of scattering on this side and thus increasing the energy of the magnetron motion on average. This situation has been discussed and demonstrated in detail in [32, 33].

Usually, it is desired to cool the axial and radial motions simultaneously. This can be achieved by one red-detuned axial laser and one offset radial laser as described above. A quantitative description of this situation involves a model that takes the

Fig. 12.5 Cartoon of radial intensity gradient cooling in a cylindrical Penning trap seen along the trap axis. The centre of the Gaussian beam is offset from the axis by y'



interactions of both lasers into account, such a model has been detailed in [5, 33]. When the cooling lasers are chosen such that the photon scattering rates are equal, and the detuning is chosen to be $\Gamma/2$, one reaches the lowest overall particle temperatures given by [33]

$$T_z = \frac{9}{20} \frac{\hbar \Gamma}{k_B} \quad (12.22)$$

$$T_- = \frac{11}{20} \frac{\hbar \Gamma}{k_B} \frac{\omega_-}{\omega_1} \frac{1}{\left(\frac{\Gamma}{2y'k_L \omega_-} \right) - 1} \quad (12.23)$$

$$T_+ = \frac{11}{20} \frac{\hbar \Gamma}{k_B} \frac{\omega_+}{\omega_1} \frac{1}{1 - \left(\frac{\Gamma}{2y'k_L \omega_+} \right)}, \quad (12.24)$$

where k_L is the wave vector of the laser and again $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$. A combination of radial laser cooling with a rotating wall application as described in Chapter 13 has been presented in [34].

Efficient laser cooling requires the photon scattering rate $\Gamma/2$ to be high enough, the most common examples come from the MHz region and above. Taking $^{40}\text{Ca}^+$ as an example, $\Gamma/2 = 7 \times 10^7/\text{s}$ which leads to a temperature limit of $T \approx 1$ mK. The intensity required to saturate the transition is $I_s \approx 470 \mu\text{W}/\text{mm}^2$, and cooling from thermal energies to energies close to the Doppler limit occurs on the time scale of seconds.

The discussion so far took place in the so-called ‘weak-binding regime’, where the ion oscillation is much slower than the optical transition ($\omega_z \ll \Gamma$). This situation changes when we look at much narrower transitions, such as for example magnetic dipole (M1) transitions in highly charged ions. The transition rate (linewidth) for a ground-state ($n = 1$) magnetic dipole transition from the excited to the lower hyperfine state is given by

$$\Gamma_{\text{HFS}} = \frac{4\alpha\omega^3 \hbar^2 I (2\kappa + 1)^2}{27m_e^2 c^4 (2I + 1)}, \quad (12.25)$$

where α is the fine structure constant, I is the nuclear spin and $\kappa = \sqrt{1 - Z^2\alpha^2}$. For $^{209}\text{Bi}^{82+}$ this rate is $\Gamma \approx 2500 \text{ s}^{-1}$, for ions such as $^{207}\text{Pb}^{81+}$ or $^{209}\text{Bi}^{80+}$ it is lower. The one- and three-electron ions with highest optical transition rates Γ are protactinium $^{233}\text{Pa}^{90+}$ (7480 s^{-1}), francium $^{212}\text{Fr}^{86+}$ (6890 s^{-1}), neptunium $^{237}\text{Np}^{92+}$ (6600 s^{-1}), protactinium $^{231}\text{Pa}^{85+}$ (3750 s^{-1}), gold $^{196}\text{Au}^{78+}$ (3550 s^{-1}) and all odd isotopes of bismuth $^{\text{odd}}\text{Bi}^{82+}$ ($2500\text{--}3700 \text{ s}^{-1}$). Optical transition like these are usually in the ‘strong-binding limit’ where the oscillation frequency is much larger than the width of the transition ($\omega_z \gg \Gamma$), which leads to a situation with resolved sidebands, as we will discuss now.

12.5 Laser Cooling with Resolved Sidebands

Resolved-sideband laser cooling of the axial motion in a Penning trap can be achieved by a red-detuned narrow-band laser directed along the trap axis [23, 35] when the laser frequency is chosen to match a lower sideband of the optical transition at ω_0 . Such motional sidebands appear if the axial frequency of the ion is much higher than the width of the optical transition ($\omega_z \gg \Gamma$). This situation is called ‘strong binding limit’, since the binding of the ion in the trap is strong and the ion motion is fast compared to the timescale of the optical transition. The ion oscillates many times during the decay process and thus sidebands appear around ω_0 (‘carrier frequency’) at integer multiples of the axial frequency ω_z . Narrow-band laser irradiation at the frequency of a sideband below the transition cools the particle motion, while irradiation at a sideband frequency above heats it [35]. Resolved-sideband cooling has been discussed in detail for example in [4, 6, 36, 37].

In principle, sidebands appear at all integer multiples of ω_z , however the number of sidebands with significant strength depends on the temperature of the axial motion and is limited. The corresponding sideband spectrum can be calculated by [35]

$$I(\omega) = \sum_{n=-\infty}^{n=+\infty} |J_n(\eta)|^2 \frac{(\Gamma/2)^2}{(\Gamma/2)^2 + [\omega_0 - (\omega + n\omega_z)]^2}, \quad (12.26)$$

where n is the line order ($n = 0$ is the carrier at $\omega = \omega_0$) and J_n are Bessel functions of the first kind [38], given by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+2k}}{2^{n+2k} k! \Gamma(n+k+1)}. \quad (12.27)$$

The value η is the so-called ‘Lamb-Dicke parameter’ (named after W. E. Lamb (1913–2008) and R. H. Dicke, 1916–1997) which measures the oscillation amplitude in units of the optical wavelength, i.e. at present we have $\eta = a_z/\lambda_0$. The limited-sideband situation is most pronounced in the so-called ‘Lamb-Dicke regime’, i.e. when the motional amplitude of the ion is smaller than the wavelength of the laser light ($a_z < \lambda_0$ and hence $\eta < 1$, ‘strong binding’). In this situation the only sidebands with relevant strengths are the ones of first or second order around the carrier, all higher-order ones being negligible [35]. Figure 12.6 shows a schematic sideband spectrum of an ion in the Lamb-Dicke regime and a quantum picture of resolved sideband cooling where N_z is the quantum number of the axial oscillation. Note, that the spectral width of the sidebands is given by the natural linewidth of the transition which in the case of magnetic dipole transitions may be kHz or below.

A rough estimate for the timescale of such cooling can be given by simple reasoning: When the cooling laser is tuned to the n th red sideband at a frequency of $\omega_0 - n\omega_z$, the average motional energy taken away from the ion by each fluorescence

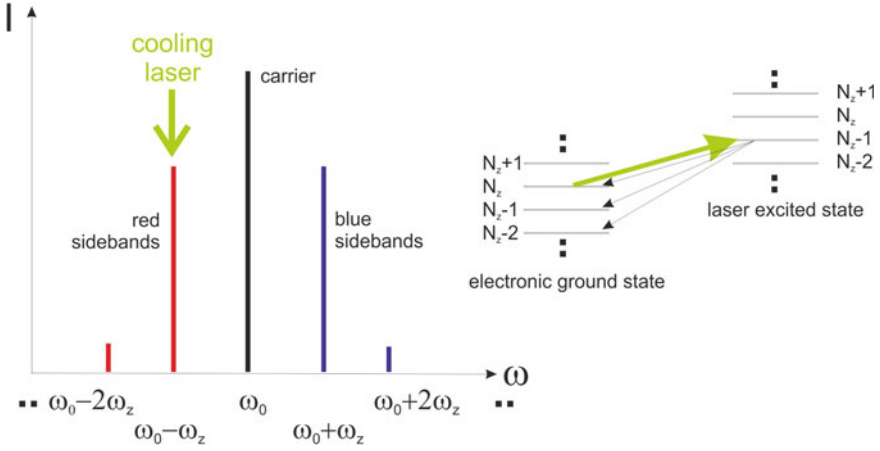


Fig. 12.6 Strong-binding limit sideband spectrum of an ion confined in a Penning trap and a quantum picture of resolved-sideband cooling where N_z is the quantum number of the axial oscillation

photon amounts to $n\hbar\omega_z$. The photon scatter rate k can be estimated by $k \approx \Gamma S_n/2$, where S_n is the relative strength of the n th sideband. S_n is given by [35]

$$S_n = \left| J_n \left(\frac{a_z}{\lambda_0} \right) \right|^2, \quad (12.28)$$

where the J_n are Bessel functions and a_z is the energy- (temperature-) dependent amplitude of the axial motion given by

$$a_z = \left(\frac{2z_0^2}{qU_0C_2} E_z \right)^{1/2}. \quad (12.29)$$

The time necessary to remove a motional energy E_z from the ion is therefore roughly given by

$$t_{cool} \approx \frac{2E_z}{n\hbar\omega_z\Gamma S_n}. \quad (12.30)$$

Resolved-sideband cooling in a Penning trap has been demonstrated for example in [4, 39] and motional temperatures about two orders of magnitude below the Doppler limit have been achieved.

Since laser cooling requires a closed 2-level system, an experiment on a single ion may require initial preparation of a suited sub-level by optical pumping. The prepared sub-state has radiative lifetimes of the order of years when the M1 microwave transition to a lower Zeeman sub-state is considered. Thus, once prepared, the system will stay in the desired state even when the following cooling and measurement times are long.

In situations where direct laser cooling is impossible or too ineffective, sympathetic cooling of highly charged ions with laser-cooled singly charged ions is a possibility to make use of the strong (electric-dipole allowed) transitions in singly charged ions [35]. This has been successfully applied to highly charged ions stored in a Penning trap, which were sympathetically cooled with laser-cooled beryllium Be^+ ions [40].

12.6 Laser Cooling in the Presence of Residual Gas

A combination of buffer gas and laser cooling can be highly efficient when particles initially have energies too high for laser cooling to be applicable, for example upon dynamic capture from a beam line. We thus have to look at a model of the combined effect of buffer gas and laser Doppler cooling. It extends the semi-classical model of Doppler cooling as presented in [41] to describe the temporal evolution of the particle energy and laser scattering rate down to energies which allow the formation process of ion Coulomb crystals in a Penning trap. Based on a rate-equation formalism, we find analytic solutions for the time dependence of the energy and thus for the scattering rate of a single particle. To this end, we add a recoil-heating term and an exponential cooling term to the rate equation. The latter accounts for the buffer gas cooling [1]. The laser frequency is scanned at a constant rate across the optical resonance used for cooling.

According to the cooling model presented in [41], the scaled energy ε of a single particle with mass m confined in a harmonic potential has a time derivative given by

$$\frac{\partial \varepsilon}{\partial \tau} = -\gamma_1(\varepsilon - \varepsilon_1) + \frac{4}{3}r \frac{1}{2\sqrt{\varepsilon r}} \text{Im}(Z) + \frac{1}{2\sqrt{\varepsilon r}} (\text{Re}(Z) + \delta \text{Im}(Z)), \quad (12.31)$$

with

$$Z = \frac{i}{\sqrt{1 - (\delta + i)^2/4\varepsilon r}}. \quad (12.32)$$

In this equation, the energy E of the particle, the laser detuning δ and the recoil energy

$$E_R = \frac{\hbar^2 k_z^2}{2m} \quad (12.33)$$

are scaled by the energy E_0 which is given by

$$E_0 \equiv \frac{\hbar \Gamma}{2} \sqrt{(1 + s_0)}. \quad (12.34)$$

Here, $\Gamma \sqrt{(1 + s_0)}$ is the power-broadened linewidth, see also the discussion in Sect. 18.4.3. The on-resonance saturation parameter s_0 is determined according to

$s_0 = I/I_s$, where I is the intensity of the laser at the position of the ions and I_s is the saturation intensity given by

$$I_s = \frac{2\pi^2 \hbar \Gamma c}{3\lambda_0^3}. \quad (12.35)$$

The time t is scaled by t_0 which is the inverse of the on-resonance fluorescence rate, such that $\tau \equiv t/t_0$, where t_0 is given by

$$t_0 = \left(\frac{\Gamma}{2} \frac{s_0}{1 + s_0} \right)^{-1}. \quad (12.36)$$

Γ is the decay rate of the excited state, k_z is the z -component of the excitation laser wave vector, γ_1 is an exponential cooling rate factor and ε_1 is the minimum energy that can be achieved by this exponential cooling mechanism. It is represented by the first term in (12.31), which for buffer gas cooling is given by [42]

$$\gamma_1 = \frac{q}{m} \frac{1}{\mu_0} \frac{p/p_0}{T/T_0}. \quad (12.37)$$

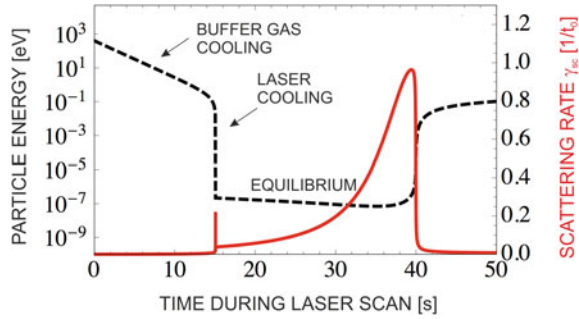
The damping coefficient γ_1 depends on the ion mobility μ_0 of the buffer gas [43–45], the residual gas pressure p and the temperature T of the buffer gas normalized by the standard pressure p_0 and the standard temperature T_0 , respectively. The second and third terms of (12.31) describe the laser-particle interaction, including stochastic heating of the particle due to photon recoil and laser Doppler cooling. The factor 4/3 in the second term is true for isotropic emission characteristics. The scaled fluorescence rate can be found to be [41]

$$\gamma_{sc} \equiv \frac{dN_{ph}}{d\tau} = \frac{1}{2\sqrt{\varepsilon r}} \text{Im}(Z). \quad (12.38)$$

The laser detuning has the form $\delta = \delta_i + \delta_m \times t$, where $\delta \equiv \omega - \omega_0$ is the detuning of the actual laser frequency ω from the resonance frequency ω_0 . δ_i is the initial laser detuning at time $t = 0$ and δ_m is the scan rate. The time $t = 0$ denotes the start of the cooling process.

This formalism is able to explain experimental observations of laser cooling in the presence of a residual gas, wherein initial energies of several hundreds of eV are reduced to the μeV region within several tens of seconds [46]. While initially, buffer gas cooling is the dominating process down to about 1 eV, from there on laser cooling dominates by far. This is shown in Fig. 12.7, where the ion energy is plotted as a function of time during the scan of the laser across the resonance. While on resonance, there is ion-laser equilibrium, after that, on the blue-detuned side of the resonance, laser heating sets in and the ion energy increases again. A detailed account on corresponding measurements is given in [46].

Fig. 12.7 Model prediction of the ion energy (dashed line) as a function of time during the cooling laser scan. The resonance frequency is reached at 40 s after which point the particle energy increases due to laser heating



12.7 More Involved Laser Cooling Techniques

In laser Doppler cooling, we have recoil heating as a limiting process to the final temperature. This is also true in case of resolved sidebands with photon recoil from the carrier.

A method that circumvents this process is the so-called ‘electromagnetically induced transparency’ (EIT) cooling, as has been suggested in [47] and experimentally demonstrated in [48, 49]. With it, ground-state cooling of confined particles has been achieved. Electromagnetically induced transparency is a quantum interference effect in which a dark resonance is created by the interference of two transitions. A bright resonance occurs at a higher frequency than this dark resonance. If the separation of the two resonances is chosen such that it equals the motional frequency of the particle, absorption will take place only on the red sideband but will be suppressed on the carrier, hence circumventing the corresponding recoil heating. For a detailed discussion see [6, 35, 50].

Another more involved laser cooling technique is polarisation-gradient cooling, also called ‘Sisyphus cooling’ after the character from Greek mythology, punished to perform the laborious and futile task of rolling a stone uphill only to see it rolling down over and over again. It is based on polarisation-dependent ac Stark shift optical pumping between ground state sublevels of a particle moving in a laser field with a strong polarization gradient. It has been thoroughly discussed in [51, 52] and experimentally demonstrated in [53]. It can similarly be applied in situations with a strong intensity gradient instead of a strong polarisation gradient [54]. A thorough description of the process can be found in [35].

As another technique to overcome the limitations of processes which involve one photon in two-level systems, namely being forced to compromise between cooling rate and final temperature, the process of stimulated Raman cooling has been brought forward. Raman cooling is based on two-photon stimulated Raman transitions that are induced by two lasers between two states of an ion, generally involving the presence of allowed dipole transitions to a third common state [35]. Sub-Doppler cooling with this method has been demonstrated, reaching temperatures as low as 100 nK [55], and the ground state of motion in a trap [56].

12.8 Buffer Gas Cooling

Collisions between confined particles and neutral species (buffer gas) represent an energy equilibration mechanism which may be used for particle cooling. One advantage of buffer gas cooling over laser cooling and resistive cooling is that it works with particles within a wide range of energies and oscillation frequencies. One disadvantage in Penning traps is the unspecific cooling of all motions, leading to an increase of the magnetron radius and eventually leading to particle loss. This, however can be circumvented by resonant coupling as discussed in the context of magnetron centring. In Penning traps, this it has first been demonstrated in [57] and has also been discussed for various applications in [35, 58].

At high energies, the cross section is dominated by the elastic-scattering cross section which is approximately given by the pure geometrical cross section of the particles

$$\sigma \approx 4\pi(r_1 + r_2)^2, \quad (12.39)$$

where r_1 and r_2 are the effective radii of the involved species. At low energies, the polarisability of the neutral species has to be taken into account (Langevin cross section, after P. Langevin, 1872–1946). The rate of Langevin collisions, i.e. collisions due to the interaction of the particle charge with the effective charge it induces in polarisable neutrals, is given by

$$k_L = 2\pi q \left(\frac{\alpha}{\mu} \right)^{1/2} n_{gas}, \quad (12.40)$$

where α is the static electric dipole polarisability, $n_{gas} = p_{gas}/(k_B T)$ is the number density of the neutral species, and μ is the reduced mass of the ion-neutral system. This equation is valid in cgs units, such that the ion charge q is given as a multiple of the elementary charge 4.8032×10^{-10} esu, α is given in cm^3 and μ is given in g. For argon atoms, $\alpha = 1.6411 \times 10^{-24} \text{ cm}^3$. For helium and hydrogen, the numbers are $\alpha = 0.21 \times 10^{-24} \text{ cm}^3$ and $\alpha = 0.66 \times 10^{-24} \text{ cm}^3$, respectively. At an assumed pressure of 10^{-10} Pa , the Langevin collision rate is of the order of 10^5 s^{-1} . Hence, for typical parameters, the Langevin collision rate is two to three orders of magnitude smaller than the Coulomb collision rate, which we will discuss below.

Aside from cooling with buffer gas, one may make use of the fact that ion species may resonantly be excited for example by irradiation of an rf dipole field at the ion oscillation frequency of interest (axial or radial). This leads to an increased collision energy with the neutral gas for just the selected ion species (depending on its charge-to-mass-ratio q/m), and may be used for selective collision-induced dissociation [59, 60] of molecular ions or cluster ions (CID). Likewise, the ion-gas collision may be used to ionize the gas [61].

In similarity to the effect of other cooling mechanisms, also buffer gas cooling can be modelled by use of a frictional force acting on the particle motion. This frictional force on a particle with mass m and velocity v is given by

$$F = m \frac{\partial}{\partial t} v = -2\gamma_b m v. \quad (12.41)$$

The axial energy of the cooled particles is damped with a rate γ_b and evolves like

$$\frac{\partial E}{\partial t} = -2\gamma_b E = -\frac{q}{m} \frac{1}{M_0} \frac{p T_0}{p_0 T}, \quad (12.42)$$

in which M_0 denotes the particles' mobility [43–45] in the buffer gas at standard conditions $p_0 = 1013 \text{ hPa}$ and $T_0 = 300 \text{ K}$. The damping coefficient γ_b is therefore proportional to the buffer gas pressure p and inversely proportional to both the temperature T of the gas and the ion mobility M_0 . Typical values for the mobility M_0 in helium gas are $20\text{--}25 \text{ cm}^2 \text{ V}^{-1} \text{ s}^{-1}$, the higher values being for light ions and the lower values for heavy ions [43–45].

When the axial particle oscillation is damped at a rate γ_b , this translates into a damping of the radial motions according to [62]

$$\gamma_{\pm} = \pm 2\gamma_b \frac{\omega_{\pm}}{2\omega_{+} - \omega_c}, \quad (12.43)$$

which means that it will reduce the reduced cyclotron radius, but increase the magnetron radius which may lead to radial particle loss.

We can see this also when we come back to (5.35) which describes the radial motions of a confined particle:

$$\ddot{u} + i\omega_0 \dot{u} - \frac{1}{2}\omega_z^2 u = 0, \quad (12.44)$$

and introduce a velocity-dependent friction term

$$\gamma = \frac{q}{m M_0}, \quad (12.45)$$

such that we have

$$\ddot{u} + (\gamma + i\omega_0)\dot{u} - \frac{1}{2}\omega_z^2 u = 0. \quad (12.46)$$

This equation is solved by [35]

$$2\omega_{\pm} = \omega_0 \pm c_{+} - i(\gamma \pm c_{-}), \quad (12.47)$$

in which

$$2c_{\pm}^2 = \sqrt{(4\omega_1^2 - \gamma^2)^2 + 4\omega_0^2 \gamma^2} \pm (4\omega_1^2 - \gamma^2) \quad (12.48)$$

and as before $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$. For small damping $\gamma \ll \omega_z$, this leads to

$$\omega_{\pm} = \frac{\omega_0 \pm 2\omega_1}{2} \left(1 \mp i \frac{\gamma}{\omega_1} \right). \quad (12.49)$$

Since the radial coordinates behave like

$$\rho_{\pm}(t) = \rho_{\pm}(0) \exp(\mp \gamma_{\pm} t), \quad (12.50)$$

we see that with time, the perturbed cyclotron radius will decrease, but the magnetron radius will increase as a result of buffer gas cooling. This limits the application in Penning traps to special cases which include motional interconversion (magnetron centring). We discuss this in detail in Sect. 11.4.

12.9 Evaporative Cooling

The term ‘evaporative cooling’ is somewhat misleading in the sense that it is rather a selection mechanism that goes along with particle loss. By allowing the ‘hottest’ particles (i.e. particles with highest kinetic energy) of a confined ensemble to leave the trap, the remaining particles equilibrate to a lower temperature (hence ‘cooling’), but are smaller in number. This is a common technique in traps for neutral particles and was brought forward in particular in connection with Bose-Einstein condensation [63, 64]. For charged particles, it finds application for example in cooling of highly charged ions in electron-beam ion traps [65, 66], and also in Penning traps [67]. This technique assumes a distribution of kinetic energies similar to or identical to a thermal distribution according to Maxwell-Boltzmann. In terms of the energy E , the probability distribution is given by

$$p(E)dE = \sqrt{\frac{4E}{\pi}} \left(\frac{1}{k_B T} \right)^{3/2} \exp\left(-\frac{E}{k_B T}\right) dE \quad (12.51)$$

with

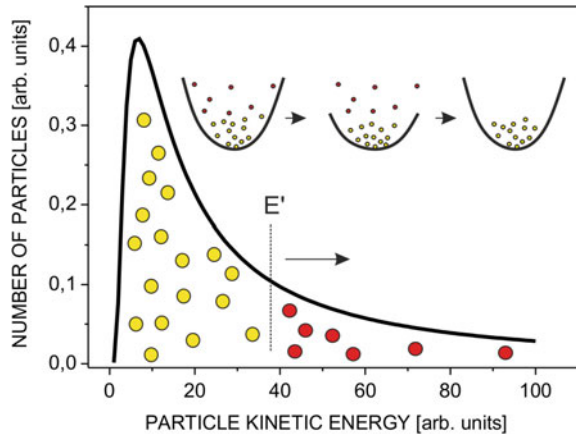
$$\int_0^{\infty} p(E)dE = 1, \quad (12.52)$$

such that the fraction f of particles above a certain kinetic energy E' is given by

$$f = \int_{E'}^{\infty} \sqrt{\frac{4E}{\pi}} \left(\frac{1}{k_B T} \right)^{3/2} \exp\left(-\frac{E}{k_B T}\right) dE. \quad (12.53)$$

The value f is hence the fraction of confined particles that may leave the trap when the potential U is lowered such that $E' = qU$. This can be used to destructively determine the ensemble temperature. Figure 12.8 schematically shows a Maxwell-Boltzmann

Fig. 12.8 Cartoon of the principle of evaporative cooling: when the trap potential is lowered, the hot particles above E' can escape, leaving a colder ensemble after confinement is restored



distribution of kinetic energies corresponding to a certain ensemble temperature. Particles above a certain energy are marked red, the others are marked yellow. The cartoon in the inset shows the general procedure of evaporative cooling. Initially, all particles are confined in a given potential well. Then, the well is shallowed such that the high-energy ions are not confined any more and leave the trap. In the last step, the well is restored to its initial state, confining the remaining particles as before, but at a lower average energy and hence at a lower temperature once they have re-equilibrated.

Note, that there are of course variations to this technique, such as for example a lowering of the trap potential only on one side, or the introduction of a short period in which the potential is switched off completely, named ‘suspended trapping’. Proper timing given, this also allows high-energy particles to leave the trap, while low-energy particles do not move far enough to leave the trap before the potential is switched on again. In particular, this may be used to eject light species such as electrons and keep heavy species such as ions.

12.10 Resistive Cooling: General Discussion

Resistive cooling of charged particles is based on the existence of induced image currents which are damped by resistances. The effect of image currents induced in electrodes by charges moving above their surface was first studied by W. B. Shockley (1910–1989, Nobel Prize in physics 1956) [68] and led to the famous Shockley-Ramo theorem [68, 69]. Looking at the axial motion as an example, the induced current from a single particle is at any time given by

$$i = \frac{q}{D} \frac{dz}{dt}, \quad (12.54)$$

where the ‘effective trap size’ D contains all information about the location and geometry of the electrodes in use, see the discussion in Sect. 12.11. In particle ensembles, the situation is complicated by particle-particle interaction, as has been discussed in [3].

Methods for resistive cooling of confined ions in general have been presented in [1, 2, 35, 40, 71], with particular focus on the analytic single-ion case. Resistive cooling of particle ensembles confined in Penning traps has been discussed in [1–3, 35, 40, 71–73], with data being presented in [2, 3, 40]. Other than for the case of a single particle, the behaviour of particle ensembles under resistive or sympathetic cooling may only be explained when ion-ion interaction is taken into account properly [3, 71, 72]. The techniques for electronic pick-up of image charges in traps have been described in [2, 70, 73]. Detailed discussions of the electronic properties of resonant detection of image currents and the combined particle-circuit systems can be found in [2, 74].

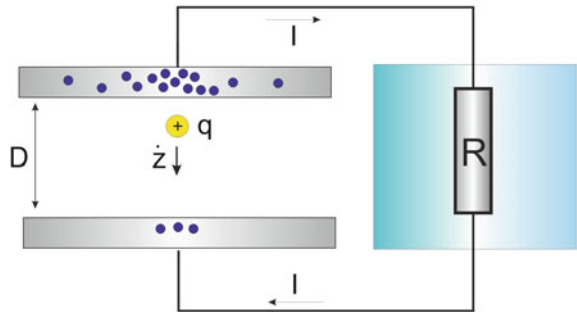
Cooling, that is damping, of a charged particle’s oscillation is possible through damping of the image current it induces in a resistance R that acts as a heat sink, see Fig. 12.9. Commonly, R is provided by a resonant RLC-circuit. In resonance, the impedance Z of an RLC-circuit is real and acts as an ohmic resistor with resistance $R = QL\omega_z$, where Q is the quality factor of the circuit and L the inductance. The cooling process finally leads to an equilibrium between the particle’s kinetic energy and the temperature of the heat bath, which is determined by the noise in the RLC-circuit [7]. Deviations from the exponential cooling behaviour may occur when the dissipated power is not proportional to the kinetic energy of the particles.

Note, that there is a limit to the possible damping with a resonant circuit of high quality factor and hence small bandwidth. Even if the damping is still under-critical, the damping rate γ shifts the frequency of the damped oscillation from the original value ω_0 to a lower value

$$\omega^2 = \omega_0^2 - \gamma^2, \quad (12.55)$$

such that the oscillation at ω may be outside the resonance of the cooling circuit.

Fig. 12.9 Schematic of resistive cooling: an ion with charge $+q$ moves between trap electrodes and induces a current I through a resistive element R that is connected to a heat sink, damping the current and in turn damping the particle motion



12.11 Effective Electrode Distance D

In the discussion of resistive cooling, we use the effective electrode distance D as a quantity. Here, we would like to have a closer look at it.

The effective electrode distance D represents the distance between two parallel infinite conducting planar electrodes that would lead to the same induced current (through a connecting resistor) as in the actual trap geometry. Close to the trap centre, it is identical to $D = 2z_0/\kappa = 2z_0/C_1$ in which the geometry factor κ is defined in [70] and is identical to the coefficient C_1 in the potential expansion (3.6). Infinite parallel plates with distance $2z_0$ have $C_1 = 1$ so that $D = 2z_0$.

In general, however, we need to consider a different geometry. Since the expansion (3.6) is valid only close to the trap centre, we need to have a closer look when it comes to particles or particle ensembles that also move further away from the trap centre, i.e. hot particles or particles on large magnetron orbits.

We hence need to consider a more general expression for the induced charge (and hence current) in the geometry given, that is valid also away from the centre. Here, we focus on a cylindrical geometry. In order to do so, the image charge $q_i(\rho, z)$ that is induced in the pick-up electrode when the particle is at position (ρ, z) is written in terms of the geometry function $\mathcal{E}(\rho, z)$ like

$$q_i(\rho, z) = -q \cdot \mathcal{E}(\rho, z), \quad (12.56)$$

where $\mathcal{E}(\rho, z)$ reads [76]

$$\mathcal{E}(\rho, z) = \frac{-1}{\rho_0 \pi} \int_0^\infty \frac{I_0(x\rho/\rho_0)}{I_0(x)} \left[\operatorname{sinc}\left(x \frac{z-z_F}{\pi\rho_0}\right) (z-z_F) - \operatorname{sinc}\left(x \frac{z-z_N}{\pi\rho_0}\right) (z-z_N) \right] dx. \quad (12.57)$$

Here, ρ_0 is the inner radius of the cylindrical electrodes, z_N and z_F are the distances from the centre of the trap to the nearest and farthest edge of the pick-up electrode, and $I_0(x)$ is the modified Bessel function of first kind. For simplification of the calculation, the Bessel function I_0 can be replaced by the approximation

$$I_0\left(x \frac{\rho}{\rho_0}\right) \approx 1 + \frac{\rho}{4\rho_0^2} x^2. \quad (12.58)$$

The effective electrode distance follows in any case from

$$D^{-1}(\rho, z) = \frac{\partial}{\partial z} \mathcal{E}(\rho, z). \quad (12.59)$$

Using as an example $\rho_0 = 8.7$ mm, $z_N = 7.5$ mm and $z_F = 21.5$ mm, the resulting effective electrode distance $D(\rho, z)$ is plotted in Fig. 12.10. Averaging D over the

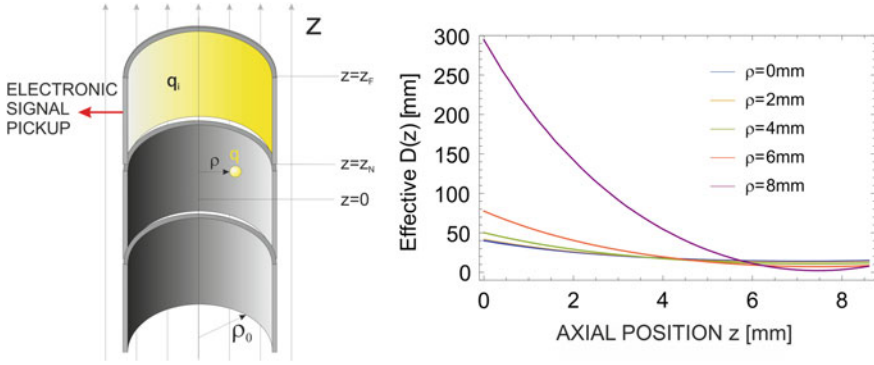


Fig. 12.10 Left: Schematic of the geometry used in the calculation of the effective electrode distance D . The induced charge q_i leads to a current against ground that is measured. Right: Effective electrode distance D as a function of the axial particle position for different radial positions

complete confinement volume in the present example leads to about 43.5 mm, which would be the relevant number for a particle or particle ensemble moving in the complete confinement region.

12.12 Resistive Cooling of a Single Particle

12.12.1 Cooling Time Constant and Final Temperature

The oscillation of a charged particle causes an image current across trap electrodes that are connected by a resistance which extenuates the current and hence the particle oscillation. When a tuned resonance circuit with an impedance $Z(\omega_z) = R$ and a quality factor Q is used to provide a resistance for cooling of the axial motion, the quality factor $Q \gg 1$ provides large R and hence efficient cooling, but limits the range of oscillation frequencies which can be cooled to a value of ω_z/Q around ω_z . The cooling may be modelled as a friction force, the equation of motion then reads

$$\omega_z^2 z - \frac{q}{m} \frac{dU(z)}{dz} - \gamma \frac{dz}{dt} = 0, \quad (12.60)$$

where γ denotes the cooling rate. The induced current from a single particle is given by

$$I(t) = \sqrt{\frac{q^2}{m}} \frac{\sqrt{E_z(t)}}{D}, \quad (12.61)$$

where the effective trap size D contains all the information about the location and the geometry of the electrodes connected by the circuit with respect to the centre of the

particle oscillation. The calculation of $D = 2z_0/\kappa = 2z_0/C_1$ has been explained in detail in [70, 77] and we give it a general treatment in Sect. 12.11. The instantaneous dissipated power is given by

$$P(t) = I^2(t)R, \quad (12.62)$$

hence the axial kinetic energy of the particle obeys the differential equation

$$\frac{dE_z}{dt} = -P(t) = -I^2(t)R \quad (12.63)$$

and therefore follows an exponential decay of the kind

$$E_z(t) = E_z(t=0) \exp(-\gamma t). \quad (12.64)$$

The single-particle cooling is characterised by the ‘cooling time constant’ τ , for the axial case given by

$$\tau_z = \gamma^{-1} = \frac{D^2}{R} \frac{m}{q^2}. \quad (12.65)$$

If the trap is electrically asymmetric and hence the trap centre is axially shifted with respect to the geometric trap centre as described by (21.17), the resulting cooling time constant is given by (12.65) modified by a factor of C_1^2 as defined in (3.6) [78].

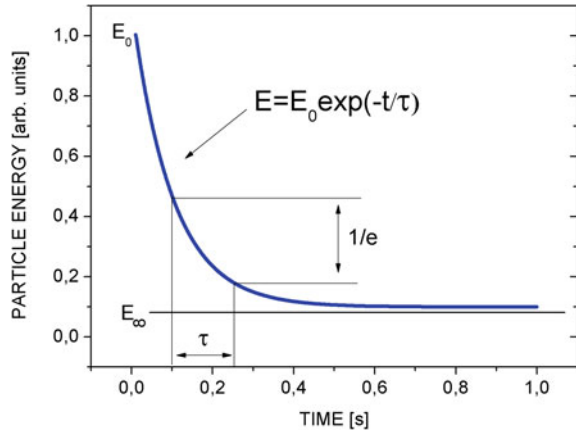
For resistive cooling of the radial motions, we similarly find

$$\tau_{\pm} = m \frac{D_{\pm}^2}{R_{\pm} q^2} \frac{\omega_+ - \omega_-}{\omega_{\pm}}, \quad (12.66)$$

where $R_{\pm}$ is the resonance resistance of the circuit used for cooling of the radial motions, and $D_{\pm}$ is the effective trap size when considering the electrodes used for radial cooling. Figure 12.11 shows the quantities involved: the energy E of a single particle as a function of time follows an exponential decay at a rate γ corresponding to a time constant $\tau = 1/\gamma$ in which the particle energy is reduced to $1/e$ of its value, in the present example $\tau = 150$ ms. The initial particle energy at time $t = 0$ is E_0 , and the limit of cooling after infinite time is $E_{\infty} \ll E_0$ which corresponds to the particle being in thermal equilibrium with the electronic noise of the cooling circuit.

While τ_z and τ_+ characterize an exponential decrease of the respective energy and hence of the motional amplitude, for the magnetron motion (which is an unstable drift in the $\mathbf{E} \times \mathbf{B}$ field) such an energy dissipation with τ_- means an *increase* in amplitude and is thus not of use when particles are to be centred in the trap and their storage time hence prolonged, see our discussion in Sect. 5.3.3. The magnetron motion can instead be ‘cooled’ (meaning a decrease of amplitude) by a coupling to

Fig. 12.11 Resistive cooling of a single particle: kinetic energy as a function of time following an exponential decay with a time constant of 150 ms in this example. A measurement of this kind has been performed for example in [14]



another motional degree of freedom. This has first been demonstrated by Wineland and Dehmelt, when a quadrupolar excitation with either the frequency $\omega_z + \omega_-$ or $\omega_+ + \omega_-$ was used to centre the magnetron motion of electrons in a Penning trap [79]. We discuss this in detail in Sect. 11.4.

In resonance, the impedance $Z(\omega)$ of an RLC-circuit is purely real and acts as an ohmic resistor with resistance $R = QL\omega$, where Q is the quality factor [80] of the circuit and L is its inductance. The bandwidth of the circuit is given by

$$\delta\omega = \frac{\omega}{Q}, \quad (12.67)$$

and for typical values like $\omega = 1$ MHz and $Q = 1000$ it is 1 kHz. Typical axial cooling time constants of single ions are below one second [3, 14]. Resistive cooling of ion ensembles is much more complicated due to the ion-ion interaction, and the existing measurements [14, 40] are not readily explained [3], as we discuss in Sect. 12.13.

The actual final particle temperature is governed by the noise of the electronics that it inflicts upon the trap electrodes. At a given ambient temperature T , the so-called ‘Johnson noise’ within the frequency bandwidth $\delta\omega$ induces a root mean square (RMS) voltage across the resistance R given by [81, 82]

$$U_T^2 = 4k_B T R \delta\omega. \quad (12.68)$$

Additional noise sources may increase this value, as we will detail in Sect. 12.17. Measurements have shown that while the ambience of a single ion was cooled to liquid helium temperature, the axial oscillation had a temperature of about 60 K which was identical to the measured noise temperature in that circuit [7]. The single ion in electromagnetic interaction with an external heat bath such as the cooling circuit is an ergodic system and hence consecutive measurements of the single ion energy lead to a Boltzmann distribution of energies to which a temperature can be assigned in the same way as to an ion ensemble, as discussed in detail in Sect. 12.2.

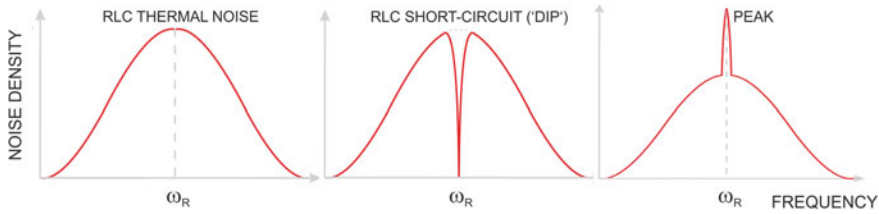


Fig. 12.12 Electronic noise density as a function of frequency of a free RLC-circuit with resonance frequency ω_R (left), the ‘dip’ (short-circuit) created by a resonant particle which is motionally colder than the electronic noise temperature (middle) and the ‘peak’ created by a resonant particle of higher motional temperature (right)

12.12.2 The Equivalence of Cooling and Detection

The detection of a particle’s oscillation and its resistive cooling are two sides of the same process. The energy dissipation by the induced image current through a resistance damps the particle motion and hence represents resistive cooling. At the same time, the image current through the cooling circuit leads to a time-dependent voltage signal that can be detected. Analysis of that signal yields the motional frequency spectrum, see Sects. 16.2 and 16.3, respectively. It is common to use an RLC circuit (sometimes called ‘resonance circuit’ or ‘tank circuit’) for resonant cooling of a motion, due to the high achievable energy dissipation into it.

Figure 12.12 (left) schematically illustrates the spectral noise density of a free RLC-circuit with resonance frequency ω_R . ‘Free’ means that the trap is empty and we hence observe only the Johnson noise corresponding to the temperature T . The middle figure shows a ‘dip’ (short-circuit) created by a particle oscillating at $\omega_z = \omega_R$ which is much colder than the electronic noise temperature. The picture on the right shows a ‘peak’ created by a resonant particle of higher motional energy.

12.12.2.1 Particle as a Parallel RLC Circuit

While the peak needs no further explanation, we should take a closer look at the dip situation. For this, it is possible to regard the particle itself as a resonant circuit [2] with a resistance r , capacitance c and inductance l given by

$$c = \frac{1}{m\omega_z^2} \frac{q^2}{D^2} \quad \text{and} \quad l = m \frac{D^2}{q^2}. \quad (12.69)$$

When this ‘particle circuit’ is connected in parallel with the resistance R_p of the detection system, the real part of the impedance of the combined circuit reads

$$\text{Re}(Z) = \frac{R_p \left(\omega l - \frac{1}{\omega c} \right)^2}{R_p^2 + \left(\omega l - \frac{1}{\omega c} \right)^2}. \quad (12.70)$$

Making use of the resonance frequency of the particle circuit

$$\omega_0 = \frac{1}{\sqrt{lc}} \quad (12.71)$$

this can be written in the form

$$\text{Re}(Z) = \frac{R_p}{1 + \frac{R_p^2}{\omega_R^2 l^2} \left(\frac{\omega_0 \omega_R}{\omega_0^2 - \omega_R^2} \right)^2}. \quad (12.72)$$

This means that on resonance ($\omega_0 = \omega_R$) the particle resonant circuit acts as a perfect short-circuit in the combined particle-trap circuit hence creating the dip in the noise spectrum.

12.12.2.2 Noise-Density Lineshape: Resonant Case

For a resonant single particle in an ideal trap, the spectral voltage density as displayed in Fig. 12.12 (middle) follows the Lorentzian shape [14]

$$U(\omega) = U(\omega_R) \frac{\omega_R^2 - \omega^2}{\omega_R^2 - i\gamma\omega - \omega^2} = \left(\frac{4k_B T R}{1 + \left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right)^2 Q^2} \right)^{1/2}, \quad (12.73)$$

from which we see that the width of the dip is given by the damping rate constant γ of the respective circuit, i.e. the inverse cooling time constant $\gamma = \tau^{-1}$. Widths of the order of 1 Hz and below can be achieved for single confined particles [14, 83].

For N particles moving coherently, i.e. with the same phase and frequency of oscillation, we need to replace γ by $N\gamma$ in (12.73), see also the discussion in Sect. 12.13.1. Additionally, we may take into account small frequency widths $\delta\omega \ll \gamma \ll \omega$ of this particle ensemble by trapping field imperfections. We express this by the quality factor Q' of the particle ensemble given by $Q' = \omega/\delta\omega$. In full similarity to (12.73) we now find [14]

$$U(\omega) = U(\omega_R) \frac{\omega_R^2 \left(1 + \frac{iN}{Q'} \right)^2 - \omega^2}{\omega_R^2 \left(1 + \frac{iN}{Q'} \right)^2 - iN\gamma\omega - \omega^2}, \quad (12.74)$$

Hence, the dip's width is proportional to the particle number N . This linearity has been experimentally shown for small numbers of highly charged ions and protons [83].

12.12.2.3 Noise-Density Lineshape: Non-Resonant Case

When the particle's motion is not in resonance with the RLC circuit ($\omega_z \neq \omega_R$), the lineshape takes a much more complicated and asymmetric shape that has been discussed in [35, 84]. The right hand side of (12.73)

$$U(\omega) = \left(\frac{4k_B T R}{1 + \left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right)^2 Q^2} \right)^{1/2} \quad (12.75)$$

then becomes modified to the form [84]

$$U(\omega) = \left(\frac{(1+a)4k_B T R}{(1+a)^2 + b} \right)^{1/2}, \quad (12.76)$$

where we have [84]

$$a = \frac{R}{r} \left[1 + \left(\frac{\omega}{\omega_z} - \frac{\omega_z}{\omega} \right)^2 Q^2 \right]^{-1} \quad \text{and} \quad b = \left[Q \left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right) - Qa \left(\frac{\omega}{\omega_z} - \frac{\omega_z}{\omega} \right)^2 \right]^2 \quad (12.77)$$

such that for $\omega_z = \omega_R$ we again obtain the Lorentzian shape as in the resonant case of (12.73) before.

The peak signal voltages U across axial and radial RLC-circuits for a single confined particle are given by

$$U_z = q \frac{2a_z Q_z}{D_z C_z} \quad \text{and} \quad U_{\pm} = q \frac{2\rho_{\pm} Q_{\pm}}{D_{\pm} C_{\pm}}, \quad (12.78)$$

respectively, where a_z and $\rho_{\pm}$ are the motional amplitudes and D are the effective trap sizes. Typical voltages are in the nV regime and hence need to be amplified in several stages. In case of a coherently moving particle ensemble, the right hand sides of these equations have to be multiplied by a factor N . Anyway, the signal increases linearly with the charge state of the particle. The same is true for the achievable signal-to-noise ratio of such a resonant pick-up given by

$$\chi_z = q \frac{2a_z}{D_z} \sqrt{\frac{\omega_z \pi}{4\delta\omega_z}} \sqrt{\frac{Q_z}{k_B T_z C_z}} \quad \text{and} \quad \chi_{\pm} = q \frac{2\rho_{\pm}}{D_{\pm}} \sqrt{\frac{\omega_{\pm} \pi}{4\delta\omega_{\pm}}} \sqrt{\frac{Q_{\pm}}{k_B T_{\pm} C_{\pm}}}, \quad (12.79)$$

where $\delta\omega$ is the frequency bandwidth of the respective resonant circuit at the oscillation frequency ω , Q is the quality factor of the circuit and C is its capacitance. The equation shows that the signal-to-noise ratio increases linearly with the charge state of the ion such that resonant detection is easiest for highly charged ions.

12.13 Resistive Cooling of Particle Ensembles

When ensembles of particles are considered, the situation is more complicated due to the larger total number of degrees of freedom and the presence of particle-particle interaction, hence, for large numbers of particles, sophisticated simulation methods have been implemented [75, 85–87].

12.13.1 Generalisation to N Particles

We first take a look at the extension of (12.61) and (12.65) for N particles. Assuming N identical particles which oscillate with the same phase and in resonance with the cooling circuit ($\omega_z = \omega_R$), the induced current is given by N times $I(t)$ from (12.61), i.e. we find

$$I_N^2(t) = N^2 I^2(t) = N^2 \frac{q^2}{m} \frac{E_z(t)}{D^2}. \quad (12.80)$$

Using that the N particles have energy $N E_z$, in analogy to (12.63) we write

$$\frac{dN E_z}{dt} = -P(t) = -N^2 I^2(t) R. \quad (12.81)$$

The cooling time constant for N particles oscillating in phase is given by

$$\tau_N = \frac{1}{N} \frac{D^2}{R} \frac{m}{q^2} = \frac{1}{N} \tau_1, \quad (12.82)$$

which is identical to τ_1/N , i.e. cooling of an ensemble under these conditions is faster than single-particle cooling by a factor of N .

If, however, the particles have arbitrary phases, we need to look at the time-averaged mean square value of the induced current due to all N particles given by [71]

$$\langle I_N^2(t) \rangle = \sum_{k=1}^N \langle I_k^2(t) \rangle + \sum_{\substack{k,l=1 \\ k \neq l}}^N \langle I_k(t) I_l(t) \rangle, \quad (12.83)$$

the first sum of which leads to

$$\langle I_N^2(t) \rangle = \frac{q^2}{m} \frac{\langle N E_z(t) \rangle}{D^2} = N \langle I^2(t) \rangle, \quad (12.84)$$

while the second sum is zero for reasons of symmetry [71]. The averaging indicated by the pointy brackets is a time-average over times significantly longer than the inverse of the corresponding oscillation frequency. This is a meaningful quantity as long as the cooling rate is much smaller than the oscillation frequency, which is a necessary requirement anyway, see the discussion at the end of Sect. 12.10. In analogy to (12.63) this dependence translates into

$$\frac{d \langle N E_z \rangle}{dt} = - \langle P(t) \rangle = -N \langle I^2(t) \rangle R. \quad (12.85)$$

The cooling time constant of N particles that oscillate with random phases is

$$\tau_N = \frac{D^2}{R} \frac{m}{q^2} = \tau_1, \quad (12.86)$$

which is identical to the single-particle value τ_1 .

The same result has been obtained from similar arguments in [1, 71]. As said, this picture is only valid for $1/\omega_z \ll \tau_1$ since the averaging needs to include more than one oscillation cycle during which time the energy must not change significantly. This, however, is fulfilled in all realistic experimental applications.

For arbitrary initial conditions concerning the distribution of particle motions, we can always separate the centre-of-charge motion from motions relative to it. For simplicity assuming only a single particle species, the centre of charge and the centre of mass are identical and may be represented by a single particle with mass Nm and charge Nq . The axial frequency of the centre of charge of N identical particles is the same as for a single particle. From (12.82) we see that the cooling time constant of the centre-of-charge oscillation is shorter than the single-ion time constant τ_1 by a factor of N , potentially providing very efficient cooling for this motion. In analogy to the equation of motion of a single particle, for the centre-of-charge coordinate Z of N particles we can write

$$\Omega_Z^2 Z - \frac{Q}{M} \frac{dV'(Z)}{dZ} - \gamma' \frac{dZ}{dt} = 0, \quad (12.87)$$

where to first order we have $\Omega_Z = \omega_z$, $V'(Z) = V(z)$, $Q = Nq$, $M = Nm$ and $\gamma' = N\gamma$. In detail, the potential $V'(z)$ includes the effect of the induced image charge of all particles on their common centre-of-charge motion. The friction constant (cooling rate) γ' of the centre-of-charge motion comprises cooling by transfer of

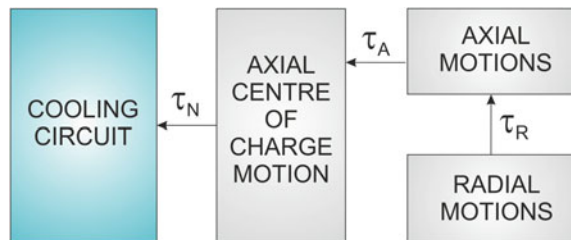
energy to the resistive circuit, but also transfer of energy to and from relative motions. Within this first order approximation the effect of image charges on the axial centre-of-charge motion can be expressed as a shift of the axial oscillation frequency [72]. Other than for image charge effects, from Newton's third law we find that the motion of the centre of charge of a particles ensemble remains unaffected when particle-particle interaction is taken into account [2].

12.13.2 Energy Flow Model

We look at a situation where each degree of freedom contains considerable kinetic energy, but direct cooling acts only on the axial centre-of-charge motion (Fig. 12.13). Hence, the other motions are cooled only by energy dissipation into that motion. The axial motions convert energy into the axial centre-of-charge motion when trapping field imperfections are present, and are indirectly cooled to that extent. The actual time constant τ_A for the cooling of axial motions is limited to the conversion time $2\pi/\delta\omega_z$ between individual axial motions and the axial centre-of-charge motion, that is given by the inverse of the axial frequency width $\delta\omega_z$ resulting from field imperfections, see the discussion in [2, 3]. We quantify this number in Sect. 12.13.4. The radial motions are cooled with a time constant τ_R given by the energy transfer into axial motions which are directly or indirectly cooled. This process occurs at the rate of thermalisation by Coulomb collisions as given by (12.119).

Hence, we generally may expect the existence of three different time constants: the centre-of-charge cooling with its time constant τ_N , a cooling of axial motions with a time constant τ_A , and a cooling of radial motions with a time constant τ_R . For N particles moving with a fixed phase, we expect $\tau_N = \tau_1/N$ from (12.82). For N particles moving with arbitrary phases, we expect $\tau_N = \tau_1$ from (12.86). These time 'constants' however change in the course of cooling, since they depend on the particle energy and density [3]. In particular, τ_A will increase because with ongoing cooling (reduction of the amplitudes a_z), the axial frequency width $\delta\omega_z$ will decrease as the particles probe less and less of the field imperfections of the trap, we discuss this in detail in Sect. 12.13.4. However, the value of τ_R will decrease with ongoing cooling [3], since the particle density will increase while the ensemble temperature will decrease, see (12.119).

Fig. 12.13 Energy flow model indicating the main flow of energy in axial resistive cooling of a hot thermalised particle ensemble



12.13.3 Observability

It is important to realize that the observability of the different components in an experiment depends crucially on the amounts of energy present in the respective motions at $t = 0$. If, for example, the particle ensemble is assumed to be completely thermalised, we expect

$$\frac{E_{cm}^{(z)}}{E} = \frac{1}{3N} \ll 1; \quad \frac{E^{(z)}}{E} = \frac{N-1}{3N} \approx \frac{1}{3} \quad \text{and} \quad \frac{E^{(R)}}{E} = \frac{2N}{3N} = \frac{2}{3} \quad (12.88)$$

in the axial centre-of-charge motion, the relative axial motions, and the radial motions, respectively, when E is the total energy present. This means that there is a negligible fraction of the total energy in the centre-of-mass motion, which will hardly be observed. Also, the time constant $\tau_N = \tau_1/N$ is typically much too fast to be resolved.

We must keep in mind that the energy flows and corresponding cooling processes occur simultaneously, and we may expect to observe only the combined action on the particle ensemble, which may not be well separated (in the time domain) if the time constants are of similar order. Let us first look at an example where the limiting process is the cooling of the axial motion itself, then look at an example where the overall cooling becomes limited by energy flow between degrees of freedom.

12.13.3.1 Fast Thermalisation: Exponential Cooling

In this example, we look at a large ensemble of highly charged argon ions in a cylindrical Penning trap with axial resistive cooling. The particles are thermalised at a temperature much higher than the cooling circuit temperature and then subjected to axial cooling. On this timescale and with particle numbers of the order of $10^3 - 10^4$, we cannot expect the centre-of-mass cooling to be resolved on either axis. The ensemble density and temperature as well as the trapping field imperfections are large enough such that thermalisation through particle-particle Coulomb collisions [see (12.119)] and the conversion rate from individual axial motions into the axial centre-of-charge motion (given by the axial frequency width $\delta\omega_z$, see also [2]) are much faster than the expected single-particle cooling rate.

Correspondingly, we observe a single exponential decay of the particle energy (Fig. 12.14) as a function of time, limited only by the direct axial cooling rate. The observed cooling time constant of about 4 s is in agreement with the calculated single-particle cooling time constant τ_1 when the effective trap size D is averaged over the populated trapping volume, see also the discussion in Sect. 12.11. Similar resistive cooling measurements have also been reported for Xe^{44+} ion ensembles in a comparable Penning trap [40], and have yielded simple exponential behaviour as well. Figure 12.15 shows the spectral width $\delta\omega_z$ of a particle species (as defined in the inset of that figure) as a function of time. It decays with roughly the same time

Fig. 12.14 Exponential cooling behaviour of a large ensemble of highly charged ions under axial resistive cooling in a Penning trap

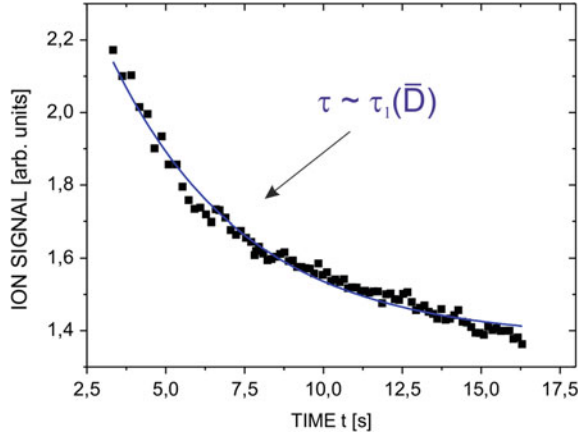
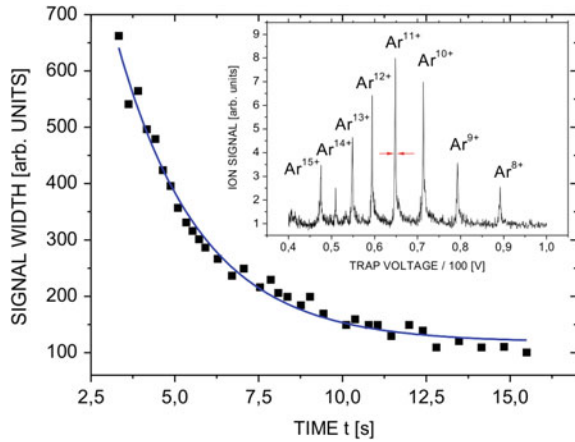


Fig. 12.15 Spectral width of the ion signal indicated in the inset as a function of time, for an ion ensemble under axial resistive cooling. The observed decay time constant agrees roughly with the observed decay of the ion energy in Fig. 12.14

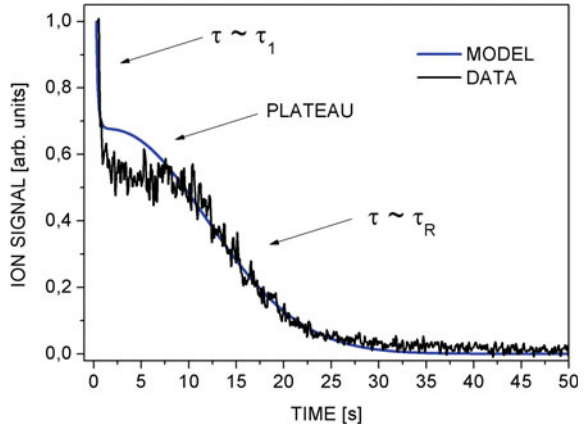


constant as the particle energy, showing that the spectral width is mainly determined by the ensemble temperature, as will be discussed in detail in Sect. 12.13.4.

12.13.3.2 Slow Thermalisation: Non-exponential Cooling

In this example, we look at a much smaller particle ensemble of 30 highly-charged carbon ions in a trap with much better tuning of the potential. Hence, both the conversion of axial energies into the axial centre-of-mass motion and the collisional thermalisation are much less efficient than in the previous example. Here, it is not the axial cooling itself that limits the overall cooling process, but the dissipation of energy between the degrees of freedom. Such behaviour has been discussed in [3, 14]. Initially, the observed cooling has a time constant τ_A comparable to the measured single-ion time constant τ_1 of 132 ms, followed by a plateau where cooling appears

Fig. 12.16 Measured data and model curve of non-exponential cooling behaviour of a small ensemble of highly charged ions. Data from [3]



to come to a halt, and a subsequent slow exponential cooling with τ_R of about 5 s, see also Fig. 12.16 and the discussion in [3]. The plateau indicates that at this point in time, the energy flow from radial motions into axial motions balances the axial cooling such that the observed axial energy keeps constant in spite of axial energy being constantly dissipated into the cooling circuit. Once this point is overcome, the observed axial cooling continues at a rate limited by the rate of inflow from the radial motions. For different initial energies, the plateau is differently pronounced, see also the discussion in [3].

In principle, a non-exponential cooling behaviour may also be observed for a single particle if the field imperfections are such that the transfer of radial energy to axial energy is initially faster than the axial cooling and then becomes slower as the particle probes less of the field imperfections due to its amplitude decreasing with ongoing cooling.

12.13.4 Thermal Width of the Oscillation Frequency Distribution

When the oscillation frequency depends on the particle energy, a distribution of energies (as in a thermal ensemble) will lead to a certain distribution of frequencies. In a non-ideal cylindrical Penning trap, the dominant contribution to an energy-dependent shift of the axial frequency comes from the z^4 -dependence of the axial trapping potential, which is measured by its coefficient C_4 as defined in [70, 77], and which leads to

$$\frac{\Delta\omega_z}{\omega_z} = \frac{3}{2} \frac{C_4}{C_2 q U_0} E_z. \quad (12.89)$$

Similarly, the axial oscillation frequency depends on the energy E_+ in the radial (perturbed cyclotron) motion when a quadratic component $B_2 z^2$ of the magnetic field is present. This dependence gives

$$\frac{\Delta\omega_z}{\omega_z} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} E_+. \quad (12.90)$$

Hence, a distribution of axial or radial kinetic energies will lead to a corresponding distribution of axial oscillation frequencies. For a thermalised particle ensemble at temperature T , the distribution of axial and radial energies is Boltzmann-like and given by

$$dp(E_{z,+}) = \frac{1}{k_B T} \exp\left(-\frac{E_{z,+}}{k_B T}\right) dE_{z,+}, \quad (12.91)$$

both with the expectation value $\langle E_{z,+} \rangle = k_B T$ and a typical width of the distribution of about $2k_B T$.

In the presence of field imperfections, we expect the thermal axial frequency width to be

$$\left\langle \frac{\delta\omega_z}{\omega_z} \right\rangle = \left(\frac{3}{qU_0} \frac{C_4}{C_2} + \frac{2}{m\omega_z^2} \frac{B_2}{B_0} \right) k_B T. \quad (12.92)$$

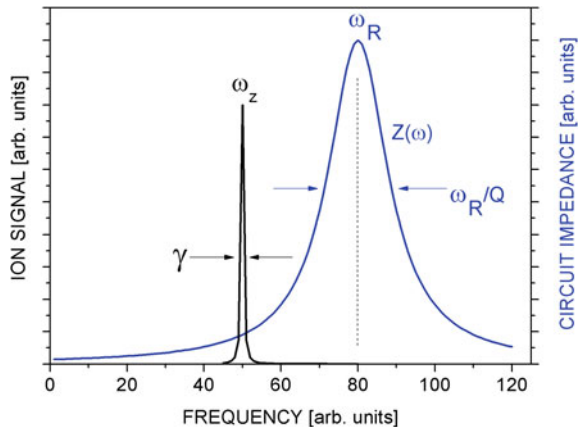
For $U_0 \approx 10$ V, $B_0 = 3.8$ T, $B_2 \approx 10$ μ T/mm² and C_4 of 10^{-5} at particle temperatures corresponding to 10 eV we expect a relative width of the order of $\langle \delta\omega_z/\omega_z \rangle \approx 10^{-5}$ which gives an absolute width of around $2\pi \times 10$ Hz.

12.13.5 Spectral Overlap Between Particle Oscillation and Circuit

Both the particle oscillation and the resonator (cooling circuit) have a centre frequency and a spectral width associated with them. The width of the cooling circuit comes from its finite quality factor Q , while the width of the particle motion comes from various sources, predominantly field imperfections, as discussed in Sect. 12.13.4. For resistive cooling to be most efficient, the two distributions need to have maximum overlap.

Let the spectral width of the particle oscillation in question be γ . For simplicity, we focus on the axial oscillation with centre frequency ω_z . A change of the particle kinetic energy (for example during cooling) in the presence of field imperfections may change this frequency significantly with respect to the frequency of the cooling circuit. This is potentially an issue as the spectral width of a resonant RLC-circuit is limited to ω_R/Q . Therefore, resistive cooling of particle ensembles may become less effective when field imperfections are significant. Note, that as the shifts are energy-

Fig. 12.17 Schematic of spectral distributions of the particle oscillation and the impedance $Z(\omega)$ of the resonant circuit. Usually, one desires $\omega_z = \omega_R$ and $\gamma \approx \omega_R/Q$ for efficient cooling



dependent, the centre-of-charge oscillation frequency changes during the cooling time and may come in or out of resonance occasionally. This is depicted qualitatively in Fig. 12.17, where the frequency overlap of a particle ensemble with a fixed resonance circuit is shown. Let the frequency-dependent circuit impedance be $Z(\omega)$ with $Z(\omega_R) = R$. As discussed in Sect. 14.6.2, field imperfections generally lead to a shifted and broadened distribution of axial frequencies of confined particles. Resistive cooling is only effective within the bandwidth of the external circuit, which may be less than the range of frequencies expected due to the distribution of particle energies and the given field imperfections.

In large or hot particle ensembles, the frequency distribution may be much broader than the cooling circuit so that we have $\gamma \gg \omega_R/Q$. In that case, we may expect the system to behave as if only the overlapping part of the particle distribution is cooled directly by the circuit, and it is expected that inter-particle collisions continuously thermalise the ensemble by redistributing energy over the ensemble's axial frequency distribution so that the resistive cooling process continues, but at a reduced rate limited by the thermalisation process. The inter-particle collision rate for a single-component particle ensemble is characterised by $k \propto q^4 m^{-1/2} n T^{-3/2}$ where q is the particle's charge, m its mass, n the particle number density and T the intrinsic temperature of the particle ensemble, see the discussion on Sect. 12.16. Therefore, thermalisation is most efficient for light, highly charged particles in a dense, cold particle ensemble. It may be expected that, in case of an efficient thermalisation process, the cooling time of a particle ensemble scales with the bandwidth of the RLC-circuit. Therefore, in practical applications, a compromise has to be found between a sufficiently small single-particle cooling time constant $\tau \propto 1/Q$ and a sufficiently large bandwidth $\delta\omega_z \propto 1/Q$.

12.14 Continuous Feedback Cooling

Feedback cooling, sometimes also called active-feedback cooling, is based on electronic pick-up of the signal induced by an oscillating particle, application of a 180° phase shift, amplification with a gain $0 < G < 1$ and returning this signal to a suited electrode. This situation is depicted in Fig. 12.18 for the axial motion of a confined particle in a Penning trap consisting of a central ring and two endcaps. The particle motion itself hence continuously produces a counter-force exactly opposing it.

This way, the resistive cooling rate γ of the cooling circuit is decreased to [88]

$$\gamma_F = (1 - G)\gamma, \quad (12.93)$$

but the final temperature T_F is also decreased to

$$T_F = \frac{\gamma_F}{\gamma} T < T, \quad (12.94)$$

so a low final temperature comes at the expense of a small cooling rate.

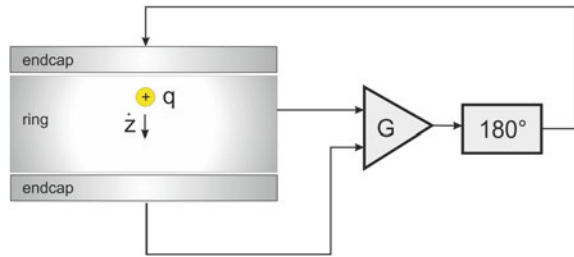
Here, T is the final temperature of cooling in absence of the feedback. In real feedback loops, there is a thermal noise voltage V_N due to the cooling impedance, and an additional noise voltage V_G due to the feedback loop. This leads to a noise voltage seen across the endcaps of [88]

$$V_F = \sqrt{(1 - G)^2 V_N^2 + G^2 V_G^2}, \quad (12.95)$$

which limits the final particle temperature to [88]

$$T_F = T \left(1 - G + \frac{G^2}{1 - G} \left(\frac{V_G}{V_N} \right)^2 \right) = T \left(1 - G + \frac{G^2}{1 - G} \frac{T_G}{T_N} \right), \quad (12.96)$$

Fig. 12.18 Simplified schematic of feedback cooling



where we have introduced the notion of a feedback gain temperature T_G . We can hence see that for $T_G \ll T$, the process of feedback cooling can be optimized by choice of $G \approx 1 - \sqrt{T_G/T}$ such that the lowest final temperature is given by [88]

$$T_F \approx 2\sqrt{TT_G}. \quad (12.97)$$

Negative-feedback cooling of a single confined electron [8] with $G \approx 0.9$ and of a single confined proton [83] have been demonstrated. Within this technique, a single particle can be electronically cooled to temperatures well below the ambient temperature, and values below 1 K have been achieved.

Let us have a look at a particle ensemble in a cylindrical open-endcap Penning trap. Let the particles induce a total image charge $Q(t)$ in the ring that is used to generate a feedback signal of voltage U_{fb} applied to the opposite endcap. The value of $Q(t)$ can be calculated as the sum of all individual induced charges $Q_{pl}(\mathbf{r}_i(t))$ which are given by

$$Q_{pl}(\mathbf{r}) = \frac{-q}{\pi R} \int_0^\infty \frac{I_0(x \cdot \frac{r}{R})}{I_0(x)} \left[\text{sinc} \left(x \frac{z - z'}{\pi R} \right) (z - z') \right]_{z'=z_r}^{z'=z_l} dx, \quad (12.98)$$

and the feedback signal voltage is the first derivative of the induced signal:

$$U_{fb}(t) = \alpha \dot{Q}(t) = \alpha \sum_i \nabla Q_{pl}(\mathbf{r}_i(t)) \cdot \mathbf{v}_i(t). \quad (12.99)$$

Here, α is a negative proportional factor that is related to the effective gain. Considering now only the z -component and the z -derivative for axialised particles, one gets:

$$U_{fb}(t) = \alpha \frac{dQ}{dt} = \alpha \frac{dQ}{dz} \frac{dz}{dt} = \alpha \frac{dQ}{dz} \dot{z}. \quad (12.100)$$

For small amplitudes it can be approximated by a linear function with slope k , so that the feedback voltage is given by

$$U_{fb}(t) = \alpha \sum_j k z_j(t) \dot{z}_j(t). \quad (12.101)$$

The resulting axial equation of motion in presence of such a feedback voltage for a particle 'i' follow from the acting forces by

$$m \ddot{z}_i = F_{el,i} + F_{pp,i} + F_{fb,i}, \quad (12.102)$$

where the force $F_{el,i}$ due to the confining electrostatic potential is given by

$$F_{el,i} = -q \frac{dU}{dz} = -q U_0 \left(C_2 \frac{z_i}{d^2} \right), \quad (12.103)$$

and the Coulomb interaction with all other particles 'j' is given by

$$F_{pp,i} = \sum_{i \neq j} \frac{1}{4\pi\epsilon_0} \frac{q^2}{((\rho_{ij}^2 + z_{ij}^2)^{3/2})} \cdot z_{ij}, \quad (12.104)$$

where the distance $z_{ij} = z_j - z_i$ between two particles is signed. The feedback force $F_{fb,i}$ follows from

$$F_{fb,i} = -qU_{fb}(t) \left(C_2 \frac{z_i}{d^2} \right). \quad (12.105)$$

To see that this kind of feedback leads to cooling, we multiply (12.102) with $\dot{z}_i$ (and neglect particle-particle interaction since it does not change the ensemble energy average), hence we find

$$m\ddot{z}_i\dot{z}_i = -2qU_0 \left(C_2 \frac{z_i}{d^2} \right) \dot{z}_i - q \sum_j k z_j(t) \dot{z}_j \left(C_2 \frac{z_i}{d^2} \right) \dot{z}_i. \quad (12.106)$$

This equation can be rewritten and sorted:

$$\frac{d}{dt} \left(\frac{1}{2} m \dot{z}_i^2 + qU_0 \left(C_2 \frac{z_i^2}{d^2} \right) \right) = \alpha \cdot k \frac{qC_2}{d^2} \sum_j z_j \dot{z}_j z_i \dot{z}_i. \quad (12.107)$$

Summing now over all indices 'i' one obtains:

$$\frac{d}{dt} E = \alpha \cdot k \frac{qC_2}{d^2} \sum_j z_j \dot{z}_j \sum_i z_i \dot{z}_i = \alpha \cdot k \frac{qC_2}{d^2} \left(\sum_i z_i \dot{z}_i \right)^2 \leq 0, \quad (12.108)$$

which is true since $\alpha < 0$, and which shows that there is always a reduction of total kinetic energy E as a function of time. Negative-feedback cooling is a form of 'sub-thermal' cooling, since it allows cooling below the ambience temperature of the trap and its electronics.

12.15 Non-continuous Feedback Cooling: Stochastic Cooling

For any type of cooling that is based on the pick-up of image charges induced in the trap electrodes, we have the issue of particle motions that we desire to cool, but that do not produce a detectable image signal. For single particles as discussed in the section about continuous feedback cooling, this is not an issue. For particle ensembles, however, it is the same issue we have discussed in the framework of resistive cooling. Relative motions of particles may largely cancel each others' induced signal (think

of two particles oscillating at the same frequency but with opposite phase), such that only the centre of charge is effectively seen by the electronics.

Stochastic cooling may be regarded as a non-continuous version of the feedback cooling described before. Like with active feedback cooling, the aim is to increase the cooling rate over the rate of pure resistive cooling, or to reduce the final temperature below the value without feedback.

In general, an image signal is detected at a time t and then later at a time t' the corresponding feedback signal is applied in a similar fashion as discussed above. For the axial oscillation, this would be done at a time $t' = t + T_z/2$ where T_z is the period of the axial oscillation. It could of course also be done at any other odd multiple of $T_z/2$. Let the feedback signal voltage be given by

$$U = -C' d\omega_z a_z, \quad (12.109)$$

where ω_z and a_z are the frequency and amplitude of the axial centre-of-mass motion, respectively, and C' is the corresponding proportionality constant for the geometry given. If all N particles have mass m and charge q , then the total kinetic energy of the ensemble will evolve like

$$E(t) = (E_0 - E_\infty) \exp(-t/\tau_S) + E_\infty, \quad (12.110)$$

where E_0 is the initial kinetic energy and E_∞ is the limit after an infinite number of feedback cycles. The time constant of cooling is then given by [71]

$$\tau_S = \frac{N \Delta t}{2g - g^2}, \quad (12.111)$$

in which Δt is the time between successive feedback pulses and g is given by

$$g = \frac{1}{m} C' dq \Delta T, \quad (12.112)$$

with ΔT being the duration of an individual feedback pulse. The limit of such cooling is reached when the ensemble has an energy E_∞ of [71]

$$E_\infty = \frac{1}{N} \frac{g^2}{2g - g^2} \tau 4k_B T B, \quad (12.113)$$

where τ is the ensemble cooling-time constant of the circuit in absence of stochastic cooling, B is the bandwidth of the cooling pulse, and T is the ambient electronic noise temperature. The term $g^2/(2g - g^2)$ in (12.113) indicates a good choice for the parameter g to be slightly below 0.5. For such values we find [71]

$$E_\infty \approx \frac{1}{(2 - g)^2} \frac{\tau}{\tau_S} 4k_B T B \Delta t \approx \frac{\tau}{\tau_S} k_B T B \Delta t. \quad (12.114)$$

This means that at a given electronic noise temperature T we can minimize E_∞ by minimizing both the cooling pulse bandwidth B and the time Δt between feedback pulses. Still, the bandwidth B needs to be comparable to the width $\delta\omega_z$ of axial oscillation frequencies of the ensemble.

This method is efficient for cooling of all degrees of freedom of a particle ensemble only if the action on the centre-of-charge motion affects also internal (relative) motions by thermalisation processes, in the same way as we have discussed for resistive cooling. When the time constant for thermalisation of the ensemble is given by τ_T , we require to have the time between subsequent feedback pulses to be larger than this, i.e. $\Delta t > \tau_T$ to give time for thermalisation.

For the specific choice of the feedback pulse bandwidth B and separation Δt according to $B = \delta\omega_z/2$ and $\Delta t = 2\tau_T = 2/\delta\omega_z$, we have a final energy of

$$E_\infty \approx \frac{g\delta\omega_z}{N} k_B T \tau \quad (12.115)$$

and a cooling time constant τ_S of

$$\tau_S = \frac{N}{g\delta\omega_z}, \quad (12.116)$$

such that we have

$$E_\infty \tau_S \approx k_B T \tau. \quad (12.117)$$

This means that the advantage of stochastic cooling over resistive cooling

$$\frac{\tau}{\tau_S} \approx \frac{E_\infty}{k_B T} \quad (12.118)$$

is given by the ratio of the obtainable final ensemble temperature and the electronic noise temperature in the system. When the initial centre-of-charge energy of the ensemble is much larger than the energy of the relative motions, we do not require the specific features of stochastic cooling over the course of many feedback cycles, but a few pulses will efficiently slow the centre of charge, and we may speak about active feedback slowing, or stochastic slowing. Stochastic cooling in Penning traps has only been demonstrated in few cases with comparatively high initial and final energies [89]. Some numerical examples are given in [71].

12.16 Sympathetic Cooling

When one fraction of confined particles is cooled by another fraction of confined species on account of their mutual Coulomb interaction, this is referred to as ‘sympathetic cooling’. It was first demonstrated with different isotopes of one element in

1980 [90], and then in 1986 between different elements [74]. It has also been applied to cooling of molecular ions [91]. Often, one species is directly cooled, e.g. by laser cooling, and sympathetically cools another species that cannot be directly cooled.

Assuming full spatial overlap of the particle distributions, the sympathetic cooling time constant is approximately given by

$$\tau \approx (4\pi\epsilon_0)^2 \frac{m_1 m_2}{q_1^2 q_2^2} \frac{1}{n_2 \ln \Lambda} \left(\frac{k_B T_1}{m_1} + \frac{k_B T_2}{m_2} \right)^{3/2}, \quad (12.119)$$

where the index ‘2’ refers to the coolant and ‘1’ refers to the particles to be cooled.

The quantity $\ln \Lambda$ is the so-called ‘Coulomb logarithm’ [92]

$$\ln \Lambda \approx 23 - \ln \left[\frac{q_1 q_2 (\mu_1 + \mu_2)}{e^2 (\mu_1 T_1 + \mu_2 T_2)} \left(\frac{n_1 q_1^2}{e^2 T_1} + \frac{n_2 q_2^2}{e^2 T_2} \right)^{1/2} \right], \quad (12.120)$$

which in case of identical particles it is given by [92]

$$\ln \Lambda \approx 23 - \ln \left(\frac{2nq^6}{e^6 T^3} \right)^{1/2}, \quad (12.121)$$

where n is given in units of cm^{-3} and T is given in units of eV. The temperature evolution of the cooled species is then given by

$$\frac{\partial T_1}{\partial t} = \frac{T_2 - T_1}{\tau}. \quad (12.122)$$

In case of several different coolants, this is generalized to

$$\frac{\partial T_1}{\partial t} = \sum_a \frac{1}{\tau^{(a)}} (T_a - T_1). \quad (12.123)$$

The slope of the sympathetic cooling time constant with q_1^2/m_1 of the cooled particles is given by

$$\frac{\partial \tau}{\partial \left(\frac{q_1^2}{m_1} \right)} = -(4\pi\epsilon_0)^2 \frac{m_2}{q_2^2} \frac{1}{n_2 \ln \Lambda} \left(\frac{k_B T_1}{m_1} + \frac{k_B T_2}{m_2} \right)^{3/2}. \quad (12.124)$$

Assuming Mg^+ ions cooled to $T_2 = 100 \text{ mK}$ and room-temperature Ar^{13+} ions both at the Brillouin density, this leads to a cooling time constant of order μs . Sympathetic cooling of highly charged ions in a Penning trap has been discussed in [3, 93]. In

such a case, cooling is very efficient as the rate is proportional to the square of the charge state. Direct laser cooling on a strong transition would be more efficient at low energies, however such transitions in the laser-accessible regime do not exist in highly charged ions [94].

12.16.1 A Special Case: Electron Cooling

One particular form of sympathetic cooling is electron (positron) cooling, as such species with very large q/m need not actively be cooled, but ‘self-cool’ via their synchrotron radiation in the trap. Electrons are readily available both from hot and cold sources (like filaments and field emission points) and can hence be confined large quantities.

Electron cooling has first been employed to cool anti-protons [95, 96]. Electron cooling of confined protons has been studied in [27, 97], and electron/positron cooling of trapped highly charged ions has been studied in [98–101].

The cooling of confined electrons or positrons (in absence of ions) through synchrotron radiation has a time constant of

$$\tau_e = \frac{3m_e c^3}{4e^2 \omega_+^2}, \quad (12.125)$$

which in a magnetic field of $B_0 = 6\text{ T}$ is about 100 ms. The final electron temperature in equilibrium is the blackbody ambient temperature T_{am} of the trap interior.

Let us assume N_i confined ions to be cooled by N_e simultaneously confined electrons such that $N_e \gg N_i$. Then (12.119) reads

$$\tau_i = (4\pi\epsilon_0)^2 \frac{mm_e}{q_1^2 e^2} \frac{1}{n_e \ln \Lambda} \left(\frac{k_B T_i}{m} + \frac{k_B T_e}{m_e} \right)^{3/2} \quad (12.126)$$

and gives the cooling time constant of the ions with the electrons. The temperature evolution of the ions for $N_e \gg N_i$ and $T_i > T_e$ is given by

$$\frac{\partial T_i}{\partial t} = -\frac{1}{\tau_i} (T_i - T_e), \quad (12.127)$$

i.e. the ion temperature approaches the electron temperature in an exponential decay.

If we cannot assume $N_e \gg N_i$, i.e. if the numbers of ions and electrons are somewhat comparable, then we need to account for the heating of electrons via Coulomb interaction with the ions (which may have considerable kinetic energy). In that case, we find the electron heating described by

$$\frac{\partial T_e}{\partial t} = \frac{1}{\tau_i} \frac{N_i}{N_e} (T_i - T_e) - \frac{1}{\tau_e} (T_e - T_{am}). \quad (12.128)$$

Considering the ongoing synchrotron cooling of the electrons, after long times an equilibrium between ions and electrons would be reached at the ambient temperature T_{am} , in reality however, the limited confinement time allows this to be reached only in part.

12.16.2 Another Special Case: Mediated Sympathetic Cooling

In some cases the particle species involved in sympathetic cooling may not be confined in the same trap. This may be due to different signs of charge, due to largely different charge-to-mass ratios leading to inconvenient oscillation frequencies, or due to a high cross section of undesired reactions such as charge exchange or annihilation. In such cases, it is still possible to employ sympathetic cooling by confining the particle species to physically separated traps, and have the Coulomb interaction take place via an electrode or wire. Figure 12.19 exemplifies this situation. Note, that the geometry may be much different from the cylindrical one displayed here. The traps may in general have largely different sizes and geometries, so long as the motions to be cooled couple efficiently. One example is the sympathetic cooling of protons and anti-protons with a common-endcap Penning trap as discussed in [102]. In this application, the axial oscillations of the involved species are coupled by the image charges induced in the common endcap between the separate traps. To quantify this situation, we can come back to the description of confined particles as an RLC-circuit, see our discussion in Sect. 12.12.2.1. We represent a confined particle ensemble of N identical particles each with mass m and charge q by

$$R = \gamma L, \quad L = \frac{4md^2}{Nq} \quad \text{and} \quad C = \frac{1}{\omega_z^2 L}, \quad (12.129)$$

where γ is the damping rate of the equivalent circuit and d is the characteristic trap size. This description holds for the particles in either trap 'A' and 'B'. The coupling is represented by the additional capacitance C_E of the common endcap set in parallel

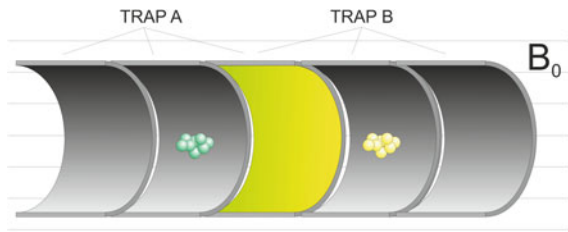


Fig. 12.19 Principle of the common-endcap Penning trap: two confinement regions share a common endcap (middle), which mediates cooling in one trap to the other via the induced image currents

with the two RLC circuits [103]. The coupling leads to an oscillation of energy between the two coupled systems, in full similarity to standard coupled oscillators. The energy exchange time between the two systems is given by [102]

$$t = \pi \omega_E C_E \sqrt{L_A L_B}, \quad (12.130)$$

where L_A and L_B are the respective inductances according to (12.129), and ω_E is the average frequency

$$\omega_E = \frac{1}{2}(\omega_A + \omega_B) \quad (12.131)$$

between the characteristic frequencies

$$\omega_A = \frac{1}{2}(\omega_A'^2 + \omega_B'^2) + \frac{1}{2} \left((\omega_A'^2 + \omega_B'^2)^2 + 4h^4 \right)^{1/2} \quad (12.132)$$

$$\omega_B = \frac{1}{2}(\omega_A'^2 + \omega_B'^2) - \frac{1}{2} \left((\omega_A'^2 + \omega_B'^2)^2 + 4h^4 \right)^{1/2}, \quad (12.133)$$

in which the value of h is given by

$$h = (L_A L_B C_E^2)^{-1/4} \quad (12.134)$$

and the two reduced resonance frequencies ω_A' and ω_B' are given by

$$\omega_A' = \left(L_A \frac{C_A C_E}{C_A + C_E} \right)^{-1/2} \quad (12.135)$$

$$\omega_B' = \left(L_B \frac{C_B C_E}{C_B + C_E} \right)^{-1/2}. \quad (12.136)$$

Thus, for the energy exchange time of mediated sympathetic cooling we obtain the expression

$$t = \pi \omega_z C_E \left(\frac{m_A m_B}{q_A^2 q_B^2} \frac{1}{N_A N_B} d_A^2 d_B^2 \right)^{1/2} \quad (12.137)$$

from which it becomes obvious that for an efficient coupling with a short energy transfer time t , we need to minimize the common-endcap capacitance C_E and use particles with a high charge-to-mass ratio. This specific form of sympathetic cooling may be called ‘mediated sympathetic cooling’ on account of a mediating stage in the Coulomb interaction.

