Jesús Crespo Cuaresma
Tapio Palokangas
Alexander Tarasyev
Editors

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Dynamic Systems, Economic Growth, and the Environment

by editors

Jesús Crespo Cuaresma

University of Innsbruck, Austria

Tapio Palokangas

University of Helsinki, Finland

and

Alexander Tarasyev

Ural Branch Russian Academy of Sciences, Russia

Springer

Professor Jesús Crespo Cuaresma
Department of Economics
University of Innsbruck
SOWI Gebäude,
Universitätsstr. 15
6020 Innsbruck
Austria
jesus.crespo-cuaresma@uibk.ac.at

Professor Tapio Palokangas
Department of Economics
P.O. Box 17 (Arkadiankatu 7)
University of Helsinki
00014 Helsinki
Finland
tapio.palokangas@helsinki.fi

Professor Alexander Tarasyev
Institute of Mathematics and Mechanics
Ural Branch Russian Academy of Sciences
S. Kovalevskoi str. 16
620219 Ekaterinburg
Russia
tam@imm.uran.ru
tarasiev@iiasa.ac.at

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Introduction

The major goal of the book is to create an environment for matching different disciplinary approaches to studying economic growth. This goal is implemented on the basis of results of the Symposium “Applications of Dynamic Systems to Economic Growth with Environment” which was held at the International Institute for Applied Systems Analysis (IIASA) on the 7th–8th of November, 2008, within the IIASA Project “Driving Forces of Economic Growth” (ECG). The symposium was organized by coordinators of the ECG project: Jesus Crespo-Cuaresma from IIASA World Population Program, and Tapio Palokangas and Alexander Tarasyev from IIASA Dynamic Systems Program.

The book addresses the issues of sustainability of economic growth in a changing environment, global warming and exhausting energy resources, technological change, and also focuses on explanations of significant fluctuations in countries’ growth rates. The chapters focus on the analysis of historical economic growth experiences in relation to environmental policy, technological change, development of transport infrastructure, population issues and environmental mortality.

The book is written in a popular-science style, accessible to any intelligent lay reader. The prime audience for the book is economists, mathematicians and engineers working on problems of economic growth and environment. The mathematical part of the book is presented in a rigorous manner, and the detailed analysis is expected to be of interest to specialists in optimal control and applications to economic modeling.

The book consists of four interrelated parts.

The first part, “Dynamic Systems in Growth Models”, comprises papers devoted to theoretical issues of economic growth modeling. It considers also the possibility of econometric forecasting of future trends of techno-economic development and its reliability including high-order precision for constructing optimal trajectories.

The first paper of this part titled “On Adequate Transversality Conditions for Infinite Horizon Optimal Control Problems—a Famous Example of Halkin”, by Sabine Pickenhain, is devoted to application of duality concept to infinite horizon optimal control problems. A duality theory is developed for proving sufficient conditions for optimality. The obtained dual solution of the problem demonstrates which kind of transversality conditions is natural.

The second paper, “Sequential Precision of Predictions in Models of Economic Growth”, by Andrey Krasovskii and Alexander Tarasyev, deals with a model of economic growth based on real time series. A distinguishing feature of the approach is that real data is analyzed not by direct statistical approximations but through formalization of the process in terms of optimal control theory. The problem of investment optimization is solved using a version of the Pontryagin maximum principle, and elements of the qualitative theory of differential equations. Numerical algorithms are proposed for constructing synthetic trajectories of economic growth. For verification of the proposed approach, several model modifications with sequential precision and case studies are presented basing on real data for economies of US, UK, and Japan.

The third paper, “High Order Precision Estimates in Algorithms for Solving Problems of Economic Growth”, by Andrey Krasovskii and Alexander Tarasyev, focuses on high order precision estimates for elaborated algorithms of constructing optimal trajectories of economic growth. The estimates establish relations between precision parameters in the phase space and precision parameters for functional indices. The results of numerical experiments illustrating different algorithm constructions are given for real data of US and Japan economies.

The second part, “Growth and Environment”, addresses results related to the application of dynamic systems to economic growth, with especial focus on issues of environmental impact.

The first paper of this part, “Growth an Climate Change: Threshold and Multiple Equilibria”, authored by Alfred Greiner, Lars Gruene, and Willi Semmler, considers a basic growth model in the presence of the effect of global warming. Two basic scenarios are studied. In the first scenario, it is assumed that abatement spending is fixed exogenously. It is shown that in this case the model may give rise to multiple equilibria and thresholds. In the second scenario, the unique social optimum is analyzed where both consumption and abatement are set optimally. It is shown that the steady state temperature is smaller and the capital stock is larger in this scenario compared to the economy with lower abatement spending. It is also demonstrated that the abatement actions of the first scenario provide the steady state in the multiple equilibria setting with parameters which are very close to that of the social optimum in the second scenario.

The second paper, “Optimal Economic Growth under Stochastic Environmental Impact: Sensitivity Analysis”, by Elena Rovenskaja, presents an approach toward the sensitivity analysis of optimal economic growth to a negative environmental impact driven by random natural hazards that damage the production output. The author develops a simplified model of GDP whose growth leads to the increase of GHG in the atmosphere provided investment in cleaning is insufficient. The hypothesis of the Poisson probability distribution is used to describe natural hazards. An optimal investment policy in production and cleaning together with optimal GDP trajectories is constructed for a standard discounted integral consumption index. The model is calibrated in the global scale and the sensitivity analysis is implemented for obtained optimal growth scenarios with respect to uncertain parameters of the Poisson distribution.

The third paper, “Optimal Economic Growth with a Random Environmental Shock”, authored by Sergey Aseev, Konstantin Besov, Simon-Erik Ollus, and Tapio

Palokangas, is devoted to the problem of balancing an old "dirty," or "polluting," technology and a new "clean" technology. At some stage in the future the usage of the old technology will be penalized. The government's incentives to invest in cleaner technologies are based on productivity of the technology and randomly increasing abatement costs for pollution in future. In contrast to the Schumpeterian model of creative destruction, both technologies can be used simultaneously. Assuming that the exogenous environmental shock follows a Poisson process, the authors use Pontryagin's maximum principle to find the optimal investment policy. Special attention is paid to conditions under which a rational government should invest all its resources in one technology, while the other is moderately run down, as well as to conditions under which it should divide the investments between the technologies in a certain ratio.

The topic of the third part, "Growth and Environmental Policy", relates to questions of policy regulation of environmental problems in order to sustain economic growth tendencies.

The first paper of this section "Prices versus Quantities Vintage Capital Model", by Thierry Bréchet, Tsvetomir Tsachev, and Vladimir Veliov, shows that the heterogeneity of the available physical capital with respect to productivity and emission intensity is an important factor for policy design, especially in the presence of emission restrictions. In a vintage capital model, reducing pollution requires to change the capital structure through investment in cleaner machines and to scrap the more polluting ones. In such a setting the authors show that emission tax and auctioned emission permits may yield contrasting outcomes. It is demonstrated also that some failures in the permits market may undermine its efficiency and that imposing the emission cap over longer periods plays a regularizing role in the market.

The second paper, "International Emission Policy with Lobbying and Technological Change", by Tapio Palokangas, examines the implementation of emission policy in a union of countries. Production in any country incurs emissions that pollute all over the union, but efficiency in production can be improved by research and development (R&D). The author compares four cases: Laissez-faire, Pareto optimal policy, and the case of a self-interested central planner that decides on nontraded and traded emission quotas. It is demonstrated that with nontraded quotas, the growth rate is socially optimal, but welfare sub-optimal. Trade in quotas speeds up growth from the initial position of laissez-faire, but slows down growth from the initial position of nontraded quotas.

The third paper, "The Role of Product Differentiation in the Producer-Targeted Promotion of Renewable Energy Technologies", by Ina Meyer and Serguei Kaniovski, explains that carbon-based technologies continue to dominate the energy sector due to their high productivity and economies of scale. This creates an obligation for the governments to provide incentives, such as taxes, subsidies and regulations, to encourage producers to implement cleaner technologies. The authors study a duopoly in which the incumbent is more efficient, has a higher propensity to invest and has a lower cost of capital. The minimal subsidy (to the entrant) or tax (on the incumbent) which is sufficient to preserve the entrant in the market in the long run is derived. The rate of the subsidy or tax depends on the underlying demand structure. The more differentiated the products and preferences of the consumers, the lower the subsidy or tax required to safeguard new entrants with innovative clean technologies.

The fourth paper, “Dynamic Oligopoly with Capital Accumulation and Environmental Externality”, by Davide Dragone, Luca Lambertini, and Arsen Palestini, deals with a differential oligopoly game of interplay between capital accumulation for production and environmental externalities. This game model is based on Ramsey dynamics. It is shown in the paper that at the social optimum it may be optimal to trade off some amount of consumer surplus in order to reduce the externality. Another interesting result demonstrates that if the external effect is proportional to the industry production then the Ramsey golden rule disappears as a stand-alone equilibrium.

In the fifth paper, “On a Decentralized Boundedly Rational Emission Reduction Strategy”, by Arkady Kryazhimskiy, the author considers a number of countries which have externality through emissions and which negotiate on marginal reductions in their emissions. While choosing a unilateral reduction in its emissions, each country trades on exchanging the value of its reduction to the reduction of the total pollution falling on its territory, which depends on the reductions of the other countries. In this setup, there is a negative externality through emissions so that a Pareto optimum requires side payments or some policy intervention (e.g. taxation). The contribution of the paper is to show that a Pareto optimum could be (approximately) attained even without taxes or side payments, if the negotiation process were organized in a specific manner.

In the fourth part, “Applications to Population and Infrastructure”, two papers discuss the possibility of application of optimal control theory to solve long-run problems arising in human population and questions of critical infrastructures.

The first paper of this section, “Environmental Mortality and Long-Run Growth”, by Ulla Lehmijoki and Elena Rovenskaja, provides a long-run consumer optimization model in which mortality is endogenous to emissions generated by production. Emissions are assumed to follow the Environmental Kuznets Curve (EKC) path, first rising and then falling along with output. It is assumed that pollution is a public good consumed by all in equal amounts, so that only its overall extent is important. The EKC hypothesis is, therefore, considered in total rather than in per capita terms. Since the emphasis of the paper is on the basic trade-off between output and deaths, many important elements such as emission-limiting policies and health-promoting medical efforts are left out of the model. The model is estimated for the European outdoor air pollution data. Economic growth will thus decrease rather than increase pollution in the future. Nevertheless, continuous population growth may increase the number of deaths in some countries.

The second paper, “Development of Transportation Infrastructure in the Context of Economic Growth”, authored by Manuel Benjamin Ortiz-Moctezuma, Denis Pivovarchuk, Jana Szolgayova, and Sabine Fuss, is devoted to a co-evolutionary model of economic output and road infrastructure. It investigates the interdependency between a country’s economic growth and the development of transportation infrastructure in this country. The model is based on the assumption that economic output (GDP) in a country depends on road infrastructure and vice versa. Based on these assumptions, an optimal control problem on infinite horizon is posed in which the growth rate of the road capacity is a control variable. This problem is solved analytically and the time-independent constant solution is obtained. It is shown that the optimal trajectories converge to the stable steady state in the space of GDP and road capacity.

We strongly believe that the results in this volume will help the reader reach a better understanding of economic growth processes via mathematical modeling, provide tools for forecasting growth trends and improving its precision. Furthermore, the models used in the book will prove to be helpful tools for policy advice when approaching the design of growth-enhancing environmental policies.

Contents

Contributors

Sergey Aseev Steklov Mathematical Institute, Gubkina str. 8, Moscow 119991, Russia; International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1, Laxenburg, 2361, Austria, aseev@mi.ras.ru

Konstantin Besov Steklov Mathematical Institute, Gubkina str. 8, Moscow 119991, Russia, kbesov@mi.ras.ru

Thierry Bréchet CORE, and Chair Lhoist Berghmans in Environmental Economics and Management, Université catholique de Louvain, Voie du Roman Pays, 34, 1348 Louvain-la-Neuve, Belgium, thierry.brechet@uclouvain.be

Davide Dragone Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy, davide.dragone@unibo.it

Sabine Fuss Forestry Program (FOR), IIASA, Schlossplatz 1, 2361 Laxenburg, Austria, fuss@iiasa.ac.at

Alfred Greiner Department of Business Administration and Economics, Bielefeld University, P.O. Box 100131, 33501 Bielefeld, Germany, agreiner@wiwi.uni-bielefeld.de

Lars Grüne Mathematical Institute, University of Bayreuth, 95440 Bayreuth, Germany, lars.gruene@uni-bayreuth.de

Serguei Kaniovski Austrian Institute of Economic Research (WIFO), P.O. Box 91, 1103 Vienna, Austria, serguei.kaniovski@wifo.ac.at

Andrey A. Krasovskii Institute of Mathematics and Mechanics, Ural Branch Russian Academy of Sciences, S. Kovalevskoi Str. 16, 620219 Ekaterinburg, Russia, ak@imm.uran.ru; Vienna Institute of Demography, Austrian Academy of Sciences, Wohllebengasse 12-14, 1040 Vienna, Austria, Andrey.Krasovskiy@oeaw.ac.at

Arkady Kryazhimskiy International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1, Laxenburg, 2361, Austria, kryazhim@iiasa.ac.at

Luca Lambertini Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy, luca.lambertini@unibo.it

Ulla Lehmijoki Department of Economics, University of Helsinki, HECER, and IZA, P.O. Box 17, 00014, Helsinki, Finland, ulla.lehmijoki@helsinki.fi

Ina Meyer Austrian Institute of Economic Research (WIFO), P.O. Box 91, 1103 Vienna, Austria, ina.meyer@wifo.ac.at

Simon-Erik Ollus University of Helsinki and HECER, Arkadiankatu 7, 00014 Helsinki, Finland, simon.ollus@fortum.com

Manuel Benjamin Ortiz-Moctezuma Dynamic Systems Program (DYN), IIASA, Schlossplatz 1, 2361 Laxenburg, Austria, moctez@iiasa.ac.at; Ciudad Victoria, Tamaulipas, Mexico

Arsen Palestini Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy, palestini@math.unifi.it

Tapio Palokangas Department of Economics, University of Helsinki and HECER, Arkadiankatu 7, P.O. Box 17, 00014 Helsinki, Finland, tapio.palokangas@helsinki.fi; International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1, 2361 Laxenburg, Austria

Sabine Pickenhain Brandenburg University of Technology Cottbus, 03013 Cottbus, Germany, sabine@math.tu-cottbus.de

Denis Pivovarchuk Dynamic Systems Program (DYN), IIASA, Schlossplatz 1, 2361 Laxenburg, Austria, pivovar@iiasa.ac.at

Elena Rovenskaya International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1, 2361 Laxenburg, Austria, rovenska@iiasa.ac.at

Willi Semmler Department of Economics, New School for Social Research, New School, 79 Fifth Ave, New York, NY 10003, USA, semmlerw@newschool.edu

Jana Szolgayova Forestry Program (FOR), IIASA, Schlossplatz 1, 2361 Laxenburg, Austria, szolgay@iiasa.ac.at

Alexander M. Tarasyev Institute of Mathematics and Mechanics, Ural Branch Russian Academy of Sciences, S. Kovalevskoi Str. 16, 620219 Ekaterinburg, Russia, tam@imm.uran.ru

Tsvetomir Tsachev Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev str., block 8, 1113 Sofia, Bulgaria, tsachev@math.bas.bg

Vladimir M. Veliov Institute of Mathematical Methods in Economics, Vienna University of Technology, 1040 Vienna, Austria, veliov@tuwien.ac.at

On Adequate Transversality Conditions for Infinite Horizon Optimal Control Problems—A Famous Example of Halkin

Sabine Pickenhain

Abstract In this paper we apply a duality concept of Klötzler (Equadiff IV. Proceedings of the Czechoslovak conference on differential equations and their applications held in Prague, 22–26 August, 1977, Lecture notes in mathematics, vol. 703, pp. 189–196, Springer, Berlin, 1979) to infinite horizon optimal control problems. The key idea is the choice of weighted Sobolev spaces as state spaces.

Different criteria of optimality are known for specific problems (Carlson et al., Infinite horizon optimal control, Springer, New York, Berlin, Heidelberg, 1991; Feichtinger and Hartl, Optimale Kontrolle ökonomischer Prozesse, de Gruyter, Berlin, New York, 1986), e.g. the overtaking criterion of von Weizsäcker (1965), the catching up criterion of Gale (1967) and the sporadically catching up criterion of Halkin (1974). Corresponding to the formulated optimality criteria we develop a duality theory and prove sufficient conditions for optimality. An example of Halkin is presented, where the solution is obtained in the framework of these weighted spaces. The obtained dual solution of the problem demonstrates which kind of transversality conditions are natural.

1 Introduction

Still at the beginning of the previous century the optimal control problems with infinite horizon became very important with regards to applications in economics and biology, where an infinite horizon seems to be a very natural phenomenon (Feichtinger and Hartl 1986; Carlson et al. 1991; Sethi and Thompson 1985). Since then these problems were treated by many authors and various optimality conditions were obtained, see for instance Blot and Cartigny (1995), Michel (1982). Nevertheless we have to observe that the theory with an integral over an unbounded interval is often represented in an incorrect or incomplete way. Most of papers and books, focused on applications, not even give a hint to different definitions of the integral-the Lebesgue—or the improper Riemann integral. In Pickenhain et al. (2006) it was demonstrated that different integral types can be useful in applications but lead to completely different theoretical results. Further, we pointed out that, for a correct setting of the problem, the choice of an appropriate state space is essential. Let us

S. Pickenhain (✉)
Brandenburg University of Technology Cottbus, 03013 Cottbus, Germany
e-mail: sabine@math.tu-cottbus.de

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mention that the Lagrange multipliers associated with the constraints belong to the dual of the space wherein the constraint set has a nonempty interior.

A lot of work has been done in the last decades to prove *necessary optimality conditions* for problems in the calculus of variations, see e.g. Blot and Cartigny (1995), and optimal control, see e.g. Carlson et al. (1991). The usual Pontryagins Maximum Principle (PMP) cannot easily be adjusted to the case of infinite horizon problems as it was first demonstrated in an example of Halkin (1979). For special problems with dominating discounts Aseev et al. (2001), where able to prove (PMP).

Considering the Problem $(P)_\infty$ with infinite horizon as a *limit* of a finite horizon problem $(P)_T$ one could expect a *natural* transversality condition for the adjoints p of the problem (P_∞):

$$\lim_{T\to\infty} p(T) = 0. \tag{1}$$

The example of Halkin shows, that even this equation (1) does not hold.

Results concerning *sufficiency conditions* were derived via Fenchel-Rockafellar duality by Rockafellar (1978), Aubin and Clarke (1979), Magill (1982), Benveniste and Scheinkman (1982). The aim of this paper is to develop a duality theory in weighted Sobolev spaces as the state space, and to obtain adjoint variables p in the dual space of the state space, such that (1) is replaced by

$$L\text{-}\int_0^\infty \langle p(t), x(t) - x^*(t)\rangle dt < \infty, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$. Our paper is organized as follows. We use the duality concept of Klötzler (1979) and a special choice of state spaces to construct a dual problem. Considering the exponential factor $e^{-\rho t}$ as a weight function we propose to choose weighted Sobolev spaces as state spaces or spaces of the adjuncts respectively, defined in the second section. The fourth section is devoted to the development of the duality theory taking some properties of weighted spaces into account. This section includes sufficiency conditions, which are proved via linear approach in the dual problem. In the last section Examples illustrate the type of transversality conditions obtained by the dual approach.

2 Problem Formulation

2.1 Problem

We deal with problems of the following type $(P)_\infty$:

Minimize the functional

$$J(x,u) = \int_0^\infty f(t, x(t), u(t))\tilde{\nu}(t)dt \tag{3}$$

with respect to all

$$[x, u] \in W^{1,n}_{p,\nu}(0, \infty) \times L^r_\infty(0, \infty) \tag{4}$$

fulfilling the

$$\text{State equations} \quad x'(t) = g(t, x(t), u(t)) \quad \text{a.e. on } (0, \infty), \tag{5}$$

$$\text{Control restrictions} \quad u(t) \in U \subseteq \text{Comp}(\mathbb{R}^r) \quad \text{a.e. on } (0, \infty), \tag{6}$$

$$\text{State constraints} \quad x(t) \in G(t) \quad \text{on } (0, \infty), \tag{7}$$

$$\text{Initial conditions} \quad x(0) = x_0. \tag{8}$$

The spaces $W^{1,n}_{p,\nu}(0, \infty)$ will be defined below.

2.2 Weighted Sobolev Spaces

We consider weighted Sobolev spaces $W^{1,n}_{p,\nu}(\Omega)$ as subspaces of weighted $L^n_{p,\nu}(\Omega)$ spaces of those absolutely continuous functions x for which both x and its derivative $\dot{x}$ lie in $L^n_{p,\nu}(\Omega)$, see Kufner (1985).

Let $\Omega = [0, \infty)$ and let $\mathcal{M}^n = \mathcal{M}(\Omega; \mathbb{R}^n)$ denote the space of Lebesgue measurable functions defined on Ω with values in $\mathbb{R}^n$. Let a *weight function* ν be given, i.e. ν is a function continuous on Ω, $0 < \nu(t) < \infty$, then we define the space $L^n_{p,\nu}(\Omega)$ with $p \geq 2$ by

$$L^n_{p,\nu}(\Omega) = \left\{ x \in \mathcal{M}^n \mid \|x\|^p_p := \int_\Omega |x(t)|^p \nu(t) dt < \infty \right\}, \tag{9}$$

for $p = \infty$

$$L^n_{\infty,\nu}(\Omega) = \left\{ x \in \mathcal{M}^n \mid \|x\|_\infty := \operatorname*{ess\,sup}_{t \in \Omega} |x(t)| \nu(t) < \infty \right\} \tag{10}$$

and the weighted Sobolev space by

$$W^{1,n}_{p,\nu}(\Omega) = \{ x \in \mathcal{M}^n \mid x \in L^n_{p,\nu}(\Omega), \dot{x} \in L^n_{p,\nu}(\Omega) \}. \tag{11}$$

Here $\dot{x}$ is the distributional derivative of x in the sense of Yosida (1974, p. 49). This space, equipped with the norm

$$\|x\|^p_{W^{1,n}_{p,\nu}(\Omega)} = \int_\Omega \{ |x(t)| + |\dot{x}(t)| \}^p \nu(t) dt, \tag{12}$$

is a Banach space.

For $x \in L^n_{p,\nu}(\Omega)$ and $y \in L^n_{q,\nu^{1-q}}(\Omega)$ the scalar product $\langle\langle x, y\rangle\rangle$ in $L^n_2(\Omega)$ defines a continuous bilinear form, since

$$\begin{aligned} |\langle\langle x, y\rangle\rangle| &\leq \int_0^\infty |x(t)|\nu^{1/p}(t)|y(t)|\nu^{-1+1/q}(t)dt \\ &\leq \|x\|_{L^n_{p,\nu}(\Omega)}\|y\|_{L^n_{q,\nu^{1-q}}(\Omega)} \end{aligned} \tag{13}$$

holds true. For the special case $p = 2$ one has $[L^n_{2,\nu}(\Omega)]^* = L^n_{2,\nu}(\Omega)$ due to the Riesz representation theorem. Therefore, we obtain the following relation between the scalar products in $L^n_{2,\nu}(\Omega)$ and $L^n_2(\Omega)$:

For $x \in L^n_{2,\nu}(\Omega)$ and $y \in L^n_{2,\nu^{-1}}(\Omega)$ there exists $\hat{y} \in L^n_{2,\nu}(\Omega)$ such that

$$\langle x, \hat{y}\rangle_{L^n_{2,\nu}(\Omega)} = \langle\langle x, y\rangle\rangle_{L^n_2(\Omega)} \tag{14}$$

and

$$\hat{y} = y/\nu. \tag{15}$$

Equation (14) is essentially used to formulate the duality theory in the sense of Klötzler (1979), in the following sections.

Remark It is well known, see Elstrodt (1996), that the inclusion $L^n_{p,\nu}(\Omega) \subseteq L^n_{q,\nu}(\Omega)$ holds true, i.e. there is a $C \in \mathbb{R}_+$ such that

$$\|x\|_{L^n_{q,\nu}} \leq C\|x\|_{L^n_{p,\nu}} \tag{16}$$

for all $p \geq q$ and $\int_0^\infty \nu(t)dt < \infty$. A weight function ν is called a density if $\int_0^\infty \nu(t)dt < \infty$.

Remark Note that here and in the proofs of other sections we abbreviate $L^n_{p,\nu}(\Omega)$ by $L^n_{p,\nu}$.

2.3 Lebesgue and Improper Riemann Integrals

Now some aspects concerning the consequences of the distinction between Lebesgue and improper Riemann integrals should be noted.