We will detail another example in which the axial motion in one trap is coupled to the radial motion in another trap by use of a connecting wire, see Sect. H.

12.17 Suppression of Unwanted Heating: Filters

The motion of confined particles may be influenced by any signal present at the trap electrodes that contains spectral noise power in bands that coincide with any motional frequency. We have seen in Sect. 12.12.1 that there is electronic noise due to thermal excitations [81, 82] in components of RLC circuits and other electronics (Johnson noise). Often, noise has a wide frequency bandwidth and is therefore capable of exciting all motions of the stored particles, both the oscillatory motions, their sidebands, but also plasma modes and potentially internal degrees of freedom. In general, there are many possible sources of signals that may lead to undesired excitations of confined particles. One obvious source are the voltage supplies for the electrostatic potential of the trap, including the wiring which may act as an antenna for radio-frequency signals. The same is true for any other conducting element that is galvanically connected with the electrodes, or efficiently coupled even in absence of a galvanic connection.

12.17.1 Types of Noise and Sources

Apart from external noise sources that may be shielded, we are concerned with internal sources of noise. Out of the many mechanisms that lead to electronic noise, we briefly discuss the main contributors in typical Penning trap experiments.

12.17.1.1 Thermal Noise

At any finite temperature, we will encounter thermal noise. It is due to the thermal motion in conductors which acts upon the conduction electrons. It is commonly called ‘Johnson-Nyquist noise’ or ‘Johnson noise’ and independent of any applied voltage [81, 82]. Thermal noise shows approximately the same behaviour as white noise, which means that its spectral power density is nearly constant throughout the radio-frequency frequency spectrum. As the name suggests, it can be reduced by a reduction of the conductor temperature. This also becomes obvious from the expression

$$U_n = \sqrt{4k_B T R B_f} \quad (12.138)$$

of the root-mean-square value of the noise voltage U_n induced in a resistance R within a frequency band B_f .

12.17.1.2 Shot Noise

There is a number of elements and devices in electronics that set discrete barriers to the flow of charges. Shot noise then results from random statistical fluctuations of

the electric current when the charge carriers traverse a barrier. The electric current is a flow of discrete charges, and the fluctuation in the arrival times of those charges on the other side of a barrier creates shot noise. The root-mean-square value of the shot noise current is given by the Schottky equation [104, 105]

$$I_n = \sqrt{2IeB_f}, \quad (12.139)$$

where I is the current and B_f is the observation bandwidth.

12.17.1.3 Flicker Noise

Flicker noise is more commonly known as ‘1/f noise’ or ‘pink noise’ due to its characteristic to decrease in magnitude with increasing frequency f . It occurs in almost all electronic devices, and results from a variety of effects that are related to a direct current. Most prominently, it determines the noise characteristics of transistor-based amplifiers, which correspondingly yield better signal-to-noise ratios for higher frequencies, hence typically the detection of the reduced cyclotron frequency is that sense easier than the detection of the axial frequency. Flicker noise usually features a ‘corner frequency’ which separates the region dominated by the low-frequency flicker noise and the higher-frequency ‘flat-band’ noise [106, 107]. Note, that sometimes ‘flicker noise’ is translated into the German word ‘Funkelrauschen’ which in fact is a different phenomenon that is proportional to $1/f^2$, the proper translation is ‘Flickerrauschen’.

12.17.1.4 Patch Noise

As discussed in Sect. 6.5, real metal surfaces have regions with different surface potentials (‘patches’) that fluctuate and give rise to radio-frequency noise with roughly the spectral characteristics of flicker noise [108–110]. This noise is thermal, however unlike Johnson noise, it does not contribute like $T^{1/2}$, but has been observed to behave like T^β with a value of β between 2 and 4 [108]. It can thus be efficiently suppressed by cooling of the surfaces, but is still a potential issue for small traps on account of the $1/d^3$ dependence of local charges on the potential in the centre of a (three-dimensional) Penning trap.

12.17.2 Filters

To avoid particle excitation by noise from outside sources, electronic filters may be used. Figure 12.20 shows generic examples of electronic filters commonly employed: low-pass, high-pass and bandpass filters. Details about the proper choice of elements and general hints for the efficient use of such filters are given in [74]. These filters

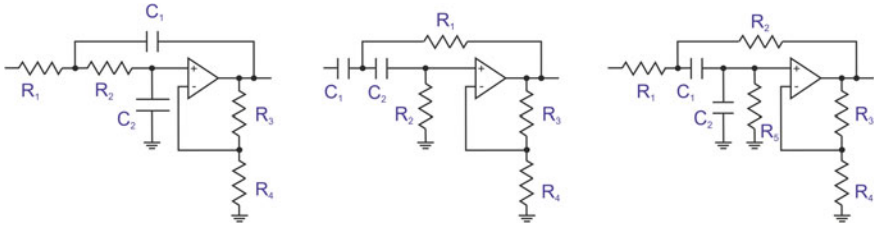


Fig. 12.20 Generic examples of electronic filters commonly used: low-pass (left), high-pass (middle) and a bandpass filter (right)

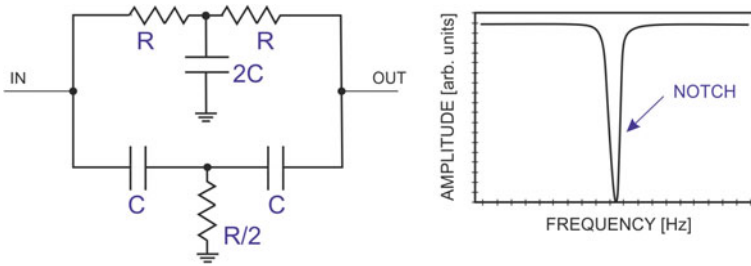


Fig. 12.21 Example of a notch filter design and illustration of the notch it cuts into a signal (amplitude as a function of frequency)

have a rather sharp cut-off characteristic but are normally not readily installed in cryogenic surroundings. In such cases, often simple RC-circuits are used, as will be discussed below. As their names suggest, they filter out high-frequency components, low-frequency components or some frequency band, respectively.

In some cases, it is required to efficiently filter out a narrow frequency band, as we have for example seen in Sect. 10.2 about resonant ejection of unwanted particles. In such situations, the use of a notch filter may be indicated. There is a number of variations on this, and we display a generic case, a passive twin-T notch filter [74]. Depending on the parameters, it can have a very sharp characteristic of the notch it cuts into a given signal, as illustrated in Fig. 12.21 (right).

As far as the dc trap voltage supplies are concerned, an obvious solution against noise is the use of a low-pass filter that allows the dc-part of the voltage to pass while high frequencies are suppressed. Electronic low-pass filters commonly have lower cut-off frequencies of the order of some kilohertz, which well above the dc-components and well below any of the oscillation frequencies.

In many cases, however, the same electrodes that are to be supplied with filtered dc voltages also need to be switched on short timescales, for example for dynamic capture, see Sect. 9.2. Especially for dynamic ion capture and ion ejection it is necessary to reach switching times well below $10\,\mu\text{s}$. In addition, we want to remain capable to resonantly excite particles by applied radio-frequency signals which is in contradiction to static low-pass filtering.

To overcome this issue, Schottky diodes are assembled in parallel to the resistor of the RC-circuit as depicted in Fig. 12.22. Schottky diodes have a very strong dependence of the current I on the voltage difference U between both ports, following an exponential function

$$I(U) = A \cdot \exp(B \cdot U), \quad (12.140)$$

which is usually valid between 0.1 and 0.6 V. Below this interval the Schottky diode has a very high resistance and is negligible compared to the resistor, while above this interval the resistance is very small, such that the electrode will be charged or discharged with a very high current. The switching times thus obtained are well below 1 μs . Applied sinusoidal signals with amplitudes above 1 V can be transmitted. Figure 12.23 shows a simulation of the transmitted voltage curves for high and low amplitude. Obviously, the low-amplitude noise signal in part (a) is suppressed, while the high-amplitude excitation signal in part (b) is transmitted.

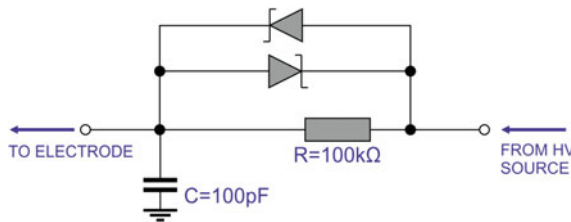


Fig. 12.22 Schematic of an RC circuit in which Schottky diodes are assembled in parallel to the resistor

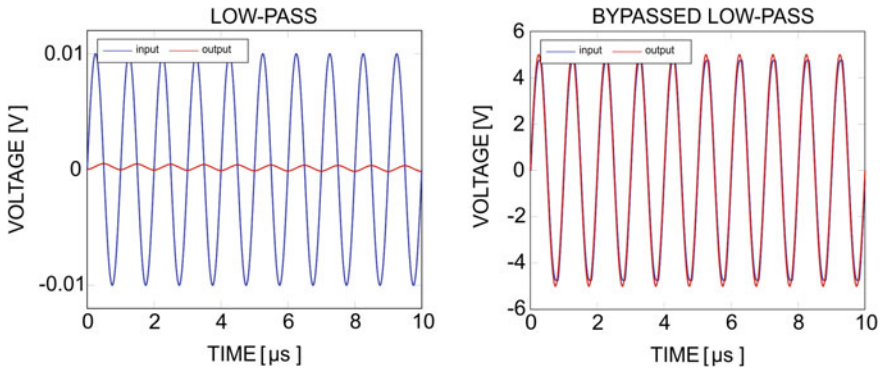


Fig. 12.23 Comparison of the behaviour of the filter circuit for an amplitude of 5 V (right), and 0.01 V (left). The signal for little amplitudes is well-suppressed, while for high amplitudes the signal is fully transferred

12.18 Adiabatic Cooling

As we have seen in (5.5), the amplitude of a motion is related to the kinetic energy of that motion. For the perturbed cyclotron and axial motions, the amplitude increases with energy, such that cooling of these motions means a decrease of both energy and amplitude. This however is only true at a fixed value of the electrostatic potential U . Let us have a look at the situation when the confining potential is time-dependent, such that we have the axial equation of motion

$$\ddot{z} + \omega_z^2(t) = 0 \quad (12.141)$$

with the solution

$$\omega_z^2(t) = \frac{qU(t)}{md^2}. \quad (12.142)$$

If now the change of the axial frequency with time is slow enough to fulfill

$$\frac{1}{\omega_z} \frac{\partial \omega_z}{\partial t} \ll \omega_z, \quad (12.143)$$

then the quantity

$$\frac{1}{\omega_z} \left(\left(\frac{\partial z}{\partial t} \right)^2 + \omega_z^2(t) z^2 \right) = \omega_z a_z^2 \quad (12.144)$$

is an adiabatic invariant of motion, i.e. the entropy of the system does not change along its trajectory in parameter space. This means that when the potential U is adiabatically decreased by a factor f , the axial energy

$$E_z = \frac{1}{2} m \omega_z^2 a_z^2 \quad (12.145)$$

is reduced by a factor $f^{1/2}$ while the amplitude a_z is actually increased by a factor $f^{1/4}$. This means that while the potential is slowly lowered, the particles assume a larger volume inside the trap, but at a lower average energy. This is somewhat equivalent to a gas that is adiabatically expanded and cools during this process. For a more detailed discussion see [111–113]. Based on the similarity to adiabatic gas expansion, a refrigeration cycle for confined particles using adiabatic cooling has been suggested [35, 114].

References

1. W.M. Itano, J.C. Bergquist, J.J. Bollinger, D.J. Wineland, Cooling methods in ion traps. *Phys. Scr.* **T59**, 106 (1995)

2. D.J. Wineland, H.G. Dehmelt, Principles of the stored ion calorimeter. *J. Appl. Phys.* **46**, 919 (1975)
3. M. Vogel et al., Resistive and sympathetic cooling of highly-charged-ion clouds in a Penning trap. *Phys. Rev. A* **90**, 043412 (2014)
4. J.F. Goodwin, G. Stutter, R.C. Thompson, D.M. Segal, Resolved-sideband laser cooling in a Penning trap. *Phys. Rev. Lett.* **116**, 143002 (2016)
5. W.M. Itano, D.J. Wineland, Laser cooling of ions stored in harmonic and Penning traps. *Phys. Rev. A* **25**, 35 (1982)
6. J. Eschner, G. Morigi, F. Schmidt-Kaler, R. Blatt, Laser cooling of trapped ions. *J. Opt. Soc. Am. B* **20**, 1003 (2003)
7. S. Djekic et al., Temperature measurement of a single ion in a Penning trap. *Eur. Phys. J. D* **31**, 451 (2004)
8. B. D'Urso, B. Odom, G. Gabrielse, Feedback cooling of a one-electron oscillator. *Phys. Rev. Lett.* **90**, 043001 (2003)
9. A. Mooser et al., Demonstration of the double Penning trap technique with a single proton. *Phys. Lett. B* **723**, 78 (2013)
10. D.A. McQuarrie, *Statistical Mechanics* (Harper and Row, New York, 1976)
11. M.J. Jensen, T. Hasegawa, J.J. Bollinger, Temperature and heating rate of ion crystals in Penning traps. *Phys. Rev. A* **70**, 033401 (2004)
12. M.J. Jensen, T. Hasegawa, J.J. Bollinger, Temperature measurements of laser-cooled ions in a Penning trap, in: *Non Neutral Plasmas V, AIP Conf. Proceedings* vol. 692 (2003), p. 193
13. E.A. Cornell, R.M. Weisskoff, K.R. Boyce, D.E. Pritchard, Mode coupling in a Penning trap: π pulses and a classical avoided crossing. *Phys. Rev. A* **41**, 312 (1990)
14. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
15. J. Verdu, Ultrapräzise Messung des elektronischen g -Faktors in wasserstoffähnlichem Sauerstoff, Ph.D. thesis, University of Mainz (2003)
16. W. Nagourney, G. Janik, H. Dehmelt, Linewidth of single laser-cooled Mg ion in radiofrequency trap. *Proc. Natl. Acad. Sci. U S A* **80**, 643 (1983)
17. W. Neuhauser, M. Hohenstatt, P.E. Toschek, H.G. Dehmelt, Optical-sideband cooling of visible atom cloud confined in parabolic well. *Phys. Rev. Lett.* **41**, 233 (1978)
18. D.J. Wineland, R.E. Drullinger, F.L. Walls, Radiation-pressure cooling of bound resonant absorbers. *Phys. Rev. Lett.* **40**, 1639 (1978)
19. F. Diedrich et al., Observation of a phase transition of stored laser-cooled ions. *Phys. Rev. Lett.* **59**, 2931 (1987)
20. J.C. Bergquist, W.M. Itano, D.J. Wineland, Recoilless optical absorption and Doppler sidebands of a single trapped ion. *Phys. Rev. A* **36**, 428 (1987)
21. F. Diedrich, J.C. Bergquist, W.M. Itano, and D.J. Wineland, Laser Cooling to the Zero-Point Energy of Motion, *Phys. Rev. Lett.* **62**, 403 (1989)
22. C. Champenois, Laser cooling techniques applicable to trapped ions, in: *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson (World Scientific, Singapore, 2016)
23. W. Demtröder, *Laser Spectroscopy* (Springer, Heidelberg, 2003)
24. H.J. Metcalf, P. Straten, *Laser Cooling and Trapping of Neutral Atoms* (Wiley Online Library, 2007)
25. R.C. Thompson, G.P. Barwood, P. Gill, Laser cooling of magnesium ions confined in a Penning trap. *Optica Acta* **33**, 535 (1986)
26. K. Dholakia et al., Investigation of ion dynamics in a Penning trap using a pulse-probe technique. *Appl. Phys. B* **60**, 375 (1995)
27. G. Birkel, S. Kassner, H. Walther, Multiple-shell structures of laser-cooled $^{24}\text{Mg}^+$ ions in a quadrupole storage ring. *Nature* **357**, 310 (1992)
28. C.J. Foot, *Atomic Physics*, Oxford Master Series in Atomic, Optical and Laser Physics, Oxford University Press (2005, reprint 2009)
29. G.Zs.K. Horvath, R.C. Thompson, Laser cooling of ions stored in a Penning trap: a phase-space picture. *Phys. Rev. A* **59**, 4530 (1999)

30. R.C. Thompson, J. Papadimitriou, Simple model for the laser cooling of an ion in a Penning trap. *J. Phys. B*, **33**, 3393 (2000)
31. W.M. Itano, L.R. Brewer, D.J. Larson, D.J. Wineland, Perpendicular laser cooling of a rotating ion plasma in a Penning trap. *Phys. Rev. A* **38**, 5698 (1988)
32. S. Mavadia, *Motional Sideband Spectra and Coulomb Crystals in a Penning Trap*, Ph.D. thesis, Imperial College London (2013)
33. J.F. Goodwin, *Sideband Cooling to the Quantum Ground State in a Penning Trap*, Ph.D. thesis, Imperial College London (2015)
34. S.B. Torrisi, J.W. Britton, J.G. Bohnet, J.J. Bollinger, Perpendicular laser cooling with a rotating-wall potential in a Penning trap. *Phys. Rev. A* **93**, 043421 (2016)
35. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
36. R.C. Thompson, Ion Coulomb crystals. *Contemp. Phys.* **56**, 63 (2015)
37. W.M. Itano, J.C. Bergquist, J.J. Bollinger, D.J. Wineland, *Laser Cooling of Trapped Ions* (North-Holland, 1992)
38. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions* (Dover, 1972)
39. S. Mavadia, G. Stutter, J.F. Goodwin, D.R. Crick, R.C. Thompson, D.M. Segal, Optical sideband spectroscopy of a single ion in a Penning trap. *Phys. Rev. A* **89**, 032502 (2014)
40. L. Gruber, J.P. Holder, D. Schneider, Formation of strongly coupled plasmas from multi-component ions in a Penning trap. *Phys. Scr.* **71**, 60 (2005)
41. J.H. Wesenberg et al., Fluorescence during Doppler cooling of a single trapped atom. *Phys. Rev. A* **76**, 053416 (2007)
42. M. Kretschmar, Particle motion in a Penning trap. *Eur. J. Phys.* **12**, 240 (1991)
43. H.W. Ellis, R.Y. Pai, E.W. McDaniel, E.A. Mason, L.A. Viehland, Transport properties of gaseous ions over a wide energy range. Part I. *At. Data Nucl. Data Tables* **17**, 177 (1976)
44. H.W. Ellis, E.W. McDaniel, D.L. Albritton, L.A. Viehland, S.L. Lin, E.A. Mason, Transport properties of gaseous ions over a wide energy range. Part I. *At. Data Nucl. Data Tables* **22**, 197 (1978)
45. H.W. Ellis, M.G. Thackston, E.W. McDaniel, E.A. Mason, Transport properties of gaseous ions over a wide energy range. Part III. *At. Data Nucl. Data Tables* **31**, 113 (1984)
46. T. Murböck et al., Rapid crystallization of externally produced ions in a Penning trap. *Phys. Rev. A* **94**, 043410 (2016)
47. G. Morigi, J. Eschner, C. Keitel, Ground state laser cooling using electromagnetically induced transparency. *Phys. Rev. Lett.* **85**, 4458 (2000)
48. C.F. Roos, D. Leibfried, A. Mundt, F. Schmidt-Kaler, J. Eschner, R. Blatt, Experimental demonstration of ground state laser cooling with electromagnetically induced transparency. *Phys. Rev. Lett.* **85**, 5547 (2000)
49. F. Schmidt-Kaler, J. Eschner, G. Morigi, C.F. Roos, D. Leibfried, A. Mundt, R. Blatt, Laser cooling with electromagnetically induced transparency: application to trapped samples of ions or neutral atoms. *Appl. Phys. B* **73**, 807 (2001)
50. S.E. Harris, Electromagnetically induced transparency. *Phys. Today* **50**, 36 (1997)
51. J. Dalibard, C. Cohen-Tannoudji, Laser cooling below the Doppler limit by polarization gradients: simple theoretical models. *J. Opt. Soc. Am. B* **6**, 2023 (1989)
52. S.M. Yoo, J. Javanainen, Polarization gradient cooling of a Trapped ion. *Phys. Rev. A* **48**, R30 (1993)
53. G. Birkel, J.A. Yeazell, R. Rückerl, H. Walther, Polarization gradient cooling of Trapped ions. *Europhys. Lett.* **27**, 197 (1994)
54. J. Dalibard, C. Cohen-Tannoudji, Dressed-atom approach to atomic motion in laser light: the dipole force revisited. *J. Opt. Soc. Am. B* **2**, 1707 (1985)
55. M. Kasevich, S. Chu, Laser cooling below a photon recoil with three-level atoms. *Phys. Rev. Lett.* **69**, 1741 (1992)
56. C. Monroe et al., Resolved-sideband Raman cooling of a bound atom to the 3D zero-point energy. *Phys. Rev. Lett.* **75**, 4011 (1995)
57. G. Savard et al., A new cooling technique for heavy ions in a Penning trap. *Phys. Lett. A* **158**, 247 (1991)

58. H.-U. Hasse et al., External-ion accumulation in a Penning trap with quadrupole excitation assisted buffer gas cooling. *Int. J. Mass Spectrom. Ion Processes* **132**, 181 (1994)
59. K.M. Ervin, Experimental techniques in gas-phase ion thermochemistry. *Chem. Rev.* **101**, 391 (2001)
60. S. Krückeberg et al., Multiple-collision induced dissociation of trapped silver clusters. *J. Chem. Phys.* **110**, 7216 (1999)
61. D. Fischer et al., Fully differential cross sections for the single ionization of helium by ion impact. *J. Phys. B.* **36**, 3555 (2003)
62. M. Kretzschmar, Calculating damping effects for the ion motion in a Penning trap. *Eur. Phys. J. D* **48**, 313 (2008)
63. W. Ketterle, N.J. van Druten, Evaporative cooling of trapped atoms. *Adv. At. Mol. Opt. Phys.* **37**, 181 (1996)
64. T. Greytak, Prospects for Bose-Einstein condensation in magnetically trapped atomic hydrogen, in *Bose-Einstein Condensation*, ed. by A. Griffin, D. Snoke, S. Stringari (Cambridge University, Cambridge, UK, 1995)
65. M.B. Schneider, M.A. Levine, C.L. Bennett, J.R. Henderson, D.A. Knapp, R.E. Marrs, in *Electron Beam Ion Sources and Their Applications, AIP Conference Proceedings*, vol. 188, ed. by A. Herscovitch (American Institute of Physics, New York, 1989)
66. B.M. Penetrante, D. Schneider, R.E. Marrs, J.N. Bardsley, Modeling the ion source performance of an electron beam ion trap. *Rev. Sci. Instrum.* **63**, 2806 (1992)
67. I. Bergström, C. Carlberg, T. Fritioff, G. Douysset, J. Schnfelder, R. Schuch, SMILETRAP Penning trap facility for precision mass measurements using highly charged ions. *Nucl. Instr. Meth. A* **487**, 618 (2002)
68. W. Shockley, Currents to conductors induced by a moving point charge. *J. Appl. Phys.* **9**, 635 (1938)
69. Z. He, Review of the Shockley-Ramo theorem and its application in semiconductor gamma-ray detectors. *Nucl. Instr. Meth. A* **463**, 250 (2001)
70. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
71. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)
72. X. Feng et al., Tank circuit model applied to particles in a Penning trap. *J. Appl. Phys.* **79**, 8 (1996)
73. D.F.A. Winters, M. Vogel, D.M. Segal, R.C. Thompson, Electronic detection of charged particle effects in a Penning trap. *J. Phys. B* **39**, 3131 (2006)
74. P. Horowitz, W. Hill, *The Art of Electronics* (Cambridge University Press, Cambridge, 1989)
75. S. van Gorp et al., Simbuca, using a graphics card to simulate Coulomb interactions in a Penning trap. *Nucl. Instr. Meth. A* **638**, 192 (2011)
76. J. Steinmann, Modellierung und Simulation der Widerstandskuehlung von hochgeladenen Ionen, Ph.D. thesis, University of Erlangen-Nürnberg (2015)
77. G. Gabrielse, L. Haarsma, S.L. Rolston, Open-endcap Penning traps for high precision experiments. *Int. J. Mass Spectrom. Ion Processes* **88**, 319 (1989)
78. G. Gabrielse, F.C. Macintosh, Cylindrical Penning traps with orthogonalized anharmonicity compensation. *Int. J. Mass. Spectrom. Ion Processes* **57**, 1 (1984)
79. D.J. Wineland, H.G. Dehmelt, Line shifts and widths of axial, cyclotron and G-2 resonances in tailored, stored electron (ion) cloud. *Int. J. Mass Spectrom. Ion Processes* **16**, 338 (1975) and **19**, 251 (1976)
80. S. Ulmer et al., The quality factor of a superconducting rf resonator in a magnetic field. *Rev. Sci. Instrum.* **80**, 123302 (2009)
81. J.B. Johnson, Thermal agitation of electricity in conductors. *Phys. Rev.* **32**, 97 (1928)
82. H. Nyquist, Thermal agitation of electric charge in conductors. *Phys. Rev.* **32**, 110 (1928)
83. S. Ulmer, C. Smorra, The magnetic moments of the proton and the antiproton, in *Fundamental Physics in Particle Traps*, Springer Tracts in Modern Physics, vol. 256 (Springer, 2014)
84. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)

85. J. Steinmann, J. Groß, F. Herfurth, G. Zwicknagel, MD simulations of resistive cooling in HITRAP using GPUs, in *AIP Conference Proceedings*, vol. 1521 (2013), p. 240
86. G. Maero et al., Numerical investigations on resistive cooling of trapped highly charged ions. *Appl. Phys. B* **107**, 1087 (2012)
87. G. Maero, *Cooling of highly charged ions in a Penning trap for HITRAP*, Ph.D. thesis, University of Heidelberg (2008)
88. B. d'Urso, *Cooling and Self-Excitation of a One-Electron Oscillator*, Ph.D. thesis, Harvard (2003)
89. N. Beverini et al., Experimental verification of stochastic cooling in Penning trap. *Phys. Scr.* **T22**, 238 (1988)
90. R.E. Drullinger, D.J. Wineland, J.C. Bergquist, High-resolution optical spectra of laser cooled ions. *Appl. Phys.* **22**, 365 (1980)
91. M.A. van Eijkelenborg, M.E.M. Storkey, D.M. Segal, R.C. Thompson, Sympathetic cooling and detection of molecular ions in a Penning trap. *Phys. Rev. A* **60**, 3903 (1999)
92. NRL Plasma Formulary, J.D. Huba, Beam Physics Branch, Plasma Physics Division, Naval Research Laboratory, Washington, DC 20375 (2013)
93. L. Gruber et al., Evidence for highly charged ion coulomb crystallization in multicomponent strongly coupled plasmas. *Phys. Rev. Lett.* **86**, 636 (2001)
94. M. Vogel, W. Quint, Trap-assisted precision spectroscopy of forbidden transitions in highly charged ions. *Phys. Rep.* **490**, 1 (2010)
95. G. Gabrielse et al., Cooling and slowing of trapped antiprotons below 100 meV. *Phys. Rev. Lett.* **63**, 1360 (1989)
96. G. Gabrielse et al., Precision mass spectroscopy of the antiproton and proton using simultaneously trapped particles. *Phys. Rev. Lett.* **82**, 3198 (1999)
97. H. Higaki et al., Electron cooling of high-energy protons in a multiring trap with a tank circuit monitoring the electron-plasma oscillations. *Phys. Rev. E* **65**, 046410 (2003)
98. N. Oshima et al., Development of a cold HCI source for ultra-slow collisions. *Nucl. Instr. Meth. B* **205**, 178 (2003)
99. A. Mohri et al., in *Proceedings of Non Neutral Physics IV*, AIP, New York, ed. by F. Anderegg et al. (2002), pp. 634640
100. J. Bernard et al., Electron and positron cooling of highly charged ions in a cooler Penning trap. *Nucl. Instr. Meth. A* **532**, 224 (2004)
101. B.E. Schultz et al., Cooling of highly-charged, short-lived ions for precision mass spectrometry at TRIUMF's Ion Trap for Atomic and Nuclear Science. *Phys. Scr.* **T156**, 014097 (2013)
102. M. Bohmann et al., Sympathetic cooling of protons and antiprotons with a common endcap Penning trap. *J. Mod. Opt.* (2017), <https://doi.org/10.1080/09500340.2017.1404656>
103. D.J. Heinzen, D.J. Wineland, Quantum-limited cooling and detection of radiofrequency oscillations by laser-cooled ions. *Phys. Rev. A* **42**, 2977 (1990)
104. W. Schottky, Über spontane Stromschwankungen in verschiedenen Elektrizitätsleitern. *Ann. Phys.* **57**, 541 (1918)
105. YaM Blanter, M. Büttiker, Shot noise in mesoscopic conductors. *Phys. Rep.* **336**, 1–166 (2000)
106. R.F. Voss, J. Clarke, Flicker (1/f) noise: equilibrium temperature and resistance fluctuations. *Phys. Rev. B* **13**, 556 (1976)
107. H.G.E. Beck, W.P. Spruit, 1/f noise in the variance of Johnson noise. *J. Appl. Phys.* **49**, 3384 (1978)
108. J. Labaziewicz et al., Temperature dependence of electric field noise above gold surfaces. *Phys. Rev. Lett.* **101**, 180602 (2008)
109. M. Brownnutt, M. Kumph, P. Rabl, R. Blatt, Ion-trap measurements of electric-field noise near surfaces. *Rev. Mod. Phys.* **87**, 1419 (2015)
110. A. Safavi-Naini, P. Rabl, P.F. Weck, H.R. Sadeghpour, Microscopic model of electric-field-noise heating in ion traps. *Phys. Rev. A* **84**, 023412 (2011)
111. G.Z. Li, R. Poggiani, G. Testera, G. Werth, Adiabatic cooling of ions in the penning trap. *Z. Phys. D* **22**, 375 (1991)
112. G. Gabrielse et al., Adiabatic cooling of antiprotons. *Phys. Rev. Lett.* **106**, 073002 (2011)

113. S.L. Rolston, G. Gabrielse, Cooling antiprotons in an ion trap. *Hyperfine Interact.* **44**, 233 (1988)
114. F.G. Major, *The Quantum Beat: the Physical Principles of Atomic Clocks* (Springer, Berlin, 1998)



Abstract The particle number density and shape of an ensemble of confined in a Penning trap can be controlled by the so-called ‘rotating wall technique’, which is a specific, non-resonant excitation of the ensemble’s rotation. Here, we briefly discuss the requirements, technical implementations, and the phenomenology of such a rotating wall, mainly when used for compression of the confined ensemble.

13.1 The Concept and Implementation

As we have discussed in Chap. 8, a strongly correlated plasma confined in a Penning trap is forced (by the presence of the magnetic field B_0) into a global rotation at a frequency $\omega_- < \omega < \omega_+$ that depends on initial conditions. Most often, this frequency ω will be close to ω_- which, as we will see, corresponds to a low particle density n . One may desire to increase this density which is possible by increasing the angular momentum of the plasma. It is defined by

$$L = m \left(\frac{\omega_c}{2} - \omega \right) \int n(\rho, z) \rho^2, d^3\rho \quad (13.1)$$

and in case of a spheroidal plasma reads [1]

$$L = \frac{3}{20} \frac{mN}{\pi n \rho_0} (\omega_c - 2\omega), \quad (13.2)$$

where N is the particle number and ρ_0 is the plasma radius. When we apply a torque $\dot{L}$ to the plasma by a rotating external field around the central axis z , we may control the particle number density of the plasma, this is done in a ‘rotating wall’. Note that alternatively, a laser perpendicular to the trap axis with an offset from the axis can be used to apply a torque [1–4].

The term ‘rotating wall’ is used for a configuration in which on top of the electrostatic potential, also a rotating electric field is used. For a thorough discussion of the topic see [5]. This is usually achieved by a segmentation of cylindrical electrodes into at least four segments (Fig. 13.1). Most often, a sinusoidal signal at a

fixed frequency ω_r is applied, with a phase shift that is different for each of the k segments, typically a symmetric situation is created by choosing the phase shift between adjacent electrodes to be $360/k$ degrees, see our discussion in Sect. 10.1. The effect of a rotating wall field is to exert a torque such that the global rotation of the confined particle ensemble is changed to the applied rotating wall frequency [6] (Fig. 13.2). Higher-order field components additionally influence the shape of the rotating ensemble [7].

For this brief discussion, we start with the expression for the potential seen by a particle within a confined particle ensemble that is a spheroid, i.e. an ellipsoid of revolution about the trap axis z in a cylindrical Penning trap, i.e. in a radially symmetric trapping potential. This is always the case under equilibrium conditions, when the ensemble has assumed a global rotation at a frequency $\omega \geq \omega_-$, which is given by the initial conditions [2]. Note, that under the influence of external forces such as a rotating quadrupole field, the spheroid, which is a bi-axial ellipsoid, can be deformed to a co-rotating tri-axial ellipsoid, the cross-section (perpendicular to

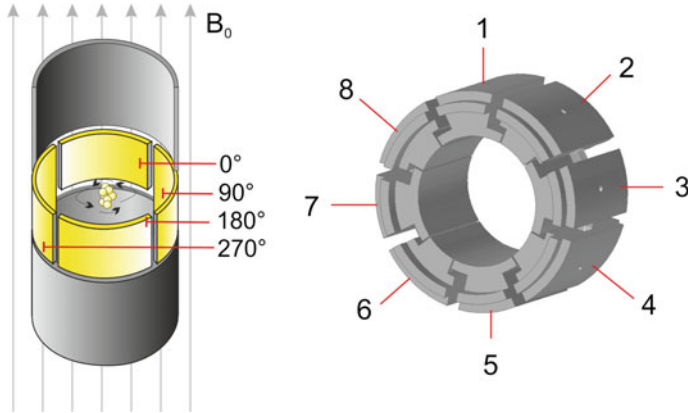


Fig. 13.1 Schematic of a rotating wall setup with one cylindrical electrode split into four equal segments (left) and eight segments (right). The numbers indicated are the phases and the order of the applied sinusoidal rotating wall voltages, respectively

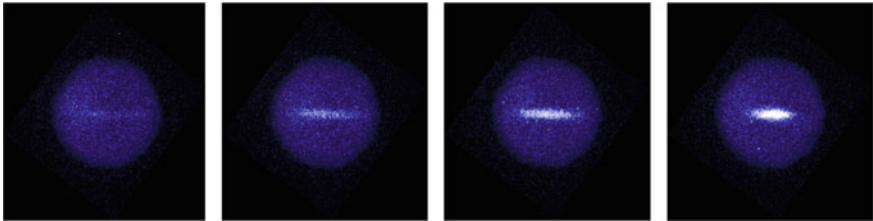


Fig. 13.2 Generic effect of a rotating wall drive as a function of time (left to right): increase of the particle density close to the trap centre seen as an increase of fluorescence light upon laser irradiation

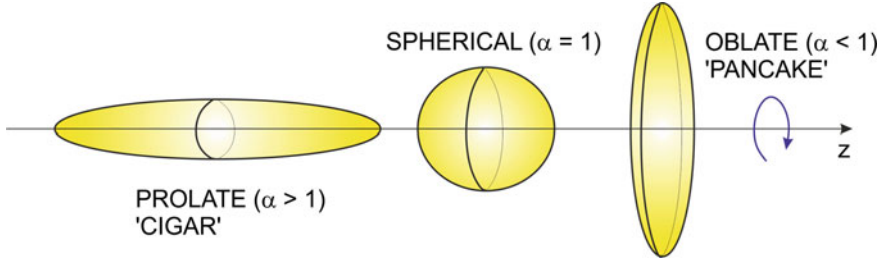


Fig. 13.3 Illustration of the three types of cases for the ensemble aspect ratio

the trap axis) of which is hence no longer a circle [7]. It is not straightforward to say how long it takes for such an equilibrium to establish, a closer examination is required. For now we take it as given, and further assume sufficiently cold particles with $k_B T \ll qU$ such that the confinement volume is within the harmonic region of the trapping potential. We may safely neglect the image charge potential when $\rho, z \ll d$, i.e. the radial and axial distances of all particles from the trap centre are much smaller than the trap size. Then the potential takes the form

$$A(z, \rho) = \frac{m}{2q} \omega (\omega_c - \omega) \rho^2 + \frac{U}{4d^2} (2z^2 - \rho^2) - \frac{qn}{6\epsilon_0} (a(\alpha)\rho^2 + b(\alpha)z^2), \quad (13.3)$$

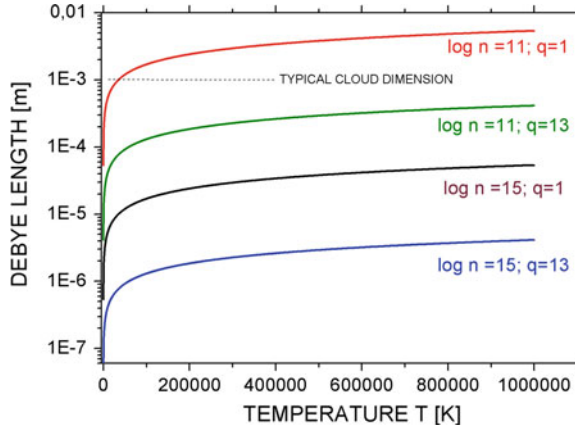
where $(m/(2q))\omega\omega_c\rho^2$ is the electric potential of a charge rotating at frequency ω on a radius ρ perpendicular to a magnetic field, $-(m/(2q))\omega^2\rho^2$ is the corresponding centrifugal potential, and $-(U/(4d^2))(2z^2 - \rho^2)$ is the trapping potential due to the applied voltage U_0 . In the ideal cylindrical case $U = C_2 U_0$. The rightmost term, the space charge potential within the particle ensemble, is written in terms of radial and axial parameters $a(\alpha)$ and $b(\alpha)$ which depend on the aspect ratio $\alpha = z/\rho$ of the ensemble (Fig. 13.3).

Under equilibrium conditions at a finite temperature T , the density n within the ensemble is constant and drops off at the boundary with a characteristic length given by the Debye length (shielding length) (Fig. 13.4)

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{2nq^2}}, \quad (13.4)$$

which leads to an overall constant density and sharp boundaries in the limit $T = 0$. The following description requires that the ensemble size is much larger than the Debye length at the given temperature, i.e. we need $\lambda_D \ll z, \rho$. In any way, a limitation of the particle number density n is given once the space charge term compensates the confining terms, i.e. when we have $A(z, \rho) = 0$ for any (z, ρ) . Strictly for this argument to hold, we require $k_B T = 0$. But since for all practical means we require a cold particle ensemble anyway, we may for the sake of simplicity

Fig. 13.4 Value of the Debye length according to (13.4) as a function of the temperature T for different ion number densities and charge states q in units of e . $\log(n) = 11$ is roughly the minimum density of $10^5/\text{cm}^3$, while $\log(n) = 15$ is roughly the Brillouin density at several Tesla of some $10^9/\text{cm}^3$



assume zero temperature and arrive at

$$\frac{qn}{6\epsilon_0} (a(\alpha)\rho^2 + b(\alpha)z^2) = \frac{m}{2q} \omega(\omega_c - \omega)\rho^2 + \frac{U}{4d^2} (2z^2 - \rho^2). \quad (13.5)$$

We can separate the axial and radial degrees of freedom to find two conditions on the maximum particle number density n :

$$n_1 = \frac{6\epsilon_0}{qa(\alpha)} \left(\frac{m}{2q} \omega(\omega_c - \omega) - \frac{U}{4d^2} \right) \quad (\text{radial}) \quad (13.6)$$

$$n_2 = \frac{6\epsilon_0}{qb(\alpha)} \frac{U}{2d^2} \quad (\text{axial}). \quad (13.7)$$

These limits are not independent, since the geometry parameters $a(\alpha)$ and $b(\alpha)$ fulfill the condition $2a(\alpha) + b(\alpha) = 3$ due to Poisson's law. The parameter $b(\alpha)$ is given by [8]

$$b(\alpha) = \frac{3}{\alpha^2 - 1} Q_1^0 \left(\frac{\alpha}{\sqrt{\alpha^2 - 1}} \right), \quad (13.8)$$

where Q_1^0 is the associated Legendre function of the second kind, in this case $Q_1^0(x) = x \tanh^{-1}(x) - 1$. In case of a prolate particle ensemble ($\alpha > 1$) it can be evaluated by

$$b(\alpha) = \frac{3}{\alpha^2 - 1} \left(\frac{k}{2} \ln \left(\frac{k+1}{k-1} \right) - 1 \right) \quad \text{with} \quad k = \frac{\alpha}{(\alpha^2 - 1)^{1/2}}, \quad (13.9)$$

while for an oblate particle ensemble ($\alpha < 1$) it is given by

$$b(\alpha) = \frac{3}{\alpha^2 - 1} \left(k \tan^{-1} \left(\frac{1}{k} \right) - 1 \right) \quad \text{with} \quad k = \frac{\alpha}{(1 - \alpha^2)^{1/2}}. \quad (13.10)$$

For a spherical ensemble we have $\alpha = 1$ and hence $a = b = 1$, while for an infinitely thin oblate we have $\alpha = 0$ and hence $a = 0$ and $b = 3$. Correspondingly, for an infinitely prolate shape, $b = 0$ and $a = 3/2$.

13.2 The Brillouin Limit

Coming back to conditions (13.6) and (13.7), we are free to force the global ensemble rotation frequency ω to the Brillouin frequency $\omega_c/2$ by the rotating wall drive. Seen in the Brillouin frame of reference as discussed in Sect. 5.4, the ensemble is stationary then. In this case, the equations (radial and axial) can (by use of $\omega_c = qB_0/m$) be written as

$$n_1 = \frac{3}{4} \frac{\varepsilon_0 B^2}{ma(\alpha)} - \frac{3\varepsilon_0 U_0 C_2}{2a(\alpha)qd^2} \quad \text{and} \quad n_2 = \frac{3\varepsilon_0 U_0 C_2}{b(\alpha)qd^2}. \quad (13.11)$$

At a voltage of

$$U_0 = \frac{1}{6C_2} \frac{qB_0^2 d^2}{m} \quad (13.12)$$

we have

$$n_1 = \frac{1}{2} \frac{\varepsilon_0 B_0^2}{ma(\alpha)} \quad \text{and} \quad n_2 = \frac{1}{2} \frac{\varepsilon_0 B_0^2}{mb(\alpha)}. \quad (13.13)$$

For a drive at the Brillouin frequency

$$\omega = \frac{\omega_c}{2} \quad (13.14)$$

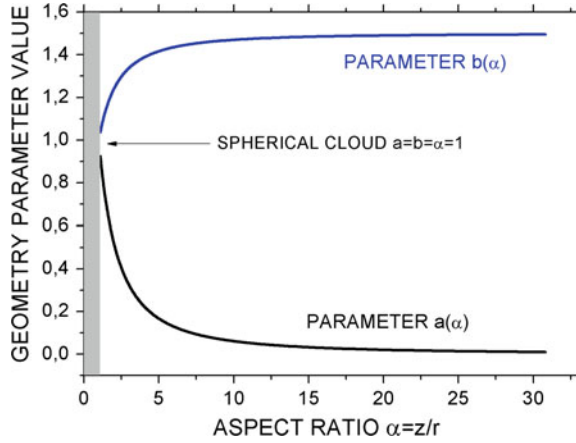
we always have the maximum particle number density given by

$$n_1 = n_2 = \frac{\varepsilon_0 B_0^2}{2m} \quad (13.15)$$

irrespective of the trap voltage.

This means that also for ensembles of many particles, like in the two-particle case discussed in Sect. 7.10, the particle-particle distance in a low-energy ensemble is highest when it globally rotates at the Brillouin frequency. For the specific voltage (13.12) we have a spherical ensemble with $a(\alpha) = b(\alpha) = 1$ and hence $\alpha = 1$.

Fig. 13.5 Values of the geometry parameters a and b as a function of the aspect ratio $\alpha = z/\rho$ of the particle ensemble in the region $\alpha \geq 1$ according to (13.9). The grey shaded region represents the oblate shapes (not calculated)



Note, that the Brillouin density is a number density that does not depend on the charge state of the ions, since q appears both in the repulsive and attractive terms and hence cancels. The ‘price’ paid for getting ion ensembles of higher charge states to the same number density (and thus higher charge density) is the higher Brillouin frequency $qB/(2m)$ necessary for the drive when compared to lower charge states.

More generally, a specific aspect ratio α can be chosen at any rotating wall frequency (if desired by the experiment), and the corresponding geometry parameters a and b follow from (13.8), see also Fig. 13.5. We use the relations

$$b(\alpha) = \frac{3\omega_z^2}{\omega_p^2} \quad \text{with} \quad \omega_p^2 = 2\omega(\omega_c - \omega) \quad (13.16)$$

and the equation for the axial oscillation frequency

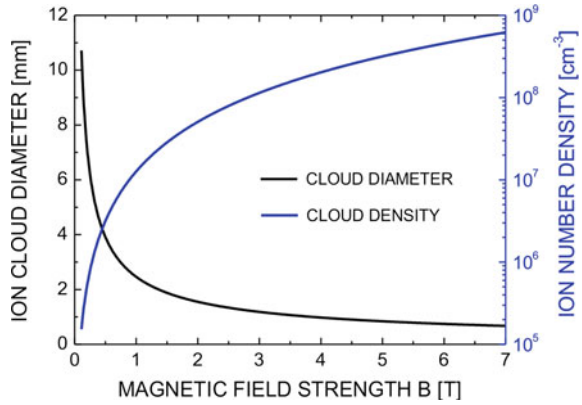
$$\omega_z^2 = \frac{qU_0C_2}{md^2}, \quad (13.17)$$

and obtain the expression

$$U_0 = \frac{2}{3} \frac{md^2}{qC_2} b(\alpha) \omega(\omega_c - \omega), \quad (13.18)$$

which yields the necessary trapping voltage U_0 at the chosen rotating wall frequency ω . Figure 13.6 shows the ion ensemble size and density as a function of the magnetic field strength for 10^7 confined charges at the Brillouin limit

Fig. 13.6 Ion ensemble size and density as a function of the magnetic field strength for 10^7 confined charges at the Brillouin limit



Under the given conditions, the particle number density is uniquely determined by ω via

$$n = \frac{2m\varepsilon_0}{q^2} \omega(\omega_c - \omega). \quad (13.19)$$

Note that this equation holds for any shape of the confining potential as long as it is axially symmetric [8].

Note also, that n only *appears* to depend on q , but that the frequency terms cancel that apparent dependence. Figure 13.7 shows the ion number density at the Brillouin limit as a function of the magnetic field strength for different ion species.

While the choice of U_0 does not affect the volume density (so long as we stay within the confinement conditions), it does change the area density of the particles, for example of the axial projection as seen by an axial laser beam. Using the relations for the volume of a spheroid with radius ρ and axial dimension $z = \alpha\rho$ we have

$$V = \frac{4}{3}\pi\rho^3\alpha = \frac{N}{n}, \quad (13.20)$$

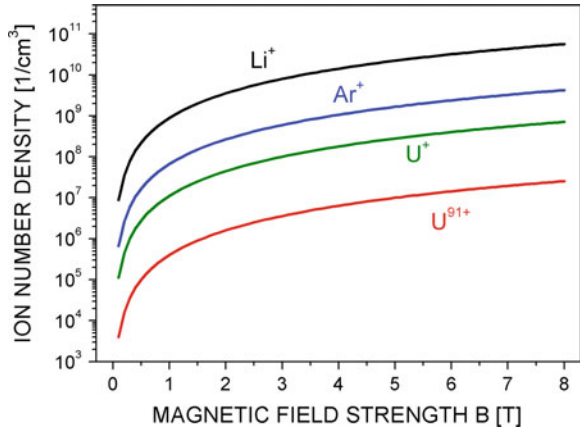
where N is the total number of particles and the volume density n is again assumed homogeneous. Inserting (13.19) and solving for the ensemble radius yields

$$\rho^3 = \frac{3}{8\pi} \frac{q^2}{m\varepsilon_0} \frac{1}{\omega(\omega_c - \omega)} \frac{N}{\alpha} \quad (13.21)$$

with a value of ρ at the Brillouin limit of

$$\rho_B^3 = \frac{3}{2\pi} \frac{m}{\varepsilon_0 B_0^2} \frac{N}{\alpha}, \quad (13.22)$$

Fig. 13.7 Ion number density at the Brillouin limit as a function of the magnetic field strength for different ion species



such that the average axial area density $\sigma = N/(\pi\rho^2)$ is given by

$$\sigma = \frac{N}{\pi} \left(\frac{8\pi}{3} \frac{m\varepsilon_0}{q^2} \omega(\omega_c - \omega) \frac{\alpha}{N} \right)^{2/3}, \quad (13.23)$$

taking the Brillouin-limit value

$$\sigma_B = \frac{N}{\pi} \left(\frac{2\pi}{3} \frac{\varepsilon_0 B_0^2}{m} \frac{\alpha}{N} \right)^{2/3}. \quad (13.24)$$

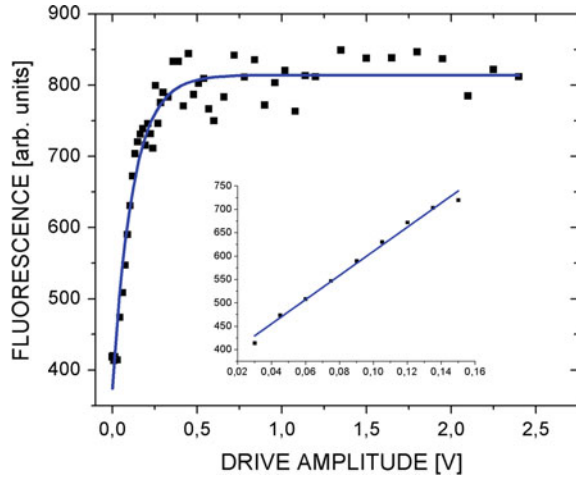
The situation of a particle density at the Brillouin limit caused by global rotation at the Brillouin frequency is often called ‘Brillouin flow’.

We need to note that the Brillouin limit applies only to stable confinement under equilibrium conditions. Away from equilibrium, for example in Penning fusion experiments [9–11], the Brillouin limit is surpassed [12] and the limiting conditions are different [13].

13.3 Amplitude and Sense of Rotation of the Drive

The required amplitude of the rotating wall drive depends on a number of experimental parameters such as size and location of the electrodes used for excitation. Note, that in a perfect trap, a pure rotating dipole is ineffective in driving the rotation of a single-component plasma. A rotating quadrupole is required by geometry, and rotating dipole excitation is possible only in the presence of trap imperfections or impurity ions [14]. The required amplitude then depends on the magnitude of the imperfections and is not readily quantified. Figure 13.8 shows an example of the observed amount of fluorescence (as a measure of the degree of compression) from

Fig. 13.8 Compression of a confined ion ensemble measured by the observed amount of fluorescence as a function of the driving voltage at the Brillouin frequency. The inset shows the linear behaviour for small values of the amplitude. Data from [6]



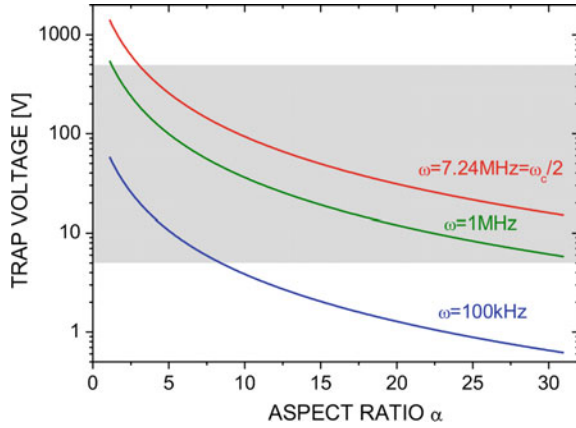
a confined pure ensemble of ions that is driven by a rotating dipole at the Brillouin frequency $\omega_c/2$ as a function of the drive amplitude. Obviously, the compression is linear with the amplitude for small values, and shows a saturation behaviour for larger amplitudes. In this case, a driving voltage of around 1 V was sufficient for an efficient drive, and the Brillouin limit of the ensemble density was reached [6].

When the excitation phases of the individual electrodes used for the rotating wall drive are chosen differently from the scheme shown in Fig. 13.1 and the discussion in Sect. 10.1, the drive is less efficient. When the sense of rotation is opposite to the actual ensemble rotation, the drive is ineffective [6]. When the ordering of the phases is such that the drive is not cyclic but has some random order, there is always a certain excitation component with the proper orientation, such that the rotation is driven, albeit with less efficiency such that a higher amplitude is required [6].

13.4 Examples

To get a feeling for the involved quantities, let us have a look at two examples. For the trap of the HILITE experiment [15], we have $C_2 = 0.592$ and $d = 0.0079$ m, while B may take any value up to 6 T. For an ensemble of singly charged argon ions with mass $m = 40$ u in a field of $B_0 = 6$ T, a trap voltage of $U_0 = 1536$ V would be required to obtain a spherical ensemble at the Brillouin limit of $n = 2.4 \times 10^9/\text{cm}^3$. For ions with higher charge states, this voltage is higher by a factor corresponding to the charge state. Reducing the magnetic field leads to a lower Brillouin limit with a quadratic scaling in B_0 , so for a field of $B_0 = 4.2$ T, the Brillouin limit of about $n = 1.2 \times 10^9/\text{cm}^3$ can be reached with $U_0 = 800$ V while the ensemble is still spherical. At this Brillouin density and assuming an ensemble size of the order of

Fig. 13.9 Trap voltage required for a specific aspect ratio at different rotating wall frequencies when considering Ar^+ ions at the HILITE experiment. The grey area indicates the accessible range of trap voltages in that experiment



mm, we require roughly $T < 4000$ K in order to fulfill $\lambda_D \ll \rho, z$. With decreasing trapping voltage, the aspect ratio $\alpha = z/\rho$ increases until either the axial extension of the ensemble leaves the region of the trapping potential or the radial extension fails to obey $\lambda_D \ll \rho, z$ (Fig. 13.9).

In the ARTEMIS experiment [16, 17], we have $C_2 = 0.5226$ and $d = 0.0077$ m, so that for $B_0 = 7$ T and boron-like argon ($m = 40u$, $q = 13e$), we have $U_0 = 30445$ V. The Brillouin limit then is $3.3 \times 10^9/\text{cm}^3$. A voltage limit of 28 V leads to an ensemble geometry of $(a, b) = (1.499, 0.002)$ at $\omega = \omega_c/2$. This geometry parameter corresponds to an aspect ratio of $\alpha \approx 80$ and an ensemble density of around $7 \times 10^6/\text{cm}^3$. Single-particle oscillation frequencies at $U_0 = 28$ V are $\omega_+ = 219.5041$ MHz, $\omega_z = 2.7714$ MHz and $\omega_- = 8.77$ kHz. For an aspect ratio of $\alpha = 1$ (spherical ensemble) we would have to have

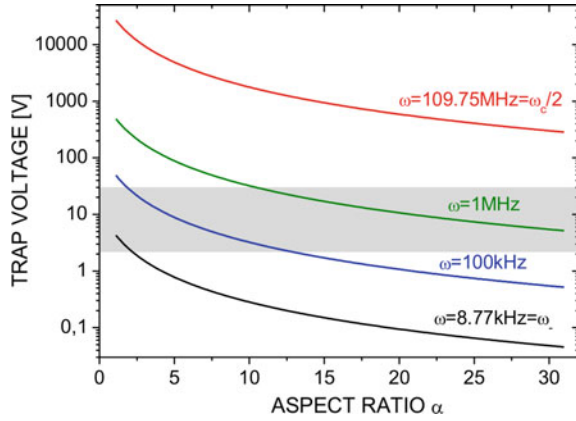
$$\omega\omega_c - \omega\omega = \frac{3U_0qC_2}{2md^2} = 1.15 \times 10^{13} \text{ Hz}^2, \quad (13.25)$$

i.e. the rotating wall frequency would have to be about 50 kHz, which is possible since it is higher than the maximum ω_- of 8.77 kHz at $U_0 = 28$ V (Fig. 13.10). It should hence be possible to obtain a minimum aspect ratio of about $\alpha = 1/6$ (oblate) at a density of $5.2 \times 10^5/\text{cm}^3$. At this density, for the condition $\lambda_D \ll z$ to hold, we require roughly $T < 70$ K if we assume an ensemble size of the order of mm.

In any case, a finite temperature $T > 0$ of the ion ensemble will decrease the density limits by some amount, since then, on average, the ‘escape’ condition will be $A - k_B T = 0$ instead of $A = 0$. Details, however, depend on the dissipation of energy within the ion ensemble by Coulomb collisions, and on the real distribution of kinetic energies, which is not necessarily a Boltzmann distribution to which a unique temperature can be assigned.

Note, that when the rotating wall frequency ω_r differs from the actual rotation frequency ω of the ion ensemble, the exerted torque is not maximal and ‘slip’ may

Fig. 13.10 Trap voltage required for a specific aspect ratio at different rotating wall frequencies when considering Ar^{13+} ions at the ARTEMIS experiment. The grey area indicates the accessible range of trap voltages in that experiment



occur in particular for large ensembles. This leads to an ensemble rotation frequency lower than ω_r , even if no internal degrees of freedom of the plasma are heated. It appears to be a specific issue with large ion plasmas, since it has been observed in [18], where only about 15% of the Brillouin density could be reached, but not for the smaller plasmas in [6], where Brillouin flow was reached.

In practice, may be advantageous to ramp up ω starting from the magnetron frequency ω_- to the desired rotation frequency. By this method, densities are increased in a controllable manner. The ideal ramping speed is a compromise between slip and undesired excitation of plasma modes. The exerted torque $\dot{L}$ has been found to follow the relationship [14]:

$$\dot{L} \propto \frac{1}{|\omega - \omega'|} \frac{V}{\sqrt{T}}, \quad (13.26)$$

where V is the amplitude of the drive, T is the plasma temperature and $|\omega - \omega'|$ is the difference between the applied and the actual rotation frequency of the ensemble.

We should also note that a rotating wall is not the only option when it comes to exert a torque on a confined ion ensemble. This can also be achieved by the force of a laser that irradiates the ensemble perpendicular to the trap axis and off-centre. It requires radial trap access, but avoids the additional electronics that comes with a rotating wall. However, control over the actual rotation frequency of the ensemble is in general more difficult to obtain.

References

1. D.J. Wineland, J.J. Bollinger, W.M. Itano, J.D. Prestage, Angular momentum of trapped atomic particles. *J. Opt. Soc. Am. B* **2**, 1721 (1985)
2. J.J. Bollinger et al., Electrostatic modes of ion-trap plasmas. *Phys. Rev. A* **48**, 525 (1993)

3. D.J. Heinzen et al., Rotational equilibria and low-order modes of a non-neutral ion plasma. *Phys. Rev. Lett.* **66**, 2080 (1991)
4. M. Asprusten, S. Worthington, R.C. Thompson, Theory and simulation of ion Coulomb crystal formation in a Penning trap. *Appl. Phys. B* **114**, 157 (2014)
5. F. Anderegg, Rotating wall technique and centrifugal separation, in *Trapped Charged Particles*, ed. by M. Knoop, N. Madsen, R.C. Thompson (World Scientific, Singapore, 2016)
6. S. Bharadia, M. Vogel, D.M. Segal, R.C. Thompson, Dynamics of laser-cooled Ca^+ ions in a Penning trap with a rotating wall. *Appl. Phys. B* **107**, 1105 (2012)
7. X.P. Huang, J.J. Bollinger, T.B. Mitchell, W.M. Itano, Phase-locked rotation of crystallized non-neutral plasmas by rotating electric fields. *Phys. Rev. Lett.* **80**, 73 (1998)
8. L.R. Brewer et al., Static properties of a non-neutral $^9\text{Be}^+$ -ion plasma. *Phys. Rev. A* **38**, 859 (1988)
9. M.M. Schauer, T.B. Mitchell, M.H. Holzschneider, D.C. Barnes, Electron Penning trap for the generation of high density non-neutral plasmas. *Rev. Sci. Instrum.* **68**, 3340 (1997)
10. D.C. Barnes, M.M. Schauer, K.R. Umstadter, L. Chacon, G.H. Miley, Electron equilibrium and confinement in a modified Penning trap and its application to Penning fusion. *Phys. Plasmas* **7**, 1693 (2000)
11. M.M. Schauer, D.C. Barnes, K.R. Umstadter, Physics of non-thermal Penning-trap electron plasma and application to ion trapping. *Phys. Plasmas* **11**, 9 (2004)
12. D.C. Barnes, T.B. Mitchell, M.M. Schauer, Beyond the Brillouin limit with the Penning fusion experiment. *Phys. Plasmas* **4**, 1745 (1997)
13. L. Chacon, D.C. Barnes, Stability of thermal ions confined by electron clouds in Penning fusion systems. *Phys. Plasmas* **7**, 4774 (2000)
14. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
15. M. Vogel, W. Quint, G. Paulus, Th Stöhlker, A Penning trap for advanced studies with particles in extreme laser fields. *Nucl. Instr. Meth. B* **285**, 65 (2012)
16. W. Quint, D. Moskovkin, V.M. Shabaev, M. Vogel, Laser-microwave double-resonance technique for g -factor measurements in highly charged ions. *Phys. Rev. A* **78**, 032517 (2008)
17. D. von Lindenfels et al., Experimental access to higher-order Zeeman effects by precision spectroscopy of highly charged ions in a Penning trap. *Phys. Rev. A* **87**, 023412 (2013)
18. X.P. Huang et al., Steady-state confinement of non-neutral plasmas by rotating electric fields. *Phys. Rev. Lett.* **78**, 875 (1997)

Chapter 14

The Concept of Image Charges



Abstract Since non-destructive particle detection and frequency measurements in a Penning trap rely on image charges, we look at the amount of image charge and current created in different situations. This discussion naturally leads to a quantitative description of space charge and image charge effects.

14.1 Method

The interaction between a charged particle and a conducting object can be described via the ‘method of images’ [1, 2]. If we assume a charge $+q$ that is surrounded by a closed conducting object of any shape, then it is intuitive that the total amount of induced image charge in the object is exactly $q' = -q$. This is also the case for any other infinite conducting object (Fig. 14.1). In arbitrary geometries, to give a relation between q and q' is a complicated problem, and there are hopes for analytical solutions only in highly symmetric situations. Let us begin by looking at some simple examples and follow the discussion in [3].

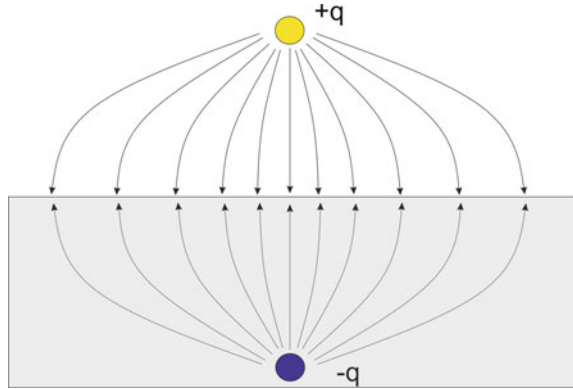
14.2 Charge Between Two Parallel Conducting Plates

A particle with a real charge q , positioned at a distance z in front of the surface of a conducting plate, experiences a force due to the induced surface charge density. In the case of infinite extension of the plate, this picture is equivalent to a point-like image charge $q' = -q$ at distance z on the opposite side of the surface, whereas for finite extension, we have $|q'| < |q|$. The induced surface charge density $\sigma(\rho, z)$ can be calculated by Gauss’ law and is given by

$$\sigma(\rho, z) = -\frac{q}{2\pi} \frac{z}{(\rho^2 + z^2)^{3/2}}, \quad (14.1)$$

where $\rho^2 = x^2 + y^2$ and (x, y) is the position on the surface. On an infinite flat surface the charge induced within a radius ρ_0 is given by [4]

Fig. 14.1 Charge $+q$ and its image charge $-q$ induced in an infinite conducting object



$$q' = 2\pi \int_0^{\rho_0} \sigma(\rho, z) \rho d\rho = -q \left(1 - \frac{z}{\sqrt{\rho_0^2 + z^2}} \right), \quad (14.2)$$

such that for $\rho_0 \rightarrow \infty$ the induced charge is $q' = -q$. A particle located between two parallel conducting plates experiences forces due to a series of images. The real charge induces an image in one plate, which in turn induces an image in the other plate and so forth. This is schematically indicated in Fig. 14.2. Positive images are located at even multiples of the plate separation d , i.e. the distances $\pm 2d, \pm 4d, \dots$. Negative images are located at odd multiples of d , i.e. the distances $\pm d, \pm 3d, \dots$. However, the effective distance between the particle and the positive images does not change as it moves between the plates. Therefore, only negative images exert a net force on the particle. The total image force on the particle in the z -direction is given by [3]

$$F'_z = \frac{-q^2}{4\pi\epsilon_0} \sum_{k=0}^{\infty} \left[\left(\frac{1}{kd - 2z} \right)^2 - \left(\frac{1}{kd + 2z} \right)^2 \right]; \quad k \text{ odd} \quad (14.3)$$

where the summation is over odd values of k only. In [5], the image force is given by $F'_z \approx -8q^2z/(4\pi\epsilon_0d^3)$, however under the assumption that the displacement z is much smaller than the separation d between the plates, (14.3) is better approximated by considering also higher-order terms in k , such that we have [3]

$$F'_z = \frac{-q^2}{4\pi\epsilon_0} \left[\frac{8z}{d^3} \left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \dots \right) \right]. \quad (14.4)$$

For larger amplitudes, this approximation is no longer accurate. Equation (14.3) is then better approximated by the power series expansion

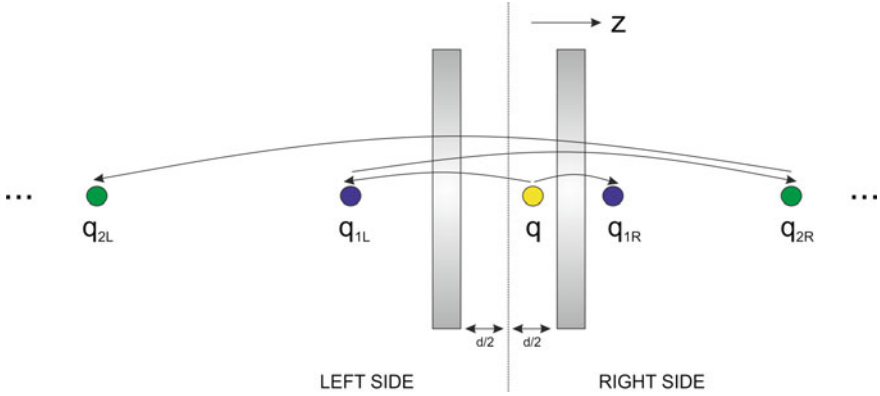


Fig. 14.2 Charge q and its image charges in two parallel plates. The numbers in the index indicate the ‘generation’ of the image, while ‘L’ and ‘R’ stand for ‘left’ and ‘right’ of the symmetry plane, respectively

$$F'_z = \frac{-q^2}{4\pi\epsilon_0} \left[\frac{8z}{d^3} + \frac{64z^3}{d^5} + \dots \right]. \quad (14.5)$$

From (14.5) it can be seen that there is also a non-negligible cubic term present. Below we will show that higher-order terms do not only occur for plate electrodes, but also for electrodes with a strong curvature.

For the case of a charge between two parallel conducting plates, the instantaneous current is given by

$$i = \frac{\partial \Delta q}{\partial t} = \frac{\partial \Delta q}{\partial z} \frac{\partial z}{\partial t} = v_z \frac{\partial \Delta q}{\partial z}, \quad (14.6)$$

where Δq is the induced charge difference between the two plates. If Δq varies linearly with z so that we have $\Delta q = Aqz$, (14.6) becomes $i = Aqv_z$, where A is a constant. However, there can be higher order (odd) terms as well. For example if $\Delta q = Aqz + Bqz^3$, it can be shown that the electronic circuit also picks up the higher order term. For simple harmonic motions of the form $z(t) = z_0 \cos(\omega_z t)$ one obtains [3]

$$\Delta q(t) = Aqz_0 \cos(\omega_z t) + \frac{Bqz_0^3}{4} [3 \cos(\omega_z t) + \cos(3\omega_z t)]. \quad (14.7)$$

The cubic term therefore induces a signal at ω_z and $3\omega_z$.

Consider the case of a single particle with mass m and charge q which is moving inside a particle ensemble of mass M and charge Q . In this case only the centre-of-mass motion of the ensemble will be detected. If the particles are all identical, then $q/m = Q/M$. If the position of q is z and the position of Q is Z , then $Z = -mz/M = qz/Q$ in order for the centre of mass to be stationary. This would be the case for small oscillations of the particle about an equilibrium position, i.e. the

normal modes of the system. The induced charge difference, with q and Q moving such that the centre of mass is stationary, is [3]

$$\Delta q = Bqz^3 \left(1 - \frac{q^2}{Q^2} \right) \approx Bqz^3. \quad (14.8)$$

Again the signal is induced by the cubic term, which gives rise to a signal at ω_z and $3\omega_z$. If the oscillation frequency of q in the environment Q is reduced from the single-particle value, then this shifted frequency will be seen in the signal around ω_z , so long as the term in B is present.

If the amplitudes of motion are larger than the inter-particle spacing, the normal mode description no longer applies. In this case the motion of an individual particle does not necessarily give rise to a compensating motion of the rest of the ensemble. Therefore, the individual oscillation frequencies of all particles are observed directly in the detected signal.

14.3 Charge Between Two Hollow Conducting Spheres

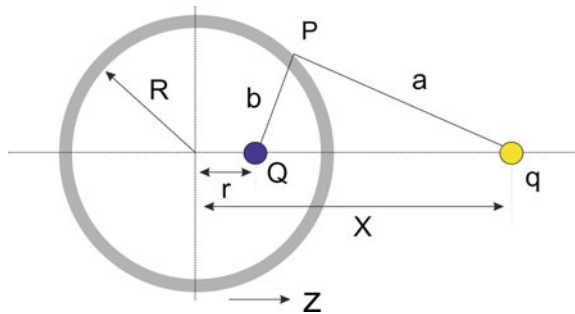
In a conventional Penning trap geometry, the endcap electrodes are better approximated by two hollow conducting spheres, ignoring the presence of the ring electrode. This geometry is slightly more complicated to take into account, but can be solved by adopting the case of a real charge inside a hollow conducting sphere [4] (Fig. 14.3). The potential at a point P on the surface of the sphere is given by

$$V = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{A} + \frac{q}{B} \right), \quad (14.9)$$

where the distances

$$A^2 = R^2 + X^2 + 2XR \cos(\alpha) \quad \text{and} \quad B^2 = R^2 + r^2 + 2rR \cos(\alpha) \quad (14.10)$$

Fig. 14.3 A real charge q inside a hollow conducting sphere and the corresponding image charge Q



are given by the cosine rule. In order to make the potential at any point P on the surface zero, i.e. a grounded surface, we need to have

$$X = R^2/r \quad \text{and} \quad Q = -qX/R. \quad (14.11)$$

The force between the charges q and Q is then given by

$$F'_r = \frac{1}{4\pi\epsilon_0} \frac{qQ}{(X-r)^2}. \quad (14.12)$$

Substituting (14.11) into (14.12) one obtains

$$F'_r = \frac{-q^2}{4\pi\epsilon_0} \frac{R}{(R^2 - r^2)^2} r \quad (r \leq R). \quad (14.13)$$

Simply by exchanging the real charge for an imaginary charge, the case of charge outside a hollow conducting sphere is obtained. Equation (14.13) can be represented by a power-series expansion [3]

$$F'_r = \frac{-q^2}{4\pi\epsilon_0} \left[\frac{r}{R^3} + 2\frac{r^3}{R^5} + \dots \right] \quad r \rightarrow z \quad (14.14)$$

The dependence of F'_r on r becomes non-linear for large values of r , which can only be described by inclusion of the cubic term in (14.14). This was also the case for the two parallel conducting plates, see (14.5). This is a clear indication that higher-order contributions also induce image charges in the endcaps.

Consider two spheres A and B as depicted in Fig. 14.4 with separation $2D$ and radius R . Then the gap between them is $2(D - R)$. In analogy with the case above, a real charge q at a displacement z from the centre induces an image charge

$$q_A = -q \frac{R}{D - z} \quad (14.15)$$

in sphere A at position

$$z_A = D - \frac{R^2}{D - z}, \quad (14.16)$$

and this in turn induces a charge in sphere B and so on. Similarly, the real charge induces an image charge

$$q_B = -q \frac{R}{D + z} \quad (14.17)$$

in sphere B at location

$$z_B = -D + \frac{R^2}{D + z}, \quad (14.18)$$

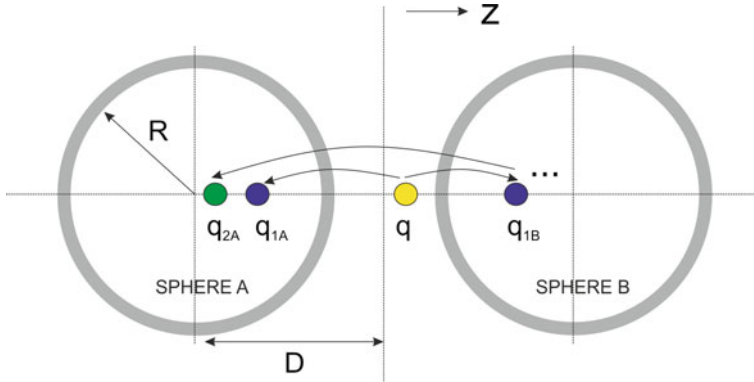


Fig. 14.4 A real charge q between two hollow conducting spheres and the corresponding image charges

and this in turn is imaged in sphere A and so forth. This gives two infinite series of image charges induced in each sphere. Their magnitudes become progressively smaller as the series is extended. The case of finite curved electrodes cannot be calculated analytically, however, a numerical procedure is suitable and converges rapidly. Typical values for modelling a trap are $D/R = 2$.

Looking at the induced charge difference Δq between the two hollow conducting spheres as a function of the relative displacement of the particle, it can be seen that for D only slightly larger than R , the variation is close to linear, as expected. For larger D there is an increasing non-linear term which is proportional to z^3 by symmetry. Note, that for $R \rightarrow \infty$ with fixed $(D - R)$ this geometry approaches the case of a particle between two infinite parallel conducting plates.

14.4 Charge Inside a Hollow Cylinder

If we have a charge $+q$ in an infinitely long cylinder (hollow with open endings) of inner radius ρ_0 , the total induced charge in the cylinder is $-q$. If the cylinder is cut into two symmetric halves (along any direction), the amount of image charge in each of them is equal to $-q/2$. If the cylinder is of finite length and extending from z_l to z_r , then some of the field created by the charge $+q$ escapes through the cylinder openings (assuming a grounded surrounding not at infinity), and the total amount of induced image charge will be less than $-q$. To be more precise, it will be

$$q' = -\frac{q}{\pi \rho_0} \int_0^\infty \frac{I_0(x \cdot \frac{\rho}{\rho_0})}{I_0(x)} \left[\text{sinc} \left(x \frac{z_1 - z}{\pi \rho_0} \right) (z_1 - z) \right]_{z=z_l}^{z=z_r} dx, \quad (14.19)$$

where z_1 axial position of the particle and $I_0(x)$ is the modified Bessel function of first kind. For simplification of the calculation, the Bessel function I_0 can be replaced by the approximation

$$I_0\left(x \frac{\rho}{\rho_0}\right) \approx 1 + \frac{\rho}{4\rho_0^2}x^2. \quad (14.20)$$

This approach works also in situations where the real charge $+q$ is outside the cylinder in the axial direction, but inside the cylinder in the radial direction. We use this for the determination of the effective electrode distance D in a cylindrical Penning trap, see Sect. 12.11.

14.5 Charge Inside a Hollow Cylinder Segment

If we take away the radial symmetry from the problem, we need to introduce an azimuthal coordinate, the angle ϕ . This is the case for example when we consider the induced charge in a segment of a cylindrical electrode with inner radius ρ_0 that is split into cylinder segments, like shown in Fig. 14.5. When we look at such a segment that has an axial extension from z_l to z_r and a radial angular extension from ϕ_- to ϕ_+ , the induced charge is given by

$$q' = \varepsilon_0 \rho_0 \int_{z_l}^{z_r} \int_{\phi_-}^{\phi_+} \left. \frac{\partial U'(\rho, \phi, z)}{\partial \rho} \right|_{\rho=\rho_0} d\phi dz, \quad (14.21)$$

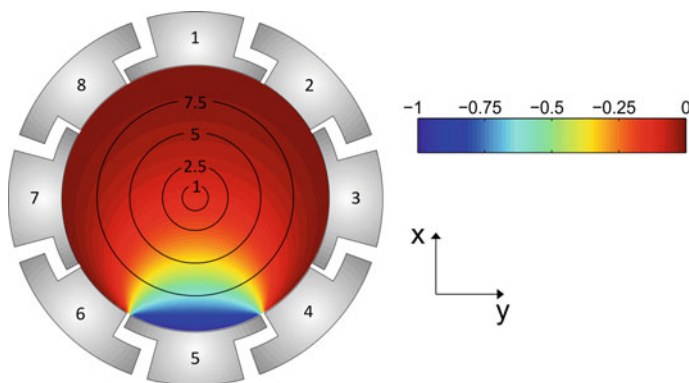


Fig. 14.5 False-colour plot of the fraction of image charge induced in electrode number 5 as a function of the particle position

where the potential $U'(\rho, \phi, z)$ for a charge q at the position (ρ_1, ϕ_1, z_1) can be calculated by [6]

$$U'(\rho, \phi, z) = \frac{q}{2\pi^2 \varepsilon_0 \rho_0} \sum_{m=0}^{\infty} (2 - \delta_{m,0}) \cos(m \cdot [\phi - \phi_1]) \int_0^{\infty} \cos\left(\omega \frac{z - z_1}{\rho_0}\right) g_m d\omega, \quad (14.22)$$

in which we have

$$g_m = \frac{I_m\left(\omega \frac{\rho_1}{\rho_0}\right)}{I_m(\omega)} \cdot \left[K_m\left(\omega \frac{\rho}{\rho_0}\right) I_m(\omega) - K_m(\omega) I_m\left(\omega \frac{\rho}{\rho_0}\right) \right]. \quad (14.23)$$

Inserting (14.22) into (14.21) we obtain the induced charge q' as function of particle position (ρ_1, ϕ_1, z_1) :

$$\begin{aligned} q'(\rho_1, \phi_1, z_1) &= \frac{q}{2\pi^2} \sum_{m=0}^{\infty} (2 - \delta_{m,0}) \int_{\phi_-}^{\phi_+} \cos(m \cdot [\phi - \phi_1]) d\phi \\ &\times \int_0^{\infty} \int_{z_l}^{z_r} \cos\left(\omega \frac{z - z_1}{\rho_0}\right) dz \frac{\partial g_m}{\partial \rho} d\omega, \end{aligned} \quad (14.24)$$

in which we have

$$\left. \frac{\partial g_m(\omega)}{\partial \rho} \right|_{\rho=\rho_0} = -\frac{I_m\left(\omega \frac{\rho_1}{\rho_0}\right)}{\rho_0 I_m(\omega)}, \quad (14.25)$$

and where the integral over the radial angle ϕ (given in degrees) takes values

$$m = 0 : \quad \frac{2\pi}{360} (|\phi_-| + |\phi_+|) \quad (14.26)$$

$$m > 0 : \quad \frac{1}{m} [\sin(m(\phi_- - \phi_1)) + \sin(m(\phi_+ + \phi_1))], \quad (14.27)$$

while the second integral takes values

$$\omega = 0 : \quad L \quad (14.28)$$

$$\omega > 0 : \quad \frac{\rho_0}{\omega} \left[\sin\left(\frac{\omega}{\rho_0} \left(\frac{L}{2} - z_1\right)\right) + \sin\left(\frac{\omega}{\rho_0} \left(\frac{L}{2} + z_1\right)\right) \right]. \quad (14.29)$$

In these, I_m and K_m are modified Bessel functions of the first and second kind, respectively. There are a number of different representations for them, perhaps the most suitable forms for our calculations are

$$I_m(x) = \left(\frac{1}{2}x\right)^m \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^m}{k! \Gamma(m+k+1)} \quad (14.30)$$

$$K_m(x) = \frac{\Gamma\left(m + \frac{1}{2}\right) (2x)^m}{\sqrt{\pi}} \int_0^{\infty} \frac{\cos t}{(x^2 + t^2)^{m+1/2}} dt. \quad (14.31)$$

Equation (14.21) produces the same values as (14.19) when integration is performed from $\phi_- = -\pi$ to $\phi_+ = \pi$.

14.6 Frequency Shifts Revisited

14.6.1 Image-Charge Shift

Besides the axial force due to the quadrupole potential, the image charge also exerts a force on a single confined charged particle (see Sect. 3.1). Therefore, the total force is given by the sum of the force due to the confining potential and the (linear term) of the image charge force (14.5). The corresponding equation of motion in the axial direction is given by

$$m\ddot{z} = -\frac{4qU_0}{\rho_0^2}z + \frac{2q^2}{\pi\epsilon_0 d^3}z, \quad (14.32)$$

and the image-charge-shifted axial frequency is hence given by

$$\omega_z'^2 = \frac{4qU_0}{m\rho_0^2} - \frac{2q^2}{\pi\epsilon_0 m d^3}, \quad (14.33)$$

such that, to first order, the shift itself is given by

$$\Delta\omega_z \equiv \omega_z' - \omega_z \approx -\frac{1}{\pi\epsilon_0} \frac{q^2}{m\omega_z d^3}. \quad (14.34)$$

In principle, there are also higher-order effects due to the non-linear terms in (14.5), but these are small and we neglect them. When there is more than one particle between the plates, the image force and thus the axial frequency shift scales with the number of particles N . A result similar to (14.34) is obtained when the particle is considered to move inside a hollow conducting sphere [7]. The image charge affects the centre-of-mass motion of a particle ensemble, because it is an induced external force acting on the ensemble as a whole.

Due to its radial motions, i.e. the magnetron and modified cyclotron motion, the charged particle also induces image charges in the ring electrode which affect the radial frequencies. Here a treatment similar to that given before can be applied, where the ring electrode is a small central section of the surface. For this case there

will also be a force like (14.13), however smaller in magnitude due to the fact that not all the field lines originating from the charge q are directed at the section. Or, in other words, the image charge that is formed is incomplete. A charged particle with, for example, a magnetron radius r will thus experience a shifted magnetron frequency due to the induced charge in the ring. The radial image charge force F'_ρ is directed radially outward, following the direction of the electric quadrupole force (for positively charged particles). The shifted radial frequencies are thus given by

$$\omega'_\pm = \frac{\omega_c}{2} \pm \sqrt{\left(\frac{\omega_c}{2}\right)^2 - \frac{1}{2}\omega_z^2 - \gamma \frac{F'_\rho}{m}}, \quad (14.35)$$

where γ is related to the ratio of the ring area to the surface area. The image-charge shift is only significant for very small traps due to its scaling with $1/d^3$. Measurements of the image charge shift dependence on the number of particles have been performed for $^1\text{H}^+$, $^2\text{H}^+$, $^3\text{He}^+$, $^3\text{He}^{2+}$, and $^{12}\text{C}^{4+}$ ions [7] and frequency shifts were observed even for small numbers of particles. The magnetron frequency was shifted upwards and the modified cyclotron frequency was shifted downwards linearly, as expected from (14.35). The relative magnitudes of the observed shifts, about 20 mHz per ion per charge, are in fair agreement with those predicted by (14.35). For the cyclotron frequency $\omega_c = \omega_+ + \omega_-$ no significant shift has been observed, as expected in this geometry, see also the discussion in Sect. 7.1.

14.6.2 Space-Charge Shift

Similarly to the image-charge effect, there is a potential due to the presence of space charge which adds to the trapping potential and thus changes the oscillation frequencies. The space-charge potential U' is the potential seen by a particle due to the presence of all other particles and can be derived in a manner similar to the quadrupole potential. As a simple example, it is assumed that the particle ensemble can be described by a perfect sphere with radius ρ'_0 and homogeneous charge density $\rho = Nq/V'$, where N is the number of particles and $V' = 4\pi\rho_0^3/3$ the volume of the sphere. Laplace's law in spherical coordinates (r, θ, ϕ) applied to this particle ensemble states that

$$\frac{\partial^2 U'}{\partial r^2} + \frac{2}{r} \frac{\partial U'}{\partial r} = -\frac{\rho}{\epsilon_0}, \quad (14.36)$$

where spherical symmetry with respect to θ and ϕ holds. Outside the sphere the potential can be described by that of a point charge, i.e.

$$U'(r \geq \rho'_0) = \frac{Nq}{4\pi\epsilon_0 r} = \frac{\rho\rho_0^3}{3\epsilon_0 r}. \quad (14.37)$$

Inside the sphere the potential varies with r^2 in order to satisfy the Laplace equation. The final form of the potential is

$$U' = \frac{\rho}{6\epsilon_0} (3\rho_0^2 - r^2). \quad (14.38)$$

With the substitution $r \rightarrow z$, the equation of motion in the z -direction for a single particle becomes

$$m\ddot{z} = -q \frac{\partial}{\partial z} (U + U') = -\frac{4qU_0}{\rho_0^2} z + \frac{\rho q}{3\epsilon_0} z. \quad (14.39)$$

Both terms on the right-hand side of (14.39) vary linearly with z , therefore the space-charge-shifted axial frequency is given by [8]

$$\omega_z'^2 = \frac{4qU_0}{m\rho_0^2} - \frac{\rho q}{3m\epsilon_0} = \omega_z^2 - \frac{\omega_p^2}{3}, \quad (14.40)$$

where $\omega_p^2 = \rho q / (m\epsilon_0)$ is the plasma frequency and $n = N / V' = \rho / q$ is the particle number density. The factor 3 in the denominator of the space-charge term in (14.40) accounts for the geometry of the particle ensemble. In a more general case, when the particle ensemble is not a sphere but rather an ellipsoid, this constant changes. If the ensemble shape is prolate the factor is smaller than 3, if the shape is oblate it is larger [8]. To first order, the absolute frequency shift is given by

$$\Delta\omega_z \equiv \omega_z' - \omega_z \approx -\frac{nq^2}{6m\epsilon_0\omega_z}. \quad (14.41)$$

The relative frequency shift

$$\frac{\Delta\omega_z}{\omega_z} \equiv \frac{\omega_z' - \omega_z}{\omega_z} \approx -\frac{nqR_0^2}{24\epsilon_0 U_0} \quad (14.42)$$

depends linearly on the charge density $\rho = nq$ and thus, for a given ensemble size, on the total number of particles $N = \rho V'$.

The space-charge limit is reached for $|U'| \geq |U|$, i.e. when the space-charge potential compensates the trap potential. In this case there is no potential minimum and confinement is lost.

If the space charge density is not homogeneous, but has a Maxwell-Boltzmann distribution, the space-charge frequency shift depends on the position of the particle relative to the centre of the ensemble [9]. This leads to a distribution of shifted axial frequencies, as we have discussed in Sect. 8.2. However, the centre-of-mass motion of a particle ensemble is not affected by intra-ensemble interactions and thus remains unshifted by space-charge effects. A similar broadening of the axial frequency distribution occurs when the trap potential is anharmonic. In that case,

the axial frequency depends on the amplitude of the motion such that a distribution of particle energies automatically leads to a distribution of axial frequencies, see Sect. 12.13.4.

The space-charge potential U' also affects the radial motions and thus the radial frequencies of the particle. Due to the difference in signs between the z - and ρ -dependences, the shift has the same sign as ω_z . The space-charge shifted radial frequencies are then given by

$$\omega'_{\pm} = \frac{\omega_c}{2} \pm \sqrt{\left(\frac{\omega_c}{2}\right)^2 - \frac{1}{2}\omega_z^2 - \frac{\omega_p^2}{3}}. \quad (14.43)$$

Thus, there is a downwards shift of the modified cyclotron frequency ω_+ and an upwards shift of the magnetron frequency ω_- with increasing charge density.

Measurements of the magnetron frequency of an ion ensemble in a Penning trap with $r_0 = 5$ mm, $z_0 = 3.5$ mm, $U_0 \approx 6$ V and $B = 1$ T have been performed [10] by use of laser investigations. For a single ion they yield a magnetron frequency of about 40 kHz, while for ensembles with ion number densities of the order of 10^7 cm $^{-3}$, the corresponding value is shifted to about 60 kHz. This is in fair accordance with the behaviour expected from (14.43).

References

1. R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, vol. II (Addison-Wesley, Reading, 1965)
2. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)
3. D.F.A. Winters, M. Vogel, D.M. Segal, R.C. Thompson, Electronic detection of charged particle effects in a Penning trap. *J. Phys. B* **39**, 3131 (2006)
4. B.I. Bleaney, B. Bleaney, *Electricity and Magnetism*, 2nd edn. Oxford (1965)
5. D.J. Wineland, H.G. Dehmelt, Principles of the stored ion calorimeter. *J. Appl. Phys.* **46**, 919 (1975)
6. J. Steinmann, Modellierung und Simulation der Widerstandskuehlung von hochgeladenen Ionen, Ph.D. thesis, University of Erlangen-Nürnberg (2015)
7. R.S. Van Dyck Jr., F.L. Moore, D.L. Farnham, P.B. Schwinberg, Number dependency in the compensated Penning trap. *Phys. Rev. A* **40**, 6308 (1989)
8. J.B. Jeffries, S.E. Barlow, G.H. Dunn, Theory of space-charge shift of ion cyclotron resonance frequencies. *Int. J. Mass Spectrom. Ion Proc.* **54**, 169 (1983)
9. C. Meis, M. Desaintfuscien, M. Jardino, Analytical calculation of the space charge potential and the temperature of stored ions in an rf quadrupole trap. *Appl. Phys. B* **45**, 59 (1988)
10. K. Dholakia et al., Investigation of ion dynamics in a Penning trap using a pulse-probe technique. *Appl. Phys. B* **60**, 375 (1995)

Chapter 15

Electronic Detection and Frequency Measurement



Abstract In this chapter, we take a look at the non-destructive detection of confined particles and the measurement of their oscillation frequencies by use of resonant electronic circuits. This includes superconducting circuits and their properties in the external magnetic field of the Penning trap.

15.1 Method

The resonant detection of image currents (moving image charges induced by oscillating particles) for non-destructive particle detection and frequency measurements in a trap is called ‘bolometric’ detection and was first brought forward and realised by Dehmelt et al. [1]. Sometimes it is understood in terms of a calorimeter [2] on account of the energy dissipation into the detection circuits. In order to experimentally determine an oscillation frequency, usually a resonant detection circuit is attached to a corresponding set of electrodes that produce the optimum difference of induced image charges connected with the particle motion. For the axial frequency, usually a resonant RLC-circuit with a high quality factor is connected to the endcaps. For radial frequencies, a radially segmented electrode is used similarly. The signal from the circuit is then amplified, often in two stages, one cryogenic amplifier close to the trap and one room-temperature amplifier outside. This is necessary as the currents i induced by the particle motion are typically of the order of pA or less, and hence easily obstructed by electronic noise. A Fourier analysis of the induced signal then yields the spectrum of oscillation frequencies. This situation is schematically depicted in Fig. 15.1. The signal-to-noise ratio Σ of such a resonant pick-up of the particle motion is given by

$$\Sigma \propto \frac{qa}{D} \sqrt{\frac{\omega}{\delta\omega}} \sqrt{\frac{Q}{k_B T C}}, \quad (15.1)$$

where q is the particle charge, a is the oscillation amplitude in the respective degree of freedom, D is the effective electrode distance of the chosen pick-up arrangement, $\delta\omega$ is the frequency bandwidth of the resonant circuit at the oscillation frequency ω ,

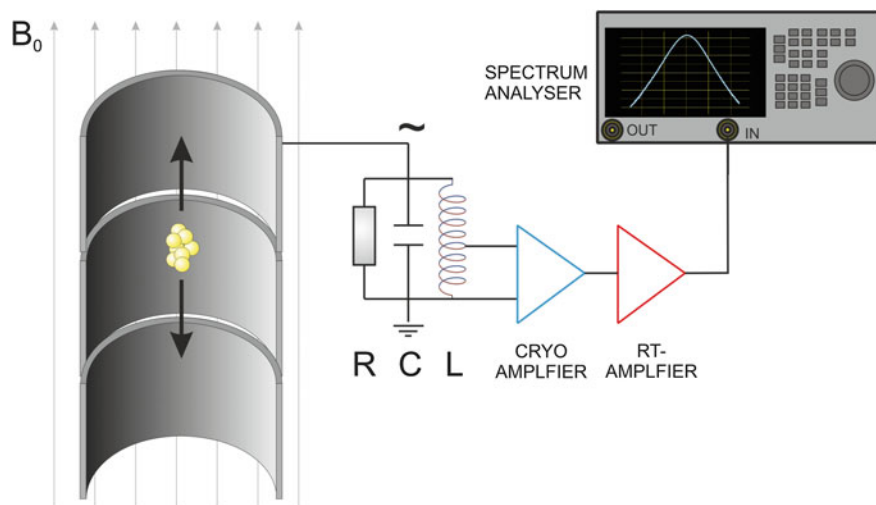


Fig. 15.1 Schematic of the signal processing for a measurement of the axial oscillation frequency. This is a so-called ‘asymmetric detection’ where the signal at one endcap against ground is measured

Q is the quality factor of the circuit and C is its capacitance. The equation shows that the signal-to-noise ratio increases linearly with the charge state of the particle such that resonant detection is easiest for highly charged ions. The linearity with the amplitude a shows that the signal can be improved by excitation of the particle motion. The effective electrode distance D in the denominator makes large detection surfaces close to the particle oscillation preferable, since then D is small. For a closer discussion of this quantity see Sect. 12.11.

Non-destructive detection systems with high signal-to-noise-ratios are often indispensable in Penning-trap measurements. As we have already stated, this commonly relies on detection by use of resonators which often are superconducting, since these usually exhibit the highest quality factors and hence the highest amplification, at least, as we will see, in the most common radio-frequency domain.

15.2 General Statements About Superconductors

We should say a few words about superconductivity in general, since the magnetic field used for confinement is usually produced by a superconducting magnet, and in cryogenic Penning traps components like superconducting resonators are widely used for non-destructive detection. Also, superconducting elements may be used to create magnetic bottles and such, acting like perfect diamagnets in the outer magnetic field.

The most common superconductor in use is an alloy of niobium and titanium, often in the form of a thin insulated wire. In a field-free configuration, NbTi becomes superconducting below 9.2 K and has an upper critical field strengths B_{c2} of about 14.5 T at zero temperature. This makes the use of NbTi (or likewise Nb₃Sn) in high magnetic field experiments advantageous over other superconductor materials in question, such as for example pure niobium, tantalum or lead, since generally, the upper critical field strength of type-II superconductors exceeds the critical field strength of type-I superconductors significantly. Type-I superconductors below their critical field strength and type-II superconductors below their lower critical field strength B_{c1} are in the so-called ‘Meissner phase’ (named after F. W. Meissner, 1882–1974), i.e. they exclude an external magnetic field. A type-II-superconductor in an external field between the lower and upper critical field strength B_{c1} and B_{c2} is the so-called ‘Shubnikov phase’ (after L. W. Shubnikov, 1901–1937), i.e. it is pervaded by the magnetic field in quanta of the magnetic flux $\Phi_0 = hc/(2e) \approx 2.067 \times 10^{-15} \text{ Tm}^2$, which create vortices in the superconductor (Fig. 15.2). These are circular super-currents around normal-conducting cores which form about the magnetic field lines. In presence of an ac-current such as a radio-frequency signal, the interaction of the vortices with the current of the ion signal gives rise to an effective resistance which leads to signal damping. This means that the energy loss per oscillation cycle is non-zero even for a superconductor. We will discuss this in relation to superconducting resonators in Sect. 15.6. Pure niobium has a critical field-free temperature of $T_c(B = 0)$ of 9.25 K [4] which is lowered in the presence of a magnetic field according to [3]

$$T_c(B_0) \approx T_c(0) \sqrt{1 - \left(\frac{B_0}{B_c}\right)}. \quad (15.2)$$

For example, at an ambient temperature of $T = 4.2 \text{ K}$, the Meissner effect in pure niobium is complete up to a magnetic field strength of about 142 mT.

Type-I superconductors like lead, vanadium and tantalum have critical temperatures slightly above typical cryogenic temperatures like that of liquid helium and may thus be interesting for applications. Their critical field strengths are limited to

Fig. 15.2 Schematic of the different superconducting phases in the (B, T) -plane. Note that the Meissner and Shubnikov phases can be left both in the B - and T -directions, leading to interdependent critical values for B and T

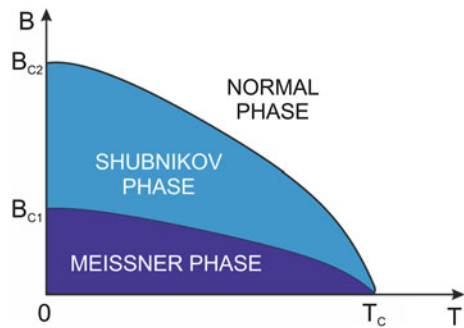


Table 15.1 Critical magnetic field strength of several relevant superconductors. Values are taken from [3–5]

Superconductor	$B_{c1}(T = 0)$	$B_{c2}(T = 0)$	$T_c(B = 0)$ (K)
Pb		81 mT	7.2
V		102 mT	5.3
Ta		83 mT	4.5
NbTi	35 mT	10.3 T	9.3
Nb ₃ Sn	19 mT	24.5 T	18
Nb	180 mT	410 mT	9.25

somewhat below 100 mT, see Table 15.1. Also type-II superconductors like niobium are interesting for our purposes. They provide a complete Meissner effect up to a critical magnetic field strength B_{c1} . Above this field strength, the Meissner effect is incomplete (‘Shubnikov phase’ [3]). Magnetic flux pinning by artificial material imperfections (doping) may help to shift the limits beyond B_{c1} , but this needs further study. Above a second critical field strength of B_{c2} , superconductivity is lost completely. Generally, the lower critical field strength depends on the actual temperature like [3]

$$B_{c1}(T) \approx B_{c1}(0) \left(1 - \left(\frac{T}{T_c} \right)^2 \right), \quad (15.3)$$

where the empirically found exponent for pure niobium is 2.13 instead of 2 [4]. The upper critical field strength has a temperature dependence given by [5]

$$B_{c2}(T) \approx B_{c2}(0) \left(1 - \left(\frac{T}{T_c} \right)^2 \right) \left(1 + \left(\frac{T}{T_c} \right)^2 \right)^{-1}. \quad (15.4)$$

While the value of B_{c2} can reach many Tesla, the lower critical field strength is typically only several tens to hundreds of mT. Table 15.1 lists the critical field strengths for several relevant superconductors.

In terms of a minimum possible size of a superconductor, we are limited by the finite values of the residual field penetration depth and of the coherence length. The London effect [3, 6] (named after brothers F. London (1900–1954) and H. London, 1907–1970) allows a certain non-vanishing magnetic field penetration into the superconductor. The penetrating magnetic field strength decays exponentially with the depth d according to

$$B(d) = B_0 \exp \left(-\frac{d}{\lambda_L} \right), \quad (15.5)$$

with a London penetration depth λ_L on the scale of several tens of nanometres (for example 39 nm for pure niobium, 37 nm for lead [7]). The same length scale is

valid for the coherence length in typical superconductors which defines the smallest region in which superconductivity can be achieved. For example, this length is 38 nm for niobium and 83 nm for lead [7]. Hence, the minimum size of well-defined arrangements is limited to micrometres and above, which does not restrict any of the applications to be discussed. The same is true for limits set by magnetic flux quantisation. The magnetic flux is quantized by nature [7, 8], however for structures of the present sizes, this quantization can be ignored due to the smallness of the quantum $\Phi = hc/(2e) \approx 2 \times 10^{-15} \text{ Tm}^2$ [7].

15.3 Particle-Resonator Combined Circuit

In an ideal Penning trap, the particle motion is equivalent to an intrinsically undamped oscillator [2], which can be described in terms of a series LC-circuit by

$$l_p \frac{dI_p}{dt} + \frac{1}{c_p} \int I_p dt = U_p, \quad (15.6)$$

where c_p and l_p are the capacitance and the inductance of the equivalent LC circuit, respectively, connected to a voltage of U_p . In addition, I_p is the induced image current on the electrodes of the Penning trap by the particle oscillation. The value of I_p is typically of the order of fA to pA, and for a charge q at any given coordinate ρ is given by

$$I_p = q \frac{\dot{\rho}}{D} = \omega \frac{q}{D_\rho} \rho, \quad (15.7)$$

ω being the oscillation frequency of the charge and D the effective electrode distance of the specific particle motion. To detect this motion, an inductor L is connected to trap electrodes suited for efficient pick-up of the image signal. Alongside with its self-capacitance C_P and the capacitance of the trap C_T it forms a parallel RLC-circuit. Figure 15.3 show this configuration and the electrical equivalent representation of the combined particle ensemble and RLC circuit. The quality factor Q of the arrangement

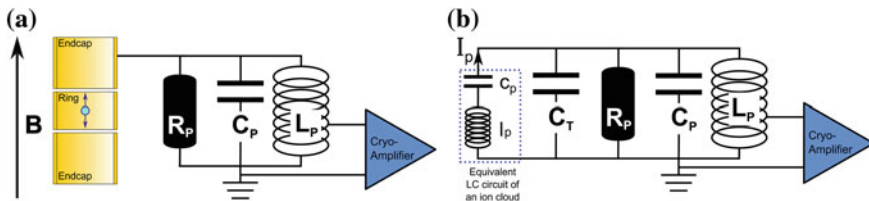
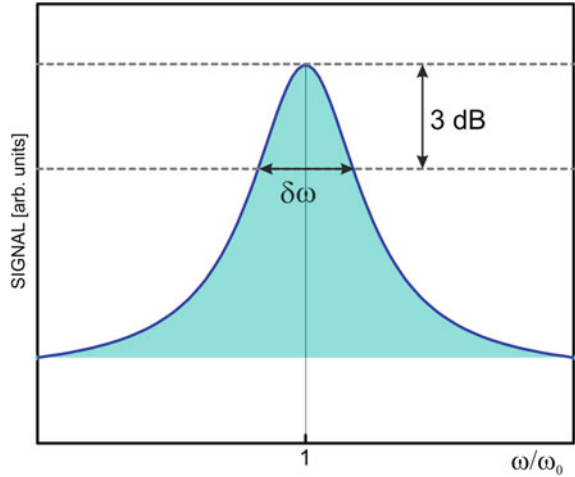


Fig. 15.3 **a** Schematic representation of a parallel resonator circuit connected to cylindrical trap with an oscillating charge q , where L , C_P and R_P are the circuit's inductance, parasitic capacitance and effective resistance, respectively. **b** Equivalent circuit of confined charge and the RLC circuit

Fig. 15.4 Typical Lorentz spectrum of the response signal of a resonant circuit, and the definition of the width $\delta\omega = 2\pi\delta\nu$ at -3 dBm from the resonance signal at $\omega/\omega_0 = 1$, see also [9]



is determined by the resonance frequency ω_0 and the width $\delta\omega$ of the resonance, see Fig. 15.4.

The quality factor Q is defined as the ratio of the stored and the dissipated energy per angular cycle, and is given by

$$Q = \frac{\omega_0}{\delta\omega}. \quad (15.8)$$

The impedance of the resonator circuit at its resonance frequency is considered as an effective parallel resistor given by

$$R_P = \omega_0 L Q. \quad (15.9)$$

When the resonance frequency of the RLC circuit coincides with the oscillation frequency of the particle, a voltage drop of

$$U_P = R_P I_P = \omega_0 L Q I_P \quad (15.10)$$

is observed. In a given experimental situation, this voltage is highest for the maximum possible quality factor and maximum possible inductance L of the arrangement. After amplification of U_P in one or several stages, the voltage signal can be Fourier-transformed to yield the motional frequency spectrum of the particle.

In case of more than one particle, the circuit will detect any total current (i.e. the instantaneous sum of all individual currents) induced within its bandwidth ω_0/Q around ω_0 . The signal will depend on the details of the particle motion, as discussed in Sect. 12.13.

15.4 Resonator and Detection Design Principles

A typical non-destructive detection system in a Penning trap experiment is usually composed of the following components and is connected to a spectrum analyser for fast-Fourier-transform (FFT) analysis:

- Trap electrodes for signal pickup
- (Superconducting) Resonator
- Varactor-diode board (if the resonance frequency needs to be variable)
- Cryogenic and room-temperature amplifiers
- Feedthroughs and cables between the components

The importance of the proper choice and geometry of the electrodes used for pickup of the signal find expression in the effective trap size D as described above. Here, we will say a few words about the resonator (which may be superconducting), and about the use of a varactor diode board for flexibility of the actual resonance frequency. We will not say much about amplifiers and the cabling, but note that they are an important part of the detection system which needs to have small insertion loss. Due to the fact that all components contribute to the overall quality factor of the detection system, we need to write [10]

$$\frac{1}{Q} = \frac{1}{Q_{Trap}} + \frac{1}{Q_{Res}} + \frac{1}{Q_{Var}} + \frac{1}{Q_{Amp}} + \frac{1}{Q_{Con}} + \dots \quad (15.11)$$

Here Q is the overall quality factor of the detection system and Q_{Trap} , Q_{Res} , Q_{Var} , Q_{Amp} , Q_{Con} are the quality factor of trap electrodes, resonator, varactor diode board, amplifier board, and connectors, respectively. To this end, it is preferable to have the resonators as close as possible to the trap electrodes. Note, that on account of (15.11), any small quality factor in the chain limits the overall quality factor of the detection system.

Figure 15.5 show the schematic overview of a typical detection system. As mentioned already in (15.10), in order to achieve optimum signal strength, it is required to

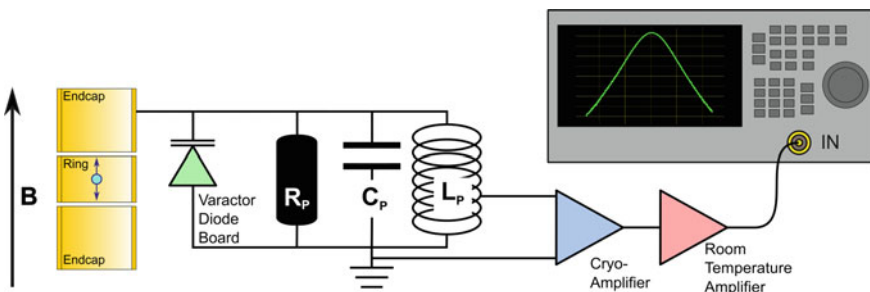


Fig. 15.5 Typical non-destructive detection setup for a measurement of the axial particle oscillation frequency in the trap

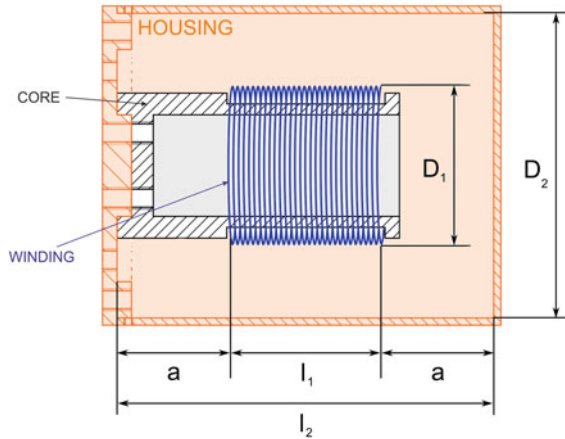


Fig. 15.6 Schematic of a helical resonator consisting of a winding on a core in a housing

maximize the Q -factor and the inductance L of the inductor coil. Two main geometries of inductors are conventionally used, helical and toroidal designs, as depicted in Figs. 15.6 and 15.7. These coils are normally placed in a grounded housing to shield them from external noise.

The design of the resonator and its housing can be guided by largely empirical formulae from studies of radio-frequency design, see for example [11, 12].

The resonance frequency of an LC-circuit is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}. \quad (15.12)$$

In case of a single-layer helical coil with air core, the inductance follows from

$$L \approx \frac{D_1^2 N^2}{18D_1 + 40l_1}. \quad (15.13)$$

Here, D_1 and l_1 are the diameter and length of the coil given in inches, respectively (Fig. 15.6) and the resulting inductance is in μH . The number of turns N for such a coil to achieve the calculated inductance is immediately apparent. If a housing as depicted in Fig. 15.6 is used, the inductance is modified to a value of

$$L_H \approx L \left[1 - \left(\frac{D_1}{D_2} \right)^3 \right] \left[1 - \left(\frac{l_1}{2l_2} \right)^2 \right]. \quad (15.14)$$

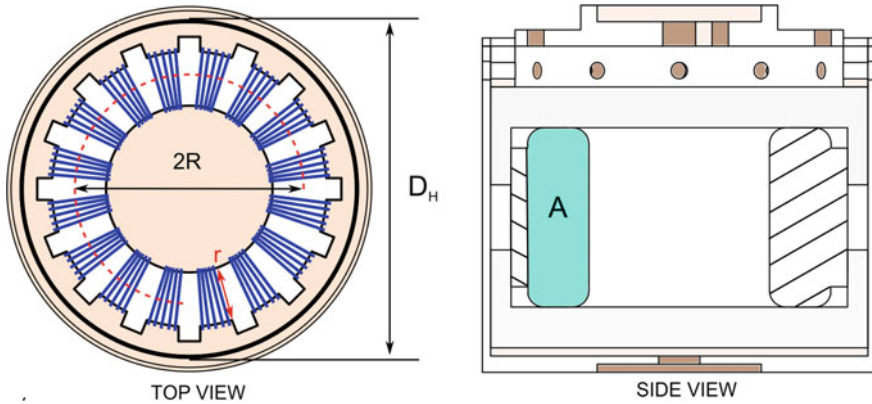


Fig. 15.7 Schematic sectional views of a toroidal resonator (left) and its housing (right)

Here, D_2 and l_2 are geometry parameters of the housing as given in Fig. 15.6. There are two empirical golden ratios for (15.14), $l_1 = 0.55D_2$ and $l_1/2l_2 = 0.377$, resulting in the minimum decrease of the coils inductance L . Often, spatial limitations demand a multi-layer coil for which the inductance can be calculated from

$$L \approx \frac{0.2 (D_1^2 N^2)}{3D_1 + 9l_1 + 10D_w}. \quad (15.15)$$

As before, the same unit considerations are valid and D_w is the thickness of the windings on top of each other (number of layers times thickness of the wire). One can estimate the Q -factor of such a resonator as $Q = 50 D_2 \sqrt{\omega_0/2\pi}$, but as mentioned before this estimation is practically difficult to be realized in the experiments environment.

The capacitance of single layer air-core resonator can be estimated for the turn-to-turn and coil-housing as

$$C_{TT} \approx \frac{\pi^2 D_1 \varepsilon_0}{\ln \left(p/2r + \sqrt{(p/2r)^2 - 1} \right)}, \quad C_{TH} \approx \frac{2\pi^2 D_1 \varepsilon_0}{\ln \left(h/r + \sqrt{(h/r)^2 - 1} \right)}, \quad (15.16)$$

where ε_0 is the vacuum permittivity, r the wire thickness, p the distance of adjacent wires, and h the distance of the wire from the housing. The result of capacity calculations in such a system is typically of the order of pF, reflecting how sensitive a resonator is to capacitance changes imposed by the environment or additional components such as wires or feed-throughs.

A toroidal structure has the advantage over the helical one that the magnetic field flux is contained within the toroid structure. This minimizes the eddy-current losses in the housing and results in an increase of the quality factor Q . The resonance frequency obeys the same formula as in the helical case, see (15.12). The inductance of a single-layer toroidal inductor can be estimated as

$$L \approx \frac{\mu N^2 A}{2\pi R}, \quad (15.17)$$

with N being the number of winding turns, A the cross-sectional area, μ the permittivity of the core of the coil and R the major radius of the toroid, see Fig. 15.7.

15.5 Resonance Frequency Shift by Varactor Diodes

The resonance frequency of a given resonator can be influenced by use of a varactor diode. It introduces a tunable capacitance which is comprised of a P-N diode that shows a decreasing capacitance C_V and a decreasing series resistance R_V with increasing reverse bias voltage V . In presence of a varactor diode, the capacitance of the resonator system can be rewritten as

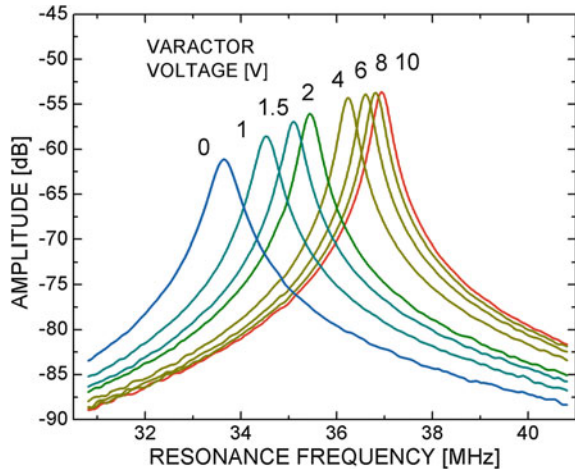
$$C(U_V) = C_R + C_V(U_V), \quad (15.18)$$

where C_R is the capacitance of the entire detection system minus the capacitance of the varactor diode board $C_V(U_V)$. The influence of the varactor diode board on the overall quality factor of the detection system can be written as

$$\frac{1}{Q(U_V)} = \frac{1}{Q_R} + \frac{1}{Q_V(U_V)}, \quad (15.19)$$

with Q_R the quality factor of the rest of the system expect the quality factor of the varactor diode board $Q_V(U_V)$. Figure 15.8 shows the effect of different varactor

Fig. 15.8 Circuit resonances for different varactor diode voltages



diode voltages on the resonance behaviour of a given resonant circuit. In addition to tunable resonance frequencies, also methods have been devised for well-defined tuning of the quality factor of superconducting resonators [13].

15.6 Superconducting Resonators in a Magnetic Field

As we have mentioned, the RLC-circuits used for detection and cooling may be constituted by normal-conducting or superconducting resonators. A type-II-superconductor in an external field between the lower and upper critical field strength B_{c1} and B_{c2} is the so-called ‘Shubnikov phase’, i.e. it is pervaded by the magnetic field in quanta of the magnetic flux $\Phi_0 = hc/(2e) \approx 2.067 \times 10^{-15} \text{ Tm}^2$, which create vortices in the superconductor. These are circular super-currents around normal-conducting cores which form about the magnetic field lines. In presence of an ac-current such as the radio-frequency ion signal, the interaction of the vortices with the current of the ion signal gives rise to an effective resistance which leads to signal loss. This means that the energy loss per oscillation cycle is non-zero even for such a superconductor. The loss scales with the cube of the frequency, such that the use of a type-II superconducting resonator is advantageous over a normal-conducting resonator only for sufficiently small frequencies, typically on the scale of several tens of MHz and below.

In principle, the resonance frequency and quality factor of a resonator can in either case be calculated from the chosen geometry and material properties, however not to high precision. This is mainly due to the influence of geometric detail on the inductance and capacitance, and is particularly true for superconducting resonators in a magnetic field and radio frequency currents.

For application in a given experiment, resonance frequency and quality factor usually need to be chosen specifically. However, as the resonator characteristics depend not only on intrinsic properties but also on experimental parameters such as temperature and in particular field strength, it is valuable to have quantitative information on these dependences. Especially in situations where the magnetic field is changed during the superconducting state, also hysteresis effects need to be considered. With superconductors, it is further important to distinguish whether the superconducting state is reached before or after a magnetic field is established. The resonance circuit is constituted by the total inductance L of the resonator and the total capacity C of it and its wiring. It is resonant at a frequency $\omega_0 = 1/\sqrt{LC}$ and in resonance represents an Ohmic resistance of $R = \omega_0 L Q$, where the quality factor Q is the ratio of the total energy in the circuit and the resistive energy loss per oscillation. In many applications, the quality factor and the resonance resistance are desired to be as high as possible, requiring C to be small, which is non-trivial due to parasitic capacitances of the setup. In-situ capacitance measurements [14] of the trap and wiring may help to estimate the achievable overall quality factor. Most often, the potentially high quality factor is the motivation to prefer superconducting rf-resonators over normal-conducting ones.

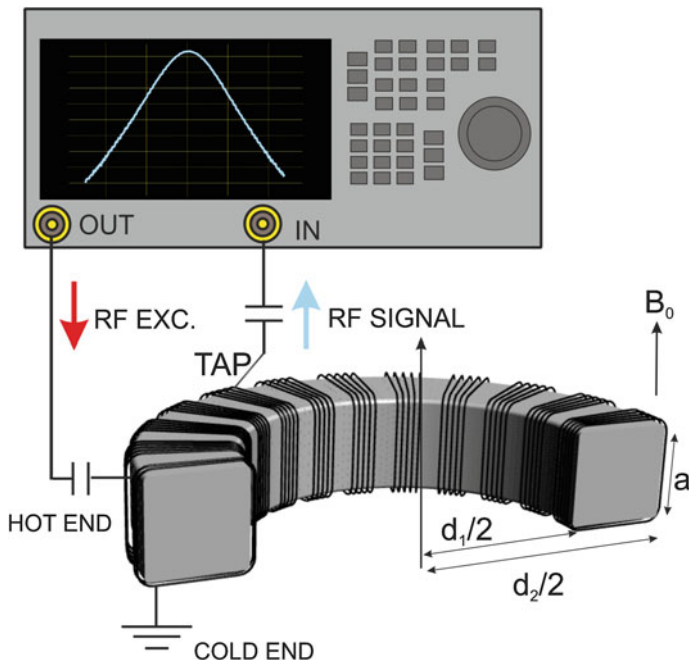


Fig. 15.9 Toroidal superconducting resonator with rectangular cross section and the setup for measurement of its properties

Usually, superconducting resonators mainly consist of a coil made from niobium-titanium (NbTi, a type-II superconductor) wire wound on an insulator core and sealed in a conductive housing, usually made from oxygen-free high-conductivity copper (OFHC copper).

To give an example, we follow the discussion in [15] and regard a resonator coil of toroidal geometry, i.e. the wire is wound around a toroidal core (Fig. 15.9). The cross section of this toroid is rectangular, the inner and outer diameters d_1 and d_2 being 23 and 38 mm, respectively, and a toroid thickness a of 22 mm. The wire is made of a NbTi conductor with a diameter of 0.075 mm. The diameter is larger (by at least two orders of magnitude) than the London penetration depth and the expected coherence length in the material, such that size effects are not expected. The inductance L_c of such a coil can be approximated by [11]

$$L_c \approx \kappa N^2 a \ln \left(\frac{d_2}{d_1} \right) \quad (15.20)$$

where $\kappa = 0.46 \times 10^{-6}$ is a geometry-specific factor and N is the total winding number, let us take an example with $N = 800$. Presently, this results in $L_c \approx 3$ mH. The coil inductance L_c constitutes the dominant part of the total resonator inductance L , apart from a small reduction by the presence of the resonator housing

[11], estimated to result in $L \approx 2.4$ mH. Similar equations exist for different coil geometries, see [11].

The self-capacitance C_c of such a coil can be approximated by [16]

$$C_c \approx \varepsilon_0 l \beta \left(\frac{\varepsilon_r \Theta}{\ln \frac{d_2}{d_w}} + \cot \frac{\Theta}{2} - 1 \right), \quad (15.21)$$

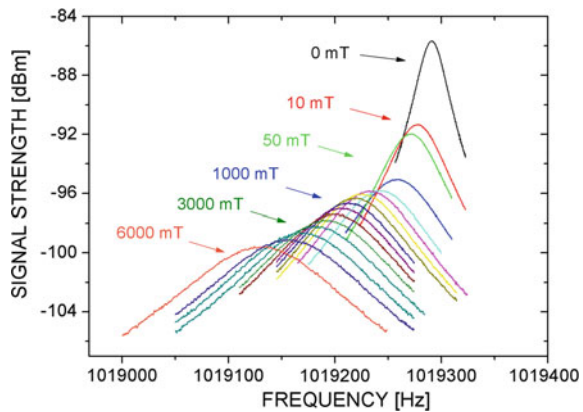
where $l \approx (d_2 - d_1) + 2a$ is the single mean-turn length, and the quantity

$$\Theta = \arccos \left(1 - \frac{1}{\varepsilon_r} \ln \frac{d_2}{d_w} \right) \quad (15.22)$$

accounts for the diameter d_w of the core including the winding. Here, $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m and $\varepsilon_r = 2.1$ are the permittivities of free space and the insulator, respectively, and $\beta \approx 1.366$ again is a geometry factor. The total capacitance C of the resonance circuit consists of the self-capacitance C_c of the coil, and the parasitic capacitances of the coil-housing arrangement and the wiring. Figure 15.10 shows measured response curves (signal strength as a function of frequency) for different values of the magnetic field strength from zero to 6 T. There is a significant change in the spectral response with respect to the resonance frequency, the signal width, and the overall magnitude of the signal. This is an undesired effect, since it makes control over the most important resonator properties such as resonance frequency and quality factor difficult. Even more, when the magnetic field strength is changed up and down, there is some hysteresis effect [15]. Note, that the observed magnitudes of these effects cannot be explained by high-frequency loss mechanisms in the superconductor, but need to be attributed to minor changes of the coil geometry under magnetic field gradient forces when the field is changed.

In particular, the capacitance is very sensitive to changes of the coil geometry, as can be seen from (15.21) (see also the discussion in [16]). When looking at the

Fig. 15.10 Measured resonator response as a function of the frequency for different values of the magnetic field strength



change of resonator capacity resulting from a change of the effective coil diameter d_w

$$\frac{\partial C_c}{\partial d_w} = \varepsilon_0 \varepsilon_r l \beta \frac{1}{d_w} \ln^{-2} \left(\frac{d_w}{d_2} \right) \arccos \left(1 - \frac{1}{\varepsilon_r} \ln \left(\frac{d_w}{d_2} \right) \right), \quad (15.23)$$

then for the present resonator it will amount to 1.1 pF/mm (which is a stronger relative change than that of the inductance, by one order of magnitude), such that a relative change of the capacity by 10^{-4} can easily result from variation of the effective coil diameter on the micrometre scale.

The lesson is that it is mandatory to design the resonator in such a way as to minimize possible geometry changes. This is true both for effects of temperature gradients and field gradients. The resonator discussed in [17] apparently has a smaller magnetic field dependence, on account of manufacturing details.

Let us note that the power spectrum of the unexcited resonator at zero field can be used to derive the overall inductance L and capacitance C of the resonator. The power signal $P(\omega)$ given in dBm (Fig. 15.10) can be converted into the corresponding voltage $u(\omega)$ across the impedance R' of the measuring device by

$$u^2(\omega) = R' \times 10^{(P(\omega)-30)/10} \times 1 \text{ W}. \quad (15.24)$$

This voltage $u(\omega)$ corresponds via the Johnson noise equation

$$u^2(\omega) = 4 k_B T B_f \text{Re}(Z(\omega)) \quad (15.25)$$

to the real part

$$\text{Re}(Z(\omega)) = R \left(1 + \frac{Q^2}{\left(\frac{\omega}{\omega_0} \right)^2} \left[\left(\frac{\omega}{\omega_0} \right)^2 - 1 \right]^2 \right)^{-1} \quad (15.26)$$

of the resonator impedance, when T is the temperature of the resonator and B_f is the chosen observation bandwidth of the spectrum analyser. A fit of u to the measured resonance curve yields the quality factor Q and $R = Z(\omega_0)$ at the known resonance frequency ω_0 , from which L and C can be calculated.

15.7 Detection of Small Frequency Differences

The application of the continuous Stern-Gerlach effect to the electron bound in a highly charged ion, and even more so in the case of the proton and anti-proton, requires to measure small differences in the oscillation frequencies of the particle due to a change of the spin state. For the axial motion, the frequency change due to a spin flip is given by [18]

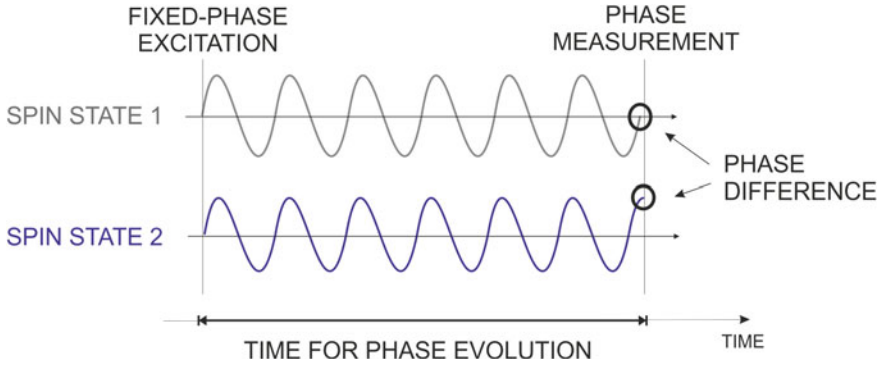


Fig. 15.11 Schematic principle of the phase-sensitive detection of frequency differences by the different phases after a defined time of free phase evolution following a fixed-phase excitation

$$\Delta\omega_z = \frac{\hbar}{m_e} \frac{\omega_+}{\omega_z} \frac{B_2}{B_0} \frac{e}{q} g_J \propto \frac{1}{\sqrt{qm}} g_J B_2, \quad (15.27)$$

see also the discussion in Sect. 17.2. This frequency change is usually significantly below one Hz, and tends to decrease with increasing ion mass m and charge state q . Also, it is proportional to the magnetic moment of the particle, which makes the measurement much more complicated for particles like the proton and anti-proton.

However, for a decision on a spin flip it is not required to make precise measurements of the absolute frequencies, but it suffices to be able to tell the oscillation frequencies apart. For this purpose, sensitive detection schemes have been conceived. They circumvent the classical Fourier limit of frequency analysis by relating only to the frequency difference. These are phase-sensitive detection schemes where the frequency difference between a particle with electron spin up and one with spin down is seen as a phase difference of the two cases after a given time of free phase evolution, as depicted in Fig. 15.11. The details of these methods (‘PnP’: Pulse and Phase, and ‘PnA’: Pulse and Amplify) have been given in [19–22], respectively. They are in some sense the non-destructive counterparts of the phase-imaging technique as described in [23]. The relative frequency resolution can be 10^{-10} and better with measurement times of the order of seconds.

References

1. H.G. Dehmelt, F.L. Walls, “Bolometric” technique for the RF spectroscopy of stored ions. *Phys. Rev. Lett.* **21**, 127 (1968)
2. D.J. Wineland, H.G. Dehmelt, Principles of the stored ion calorimeter. *J. Appl. Phys.* **46**, 919 (1975)
3. M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996)

4. D.K. Finnemore, T.F. Stromberg, C.A. Swenson, Superconducting properties of high-purity niobium. *Phys. Rev.* **149**, 231 (1966)
5. M. Tinkham, Effect of fluxoid quantization on transitions of superconducting films. *Phys. Rev.* **129**, 2413 (1963)
6. F. London, H. London, The electromagnetic equations of the supraconductor. *Proc. Roy. Soc. (London)* **A149**, 866 (1935)
7. C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1986)
8. R. Doll, M. Näbauer, Experimental proof of magnetic flux quantization in a superconducting ring. *Phys. Rev. Lett.* **7**, 51 (1961)
9. J.J. Carr, *Secrets of RF Circuit Design*, 3rd edn. (McGraw-Hill, New York, 2000). ISBN 978-0071370677
10. D.M. Pozar, *Microwave Engineering* (Wiley, New York, 2011). ISBN 9780470631553
11. F. Terman, *Radio Engineers' Handbook* (McGraw-Hill, New York, 1943)
12. D.H. Staelin, *Electromagnetics and Applications* (Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA, 2011)
13. H. Nagahama et al., Highly sensitive superconducting circuits at 700 kHz with tunable quality factors for image-current detection of single trapped antiprotons. *Rev. Sci. Inst.* **87**, 113305 (2016)
14. A. Reza et al., An in situ trap capacitance measurement and ion-trapping detection scheme for a Penning ion trap facility. *Rev. Sci. Inst.* **88**, 034705 (2017)
15. M.S. Ebrahimi et al., Superconducting radio-frequency resonator in magnetic fields up to 6 Tesla. *Rev. Sci. Inst.* **87**, 075110 (2016)
16. S.W. Pasko, M.K. Kazimierczuk, B. Grzesik, Self-capacitance of coupled toroidal inductors for EMI filters. *IEEE Trans. Electromagn. Compat.* **57**, 216 (2015)
17. S. Ulmer et al., The quality factor of a superconducting RF resonator in a magnetic field. *Rev. Sci. Inst.* **80**, 123302 (2009)
18. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
19. S. Stahl et al., Phase-sensitive measurement of small frequency differences. *J. Phys. B* **38**, 297 (2005)
20. E.A. Cornell et al., Single-ion cyclotron resonance measurement of $M(\text{CO}^+)/M(\text{N}_2^+)$. *Phys. Rev. Lett.* **63**, 1674 (1989)
21. S. Sturm, A. Wagner, B. Schabinger, K. Blaum, Phase-sensitive cyclotron frequency measurements at ultralow energies. *Phys. Rev. Lett.* **107**, 143003 (2011)
22. F. Köhler et al., The electron mass from g-factor measurements on hydrogen-like carbon $^{12}\text{C}^{5+}$. *J. Phys. B.* **48**, 144032 (2015)
23. S. Eliseev et al., Phase-imaging ion-cyclotron-resonance measurements for short-lived nuclides. *Phys. Rev. Lett.* **110**, 082501 (2013)

Chapter 16

Radio-Frequency Spectroscopy: Penning-Trap Mass Spectrometry



Abstract This chapter takes a short look at mass spectrometry in Penning traps, which to some extent is one specific application of radio-frequency spectroscopy of the particle oscillations in the trap. We have a brief look at precision mass spectrometry, and then discuss mass spectrometry as an analytical tool for a quantitative determination of the trap content.

16.1 Penning Traps as Mass Spectrometers

Penning traps are well-suited instruments for mass spectrometry. While the precision of Paul traps and other radio-frequency devices is limited by the stability of the electric field, this caveat is not present in Penning traps. The frequencies of all three oscillatory motions of a confined particle in a Penning trap depend on the particle's mass-to-charge ratio. We can hence solve (5.13) for the mass m or likewise the ratio m/q , so that on the right hand side of the equations we have known values and frequencies that can potentially be measured with high accuracy. Ideally, this is done with a single particle in order to avoid influences of particle-particle interaction. A measurement using only the axial oscillation would not require the magnetic field to be known or constant, but is limited to first order by the accuracy of the voltage source [1]. A measurement using radial frequencies requires knowledge of the magnetic field strength, which is possible via $\omega_c = qB_0/m$ and the invariance theorem $\omega_c^2 = \omega_+^2 + \omega_z^2 + \omega_-^2$ when all three oscillation frequencies are measured with a well-known particle. As an alternative, an optical spectroscopy technique (laser-fluorescence mass spectroscopy) can be used to replace the electronic measurement of the particles' radio-frequency oscillations [2].

A very common technique to gain high accuracies is to not perform an absolute measurement, but to compare frequencies against one another [3–9] or to those of a known mass standard such as ^{12}C [10, 11]. This is preferably done with simultaneously confined particles, to minimise systematic uncertainties [12]. Most prominent examples of mass comparisons are the ^3He – ^3H mass difference [13–15] (relevant for the neutrino mass determination), the mass difference between the α -particle and ^4He [16] (relevant in several contexts within high-precision mass spectrometry), the

proton-antiproton mass ratio $m(p)/m(\bar{p})$ [6, 7] (relevant for tests of fundamental symmetries like CPT), and the electron-proton mass ratio $m(e)/m(p)$ (relevant in several metrological aspects [17] and also for astrophysics [18, 19]). Of further significance are the masses of silicon ^{28}Si (in the framework of a definition of the kg [17]), the mass of caesium ^{133}Cs (in the framework of the link between the Rydberg constant R_∞ and the fine structure constant α [17]), and further the masses of ^{87}Rb , ^{85}Rb , and ^{23}Na [20]. For a thorough discussion of these cases see for example [17, 21].

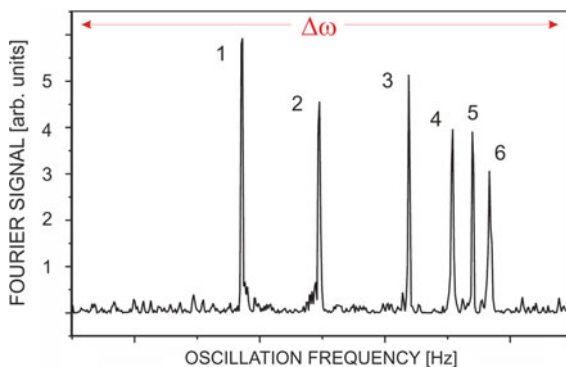
There is a huge range of techniques and applications of high-precision mass spectrometry, as it does not only allow to determine atomic masses, but via the relation $m = E/c^2$ also binding energies and other properties of nuclei and the like [22–25]. In turn, highly precise mass measurements in Penning traps can be used as a sensitive test of relativity [26]. Mass spectrometry with or in Penning traps comprises a large number of techniques, including time-of-flight spectrometry upon mass-selective excitation and ejection from a Penning trap (TOF-ICR, see also Sect. 10.4) [27], parametric excitation of confined particles [28, 29], the application of Ramsey techniques to Penning traps [30–33], laser fluorescence mass spectroscopy [2] and many more. Penning traps can also act as beam or bunch purification devices on account of their mass selectivity [34, 35]. A comprehensive treatment of Penning trap mass spectrometry can be found in [36–38]. A similar text with particular focus on studies of radio-nuclides such as [39–41] can be found in [42, 43].

Here we focus on the spectrometry of particle species contained in the trap, as this is a common requirement. Most particle creation/capture techniques are not selective enough to produce only one species of interest, but a variety of species with different mass-to-charge ratios m/q . Since the oscillation frequencies in the trap are dependent on this value, radio-frequency spectroscopy can be used to obtain mass-to-charge spectra of the particles confined in the trap, based on non-destructive (electronic) detection. This is usually possible in two main ways.

16.2 Transient Recording: FT-ICR

One well-established technique is Fourier-Transform Ion-Cyclotron Resonance (FT-ICR) in which the (commonly radial) particle motion (at considerable energy) is recorded within a transient time T and subsequently analysed for its frequency content by Fourier analysis within a bandwidth $\Delta\omega$. The frequency resolution within this bandwidth is Fourier-limited by $1/T$. This gives simultaneous information about the radial frequencies (and hence mass-to-charge ratios) present in the trap, provided that the magnetic field is known or that at least one of the particle species has a well-known mass-to-charge ratio. Figure 16.1 shows a generic example of six particle species with different values of m/q in an FT-ICR spectrum within a frequency bandwidth $\Delta\omega$. At the same time, the signal intensities are proportional to the

Fig. 16.1 Generic example of six particle species with different m/q in an FT-ICR spectrum within a frequency bandwidth $\Delta\omega$



particle number and energy, such that also quantitative information is obtained. The FT-ICR technique was first introduced in 1974 by Comisarow and Marshall [44] and a detailed overview of the technique and applications can be found in [45–48].

16.3 Narrow-Band Scan: ARES

A scan of a broad range of radial frequencies using a narrow-band detection is commonly impossible since the magnetic field usually cannot be scanned in an appropriate fashion. This, however is possible with the trap voltage the produces the confining potential well. Since the axial frequency depends to first order on that voltage, a voltage scan can be combined with a narrow-band electronic detection at a fixed frequency, for example by a resonant detection circuit attached to an endcap. This technique is called ‘Axial REsonance Spectroscopy’ (ARES). When the trap voltage is scanned, particles with different mass-to-charge ratios come into resonance with the circuit subsequently and thus the scan produces a spectrum of all mass-to-charge ratios within the scan region. The result of such a voltage scan is shown in Fig. 16.2. It shows the detected signal (power density recorded by the resonant circuit within its natural bandwidth) as a function of trap voltage. Again, the signals can be identified with specific mass-to-charge ratios once the trap parameters are known or at least one of the species contained is known. The technique has been described for example in [49].

Since the detection system records a spectral power density in a certain frequency band for a certain scan time, each peak area is proportional to an energy. To be more precise, it is proportional to the individual particle energy times the number of particles in that peak. This means that at constant particle number, the area measures the individual particle energy. For measurements of cooling, it is hence necessary to be able to rule out particle loss for example by charge exchange during the measurement. An example is shown in Fig. 16.3: it shows the exponential cooling of a selected (pure) ensemble of Ar^{13+} ions, where no product of a charge exchange

Fig. 16.2 ARES-Spectrum of ions with different q/m in a Penning trap

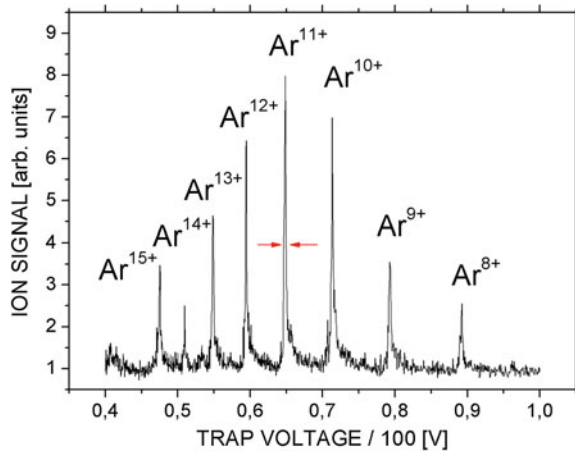
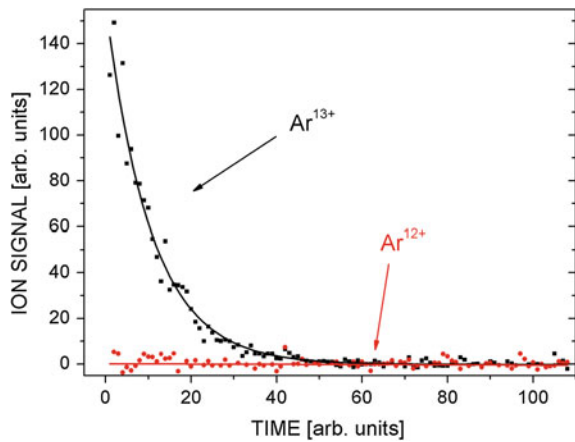


Fig. 16.3 Resistive cooling of a pure ensemble of Ar^{13+} ions in a trap, with no indication of ion loss by charge exchange, which would result in an increase of the Ar^{12+} signal as a function of time



appears as a function of time. In presence of charge exchange, the observed decay of Ar^{13+} would (at least partially) be due loss of the charge state, but then the Ar^{12+} or other low charge state signals would increase as a function of time.

In turn, for fixed energy, the peak area measures the particle number in units of some energy. If the individual particle energy and complete transfer function of the signal pick-up and amplification is known, this yields absolute particle numbers.

The spectral widths of the peaks is commonly larger than the spectral resolution of the detection system and thus not limited by it. Then, it is mainly given by the distribution of particle energies which leads to a distribution of oscillation frequencies due to imperfections of the confining fields, see the discussion in Sect. 12.13.4.

16.4 Energy Dependence of Spectral Features

In spectra such as shown in the previous examples, the individual peaks are mainly characterised by their area and their position within the spectrum, corresponding to their ratio of charge to mass. We assume that the spectral width of the detection circuit used for the recording of a spectrum is negligible with respect to the width of the individual peaks, i.e. the quality factor Q of the circuit is significantly larger than the quality factor of each particle species. This simplifies the discussion as we do not need to consider a de-convolution of the two distributions, but may use the detected signals directly.

We have discussed that in absence of particle loss, each peak area is a measure for the corresponding ensemble kinetic energy. In the presence of imperfections of the confining fields as discussed in Chap. 6, the width of a signal peak is mainly due to the distribution of oscillation frequencies corresponding to the distribution of particle kinetic energies, see also the discussion in Sect. 12.13.4. This width is usually mainly linear in the ensemble temperature T , as for small energies commonly the electrostatic term in C_4 is the dominant imperfection. It may, however, have higher-order contributions (in E^2 etc.) as well, in case the higher-order coefficients such as C_6 are not negligible and the particle energies are high, see the discussion in Sect. 6.2. The particle kinetic energies at a given temperature follow a Boltzmann distribution which is asymmetric. Correspondingly, a reduction of the ensemble temperature results in a reduced area and width of the frequency distribution and to a shift of the centre of gravity of the peaks.

Figure 16.4 (left) gives the expected energy distribution of a thermalised ensemble of a single particle species at two different temperatures $T_1 > T_2$. The median energy in the regarded degree of freedom (for example the axial oscillation) at temperature T is $k_B T/2$, and the expected width of the distribution is about $2k_B T$. The right part of the figure shows the corresponding spectral appearance of the detected signal at

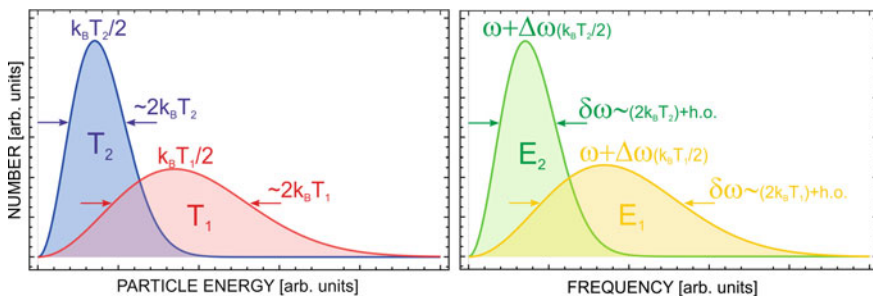


Fig. 16.4 Left: energy distribution of a thermalised ensemble of a single particle species at two different temperatures $T_1 > T_2$ with the median value and the expected width indicated. Right: the corresponding spectral appearances (signal intensity as a function of frequency) characterised by energy-dependent frequency shifts with respect to the ‘ideal’ oscillation frequency ω and temperature-dependent signal widths

the two temperatures, characterised by the peak area proportional to the ensemble energy, a temperature-dependent median frequency shift (with respect to the ‘ideal’ frequency in absence of field imperfections, or at zero kinetic energy) and a signal width that depends on the width of the particle energy distribution at this temperature, possibly including higher-order contributions.

The display of the expected signal at two different temperatures illustrates the evolution of an observed signal during a cooling process. As the energy decreases, so does the peak area. At the same time, the signal width decreases as the particles probe less of the field imperfections and the peak centre shifts in frequency accordingly. So the spectral line shape evolution of the signal during cooling is characterized by the time derivatives of the three main parameters: the particle energy E , the width of the signal spectrum $\delta\omega$ and the corresponding centre frequency shift $\Delta\omega$, which in common cases are expected to be similar in value.

16.5 Destructive Detection Techniques

Let us briefly discuss the possibilities for a destructive detection and spectroscopy of particles in a Penning trap upon ejection. The trap content can be non-resonantly ejected by lowering of the trap potential, in the inverse sense of dynamic particle capture as discussed in Sect. 9.2, and an additional acceleration voltage can be applied. Usually, this is done along the magnetic field axis such that the particles drift towards some detector on or along the axis, which may be used for time-of-flight mass spectrometry (TOF-MS) [50] and TOF-ICR [27].

The most simple detection device is a Faraday cup [51], which is a charge collector with a subsequent charge amplifier that produces a signal in a sensitive ampere-meter or a voltage signal, for example on an oscilloscope. This is a well-suited device for dc currents or large bunches of charged particles, respectively.

Higher sensitivities are achieved with micro-channel-plate detectors (MCPs) [52] which come in a huge variety of specifications, and which are also available with resolution of the arrival position (position sensitivity).

A variation of this is the channeltron [53, 54], which follows the same idea of charge avalanches (secondary emission) produced when the impinging particle releases electrons from hit surfaces in a high-voltage arrangement, but this form of detector can also be placed off-axis, which is of advantage when for example there is a laser beam on the trap axis.

The conversion of charged-particle kinetic energy into optical photons in scintillators [55] can be used for a position-sensitive detection with energy resolution. Read-out is achieved with a simple light imaging such as a CCD camera, and these devices have been shown to be well-suited also for bunches of highly charged ions [56]. On average, for ions, the scintillation light yield is roughly 50–70% of the yield for electrons, and about 25–50% of the yield for photons [57]. Scintillator materials like P43 (amorphous $\text{Gd}_2\text{O}_2\text{S:Tb}$) emit on average 36 photons per keV deposited kinetic energy, P46 ($\text{Y}_3\text{Al}_5\text{O}_{12}\text{:Ce}$) emits about half of that [57]. P43 emits light

between 360 and 680 nm with a spectral peak wavelength of about 545 nm (green) and has an average decay time (90–10% intensity) of 1.0 ms. P46 also emits green light with a spectral peak wavelength of 550 nm and has an average decay time of 70 ns. Several other types of scintillators suitable for UHV are available. A detailed account on particle detection devices can be found for example in [58].

References

1. D.B. Pinegar, K. Blaum, T.P. Biesiadzinski, S.L. Zafonte, R.S. Van Dyck Jr., Stable voltage source for Penning trap experiments. *Rev. Sci. Instr.* **80**, 064701 (2009)
2. D.J. Wineland, J.J. Bollinger, W.M. Itano, Laser-fluorescence mass spectroscopy. *Phys. Rev. Lett.* **50**, 628 (1983)
3. E.A. Cornell et al., Single-ion cyclotron resonance measurement of $M(\text{CO}^+)/M(\text{N}_2^+)$. *Phys. Rev. Lett.* **63**, 1674 (1989)
4. M. Redshaw, J. McDaniel, E.G. Myers, Dipole moment of PH^+ and the atomic masses of ^{28}Si , ^{31}P by comparing cyclotron frequencies of two ions simultaneously trapped in a Penning trap. *Phys. Rev. Lett.* **100**, 093002 (2008)
5. D.L. Farnham, R.S. van Dyck, P.B. Schwinberg, Determination of the electron's atomic mass and the proton/electron mass ratio via Penning trap mass spectroscopy. *Phys. Rev. Lett.* **75**, 3598 (1995)
6. S. Ulmer et al., High-precision comparison of the antiproton-to-proton charge-to-mass ratio. *Nature* **524**, 196 (2015)
7. G. Gabrielse, A. Khabbaz, D.S. Hall, C. Heimann, H. Kalinowsky, W. Jhe, Precision mass spectroscopy of the antiproton and proton using simultaneously trapped particles. *Phys. Rev. Lett.* **82**, 3198 (1999)
8. F. DiFilippo, V. Natarajan, M. Bradley, F. Palmer, D.E. Pritchard, Accurate atomic mass measurements from Penning trap mass comparisons of individual ions. *Phys. Scr.* **T59**, 144 (1995)
9. S. Brunner, T. Engel, A. Schmitt, G. Werth, Helium and deuterium mass ratios in a room temperature Penning trap. *AIP Conf. Proc.* **457**, 125 (1999)
10. K. Blaum et al., Carbon clusters for absolute mass measurements at ISOLTRAP. *Eur. Phys. J. A* **15**, 245 (2002)
11. A. Kellerbauer et al., From direct to absolute mass measurements: a study of the accuracy of ISOLTRAP. *Eur. Phys. J. D* **22**, 53 (2003)
12. S. Rainville et al., An ion balance for ultra-high-precision atomic mass measurements. *Science* **303**, 334 (2004)
13. R.S. Van Dyck Jr. et al., The UW-PTMS: systematic studies, measurement progress, and future improvements. *Int. J. Mass Spectr.* **251**, 231 (2006)
14. D.B. Pinegar, Tools for a precise tritium to helium-3 mass comparison, Ph.D. thesis, University of Washington, Seattle (2007)
15. C. Diehl et al., Progress with the MPIK/UW-PTMS in Heidelberg. *Hyp. Int.* **199**, 291 (2011)
16. R.S. Van Dyck Jr., S.L. Zafonte, S. Van Liew, D.B. Pinegar, P.B. Schwinberg, Ultraprecise atomic mass measurement of the α Particle and ^4He . *Phys. Rev. Lett.* **92**, 220802 (2004)
17. P.J. Mohr, D.B. Newell, B.N. Taylor, *Rev. Mod. Phys.* **88**, 035009 (2016)
18. J. King, J. Webb, M. Murphy, R. Carswell, Stringent null constraint on cosmological evolution of the proton-to-electron mass ratio. *Phys. Rev. Lett.* **101**, 251304 (2008)
19. M. Murphy, V. Flambaum, S. Muller, C. Henkel, Strong limit on a variable proton-to-electron mass ratio from molecules in the distant universe. *Science* **320**, 1611 (2008)
20. M.P. Bradley, J.V. Porto, S. Rainville, J.K. Thompson, D.E. Pritchard, Penning trap measurements of the masses of ^{133}Cs , $^{87,85}\text{Rb}$, and ^{23}Na with uncertainties < 0.2 ppb. *Phys. Rev. Lett.* **83**, 4510 (1999)

21. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
22. M. Block et al., Discovery of a nuclear isomer in ^{65}Fe with Penning trap mass spectrometry. *Phys. Rev. Lett.* **100**, 132501 (2008)
23. K. Blaum et al., Population inversion of nuclear states by a Penning trap mass spectrometer. *Europhys. Lett.* **67**, 586 (2004)
24. T. Eronen et al., Mass and QEC value of ^{26}Si . *Phys. Rev. C* **79**, 032802 (2009)
25. A. Kellerbauer et al., Direct mass measurements on the superallowed emitter ^{74}Rb and its daughter ^{74}Kr : isospin-symmetry-breaking correction for standard-model tests. *Phys. Rev. Lett.* **93**, 072502 (2004)
26. S. Rainville et al., World year of physics: a direct test of $E = mc^2$. *Nature* **438**, 1096 (2005)
27. G. Bollen et al., The accuracy of heavy-ion mass measurements using time of flight-ion cyclotron-resonance in a Penning trap. *J. Appl. Phys.* **68**, 4355 (1990)
28. L. Schweikhard, M. Lindinger, H.-J. Kluge, Parametric mode excitation/dipole mode detection Fourier transform ion cyclotron resonance spectrometry. *Rev. Sci. Instr.* **61**, 1055 (1990)
29. D.L. Rempel, E.B. Ledford, S.K. Huang, M.L. Gross, Parametric mode operation of a hyperbolic Penning trap for Fourier transform mass spectrometry. *Anal. Chem.* **59**, 2527 (1987)
30. S. George et al., Ramsey method of separated oscillatory fields for high-precision Penning trap mass spectrometry. *Phys. Rev. Lett.* **98**, 162501 (2007)
31. S. George et al., The Ramsey method in high-precision mass spectrometry with Penning traps: experimental results. *Int. J. Mass Spectr.* **264**, 110 (2007)
32. M. Kretschmar, The Ramsey method in high-precision mass spectrometry with Penning traps: theoretical foundations. *Int. J. Mass Spectr.* **264**, 122 (2007)
33. M. Heck et al., One- and two-pulse quadrupolar excitation schemes of the ion motion in a Penning trap investigated with FT-ICR detection. *Appl. Phys. B* **107**, 1019 (2012)
34. P. Ascher et al., PIPERADE: a Penning-trap isobar separator for the DESIR low-energy facility of SPIRAL2. *EPJ Web of Conferences* **66**, 11002 (2014)
35. E. Minaya Ramirez et al., Conception of PIPERADE: a high-capacity Penning-trap mass separator for high isobaric contamination at DESIR. *Nucl. Inst. Meth. B* **376**, 298 (2016)
36. K. Blaum et al., Penning traps as a versatile tool for precise experiments in fundamental physics. *Contemp. Phys.* **51**, 149 (2010)
37. E.G. Myers, The most precise atomic mass measurements in Penning traps. *Int. J. Mass Spectr.* **349–350**, 107 (2013)
38. K. Blaum, High-accuracy mass spectrometry with stored ions. *Phys. Rep.* **425**, 1 (2006)
39. G. Bollen et al., Experiments with thermalized rare isotope beams from projectile fragmentation: a precision mass measurement of the superallowed β emitter ^{38}Ca . *Phys. Rev. Lett.* **96**, 152501 (2006)
40. M. Smith et al., First Penning-trap mass measurement of the exotic halo nucleus ^{11}Li . *Phys. Rev. Lett.* **101**, 202501 (2008)
41. E. Minaya Ramirez et al., Direct mapping of nuclear shell effects in the heaviest elements. *Science* **337**, 1207 (2012)
42. M. Block, High-precision mass measurements of radionuclides in Penning traps, in: *Fundamental Physics in Particle Traps*, Springer Tracts in Modern Physics, vol 256 (Springer, Berlin, 2014)
43. M. Block, Mass measurements and ion-manipulation techniques applied to the heaviest elements, Nobel Symposium NS160 - Chemistry and Physics of Heavy and Superheavy Elements, EPJ Web of Conferences, vol 131, 05003 (2016)
44. M.B. Comisarow, Fourier transform ion cyclotron resonance spectroscopy. *Chem. Phys. Lett.* **25**, 282 (1974)
45. A.G. Marshall, Fourier transform ion cyclotron resonance detection: principles and experimental configurations. *Int. J. Mass Spectr.* **215**, 59 (2002)
46. A.G. Marshall et al., Fourier transform ion cyclotron resonance mass spectrometry: a primer. *Mass Spectrom. Rev.* **17**, 1 (1998)
47. D. Rodriguez et al., Broad-band FT-ICR MS for the Penning-trap mass spectrometer MATS. *AIP Conf. Proc.* **1265**, 483 (2010)

48. M. Ubieto-Diaz, A broad-band FT-ICR Penning trap system for KATRIN. *Int. J. Mass. Spectr.* **288**, 1 (2009)
49. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
50. W.C. Wiley, I.H. McLaren, Time-of-flight mass spectrometer with improved resolution. *Rev. Sci. Instr.* **26**, 1150 (1955)
51. K.L. Brown, G.W. Tautfest, Faraday-cup monitors for high-energy electron beams. *Rev. Sci. Instr.* **27**, 696 (1956)
52. J. Wiza, Microchannel plate detectors. *Nucl. Inst. Meth.* **162**, 587 (1979)
53. J.S. Allen, The detection of single positive ions, electrons and photons by a secondary electron multiplier. *Phys. Rev.* **55**, 966 (1939)
54. J.S. Allen, An improved electron multiplier particle counter. *Rev. Sci. Instr.* **18**, 739 (1947)
55. S.C. Curran, *Counting Tubes, Theory and Applications* (Academic Press, New York, 1949)
56. M. Vogel, D.F.A. Winters, H. Ernst, O. Kester, H. Zimmermann, Scintillation light produced by slow, highly-charged ions. *Nucl. Inst. Meth. B* **263**, 518 (2007)
57. W.R. Leo, *Techniques for Nuclear and Particle Physics Experiments* (Springer, Berlin, 1994)
58. W. Göpel, J. Hesse, J.N. Zemel, *Sensors, Optical Sensors* (Wiley, New York, 2008). ISBN 978-3-527-62070-8

Chapter 17

Radio-Frequency Spectroscopy: Outreach



Abstract This chapter gives a short review of the spectroscopic techniques that are specific to Penning traps, since they make dedicated use of the confining fields and their controllable properties. In particular, we show how precise measurements of the oscillation frequencies of particles in the radio-frequency domain may be used to infer spectroscopic information in the optical and microwave domains.

17.1 Expressions for Frequency Shifts

We consider the combined effect of the lowest-order dependences given in (6.25), (6.55) and (7.14). This leads us to twelve equations for frequency shifts (reduced cyclotron, axial, magnetron and Larmor motions each as a function of reduced cyclotron, axial and magnetron energy), each consisting of three additive contributions due to electrostatic imperfections, magnetostatic imperfections and the relativistic effect. We now focus on the reduced cyclotron and axial motions, leaving the magnetron motion aside, since it is not of relevance for the present discussion. Apart from the discussion of the Larmor frequency this leaves four equations. They describe the relative frequency shifts as a function of oscillation energies for the axial and perturbed cyclotron motion:

$$\frac{\Delta\omega_+}{\omega_+}(E_z) = \left(\frac{-6C_4}{C_2} \frac{1}{qU} \frac{\eta^2}{2} - \frac{1}{2m\omega_- \omega_+} \frac{B_2}{B_0} - \frac{1}{2mc^2} \right) E_z \quad (17.1)$$

$$\frac{\Delta\omega_+}{\omega_+}(E_+) = \left(\frac{6C_4}{C_2} \frac{1}{qU} \frac{\eta^4}{4} - \frac{1}{2m\omega_- \omega_+} \frac{B_2}{B_0} \eta^2 - \frac{1}{mc^2} \right) E_+ \quad (17.2)$$

and

$$\frac{\Delta\omega_z}{\omega_z}(E_z) = \left(\frac{3C_4}{2C_2} \frac{1}{qU} - \frac{3}{8mc^2} \right) E_z \quad (17.3)$$

$$\frac{\Delta\omega_z}{\omega_z}(E_+) = \left(\frac{-6C_4}{C_2} \frac{1}{qU} \frac{\eta^2}{2} + \frac{1}{2m\omega_- \omega_+} \frac{B_2}{B_0} - \frac{1}{2mc^2} \right) E_+, \quad (17.4)$$

where the first term describes the electric shift, the second term the magnetic shift, and the third term the relativistic effect, respectively. For the Larmor frequency we have the additional two dependences

$$\frac{\Delta\omega_L}{\omega_L}(E_z) = \frac{1}{2m\omega_-\omega_+} \frac{B_2}{B_0} E_z \quad (17.5)$$

$$\frac{\Delta\omega_L}{\omega_L}(E_+) = -\eta^2 \frac{1}{2m\omega_-\omega_+} \frac{B_2}{B_0} E_+. \quad (17.6)$$

All terms of order η^2 or higher may be considered inept for the applications we will discuss below, since $\eta^2 = \omega_z^2/\omega_+^2$ is typically of the order of 10^{-2} or smaller, such that the achievable effect in an experiment is smaller by that number as well. This however is not necessarily an obstruction.

For the upcoming discussion, it is useful to look at a quantum description of the particle motion in the confining potential of the trap. We will restrict the discussion to effects in a magnetic bottle and therefore look at a situation in which electric imperfections have been tuned out by corresponding choice of U_C/U_0 in (3.8). A quantum description of effects due to electric imperfections is given in detail in [1, 2].

17.2 Explicit Expressions: Quantum Picture

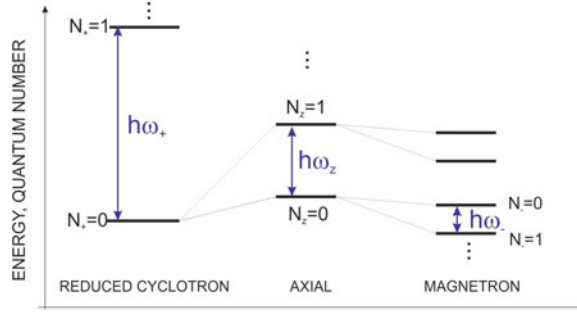
In the context of many experiments, the classical picture of confined motion is a very good description of reality. For confined particles at very low energies, the quantisation of energy cannot be neglected and we need to speak about the quantum number N of each individual motion. While light particles such as the electron are close to the ground state at liquid-helium temperature, ions have large quantum numbers $N \gg 1$ such that the relative difference between adjacent quantum numbers is negligible and the classical picture holds. It is still instructive to look at these equations in the quantised picture, especially for the discussion of the continuous Stern-Gerlach effect and particle spin flips in the presence of a magnetic bottle.

Figure 17.1 schematically shows the level scheme of a confined particle with quantum numbers N_+ , N_z and N_- for the reduced cyclotron, axial and magnetron motion, respectively. The kinetic energies of the motions are given by the usual

$$E_k = \hbar\omega_k \left(N_k + \frac{1}{2} \right). \quad (17.7)$$

Note, however, that the magnetron energy decreases with increasing quantum number due to the unstable nature of the motion as discussed in Sect. 5.3.3.

Fig. 17.1 Schematic of the level scheme of a confined particle with quantum numbers N_+ , N_z and N_- for the reduced cyclotron, axial and magnetron motion, respectively



Following the discussion in [2], we employ quantum mechanical first-order perturbation theory to describe the energy shift of the confined particle due to the presence of the magnetic bottle by

$$\Delta E(N_+, N_z, M_S) = \langle N_+ N_z M_S | H' | N_+ N_z M_S \rangle, \quad (17.8)$$

where H' is the perturbation Hamiltonian describing the influence of the magnetic bottle and N_+ and N_z are the quantum numbers of the modified cyclotron and the axial motion, respectively. M_S is the spin quantum number corresponding to the particle spin $\mathbf{S}$. The oscillatory motion of the particle in the potential of the trap is quantized such that e.g. $N_z = 0$ denotes the ground state axial motion corresponding to an energy of $E_z = \hbar\omega_z/2$. Details about this formalism can be found in [1, 2]. The perturbation by the magnetic bottle is given by

$$H' = g\mu_B \frac{\mathbf{S}}{\hbar} \mathbf{B}' + \frac{q\mathbf{v}}{c} \mathbf{A}', \quad (17.9)$$

where $\mathbf{A}'$ is the vector potential corresponding to the magnetic bottle field

$$\mathbf{B}' = B_2 \left(\left(z^2 - \frac{\rho^2}{2} \right) \mathbf{e}_z - z\rho \mathbf{e}_\rho \right). \quad (17.10)$$

g is the g -factor of the particle measuring its magnetic moment in units of the Bohr magneton $\mu_B = e\hbar/(2m_e)$. The vector potential due to the bottle contribution is given by [2]

$$\mathbf{A}' = \frac{1}{2} B_2 \left(z^2 + \frac{\rho^2}{4} \right) \mathbf{e}_z \times \rho \mathbf{e}_\rho, \quad (17.11)$$

and yields a total perturbation energy of [2]

$$\begin{aligned}
E(N_+, N_-, N_z, M_S) = & g\mu_B B_2 \frac{\hbar}{m} \left(\frac{N_z + \frac{1}{2}}{\omega_z} - \frac{N_+ + N_- + 1}{\omega_+ - \omega_-} \right) M_S \quad (17.12) \\
& + \frac{q\hbar^2}{m^2 c} B_2 \frac{N_z + \frac{1}{2}}{\omega_z} \frac{1}{\omega_+ - \omega_-} \left[\omega_+ \left(N_+ + \frac{1}{2} \right) + \omega_- \left(N_- + \frac{1}{2} \right) \right] \\
& + \frac{q\hbar^2}{m^2 c} B_2 \frac{N_+ + N_- + 1}{2(\omega_+ - \omega_-)} \frac{1}{\omega_+ - \omega_-} \left[\omega_+ \left(N_+ + \frac{1}{2} \right) + \omega_- \left(N_- + \frac{1}{2} \right) \right] \\
& - \frac{q\hbar^2}{m^2 c} B_2 \frac{\omega_+ + \omega_-}{(\omega_+ - \omega_-)^2} \left(N_+ + \frac{1}{2} \right) \left(N_- + \frac{1}{2} \right).
\end{aligned}$$

The shift of individual frequencies is then obtained by writing down the perturbation energy difference of adjacent energy levels in the corresponding degree of freedom, for example looking at the axial motion we have

$$\Delta\hbar\omega_z(N_+, N_-, N_z, M_S) = E(N_+, N_-, N_z + 1, M_S) - E(N_+, N_-, N_z, M_S). \quad (17.13)$$

Using this and including the relativistic shift in terms of quantum numbers, the corresponding frequency shifts can be written as

$$\Delta\omega_+(N_z) = \left[\frac{\hbar}{m} \frac{B_2}{B_0} \frac{\omega_+}{\omega_z} - \frac{\hbar\omega_+}{2mc^2} \omega_z \right] \left(N_z + \frac{1}{2} \right) \quad (17.14)$$

$$\Delta\omega_+(N_+) = \frac{\hbar}{m} \frac{B_2}{B_0} \left[\omega_+(N_+ + 3) + \omega_-(N_+ + \frac{1}{2}) \right] - \frac{\hbar\omega_+}{2mc^2} \omega_+ \left(N_+ + \frac{1}{2} \right) \quad (17.15)$$

and

$$\Delta\omega_z(N_z) = -\frac{3\hbar\omega_z}{8mc^2} \omega_z \left(N_z + \frac{1}{2} \right) \quad (17.16)$$

$$\Delta\omega_z(N_+) = \left[\frac{\hbar}{m} \frac{B_2}{B_0} \frac{\omega_+}{\omega_z} - \frac{\hbar\omega_z}{2mc^2} \omega_+ \right] \left(N_+ + \frac{1}{2} \right), \quad (17.17)$$

again using the frequency hierarchy $\omega_- \ll \omega_z \ll \omega_+ \approx \omega_c$.

For the dependence of the spin precession (Larmor) frequency $\omega_L = g\mu_B B_0/\hbar$ on the axial and radial quantum numbers we find

$$\Delta\omega_L(N_z) = \frac{g\mu_B B_2}{m} \left(\frac{N_z + \frac{1}{2}}{\omega_z} \right) \quad (17.18)$$

$$\Delta\omega_L(N_+) = -\frac{g\mu_B B_2}{m} \left(\frac{N_+ + 1}{\omega_+ - \omega_-} \right). \quad (17.19)$$

We will also use the dependence of the oscillation frequencies on the spin orientation of the trapped particle, since it is of relevance for example when the continuous Stern-Gerlach effect in a magnetic bottle is used to determine the magnetic moment of the bound electron, as will be discussed in Sect. 22.4.1.

Using the spin orientation energy $E_S = g\mu_B B_0 M_S$, we find the dependence of the axial and modified cyclotron frequencies on the spin orientation by

$$\Delta\omega_z(\Delta M_s) = \frac{\hbar}{2m} \frac{\omega_+}{\omega_z} \frac{B_2}{B_0} \frac{e}{q} g \Delta M_s \quad (17.20)$$

$$\Delta\omega_+(\Delta M_s) = \frac{\hbar}{2m} \frac{\omega_+}{\omega_+ - \omega_-} \frac{B_2}{B_0} \frac{e}{q} g \Delta M_s \quad (17.21)$$

As we will see in the discussion of applications of this dependence, the relative change of the axial frequency is typically much larger than the relative change of the radial motions on account of

$$\frac{\omega_+}{\omega_z} > \frac{\omega_+}{\omega_+ - \omega_-}, \quad (17.22)$$

such that the axial motion is commonly used to detect a change of the spin orientation. Still, for the electron bound in an ion, or for an unbound proton / anti-proton, this frequency change is typically below 1 Hz out of MHz.

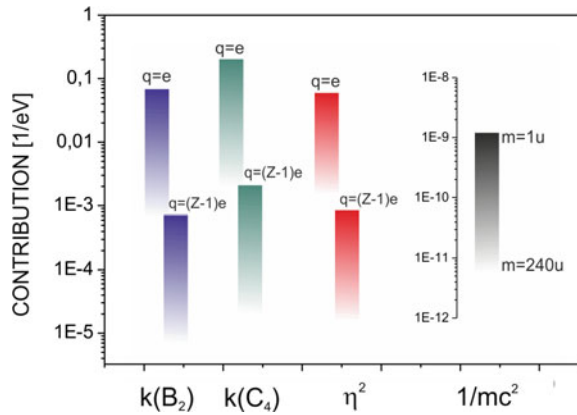
17.3 Magnitudes of the Involved Parameters

Figure 17.2 illustrates the magnitudes of the involved quantities as discussed in the above sections.