Let us remind that

$$R\text{-}\int_0^\infty f(t)dt := \lim_{T\to\infty} R\text{-}\int_0^T f(t)dt \tag{17}$$

where $f: \mathbb{R} \to \mathbb{R}$ has to be R-integrable over any closed interval $[0, T] \subset \mathbb{R}$. If, under this assumption, the Lebesgue integral converges absolutely, i.e.

$$L\text{-}\int_0^\infty |f(t)|dt < \infty, \tag{18}$$

then the Lebesgue and the improper Riemann integral coincide,

$$L\text{-}\int_0^\infty f(t)dt = R\text{-}\int_0^\infty f(t)dt = \lim_{T\to\infty} L\text{-}\int_0^T f(t)dt, \tag{19}$$

(see Elstrodt 1996, p. 151 f., Theorem 6.3). It may happen, however, as in the famous example with $f(t) = \sin t/t$, that the improper Riemann integral

$$R\text{-}\int_0^\infty \frac{\sin t}{t}dt \tag{20}$$

converges conditionally (i.e., the corresponding series converges non-absolutely, while the Lebesgue integral over the same domain does not exist (see Elstrodt 1996, p. 152).

2.4 Problems with Lebesgue and Improper Riemann Integral

As mentioned before, the infinite horizon control problem (P_∞) is not well defined since the interpretation of the integral within the objective is ambiguous.

In order to make this formulation precise, we denote the set of pairs (x, u) satisfying (4)–(8) by $\mathcal{A}$ and formulate the following basic problems:

$$(P)_\infty^L\colon \quad J_L(x,u) = L\text{-}\int_0^\infty f(t,x(t),u(t))\tilde{\nu}(t)dt \longrightarrow \text{Min!}, \tag{21}$$

$$(x,u) \in \mathcal{A} \cap \mathcal{A}_L, \tag{22}$$

where the integral in the objective is understood as a Lebesgue integral, and $\mathcal{A}_L$ consists of all processes $(x, u) \in \mathcal{A}$, which make the Lebesgue integral in (21) convergent. In the second problem,

$$(P)_\infty^R\colon \quad J_R(x,u) = R\text{-}\int_0^\infty f(t,x(t),u(t))\tilde{\nu}(t)dt \longrightarrow \text{Min!}, \tag{23}$$

$$(x,u) \in \mathcal{A} \cap \mathcal{A}_R, \tag{24}$$

the integral in the objective is understood as an improper Riemann integral, and $\mathcal{A}_R$ consists of all processes $(x, u) \in \mathcal{A}$, which make the improper Riemann integral in (23) (at least conditionally) convergent.

Throughout the paper we assume that the data satisfy the following regularity conditions:

1. The functions f, g are continuously differentiable in all arguments.
2. The control set U is assumed to be compact.
3. The functions ν and $\tilde{\nu}$ are weight functions in the sense explained below.

Remark The feasible domains $\mathcal{A}_L$ and $\mathcal{A}_R$ are, in general, incomparable. Applying the Lebesgue integral, we exclude from $\mathcal{A}$ all feasible trajectories, which make the improper Riemann integral non-absolutely convergent. On the other hand, taking the improper Riemann integral, all trajectories from $\mathcal{A}$, which are Lebesgue integrable but not Riemann integrable even on compact sets, will get lost. For these reasons, it is very important to formulate an infinite horizon problem with the proper integral notion reflecting the situation behind the model in an appropriate way. As it was shown in Pickenhain et al. (2006) the problems with distinct integral types require a completely different mathematical treatment.

3 Optimality Criteria

In the case of infinite horizon optimal control problems we can find several optimality criteria, which are adopted either to problems $(P)^R_\infty$ or to $(P)^L_\infty$.

We first introduce global optimality criteria for the problem $(P)^L_\infty$.

Definition 1 Let a process $(x, u) \in \mathcal{A}_L$ be given. We define

$$\Delta_L(T) = L\text{-}\int_0^T f(t, x(t), u(t))\tilde{\nu}(t)dt - L\text{-}\int_0^T f(t, x^*(t), u^*(t))\tilde{\nu}(t)dt. \quad (25)$$

Then the pair $(x^*, u^*) \in \mathcal{A}_L$ is called optimal for $(P)^L_\infty$ in the sense of

criterion L1 if for any pair $(x, u) \in \mathcal{A}_L$ we have $\lim_{T\to\infty} \Delta_L(T) \geq 0$,

criterion L2 if for any admissible pair $(x, u) \in \mathcal{A}_L$ there exists a moment τ such that for all $T \geq \tau$ we have $\Delta_L(T) \geq 0$ (*uniform strong optimality*).

Remark that optimality in the sense of **L2** is stronger then optimality with respect to **L1**.

In the case of problem $(P)^R_\infty$ we have the following optimality criteria, see Carlson et al. (1991).

Definition 2 Let a process $(x, u) \in \mathcal{A}_R$ is given. We define

$$\Delta_R(T) = R\text{-}\int_0^T f(t, x(t), u(t))\tilde{\nu}(t)dt - R\text{-}\int_0^T f(t, x^*(t), u^*(t))\tilde{\nu}(t)dt. \quad (26)$$

Then the pair $(x^*, u^*) \in \mathcal{A}_R$ is called optimal for $(P)^R_\infty$ in the sense of

criterion R1 if for any pair $(x, u) \in \mathcal{A}_R$ we have $\lim_{T\to\infty} \Delta_R(T) \geq 0$,

criterion R2 if for any admissible pair $(x, u) \in \mathcal{A}_R$ there exists a moment τ such that for all $T \geq \tau$ we have $\Delta_R(T) \geq 0$ (uniform strong optimality),

criterion R3 if for any admissible pair $(x, u) \in \mathcal{A}_R$ we have $\underline{\lim}_{t\to\infty}\Delta_R(t) \geq 0$, i.e. if $\forall \varepsilon > 0\ \exists \tau : [\forall T \geq \tau\, \Delta_R(T) + \varepsilon \geq 0]$ (uniform weak overtaking optimality, catching up criterion (Gale, (67))),

criterion R4 if for any admissible pair $(x, u) \in \mathcal{A}_R$ $\overline{\lim}_{t\to\infty}\Delta_R(t) \geq 0$, i.e. $\forall \varepsilon > 0\ \forall \tau\ \exists T \geq \tau : \Delta_R(T) + \varepsilon \geq 0$ (sporadically catching up criterion (Halkin, (74))).

4 Duality Theory

Before formulating a duality theory for infinite horizon optimal control problems of Lebesgue type $(P)_\infty^L$ we provide some useful properties of functions in weighted Sobolev spaces.

Lemma 1 *Let $x^* \in W_{p,\nu}^{1,n}(\Omega)$ with $x^*(0) = x_0$ and $S : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a function of the form*

$$S(t,\xi) = a(t) + \langle y(t), \xi - x^*(t)\rangle, \tag{27}$$

having $a \in W_1^1(\Omega)$; $y \in W_{q,\nu^{1-q}}^{1,n}(\Omega)$.

Then, for any $x \in W_{p,\nu}^{1,n}(\Omega)$ with $x(0) = x_0$, it holds:

$$\lim_{T\to\infty} S(T, x(T)) = 0, \tag{28}$$

$$\int_0^\infty \frac{d}{dt} S(t, x(t))dt = -S(0, x_0). \tag{29}$$

Proof We observe

$$\int_0^\infty |S(t,x(t))|dt \le \int_0^\infty |a(t)|dt + \int_0^\infty |\langle y(t), x(t) - x^*(t)\rangle|dt. \tag{30}$$

Applying Hölder's inequality we obtain

$$\int_0^\infty |S(t,x(t))|dt \le \|a\|_{W_1^1} + \left(\int_0^\infty |y(t)|^q \nu^{1-q}(t)dt\right)^{1/q} \times \left(\int_0^\infty |x(t) - x^*(t)|^p \nu(t)dt\right)^{1/p} \tag{31}$$

or

$$\int_0^\infty |S(t,x(t))|dt \le \|a\|_{W_1^1} + \|y\|_{L^n_{q,\nu^{1-q}}} \cdot \|x - x^*\|_{L^n_{p,\nu}} < \infty. \tag{32}$$

The convergence of $\int_0^\infty |S(t,x(t))|dt$ yields (28), since

$$\begin{aligned}\lim_{T\to\infty}\int_0^T S(t,x(t))dt &= \lim_{T\to\infty}\left(\int_0^{T-1} S(t,x(t))dt + \int_{T-1}^T S(t,x(t))dt\right)\\ &= \lim_{T\to\infty}\int_0^T S(t,x(t))dt + \lim_{\tau\to\infty} S(\tau, x(\tau)),\end{aligned} \tag{33}$$

whereby τ is an element in $[T-1, T]$. Equation (29) can now easily be derived applying (28). $\square$

We introduce the Hamiltonian as

$$\mathcal{H}(t, \xi, \eta) = \sup_{v \in U} H(t, \xi, v, \eta) \tag{34}$$

with

$$H(t, \xi, v, \eta) = -f(t, \xi, v) + \frac{1}{\tilde{\nu}(t)} \langle \eta, g(t, \xi, v) \rangle, \tag{35}$$

where H represents the Pontryagin function. Further, for $i = 1, \ldots, k$ we define the sets

$$\Omega_i := [\tau_i, \tau_{i+1}), \qquad \tau_1 = 0, \qquad \tau_{k+1} = \infty, \tag{36}$$

where $\{\Omega_1, \ldots, \Omega_k\}$ is a finite decomposition of Ω,

$$X_i := \{(t, \xi) \mid t \in \Omega_i, \xi \in G(t)\}, \qquad X := \cup X_i \tag{37}$$

and

$$Y = \left\{ S : X \to \mathbb{R} \;\middle|\; \begin{array}{l} S(t, \xi) = a(t) + \langle y(t), \xi - x^*(t) \rangle \\ \quad a \in W_1^1(\overline{\Omega_i}),\, y \in W^{1,n}_{q, \nu^{1-q}}(\overline{\Omega_i}), \\ \frac{1}{\tilde{\nu}(t)} \partial_t S(t, \xi) + \mathcal{H}(t, \xi, \partial_\xi S(t, \xi)) \le 0 \\ \quad \forall (t, \xi) \in X_i. \end{array} \right\} \tag{38}$$

Using the scheme described in Klötzler (1979) we construct a dual problem $(D)_\infty^L$ and prove

Theorem 1 (Weak duality) *Let a problem $(P)_\infty^L$ be given. Then the problem $(D)_\infty^L$:*

$$g_\infty^L(S) := \sum_{i=2}^{k} \inf_{Q_i} \{S(\tau_i - 0, \xi_i) - S(\tau_i + 0, \xi_i)\} - S(0, x_0) \to \sup! \tag{39}$$

$$\textit{with respect to } S \in Y, \tag{40}$$

where the infimum in (39) is taken over Q_i,

$$Q_i := \{\xi_i \in \mathbb{R}^n \mid \xi_i \in G(\tau_i), i = 1, \ldots, k\}, \tag{41}$$

is a dual problem to $(P)_\infty^L$, i.e. the weak duality relation

$$\inf (P)_\infty^L \ge \sup (D)_\infty^L \tag{42}$$

holds.

Proof Let $(x, u) \in \mathcal{A}_L$ and S be admissible for $(D)^L_\infty$. Then we have

$$
\begin{aligned}
J(x,u) &= \int_0^\infty f(t,x(t),u(t))\tilde{\nu}(t)dt \\
&= \sum_{i=1}^k \int_{\Omega_i} (-H(t,x(t),u(t),\partial_\xi S(t,x(t))))\tilde{\nu}(t)dt \\
&\quad + \sum_{i=1}^k \int_{\Omega_i} \left\langle \frac{\partial_\xi S(t,x(t))}{\tilde{\nu}(t)}, g(t,x(t),u(t)) \right\rangle \tilde{\nu}(t)dt \\
&= \sum_{i=1}^k \int_{\Omega_i} \left(-H(t,x(t),u(t),\partial_\xi S(t,x(t))) - \frac{\partial_t S(t,x(t))}{\tilde{\nu}(t)} \right) \tilde{\nu}(t)dt \\
&\quad + \sum_{i=1}^k \int_{\Omega_i} \left(\frac{\partial_t S(t,x(t))}{\tilde{\nu}(t)} + \left\langle \frac{\partial_\xi S(t,x(t))}{\tilde{\nu}(t)}, \dot{x}(t) \right\rangle \right) \tilde{\nu}(t)dt \\
&\geq -\sum_{i=1}^k \int_{\Omega_i} \left(\mathcal{H}(t,x(t),\partial_\xi S(t,x(t))) + \frac{\partial_t S(t,x(t))}{\tilde{\nu}(t)} \right) \tilde{\nu}(t)dt \\
&\quad + \sum_{i=1}^k \int_{\Omega_i} (\partial_t S(t,x(t)) + \langle \partial_\xi S(t,x(t)), \dot{x}(t) \rangle)dt \\
&\geq -\sum_{i=1}^k \int_{\Omega_i} \sup_{\xi \in G(t)} \left\{ \mathcal{H}(t,\xi,\partial_\xi S(t,\xi)) + \frac{\partial_t S(t,\xi)}{\tilde{\nu}(t)} \right\} \tilde{\nu}(t)dt \\
&\quad + \sum_{i=1}^k \int_{\Omega_i} (\partial_t S(t,x(t)) + \langle \partial_\xi S(t,x(t)), \dot{x}(t) \rangle)dt \\
&\geq \sum_{i=1}^k \int_{\Omega_i} \frac{d}{dt} S(t,x(t))dt \\
&\geq \sum_{i=2}^k \inf_{Q_i} \{S(\tau_i - 0, \xi_i) - S(\tau_i + 0, \xi_i)\} + \lim_{T\to\infty} S(T,x(T)) - S(0,x_0) \\
&\geq \sum_{i=2}^k \inf_{Q_i} \{S(\tau_i - 0, \xi_i) - S(\tau_i + 0, \xi_i)\} - S(0,x_0).
\end{aligned}
\tag{43}
$$

□

The next two corollaries provide sufficiency conditions for global optimality in the sense of criterion **L1** and criterion **L2**, respectively.

Corollary 1 (The generalized maximum principle for $(P)^L_\infty$, criterion **L1**) *An admissible pair (x^*, u^*) is a global minimizer of $(P)^L_\infty$ (in the sense of criterion* **L1**), *if there exists an admissible S for $(D)^L_\infty$, $S \in Y$, such that the following conditions are fulfilled for almost all $t > 0$ and, $i = 2, \dots, k$:*

$$(\mathbf{M}) \quad \mathcal{H}(t, x^*(t), \partial_\xi S(t, x^*(t))) = H(t, x^*(t), u^*(t), \partial_\xi S(t, x^*(t))), \tag{44}$$

$$(\mathbf{HJ}) \quad \frac{1}{\nu(t)}\partial_t S(t, x^*(t)) + \mathcal{H}(t, x^*(t), \partial_\xi S(t, x^*(t))) = 0, \tag{45}$$

$$(\mathbf{B_i}) \quad \inf_{Q_i}\{S(\tau_i - 0, \xi_i)) - S(\tau_i + 0, \xi_i)\} = S(\tau_i - 0, x^*(\tau_i)) - S(\tau_i + 0, x^*(\tau_i)). \tag{46}$$

Proof This follows immediately from the weak duality relation (42) and the property that all inequalities in (43) become equations, if (**M**), (**HJ**) and ($\mathbf{B_i}$) are satisfied. □

Conclusion 1 *The boundary condition*

$$(\mathbf{B}_\infty) \qquad \lim_{T\to\infty} S(T, x^*(T)) = 0 \tag{47}$$

is automatically satisfied due to Lemma 1. Admissibility of S to the dual problem means especially $y \in W^{1,n}_{q,\nu^{1-q}}(\overline{\Omega_k})$, or

$$\int_{\tau_k}^{\infty} |y(t)|^q \nu^{1-q}(t)dt < \infty \tag{48}$$

and

$$\int_{\tau_k}^{\infty} |y'(t)|^q \nu^{1-q}(t)dt < \infty. \tag{49}$$

This results in the transversality conditions

$$\lim_{T\to\infty} |y(T)|^q \nu^{1-q}(T) = 0 \tag{50}$$

and

$$\lim_{T\to\infty} |y'(T)|^q \nu^{1-q}(T) = 0. \tag{51}$$

Corollary 2 (The generalized maximum principle for $(P)^L_\infty$, criterion **L2**) *An admissible pair (x^*, u^*) is a global minimizer of $(P)^L_\infty$ (in the sense of criterion* **L2**), *if there exists a family $\{(S_T)\}_{T\geq\tau} \subset Y$, for a sufficiently large τ, such that the following conditions are fulfilled for almost all $t \in (0, T)$, $i = 1, \dots, k+1$:*

$$\begin{aligned}(\mathbf{M_T}) \quad &\mathcal{H}(t, x^*(t), \partial_\xi (S_T)(t, x^*(t))) \\ &= H(t, x^*(t), u^*(t), \partial_\xi (S_T)(t, x^*(t))),\end{aligned} \tag{52}$$

$$(\mathbf{HJ_T})\quad \frac{1}{\nu(t)}\partial_t(S_T)(t,x^*(t))+\mathcal{H}(t,x^*(t),\partial_\xi(S_T)(t,x^*(t)))=0, \tag{53}$$

$$(\mathbf{B_i})\quad \inf_{\xi_i\in Q_i}\{S(\tau_i-0,\xi_i))-S(\tau_i+0,\xi_i)\}$$
$$=S(\tau_i-0,x^*(\tau_i))-S(\tau_i+0,x^*(\tau_i)), \tag{54}$$

$$(\mathbf{B_T})\quad \inf_{\xi_T\in Q_T} S(T-0,\xi_T)=S(T-0,x^*(T)). \tag{55}$$

Proof According to criterion **L2**, we obtain the following inequalities for all $T \geq \tau \geq \tau_k$ and $S_T \in Y$:

$$\begin{aligned}
J_T(x,u) &= \int_0^T f(t,x(t),u(t))\tilde{\nu}(t)dt\\
&= \sum_{i=1}^{k-1}\int_{\Omega_i}\left(-H(t,x(t),u(t),\partial_\xi S_T(t,x(t)))-\frac{\partial_t S_T(t,x(t))}{\tilde{\nu}(t)}\right)\tilde{\nu}(t)dt\\
&\quad+\int_{\tau_k}^T\left(-H(t,x(t),u(t),\partial_\xi S_T(t,x(t)))-\frac{\partial_t S_T(t,x(t))}{\tilde{\nu}(t)}\right)\tilde{\nu}(t)dt\\
&\quad+\sum_{i=1}^{k-1}\int_{\Omega_i}\left(\frac{\partial_t S_T(t,x(t))}{\tilde{\nu}(t)}+\left\langle\frac{\partial_\xi S_T(t,x(t))}{\nu(t)},\dot{x}(t)\right\rangle\right)\tilde{\nu}(t)dt\\
&\quad+\int_{\tau_k}^T\left(\frac{\partial_t S_T(t,x(t))}{\tilde{\nu}(t)}+\left\langle\frac{\partial_\xi S_T(t,x(t))}{\tilde{\nu}(t)},\dot{x}(t)\right\rangle\right)\tilde{\nu}(t)dt\\
&\geq -\sum_{i=1}^{k-1}\int_{\Omega_i}\left(\mathcal{H}(t,x(t),\partial_\xi S_T(t,x(t)))+\frac{\partial_t S_T(t,x(t))}{\tilde{\nu}(t)}\right)\tilde{\nu}(t)dt\\
&\quad-\int_{\tau_k}^T\left(\mathcal{H}(t,x(t),\partial_\xi S_T(t,x(t)))+\frac{\partial_t S_T(t,x(t))}{\tilde{\nu}(t)}\right)\tilde{\nu}(t)dt\\
&\quad+\sum_{i=1}^{k-1}\int_{\Omega_i}(\partial_t S_T(t,x(t))+\langle\partial_\xi S_T(t,x(t)),\dot{x}(t)\rangle)dt\\
&\quad+\int_{\tau_k}^T(\partial_t S_T(t,x(t))+\langle\partial_\xi S_T(t,x(t)),\dot{x}(t)\rangle)dt\\
&\geq -\sum_{i=1}^{k}\int_{\Omega_i}\sup_{\xi\in G(t)}\left\{\left(\mathcal{H}(t,\xi,\partial_\xi S_T(t,\xi))+\frac{\partial_t S_T(t,\xi)}{\tilde{\nu}(t)}\right)\right\}\tilde{\nu}(t)dt\\
&\quad-\int_{\tau_k}^T\sup_{\xi\in G(t)}\left\{\left(\mathcal{H}(t,\xi,\partial_\xi S_T(t,\xi))+\frac{\partial_t S_T(t,\xi)}{\nu(t)}\right)\right\}\nu(t)dt
\end{aligned}$$

$$+ \sum_{i=2}^{k} \inf_{\xi_i \in Q_i} \{S_T(\tau_i - 0, \xi_i) - S_T(\tau_i + 0, \xi_i)\}$$

$$+ \inf_{\xi \in Q_T} \{S_T(T - 0, \xi_T)\} - S_T(0, x_0). \tag{56}$$

All inequalities in (56) become equations if the conditions ($\mathbf{M_T}$), ($\mathbf{HJ_T}$), ($\mathbf{B_i}$) and ($\mathbf{B_T}$) are satisfied for the pair (x^*, u^*). This means that for all $T \geq \tau$ the strong duality relation for problems with finite horizon is fulfilled, see Pickenhain and Tammer (1991), i.e.

$$J_T(x^*, u^*) = \sum_{i=2}^{k} \inf_{Q_i}\{S_T(\tau_i - 0, x(\tau_i)) - S_T(\tau_i + 0, x(\tau_i))\}$$

$$+ \inf_{\xi \in Q_T} \{S_T(T - 0, \xi_T)\} - S_T(0, x_0). \tag{57}$$

holds. Having in mind the definition of criterion **L2**, we can easily see that the pair (x^*, u^*) is an optimal solution of the problem $(P)_\infty^L$ in the sense of criterion **L2**. □

Conclusion 2 *Considering the case without state constraints*, $G(t) = \mathbb{R}^n$, *it follows from* (57) *that the transversality conditions*

$$y_T(T - 0) = 0 \quad \textit{for all } T \geq \tau. \tag{58}$$

have to be satisfied.

5 Examples

5.1 The Example of Halkin

Consider the problem, treated in Halkin (1979):

$$(P_0)_\infty^L: \quad J_L(z, u) = L\text{-}\int_0^\infty (-(1 - z(t))u(t))dt \longrightarrow \text{Min!}, \tag{59}$$

$$(z, u) \in W_2^1(0, \infty) \times L_\infty(0, \infty), \tag{60}$$

$$z'(t) = (1 - z(t))u(t) \quad \text{a.e. on } [0, \infty), \tag{61}$$

$$z(0) = 0, \tag{62}$$

$$u(t) \in U = [0, 1] \quad \text{a.e. on } [0, \infty). \tag{63}$$

By the transformation $x(t) = z(t) - 1$, this problem is equivalent to

$$(P_0)_\infty^L: \quad J_L(x, u) = L\text{-}\int_0^\infty (x(t)u(t))dt \longrightarrow \text{Min!}, \tag{64}$$

$$(x,u) \in W_2^1(0,\infty) \times L_\infty(0,\infty), \tag{65}$$

$$x'(t) = -x(t)u(t) \quad \text{a.e. on } [0,\infty), \tag{66}$$

$$x(0) = -1, \tag{67}$$

$$u(t) \in U = [0,1] \quad \text{a.e. on } [0,\infty). \tag{68}$$

The optimal solution of this problem can be found directly. Integrating the state equations with the initial condition $x(0) = -1$, we obtain $x(t) = -e^{-F(t)}$ with $F(t) = \int_0^t u(\tau)d\tau$. It follows that any control $u \in L_\infty(0,\infty)$, with $u(t) \in [0,1]$ and $F(t) \to \infty$ for $t \to \infty$ is optimal with respect to criterion **L1**. This follows from the estimate:

$$\begin{aligned} J_L(x,u) &= L\text{-}\int_0^\infty (x(t)u(t))dt \\ &= L\text{-}\int_0^\infty (-x'(t))dt = -1 + \lim_{T\to\infty} e^{-F(T)} \ge -1. \end{aligned} \tag{69}$$

In Pickenhain et al. (2008) it was shown that the objective with the Lebesgue integral is neither strongly nor weakly lower semicontinuous within the spaces $W^1_{p,\nu}(0,\infty) \times L_{p,\nu}(0,\infty)$, for an arbitrary density function ν and $1 \le p < \infty$. Let us further note that the replacement of the control set $U = [0,1]$ by $U = [\alpha, 1]$ with $0 < \alpha < 1$ leads to a problem $(P_\alpha)^L_\infty$, where the objective is constant and therefore, continuous with respect to any topology on the feasible domain.