Typical values for the terms

$$k(C_4) = \frac{6C_4}{C_2} \frac{1}{qU} \quad (17.23)$$

Fig. 17.2 Typical values of $k(C_4)$, $k(B_2)$ and η^2 for singly charged ions ($q = e$) and hydrogen-like ions with $q = (Z - 1)e$. The relativistic term $1/(mc^2)$ is shown for masses from $m=1$ u to $m=240$ u



and

$$k(B_2) = \frac{B_2}{2m\omega_- \omega_+ B_0} \quad (17.24)$$

in (6.25) and (6.55) range from $10^{-5}/\text{eV}$ to $10^{-3}/\text{eV}$ in highly charged ions and from $10^{-3}/\text{eV}$ to $10^{-1}/\text{eV}$ in singly charged ions. The value of $\eta^2 = \omega_z^2/\omega_+^2$ can be approximated by

$$\eta^2 \approx \frac{mU}{qd^2 B_0^2} \quad (17.25)$$

and typically ranges from 10^{-5} to 10^{-3} in highly charged ions and from 10^{-3} to 10^{-1} in singly charged ions. The relativistic term $1/(mc^2)$ is of order $10^{-10}/\text{eV}$ for light ions and of order $10^{-11}/\text{eV}$ for heavy ions. A graphic representation of the typical magnitude of these shift terms is given in Fig. 17.2. Here, the ion mass has been varied between $m = 1 \text{ u}$ and $m = 240 \text{ u}$, the trap potential U between 10 and 1000 V, magnetic fields B_0 between 1 and 10 T, and values of $d = 1 \text{ cm}$ and $B_2 = 10 \text{ mT/mm}^2$ have been assumed.

17.4 Classification of Field Effects

When we use the information about energy-dependent shifts of oscillation frequencies as discussed above and detailed in Sects. 6.1 and 6.3, we find the non-destructive measurement of frequencies or small frequency differences to be a tool for optical and microwave spectroscopy, since the shifts may tell us about the success of electromagnetic absorption or emission, and thus about the location of the desired resonance. This hence forms a link between radio-frequency spectroscopy of particle oscillations in the trap, and optical and microwave spectroscopy of internal transitions in confined particles. This link is specific to the confinement situation and depends on the confining fields and their imperfections as parameterised by the coefficients C_k and B_k , see above and Sects. 6.1 and 6.3.

When we look at the explicit equations for energy-dependent oscillation frequency shifts, we find 10 energy-dependent frequency shift terms when we exclude the relativistic corrections. Out of those, four terms are of significant magnitude (i.e. non-zero and not of order η^2 or higher). They can be used for a detection of changes of the oscillation energy by observation of the corresponding shifts in the oscillation frequencies. These terms are

$$\Delta\omega_+(E_z, B_2), \quad \Delta\omega_z(E_z, C_4), \quad \Delta\omega_z(E_+, B_2) \quad (\text{group 1}) \quad \text{and} \quad \Delta\omega_L(E_z, B_2). \quad (17.26)$$

Additionally, we have two terms

$$\Delta\omega_z(M_s, B_2) \quad \text{and} \quad \Delta\omega_+(M_s, B_2) \quad (\text{group 2}), \quad (17.27)$$

and the four relativistic correction terms of order 1 in E/mc^2 (group 3).

- Terms in group 1 can be used to detect changes in the oscillation energy of confined ions, e.g. due to laser cooling or heating, and thus serve as an electronic detector for optical photons.
- Terms in group 2 allow to determine a spin change of the system, e.g. of a single electron bound in an ion, and therefore can serve as an electronic detector for microwave photons which induce spin transitions. This is part of the measurement of magnetic moments by use of the continuous Stern-Gerlach effect.
- The relativistic terms in group 3 make oscillation frequencies dependent on kinetic energies even for ideal confining fields, but are usually too small in magnitude for spectroscopic purposes. However, the relativistic mass effect due to $\Delta m = \Delta E/c^2$ allows to weigh internal excitation energies by the corresponding frequency shift, e.g. of nuclear isomeric states in ions.

Additionally, the dependence $\Delta\omega_L(E_z, E_+)$ can be used for a manipulation of the Larmor frequency, which may be of use in spectroscopy as in group 2. The dependences in group 1 can alternatively be used to measure magnetic bottle strengths by electronic means: for known trap geometry and confining fields, the electric anharmonicity C_4 can be chosen by variation of the voltage ratio U_C/U_0 such that it cancels the effect of the B_2 term. A scan of U_C/U_0 such that the total energy-dependence of the oscillation frequency vanishes, yields the corresponding B_2 .

All of these possible applications rely on a detection of the corresponding oscillation frequency shift. For cooled ions under suitable trapping conditions, relative frequency shifts of 10^{-10} and below can be detected by application of a phase-sensitive detection scheme as outlined in [3, 4]. When the frequencies themselves are not measured, but only a shift is to be detected, one can thereby circumvent the Fourier limit $\sigma = 1/T$ and detect sub-Hertz changes in sub-second times T [3]. With novel variants of this method, it is not necessary to artificially excite the ion for a measurement [4].

17.5 Group-1-Terms: Optical Spectroscopy

17.5.1 Transition Energies

Assume an ion stored in a magnetic bottle with $B_2 \neq 0$ superimposed to the magnetic trapping field B_0 . The terms $\Delta\omega_+(E_z, B_2)$ and $\Delta\omega_z(E_+, B_2)$ then describe the dependence of the radial frequency ω_+ on the axial energy E_z , and the dependence of the axial frequency ω_z on the radial energy E_+ of the ion, respectively. The energies can individually be set to well-defined values by application of initial cooling, e.g. by resistive cooling to the cryogenic ambience temperature using a resonance circuit. The effect of laser cooling (or heating) on a transition of interest can then be observed by a frequency shift corresponding to the expressions $\Delta\omega_+(E_z)$ or

$\Delta\omega_z(E_+)$. Scanning a narrow-band laser over the transition of interest, the resonance is found as a maximum shift of the corresponding ion oscillation frequency, which is detected electronically. The applicability and potential of such a scheme has been described in detail in [5]. There, the focus has been set on the precise determination of magnetic dipole transition frequencies in highly charged ions (fine structure and hyperfine structure transitions), however, the concept is *per constructionem* applicable to any particle suited for laser cooling. The potential precision lies beyond the part per billion region due to the low ion velocities and details of the confinement [5]. The concept is applicable also to cases where ion production is difficult since only a single ion is needed. Also, it can be applied in transition frequency domains where suited photon detectors are unavailable, especially in the infra-red.

17.5.2 Transition Rates

In a situation as described above, there are two mechanisms which can be used to change the energy of the ion oscillation in a well-defined way. One is the cooling or heating by the detuned laser, the other is the cooling or heating by a resonance circuit as used for initial resistive cooling. A balance between any two opposing mechanisms, e.g. laser heating against resistive cooling, results in a vanishing shift of the oscillation frequency as a function of time and may be used to determine the desired rate (inverse lifetime) Γ of the used optical transition. The power transferred to the ion by the laser is given by

$$P_{opt} = \hbar\Delta\omega \frac{S}{1 + 2S + (\Delta\omega/\Gamma)^2} \Gamma, \quad (17.28)$$

where $\Delta\omega$ is the laser detuning with respect to the atomic transition frequency and $S = |\Omega|^2/\Gamma^2$ is the saturation parameter which is proportional to the square of the on-resonance Rabi frequency Ω . $S/(1+2S)$ is equal to 1/2 for a fully saturated transition [6], and for sufficiently small laser detuning $(\Delta\omega/\Gamma)^2 \ll 1$, (17.28) simplifies to

$$P_{opt} = \hbar\Delta\omega\Gamma/2. \quad (17.29)$$

For simplicity, we will use this relation for the further calculation. Depending on the sign of $\Delta\omega$ this power transfer is positive or negative and can be balanced either by the (negative) power transfer of resistive cooling or the (positive) power transfer P_E of electronic excitation via a resonant electric circuit, thus yielding the value of Γ by

$$\Gamma = \frac{2P_E}{\hbar\Delta\omega}. \quad (17.30)$$

Another possibility, independent from electronic power transfer, makes use of the fact that the rate at which the observed frequency ω_+ in the above example shifts is directly proportional to the desired transition rate Γ , since

$$\frac{d}{dt}\omega_+ \propto \frac{d}{dt}E_z = P_{opt} \propto \Gamma. \quad (17.31)$$

Thus, for known detuning, saturation and magnetic bottle strength in a given trap, the desired transition rate can be determined by the rate of the frequency shift using

$$\Gamma = \frac{1}{\hbar\Delta\omega} \frac{mU_0C_2}{2B_2d^2} \frac{d}{dt}\Delta\omega_+. \quad (17.32)$$

If these parameters are not known to sufficient accuracy, a system with well-known lifetime can be used to gauge the above relation and experimentally determine the proportionality factor between Γ and the frequency shift rate. The potential accuracy of this kind of lifetime measurement is limited by the accuracy to which the frequency shift rate can be measured, since all other parameters are well-controllable.

Alternatively, one can make use of the fact that the light pressure of laser cooling shifts the ion axially from the trap centre by an amount much larger than the motional amplitude. In the presence of a magnetic inhomogeneity, this results in a shift of the radial frequencies which can be measured. An axially asymmetric trapping potential may be used to restore the ion position and thus its radial frequencies, yielding the value of the shift. This directly determines the desired value of the transition rate Γ . The shift Δz of the axial ion position is obtained from balancing the force F_L of the laser with the restoring force F_E due to the electrostatic trapping potential, i.e.

$$F_L = F_E \rightarrow \frac{\hbar\Delta\omega\Gamma}{2c} = \frac{qC_2U_0\Delta z}{d^2}. \quad (17.33)$$

The result is

$$\Delta z = \frac{\hbar\Delta\omega\Gamma d^2}{2cqC_2U_0}. \quad (17.34)$$

This shift of the axial position (and thus the shift of the measured radial frequencies) can be restored if the electrostatic trap potential is made asymmetric by introducing an additional voltage U_A to one endcap. The effect of this is a shift of the axial position by

$$\Delta z = \frac{1}{2} \frac{d}{z_0} \frac{U_A}{U_0} \frac{C_1}{C_2} d, \quad (17.35)$$

where C_1 is the first term in the expansion of the trapping potential as given by (3.6) and is non-zero due to the introduced asymmetry. Equating the latter two expressions (17.34) and (17.35) yields

$$\Gamma = \frac{cqC_1}{\hbar\Delta\omega z_0} U_A, \quad (17.36)$$

which is the desired transition rate expressed by of the voltage U_A applied to make the laser-induced shift of the axial position (and correspondingly of the radial frequency) vanish. Equation (17.36) has the nice feature that on the right hand side, we find only constants and well-controllable parameters. Also, (17.36) is independent of the electric trapping potential given by U_0 such that one is free to choose the axial frequency.

17.6 Group-2 Terms: Microwave Spectroscopy

When an ion is confined in the presence of a magnetic bottle, the terms in group 2 (i.e. $\Delta\omega_z(M_s, B_2)$ and $\Delta\omega_+(M_s, B_2)$) provide that the ion oscillation frequencies depend on the spin orientation of an electron bound in the ion relative to the magnetic field. Especially for hydrogen-like ions, this so-called ‘continuous Stern-Gerlach effect’ offers a possibility to determine the spin orientation of the electron (which is an intrinsic ion property described by the magnetic spin quantum number M_s) by a radio-frequency measurement of a macroscopic oscillatory motion of the ion. For typical confinement parameters, i.e. for magnetic fields B_0 of several Tesla strength, the Larmor frequency of electrons is in the microwave domain at typically 100 GHz, corresponding to photon energies of order meV. For protons, this number is still smaller by a factor of about 568, such that photon energies are of order μeV and below. Irradiating such microwaves around the Larmor frequency of the spin precession around the magnetic field and scanning the microwave frequency across the Larmor resonance, the Larmor frequency can be found as a maximum spin transition rate [7].

Alternatively, the term $\Delta\omega_L(E_z, E_+)$ (17.5) and (17.6) may be used to scan the Larmor itself while keeping the irradiated microwave frequency fixed. The resulting resonance is equivalent to before, however this kind of ion manipulation is expected to be more difficult experimentally and limited in accuracy due to other energy-dependent effects coming into play. It may nevertheless be an option in selected cases, where the microwave frequency scan is impossible.

Using the terms in group 2, the magnetic moment of the bound electron (and thus its g -factor g_J) can be determined with a relative precision on the part per billion scale. Corresponding measurements have previously been performed on the hydrogen-like ions, for an overview see [8].

17.7 Group-3 Terms: ‘Weighing’ Photons

The relativistic mass effect provides a change of the oscillation frequencies due to the mass change of the confined ion when its energy content is changed. The relativistic mass shift $\Delta m = \Delta E/c^2$ changes the oscillation frequencies, such that the absorption or emission of a photon with energy ΔE can be monitored by an

oscillation frequency measurement. Note, that it is not necessary to employ trapping field imperfections for a detection of relativistic effects such that this ‘weighing’ of excitations is equivalent to high-precision mass measurements and has the same potential precision [9]. The relative frequency shift of order $10^{-10}/\text{eV}$ for light ions is for optical spectroscopy at the limit of the current resolution, however, this does not restrict the principle idea. Absorption of a photon of several eV energy by a light ion may serve as a proof of principle. Given a sensitivity higher by an order of magnitude, the relativistic frequency shift could be a valuable tool in finding the famous low-lying nuclear transition in ^{229}Th [10].

In highly charged ions, electronic excitation energies are much higher (up to the order of 100 keV) and could thus be detected much more easily, however the corresponding lifetimes of the excited states are extremely short. In few-electron ions, the upper state lifetime scales with the nuclear charge number Z as Z^{-4} for electric dipole, as Z^{-6} for magnetic dipole and as Z^{-10} for electric quadrupole transitions, such that only for $Z < 5$ lifetimes of order seconds are possible [11]. One well-known example is the metastable 2^3S_1 state in Li^+ with a lifetime of about 50 s and a decay energy of about 60 eV [12].

In the study of nuclear de-excitations, however, both the upper state lifetime and the photon energies are potentially high. With photon energies in the keV to MeV region, the expected relative frequency shifts are of order 10^{-7} to several 10^{-5} and thus easily detectable. The corresponding ion recoil energy due to photon emission is given by $p^2/(2m)$, where $p = \hbar\omega/c$ is the emitted photon momentum, and is of order eV for the highest relevant photon energies such that the confinement is not influenced significantly. The radioactive decay of isotopes is followed by a discontinuous change of the mass-to-charge ratio of the ion (due to α or β emission) and can be detected as a corresponding frequency shift simply due to the normal equations of motion.

The radiative decay of long-lived nuclear isomers, however, does not change the mass-to-charge ratio and can therefore only be seen by the relativistic mass shift corresponding to the emitted photon energy. Corresponding measurements of long-lived nuclear isomers have been performed, e.g. on $^{65\text{m}}\text{Fe}$ [14] and $^{68\text{m}}\text{Cu}$ [15]. Also, the Q_{EC} value of the super-allowed β -emitter ^{26}Si has been determined by such a trap-assisted measurement with a relative accuracy of about 10^{-5} [16]. Suitable systems for trap-assisted measurements need to have isomeric lifetimes which are longer than the required frequency measurement time and are limited by the charge exchange lifetime τ of the ion due to residual gas in the trap, since a charge exchange alters the mass-to-charge ratio and makes a determination of the relativistic shift difficult or impossible. At typical cryogenic vacua, a lifetime of order 1000 s is commonly achieved. Figure 17.3 shows all known metastable nuclear isomers with transition lifetimes between 1 and 1000 seconds, for which either no reliable energy measurement exists and / or the transition energy is not known to better than 1 keV. This includes also the isomers for which no transition energy uncertainty is given. Together, these are about 170 out of 350 known isomers in the given lifetime interval. The data have been taken from [13]. The left hand scale gives the transition energy in keV, the right hand scale shows the corresponding relative frequency shift due to the relativistic mass effect. The encircled ions are particularly good candidates for

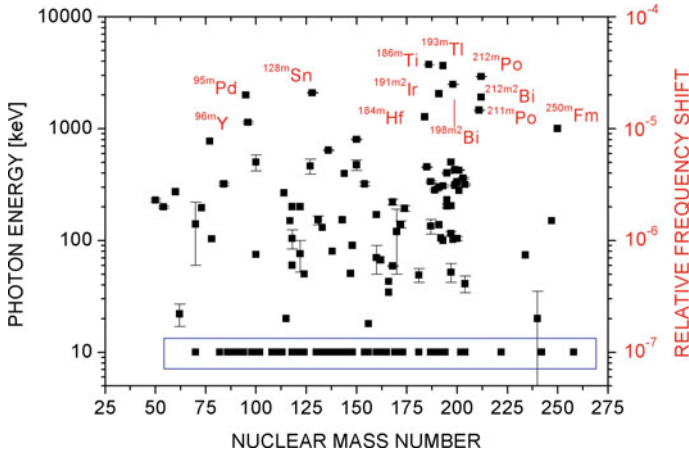


Fig. 17.3 Metastable nuclear isomers with transition lifetimes between 1 and 1000 s, for which either no reliable energy measurement exists and / or the transition energy is not known to better than 1 keV. Data taken from [13]. The indicated ions are particularly good candidates for trap-assisted measurements due to their large relative frequency shift of up to several 10^{-5} . For the candidates in the box, no measurements exist

trap-assisted measurements due to their large frequency shift of up to several 10^{-5} . Assuming a frequency resolution of some 10^{-10} , the transition energies of these nuclei can be measured with a relative accuracy of about 10^{-5} , which is substantially more precise than any of the measurements shown in Fig. 17.3. Due to the system-unspecific nature of the measurement method which does not require certain energy level schemes or a detection of the emitted photon(s), it is suited for any de-excitation process within the given lifetime region.

17.8 Spectroscopic Use of Trap Properties: Overview

Both the measurement of magnetic moments by use of the continuous Stern-Gerlach effect as described in Sect. 21.2, and laser-microwave double-resonance spectroscopy as presented in Chap. 20 make use of the predictable influence of an unobservable experimental quantity on an observable one, through effects of the confining fields and / or their imperfections.

In case of the ‘classical’ Stern-Gerlach effect, the spin orientation M_S is translated via a magnetic field gradient B_1 into a change Δz in position, hence we may say that

$$\Delta M_S \rightarrow \Delta z. \quad (17.37)$$

In the case of the continuous Stern-Gerlach effect, the change of the orientation M_J of the bound-electron spin is translated into an observable frequency difference $\Delta\omega_z$ of the axial ion oscillation by the presence of a magnetic bottle B_2 such that

$$\Delta M_J \rightarrow \Delta\omega_z. \quad (17.38)$$

In case of the laser-microwave double-resonance spectroscopy it is the resonant Zeeman transition which is found as a minimum of observed fluorescence light I when the irradiated microwave frequency is varied in the presence of the constant homogeneous magnetic field B_0 , so we may say

$$\Delta\omega_{MW} \rightarrow \Delta I. \quad (17.39)$$

We have seen for the decay of metastable nuclear states that the emission (or absorption) of photons with energy E can be measured by the relativistic mass equivalent $\Delta m = \Delta E/c^2$, by which the ion mass is changed during the process. In a trap, this is observable as a change of the cyclotron frequency, i.e.

$$\Delta m \rightarrow \Delta\omega_c. \quad (17.40)$$

Taking the well-known relations of the ion oscillation frequencies for specific non-ideal confining fields as given in Sects. 6 and 21.2 for granted, well-defined (artificial) field imperfections can be used to evoke particle oscillation frequency changes upon absorption or emission of optical or X-ray photons. For example, when a confined ion is laser-cooled, the change in motional energy may be translated via a magnetic bottle B_2 or an electrostatic anharmonicity C_4 into a change of an oscillation frequency, for example axial cooling or heating changing the radial oscillation frequency, which may for either case be written as

$$\Delta E_z \rightarrow \Delta\omega_+ \quad (17.41)$$

In such experiments, classical spectroscopy can be performed without the need for corresponding fluorescence detectors, since the ion oscillation frequency serves as the measurable quantity. Hence, such specific field distortions may be seen as a kind of ‘transducer’ from the optical and microwave domains into the radio-frequency domain where non-destructive detection is possible.

Apart from the influence on oscillation frequencies, also spatial information about confined particles may be used, for example when resonant photo-ionisation changes the charge state of a particle and this effect is either seen as a significant discrete change of the oscillation frequency, but also of the amplitudes, as we have discussed in Sect. 7.9. We would hence have both

$$\Delta q \rightarrow \Delta\omega_{z,\pm} \quad \text{and} \quad \Delta q \rightarrow \Delta(a_z, \rho_{\pm}). \quad (17.42)$$

References

1. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
2. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
3. S. Stahl et al., Phase-sensitive measurement of small frequency differences. *J. Phys. B* **38**, 297 (2005)
4. S. Sturm, A. Wagner, B. Schabinger, K. Blaum, Phase-sensitive cyclotron frequency measurements at ultralow energies. *Phys. Rev. Lett.* **107**, 143003 (2011)
5. M. Vogel, W. Quint, Laser spectroscopy by a radiofrequency measurement on a single ion in a Penning trap. *New J. Phys.* **11**, 013024 (2009)
6. W. Demtröder, *Laser Spectroscopy* (Springer, Heidelberg, 2003)
7. H. Häffner et al., High-accuracy measurement of the magnetic moment anomaly of the electron bound in hydrogen-like carbon. *Phys. Rev. Lett.* **85**, 5308 (2000)
8. S. Sturm et al., Experiments for high-precision measurements of the bound electron's magnetic moment. *Atoms* **5**, 4 (2017)
9. K. Blaum, High-accuracy mass spectrometry with stored ions. *Phys. Rep.* **425**, 1 (2006)
10. E. Peik, C. Tamm, Nuclear laser spectroscopy of the 3.5 eV transition in Th-229. *Europhys. Lett.* **61**, 181 (2003)
11. H.F. Beyer, H.J. Kluge, V.P. Shevelko, *X-Ray Radiation of Highly-Charged Ions* (Springer, Heidelberg, 1997)
12. R.D. Knight, M.H. Prior, Radiative lifetime of metastable 2^3S_1Li^+ . *Phys. Rev. A* **21**, 179 (1980)
13. S.Y.F. Chu, L.P. Ekström, R.B. Firestone, WWW Table of Radioactive Isotopes, <http://nucleardata.nuclear.lu.se/nucleardata/toi/>
14. M. Block et al., Discovery of a nuclear isomer in ^{65}Fe with Penning trap mass spectrometry. *Phys. Rev. Lett.* **100**, 132501 (2008)
15. K. Blaum et al., Population inversion of nuclear states by a Penning trap mass spectrometer. *Europhys. Lett.* **67**, 586 (2004)
16. T. Eronen et al., Mass and Q_{EC} value of ^{26}Si . *Phys. Rev. C* **79**, 032802 (2009)

Chapter 18

Optical Spectroscopy



Abstract Confinement of particles in Penning traps yields two major advantages for spectroscopy, namely localisation for extended periods (i.e. the particle position is constant and well-known) and cooling (which leads to small Doppler shifts and broadening of transitions). Also, the particle ensemble can often be treated as a point-like source, which facilitates detection. We give a short account of the important aspects to optical spectroscopy when performed in a Penning trap.

18.1 Applications of Optical Spectroscopy in Penning Traps

An outline of spectroscopy of singly charged ions in traps has been given in [1, 2]. This includes a discussion of frequency standards with ions, for which Penning traps are suited [3], although many such standards have been realised in radio-frequency traps [4]. A compilation of techniques for optical precision spectroscopy including ions in traps is given in [5, 6]. Here, we concentrate on optical spectroscopy of the fine structure and hyperfine structure of highly charged ions, also with respect to its combination with microwave spectroscopy that allows one to determine electronic and nuclear magnetic moments by laser-microwave double-resonance spectroscopy of highly charged ions.

Optical spectroscopy of confined particles (apart from optical clocks and such) is applied mainly to electric dipole (E1) transitions in singly charged ions, and to magnetic dipole (M1) transitions in highly charged ions, where for a number of species either the fine structure or the hyperfine structure is in the optical domain.

18.2 Light Source and Fluorescence Detection

For optical spectroscopy it is commonly necessary to bring light from its source to the confinement volume, and to collect light emitted from there and guide it to a suited photon detector. Laser light can be guided to the trap either in a sequence of straight optical paths by use of mirrors or prisms, or it can be guided in fibers or

light conduits that work on the basis of internal total reflection. The latter is possible only for a certain part of the optical spectrum, but is an interesting alternative for complicated optical paths. Light collection can be achieved in the same way, and it is often a nuisance that almost all photon detection devices cannot be placed in close proximity to the trap since they do not agree with the combination of strong magnetic fields and cryogenic temperatures. This is true at least for the most common devices such as photo-multiplier tubes (PMTs), avalanche photo-diodes (APDs) [7], charged-coupled devices (CCDs) and enhanced-sensitivity variants such as EMCCDs which may have single-photon sensitivity at low dark count rates. Imaging of light emitted from confined particles can either be achieved by straight projection onto a camera device like in [8–12], or by use of an image conduit which is a flexible bundle of optical fibers with a one-to-one correspondence of the fiber positions on either end as used in [13–15].

The choice of detector depends mostly on the wavelength region in question, commonly one distinguishes three domains, the ultra-violet (UV) region (roughly 200–400 nm), the visible to near-infrared (NIR) region (roughly 400–1000 nm) and the near-infrared region from about 1000 to 1650 nm. Often, photo-multiplier tubes are a good choice in the UV domain, where detection quantum efficiencies are high and dark count rates are low. In the visible to near-infrared region, often avalanche photo-diodes or CCD devices are suited [16], while detection in the infrared, for example with specialised NIR-APD detectors, is usually complicated by thermal background and correspondingly high dark-count rates d [16, 17]. These detectors hence need to be cooled, but do not work at temperatures as low as 4 K.

Recent years have seen efforts to introduce optical detection devices such as UV-sensitive superconducting nanowire single-photon detectors [18] or light guides that are integrated in trap structures [19], which operate at cryogenic temperatures and close to the particle confinement region, this however so far appears to be true only for situations without a strong magnetic field.

18.3 Photon Detection Rate and Counting Statistics

When we assume a situation with N confined particles that each emit photons at a rate Γ , then the total photon yield is $N\Gamma$, since for the comparatively low particle number densities in Penning traps there is negligible shielding or re-absorption and transfer into other de-excitation processes. Apart from deviations due to possible preferred emission characteristics as discussed in Sect. 18.5, these photons are commonly emitted isotropically into the full solid angle of 4π . Depending on the type of trap in use, the detectable solid angle Ω can be anywhere between zero and almost 4π . Optical access to confined particles has been a good part of the motivation for the development of planar Penning traps [20–24] as well as of various other types of traps such as wire Penning traps [25], stylus traps [26] and others [27–29], see also our discussion in Chap. 4. The number $\Omega/4\pi$ is often called the ‘light collection efficiency’ of the trap. This light is guided to the detector, and the light transport

efficiency is quantified by a factor $a < 1$, which is often not much below 1. The conversion from photons to an electronic signal in the detector has a quantum efficiency $q < 1$, which strongly depends on the wavelength region and the details of the detector. These numbers can be close to unity, but for low-energy photons can be in the range of 10^{-4} or below. Hence, the total signal rate in the detector is given by

$$R = N\Gamma \frac{\Omega}{4\pi} aq, \quad (18.1)$$

which needs to be optimized with respect to the detector dark count rate, in order to produce the optimal signal-to-noise ratio of the detection.

Note that the rate R does not necessarily have to be bigger than the detector dark count rate in order to make a statement on the presence of the signal. Assuming again N confined particles each emitting fluorescence photons at a rate Γ , the observation time t necessary to detect a signal at a given confidence level k_σ (measured in units of a standard deviation) obeys the relation [17]

$$Rt = N\Gamma \frac{\Omega}{4\pi} aqt \geq k_\sigma \sqrt{td}, \quad (18.2)$$

where d is the detector dark count rate and $aq\Omega/4\pi$ is the total detection efficiency taking into account solid angles and quantum efficiencies. One example: we assume Ar^{13+} fine-structure spectroscopy on the 441 nm line, with typical numbers of $N = 10^3$, $\Gamma = 10^2 \text{ s}^{-1}$, $aq\Omega/4\pi = 10^{-4}$ and $d = 100 \text{ s}^{-1}$ such that a confidence level of $k_\sigma = 5$ is reached within seconds, although the signal rate just equals the dark count rate.

18.4 Transition Line Broadening Effects

18.4.1 Natural Broadening

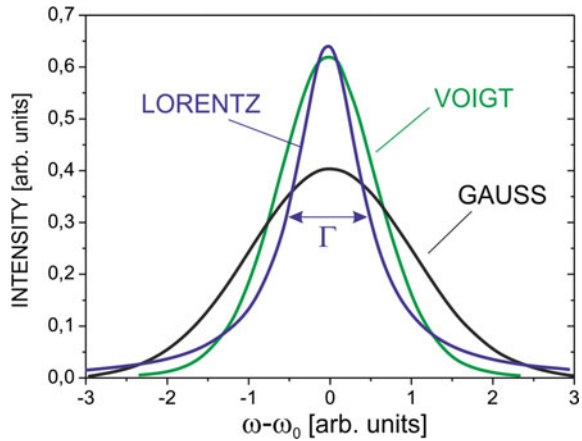
The natural linewidth Γ of an electromagnetic transition at frequency ω_0 is finite due to the Heisenberg uncertainty principle and creates an intensity line profile $I(\omega)$ of Lorentzian shape [6]

$$\frac{I(\omega)}{I_0} = \frac{\Gamma/2\pi}{(\omega - \omega_0)^2 + \left(\frac{\Gamma}{2}\right)^2}, \quad (18.3)$$

where Γ is the full width of the line at half maximum (FWHM), and the profile is normalised such that

$$\int_0^\infty \frac{I(\omega)}{I_0} d\omega = 1, \quad (18.4)$$

Fig. 18.1 Comparison of line shapes: Lorentzian distribution, Gaussian distribution, and their convolution, the Voigt distribution. All distributions normalised to unit area



see Fig. 18.1. This natural line shape and width is altered by additional line broadening effects such as collisional broadening, power broadening and Doppler broadening.

18.4.2 Collisional Broadening

In Penning traps we may need to consider collisional broadening (pressure broadening) which maintains the Lorentzian lineshape but broadens the value of Γ in (18.3) to

$$\Gamma' = \Gamma + 2\nu, \quad (18.5)$$

where ν is the rate of collisions with residual gas that are able to induce a transition on account of their mutual electromagnetic interaction during the collision, see also the discussion of collision rates in Sect. 12.8. The real rate ν depends on details of the involved particles and is particularly high for molecular ions because of their many close-lying levels [6]. In any case, ν depends on the confined particles' velocity such that cooling reduces the broadening. It also depends on the residual gas temperature and pressure, such that the effect is typically negligibly small when Penning trap vacua are considered, particularly in cryogenic vacua.

18.4.3 Power Broadening

In laser spectroscopy it is commonly not desirable to power-broaden the line of interest by too much laser intensity. Details of the mechanism of power broadening (sometimes 'saturation broadening') are given in [6, 30]. A natural linewidth Γ is power-broadened to a linewidth Γ' given by

$$\Gamma' = \Gamma \sqrt{1 + \frac{I}{I_s}}, \quad (18.6)$$

when I is the intensity used and I_s is the saturation intensity. The line shape is still a Lorentzian with the increased width $\Gamma' > \Gamma$.

The saturation intensity I_s of an electric dipole transition with wavelength λ_0 is given by [6]

$$I_s = \frac{2\pi^2 \hbar \Gamma c}{3\lambda_0^3}. \quad (18.7)$$

For electric dipole transitions with transition rates Γ of up to $10^8/\text{s}$, this value can exceed 100 W/m^2 . However, the laser intensity needed to saturate a given transition is also determined by the ratio of spectral widths of the laser and the transition,

$$I_L = I_s \cdot \frac{\Gamma_L}{\Gamma}, \quad (18.8)$$

where Γ is the natural linewidth of the transition and Γ_L is the laser linewidth.

18.4.4 Doppler Broadening

A very common situation is to have an ensemble of particles at some temperature T confined in the trap, and to perform spectroscopy on an electromagnetic transition with resonance frequency ω_0 . In this situation, the velocity distribution (in the direction of observation) of the particles connected with the finite temperature T creates a Gaussian line profile (see Fig. 18.1) given by

$$I(\omega) = I(\omega_0) \exp \left[- \left(\frac{\omega - \omega_0}{\sqrt{4 \ln 2} \delta \omega_D} \right)^2 \right]. \quad (18.9)$$

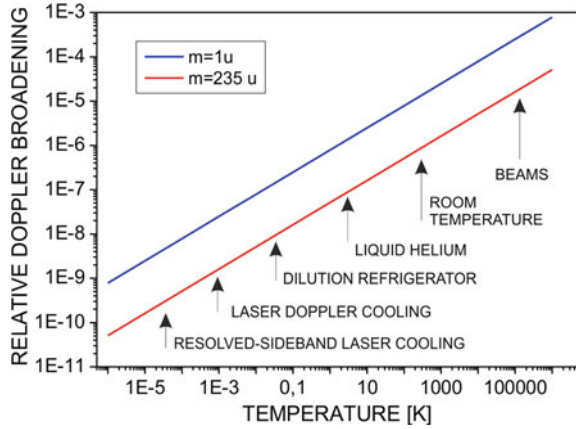
This distribution is observed when the transition is excited with broad-band radiation and the fluorescence is spectrally resolved. For narrow-band excitation, this shape determines the excitation probability and hence the amount of fluorescence detected at a frequency ω .

The Doppler broadening of the linewidth of the transition in such an ensemble is given by [6]

$$\frac{\delta \omega_D}{\omega_0} = \frac{2\sqrt{2 \ln 2}}{c} \sqrt{\frac{k_B T}{m}}. \quad (18.10)$$

In particular for narrow transitions such as M1 transitions, this broadening dominates the natural and pressure broadened linewidths by far, even at low temperatures.

Fig. 18.2 Relative Doppler width of an optical transition in a confined particle for typical trap parameters as a function of the particle ensemble temperature for two different particle masses. The arrows indicate temperatures of special interest as obtained by different cooling mechanisms



The Doppler-broadened linewidth is typically of the order of 10 MHz for ions at $T \approx 4$ K. Correspondingly, at a transition frequency of the order of 10^{14} Hz, the spectral resolution is about $\delta\omega/\omega_0 \approx 10^{-7}$. Figure 18.2 shows the relative Doppler width of an optical transition in a confined ion for typical trap parameters as a function of the ion ensemble temperature for two different particle masses. The arrows indicate temperatures of special interest as obtained by different cooling mechanisms. The observed lineshape is hence often close to a Gaussian, since the Doppler broadening dominates the natural Lorentzian contributions by far, however the true line shape in this situation is a Voigt line shape, which is the convolution of a Lorentzian with a Gaussian distribution, see Fig. 18.1 and the discussions in [6] and Sect. 12.3.2. This is true since the particles moving at a velocity v (in the direction of observation) do not all emit at the same Doppler-shifted frequency

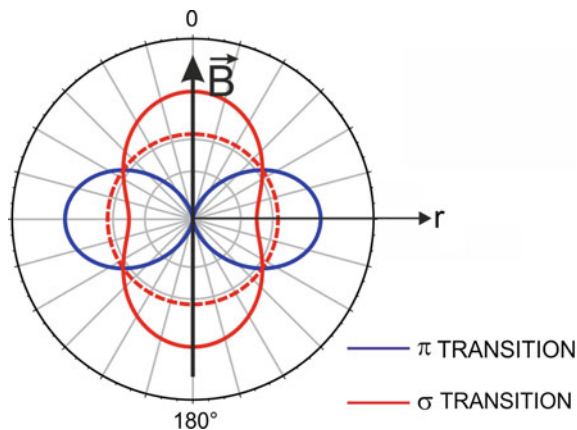
$$\omega'_0 = \omega_0 \left(1 + \frac{v}{c}\right), \quad (18.11)$$

but individually obey the Lorentz distribution (18.3) corresponding to their natural line width. This results in the overall distribution [6]

$$I(\omega) = I(\omega_0) \Gamma \frac{1}{2\pi^{3/2} \omega_0 v/c} \int_0^\infty d\omega' \frac{1}{(\omega - \omega'_0)^2 + \left(\frac{\Gamma}{2}\right)^2} \exp\left(-\frac{(\omega - \omega'_0)^2}{(\omega'_0 v/c)^2}\right). \quad (18.12)$$

This shape hence contains information about the thermal contribution to the broadening and can therefore be used to derive the particle temperature T from a spectral measurement, as we detail in Sect. 12.3.2.

Fig. 18.3 Spatial emission characteristics of π and σ transitions in a magnetic field



18.5 Light Emission Characteristics

In the strong magnetic field of a Penning trap, we have to expect a pronounced Zeeman splitting of optical transitions and a different spatial emission characteristic than in the absence of a magnetic field. This is displayed in Fig. 18.3 which shows the different spatial emission of π and σ transitions in a magnetic field. These characteristics may be important for efficient light collection, as has been discussed in Sect. 3.4. A thorough discussion of emission characteristics can be found for example in [6, 31].

For the implementation of highly precise optical spectroscopy such as for example in frequency standards, a series of effects due to the presence of the electric and magnetic fields used for confinement and their spatial and temporal gradients need to be considered. Predominantly, this is the electric quadrupole shift, higher-order Zeeman shifts in the strong magnetic field, quadratic Stark shifts, the relativistic Doppler shift and gravitational red-shifts. A detailed account on these effects is given in [1].

References

1. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
2. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
3. J.J. Bollinger, S.L. Gilbert, W.M. Itano, D.J. Wineland, Frequency standards utilizing penning traps, in *Frequency Standards and Metrology* (Springer, Berlin, 1989)
4. F. Riehle, *Frequency Standards: Basics and Applications* (Wiley VCH, Weinheim, 2006)
5. R.C. Thompson, High-resolution laser spectroscopy of atomic systems. *Rep. Prog. Phys.* **48**, 531 (1985)
6. W. Demtröder, *Laser Spectroscopy* (Springer, Heidelberg, 2003)
7. I. Prochazka, K. Hamal, B. Sopko, Recent achievements in single photon detectors and their applications. *J. Mod. Opt.* **51**, 1289 (2004)

8. L. Gruber, J.P. Holder, D. Schneider, Formation of strongly coupled plasmas from multi-component ions in a Penning trap. *Phys. Scr.* **71**, 60 (2005)
9. T. Murböck et al., Rapid crystallization of externally produced ions in a Penning trap. *Phys. Rev. A* **94**, 043410 (2016)
10. S. Bharadia, M. Vogel, D.M. Segal, R.C. Thompson, Dynamics of laser-cooled Ca^+ ions in a Penning trap with a rotating wall. *Appl. Phys. B* **107**, 1105 (2012)
11. L. Gruber et al., Evidence for highly charged ion coulomb crystallization in multicomponent strongly coupled plasmas. *Phys. Rev. Lett.* **86**, 636 (2001)
12. S. Mavadia et al., Control of the conformations of ion Coulomb crystals in a Penning trap. *Nat. Comm.* **4**, 2571 (2013)
13. D. von Lindenfels et al., Half-open Penning trap with efficient light collection for precision laser spectroscopy of highly charged ions. *Hyp. Int.* **227**, 197 (2014)
14. D. Von Lindenfels et al., Bound electron g -factor measurement by double-resonance spectroscopy on a fine-structure transition. *Can. J. Phys.* **89**, 79 (2011)
15. D. von Lindenfels et al., Experimental access to higher-order Zeeman effects by precision spectroscopy of highly charged ions in a Penning trap. *Phys. Rev. A* **87**, 023412 (2013)
16. R. Jöhren et al., APDs as single-photon detectors for visible and near-infrared wavelengths down to Hz rates. *J. Instrum.* **7**, P02015 (2012)
17. R. Jöhren, Spectroscopy of the hyperfine transition in lithium-like bismuth at the ESR at GSI and an APD-based single-photon detector for laser spectroscopy on highly charged ions, Ph.D. thesis, University of Münster (2013)
18. D.H. Slichter et al., UV-sensitive superconducting nanowire single photon detectors for integration in an ion trap. *Opt. Express* **25**, 8705 (2017)
19. L. Jiang, W.B. Whitten, S. Pau, A planar ion trapping microdevice with integrated waveguides for optical detection. *Opt. Express* **19**, 3037 (2011)
20. S. Stahl et al., A planar Penning trap. *Eur. Phys. J. D* **32**, 139 (2005)
21. J.R. Castrejon-Pita, R.C. Thompson, Proposal for a planar Penning ion trap. *Phys. Rev. A* **72**, 013405 (2005)
22. F. Galve, P. Fernandez, G. Werth, Operation of a planar Penning trap. *Eur. Phys. J. D* **40**, 201 (2006)
23. P. Bushev, S. Stahl, R. Natali, G. Marx, E. Stachowska, G. Werth, M. Hellwig, F. Schmidt-Kaler, Electrons in a cryogenic planar Penning trap and experimental challenges for quantum processing. *Eur. Phys. J. D* **50**, 97 (2008)
24. J. Goldmann, G. Gabrielse, Optimized planar Penning traps for quantum information studies. *Phys. Rev. A* **81**, 052335 (2010)
25. J.R. Castrejon-Pita et al., Novel designs for Penning ion traps. *J. Mod. Opt.* **11**, 1581 (2007)
26. R. Maiwald, D. Leibfried, J. Britton, J.C. Bergquist, G. Leuchs, D.J. Wineland, Stylus ion trap for enhanced access and sensing. *Nat. Phys.* **5**, 551 (2009)
27. N. Yu, W. Nagourney, H. Dehmelt, Demonstration of new Paul-Straubel trap for trapping single ions. *J. Appl. Phys.* **69**, 3779 (1991)
28. C. Schrama, E. Peik, W. Smith, H. Walther, Novel miniature ion traps. *Opt. Commun.* **101**, 32 (1993)
29. L. Deslauriers et al., Scaling and suppression of anomalous heating in ion traps. *Phys. Rev. Lett.* **97**, 103007 (2006)
30. M.J. Goeckner, J. Goree, T.E. Sheridan, Saturation broadening of laser-induced fluorescence from plasma ions. *Rev. Sci. Instr.* **64**, 996 (1993)
31. D. Budker, D.F. Kimball, D.P. DeMille, *Atomic Physics* (Oxford University Press, Oxford, 2004)

Chapter 19

Microwave Spectroscopy



Abstract Microwave spectroscopy of confined particles in Penning traps is mainly applied to magnetic dipole transitions in various contexts. The strong magnetic field of the Penning trap gives rise to significant spin orientation energies and to Zeeman splitting of lines, which can be used for precision spectroscopy of magnetic dipole transitions and the determination of magnetic moments. Prominent examples are the precision measurements of the magnetic moments (g -factors) of the free (unbound) electron, proton and their anti-particles, as well as of the bound electron by Larmor frequency measurements, and in general the application to the Zeeman splitting of levels that occurs due to the presence of magnetic field used for confinement. We will discuss these applications in detail below.

19.1 Applications of In-Trap Microwave Spectroscopy

Microwave spectroscopy has been used to determine the electronic g -factors of singly charged atomic and molecular ions such as Be^+ [1], Mg^+ [1], Ca^+ [2, 3], Ba^+ [4, 5], Hg^+ [6] and H_2^+ [7], as well as nuclear g -factors in singly charged ions such as Mg^+ [8], Ba^+ [4] and Eu^+ [3]. For an overview and detailed discussion see [9]. It has further been used to measure the ground-state hyperfine splittings in singly charged ions, and to implement microwave frequency standards with confined ions, for a thorough treatment see [9, 10]. In the following, we specialise the discussion to microwave spectroscopy of Zeeman sublevels of the fine structure and hyperfine structure in highly charged ions.

19.2 Microwave Source and Saturation Intensity

19.2.1 Microwave Source and Guide

The most common source of microwaves is a microwave generator in form of a table-top device that usually employs yttrium-iron garnet (YIG) oscillators for generation

of the frequency signal. For increased frequency stability, it is often connected to a frequency standard and/or has a GPS signal input. A typical output is a signal at a frequency of up to around 100 GHz (possibly involving passive or active frequency multipliers) with high spectral resolution and stability.

Depending on the actual frequency, the signal can be guided to the trap in different ways, connected with different loss behaviour. At the time of writing of this text, commercial coaxial cables exist for frequencies up to about $\omega = 2\pi \times 65$ GHz. The signal damping α in a coaxial cable with central conductor diameter d and inner diameter D of the outer conductor is given by

$$\alpha(\omega) = k \left(\frac{\omega \mu_0 \varepsilon_r}{\pi} \right)^{1/2} \left(\frac{(\mu_1 \rho_1)^{1/2}}{D} + \frac{(\mu_2 \rho_2)^{1/2}}{d} \right) \frac{1}{\log \left(\frac{D}{d} \right)}, \quad (19.1)$$

where ε_r is the dielectric constant of the insulator between inner and outer conductor, μ_2 (μ_1) is the permeability of the central (outer) conductor, ρ_2 (ρ_1) is the resistivity of the central (outer) conductor, and $k = 0.0315$ is a numerical constant to achieve the result in dB/m.

At higher frequencies, the signal attenuation in cables is too high and hollow conductors need to be used. These are usually hollow rectangular metal guides, if needed with a high-conductivity coating such as gold on the inside. Details of signal propagation in such hollow conductors can be found in [11, 12]. In rectangular hollow guides, the relevant radiation mode is the TE₁₀-mode. The resistive signal loss α (given in dB/m) for this mode in a guide with cross section $a \times b$ (with $a > b$) is given by

$$\alpha(\omega) = \frac{k R_s}{b \sqrt{1 - (\omega'/\omega)^2}} \left(1 + 2 \frac{b}{a} \left(\frac{\omega'}{\omega} \right)^2 \right), \quad (19.2)$$

where $k = 0.0023/\Omega$ is a constant, R_s is the conductor's inner surface resistance, and $\omega' = \pi/(a\sqrt{\mu\varepsilon})$ is the cutoff frequency. The surface resistance in Ohms is given by

$$R_s = 0.00825 \sqrt{\frac{\sigma'}{\sigma}} \sqrt{\omega}, \quad (19.3)$$

when σ'/σ is the electric bulk conductivity ratio between copper and the inner surface material and the frequency ω is given in units of GHz.

For applications with low thermal conductivity, i.e. for paths from room temperature to a cryogenic trap, stainless steel coaxial cables are a good choice. Also microwave guides from thin stainless steel with interior gold coating exist.

19.2.2 Saturation Intensity

The saturation intensity I_s of an electromagnetic transition is defined by [13]

$$\frac{I}{I_s} = 2 \left(\frac{\Omega}{\Gamma} \right)^2 \quad (19.4)$$

where Γ is the natural linewidth of the transition and Ω is the Rabi oscillation frequency driven by the actual intensity I . This expression is equivalent to the saturation intensity given by (18.7) [14]. Thus, at the saturation intensity, the induced emission has the same rate as the spontaneous emission. I is given by

$$I = \frac{1}{2} c \varepsilon_0 E^2 \quad (19.5)$$

with E being the electric field strength of the exciting radiation. The Rabi oscillation frequency is related to the electric dipole operator $\mathbf{d}$ via $\Omega := -\mathbf{d} \cdot \mathbf{E}/\hbar$ with $\mathbf{d} = \hat{d}\mathbf{d}/\hbar$ in which $\hat{d}$ is the unit vector of the electric field and d is the electric dipole moment of the particle. From this, we obtain

$$I_s = \frac{c \varepsilon_0 \Gamma^2 \hbar^2}{4(\hat{d}d)^2}. \quad (19.6)$$

We replace the electric dipole operator $\mathbf{d}$ by the magnetic dipole operator $\mathbf{m}$

$$\frac{1}{\varepsilon_0} |\langle A | \mathbf{d} | B \rangle|^2 \rightarrow \mu_0 |\langle A | \mathbf{m} | B \rangle|^2. \quad (19.7)$$

Then, (19.6) becomes

$$I_s = \frac{c \Gamma^2}{4\mu_0(\hat{m}m)^2}, \quad (19.8)$$

where now $\hat{m}$ is the unit vector of the magnetic field and m is the magnetic dipole moment which we can denote by its g -factor using $m = g\mu_B$. We now use

$$\frac{c}{\mu_0} = \frac{c^2}{\sqrt{\frac{\mu_0}{\varepsilon_0}}}. \quad (19.9)$$

For the saturation intensity of the magnetic dipole transition we obtain

$$I_s = \frac{c^2 \Gamma^2}{4\sqrt{\frac{\mu_0}{\varepsilon_0}}(g\mu_B)^2}. \quad (19.10)$$

In other words, up to the factor 4 due to the definition (see also 19.24), at the saturation intensity, the ratio of the magnetic field component

$$B^2 = \frac{1}{c^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} I \quad (19.11)$$

of the driving radiation and the homogeneous magnetic field B_0 is the same as the ratio of the magnetic dipole transition linewidth Γ and the Larmor frequency ω_L of the spin precession around B_0 . See also the discussion of the power shift of the Larmor frequency in Sect. 19.4.3.

19.3 Microwave Zeeman Transitions in Highly Charged Ions

The Zeeman splitting of the hyperfine levels in a highly charged ion gives access to the magnetic moment of the ion. Thus, for highly charged ions confined in a Penning trap, the Zeeman splitting of the hyperfine structure due to the external field is of special interest. In an external magnetic field B_0 , hyperfine levels with quantum number F split into $(2F + 1)$ Zeeman sublevels with magnetic quantum numbers $M_j = -F, -F + 1, \dots, F - 1, F$, which have a separation of $\Delta E = g_F \mu_B B_0$. Here, g_F is the atomic or ionic g -factor and μ_B is the Bohr magneton. In highly charged ions, transitions between Zeeman sublevels of the same fine structure multiplet are typically in the microwave domain and therefore accessible by precision microwave spectroscopy.

When the value of the magnetic field B_0 is known, a measurement of ΔE yields the ionic g -factor for a certain F . This will be made use of in the determination of electronic and nuclear magnetic moments, as we will discuss in Chap. 20.

In general, the Breit-Rabi equation describes the evolution of the Zeeman substates with the magnetic field. When considering hydrogen-like ions in their electronic ground state and confined by an external field B_0 , the energy of a Zeeman level with magnetic quantum number m_F is given by [15]

$$\frac{\Delta E_M}{\Delta E_{\text{HFS}}} = a_1 m_F x \pm \frac{1}{2} \sqrt{1 + \frac{4m_F}{2I + 1} c_1 x + c_2 x^2}, \quad (19.12)$$

where we have

$$x = \frac{\mu_0 B_0}{\Delta E_{\text{HFS}}} \quad (19.13)$$

$$a_1 = -g'_I = -\frac{m_e}{m_p} (g_I + \Delta g_{\text{rec}}) \quad (19.14)$$

$$c_1 = g_j + g'_I \quad \text{and} \quad c_2 = (g_j + g'_I)^2, \quad (19.15)$$

when g_j is the electronic g -factor and Δg_{rec} is the nuclear recoil correction to the bound-electron g factor [15]. This energy ΔE_M is measured with respect to the

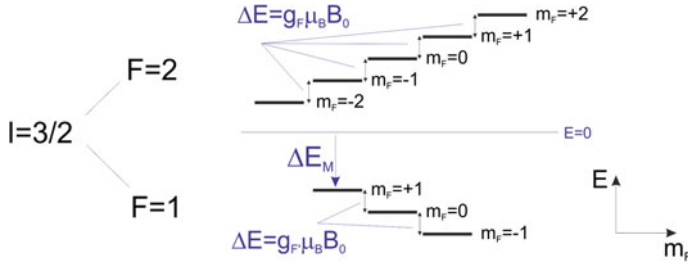


Fig. 19.1 Level scheme showing the Zeeman splitting of hyperfine levels in the presence of a magnetic field B . An ion with nuclear spin $I = 3/2$ and positive nuclear magnetic moment $\mu > 0$ has been chosen for this example

mean energy of the hyperfine levels $E = (E_F + E_{F'})/2$. Note, that both the ionic and electronic g -factors contain contributions from bound-state QED. Figure 19.1 schematically shows the Zeeman splitting of hyperfine levels in the presence of a magnetic field B . An ion with nuclear spin $I = 3/2$ and positive nuclear magnetic moment $\mu > 0$ has been chosen for this example.

19.4 Microwave Larmor Frequency Measurement

19.4.1 Measurement of Spin Flips

The Larmor frequency ω_L as given by $\hbar\omega_L = g_J \mu_B B_0$ is the precession frequency of the spin of a particle in the external magnetic field, and thus it does not produce an image charge by which it could be measured directly. It is instead determined from the probability of spin state transitions ('spin flips') during a frequency scan of the driving microwave radiation. Spin state analysis is in this case done via the continuous Stern-Gerlach effect in a magnetic bottle. We discuss the application to the measurement of magnetic moments in Sect. 22.1. Microwaves of a known frequency ω_{MW} are irradiated for a defined time span t . The magnetic field component B of the microwave radiation drives the precession of the electron spin and Rabi oscillations of the spin state occur at a frequency

$$\Omega = \omega_L \frac{B}{B_0} \quad (19.16)$$

where B is given by

$$B^2 = \frac{1}{c^2} \sqrt{\frac{\mu_0}{\epsilon_0}} I, \quad (19.17)$$

in which I is the microwave intensity. The probability for a spin flip as a function of the irradiated frequency ω is given by [16]

$$P(\omega) = \frac{\pi}{2} \Omega^2 t \chi(\omega), \quad (19.18)$$

where χ is the lineshape of the resonance which in experimental situations is asymmetric and broadened in a non-trivial fashion by a number of effects such as the finite motional temperature of the ion, the residual field inhomogeneities and the power of the microwave radiation as discussed in detail for example in [16, 17].

19.4.2 Saturation Behaviour

Experimentally, we can only distinguish between two states ('spin up' and 'spin down') after a given irradiation time $t > \Omega^{-1}$. This means that it is only possible to distinguish between an odd and an even number of spin flips which have occurred during the irradiation time t . This leads to a difference between the true rate of spin flips R from the observed rate R' . Looking at the general rate equations for such a two-state system with populations of the two states n_u ('spin up') and n_d ('spin down')

$$\dot{n}_u = -Rn_u + Rn_d \quad (19.19)$$

$$\dot{n}_d = Rn_u - Rn_d, \quad (19.20)$$

which can be solved by the ansatz $n(t) = n(t=0) \exp(\lambda t)$ and the characteristic equation

$$(-R - \lambda)^2 - R^2 = \lambda(\lambda + 2R) = 0, \quad (19.21)$$

for which we find solutions $\lambda = 0$ and $\lambda = -2R$. Correspondingly, the relation between the true spin-flip rate R and the observed rate R' is given by [16]

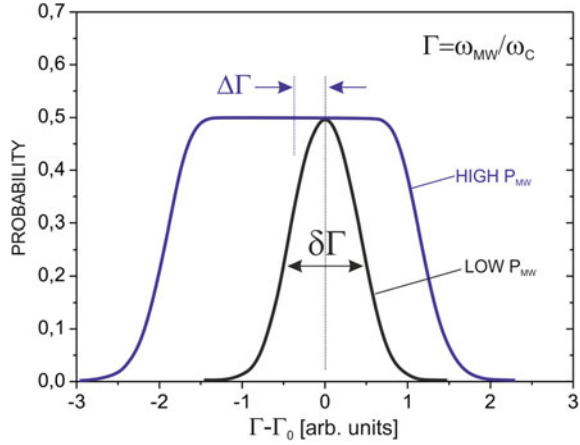
$$R' = \frac{1}{2t} (1 - e^{-2Rt}). \quad (19.22)$$

Combining this saturation correction with (19.18), we find the experimentally observed probability for a spin flip to be given by

$$P'(\omega) = \frac{1}{2} (1 - \exp(-\pi \Omega^2 t \chi(\omega))). \quad (19.23)$$

This obviously limits the maximum observable spin-flip probability to 1/2.

Fig. 19.2 Schematic of a Larmor resonance and its broadening and shift when the microwave transition is over-saturated by too much microwave power



When the microwave power over-saturates the transition, the resonance of the spin-flip probability as a function of microwave frequency ω_{MW} (scaled by the cyclotron frequency ω_c , the ' Γ -resonance', see also Sect. 19.5) is flattened out and shifted, as has been experimentally demonstrated in [16, 18, 19]. This is shown in Fig. 19.2: it compares a resonance obtained at low power with a resonance at much higher power, which becomes broadened and shifted. Under ideal conditions, the value of $\Gamma = \omega_{\text{MW}}/\omega_c$ at resonance is $\Gamma_0 = \omega_L/\omega_c$, from which the g -factor can be determined, see the discussion in Sect. 22.1.

19.4.3 Power Shift of the Larmor Frequency

As the Larmor frequency $\hbar\omega_L = g_J\mu_B B_0$ is the precession frequency of the particle spin in the external magnetic field, any contribution which adds to the confining field B_0 will shift it. In principle, this is true also for the magnetic field component of the microwave radiation used to find the true Larmor frequency as the maximum of the spin transition probability as a function of the microwave frequency. The Larmor resonance is shifted by the magnetic field component B_{MW} of the microwave signal ('power shift') according to [20]

$$\frac{\Delta\omega_L}{\omega_L} = \left(\frac{B_{\text{MW}}}{2B_0}\right)^2 \quad \text{with} \quad B_{\text{MW}}^2 = \frac{1}{c^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{P_{\text{MW}}}{A_{\text{MW}}}. \quad (19.24)$$

Typical irradiated areas A_{MW} are of order mm^2 . The microwave power in typical experiments [21–24] is limited to values of the order of μW and below to avoid

undesired broadening and distortion of the lineshape of the Larmor resonance. At these values, the resulting relative shift of the Larmor frequency is of order 10^{-15} and hence outside of the present experimental accuracy.

19.5 Stern-Gerlach-Type g -Factor Measurements

Experimentally, the magnetic moment (g -factor) of a confined particle can be determined by a measurement of the precession frequency of the particle spin in an external magnetic field B_0 . This ‘Larmor frequency’ ω_L is given by

$$\hbar\omega_L = \mu B_0 = g_J \mu_B B_0. \quad (19.25)$$

Hence, a determination of the magnetic moment (or g -factor) requires knowledge of the magnetic field strength B_0 at the position and time of the measurement (Fig. 19.3). This is possible via a determination of the cyclotron frequency ω_c , usually by application of the invariance theorem (5.27). The determination of the Larmor frequency ω_L , at which the electron spin flip occurs resonantly under microwave irradiation, follows the idea of the ‘continuous Stern-Gerlach effect’ as previously discussed. With the knowledge of ω_c and ω_L , the g -factor can readily be calculated, however there is one difference between an unbound particle and the bound electron: the measured cyclotron frequency in the case of the bound electron refers to the ion as a whole. Hence, we need to know the electron-ion mass ratio m_e/m , or, to be more precise, the charge-to-mass ratios.

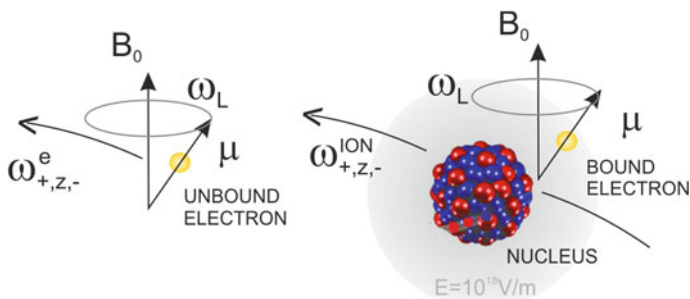


Fig. 19.3 Schematic of the motions in the external magnetic field for the unbound electron (left) and the bound electron (right) spin precession around the external magnetic field B_0 at the Larmor frequency ω_L . The grey shaded region indicates the extreme field of the binding

The g -factors in the unbound and bound situations can be calculated by

$$\frac{g_s}{2} = \frac{\omega_L}{\omega_c} \quad \text{and} \quad \frac{g_j}{2} = \frac{\omega_L}{\omega_c} \frac{q}{e} \frac{m_e}{m}, \quad (19.26)$$

respectively. In literature, the shorthand $\omega_L/\omega_c \equiv \Gamma_0$ is common.

References

1. J.J. Bollinger, et al., *Laser Spectroscopy VI*, ed. by H.P. Weber, W. Luthy (Springer, Berlin, 1983)
2. G. Tommaseo et al., The g_J -factor in the ground state of Ca^+ . *Eur. Phys. J. D* **25**, 113 (2003)
3. M. Chwalla et al., Absolute frequency measurement of the $^{40}\text{Ca}^+$ $4s2S_{1/2} - 3d2D_{5/2}$ clock transition. *Phys. Rev. Lett.* **102**, 023002 (2009)
4. G. Marx, G. Tommaseo, G. Werth, Precise g_J - and g_I -factor measurements of Ba^+ isotopes. *Eur. Phys. J. D* **4**, 279 (1998)
5. K.H. Knoell et al., Experimental g_J factor in the metastable $5D_{3/2}$ level of Ba^+ . *Phys. Rev. A* **54**, 1199 (1996)
6. R.J. Rafac et al., Sub-dekahertz ultraviolet spectroscopy of $^{199}\text{Hg}^+$. *Phys. Rev. Lett.* **85**, 2462 (2000)
7. R. Loch, R. Stengler, G. Werth, Measurement of the electronic g factor of H_2^+ . *Phys. Rev. A* **38**, 5484 (1988)
8. W.M. Itano, D.J. Wineland, Precision measurement of the ground-state hyperfine constant of $^{25}\text{Mg}^+$. *Phys. Rev. A* **24**, 1364 (1981)
9. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
10. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
11. W. Demtröder, *Elektrizität und Optik* (Springer, Heidelberg, 2006)
12. D.M. Pozar, *Microwave Engineering*, 4th edn. (Wiley, Hoboken, NJ, 2005)
13. W. Demtröder, *Laser Spectroscopy* (Springer, Heidelberg, 2003)
14. C.J. Foot, *Atomic Physics* (Oxford University Press, Oxford, 2009)
15. D.L. Moskovkin, V.M. Shabaev, Zeeman effect of the hyperfine-structure levels in hydrogenlike ions. *Phys. Rev. A* **73**, 052506 (2006)
16. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
17. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
18. R.S. Van Dyck, Jr., P.B. Schwinberg, H.G. Dehmelt, in *Atomic Physics 9*, ed. by R.S. Van Dyck, Jr., E.N. Fortson (World Scientific, Singapore, 1984)
19. H. Häffner, Präzisionsmessung des magnetischen Moments des Elektrons in wasserstoffähnlichen Kohlenstoff, Ph.D. thesis, University of Mainz, 2000
20. A. Abragam, *The Principles of Nuclear Magnetism* (Clarendon Press, Oxford, 1961)
21. H. Häffner et al., High-accuracy measurement of the magnetic moment anomaly of the electron bound in Hydrogen-like Carbon. *Phys. Rev. Lett.* **85**, 5308 (2000)
22. J. Verdú et al., Electronic g factor of Hydrogen-like Oxygen $^{16}\text{O}^{7+}$. *Phys. Rev. Lett.* **92**, 093002 (2004)
23. J. Verdú et al., Determination of the g -factor of single Hydrogen-like ions by mode coupling in a Penning trap. *Phys. Scripta* **T112**, 68 (2004)
24. S. Sturm et al., g Factor of Hydrogen-like $^{28}\text{Si}^{13+}$. *Phys. Rev. Lett.* **107**, 023002 (2011)

Chapter 20

Laser-Microwave Double-Resonance Spectroscopy



Abstract In special situations, optical and microwave spectroscopy can be combined to constitute a powerful tool that uses the high spectral resolution of the microwaves and the good detection properties of optical light. Here, we briefly discuss its application to the determination of magnetic moments of the nucleus and the electron bound in highly charged ions.

20.1 Application of Laser-Microwave Double-Resonance Spectroscopy

In atoms and singly charged ions, double-resonance spectroscopy has for example been used to perform precision measurements of the hyperfine transitions and Zeeman transitions for Hg^+ , Ba^+ , Pb^+ , Yb^+ and Be^+ [1–6]. Such measurements have yielded electronic and nuclear magnetic moments of a number of singly charged ions. For a thorough discussion of this application and an overview of the measurements, see [7, 8].

The same concept can be applied to highly charged ions confined in a Penning trap, and to the determination of magnetic moments and Zeeman transitions in general. This includes measurements of the magnetic moments of the ionic nucleus and the of the bound electron. While measurements of the bound electron magnetic moment by the continuous Stern-Gerlach effect combine microwave probing of the Larmor frequency with a radio-frequency detection of the success by an ion oscillation frequency measurement in the trap, double-resonance spectroscopy combines the microwave probing of the Larmor frequency with optical detection of the success by laser spectroscopy of the fine or hyperfine structure of the ion. As the laser and microwave excitations are performed simultaneously, this technique is called laser-microwave double-resonance spectroscopy. It circumvents the need for a magnetic bottle and trades this for the availability of an appropriate laser and optical detection.

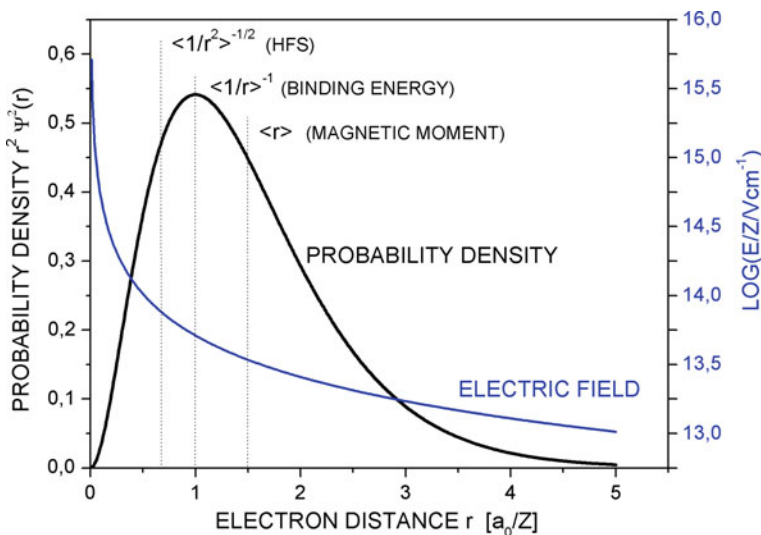


Fig. 20.1 Probability density for the $1s$ -electron in hydrogen-like ions with nuclear charge number Z as a function of the electron distance r to the nucleus (left hand scale), and the electric field strength E of the nuclear charge eZ (right hand scale)

20.2 Optical Transitions in Highly Charged Ions

In highly charged ions, the energy of fine structure and hyperfine structure transitions are shifted to much higher values as compared to singly charged ions by the extreme electromagnetic fields in the vicinity of the nucleus. To illustrate the relevant differences between singly charged ions and highly charged ions, and in preparation of the discussion to come, Fig. 20.1 shows the probability density for the $1s$ -electron in hydrogen-like ions with nuclear charge number Z as a function of the electron distance r to the nucleus (left hand scale). The value of r is given in units of a_0/Z where a_0 is the Bohr radius of about 5.29×10^{-11} m. The same figure indicates the electric field strength E of the nuclear charge eZ (right hand scale), and three characteristic expectation values of the electron location, $\langle 1/r^2 \rangle^{-1/2}$, $\langle 1/r \rangle^{-1}$ and $\langle r \rangle$. The prior is relevant for the hyperfine structure splitting in the electronic level scheme, the second is relevant for the electron binding energy and the latter for the bound electron magnetic moment (g -factor). It shows that the hyperfine structure is the most strongly subjected to nuclear field effects while the magnetic moment is the least so. However, the electric field strength per nuclear charge number E/Z at these characteristic distances is of about the same magnitude, such that also the magnetic moment of the electron is significantly influenced by quantum electrodynamics of the bound state (BS-QED). The electric field strengths reach about $10^{13.5}$ V/cm per nuclear charge, such that for heavy hydrogen-like ions values of up to 10^{16} V/cm are reached, which is far beyond what is possible in macroscopic regions like laser foci. Moreover, as

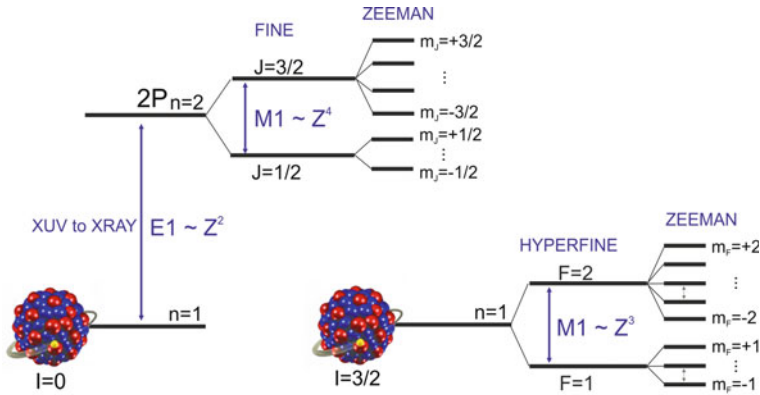


Fig. 20.2 Left: Level scheme of a highly charged ion with spin-less nucleus ($I = 0$) and a fine structure of the $2P$ -level with Zeeman sublevels due to the magnetic field of the trap (left). Right: Level scheme of a highly charged ion with non-vanishing nuclear spin ($I = 3/2$) and a ground-state hyperfine splitting with Zeeman sublevels

the typical electron distance to the nucleus in a hydrogen-like ion a_0/Z is Z times smaller than in hydrogen, the bound electron becomes a sensitive probe of nuclear properties. Generic examples of resulting level schemes are depicted in Fig. 20.2 for a highly charged ion with spin-less nucleus ($I = 0$) and a fine structure of the $2P$ -level with Zeeman sublevels due to the magnetic field of the trap (left), and for a highly charged ion with non-vanishing nuclear spin ($I = 3/2$) and a ground-state hyperfine splitting with Zeeman sublevels. The fine structure transitions are magnetic dipole transitions (M1) which are in the optical regime for certain medium-heavy ions, see below. The hyperfine structure transitions are magnetic dipole transitions as well, which are in the optical regime for certain heavy ions, see also below. The transitions between Zeeman sublevels are as well magnetic dipole transitions which depend on the choice of the external magnetic field, but are commonly microwave transitions.

20.2.1 Hyperfine Transition Energies

In hydrogen-like ions, the energy of the ground-state hyperfine structure (HFS) splitting depends as Z^3 on the nuclear charge number, for other few-electron ions the dependence is similar. It can be seen from the r^{-3} -dependence of the magnetic field of a dipole keeping in mind the length scale for r is given by a_0/Z , so $E_{\text{HFS}} \propto \langle r^{-3} \rangle \propto Z^3$. Ignoring relativistic, nuclear and QED effects, the ‘classical’ ground-state HFS transition energy in a hydrogen-like ion can be used as a fair approximation of the real value and reads [9]

$$E_{\text{HFS}} \approx \frac{4}{3} \alpha (Z\alpha)^3 g_I \frac{m_e}{m_p} \frac{2I+1}{2} m_e c^2 \propto Z^3, \quad (20.1)$$

where g_I is the nuclear g -factor which measures the nuclear magnetic moment in units of the nuclear magneton $\mu_N = e\hbar/2m_p = 5.05078324(13) \times 10^{-27}$ J/T. The same equation applies also to other charge states, the factor $4/3$ then changes to $1/6$ for lithium-like ions, to $1/18$ for boron-like ions and so forth [9].

To be more precise, for the hyperfine structure of the $(1s)^2 S_{1/2}$ ground state of hydrogen-like ions, the transition energy is given by [9]:

$$E_{\text{HFS}} = \frac{4}{3} \alpha (Z\alpha)^3 g_I \frac{m_e}{m_p} \frac{2I+1}{2} m_e c^2 A_{1s} (1 - \delta_{1s}), \quad (20.2)$$

where

$$A_{1s} = \frac{1}{\kappa(2\kappa - 1)} \quad \text{and} \quad \kappa = \sqrt{1 - (Z\alpha)^2}. \quad (20.3)$$

This energy value represents the normal ground state hyperfine splitting energy multiplied by a correction A_{1s} for the relativistic energy of the $1s$ electron, where κ is related to the angular momentum j of the electron. To account for the charge distribution of the nucleus, a first-order approximation is made by assuming that the charge is evenly distributed over the volume of the spherical nucleus. This finite-size correction is given by the factor $(1 - \delta_{1s})$.

Similarly, the hyperfine transition energy of the $(1s^2 2s)^2 S_{1/2}$ ground state for lithium-like ions is given by [9]

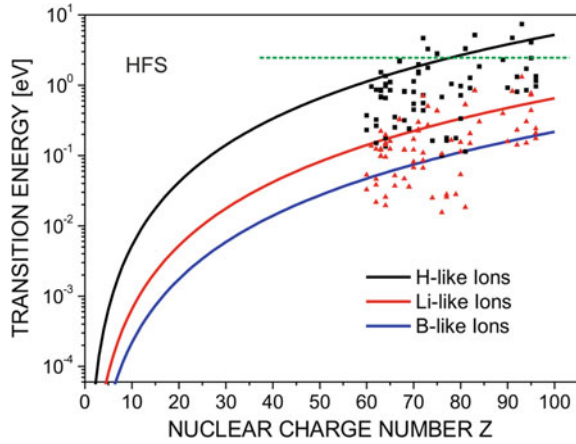
$$E_{\text{HFS}} = \frac{1}{6} \alpha (Z\alpha)^3 g_I \frac{m_e}{m_p} \frac{2I+1}{2} m_e c^2 A_{2s} (1 - \delta_{2s}) \quad (20.4)$$

with the relativistic correction

$$A_{2s} = 2 \frac{2(1 + \kappa) + \sqrt{2(1 + \kappa)}}{(1 + \kappa)^2 \kappa (4\kappa^2 - 1)}. \quad (20.5)$$

Figure 20.3 shows the HFS transition energies as a function of nuclear charge number for H-like, Li-like and B-like ions. The solid lines assume a nuclear spin of $I = 5/2$ and a nuclear magnetic moment of $1 \mu_N$. The data points show the result when the proper values of I and g_I are used in (20.1). We have restricted ourselves to $Z > 60$ as below that, the transitions are far outside the optical regime. The ground-state hyperfine splitting in hydrogen-like ions has been measured in $^{165}\text{Ho}^{66+}$ [10], $^{185,187}\text{Re}^{74+}$ [11], $^{203,205}\text{Tl}^{80+}$ [12], $^{207}\text{Pb}^{81+}$ [13, 14] and $^{209}\text{Bi}^{82+}$ [14, 15] as well as in $^{209}\text{Bi}^{80+}$ [16]. Calculations of the hyperfine transition energies in highly charged ions have been performed, see for example [17–26].

Fig. 20.3 HFS transition energies as a function of nuclear charge number for H-like, Li-like and B-like ions. The dotted line indicates the photon energy of a green laser



20.2.2 Fine Structure Transition Energies

For the fine structure (FS) splitting, the Z -dependence is even stronger and scales with Z^4 . This is obvious from the fine structure energy $E_{\text{FS}} \propto \mu B \propto Z^2/r^2$ where $\langle r^{-2} \rangle \propto Z^2$ such that we obtain $E_{\text{FS}} \propto Z^4$, see for example [27]. The energy of a fine-structure transition can be approximated by use of the Dirac equation for the energy of the single electron in hydrogen-like systems with quantum numbers n and j . For example, the energy of a fine structure transition in a hydrogen-like ion from a state (n, j) to $(n, j - 1)$ is given by $E_{\text{FS}} = E(n, j) - E(n, j - 1)$ with the energy of a state (n, j) given by the equation

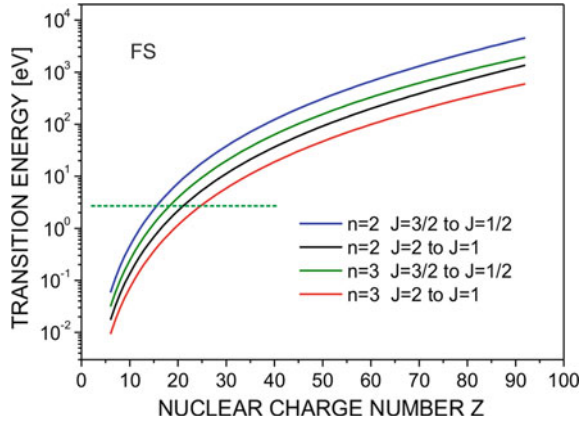
$$E(n, j) \approx \left[1 + Z^2 \alpha^2 \left(n - j - \frac{1}{2} + \sqrt{\left(j + \frac{1}{2} \right)^2 - Z^2 \alpha^2} \right)^{-2} \right]^{-1/2} m_e c^2. \quad (20.6)$$

Figure 20.4 (right) shows the FS transition energies as a function of nuclear charge number for different states according to (20.6). This leads to the splittings being shifted to the laser-accessible regime for certain values of Z . For fine structure transitions, the optical frequency regime is reached for medium charge states, while the weaker dependence of the hyperfine transition energy requires higher charge states. For a number of such ions, laser spectroscopic measurements of energy splittings have been performed, an overview is given for example in [28].

20.2.3 Optical Transition Lifetimes

Fine and hyperfine structure transitions are both magnetic dipole (M1) transitions with correspondingly long lifetimes and hence much narrower linewidths than the

Fig. 20.4 Fine structure transition energies as a function of nuclear charge number. The dotted line indicates the photon energy of a green laser



principal transitions, which are electric dipole (E1) transitions and for highly charged ions in the XUV to x-ray regime of energies.

Magnetic dipole transitions are caused by the oscillating magnetic field $\mathbf{B} = \mathbf{B}_0 \cos \omega t$ of the laser radiation. The transition matrix element between two levels 1 and 2 is given by $\mu_{21} \propto \langle 2 | \boldsymbol{\mu} \cdot \mathbf{B} | 1 \rangle$ where $\boldsymbol{\mu}$ is the magnetic dipole operator. The transition rate of spontaneous decay is given by $A_{21} \propto \omega^3 |\mu_{21}|^2$. It is smaller than the rate of an electric dipole (E1) transition by a factor of about

$$\frac{|\langle 2 | \boldsymbol{\mu} \cdot \mathbf{B} | 1 \rangle|^2}{|\langle 2 | e \mathbf{r} \cdot \mathbf{E} | 1 \rangle|^2} \approx \left(\frac{\mu_B / c}{ea_0 / Z} \right)^2 \approx (Z\alpha)^2. \quad (20.7)$$

The transition rate (linewidth) for a ground-state ($n = 1$) magnetic dipole transition from the excited to the lower hyperfine state in a hydrogen-like ion is given by [9]

$$\Gamma_{\text{HFS}} = \frac{4\alpha\omega^3 \hbar^2 I (2k + 1)^2}{27m_e^2 c^4 (2I + 1)}, \quad (20.8)$$

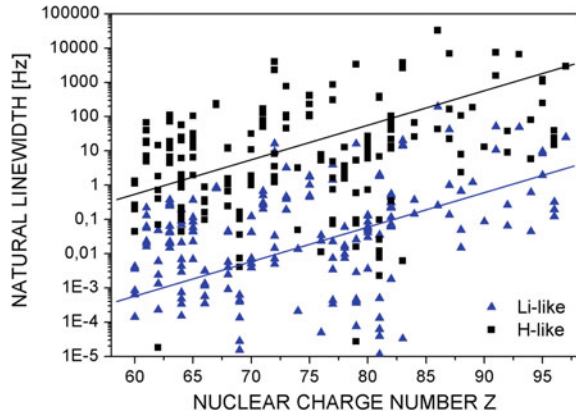
where α is the fine structure constant, I is the nuclear spin and $k = \sqrt{1 - Z^2\alpha^2}$ (Fig. 20.5).

For the total decay rate (linewidth) of an excited state with quantum numbers (n, l) in hydrogen-like ions, a semi-classical calculation [29] yields the expression

$$\Gamma_{\text{FS}} \approx \frac{2Z^4 \alpha^5 m_e c^2}{3\hbar n^5} \frac{n^2}{l(l+1)} \quad (20.9)$$

for principal quantum numbers $n \geq 2$ and non-vanishing orbital quantum number l . It is especially well-suited for high quantum numbers n and l . More detailed calculations employing full quantum mechanics have been performed [30] and lead to values with accuracies of some hundreds of ppm. They also give a better approximation to

Fig. 20.5 Natural linewidths of the (magnetic dipole) ground-state hyperfine transitions in hydrogen- and lithium-like ions as a function of the nuclear charge number Z



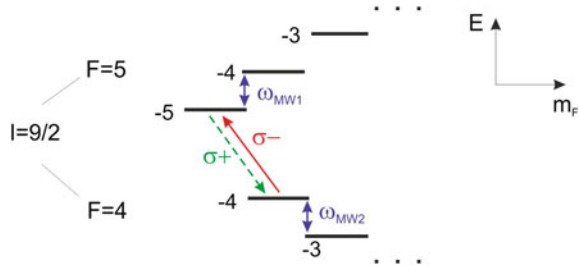
the one shown in (20.9) by replacing $l(l+1)$ by $l^2 + l + (l+1)/(8l)$. A more detailed discussion can be found in [30]. The linewidth of Zeeman sub-state transitions is typically of order mHz and below, as discussed in Sect. 7.8. Generally, the linewidth calculation of a specific transition like a fine structure transition from $2p_{3/2}$ to $2p_{1/2}$ is quite involved, see for example [31]. We can thus make no general statement apart from the scaling.

20.3 Microwave Zeeman Transitions and Magnetic Moments

In a Penning trap, the fine structure or hyperfine structure levels are subject to the Zeeman effect which lifts the degeneracy with respect to the magnetic quantum numbers of the states. The result is a splitting into $2J + 1$ fine structure sub-states, or, if present, $2F + 1$ hyperfine sub-levels where $F = I \pm J$. This Zeeman splitting is nearly uniform within a given state and well within the microwave regime for typical magnetic fields of a few Tesla. To first order, the splitting is given by $\hbar\omega_{MW} = g\mu_B B_0$, where $g\mu_B$ is the ion magnetic moment. Thus, a measurement of the microwave frequency is a measurement of the ionic magnetic moment for a given value of the magnetic field strength. Experimentally, this can again be determined from a measurement of the free ion cyclotron frequency by use of the invariance theorem as discussed in Sects. 5.3.2 and 6.6.

The concept of double-resonance spectroscopy uses a change in the optical fluorescence signal from the FS or HFS transition as an indication for the irradiated microwave frequency to be in resonance with the Zeeman transition. The ionic g -factors g_F and $g_{F'}$ are determined from a scan of the irradiated microwave frequencies ω_{MW1} and ω_{MW2} while the closed optical cycle at frequency ω is observed. When one of the microwave transitions is in resonance, the optical signal is reduced due

Fig. 20.6 Schematic of the hyperfine level scheme of $^{209}\text{Bi}^{82+}$ ($I = 9/2$) with the optical and microwave transitions used for double-resonance spectroscopy indicated



to population being extracted from the otherwise closed cycle (Fig. 20.6). Hence, a minimum in the optical signal indicates the Zeeman resonance and allows determinations of the ionic magnetic moments. Details and further applications are given in [32].

It can be shown that for ions with non-vanishing nuclear spin, the ionic magnetic moment is a non-trivial composition of the magnetic moments of both the bound electron and the nucleus. Generally,

$$g_F = g_J \frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)} - \frac{m_e}{m_p} g_I \frac{F(F+1) + I(I+1) - J(J+1)}{2F(F+1)} \quad (20.10)$$

describes the general relation between the g_F -factor, the g_J -factor of the bound electron and the nuclear g -factor $g_I = \mu/(\mu_N I)$ where μ_N is the nuclear magneton and m_e and m_p are the electron and proton mass, respectively. The g -factors of the bound electron g_J and of the nucleus g_I can be disentangled to a high degree by use of theory [32]. To this end, the ionic magnetic moments need to be measured in two different hyperfine states F and F' .

If two g -factors g_F and $g_{F'}$ are measured for states with different F in one ion ($F = I - 1/2$ and $F' = F + 1$), the bound electron and nuclear g -factors g_J and g_I can be expressed independently in terms of the experimentally obtained values by means of [32]

$$g_J = (I+1)g_{F'} - Ig_F - \delta_Q Q \left(\frac{m_e c}{\hbar} \right)^2 \frac{2(I+1)}{2I-1} \quad (20.11)$$

and

$$g_I = -\frac{m_p}{m_e} \frac{g_{F'} + g_F + \delta_Q Q \left(\frac{m_e c}{\hbar} \right)^2 \frac{3}{I(2I-1)}}{2(1-\delta_\mu)}, \quad (20.12)$$

where Q is the electric quadrupole moment of the nucleus and δ_μ and δ_Q are small corrections which can be obtained from theory [32].

This concept, however, can only be applied to ions with $I > 1/2$ since for an ion with $I = 1/2$ the lower HFS level has $F = 0$ and does not split in the magnetic field.