We consider now the case $U = [\alpha, 1], 0 < \alpha < 1$. In particular, $u^*(t) = 1$ and $x^*(t) = -e^{-t}$ is an optimal solution. Since $x(t)u(t) \le 0$ the inclusion

$$\mathcal{A}_R \subseteq \mathcal{A}_L \tag{70}$$

holds and of course $(x^*, u^*) \in \mathcal{A}_R$. By definition this solution is optimal with respect to the optimality criteria **L1** and **R1**. We proof that $(x^*, u^*) \in \mathcal{A}_L$ is the only uniform strong solution of the problem $(P_\alpha)^L_\infty$ with respect to the criterion **L2**. We apply Corollary 2 and verify the conditions ($\mathbf{M_T}$), ($\mathbf{HJ_T}$) and ($\mathbf{B_T}$) for a family $\{(S_T)\}_{T\ge\tau} \subset Y$.

The Hamiltonian of the problem is

$$\mathcal{H}(t,\xi,\eta) = \begin{cases} (-\xi)(1+\eta) & (1+\eta) > 0, \\ 0 & (1+\eta) \le 0 \end{cases} \tag{71}$$

where the maximum in (71) is attained for

$$v^* = \begin{cases} 1 & (1+\eta) > 0, \\ \tilde{v} \in [\alpha, 1] & (1+\eta) = 0, \\ 0 & (1+\eta) < 0. \end{cases} \tag{72}$$

Following condition ($\mathbf{B_T}$) of Corollary 2, y_T satisfies the transversality condition

$$y_T(T) = 0. \tag{73}$$

In case of concave Hamiltonian condition $(\mathbf{HJ_T})$ is equivalent to the adjoint equation

$$y_T'(t) = -\mathcal{H}_\xi(t, x^*(t), y(t)) = 1 + y_T(t) \tag{74}$$

for $t \in [\tau, T]$ thus

$$y(t) = -1 + e^{t-T}. \tag{75}$$

Since $y(\tau) = -1 + e^{\tau - T} + 1 > 0 \; \forall \tau \geq 0$ we obtain $u^*(t) = 1$ for all $0 \leq t \leq T$.

Now we apply the duality theory in Weighted spaces to prove sufficient optimality conditions for $(P_\alpha)^L_\infty$. First we introduce an adequate state space. An admissible x for $(P_\alpha)^L_\infty$ satisfies

$$-e^{-\alpha t} \leq x(t) \leq -e^{-t}. \tag{76}$$

For the weight function ν we choose

$$\nu(t) = e^{\beta t} \tag{77}$$

with $0 < \beta < 2\alpha$ and obtain for an admissible x to $(P_\alpha)^L_\infty$:

$$\int_0^\infty |x(t)|^2 \nu(t) dt \leq \int_0^\infty e^{-2\alpha t} e^{\beta t} dt < \infty \tag{78}$$

and

$$\int_0^\infty |x'(t)|^2 \nu(t) dt \leq \int_0^\infty |x(t)|^2 \nu(t) dt < \infty \tag{79}$$

and therefore $x \in W^{1,1}_{2,\nu}(0, \infty)$.

Let now

$$S(t, \xi) = a(t) + y(t)(\xi - x^*(t)), \tag{80}$$

$$x^*(t) = -e^{-t}, \quad u^*(t) = 1, \quad t \in \Omega. \tag{81}$$

In the dual problem we choose $\kappa = 1$, $\Omega_\kappa = \Omega$, $\tilde{\nu} = 1$ and $p = 2$. The admissible set for the dual problem $(D)^L_\infty$ is then given by

$$Y = \left\{ S : X \to \mathbb{R} \;\middle|\; \begin{array}{l} S(t, \xi) = a(t) + \langle y(t), \xi - x^*(t) \rangle \\ \quad a \in W^{1,1}_1(\overline{\Omega}),\; y \in W^{1,1}_{2,\nu^{-1}}(\overline{\Omega}), \\ \partial_t S(t, \xi) + \mathcal{H}(t, \xi, \partial_\xi S(t, \xi)) \leq 0 \\ \quad \forall (t, \xi) \in X \end{array} \right\} \tag{82}$$

where

$$X := \{(t, \xi) \mid t \in \Omega, \xi \in G(t)\} \tag{83}$$

and

$$G(t) = \{\xi \in \mathbb{R} \mid \xi \leq -e^{-t}\}. \tag{84}$$

Admissibility for S means that the Hamilton-Jacobi-Inequality

$$\Lambda(t,\xi) := S_t(t,\xi) + \mathcal{H}(t,\xi,y(t)) \leq 0 \tag{85}$$

has to be satisfied for all $\xi \in G(t),\ t \in \Omega$. The condition **(HJ)** means

$$\Lambda(t, x^*(t)) = 0 \quad \forall t \in \Omega. \tag{86}$$

The Maximum in the Pontryagin-function is attained by $u^*(t) = 1$ for $1 + \eta \geq 0$ and in this case the Hamiltonian is

$$\mathcal{H}(t,\xi,\eta) = (-\xi)(1+\eta). \tag{87}$$

Equations (85) and (86) are satisfied, if $x^*(t)$ solves the parametric optimization problem

$$\Lambda(t,\xi) \longrightarrow \max! \quad \text{with respect to } \xi \in G(t). \tag{88}$$

Equation (88) is a linear optimization problem. Thus the following condition is necessary and sufficient for optimality of $x^*(t)$

$$\Lambda_\xi(t, x^*(t)) = y'(t) - y(t) - 1 \geq 0. \tag{89}$$

For any $c \in \mathbb{R}$ the function $y(t) = -1 + ce^t$ solves the equation $y'(t) - y(t) - 1 = 0$. This solution y belongs to the Sobolev space $W^1_{2,\nu^{-1}}(\overline{\Omega})$ iff $c = 0$. Finally we determine the function a by **(HJ)**,

$$\begin{aligned} \Lambda(t, x^*(t)) &= a'(t) - y(t)x^*(t) + (-\xi)(y(t)+1) \\ &= a'(t) - e^{-t} = 0 \end{aligned} \tag{90}$$

and

$$a(t) = -e^{-t}, \quad a \in W^{1,1}_1(\overline{\Omega}). \tag{91}$$

Summarized we have shown that

$$S(t,\xi) = a(t) + y(t)(\xi - x^*(t)) = -\xi - 2e^{-t} \tag{92}$$

solves the dual problem $(D_\alpha)^L_\infty$. The corresponding transversality condition is:

$$\lim_{T\to\infty} |y(T)|^2 \nu^{-1}(T) = \lim_{T\to\infty} |(-1)|^2 e^{-\beta t}(T) = 0 \tag{93}$$

and by Corollary 1 (x^*, u^*) is a global minimizer of $(P_2)^L_\infty$.

5.2 A Resource Allocation Problem

Consider the problem:

$$(P_2)_\infty^L: \quad J_L(x,u) = L\text{-}\int_0^\infty (-x(t)(1-u(t))e^{-\rho t})dt \longrightarrow \text{Min!}, \tag{94}$$

$$(x,u) \in W^{1,1}_{2,\nu}(0,\infty) \times L_\infty(0,\infty), \tag{95}$$

$$x'(t) = x(t)u(t) \quad \text{a.e. on } [0,\infty), \tag{96}$$

$$x(t) \in G(t) := \{\xi \in \mathbb{R} \mid x_0 \le \xi \le Ce^{\hat{\alpha}t}\}, \quad \hat{\alpha} < \rho, \tag{97}$$

$$x(0) = x_0, \quad C > x_0, \tag{98}$$

$$u(t) \in U = [0,1] \quad \text{a.e. on } [0,\infty). \tag{99}$$

We show that the control

$$u^*(t) = \begin{cases} 1, & t \le \tau, \\ \hat{\alpha}, & t > \tau \end{cases} \tag{100}$$

is optimal for some switching point τ. The corresponding state trajectory is

$$x^*(t) = \begin{cases} x_0 e^t, & t \le \tau, \\ x_0 e^{(1-\hat{\alpha})\tau} e^{\hat{\alpha}t}, & t > \tau. \end{cases} \tag{101}$$

This solution satisfies the state constraint (97) if $x_0 e^\tau = Ce^{\hat{\alpha}\tau}$, which means that $\tau = \frac{1}{1-\hat{\alpha}} \ln(\frac{C}{x_0}) \ge 0$.

Now we apply the duality theory in Weighted Spaces to prove sufficient optimality conditions for $(P_2)_\infty^L$. First we introduce an adequate state space. An admissible x for $(P_2)_\infty^L$ satisfies

$$x \le Ce^{\hat{\alpha}t}. \tag{102}$$

For the weight functions ν we choose

$$\nu(t) = e^{-\alpha^* t}, \tag{103}$$

with $0 < 2\hat{\alpha} < \alpha^* < \rho$ and obtain for an admissible x to $(P_2)_\infty^L$:

$$\int_0^\infty |x(t)|^2 \nu(t)dt \le C^2 \int_0^\infty e^{(2\hat{\alpha}-\alpha^*)t} dt < \infty \tag{104}$$

and

$$\int_0^\infty |x'(t)|^2 \nu(t)dt \le \int_0^\infty |x(t)|^2 \nu(t)dt < \infty. \tag{105}$$

Let now again

$$S(t,\xi)=a(t)+y(t)(\xi-x^*(t)). \tag{106}$$

In the dual problem we choose $\kappa=2$, $\tilde{\nu}=e^{-\rho t}$ and $p=2$. The admissible set for the dual problem $(D_2)^L_\infty$ is then given by

$$Y=\left\{ S:X\to\mathbb{R} \;\middle|\; \begin{array}{l} S(t,\xi)=a(t)+\langle y(t),\xi-x^*(t)\rangle \\ \quad a\in W_1^{1,1}(\overline{\Omega_i}),\, y\in W^{1,1}_{2,\nu^{-1}}(\overline{\Omega_i}), \\ \dfrac{1}{\tilde{\nu}(t)}\partial_t S(t,\xi)+\mathcal{H}(t,\xi,\partial_\xi S(t,\xi))\le 0 \\ \quad \forall (t,\xi)\in X_i,\ i=1,2 \end{array} \right\} \tag{107}$$

with $\Omega_1=[0,\tau],\ \Omega_2=[\tau,\infty],\ X_i=\{(t,\xi)|t\in\Omega_i,\xi\in G(t)\}$. The Hamiltonian for the problem $(P_2)^L_\infty$ is defined by

$$\mathcal{H}(t,\xi,\eta)=\max_{v\in U}H(t,\xi,v,\eta)=\begin{cases} \eta\xi e^{\rho t} & \xi(\eta e^{\rho t}-1)>0, \\ \xi & \xi(\eta e^{\rho t}-1)=0, \\ \xi & \xi(\eta e^{\rho t}-1)<0. \end{cases} \tag{108}$$

We consider the parametric optimization problem of maximizing the defect within the Hamilton-Jacobi inequality:

$$\begin{aligned} \Lambda(t,\xi):=\frac{1}{\tilde{\nu}(t)}\partial_t S(t,\xi)+\mathcal{H}(t,\xi,\partial_\xi S(t,\xi))&\to \max! \\ \text{with respect to } \xi\in G(t).& \end{aligned} \tag{109}$$

The defect $\Lambda(t,\xi)$ from the last formula can be rewritten as

$$\begin{aligned} \Lambda(t,\xi)=&\frac{1}{\tilde{\nu}(t)}\big(a'(t)+\langle y'(t),\xi-x^*(t)\rangle\big)-\langle y(t)x^{*\prime}(t)\rangle) \\ &+\mathcal{H}(t,\xi,\partial_\xi S(t,\xi)). \end{aligned} \tag{110}$$

The component $a(t)$ of the dual variable $S(t,\xi)$ will be chosen from the Hamilton-Jacobi equation $\Lambda(t,x^*(t))=0$ and we obtain

$$a'(t)=y(t)^T x^{*\prime}(t)-\mathcal{H}(t,x^*(t),y(t))\tilde{\nu}(t). \tag{111}$$

With the Lagrangian

$$L(t,\xi)=-\Lambda(t,\xi)+\lambda(t)(x_0-\xi)+\mu(t)(\xi-Ce^{\alpha t}) \tag{112}$$

the Karush-Kuhn-Tucker conditions for the problem (109) are

$$\partial_\xi L(t, x^*(t)) = -e^{\rho t} y'(t) - \partial_\xi \mathcal{H}(t, x^*(t), y(t), 1) - \lambda(t) + \mu(t) = 0, \tag{113}$$

$$\lambda(t)(x_0 - x^*(t)) = 0, \quad \lambda(t) \geq 0, \tag{114}$$

$$\mu(t)(x^*(t) - Ce^{\alpha t}) = 0, \quad \mu(t) \geq 0. \tag{115}$$

Since $u^*(t) = 1$ for all $t \in (0, \tau)$, the strict inequalities $x_0 < x^*(t) < Ce^{\hat{\alpha} t}$ hold for all t in this interval, which means together with complementary conditions (114), (115) that $\lambda(t) = \mu(t) = 0$ on $(0, \tau)$. In this case the condition (113) reduces to

$$y'(t) = -y(t), \quad t \in (0, \tau) \tag{116}$$

and, consequently, $y(t) = De^{-t}$, $D \in \mathbb{R}$. From the condition $De^{-t} > e^{-\rho t}$ $\forall t \in (0, \tau)$ we choose the constant $D := e^{(1-\rho)\tau}$.

The fact that $u^*(t) = \hat{\alpha} \in (0, 1)$ for all $t \in (\tau, \infty)$ implies $\mathcal{H}(t, \xi, \eta) = \xi$ and $\lambda(t) = 0, t \in (\tau, \infty)$. For all $t \in (\tau, \infty)$ we rewrite the condition (113) in the form

$$y'(t) = -e^{-\rho t} + \mu(t)e^{-\rho t}. \tag{117}$$

Together with the condition $y(t) = e^{-\rho t}$ for all $t \in (\tau, \infty)$ we obtain the equation with respect to the multiplier $\mu(t)$:

$$-\rho e^{-\rho t} = -e^{-\rho t} + \mu(t)e^{-\rho t}, \tag{118}$$

whose solution $\mu(t) = 1 - \rho$ is strictly positive.

The continuity of the adjoint function $y(t)$ at the point τ indicates that the boundary condition (46) of Corollary 1 is satisfied by the triple (x^*, u^*, S).

Finally, we prove that the adjoint function $y(t)$ belongs to the space $W^1_{2,e^{\alpha^* t}}(\Omega_i)$:

$$\begin{aligned} \|y\|_{L_{2,e^{\alpha^* t}}}(\mathbb{R}^+) &= \int_0^\infty |y(t)|^2 e^{\alpha^* t} dt \\ &= e^{2(1-\rho)\tau} \int_0^\tau e^{-(2-\alpha^*)t} dt + \int_\tau^\infty e^{-(2\rho-\alpha^*)t} dt < \infty. \end{aligned} \tag{119}$$

We analogously show that $\|y'\|_{L_{2,e^{\alpha^* t}}}(\mathbb{R}^+) < \infty$ holds. It remains to verify, whether $a(t)$ from (111) belongs to the space $W^{1,1}_1(\mathbb{R}^+)$. Indeed, the function $a(t)$, given by

$$a(t) = \begin{cases} x_0 \frac{\hat{\alpha}-1}{\hat{\alpha}-\rho} e^{(1-\rho)\tau} & t < \tau, \\ x_0 \frac{\hat{\alpha}-1}{\hat{\alpha}-\rho} e^{(1-\hat{\alpha})\tau} e^{(\hat{\alpha}-\rho)t} & t > \tau \end{cases} \tag{120}$$

is obviously an element of the $W^1_1(\mathbb{R}^+)$.

Thus, all the conditions of Corollary 1 are fulfilled and we conclude that the pair (x^*, u^*) is the global minimizer of the problem $(P_2)^L_\infty$.

6 Conclusions

In the present paper we considered a two economical models, formulated on an unbounded time interval. Using the weight functions approach we have succeeded in proving sufficient optimality conditions using a duality concept of Klötzler. The formulated sufficiency conditions show, how natural transversality conditions look like.

However, an equivalent of Pontryagin's maximum principle for the problem of this type with adjoint variables belonging to dual spaces of the state spaces, including correct transversality conditions is still missing and represents a future challenge.

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Sequential Precision of Predictions in Models of Economic Growth

Andrey A. Krasovskii and Alexander M. Tarasyev

Abstract The research deals with the model of economic growth based on the real time series. The methodology for analysis of a country's macroeconomic parameters is proposed. A distinguishing feature of the approach is that real data is analyzed not by direct statistical approximations but through formalization of the process in terms of optimal control theory. The econometric analysis is used only at the stage of calibration of initial parameters of the model. This feature helps to analyze the dynamism in growth of economic factors which drive the economic growth. The study is focused on the gross domestic product (GDP) of a country. There are three production factors in the model: capital, labor and useful work. Several production functions (Cobb-Douglas, modifications of LINEX) are implemented in the model to express the relationship between factors of production and the quantity of output produced. The problem of investments optimization is solved using the version of the Pontryagin maximum principle, elements of the qualitative theory of differential equations and methods of differential games. Numerical algorithm is proposed for constructing synthetic trajectories of economic growth. Numerical experiments are fulfilled via elaborated software. For verification of the proposed approach several model modifications and case studies are presented. By means of comparison of obtained model trajectories with real data one can judge on the forecasting capacity of the model. As time goes by real data is collected and can be compared to forecast. At some stage it is necessary to make the forecast more precise. Using the data updates one restarts the model from the very beginning. Based on the model restart the new forecast is obtained which makes the previous one more accurate. Extensive simulations are done which realized the suggested methodology. They show that based on several data updates a series of forecasting trajectories demonstrate sequential preci-

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A.A. Krasovskii (✉)
Institute of Mathematics and Mechanics, Ural Branch Russian Academy of Sciences, S. Kovalevskoi Str. 16, 620219 Ekaterinburg, Russia
e-mail: Andrey.Krasovskiy@oeaw.ac.at

A.A. Krasovskii
Vienna Institute of Demography, Austrian Academy of Sciences, Wohllebengasse 12-14, 1040 Vienna, Austria

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sion of predictions property. Numerical results are based on real data for economies of US, UK, and Japan.

1 Introduction

The research is focused on the analysis of trajectories generated by nonlinear stabilizers for a country economic growth. A methodology for analysis of the macroeconomic data time series is proposed. The historical data is analyzed not by direct statistical approximations but by considering economy's development as a dynamic process. Thus the research deals with models of economic growth risen in papers by Arrow (1985), Intriligator (1971), Ramsey (1928), Shell (1969), Solow (1970). The distinguishing feature of the model is the exogenous growth of useful work which shows the impact of energy resources on the economic growth. The factor of useful work is implemented into the model by means of the (linear-exponential) LINEX production function (see Ayres and Warr 2005; Ayres and Martinás 2005).

Based on the mathematical formalization of the model the optimal control problem is posed. The solution of the investments optimization problem is obtained in the framework of the maximum principle of Pontryagin et al. (1962). It is developed using necessary and sufficient optimality conditions for problems on infinite horizon (see Aseev and Kryazhimskiy 2007; Krasovskii and Tarasyev 2007, 2008; Tarasyev and Watanabe 2001). In the framework of the optimal stabilization theory (see Krasovskii 1963; Malkin 1966; Letov 1961; Krasovskii and Krasovskii 1995) a system of nonlinear regulators is elaborated for construction of synthetic trajectories of economic growth. The algorithm is realized in the elaborated software.

Comparison of synthetic trajectories with the real data is implemented for macroeconomic indicators of the US economy. The results of model simulation are given.

A special section of the paper is devoted to a methodological scheme for verification of the proposed approach. This scheme is based on application of the model to time series of different sizes. An algorithm of sequential precision of predictions is proposed for models of economic growth. The sequential precision of predictions algorithm provides adjustment of trajectories forecasting optimal growth trends according to data updates.

2 Methodological Scheme of the Research

The methodological scheme starts from the block of data. Initially there is data on the country's economy which is presented by time series for GDP and three production factors: capital, labor and useful work. Using methods of econometric analysis, data is calibrated for identification of LINEX production function and the model parameters: rate of capital depreciation, labor growth rate, discount parameter and exogenous growth of useful work. On the next step the model of economic growth is analyzed basing on calibrated parameters. The control problem of dynamic optimization of investments is posed for a nonlinear dynamics of economic growth. The solution of control problem is constructed using nonlinear stabilizers. Finally, one comes back to real data to make a comparison of the synthetic model trajectories with real data trends.

Let us note that in this research we use the model of a representative agent to explain the development patterns of real economies. There are two possibilities to motivate and explain the use of such models:

(a) The principle of perfect markets. If all households and firms in the economy are price takers, then they as a group behave as if they were a single agent.
(b) The principle of a representative government. If governments are benevolent and have enough instruments (e.g. taxation) to determine the allocation of resources in the economy, then they will optimize the representative household's welfare given the resources of the economy.

If markets are more or less perfect (cf. (a) holds), or if the governments correct the effects of market imperfections by public policy (cf. (b) holds), then one may assume that a single economy behaves (roughly) as if there were a fictitious representative agent that would control all the resources in that economy.

3 Model of Optimal Economic Growth

A model is focused on the analysis of gross domestic product (GDP) of a country which is defined as the market value of all final goods and services produced within a country in a year. Two production factors are considered in a model. If symbols $K(t)$, $L(t)$ and $U(t)$ denote stocks of capital, labor and useful work then the output $Y(t)$ at time t is given by equation

$$Y(t) = F[K(t), L(t), U(t)]. \tag{1}$$

Here the symbol F denotes production function. Using the fact that the LINEX production function is homogenous of degree one it is possible to fix relation between quantity of output per worker and quantities of capital per worker. Introducing per worker notations $y = Y/L$ for GDP, $k = K/L$ for capital, and $u = U/L$ for useful work one can consider a per worker production function

$$y(t) = f(k(t), u(t)) = F\left[\frac{K(t)}{L(t)}, \frac{U(t)}{L(t)}, 1\right]. \tag{2}$$

Let symbols $C(t) \geq 0$ and $I(t) \geq 0$ denote rates at time t of consumption and investment, respectively, and the symbol $s(t)$, $0 \leq s(t) \leq 1$, denotes the fraction of output which is saved and invested. Then the national income is defined by the formula

$$Y(t) = C(t) + I(t) = (1 - s(t))Y(t) + s(t)Y(t). \tag{3}$$

It is assumed that the model characterizes growth in an aggregative closed economy.

3.1 Dynamics of Capital and Labor

The capital stock is accumulated according to equation

$$\dot{K}(t) = s(t)Y(t) - \mu K(t). \tag{4}$$

Here parameter $\mu > 0$ is the rate of capital depreciation.

It is assumed that the labor input grows exponentially

$$\frac{\dot{L}(t)}{L(t)} = n, \tag{5}$$

with a constant growth rate $n > 0$. Then dynamics of capital per worker is described by equation

$$\dot{k}(t) = s(t)y(t) - \lambda k(t), \tag{6}$$

where $\lambda = \mu + n$ is capital decay, and n is capital dilution.

Let us assume that function $(k, u) \mapsto f(k, u)$ has the following properties

$$\frac{\partial f}{\partial k}(k, u) > 0 \quad \text{and} \quad \frac{\partial f}{\partial u}(k, u) > 0 \quad \text{for } k \in (0, +\infty). \tag{7}$$

Here $\partial f / \partial k$ is the marginal productivity of capital per worker. It is assumed that there exists a convex domain $K^0 \subset (0, +\infty)$ in which the Hessian matrix of function $f(k, u)$

$$G(f(k, u)) = \begin{pmatrix} \frac{\partial^2 f(k,u)}{\partial k^2} & \frac{\partial^2 f(k,u)}{\partial k \partial u} \\ \frac{\partial^2 f(k,u)}{\partial k \partial u} & \frac{\partial^2 f(k,u)}{\partial u^2} \end{pmatrix} \tag{8}$$

is negatively definite.

It is also assumed that function $k \mapsto f(k, u)$ satisfies Inada's limit conditions (see Intriligator 1971).

3.2 Dynamics of Useful Work per Worker

We assume that useful work in the model is presented by an exogenous dynamics. Useful work is a factor showing an impact of available energy (exergy) resources on economy. This factor also takes into account the efficiency of energy use which is subject to technological change in economic growth.

Based on the analysis of data trends for the growth of useful work per worker one can present this process by a logistic curve. It is reasonable to assume that in per worker quantities energy resources have a saturation level due to natural restrictions on their availability.

Let us note that useful work is a aggregated factor representing an impact of available energy on economic growth. One can similarly model economic growth with a particular exhausting energy resource, i.e. gas, oil, metals, etc.

The useful work in the model is subject to the dynamics

$$\dot{u}(t) = \upsilon u(t)\left(1 - \frac{u(t)}{\rho}\right). \tag{9}$$

Differential equation (9) is the Verhulst equation. Parameter υ defines constant growth rate, and symbol ρ indicates saturation level of growth. The solution of (9) is given by the logistic curve

$$u(t) = \frac{\rho u^0 e^{\upsilon t}}{\rho + u^0(e^{\upsilon t} - 1)}, \tag{10}$$

where u^0 is the initial value of useful work per worker.

4 Optimal Control Design of the Model

Let us present the utility function of the model by formula

$$J = \int_0^{+\infty} \big[\ln f(k(t), u(t)) + \ln(1 - s(t))\big] e^{-\delta t} dt, \tag{11}$$

which describes the integral of logarithmic consumption index discounted on infinite horizon. Here parameter $\delta > 0$ stands for a constant discount coefficient.

Let us note that in the utility theory the logarithmic function describes the relative increment (of consumption in the case) in unit time. Under uncertainty, the logarithmic function defines constant *Arrow-Pratt measure of relative risk-aversion* (see Arrow 1971).

The problem is to maximize the utility function (12) by controlling the investment variable $s(t)$ in the dynamic process of economic growth starting from initial position of factors k and u.

4.1 Optimal Control Problem

We deal with the following optimal control problem

$$J = \int_0^{+\infty} \big[\ln f(k(t), u(t)) + \ln(1 - s(t))\big] e^{-\delta t} dt \underset{(k(\cdot), u(\cdot), s(\cdot))}{\longrightarrow} \max \tag{12}$$

under conditions

$$\begin{aligned} &\dot{k}(t) = s(t) f(k(t), u(t)) - \lambda k(t), \qquad \dot{u}(t) = \upsilon u(t)\left(1 - \frac{u(t)}{\rho}\right), \\ &k(0) = k^0, \qquad u(0) = u^0, \qquad s \in [0, a], \quad a < 1. \end{aligned} \tag{13}$$

Here parameters δ, $\lambda = n + \mu$, k^0, u^0 are given positive numbers and $s(t)$ is control variable measurable in time t. Parameter $0 < a < 1$ is a positive number which separates the right bound of control parameter from unit.

Let us apply Pontryagin's maximum principle to problem (12)–(13). The Hamiltonian of the problem is given by expression

$$\tilde{H}(t, s, k, u, \tilde{\psi}_1, \tilde{\psi}_2) = \big[\ln f(k, u) + \ln(1 - s)\big]e^{-\delta t} \\ + \tilde{\psi}_1(sf(k, u) - \lambda k) + \tilde{\psi}_2 \upsilon u\left(1 - \frac{u}{\rho}\right), \tag{14}$$

where adjoint variables ψ_1 and ψ_2 are representing shadow prices for capital and useful work respectively. To exclude time dependent exponential term from expression (14) let us introduce new variables:

$$\psi_1 = \tilde{\psi}_1 e^{\delta t}, \qquad \psi_2 = \tilde{\psi}_2 e^{\delta t}, \qquad H(s, k, \psi) = e^{\delta t}\tilde{H}(s, k, t, \psi), \tag{15}$$

and consider the Hamiltonian in the following form

$$H(s, k, u, \psi_1, \psi_2) = \ln f(k, u) + \ln(1 - s) + \psi_1(sf(k, u) - \lambda k) \\ + \psi_2 \upsilon u\left(1 - \frac{u}{\rho}\right). \tag{16}$$

Let us consider necessary condition of maximum of the Hamiltonian

$$\frac{\partial H}{\partial s} = -\frac{1}{1 - s} + \psi_1 f(k, u) = 0. \tag{17}$$

One can derive a structure of optimal control

$$s^0 = 1 - \frac{1}{\psi_1 f(k, u)}. \tag{18}$$

For shadow prices one can compose the dynamics of adjoint equations

$$\begin{aligned} \dot{\psi}_1 &= \delta\psi_1 - \frac{\partial H}{\partial k}, \\ \dot{\psi}_2 &= \delta\psi_2 - \frac{\partial H}{\partial u}, \end{aligned} \tag{19}$$

which balance the increment in flow and the change in price.

The necessary optimality conditions of the maximum principle are expressed in the Hamiltonian system of equations

$$\begin{cases} \dot{k} = f(k,u) - \dfrac{1}{\psi_1} - \lambda k, \\ \dot{\psi}_1 = \psi_1 \left(\delta + \lambda - \dfrac{\partial f(k,u)}{\partial k} \right), \\ \dot{u} = \upsilon u \left(1 - \dfrac{u}{\rho} \right), \\ \dot{\psi}_2 = \psi_2 \left(\delta - \upsilon + 2\upsilon \dfrac{u}{\rho} \right) - \psi_1 \dfrac{\partial f(k,u)}{\partial u}. \end{cases} \tag{20}$$

To resolve peculiarities of the system we introduce variables

$$z_1 = \psi_1 k, \qquad z_2 = \psi_2 u, \tag{21}$$

which can be interpreted as costs of capital and useful work.

One can present the Hamiltonian system in variables (k, u, z_1, z_2)

$$\begin{cases} \dot{k} = f(k,u) - \dfrac{k}{z_1} - \lambda k, \\ \dot{z}_1 = z_1 \left(\delta - \dfrac{\partial f(k,u)}{\partial k} + \dfrac{f(k,u)}{k} \right) - 1, \\ \dot{u} = \upsilon u \left(1 - \dfrac{u}{\rho} \right), \\ \dot{z}_2 = z_2 \left(\delta + \upsilon \dfrac{u}{\rho} \right) - u \dfrac{z_1}{k} \dfrac{\partial f(k,u)}{\partial u}. \end{cases} \tag{22}$$

Due to the fact that saturation level of exogenous useful work growth (9) is given it is possible to calculate steady state of the system (22).

Based on properties of function $f(k,u)$ one can show that the steady state is unique. Let us estimate coordinates of the steady state as follows:

$$\begin{cases} u^* = \rho, \\ \dfrac{\partial f(k^*,u^*)}{\partial k} = \delta + \lambda, \\ \dfrac{1}{z_1^*} = \dfrac{f(k^*,u^*)}{k^*} - \lambda, \\ z_2^* = \dfrac{z_1^* \rho}{k^*(\delta + \upsilon)} \dfrac{\partial f(k^*,u^*)}{\partial u}. \end{cases} \tag{23}$$

4.2 Saddle Character of the Steady State

Let us linearize the Hamiltonian system in the neighborhood of the steady state (see Hartman 1964). We denote the right-hand sides (22) of equations by functions

$$\begin{cases} F_1(k,u,z_1,z_2) = f(k,u) - \dfrac{k}{z_1} - \lambda k, \\ F_2(k,u,z_1,z_2) = z_1\left(\delta - \dfrac{\partial f(k,u)}{\partial k} + \dfrac{f(k,u)}{k}\right) - 1, \\ F_3(k,u,z_1,z_2) = \upsilon u\left(1 - \dfrac{u}{\rho}\right), \\ F_4(k,u,z_1,z_2) = z_2\left(\delta + \upsilon\dfrac{u}{\rho}\right) - u\dfrac{z_1}{k}\dfrac{\partial f(k,u)}{\partial u}. \end{cases} \tag{24}$$

The linearized system can be presented in the following matrix form

$$\begin{pmatrix} \dot{k} \\ \dot{z}_1 \\ \dot{u} \\ \dot{z}_2 \end{pmatrix} = A\begin{pmatrix} k - k^* \\ z_1 - z_1^* \\ u - u^* \\ z_2 - z_2^* \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \tag{25}$$

where elements a_{ij}, $i = 1,\dots,4$, $j = 1,\dots,4$, are calculated as follows

$$a_{i1} = \frac{\partial F_i(k^*,u^*,z_1^*,z_2^*)}{\partial k}, \qquad a_{i2} = \frac{\partial F_i(k^*,u^*,z_1^*,z_2^*)}{\partial z_1^*},$$

$$a_{i3} = \frac{\partial F_i(k^*,u^*,z_1^*,z_2^*)}{\partial u}, \qquad a_{i4} = \frac{\partial F_i(k^*,u^*,z_1^*,z_2^*)}{\partial z_2^*}.$$

Lemma 1 *The eigenvalues of the matrix A are real. Two of them are positive and two are negative. Positive eigenvalues are larger than the discount parameter δ.*

Proof Let us compile characteristic equation for the eigenvalues of matrix A (25).

$$(a_{44} - \chi)(a_{33} - \chi)(\chi^2 - (a_{11} + a_{22})\chi + (a_{11}a_{22} - a_{12}a_{21})) = 0. \tag{26}$$

One can obtain the following relations for eigenvalues

$$\chi_1 = a_{44} = \frac{\partial F_4(k^*,u^*,z_1^*,z_2^*)}{\partial z_2} = \delta + \upsilon > \delta > 0, \tag{27}$$

$$\chi_2 = a_{33} = \frac{\partial F_3(k^*,u^*,z_1^*,z_2^*)}{\partial u} = -\upsilon < 0, \tag{28}$$

$$\chi_3 = \frac{\delta - \sqrt{(\delta)^2 - 4d}}{2} < 0, \tag{29}$$

$$\chi_4 = \frac{\delta + \sqrt{(\delta)^2 - 4d}}{2} > \delta > 0, \tag{30}$$

where

$$d = a_{11}a_{22} - a_{12}a_{21} = \frac{k^*}{z_1^*} f''(k^*, u^*, z_1^*, z_2^*) < 0. \tag{31}$$

□

Let us note that results of Lemma 1 show, roughly speaking, that velocities of trajectories which do not converge to the steady state are greater than the discount factor δ, and, thus, explain why the transversality condition is not fulfilled for them.

5 Nonlinear Stabilizers

In this section we consider several versions of stabilizers for the nonlinear dynamics (13) whose constructions are based on elements of the linearized Hamiltonian system at the steady state.

5.1 Stabilizer of a Steady State

To construct a regulator of the steady state let us consider the value of optimal control (18) at the steady state

$$s^0(k^*, u^*) = \lambda \frac{k^*}{f(k^*, u^*)}. \tag{32}$$

We substitute the value of control (32) into dynamics of a system (13)

$$\begin{aligned} \dot{k}(t) &= \lambda \frac{k^*}{f(k^*, u^*)} f(k(t), u(t)) - \lambda k(t), \\ \dot{u}(t) &= \upsilon u(t)\left(1 - \frac{u(t)}{\rho}\right). \end{aligned} \tag{33}$$

Lemma 2 *Position* (k^*, u^*) *is an equilibrium point of the dynamic system* (33).

Proof Substituting $k(t) = k^*, u(t) = u^*$ to the right-hand side of the system (33) one can see that it vanishes. □

Lemma 3 *Regulator* (32) *stabilizes the system at the steady state.*

Proof Let us compile the Jacobi matrix for the dynamic system (33)

$$J_1 = \begin{pmatrix} \lambda \frac{k^*}{f(k^*,u^*)} \frac{\partial f(k^*,u^*)}{\partial k} - \lambda & \lambda \frac{k^*}{f(k^*,u^*)} \frac{\partial f(k^*,u^*)}{\partial u} \\ 0 & \upsilon - 2\upsilon \frac{u^*}{\rho} \end{pmatrix}.$$

It is clear that eigenvalues of the Jacobi matrix are equal to the diagonal elements. Due to properties of production function (7)–(8) one can obtain the following estimates for the eigenvalues

$$\begin{aligned} \xi_1 &= \lambda \frac{k^*}{f(k^*, u^*)} \frac{\partial f(k^*, u^*)}{\partial k} - \lambda < 0, \\ \xi_2 &= -\upsilon < 0. \end{aligned} \tag{34}$$

□

5.2 Stabilizers of the Hamiltonian System

Let us introduce the following notations for eigenvectors corresponding to negative eigenvalues

$$h^1 = \begin{pmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \\ h_4^1 \end{pmatrix}, \qquad h^2 = \begin{pmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \\ h_4^2 \end{pmatrix}. \tag{35}$$

We construct the plane which contains these vectors. This plane is the intersection of two hyperplanes belonging to the orthogonal complement. It means that this plane is defined by the system of two linear equations

$$\begin{cases} a_1^1(k - k^*) + a_2^1(u - u^*) + a_3^1(z_1 - z_1^*) + a_4^1(z_2 - z_2^*) = 0, \\ a_1^2(k - k^*) + a_2^2(u - u^*) + a_3^2(z_1 - z_1^*) + a_4^2(z_2 - z_2^*) = 0. \end{cases}$$

Where $a_i^1, a_i^2, i = 1, \ldots, 4$, are coordinates of the basis vectors of the orthogonal complement.

Let us find these vectors. Each vector of a^1, a^2, should be orthogonal to both eigenvectors h^1 and h^2 and, thus, satisfy the system of equations:

$$\begin{cases} a_1^j h_1^1 + a_2^j h_2^1 + a_3^j h_3^1 + a_4^j h_4^1 = 0, \\ a_1^j h_1^2 + a_2^j h_2^2 + a_3^j h_3^2 + a_4^j h_4^2 = 0, \end{cases} \tag{36}$$

where $j = 1, 2$.

Without restriction of generality one can assume that determinant of two first columns differs from zero

$$\Delta = \begin{vmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{vmatrix} | = h_1^1 h_2^2 - h_1^2 h_2^1 \neq 0. \tag{37}$$

We choose the following values for independent variables $a_3^1 = -1, a_4^1 = 0$. Then values of dependent variables a_1^1 and a_2^1 are determined by the following equations

$$\begin{cases} a_1^1 h_1^1 + a_2^1 h_2^1 = h_3^1, \\ a_1^1 h_1^2 + a_2^1 h_2^2 = h_3^2. \end{cases} \tag{38}$$

These equations can be resolved according to Cramer's rule

$$a_1^1 = \frac{1}{\Delta}\begin{vmatrix} h_1^1 & h_3^1 \\ h_1^2 & h_3^2 \end{vmatrix} = \frac{1}{\Delta}(h_1^1 h_3^2 - h_1^2 h_3^1), \tag{39}$$

$$a_2^1 = \frac{1}{\Delta}\begin{vmatrix} h_3^1 & h_2^1 \\ h_3^2 & h_2^2 \end{vmatrix} = \frac{1}{\Delta}(h_3^1 h_2^2 - h_3^2 h_2^1). \tag{40}$$

Analogously, one can select the following values for independent variables $a_3^2 = 0$, $a_4^2 = -1$. Then values of dependent variables a_1^2 and a_2^2 are determined by the following equations

$$\begin{cases} a_1^2 h_1^1 + a_2^2 h_2^1 = h_4^1, \\ a_1^2 h_1^2 + a_2^2 h_2^2 = h_4^2. \end{cases} \tag{41}$$

These equations can be resolved according to Cramer's rule

$$a_1^2 = \frac{1}{\Delta}\begin{vmatrix} h_1^1 & h_4^1 \\ h_1^2 & h_4^2 \end{vmatrix} = \frac{1}{\Delta}(h_1^1 h_4^2 - h_1^2 h_4^1), \tag{42}$$

$$a_2^2 = \frac{1}{\Delta}\begin{vmatrix} h_4^1 & h_2^1 \\ h_4^2 & h_2^2 \end{vmatrix} = \frac{1}{\Delta}(h_4^1 h_2^2 - h_4^2 h_2^1). \tag{43}$$

Finally, we obtain a basis of the orthogonal complement which includes the following vectors

$$a^1 = \begin{pmatrix} a_1^1 \\ a_2^1 \\ -1 \\ 0 \end{pmatrix}, \qquad a^2 = \begin{pmatrix} a_1^2 \\ a_2^2 \\ 0 \\ -1 \end{pmatrix}. \tag{44}$$

The plane generated by eigenvectors h^1, h^2 is defined by the system of linear equations

$$\begin{cases} a_1^1(k - k^*) + a_2^1(u - u^*) - (z_1 - z_1^*) = 0, \\ a_1^2(k - k^*) + a_2^2(u - u^*) - (z_2 - z_2^*) = 0. \end{cases}$$

Resolving this system with respect to dependent variables $z_1 = z_1(k, u)$ and $z_2 = z_2(k, u)$ through independent variables k, u, we obtain the following relations for feedback constructions

$$\begin{cases} z_1 = z_1^* + a_1^1(k - k^*) + a_2^1(u - u^*), \\ z_2 = z_2^* + a_1^2(k - k^*) + a_2^2(u - u^*). \end{cases} \tag{45}$$

One can substitute the relation for variable z_1 to the expression of optimal control (18) and obtain the following feedback

$$\begin{aligned} s = s(k,u) &= 1 - \frac{k}{z_1 f(k,u)} \\ &= 1 - \frac{k}{(z_1^* + a_1^1(k-k^*) + a_2^1(u-u^*))f(k,u)}. \end{aligned} \tag{46}$$

Basing on this feedback one can derive the feedback dynamics for the system

$$\begin{aligned} \dot{k} &= \left(f(k,u) - \frac{k}{(z_1^* + a_1^1(k-k^*) + a_2^1(u-u^*))} \right) - \lambda k, \\ \dot{u}(t) &= \upsilon u(t)\left(1 - \frac{u(t)}{\rho}\right). \end{aligned} \tag{47}$$

One can prove for dynamics (47) the stabilizing property. Let us linearize the right-hand side of dynamics (47) in a neighborhood of the steady state (k^*, u^*).

$$J_2 = \begin{pmatrix} \delta - \frac{1}{z_1^*}(1 + a_1^1 \frac{k^*}{z^*}) & \frac{\partial f(k^*,u^*)}{\partial u} + \frac{k a_2^1}{(z_1^*)^2} \\ 0 & \upsilon - 2\upsilon \frac{u^*}{\rho} \end{pmatrix}. \tag{48}$$

It is clear that eigenvalues of the Jacobi matrix are equal to the diagonal elements

$$\begin{aligned} \xi_1 &= \delta - \frac{1}{z_1^*}\left(1 + a_1^1 \frac{k^*}{z^*}\right), \\ \xi_2 &= -\upsilon < 0. \end{aligned} \tag{49}$$

One can show that eigenvalue ξ_1 is strictly negative.

6 Simulation of the Model

To construct the trajectories of growth generated by the nonlinear regulators a numerical experiment was fulfilled based on the data on macroeconomic parameters of the US economy.

6.1 Econometric Analysis

Parameters of the exogenous growth of useful work (9) were calibrated using econometric analysis of real time series for 100 years (1900–2000). Their values are identified on the following levels: $\upsilon = 0.0402$, $\rho = 13.346$.

The following LINEX production function was implemented in the numerical experiment

$$f(k,u) = u \exp\left(-0.166 \frac{u}{k}\right). \tag{50}$$

The model was simulated for the following values of parameters: $\delta = 0.22$, $\lambda = 0.22$. The initial conditions for phase variables were chosen on the level of 1950: $(k^0, u^0) = (2.087, 5.496)$.

The steady state (23) of the Hamiltonian system was calculated

$$\begin{pmatrix} k^* \\ u^* \\ z_1^* \\ z_2^* \end{pmatrix} = \begin{pmatrix} 6.997 \\ 13.346 \\ 0.855 \\ 3.12 \end{pmatrix}. \tag{51}$$

The Hamiltonian system was linearized in the neighborhood of the steady state and the following eigenvalues were calculated

$$(\chi_1, \chi_2, \chi_3, \chi_4) = (0.26, -0.04, -0.8273, 1.047). \tag{52}$$

The coordinates of eigenvectors (35) corresponding to negative eigenvalues are given by numbers

$$h^1 = \begin{pmatrix} 0.999 \\ 0 \\ 0.013 \\ -0.012 \end{pmatrix}, \qquad h^2 = \begin{pmatrix} -0.484 \\ -0.874 \\ 0 \\ 0.028 \end{pmatrix}.$$

Using these eigenvectors one can obtain the values of vectors a^1 and a^2 (44)

$$a^1 = \begin{pmatrix} -0.006 \\ 0.013 \\ -1 \\ 0 \end{pmatrix}, \qquad a^2 = \begin{pmatrix} -0.025 \\ -0.012 \\ 0 \\ -1 \end{pmatrix}.$$

Based on these values one can construct the stabilizer (47) of the Hamiltonian system.

6.2 Comparison with Real Data

Basing on the elaborated software extensive experiments have been implemented for the model with the data on macroeconomic indicators of the US economy. On Figs. 1–3 the results of these experiments are presented. Synthetic model trajectories generated by the nonlinear stabilizers are shown by solid lines and time series of real data are presented by dashed lines. The experiments demonstrate on Figs. 1, 3 that trajectories generated by nonlinear stabilizers provide greater levels of capital and GDP in comparison with real data. Graphs show that the growth trends for capital and GDP per worker have saturation levels which can be reached in the nearest

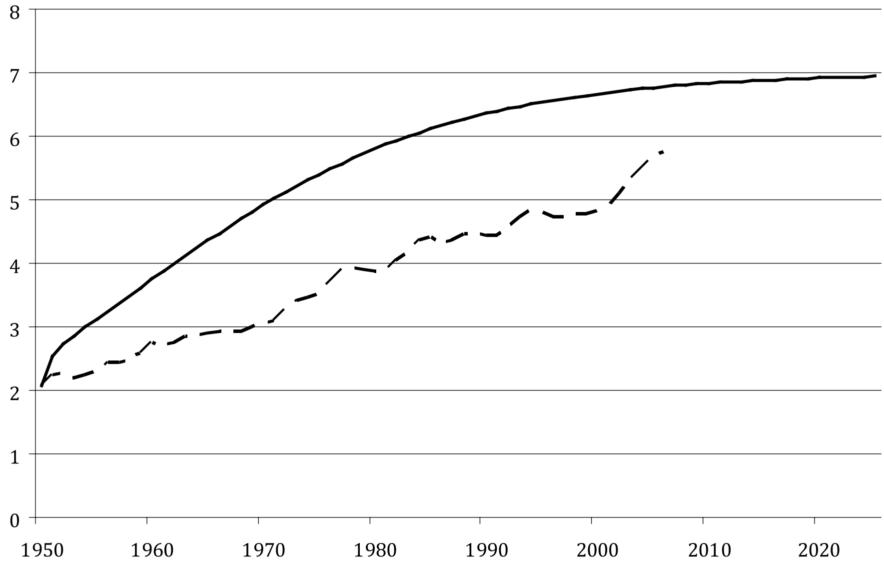

Fig. 1 Dynamics of capital, $k_{1900} = 1$

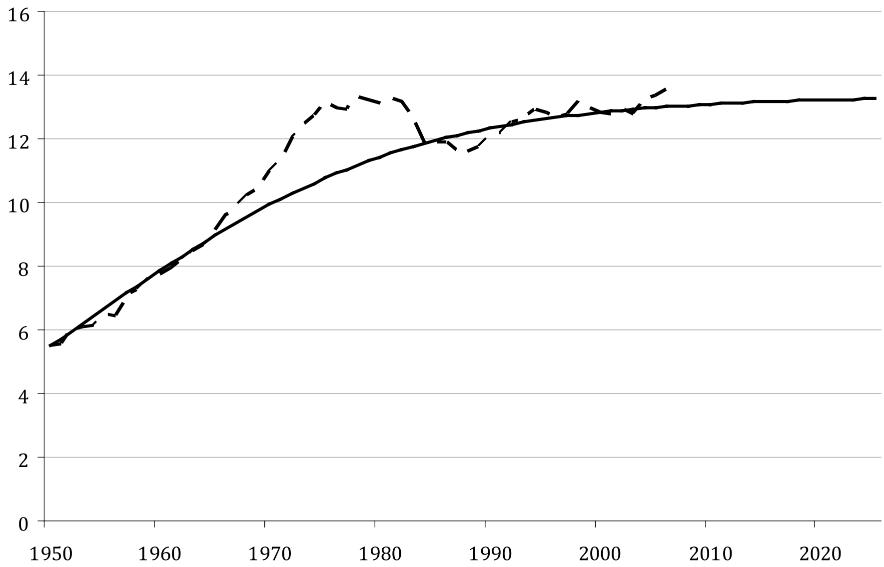

Fig. 2 Dynamics of useful work, $u_{1900} = 1$

future. At the same time Fig. 2 demonstrates a good fitness for growth trends of the useful work. On Fig. 4 declining trends of investment levels generated by nonlinear stabilizers are presented at the saturation level around 15 percent.