Due to the factor m_e/m_p in (20.10) the obtainable accuracy in g_I is about three orders of magnitude smaller than the ones in g_J and g_F . However, given an assumed relative accuracy of the microwave frequency measurement of ppb, a determination of the nuclear magnetic moment is possible at the ppm level of accuracy and thus competitive to most other measurements.

Note, that in this kind of measurement, diamagnetic shielding of the nucleus is absent due to only one or few electrons being present. This is an advantage with respect to nuclear magnetic resonance (NMR) measurements and allows a benchmarking of shielding models. The shielding of the external magnetic field B_0 due to the presence of the electrons in atoms or singly charged ions is expressed by a shielding parameter σ that modifies the magnetic field at the position of the nucleus to

$$B = B_0(1 - \sigma), \quad (20.13)$$

where σ takes values up to 0.03 for many-electron elements [33]. This effect is thus well within experimental accuracies and needs to be accounted for. Unfortunately, the available theory values differ significantly [34], such that a model-independent measurement would give theory a benchmark.

20.4 Double-Resonance and Higher-Order Zeeman Effect

The concept of double-resonance spectroscopy can also be applied to certain level schemes that yield access to higher-order Zeeman effects. Since the initial discovery of the quadratic Zeeman effect by Segré and Jenkins in the 1930s [35, 36], there have been numerous studies, both experimental and theoretical, of higher-order Zeeman contributions in atoms, molecules, and singly charged ions in laboratory magnetic fields, see for example [37–39]. However, higher-order contributions to the Zeeman effect have not yet been observed in highly charged ions. Among others, boron-like argon is a good candidate for such studies, since its P-doublet structure offers experimental access and fair theoretical prediction (Fig. 20.7).

When writing the energy of the Zeeman splitting in the form

$$\Delta E_A = \Delta E_A^{(1)} + \Delta E_A^{(2)} + \Delta E_A^{(3)} + \dots, \quad (20.14)$$

the first term

$$\Delta E_A^{(1)}(B_0) = g_J \mu_B B_0 \quad (20.15)$$

is the usual linear Zeeman splitting with its g-factor g_J . The second- and third-order terms in the Zeeman splitting can be represented by

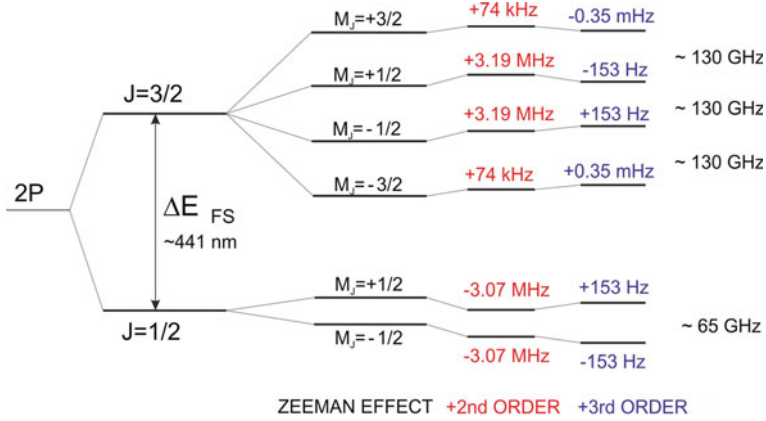


Fig. 20.7 Predicted Zeeman effect in the fine structure of boron-like argon, with quadratic and cubic shifts indicated

$$\Delta E_A^{(2)}(B_0) = \frac{1}{E_0} g_J^{(2)}(M_J)(\mu_B B_0)^2 \quad (20.16)$$

$$\Delta E_A^{(3)}(B_0) = \frac{1}{E_0^2} g_J^{(3)}(M_J)(\mu_B B_0)^3, \quad (20.17)$$

where $E_0 = mc^2$ is the electron rest energy and $g_J^{(2)}$ and $g_J^{(3)}$ are again dimensionless coefficients. A fully relativistic calculation in the framework of quantum electrodynamics yields [40]

$$\frac{\Delta E_A^{(2)}(B_0)}{\Delta E_A^{(1)}(B_0)} \approx \frac{\Delta E_A^{(1)}(B_0)}{\Delta E_{FS}} \approx 10^{-4} \quad (20.18)$$

$$\frac{\Delta E_A^{(3)}(B_0)}{\Delta E_A^{(1)}(B_0)} \approx \left(\frac{\Delta E_A^{(1)}(B_0)}{\Delta E_{FS}} \right)^2 \approx 10^{-8}, \quad (20.19)$$

where the quantity

$$\Delta E_{FS} = E_{3/2}^{(0)} - E_{1/2}^{(0)} \quad (20.20)$$

is the optical fine-structure interval. Hence, the second-order Zeeman effect has a 10^{-4} contribution to the microwave Zeeman splitting and thus needs to be taken into account in such measurements. Also the 10^{-8} contribution of the third order may be within the experimental resolution. Details of such an investigation have been given in [40].

References

1. F.G. Major, G. Werth, High-resolution magnetic hyperfine resonance in harmonically bound ground-state ^{199}Hg ions. *Phys. Rev. Lett.* **30**, 1155 (1973)
2. M. McGuire, R. Petsch, G. Werth, Precision determination of the ground-state hyperfine separation in $^{199}\text{Hg}^+$ using the ion-storage technique. *Phys. Rev. A* **17**, 1999 (1978)
3. R. Blatt, G. Werth, Precision ground state Hfs-separation of ^{137}Ba . *Z. Phys. A* **299**, 93 (1981)
4. R. Blatt, H. Schnatz, G. Werth, Ultrahigh-resolution microwave spectroscopy on trapped $^{171}\text{Yb}^+$ ions. *Phys. Rev. Lett.* **48**, 1601 (1982)
5. X. Feng, G.Z. Li, G. Werth, High-precision hyperfine spectroscopy in M1–M1 double-resonance transitions on trapped $^{207}\text{Pb}^+$. *Phys. Rev. A* **46**, 2959 (1992)
6. T. Nakamura et al., Precision spectroscopy of the Zeeman splittings of the $^9\text{Be}^+$ $2^2\text{S}_{1/2}$ hyperfine structure for nuclear structure studies. *Opt. Commun.* **205**, 329 (2002)
7. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
8. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
9. T. Beier, The g_J -factor of a bound electron and the hyperfine structure splitting in hydrogenlike ions. *Phys. Rep.* **339**, 79 (2000)
10. J.R. Crespo López-Urrutia, P. Beiersdorfer, D.W. Savin, K. Widmann, Direct observation of the spontaneous emission of the hyperfine transition $F = 4$ to $F = 3$ in ground state hydrogenlike $^{165}\text{Ho}^{66+}$ in an electron beam ion trap. *Phys. Rev. Lett.* **77**, 826 (1996)
11. J.R. Crespo López-Urrutia et al., Nuclear magnetization distribution radii determined by hyperfine transitions in the 1s level of H-like ions $^{185}\text{Re}^{74+}$ and $^{187}\text{Re}^{74+}$. *Phys. Rev. A* **57**, 879 (1998)
12. P. Beiersdorfer et al., Hyperfine structure of hydrogenlike thallium isotopes. *Phys. Rev. A* **64**, 032506 (2001)
13. P. Seelig et al., Ground state hyperfine splitting of hydrogenlike $^{207}\text{Pb}^{81+}$ by laser excitation of a bunched ion beam in the GSI experimental storage ring. *Phys. Rev. Lett.* **81**, 4824 (1998)
14. S. Borneis et al., Ground state hyperfine structure of heavy hydrogen like ions. *Hyp. Int.* **127**, 305 (2000)
15. I. Klaft et al., Precision laser spectroscopy of the ground state hyperfine splitting of hydrogenlike $^{209}\text{Bi}^{82+}$. *Phys. Rev. Lett.* **73**, 2425 (1994)
16. P. Beiersdorfer, A.L. Osterheld, J.H. Scofield, J.R. Crespo López-Urrutia, K. Widmann, Measurement of QED and hyperfine splitting in the $2s_{1/2} - 2p_{3/2}$ X-ray transition in Li-like $^{209}\text{Bi}^{80+}$. *Phys. Rev. Lett.* **80**, 3022 (1998)
17. A.N. Artemyev, V.M. Shabaev, G. Plunien, G. Soff, V.A. Yerokhin, Vacuum-polarization corrections to the hyperfine splitting in heavy ions and to the nuclear magnetic moments. *Phys. Rev. A* **63**, 062504 (2001)
18. V.M. Shabaev, A.N. Artemyev, V.A. Yerokhin, O.M. Zherebtsov, G. Soff, Towards a test of QED in investigations of the hyperfine splitting in heavy ions. *Phys. Rev. Lett.* **86**, 3959 (2001)
19. V.A. Yerokhin, A.N. Artemyev, V.M. Shabaev, G. Plunien, All-orders results for the one-electron QED correction to the hyperfine structure in light H-like ions. *Phys. Rev. A* **72**, 052510 (2005)
20. A.A. Elizarov, V.M. Shabaev, N.S. Oreshkina, I.I. Tupitsyn, T. Stoehlker, The hyperfine structure of heavy hydrogen-like ions: calculation based on experimental data on muonic atoms. *Opt. Spectrosc.* **100**, 361 (2006)
21. D.L. Moskovkin, V.M. Shabaev, Zeeman effect of the hyperfine-structure levels in hydrogenlike ions. *Phys. Rev. A* **73**, 052506 (2006)
22. D.L. Moskovkin, V.M. Shabaev, W. Quint, Zeeman effect of the hyperfine structure levels in lithiumlike ions. *Phys. Rev. A* **77**, 063421 (2008)
23. A.V. Volotka, D.A. Glazov, I.I. Tupitsyn, N.S. Oreshkina, G. Plunien, V.M. Shabaev, Ground-state hyperfine structure of H-, Li-, and B-like ions in the intermediate-Z region. *Phys. Rev. A* **78**, 062507 (2008)

24. N.S. Oreshkina, D.A. Glazov, A.V. Volotka, V.M. Shabaev, I.I. Tupitsyn, G. Plunien, Radiative and interelectronic-interaction corrections to the hyperfine splitting in highly charged B-like ions. *Phys. Lett. A* **372**, 675 (2008)
25. D.A. Glazov, A.V. Volotka, V.M. Shabaev, I.I. Tupitsyn, G. Plunien, Evaluation of the screened QED corrections to the g factor and the hyperfine splitting of lithiumlike ions. *Phys. Rev. A* **81**, 062112 (2010)
26. A.V. Volotka, D.A. Glazov, O.V. Andreev, V.M. Shabaev, I.I. Tupitsyn, G. Plunien, Test of many-electron QED effects in the hyperfine splitting of heavy high-Z ions. *Phys. Rev. Lett.* **108**, 073001 (2012)
27. D. Budker, D.F. Kimball, D.P. DeMille, *Atomic Physics* (Oxford University Press, Oxford, 2004)
28. M. Vogel, W. Quint, Trap-assisted precision spectroscopy of forbidden transitions in highly charged ions. *Phys. Rep.* **490**, 1 (2010)
29. H. Marxer, L. Spruch, Semiclassical estimation of the radiative mean lifetimes of hydrogenlike states. *Phys. Rev. A* **43**, 1268 (1991)
30. M.W. Horbatsch, M. Horbatsch, E.A. Hessels, A universal formula for the accurate calculation of hydrogenic lifetimes. *J. Phys. B* **38**, 1765 (2005)
31. A.N. Artemyev et al., Ab initio calculations of the 2 p_{3/2}-2 p_{1/2} fine-structure splitting in boronlike ions. *Phys. Rev. A* **88**, 032518 (2013)
32. W. Quint, D. Moskovkin, V.M. Shabaev, M. Vogel, Laser-microwave double-resonance technique for g-factor measurements in highly charged ions. *Phys. Rev. A* **78**, 032517 (2008)
33. P. Raghavan, *At. Data Nucl. Data Tables* **42**, 189 (1989)
34. M.G.H. Gustavsson, A.-M. Martensson-Pendrill, Need for remeasurements of nuclear magnetic dipole moments. *Phys. Rev. A* **58**, 3611 (1998)
35. F.A. Jenkins, E. Segré, The quadratic Zeeman Effect. *Phys. Rev.* **55**, 52 (1939)
36. L.I. Schiff, H. Snyder, Theory of the quadratic Zeeman Effect. *Phys. Rev.* **55**, 59 (1939)
37. W.R.S. Garton, F.S. Tomkins, Diamagnetic Zeeman Effect and magnetic configuration mixing in long spectral series of BA I. *Astrophys. J.* **158**, 839 (1969)
38. G. Feinberg, A. Rich, J. Sucher, Quadratic Zeeman effect in positronium. *Phys. Rev. A* **41**, 3478 (1990)
39. M. Raoult, S. Guizard, D. Gauyacq, A. Matzkin, Quadratic Zeeman effect in Rydberg states of NO. *J. Phys. B.* **38**, S171 (2005)
40. D. von Lindenfels et al., Experimental access to higher-order Zeeman effects by precision spectroscopy of highly charged ions in a Penning trap. *Phys. Rev. A* **87**, 023412 (2013)

Chapter 21

Magnetic Bottles as Implemented in Penning Traps



Abstract This chapter takes a look at the effects and possible implementations of specific magnetic field geometries, mainly of so-called ‘magnetic bottles’ which are a key ingredient to the application of the continuous Stern-Gerlach effect in Penning traps.

21.1 Applications of Specific Magnetic Field Geometries

Magnetic field gradients find quite a number of applications, such as in magnetic mirrors [1], magnetic-bottle spectrometers [2, 3], and Zeeman slowers [4, 5]. In the context of particle traps, the shaping of fields by introduction of defined gradients has two main aspects. One is that in a position-dependent field one may choose the absolute value of the field experienced by the particles by according positioning. In presence of a field gradient, a shift of the position results in a change of the absolute value of the magnetic field at the position of the particle and thus changes the value of the Zeeman splitting of spectral lines and so forth, which finds application for example in (Zeeman-tuned) level-crossing spectroscopy [6–12], magneto-optical traps [13] and individual addressing of trapped particles via a position-dependent magnetic field [14], particularly in quantum logic with confined particles [15].

The other important aspect is the geometry of the field itself which can give rise to desired effects, for example magnetic gradient forces on dipoles [16] as used in time orbiting potential (TOP) traps [17], in Ioffe-Pritchard traps [18], and various magnetic micro-traps [19–22]. Quadratic distortions of the magnetic field in the shape of a so-called ‘magnetic bottle’ are valuable to Penning-trap experiments, for example when the continuous Stern-Gerlach effect [23] is used for measurements of magnetic moments of unbound electrons [24–26], unbound protons and anti-protons [27, 28], and bound electrons in highly-charged ions [29–35], or when defined couplings amongst oscillatory degrees of freedom in the Penning trap are used for spectroscopic

purposes [36, 37]. Nested Penning traps [38] can benefit from introduced magnetic field gradients, regarding for example Penning-Ioffe traps as used in anti-hydrogen research [39]. There have been considerations of magnetic bottles of variable strength [40] and of switchable superconducting magnetic bottles [41].

21.2 Magnetic Bottles

Let the symmetry axis $\mathbf{e}_z$ of the trap be parallel to the magnetic field B_0 and the radial coordinate be ρ . The value of the magnetic field strength near the trap centre can then be written as the expansion (see the discussion in Sect. 6.3.2)

$$B_z(z, \rho) = B_0 - 2B_1z + B_2 \left(z^2 - \frac{1}{2}\rho^2 \right) + \dots \quad (21.1)$$

$$B_\rho(z, \rho) = -\frac{1}{2}B_1\rho - B_2z\rho + \dots \quad (21.2)$$

When we collect the terms with B_2 , we have isolated a so-called ‘magnetic bottle’.

Hence, a magnetic bottle is magnetic field inhomogeneity of the kind

$$\mathbf{B}(z, \rho) = B_2 \left(\left(z^2 - \frac{\rho^2}{2} \right) \mathbf{e}_z - \rho z \mathbf{e}_\rho \right) \quad (21.3)$$

superimposed on the magnetic trapping field $B_0\mathbf{e}_z$ with radial symmetry around the central trap axis.

This configuration is sometimes also called a ‘Lawrence bottle’ [42]. The field contribution of the bottle is the source of an axial force

$$F_z = -\frac{\partial}{\partial z}(\boldsymbol{\mu} \cdot \mathbf{B}(z, \rho)) = -2\mu_z B_2 z \quad (21.4)$$

that acts on a particle with a non-zero magnetic moment and thus changes its axial oscillation frequency, which hence becomes a sensor for the spin state of the particle (‘continuous Stern-Gerlach effect’).

For the following, it is useful to visualise the resulting magnetic field lines. Concerning such a disc configuration and an external homogeneous magnetic field, we distinguish four cases:

- **A:** a ferromagnetic disc in an external magnetic field
- **B:** a disc is made superconducting within an already existing external magnetic field

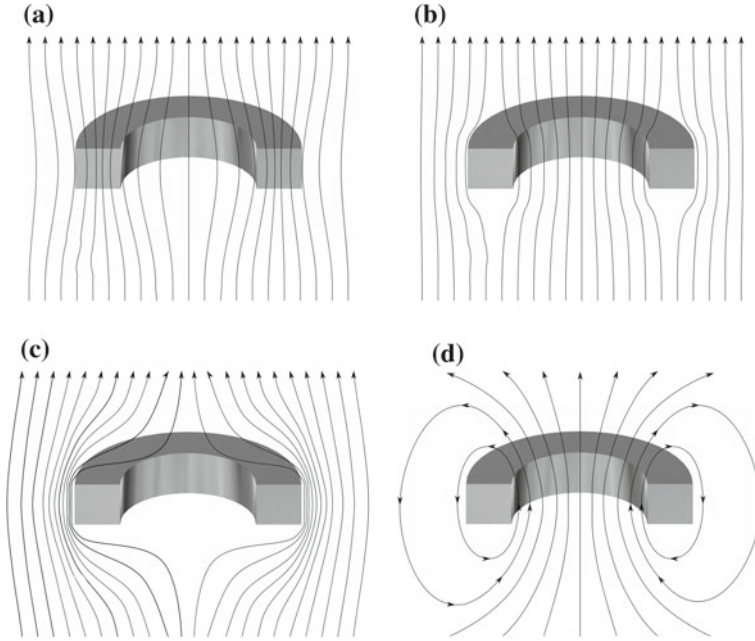


Fig. 21.1 Magnetic field configurations of different magnetic bottle situations

- **C:** a magnetic field is established when a flux-free superconducting disc is already present
- **D:** a flux-pervaded superconducting disc as in case B after the external magnetic field is removed again

Figure 21.1 gives a qualitative picture of the field configurations in these cases. As type A is the most common one, we focus the following discussion on it. The other cases will be discussed in more detail below.

21.3 Magnetic Bottle and the Invariance Theorem

The invariance theorem $\omega_c^2 = \omega_+^2 + \omega_z^2 + \omega_-^2$ [43] is not exactly valid in the presence of a magnetic bottle. The individual oscillation frequencies are shifted by finite motional energies E_+ , E_z , E_- or by M_J -transitions in a way which does not to all orders fulfil the invariance theorem. Expressing the effect of non-zero motional energies by the perturbed cyclotron frequency shift $\Delta\omega_+$ as obtained from (6.55), we have

$$\Delta\omega_- = -\Delta\omega_+ \quad \text{and} \quad \Delta\omega_z = -\Delta\omega_+ \frac{(\omega_+ - \omega_-)}{\omega_z}. \quad (21.5)$$

Inserting this into the invariance theorem yields a second-order cyclotron frequency shift given by

$$\frac{\Delta\omega_c}{\omega_c}(\Delta\omega_+) \approx \left[1 + \frac{1}{2} \left(\frac{\omega_+ - \omega_-}{\omega_z} \right)^2 \right] \left(\frac{\Delta\omega_+}{\omega_+} \right)^2, \quad (21.6)$$

of which the leading contribution expressed in terms of experimental quantities is

$$\frac{\Delta\omega_c}{\omega_c} \approx \frac{\omega_+^2}{2\omega_z^2} \frac{1}{m^2\omega_z^4} \frac{B_2^2}{B_0^2} (E_z^2 + 2E_z E_- + 4E_-^2). \quad (21.7)$$

The relative shift of the cyclotron frequency in the presence of a magnetic bottle with $B_2 \neq 0$ can be linearly approximated in terms of the motional amplitudes by [43]

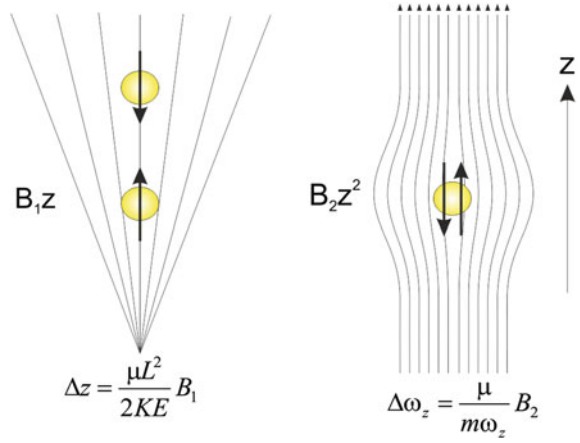
$$\frac{\Delta\omega_c}{\omega_c} \approx \frac{B_2}{B_0} \left[z^2 - \frac{\rho_+^2}{4} \left(1 + \frac{\omega_c}{\omega_+ - \omega_-} \right) - \frac{\rho_-^2}{4} \left(1 + \frac{\omega_c}{\omega_+ - \omega_-} \right) \right]. \quad (21.8)$$

For particles cooled to liquid-helium temperature or below, this effect is usually outside the spectroscopic resolution. With the advent of detection techniques which do not require significant excitation of the ion motion [44], this effect will be rendered irrelevant as the energies on the right hand side will be negligible.

21.4 Magnetic Bottles and Continuous Stern-Gerlach Effect

The continuous Stern-Gerlach effect is named after its earlier counterpart, the Stern-Gerlach effect [45]. According to Dehmelt ‘**The famous Stern-Gerlach experiment on a beam of silver atoms passing through a transverse inhomogeneous magnetic field had shown that, when a parameter of the trajectory (the deflection) becomes known accurately enough to deduce from it the z -component of the magnetic moment, the values found are only $\pm\mu_B$. In our experiment on an individual electron we greatly increased the detection sensitivity for changes in the classical trajectory, which is now essentially parallel to the magnetic field. This has been achieved by electronic techniques and by making the trajectory periodic in an axial electric parabolic trapping potential and very long. A weak, auxiliary magnetic bottle -known from Lawrence’s cyclotron- produced by a nickel wire ring magnetized to saturation causes the axial oscillation frequency, to be slightly spin dependent**’ [23].

Fig. 21.2 Schematic comparison of the Stern-Gerlach effect (left) which separates different spin orientations in space, the continuous Stern-Gerlach effect in a magnetic bottle separates the two spin orientations in frequency space



This means that other than the magnetic field gradient spatially separating different spin orientations, the continuous Stern-Gerlach effect separates the two spin orientations in frequency space, see Fig. 21.2. Of particular interest for a number of applications is the fact that in the presence of a magnetic bottle the intrinsic spin orientation is linked to external degrees of freedom of a particle, namely to its oscillation frequencies in the trap.

Using the orientation energy $E_J = g_J \mu_B B_0 M_J$, we find the shift of the axial frequency resulting from a change of the angular quantum number M_J by [36]

$$\Delta \omega_z(\Delta M_J) \approx \frac{\hbar}{2m} \frac{\omega_+}{\omega_+ - \omega_-} \frac{B_2}{B_0} \frac{e}{q} g_J \Delta M_J. \quad (21.9)$$

Likewise, for the radial frequencies we have

$$\Delta \omega_+(\Delta M_J) \approx -\frac{\hbar}{2m} \frac{\omega_+}{\omega_+ - \omega_-} \frac{B_2}{B_0} \frac{e}{q} g_J \Delta M_J \approx \frac{\omega_z}{\omega_+} \Delta \omega_z(\Delta M_J) \quad (21.10)$$

$$\Delta \omega_-(\Delta M_J) \approx \frac{\hbar}{2m} \frac{\omega_+}{\omega_+ - \omega_-} \frac{B_2}{B_0} \frac{e}{q} g_J \Delta M_J \approx -\Delta \omega_+(\Delta M_J), \quad (21.11)$$

where g_J is the g -factor corresponding to the magnetic moment of interest and M_J is the angular momentum quantum number. Commonly, the axial motion is used as the effect is highest. In typical experiments, also the relative shifts $\Delta \omega_+/\omega_+$ and $\Delta \omega_z/\omega_z$ of the oscillation frequencies ω_+ and ω_z

$$\begin{pmatrix} \Delta \omega_+/\omega_+ \\ \Delta \omega_z/\omega_z \end{pmatrix} = \frac{1}{m \omega_z^2} \frac{B_2}{B_0} \begin{pmatrix} -(\omega_z/\omega_+)^2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta E_+ \\ \Delta E_z \end{pmatrix} \quad (21.12)$$

due to corresponding finite motional energies E_+ and E_z are relevant, either as an unwanted residual effect or as a working principle for measurement techniques.

21.5 Implementation of a Magnetic Bottle

The usual way to implement such a magnetic bottle in a Penning trap experiment like presently discussed is to use a ferromagnetic central trap electrode in the shape of an annular disc with inner radius r_1 , outer radius r_2 and a thickness of $2a$ in an external homogeneous magnetic field B_0 , as depicted in Fig. 21.3. Such an arrangement distorts the homogeneity of the external magnetic field used for confinement to form a magnetic bottle of the kind given in (21.3). The coefficient B_2 has a unit of Tesla per square metre. It can be calculated by considering the scalar magnetic potential Φ with $\Delta\Phi = -4\pi\rho_M = 4\pi\nabla \cdot \mathbf{M}$, where $\mathbf{M}$ is the magnetisation which is assumed to be homogeneous over the ring volume and has the value M_0 .

Inserting the ring geometry and comparing coefficients with the general multipole expansion of the magnetic field [43], the strength B_2 of the magnetic bottle at its centre position ($z = 0$, $\rho = 0$) is given by

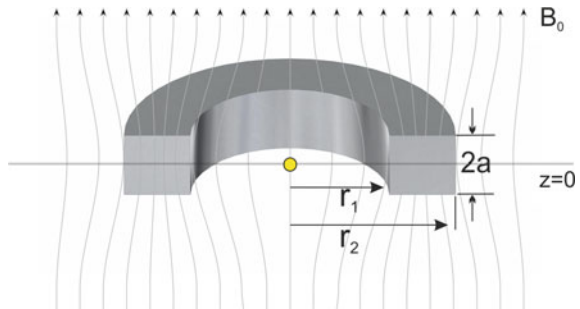
$$B_2 = 3\mu_0 M_0 \left(\frac{ar_1^2}{2(a^2 + r_1^2)^{5/2}} - \frac{ar_2^2}{2(a^2 + r_2^2)^{5/2}} \right). \quad (21.13)$$

The linear gradient B_1 is zero at the centre position. The homogeneous part B_0 of the field at the centre position is modified by an amount

$$\Delta B_0 = -\frac{\mu_0 M_0}{2} \left(\frac{2a}{a^2 + r_1^2} - \frac{2a}{a^2 + r_2^2} \right). \quad (21.14)$$

Typical materials for ring electrodes to form a magnetic bottle come from the Nickel, Cobalt, Iron, CoFe, SmCo, NdFeB and AlNiCo families. The saturation field strength

Fig. 21.3 Schematic view of a magnetic bottle formed by a ferromagnetic ring electrode of a Penning trap with inner radius r_1 , outer radius r_2 and a thickness of $2a$ inside an otherwise homogeneous magnetic field along the z -direction



$\mu_0 M_0$ of specific high-permeability iron alloys such as Hisat50 reaches up to 2.44 T for a specific CoFe alloy with vanadium. With a CoFe alloy, values of B_2 up to around 400 mT/mm² have been achieved by use of optimized ring geometries [46–48].

For the ferromagnetic annular disc ring electrode, and at a given magnetisation M_0 , the maximum of the magnetic bottle strength B_2 is reached when the ratio a/r_1 is chosen such that the Legendre polynomial

$$P_4(\cos \beta) = \frac{35}{8} \cos^4 \beta - \frac{30}{8} \cos^2 \beta + \frac{3}{8} \quad (21.15)$$

with

$$\beta = \tan^{-1} \left(\frac{a}{r_1} \right) \quad (21.16)$$

vanishes, see also the discussion in [43]. This leads to the condition $a/r_1 \approx 0.577$ (and $r_2 \gg r_1$). Higher values of B_2 can be reached for geometries which more closely follow the conditions imposed by the behaviour of P_4 , for details see the discussion in [43]. One such design is the so-called ‘hybrid Penning trap’, which is a cylindrical Penning trap with a ferromagnetic toroidal ring that forms the magnetic bottle. The toroidal shape improves on the ‘efficiency’ of creating a magnetic bottle by more closely fulfilling the above geometry condition. Despite the discontinuous geometry, the properties of such a magnetic bottle and overall trap can be calculated analytically. This design has been discussed in detail for example in [48].

We should note that also non-ferromagnets such as the often-used copper and Macor have finite magnetisations in an external magnetic field. At a fields strengths of 6 T and at liquid-helium temperature, copper has an M_0 of $-50 \mu\text{T}$ and Macor of $780 \mu\text{T}$ [43]. In principle, when calculating the total magnetic bottle strength, these need to be taken into account as well, however, at the present machining precisions and experimental resolutions of magnetic bottle strength measurements, these contributions may well be neglected. The same is true for the temperature-dependence of the ring’s saturation magnetisation, which e.g. for nickel is $\Delta M_0/M_0 \approx 2.4 \times 10^{-5}/\text{K}$, and for typical temperature fluctuations of much below one Kelvin leads to relative changes of the magnetic field at the position of the oscillation frequency measurement of much below 10^{-10} .

21.6 Measurement of the Magnetic Bottle Strength

An experimental determination of the magnetic bottle strength B_2 can be performed by a position-dependent measurement of the magnetic field strength, for example a measurement of the cyclotron frequency as a function of the axial ion position. To this end, the trap can be made electrically asymmetric by applying an additional small voltage U_a across the endcaps. This adds a small uniform axial electric field

$E_z = C_1 U_a / 2z_0$ close to the trap centre and shifts the centre position of the axial oscillation by an amount Δz given by [49]

$$\Delta z = \frac{1}{2} \frac{d^2}{z_0} \frac{C_1}{C_2} \frac{U_a}{U_0}, \quad (21.17)$$

where C_1 measures the electric field gradient (dipole term) as we have detailed in Sect. 6.3. A non-zero value of C_1 shifts the axial frequency ω_z by an amount

$$\frac{\Delta\omega_z}{\omega_z} = -\frac{3}{4} \frac{d^4}{z_0^4} C_1 C_3 \left(\frac{U_a}{U_0} \right)^2, \quad (21.18)$$

where C_3 is the hexapolar field geometry coefficient which for common trap geometries obeys $C_1 + C_3 \approx 1$ [43, 49]. This shift needs to be considered when the magnetic field strength is deduced from the measured oscillation frequencies. Also, the range of possible centre shifts Δz is limited by the potential asymmetry U_a giving rise to additional anharmonicities. The leading contribution is given as a change of the C_4 -coefficient (see Sect. 6) where we have [50]

$$C_4 \rightarrow C_4 - \frac{5}{4} C_3^2 \left(\frac{U_a}{U_0} \right)^2. \quad (21.19)$$

The general recipe for the calculation of such potential influences is given in [43]. A measurement of the magnetic bottle strength B_2 for the trap in [30–32] has been performed [31] and yields results for the actual B_2 in fair agreement with the prediction by (21.13).

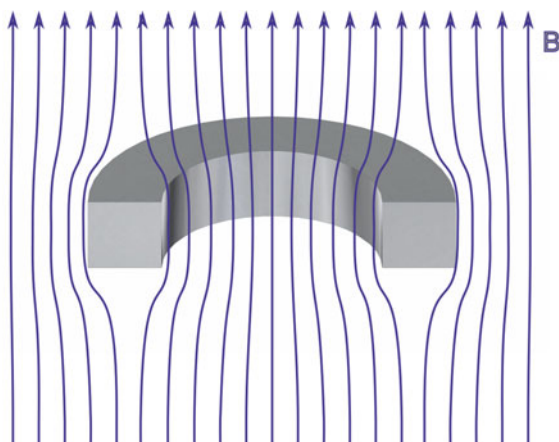
21.7 Superconducting Magnetic Bottle

Instead of a ferromagnetic material, we may choose a superconductor to create a magnetic bottle. In principle, it behaves like a perfect diamagnet, but has additional features.

Initial experiments to create a magnetic bottle from a superconducting current loop have been successful and produced variable field gradients up to $B_2 = 2 \mu\text{T/mm}^2$ [40], but have not found experimental application so far.

In the superconducting phase, the Meissner-Ochsenfeld effect [51] provides that the interior of a superconducting material is field-free. Given the same geometry, it hence distorts an outer magnetic field to form a magnetic bottle similarly to the ferromagnet, however of opposite sign and limited strength as will be discussed below. Other than for the ferromagnetic disc discussed as case A in Sect. 21.2, for such a superconducting bottle we need to distinguish the three different cases B, C and D from Fig. 21.1 in Sect. 21.2.

Fig. 21.4 Magnetic field configuration of a superconducting magnetic bottle that became superconducting in the presence of the field



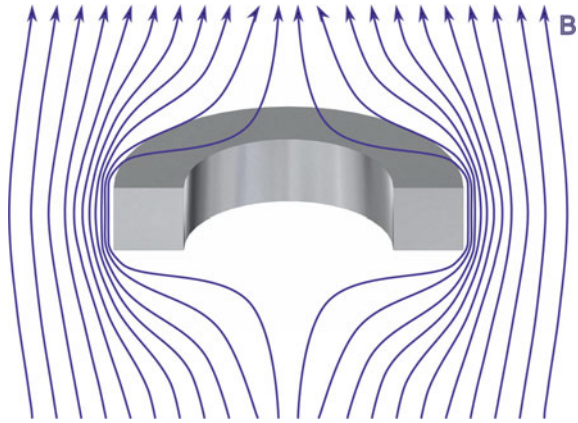
21.7.1 *Field Before Superconductivity*

This is case B from Fig. 21.1 in Sect. 21.2. The external magnetic field penetrates the whole arrangement without any significant distortion as long as the annulus is normal-conducting. When it is cooled below its critical temperature, the bulk expels the magnetic field, but the bore is still flux-pervaded, hence creating a magnetic bottle (Fig. 21.4). This case is opposed to the ferromagnetic case A in that the field lines concentrate towards the central axis. It is possible to ‘trap’ magnetic field lines when e.g. a circular disc is cooled radially from the outside to the inside and gradually becomes superconducting from the outside to the inside. Then all field lines which initially penetrated the whole disc are forced into the non-superconducting centre. If this centre is well-defined for example by different choice of material, the field shape and strength in this centre are well-defined as well, and it is possible to produce magnetic fields up to the critical field strength even in the absence of an outer magnet, simply by ‘compression’ of the Earth’s magnetic field [41].

21.7.2 *Superconductivity Before Field*

When the disc is superconducting before the external magnetic field is established, magnetic flux is expelled from the entire annular disc including the bore (Fig. 21.5). This is case C from Fig. 21.1. It is a consequence of the fact, that while in the superconducting state, the flux through the whole superconductor area cannot change as persistent currents in its outer surface shield the interior from the external flux exactly. However, this is true only up to the point when residually penetrating fields exceed the critical field strength of the superconductor. For a given field, this determines

Fig. 21.5 Magnetic field configuration of a superconducting magnetic bottle that was superconducting before the field was established



the diameter of the inner disc region which stays superconducting. A more detailed discussion of this is given in [41].

We should note that the induced persistent currents which maintain the magnetic flux within the superconductor represent a magnetic moment μ which in presence of an outer magnetic field gradient $\nabla \mathbf{B}$ leads to a force $\mu \cdot \nabla \mathbf{B}$. Simplifying the disc to a current loop of radius R and picking a one-dimensional gradient in z -direction, the force F_z induced by an outer magnetic field gradient $\partial B_0 / \partial z$ can be estimated by

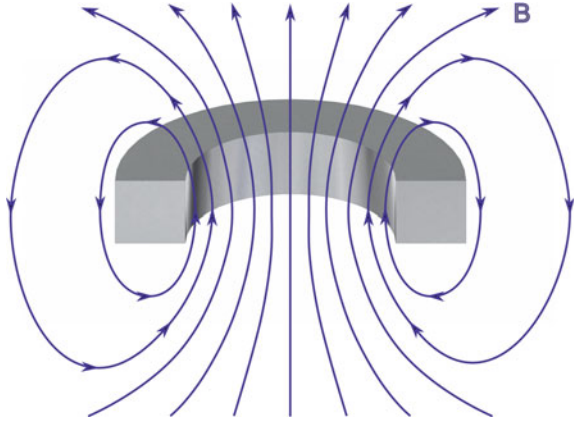
$$F_z \approx \frac{2\pi R^3 B_0}{\mu_0} \frac{\partial B_0}{\partial z} \quad (21.20)$$

which for $R = 10$ mm at a field of $B_0 = 1$ T and a gradient of $\partial B_0 / \partial z = 1$ T/m would be 5 N. For an annulus in a field with a gradient the real situation is more complicated, but the order of magnitude of the force remains true. This force can either be seen as mechanical stress on immovable parts or as a means to exert a force e.g. to remotely operate a mechanical switch or perform another motion in high magnetic field at low temperatures simply by heating or not heating e.g. with a laser. In a homogeneous outer magnetic field as typical for Penning trap experiments, however, this effect does not occur as the gradient is close to zero.

21.7.3 Magnetic Field Removal from a Superconducting Bottle

When the external magnetic field is switched off while the superconducting disc is flux-pervaded, this results in magnetic flux trapping and the disc maintains the magnetic bottle field forever even in absence of an external field (Fig. 21.6). This is case D from Fig. 21.2. This may prove valuable to experiments which need a

Fig. 21.6 Magnetic field configuration of a superconducting magnetic bottle that when the external field has been switched off



local magnetic bottle, but are otherwise sensitive to magnetic fields. One can hence ‘load’ the bottle initially, and then remove the outer field for the experiment to be performed. This is also an ideal tool to create arrays of permanent magnetic micro-bottles of identical strength, some of which can then individually be switched off again by short heating laser irradiation and will remain switched off even after cooling back. This allows to manufacture complicated arrangements of planar (micro-)traps with and without magnetic bottles and may find application in quantum information processing, see the discussion in [41].

21.8 Residual Field of a Magnetic Bottle

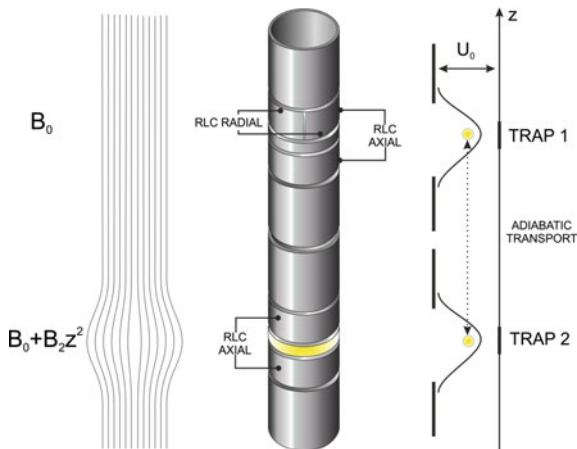
For means of an easy analysis of the spin direction, it is advantageous to choose the strength B_2 of the magnetic bottle as high as possible. This however limits the overall accuracy which can be reached by use of the continuous Stern-Gerlach effect, since from (6.55) it follows that the cyclotron frequency depends on the motional energies (and hence motional amplitudes a_z , $\rho_{\pm}$) in the presence of a magnetic bottle with strength B_2 according to

$$\frac{\Delta\omega_c}{\omega_c} = \frac{B_2}{B_0} \left[a_z^2 - \frac{\rho_+^2}{4} \left(1 - \frac{\omega_+ + \omega_-}{\omega_+ - \omega_-} \right) - \frac{\rho_-^2}{4} \left(1 + \frac{\omega_+ + \omega_-}{\omega_+ - \omega_-} \right) \right], \quad (21.21)$$

in which the relations between motional energy and motional amplitude are given by

$$a_z^2 = \frac{E_z}{qC_2U_0} d^2, \quad \rho_+^2 = \frac{2E_+}{m(\omega_+^2 - \omega_z^2/2)} d^2 \quad \text{and} \quad \rho_-^2 = \frac{2E_-}{m(\omega_-^2 - \omega_z^2/2)} d^2. \quad (21.22)$$

Fig. 21.7 Schematic of the implementation of the double-trap technique separating a trap with a magnetic bottle for application of the continuous Stern-Gerlach effect from a trap for precision frequency measurements



Therefore, thermal and any other fluctuations of these energies (amplitudes) broaden the Larmor-to-cyclotron resonance and hence increase the uncertainty of the resulting g -factor. Proposals to solve this problem by use of an adjustable magnetic bottle have been brought forward and initial tests were successful [40], however this idea was not used in such experiments. A measurement of the magnetic moment of the electron bound in hydrogen-like carbon $^{12}\text{C}^{5+}$ in [29] was limited due to the continuous presence of the magnetic bottle with $B_2 \approx 10 \text{ mT/mm}^2$ and resulted in $g_J = 2.001\,042(2)$ which corresponds to a relative accuracy of about 1×10^{-6} .

In succeeding experiments, the location of the precision oscillation frequency measurement was separated from the location of the spin analysis ('double-trap technique') which has shifted the relative accuracy into the region of 10^{-9} and better [30–35] (Fig. 21.7).

The spatial separation of spin analysis and precision oscillation frequency measurement has first been used in [30] and resulted in an improvement of the bound-electron magnetic moment in $^{12}\text{C}^{5+}$ by about three orders of magnitude [30]. For the application of the double-trap technique, the Penning trap basically features two trapping regions separated by transport electrodes. One trapping region is the centre of a magnetic bottle, while the other is designed for high precision of the oscillation frequency measurement. Ideally, in this trap the magnetic inhomogeneity is zero. However, the remaining field distortion of the magnetic bottle at the position of the precision trap does not vanish. When the axial separation of the two trap centres is given by Δz , the residual axial magnetic field component of the bottle at the position of the precision trap reduces the effective axial magnetic field there. The magnitude of this contribution is given by

$$B_z(\Delta z) = \frac{\mu_0 M_0}{2} \left(\frac{z_+}{\sqrt{r_2^2 + z_+^2}} - \frac{z_-}{\sqrt{r_2^2 + z_-^2}} - \frac{z_+}{\sqrt{r_1^2 + z_+^2}} + \frac{z_-}{\sqrt{r_1^2 + z_-^2}} \right), \quad (21.23)$$

where $z_{\pm} = \Delta z \pm a$. The higher-order components of this residual distortion are given by

$$B_{(n)}(\Delta z) = \frac{\partial^n B_z(\Delta z)}{\partial z^n} \quad (21.24)$$

and cause a linear gradient $B_1 \neq 0$ and a residual magnetic bottle $B_2 \neq 0$ at the position of the precision trap. The first derivative reads

$$B_1(\Delta z) = \frac{\partial B}{\partial z}(\Delta z) = -\frac{\mu_0 M_0}{2} G_1(\Delta z), \quad (21.25)$$

where the geometry factor $G_1(\Delta z)$ is given by

$$G_1(\Delta z) = \frac{r_1^2}{(r_1^2 + z_+^2)^{3/2}} - \frac{r_1^2}{(r_1^2 + z_-^2)^{3/2}} + \frac{r_2^2}{(r_2^2 + z_-^2)^{3/2}} - \frac{r_2^2}{(r_2^2 + z_+^2)^{3/2}}, \quad (21.26)$$

where we use $z_{\pm} = a \pm \Delta z$. As discussed above, for $z = 0$ the linear gradient B_1 vanishes as the geometry factor $G_1(z = 0)$ vanishes. The residual magnetic bottle strength B_2 at the position Δz along the central axis is given by the second derivative of B with respect to z

$$B_2(\Delta z) = \frac{\partial^2 B}{\partial z^2}(\Delta z) = -\frac{\mu_0 M_0}{2} G_2(\Delta z) \quad (21.27)$$

where $G_2(\Delta z)$ is given by

$$G_2(\Delta z) = -\frac{3r_1^2 z_+}{(r_1^2 + z_+^2)^{5/2}} - \frac{3r_1^2 z_-}{(r_1^2 + z_-^2)^{5/2}} + \frac{3r_2^2 z_-}{(r_2^2 + z_-^2)^{5/2}} + \frac{3r_2^2 z_+}{(r_2^2 + z_+^2)^{5/2}}. \quad (21.28)$$

For typical trap separations Δz of a few cm, the residual magnetic bottle strength there is suppressed by three to four orders of magnitude. As an example, in [30] and [32] the magnetic bottle of one trap with $B_2 \approx 10 \text{ mT/mm}^2$ created a residual magnetic bottle at the position of the adjacent trap with $B_2 \approx 4 \mu\text{T/mm}^2$, a linear gradient B_1 of about $66 \mu\text{T/mm}$ and a fourth-order term B_4 of about 11 nT/mm^4 .

The effect of a non-zero term B_2 has been discussed in (21.2) and residual magnetic bottles at the position of the cyclotron frequency measurement typically are the dominating source of systematic uncertainties over residual contributions B_1 and B_4 . The effect of a linear magnetic field gradient B_1 on the cyclotron frequency is given by

$$\frac{\Delta\omega_c}{\omega_c} \approx -\frac{1}{m\omega_z^2} \left(\frac{B_1}{B_0} \right)^2 E_c. \quad (21.29)$$

For the present parameters, this effect is much smaller than the residual magnetic bottle effect due to the B_1^2 -dependence. The corresponding effect of a non-zero B_4 term on an ion with motional radius ρ_c is given by

$$\frac{\Delta\omega_c}{\omega_c} \approx \frac{3}{8} \left(\frac{B_4}{B_0} \right) \rho_c^4 = \frac{3}{2m^2\omega_c^4} \left(\frac{B_4}{B_0} \right) E_c^2. \quad (21.30)$$

and can often be neglected due to the small value of B_4 .

During the measurement cycle, the particle is transported adiabatically between the two regions, such that the spin state is preserved, see also our discussion of adiabaticity in Sect. 9.3. A detailed account of the experimental procedures is given in [31, 52]. This method has been used in a number of experiments with ions [32, 34, 35], and with protons and anti-protons [52].

References

1. S. Granger, G.W. Ford, Electron spin motion in a magnetic mirror trap. *Phys. Rev. Lett.* **28**, 1479 (1972)
2. T. Tsuboi et al., Magnetic bottle electron spectrometer using permanent magnets. *Rev. Sci. Inst.* **59**, 1357 (1988)
3. H. Handschuh, G. Ganteföhr, W. Eberhardt, Vibrational spectroscopy of clusters using a magnetic bottle electron spectrometer. *Rev. Sci. Inst.* **66**, 3838 (1995)
4. C.J. Dedman et al., Optimum design and construction of a Zeeman slower for use with a magneto-optic trap. *Rev. Sci. Inst.* **75**, 5136 (2004)
5. M.A. Joffe, W. Ketterle, A. Martin, D.E. Pritchard, Transverse cooling and deflection of an atomic beam inside a Zeeman slower. *J. Opt. Soc. Am. B* **10**, 2257 (1993)
6. W. Demtröder, *Laser Spectroscopy* (Springer, Heidelberg, 2003)
7. P. Hannaford, Oriented atoms in weak magnetic fields. *Phys. Scr.* **T70**, 117 (1997)
8. S. Rydberg, S. Svanberg, Investigation of the $np^2P_{3/2}$ level sequence in the Cs I spectrum by level crossing spectroscopy. *Phys. Scr.* **5**, 209 (1972)
9. J. Alnis, K. Blushs, M. Auzinsh, S. Kennedy, N. Shafer-Ray, E.R.I. Abraham, The Hanle effect and level crossing spectroscopy in Rb vapour under strong laser excitation. *J. Phys. B* **36**, 1161 (2003)
10. B. Budick, S. Marcus, R. Novick, Level-crossing spectroscopy with an electric field: stark shift of the 3^2P term in lithium. *Phys. Rev.* **140**, A1041 (1965)
11. W. Hogervorst, S. Svanberg, Stark effect investigation of D states in ^{85}Rb and ^{133}Cs using level crossing spectroscopy with a CW Dye Laser. *Phys. Scr.* **12**, 67 (1975)
12. A.C. Luntz, R.G. Brewer, Zeeman-tuned level crossing in $1\Sigma\text{CH}_4$. *J. Chem. Phys.* **53**, 3380 (1970)
13. S. Chu, Nobel lecture: the manipulation of neutral particles. *Rev. Mod. Phys.* **70**, 685 (1998)
14. S.X. Wang, J. Labaziewicz, Y. Ge, R. Shewmon, I.L. Chuang, Individual addressing of ions using magnetic field gradients in a surface-electrode ion trap. *Appl. Phys. Lett.* **94**, 094103 (2009)
15. F. Mintert, C. Wunderlich, Ion-trap quantum logic using long-wavelength radiation. *Phys. Rev. Lett.* **87**, 257904 (2001)
16. T.H. Boyer, The force on a magnetic dipole. *Am. J. Phys.* **56**, 688 (1988)
17. W. Petrich et al., Stable, tightly confining magnetic trap for evaporative cooling of neutral atoms. *Phys. Rev. Lett.* **74**, 3352 (1995)

18. Y.V. Gott, M.S. Ioffe, V.G. Telkowskii, Some results on confinement in magnetic trapping. Nucl. Fusion, Suppl. 2, Pt. 3, 1045 (1962)
19. E.A. Hinds, I.G. Hughes, Magnetic atom optics: mirrors, guides, traps, and chips for atoms. J. Phys. D **32**, R119 (1995)
20. J. Reichel, Microchip traps and Bose-Einstein condensation. Appl. Phys. B **75**, 469 (2002)
21. R. Folman et al., Microscopic atom optics: from wires to an atom chip. Adv. At. Mol. Opt. Phys. **48**, 263 (2002)
22. J. Fortágh, C. Zimmermann, Magnetic microtraps for ultracold atoms. Rev. Mod. Phys. **79**, 235 (2007)
23. H. Dehmelt, Continuous Stern-Gerlach effect: principle and idealized apparatus. Proc. Natl. Acad. Sci. USA **83**, 2291 (1986)
24. R.S. van Dyck, P.B. Schwinberg, H.G. Dehmelt, New high-precision comparison of electron and positron g factors. Phys. Rev. Lett. **59**, 26 (1987)
25. G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, B. Odom, New determination of the fine structure constant from the electron g value and QED. Phys. Rev. Lett. **97**, 030802 (2006); Erratum ibidem **99**, 039902 (2007)
26. D. Hanneke, S. Fogwell, G. Gabrielse, New measurement of the electron magnetic moment and the fine structure constant. Phys. Rev. Lett. **100**, 120801 (2008)
27. S. Ulmer et al., Observation of spin flips with a single trapped proton. Phys. Rev. Lett. **106**, 253001 (2011)
28. C. Smorra et al., A parts-per-billion measurement of the antiproton magnetic moment. Nature **550**, 371 (2017)
29. N. Hermanspahn et al., Observation of the continuous Stern-Gerlach effect on an electron bound in an atomic ion. Phys. Rev. Lett. **84**, 427 (2000)
30. H. Häffner et al., High-accuracy measurement of the magnetic moment anomaly of the electron bound in Hydrogen-like Carbon. Phys. Rev. Lett. **85**, 5308 (2000)
31. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. Eur. Phys. J. D **22**, 163 (2003)
32. J. Verdú et al., Electronic g factor of Hydrogen-like Oxygen $^{16}\text{O}^{7+}$. Phys. Rev. Lett. **92**, 093002 (2004)
33. J. Verdú et al., Determination of the g -factor of single Hydrogen-like ions by mode coupling in a Penning trap. Phys. Scr. **T112**, 68 (2004)
34. S. Sturm et al., g factor of Hydrogen-like $^{28}\text{Si}^{13+}$. Phys. Rev. Lett. **107**, 023002 (2011)
35. A. Wagner et al., g factor of Lithium-like Silicon $^{28}\text{Si}^{11+}$. Phys. Rev. Lett. **110**, 033003 (2013)
36. M. Vogel, W. Quint, W. Nörtershäuser, Trapped ion oscillation frequencies as sensors for spectroscopy. Sensors **10**, 2169 (2010)
37. M. Vogel, W. Quint, Laser spectroscopy by a radiofrequency measurement on a single ion in a Penning trap. New J. Phys. **11**, 013024 (2009)
38. G. Gabrielse, S.L. Rolston, L. Haarsma, W. Kells, Antihydrogen production using trapped plasmas. Phys. Lett. A **129**, 38 (1988)
39. G. Gabrielse et al., Antihydrogen production within a Penning-Ioffe trap. Phys. Rev. Lett. **100**, 113001 (2008)
40. R.S. Van Dyck Jr., F.L. Moore, D.L. Farnham, P.B. Schwinberg, Variable magnetic bottle for precision geonium experiments. Rev. Sci. Instr. **57**, 593 (1986)
41. M. Vogel et al., Switchable magnetic bottles and field gradients for particle traps. Appl. Phys. B. **114**, 63 (2013)
42. R.S. Van Dyck Jr., P.B. Schwinberg, H.G. Dehmelt, Electron magnetic moment from geonium spectra: early experiments and background concepts. Phys. Rev. D **34**, 722 (1986)
43. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. Rev. Mod. Phys. **58**, 233 (1986)
44. S. Sturm, A. Wagner, B. Schabinger, K. Blaum, Phase-sensitive cyclotron frequency measurements at ultralow energies. Phys. Rev. Lett. **107**, 143003 (2011)
45. O. Stern, W. Gerlach, Das magnetische moment des silberatoms (The magnetic moment of the silver atom) Z. Phys. **9**, 349 (1922) and Z. Phys. **9**, 353 (1922)

46. J. DiSciacca, G. Gabrielse, Direct measurement of the proton magnetic moment. *Phys. Rev. Lett.* **108**, 153001 (2012)
47. A. Mooser et al., Resolution of single spin flips of a single proton. *Phys. Rev. Lett.* **110**, 140405 (2013)
48. J. Verdú et al., Calculation of electrostatic fields using quasi-Green's functions: application to the hybrid Penning trap. *New J. Phys.* **10**, 103009 (2008)
49. G. Gabrielse, F.C. Macintosh, Cylindrical Penning traps with orthogonalized anharmonicity compensation. *Int. J. Mass. Spec. Ion Proc.* **57**, 1 (1984)
50. G. Gabrielse, L. Haarsma, S.L. Rolston, Open-endcap Penning traps for high precision experiments. *Int. J. Mass Spectr. Ion Proc.* **88**, 319 (1989)
51. W. Meissner, R. Ochsenfeld, Ein neuer Effekt bei Eintritt der Supraleitfähigkeit. *Naturwissenschaften* **21**, 787 (1933)
52. A. Mooser et al., Demonstration of the double Penning trap technique with a single proton. *Phys. Lett. B* **723**, 78 (2013)

Chapter 22

Application of the Continuous Stern Gerlach Effect: Magnetic Moments



Abstract This chapter briefly reviews the measurements of magnetic moments that have been performed by application of the continuous Stern-Gerlach effect to a single particle confined in a Penning trap with a magnetic bottle.

22.1 Magnetic Moment Measurements

Penning traps that feature a magnetic bottle to make use of the continuous Stern-Gerlach effect allow to precisely measure the magnetic moments of confined individual particles such as the electron/positron, proton/anti-proton or the electron bound in highly charged ions. These measurements allow us to test predictions by quantum electrodynamics (QED) in the absence and in the presence of extreme electromagnetic fields and with high stringency. They also yield access to fundamental numbers like the fine structure constant and the electron mass. Comparisons of magnetic moments of particles and their anti-particles further represent tests of fundamental symmetries such as CPT invariance. These aspects have been discussed in detail in [1–4].

The magnetic moment is often expressed by the ‘g-factor’ which is the dimensionless proportionality constant between a particle’s total angular momentum $\mathbf{j}$ and its corresponding magnetic moment $\boldsymbol{\mu}$ such that

$$\boldsymbol{\mu} = g\mu_B \frac{\mathbf{j}}{\hbar}, \quad (22.1)$$

where μ_B is the Bohr magneton $\mu_B = e\hbar/(2m_e) \approx 9.274 \times 10^{-24}$ J/T.

The constant g is sometimes called ‘Landé g-factor’ after Alfred Landé [5] or simply ‘g-factor’. For a free (unbound) particle, the angular momentum $\mathbf{j}$ consists of the particle spin $\mathbf{s}$, while for a bound particle spin and orbital angular momentum $\mathbf{l}$ need to be added: $\mathbf{j} = \mathbf{l} + \mathbf{s}$. Thus, for free particles, g is often referred to as g_s , while for

bound particles, g is referred to as g_j [6]. For a classical (spin-less) particle we have $g = 1$. It is a main result of Dirac theory that spin-1/2-particles have exactly $g = 2$ [7]. Measurements by Kusch and Foley [8, 9] performed around the year 1947 found a deviation of g of the electron from exactly 2, and yielded $g = 2(1.00119 \pm 0.00005)$. This deviation was successfully explained by Schwinger with QED effects due to self-energy and vacuum polarisation to first order in α/π [10].

22.2 Electron and Positron Magnetic Moments

As stated in the history introduction, from about 1973 on Dehmelt et al. conceived and implemented concepts for the storage and cooling of single electrons in a Penning trap [11], including the quantum jump method that Dehmelt called ‘continuous Stern-Gerlach effect’ [12] and which is based on the field of a magnetic bottle in a Penning trap. This was used to measure the magnetic moment anomaly of the electron. We will discuss the measurement process below when we compare unbound and bound particles.

The anomalous magnetic moment a is the deviation of the true magnetic moment from the Dirac prediction $g = 2$ and is defined by

$$a = \frac{\mu}{\mu_B} - 1 = \frac{g - 2}{2}. \quad (22.2)$$

Measurements of the electron and positron magnetic moment anomalies were carried out in 1987 and have yielded the values [13]

$$a_{e^-} = 0.001\,159\,652\,188\,4(43) \quad \text{and} \quad a_{e^+} = 0.001\,159\,652\,187\,9(43). \quad (22.3)$$

The measurements yield an upper limit on the ratio of the electronic and positronic magnetic moment of [13]

$$\frac{g_{e^-}}{g_{e^+}} = 1 + (0.5 \pm 2.1) \times 10^{-12}. \quad (22.4)$$

This result serves as a test of CPT invariance for the electron-positron system. Assuming exact CPT invariance, the two values can be combined to give

$$g = 2.002\,319\,304\,376\,6(84) \quad \text{and} \quad a = 0.001\,159\,652\,188\,3(42). \quad (22.5)$$

For about two decades, this value of the magnetic moment was the most precise one available, until a new determination of the electron anomaly again using a single electron confined in a Penning trap has been carried out, giving the values [14]

$$g = 2.002\,319\,304\,361\,70\,(15) \quad \text{and} \quad a = 0.001\,159\,652\,180\,85(76), \quad (22.6)$$

which have an uncertainty that is nearly six times smaller than the previous measurement. In 2008, there has been an improved measurement using the same techniques [15] which is the most precise measurement to date. It reads

$$g = 2.002\,319\,304\,361\,460\,(56) \quad \text{and} \quad a = 0.001\,159\,652\,180\,73(28). \quad (22.7)$$

By use of the theory link between the electron magnetic moment and the fine structure constant α , this is also the most precise determination of α so far [15]. It has resulted in the value

$$\alpha^{-1} = 137.035\,999\,084\,(51), \quad (22.8)$$

which is in agreement with the best value

$$\alpha^{-1} = 137.035\,999\,037\,(91) \quad (22.9)$$

from an unrelated experiment based on the measurement of the recoil momentum of photons that are absorbed by atoms [16]. Together, these findings are commonly regarded as the most convincing corroborations of the validity of quantum electrodynamics theory in the absence of external fields.

22.3 Proton and Anti-proton Magnetic Moments

The magnetic moment of the proton has initially been measured by Winkler et al. in 1972 via the splitting and shift of the hyperfine levels in atomic hydrogen [17]. From this measurement, the ratio of the electron- to proton magnetic moment has been determined to be

$$\frac{\mu_e}{\mu_p} = 658.210\,705\,8\,(66). \quad (22.10)$$

The magnetic moment of the proton is measured in units of the nuclear magneton

$$\mu_N = \frac{e\hbar}{2m_p} = 5.050\,783\,24\,(13) \times 10^{-27} \text{ J/T} \quad (22.11)$$

which gives the value of the magnetic moment of a proton with angular momentum $|\mathbf{l}| = \hbar$. Correspondingly, it is smaller than the Bohr magneton by a factor m_p/m_e . The value found for the protonic g -factor is $g_p = 5.585\,694\,706\,(56)$. For the anti-proton, a measurement of the magnetic moment has been performed by Kreissl et al. in 1988 [18]. It measures the magnetic moment of the anti-proton in a scattering

experiment and is limited in accuracy to 3×10^{-3} for the resulting value of $g_{\bar{p}} = 5.601(18)$. A Stern-Gerlach type experiment with a single, stored proton or anti-proton contains the challenge that the magnetic moment is smaller than that of an electron or positron by a factor of about 658, see (22.10). This translates into a continuous Stern-Gerlach effect smaller in magnitude by that factor when compared to the electron case. In 1993, this frequency resolution was deemed experimentally too demanding and an alternative method ('zig-zag' method) was brought forward [19]. In 2004, the use of much stronger magnetic bottles and spatial separation of spin state analysis from precision frequency measurement ('double-trap technique') was suggested for proton and anti-proton measurements [20].

The high accuracy of such a measurement is meaningful not only in terms of a comparison with theory values, but for a comparison between the magnetic moments of the proton and the anti-proton. Such a comparison represents a highly sensitive test of the CPT theorem [1]. According to models, the sensitivity to CPT violation of such a test is significantly higher than reached in a comparison between the proton and anti-proton masses and higher than the sensitivity of a g -factor comparison between electron and positron [2].

In single-particle measurements in a Penning trap, the Larmor frequency of the proton has been measured with a relative uncertainty of 1.8×10^{-6} and the magnetic moment has been determined with a relative uncertainty of 8.9×10^{-6} . A g -factor of $g_p = 5.585696(50)$ has been obtained, corresponding to [21]

$$\frac{\mu_p}{\mu_N} = 2.792\,849\,(25). \quad (22.12)$$

Using the same technique in a separate experiment, the value [22]

$$\frac{\mu_p}{\mu_N} = 2.792\,846\,(7) \quad (22.13)$$

was obtained. Employing the double-trap technique as discussed in Sect. 9.3, the value was improved to [23, 24]

$$\frac{\mu_p}{\mu_N} = 2.792\,847\,350\,(9) \quad \text{and later} \quad \frac{\mu_p}{\mu_N} = 2.792\,847\,344\,62\,(82). \quad (22.14)$$

In 2013, the first one-particle measurement of the magnetic moment (in units of the nuclear magneton) of the anti-proton in a Penning trap has been reported [25], with the result

$$\frac{\mu_{\bar{p}}}{\mu_N} = 2.792\,845\,(12). \quad (22.15)$$

This measurement, however, was not based on resolved individual spin flips by the continuous Stern-Gerlach effect. In 2017, this has been achieved and led to the value [26]

$$\frac{\mu_{\bar{p}}}{\mu_N} = 2.792\,847\,344\,1\,(42). \quad (22.16)$$

22.4 Bound Electron Magnetic Moment

22.4.1 *Applicability of Stern-Gerlach Experiments*

Before the year 1973, there had been an ongoing debate as to whether the Stern-Gerlach effect [27, 28] is limited to measurements with neutral atoms, since it was unclear if the Lorentz force acting on a moving charged particle in a magnetic field would mask the force due to the magnetic moment. It had been proposed to separate the spin directions by the acceleration of electrons when moving along the field lines of an inhomogeneous magnetic field [29]. This idea, however, was rejected both by Bohr [30] and Pauli [31] who argued with the Heisenberg uncertainty principle. Attempts to separate the spin states in an electron beam by use of the different sign of the force on the spin as produced by a longitudinal inhomogeneous magnetic field have been unsuccessful [32]. New proposals have been brought forward to perform Stern-Gerlach experiments on electron beams which would result in a high degree of spin separation under specific initial conditions [33–35]. A detailed account of the debate is given in [36].

The requirements for a measurement of the electron magnetic moment in a constant homogeneous magnetic field by inducing transitions between particular states of the electron's motion with a perturbing electric field were investigated by Byrne in 1963. He first showed the necessity of 'some form of trapping device' that provides a sufficiently long time period for the transitions to be detected [37].

Then in 1973, Dehmelt and Eckstrom finally proposed to use the force of an inhomogeneous magnetic field on the spin of an electron confined in a Penning trap [9]. They found that confining a charged particle by electromagnetic fields provides a way to circumvent Bohr's and Pauli's argument since for a particle which oscillates in a parabolic potential well, the presence of a magnetic inhomogeneity leads to a measurable difference of its oscillation frequency for different orientations of the spin [38]. Therefore, a precise measurement of the oscillation frequency yields information on the spin direction. Dehmelt et al. used this effect for the detection of induced changes of the spin direction of an electron by observing the corresponding changes in the electron's oscillation frequency in a Penning trap. Since the confined particle's spin direction is monitored continuously, Dehmelt called this the 'continuous Stern-Gerlach effect'. It has later been applied to single confined protons [39], anti-protons [26], and to electrons bound in highly charged ions [40].

22.4.2 *Difference Between the Unbound and the Bound Electron*

For the unbound electron, there is no coupling to the field of an ionic nucleus, and the theory value of the g -factor can be written as [41]

$$g_f = 2 \left[1 + K_2 \cdot \left(\frac{\alpha}{\pi} \right) + K_4 \cdot \left(\frac{\alpha}{\pi} \right)^2 + K_6 \cdot \left(\frac{\alpha}{\pi} \right)^3 + \dots \right], \quad (22.17)$$

where the first three coefficients

$$\begin{aligned} K_2 &= 0.5, \\ K_4 &= 0.328\,478\,965\,579\,193, \\ K_6 &= 1.181\,241\,456 \end{aligned}$$

have been calculated analytically, while

$$\begin{aligned} K_8 &= -1.912\,98\,(84), \\ K_{10} &= 7.795\,(336) \end{aligned}$$

are known from numerical investigations [41, 42]. There is a newer, semi-analytical value of $K_8 = -1.912\,245\,764\,9\dots$ [43] from a computation using a different set of methods [44]. Expression (22.17) is an expansion in the fine structure constant $\alpha \approx 1/137$ that converges sufficiently fast due to $\alpha \ll 1$. It establishes a very well-defined relation between g_f and α that can be used for stringent tests of free-QED calculations when α is measured separately, like for example in [16]. In turn, when the relation is taken for granted, a measurement of g_f represents a measurement of α [15].

For the bound electron the situation is different. In atoms and ions, binding effects significantly contribute to the value of the electronic g -factor. Especially in hydrogen-like ions, the bound electron experiences extremely strong electric and magnetic fields which influence the properties of the electron. These fields are by far the strongest available to experimental physics and exceed 10^{18} V/m and 10^5 T for the heaviest systems [45]. Formally, a similar expansion as in the free-electron case can be used, however it is now an expansion in both α and $Z\alpha$:

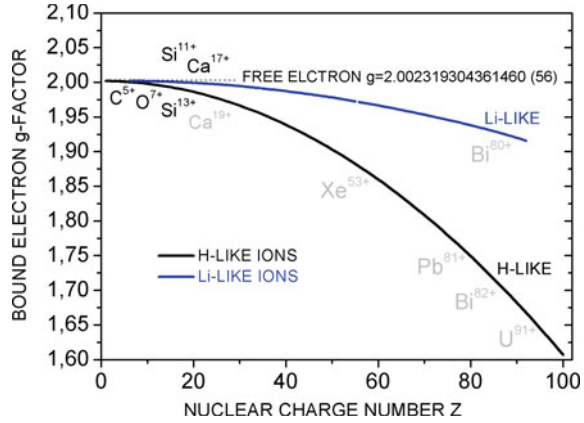
$$g_b = 2 \left[K_0(Z\alpha) + K_2(Z\alpha) \cdot \left(\frac{\alpha}{\pi} \right) + K_4(Z\alpha) \cdot \left(\frac{\alpha}{\pi} \right)^2 + K_6(Z\alpha) \cdot \left(\frac{\alpha}{\pi} \right)^3 + \dots \right] \quad (22.18)$$

since the coefficients K_n now depend on the nuclear charge Z and are expressed with $Z\alpha$ as the expansion parameter. In contrast to the free electron, the expansion parameter $Z\alpha$ is, at least for large Z , no longer small compared to 1 and the series converges much less rapidly. Thus, non-perturbative methods need to be applied [46]. An overview of the existing calculations of expansion coefficients and their results is given in [17], see also Fig. 22.1.

The main contribution to the anomaly of the g -factor of a bound electron comes from relativistic effects. In 1928, Breit found an analytical solution for the g -factor of the electron bound in the ground state of hydrogen-like systems with nuclear charge Z in the form [47]

$$g_b = \frac{2}{3} \left(1 + 2\sqrt{1 - (Z\alpha)^2} \right), \quad (22.19)$$

Fig. 22.1 Magnetic moment (g -factor) of an electron bound in hydrogen-like and lithium-like ions as a function of nuclear charge number Z . The solid lines represent the theoretical prediction. The ions with charge states below 19 have been measured so far, the others are selected candidate systems



which can be expanded in the parameter $Z\alpha$ as

$$g_b = 2 \left[1 - \frac{1}{3}(Z\alpha)^2 - \frac{1}{12}(Z\alpha)^4 - \frac{1}{24}(Z\alpha)^6 \dots \right]. \quad (22.20)$$

For the electron bound in a hydrogen-like ion, the ratio of the bound electron g -factor g_b to the g -factor g_f of the free electron can thus be written as

$$\frac{g_b}{g_f} = 1 - \frac{(Z\alpha)^2}{3} + \frac{\alpha(Z\alpha)^2}{4\pi} + O(Z\alpha)^4, \quad (22.21)$$

from which it is obvious that the deviation from the free case is significant in particular for high Z , i.e. when the electron is bound in a heavy, hydrogen-like ion. It is then subjected to relativistic effects due to its motion in the strong binding, to effects of bound-state quantum electrodynamics (BS-QED) in the presence of the strong electromagnetic fields in the vicinity of the nucleus [45], and becomes sensitive to nuclear properties [48] as the tight binding localises the electron closer to the nucleus by a factor of Z [49]. The effects of bound-state QED scale roughly like Z^2 such that heavy systems display the largest effects and are promising candidates for tests of corresponding calculations [3, 50].

The bound electron in a hydrogen-like ion is also a promising system for a determination of α from g_b due to [51]

$$\frac{\delta\alpha}{\alpha} \approx \frac{1}{(Z\alpha)^2} \frac{\delta g}{g}, \quad (22.22)$$

but so far the relation between them is less well-known than in the free-electron case. However, measurements of g_b can be used for a determination of the electron's atomic mass, which is in particular interesting in the case of the ion $^{12}\text{C}^{5+}$ since ^{12}C is an atomic mass standard. The currently best value of m_e in atomic units has

been derived from such a g -factor measurement with a single ion in a Penning trap [52, 53].

The measurement routines and details have been given in the references listed in the table. Briefly, a single ion is stored in a Penning trap with an artificial inhomogeneity B_2 of the magnetic field (‘magnetic bottle’). As we have seen in (21.9), the oscillation frequencies of the ion depend on the spin state of the electron, such that an ion oscillation frequency measurement, for example of the axial frequency, tells apart the different spin states. In this type of experiment, the Larmor frequency is a microwave frequency, and the measurement performed is one of the spin-flip transition probability determined in a repeated axial frequency measurement as a function of the irradiated microwave frequency. The maximum of this curve indicates the Larmor frequency which yields the magnetic moment. The exact lineshape of the F -resonance depends on many experimental parameters and is subject to a multitude of systematic effects, as has been discussed for example in [57], see also Sect. 19.4.

Table 22.1 compiles the values of magnetic moments obtained from such experiments so far. Note that the values in the middle section of Table 22.1 are not fully independent due to their common experimental outline. For the measurements with $^{12}\text{C}^{5+}$ [56] and $^{16}\text{O}^{7+}$ [58], for example, a correlation coefficient of 0.035 has been given [65], see also the discussion in [41]. One further measurement of this kind with the $^{12}\text{C}^{5+}$ ion has been performed, and was used to derive a value for the atomic mass of the electron [52], hence, it does not appear in the above table since correspondingly, no electron magnetic moment was derived from it.

Table 22.1 Measured values of the electron magnetic moment. Apart from the case of the free electron, the middle section lists all values obtained by application of the continuous Stern-Gerlach effect. The measurement [52] is not listed, as it was used to derive the atomic mass of the electron

Z	System	Magnetic moment (g_J)	References
0	Free electron	2.002 319 304 361 460 (56)	[15]
1	Hydrogen ^1H	2.002 283 853 (26)	[54]
4	Helium $^4\text{He}^+$	2.002 259 33 (60)	[55]
6	Carbon $^{12}\text{C}^{5+}$	2.001 042 (2)	[40]
6	Carbon $^{12}\text{C}^{5+}$	2.001 041 596 3 (10) (44)	[56, 57]
8	Oxygen $^{16}\text{O}^{7+}$	2.000 047 026 0 (15) (44)	[58]
8	Oxygen $^{16}\text{O}^{7+}$	2.000 047 020 8 (24) (44)	[59]
14	Silicon $^{28}\text{Si}^{13+}$	1.995 348 958 77 (50)(30)(80)	[60, 63]
14	Silicon $^{28}\text{Si}^{11+}$	2.000 889 889 9 (19)(5)(8)	[61]
20	Calcium $^{40}\text{Ca}^{17+}$	1.999 202 040 55 (10)(12)(110)	[62]
20	Calcium $^{48}\text{Ca}^{17+}$	1.999 202 028 85 (12)(13)(80)	[62]
82	Lead $^{207}\text{Pb}^{81+}$	1.78 (12)	[45, 64]
83	Bismuth $^{209}\text{Bi}^{82+}$	1.734 1 (35)	[45, 64]

References

1. R. Bluhm, V.A. Kostelecky, N. Russell, Testing CPT with anomalous magnetic moments. *Phys. Rev. Lett.* **79**, 1432 (1997)
2. R. Bluhm, V.A. Kostelecky, N. Russell, CPT and Lorentz tests in Penning traps. *Phys. Rev. D* **57**, 3922 (1998)
3. S. Sturm et al., Experiments for high-precision measurements of the bound electron's magnetic moment. *Atoms* **5**, 4 (2017)
4. M. Vogel, W. Quint (eds.) *Fundamental Physics in Particle Traps*, Springer Tracts in Modern Physics, vol. 256 (Springer, 2014)
5. A. Landé, Termstruktur und Zeemaneffekt der multipletts. *Zeitschrift für Phys.* **15**, 189 (1923)
6. M. Vogel, The anomalous magnetic moment of the electron. *Contemp. Phys.* **50**, 437 (2009)
7. P.A.M. Dirac, The quantum theory of the electron. *Proc. R. Soc. A: Math. Phys. Eng. Sci.* **117**, 610 (1928)
8. P. Kusch, H.M. Foley, Precision measurement of the ratio of the atomic g values in the $^2P_{3/2}$ and $^2P_{1/2}$ states of Gallium. *Phys. Rev.* **72**, 1256 (1947)
9. P. Kusch, H.M. Foley, On the intrinsic moment of the electron. *Phys. Rev.* **74**, 250 (1948)
10. J. Schwinger, On quantum-electrodynamics and the magnetic moment of the electron. *Phys. Rev.* **73**, 416 (1948)
11. H. Dehmelt, P. Ekstrom, Proposed $g - 2$ experiment on stored single electron or positron. *Bull. Am. Phys. Soc.* **18**, 727 (1973)
12. H. Dehmelt, Continuous Stern Gerlach Effect: principle and idealized apparatus. *Proc. Natl. Acad. Sci. USA* **83**, 2291 (1986) and **83**, 3074 (1986)
13. R.S. van Dyck, P.B. Schwinberg, H.G. Dehmelt, New high-precision comparison of electron and positron g factors. *Phys. Rev. Lett.* **59**, 26 (1987)
14. G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, B. Odom, New determination of the fine structure constant from the electron g value and QED. *Phys. Rev. Lett.* **97**, 030802 (2006); Erratum ibidem **99**, 039902 (2007)
15. D. Hanneke, S. Fogwell, G. Gabrielse, New measurement of the electron magnetic moment and the fine structure constant. *Phys. Rev. Lett.* **100**, 120801 (2008)
16. R. Bouchendira et al., New determination of the fine structure constant and test of the quantum electrodynamics. *Phys. Rev. Lett.* **106**, 080801 (2011)
17. P.F. Winkler, D. Kleppner, T. Myint, F.G. Walther, Magnetic moment of the proton in Bohr magnetons. *Phys. Rev. A* **5**, 83 (1972)
18. X. Kreissl et al., Remeasurement of the magnetic moment of the antiproton. *Z. Phys. C* **37**, 557 (1988)
19. W. Quint, G. Gabrielse, The magnetic moment of the antiproton. *Hyp. Int.* **76**, 379 (1993)
20. W. Quint et al., Continuous SternGerlach effect and the magnetic moment of the antiproton. *Nucl. Inst. Meth. B* **214**, 207 (2004)
21. C.C. Rodegheri et al., An experiment for the direct determination of the g -factor of a single proton in a Penning trap. *New J. Phys.* **14**, 063011 (2012)
22. N. Guise, J. DiSciaccia, G. Gabrielse, Self-excitation and feedback cooling of an isolated proton. *Phys. Rev. Lett.* **104**, 143001 (2010)
23. A. Mooser et al., Direct high-precision measurement of the magnetic moment of the proton. *Nature* **509**, 596 (2014)
24. G. Schneider et al., Double-trap measurement of the proton magnetic moment at 0.3 parts per billion precision. *Science* **358**, 1081 (2017)
25. J. DiSciaccia, M. Marshall, K. Marable, G. Gabrielse, One-particle measurement of the antiproton magnetic moment. *Phys. Rev. Lett.* **110**, 130801 (2013)
26. C. Smorra et al., A parts-per-billion measurement of the antiproton magnetic moment. *Nature* **550**, 371 (2017)
27. O. Stern, W. Gerlach, Das magnetische Moment des Silberatoms (The magnetic moment of the silver atom). *Z. Phys.* **9**, 349 (1922) and *Z. Phys.* **9**, 353 (1922)

28. W. Gerlach, O. Stern, Über die Richtungsquantelung im magnetfeld. *Ann. Phys.* **379**, 673 (1924)
29. L. Brillouin, Is it possible to test by a direct experiment the hypothesis of the spinning electron? *Proc. Natl. Acad. Sci. USA* **14**, 755 (1928)
30. N. Bohr, in *Collected Works of Niels Bohr*, ed. by J. Kalckar, vol 6 (North-Holland, Amsterdam, 1996)
31. W. Pauli, in *Handbuch der Physik, Band 5: Prinzipien der Quantentheorien*, ed. by S. Flügge (Springer, Berlin, 1958), p. 167
32. F. Bloch, Experiments on the g -factor of the electron. *Physica* **19**, 821 (1953)
33. H. Batelaan, T.J. Gay, J.J. Schwendiman, Stern-Gerlach Effect for electron beams. *Phys. Rev. Lett.* **79**, 4517 (1997)
34. B.M. Garraway, S. Stenholm, Observing the spin of a free electron. *Phys. Rev. A* **60**, 63 (1999)
35. G.A. Gallup, H. Batelaan, T.J. Gay, Quantum-mechanical analysis of a longitudinal Stern-Gerlach Effect. *Phys. Rev. Lett.* **86**, 4508 (2001)
36. B.M. Garraway, S. Stenholm, Does a flying electron spin? *Contemp. Phys.* **43**, 147 (2002)
37. J. Byrne, Study of a proposal for determining the g -factor anomaly for electrons by resonance excitation in a magnetic field. *Can. J. Phys.* **41**, 1571 (1963)
38. H.G. Dehmelt, New continuous Stern-Gerlach effect and a hint of “the” elementary particle. *Z. Phys. D* **10**, 127 (1988)
39. A. Mooser et al., Resolution of single spin flips of a single proton. *Phys. Rev. Lett.* **110**, 140405 (2013)
40. N. Hermanspahn et al., Observation of the continuous Stern-Gerlach Effect on an electron bound in an atomic ion. *Phys. Rev. Lett.* **84**, 427 (2000)
41. P.J. Mohr, D.B. Newell, B.N. Taylor, CODATA recommended values of the fundamental physical constants: 2014. *Rev. Mod. Phys.* **88**, 035009 (2016)
42. T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Tenth-order electron anomalous magnetic moment: contribution of diagrams without closed lepton loops. *Phys. Rev. D* **91**, 033006 (2015)
43. S. Laporta, High-precision calculation of the 4-loop contribution to the electron $g-2$ in QED. *Phys. Lett. B* **772**, 232 (2017)
44. S. Volkov, New method of computing the contributions of graphs without lepton loops to the electron anomalous magnetic moment in QED. *Phys. Rev. D* **96**, 096018 (2017)
45. T. Beier, The g_J -factor of a bound electron and the hyperfine structure splitting in hydrogenlike ions. *Phys. Rep.* **339**, 79 (2000)
46. A.V. Volotka, D.A. Glazov, G. Plunien, V.M. Shabaev, Progress in quantum electrodynamics theory of highly charged ions. *Ann. Phys.* **525**, 636 (2013)
47. G. Breit, The magnetic moment of the electron. *Nature* **122**, 649 (1928)
48. V.M. Shabaev, V.A. Yerokhin, Recoil correction to the bound-electron g -factor in H-like atoms to all orders in αZ . *Phys. Rev. Lett.* **88**, 091801 (2002)
49. M. Vogel, W. Quint, Aspects of fundamental physics in precision spectroscopy of highly charged ions in Penning Traps. *Ann. Phys.* **525**, 505 (2013)
50. S. Sturm et al., g -factor measurement of hydrogen-like $^{28}\text{Si}^{13+}$ as a challenge to QED calculations. *Phys. Rev. A* **87**, 030501(R) (2013)
51. V.M. Shabaev, D.A. Glazov, N.S. Oreshkina, A.V. Volotka, G. Plunien, H.J. Kluge, W. Quint, g -factor of heavy ions: a new access to the fine structure constant. *Phys. Rev. Lett.* **96**, 253002 (2006)
52. S. Sturm et al., High-precision measurement of the atomic mass of the electron. *Nature* **506**, 467 (2014)
53. F. Köhler et al., The electron mass from g -factor measurements on hydrogen-like carbon $^{12}\text{C}^{5+}$. *J. Phys. B* **48** (2015)
54. J.S. Tiedeman, H.G. Robinson, Determination of $g_J(^1\text{H}, ^2S_{1/2})/g_s(e)$: test of mass-independent corrections. *Phys. Rev. Lett.* **39**, 602 (1977)
55. C.E. Johnson, H.G. Robinson, g_J factor of an ion: determination of $g_J(^4\text{He}^+, ^2S_{1/2})/g_J(^4\text{He}, ^2^3S_1)$. *Phys. Rev. Lett.* **45**, 250 (1980)

56. H. Häffner et al., High-accuracy measurement of the magnetic moment anomaly of the electron bound in Hydrogen-like Carbon. *Phys. Rev. Lett.* **85**, 5308 (2000)
57. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
58. J. Verdú et al., Electronic g factor of Hydrogen-like Oxygen $^{16}\text{O}^{7+}$. *Phys. Rev. Lett.* **92**, 093002 (2004)
59. J. Verdú et al., Determination of the g -factor of single Hydrogen-like ions by mode coupling in a Penning Trap. *Phys. Scr.* **T112**, 68 (2004)
60. S. Sturm et al., g factor of hydrogen-like $^{28}\text{Si}^{13+}$. *Phys. Rev. Lett.* **107**, 023002 (2011)
61. A. Wagner et al., g factor of Lithium-like Silicon $^{28}\text{Si}^{11+}$. *Phys. Rev. Lett.* **110**, 033003 (2013)
62. F. Köhler et al., Isotope dependence of the Zeeman effect in lithium-like calcium. *Nat. Comm.* **7**, 10246 (2016)
63. B. Schabinger et al., Experimental g factor of hydrogenlike silicon-28. *Eur. Phys. J. D* **66**, 71 (2012)
64. V.M. Shabaev, Transition probability between the hyperfine structure components of hydrogenlike ions and bound-electron g -factor. *Can. J. Phys.* **76**, 907 (1998)
65. G.W.F. Drake (ed.), *Handbook of Atomic, Molecular and Optical Physics* (Springer, Heidelberg, 2006)

Appendix A

Early History of Charged-Particle Traps

Abstract Today, we see a broad variety of particle traps and related devices for confinement and study of charged particles in various fields of science and industry (Werth, *Charged Particle Traps*, Springer, Heidelberg; Werth, *Charged Particle Traps II*, Springer, Heidelberg; Ghosh, *Ion Traps*, Oxford University Press, Oxford; Stafford, *Am Soc Mass Spectrom.* 13(6):589). The multitude of realisations and applications of electromagnetic confinement cannot be easily overlooked as the variability of particle confinement and the vast number of experimental techniques available has allowed traps and related instrumentation to enter many different fields in physics and chemistry as well as in industrial processes and applied analysis, mainly in the form of mass spectrometry (Stafford, *Am Soc Mass Spectrom.* 13(6):589; March and Todd, *Practical Aspects of Ion Trap Mass Spectrometry*, CRC Press, Boca Raton). Here, we would like to give a brief account of the main historical aspects of charged-particle confinement, of the principles, technical foundations and beginnings that led to the realisation of traps and to the 1989 Nobel Prize in Physics awarded to Hans Georg Dehmelt and Wolfgang Paul ‘for the invention of the ion trap technique’, at which point we choose to end our discussion. Mainly, we will look at the work of the people who pioneered the field and paved the way to the first working traps. These can be seen as belonging to two different families of traps, Penning traps and Paul traps, each of these families nowadays with numerous variations of the respective confinement concepts. Here, however, we just have a brief look at the developmental history of the trap concepts.