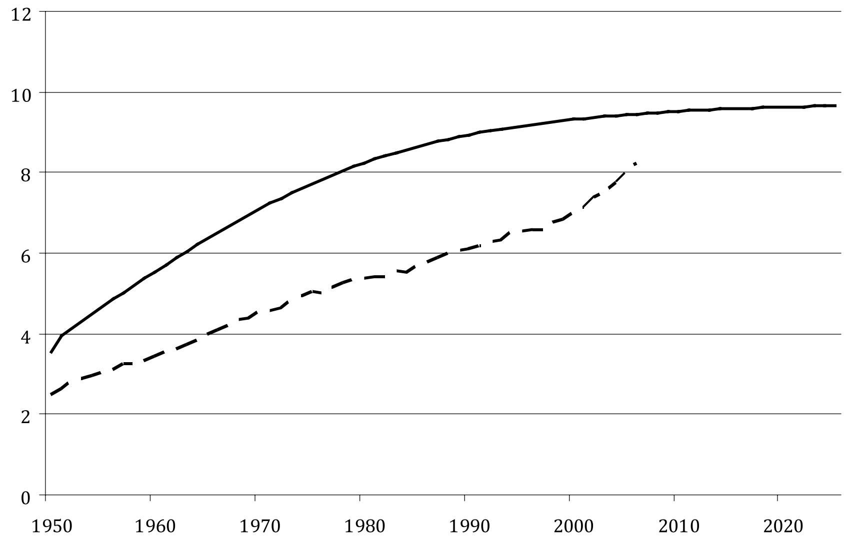

Fig. 3 Dynamics of GDP, $y_{1900} = 1$

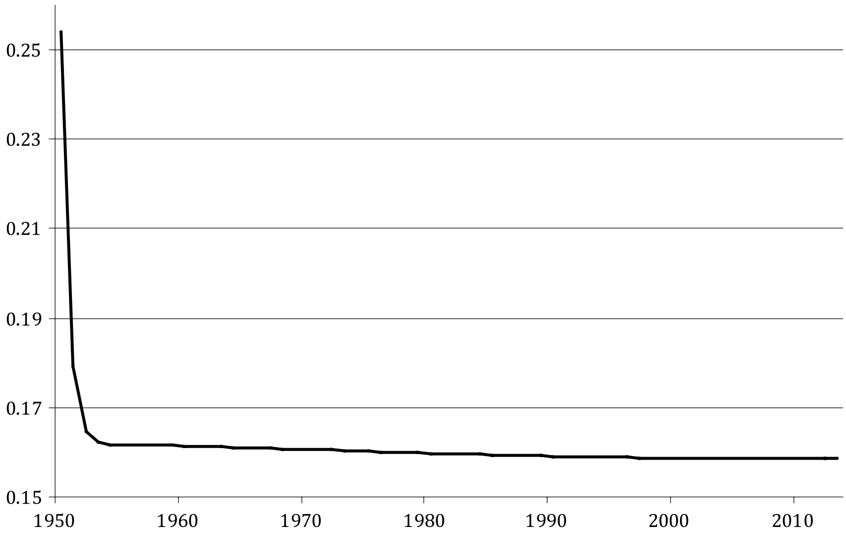

Fig. 4 Dynamics of investments, in percentage

7 Sequential Precision of Predictions Algorithm

7.1 The Case-Study

Let us consider an application of the proposed methodology to analysis of macroeconomic indices of the UK economy. In this case-study we use time series on capital, labor, and GDP in the period of 1901 to 2004. At the first stage of analysis, the data is adjusted for specification of the input model parameters. One of the basic constructions of the model is presented by the production. For illustration we introduce a two factor model with capital and labor factors. For identification of productivity in per capita one can use the Cobb-Douglas production function. On Fig. 5 results of the calibration procedure are presented for per worker quantities. The production function is defined by the following formula:

$$f(k) = Ak^{\alpha}, \quad A = 1.03, \; \alpha = 0.66. \tag{53}$$

For construction of the optimal economic growth trajectory we solve the problem of optimal control with the infinite horizon. The solution of the problem is constructed numerically by sewing the Hamiltonian systems which correspond to various control regimes (see Krasovskii and Tarasyev 2008). Numerical results of construction of optimal trajectories and sewing curves in coordinates (k, z) are given on Fig. 6.

For completing of modeling we compare the optimal trajectory with the real data and fulfill time scaling. Comparison results are given on Fig. 7. By the dashed line the optimal trajectory is shown and by the full line real data on capital per worker is depicted for the UK economy in the period from 1951 to 2004. One can see that the optimal trajectory follows the data quite adequately. It has S-shape and demonstrates the saturation level at the capital steady state $k^* = 18.857$. It is worth to note that in the modeling process of economic growth based on the Cobb-Douglas production

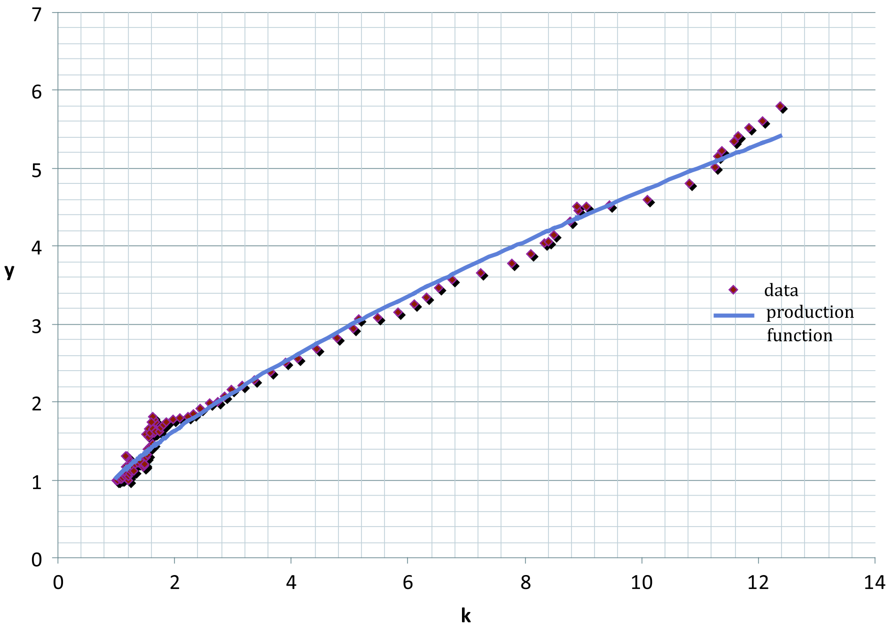

Fig. 5 Calibration of the production function, UK data

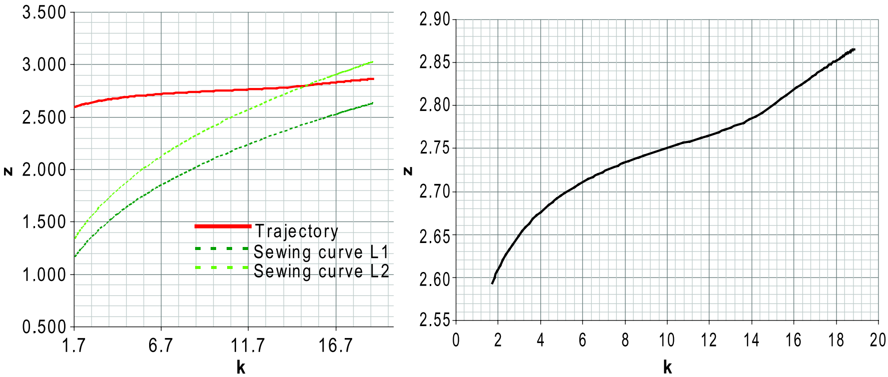

Fig. 6 Construction of optimal trajectories

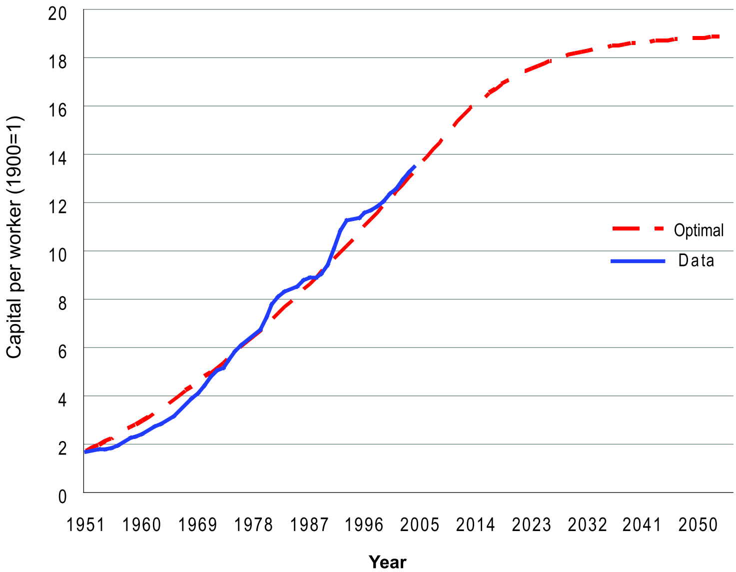

Fig. 7 Comparison of the optimal trajectory with real data

function good coincidence with data is achieved by introducing constraints on the control variable of investments. In an economy, these constraints correspond to restrictions on investments into capital formation and are expressed in GDP fraction. Let us show that overwhelming of these constraints leads to qualitative changes in results of modeling of optimal trajectories. Fig. 8 demonstrates plots of optimal investment level in the modeling process with constraints on investments and compares them with plots of optimal investment levels obtained in the unconstraint

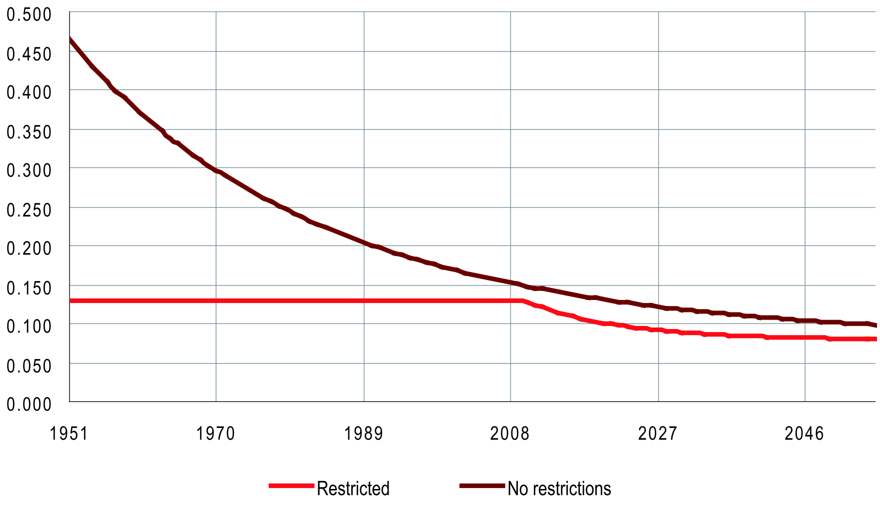

Fig. 8 Optimal investments in model with constraints and in model without constraints

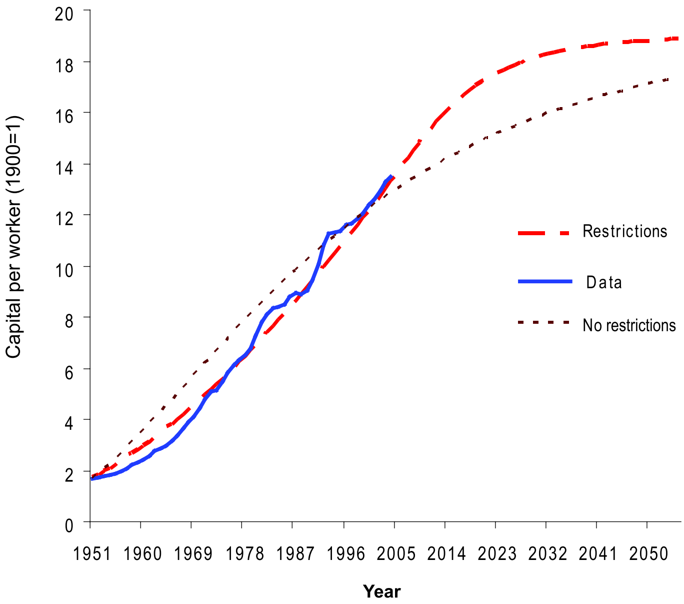

Fig. 9 Optimal trajectories depending on the constraints

model. On Fig. 9 the corresponding curves of optimal growth trajectories are shown in comparison with the real data. One can see that presence of constraints provides qualitatively better results.

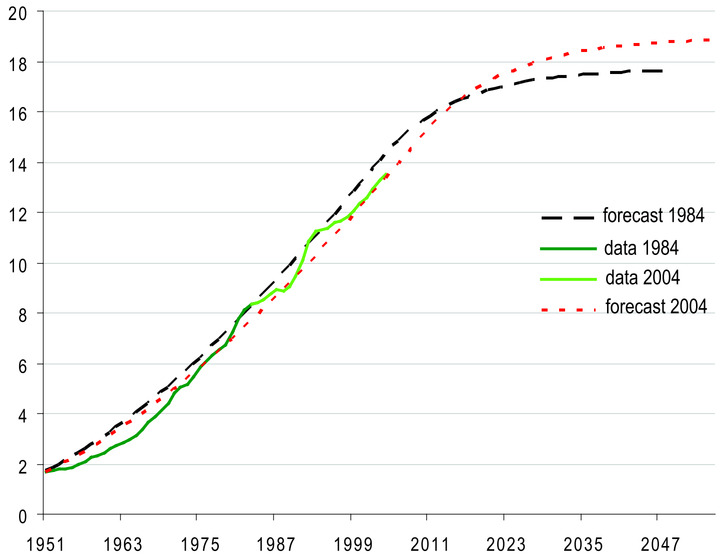

Fig. 10 Verification of the model

7.2 Model Verification

Verification of the model is presented by the following procedure. Assume that the proposed approach is applied to a fixed time interval in the given data time series. If comparison of the optimal growth trajectory with the real data is satisfactory at this fixed time interval then it can be used for forecasting.in the course of time, the data is updated and one can compare the forecast with the new data. Over a time period one can adjust the forecast based on the upgraded data. For this purpose, the model is restarted for an extended time series on a longer time interval. Then, all stages of modeling are fulfilled and the new optimal trajectory is constructed. Depending on consistency of the new forecast with the previous prognosis a conclusion is made about robustness of the forecasting procedure. For realization of the verification approach the data is split in several time intervals.

For example, assume that we have data till 1984. Using this data we model the optimal growth trajectory according to the proposed methodology. To this end, the production function is calibrated $f(k) = 1.066k^{0.647}$ on the fixed time interval, and values for model parameters are identified. As a result of modeling, a new forecast is obtained for the optimal development. This optimal trajectory indicates saturation level corresponding to the steady state $k^* = 17.772$. Next, the data is collected for the time period up to 2004. The model is restarted and the new optimal trajectory is calculated. Let us note that this trajectory is described already in Sect. 7.1. Comparison of these two optimal trajectories one of which corresponds to the time period (1951–1984) and another one corresponds to the time period (1951–2004) is given on Fig. 10. One can see that the first trajectory forecasts quite adequately data

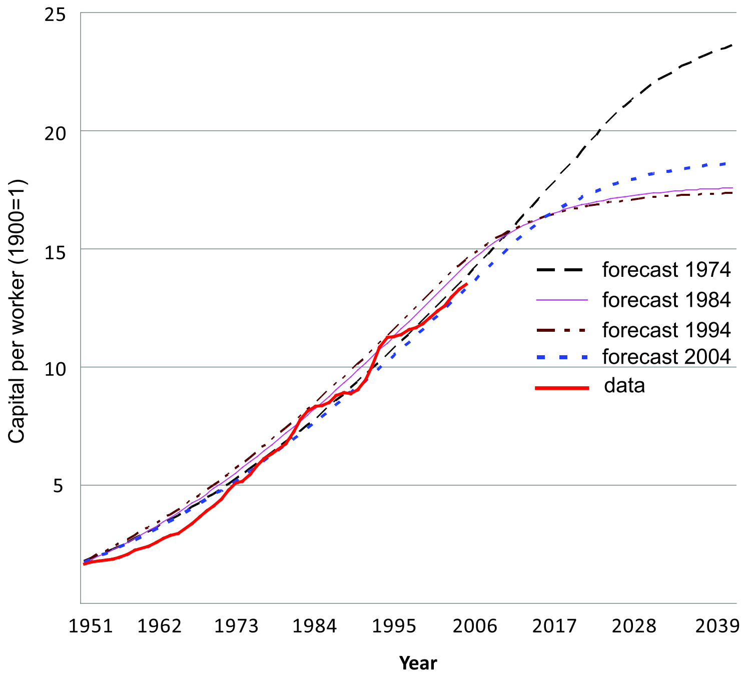

Fig. 11 (Color online) Sequential precision of predictions algorithm

trends in the period from 1984 to 2004 and the second trajectory adjusts the future prognosis after 2004.

7.3 *Sequential Precision of Predictions Algorithm*

A sequential precision of predictions algorithm updates optimal trajectories according to the proposed methodology in a sequential data accumulation. On Fig. 11 series of the optimal growth trajectories of capital for the economy of the United Kingdom are depicted. These trajectories are constructed in the process of sequential precision of predictions of the model on the basis of the available data from 1950 to 1974, 1984, 1994, and 2004. The changing trends of the model trajectories show that with the availability of more accurate data they follow it dynamically more precisely and demonstrate qualitatively the convergence of S-shaped forecasting trends. Real data is depicted in red and the model trajectories are colored in accordance with the time periods—dark blue, lavender, orange, and blue.

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High Order Precision Estimates in Algorithms for Solving Problems of Economic Growth

Andrey A. Krasovskii and Alexander M. Tarasyev

Abstract The research is devoted to analysis of optimal control problems arising in models of economic growth. The Pontryagin maximum principle is applied for analysis of the optimal investment problem. Specifically, the research is based on existence results and necessary conditions of optimality in problems with infinite horizon. Properties of Hamiltonian systems are examined for different regimes of optimal control. The existence and uniqueness result is proved for a steady state of the Hamiltonian system. Analysis of properties of eigenvalues and eigenvectors is completed for the linearized system in a neighborhood of the steady state. Description of behavior of the nonlinear Hamiltonian system is provided on the basis of results of the qualitative theory of differential equations. This analysis allows us to outline proportions of the main economic factors and trends of optimal growth in the model. A numerical algorithm for construction of optimal trajectories of economic growth is elaborated on the basis of constructions of backward procedures and conjugation of an approximation linear dynamics with the nonlinear Hamiltonian dynamics. High order precision estimates are obtained for the proposed algorithm. These estimates establish connection between precision parameters in the phase space and precision parameters for functional indices. The results of numerical experiments illustrating algorithm's constructions are given for real data of US and Japan economies.

1 Introduction

The paper is devoted to analysis of optimal control problems with infinite horizon. The focus is on elaboration of algorithms for constructing optimal trajectories in

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A.A. Krasovskii (✉)
Institute of Mathematics and Mechanics, Ural Branch Russian Academy of Sciences, S. Kovalevskoi Str. 16, 620219 Ekaterinburg, Russia
e-mail: ak@imm.uran.ru

A.A. Krasovskii
Vienna Institute of Demography, Austrian Academy of Sciences, Wohllebengasse 12-14, 1040 Vienna, Austria

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these problems and estimating precision of constructions in algorithms. The goal of the research is explained by several interacting reasons. The first one is connected with the fact that solutions of nonlinear control problems with infinite horizon can be constructed only numerically as a rule since derivation of analytical solutions is very complicated even for problem of small dimensions (see Bardi and Dolcetta 1997; Falcone 1987; Feichtinger and Wirl 2000; Feichtinger et al. 2006; Rockafellar 2004). The second point deals with the problem of integration of stiff equations in the Hamiltonian system of the Pontryagin maximum principle (see Arnold 1983; Hairer and Wanner 2004; Kuznetsov 2004; Lambert 2000). The third problem is connected with the balanced partition of the utility integral into two terms: a finite integral and an infinite horizon "tail" (see Adiatulina and Tarasyev 1987; Bardi and Dolcetta 1997; Lions 1982; Souganidis 1985).

Problems of optimal control with infinite horizon have a background in models of economic growth (see Arrow 1968; Intriligator 1971; Ramsey 1928; Shell 1969; Solow 1970). The phase variables in these problems can be interpreted as factors of production, and control parameters at each moment of time are investments in factors of production. The production output is described by a production function. The objective functional is given by an integral characteristic of the discounted consumption index on the infinite horizon.

The research is implemented in the framework of the optimal control theory (see Pontryagin et al. 1962; Krasovskii and Krasovskii 1995). Specifically, methods developed for the problems with infinite horizon (see Aseev and Kryazhimskiy 2007) are used to justify the existence result and necessary conditions of optimality.

The main goal of the paper is to elaborate an algorithm for constructing optimal trajectories in problems with infinite horizon. For this purpose, results of papers (see Krasovskii and Tarasyev 2007, 2008; Tarasyev and Watanabe 2001) are developed in the direction of obtaining precision estimates. The special focus of this analysis is to raise the accuracy of algorithms.

2 Outline of Optimal Control Problem for Economic Growth Model

The paper deals with the following optimal control problem which arises in models of optimal economic growth.

Stated specifically, the problem is to maximize the functional

$$J = \int_0^{+\infty} [\ln f(k(t)) + \ln(1 - s(t))] e^{-\delta t} dt \underset{(k(\cdot), s(\cdot))}{\longrightarrow} \max \tag{1}$$

under the following dynamic constraints

$$\begin{aligned} &\dot{k}(t) = s(t) f(k(t)) - \lambda k(t), \\ &k(0) = k^0, \qquad s \in [0, a], \quad a < 1, \end{aligned} \tag{2}$$

where the phase variable k denotes capital per worker, symbol $f(k)$ stands for the production function, investments s is a control variable measurable in time, parameters δ, $\lambda = n + \mu$, k^0 are given positive numbers. Parameter $0 < a < 1$ is a positive number which separates the right bound of control parameter from unit.

Results of the paper are formulated for a class of control problems with concave production functions. More precisely, we assume that function $f(k)$ should satisfy the following conditions

$$f'(k) > 0 \quad \text{and} \quad f''(k) < 0 \quad \text{for } k \in (0, +\infty). \tag{3}$$

Here $f'(k) = \partial f(k)/\partial k$ is the marginal productivity of capital per worker (Intriligator 1971). Also it is assumed that function $f(k)$ satisfies the "Inada's limit conditions"

$$\begin{cases} \lim_{k\downarrow 0} f(k) = 0, & \lim_{k\uparrow +\infty} f(k) = +\infty, \\ \lim_{k\downarrow 0} f'(k) = +\infty, & \lim_{k\uparrow +\infty} f'(k) = 0. \end{cases} \tag{4}$$

2.1 Hamiltonians in the Pontryagin Maximum Principle

Let us apply the Pontryagin maximum principle to the problem (1)–(2). Introducing the adjoint variable $\tilde{\psi} = \tilde{\psi}(t)$, interpreted in economy as a shadow price of capital, one can compile the Hamiltonian of the problem

$$\tilde{H}(s, k, t, \tilde{\psi}) = [\ln(1 - s) f(k)] e^{-\delta t} + \tilde{\psi}(s f(k) - \lambda k). \tag{5}$$

To exclude the exponential term depending on time from the Hamiltonian let us introduce new variables

$$\psi = \tilde{\psi} e^{\delta t}, \quad H(s, k, \psi) = e^{\delta t} \tilde{H}(s, k, t, \psi), \tag{6}$$

and consider the stationary form of the Hamiltonian

$$H(s, k, \psi) = \ln f(k) + \ln(1 - s) + \psi(s f(k) - \lambda k). \tag{7}$$

2.2 Concavity Properties of Hamiltonians

Let us analyze properties of the Hamiltonian (7).

Lemma 1 *The Hamiltonian $H(s, k, \psi)$ (7) is a strictly concave function in variable s.*

The proof follows immediately from strict negativity of the second derivative of the Hamiltonian (7) in s.

Let us introduce the necessary maximum condition for the Hamiltonian $H(s,k,\psi)$ (7) in the absence of restrictions

$$\frac{\partial H}{\partial s} = -\frac{1}{1-s} + \psi f(k) = 0. \tag{8}$$

This equation implies the following expression for the optimal investment level

$$s^0 = 1 - \frac{1}{\psi f(k)}. \tag{9}$$

Let us introduce the construction of the maximized Hamiltonian in presence of restrictions on control variable s

$$\hat{H}(k,\psi) = \max_{s\in[0,a]} H(s,k,\psi). \tag{10}$$

One can prove that the maximized Hamiltonian $\hat{H}(k,\psi)$ is constructed basing on location of the maximum point s^0 according to the following algorithm:

1. If $s^0 \in [0,a]$ then $\hat{H}(k,\psi) = H(s^0,k,\psi)$.
2. If $s^0 < 0$ then $\hat{H}(k,\psi) = H(0,k,\psi)$.
3. If $s^0 > a$ then $\hat{H}(k,\psi) = H(a,k,\psi)$.

The following results are valid. The maximized Hamiltonian $\hat{H}(k,\psi)$ is smoothly pasted out of branches $H_i(k,\psi)$, $i = 1,2,3$, in variables (k,ψ) on sewing curves L_i, $i = 1,2$. The maximized Hamiltonian $\hat{H}(k,\psi)$ is a strictly concave function in variable k for all $\psi > 0$. Basing on these two properties one can obtain the sufficient result for optimality conditions of the Pontryagin maximum principle (see Krasovskii and Tarasyev 2008).

2.3 Hamiltonian Systems in the Pontryagin Maximum Principle

Let us introduce the following notation $z(t) = k(t)\psi(t)$ for the cost of capital.

Three optimal control regimes generate three Hamiltonian systems. The Hamiltonian system for the zero control is defined by the following system of differential equations

$$\begin{cases} \dot{z} = \delta z - \dfrac{kf'(k)}{f(k)}, \\ \dot{k} = -\lambda k, \end{cases} \tag{11}$$

in the domain D_1 described by relations

$$D_1 = \left\{ (k,z) : z \le \frac{k}{f(k)},\ k > 0, z > 0 \right\}. \tag{12}$$

The Hamiltonian system generated by the intensive optimal control $s^0 = a$ is presented by equations

$$\begin{cases} \dot{z} = z\left(\delta + a\dfrac{f(k)}{k} - af'(k)\right) - \dfrac{kf'(k)}{f(k)}, \\ \dot{k} = af(k) - \lambda k, \end{cases} \tag{13}$$

in the domain D_3 defined by relations

$$D_3 = \left\{(k, z) : z \geq \frac{k}{(1-a)f(k)},\ k > 0, z > 0\right\}. \tag{14}$$

Let us note that the Hamiltonian systems (11), (13) have no steady states.