A.1 Motion of Charges in Electric and Magnetic Fields

Since traps for charged particles work on account of electric and magnetic fields, we briefly summarize the corresponding background before which the first steps towards particle confinement and traps have been made.

By the end of the 19th century, the effects of electric and magnetic fields on the motion of charges had been expounded in a modern sense. James Clerk Maxwell’s

(1831–1879) [6] famous twenty equations, in their modern form of four partial differential equations, appeared in ‘A Treatise on Electricity and Magnetism’ in 1873 [7] following a less compact presentation in ‘A Dynamical Theory of the Electromagnetic Field’ in 1865 [8]. Already before that, he had established a field-line picture in his four-article series ‘On physical lines of force’ in 1861 [9]. According to this picture, electric test charges follow electric fields lines that are created by charges or time-varying magnetic fields. In modern notation, this means we have a force $\mathbf{F}_E$ acting on a test particle with electric charge q given by

$$\mathbf{F}_E = q\mathbf{E}, \quad (\text{A.1})$$

when $\mathbf{E}$ is the vector describing the strength and direction of the electric field. The electric fields created by charges are given by Gauss’s law, published in 1813 by Carl Friedrich Gauss (1777–1885) [10], which became part of Maxwell’s equations (commonly the first one). The electric fields created by time-varying magnetic fields are given by Faraday’s law of induction, introduced by Michael Faraday (1791–1867) in 1831/32 [11, 12], which also became one of Maxwell’s laws, usually the third one.

The motion of test charges in a magnetic field is governed by the Lorentz force, named after Hendrik Antoon Lorentz (1853–1928, Nobel Prize in 1902 together with Pieter Zeeman) [13]. It is a force perpendicular to the particle motion and to the magnetic field lines, such that in principle, a charged particle can perform a closed circular motion in a plane perpendicular to the axis of a homogeneous magnetic field, a cyclotron motion. The Lorentz force $\mathbf{F}_B$ is given by

$$\mathbf{F}_B = q(\mathbf{v} \times \mathbf{B}), \quad (\text{A.2})$$

when q again is the electric charge of the particle, $\mathbf{v}$ is its velocity and $\mathbf{B}$ is the magnetic field. The first derivation of the Lorentz force on a charged particle in a magnetic field is commonly attributed to Oliver Heaviside (1850–1925) in 1889 [14] although historians argue about an earlier origin in a paper by James Clerk Maxwell in 1865 [15]. In any case, Hendrik Lorentz derived it independently in 1895, a few years after Heaviside [16], and he became the namesake.

A.2 Samuel Earnshaw and His Theorem

Already before the work of Maxwell, British clergyman and mathematician Samuel Earnshaw (1805–1888) [17] in his text ‘On the nature of the molecular forces which regulate the constitution of the luminiferous ether’ [18] had proved a theorem from which it follows that it is not possible to create a static electromagnetic field that confines a test charge in all three dimensions. The proof argues with Laplace’s equation (after Pierre-Simon Laplace, 1749–1827), given by (Fig. A.1)

$$\Delta\Phi = \nabla \cdot \mathbf{E} = 0, \quad (\text{A.3})$$

$$2(C - C') = d_x^2 V \cdot x^2 + d_y^2 V \cdot y^2 + d_z^2 V \cdot z^2,$$

which for indefinitely small values of x , y , z may be regarded as the equation of the parametric surface. It must be remembered that the coefficients of x^2 , y^2 , z^2 are subject to the condition,

$$0 = d_x^2 V + d_y^2 V + d_z^2 V;$$

and because at least one of these coefficients will be negative, and one positive, the equation is that of an hyperboloid.

Fig. A.1 Part of Earnshaw's 1842 statement from 'On the nature of the molecular forces which regulate the constitution of the luminiferous ether' [18]

which states that the divergence $\nabla \cdot \mathbf{E}$ of the electric field in absence of charges is zero, and so is the second spatial derivative $\Delta \Phi$ of the corresponding electric potential Φ . Confining a charge in all spatial dimensions (x , y , z) requires a restoring force for any dislocation from a central point in space, and hence requires the potential Φ to have a minimum there. For simplicity (and without loss of generality) looking at a linear restoring force, the corresponding potential has the harmonic form

$$\Phi \propto (ax^2 + by^2 + cz^2) \quad (\text{A.4})$$

with real-valued positive coefficients a , b , c . Inserting this potential into Laplace's equation directly leads to the relation $a + b + c = 0$, which has obviously no solution for positive a , b and c . This means that at least one coefficient must be negative, implying a repulsive force in this direction. Hence, the potential has the shape of a saddle, a two-dimensional surface with both positive and negative curvatures embedded in three-dimensional space.

This destroys any dream of a trap for charged particles that uses a single static field for stable confinement. Obviously, there are two possible solutions:

1. Use more than one static field. This is the way to the Penning trap which employs a static electric field and a static magnetic field.
2. Use a static and a non-static field. This is the way to the Paul trap which employs a time-varying electric field that consists of a constant and a time-dependent component.

The solution can also be more complicated than this, like for example in case of the combined trap which uses all of the above fields. Another way to approach the problem is not to go for stable confinement with a static equilibrium of charged particles in the trap potential, but to implement dynamical confinement which puts up requirements for the particle motion. One example of such a trap is the Kingdon trap.

A.3 Kenneth Hay Kingdon and Electric ‘Imprisonment’

One case of dynamical confinement is the nowadays so-called ‘Kingdon trap’, sometimes ‘Kingdon cage’, named after Kenneth Hay Kingdon (1894–1982), who had studied at the University of Toronto and worked mainly on the topic of electric discharges for the General Electric company in Schenectady, New York [19]. In his 1923 publication ‘A Method for the Neutralization of Electron Space Charge by Positive Ionization at Very Low Gas Pressures’ he writes [20]

‘If a very small filament, diameter 0.01 cm, is run axially through a cylindrical anode with closed ends, positive ions formed between the electrodes can only rarely escape and will describe orbits around the filament until they lose sufficient energy by collision with gas molecules to enable them to fall into the cathode. The imprisoned ions, during their lives, neutralize a certain amount of the space charge between the electrodes.’

Kingdon does not use the word ‘trap’ in his publication, rather the first term used for some localisation of charged particles was ‘imprisonment’. He was not interested in the localisation of the ions, but to find an efficient way to counteract space charge created by the electrons in discharge devices. Hence, he did not study motional frequencies and trajectories in detail.

As described by Kingdon, the device is a hollow cylinder (A in Fig. A.2) with a thin wire (C in Fig. A.2) along the central axis and a pair of flat caps (G in Fig. A.2) that close the cylinder. There are different voltages set to each of these three components (cylinder, wire and caps). The values of these voltages can be chosen such that particles with a finite angular momentum with respect to the central wire experience a restoring force in the radial plane, hence dynamically stabilising the radial motion. At the same time, the voltages set to the caps can act repulsive on the particles, such that they provide axial confinement. Overall, the particle trajectory is then limited to a certain part of the cylindrical device, however confinement is restricted to a certain range of dynamical parameters.

A METHOD FOR THE NEUTRALIZATION OF ELECTRON SPACE CHARGE BY POSITIVE IONIZATION AT VERY LOW GAS PRESSURES.

BY K. H. KINGDON.

ABSTRACT.

Neutralization of space charge around a hot filament by imprisoned positive ions in gas at very low pressures.—(1) Design of tube. If a very small filament, diameter 0.01 cm, is run axially through a cylindrical anode with closed ends, positive ions formed between the electrodes can only rarely escape and will describe orbits around the filament until they lose sufficient energy by collision with gas molecules to enable them to fall into the cathode. The imprisoned ions, during their lives, neutralize a certain amount of the space charge between the electrodes. The effect should increase with the absolute

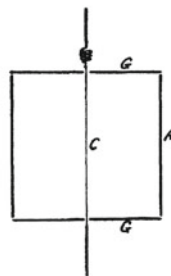


Fig. 1. Design of experimental tube.

Fig. A.2 Title and abstract from ‘A Method for the Neutralization of Electron Space Charge by Positive Ionization at Very Low Gas Pressures’ [20] from 1923 and the setup presented therein

The potential close to the central wire is logarithmic, but more complicated for larger orbits and closer to the caps. The overall spatial distribution of the resulting potential is difficult to calculate and there is no general analytic solution. The motion has been calculated, however, under certain simplifying assumptions [21, 22].

Initial ‘imprisonment’ times were severely limited, as they were not seen as a virtue by themselves. Kingdon achieved imprisonment for several hundreds of orbits which corresponds to periods of a few milli-seconds. Rather, this kind of device was widely used as a detector in the 1960s, for example for molecular beams [23], or as the basis of a vacuum pump [24]. In more recent experiments in the 1980s, confinement times of up to several tens of milli-seconds have been achieved [25]. In the 1990s and 2000s, more refined designs like the dynamic Kingdon trap [20] or the so-called ‘orbitrap’ [27, 28], have been brought forward, however Kingdon traps in their original configuration have also been used for modern measurements such as lifetime measurements [29] and spectroscopy of highly charged ions [30].

Electrostatic devices of various other geometries are also in use, based on static electric fields that create complicated escape paths for charged particles. This situation is neither meant to represent stable nor dynamical confinement, but rather a sophisticated deflection to the end of forcing the particles to cover a long distance within a defined volume. Such devices are typically used in mass spectrometry [31–34].

A.4 Frans Michel Penning and Paths in a Magnetic Field

Frans Michel Penning (1894–1953) had studied at the University of Leiden and graduated in 1923 with a thesis on ‘Measurements on Isometric Density Lines of Gases at Low Temperatures’ in the group of Heike Kamerlingh Onnes (1853–1926, Nobel Prize in 1913) before he began working on low-pressure gas discharges at the Philips Laboratory in Eindhoven.

At the time, vacuum pumps were increasing in their capability of achieving ever lower pressures, and there was a need for a reliable measurement at such low pressures. Pirani gauges (after Marcello Pirani, 1880–1968, who headed the Siemens & Halske research labs in Berlin from 1904 to 1919), were robust and readily available, but lacked low-pressure capabilities below 10^{-4} mbar.

Penning used the electric discharge current at high voltage as a measure for the pressure in a vacuum tube, but with little success at low pressures. Vacuum pioneer Wolfgang Gaede (1878–1945) had earlier indicated that a magnetic field can increase the length of electron trajectories in a vacuum, and Penning made according changes to the geometry of his vacuum tube by using an annular cathode with an anode plates on either side in a magnetic field. Hence, electrons leaving the annulus travelled between the electrodes in circular orbits instead of a straight line. This significantly increased the ionisation yield and resulted in a large amplification of the measured current [35, 36]. He published this work in two articles ‘Die Glimmentladung bei niedrigem Druck zwischen coaxialen Zylindern in einem axialen Magnetfeld’ (1936)

DIE GLIMMENTLADUNG BEI NIEDRIGEM DRUCK ZWISCHEN KOAXIALEN ZYLINDERN IN EINEM AXIALEN MAGNETFELD

von F. M. PENNING

Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken Eindhoven-Holland

Summary

Experiments are described concerning the starting potential and the characteristic of the glow discharge between coaxial cylinders in an axial magnetic field at low gas pressures. With a magnetic field H of sufficient strength electrons, leaving the cathode, describe a cycloidal path of considerable length before reaching the anode. So the magnetic

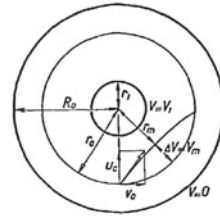


Fig. 2. Bezeichnungen zur Beschreibung der Elektronenbewegung im Magnetfeld.

Fig. A.3 Title and abstract from ‘Die Glimmentladung bei niedrigem Druck zwischen koaxialen Zylindern in einem axialen Magnetfeld’ [Glow discharge at low pressures between co-axial cylinders in an axial magnetic field] and the depicted use of an axial magnetic field [35]

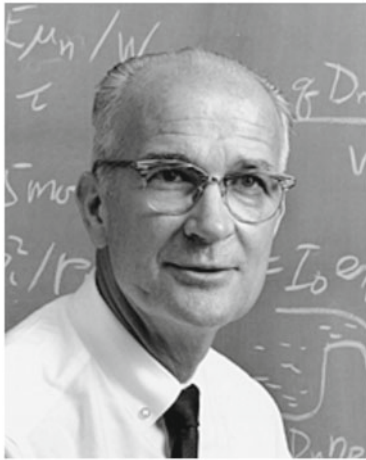
[35] (Fig. A.3) and ‘Ein neues Manometer für niedrige Gasdrücke, insbesondere zwischen 10^{-3} und 10^{-5} mm’ (1937) [36].

‘With a magnetic field H of sufficient strength electrons, leaving the cathode, describe a cycloidal path of considerable length before reaching the anode [...]. If there is a sufficient number of gas molecules in the chamber, an electron can collide with these molecules. If it loses energy in these collisions the return to the cathode is impossible and the electron will describe a significantly longer path before eventually impinging onto the anode...’

Penning was possibly also inspired to this use of a magnetic field by the contemporary work on and with cyclotrons. These are accelerator devices which use a homogeneous magnetic field to force electrically accelerated particles on a spiral trajectory, hence allowing a compact form still with a long acceleration section. This kind of device had been conceived in the 1920s in Europe, and first built and put into operation by Ernest Lawrence (1901–1957, Nobel Prize in Physics for this work in 1939) at the University of California in Berkeley [37]. Lawrence published the well-received article ‘The Production of High Speed Light Ions Without the Use of High Voltages’ in 1932 [38]. Cyclotrons and the particle and nuclear physics possible with them were a topic of the times around which Penning worked on his gauges and may hence well have been an inspiration. The above-mentioned circular orbit of a charged particle in a magnetic field on account of the Lorentz force is named ‘cyclotron motion’ after the cyclotron device, as is the ‘cyclotron frequency’.

A.5 William Bradford Shockley and Image Currents

William Bradford Shockley Jr. (1910–1989) earned a Ph.D. with work on ‘Electronic Bands in Sodium Chloride’ from MIT in 1936. He joined Bell Labs in New Jersey and later headed a research group that included John Bardeen and Walter Brattain, together with whom he was awarded the 1956 Nobel Prize in Physics for ‘researches on semiconductors and their discovery of the transistor effect’ [39]. Shockley had a very productive phase upon joining Bell Labs, part of which is his 1938 work on ‘Currents to Conductors Induced by a Moving Point Charge’ [40]. Together with



We are thus led to consider the charges and currents induced on a system of grounded stationary conductors by the motion of a point charge. If we have a system of grounded conductors, perhaps as illustrated in Fig. 1, num-

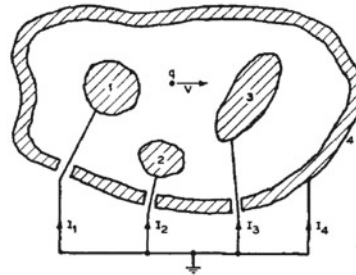


FIG. 1. Schematic representation of conductors and currents.

Fig. A.4 Portrait of Shockley from 1975 and part of his 1938 publication ‘Currents to Conductors Induced by a Moving Point Charge’ [40]. *Source* Wikimedia, public domain

Simon Ramo’s (1913–2016) ‘Currents Induced by Electron Motion’ from 1939 [41] (Fig. A.4), it led to the so-called ‘Shockley-Ramo theorem’ [42] which would later become the foundation of non-destructive detection techniques of charged-particle oscillations in traps, which represents a very powerful diagnostic tool and enables precision frequency measurements and cooling of particle motions, a method Hans Dehmelt in 1968 called the ‘bolometric technique’ [43].

A.6 John Robinson Pierce and the First Mention of ‘Trap’

John Robinson Pierce (1910–2002) earned a Ph.D. in electrical engineering and physics at Caltech in 1936, advised by Francis Maxstadt. He then joined Bell Telephone Laboratories in Murray Hill, New Jersey, and worked on the design of high-frequency vacuum tubes. During his long career he was involved in many important developments within what is today called the ‘golden age of research in communication principles and science’ [44]. It is, however, often unnoted that one of his early works involved the first mention of what was later to become the ‘Penning trap’, a device which confines charged particles by a static electric field of a certain shape and a superimposed homogeneous static magnetic field. Among the several books he has published, the first one is ‘Theory and Design of Electron Beams’ (1949) [45] and we will later see that this work inspired H. G. Dehmelt (1922–2017) to build and first use such a kind of device for electron storage in 1959. Dehmelt named this kind of trap a ‘Penning trap’ after Frans Michel Penning. Pierce was the first person to use the word ‘trapping’ in its modern sense in this text. He pointed out that it is possible to obtain a sinusoidal motion of electrons trapped in a combination of a quadrupole electric and axial magnetic field.

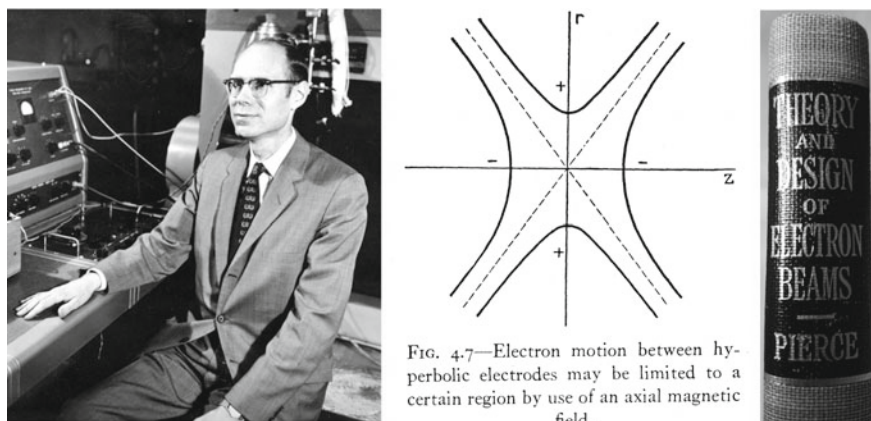


Fig. A.5 Portrait of Pierce, the back of his 1949 book ‘Theory and design of electron beams’ and his hyperbolic trap design on page 41 of that book [44]. *Source* NASA, public domain

‘There is one particular field which perhaps merits individual attention [...]. This field can be produced by hyperbolic electrodes [...]. Suppose we have also a uniform magnetic field of strength B in the z direction [...]. We see that it is possible to obtain a pure sinusoidal motion of electrons trapped in this combination of electric and magnetic fields.’

His text includes a figure of the hyperbolic shape of the electrodes (Fig. A.5) and gives expressions for the potential and a stability condition. From the way it is presented it may as well stem from a modern textbook on the topic. Apparently, Pierce was never officially credited for this concept, that is none of the many awards and citations he received directly related to the Penning trap, apart from the credit he was later given by Dehmelt in his Nobel biographical.

A.7 Wolfgang Paul and Electrodynamic Containment

Wolfgang Paul (1913–1993) earned his Ph.D. in physics in Berlin in 1939, and became a professor of physics at the University of Bonn in 1952. Before this, while still in Göttingen with Wilhelm Walcher, he had worked on mass spectrometry, which would later become part of his research in Bonn and led, among others, to the invention of the Paul trap.

‘The best known results of this research are the electric mass filter, widely used as a residual gas analyzer, and the electrodynamic ion trap, which turned out many years later to be one of the essential ingredients for the isolation and detection of single atomic ions.’ [46]

Novel methods for mass spectrometry had been a topic already, and the mass dependence of characteristic motions in external fields were exploited to that end. Sommer et al. used the cyclotron frequency of particles in a magnetic field for a

determination of the proton magnetic moment and the electron-to-proton mass ratio [47]. Paul came up with a mass spectrometer that replaced the magnetic field by a radio-frequency electric field and in 1953 published the article ‘Ein neues Massenspektrometer ohne Magnetfeld’ [48], and filed a patent in Germany [49] and in the United States [50].

In his 1989 Nobel biographical he mentions the trap only briefly:

‘[After becoming director of the Bonn physics institute in 1952] Here we started new activities: molecular beam physics, mass spectrometry and high energy electron physics. It was a scanty period after the war. But in order to become in a few years competitive with the well advanced physics abroad we tried to develop new methods and instruments in all our research.[...] The quadrupole mass spectrometer and the ion trap were conceived and studied in many respects by research students.’

Paul worked with a linear radio-frequency mass filter, and the two-dimensional dynamic stabilisation of the particle motion was understood to be applicable to the three-dimensional case as well. The linear case corresponds to the choice $a = -b$ and $c = 0$ in (A.4), while the three-dimensional case is given for $a = b$ and $c = -2a = -2b$. Conveniently, the differential equations describing the particle motions in this situation, the Mathieu equations, had been known and solved earlier [51, 52]. The three-dimensional Paul trap uses hyperbolic electrodes in the same way as described by Pierce [45], but particle escape along the repulsive part of the potential is avoided by the radio-frequency component of the applied voltage, which effectively rotates the saddle-shaped potential surface such that particles have no time to escape (Fig. A.6). This is sometimes depicted by the mechanical analogue, a ball on a rotating saddle surface, although the validity of this picture is subject to discus-

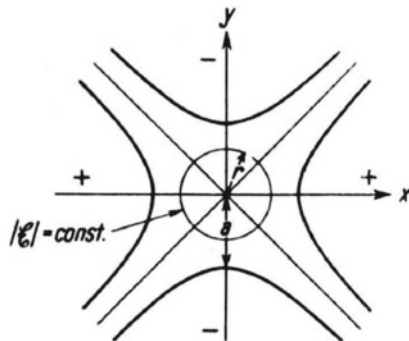


Abb. 1. Elektrisches 4-Polfeld, in dem $|\mathcal{E}| = \frac{U + V \cos \omega t}{a^2} \cdot r$ ist. Äquipotentialflächen (Elektroden) dick ausgezogen. Die eingezeichneten + und - Zeichen bedeuten $+\frac{1}{2}(U + V \cos \omega t)$ bzw. $-\frac{1}{2}(U + V \cos \omega t)$.

Fig. A.6 Paul’s quadrupolar (4-pole) trap design from ‘Ein neues Massenspektrometer ohne Magnetfeld’ [A novel mass spectrometer without a magnetic field] from 1953 [48]

sion [53]. The achieved particle confinement gave the device the name ‘Ionen-Käfig’ (ion cage) [54]. Similar work around that time by Wuerker et al. [55–57] acknowledged Paul’s achievements, and used the word ‘electrodynamic containment’ [56] for the confinement situation. The Paul trap was first used for spectroscopy in 1962 in measurements on $^4\text{He}^+$ [58] and H_2^+ [59] by Dehmelt.

A.8 Hans Georg Dehmelt and the Hyperbolic Penning Trap

Hans Georg Dehmelt (1922–2017) earned a Ph.D. in physics from the University of Göttingen, Germany, in 1949. After a short time at Duke University (Durham, North Carolina) he went to the University of Washington, Seattle in 1952. In 1955, he realised the advantages of particle confinement for spectroscopy [60] (Fig. A.7).

‘I put these [...] ideas to good use in 1956 in Seattle in an experiment entitled ‘Paramagnetic Resonance Reorientation of Atoms and Ions Aligned by Electron Impact’. In this paper I first pointed out the usefulness of ion trapping for high resolution spectroscopy and mentioned the 1923 Kingdon trap as a suitable device.’

This and following work such as ‘Spin Resonance of Free Electrons Polarized by Exchange Collisions’ required electron trapping and spin transfer under well-defined conditions. In his 1989 Nobel Prize biographical, Dehmelt stated:

‘I was not satisfied with the plasma trapping scheme used for the electrons and asked my student, Keith Jefferts, to study ion trapping in an electron beam traversing a field free vacuum space between two grids. Also, I began to focus on the magnetron/Penning discharge geometry, which, in the Penning ion gauge, had caught my interest already at Göttingen and at Duke. In their 1955 cyclotron resonance work on photoelectrons in vacuum Franken and Liebes had reported undesirable frequency shifts caused by accidental electron trapping. Their analysis made me realize that in a pure electric quadrupole field the shift would not depend on the location of the electron in the trap. This is an important advantage over many other traps that I decided to exploit. A magnetron trap of this type had been briefly discussed in J.R. Pierce’s 1949 book, and I developed a simple description of the axial, magnetron, and cyclotron motions of an electron in it. With the help of the expert glassblower of the Department, Jake Jonson,

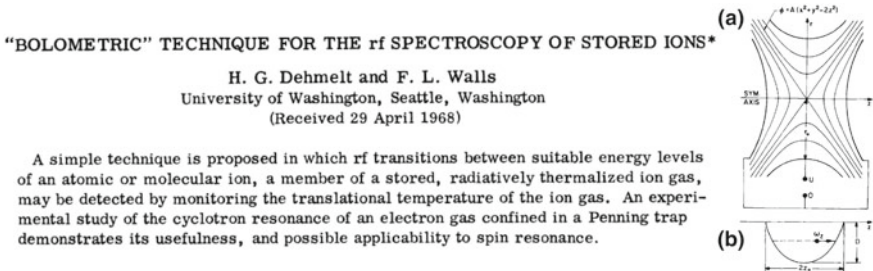


Fig. A.7 Title of “‘Bolometric” Technique for the rf Spectroscopy of Stored Ions’ from 1968 with the first prominent mentioning of the name ‘Penning trap’ [43] and apparatus drawing from [61]

I built my first high vacuum magnetron trap in 1959 and was soon able to trap electrons for about 10 sec and to detect axial, magnetron and cyclotron resonances.'

In the following years, he and his co-workers performed a number of spectroscopic measurements with stored ions and protons [43, 61–65]. In 1973, they were able to store a single electron (see the section about David Wineland), and Dehmelt and Eckstrom proposed to use the force of an inhomogeneous magnetic field on the spin of an electron confined in a Penning trap [66]. This way, a precise measurement of the oscillation frequency yields information on the spin direction. They used this effect for the detection of induced changes of the spin direction of an electron by observing the corresponding changes in the electron's oscillation frequency in a Penning trap. Since the confined particle's spin direction is monitored continuously, Dehmelt called this the 'continuous Stern-Gerlach effect' [67]. It has since been successfully applied to a number of magnetic moment studies on both free and bound electrons, the positron, as well as on protons and anti-protons.

Dehmelt was awarded the 1989 Nobel Prize in Physics together with Wolfgang Paul for 'the development of the ion trap technique'. The prize was shared with Norman F. Ramsey 'for the invention of the separated oscillatory fields method and its use in the hydrogen maser and other atomic clocks'. See also the Nobel lectures of Paul and Dehmelt [68, 69].

The theory of single charged particles confined in a Penning trap was called 'Geonium theory' by Dehmelt, to point out the notion of the confined particle and the rest of the world constituting a 'Geonium atom'. This spirit is also captured in the title 'Single Elementary Particles at Rest in Free Space' of Dehmelt's 1979 publication [70].

A.9 David Jeffrey Wineland and the Stored-Ion Calorimeter

David Jeffrey Wineland earned his Ph.D. in physics from Harvard University in 1970 on 'The Atomic Deuterium Maser' supervised by Norman Foster Ramsey Jr., who would later share the 1989 Nobel Prize in Physics with Hans Dehmelt and Wolfgang Paul. He then joined the group of Hans Dehmelt at the University of Washington, where he worked on particle trapping before joining NIST (at that time still called 'National Bureau of Standards') in Boulder, Colorado. He started working on various areas of particle confinement techniques and applications, including laser cooling (following earlier proposals from 1975 [71], and succeeding in 1978 [72] in parallel to Toschek and Dehmelt in Heidelberg [73]), and quantum information research for which he was awarded the 2012 Nobel Prize in Physics together with Serge Haroche, 'for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems' [74].

While working with Dehmelt, he achieved the confinement and study of a single electron ('Monoelectron Oscillator' [75]) and contributed significantly to the understanding of the collective (thermal) behaviour of confined particles and the electronic

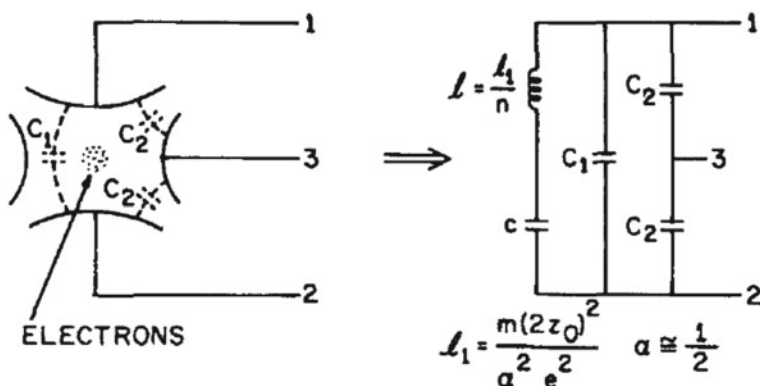


Fig. A.8 Part of ‘Principles of the stored ion calorimeter’ from 1975, illustrating the correspondence of stored electrons and an electronic circuit [76]

circuits they interact with (via the image charges in trap electrodes), including a picture in which the confined particles themselves are understood as a resonant electronic circuit such that the overall system can be described in terms of electronics. This work is mainly reflected in his 1975 text ‘Principles of the stored ion calorimeter’ [76] together with Dehmelt, and this picture is still widely used for example in the description of electronic (resistive) particle cooling (Fig. A.8).

A.10 Gabrielse, Tan and the Cylindrical Penning Trap

Gerald Gabrielse earned his Ph.D. in physics from the University of Chicago in 1980, supervised by Henry Gordon Berry, before joining the group of Hans Dehmelt at the University of Washington in Seattle in 1978.

For about a decade after the achievement of single electrons in a Penning trap and the introduction of the continuous Stern-Gerlach effect around 1973, hyperbolic Penning traps of the shape initially used by Dehmelt had been used for a number of important measurements, including precision determinations of the magnetic moments of the electron and the positron as well as of the electron-to-proton mass ratio [77–84]. In his 1984 paper ‘Cylindrical Penning Traps with Orthogonalized Anharmonicity Compensation’ Gabrielse came to say [85]:

‘All of the precision experiments mentioned were made in compensated Penning traps with hyperbolic ring and endcap electrodes. The electrostatic properties of such traps and the effect of radio-frequency potentials on the endcaps have already been investigated via relaxation calculations. These calculations showed that a judicious choice of hyperbolic endcap and ring electrodes makes the axial oscillation frequency of a trapped particle independent of changes in the compensation potential. A better understanding of the electrostatic properties of hyperbolic traps suggested that the ratio of minimum distances between the endcap and ring electrodes was much more important than the hyperbolic contours themselves. It seemed

that hyperbolic electrodes might be less necessary for precision work than had been earlier assumed.'

The mentioned relaxation calculation of the hyperbolic shape is the one presented in 'Relaxation calculation of the electrostatic properties of compensated Penning traps with hyperbolic electrodes' in 1983 [86]. At the time, the hyperbolic shape was indeed difficult to machine, as also obvious from the statement in Fig. A.9. So a favourable way to circumvent this issue was to use a cylindrical shape for a Penning trap that is easier and more precise to produce and yet allows the potential to be compensated by means of tuning of voltages.

This transformation is not a change of topology, just of geometry. The hyperbolic ring is 'inflated' and straightened to become a cylinder, while the hyperbolic endcaps are flattened out to become disks like in the Kingdon trap. This deformation distorts the potential inside the trap and makes it anharmonic, but that effect can be compensated either by careful choice of the relative dimensions and/or by introduction of additional compensation electrodes to which another voltage is applied so that the trap potential becomes harmonic at least close to the centre [85].

Still, such traps were geometrically closed to a large extent, which made the introduction of external particles and beams difficult, and hindered optical detection of trapped particles' radiation. This obstacle was overcome with the introduction of the

Highest precision mass spectroscopy and other highly accurate measurements have been carried out in Penning traps with metal electrodes painstakingly shaped along hyperbolic contours. A single electron has now been observed in a much simpler, cylindrical trap with equally good signal-to-noise ratio, thus demonstrating the possibility of conducting such experiments in a more readily constructed apparatus and opening the possibility of some new experiments. An essential requirement is the careful choice of electrode lengths to make the size of the electric quadrupole potential insensitive to adjustments which minimize deviations from an electric quadrupole.

Fig. A.9 Abstract from 'One electron in an orthogonalized cylindrical Penning trap' from 1989, describing the machining of hyperbolic traps as 'painstaking' [87]

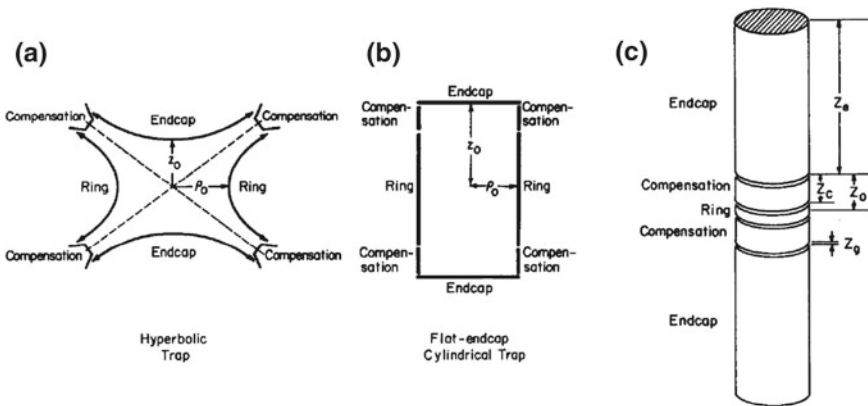


Fig. A.10 Comparison of hyperbolic traps (left) and closed cylindrical traps (middle) with open-endcap cylindrical Penning traps (right) as depicted in [88]

open-endcap cylindrical Penning trap in 1989 [88], which replaced the flat endcaps by open cylinders of considerable length such that they electrostatically appear closed from positions near the trap centre. Figure A.10 shows the comparison of hyperbolic traps (left) and closed cylindrical traps (middle) with open-endcap cylindrical Penning traps (right) as it is depicted in [88]. Cylindrical open-endcap Penning traps and variations have been used and are being used in numerous experiments, mainly on account of their comparatively simple geometry, good tunability, and axial access to the trap centre. The cylindrical geometry is also simple to extend to form nested traps, double-trap arrangements and a number of variations such as the Penning-Malmberg trap, the Penning-Ioffe trap, planar cylindrical Penning traps and the like.

A.11 Timeline

To somewhat summarise, the below timeline gives an overview account of some of the important steps in trap development within our chosen period, from the beginnings to the Nobel Prize in 1989 (Fig. A.11)

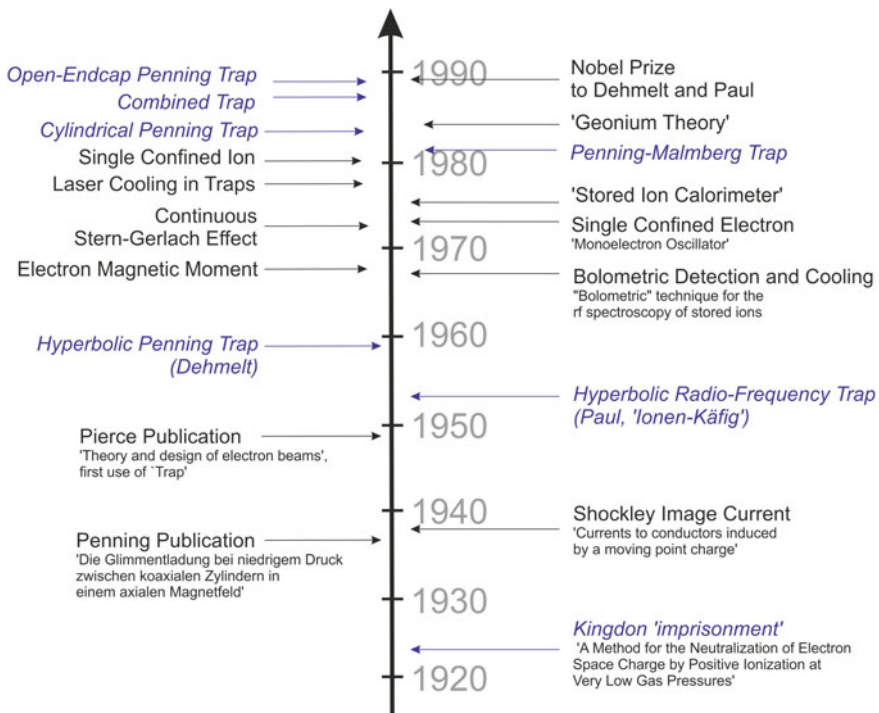


Fig. A.11 Timeline of some important steps in trap development within our chosen period, from the beginnings to the Nobel Prize in 1989

References

1. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)
2. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps II* (Springer, Heidelberg, 2009)
3. P. Ghosh, *Ion Traps* (Oxford University Press, Oxford, 1995)
4. G. Stafford Jr., Ion trap mass spectrometry: a personal perspective. *J. Am. So. Mass Spectrom.* **13**(6), 589 (2002)
5. R.E. March, J.F.J. Todd (eds.), *Practical Aspects of Ion Trap Mass Spectrometry*, vols. 1, 2 and 3 (CRC Press, Boca Raton, FL, 1995)
6. B. Mahon, *The Man Who Changed Everything: The Life of James Clerk Maxwell* (Wiley, Chichester, 2004)
7. J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Clarendon Press Series Vol. II (Oxford at the Clarendon Press, Oxford, 1873)
8. J.C. Maxwell, A dynamical theory of the electromagnetic field. *Philos. Trans. R. Soc. Lond.* **155**, 459 (1865)
9. J.C. Maxwell, On physical lines of force. *Philos. Mag.* **90**, 11 (1861)
10. C.F. Gauss, *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata, Commentationes societatis regiae scientiarum Gottingensis recentiores*, vol. II (Göttingen, 1813), reprinted in: C.F. Gauss, *Werke*, Band 5 (Springer, Berlin, Heidelberg, 1877), pp. 279–286
11. M. Faraday, Experimental researches in electricity, *Philos. Trans. R. Soc. Lond.* **122**, 125 (1832)
12. M. Faraday, *Experimental Researches in Electricity*, vol. 1 (Richard & John Edward Taylor, 1839) [side note: The book is compiled from articles published in the Philosophical Transactions of the Royal Society from 1831–1838]
13. F. Berends, Hendrik Antoon Lorentz: his role in physics and society. *J. Phys.: Condens. Matter* **21**, 164223 (2009)
14. P.J. Nahin, *Oliver Heaviside* (Johns Hopkins University Press, Baltimore, 2002)
15. P.G. Huray, *Maxwell's Equations* (Wiley-IEEE, Hoboken, 2010). ISBN 0-470-54276-4
16. P.F. Dahl, *Flash of the Cathode Rays: A History of J. J. Thomson's Electron* (CRC Press, Boca Raton, 1997)
17. W.T. Scott, Who was Earnshaw? *Am. J. Phys.* **27**, 418 (1959)
18. S. Earnshaw, On the nature of the molecular forces which regulate the constitution of the luminiferous ether. *Trans. Camb. Philos. Soc.* **7**, 97 (1842)
19. K.H. Kingdon published 'Notes on Electricity and Magnetism' together with R. C. Dearle in the University of Toronto Press (No. 191), reprinted in the Leopold Classic Library [side fact: His daughter Elisabeth (*1928) became a famous opera singer (soprano) in Germany]
20. K.H. Kingdon, A Method for the neutralization of electron space charge by positive ionization at very low gas pressures. *Phys. Rev.* **21**, 408 (1923)
21. R.R. Lewis, Motion of ions in the Kingdon trap. *J. Appl. Phys.* **53**, 3975 (1982)
22. R.H. Hooverman, Charged particle orbits in a logarithmic potential. *J. Appl. Phys.* **34**, 3505 (1963)
23. P.R. Brooks, D.R. Herschbach, Kingdon cage as a molecular beam detector. *Rev. Sci. Instr.* **35**, 1528 (1964)
24. R.A. Douglas, J. Zabritski, R.G. Herb, Orbitron vacuum pump. *Rev. Sci. Instr.* **36**, 1 (1965)
25. C.R. Vane, M.H. Prior, R. Marrus, Electron capture by Ne^{10+} trapped at very low energies. *Phys. Rev. Lett.* **46**, 107 (1981)
26. H. Blümel, Dynamic Kingdon trap. *Phys. Rev. A* **51**, R30 (1995)
27. Q. Hu et al., The orbitrap: a new mass spectrometer. *J. Mass Spectrom.* **40**, 430 (2005)

28. R.H. Perry, R.G. Cooks, R.J. Noll, Orbitrap mass spectrometry: instrumentation, ion motion and applications. *Mass. Spectrom. Rev.* **27**, 661 (2008)
29. D.P. Moehs, D.A. Church, Kingdon trap apparatus and technique for precise measurement of the lifetimes of metastable levels of ions. *Rev. Sci. Instr.* **69**, 1991 (1998)
30. N. Numadate, K. Okada, N. Nakamura, H. Tanuma, Development of a Kingdon ion trap system for trapping externally injected highly charged ions. *Rev. Sci. Instr.* **85**, 103119 (2014)
31. H.T. Schmidt, H. Cederquist, J. Jensen, A. Fardi, Conetrap: a compact electrostatic ion trap. *Nucl. Instrum. Methods Phys. Res. B* **173**, 523 (2001)
32. M. Dahan et al., A new type of electrostatic ion trap for storage of fast ion beams. *Rev. Sci. Instr.* **69**, 76 (1998)
33. M. Wang, A.G. Marshall, A “screened” electrostatic ion trap for enhanced mass resolution, mass accuracy, reproducibility, and upper mass limit in Fourier-transform ion cyclotron resonance mass spectrometry. *Anal. Chem.* **61**, 1288 (1989)
34. A. Makarov, Electrostatic axially harmonic orbital trapping: a high-performance technique of mass analysis. *Anal. Chem.* **72**, 1156 (2000)
35. F.M. Penning, Die Glimmentladung bei niedrigem Druck zwischen koaxialen Zylindern in einem axialen Magnetfeld. *Physica (Utrecht)* **3**, 873 (1936)
36. F.M. Penning, Ein neues Manometer für niedrige Gasdrücke, insbesondere zwischen 10^{-3} und 10^{-5} mm. *Physica* **4**, 71 (1937)
37. P.F. Dahl, *From Nuclear Transmutation to Nuclear Fission 1932–1939* (CRC Press, Boca Raton, 2002). ISBN 978-0-7503-0865-6
38. E.O. Lawrence, M.S. Livingston, The production of high speed light ions without the use of high voltages. *Phys. Rev.* **40**, 19 (1932)
39. 1956 Nobel Prize in Physics, *Physics Today* **10**(1), 16 (1957)
40. W. Shockley, Currents to conductors induced by a moving point charge. *J. Appl. Phys.* **9**, 635 (1938)
41. S. Ramo, Currents induced by electron motion. *Proc. IRE* **27**(9), 584 (1939) [side note: IRE stands for ‘Institute of Radio Engineers’, founded 1912, joined with the ‘American Institute of Electrical Engineers’ (AIEE) in 1963 to become the ‘Institute of Electrical and Electronics Engineers’ (IEEE)]
42. Z. He, Review of the Shockley Ramo theorem and its application in semiconductor gamma-ray detectors. *Nucl. Instr. Meth. A* **463**, 250 (2001)
43. H.G. Dehmelt, F.L. Walls, “Bolometric” technique for the rf spectroscopy of stored ions. *Phys. Rev. Lett.* **21**, 127 (1968)
44. M. Mathews, *John Robinson Pierce*, *Physics Today* **56** (12), 88 (2003)
45. J.R. Pierce, *Theory and design of electron beams* (Van Nostrand, Princeton, 1949)
46. P.E. Toschek, *Wolfgang Paul*, *Physics Today* **47** (7), 76 (1994)
47. H. Sommer, H.A. Thomas, J.A. Hipple, The measurement of e/M by Cyclotron Resonance, *Phys. Rev.* **82**, 697 (1951)
48. W. Paul, H. Steinwedel, Ein neues Massenspektrometer ohne Magnetfeld. *Z. Naturforschg.* **8a**, 448 (1953)
49. Patent DE 944900, Verfahren zur Trennung bzw. zum getrennten Nachweis von Ionen verschiedener spezifischer Ladung, ed. by W. Paul, H. Steinwedel, filed on December 24, 1953, priority December 23, 1953
50. W. Paul, H. Steinwedel, Apparatus for separating charged particles of different specific charges. Patent US 2939952 A
51. G. Kotowski, Solutions of inhomogeneous Mathieu differential equation with periodic perturbation of arbitrary frequency. *Z. Angew. Math. Mech.* **23**, 213 (1943)
52. N.W. MacLachlan, *Theory and applications of Mathieu functions* (Clarendon Press, Oxford, 1947)

53. R.I. Thompson, T.J. Harmon, M.G. Ball, The rotating-saddle trap: a mechanical analogy to RF-electric-quadrupole ion trapping? *Can. J. Phys.* **80**, 1433 (2002)
54. W. Paul, O. Osberghaus, E. Fischer, Ein Ionenkäfig, *Forschungsberichte des Wirtschafts- und Verkehrsministeriums Nordrhein-Westfalen*, vol. 415 (Westdeutscher Verlag, Köln und Opladen, 1958)
55. H. Shelton, R.F. Wuerker, R.V. Langmuir, Electrodynamical containment of charged particles. *Bull. Am. Phys. Soc.* **2**, 375 (1957)
56. R.F. Wuerker, H. Shelton, R.V. Langmuir, Electrodynamical containment of charged particles. *J. Appl. Phys.* **30**, 342 (1959)
57. R.F. Wuerker, H.M. Goldenberg, R.V. Langmuir, Electrodynamical containment of charged particles by three-phase voltages. *J. Appl. Phys.* **30**, 441 (1959)
58. H.G. Dehmelt, F.G. Major, Orientation of $(\text{He}_4)^+$ ions by exchange collisions with cesium atoms. *Phys. Rev. Lett.* **8**, 213 (1962)
59. H.G. Dehmelt, K.B. Jefferts, Alignment of the He_2^+ Molecular ion by selective photodissociation. *Phys. Rev.* **125**, 1318 (1962)
60. H.G. Dehmelt, Paramagnetic resonance reorientation of atoms and ions aligned by electron impact. *Phys. Rev.* **103**, 1125 (1956)
61. H.G. Dehmelt, Radiofrequency spectroscopy of stored ions I: storage advances in atomic. *Mol. Opt. Phys.* **3**, 53 (1968)
62. E.N. Fortson, F.G. Major, H.G. Dehmelt, Ultrahigh resolution $F = 0, 1 \text{ } ^3\text{He}^+$ Hfs spectra by an ion-storage collision technique. *Phys. Rev. Lett.* **16**, 221 (1966)
63. F.G. Major, H.G. Dehmelt, Exchange-collision technique for the rf spectroscopy of stored ions. *Phys. Rev.* **170**, 91 (1968)
64. D.A. Church, H.G. Dehmelt, Radiative cooling of an electrostatically contained proton gas. *J. Appl. Phys.* **40**, 3421 (1969)
65. H.A. Schuessler, E.N. Fortson, H.G. Dehmelt, Hyperfine structure of the ground state of $^3\text{He}^+$ by the ion-storage exchange-collision technique. *Phys. Rev.* **187**, 5 (1969)
66. H. Dehmelt, P. Ekstrom, Proposed $g - 2$ experiment on stored single electron or positron. *Bull. Am. Phys. Soc.* **18**, 727 (1973)
67. H. Dehmelt, Continuous Stern-Gerlach effect: principle and idealized apparatus. *Proc. Natl. Acad. Sci. USA* **83**, 2291 (1986)
68. H.G. Dehmelt, Experiments with an isolated subatomic particle at rest. *Rev. Mod. Phys.* **62**, 525 (1990)
69. W. Paul, Electromagnetic traps for charged and neutral particles. *Rev. Mod. Phys.* **62**, 531 (1990)
70. H. Dehmelt, R.S. Van Dyck, P.B. Schwinberg, G. Gabrielse, Single elementary particles at rest in free space. *Bull. Am. Phys. Soc.* **24**, 757 (1979)
71. D.J. Wineland, H. Dehmelt, *Bull. Am. Phys. Soc.* **20**, 637 (1975)
72. D.J. Wineland, R.E. Drullinger, F.L. Walls, Radiation-pressure cooling of bound resonant absorbers. *Phys. Rev. Lett.* **40**, 1639 (1978)
73. W. Neuhauser, M. Hohenstatt, P.E. Toschek, H.G. Dehmelt, Optical-sideband cooling of visible atom cloud confined in parabolic well. *Phys. Rev. Lett.* **41**, 233 (1978)
74. D.J. Wineland, Nobel lecture: superposition, entanglement, and raising Schrödingers cat. *Rev. Mod. Phys.* **85**, 1103 (2013)
75. D.J. Wineland, P. Ekstrom, H. Dehmelt, Monoelectron oscillator. *Phys. Rev. Lett.* **31**, 1279 (1973)
76. D.J. Wineland, H.G. Dehmelt, Principles of the stored ion calorimeter. *J. Appl. Phys.* **46**, 919 (1975)
77. R.S. Van Dyck Jr., P.B. Schwinberg, H.G. Dehmelt, Electron magnetic moment from geonium spectra: early experiments and background concepts. *Phys. Rev. D* **34**, 722 (1986)

78. R.S. van Dyck, P.B. Schwinberg, H.G. Dehmelt, New high-precision comparison of electron and positron g factors. *Phys. Rev. Lett.* **59**, 26 (1987)
79. R.S. van Dyck Jr., F.L. Moore, D.L. Farnham, P.B. Schwinberg, New measurement of the proton-electron mass ratio. *Int. J. Mass Spectrom. Ion Proc.* **66**, 327 (1985)
80. P.B. Schwinberg, R.S. Van Dyck Jr., H.G. Dehmelt, New comparison of the positron and electron g factors. *Phys. Rev. Lett.* **47**, 1679 (1981)
81. R.S. Van Dyck Jr., P.B. Schwinberg, Preliminary proton/electron mass ratio using a compensated quadring Penning trap. *Phys. Rev. Lett.* **47**, 395 (1981)
82. P.B. Schwinberg, R.S. Van Dyck Jr., H.G. Dehmelt, Trapping and thermalization of positrons for geonium spectroscopy. *Phys. Lett. A* **81**, 199 (1981)
83. R.S. Van Dyck Jr., P.B. Schwinberg, H.G. Dehmelt, Precise measurements of axial, magnetron, cyclotron, and spin-cyclotron-beat frequencies on an isolated 1 meV electron. *Phys. Rev. Lett.* **38**, 310 (1977)
84. R.S. Van Dyck Jr., D.J. Wineland, P.A. Ekstrom, H.G. Dehmelt, High mass resolution with a new variable anharmonicity Penning trap. *Appl. Phys. Lett.* **28**, 446 (1976)
85. G. Gabrielse, F.C. Macintosh, Cylindrical Penning traps with orthogonalized anharmonicity compensation. *Int. J. Mass. Spectrom. Ion Proc.* **57**, 1 (1984)
86. G. Gabrielse, Relaxation calculation of the electrostatic properties of compensated Penning traps with hyperbolic electrodes. *Phys. Rev. A* **27**, 2277 (1983)
87. J. Tan, G. Gabrielse, One electron in an orthogonalized cylindrical Penning trap. *Appl. Phys. Lett.* **55**, 2144 (1989)
88. G. Gabrielse, L. Haarsma, S.L. Rolston, Open-endcap Penning traps for high precision experiments. *Int. J. Mass Spectrom. Ion Proc.* **88**, 319 (1989)

Appendix B

Penning Trap Confinement: The Brief Version

This chapter gives a brief summary of the most important statements and equations concerning particle confinement in Penning traps. We first treat the general properties that apply to single particles as well as to particle ensembles, then we have a look only at confinement of particle ensembles.

B.1 General Statements about Confinement

In a Penning trap, confinement of charged particles is due to a superposition of a (homogeneous) magnetostatic field B_0 with a (harmonic) electrostatic potential U . The electrostatic field of the Penning trap $\mathbf{E} = -\nabla U$ created by a three-dimensional quadrupolar potential U of the shape $(2z^2 - x^2 - y^2)$ represents a harmonic potential well along the axial direction z for a charged particle with $qU > 0$, but at the same time a repulsive potential in the radial direction ρ which lies in the (x, y) -plane. Escape along this direction is hindered by the strong magnetic field $\mathbf{B} = (0, 0, B_0)$ in the axial direction which forces a particle on a radial orbit on account of the Lorentz force $\mathbf{F} = -q\nabla U + q(\mathbf{v} \times \mathbf{B}_0)$. The ideal (hyperbolic) **electrostatic potential** is given by

$$U = \frac{U_0}{2d^2} (2z^2 - x^2 - y^2), \quad (\text{B.1})$$

where d is the so-called ‘**characteristic trap size**’ given by

$$d^2 = \frac{1}{2} \left(z_0^2 + \frac{\rho_0^2}{2} \right). \quad (\text{B.2})$$

It leads to Cartesian **equations of motion** of a single particle

$$\ddot{x} = \omega_c \dot{y} + \frac{\omega_z^2}{2} x; \quad \ddot{y} = -\omega_c \dot{x} + \frac{\omega_z^2}{2} y; \quad \ddot{z} = -\omega_z^2 z, \quad (\text{B.3})$$

where we have the characteristic frequencies (**‘eigen-frequencies’**)

$$\omega_z^2 = \frac{qU}{md^2} \quad \text{and} \quad \omega_{\pm} = \frac{\omega_c}{2} \pm \left(\frac{\omega_c^2}{4} - \frac{\omega_z^2}{2} \right)^{1/2}, \quad (\text{B.4})$$

in which the so-called **‘free-particle’ cyclotron frequency** ω_c

$$\omega_c = \frac{qB_0}{m}, \quad (\text{B.5})$$

is the value of ω_+ for $\omega_z = 0$ (i.e. for $U = 0$). These eigenfrequencies fulfill the identities

$$\omega_+ + \omega_- = \omega_c \quad \text{and} \quad \omega_+ \omega_- = \frac{\omega_z^2}{2}, \quad (\text{B.6})$$

and the so-called **‘invariance theorem’**

$$\omega_c^2 = \omega_-^2 + \omega_+^2 + \omega_z^2, \quad (\text{B.7})$$

which has its name from the fact that a number of small imperfections do affect the individual eigen-frequencies, but leave ω_c invariant. Typically, the oscillation frequencies obey the **frequency hierarchy**

$$\omega_c > \omega_+ \gg \omega_z \gg \omega_-, \quad (\text{B.8})$$

although we can have $\omega_z > \omega_+$ for uncommonly large trap depths

$$U > \frac{4qd^2B_0^2}{9m} = \frac{8}{9}U_{\text{MAX}}, \quad (\text{B.9})$$

where U_{MAX} is the **maximum trap potential** before radial confinement is lost. When the oscillation frequencies fulfill the condition

$$N_+\omega_+ + N_-\omega_- + N_z\omega_z = 0, \quad (\text{B.10})$$

with integer coefficients N_+ , N_- and N_z , confinement can become unstable (**‘non-linear resonances’**) since energy can be resonantly transferred between the degrees of freedom.

There is a large number of possible **geometric imperfections** of both the electrostatic potential and the magnetic field. To quantify the geometric imperfections of the **electrostatic potential**, we can expand it in cylindrical coordinates (ρ, z) like

$$U = U_0 \sum_{n=1}^{\infty} \frac{C_n}{d^n} \sum_{j=0}^{n'} A_n^{(j)} z^{n-2j} \rho^{2j}, \quad (\text{B.11})$$

where the $A_n^{(j)}$ are geometry parameters given by

$$A_n^{(j)} = \frac{(-1)^{2n-j} n!}{2^{2j} (j!)^2 (n-2j)!}. \quad (\text{B.12})$$

This introduces **expansion coefficients** C_n which measure the strength of the respective contribution to the total potential. The coefficient C_0 is an overall potential offset and hence irrelevant to the particle motion. C_2 represents the desired quadrupole term relevant for electrostatic confinement and reflects the ‘efficiency’ of the trap in creating a potential well from the applied voltages. Its value chiefly determines the actual oscillation frequencies of particles in the trap. A hyperbolic Penning trap has $C_2 = 1$ by design. For cylindrical Penning traps C_2 is typically about 0.5, and the dominant electric imperfection is characterized by the term C_4 , the octupole component of the electrostatic potential. For a trap with mirror symmetry about the ρ -plane through $z = 0$, terms with odd n do not appear in the potential U . The **total electrostatic potential** then has the explicit form

$$\begin{aligned} \frac{U}{U_0} = & \frac{C_2}{d^2} \left(z^2 - \frac{1}{2} \rho^2 \right) + \frac{C_4}{d^4} \left(z^4 - 3z^2 \rho^2 + \frac{3}{8} \rho^4 \right) \\ & + \frac{C_6}{d^6} \left(z^6 - \frac{15}{2} z^4 \rho^2 + \frac{45}{8} z^2 \rho^4 - \frac{5}{16} \rho^6 \right) \\ & + \frac{C_8}{d^8} \left(z^8 - \frac{21}{2} z^6 \rho^2 + \frac{210}{8} z^4 \rho^4 - \frac{140}{16} z^2 \rho^6 + \frac{35}{128} \rho^8 \right) + \dots \end{aligned} \quad (\text{B.13})$$

and the leading **shifts of the oscillation frequencies** are given by

$$\frac{\Delta\omega_+}{\omega_+} = \frac{C_4}{C_2} \frac{1}{qU} \left(\frac{\omega_z}{\omega_+} \right)^2 \left[-3E_z + \frac{3}{2} \left(\frac{\omega_z}{\omega_+} \right)^2 E_+ - 6E_- \right] \quad (\text{B.14})$$

$$\frac{\Delta\omega_z}{\omega_z} = \frac{C_4}{C_2} \frac{1}{qU} \left[\frac{3}{2} E_z - 3 \left(\frac{\omega_z}{\omega_+} \right)^2 E_+ + 6E_- \right] \quad (\text{B.15})$$

$$\frac{\Delta\omega_-}{\omega_-} = \frac{C_4}{C_2} \frac{1}{qU} \left[6E_z - 6 \left(\frac{\omega_z}{\omega_+} \right)^2 E_+ + 6E_- \right]. \quad (\text{B.16})$$

A **modulation** of the trap voltage U_0 by an additional voltage $V \cos \omega t$ between ring and endcaps leads to perturbed oscillation frequencies given by

$$\omega'_\pm = \omega_\pm \mp \frac{1}{2} \left(\frac{qC_2}{2m\omega_1} \right)^2 \frac{2\omega_1}{4\omega_1^2 + \omega^2} \quad (\text{B.17})$$

$$\omega'_z = \omega_z - \left(\frac{qC_2}{m\omega_z} \right)^2 \frac{\omega_z}{\omega_z^2 + \omega^2}, \quad (\text{B.18})$$

where $4\omega_1^2 = \omega_c^2 - 2\omega_z^2$, and only the lowest-order contribution to the frequency shift has been taken into account.

For the geometric imperfections of the magnetic field, we can write similar expansions as in the case of the electric potential. Since we speak about a field rather than about a potential, we need to treat its axial and radial components separately. We can write the **axial component** of the **magnetic field** in the form

$$B_z(\rho, z) = \sum_{n=0}^{\infty} B_n \sum_{j=0}^{n'} A_n^{(j)} z^{n-2j} \rho^{2j}, \quad (\text{B.19})$$

where n' is the floor of $n/2$, and the same geometry functions as above. The **radial component** of the **magnetic field** can be written as

$$B_\rho(\rho, z) = \sum_{n=0}^{\infty} B_n \sum_{j=1}^{n''} A_n^{(j)} z^{n-2j+1} \rho^{2j-1}, \quad (\text{B.20})$$

where n'' is the floor of $(n+1)/2$ and the geometry coefficients of the radial magnetic field components are given by

$$A_n^{(j)} = \frac{(-1)^j j n!}{2^{2j-1} (n-2j+1)! (j!)^2}. \quad (\text{B.21})$$

Looking at the total magnetic field, for symmetry about the (x, y) -plane through $z = 0$ we have a geometry with even terms only, as in the electrostatic case. Hence, the **total magnetic field** has the form

$$\begin{aligned} \mathbf{B}(\rho, z) = & B_0 \mathbf{e}_z + B_2 \left[\left(z^2 - \frac{1}{2} \rho^2 \right) \mathbf{e}_z + (-z\rho) \mathbf{e}_\rho \right] \\ & + B_4 \left[\left(z^4 - 3z^2 \rho^2 + \frac{3}{8} \rho^4 \right) \mathbf{e}_z + \left(-2z^3 \rho + \frac{3}{2} z \rho^3 \right) \mathbf{e}_\rho \right] + \dots \end{aligned} \quad (\text{B.22})$$

In this representation, the first term is the homogeneous part of the magnetic field, which only has an axial component along $\mathbf{e}_z$ of strength B_0 , and the second term characterises a magnetic bottle of strength B_2 . It affects the oscillation frequencies like

$$\frac{\Delta\omega_+}{\omega_+} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} \left[-\left(\frac{\omega_z}{\omega_+} \right)^2 E_+ + E_z - 2E_- \right] \quad (\text{B.23})$$

$$\frac{\Delta\omega_z}{\omega_z} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} [E_+ - E_-] \quad (\text{B.24})$$

$$\frac{\Delta\omega_-}{\omega_-} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} [2E_+ - E_z - 2E_-] \quad (\text{B.25})$$

$$\frac{\Delta\omega_L}{\omega_L} = \frac{1}{m\omega_z^2} \frac{B_2}{B_0} \left[-\left(\frac{\omega_z}{\omega_+}\right)^2 E_+ + E_z + 2E_- \right], \quad (\text{B.26})$$

where in addition to the eigen-frequencies of oscillation, we have ω_L , the **Larmor frequency** (spin precession frequency) of the particle. It is not a classical oscillation but can be described within the same formalism.

Apart from these radially symmetric imperfections of the fields, there can additionally be an **elliptic component** of the electrostatic potential which breaks the radial field symmetry by adding a term proportional to $-\varepsilon(x^2 - y^2)/2$ such that the total electrostatic potential reads

$$U_\varepsilon = \frac{U_0}{2d^2} \left(z^2 - \frac{1}{2}(x^2 + y^2) - \frac{1}{2}\varepsilon(x^2 - y^2) \right). \quad (\text{B.27})$$

Also, there can be a **tilt** of the magnetic field axis with respect to the trap axis, such that the magnetic field as seen from the trap aligned with the ellipticity is no longer $\mathbf{B} = (0, 0, B_0)$ but takes the form

$$\mathbf{B} = B_0(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (\text{B.28})$$

The oscillation frequencies in the presence of a small ellipticity ε and a small tilt can be written as

$$\omega'_\pm \approx \omega_\pm + \frac{1}{2}\omega_- \sin^2 \theta (3 + \varepsilon \cos 2\phi) \quad (\text{B.29})$$

$$\omega'_z \approx \omega_z \left(1 - \frac{1}{4} \sin^2 \theta (3 + \varepsilon \cos 2\phi) \right), \quad (\text{B.30})$$

where the quantity η is the so-called **misalignment parameter** given by

$$\eta = \frac{1}{2} \sin^2 \theta (3 + \varepsilon \cos 2\phi). \quad (\text{B.31})$$

Importantly, when we insert the shifted frequencies into the **invariance theorem**, we find

$$\omega_+^{\prime 2} + \omega_z^{\prime 2} + \omega_-^{\prime 2} \approx \omega_c^2, \quad (\text{B.32})$$

which means that the invariance theorem is valid in the presence of small tilts and ellipticities, and which is the origin of its name.

The magnitude of the effect of geometric field imperfections on the particle motion depends on the amplitude and hence on the energy of the particle. The motional **amplitudes** ρ_+ , ρ_- and a_z are related to the corresponding kinetic energies E_+ , E_- and E_z by

$$\rho_+^2 = \frac{2E_+}{m(\omega_+^2 - \omega_z^2/2)} d^2 \quad (\text{B.33})$$

$$\rho_-^2 = \frac{2E_-}{m(\omega_-^2 - \omega_z^2/2)} d^2 \quad (\text{B.34})$$

$$a_z^2 = \frac{E_z}{qU} d^2. \quad (\text{B.35})$$

To minimize the effect of geometric field imperfections, particles can be motionally cooled, which means a reduction of the particle's kinetic energy and hence a reduction of the amplitude of oscillation. This is true for the axial and perturbed cyclotron motions. For the magnetron motion, a more careful discussion is in place, since its total energy is always negative and a reduction leads to an increase of the radial amplitude. The amplitude of the magnetron motion can be reduced by **resonant coupling** to the perturbed cyclotron motion that is cooled by some means. This is achieved by an azimuthal quadrupolar excitation at the upper sideband frequency $\omega_+ + \omega_- = \omega_c$ ('**magnetron centring**'). When the perturbed cyclotron motion is cooled at a rate γ , one observes a reduction of the magnetron radius according to

$$\rho_{\pm}(t) = \rho_{\pm}(t=0) \exp\left(-\frac{\omega_{\pm}}{\omega_+ - \omega_-} \gamma t\right). \quad (\text{B.36})$$

In general, a multitude of cooling techniques for confined particles have been developed, most notable are resistive cooling and several forms of laser cooling.

In **resistive cooling**, the oscillation of a charged particle causes an image current across trap electrodes that are connected by a resistance R which extenuates the current and hence the particle oscillation. The cooling may be modelled as a friction force, for example the axial equation of motion then reads

$$\omega_z^2 z - \frac{q}{m} \frac{dU(z)}{dz} - \gamma \frac{dz}{dt} = 0, \quad (\text{B.37})$$

where γ denotes the cooling rate. The particle energy follows an exponential decay of the kind

$$E_z(t) = E_z(t=0) \exp(-\gamma t), \quad (\text{B.38})$$

where the single-particle cooling rate is given by

$$\gamma^{-1} = \frac{D^2}{R} \frac{m}{q^2}, \quad (\text{B.39})$$

where D is the **effective electrode distance** for the arrangement of electrodes in use. The quantity D represents the distance between two parallel infinite conducting planar electrodes that would lead to the same induced current (through the connecting impedance) as in the actual trap geometry. Close to the trap centre, it is identical to $D = 2z_0/C_1$. Infinite parallel plates with distance $2z_0$ have $C_1 = 1$ so that $D = 2z_0$.

The same rate applies to the situation with a particle ensemble that oscillates with randomly distributed phases. In any case, the final temperature is limited by the electronic noise temperature which may be as low as the ambient temperature that is commonly on the Kelvin scale.

In **laser cooling**, we employ a red-detuned laser beam on particles with a convenient level scheme and a sufficiently large transition width Γ . Each absorption and spontaneous emission cycle removes kinetic energy from the particle, hence cooling its oscillatory motion. The final temperature assumes a minimum for a laser detuning of $\delta = \Gamma/2$, the so-called ‘**Doppler cooling limit**’

$$k_B T = \frac{\hbar \Gamma}{2}, \quad (\text{B.40})$$

which commonly is on the mK scale.

Apart from field imperfections as above, which can in principle be avoided by careful choice of geometries and potentials, there are also effects that are intrinsic to the confinement situation, i.e. they cannot be avoided even in principle.

One such effect is the so-called ‘**image-charge effect**’. The presence of the induced image charges in the conducting electrodes of the trap effectively changes the confining potential at the particle position and thus alters its oscillation frequencies. In the general situation we have frequency shifts due to the image charge given by

$$\Delta\omega_z = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{m} \frac{P_z}{2\omega_z} \quad (\text{B.41})$$

$$\Delta\omega_{\pm} = \mp \frac{1}{4\pi\epsilon_0} \frac{q^2}{m} \frac{P_{\rho}}{\omega_+ - \omega_-} \approx \mp \frac{q}{e} \frac{P_{\rho}}{B_0}, \quad (\text{B.42})$$

in which the P_z and P_{ρ} are geometry-dependent electric field gradients that scale roughly like $1/d^3$, hence this effect is pronounced in small traps.

There is also an **image-current effect** due to a non-vanishing imaginary component of the impedance $Z(\omega)$ that is used for particle cooling and detection. The corresponding relative frequency shifts are given by

$$\frac{\Delta\omega_+}{\omega_+} \approx -\frac{q}{m} \frac{1}{2\omega_+ D_+^2} \text{Im}(Z_+(\omega_+)) \quad (\text{B.43})$$

$$\frac{\Delta\omega_z}{\omega_z} \approx -\frac{q}{m} \frac{1}{2\omega_z D_z^2} \text{Im}(Z_z(\omega_z)). \quad (\text{B.44})$$

When we look at a particle that has a non-vanishing electric **dipole polarisability** α , then the dipole moment in its rotation with the cyclotron motion in the trap leads to a shift of that frequency. This situation results in a constant frequency shift of

$$\frac{\Delta\omega_+}{\omega_+} = -\alpha \frac{B_0^2}{m}. \quad (\text{B.45})$$

Radiative damping of particle motions, i.e. kinetic energy loss due to radiation of the accelerated charge, is connected with a confined particle's oscillation. It leads to exponential energy loss according to

$$E(t) = E_0 \cdot \exp(-\gamma_R t), \quad (\text{B.46})$$

where the constants γ_R of motions are given by

$$\gamma_{R,+} = \frac{4q^2\omega_+^2}{3mc^3} ; \quad \gamma_{R,z} = \frac{2q^2\omega_z^2}{3mc^3} ; \quad \gamma_{R,-} = \frac{4q^2\omega_-^2}{3mc^3}. \quad (\text{B.47})$$

This radiative damping can in principle lead to magnetron loss of particles, but the rate $\gamma_{R,-}$ is extremely small even for particles with high q/m . If the oscillation frequencies and the trap size have suitable values, the trap acts as a **cavity** and may **inhibit spontaneous emission**, such as observed for electrons where the trap forms a microwave cavity. The spontaneous radiation from the perturbed cyclotron motion of light particles such as electrons has values which make the associated 'self-cooling' a workable technique in **sympathetic cooling** that is called '**electron cooling**'.

The **relativistic shifts** of the frequencies in the external degrees of freedom (oscillation frequencies) can be understood in terms of the relativistic mass effect due to the motional energy of the particle. The relative frequency shifts are given by

$$\frac{\Delta\omega_+}{\omega_+} = -\frac{1}{mc^2} \left[E_+ + \frac{1}{2}E_z - \left(\frac{\omega_z}{\omega_+} \right)^2 E_- \right] \quad (\text{B.48})$$

$$\frac{\Delta\omega_z}{\omega_z} = -\frac{1}{mc^2} \left[\frac{1}{2}E_+ + \frac{3}{8}E_z - \frac{1}{4} \left(\frac{\omega_z}{\omega_+} \right)^2 E_- \right] \quad (\text{B.49})$$

$$\frac{\Delta\omega_-}{\omega_-} = -\frac{1}{mc^2} \left[-\left(\frac{\omega_z}{\omega_+} \right)^2 E_+ - \frac{1}{4} \left(\frac{\omega_z}{\omega_+} \right)^2 E_z - \frac{1}{4} \left(\frac{\omega_z}{\omega_+} \right)^4 E_- \right], \quad (\text{B.50})$$

and for the Larmor frequency we find

$$\frac{\Delta\omega_L}{\omega_L} = -\frac{1}{mc^2} \left[\frac{2}{9}E_+ + \frac{1}{2}E_z - \left(\frac{\omega_z}{\omega_+} \right)^2 E_- \right]. \quad (\text{B.51})$$

Coming back to the magnetic imperfections, the **magnetic bottle** is of particular interest for a number of applications since the spin orientation of the particle is linked to its oscillation frequencies in the trap. Using the orientation energy $E_J = g_J \mu_B B_0 M_J$, we find the shift of the axial frequency resulting from a change of the angular quantum number M_J by

$$\Delta\omega_z(\Delta M_J) \approx \frac{\hbar}{2m} \frac{\omega_+}{\omega_z} \frac{B_2}{B_0} \frac{e}{q} g_J \Delta M_J. \quad (\text{B.52})$$

This is used in the **continuous Stern-Gerlach effect**, where a flip of the particle's spin is detected via a change of the axial frequency.

The usual way to implement such a magnetic bottle in a Penning trap experiment is to use a ferromagnetic central trap electrode in the shape of an annular disc with inner radius r_1 , outer radius r_2 and a thickness of $2a$. This distorts the homogeneity of the external magnetic field used for confinement and forms a magnetic bottle of strength

$$B_2 = 3\mu_0 M_0 \left(\frac{ar_1^2}{2(a^2 + r_1^2)^{5/2}} - \frac{ar_2^2}{2(a^2 + r_2^2)^{5/2}} \right), \quad (\text{B.53})$$

that allows us to perform precision measurements of magnetic moments in the form of microwave spectroscopy to determine the **Larmor frequency** of the spin precession around the magnetic field, from which the **magnetic moments** can be derived.

In this context, the magnetic field strength needs to be determined as well, usually from the measured oscillation frequencies and application of the invariance theorem. However, the **invariance theorem** does not hold in the presence of a magnetic bottle. The relative shift of the cyclotron frequency for a magnetic field with $B_2 \neq 0$ can be linearly approximated in terms of the motional amplitudes by

$$\frac{\Delta\omega_c}{\omega_c} \approx \frac{B_2}{B_0} \left[z^2 - \frac{\rho_+^2}{4} \left(1 + \frac{\omega_c}{\omega_+ - \omega_-} \right) - \frac{\rho_-^2}{4} \left(1 + \frac{\omega_c}{\omega_+ - \omega_-} \right) \right]. \quad (\text{B.54})$$

This usually makes it necessary to spatially separate the measurement of the spin orientation and the measurement of the magnetic field, which is done by the so-called '**double-trap technique**'.

The double-trap technique requires adiabatic particle transport between confinement regions, i.e. transport through the magnetic field without a change of the spin state. This is connected with the **adiabaticity condition**

$$\frac{1}{B} \left| \frac{\partial B}{\partial t} \right| = \frac{v}{B} \left| \frac{\partial B}{\partial z} \right| \ll \frac{\mu B}{\hbar} = \omega_L, \quad (\text{B.55})$$

where v is the velocity of the transport and the Larmor frequency ω_L is the spin precession frequency in the magnetic field B .

B.2 Statements Specific to Particle Ensembles

This section treats behaviour that is specific to ensembles only. When many particles are confined, the corresponding charge distribution locally lifts the trap potential and changes the oscillation frequencies. This also sets a limit to the amount of charge

that can be confined in a given trap. Assuming cold particles in a nearly spherical ensemble, the space-charge potential has the same quadratic shape as the external confining potential. The presence of **space charge** results in a shift of the oscillation frequencies given by

$$\omega'_z = \omega_z \sqrt{1 - \frac{\omega_p^2}{3\omega_z^2}} \quad \text{and} \quad \omega'_\pm = \frac{\omega_c}{2} \left(1 \pm \sqrt{1 - \left(1 + \frac{2\omega_p^2}{3\omega_z^2}\right) \frac{2\omega_z^2}{\omega_c^2}} \right), \quad (\text{B.56})$$

where ω_p is the **plasma frequency** (characteristic frequency of charge oscillation in the plasma) given by

$$\omega_p^2 = \frac{q^2 n}{\varepsilon_0 m}. \quad (\text{B.57})$$

When a confined particle ensemble is dense enough such that the **Debye length**

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{2nq^2}} \quad (\text{B.58})$$

is much smaller than any of the ensemble dimensions, the particles are coupled by their mutual Coulomb forces such that their dynamics cannot be described by a single-particle approach, but instead by terms of a single-component plasma. A useful quantity for the characterization of ion plasmas is the **plasma parameter** Γ_p which measures the Coulomb energy between ions relative to their thermal energy. It is defined by

$$\Gamma_p \equiv \frac{q^2}{4\pi \varepsilon_0 a_{\text{WS}} k_B T}, \quad (\text{B.59})$$

where q is the ion charge, T is the ensemble temperature and $a_{\text{WS}} = (4/3 \pi n)^{1/3}$ is the Wigner-Seitz radius measuring the effective ion-ion distance at a given ion number density n . Commonly, one speaks of a weakly correlated plasma (a gas-like state) for $\Gamma_p \ll 1$, and of a strongly correlated plasma for $\Gamma_p > 1$. Theoretical studies predict a fluid-like behaviour for $174 > \Gamma_p > 2$ and a crystal-like behaviour for $\Gamma_p > 174$. At sufficiently low kinetic energies, i.e. for sufficiently large plasma parameters $\Gamma_p \geq 175$, confined particles ‘crystallise’ into well-defined structures given by their mutual Coulomb repulsion in the presence of the confining trap potential (**‘ion crystals’**).

When more than one species is present, the particles may undergo a spatial radial separation. The criterion for this **centrifugal separation** between particle species ‘1’ and ‘2’ is

$$e \left| \frac{m_1}{q_1} - \frac{m_2}{q_2} \right| \omega_r^2 R_p^2 > k_B T, \quad (\text{B.60})$$

where ω_r is the global rotation frequency of the ensemble and R_p is the radial extension of the plasma. A particle ensemble that may be regarded as a plasma possesses many possible modes of internal oscillation, the so-called **‘plasma modes’**.

These can be resonantly excited by irradiation of electromagnetic radiation of the proper geometry. The general situation for **rotating multipoles** can be expressed by

$$U_k(t) = U \cos(m(\phi_k - \omega t)), \quad (\text{B.61})$$

where the voltage $U_k(t)$ is applied to the k th electrode (along the sense of rotation), and where the phase ϕ_k is given by

$$\phi_k = 2\pi \frac{k}{n}, \quad (\text{B.62})$$

in which n is the number of segments in use, and the quantity m represents dipole excitation ($m = 1$), quadrupole excitation ($m = 2$) and so forth.

One application of such a rotating multipole is the so-called ‘**rotating wall technique**’, which allows one to apply a torque to a confined low-energy plasma and to drive its global rotation frequency to the applied rotation frequency ω . The particle number density n is then uniquely determined by ω via

$$n = \frac{2m\varepsilon_0}{q^2} \omega(\omega_c - \omega), \quad (\text{B.63})$$

which assumes the maximum density, the **Brillouin flow**, for $\omega = \omega_c/2$.

The **shape of the confined plasma** is that of a spheroid, a bi-axial ellipsoid, with an **aspect ratio** $\alpha = z/\rho$ that depends on geometry parameters $a(\alpha)$ and $b(\alpha)$ implicitly given by

$$\frac{qn}{6\varepsilon_0} (a(\alpha)\rho^2 + b(\alpha)z^2) = \frac{m}{2q} \omega(\omega_c - \omega)\rho^2 + \frac{U}{4d^2} (2z^2 - \rho^2), \quad (\text{B.64})$$

that fulfill the condition $2a(\alpha) + b(\alpha) = 3$ due to Poisson’s law. For a given particle species in a given trap, the plasma particle number density n is hence determined by the rotation frequency ω , and at that rotation, the shape of the plasma α is determined by the value of the trap potential. For small potentials, the plasma is **prolate** (‘cigar-shaped’, $\alpha > 1$) along the trap axis, while for large values of the trap potential, it is **oblate** (‘pancake-shaped’, $\alpha < 1$). Inbetween, for $\alpha = 1$ we have the case of a spherical plasma.

Appendix C

Magnetic Field Creation and Control

For a number of applications, the absolute value of the magnetic field needs to be set and/or known with high accuracy. For applications like microwave spectroscopy of Zeeman transitions or level-crossing spectroscopy we additionally wish to have the possibility to scan over a certain range of the field strength in a fast and reproducible fashion. Coarse tuning of the magnetic field strength is possible with the current of the main magnet solenoids, which typically can be done with an accuracy of better than 1%. At a field strength of several Tesla, this leaves a region of the order of 10 mT for fine-tuning.

Such fine-tuning can be achieved for example by a small independent solenoid around the region of interest. This can be a Helmholtz arrangement as with the currents applied in the same sense of rotation, as we will see below. With a typical current of a few Amperes, the range of several tens of mT can be covered with a resolution of 0.1% or better, limited by the accuracy of the current source. Hence, the total magnetic field strength can be set with an accuracy of the order of $10\text{ }\mu\text{T}$. For further fine-tuning of this value and a scan across a certain region of field strengths, one can employ the position dependence of the effective field strength in the presence of non-zero field gradients. For typical Penning trap parameters, the resulting axial shift Δz is usually of the order of 1 mm/V . Considering the contribution from terms in B_1 (linear magnetic field gradient along z) results in a voltage-dependent magnetic field strength contribution given by

$$\Delta B = -B_1 z_0 d^2 \frac{C_1}{C_2} \frac{U_a}{U_0} \quad (\text{C.1})$$

which can cover several hundreds of μT . Such a gradient B_1 can well be produced by a Maxwell arrangement of current loops, as we will see below. The sign of ΔB can be chosen by the sign of U_a . Assuming a typical commercially available accuracy of 10^{-6} for the voltage U_a , the corresponding accuracy of ΔB is of the order of 1 nT or better, which represents a relative accuracy on the ppb level or below. This is comparable to the short-term field fluctuations of a typical superconducting magnet, to the accuracy to which particle oscillation frequencies can usually be measured

electronically, and to the natural fluctuation of the Earth's magnetic field within a time of one hour [1].

Calibration of the solenoid current and electrode voltage settings to the absolute value of the obtained magnetic field strength can be achieved by an electronic measurement of the cyclotron frequency of a well-known test particle such as $^{12}\text{C}^{5+}$ in that field. Such a measurement usually would make use of the identities (5.15) or the invariance theorem (5.27).

Apart from applications in double-resonance and radio-frequency spectroscopy, such magnetic field control may also be used for level-crossing spectroscopy, which has extensively been used to investigate properties of neutral atoms and singly charged atomic and molecular ions even before the invention of lasers. An overview of applied techniques and performed measurements can be found in [2, 3]. Typically, particles are confined in gas cells or traps and a variable homogeneous magnetic (or electric) field is applied, see for example [4–7]. Laser light is used to excite an electronic transition and the corresponding fluorescence is observed as a function of the external magnetic field strength. When a level crossing of the observed levels with other field-dependent levels occurs, e.g. Zeeman sublevels of the fine or hyperfine structure, the fluorescence is increased and thereby the corresponding value of the magnetic field is determined. For atoms and singly charged ions, this value is typically of the order of mT and can thus be readily produced by electromagnets. Level-crossing spectroscopy yields access to the polarisability of states and to the magnetic dipole and electric quadrupole interaction constants as well as to the electronic Landé-factor g_J [2, 8].

In principle, the magnetic field configuration produced by any arrangement of currents can be determined by integrating Biot-Savart's law

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I d\mathbf{l} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \quad (\text{C.2})$$

over the given geometry. This, however, yields analytical results only for very special cases and in general leads to expressions that cannot be evaluated easily. We would like to find some simple analytic expressions for the magnetic field of finite-length solenoids. In general, analytic solutions to this problem are approximations only, but may prove of value nonetheless.