The Hamiltonian system generated by the transient regime (9) is adjoined to the steady state

$$\begin{cases} \dot{z} = z\left(\dfrac{f(k)}{k} + \delta - f'(k)\right) - 1, \\ \dot{k} = f(k) - \lambda k - \dfrac{k}{z}, \end{cases} \tag{15}$$

in the domain D_2 given by relations

$$D_2 = \left\{(k, z) : \frac{k}{f(k)} \leq z \leq \frac{k(1-a)^{-1}}{f(k)}, k > 0, z > 0\right\}. \tag{16}$$

Let us note that domains D_1 and D_2 are pasted together at points of the curve

$$L_1 = \left\{(k, z) : z = \frac{k}{f(k)},\ k > 0, z > 0\right\}. \tag{17}$$

The pair of domains D_2 and D_3 are pasted together at points of the curve

$$L_2 = \left\{(k, z) : z = \frac{k}{(1-a)f(k)},\ k > 0, z > 0\right\}. \tag{18}$$

Let us note that all basic elements of the algorithm for construction of optimal trajectories are connected namely with the Hamiltonian system (15), its steady state and properties of eigenvalues and eigenvectors of the linearized Hamiltonian system at the steady state.

The steady state of the Hamiltonian system (15) is defined by the system of equations

$$\begin{cases} \left(\dfrac{f(k)}{k} + \delta - f'(k)\right) - 1 = 0, \\ f(k) - \lambda k - \dfrac{k}{z} = 0. \end{cases} \tag{19}$$

There exists the unique steady state (k^*, z^*) for which the following estimates are valid

$$k^* > 0, \qquad 0 < z^* < \frac{1}{\delta}. \tag{20}$$

One can prove that the unique steady state (k^*, z^*) of the Hamiltonian system (15) possesses the saddle property: eigenvalues of the linearized Hamiltonian system are real and have different signs. Based on the Grobman-Hartman theorem (see Hartman 1964) one can show that the optimal trajectory starting from the initial condition k^0 converges to the steady state and it is tangent to the eigenvector corresponding to the negative eigenvalue of the linearized Hamiltonian system (15). Only this trajectory satisfies the transversality condition of the Pontryagin maximum principle

$$\lim_{t\to\infty} z(t) = 0. \tag{21}$$

3 Numerical Algorithm

The algorithm for construction of the optimal trajectory includes the following steps.

1. Numerical estimation of the steady state (k^*, z^*).
2. Linearization of the Hamiltonian system (15) in the neighborhood of the steady state (k^*, z^*).
3. Calculation of eigenvalues and eigenvectors of the linearized Hamiltonian system.
4. Fixation of the precision parameter $\varepsilon > 0$ and calculation of the characteristic point $(k_\varepsilon, z_\varepsilon)$ at the ε-neighborhood of the steady state (k^*, z^*) in the direction of the eigenvector corresponding to the negative eigenvalue.
5. Integration of the Hamiltonian system (15) in the reverse time starting from the characteristic point $(k_\varepsilon, z_\varepsilon)$. Integration is performed until one of two alternatives: (1) if the integrated trajectory reaches the initial point k^0 in domain D_2 then the algorithm is stopped and the trajectory is built; (2) if the integrated trajectory reaches sewing curves L_i, $i = 1, 2$, before it reaches the initial point k^0 then the Hamiltonian system (15) is switched either to the Hamiltonian system (11) at points of the sewing curve L_1, or to the Hamiltonian system (13) at points of the sewing curve L_2.
6. Expansion of the integrated trajectory in the direct time and its time scaling.

4 Precision Estimates of the Algorithm

In this paragraph precision estimates for the accuracy of the proposed numerical algorithm are obtained. One can prove that the following result takes place.

Theorem 1 *The accuracy of algorithm expressed in functional indices is estimated by the precision parameter ε of initial conditions of the algorithm depending on relations between the estimates of growth parameters. Three cases are possible*:

- *in the case when the Lipschitz module of the system dynamics is strictly less than the discount parameter, the precision estimate of the algorithm in functional indices is of order ε^2*;
- *in the case when the Lipschitz module of the system dynamics coincides with the discount parameter, the precision estimate of the algorithm in functional indices has the order $\varepsilon^2 \ln \frac{1}{\varepsilon^2}$*;
- *in the case when the Lipschitz module of the system dynamics is strictly larger than the discount parameter, the precision estimate of the algorithm in functional indices is expressed by the order $\varepsilon^{\frac{2}{\beta+1}}$, $\beta > 0$.*

Proof Let us analyze the utility functional in the optimal control problem (1)–(2). It can be presented as a sum of two integrals

$$\begin{aligned} J &= \int_0^{+\infty} e^{-\delta t}\big[\ln y(t) + \ln(1 - s(t))\big]dt \\ &= \int_0^{T} e^{-\delta t}\big[\ln y(t) + \ln(1 - s(t))\big]dt \\ &\quad + \int_T^{+\infty} e^{-\delta t}\big[\ln y(t) + \ln(1 - s(t))\big]dt, \quad 0 \le T < +\infty. \end{aligned} \tag{22}$$

Let us consider two integrals which are obtained for different regimes of control. One can denote by the symbol J_1 the integral corresponding to a pair (y_1, s_1):

$$\begin{aligned} J_1 &= \int_0^{+\infty} e^{-\delta t}\big[\ln y_1(t) + \ln(1 - s_1(t))\big]dt \\ &= \int_0^{T} e^{-\delta t}\big[\ln y_1(t) + \ln(1 - s_1(t))\big]dt \\ &\quad + \int_T^{+\infty} e^{-\delta t}\big[\ln y_1(t) + \ln(1 - s_1(t))\big]dt. \end{aligned} \tag{23}$$

By the symbol J_2 we denote the integral corresponding to a pair (y_2, s_2):

$$J_2 = \int_0^{+\infty} e^{-\delta t}\big[\ln y_2(t) + \ln(1 - s_2(t))\big]dt \tag{24}$$

$$\begin{aligned} &= \int_0^{T} e^{-\delta t}\big[\ln y_2(t) + \ln(1 - s_2(t))\big]dt \\ &\quad + \int_T^{+\infty} e^{-\delta t}\big[\ln y_2(t) + \ln(1 - s_2(t))\big]dt. \end{aligned} \tag{25}$$

Let us consider the module of difference of two integrals

$$
\begin{aligned}
|J_1 - J_2| \le & \int_0^{+\infty} e^{-\delta t}\big[|\ln y_1(t) - \ln y_2(t)|\big]dt \\
& + \int_0^{+\infty} e^{-\delta t}\big[|\ln(1 - s_1(t)) - \ln(1 - s_2(t))|\big]dt \\
= & \, I_1 + I_2.
\end{aligned} \tag{26}
$$

Here integrals I_1 and I_2 are calculated as follows

$$
\begin{aligned}
I_1 = & \int_0^{T} e^{-\delta t}\big[|\ln y_1(t) - \ln y_2(t)|\big]dt \\
& + \int_0^{T} e^{-\delta t}\big[|\ln(1 - s_1(t)) - \ln(1 - s_2(t))|\big]dt, \\
I_2 = & \int_T^{+\infty} e^{-\delta t}\big[|\ln y_1(t) - \ln y_2(t)|\big]dt \\
& + \int_T^{+\infty} e^{-\delta t}\big[|\ln(1 - s_1(t)) - \ln(1 - s_2(t))|\big]dt.
\end{aligned} \tag{27}
$$

We analyze two terms in the right-hand side of the estimate (26). Let us start the estimate with the second term I_2. One can obtain the following inequality

$$
\begin{aligned}
I_2 \le & \int_T^{+\infty} e^{-\delta t}\big[|\ln y_1(t)| + |\ln y_2(t)|\big]dt \\
& + \int_T^{+\infty} e^{-\delta t}\big[|\ln(1 - s_1(t))| + |\ln(1 - s_2(t))|\big]dt.
\end{aligned} \tag{28}
$$

In accordance with the dynamics of the system one can choose parameters $B > 1$ and $b > 0$ such that

$$
\begin{aligned}
y_1(t) \le Be^{bt}, \qquad & y_2(t) \le Be^{bt}, \\
1 - s_1(t) \le Be^{bt}, \qquad & 1 - s_2(t) \le Be^{bt}.
\end{aligned} \tag{29}
$$

Then, one can continue the estimate of integral I_2 in the following way

$$
\begin{aligned}
I_2 \le & \, 4\int_T^{+\infty} e^{-\delta t} \ln(Be^{bt})dt \\
= & \, 4b\int_T^{+\infty} te^{-\delta t}dt + 4\int_T^{+\infty} \ln Be^{-\delta t}dt \\
= & \, 4b\bigg(-\frac{te^{-\delta t}}{\delta}\bigg|_T^{+\infty} + \int_T^{+\infty} \frac{e^{-\delta t}}{\delta}dt\bigg) - 4\ln B\frac{e^{-\delta t}}{\delta}\bigg|_T^{+\infty}
\end{aligned}
$$

$$
\begin{aligned}
&= 4b\left(\frac{Te^{-\delta T}}{\delta} - \frac{e^{-\delta t}}{\delta^2}\bigg|_T^{+\infty}\right) + 4\ln B\frac{e^{-\delta T}}{\delta} \\
&= 4b\left(\frac{Te^{-\delta T}}{\delta} + \frac{e^{-\delta T}}{\delta^2}\right) + 4\ln B\frac{e^{-\delta T}}{\delta} \\
&= 4b\frac{e^{-\delta T}}{\delta}\left(T + \frac{1}{\delta}\right) + 4\ln B\frac{e^{-\delta T}}{\delta}. \qquad (30)
\end{aligned}
$$

Let us choose parameter ϕ from the condition $\delta > \phi > 0$ (i.e. $\delta - \phi > 0$) and parameter $G > 0$ are such that the inequality is fulfilled $(T + 1/\delta) \leq Ge^{\xi T}$. Then, integral I_2 is estimated by the relation

$$
\begin{aligned}
I_2 &\leq \frac{4b}{\delta}e^{-(\delta-\phi)T} + 4\ln B\frac{e^{-\delta T}}{\delta} \\
&\leq \frac{4b}{\delta}Ge^{-(\delta-\phi)T} + 4\ln B\frac{e^{-(\delta-\phi)T}}{\delta} \\
&= \frac{4}{\delta}e^{-(\delta-\phi)T}(bG + \ln B). \qquad (31)
\end{aligned}
$$

We introduce parameter $\rho > 0$ in order to estimate the module of difference (26). Let us find the moment of time T from the condition

$$
I_2 \leq \frac{4}{\delta}e^{-(\delta-\phi)T}(bG + \ln B) \leq \rho. \qquad (32)
$$

Solving this equation with respect to time T, we obtain the chain of relations

$$
\begin{aligned}
e^{-(\delta-\phi)T} &\leq \frac{\rho\delta}{4(bG + \ln B)}, \\
-(\delta-\phi)T &\leq \ln\frac{\rho\delta}{4(bG + \ln B)}.
\end{aligned} \qquad (33)
$$

From this chain one can obtain the estimate for the moment of time T

$$
T \geq \ln\left(\frac{1}{\rho}\right)^{(\delta-\phi)} + \ln\left(\frac{4(bG + \ln B)}{\delta}\right)^{(\delta-\phi)}. \qquad (34)
$$

Let us consider the first term I_1 in the estimate of the module of difference (26).

One can estimate the distance between initial points y_1^0 and y_2^0 of the ideal optimal trajectory $y_1(t)$ and the trajectory $y_2(t)$ generated by the algorithm. Since the trajectory of algorithm is tangent to the ideal trajectory at the steady state of the Hamiltonian system and the initial points are chosen in the ε-neighborhood of the steady state, then for the same values of variable k the distance of these two points is of order ε^2

$$
|y_1^0 - y_2^0| \leq \ell\varepsilon^2. \qquad (35)
$$

In what follows, these trajectories are integrated according to the same nonlinear dynamics of the Hamiltonian system in the Pontryagin maximum principle. Let us remind that the right-hand side of the Hamiltonian system satisfies the Lipschitz property. Thus, according to the theorem of continuity of trajectories relative to initial conditions, we obtain the following estimates

$$|\ln y_1(t) - \ln y_2(t)| \le \varepsilon^2 P e^{\nu t}, \tag{36}$$

$$|\ln(1 - s_1(t)) - \ln(1 - s_2(t))| \le \varepsilon^2 M e^{\nu t}. \tag{37}$$

Here parameters $P > 0$, $M > 0$ depend on the distance parameter ℓ between initial points. The exponential parameter $\nu > 0$ depends on the Lipschitz constant of the right-hand side of the Hamiltonian system. The parameter ε^2 stands for the precisions estimate of the trajectory generated by the algorithm.

For the first term we have the following accuracy inequalities

$$\begin{aligned} I_1 &\le \int_0^T e^{-\delta t} e^{\nu t} \varepsilon^2 (P + M) dt \\ &= \varepsilon^2 (P + M) \int_0^T e^{(\nu - \delta)t} dt \qquad (38) \\ &= \varepsilon^2 (P + M) \frac{e^{(\nu-\delta)t}}{(\nu - \delta)} \bigg|_0^T \\ &= \varepsilon^2 (P + M) \left(\frac{e^{(\nu-\delta)T} - 1}{(\nu - \delta)} \right). \qquad (39) \end{aligned}$$

Let us continue the estimate depending on relations between parameters.

Case 1. Let the inequality $\nu < \delta \Rightarrow \nu - \delta < 0$ be fulfilled, then integral I_1 is estimated in the following way

$$I_1 \le \varepsilon^2 (P + M) \left(\frac{1 - e^{-(\nu-\delta)T}}{(\nu - \delta)} \right) \le \frac{\varepsilon^2 (P + M)}{(\nu - \delta)}. \tag{40}$$

Case 2. Let parameters ν and δ coincide, $\nu = \delta \Rightarrow \nu - \delta = 0$. In this case, integral I_1 can be estimated according to the following rule

$$\begin{aligned} I_1 &\le \varepsilon^2 (P + M) T \le \varepsilon^2 (P + M) \\ &\quad \times \left(\ln\left(\frac{1}{\rho}\right)^{(\delta-\xi)} + \ln\left(\frac{4(bG + \ln B)}{\delta}\right)^{(\delta-\xi)} \right). \end{aligned} \tag{41}$$

Case 3. Let the strict inequality $\nu > \delta \Rightarrow \nu - \delta > 0$ be valid.

In this case integral I_1 is estimated according to the following chain of inequalities

$$I_1 \le \varepsilon^2 (P + M) \left(\frac{e^{(\nu-\delta)T} - 1}{(\nu - \delta)} \right)$$

$$\leq \frac{\varepsilon^2(P+M)}{(\nu-\delta)}\left(\frac{4(bG+\ln B)}{\delta}\right)^{(\delta-\xi)(\nu-\delta)} \times \frac{1}{\rho^{(\delta-\xi)(\nu-\delta)}}. \tag{42}$$

Finally, for the estimate of the module of difference of integrals one has inequalities depending on a particular case among previous three. In case 1 the following inequality takes place

$$I_1 + I_2 \leq \frac{\varepsilon^2(P+M)}{(\delta-\nu)} + \rho = \varepsilon^2\left(\frac{P+M}{\delta-\nu} + 1\right). \tag{43}$$

Here $\rho = \varepsilon^2$.

In case 2 one has to minimize the estimate with respect to the precision parameter ρ

$$\begin{aligned} I_1 + I_2 &\leq \varepsilon^2(P+M)T + \rho \\ &\leq \varepsilon^2(P+M) \\ &\quad \times \left(\ln\left(\frac{1}{\rho}\right)^{(\delta-\xi)} + \ln\left(\frac{4(bG+\ln B)}{\delta}\right)^{(\delta-\xi)}\right) + \rho \to \min_{\rho>0}. \end{aligned}$$

The necessary condition of minimum can be expressed by the following equation

$$-\frac{\varepsilon^2(P+M)(\delta-\xi)}{\rho} + 1 = 0. \tag{44}$$

Then, the point of minimum is determined by the relation

$$\rho = \varepsilon^2(P+M)(\delta-\xi). \tag{45}$$

Substituting the minimum point into estimate (26) one obtains the following result

$$\begin{aligned} I_1 + I_2 &\leq \varepsilon^2(P+M) \\ &\quad \times (\delta-\xi)\ln\left(\frac{4(bG+\ln B)}{\varepsilon^2(P+M)\delta(\delta-\xi)}\right) + \varepsilon^2(P+M)(\delta-\xi) \\ &= \varepsilon^2(P+M)(\delta-\xi)\ln\left(\frac{4(bG+\ln B)}{\varepsilon^2(P+M)\delta(\delta-\xi)}\right). \end{aligned} \tag{46}$$

For parameter ε^2 tending to zero one can check basing on the L'Hôpital rule that the estimate function tends to zero with the declining rate $\varepsilon^2 \ln \frac{1}{\varepsilon^2}$ for $\varepsilon^2 \to 0$.

In case 3 it is also necessary to minimize the estimate with respect to the precision parameter ρ. To shorten calculations let us introduce the following notations

$$D = \frac{(P+M)}{(\nu-\delta)}\left(\frac{4(bG+\ln B)}{\delta}\right)^{(\delta-\xi)(\nu-\delta)}, \qquad \beta = (\delta-\xi)(\nu-\delta) > 0. \tag{47}$$

One should solve the problem of the estimate minimization

$$I_1 + I_2 \le D\varepsilon^2\rho^{-\beta} + \rho \underset{\rho}{\longrightarrow} \min. \tag{48}$$

We have the following necessary condition of minimum

$$-\beta D\varepsilon^2\rho^{-\beta-1} + 1 = 0. \tag{49}$$

The minimum point is defined by relation

$$\rho = (\beta D)^{\frac{1}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}}. \tag{50}$$

Substituting the minimum point into estimate (26) we obtain the chain of relations

$$\begin{aligned} I_1 + I_2 &\le D\varepsilon^2(\beta D)^{-\frac{\beta}{\beta+1}}\varepsilon^{-\frac{2\beta}{\beta+1}} + (\beta D)^{\frac{1}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}} \\ &= D^{\frac{1}{\beta+1}}\beta^{-\frac{\beta}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}} + D^{\frac{1}{\beta+1}}\beta^{\frac{1}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}} \\ &= D^{\frac{1}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}}\left(\beta^{-\frac{\beta}{\beta+1}} + \beta^{\frac{1}{\beta+1}}\right) \\ &= D^{\frac{1}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}}\beta^{\frac{1}{\beta+1}}\left(\frac{1}{\beta}+1\right). \end{aligned} \tag{51}$$

The expression of estimates shows that its order is determined by the power function $\varepsilon^{\frac{2}{\beta+1}}$.

Finally, we obtain that in all three cases the accuracy of the algorithm expressed in functional indices is estimated by the accuracy of approximation of initial conditions in the algorithm: in the first case the accuracy of the algorithm is of degree ε^2, in the second case—of degree $\varepsilon^2 \ln\frac{1}{\varepsilon^2}$, and in the third case—of degree $\varepsilon^{\frac{2}{\beta+1}}$. □

5 Numerical Experiments

In this section we present numerical experiments which realize an algorithm proposed in the paper. Simulations are performed with precision parameter $\varepsilon = 0.001$ and integration step $\Delta t = 0.0001$. Model parameters are identified from the real data on the economies of Japan and US (Ayres and Martinás 2005).

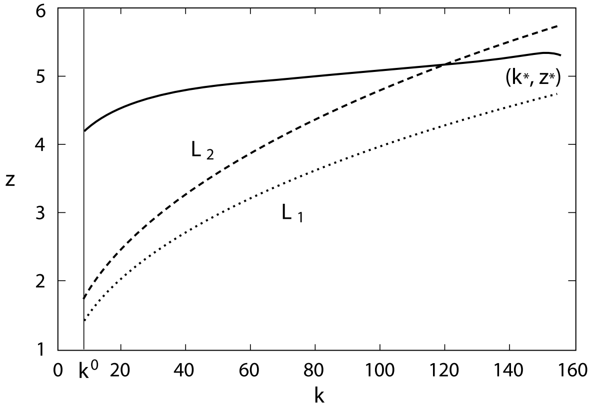

Fig. 1 Construction of the synthetic optimal trajectory (Japan)

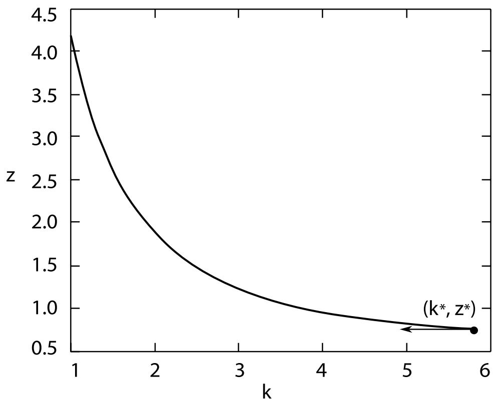

Fig. 2 Construction of the synthetic optimal trajectory (US)

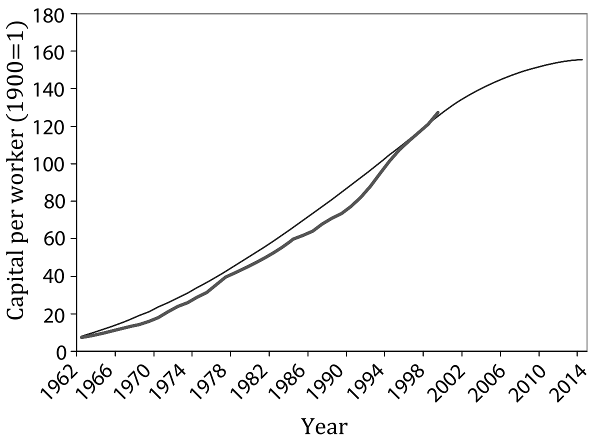

Fig. 3 Comparison of optimal trajectory with data (Japan)

On Fig. 1 results of construction of the optimal trajectory based on the data for economy of Japan is presented. The trajectory is integrated in the reverse time starting at the ε-neighborhood of the steady state (k^*, z^*). First part of the trajectory is

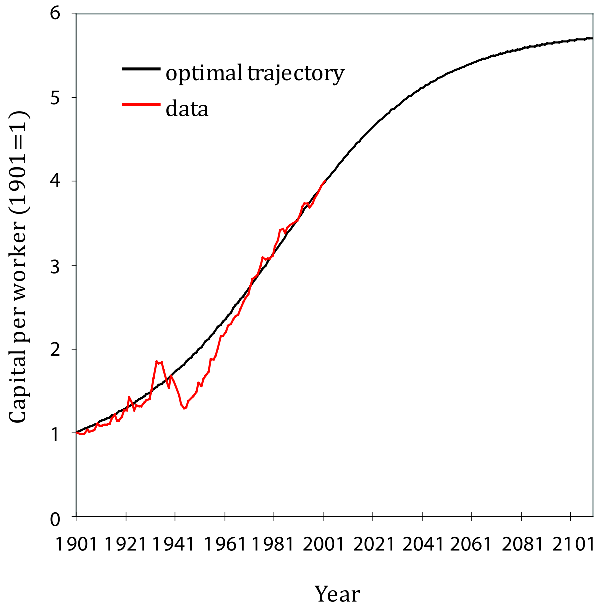

Fig. 4 Comparison of optimal trajectory with data (US)

integrated in the domain D_2 (16). Then trajectory reaches the sewing curve L_2, and second part of the trajectory is integrated in the domain D_3 (14) till the stopping criterion k^0. Sewing curve L_2 is generated by the investment constraint parameter $a = 0.17$.

Results of construction of optimal trajectory for the US data is given on Fig. 2. This figure depicts steady state of the system (k^*, z^*), eigenvector corresponding to negative eigenvalue of the linearized Hamiltonian system, and the synthetic optimal trajectory which is integrated in domain D_2.

Comparison results of synthetic optimal trajectories with real data on macroeconomic indicators of economies of US and Japan are given on Figs. 3–4 in real time scale. These figures show that synthetic optimal trajectories constructed using elaborated high-precision algorithm adequately describe trends of real time-series.

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Growth and Climate Change: Threshold and Multiple Equilibria

Alfred Greiner, Lars Grüne, and Willi Semmler

Abstract In this paper we analyze a basic growth model where we allow for global warming. As concerns global warming we assume that the climate system is characterized by feedback effects such that the ability of the earth to emit radiation to space is reduced as the global surface temperature rises. We first study the model assuming that abatement spending is fixed exogenously and demonstrate with the use of numerical examples that the augmented model may give rise to multiple equilibria and thresholds. Then, we analyze the social optimum where both consumption and abatement are set optimally and show that the long-run equilibrium is unique in this case. In the context of our model with multiple equilibria initial conditions are more important for policy actions than discount rates. Our analysis thus supports the view that policy actions against global warming are urgently needed.

1 Introduction

Meanwhile, it is widely accepted that the emission of greenhouse gases (GHGs), such as carbon dioxide (CO_2) or methane (CH_4) just to mention two, considerably affects the atmosphere of the earth and, thus, climate on earth. One consequence of a higher concentration of GHGs in the atmosphere is an increase in the global average surface temperature on the earth. According to the Intergovernmental Panel on Climate Change (IPCC) it is certain that the global average surface temperature of the earth has increased since 1861. Over the 20th century the temperature has risen by about 0.6 degree Celsius, and it is very likely that the 1990s was the warmest decade since 1861 (IPCC 2001, p. 26), where very likely means that the level of confidence is between 90–99 percent. Eleven of the twelve years from 1995–2006 rank among the 12 warmest years since 1850 and for the next one hundred years the IPCC expects that the mean temperature will rise by about 3 degrees Celsius (IPCC 2007). Besides the increase of the average global surface temperature, heavy and extreme weather events, primarily in the Northern Hemisphere have occurred more frequently. It is true that changes in the climate may occur as a result of both internal variability within the climate system and as a result of external factors where the latter can be natural or anthropogenic. But, there is strong evidence that most of the

A. Greiner (✉)
Department of Business Administration and Economics, Bielefeld University,
P.O. Box 100131, 33501 Bielefeld, Germany
e-mail: agreiner@wiwi.uni-bielefeld.de

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climate change observed over the last 50 years is the result of human activities such as the emission of greenhouse gases.

The rise in the average global surface temperature will not only have immediate effects for the natural environment but it will also affect economies. This holds because, on the one hand, agricultural production has been adapted to the current climatic situation and deviations from it will be associated with costs. On the other hand, more extreme weather events will cause immediate damages that imply costs and may reduce GDP. Therefore, economists have constructed models that incorporate climatic interrelations with the economic subsystem. Examples for this type of models are CETA (see Peck and Teisberg 1992), FUND (see Tol 1999), RICE and DICE (see Nordhaus and Boyer 2000 and Nordhaus 2008), WIAGEM (see Kemfert 2001) or DART (see Deke et al. 2001). The aim of these studies is to evaluate different abatement scenarios as to economic welfare and as to their effects on GHG emissions. However, it must be pointed out that the results partly are very sensitive with respect to the assumptions made. Popp (2003), for example, demonstrates that the outcome in Nordhaus and Boyer (2000) changes considerably when technical change is taken into account.

A recent study that has received great attention in the economics literature as well as in press media is the report by Stern (2006–2007). Stern strongly argues that decisive actions should be undertaken now that aim at reducing GHG emissions in order to avoid catastrophic possibilities that could along with major economic costs. Otherwise, future generations will suffer from extremely high costs that are much larger than costs of avoiding GHG emissions today in present values. But the Stern review has also bee in part heavily criticized. Weitzman (2007) argues that the outcome obtained by the Stern review heavily depends on the low discount rate that is resorted to in the latter report. In addition, there is large uncertainty about structural parameters such that makes the predictions of the Stern review rather uncertain.

Another important research direction, undertaken by scientists, studies the impact of greenhouse gas emissions on climate change through the change of ocean circulations. The papers by Deutsch et al. (2002) and Keller et al. (2000), for example, describe how the gulf stream and the North Atlantic current, part of the North Atlantic thermohaline circulation (THC), transport a large amount of heat from warm regions to Europe. As those papers show, due to the heating up of surface water, the currents could suddenly change and trigger a change in temperature. The THC collapse and the sudden cooling of regions would most likely have a strong economic impact on Europe and Africa. An event like this would have an impact on the climate in these regions and would also likely affect economic growth. Further results on THC mechanisms are given in Broecker (1997). Although a breakdown of the gulf stream is to be considered as rather unlikely meanwhile, this does not hold for the existence of feedback effects of a change in the global climate that affect the ability of earth to emit radiation to space.

The goal of our contribution is different from the above economic studies and we do not intend to evaluate abatement policies as to their welfare effects. We want to study, in the context of a basic growth model, the long-run effects of the interaction of global warming and economics and, in particular, the transitions dynamics that might occur with global warming. More specifically, we want to study the question of whether there possibly exist multiple equilibria and thresholds that separate basins of attraction for optimal paths to some long-run steady state. In order to study such a problem, we take a basic growth model and integrate a simple climate model. Our approach is related to the one presented in Greiner and Semmler (2005) where an endogenous growth model is studied. However, in contrast to the latter we analyze an exogenous growth model and we rigorously prove that initial conditions can be decisive as concerns the question of to which equilibrium the economy converges in the long-run. In our context the urgency of actions is given less by a low discount rate but rather by initial conditions.

The remainder of the paper is organized as follows. In the next section we present the model with non-optimal abatement spending and analyze its dynamics. In Sect. 3, we study the social optimum where both consumption and abatement are chosen optimally and Sect. 4, finally, concludes.

2 A Basic Growth Model with Non-optimal Abatement Spending

In this section we present the neoclassical growth model where we integrate a climate system of the earth and where abatement is not chosen optimally. First, we present the structure of the model and, then, we analyze its dynamics.

2.1 The Structure of the Model

Our economy is represented by one household with household production that chooses consumption in order to maximize a discounted stream of utility over an infinite time horizon subject to its budget constraint.

Economic activities of the household generate emissions of GHGs. As regards emissions of GHGs we assume that these are a by-product of capital used in production and expressed in CO_2 equivalents. Hence, emissions are a function of per-capita capital, K, relative to per-capita abatement activities, A. This implies that a higher capital stock goes along with higher emissions for a given level of abatement spending. This assumption is frequently encountered in environmental economics (see e.g. Smulders 1995, or Hettich 2000). We should also like to point out that the emission of GHGs does not affect utility and production directly but only indirectly by affecting the climate of the earth which leads to a higher surface temperature and to more extreme weather situations. Formally, emissions are described by

$$E = \left(a \frac{LK}{LA} \right)^{\gamma}, \tag{1}$$

with L the amount of labour, $\gamma > 0$ and $a > 0$ are constants. The parameter a can be interpreted as a technology index describing how polluting a given technology is.

For large values of a a given stock of capital (and abatement) goes along with high emissions implying a relatively polluting technology and vice versa.

The effect of emissions is to raise the GHG concentration, M, in the atmosphere. The concentration of GHGs evolves according to the following differential equation

$$\dot{M} = \beta_1 E - \mu M, \qquad M(0) = M_0, \tag{2}$$

where μ is the inverse of the atmospheric lifetime of CO_2. As to the parameter μ we assume a value of $\mu = 0.1$.[1] β_1 captures the fact that a certain part of GHG emissions are taken up by oceans and do not enter the atmosphere. According to IPCC $\beta_1 = 0.49$ for the time period 1990 to 1999 for CO_2 emissions (IPCC 2001, p. 39).

The evolution of per-capita capital is described by the following differential equation that gives the budget constraint of the household,

$$\dot{K} = Y - C - A - (\delta + n)K, \qquad K(0) = K_0, \tag{3}$$

with Y per-capita production, K per-capita capital, A per-capita abatement activities and δ is the depreciation rate of capital. L is labour, which grows at rate n.

As concerns abatement activities we assume that these are determined exogenously. One can assume that the government levies a non-distortionary tax, like a lump-sum tax or a tax on consumption in our model, and uses its revenue to finance abatement spending.[2]

The production function giving per-capita output is given by

$$Y = BK^{\alpha} D(T - T_0), \tag{4}$$

with $\alpha \in (0, 1)$ the capital share and B is a positive constant. $D(T - T_0)$ is the damage due to deviations from the normal pre-industrial temperature T_o. As concerns the damage function $D(\cdot)$ we assume the function

$$D(\cdot) = (a_1 (T - T_o)^2 + 1)^{-\psi}, \tag{5}$$

with $a_1 > 0$, $\psi > 0$. This function shows that the damage is the higher the higher the deviation of the actual temperature, T, from the pre-industrial temperature T_o.

To model the climate system of the earth we use the simplest way and resort to a so-called energy balance models (EBM). According to an EBM the change in the average surface temperature on earth is described by[3]

$$\frac{dT(t)}{dt} c_h \equiv \dot{T}(t) c_h = S_E - H(t) - F_N(t), \qquad T(0) = T_0, \tag{6}$$

[1]The range of μ given by IPCC is $\mu \in (0.005, 0.2)$, see IPCC (2001, p. 38).

[2]We are not interested in distortions arising from taxation but in the dynamics of the model. Therefore, we can limit our considerations to the income effect of taxation.

[3]This subsection follows Roedel (2001), Chaps. 10.2.1 and 1 and Henderson-Sellers and McGuffie (1987), Chaps. 1.4 and 2.4. See also Gassmann (1992) and Harvey (2000).

with $T(t)$ the average global surface temperature measured in Kelvin[4] (K), c_h the heat capacity[5] of the earth with dimension $\mathrm{J\,m^{-2}\,K^{-1}}$ (Joule per square meter per Kelvin)[6] which is considered a constant parameter. Since most of the earth's surface is covered by seawater, c_h is largely determined by the oceans. Therefore, the heat capacity of the oceans is used as a proxy for that of the earth. The numerical value of this parameter[7] is $c_h = 0.1497\ \mathrm{J\,m^{-2}\,K^{-1}}$. S_E is the solar input, $H(t)$ is the non-radiative energy flow, and $F_N(t) = F{\uparrow}(t) - F{\downarrow}(t)$ is the difference between the outgoing radiative flux and the incoming radiative flux. S_E, $H(t)$ and $F_N(t)$ have the dimension Watt per square meter ($\mathrm{W\,m^{-2}}$). t is the time argument which will be omitted in the following as long as no ambiguity can arise. $F{\uparrow}$ follows the Stefan-Boltzmann-Gesetz, which is

$$F{\uparrow} = \epsilon \sigma_T T^4, \tag{7}$$

with ϵ the emissivity that gives the ratio of actual emission to blackbody emission. Blackbodies are objects that emit the maximum amount of radiation and that have $\epsilon = 1$. For the earth ϵ can be set to $\epsilon = 0.95$. σ_T is the Stefan-Boltzmann constant that is given by $\sigma_T = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$. Further, the flux ratio $F{\uparrow}/F{\downarrow}$ is given by $F{\uparrow}/F{\downarrow} = 109/88$. The difference $S_E - H$ can be written as $S_E - H = Q(1-\alpha_1(T))/4$, with $Q = 1367.5\ \mathrm{W\,m^{-2}}$ the solar constant, $\alpha_1(T)$ the planetary albedo, determining how much of the incoming energy is reflected to space.

According to Henderson-Sellers and McGuffie (1987) and Schmitz (1991) the albedo $\alpha_1(T)$ is a function that negatively depends on the temperature on earth. This holds because deviations from the equilibrium average surface temperature have feedback effects that affect the reflection of incoming energy. Examples of such feedback effects are the ice-albedo feedback mechanism and the water vapour 'greenhouse' effect (see Henderson-Sellers and McGuffie 1987, Chap. 1.4). With higher temperatures a feedback mechanism occurs, with the areas covered by snow and ice likely to be reduced.[8] This implies that a smaller amount of solar radiation is reflected when the temperature rises tending to increase the temperature on earth further. Therefore, Henderson-Sellers and McGuffie (1987, Chap. 2.4) and Schmitz (1991, 194) propose a function as shown in Fig. 1.

Figure 1 shows $1-\alpha_1(T)$, that part of energy that is not reflected by earth. For the average temperature smaller than T_l the albedo is a constant, then the albedo declines linearly, so that $1-\alpha_1(T)$ rises until the temperature reaches T_u from which point on, the albedo is constant again. Here, we should like to point out that other feedback effects may occur, such as a change in the flux ratio of outgoing to incom-

[4] 273 Kelvin are 0 degree Celsius.

[5] The heat capacity is the amount of heat that needs to be added per square meter of horizontal area to raise the surface temperature of the reservoir by 1 K.

[6] 1 Watt is 1 Joule per second.

[7] For more details concerning the calculation of this parameter see Harvey (2000).

[8] For a further detailed discussion of positive feedback effects from temperature to higher temperature, see Lovelock (2006).

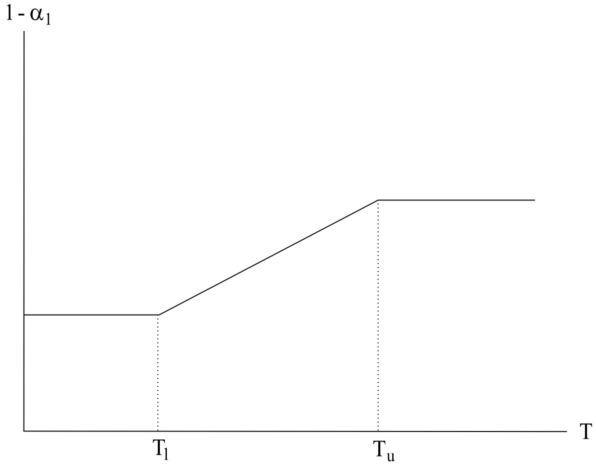


Fig. 1 Albedo as a function of the temperature

ing radiative flux for example. However, we do not take into account these effects since the qualitative result would remain the same.

The effect of emitting GHGs is to raise the concentration of GHGs in the atmosphere according to (2). The effect of a higher concentration of GHGs on the temperature is obtained by calculating the so-called radiative forcing, which is a measure of the influence a GHG, such as CO_2 or CH_4, has on changing the balance of incoming and outgoing energy in the earth-atmosphere system. The dimension of the radiative forcing is $\mathrm{W\,m^{-2}}$. For example, for CO_2 the radiative forcing, which we denote by F, is approximately given by

$$F = 6.3 \ln \frac{M}{M_o}, \tag{8}$$

with M the actual CO_2 concentration, M_o the pre-industrial CO_2 concentration and ln the natural logarithm (see IPCC 1996, pp. 52–53).[9] For other GHGs other formulas can be given describing their respective radiative forcing and these values can be converted in CO_2 equivalents.

Incorporating (8) in (6) gives

$$\begin{aligned} \dot{T}(t)c_h = & \frac{1367.5}{4}(1-\alpha_1(T)) - 0.95(5.67\times 10^{-8})(21/109)T^4 \\ & + (1-\xi)6.3\ln\frac{M}{M_o}, \\ T(0) = & \, T_0. \end{aligned} \tag{9}$$

[9]The CO_2 concentration is given in parts per million (ppm).

The parameter ξ captures the fact that a certain part of the warmth generated by the greenhouse effect is absorbed by the oceans which transport the heat from upper layers to the deep sea. We set $\xi = 0.23$.

According to Roedel (2001), $(1-\alpha_1(T)) = 0.21$ holds in equilibrium, for $\dot{T} = 0$ with $M = M_o$, giving a surface temperature of about 288 Kelvin which is about 15 degree Celsius.

2.2 The Dynamics of the Model

In order to analyze the dynamics of our model, we first have to solve the optimization problem of the household. The household maximizes a discounted stream of utility arising from per-capita consumption, C, times the number of household members subject to the budget constraint and taking into account that emissions affect the climate. As to the utility function we assume a logarithmic function $U(C) = \ln C$.

Thus, the agent's optimization problem can be written as

$$\max_C \int_0^\infty e^{-\rho t} L_0 e^{nt} \ln C \, dt, \tag{10}$$

subject to (2), (3) and (9). ρ in (10) is the subjective discount rate, and L_0 is labour supply at time $t = 0$.

To find the optimal solution we form the current-value Hamiltonian[10] which is

$$\begin{aligned} H(\cdot) = {} & \ln C + \lambda_1 (B K^\alpha D(T - T_o) - C - A - (\delta + n)K) \\ & + \lambda_2 (\beta_1 a^\gamma K^\gamma A^{-\gamma} - \mu M) + \lambda_3 (c_h)^{-1} \left(\frac{1367.5}{4} (1 - \alpha_1(T)) \right. \\ & \left. - (5.6710^{-8})(19.95/109) T^4 + (1-\xi) 6.3 \ln \frac{M}{M_o} \right), \end{aligned} \tag{11}$$

where λ_i, $i = 1, 2, 3$, are the shadow prices of K, M and T, respectively, and $E = a^\gamma K^\gamma A^{-\gamma}$. Note that λ_1 is positive while λ_2 and λ_3 are negative.

As to the albedo, $\alpha_1(T)$, we use a function as shown in Fig. 1. We approximate the function shown in Fig. 1 by a differentiable function. More concretely, we use the function

$$1 - \alpha_1(T) = k_1 \left(\frac{2}{\Pi} \right) \mathrm{ArcTan} \left(\frac{\Pi (T - 293)}{2} \right) + k_2. \tag{12}$$

k_1 and k_2 are parameters that are set to $k_1 = 5.6 \times 10^{-3}$ and $k_2 = 0.2135$.

[10]For an introduction to the optimality conditions of Pontryagin's maximum principle, see Feichtinger and Hartl (1986) or Seierstad and Sydsaeter (1987).

The necessary optimality conditions, then, are obtained as

$$\frac{\partial H(\cdot)}{\partial C} = C^{-1} - \lambda_1 = 0, \tag{13}$$

$$\dot{\lambda}_1 = (\rho+\delta)\lambda_1 - \lambda_1 \alpha K^{\alpha-1} B D(\cdot) - \lambda_2 \beta_1 \gamma a^{\gamma} K^{\gamma-1} A^{-\gamma}, \tag{14}$$

$$\dot{\lambda}_2 = (\rho-n)\lambda_2 + \lambda_2\mu - \lambda_3(1-\xi)6.3c_h^{-1}M^{-1}, \tag{15}$$

$$\begin{aligned}\dot{\lambda}_3 = {} & (\rho-n)\lambda_3 - \lambda_1 B K^{\alpha} D'(\cdot) + \lambda_3 (c_h)^{-1} 341.875 \alpha_1'(\cdot) \\ & + \lambda_3 (5.67\times 10^{-8}(19.95/109)4T^3)(c_h)^{-1},\end{aligned} \tag{16}$$

with $\alpha_1' = -k_1(1+0.25\Pi^2(T-293)^2)^{-1}$. Further, the limiting transversality condition $\lim_{t\to\infty} e^{-(\rho+n)t}(\lambda_1 K + \lambda_2 T + \lambda_3 M) = 0$ must hold.

Combining (13) and (14) the economy is completely described by the following differential equations:

$$\dot{C} = C(B\alpha K^{\alpha-1} D(\cdot) + \lambda_2 \beta_1 \gamma a^{\gamma} K^{\gamma-1} A^{-\gamma} - (\rho+\delta)), \tag{17}$$

$$\dot{K} = BK^{\alpha} D(\cdot) - C - A - (\delta+n)K, \qquad K(0) = K_0, \tag{18}$$

$$\dot{M} = \beta_1 a^{\gamma} K^{\gamma} A^{-\gamma} - \mu M M, \qquad M(0) = M_0, \tag{19}$$

$$\begin{aligned}\dot{T} = {} & c_h^{-1}\bigg(341.875(1-\alpha_1(T)) - 5.67\times 10^{-8}(19.95/109)T^4 \\ & + 6.3(1-\xi)\ln\frac{M}{M_o}\bigg), \qquad T(0) = T_0,\end{aligned} \tag{20}$$

$$\dot{\lambda}_2 = (\rho-n)\lambda_2 + \lambda_2\mu - \lambda_3(1-\xi)6.3c_h^{-1}M^{-1}, \tag{21}$$

$$\begin{aligned}\dot{\lambda}_3 = {} & (\rho-n)\lambda_3 - \lambda_1 B K^{\alpha} D'(\cdot) + \lambda_3 (c_h)^{-1} 341.875 \alpha_1'(\cdot) \\ & + \lambda_3 (5.67\times 10^{-8}(19.95/109)4T^3)(c_h)^{-1},\end{aligned} \tag{22}$$

where $C(0)$, $\lambda_2(0)$ and $\lambda_3(0)$ can be chosen by society. A rest point of the dynamic system (17)–(22) gives a steady state for our economy, where we are only interested in solutions with $M^{\star} \geq M_o$.[11] In order to get additional insight we resort to a numerical analysis where we use the following parameter values.

We consider one time period to comprise one year. The discount rate is set to $\rho = 0.035$, the population growth rate is assumed to be $n = 0.03$, and the depreciation rate of capital is $\delta = 0.075$. The pre-industrial level of GHGs is normalized to one (i.e. $M_o = 1$) and we set $\gamma = 1$. ξ is set to $\xi = 0.23$ (see the previous subsection) and the capital share is set to $\alpha = 0.18$. The parameter a is set to $a = 3.5 \times 10^{-4}$, abatement is $A = 0.0012$ and B is normalized to one, i.e. $B = 1$. As concerns the parameters in the damage function $D(\cdot)$, specified in (5), we assume $a_1 = 0.025$ and $\psi = 0.025$.

[11] The $\star$ denotes steady state values.

Fig. 2 $\dot{T} = 0$ isocline ($Q1$) and $\dot{\lambda}_3 = 0$ isocline ($Q2$) in the $(T - K)$ plane

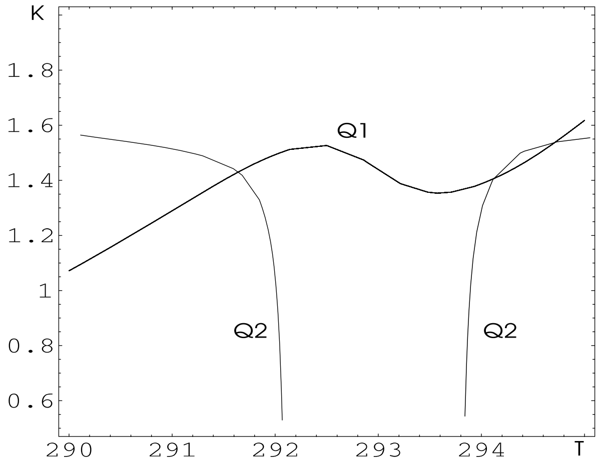


Table 1 Steady state values and eigenvalues of the Jacobian matrix

Steady state	$T^{\star}$	$K^{\star}$	$C^{\star}/Y^{\star}$	Eigenvalues
I	291.9	1.47	85.4%	$3.88, -3.88, 0.202 \pm 0.059i, -0.197 \pm 0.059i$
II	294.1	1.4	85.8%	$3.71, -3.70, 0.3, -0.3, 0.003 \pm 0.115i$
III	294.6	1.5	84.9%	$5.39, -5.39, 0.25, -0.25, 0.079, -0.074$

In order to find rest points of the system (17)–(22) we first solve $\dot{\lambda}_2 = 0$ with respect to M giving $M = M(\lambda_2, \lambda_3, \cdot)$ and $\dot{M} = 0$ with respect to λ_3 that yields $\lambda_3 = \lambda_3(K, \lambda_2, \cdot)$. Next, we solve $\dot{C}/C = 0$ with respect C leading to $C = C(K, T, \lambda_2, \cdot)$ and setting $\dot{K} = 0$ gives $\lambda_2 = \lambda_2(K, T, \cdot)$. Thus, we end up with the two differential equations $\dot{T}$ and $\dot{\lambda}_3$ that only depend on the two variables K and T and a solution $\dot{T} = \dot{\lambda}_3 = 0$ with respect to K and T gives a steady state for our economy. In order to find possible steady states we plot the $\dot{T} = 0$ isocline, denoted by $Q1$, and the $\dot{\lambda}_3 = 0$ isocline, denoted by $Q2$, in the $(T - K)$ plane. A point where the isoclines intersect gives a rest point for our dynamical system (17)–(22) and, thus, a steady state for our economy.

Figure 2 shows the $Q1$ and the $Q2$ isoclines in the $(T - K)$ plane. One realizes that there are 3 solutions for $Q1 = Q2$.

Table 1 gives the steady state values for $T^{\star}$, $K^{\star}$ and $C^{\star}/Y^{\star}$ as well as the eigenvalues of the Jacobian matrix evaluated at the corresponding rest point of (17)–(22).

This table shows that the first and third long-run steady states (I and III) are saddle point stable, while the second is unstable, with the exception of a two-dimensional stable manifold. Thus, there are two possible long-run steady states to which the economy can converge where the initial values of consumption, $C(0)$, of the shadow price of GHGs, $\lambda_2(0)$, and of the shadow price of the temperature, $\lambda_3(0)$, must be chosen such that these values lie on the stable manifold leading either to the first or to the third steady state. The first steady state implies a temperature increase of about 3.9 degrees and a steady state consumption share of 85.4 percent;

the third steady state corresponds to a temperature increase of about 6.6 degrees and a steady state consumption share of 84.9 percent. The GHG concentration associated with the first steady state is 2.1 and that associated with the third steady state is 2.16.