C.1 Current Loop

We start with a single current loop of radius a , that is oriented perpendicular to the z -axis and runs a dc current I according to the right-hand rule. $z = 0$ is in the (x, y) -plane in which the loop lies. The magnetic potential in this situation is given by [9]

$$A(r, \theta) = \frac{\mu_0 a I}{4\pi} \int_0^{2\pi} \frac{\cos \phi'}{\sqrt{a^2 + r^2 - 2ar \sin \theta \cos \phi'}} d\phi' \quad (\text{C.3})$$

where (r, θ, ϕ) are the common spherical coordinates, i.e. r is the distance from the loop centre to the point of interest, and θ is the angle of the connecting line to the plane in which the loop lies. Since the problem has radial symmetry (with respect to the z -axis), the dependence on the radial angle ϕ integrates out. In spherical coordinates, the components of the magnetic field are then given by

$$\begin{aligned} B_r &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A \sin \theta) \\ B_\theta &= -\frac{1}{r} \frac{\partial}{\partial r} (Ar) \\ B_\phi &= 0, \end{aligned}$$

where B_r is the magnetic field component along the vector $\mathbf{r}$ and B_θ is the field component perpendicular to B_r , both at position $\mathbf{r} = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$. Let us define the following quantities, for simplicity of further notation:

$$k^2 \equiv \frac{4ar \sin \theta}{r^2 + a^2 + 2ar \sin \theta} \equiv 1 - \frac{\alpha^2}{\beta^2}, \quad (\text{C.4})$$

such that we use

$$\begin{aligned} \alpha^2 &\equiv r^2 + a^2 - 2ar \sin \theta \\ \beta^2 &\equiv r^2 + a^2 + 2ar \sin \theta. \end{aligned}$$

The k^2 we have introduced is the argument of the elliptic functions that will be needed to express the field components. Let us first stay in spherical coordinates:

$$\begin{aligned} B_r &= \frac{\mu_0 I a^2 \cos \theta}{\pi \alpha^2 \beta} E(k^2) \\ B_\theta &= \frac{\mu_0 I}{2\pi \alpha^2 \beta \sin \theta} \left((r^2 + a^2 \cos 2\theta) E(k^2) - \alpha^2 K(k^2) \right), \end{aligned}$$

in which $E(k^2)$ and $K(k^2)$ are the elliptic integrals as defined in [10]. For practical purposes, they can be evaluated by their sum representations

$$\begin{aligned} E(k) &= \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n} (n!)^2} \right]^2 \frac{k^{2n}}{1 - 2n} \\ K(k) &= \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n} (n!)^2} \right]^2 k^{2n}. \end{aligned}$$

Close to the axis ($\theta \ll 1$) we can avoid the use of elliptic integrals by expansion of the expressions like [9]

$$B_r \approx \frac{\mu_0 I a^2 \cos \theta}{2(a^2 + r^2)^{3/2}} (1 + P + \dots)$$

$$B_\theta \approx \frac{\mu_0 I a^2 \sin \theta}{4(a^2 + r^2)^{5/2}} \left(r^2 - 2a^2 - \frac{1}{2} P (4a^2 - 3r^2) + \dots \right),$$

in which we have used a dimensionless parameter

$$P \equiv \frac{15a^2 r^2 \sin^2 \theta}{4(a^2 + r^2)^2}. \quad (\text{C.5})$$

We can write the solutions also in axial and radial components using cylindrical coordinates, for simplicity of expression we re-define $k^2 \equiv 4ar/((a+r)^2 + z^2)$ and find

$$B_z = \frac{\mu_0 I}{\sqrt{(a+r)^2 + z^2}} \left(\frac{a^2 - r^2 - z^2}{(a-r)^2 + z^2} E(k) + K(k) \right)$$

$$B_\rho = \frac{\mu_0 I z / \sqrt{r^2 - z^2}}{\sqrt{(a+r)^2 + z^2}} \left(\frac{a^2 + r^2 + z^2}{(a-r)^2 + z^2} E(k) - K(k) \right).$$

Coming to the magnetic field components in Cartesian coordinates, we re-define our notation for simplicity of final expression, i.e. we use the common $r^2 = x^2 + y^2 + z^2$ and $\rho^2 = x^2 + y^2$, as well as

$$\alpha^2 \equiv r^2 + a^2 - 2a\rho$$

$$\beta^2 \equiv r^2 + a^2 + 2a\rho,$$

such that $k^2 \equiv 1 - \alpha^2/\beta^2$ still holds. We can now write [11]

$$B_x = \frac{\mu_0 I x z}{2\pi \alpha^2 \beta \rho^2} ((a^2 + r^2) E(k^2) - \alpha^2 K(k^2))$$

$$B_y = \frac{\mu_0 I y z}{2\pi \alpha^2 \beta \rho^2} ((a^2 + r^2) E(k^2) - \alpha^2 K(k^2))$$

$$B_z = \frac{\mu_0 I}{2\pi \alpha^2 \beta} ((a^2 - r^2) E(k^2) + \alpha^2 K(k^2)).$$

From this, we find as special cases: The magnetic field on the central axis of the loop at a distance z from the (x, y) -plane is given by

$$B_z = \frac{\mu_0 a^2 I}{(a^2 + z^2)^{3/2}}, \quad (\text{C.6})$$

such that in the centre of the loop we have $B_z = \mu_0 I/a$, while for symmetry reasons, on the z -axis we have $B_x = 0$ and $B_y = 0$ for all values of z . Further, close to the

axis, i.e. for $(x, y) \ll a$, we have

$$B_x = \frac{3\mu_0 a^2 I x z}{4(a^2 + z^2)^{5/2}}, \quad (\text{C.7})$$

and for reasons of rotational symmetry

$$B_y = \frac{3\mu_0 a^2 I y z}{4(a^2 + z^2)^{5/2}}. \quad (\text{C.8})$$

Far away from the loop (i.e. for $r \gg a$) we have a configuration that can be regarded like a dipole, hence we have

$$\begin{aligned} B_r &= \frac{\mu_0 a^2 I \cos \theta}{2} \frac{1}{r^3} \\ B_\theta &= \frac{\mu_0 a^2 I \sin \theta}{4} \frac{1}{r^3}, \end{aligned}$$

or in cylindrical coordinates

$$\begin{aligned} B_z &= \mu_0 a^2 I \frac{3 \cos^2 \theta - 1}{r^3} \\ B_\rho &= \mu_0 a^2 I \frac{3 \cos \theta \sin \theta}{r^3}. \end{aligned}$$

C.2 Helmholtz and Maxwell Loops

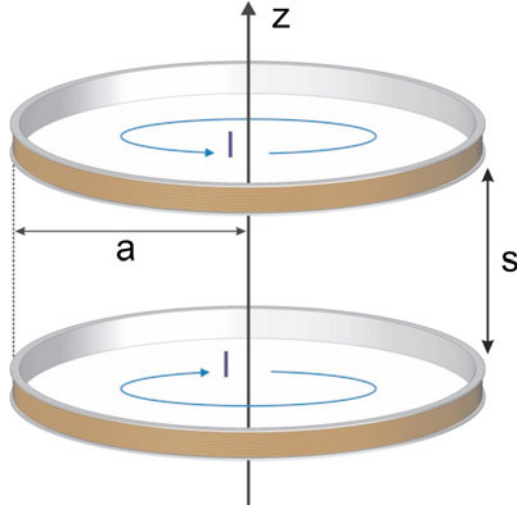
As a sidenote, we regard an arrangement of two coplanar current loops which have their centres on the z -axis and are separated along z by a distance s . If they have the same radius a and current I , this is commonly referred to as a ‘Helmholtz Coil’. The combined magnetic field is of course a superposition of the individual fields, such that (C.6) holds true for each contribution on the z -axis. A Taylor expansion of the superposition yields

$$B_z = \mu_0 a^2 I \left(\frac{1}{\left(a^2 + \frac{s^2}{4}\right)^{3/2}} - \frac{3(a^2 - s^2)}{\left(a^2 + \frac{s^2}{4}\right)^{7/2}} z^2 + \dots \right) \quad (\text{C.9})$$

which for the special case $a = s$ leads to

$$B_z(z = 0) = \frac{\mu_0 I}{\left(\frac{5}{4}\right)^{3/2} a} \quad (\text{C.10})$$

Fig. C.1 Maxwell arrangement of two co-planar circular solenoids with counter-propagating currents



A pair of coplanar concentric current loops with counter-propagating currents I is called a ‘Maxwell configuration’, sometimes also labelled as ‘anti-Helmholtz configuration’. Such an arrangement is depicted in Fig. C.1. For loops with a radius a and a separation of s along the z -axis, the axial field is given by

$$B_z = \frac{\mu_0 I a^2}{2 \left[a^2 + \left(\frac{s}{2} - z \right)^2 \right]^{3/2}} - \frac{\mu_0 I a^2}{2 \left[a^2 + \left(\frac{s}{2} + z \right)^2 \right]^{3/2}}. \quad (\text{C.11})$$

For $z = 0$ (in the centre of the arrangement) the magnetic field $B(z = 0)$ is zero. For symmetry reasons, this is also true for all even expansion terms $B_2, B_4, \dots$

The odd terms are given by

$$B_1 = \frac{\partial B_z}{\partial z} = 3\mu_0 I a^2 \frac{s/2}{\left[a^2 + \frac{s^2}{4} \right]^{5/2}} \quad (\text{C.12})$$

and

$$B_3 = \frac{\partial^3 B_z}{\partial z^3} = 15\mu_0 I a^2 \frac{s^3/2 - 3sa^2/2}{\left[a^2 + \frac{s^2}{4} \right]^{9/2}}. \quad (\text{C.13})$$

The cubic term vanishes for a geometry with $s = a\sqrt{3}$. In this case, the even terms are exactly zero and all odd terms except B_1 vanish or are negligible (of order z^5 or higher). This leaves a linear magnetic field gradient along the z axis of

$$B_1 = \frac{3^{3/2} \mu_0 I}{2 \left(\frac{7}{4}\right)^{5/2} a^2}. \quad (\text{C.14})$$

C.3 Finite Single-Layer Solenoid

Let us now make the transition to a cylindrical solenoid that can be regarded as a succession of current loops along the z -axis, again with radius a , and with a finite length L such that the solenoid extends from $-L/2$ to $+L/2$. The total number of windings is N . For the finite solenoid, it is again possible to start with the vector potential of the arrangement and find the magnetic field components. For this, we re-define our quantity k from above by

$$k \equiv \frac{4ar \sin \theta}{a^2 + r^2 + 2r \sin \theta + \zeta} = \frac{4a\rho}{(a + \rho)^2 + \zeta} \quad (\text{C.15})$$

and define $h \equiv k(\zeta = 0)$ and $\zeta_{\pm} = z \pm L/2$. Using the same conventions for E and K as above, the vector potential of the solenoid is given by

$$A = \frac{\mu_0 I N}{2\pi} \sqrt{\frac{a}{L^2 \rho}} \left[\frac{k^2 + h^2 - h^2 k^2}{h^2 k^2} K(k) - \frac{E(k)}{k^2} + \frac{h^2 - 1}{h^2} \Pi(h^2, k) \right]_{\zeta_-}^{\zeta_+} \quad (\text{C.16})$$

in which $\Pi(h^2, k)$ is the elliptic integral of the third kind given by

$$\Pi(h^2, k) = \int_0^{\pi/2} \left((1 - h^2 \sin^2 \theta) \sqrt{1 - k \sin^2 \theta} \right)^{-1} d\theta. \quad (\text{C.17})$$

Unfortunately, there seems to be no sum representation of the integral. From this vector potential, in similarity to above, we obtain the magnetic field components perpendicular to the axis

$$B_{\rho} = \frac{\mu_0 I N}{2\pi L} \sqrt{\frac{a}{\rho}} \left[\frac{k^2 - 2}{k} K(k) + \frac{2}{k} E(k) \right]_{\zeta_-}^{\zeta_+} \quad (\text{C.18})$$

and parallel to the axis

$$B_z = \frac{\mu_0 I N}{4\pi L} \sqrt{\frac{1}{a\rho}} \left[\zeta k \left(K(k) + \frac{a - \rho}{a + \rho} \Pi(h^2, k) \right) \right]_{\zeta_-}^{\zeta_+}. \quad (\text{C.19})$$

This solution is fine, however, it is hard to derive from it simple approximate equations. A second option is to start from the scalar potential and expand into a series that can be taken up to a certain order. For simplicity of notation, let us define a

quantity

$$C = \frac{\mu_0 I N}{L} \quad (\text{C.20})$$

and start with the scalar potential along the z -axis

$$\Phi(z) = -C \left(\sqrt{a^2 + \zeta_+^2} - \sqrt{a^2 + \zeta_-^2} \right). \quad (\text{C.21})$$

Correspondingly, from $B_z = -\partial\Phi(z)/\partial z$ we find the magnetic field along the z -axis to be given by

$$B_z = C \left(\frac{\zeta_+}{\sqrt{a^2 + \zeta_+^2}} - \frac{\zeta_-}{\sqrt{a^2 + \zeta_-^2}} \right) \quad (\text{C.22})$$

which in the centre ($z = 0$) simplifies to

$$B_z(z = 0) = \frac{\mu_0 I N}{\sqrt{a^2 + \frac{L^2}{4}}}. \quad (\text{C.23})$$

For a slim solenoid ($a \ll L$) this further simplifies to

$$B_z(z = 0) = \frac{\mu_0 I N}{L}, \quad (\text{C.24})$$

and in that case is exactly twice the value it takes at the end faces of the solenoid, i.e. on the axis at $z = -L/2$ and $z = +L/2$:

$$B_z(z = \pm L/2) = \frac{\mu_0 I N}{2L}. \quad (\text{C.25})$$

Again, like for the current loop, the magnetic field on the central axis (z -axis) only has axial components, while $B_x = B_y = B_\rho = 0$. As long as we are interested in the magnetic field inside the coil only, we may expand the scalar potential in a Taylor series to get [12]

$$\begin{aligned} \Phi(r, \theta) = & -C \frac{Lr \cos \theta}{\left(a^2 + \frac{L^2}{4}\right)^{1/2}} \\ & + C \left(\frac{5}{2} \cos^3 \theta + \frac{3}{2} \cos \theta \right) \left(\frac{L}{2 \left(a^2 + \frac{L^2}{4}\right)^{3/2}} - \frac{L^3}{8 \left(a^2 + \frac{L^2}{4}\right)^{5/2}} \right) r^3 + \dots \end{aligned} \quad (\text{C.26})$$

The components of the magnetic field are then given by

$$\begin{aligned}
B_r &= -\frac{\partial}{\partial r} \Phi(r, \theta) \\
B_\theta &= -\frac{1}{r} \frac{\partial}{\partial \theta} \Phi(r, \theta) \\
B_\phi &= 0,
\end{aligned}$$

namely

$$\begin{aligned}
B_r &= C \frac{L \cos \theta}{\left(a^2 + \frac{L^2}{4}\right)^{1/2}} \\
&\quad - C \frac{3}{2} r^2 (5 \cos^3 \theta + 3 \cos \theta) \left(\frac{L}{2 \left(a^2 + \frac{L^2}{4}\right)^{3/2}} - \frac{L^3}{8 \left(a^2 + \frac{L^2}{4}\right)^{5/2}} \right) + \dots
\end{aligned} \tag{C.27}$$

and perpendicular to that

$$\begin{aligned}
B_\theta &= -C \frac{L \sin \theta}{\left(a^2 + \frac{L^2}{4}\right)^{1/2}} \\
&\quad - C \left(\frac{3}{2} \sin \theta + \frac{15}{2} \cos^2 \theta \sin \theta \right) \left(\frac{L}{2 \left(a^2 + \frac{L^2}{4}\right)^{3/2}} - \frac{L^3}{8 \left(a^2 + \frac{L^2}{4}\right)^{5/2}} \right) r^2 + \dots
\end{aligned} \tag{C.28}$$

These statements can be converted into statements in axial and radial directions, respectively, by

$$\begin{aligned}
B_\rho &= B_r \sin \theta + B_\theta \cos \theta \\
B_z &= B_r \cos \theta - B_\theta \sin \theta.
\end{aligned}$$

Making use of the coordinate relations $z = r \cos \theta$, $\rho = r \sin \theta$ and $r^2 = z^2 + \rho^2$, this leads to

$$B_\rho \approx C \frac{3La^2r^2}{2 \left(a^2 + \frac{L^2}{4}\right)^{5/2}} \sin \theta \cos \theta = C \frac{3La^2\rho z}{2 \left(a^2 + \frac{L^2}{4}\right)^{5/2}} = \frac{3\mu_0 I N a^2 \rho z}{2 \left(a^2 + \frac{L^2}{4}\right)^{5/2}} \tag{C.29}$$

for the radial field component, which for slim coils ($a \ll L$) simplifies to

$$B_\rho = \frac{48\mu_0 I N a^2 \rho z}{L^5}. \tag{C.30}$$

The axial field component B_z is given by

$$B_z = \frac{2CL}{(4a^2 + L^2)^{1/2}} \left(1 - \frac{6a^2r^2}{(4a^2 + L^2)^2} - \frac{18a^2r^2 \cos 2\theta}{(4a^2 + L^2)^2} \right), \quad (\text{C.31})$$

which by use of $\cos 2\theta = 2 \cos^2 \theta - 1$ and $z = r \cos \theta$ can be written as

$$\begin{aligned} B_z &= \frac{2CL}{(4a^2 + L^2)^{1/2}} \left(1 + \frac{12a^2r^2}{(4a^2 + L^2)^2} - \frac{36a^2z^2}{(4a^2 + L^2)^2} \right) \\ &= \frac{2\mu_0 IN}{(4a^2 + L^2)^{1/2}} \left(1 + \frac{12a^2(\rho^2 + z^2)}{(4a^2 + L^2)^2} - \frac{36a^2z^2}{(4a^2 + L^2)^2} \right), \end{aligned}$$

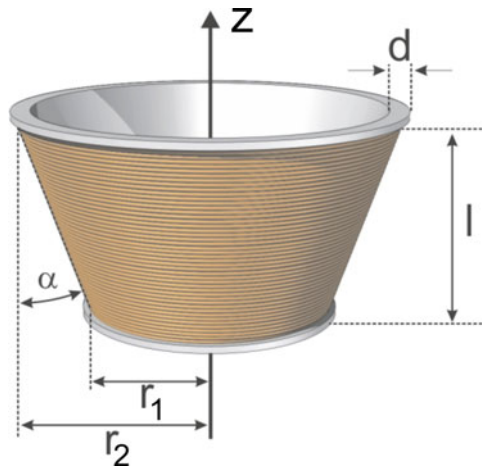
which for slim coils ($a \ll L$) simplifies to

$$B_z \approx \frac{2\mu_0 IN}{L} \left(1 + \frac{12a^2(\rho^2 + z^2)}{L^4} - \frac{36a^2z^2}{L^4} \right). \quad (\text{C.32})$$

C.4 Conical Single-Layer Solenoid

We assume a conical solenoid of length l , radii r_1 and r_2 on the respective ends and N windings with a current I . Let the centre of the cone be at $z = 0$ such that the ends are at $z = \pm l/2$ respectively, and $a = r(z = 0) = (r_1 + r_2)/2$, see Fig. C.2. This geometry defines an opening angle α given by $\alpha = \tan^{-1}((r_2 - r_1)/l)$.

Fig. C.2 Conical solenoid with an opening angle α with respect to a cylindrical solenoid



Starting with the on-axis field of N current loops

$$B_z(z) = \frac{IN\mu_0 r^2}{2(r^2 + z^2)^{3/2}} \quad (\text{C.33})$$

and integrating all contributions from $z = -L/2$ to $z = L/2$ along the conical geometry given by $r(z) = a + z \tan \alpha$ to the on-axis field in the centre ($z = 0$) yields an expression which can be expanded in terms of the opening angle α . Neglecting higher-order contributions (since the opening angle is assumed small), one obtains the expression

$$B_1(z = 0) = \frac{\partial B_z}{\partial z}(z = 0) \approx \frac{12IN\mu_0 L^2 a \tan \alpha}{(L^2 + 4a^2)^{5/2}} \quad (\text{C.34})$$

C.5 Finite Multi-layer Solenoid

By laws of linear superposition, the field configuration of a multi-layer solenoid can be obtained from summation over (single-layer) solenoids as discussed above. For a multi-layer solenoid of homogeneous winding density with inner radius a and outer radius b and again using $\zeta_{\pm} = z \pm L/2$, we find an expression for the on-axis axial field strength by integration

$$B_z(z = 0) = \int_a^b \frac{1}{b-a} \int_{-L/2}^{+L/2} B_z(z, z') dz' dr \quad (\text{C.35})$$

of axial magnetic field contributions

$$B(z, z') = \mu_0 IN \frac{r^2}{((z - z')^2 + r^2)^{3/2}}, \quad (\text{C.36})$$

which results in

$$B_z(z = 0) = \frac{\mu_0 IN}{L} \frac{1}{b-a} \left[\zeta_+ \ln \frac{b + \sqrt{b^2 + \zeta_+^2}}{a + \sqrt{a^2 + \zeta_+^2}} - \zeta_- \ln \frac{b + \sqrt{b^2 + \zeta_-^2}}{a + \sqrt{a^2 + \zeta_-^2}} \right]. \quad (\text{C.37})$$

We see that with increasing complication of geometry, simple expressions are available only for the limited parameter space along the central axis (z -axis) or even only at the origin.

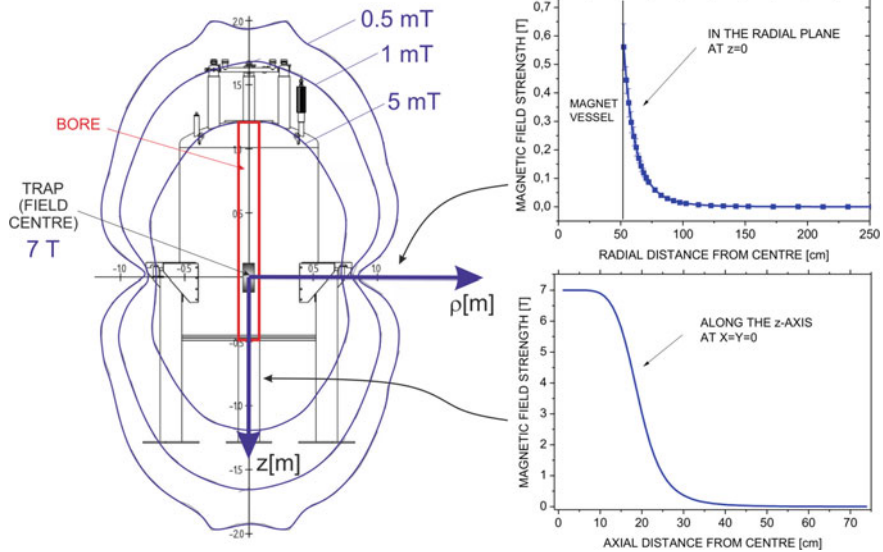


Fig. C.3 Calculated and measured magnetic field of a superconducting magnet

We therefore stop here and leave the rest to finite-element codes and the like. Figure C.3 shows the result of a finite-element calculation of a real superconducting magnet with finite multi-layer solenoids.

References

1. F.D. Stacey, P.M. Davis, *Physics of the Earth*, 4th edn. (Cambridge University Press, Cambridge, 2008)
2. W. Demtröder, *Laser Spectroscopy* (Springer, Heidelberg, 2003)
3. P. Hannaford, Oriented atoms in weak magnetic fields. *Physica Scripta* **T70**, 117 (1997)
4. S. Rydberg, S. Svanberg, Investigation of the np 2P_{3/2} level sequence in the Cs I spectrum by level crossing spectroscopy. *Physica Scripta* **5**, 209 (1972)
5. J. Alnis, K. Blushs, M. Auzinsh, S. Kennedy, N. Shafer-Ray, E.R.I. Abraham, The Hanle effect and level crossing spectroscopy in Rb vapour under strong laser excitation. *J. Phys. B* **36**, 1161 (2003)
6. D. Budker, D.F. Kimball, D.P. DeMille, *Atomic Physics* (Oxford University Press, Oxford, 2004)
7. W. Hogervorst, S. Svanberg, Stark effect investigation of D states in ⁸⁵Rb and ¹³³Cs using level crossing spectroscopy with a CW dye laser. *Physica Scripta* **12**, 67 (1975)
8. J. Bengtsson, J. Larsson, S. Svanberg, C.G. Wahlström, Hyperfine-structure study of the 3d10-5p 2P_{3/2} level of neutral copper using pulsed level-crossing spectroscopy at short laser wavelengths. *Phys. Rev. A* **41**, 233 (1990)
9. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)

10. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover (1972)
11. J. Simpson, J. Lane, C. Immer, R. Youngquist, Simple analytic expressions for the magnetic field of a circular current loop. nasa techdoc 20010038494
12. S.R. Muniz, V.S. Bagnato, M. Bhattacharya, Analysis of off-axis solenoid fields using the magnetic scalar potential: an application to a Zeeman-slower for cold atoms. *Am. J. Phys.* **83**, 513 (2015)

Appendix D

Equivalence of Results for Magnetic Bottle Strength

In literature, we find two approaches to the problem of calculating the coefficient B_2 of an annular magnetic bottle [1–3] which at first glance seem not to be equivalent. Here, we take a closer look to find that indeed they are, and lead to the same result, as they should.

Assume again a ferromagnetic ring electrode with inner radius r_1 , outer radius r_2 and a thickness of $2a$ in an external homogeneous magnetic field. Such an arrangement will distort the homogeneity of the magnetic field to form a magnetic bottle of the kind $\mathbf{B} = B_2 ((z^2 - \rho^2/2)\mathbf{e}_z - z\rho\mathbf{e}_\rho)$ with $B_2 \neq 0$ superimposed on the magnetic trapping field B_0 . $\mathbf{e}_z$ is the unit vector along the z -axis (field axis) and $\mathbf{e}_\rho$ is the radial coordinate such that the inhomogeneity has radial symmetry around the ring centre. The strength of the magnetic bottle is characterised by the coefficient B_2 which has a unit of Tesla per square metre.

It can be calculated by starting with Maxwell's equations in the form $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times (\mathbf{H} + 4\pi\mathbf{M}) = 0$ such that for the scalar magnetic potential Φ we have $\Delta\Phi = -4\pi\rho_M = 4\pi\nabla \cdot \mathbf{M}$, $\mathbf{M}$ being the magnetisation which will be assumed homogeneous and equal to M_0 , given in Tesla. Following the discussion in [1, 2], the scalar magnetic potential written in terms of the magnetic surface charge density $\sigma_M = \mathbf{n} \cdot \mathbf{M}$ on the z -axis is thus given by the integral over the closed surface

$$\Phi(z) = \oint_S \frac{\sigma_M}{|\mathbf{x} - \mathbf{x}'|} da', \quad (\text{D.1})$$

which for our homogeneously magnetised ring is

$$M_0 \int_0^{2\pi} \int_{r_1}^{r_2} \frac{\rho'}{\sqrt{(z-a)^2 + \rho'^2}} - \frac{\rho'}{\sqrt{(z+a)^2 + \rho'^2}} d\phi' d\rho'. \quad (\text{D.2})$$

Defining for convenience $X_{\pm}^{(1,2)} \equiv ((z \pm a)^2 + r_{(1,2)}^2)^{1/2}$, this integration results in

$$\Phi(z) = 2M_0 \left(X_-^{(2)} - X_-^{(1)} - X_+^{(2)} + X_+^{(1)} \right). \quad (\text{D.3})$$

The magnetic field on the axis is given as the corresponding derivative of the magnetic potential, i.e.

$$\frac{d}{dz}\Phi = H(z) = 2\pi M_0 \left(\frac{z-a}{X_-^2} - \frac{x-a}{X_-^1} - \frac{z+a}{X_+^2} + \frac{z+a}{X_+^1} \right). \quad (\text{D.4})$$

Defining further $Y_{1,2} \equiv (a^2 + r_{(1,2)}^2)^{1/2}$, using that in vacuum $\mathbf{H} = \mathbf{B}$, and performing a series expansion of the square roots yields the linear and quadratic term of the field in the ring centre ($z = 0$) to be

$$B_0 = \frac{M_0}{2} \left(\frac{2a}{Y_1^2} - \frac{2a}{Y_2^2} \right) \quad (\text{D.5})$$

and

$$B_2 = \frac{M_0}{2} \left[\frac{3}{Y_1} \left(-\frac{a}{Y_1^2} + \frac{a^3}{Y_1^4} \right) + \frac{3}{Y_2} \left(\frac{a}{Y_2^2} - \frac{a^3}{Y_2^4} \right) \right]. \quad (\text{D.6})$$

This is the final result in [1, 2]. Coming back to the definitions of Y , it can, however, be further simplified to

$$B_2 = 3M_0 \left(\frac{ar_1^2}{2(a^2 + r_1^2)^{5/2}} - \frac{ar_2^2}{2(a^2 + r_2^2)^{5/2}} \right). \quad (\text{D.7})$$

Brown and Gabrielse [3] also start with the Maxwell equations and write the scalar magnetic potential on the z -axis in the coordinates of the ring. They construct infinitesimal rings of uniformly distributed magnetic dipoles. Taking account for the factor $\mu_0/4\pi$ between unit systems, their expression (6.14) in [3]

$$\Phi(z) = \frac{M_0}{2} \int \frac{\partial}{\partial z} (\rho'^2 + (z' - z)^2)^{-1/2} \rho' d\rho' dz' \quad (\text{D.8})$$

is a volume integral summing up the magnetic scalar potential contributions from all infinitesimal rings and is equivalent to the surface integral (D.2). At the same time, the scalar magnetic potential can be written as a general multipole expansion of the form

$$\Phi(z) = \sum_{l=1}^{\infty} l^{-1} B_{l-1} r^l P_l(\cos \theta). \quad (\text{D.9})$$

Expanding (D.8) in powers of z and identifying the coefficients B_l in the general multipole expansion (D.9), one finds

$$B_l = (l+1)(l+2) \frac{M_0}{2} \int r'^{l-3} P_{l+2}(\cos \theta') \rho' d\rho' dz', \quad (\text{D.10})$$

such that

$$B_2 = 6M_0 \int r'^{-5} P_4(\cos \theta') \rho' d\rho' dz'. \quad (\text{D.11})$$

Taking the Legendre polynomial $P_4(x) = (3 - 30x^2 + 35x^4)/8$ for $x = \cos(\theta)$ and substituting $\cos(\theta)$ by $z/(\rho^2 + z^2)^{1/2}$ according to the geometry, this equation becomes

$$B_2 = 6M_0 \int_{r_1}^{r_2} \int_{-a}^{+a} P_4\left(\frac{z'}{\sqrt{\rho'^2 + z'^2}}\right) \frac{1}{(\rho'^2 + z'^2)^{5/2}} \rho' d\rho' dz', \quad (\text{D.12})$$

which can be analytically integrated to give

$$B_2 = 6M_0 \left(\frac{ar_1^2}{4(a^2 + r_1^2)^{5/2}} - \frac{ar_2^2}{4(a^2 + r_2^2)^{5/2}} \right), \quad (\text{D.13})$$

which is equivalent to the result in (D.7).

We also wish to derive equation (21.6) that shows the deviation from the invariance theorem in presence of such a magnetic bottle. Starting with the invariance theorem

$$\omega_c^2 = \omega_+^2 + \omega_z^2 + \omega_-^2 \quad (\text{D.14})$$

we introduce shifts to all eigenfrequencies to see the resulting shift $\Delta\omega_c$ of the unperturbed cyclotron frequency ω_c . So, we begin with

$$(\omega_c + \Delta\omega_c)^2 = (\omega_+ + \Delta\omega_+)^2 + (\omega_z + \Delta\omega_z)^2 + (\omega_- + \Delta\omega_-)^2. \quad (\text{D.15})$$

We may now use the relations (21.5)

$$\Delta\omega_z = -\frac{\omega_+ - \omega_-}{\omega_z} \Delta\omega_+ \quad \text{and} \quad \Delta\omega_- = -\Delta\omega_+ \quad (\text{D.16})$$

when we detail both sides of equation (D.15):

$$\begin{aligned} \omega_c^2 + 2\omega_c \Delta\omega_c + \Delta\omega_c^2 &= \omega_+^2 + 2\omega_+ \Delta\omega_+ + \Delta\omega_+^2 \\ &+ \omega_z^2 - 2\omega_z \frac{\omega_+ - \omega_-}{\omega_z} \Delta\omega_+ + \left(\frac{\omega_+ - \omega_-}{\omega_z} \right)^2 \Delta\omega_+^2 \\ &+ \omega_-^2 - 2\omega_- \Delta\omega_+ + \Delta\omega_+^2 \end{aligned} \quad (\text{D.17})$$

Again using the invariance theorem, this simplifies to

$$\begin{aligned}
2\omega_c \Delta\omega_c + \Delta\omega_c^2 &= 2\omega_+ \Delta\omega_+ + \Delta\omega_+^2 \\
&\quad - 2(\omega_+ - \omega_-) \Delta\omega_+ + \left(\frac{\omega_+ - \omega_-}{\omega_z} \right)^2 \Delta\omega_+^2 \\
&\quad - 2\omega_- \Delta\omega_+ + \Delta\omega_+^2.
\end{aligned} \tag{D.18}$$

We now divide either side by ω_c^2 to get

$$\begin{aligned}
2\frac{\Delta\omega_c}{\omega_c} + \frac{\Delta\omega_c^2}{\omega_c^2} &= 2\frac{\omega_+}{\omega_c^2} \Delta\omega_+ + \frac{\Delta\omega_+^2}{\omega_c^2} \\
&\quad - 2\frac{\omega_+ - \omega_-}{\omega_c^2} \Delta\omega_+ + \left(\frac{\omega_+ - \omega_-}{\omega_z} \right)^2 \frac{\Delta\omega_+^2}{\omega_c^2} \\
&\quad - 2\frac{\omega_-}{\omega_c^2} \Delta\omega_+ + \frac{\Delta\omega_+^2}{\omega_c^2}.
\end{aligned} \tag{D.19}$$

For the typical frequency hierarchy $\omega_c \approx \omega_+ \gg \omega_z \gg \omega_-$ we can ignore the term $\Delta\omega_c^2/\omega_c^2$ on the left hand side and use $\Delta\omega_+^2/\omega_c^2 \approx \Delta\omega_+^2/\omega_+^2$ on the right hand side. Looking at the right hand side, we also find that

$$2\frac{\omega_+}{\omega_c^2} \Delta\omega_+ - 2\frac{\omega_+ - \omega_-}{\omega_c^2} \Delta\omega_+ - 2\frac{\omega_-}{\omega_c^2} \Delta\omega_+ = 0. \tag{D.20}$$

With this, equation (D.19) becomes

$$2\frac{\Delta\omega_c}{\omega_c} = 2\frac{\Delta\omega_+^2}{\omega_+^2} + \left(\frac{\omega_+ - \omega_-}{\omega_z} \right)^2 \frac{\Delta\omega_+^2}{\omega_+^2} \tag{D.21}$$

which yields the stated relation

$$\frac{\Delta\omega_c}{\omega_c} = \left(1 + \frac{1}{2} \left(\frac{\omega_+ - \omega_-}{\omega_z} \right)^2 \right) \frac{\Delta\omega_+^2}{\omega_+^2}. \tag{D.22}$$

References

1. N. Hermanspahn, Aufbau eines Tieftemperaturkryostaten zum Betrieb einer Penningfalle. Diploma thesis, University of Mainz, 1996
2. N. Hermanspahn, Das magnetische Moment des gebundenen Elektrons in wasserstoffartigem Kohlenstoff C^{5+} . Ph.D. thesis, University of Mainz, 1999
3. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. Rev. Mod. Phys. **58**, 233 (1986)

Appendix E

Cryogenics

E.1 Heat Load Considerations

As far as the use of superconducting devices is concerned, we have seen that it is necessary to reach temperatures of or close to liquid-helium temperature around 4 K. This is most often achieved by evaporation of liquid helium from dewars, or by operation of closed-cycle systems which also use helium, such as pulse-tube cryo-coolers or re-condensing systems.

When liquid helium is used, as common in precision experiments that need to avoid mechanical vibrations, the extremely small vaporisation heat of liquid helium of only 2.6 kJ per litre makes it necessary to properly isolate all components that need to be at that temperature. This number corresponds to a power of 30 mW if one litre of liquid helium is to be used up in the course of one day. The so-called ‘heat-load’, the energy dissipation into the cooled region, needs to be minimized. There are three different contributions to the heat load of a given system:

- **Convection**, that is transport of heat by matter flow such as by residual gas. The heat load due to this mechanism is proportional to the gas temperature and the amount of gas. At cryogenic temperatures in closed or nearly closed systems this is negligible mainly due to freeze-out. In other cases, an isolation vacuum in the region of 10^{-4} hPa or better is usually used.
- **Heat conduction**, the unavoidable influx of thermal energy via vibrational excitations in solids that connect the cooled region to the rest of the apparatus and the outside world. It is proportional to the temperature difference and the cross section of the connections, and inversely proportional to the length of the connecting paths. This load is usually minimized by proper choice of materials with low heat conduction coefficients, and by minimizing the cross sections of those elements while maximizing the path of energy conduction. Here, often thin-walled tubes of stainless steel, Kevlar wires, Nylon or PEEK spacers and the like are used. Also cables with conductors other than copper are used, like aluminium, brass or stainless steel.

- **Thermal radiation.** The heat load due to thermal radiation is given by the difference of power radiated inward and radiated outward. According to Stefan-Boltzmanns law the power P emitted per unit surface S at a temperature T is given by

$$\frac{\partial P}{\partial S} = \varepsilon \sigma T^4, \quad (\text{E.1})$$

where ε is the emissivity of the surface and $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2} \text{ K}^{-4}$ is the Stefan-Boltzmann constant. Due to the dependence on T^4 it is important to keep the temperature difference between the respective surfaces small. This is usually realized by thermal shielding in different temperature stages, such as for example a liquid-nitrogen shield at 77 K which in comparison to a room temperature surrounding reduces the heat load already by a factor of $(300/77)^4 \approx 230$. Primary stages of cryo-coolers often work at 45 K, in that case we even have $(300/45)^4 \approx 1975$.

E.2 Low-Temperature Material Properties

E.2.1 Thermal Conductivity

We should note that the thermal conductivity of materials at 4 K is much different from the corresponding value at room temperature. Table E.1 lists the thermal conductivities of the material most common to cryogenic Penning trap experiments.

The thermal conductivity C is given in $\text{W}/(\text{m} \cdot \text{K})$. The linear relative thermal expansion L between 4 and 300 K is defined as $(L_{300} - L_4)/L_{300}$ and given in 10^{-4} . Bulk silver and gold are rarely used in Penning trap experiments, but in general have roughly the same properties as copper.

Table E.1 Comparison of thermal conductivities C of various materials at 300 and at 4 K, as well as relative thermal expansion coefficients L [1]. C is given in $\text{W}/(\text{m} \cdot \text{K})$, L is defined as $(L_{300} - L_4)/L_{300}$ and given in 10^{-4}

Material	C @ 4 K	C @ 300 K	L
Copper (OFHC)	2000	400	32.6
Aluminium (6061)	6	170	41.4
Copper-Beryllium (CuBe)	2	10	32.4
Stainless steel (304)	0.3	15	29
Teflon	0.05	0.3	214
Kevlar	0.03	2	
Sapphire	10	30	7.9
PEEK	0.025	0.25	

E.2.2 Electrical Resistivity

With decreasing temperature, the the electrical conductivity of metals decreases. We need to stress that in the low-temperature regime, the exact value of the resistivity depends crucially on impurities. This fact is usually reflected in the so-called ‘residual resistivity ratio’ (RRR), which can easily span a few orders of magnitude for different available purities and standards of for example copper. It is defined as $RRR = \rho(300\text{ K})/\rho(4\text{ K})$. Commercially pure copper wire has an RRR of 50–500, whereas very high-purity copper, well-annealed, can have an RRR of around 2000. Special measures to reduce the effectiveness of electron scattering centres, such as oxygen annealing, can raise the RRR to 50,000 [2].

In the presence of a magnetic field as used in Penning trap experiments, the effective value of RRR is decreased due to magneto-resistance, as expressed in Kohler plots [3]. For a typical example with copper, an initial RRR of 100 can be effectively decreased to about 25 when a magnetic field of 10 T is present.

E.2.3 Specific Heat

The specific heat decreases significantly when comparing 300 to 4 K, for example from 390 to 0.1 J/(kgK) in case of copper and from about 900 J/(kgK) to about 3 J/(kgK) in the case of aluminium. This leads to large temperature changes for comparatively small energy influx at low temperatures. It especially means that the superconducting state may be easily broken by small changes of the cooling power.

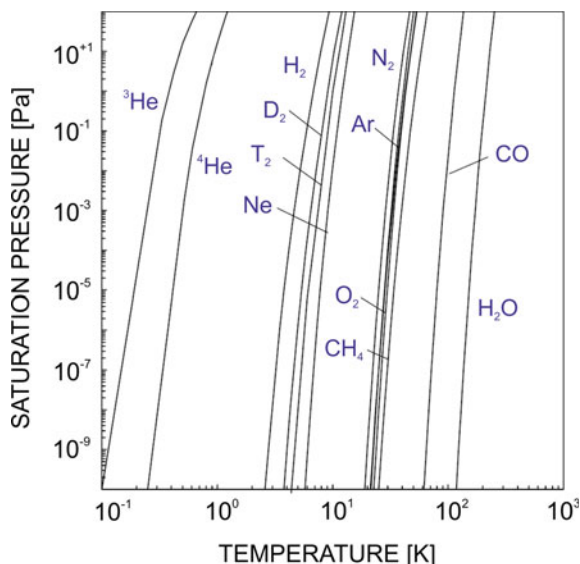
E.3 Cryopumping

The main processes that constitute cryo-pumping are cryo-condensation, cryo-sorption and cryo-trapping.

Cryo-condensation describes both the processes of gas liquefaction on a cold surface and re-sublimation of gas on a cold surface, i.e. the solidification from gas without a transient liquid phase. The achievable residual gas pressure is determined by the saturation pressure at the temperature chosen for the cold surface. This is not specific to the material or structuring of the cold surface. Figure E.1 shows the saturation pressures of the most common residual gases as a function of temperature. It becomes obvious that for temperatures below roughly 20 K, all gases with the exception of helium are efficiently cryo-condensed on cold walls.

Cryo-sorption is based on the fact that gas particles impinging on a surface of sufficiently low temperature lose so much of their incident kinetic energy that they stay attached to the cold surface by weak intermolecular forces, resulting in significantly higher molecular concentration on the surface than in the gas phase. This

Fig. E.1 Saturation pressures of gases as a function of temperature



phenomenon is also called 'physical adsorption' or 'physi-sorption', and is obviously dependent on the involved material and its microscopic structure. Cryosorption denotes the physical adsorption process under vacuum conditions and low temperatures. The equilibrium pressure of adsorbed gas particles is significantly lower than the corresponding saturation pressure for cryo-condensation. This is due to the fact that the dispersion forces between the gas molecule and the surface are greater than between the gas molecules themselves in the condensed state. Hence, gas can be retained by adsorption even in a sub-saturated state, i.e. at considerably higher temperatures than would be required for condensation. This fact is essential in cryo-pumping helium, hydrogen, and neon, which are difficult to condense, see Fig. E.1.

Cryo-trapping is the process of spatially confining a residual gas that would otherwise not condensate or physi-sorb by another gas that is intermixed in the gas phase and does get condensed or adsorbed efficiently.

References

1. E.D. Marquardt, J.P. Le, R. Radebaugh, *Cryogenic Material Properties Database* (National Institute of Standards and Technology, Boulder, 2000)
2. N.J. Simon, E.S. Drexler, R.P. Reed, Properties of copper and copper alloys at cryogenic temperatures. NIST Monogr. **177** (1992)
3. A.B. Pippard, *Magnetoresistance in metals* (Cambridge University Press, Cambridge, 1989)

Appendix F

Collisional Effects in Penning Traps

We consider the interactions of confined ions in a Penning trap with either neutral species (residual gas), with other ions, and with electron beams.

The interaction of ions with neutral species is mainly of interest for the determination of the charge state lifetime of the confined ions against electron capture from residual gas, particularly relevant to highly charged ions.

The ion-ion interaction gives rise to numerous collective phenomena linking the topic, amongst others, to the behaviour of non-neutral plasmas. Important questions are for example the behaviour of stored ion ensembles under resistive and sympathetic cooling, or the application of a rotating wall for control over the charge density, as well as the influence of charge effects on the experimental possibilities and requirements.

Interaction with an electron beam is relevant mostly with respect to electron impact ionisation (EII) as used for charge breeding of highly charged ions, and electron impact excitation (EIE), particularly in the optical domain, as used for emission spectroscopy.

F.1 Ion-Neutral Collisions

Generally, ions, particularly highly charged ions, perform charge exchange with ambient neutral gas. The cross section for electron capture from the neutral gas consists of contributions from processes in which k electrons are captured simultaneously. An initial ion charge state q thus decays with different rate constants into all lower charge states, both through sequential single-electron capture and many-electron captures of all orders k . For many applications, however, it will be sufficient to look only at $k = 1$ or sometimes $k = 1$ and $k = 2$, as the higher orders do not contribute significantly in most cases, as we will see in (F.14).

F.1.1 Charge-State Lifetime

We intend to determine the expected charge state lifetime constant t_c of a specific ion from the residual gas pressure p by use of the relation

$$t_c = \frac{1}{\sigma p} \sqrt{\frac{k_B T \mu_m}{3}} \quad \text{with} \quad \mu_m = \frac{m_R m}{m_R + m} \quad (\text{F.1})$$

where m_R is the mass of the residual gas atom or molecule, m is the ion mass, p is the residual gas pressure, T is the temperature which is assumed equal for ions and gas and k_B is the Boltzmann constant. If several gases are present, then (F.1) is to be used with the respective partial gas pressures and corresponding cross sections and we have to sum over the gases present.

t_c is the time constant of the decay of the ion charge state to any lower charge state. As we will see in Sect. F.1.3, the cross section for single electron capture ($k = 1$) is usually highest, higher-order captures are statistically suppressed to some extent, but in principle all orders k of the charge decay process



occur simultaneously. So when the initial ion sample is pure (only X^{q+} and maybe lower charge states), the charge state lifetime that will be observed in the experiment can be determined just by summing over all cross sections. If it is not pure (also higher charge states than q present), we need to solve the system of coupled rate equations for all orders k and all charge states above and including the ion of interest, in similarity to the evaluation in [1].

F.1.2 Lifetime as a Pressure Gauge

In cryogenic set-ups, direct pressure measurements are typically not possible. However, when one is able to measure the number of ions of a specific charge state at different times, its lifetime against electron capture from neutral residual gas yields fair information on the residual gas pressure. This is best done with a pure ensembles of ions. For this, we just use (F.1) in the form

$$p = \frac{1}{\sigma t_c} \sqrt{\frac{k_B T \mu_m}{3}} \quad (\text{F.3})$$

and put in the observed charge-state lifetime t_c . The temperature T is commonly known well enough in such experiments, and since it enters under the square root, its uncertainty will not contribute critically. At cryogenic temperatures of liquid helium, the only significant contribution to the residual gas is helium itself and possibly

atomic and molecular hydrogen. All other gases are frozen out to a large extent, see Fig. E.1.

For large numbers of ions, the process is seen as an exponential decay of the ion number in a specific charge state with time. If at time $t_0 = 0$ the number of ions is n_0 , and at a later time t_1 the observed number is n_1 , then the decay time constant t_c is given by

$$t_c = \frac{t_1}{\ln \frac{n_0}{n_1}}. \quad (\text{F.4})$$

For a spectrum of different charge states, this is not so easy since any specific species A will be populated by higher charge states being converted to A, while it will be de-populated by itself being converted to lower charge states. Hence, to be precise, we are obliged to solve a set of coupled rate equation describing this situation. Figure F.1 gives an example of such a situation. It shows a spectrum of different charge states of argon confined in a Penning trap shortly after ion creation and two days later. One can see that at the later time, higher charge states have reduced in number, while lower charge states have gained in number, due to recombination with residual gas. To derive ion numbers from the spectrum, it is necessary to keep the ions at constant energy, since the area under a spectral peak is proportional to the ion number times the ion energy, so for counting we require constant energy.

If, however, $n_1 = n_0$, i.e. no ions have decayed, then (F.4) cannot be applied. Instead, a lower limit for the decay time constant can be derived. As this case is only realistic for small ion numbers n_0 , binomial statistics needs to be applied. Let x be the probability of a single ion to not decay within the time between $t_0 = 0$ and t_1 , then the probability of k ions to have decayed during this time is given by

$$P_k^{n_0} = \binom{n_0}{k} (1-x)^k x^{n_0-k} \quad (\text{F.5})$$

which particularly means that the probability of zero ions to have decayed is given by

$$P_0^{n_0} = x^{n_0}. \quad (\text{F.6})$$

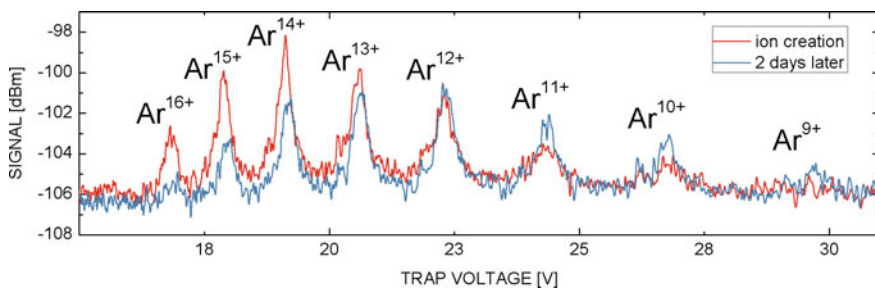


Fig. F.1 Spectrum of different charge states of argon confined in a Penning trap shortly after ion creation, and two days later

To obtain a meaningful expression for the lifetime limit t_c , we now need to set a certain confidence level which leads us to an expectation value $\langle n \rangle$ for the binomial distribution in case zero decays have been recorded. Typically, one uses a 95% confidence level ($c = 0.95$), such that we obtain

$$\langle n \rangle = n_0(1 - c)^{1/n_0}. \quad (\text{F.7})$$

This is the upper limit for the expected number of decays (assuming binomial distribution and setting the confidence level c) if zero decays out of n_0 ions have been observed within a time t_1 . We come back to (F.4) and write

$$t_c \geq \frac{t_1}{\ln(\frac{n_0}{\langle n \rangle})}. \quad (\text{F.8})$$

Hence, we have found a lower limit for the decay time constant at a given confidence level c .

Example: an ensemble of $n_0 = 34$ H-like carbon ions was stored for $t_1 = 16$ days without loss. From this, one finds with 95% confidence that for a single ion t_c is 182 days [2].

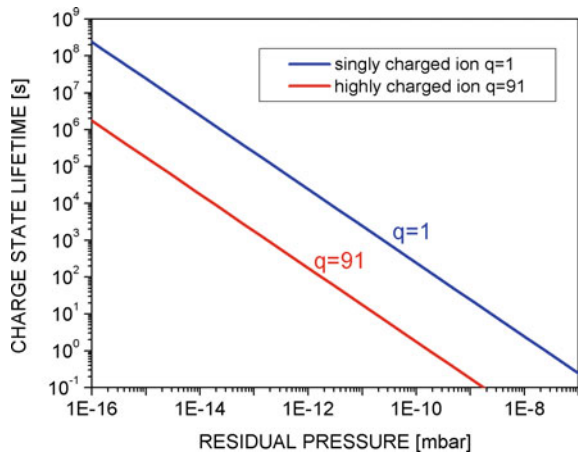
Figure F.2 shows the expected lifetime of the two extreme charge states of uranium as a function of the residual gas pressure it undergoes electron capture with.

The mean free path λ of an ion is calculated from the cross section σ and the gas particle number density by

$$\lambda = \frac{1}{(n\sigma)}. \quad (\text{F.9})$$

Within the laws of ideal gases, the relation between number density n and pressure p is given by

Fig. F.2 Expected lifetime of two different charge states of uranium as a function of the residual gas pressure it undergoes electron capture with



$$n = \frac{p}{(k_B T)}. \quad (\text{F.10})$$

Correspondingly, the fractional loss per metre L of an ion beam is given by

$$L = n\sigma = \frac{\sigma p}{k_B T}. \quad (\text{F.11})$$

F.1.3 Cross Section

So far, we have not discussed the values of the cross section σ . The simplest model describing electron capture processes in low-energy ion-neutral collisions is the ‘Classical Over-the-Barrier Model’ (CBM) which assumes the cross section

$$\sigma = \frac{1}{2} \pi R_C^2, \quad (\text{F.12})$$

where R_C is the effective collision radius given by

$$R_C = \frac{27.2a_0(2\sqrt{\hat{q}} + 1)}{I}, \quad (\text{F.13})$$

in which $a_0 = 5.29177 \times 10^{-11}$ m is the Bohr radius (atomic length unit), I is the ionisation potential of the neutral species given in eV and $\hat{q} = q/e$ is the ion charge state. For helium, $I = 24.587$ eV, for atomic hydrogen $I = 13.598$ eV, for molecular hydrogen H_2 , $I = 15.426$ eV.

A comprehensive review of theory data is given by R. K. Janev and J. W. Gallagher which can be found at the NIST homepage. Unfortunately, cross section measurements for highly charged ions have been performed for high and medium energies, only. To our knowledge, only sparse data exists for the energy region around eV and below. Nevertheless, there is sufficient data for fair extrapolations. Generally, the cross section increases with the charge state of the highly charged ion slightly stronger than linearly, increases strongly with decreasing ionisation potential of the neutral species, and increases slightly with decreasing collision energy.

F.1.3.1 Measurements at Energies Above keV

In [3], experimental cross sections for single-electron capture in highly charged oxygen and gold ions colliding with helium atoms are presented for medium to high energies in the low keV to MeV range [3].

Single electron capture and loss cross sections for boron and carbon ions ranging from singly ionized to bare species, measurements in the energy range 100–2500 keV for H and H_2 targets [5].

Cross sections for charge-exchange processes of Br^{q+} and I^{q+} ions in collisions with H_2 and He in the energy range 6–15 MeV [6].

Single electron capture cross sections for He- and Li-like boron, carbon, nitrogen and oxygen ions on molecular and atomic hydrogen targets in the energy range 4–25 keV/q [7].

Single and double electron capture cross sections for Fe^{q+} ions ($3 < q < 25$) on molecular hydrogen at energies between 100 and 3400 keV/u [8, 9]. Similar data exists for electron capture by C^{2+} and Ti^{2+} ions in H and H_2 [10].

Absolute charge-exchange cross sections for Xe^{q+} ($15 < q < 43$) on neutral He, Ar, and Xe at an energy of 3.8 keV/q [11].

Cederquist et al. have measured n-state-resolved energy-gain distributions for single-electron capture in Ar^{q+} on He collisions ($q = 15\text{--}18$) at energies in the vicinity of 3.35 keV/q [12].

Müller and Salzborn have compiled data in the keV-regime and fitted a semi-empirical function to it. According to this, the electron capture cross section σ may for highly charged ions be calculated according to [13]

$$\begin{aligned} \sigma [\text{cm}^2] \approx & 1.43 \times 10^{-12} \hat{q}^{1.17} I[\text{eV}]^{-2.76} \\ & + 1.08 \times 10^{-12} \hat{q}^{0.71} I[\text{eV}]^{-2.80} \\ & + 5.50 \times 10^{-14} \hat{q}^{2.10} I[\text{eV}]^{-2.89} \\ & + 3.57 \times 10^{-16} \hat{q}^{4.20} I[\text{eV}]^{-3.03}, \end{aligned} \quad (\text{F.14})$$

where $\hat{q} = q/e$ is the ion charge state and I is the ionisation potential of the residual gas given in eV. The first term describes single electron capture, and the second one double-electron capture and so forth. A similar semi-empirical scaling law for the cross section is presented in [14].

F.1.3.2 Measurements at Energies Below keV

State-selective electron capture cross section for molecular hydrogen targets are presented for highly-charged nitrogen and oxygen ions in the energy range 5–4000 eV/u [15].

Total electron capture cross section for O^{q+} , $1 < q < 7$ and for C^{q+} , $2 < q < 7$ with atomic and molecular hydrogen in the energy range 0.01–10 keV/u are given in [16].

Total electron-capture cross sections have been measured for collisions of Fe^{q+} ions ($2 \leq q \leq 15$) with H and H_2 at energies in the 10–95 eV/amu range [17].

Single and double electron capture cross sections for Ne^{q+} ($1 < q < 8$) and for Ar^{q+} ($1 < q < 11$) with atomic and molecular hydrogen in the energy range 50–3000 eV/q are given in [18].

Cross sections for one-electron capture by various doubly-charged ions in both H and H_2 have been determined in the energy range 0.8–40 keV [19].

Experimental cross sections are reported for electron capture and transfer ionisation for Ne, Ar, Kr, and Xe projectiles on He for charge states q between 2 and 13 and for projectile energies between 250 and 1000 eV/ q by Justiniano et al. [20].

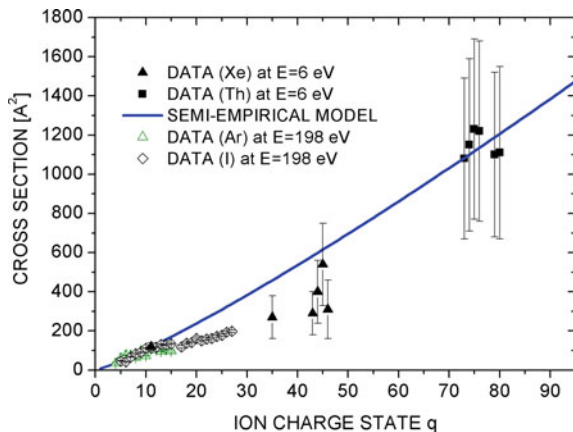
Total one-electron capture cross sections for Ar^{q+} ($4 < q < 15$) and I^{q+} ($5 < q < 27$) ions in slow collisions ($E = 198$ eV/ q) with H_2 and He have been measured by Mann [21]. These can fairly be described by the Classical Over-the-Barrier model. The data is plotted in Fig. F.3

In-trap measurements have been performed of total cross sections for electron capture from H_2 to Xe^{q+} ions ($35 < q < 46$) and Th^{q+} ions ($73 < q < 80$) by Weinberg et al. [23]. The collision energy here was as low as 6 eV. This data is plotted in Fig. F.3.

Given the energy-dependence of the charge exchange cross section as observed in similar ions [3, 4], (F.14) may underestimate the value of σ at liquid-helium kinetic energies by a factor of two to three. In the range below a few keV, the cross section has only small dependence on the collision energy, at least in the regime explored so far. As already mentioned, there is little data for eV energies and below. From the present knowledge, it appears feasible to estimate the cross section and hence the residual gas pressure within a factor of two to three even when energies as low as few Kelvin are considered.

Coming back to the example: From the total observed lifetime of a single C^{5+} ion of longer than 273 days, and from an assumed charge exchange cross section of $\sigma = 1.35 \times 10^{-14} \text{ cm}^2$ with residual helium, an upper limit for the residual gas pressure of about 9×10^{-17} hPa has been derived for the experiments performed at the University of Mainz [2].

Fig. F.3 Comparison of the data in [21, 22] with the semi-empirical equation (F.14)



F.1.4 Ions of Specific Interest

Argon Ar^{13+} Let us look at Ar^{13+} in helium. The CBM gives the value $\sigma = 3.6 \times 10^{-15} \text{ cm}^2$. The Müller-Salzborn-fit yields $\sigma = 5.03 \times 10^{-15} \text{ cm}^2$. From measured data on argon with $q = 5, 6, 70$ at $E = 100 \text{ eV}$ and a cross section per charge of about $5 \times 10^{-16} \text{ cm}^2$ [3] one would obtain a value of $\sigma = 6.5 \times 10^{-15} \text{ cm}^2$. The direct measurement by Mann et al. at an energy of 198 eV/q yields $\sigma = (3.25 \pm 0.25) \times 10^{-15} \text{ cm}^2$.

Anomaly with Argon Ar^{2+} and Ar^{3+} Direct measurements at a collision energy of 80 eV/q exist for argon with $q = 2$ and 3 (and higher charge states) in helium [20]. Surprisingly, the cross section for Ar^{2+} ($\sigma = 2.5 \times 10^{-16} \text{ cm}^2$) is higher than the cross section for Ar^{3+} ($\sigma = 4.5 \times 10^{-17} \text{ cm}^2$). For all other noble gases, this anomaly does not occur.

H-like Lead and Bismuth and Uranium: Pb^{81+} , Bi^{82+} and U^{91+} The Weinberg measurements of Th^{80+} in molecular hydrogen yield a cross section at $E = 6 \text{ eV}$ of $\sigma = 1.1 \times 10^{-13} \text{ cm}^2$. The corresponding data set seems to be well-represented by the Müller-Salzborn fit. Taking the $I^{-2.76}$ -dependence on the ionisation potential of the neutral species for granted, this value would read $\sigma = 3.0 \times 10^{-14} \text{ cm}^2$ in a helium surrounding. Hence, for hydrogen-like lead and bismuth the corresponding value would be about 2% larger than this number (which vanishes in the uncertainties). For U^{91+} it would be 16% larger.

F.1.5 Derivation of the CBM Equation

Let N_1 be a highly charged ion of charge q , and N_2 be a singly charged atomic core ion. An electron e is bound to N_2 , making it neutral. In this classical model, the electron orbits around the core at distance r_2 , while the distance between the neutral atom and N_1 is R . The electron is at distance r_1 from the ion N_1 . The electron feels the coulomb force

$$|F_1| = \frac{qe}{4\pi\epsilon_0 r_1^2} \quad (\text{F.15})$$

from the HCI and

$$|F_2| = \frac{e^2}{4\pi\epsilon_0 r_2^2} \quad (\text{F.16})$$

from the core ion, both approximated by point charges.

The atom need not be hydrogen: The model is generalized by taking into account an arbitrary ionisation potential I for the electron. The potential energy of the electron is

$$V = -\frac{e^2}{4\pi\epsilon_0 r_2} = -2I, \quad (\text{F.17})$$

according to the virial theorem. We note that $\frac{e^2}{4\pi\epsilon_0} = \alpha\hbar c = 2a_0I_H$, where $a_0 = 5.3 \times 10^{-11}$ m is the Bohr radius and $I_H = 13.6$ eV is the ionisation potential of atomic hydrogen ($2I_H = 27.2$ eV). We find an expression for r_2

$$r_2 = \frac{2a_0I_H}{2I}, \quad (\text{F.18})$$

and then the force can be expressed in terms of the ionisation potential:

$$|F_2| = \frac{4I^2}{2a_0I_H}. \quad (\text{F.19})$$

Of course, the relation between I and F is slightly different in case core electrons modify the potential.

The ionisation barrier is reached, when the forces are balanced, $|F_1| = |F_2|$. Rearranging (F.15) leads to

$$r_1^2 = \frac{q}{e} \frac{2a_0I_H}{|F_1|} = \frac{q}{4eI^2} (2a_0I_H)^2. \quad (\text{F.20})$$

Then we add the distance

$$r_1 = a_0 \frac{I_H}{I} \sqrt{\frac{q}{e}} \quad (\text{F.21})$$

to the electron orbital radius and arrive at

$$R = \frac{a_0I_H}{I} \left(\sqrt{\frac{q}{e}} + 1 \right), \quad (\text{F.22})$$

which is identical to the literature (F.13),

$$R_C = \frac{2a_0I_H}{I} \left(2\sqrt{\frac{q}{e}} + 1 \right). \quad (\text{F.23})$$

F.2 Particle-Particle Collisions

The characteristics and effects of collisions between charged particles are very different from those of the more commonly understood collisions of neutral particles. As a charged test particle moves through an ion ensemble, it simultaneously experiences the weak Coulomb electric field forces surrounding all the charged particles, and its direction of motion is deflected as it passes by each of them, resulting in a quasi-continuous change of motion with small impact factors. Such interactions are

called ‘Coulomb collisions’, albeit these ‘collisions’ within ensembles of confined ions are nearly continuous interactions.

F.2.1 Ion-Ion Interaction in the Collision Picture

The dominant collision process amongst ions stored under equilibrium conditions are Coulomb collisions. They by far dominate Langevin collisions since the static electric dipole polarisability of an ion never allows an induced electric charge of similar magnitude as the actual ion charge. The rate of such Coulomb collisions is given by the equation

$$k = \frac{4}{3} \sqrt{\pi} n \frac{q^4}{m^{1/2} (k_B T)^{3/2}} \frac{\ln \Lambda}{(4\pi \epsilon_0)^2} \quad (\text{F.24})$$

where n is the ion number density, q is the ion charge, m is the ion mass, T the ion temperature, ϵ_0 the permittivity of free space and Λ the Coulomb parameter we will discuss below. For collisions between ions of different charge states q_1 and q_2 , the q^4 in this equation would have to be replaced by $q_1^2 q_2^2$ and the remaining ionic parameters indexed with ‘1’ or ‘2’.

The ion number density n may in principle take any value between the finite minimum density as given by the ion number and confinement parameters, for a plasma at a given ion number the density is lowest for a global rotation at the magnetron frequency [32], and highest at the Brillouin limit

$$n_{\max} = \frac{\epsilon_0 B_0^2}{2m} \quad (\text{F.25})$$

where B_0 is the magnetic field strength. Talking about Ar^{13+} in a 6 T field, this is about 10^9 charges per cm^3 , i.e. about 10^8 ions per cm^3 . Note, however, that the following discussion is only valid in weakly coupled surroundings (and not e.g. for ion crystals), i.e. at low density and/or temperatures significantly above the motional ground state.

In a collision between ions of the same charge q , the net collisional energy exchange parallel to the ion motion is given by

$$\Delta E = \frac{q^4}{b^2 E} \frac{1}{(4\pi \epsilon_0)^2} \quad (\text{F.26})$$

where b is the impact parameter of the collision. We treat the ions as a thermal ensemble, so we use $E = k_B T/2$ in every degree of freedom. In principle, the maximum impact parameter is given by the size of the confinement space, however in thermalised situations as discussed here, the maximum impact parameter $b_{\max}$ is given by the Debye length of the ion since the net effect of ions outside the Debye

sphere cancels almost completely,

$$b_{max} = \lambda_D = \sqrt{\frac{\epsilon_0 E}{2nq^2}} \quad (\text{F.27})$$

while the minimum impact parameter b_{min} is given for 90° deflection by

$$b_{min} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{E}. \quad (\text{F.28})$$

According to (F.26), the collisional energy transfer is highest for the minimum impact parameter, so inserting b_{min} into (F.26) yields

$$\Delta E_{max} = E, \quad (\text{F.29})$$

while for the maximum impact parameter we get the minimum energy transfer to be

$$\Delta E_{min} = \frac{1}{(4\pi)^2} \frac{q^6 n}{E^2 \epsilon_0^3}. \quad (\text{F.30})$$

For thermal Ar^{13+} ions at 4 K the ΔE_{max} is of the order of 10^{-4} eV. In a plasma in equilibrium, the minimum ion ensemble density n_{min} corresponds to a global ensemble rotation at the magnetron frequency ω_m , and inserting into (F.30) leads to

$$\Delta E_{min} = \frac{2}{(4\pi)^2} \frac{mq^4}{E^2 \epsilon_0^2} \omega_m (\omega_c - \omega_m), \quad (\text{F.31})$$

which for the above mentioned parameters ($n_{min} \approx 10^6/\text{cm}^3$) is of the order of 10^{-9} eV.

Coming back to an estimate of the total Coulomb collision rate, we need to cumulate the effects of all possible impact parameters. The Coulomb logarithm Λ is a quantity that basically represents the ratio of the maximum to the minimum collision parameter possible under the given conditions, i.e. it represents the sum or cumulative effects of all Coulomb collisions within a Debye sphere for impact parameters ranging from b_{min} to $b_{max} = \lambda_D$. For ion-ion collisions it reads

$$\ln \Lambda = 23 - \ln \left[\frac{q_1 q_2 (\mu_1 + \mu_2)}{e^2 (\mu_1 T_1 + \mu_2 T_2)} \left(\frac{n_1 q_1^2}{e^2 T_1} + \frac{n_2 q_2^2}{e^2 T_2} \right)^{1/2} \right] \quad (\text{F.32})$$

where $\mu = m/m_p$ (m_p being the proton mass), the densities n are given in cm^{-3} and the temperatures T are given in eV [23]. In case of collision amongst identical ions this equation reduces to

$$\ln \Lambda = 23 - \ln \left(\frac{2nq^6}{e^6 T^3} \right)^{1/2} \quad (\text{F.33})$$

which for $^{40}\text{Ar}^{13+}$ ions at 4 K and 10^6 cm^{-3} yields a Coulomb logarithm of $\ln \Lambda \approx 6$.

F.3 Excitation of Optical Transitions by Electrons

Within the field of ion-electron-beam interaction, two processes are of fundamental interest: electron impact ionisation (EII) and electron impact excitation (EIE). The former is the main process for charge breeding of highly charged ions, while the latter is the main excitation process in emission spectroscopy, like for example in electron beam ion traps. During in-trap ion creation by charge breeding with an electron beam, electron impact is the mechanism for ionisation, but also leads to electronic excitation of optical transitions. We are interested in an estimation of the optical excitation rate as a function of electron beam parameters for a given ion species. In the presence of several ion species during creation, the temporal change of the distribution of species due to the charge breeding complicates the isolation of a single source for optical photons, but by use of narrow-band filters, this situation may be controlled.

Let us write the rate r of excitation like

$$r(E_b) = n\sigma(E_b)j, \quad (\text{F.34})$$

where n is the number of ions under consideration (i.e. the number of ions interacting with the electron beam), $\sigma(E_b)$ is the excitation cross section at a given electron beam energy E_b , and j is the surface number density of the electron beam in axial direction (incoming electrons per square metre as seen by the ions). For simplicity, we assume all of these quantities constant with time, and the electron beam to have constant current density throughout.

F.3.1 Electric Dipole Transitions (E1)

The semi-empirical van Regemorter formula [24, 25] gives us the cross section σ in terms of ion and beam properties. We use the formulation by Fisher et al., which reads [26]

$$\sigma = \frac{8\pi^2}{\sqrt{3}} a_0^2 f R^2 \frac{1}{E^2} \frac{G(x)}{x}, \quad (\text{F.35})$$

where E is the energy of the optical transition of interest, $x = E_b/E$ is the electron beam energy E_b divided by E , and $G(x)$ is the Gaunt factor given by

$$G(x) \approx 0.349 \ln(x) + 0.0988 + 0.455/x. \quad (\text{F.36})$$

Further, a_0 is the Bohr radius given by

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 0.529 \times 10^{-10} \text{ m}, \quad (\text{F.37})$$

and R is the Rydberg energy given by

$$R = \frac{m_e e^4}{(4\pi\epsilon_0)^2 2\hbar^2} = 13.605 \text{ eV}. \quad (\text{F.38})$$

The quantity f is the oscillator strength of the optical transition and is related to the transition rate coefficient A by

$$A = \frac{2\pi e^2}{m_e c \epsilon_0 \lambda^2} \frac{g_i}{g_f} f. \quad (\text{F.39})$$

The values for the statistical factors $g_{i,f}$, the oscillator strength f and the rate coefficient A can be found in tables such as the NIST database. For allowed (E1) transitions, $g_i/g_f \cdot f$ is typically not far below unity and A is of the order of $10^6/\text{s}$ to $10^8/\text{s}$. Inserting into (F.35) yields a cross section of about 10^{-15} cm^2 for an electron energy of $E_b = 1855 \text{ eV}$ and an optical transition around 441 nm . Let us assume an average electron current density j of about $1 \mu\text{A}/\text{cm}^2$ (which equals about $10^{13} \text{ s}^{-1} \text{ cm}^{-2}$) and a number of ions within the electron beam region of the order of 10^6 , the resulting rate of excitation would be around $r \approx 10^4/\text{s}$ for any given allowed transition in the optical region. Measurements for argon have been performed in [27].

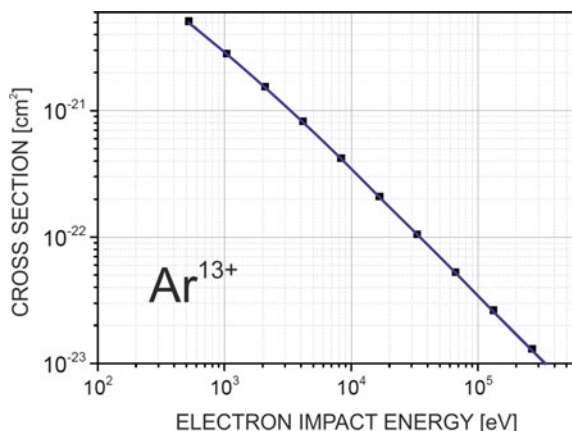
F.3.2 Magnetic Dipole Transitions (M1)

For other types of transitions, the situation is more complicated, as has been detailed for example in [28]. Often, reliable information requires calculations which are not fully analytic, as has been detailed in [29]. For Ar^{13+} such a calculation results in the following cross section as a function of electron impact energy, see Fig. F.4.

Using this, the cross section σ at an electron energy of 1855 eV is about $2 \times 10^{-21} \text{ cm}^2$. Let us again assume an average electron current density j of about $1 \mu\text{A}/\text{cm}^2$ (which equals about $10^{13} \text{ s}^{-1} \text{ cm}^{-2}$) and a number of Ar^{13+} ions within the electron beam region of the order of 10^4 , the resulting rate of excitation would be around $r \approx 10^{-4}/\text{s}$ and hence undetectable for all practical purposes.

The situation is different for dedicated set-ups, where both the electron current density and energy are optimized for electron impact excitation, which is a common base for emission spectroscopy experiments as routinely performed in electron beam ion traps and such.

Fig. F.4 Electron excitation cross section as a function of electron impact energy for the 441 nm M1 transition in Ar^{13+}



References

1. U. Rieth et al., Ion-molecule reactions of Ru^+ and Os^+ with oxygen in a Penning trap. *Radiochim. Acta* **90**, 337 (2002)
2. H. Häffner et al., Double Penning trap technique for precise g factor determinations in highly charged ions. *Eur. Phys. J. D* **22**, 163 (2003)
3. H. Knudsen, H.K. Haugen, P. Hvelplund, Single-electron-capture cross section for medium- and high-velocity, highly charged ions colliding with atoms. *Phys. Rev. A* **23**, 597 (1981)
4. S. Kravis et al., Single- and double-charge-exchange cross sections for $\text{Ar}^{q+} + \text{H}_2$ ($q = 6, 7, 8, 9$, and 11) collisions from 6 to 11 keV. *Phys. Rev. A* **52**, 1206 (1995)
5. T.V. Goffe, M.B. Shah, H.B. Gilbody, One-electron capture and loss by fast multiply charged boron and carbon ions in H and H_2 . *J. Phys. B* **12**, 3763 (1979)
6. H.D. Betz et al., Cross sections for electron capture and loss by fast bromine and iodine ions traversing light gases. *Phys. Rev. A* **3**, 197 (1971)
7. D.H. Crandall et al., Electron capture by slow multicharged ions in atomic and molecular hydrogen. *Phys. Rev. A* **19**, 504 (1979)
8. K.H. Berkner et al., Electron-capture and impact-ionisation cross sections for partially stripped iron ions colliding with atomic and molecular hydrogen. *J. Phys. B* **11**, 875 (1978)
9. K.H. Berkner et al., Electron-capture, electron-loss, and impact-ionization cross sections for 103- to 3400-keV/amu multicharged iron ions colliding with molecular hydrogen. *Phys. Rev. A* **23**, 2891 (1981)
10. W.L. Nutt et al., Electron capture by C^{2+} and Ti^{2+} ions in H and H_2 . *J. Phys. B* **11**, L181 (1978)
11. N. Selberg et al., Absolute charge-exchange cross sections for the interaction between slow Xe^{q+} ($15 < q < 43$) projectiles and neutral He, Ar, and Xe. *Phys. Rev. A* **56**, 4623 (1997)
12. H. Cederquist et al., Measurements of translational energy gain for one- and two-electron transfer in slow Ar^{q+} -He ($q = 15$ – 18) collisions. *Phys. Rev. A* **51**, 2169 (1995)
13. A. Müller, E. Salzborn, Scaling of cross sections for multiple electron transfer to highly charged ions colliding with atoms and molecules. *Phys. Lett. A* **62**, 391 (1977)
14. A.S. Schlachter et al., Electron capture for fast highly charged ions in gas targets: an empirical scaling rule. *Phys. Rev. A* **27**, 3372 (1983)

15. G. Lubinski, et al., State-selective electron-capture cross section measurements for low-energy collisions of He-like ions on H₂. J. Phys. B **33**, 5275 (2000)
16. R.A. Phaneuf et al., Electron capture in low-energy collisions of C^{q+} and O^{q+} with H and H₂. Phys. Rev. A **26**, 1892 (1982)
17. R.A. Phaneuf, Electron capture by slow Fe^{q+} ions from hydrogen atoms and molecules. Phys. Rev. A **28**, 1310 (1983)
18. C. Can et al., Electron-capture cross sections for low-energy highly charged neon and argon ions from molecular and atomic hydrogen. Phys. Rev. A **31**, 72 (1985)
19. R.W. McCullough et al., One-electron capture by slow doubly charged ions in H and H₂. J. Phys. B **12**, 4159 (1979)
20. E. Justiniano et al., Total cross sections for electron capture and transfer ionization by highly stripped, slow Ne, Ar, Kr, and Xe projectiles on helium. Phys. Rev. A **29**, 1088 (1984)
21. R. Mann, Total one-electron capture cross sections for Ar^{q+} and I^{q+} ions in slow collisions on H₂ and He. Z. Phys. D **3**, 85 (1986)
22. G. Weinberg et al., Electron capture from H₂ to highly charged Th and Xe ions trapped at center-of-mass energies near 6 eV. Phys. Rev. A **57**, 4452 (1998)
23. J.D. Huba, *NRL Plasma Formulary* (Beam Physics Branch, Plasma Physics Division, Naval Research Laboratory, Washington, DC, 2013), p. 20375
24. H. Van Regemorter, Rate of collisional excitation in stellar atmospheres. Astrophys. J. **136**, 906 (1962)
25. D.H. Sampson, H.L. Zhang, Use of the Van Regemorter formula for collision strengths or cross sections. Phys. Rev. A **45**, 1556 (1992)
26. V. Fisher, V. Bernshtam, H. Golten, Y. Maron, Electron-impact excitation cross sections for allowed transitions in atoms. Phys. Rev. A **53**, 2425 (1996)
27. J.K. Ballou, C.C. Lin, Electron-impact excitation of the argon atom. Phys. Rev. A **8**, 1797 (1973)
28. V.P. Shevelko, E.A. Yukov, A model potential for dipole and quadrupole transitions induced by electron impact. Physica Scripta **31**, 265 (1985)
29. H.L. Zhang, D.H. Sampson, At. Data Nucl. Data Tables **56**, 41 (1994)

Appendix G

General Design Considerations

G.1 The Geometry and Absolute Size of a Trap

It is difficult to make general statements about the type and exact geometry of the trap best suited for a given experimental application. Many aspects are obvious from our discussions in Sect. 2.1 and we need not discuss them further. Sometimes, there are non-obvious details, such as the fact that despite the optimum geometry, hyperbolic traps in practice usually have larger field imperfections than cylindrical traps that allow tuning. Or that when magnetic field variations due to the temperature-dependent paramagnetism of copper should be avoided, silver may be a better material choice.

With regard to the absolute size of a trap, we can however make some quantitative statements that should be considered.

- The voltages required to produce a specific field strength for any kind of application (such as excitation) in the confinement volume scale roughly linearly with the trap size d .
- The undesired effects of surface patches on the trap electrodes is more pronounced in small traps and scales roughly like $1/d$.
- The effect of image charges on the confining potential scales with the characteristic trap size like $1/d^3$.
- The relative effect of machining imperfections is more pronounced in small traps and scales roughly like $1/d$.
- The maximum achievable ion number density, or in fact any ion number density achieved under the circumstances given, translates via the confinement volume into an absolute number of particles that can be confined simultaneously, this is roughly proportional to d^3 .
- The dimension of the trap determines the dimension of the required magnetic field and hence of the magnet (bore). The cost of a magnet usually scales more than linearly with the bore diameter.

- For traps below a certain size, cavity effects may not be negligible. This has been discussed in detail in [1]. The cavity effects are easier to calculate in a cylindrical trap than in a hyperbolic trap [2].

G.2 Practical Aspects

Here we list a number of general design considerations and effects that often have to be taken into account.

- Machining imperfections are fundamentally unavoidable and the system should allow to correct for them at runtime, tuning of the trap potential should be possible without large efforts.
- Depending on the quality of the electrode surface, the maximum applicable voltage can be limited to below design values by the unwanted onset of field emission of electrons from rough edges, thus ionizing residual gas or charging up insulators. Polishing is often advisable.
- The visibility of non-conducting material from the trap centre and paths of charged particles should be zero, in order to rule out charging up of elements that affect the confining potential.
- The design needs to take into account the integrated thermal expansion coefficients of the materials used between room temperature and operation temperature.
- It is worth noting that the heat capacity of most materials, especially of metals, decreases drastically with decreasing temperature. This often makes it more difficult to keep certain elements at a design temperature in the presence of heating.
- The critical temperature of superconductors decreases in the presence of an external magnetic field like the one used for confinement. For example, NbTi loses superconductivity above 9.2 K at $B_0 = 0$, but above 6.8 K at $B_0 = 6$ T.
- Cabeling needs to be properly shielded, as oscillation frequencies often are in the radio-frequency domain where powerful sources may be present that disturb measurements. Ground loops need to be avoided. The trap should be galvanically decoupled from the equipment where possible. dc voltage supply lines should have low-pass filters. ac inputs should have high-pass filters.
- To make a good thermal connection between components, often tight fitting with strong forces/high pressure is necessary.
- In cryogenic traps, out-gassing of materials is a far smaller issue than in room-temperature traps, since all impurities apart from helium and partly hydrogen freeze out nearly completely.
- The electric conductivities of different kinds (i.e. industry standards) of copper (and other group-11-metals) differ only slightly at room temperature, but usually differ significantly at cryogenic temperatures.
- Cooling down from room temperature to e.g. liquid-helium temperature (and the way back) may cause significant mechanical stress in components such as electronics (resistors, capacitors, diodes, etc.) such that their compatibility should be tested

in advance. Sometimes the inner makings of apparently identical components are different, which then becomes important.

- It is advantageous to plate copper electrodes with a noble metal such as gold, in order to avoid oxidation ($1\text{ }\mu\text{m}$ is enough). It is necessary to add a diffusion barrier in-between such that the gold cannot diffuse into the copper bulk, which it does rather quickly even at room temperature. The common nickel is ferromagnetic and hence influences the homogeneity of the magnetic field, such that $10\text{ }\mu\text{m}$ of silver is often used.

References

1. G. Gabrielse et al., Precise matter and antimatter tests of the standard model, in *Fundamental Physics in Particle Traps*, Springer Tracts in Modern Physics, vol. 256 (Springer, Heidelberg, 2014)
2. L.S. Brown et al., Cyclotron motion in a Penning trap microwave cavity. *Phys. Rev. A* **37**, 4163 (1988)

Appendix H

A More Detailed Look at the Potential

The electrostatic potential near the centre of the Penning trap can be written as a general solution to Laplace's equation in spherical polar coordinates (r , θ and ϕ). Since the electrostatic potential has azimuthal symmetry i.e. rotation symmetry about z -axis, there will no ϕ -dependence in potential. Assuming that the confined particles have no effect on the potential created by the electrodes, and taking the origin along the azimuthal symmetry where the $z = 0$ plane bisects the ring electrode, the electrostatic potential can be expanded in Legendre polynomials [1]

$$\Phi(r, \theta) = \sum_{k=0}^{\infty} A_k r^k P_k(\cos \theta) \quad (\text{H.1})$$

where A_k is expansion coefficients and $P_k(\cos \theta)$ is Legendre polynomial. Since the cylindrical electrodes have reflection symmetry across the $z = 0$ plane, it imposes the condition that the potential ϕ must be even in z , as a result the coefficient A_k is non-zero for even values of k and zero for odd k . If a voltage U_0 is applied between the endcaps and the ring electrode, the potential field inside then trap can be written as (even k)

$$U(r, \theta) = U_0 \Phi(r, \theta) = U_0 \sum_{k=0}^{\infty} A_k r^k P_k(\cos \theta) \quad (\text{H.2})$$

Complete information of the potential U over a larger volume is not required because particle confinement is to take place within a small volume about the trap centre.

If we define the characteristic trap dimension in terms of the minimum axial and radial distances to the trap electrodes z_0 and ρ_0 as

$$d^2 = \frac{1}{2} \left(z_0^2 + \frac{\rho_0^2}{2} \right), \quad (\text{H.3})$$

then for $r \ll d$ the potential U of (H.2) can be written as (even k) [2–4]

$$U(r, \theta) = U_0 \Phi(r, \theta) = \frac{1}{2} U_0 \sum_{k=0}^{\infty} C_k \left(\frac{r}{d}\right)^2 P_k(\cos \theta), \quad (\text{H.4})$$

where $C_k = 2d^k A_k$. For a particle trapped near the centre of trap, where r is small, lowest-order terms in (H.4) will dominate. Keeping only the first two terms and ignoring higher-order terms, the potential field in cylindrical coordinates (ρ, ϕ, z) can be written as

$$U(\rho, z) = \frac{1}{2} U_0 C_0 + U_0 \left(\frac{z^2 - \rho^2/2}{2d^2} \right) C_2. \quad (\text{H.5})$$

The coefficient C_0 represents an absolute potential offset and can be ignored. The coefficient C_2 represents the quadrupolar contribution and hence the depth of the potential well. In a perfect quadrupole trapping field the coefficient $C_2 = 1$ and all higher-order coefficients are zero. The ions with kinetic energy less than the potential depth are trapped and execute rapid harmonic oscillations along the trap axis. The axial frequency of an ion with mass m and electric charge q in harmonic potential well is

$$\omega_z = \sqrt{\frac{q V_0}{m d^2} C_2} \quad (\text{H.6})$$

In a real cylindrical Penning trap, higher-order coefficients such as C_4 , C_6 and C_8 are generally non-zero, which is often undesirable. In a perfect harmonic potential, the frequencies are independent of the amplitudes (energies) of oscillation. The amount of frequency shift is proportional to the expansion coefficients, where usually the largest contribution comes from C_4 . In a generic cylindrical Penning trap, C_4 and C_6 will certainly have non-zero values because of the cylindrical shape of electrodes, machining imperfection, misalignment etc. One method to eliminate anharmonicity coefficients is to introduce compensation electrodes between ring and endcap electrodes. Choosing the length, the length-to-radius ratio and a suitable potential on compensation electrodes, it is possible to make C_4 and C_6 zero simultaneously. Instead of such electrical compensation, it is also possible to make C_4 zero through mechanical compensation in which tuning is achieved by a suitable combination of ring electrode radius and endcap position from the trap centre, i.e. by adjusting the ratio $\frac{\rho_0}{z_0}$. This is a one-time process and cannot be adjusted during an experiment, but has the advantage of featuring a longer ring electrode which is instrumental for efficient signal pick-up near the trap centre or application of techniques such as a rotating wall.

Since the potential is already expressed in spherical polar coordinates, now in order to obtain exact boundary conditions on electrodes, we can solve the Laplace's equation in cylindrical coordinates. By considering the rotational symmetry about the z -axis, reflection symmetry about the xy -plane, and assuming again no effect of confined particles on the potential, the general solution of Laplace's equation in cylindrical coordinates (ρ, ϕ, z) is [1]

$$\Phi(\rho, \phi, z) = \Phi(\rho, z) = AJ_0(ik\rho) \cos(kz), \quad (\text{H.7})$$

where A and k are constants and J_0 is the zero-order Bessel function of the first kind. The Bessel function with purely imaginary arguments represents a solution to the modified Bessel differential equation known as modified Bessel function and is expressed as

$$I_0(k_n \rho_0) = J_0(ik_n \rho_0) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{k_n \rho_0}{2} \right)^{2m}. \quad (\text{H.8})$$

The value of the constant k can be determined from the boundary conditions at the end of the trap i.e. at $z = \pm L$, where $L = z_0 + z_e$ is half the length of the trap. The reflection symmetry about the xy plane demands that potential must be the same at $z = L$ and $z = -L$ so $\Phi(\rho, L) = \Phi(\rho, -L)$, and as a result the cosine factor in (H.7) must be the same at the ends of the endcap electrodes, i.e.

$$\cos(k(z_0 + z_e)) = \cos(-k(z_0 + z_e)). \quad (\text{H.9})$$

This equality holds when

$$\exp(ik(z_0 + z_e)) = \exp(-ik(z_0 + z_e)) \quad (\text{H.10})$$

or

$$\exp(2ik(z_0 + z_e)) = 1, \quad (\text{H.11})$$

which is satisfied when $2k(z_0 + z_e) = 2n\pi$. The constant k is given by

$$k_n = \frac{n\pi}{z_0 + z_e}; \quad n = 0, 1, 2, 3 \dots \quad (\text{H.12})$$

This give linear independent solutions for each value of n . The general solution of Laplace's equation is

$$\Phi(\rho, z) = \sum_{n=0}^{\infty} A_n I_0(k_n \rho_0) \cos(k_n z). \quad (\text{H.13})$$

With applied voltage of U_0 , the potential field can be written as

$$U = U_0 \Phi(\rho, z) = U_0 \sum_{n=0}^{\infty} A_n I_0(k_n \rho_0) \cos(k_n z) \quad (\text{H.14})$$

In order to determine the relation between coefficients A_n and C_k , (H.4) and (H.13) can be evaluated along the z -axis (where $\rho = 0$) at $r = \pm z$ with $\theta = 0$. The coefficients C_k can be written for even k as

$$C_k = \frac{(-1)^{k/2}}{k!} \frac{\pi^k}{2^{k-1}} \left(\frac{d}{z_0 + z_e} \right)^k \sum_{n=0}^{\infty} (2n)^k A_n. \quad (\text{H.15})$$

The coefficients A_n can be determined by Fourier expansion of potential function $\Phi(\rho, z)$ of (H.13) (which is periodic on an interval $[-L, L]$) in terms of an infinite cosine series at $\rho = \rho_0$

$$\Phi(\rho_0, z) = A_0 + \sum_{n=1}^{\infty} A_n I_0(k_n \rho_0) \cos(k_n z) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} z\right), \quad (\text{H.16})$$

where $A_0 = a_0$ and

$$A_n = \frac{a_n}{I_0(k_n \rho_0)}. \quad (\text{H.17})$$

Fourier coefficients are computed by integrating over the half interval $[0, L]$ as

$$a_n = \frac{2}{L} \int_0^L \Phi(\rho_0, z) \cos(k_n z); n = 1, 2, 3, \dots \quad (\text{H.18})$$

In order to evaluate the integral (H.18), the value of the potential $\Phi_0(\rho_0, z)$ is required within the range $z = 0$ to $z = L$. This demands prior knowledge of the boundary conditions for the solution to Laplace's equations. Since the potential difference between endcap and ring electrode is U_0 , the difference between $\Phi_0(\rho_0, z)$ on ring and endcap electrodes should be the same. By assuming that the gap between the endcap and ring electrodes are negligibly small, Φ_0 can be specified as $-\frac{1}{2}$ for the ring electrode and $+\frac{1}{2}$ for the endcap electrode. Therefore we have

$$a_n = \frac{2}{L} \int_0^{z_0} \Phi_0(\rho_0, z) \cos(k_n z) + \frac{2}{L} \int_{z_0}^{z_0+z_e} \Phi_0(\rho_0, z) \cos(k_n z) = -\frac{2}{n\pi} \sin(k_n z_0). \quad (\text{H.19})$$

The coefficient A_n becomes

$$A_n = -\frac{2}{n\pi} \frac{\sin(k_n z_0)}{I_0(k_n \rho_0)}, \quad (\text{H.20})$$

and coefficients C_k can be written as

$$C_k = \frac{(-1)^{\frac{k}{2}+1}}{k!} \frac{\pi^{k-1}}{2^{k-3}} \left(\frac{d}{z_0 + z_e} \right)^k \sum_{n=1}^{\infty} (2n)^{k-1} \frac{\sin(k_n z_0)}{I_0(k_n \rho_0)}. \quad (\text{H.21})$$

Unlike the simple Bessel function which is oscillatory in nature, the modified Bessel function increases exponentially, as a result, series in (H.21) converges rapidly for large arguments.

References

1. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)
2. X. Fei, W. Snow, Cylindrical Penning traps with dynamic orthogonalized anharmonicity compensation for precision experiments. *Nucl. Inst. Meth. A* **425**, 431 (1999)
3. K. Farrar, Calculation of the electrostatic potential field for an open-endcap cylindrical Penning trap. *Nucl. Inst. Meth. A* **485**, 780 (2002)
4. A. Sikdar, A. Ray, P. Das, A. Reza, Design of a dynamically orthogonalized penning trap with higher order anharmonicity compensation. *Nucl. Inst. Meth. A* **712**, 174 (2013)

Appendix I

A Special Case of Mediated Cooling

In the following, we want to bring together a number of elements discussed so far, i.e. ‘self-cooling’ of electrons via synchrotron radiation, mediated sympathetic cooling, and resonant motional coupling of different degrees of freedom. The special case here is that the coupling is established between different traps that are tuned such that the magnetron motion of electrons confined in one trap is resonant with the axial motion of ions in another trap. We wish to discuss the cooling of those ions by the electrons. This is a situation similar to the sympathetic cooling in a common-endcap setup as discussed in Sect. 12.16.2, and the present discussion goes back to the work in [1]. The present example may be of practical relevance, but in any case it demonstrates many of the techniques and effects we have encountered so far, and may thus be seen as a demonstration of principles.

In such a situation, when proximity of different species is not required, problems of recombination, electron capture or other reactions are avoided *per constructionem*. Hence, the sign of the charges in either trap is irrelevant, and the term ‘ions’ will be used for all particles which may be cooled within the present scheme, i.e. singly or multiply charged atomic or molecular cations and anions, but conceptionally also protons and anti-protons. Further, as the cooling of large particle ensembles depends on imperfections of the confining fields [2, 3], one may individually choose different trapping conditions for either trap, thus optimizing the overall cooling process. It is also possible to separately optimize the respective trap geometries for the desired confinement and cooling properties. The overall number of electrodes and components can be kept comparatively small, which is advantageous specifically in cryogenic environments.

I.1 Setup and Working Principle

The envisaged setup for the present cooling scheme consists of two neighbouring cylindrical open-endcap Penning traps along a common z -axis, one of which is optimized for efficient pick-up of the axial ion oscillation signal, while the other is

optimized for efficient pick-up of the electron's radial oscillation signal. For simplicity, from here on we use quantities with a bar for description of the electrons, and those without a bar for ions. The two pick-ups are connected by a wire. When the axial frequency of the ions

$$\omega_z = \left(\frac{qU_0C_2}{md^2} \right)^{1/2} \quad (I.1)$$

matches the magnetron frequency of the electrons

$$\bar{\omega}_- = \frac{\bar{\omega}_c}{2} - \left(\frac{\bar{\omega}_c^2}{4} - \frac{e\bar{U}_0\bar{C}_2}{2\bar{m}\bar{d}^2} \right)^{1/2} \quad (I.2)$$

by appropriate choice of the respective trapping potentials C_2U_0 and $\bar{C}_2\bar{U}_0$, the two motions are resonantly coupled and exchange energy. Despite the fact that the magnetron motion is an unstable drift motion, in a sufficiently collision-less environment (ultra-high vacuum), electron ensembles can be stored for days [2]. Resonant dipole excitation of the magnetron motion will increase its radius, but conditions can be chosen such that particle loss is prevented, we will discuss this below. Here, $\bar{\omega}_c$ is the free cyclotron frequency of the electrons $\bar{\omega}_c = eB/\bar{m}$ in the magnetic field B , and d is the characteristic size of the respective trap given by $d^2 = z_0^2/4 + \rho_0^2/2$ where $2z_0$ is the endcap separation and ρ_0 is the trap's inner radius. The electrons act as a heat sink continuously cooled by their synchrotron radiation which dissipates axial energy from the ions via the connecting wire.

The cooling rate connected with synchrotron radiation of each individual electron is approximately given by

$$\bar{\gamma}_j = \frac{4\bar{r}}{3c} \bar{\omega}_j^2 \quad (I.3)$$

where $\bar{r} = e^2/(4\pi\epsilon_0\bar{m}c^2) \approx 2.8 \times 10^{-15} \text{ m}$ is the classical electron radius, c is the speed of light, and $\bar{\omega}_j$ is the oscillation frequency, where the index j refers to each of the three motional degrees of freedom of the single electron, namely the (reduced) cyclotron motion, the axial motion and the magnetron motion [4]. Hence, the cyclotron cooling rate is inversely proportional to the cube of the particle mass ($\gamma_c \propto 1/m^3$), and thus the mechanism is effective for electrons (and positrons) only. The rate $\bar{\gamma}_j$ determines the decrease of energy via synchrotron radiation in that degree of freedom, the decrease is an exponential decay $\bar{E}_j(t) = \bar{E}_j(0) \cdot \exp(-\bar{\gamma}_j t)$.

Since for typical trapping conditions we have $\bar{\omega}_c \approx \bar{\omega}_+ \gg \bar{\omega}_z \gg \bar{\omega}_-$, the cooling rate of the (reduced) cyclotron motion at frequency $\bar{\omega}_+$ dominates over the other two rates by far. Hence, the axial and magnetron motions are cooled by their respective synchrotron radiations only to a small extent, but very efficiently dissipate energy to the cyclotron motion by trapping field imperfections and electron-electron interaction [2, 3]. Measurements with stored ensembles of about 10^8 electrons show that the observed cooling rate for the axial motion is identical to the expected cooling rate

for the cyclotron motion [5]. This indicates that the dissipation of energy amongst the degrees of freedom is quasi-instantaneous compared to the cooling time constants of milli-seconds to seconds [6]. This is also in agreement with earlier observations [2] and electron plasma theory [7].

The same mechanism of energy dissipation amongst degrees of freedom applies to the ions as well: while the coupling to the electron trap directly cools the axial ion motion only, energy from the remaining ionic degrees of freedom is dissipated into the axial ion motions and they are hence cooled as well. This situation has been studied in [3]. Efficient thermalisation amongst all motional degrees of freedom requires the presence of trapping field imperfections and a high rate of Coulomb collisions between ions, as will be discussed below. Specific to the present concept is the question, how fast the energy is dissipated between the traps when they are connected as proposed. In a simple picture, the proposed coupling is similar to active feedback cooling, where the ion motion induces an electron motion which couples back to the ion motion with a phase shift of π , such that the ion motion is damped. As there is a heat sink on the electron side, the coupling is used to drain the energy from the ion side. To study the situation in detail, we have devised a model for the dissipation of energy through the induced currents between the connected traps.

I.2 Model and Calculations

I.2.1 Single Particles

In order to obtain a quantitative statement about the expected ion cooling behaviour, we calculate the induced charges in the connected electrodes of either trap, the corresponding currents through the wire (represented by an impedance with residual R and C), and the following electric potentials in the two traps. Thanks to the existing symmetries, this is still possible analytically. The scheme is depicted in Fig. I.1. We first study the case of a single ion with charge q and mass m in the ion trap and a single electron with charge $-e$ and mass $\bar{m}$ in the electron trap. The cylindrical ion trap is symmetric around the z -axis and has inner radius ρ_0 .

I.2.1.1 Induced Charges and Potentials

The charge Q induced by the charge at cylindrical coordinates (r_1, z_1, ϕ_1) on the electrode of interest (i.e. the one connected by the impedance to the electron trap) is given by

$$Q = \frac{q}{\pi} \int_0^\infty \frac{I_0\left(x \frac{r_1}{\rho_0}\right)}{x I_0(x)} \cdot \left[\sin\left(x \frac{z_1 - z}{\rho_0}\right) \right]_{z=z_L}^{z=z_R} dx, \quad (\text{I.4})$$

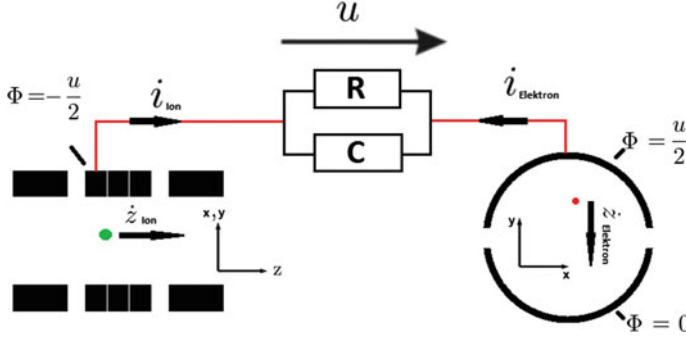


Fig. I.1 Schematic of the trap configuration. The axial ion motion in the left trap is coupled to the radial motion of the electron in the right trap

where I_0 is the modified Bessel function of order 0, and the electrode of radius ρ_0 extends axially from coordinates z_L to z_R . This can be linearised to give

$$\underline{Q} \approx q [\kappa_1 \dot{z} + \kappa_0] \quad (\text{I.5})$$

and the corresponding current $\underline{i} \approx q \kappa_1 \dot{z}$, where the underline indicates a linearised quantity from here on. The parameters κ are given by

$$\kappa_0 \approx -\frac{1}{2} \sum_{k=0}^{k_{\max}} \alpha_k \left[\operatorname{erf} \left(\frac{x}{\gamma_k} \right) \right]_{x=z_L}^{x=z_R} \quad (\text{I.6})$$

$$\kappa_1 \approx \frac{1}{\sqrt{\pi}} \sum_{k=0}^{k_{\max}} \frac{\alpha_k}{\gamma_k} \left[\exp \left(-\frac{x^2}{\gamma_k^2} \right) \right]_{x=z_L}^{x=z_R} \quad (\text{I.7})$$

$$\gamma_k = 2\beta_k \rho_0. \quad (\text{I.8})$$

For the electron trap, the charge $\overline{Q}_S$ induced in the split central electrode (i.e. the upper half connected by the impedance to the ion trap) is given by

$$\overline{Q}_S = \overline{Q}_S^{(0)} + \overline{Q}_S^{(m)} \quad \text{where} \quad (\text{I.9})$$

$$\overline{Q}_S^{(0)} = -\frac{q}{\pi} \int_0^\infty \frac{I_0(\eta \frac{\tilde{r}_1}{\rho_0})}{I_0(\eta)} \frac{\sin(\eta \frac{\tilde{z}_R}{\rho_0}) \cos(\eta \frac{\tilde{z}_1}{\rho_0})}{\eta} d\eta \quad (\text{I.10})$$

$$\overline{Q}_S^{(m)} = -\frac{2q}{\pi^2} \sum_{m=1}^\infty \frac{[\sin(m \{\bar{\phi} - \bar{\phi}_1\})]_0^\pi}{m} \quad (\text{I.11})$$

$$\times \int_0^\infty \frac{I_m(\eta \frac{\tilde{r}_1}{\rho_0})}{I_m(\eta)} \frac{\sin(\eta \frac{\tilde{z}_R}{\rho_0}) \cos(\eta \frac{\tilde{z}_1}{\rho_0})}{\eta} d\eta. \quad (\text{I.12})$$

We choose coordinates such that the splitting is symmetric to $\bar{z} = 0$, so that we have $\bar{z}_L = -\bar{z}_R$. The linearisation requires that $|\bar{z}_1| \ll \bar{z}_R$ and $\bar{\rho}_0$ sufficiently small so that the electrode carries the dominant part of the induced charge irrespective of $\bar{z}_1$, and that the charge induced in one segment of the electrode depends only on $\bar{y}$. This means that for the charge displacement we have $\bar{\phi}_1 = \frac{\pi}{2} \Leftrightarrow \bar{y} \geq 0$ resp. $\bar{\phi}_1 = -\frac{\pi}{2} \Leftrightarrow \bar{y} < 0$. The result of the linearisation then is

$$\begin{aligned}\bar{\underline{Q}}_S &= \bar{\underline{Q}}^{(0)} + \bar{\underline{Q}}^{(m)} = q [\delta_0 + \bar{y}_1 \delta_1] \quad \text{where} \\ q \delta_0 &= \bar{\underline{Q}}_S^{(0)} = -\frac{q}{\pi} \int_0^\infty \frac{\sin(\eta \frac{\bar{z}_R}{\bar{\rho}_0})}{\eta I_0(\eta)} d\eta, \\ q \bar{y}_1 \delta_1 &= \bar{\underline{Q}}_S^{(m)} = -\frac{2q \bar{y}_1}{\bar{\rho}_0 \pi^2} \int_0^\infty \frac{\sin(\eta \frac{\bar{z}_R}{\bar{\rho}_0})}{I_1(\eta)} d\eta.\end{aligned}\tag{I.13}$$

We now need to find the confining potentials corresponding to the induced charges. First, Φ and $\bar{\Phi}$ are the usual electric trapping potentials

$$\Phi = \frac{m}{2q} \omega_z^2 \left[z^2 - \frac{r^2}{2} \right] \tag{I.14}$$

$$\bar{\Phi} = -\frac{\bar{m}}{2e} \bar{\omega}_z^2 \left[\bar{z}^2 - \frac{\bar{r}^2}{2} \right] \tag{I.15}$$

of the ion and the electron, respectively, in absence of coupling.

Then, let $\underline{\Phi}_B$ denote the (linearised) additional potential in the ion trap due to the voltage induced by the electron motion in the connected electron trap, and $\bar{\underline{\Phi}}_S$ the (linearised) additional potential in the electron trap induced by the ion motion in the connected ion trap. For the latter, we have

$$\begin{aligned}\bar{\underline{\Phi}}_S(\bar{r}, \bar{z}, \bar{\varphi}) &= \frac{16u}{\pi^2 \bar{d}_1 \bar{d}_2} \sum_{j=0}^\infty \bar{K}_1 \bar{K}_2 \quad \text{with} \\ \bar{K}_1 &= \frac{\sin(\frac{\bar{d}_1}{2} [2j+1]) \sin(\bar{\varphi} [2j+1])}{[2j+1]^2}, \\ \bar{K}_2 &= \int_0^\infty \frac{\sin(\eta \frac{\bar{L}_2 + \bar{d}_2}{2}) \sin(\eta \frac{\bar{d}_2}{2}) I_{2j+1}(\eta \bar{r}) \cos(\eta \bar{z})}{\eta^2 I_{2j+1}(\eta \bar{\rho}_0)} d\eta,\end{aligned}\tag{I.16}$$

where $\bar{d}_1$ is the gap between the two segments of the split electrode, and $\bar{d}_2$ is the electrode separation. The linearisation $\bar{\underline{\Phi}}_S$ is obtained by evaluating

$$\overline{\Phi}_S(\bar{y}) \approx \bar{y} \frac{\bar{u}}{2} \frac{8 \sin(\frac{\bar{d}_1}{2})}{\pi^2 \bar{d}_1 \bar{d}_2} \int_0^\infty \frac{\sin(\eta \left[\frac{\bar{d}_2 + \bar{L}_2}{2} \right]) \sin(\eta \frac{\bar{d}_2}{2})}{\eta I_1(\eta \bar{\rho}_0)} d\eta \quad (\text{I.17})$$

at $(\bar{z} = 0; \bar{\varphi} = \frac{\pi}{2}, \bar{u} = 1)$, where we define the geometry parameter ρ_1 by

$$\rho_1 = \frac{2}{\bar{y}\bar{u}} \overline{\Phi}_S(\bar{y}). \quad (\text{I.18})$$

When we denote the length of the connected electrode by $\bar{L}_3$ and its axial centre coordinate by $\bar{z}_P$, the potential $\overline{\Phi}_B$ is given by

$$\begin{aligned} \overline{\Phi}_B(\bar{r}, \bar{z}) = & -\frac{2\bar{u}\bar{\rho}_0}{\pi \bar{d}_2} \\ & \times \int_0^\infty \frac{I_0(\eta \frac{\bar{r}}{\bar{\rho}_0}) \sin(\eta \frac{\bar{L}_3 + \bar{d}_2}{2\bar{\rho}_0}) \sin(\eta \frac{\bar{d}_2}{2\bar{\rho}_0}) \cos(\eta \frac{\bar{z} - \bar{z}_P}{\bar{\rho}_0})}{I_0(\eta) \eta^2} d\eta \end{aligned} \quad (\text{I.19})$$

with the corresponding linearisation

$$\begin{aligned} \overline{\Phi}_B(\bar{r} = 0, \bar{z}) = & -\frac{\bar{u}}{2} [\xi_0 + \xi_1 \bar{z}] \quad \text{where} \\ \xi_0 = & \frac{4\bar{\rho}_0}{\pi \bar{d}_2} \int_0^\infty \frac{\sin(\eta \frac{\bar{L}_3 + \bar{d}_2}{2\bar{\rho}_0}) \sin(\eta \frac{\bar{d}_2}{2\bar{\rho}_0}) \cos(\eta \frac{\bar{z}_P}{\bar{\rho}_0})}{I_0(\eta) \eta^2} d\eta \\ \xi_1 = & \frac{4}{\pi \bar{d}_2} \int_0^\infty \frac{\sin(\eta \frac{\bar{L}_3 + \bar{d}_2}{2\bar{\rho}_0}) \sin(\eta \frac{\bar{d}_2}{2\bar{\rho}_0}) \sin(\eta \frac{\bar{z}_P}{\bar{\rho}_0})}{I_0(\eta) \eta} d\eta. \end{aligned} \quad (\text{I.20})$$

I.2.1.2 Equations of Motion

We now have all relevant potentials and may write down the equations of motion. For the particle motions, we find the differential equations

$$\ddot{\mathbf{r}} = \frac{q}{m} (-\nabla \Phi - \nabla \underline{\Phi}_B + \mathbf{P} \times \mathbf{B}) \quad (\text{I.21})$$

$$\ddot{\mathbf{N}} = \frac{-e}{\bar{m}} (-\nabla \overline{\Phi} - \nabla \underline{\Phi}_S + \mathbf{N} \times \mathbf{B}), \quad (\text{I.22})$$

in which $\mathbf{B} = B\mathbf{e}_z$ is the magnetic field at position $\mathbf{r}$. Inserting the potentials, these equations can be written out as

$$\ddot{x} = \frac{\omega_z^2}{2}x + \omega_c \dot{y} \quad (\text{I.23})$$

$$\ddot{y} = \frac{\omega_z^2}{2}y - \omega_c \dot{x} \quad (\text{I.24})$$

$$\ddot{z} = -\omega_z^2 z + \frac{q\xi_1}{2m}(u + \bar{u}) \quad (\text{I.25})$$

$$\ddot{\bar{x}} = \frac{\bar{\omega}_z^2}{2}\bar{x} - \bar{\omega}_c \dot{\bar{y}} \quad (\text{I.26})$$

$$\ddot{\bar{y}} = \frac{\bar{\omega}_z^2}{2}\bar{y} + \bar{\omega}_c \dot{\bar{x}} - \frac{e\rho_1}{2\bar{m}}(u + \bar{u}) \quad (\text{I.27})$$

$$\ddot{\bar{z}} = -\bar{\omega}_z^2 \bar{z}, \quad (\text{I.28})$$

where the first three equations describe the ion motion and the other three describe the electron motion. Equations (I.25)–(I.27) describe the desired coupling between ion and electron and can be separated from the remaining equations.

The coupling leads to an equilibration of the motions in the coordinates $\bar{x}$, $\bar{y}$ of the electron and z of the ion. By the geometry of the coupling, the axial motion of the ion excites the radial motion of the electron by a force in the y -direction. This translates into an excitation of both cyclotron and magnetron motion, in the present case of resonant coupling at $\bar{\omega}_-$, only the magnetron motion is excited. We nonetheless treat the case in its generality with $\dot{\bar{y}} = \dot{R}_+^{\bar{y}} + \dot{R}_-^{\bar{y}}$ and discuss the specific case later.

Therefore, the voltage u induced across the impedance by the ion motion depends on $\dot{z}$, and the (counter-) voltage $\bar{u}$ induced by the electron motion depends on the y -component of the electron velocity in the magnetron motion $\dot{R}_-^{\bar{y}}$. These voltages can be given by

$$u \approx q\mathcal{R}\kappa_1 \dot{z} \quad (\text{I.29})$$

$$\bar{u} \approx -e\mathcal{R}\delta_1 \dot{R}_\pm^{\bar{y}} \quad (\text{I.30})$$

When we write the homogeneous solution of (I.26) and (I.27) as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} R_+^{\bar{x}} \\ R_+^{\bar{y}} \end{pmatrix} + \begin{pmatrix} R_-^{\bar{x}} \\ R_-^{\bar{y}} \end{pmatrix} \quad (\text{I.31})$$

then, for the radial electron motion, we find the equations of motion

$$\ddot{R}_\mp^{\bar{x}} = \frac{\bar{\omega}_z^2}{2}R_\mp^{\bar{x}} - \bar{\omega}_c \dot{R}_\mp^{\bar{y}} \quad (\text{I.32})$$

$$\ddot{R}_\mp^{\bar{y}} = \frac{\bar{\omega}_z^2}{2}R_\mp^{\bar{y}} + \bar{\omega}_c \dot{R}_\mp^{\bar{x}} \quad (\text{I.33})$$

$$\ddot{R}_{\pm}^{\bar{x}} = \frac{\bar{\omega}_z^2}{2} R_{\pm}^{\bar{x}} - \bar{\omega}_c \dot{R}_{\pm}^{\bar{y}} \quad (\text{I.34})$$

$$\ddot{R}_{\pm}^{\bar{y}} = \frac{\bar{\omega}_z^2}{2} R_{\pm}^{\bar{y}} + \bar{\omega}_c \dot{R}_{\pm}^{\bar{x}} - 2\alpha_1 \dot{R}_{\pm}^{\bar{y}} + D_1 \cos(\bar{\omega}_{\pm}t + \varphi_0) \quad (\text{I.35})$$

in which we have the parameters

$$\alpha_1 = -\frac{e^2 \rho_1 \mathcal{R} \delta_1}{4\bar{m}} \quad (\text{I.36})$$

$$D_1 = -\frac{eq\mathcal{R}\kappa_1\rho_1}{2\bar{m}} \sqrt{\frac{2\langle E_z \rangle}{m}}. \quad (\text{I.37})$$

The quantity α_1 is the rate of equilibration between the voltages u and $\bar{u}$ induced by the coupled motions, and $2D_1\bar{m}/\rho_1$ is the amplitude of the induced voltage. The radial electron motion can be written in the form

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \bar{\mathbf{R}}_{\mp} + \bar{\mathbf{R}}_{\pm} + \bar{\mathbf{R}}_{\pm}^{(\infty)}, \quad (\text{I.38})$$

in which we have

$$\begin{aligned} \bar{\mathbf{R}}_{\mp} &= \bar{R}_{\mp}^{(0)} \begin{pmatrix} \cos(\bar{\omega}_{\mp}t + \varphi_1) \\ \sin(\bar{\omega}_{\mp}t + \varphi_1) \end{pmatrix} \\ \bar{\mathbf{R}}_{\pm} &\approx \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] e^{-\alpha_1 t} \begin{pmatrix} \cos(\bar{\omega}_{\pm}t + \varphi_0) \\ \sin(\bar{\omega}_{\pm}t + \varphi_0) \end{pmatrix} \\ \bar{\mathbf{R}}_{\pm}^{(\infty)} &= 2\bar{R}_{\pm}^{(\infty)} \begin{pmatrix} \cos(\bar{\omega}_{\pm}t + \varphi_0) \\ \sin(\bar{\omega}_{\pm}t + \varphi_0) \end{pmatrix}. \end{aligned}$$

Here, $\bar{R}_{\mp}^{(0)}$ is the initial radius of the electron motion, $2\bar{R}_{\pm}^{(\infty)} = D_1/(\alpha_1\bar{\omega}_{\pm})$ is the final radius (after infinite time), and α_1 is the rate of transition between those. For an initially centred electron with $\bar{R}_{\pm}^{(0)} = 0$, we can write the radial electron trajectory as

$$\begin{aligned} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} &= \bar{R}_{\mp}^{(0)} \begin{pmatrix} \cos(\bar{\omega}_{\mp}t + \varphi_1) \\ \sin(\bar{\omega}_{\mp}t + \varphi_1) \end{pmatrix} \\ &+ \left\{ \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] e^{-\alpha_1 t} + 2\bar{R}_{\pm}^{(\infty)} \right\} \begin{pmatrix} \cos(\bar{\omega}_{\pm}t + \varphi_0) \\ \sin(\bar{\omega}_{\pm}t + \varphi_0) \end{pmatrix} \end{aligned} \quad (\text{I.39})$$

while for the radial velocities by differentiation we find

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \bar{\omega}_{\pm} \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] e^{-\alpha_1 t} \\
&+ 2\omega_{\pm} \bar{R}_{\pm}^{(\infty)} \begin{pmatrix} -\sin(\bar{\omega}_{\pm} t + \varphi_0) \\ \cos(\bar{\omega}_{\pm} t + \varphi_0) \end{pmatrix} \\
&+ \bar{\omega}_{\mp} \bar{R}_{\mp}^{(0)} \begin{pmatrix} -\sin(\bar{\omega}_{\mp} t + \varphi_1) \\ \cos(\bar{\omega}_{\mp} t + \varphi_1) \end{pmatrix} \\
&- \alpha_1 \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] e^{-\alpha_1 t} \begin{pmatrix} \cos(\bar{\omega}_{\pm} t + \varphi_0) \\ \sin(\bar{\omega}_{\pm} t + \varphi_0) \end{pmatrix}.
\end{aligned} \tag{I.40}$$

For the axial ion motion, we find

$$z \approx \frac{1}{2\bar{\omega}_{\pm}} \sqrt{\frac{2\langle E_z \rangle}{m}} (e^{-\alpha_1 t} + 1) \sin(\bar{\omega}_{\pm} t + \varphi_0) \tag{I.41}$$

I.2.1.3 Energy Consideration

Let us calculate the work done on the electron by the coupling to the ion. The work along a path element

$$d\mathbf{s} = \mathbf{e}_2 \dot{R}_{\pm}^{(\bar{y})} dt \tag{I.42}$$

by the force on the electron

$$\begin{aligned}
\mathbf{F} &= -e\mathbf{E} = e\nabla\Phi_s = \frac{e\rho_1}{2} [u + \bar{u}] \mathbf{e}_2 \\
&= \bar{m} \left[-D_1 \cos(\bar{\omega}_{\pm} t + \varphi_0) + 2\alpha_1 \dot{R}_{\pm}^{(\bar{y})} \right] \mathbf{e}_2
\end{aligned}$$

is given by

$$\begin{aligned}
dW_1 &= \mathbf{F} d\mathbf{s} \\
&= \bar{m} \left\{ -D_1 \dot{R}_{\pm}^{(\bar{y})} \cos(\bar{\omega}_{\pm} t + \varphi_0) + 2\alpha_1 \left[\dot{R}_{\pm}^{(\bar{y})} \right]^2 \right\} dt.
\end{aligned}$$

Under the reasonable assumption that $\alpha_1 \ll \bar{\omega}_{\pm}$, we can make an approximation for $\dot{R}_{\pm}^{(\bar{y})}$ by

$$\dot{R}_{\pm}^{(\bar{y})} \approx \bar{\omega}_{\pm} \left\{ \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] e^{-\alpha_1 t} + 2\bar{R}_{\pm}^{(\infty)} \right\} \cos(\bar{\omega}_{\pm} t + \varphi_0). \tag{I.43}$$

The work over one period of oscillation $T = \frac{2\pi}{\bar{\omega}_{\pm}}$ can be approximated by

$$T \frac{dW_1}{dt} \approx \alpha_1 \bar{m} T \bar{\omega}_{\pm}^2 \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right]^2 e^{-2\alpha_1 t} + \alpha_1 \bar{m} T \bar{\omega}_{\pm}^2 2\bar{R}_{\pm}^{(\infty)} \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] e^{-\alpha_1 t}. \quad (\text{I.44})$$

The total work done by the coupling field to get the charge $-e$ from radius $\bar{R}_{\pm}^{(0)}$ to $2\bar{R}_{\pm}^{(\infty)}$ is the integral

$$W_1 = \alpha_1 \bar{m} \bar{\omega}_{\pm}^2 \int_0^{\infty} K_e(t) dt \quad (\text{I.45})$$

where the integrand $K_e(t)$ is given by

$$\left\{ \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right]^2 e^{-2\alpha_1 t} + 2\bar{R}_{\pm}^{(\infty)} \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] e^{-\alpha_1 t} \right\} \quad (\text{I.46})$$

This energy integral can be evaluated to give

$$W_1 = \frac{1}{2} \bar{m} \bar{\omega}_{\pm}^2 \left\{ \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right]^2 + 2\bar{R}_{\pm}^{(\infty)} \left[\bar{R}_{\pm}^{(0)} - 2\bar{R}_{\pm}^{(\infty)} \right] \right\} = \frac{1}{2} \bar{m} \bar{\omega}_{\pm}^2 \left\{ \left[\bar{R}_{\pm}^{(0)} \right]^2 - \left[2\bar{R}_{\pm}^{(\infty)} \right]^2 \right\}. \quad (\text{I.47})$$

Now looking at the magnetron motion only, when the electron is initially centred ($\bar{R}_{-}^{(0)} = 0$) then this is just the common equation for the kinetic magnetron energy. In other words, when the coupling to the ion motions does a work

$$W_1 = -\frac{1}{2} \bar{m} \bar{\omega}_{-}^2 2^2 \bar{R}_{-}^{(\infty)2} \quad (\text{I.48})$$

then the electron obtains a radius of $2\bar{R}_{-}^{(\infty)}$. For a radius of order mm, the energy is typically of order meV.

1.2.2 Particle Ensembles

1.2.2.1 Coupling of Particle Ensembles

First, we generalize the single-particle result of the previous section to particle ensembles. We assume a thermalised hot ensemble of ions in one trap and an initially cold and centred electron ensemble in the other trap. This means that the phases of the ions' axial motions are distributed randomly, while the induced motion of the electrons initially has a fixed phase such that it may be considered a pure centre-of-mass

motion. Then, in similarity to (I.29) and (I.30) the induced voltages are given by

$$u \approx q\mathcal{R}\kappa_1 \sum_{k=1}^N \dot{z}_k \approx q\sqrt{N}\mathcal{R}\kappa_1 \sqrt{\frac{2\langle E_z \rangle}{m}} \cos(\omega_z t + \phi) \quad (\text{I.49})$$

$$\bar{u} \approx -e\bar{N}\mathcal{R}\delta_1 \dot{\bar{R}}_{\pm}, \quad (\text{I.50})$$

in which N denotes the number of ions and $\bar{N}$ denotes the number of electrons. The equations of motion from the previous sections hold true for the respective centre of masses when the coefficients α_1 and D_1 given by (I.36) and (I.37) are replaced by

$$\alpha_N = -\frac{e^2\bar{N}\rho_1\delta_1\mathcal{R}}{4\bar{m}} = \bar{N}\alpha_1 \quad (\text{I.51})$$

$$D_N = -\frac{eq\mathcal{R}\sqrt{N}\kappa_1\rho_1}{2\bar{m}} \sqrt{\frac{2\langle E_z \rangle}{m}} = \sqrt{N}D_1. \quad (\text{I.52})$$

This has mainly two consequences:

- (1) Since $\alpha_N = \bar{N}\alpha_1$, the centre-of-mass equilibration between ensembles is much faster than between single particles, namely by a factor of $\bar{N}$ which can be very large up to the point where the assumptions break down, which is expected as α_N approaches $\omega_z = \bar{\omega}_-$ since then the damping of the ion oscillation shifts the oscillation frequency significantly.
- (2) The final radius $\bar{R}_{c.m.-}^{(\infty)}$ of the electron centre of mass depends on D_N/α_N for which we have

$$\bar{R}_{c.m.-}^{(\infty)} = \frac{D_N}{2\alpha_N\bar{\omega}_-} = \frac{\sqrt{N}}{\bar{N}} \frac{D_1}{2\alpha_1\bar{\omega}_-} \ll \frac{D_1}{2\alpha_1\bar{\omega}_-} = \bar{R}_-^{(\infty)}. \quad (\text{I.53})$$

Hence, the magnetron radius $\bar{R}_{c.m.-}^{(\infty)}$ after equilibration with the axial centre-of-mass motion of the ion ensemble is smaller than in the single-particle case due to the factor $\sqrt{N}/\bar{N} \ll 1$. If the electron motion de-phases on the equilibration time scale due to field imperfections, this factor needs to be replaced by $N/\bar{N}$ (i.e. the factor $\bar{N}$ in (I.50) becomes $\sqrt{\bar{N}}$). Such a de-phasing is expected under experimental conditions when field imperfections result in a width Δ of the oscillation frequencies on the time scale $1/\Delta$.

The energy transfer from the ions to the electrons is obviously most efficient when parameters are chosen, such that the magnetron excitation uses the available radius in the given trap. These parameters can be found from relating the induced voltages u and $\bar{u}$ and setting the final radius equal to the available radius in the trap (i.e. $\bar{R}_{c.m.-}^{(\infty)} = \bar{\rho}_0$). When we assume a fast de-phasing of the electron motions due to field imperfections and allow a maximum radius $\bar{\rho}_0$ for the motion, we find

$$E_z(\bar{\rho}_0) = \frac{m}{2} \frac{e^2}{q^2} \frac{\delta_1^2}{\kappa_1^2} \frac{\bar{N}}{N} \bar{\rho}_0^2 \bar{\omega}_-^2 \quad (I.54)$$

as the maximum axial energy per ion such that the coupling does not lead to electron loss. The total energy transferred to the electron motion is then given by the kinetic magnetron energy of $\bar{N}$ electrons at that radius $\bar{\rho}_0$

$$\bar{E}_-(\bar{\rho}_0) = \frac{1}{2} \bar{m} \bar{N} \bar{\rho}_0^2 \bar{\omega}_-^2. \quad (I.55)$$

This is a fraction f of the total ion energy $NE_z(\bar{\rho}_0)$ of

$$f = \frac{\bar{E}_-}{NE_z(\bar{\rho}_0)} = \frac{\bar{m}}{m} \frac{q^2}{e^2} \frac{\kappa_1^2}{\delta_1^2} \quad (I.56)$$

which only depends on the mass-to-charge ratios of the particles and on the geometry. Ideally, we have $f \geq 1$. After several times $1/\alpha_N$ the induces signals u and $\bar{u}$ have equilibrated and there is no net energy transfer. The ion energy can be further reduced when the electrons' magnetron motion is re-cooled, i.e. coupled to their axial or cyclotron motions, such that a flow of energy from the ions' axial centre-of-mass motion to the electrons' axial or cyclotron motions is sustained. In this case, the ions' axial centre of mass is equilibrated with electron motions that are coupled to the surrounding heat bath, such that it is cooled close to that temperature. We will discuss the model in more detail below. First, we describe how ion motions other than the axial centre-of-mass motion are cooled by this. Clearly, it requires flow of energy into the axial centre-of-mass motion from all other motions, which is possible by trapping field imperfections and by Coulomb collisions.

I.2.2.2 Trapping Field Imperfections and Collisions

Equilibration amongst motional degrees of freedom within a stored particle ensemble occurs through conversion of motions relative to their centre-of-mass motion with that centre-of-mass motion within each degree of freedom, and through particle-particle interaction by the Coulomb force (Coulomb collisions).

Let us first discuss the motional conversion. It has been treated in detail for the axial degree of freedom [2], and the rate of conversion between centre-of-mass motion and relative motions is given by the width of the respective frequency distribution [2]. This means that in situations, where only the centre of mass of a motion is directly cooled (like presently and also in resistive cooling), the remaining motions relative to it can only be cooled when they convert energy into the centre-of-mass motion, which requires the presence of trapping field imperfections that produce a finite frequency width within that degree of freedom.

Commonly, the main source of field imperfections is a deviation of the electric potential from a harmonic well. Ignoring the small contribution from the cyclotron

energy [4], we find a shift of the axial and magnetron frequency for finite axial energy E_z and magnetron energy E_- to be given by

$$\frac{\Delta\omega_z}{\omega_z} \approx \frac{3}{4} \frac{C_4}{C_2^2} \frac{E_z + 4|E_-|}{qU_0} + \frac{15}{16} \frac{C_6}{C_2^3} \frac{E_z^2 + 4|E_-|^2}{(qU_0)^2} \quad (I.57)$$

$$\frac{\Delta\omega_-}{\omega_-} \approx \frac{3C_4}{C_2^2} \frac{E_z + |E_-|}{qU_0} + \frac{15}{4} \frac{C_6}{C_2^3} \frac{E_z^2 + |E_-|^2}{(qU_0)^2}. \quad (I.58)$$

The coefficients C_4 and C_6 depend on the trap geometry and voltages, as detailed in Sect. 6.1. Assuming a thermal distribution of energies, the frequency shifts as given by (I.57) and (I.58) can be translated into frequency widths δ by replacing the energies with the typical energy widths $2k_B T$ of a thermal distribution, such that

$$\delta_z \approx \Delta\omega_z(2k_B T) \quad (I.59)$$

$$\delta_- \approx \Delta\omega_-(2k_B T). \quad (I.60)$$

This means that the widths can be adjusted by choice of trap voltages that change C_4 and C_6 . In a carefully tuned trap, we may have C_4 and C_6 of the order of 10^{-6} and 10^{-3} , respectively [3], such that the expected relative frequency width for an ion temperature of 10 eV is around 10^{-5} . Slight detunings of the trap can increase this number by two orders of magnitude or more, without changing the confinement or resonance conditions significantly. This is true when the trap is orthogonal by its geometry, since then C_2 (and hence ω_z) remains constant under trap detuning [8]. The axial frequency distribution can be controlled both with respect to its centre frequency and to its width by tuning of voltages according to [8, 9]. This is relevant for two reasons:

- The axial frequency width determines the conversion of axial motions into the axial centre-of-mass motion (which for the ions is the only directly cooled motion)
- For the ions, the axial centre frequency and width need to be chosen such that the resonance with the connecting circuit is optimal. The method assumes coupling of motions by a resonant circuit at $\omega = \omega_z = \bar{\omega}_-$. When the circuit has a given quality factor Q , the effective bandwidth of the coupling is limited to a value of about ω/Q . Oscillations of either ions or electrons outside of this frequency band would not be coupled.

Let us now discuss the equilibration of motions through Coulomb collisions. In similarity to [10] we quantify the time scale for equilibration due to Coulomb interaction by the thermalisation time constant (‘Spitzer self-collision time’), given as [7]

$$\gamma_T^{-1} = \tau_T \approx (4\pi\epsilon_0)^2 \frac{3\sqrt{m} (k_B T)^{3/2}}{4\sqrt{\pi} n q^4 \ln \Lambda}, \quad (I.61)$$

where $\ln \Lambda$ is the so-called ‘Coulomb logarithm’ which represents the ratio of the maximum to the minimum collision parameter possible under the given conditions,

i.e. it represents the cumulative effects of all Coulomb collisions. In case of collisions amongst identical particles it is given by [7]

$$\ln \Lambda = 23 - \ln \left(\frac{2nq^6}{e^6 T^3} \right)^{1/2}, \quad (\text{I.62})$$

where n is given in units of cm^{-3} and T is given in units of eV. More sophisticated theory finds a modification of (I.61) by a factor $6/5$ [10], and this equilibration rate has experimentally been verified to within few percent with stored electrons over two decades of density and temperature [11]. For the present parameters, $\ln \Lambda$ is about 11 for the electrons and about 13 for the ions. For now assuming a number density n of $10^{14}/\text{m}^3$ for electrons and $10^{10}/\text{m}^3$ for the ions, the equilibration times are about equal and for an assumed temperature of 0.1 eV are on the μs scale. For initially hot ions at 10 eV temperature and correspondingly lower density, the time is on the scale of seconds [3].

I.2.2.3 Motional Sideband Coupling

Any motional degrees of freedom can be actively coupled by irradiation of a sideband frequency, as discussed in Sect. 11.4. As mentioned above, this is of particular interest for the magnetron motion of the electrons. We have seen that the coupling to the hot ions increases the magnetron radius of the electrons, potentially to a point where they reach the trap radius and are lost. Also, to efficiently cool the ions, it is necessary to centre the electrons' magnetron motion. This is possible by motional sideband coupling which leads to interconversion of the magnetron motion with the (cooled) cyclotron or axial motion [12, 13]. To that end, an azimuthal quadrupole field at frequency $\omega_z + \omega_-$ is applied, which under the present conditions is a convenient microwave frequency of few GHz for the electrons and few hundreds of MHz for the ions. To that end, a fourfold split electrode in the respective trap connected to a signal generator can be used. The time for each interconversion cycle is given by

$$T_c = \frac{4\pi r_-^2 B}{U_c}, \quad (\text{I.63})$$

which typically is on the μs scale. For a confined particle ensemble, the magnetron centre-of-mass radius can thus be decreased to

$$r_- = 2 \left(\frac{\omega_-}{\omega_z} \langle r_z^2 \rangle \right)^{1/2} \quad (\text{I.64})$$

where $\langle r_z^2 \rangle^{1/2}$ is the axial centre-of-mass amplitude, and the factor $(\omega_-/\omega_z)^{1/2} \approx 35$ is sufficiently large to make this efficient. Extremely long ensemble lifetimes have been observed using this technique [12, 14]. The coupling of the magnetron centre-

of-mass motion to the relative magnetron motions due to field imperfections has been found to also compress the ensemble radially [15]. The cooling rate for the magnetron centre-of-mass motion (decrease of the c.m radius) connected with sideband coupling is given by [12]

$$\gamma_- \approx \frac{e^2 U_c^2}{4m^2 \omega_z \gamma_z (\omega_c - 2\omega_-)} \quad (\text{I.65})$$

where U_c is the amplitude of the coupling drive and γ_z is the axial centre-of-mass cooling rate which is assumed much larger than γ_- . It would hence be beneficial to actively couple the magnetron frequency to the axial frequency in both the case of the electrons and of the ions. This is even more true, as the collisional equilibration may be expected to be anisotropic: in a high magnetic field, electrons are found to thermally equilibrate along the field lines faster than across the field [11], although transport across the field towards an equilibrium density distribution has been found to be significantly faster than expected from simple theory [10]. Coupling of radial to axial motion would hence also help to make thermalisation isotropic.

I.2.2.4 Energy Balance

The overall energy balance of the coupled ensembles is depicted in Fig. I.2. It shows the ionic side on the left (red) and the electronic side on the right (blue).

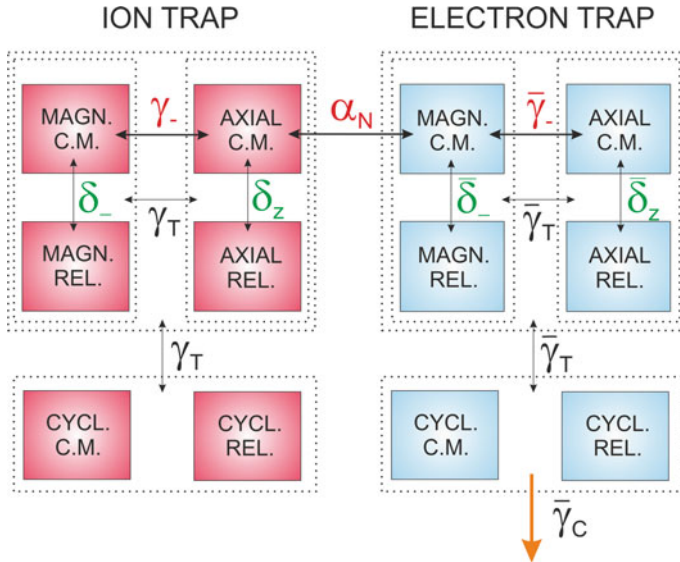


Fig. I.2 Overall cooling scheme with rates indicated. Couplings in red are active, such in green are passive but adjustable and such in black are not adjustable

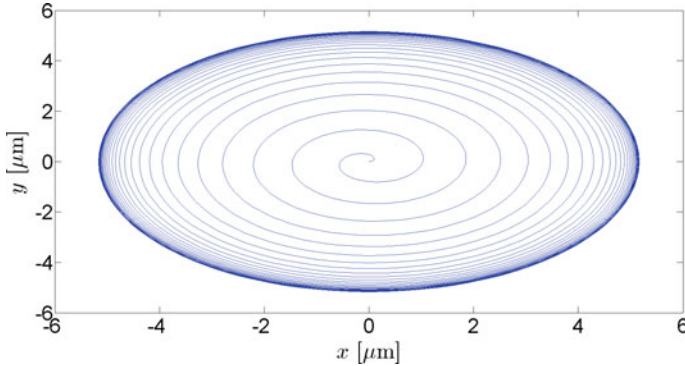


Fig. I.3 Induced magnetron centre-of-mass motion according to (I.39) for $\bar{N} = 1000$ electrons heated by $N = 1$ $^{238}\text{U}^{91+}$ ion with $\langle E_z \rangle = 1$ eV; $\mathcal{R} = 10 \text{ M}\Omega$; $0 \leq t \leq 5 \mu\text{s}$; $\bar{\omega}_- = 2\pi \text{ MHz}$

Here, α_N (I.51) describes the resonant coupling via the impedance. It can be controlled by choice of $\mathcal{R}$ and $\bar{N}$.

The rates γ_- and $\bar{\gamma}_-$ result from the active coupling of ionic and electronic magnetron centre-of-mass motion to axial centre-of-mass motion [(I.65) for ions and electrons, respectively]. They can be controlled by choice of the sum frequency irradiation intensity.

The rates δ_- , δ_z , $\bar{\delta}_-$ and $\bar{\delta}_z$ [(I.59) and (I.60) for ions or electrons] represent the respective conversion of relative to centre-of-mass motion by finite oscillation frequency widths. These can be controlled by choice of the trap voltages which determine the oscillation frequency widths.

The remaining rates γ_T and $\bar{\gamma}_T$ are the collisional thermalisation rates given by equation and (I.61) for ions and electrons, respectively.

Finally, a heat sink is given by the synchrotron radiation of the electronic cyclotron motion at a rate of $\bar{\gamma}_c$ as given by (I.3).

I.3 Example

To give an example, we choose a U^{91+} ion cooled by $\bar{N} = 1000$ electrons at a coupling frequency of $\omega = \omega_z = \bar{\omega}_- = 2\pi \times 1 \text{ MHz}$ and assume $\mathcal{R} = 10 \text{ M}\Omega$, which is a typical value for such resonant circuits. For the present geometry we have $\delta_1 \approx -55/\text{m}$, $\rho_1 = 57/\text{m}$ and $\kappa_1 \approx -22/\text{m}$, such that the equilibration rate via the impedance is $\alpha_N \approx \bar{N} \times 220/\text{s}$. Figure I.3 shows that the axial motion of one $^{238}\text{U}^{91+}$ ion with 1 eV energy excites the magnetron centre-of-mass orbit of an ensemble of 1000 electrons to a radius of $\bar{R}_-^{(\infty)} = D_N / (2\alpha_N \bar{\omega}_-)$ of below $10 \mu\text{m}$ and equilibrates with axial ion motion within about $1 \mu\text{s}$. Under experimental conditions with N of

order 10^4 and $\bar{N}$ of order 10^8 , the factor $\bar{N}/\sqrt{\bar{N}}$ is 10^6 , such that each ion could have an axial energy of about 1 MeV to result in the same electron excitation as in Fig. I.3.

So far, we have only looked at the centre-of-mass motion and treated the electron ensemble point-like. We also need to see whether the real ensemble size imposes problems. The spatial extension of an electron ensemble in equilibrium with a liquid hydrogen environment is dominated by the space charge rather than by motional amplitudes. This is particularly true for initial magnetron cooling by active coupling. Assuming a roughly spherical ensemble, the volume containing $\bar{N}$ electrons given by space charge is

$$V \approx \frac{\bar{N} e d^2}{4\pi \bar{C}_2 \epsilon_0 \bar{U}_0}, \quad (\text{I.66})$$

which agrees with the value from plasma theory

$$V \approx \frac{2\bar{m}\bar{N}\epsilon_0}{e^2} \bar{\omega}_c (\bar{\omega}_c - \bar{\omega}_-), \quad (\text{I.67})$$

when assuming a global ensemble rotation close to the magnetron frequency. Using this, for the present parameters, an ensemble of 10^8 electrons would extend a few mm, which is still small when compared to the trap dimensions.

In a magnetic field of $B = 6$ T and at 1 V amplitude of the magnetron cooling drive, the interconversion rate γ_- according to (I.65) between the magnetron centre-of-mass and axial centre-of-mass motion is $\bar{\gamma}_- \approx 125/\text{s}$ for the electrons and $1/\text{s}$ for the ions. For the ions, this is not necessarily an efficient equilibration process, but useful to spatial particle centring.

The conversion frequencies δ between relative and centre-of-mass motion due to field imperfections depend on the trap geometry and detuning and are commonly on the kHz scale.

The collisional equilibration rates γ_T depend strongly on the actual particle density and temperature, expected to be on the kHz scale (I.61), and measured to be much faster than $\bar{\gamma}_c \approx 10/\text{s}$.

Finally, the rate of synchrotron radiation from the cyclotron motion is $\bar{\gamma}_c \approx 10/\text{s}$, which means that the long-term cooling of the overall system occurs at a rate of $\gamma \approx 10/\text{s}$, the minimum rate in the equilibration chain. This, however does not limit the efficient cooling of the ions as long as the electrons are initially cold and their total energy capacity is large enough which we have shown above for $\bar{N} \gg N$.

I.4 Experimental Aspects and Requirements

The end point of cooling is reached when the electrons are in equilibrium with their environment, including the resonant circuit to which they are coupled. Since in the present case there are no active electronic components in contact with the electrodes of interest, the electronic noise is predominantly given by the Johnson noise corre-

sponding to the physical temperature of the circuit. Ideally, both the traps and circuit are cooled to liquid helium temperature which defines the final ion temperature and provides effective cryo-pumping.

Possible pitfalls in stable electron confinement are trapping instabilities due to incidental resonant coupling of motions [16]. This has also been carefully studied by [17] indicating that a situation with $k\bar{\omega}_+ + l\bar{\omega}_z + m\bar{\omega}_- = 0$ (integer k, l, m) has to be avoided [18], see also our discussion in Sect. 8.3. It has also been found that in practice, stable trapping becomes difficult when the required trapping potential exceeds about half of the maximum value given by $U_0 < ed^2B^2/(4\bar{m})$ [19]. It is further imperative that residual gas is minimized, as electron-gas collisions lead to electron loss, and electron capture from residual gas leads to a decay of the ion charge state. For the required cooling time scales of seconds, the pressure should be on the 10^{-13} mbar level or below, which is commonly reached in cryogenic trap environments.

When the proposed scheme is realized, we may expect that initially cold and centred electron ensembles in one Penning trap allow efficient cooling of particles in another Penning trap by equilibration via a resonant connection. The electron trap can be loaded and the electrons prepared by motional coupling between magnetron and axial motion, thus cooling and centring the whole ensemble. Initially hot particles in the connected trap can be thus cooled efficiently, and without restrictions due to undesired particle-electron interactions such as recombination.

References

1. J. Steinmann, Modellierung und Simulation der Widerstandskuehlung von hochgeladenen Ionen, Ph.D. thesis, University of Erlangen-Nürnberg, 2015
2. D.J. Wineland, H.G. Dehmelt, Principles of the stored ion calorimeter. *J. Appl. Phys.* **46**, 919 (1975)
3. M. Vogel et al., Resistive and sympathetic cooling of highly-charged-ion clouds in a Penning trap. *Phys. Rev. A* **90**, 043412 (2014)
4. L.S. Brown, G. Gabrielse, Geonium theory: physics of a single electron or ion in a Penning trap. *Rev. Mod. Phys.* **58**, 233 (1986)
5. Z. Andelkovic et al., Status of deceleration and laser spectroscopy of highly charged ions at HITRAP. *Hyp. Int.* **235**, 37 (2015)
6. Z. Andelkovic et al., Status of deceleration and laser spectroscopy of highly charged ions at HITRAP. *Hyp. Int.* (2015). <https://doi.org/10.1007/s10751-015-1199-8>
7. J.D. Huba, NRL Plasma Formulary (Beam Physics Branch, Plasma Physics Division, Naval Research Laboratory, Washington, DC, 2013), p. 20375
8. G. Gabrielse, L. Haarsma, S.L. Rolston, Open-endcap Penning traps for high precision experiments. *Int. J. Mass Spectrom. Ion Proc.* **88**, 319 (1989)
9. G. Gabrielse, F.C. Macintosh, Cylindrical Penning traps with orthogonalized anharmonicity compensation. *Int. J. Mass. Spectrom. Ion Proc.* **57**, 1 (1984)
10. C.F. Driscoll, J.H. Malmberg, K.S. Fine, Observation of transport to thermal equilibrium in pure electron plasmas. *Phys. Rev. Lett.* **60**, 1290 (1988)
11. A.W. Hyatt, C.F. Driscoll, J.H. Malmberg, Measurement of the anisotropic temperature relaxation rate in a pure electron plasma. *Phys. Rev. Lett.* **59**, 2975 (1987)

12. W.M. Itano, J.C. Bergquist, J.J. Bollinger, D.J. Wineland, Cooling methods in ion traps. *Physica Scripta* **T59**, 106 (1995)
13. G. Savard et al., A new cooling technique for heavy ions in a Penning trap. *Phys. Lett. A* **158**, 247 (1991)
14. G. Gabrielse et al., Thousandfold improvement of the measured antiproton mass. *Phys. Rev. Lett.* **65**, 1317 (1990)
15. C.S. Weimer, J.J. Bollinger, F.L. Moore, D.J. Wineland, Electrostatic modes as a diagnostic in Penning-trap experiments. *Phys. Rev. A*. **49**, 3842 (1994)
16. J. Yu, M. Desaintfuscien, F. Plumelle, Ion density limitation in a Penning trap due to the combined effect of asymmetry and space charge. *Appl. Phys. B* **48**, 51 (1989)
17. P. Paasche et al., Individual and center-of-mass resonances in the motional spectrum of an electron cloud in a Penning trap. *Eur. Phys. J. D* **18**, 295 (2002)
18. M. Kretzschmar, Ideal gas approximation for an ion cloud in a Penning trap. *Z. Naturf.* **45a**, 965 (1990)
19. G. Werth, V.N. Gheorghe, F.G. Major, *Charged Particle Traps* (Springer, Heidelberg, 2005)

Index

Symbols

α -particle, 265

π -pulse, 162

1/f noise, 215

A

Acoustic trap, 13

Active feedback cooling, 205

Adiabatic cooling, 218

Adiabatic invariant, 139

Adiabaticity condition, 373

Adiabatic transport, 139, 142, 143

Admixture ions, 117

Admixture particles, 117

Aluminium, 324, 396

Aluminium nitride, 15

Anti-Helmholtz loops, 382

Antimony tin oxide, 28

Anti-proton, 119, 262, 263, 338

Anti-proton magnetic moment, 337

APD detectors, 290

ARES, 267

Aspect ratio, 230

Asymmetric loading, 136

ATO, 28

Avalanche photo-diodes, 290

Avoided crossing, 162

Axial frequency, 6, 48

Axial laser cooling, 177

Axial magnetic field, 368

Axial motion, 50

Axialisation, 165

B

Baffler, 135

Bandpass filter, 214

Barium, 297, 307

Battery-based confinement, 13

Bcc structure, 116

Beryllium, 119, 307

Bessel function, 63, 64

Bi-axial ellipsoid, 226

Biot-Savart's law, 378

Bismuth, 311

Bloch, F., 29

Bloch-Grüneisen law, 29

Body-centred cubic structure, 116

Bohr magneton, 300, 335

Bohr, N.A., 339

Bohr radius, 308

Bolometric detection, 249

Bolometric technique, 353

Boltzmann distribution, 170

Boron, 324

Boron-like ions, 310

Breit, G., 340

Brillouin density, 6, 230

Brillouin flow, 232, 375

Brillouin frame of reference, 6, 56, 58, 101

Brillouin frequency, 6, 230

Brillouin limit, 106

Brillouin, L.N., 57

Buffer gas cooling, 165, 185

Byrne, J., 339

C

Cadmium, 118

Caesium, 266

Calcium, 297

Cardioid, 52

Cavity effects, 94, 416

CBM model, 403
 CCD detector, 290
 Centrifugal separation, 118, 374
 Channeltron, 270
 Characteristic trap size, 25, 365
 Charge breeding, 133
 Charge-changing effect, 97
 Charged-coupled devices, 290
 Charge-exchange reactions, 155
 Charge-state lifetime, 400
 CID, 154
 Classical dressed states, 162
 Classical over-the-barrier model, 403
 Cleaning, 151
 Cobalt, 15, 324
 Coherence length, 252, 253, 260, 261
 Coherent motion, 195
 Coil inductance, 260, 261
 Coil self-capacitance, 261, 262
 Cold-bore magnet, 17
 Collisional activation, 155
 Collisional broadening, 292
 Collision-induced dissociation, 154
 Collision-induced excitation, 154
 Combined Penning trap, 33
 Combined trap, 57
 Common-endcap Penning trap, 212
 Conductive coating, 28
 Confinement, 5
 Conical-endcap trap, 84
 Continuous Stern-Gerlach effect, 2, 262, 263, 278, 281, 284, 287, 304, 319, 322, 335, 336, 339, 357
 Convection, 395
 Conversion time, 165, 166
 Cooling time constant, 192
 Coplanar-waveguide Penning trap, 33, 36
 Copper, 285, 396
 Copper-beryllium, 396
 Corner frequency, 215
 Correlation energy, 114
 Coulomb collisions, 185, 408
 Coulomb logarithm, 437
 Coulomb parameter, 408
 CPT invariance, 335, 336
 CPT theorem, 338
 CPT violation, 338
 CPW-trap, 36
 Cryo-condensation, 397
 Cryogenic gas source, 135
 Cryo-pumping, 16, 397
 Cryo-sorption, 397
 Cryo-trapping, 397

Cubic Penning trap, 22, 147
 Cubic Zeeman effect, 315
 Cyclotron, 352
 Cyclotron frequency, 6, 48, 352, 366
 Cylindrical Penning trap, 22, 24, 26, 358

D

Damped oscillator, 189
 Dark count rate, 291
 Debye length, 119, 227, 374, 408
 Debye, P., 107
 Debye sphere, 409
 Dehmelt, H.G., 2, 193, 339, 356
 Diamagnetic shielding, 315
 Dicke, R.H., 180
 Dilution refrigerator, 17
 Dipole excitation, 149
 Dipole term, 326
 Dipole trap, 14
 Dirac equation, 134
 Dirac, P.A.M., 336
 Dodecapole term, 67
 Doppler broadening, 293
 Doppler cooling, 177
 Doppler cooling limit, 371
 Doppler regime, 172
 Doppler width, 294
 Double-resonance spectroscopy, 307
 Double-trap technique, 138, 329, 330, 338, 373
 Dressed states, 162
 Dynamical confinement, 350
 Dynamic Kingdon trap, 5, 351
 Dynamic particle capture, 135, 136

E

Earnshaw's theorem, 9, 348
 Earnshaw, S., 9, 348
 EBIS, 132
 Eckstrom, P., 2, 339, 357
 ECRIS, 154
 Effective electrode distance, 188, 190, 370
 Eigen-frequencies, 47, 366
 EIT cooling, 184
 Electrical compensation, 23
 Electric dipole moment, 299
 Electric dipole operator, 299
 Electric dipole polarisability, 408
 Electric dipole transition, 293, 312
 Electric dodecapole term, 69
 Electric hexadecapole term, 69
 Electric octupole term, 69

Electric quadrupole moment, 314
 Electric quadrupole shift, 295
 Electrode truncation, 22
 Electrodynamic containment, 356
 Electromagnetically induced transparency, 184
 Electron, 94, 336
 Electron-beam ion trap, 133
 Electron cooling, 211, 372
 Electron detachment, 155
 Electron impact excitation, 410
 Electron impact ionisation, 131, 410
 Electron-ion interaction, 410
 Electron magnetic moment, 336
 Electron mass, 335
 Electron recombination, 155
 Electrostatic ion traps, 9
 Ellipsoid of revolution, 226
 Elliptical Penning trap, 36, 84
 Ellipticity, 369
 Ellipticity parameter, 36
 Elongated epitrochoid, 52
 EMCCD detector, 290
 Energy quantisation, 276
 Ensemble thermalisation, 153
 Epicycloid, 52
 Epitrochoid, 51
 Ergodic system, 170
 Europium, 297
 Evaporative cooling, 187
 Expansion coefficients, 65, 73, 367

F

Face-centred cubic structure, 116
 Faraday cup, 270
 Faraday, M., 348
 Fcc structure, 116
 Feedback cooling, 205
 Feedback gain temperature, 206
 Feedback temperature, 206
 Ferromagnets, 15
 Field anharmonicities, 61
 Field ionisation, 131
 Field misalignments, 80
 Fine structure, 311
 Fine structure constant, 266, 335
 Finnegan scan, 152
 First adiabatic invariant, 139
 Flicker noise, 215
 Flickerrauschen, 215
 Fluorine tin oxide, 28
 Flux creep, 79

Fourier limit, 281
 Frequency ω_0 , 6, 57
 Frequency ω_1 , 6, 57
 Frequency hierarchy, 48, 366
 Frequency standards, 295
 FT-ICR, 266
 FTO, 28
 Fully de-neutralised plasma, 110
 Fully ionised plasma, 110
 Funkelrauschen, 215

G

Gabrielse, G., 3, 358
 Gain noise voltage, 205
 Gas injection, 135
 Gauss, C.F., 348
 Gaussian line shape, 172, 293
 Gauss' law, 237
 Geometric compensation, 23
 Geometric imperfections, 366
 Geonium atom, 2, 357
 Geonium chip, 36
 Geonium theory, 2, 357
 Gold, 15, 298
 Gravitational red-shift, 295
 Grüneisen, E., 29
 Guiding-centre approximation, 53, 54

H

Haroche, S., 4, 357
 Hcp structure, 116
 Heat conduction, 395
 Heaviside, O., 348
 Heisenberg uncertainty principle, 291
 Heisenberg, W., 339
 Helium, 265
 Helium-3, 265
 Helium-4, 265
 Helmholtz loops, 17, 381
 Hermite polynomials, 66
 Hexadecapole term, 69
 Hexagonal close-packed structure, 116
 Hexapole term, 326
 Higher-order Zeeman effect, 315
 Highly charged ions, 117, 308
 High-pass filter, 214
 Holmium, 311
 Horizontal magnet, 17
 Hybrid Penning trap, 325
 Hydrogen, 297
 Hydrogen-like ions, 310

Hyperbolic Penning trap, 21
 Hyperfine structure, 309

I

ICR cells, 22
 Image charge, 188, 237
 Image conduit, 290
 Image-charge effect, 89, 371
 Image-charge shift, 245
 Image current, 188, 194, 237
 Image-current effect, 91, 371
 Imprisonment, 5, 350
 Indium tin oxide, 28
 In-flight capture, 5, 135, 136
 Inhibited spontaneous emission, 94
 Insulators, 15
 Intensity-gradient cooling, 178
 Invariance theorem, 53, 83, 265, 321, 366, 369
 Ioffe, A.F., 37
 Ioffe bars, 37
 Ioffe, M.S., 37
 Ioffe-Pritchard trap, 37, 319
 Ioffe trap, 37
 Ion Coulomb crystal, 117
 Ion crystal, 112, 374
 Ion crystal melting, 112
 Ion ensemble entropy, 112
 Ionen-Käfig, 356
 Ion fluid, 112
 Ion injection, 117
 Ion-ion collisions, 407
 Ion mobility, 183
 Ion-molecule reactions, 155
 Ion-neutral collisions, 399
 Ion production, 131
 Iron, 15, 285, 324
 ITO, 28

J

Johnson, J.B., 171
 Johnson noise, 193, 214, 262, 263
 Johnson-Nyquist noise, 214

K

Kevlar, 395, 396
 Kick, 158
 Kingdon, K.H., 3, 350
 Kingdon trap, 3, 356
 Kohler plot, 397
 Kusch, P., 336

L

Lamb-Dicke parameter, 180
 Lamb-Dicke regime, 172, 180
 Lamb, W.E., 180
 Landé, A., 335
 Langevin collision, 185, 408
 Langevin cross section, 185
 Langevin, P., 185
 Langmuir, I., 113
 Langmuir oscillations, 112
 Laplace equation, 5
 Laplace, P.S., 348
 Laplace's law, 246
 Large ion crystals, 116
 Larmor frequency, 75, 95, 278, 281, 284, 301, 304, 307, 338
 Laser ablation, 131
 Laser cooling, 176, 371
 Laser-fluorescence mass spectroscopy, 265
 Laser ionisation, 131
 Lawrence bottle, 320
 Lead, 307, 311
 Leibnitz harmonic triangle, 74
 Light conduit, 290
 Limacon, 52
 Linear combined trap, 34
 Linear quadrupole, 5
 Liquid helium, 19, 395
 Liquid helium vaporisation heat, 395
 Liquid nitrogen, 19
 Lithium-like ions, 310
 Localisation, 289
 London effect, 252
 London, F., 252
 London, H., 252
 London penetration depth, 252, 260, 261
 Long-range ordering, 116
 Lorentz, H.A., 348
 Lorentz line shape, 172, 292
 Lorentz steerer, 158
 Lotz formula, 133
 Low-pass filter, 214
 Lyapunov exponent, 112

M

Macor, 15, 325
 Madelung energy, 115, 117
 Magnesium, 119, 297
 Magnet bore, 17
 Magnetic bottle, 71, 278, 319, 320, 325, 329
 Magnetic bottle strength, 373
 Magnetic dipole moment, 299

Magnetic dipole operator, 299
 Magnetic dipole transition, 179, 312
 Magnetic field gradient, 142
 Magnetic field homogeneity, 17
 Magnetic field imperfections, 70
 Magnetic flux quantum, 251
 Magnetic micro-traps, 319
 Magnetic mirror, 143
 Magnet quench, 18
 Magnetron centring, 56, 164, 370
 Magnetron cooling, 56, 193
 Magnetron frequency, 6, 48
 Magnetron motion, 54
 Magnetron orbit, 81
 Magnetron-free operation, 34
 Magnet training, 18
 Majorana transition, 139
 Malmberg, J.H., 38
 Marginally stable Penning trap, 110
 Mass limit, 50
 Mass spectrometry, 155, 265
 Mathieu equation, 58
 Maximum trap potential, 366
 Maxwell, J.C., 348
 Maxwell loops, 377, 381
 Maxwell's equations, 391
 MCP, 270
 Mediated sympathetic cooling, 213, 425
 Meissner effect, 251
 Meissner, F.W., 251
 Meissner-Ochsenfeld effect, 326
 Meissner phase, 251
 Mercury, 118, 297, 307
 Mesoscopic ion crystals, 113, 114
 Method of images, 237
 Micro-channel-plate detector, 270
 Microwave cavity, 94
 Microwave frequency standard, 289
 Microwave power, 298
 Microwave power shift, 303
 Microwave spectroscopy, 297, 300
 Minimum-B trap, 38
 Misalignment parameter, 83
 Modified Bessel function, 64
 Modified Bessel function of the first kind, 243, 244
 Modified Bessel function of the second kind, 244
 Modified cyclotron frequency, 48
 Mono-electron oscillator, 157, 357
 Motional spectrum, 86
 Müller-Salzborn-fit, 406
 Multi-component plasma, 118

Multi-Species ion crystals, 117

N

Natural linewidth, 291
 Nb₃Sn, 17
 NbTi, 17
 Negative feedback cooling, 206
 Neodymium, 324
 Nested Penning traps, 35
 Nickel, 15, 324
 Niobium, 252
 Niobium-tin, 17
 Niobium-titanium, 17, 260, 261
 Nobel Prize, 2, 4, 348, 351, 352, 356, 357
 Noise sources, 214
 Noise temperature measurement, 171
 Non-exponential cooling, 201
 Non-linear resonances, 108, 366
 Non-linear Zeeman effect, 315
 Non-neutral plasma, 110
 Non-resonant ejection, 152
 Non-resonant motional coupling, 166
 Notation, 6
 Notch filter, 152, 216
 Nuclear magnetic moment, 310
 Nuclear magneton, 310, 314, 337
 Numerical eccentricity, 81
 Nylon, 15, 395

O

Oblate, 229, 247
 Octupole excitation, 147
 Octupole term, 67
 OFHC copper, 15
 Onion trap, 33
 Optical dipole trap, 13
 Optical frequency standard, 289
 Optical spectroscopy, 289
 Optical trap, 14
 Orbitrap, 5, 351
 Orbitron trap, 3
 Oscillation amplitudes, 46
 Oscillation frequencies, 47
 Oscillation frequency distribution, 202

P

Pad trap, 39
 Parametric coupling, 161
 Parametric oscillator, 157
 Particle accumulation, 136
 Particle cooling, 169

Particle ensembles, 105
 Particle number density, 6
 Particle polarisability, 371
 Particle stacking, 136
 Particle temperature, 169
 Particle temperature measurement, 171
 Particle trajectory, 45
 Pascal, B., 52
 Pascal, E., 52
 Patch noise, 215
 Patch potentials, 78
 Paul trap, 3, 355
 Paul, W., 2, 354
 Pauli, W., 339
 PEEK, 15, 395, 396
 Penning, F.M., 1, 351
 Penning-Ioffe traps, 37
 Penning-Malmberg traps, 38
 Penning-trap fusion, 232
 Permanent-magnet Penning trap, 36
 Permittivity of free space, 106
 Persistent mode, 18
 Perturbed cyclotron motion, 51
 Phase diagram, 111
 Photo-dissociation, 155
 Photo-multiplier tubes, 290
 Physical adsorption, 398
 Physi-sorption, 398
 Pierce, J.R., 1, 353
 Pink noise, 215
 Pitch angle, 145
 Pixel trap, 33
 Planar Penning trap, 31
 Planar shell model, 113, 114
 Plasma angular momentum, 225
 Plasma frequency, 374
 Plasma modes, 110, 120, 374
 Plasma parameter, 111, 374
 Plasma vortex frequency, 122
 Plate trap, 39
 PMT detector, 290
 PnA method, 263, 264
 PnP method, 263, 264
 Poisson's equation, 64
 Poisson's law, 228
 Polarisability shift, 92
 Polarisation-gradient cooling, 184
 Positron, 94, 119, 336
 Positron magnetic moment, 336
 Potential asymmetry, 139, 140
 Potential modulation, 156
 Power broadening, 182, 292
 Pressure broadening, 292

Pritchard, D.E., 37
 Prolate, 38, 229, 247
 Proton, 262, 263, 284
 Proton magnetic moment, 337
 P43 scintillator, 270
 P46 scintillator, 270
 Pulsed drift tube, 141
 Purcell, E.M., 89

Q

Quadratic Stark shift, 295
 Quadratic Zeeman effect, 315
 Quadrature excitation, 147
 Quadrupole excitation, 151
 Quadrupole Penning trap, 21
 Quality factor, 193
 Quantum electrodynamics, 335
 Quantum information processing, 118
 Quantum jump method, 336
 Quantum numbers of motion, 276

R

Rabi frequency, 162, 299
 Rabi, I.I., 162
 Rabi oscillations, 301
 Radial laser cooling, 178
 Radial magnetic field, 368
 Radiative damping, 93, 372
 Raman cooling, 184
 Ramo, S., 353
 Rapid adiabatic passage, 162
 Recoil heating, 183, 184
 Reduced cyclotron frequency, 6, 51
 Reflectron mode, 133
 Relativistic Doppler shift, 295
 Relativistic frequency shifts, 95, 372
 Relativistic mass effect, 281
 Reservoir trap, 5
 Residual resistivity ratio, 397
 Resistive cooling, 188, 197, 370
 Resolved-Sideband cooling, 180
 Resonance order, 108
 Resonant circuit, 189
 Resonant coupling, 161, 370
 Resonant motional coupling, 161
 Resonant particle ejection, 152
 Resonant particle excitation, 154
 Resonant particle loss, 108
 Rhenium, 132, 311
 Richardson constant, 132
 Richardson-Dushman equation, 132

Ring of charge, 100
 RLC circuit, 189
 Rotating dipole field, 147
 Rotating dipole torque, 235
 Rotating multipoles, 375
 Rotating quadrupole field, 148
 Rotating wall, 225
 Rotating wall slip, 235
 Rotating-wall technique, 375
 Roulette, 51
 RRR, 397
 Rubidium, 266
 Rydberg constant, 266
 Rydberg energy, 411
 Rydberg states, 131

S

Saddle potential, 9
 Samarium, 15, 324
 Sapphire, 396
 Saturation broadening, 183
 Saturation intensity, 183, 293, 299
 Saturation magnetisation, 324
 Saturation parameter, 183, 282
 Schottky diodes, 217
 Schottky equation, 215
 Schwinger field, 96
 Schwinger, J., 96, 336
 Scintillators, 270
 Selective instability, 152
 Self-cooling, 94, 372
 Self-Shielding, 78
 Separation length, 119
 Shapal, 15
 Sheet resistance, 28
 Shielding length, 227
 Shielding parameter, 315
 Shim coils, 17
 Shockley-Ramo theorem, 188, 353
 Shockley, W.B., 188, 352
 Shortened epitrochoid, 52
 Shot noise, 214
 Shubnikov, L.W., 251
 Shubnikov phase, 251
 Sideband coupling, 161
 Sideband detection, 156
 Sidekick, 158
 Silicon, 266, 285
 Silver, 15
 Single-component plasma, 110, 111
 Sisyphus cooling, 184
 Skin depth, 29

Small ion crystals, 113
 Sodium, 266
 Sokolov, A.A., 96
 Sokolov-Ternov effect, 96
 Solenoid magnetic field, 377
 Solid angle of detections, 28
 Space charge, 106, 108
 Space-charge effect, 105, 374
 Space charge shift, 246
 Specific heat, 397
 Spheroid, 226
 Spin flip probability, 302
 Spirograph, 52
 Spitzer self-collision time, 153, 437
 Stability criterion, 6, 49, 110
 Stability parameter, 6, 49
 Stainless steel, 298, 395, 396
 Stark shift, 184
 Stefan-Boltzmann constant, 396
 Stefan-Boltzmanns law, 396
 Stimulated Raman cooling, 184
 Stochastic cooling, 207
 Stochastic heating, 183
 Stored ion calorimeter, 249
 Strong-binding limit, 180
 Strongly correlated plasma, 111
 Sub-Doppler cooling, 184
 Sub-thermal cooling, 207
 Superconducting magnetic bottle, 320, 326
 Superconducting resonator, 259, 260
 Superconducting switch, 18
 Superconductivity, 250
 Superconductor, 17
 Surface patches, 78, 415
 Suspended trapping, 188
 SWIFT, 152
 Symmetric loading, 136
 Sympathetic cooling, 118, 209, 372
 Sympathetic crystallisation, 117
 Synchrotron radiation, 94, 211

T

Teflon, 396
 Temporal field imperfections, 77
 Ternov, I.M., 96
 Thallium, 311
 Thermal conductivity, 396
 Thermal expansion coefficient, 396
 Thermal noise, 214
 Thermal noise voltage, 205
 Thermal radiation, 396
 Thermionic emission, 131

Thorium, 132
Three-dimensional quadrupole, 5
Tickle pulses, 158
Tilt, 369
Time orbiting potential traps, 319
TOF-ICR, 155, 270
TOF-MS, 270
Toroidal Penning trap, 325
Total electrostatic potential, 367
Total magnetic field, 368
Transfer function, 171
Transition linewidth, 312
Transition matrix, 312
Transparent coating, 28
Trap instabilities, 108
Trapping parameter, 6, 110
Trap voltage modulation, 367
Triangular numbers, 66
Tri-axial ellipsoid, 226
Tritium, 265
Tri-triangular numbers, 66
Trivelpiece-Gould modes, 110
Tungsten, 132
Tuning ratio, 174

U

Uncertainty principle, 291
Unitary Penning trap, 13, 36
Unruh effect, 96

V

Van Regemorter formula, 410

Varactor diode, 258
Variable magnetic bottle, 320
Vertical magnet, 17
Voigt line shape, 172, 294
Voigt, W., 172
Voltage modulation, 156
Voltage stability, 78
Von Weizsäcker, C.F., 5
Vortex frequency, 122

W

Warm-bore magnet, 17
Weak-binding regime, 179
Wigner-Seitz radius, 111, 374
Wineland, D.J., 3, 193, 357
Wire trap, 39

Y

YBCO, 17
YIG, 297
Ytterbium, 307
Yttrium barium copper oxide, 17
Yttrium iron garnet, 297

Z

Zeeman effect, 313
Zeeman splitting, 300
Zeeman sublevels, 300
Zig-zag method, 338