Before we calculate the value function (10) in order to see which of the two saddle point stable steady states is optimal, we want to study how variations in the abatement spending A affects the outcome. When we reduce abatement spending the qualitative picture as shown in Fig. 2 does not change. That means there still exist three steady states, where we let looked at the range $A \in [7 \times 10^{-4}, 1.21 \times 10^{-3}]$. But the steady state value of the temperature becomes larger both for the first and for the third steady state. For example, with $A = 7 \times 10^{-4}$ the temperature increase at the first steady state is 4.3 degree Celsius and it is 8.5 degree Celsius at the third steady state. Both steady states are again saddle point stable. When we increase abatement spending the left branch of the $Q2$ isocline in Fig. 2 moves to the left and the right branch of the $Q2$ isocline moves to the right and, once abatement spending exceeds a certain threshold, only the left intersection point of the $Q2$ isocline with the $Q1$ isocline remains. For $A \in (1.21 \times 10^{-3}, 3 \times 10^{-3}]$ the steady state is unique and saddle point stable, where $A = 3 \times 10^{-3}$ was the largest value we looked at. For example, setting $A = 3 \times 10^{-3}$ gives a temperature increase of 0.3 degree Celsius in steady state.

It should also be noted $1 - \alpha_1(\cdot)$ takes the value 0.2098 for $T^\star = 291.9$ and 0.2178 for $T^\star = 294.6$ demonstrating that the quantitative decrease in the albedo does not have to be large for the occurrence of multiple equilibria.

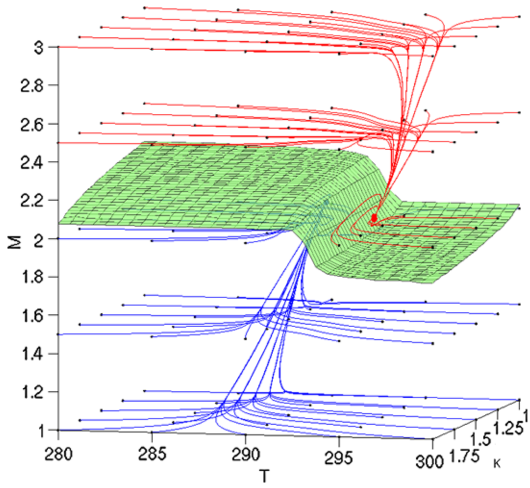
Our result suggests that there exists a threshold such that the initial conditions determine whether it is optimal to converge to steady state I or III. In order to answer the questions of for which initial values of the capital stock, of the GHG concentration and of the temperature it is optimal to converge to the first or to third steady state, respectively, we numerically compute the value function (10).

Doing so allows to calculate the so-called Skiba plane that separates the domains of attraction of the two steady states. The trajectories were computed using a dynamic programming algorithm with adaptive grid as described in Grüne (1997) and Grüne and Semmler (2004). Note that the adaptive gridding technique is particularly suited to compute the domains of attractions of multiple optimal equilibria, see also the example in Grüne and Semmler (2004), Sect. 5.2. The boundaries of the domains of attraction have been computed from the numerically simulated optimal trajectories using bisection for 50 K-values in the 2d example and for 1024 (K, T)-values in the 3d example. Figure 3 shows the Skiba plane in the $(T - K - M)$ space.

According to IPCC estimations, most projections predict that the GHG concentration in the atmosphere will stabilize at values between 450 ppm and 750 ppm (see e.g. Metz et al. 2007, p. 12). Normalizing the value of pre-industrial GHGs to one, i.e. $M_o = 1$ as in our model, this implies that GHGs stabilize at values of M between 1.6 and 2.7. A GHG concentration of 1.6 implies a temperature increase of about 1.1 to 2.9 degree Celsius and a concentration of 2.7 goes along with a rise in the average global surface temperature of 2.4 to 6.4 degree Celsius.

Figure 3 shows that for initial values of GHGs smaller than about 1.7, convergence to steady state I, with the relatively low temperature increase and the relatively

Fig. 3 Skiba plane in the $(T - K - M)$ space



high capital stock, will be the long-run outcome, independent of which temperature increases is associated with these levels of GHGs and independent of the initial physical capital stock. On the other hand, Fig. 3 also demonstrates that for initial values of GHGs larger than about 2.4, convergence to steady state III, with the relatively high temperature increase and the relatively low capital stock, will be the long-run outcome, independent of which temperature increases is associated with these levels of GHGs and independent of the initial physical capital stock.

It should also be noted that our model has important policy implications. If the government waits too long with actions against GHG emissions, the GHG concentration may rise above the threshold so that the initial condition $M(0)$ is above the Skiba plane in Fig. 3. If $M(0)$ is above the threshold, private agents will find it optimal to consume, save and invest in a way such that the economy converges to steady state III, when the government starts to take actions against GHG emissions. However, when the government now takes measures against GHG emissions, as long as the level of GHGs is below the threshold, so that the economy will stabilize at a GHG level below the threshold, the economy will converge to steady state I where the long-run temperature is smaller and production is higher, leading to higher welfare. Hence, governments should not wait too long with taking actions against global warming. Thus, the urgency of policy actions is defined more by initial conditions than by a low discount rate.

Only if stabilization of GHGs occurs between about 1.7 and 2.4, the temperature associated with a certain GHG concentration and, possibly, the initial condition with respect to physical capital may be crucial as concerns the question of to which steady state the economy finally converges. Thus, for a certain range of GHGs, it will be the climate sensitivity[12] that is decisive as to whether the economy converges to steady

[12]The climate sensitivity determines by how much the average surface temperature rises as a result of a higher GHG concentration in the atmosphere.

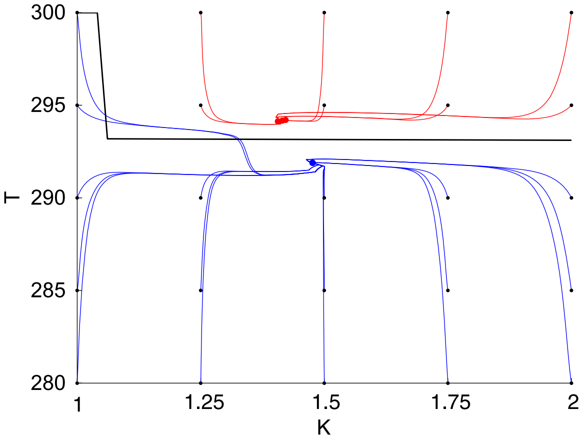


Fig. 4 Skiba curve in the $(T - K)$ plane with $M(0) = 2$

state I or to steady state III. In order to see this, we assume a doubling of GHGs and set $M(0) = 2$ which is in between the boundaries of the IPCC estimates. For that value, Fig. 4 shows the Skiba curve, drawn as the solid black line, that separates the domains of attraction of the two steady states in the $(T - K)$ plane.

From Fig. 4 it can be realized that for values of physical capital, K, smaller than about 1.05 convergence to steady state I is always optimal because the Skiba curve becomes almost vertical at $K = 1.05$. This implies that for relatively small initial capital stocks the economy will always converge to the steady state with the relatively small temperature increase and the relatively high capital stock. If the capital stock is larger than about 1.05 it is the temperature increase going along with a doubling of GHGs that determines whether the economy will converge to steady state I or to steady state III. Hence, if a doubling of GHGs implies a temperature larger than 293 Kelvin the economy converges to steady state III with a relatively small capital stock and a relatively large temperature increase. If the temperature is smaller than about 293 Kelvin the economy converges to steady state I with the relatively large capital stock and the relatively small temperature increase.

3 The Social Optimum

In formulating the optimization problem for the social optimum, a social planner needs to take account that both consumption and abatement have to be set optimally. Consequently, the optimization problem is

$$\max_{C,A} \int_0^\infty e^{-\rho t} L_0 e^{nt} \ln C dt, \tag{23}$$

subject to (2), (3) and (9).

To find necessary optimality conditions we formulate the current-value Hamiltonian which is

$$
\begin{aligned}
H(\cdot) = {} & \ln C + \lambda_4(BK^{\alpha}D(T-T_o) - C - A - (\delta+n)K) \\
& + \lambda_5(\beta_1 a^{\gamma}K^{\gamma}A^{-\gamma} - \mu M) + \lambda_6(c_h)^{-1}\bigg(\frac{1367.5}{4}(1-\alpha_1(T)) \\
& - (5.67\times 10^{-8})(19.95/109)T^4 + (1-\xi)6.3\ln\frac{M}{M_o}\bigg),
\end{aligned}
\tag{24}
$$

where λ_i, $i = 4, 5, 6$, are the shadow prices of K, M and T with $\alpha_1(T)$ given by (12) and where $D(\cdot)$ is again given by (5). Again λ_4 is positive while λ_5 and λ_6 are negative.

The necessary optimality conditions are obtained as

$$\frac{\partial H(\cdot)}{\partial C} = C^{-1} - \lambda_4 = 0, \tag{25}$$

$$\frac{\partial H(\cdot)}{\partial A} = -\lambda_5\beta_1 a^{\gamma}K^{\gamma}\gamma A^{-\gamma-1} - \lambda_4 = 0, \tag{26}$$

$$\dot{\lambda}_4 = (\rho+\delta)\lambda_4 - \lambda_4\alpha K^{\alpha-1}BD(\cdot) - \lambda_5\beta_1\gamma a^{\gamma}K^{\gamma-1}A^{-\gamma}, \tag{27}$$

$$\dot{\lambda}_5 = (\rho-n)\lambda_5 + \lambda_5\mu - \lambda_6(1-\xi)6.3c_h^{-1}M^{-1}, \tag{28}$$

$$
\begin{aligned}
\dot{\lambda}_6 = {} & (\rho-n)\lambda_6 - \lambda_5 BK^{\alpha}D'(\cdot) + \lambda_6(c_h)^{-1}341.875\alpha_1'(\cdot) \\
& + \lambda_6(5.67\times 10^{-8}(19.95/109)4T^3)(c_h)^{-1},
\end{aligned}
\tag{29}
$$

with $\alpha_1' = -k_1(1+0.25\Pi^2(T-293)^2)^{-1}$. Further, the limiting transversality condition $\lim_{t\to\infty} e^{-(\rho+n)t}(\lambda_4 K + \lambda_5 T + \lambda_6 M) = 0$ must hold.

From (25) and (26) we get the optimal abatement spending as,

$$A = (a^{\gamma}\beta_1 C\gamma K^{\gamma}(-\lambda_5))^{1/(1+\gamma)}. \tag{30}$$

The dynamics of the social optimum is described by (17)–(22) where abatement spending is replaced by its optimal value given in (30). As for the non-optimal economy a steady state is given for variables $C^{\star}$, $K^{\star}$, $T^{\star}$, $M^{\star}$, $\lambda_5^{\star}$ and $\lambda_6^{\star}$ such that $\dot{C} = \dot{K} = \dot{T} = \dot{M} = \dot{\lambda}_5 = \dot{\lambda}_6 = 0$ holds.

To find steady states for the social optimum we recursively solve system (17)–(22), with A given by (30), and end up with the three differential equations $\dot{C}$, $\dot{K}$ and $\dot{\lambda}_6$ that are nonlinear functions of the variables C, K and λ_5. A rest point of these equations then yields a steady state for the social optimum. Analyzing that system demonstrates that there exists a unique steady state with the values given in Table 2.

Table 2 demonstrates that the value of optimal abatement spending is $A = 2.74\times 10^{-3}$. If abatement spending is less than that value, as it was the case in the last section, the rise in the average temperature is larger than the socially optimal increase which is about 0.4 degree Celsius and there may be more than one

Table 2 Steady state values and eigenvalues of the Jacobian matrix

$T^\star$	$K^\star$	$A^\star$	$C^\star/Y^\star$	Eigenvalues
288.4	1.79	0.00274	82.8%	6.411, −6.406, 0.263, −0.258, 0.221, −0.216

steady state as also demonstrated above. If abatement spending is larger than the socially optimal value the steady state is unique and the increase in the temperature is smaller than in the social optimum. It should also be pointed out that the consumption share in the social optimum is smaller than in the non-optimal economy where abatement spending is below its optimum, implying that a higher share of GDP is invested in the social optimum.

4 Conclusion

In this paper we have analyzed a basic growth model with global warming. In modelling the climate change we allowed for feedback effects going along with a higher average global surface temperature, implying that the ability of the earth to emit radiation to space decreases as the average surface temperature on earth rises.

Assuming that greenhouse gases stabilize at values between 450 ppm and 750 ppm, which is plausible according to the IPCC, we could show that the initial condition with respect to the GHG concentration can be crucial as regards the questions of to which steady state the economy converges in the long-run. This outcome can be observed if abatement spending is set to a value smaller than the socially optimal value. In this case, multiple equilibria can emerge and there may exist a threshold determining whether the economy converges to the steady state with a relatively low increase in the average global surface temperature or whether it converges to the steady state with a large rise in the temperature. If GHGs stabilize within a certain corridor the climate sensitivity will be decisive to which steady state the economy converges in the long-run, independent of government policy.

Our model has also important policy implications. When governments wait too long with taking actions against GHG emissions, the GHG concentration may reach a level so that the economy always converges to the steady state with the higher temperature and with a small capital stock and low production. On the contrary, when governments act soon and achieve a stabilization of GHGs below the critical value, the economy will converge to the steady state with a more moderate temperature increase and with a higher capital stock and higher production. The latter scenario also yields higher welfare because this outcome is closer to that of the social optimum. Hence, our analysis, even without reference to a low discount rate, gives support to the policy recommendation reached by the Stern report (2006–2007) that measures against global warming should be taken soon.

Further, we could also demonstrate that multiple equilibria and thresholds cannot be observed in the social optimum. In this case, the steady state is unique and saddle point stable. In addition, the steady state temperature is smaller and the capital stock is larger compared to the economy with lower abatement spending.

Comparing our results with those obtained for an endogenous growth model, as studied in Greiner and Semmler (2005), one realizes that the outcomes are the same from a qualitative point of view. In the latter model, the social optimum is also characterized by a unique steady state,[13] but of course with ongoing growth, whereas the market economy with non-optimal abatement spending may give rise to multiple equilibria. Hence, independent of whether the long-run growth rate is an exogenous or an endogenous variable, multiple equilibria and thresholds may emerge when abatement spending is set lower than its socially optimal value.

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[13]Unless damages of the temperature increase are extremely small such that damages are virtually non-existent; a case that was not analyzed here.

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Optimal Economic Growth Under Stochastic Environmental Impact: Sensitivity Analysis

Elena Rovenskaya

Abstract In this work we present an approach toward the sensitivity analysis of optimal economic growth to a negative environmental impact driven by random natural hazards that damage the production output. We use a simplified model of the GDP growth. We assume that production leads to the increase of the atmospheric GHG provided investment in cleaning is insufficient. The hypothesis of the Poisson probability distribution of the frequency of natural hazards is used at the this research stage. We apply the standard utility function—the discounted integral consumption and construct an optimal investment policy in production and cleaning together with optimal GDP trajectories. We calibrate the model in the global scale and analyze the sensitivity of obtained optimal growth scenarios with respect to uncertain parameters of the Poisson distribution.

1 Introduction

Uncertainty arising in assessment of economic growth in relation to climate change creates enormous hurdles for scientists, stakeholders and policy makers (see, e.g., Obersteiner et al. 2001). One of the key issues is how policy choices can balance uncertainty in costs and benefits in situations when one is unsure what constraints on the atmospheric concentration of GHG are sufficient for preventing dangerous interference with the climate system, and what the degree of danger from exceeding a "safe" level of the GHG concentration is.

In this context, a dilemma arises: either to invest in abatement efforts today in order to prevent still unknown negative effects that may or may not occur in the future, or to delay investment until better knowledge on the feedback between the economy and environment is gained. A basic social goal is to minimize both the social cost of carbon emission and the abatement cost.

Modelers of socio-economic and environmental processes are challenged to create tools for finding optimal strategies for global development under an uncertain impact of climate change on human's production. Well-known DICE-type models (Nordhaus 1994; Nordhaus and Boyer 2001) tie up the neoclassical economic growth theory and global warming theory. These models view investment in economy sectors as variable control inputs. Using different investment scenarios, one

E. Rovenskaya (✉)
International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1,
2361 Laxenburg, Austria
e-mail: rovenska@iiasa.ac.at

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generates future projections for key economic and environmental indicators. Assuming that the values of model's parameters are given, one finds the optimal investment strategy that maximizes the utility, i.e., the social welfare. A number of studies initiated by Nordhaus's approach are aimed at economic assessment of GHG limitation under different types of uncertainty (see Kainuma 2006; Keller et al. 2004, and Toll 1994, and also the author's works Rovenskaya 2005, 2006).

At this stage of research, it is reasonable to complement the original purely deterministic DICE model by stochastic DICE-type models which could better represent the nature of the environmental impact on the economy. In this context, recent IIASA works, e.g., O'Neill et al. (2005), Kryazhimskiy et al. (2008) should be mentioned. The former develops a simplified stochastic "act then learn" model; and the latter suggests a dynamical multi-stage model assuming that climate provides a stochastic damaging impact on the world capital stock.

The stochastic properties of a feedback between the environmental quality and the economy are being widely discussed nowadays. Such studies as Keller et al. (2004), Hare and Meinshausen (2006), Meinshausen et al. (2006) are mainly focused on possible distributions of climate sensitivity. Keller et al. (2004) indicates that "the probability distribution of the threshold-specific damages seems at this time unknown." In this context the analysis of the sensitivity of the model's output to variations in the parameters of the probability distribution may help to understand the degree of importance of that quantitative information for decision-making.

2 Model

2.1 Economy

We consider a one-sector growth model with the so called production technology as the key driver of the world economy. Let T stand for the production technology stock used for producing public goods, and C stand for the cleaning technology stock used for barring greenhouse gases emissions that result from human production activity and go to the atmosphere. Let Y be the current GDP value. We assume that the constant fraction of the GDP $u_* \in [0, 1]$ is yearly available for developing both production and cleaning technologies. The rest fraction of the GDP is consumed by the society. Let $u \in [0, u^*]$ be a time-varying fraction of the GDP yearly allocated for developing the production technology stock whose dynamics is given by

$$\dot{T} = uY - \mu T, \qquad T(0) = T_0. \tag{1}$$

The dynamics of the cleaning technology stock is given by

$$\dot{C} = (u_* - u)Y - \mu C, \qquad C(0) = C_0. \tag{2}$$

In (1) and (2) $T_0 > 0$, $C_0 > 0$ are given initial values for T and C, μ refers to depreciation. In the dynamics (1), (2) the production ratio $u(\cdot)$ is viewed as a time-varying control.

Let E be the greenhouse gases stock accumulated in the atmosphere. We assume that GHG emissions are generated by production and restrained by implementation of cleaning technology. In other words, the growth rate of the increase of E is positively related to the current production technology stock T and negatively related to the current cleaning technology stock C:

$$\dot{E} = \max\{\beta T - \gamma C, 0\}, \qquad E(0) = E_0,$$

where $E_0 > 0$ is a given initial value for E. For the reason of simplicity we do not take into account the natural depreciation. The suggested form for the dynamics of the atmospheric GHG implies that the role of the cleaning technology is to decrease a rate of concentration growth but not to decrease the concentration itself: even if all admissible resources is invested in cleaning, the GHG concentration will not decrease.

In further analysis we will distinguish two states of the environment: we will say that the system is functioning in a "safe" mode if the current value of atmospheric GHG $E(t)$ does not exceed a critical level $E_* > E_0$ and that the system is functioning in an "unsafe" mode otherwise. We will specify the meaning of these terms in the next section.

In order to avoid difficulties with an eventual predominance of cleaning technology stock we assume a gap between E_0 and E_* to be small enough, namely,[1]

$$E_* - E_0 \leq \frac{(\beta T_0 - \gamma C_0)^2}{2\gamma u_* Y_0}. \tag{3}$$

Remark 1 Given assumption (3) we have increasing emissions trajectories in a "safe" zone for all admissible controls.

Hence if $E(t) \leq E_*$ can let

$$\dot{E} = \beta T - \gamma C, \qquad E(0) = E_0. \tag{4}$$

Figure 1 illustrates the set of values of E_* and γ satisfying to (3).

2.2 *Natural Hazards*

In the line with numerous speculations and works on modeling of the feedback between the environment and economic growth (e.g., Nordhaus and Boyer 2001), we assume the negative impact of the increasing atmospheric GHG on the economy.

[1] See details in Appendix A.

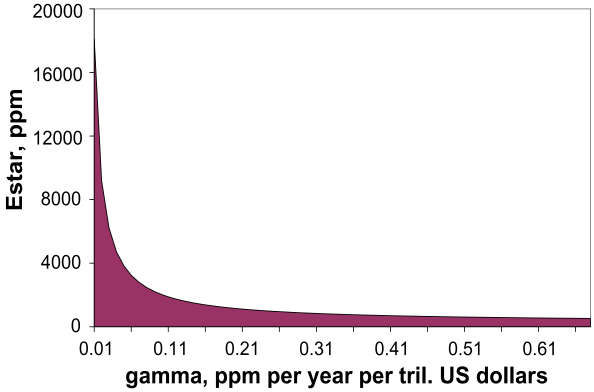

Fig. 1 Couples (γ, E_*) satisfying to (3) lie in a lilac area

Namely we believe that provided the atmospheric GHG stock does not exceed a certain critical level E_* the global economy is functioning in a "safe" mode. In this case we assume the simplest form of the Cobb-Douglas production function

$$Y = AT, \tag{5}$$

where A is the efficiency coefficient which is supposed to be constant on the considered time horizon. In the "safe" mode the aggregated GHG emission is growing due to growing production in accordance with (4).

However as soon as the GHG stock exceeds the critical limit E_* the economy enters an "unsafe" zone in which the climate change issues become significant. We assume the feedback of atmospheric GHG on the economy through global warming and caused by it natural hazards. We guess that in this case yearly the fraction of the GDP equal to $\Omega \in [0, 1]$ is damaged by natural hazards caused by climate change which leads to

$$Y = \Omega AT. \tag{6}$$

More specifically we introduce a variable hazard index $\zeta(t)$ that takes value 1 if a hazard occurs at time t and value 0 otherwise. We set

$$\Omega(t) = \begin{cases} 1, & \text{if } E(t) \le E_*, \\ 1, & \text{if } E(t) > E_* \text{ and } \zeta(t) = 0, \\ 0, & \text{if } E(t) > E_* \text{ and } \zeta(t) = 1. \end{cases} \tag{7}$$

We assume that all hazards are equal in strength and set $a \in (0, 1]$ to be a parameter characterizing the strength of a single hazard. Also we assume that at each point in time no more than one hazard may occur. We believe that the hazard index $\zeta(t)$ is a generator of the standard Poisson process describing the evolution of the number of hazards occurring over the expanding time interval $[0, t]$. Namely, for each $t \ge 0$ and each $h \in [0, \tau]$ we denote by $\eta(t, h)$ the number of hazards occurring on the time interval $(t - h, t]$ (or, equivalently, the number if instants $\tau \in (t - h, t]$ such that $\zeta(\tau) = 1$) and assume that for each $j = 0, 1, \ldots$ the probability for $\eta(t, h) = j$

is given by

$$P[\eta(t,h)=j]=\frac{(\lambda h)^j}{j!}e^{-\lambda h}. \tag{8}$$

2.3 Utility

We suppose that the society is guided by the standard utility counting the discounted integral consumption over the infinite[2] time horizon

$$\int_0^\infty e^{-\rho t}\ln[(1-u_*)Y(t)]dt$$

or, getting rid from the additive constant, equivalently

$$J[u]=\int_0^\infty e^{-\rho t}\ln Y(t)dt. \tag{9}$$

Due to the stochastic nature of $Y(t)$ (see (6) we understand an *optimal control problem* as follows: to find a piece-wise continuous control $u=u(\cdot)$: $u(t)\in[0,u_*]$ $(t\in[0,\infty))$ such that maximizes the *expected utility*, i.e.,

$$W[u]=\mathbf{E}[J[u]]\to\max_u \tag{10}$$

under (9) and dynamics (1)–(6).

Let us specify the form of the expected utility W. Obviously the life of system (1)–(6) is split into two periods: the *pre-perturbed* period $[0,\tau]$ on which $E(t)\le E_*$, catastrophes do not occur and dynamics (1)–(6) is deterministic; and the *perturbed* period (τ,∞) on which $E(t)>E_*$ and thanks to random natural catastrophes dynamics (1)–(6) becomes stochastic. Accordingly, we represent a control $u(\cdot)$ in problem (10) as a piece-wise function of the form

$$u(t)=\begin{cases}u^0(t), & t\in[0,\tau],\\ u^1(t), & t\in(\tau,\infty).\end{cases} \tag{11}$$

Consequently the expected utility W can be represented as two additive terms corresponding to these two periods:

$$W[u]=W[\tau,u^0,u^1]=J^0[u^0]+\mathbf{E}[J^1[u^1]]. \tag{12}$$

[2]One can consider this utility on the finite time horizon $[0,\theta]$. All conclusions made in what follows for the infinite time horizon case will remain for the finite time horizon case; formulas will have modified forms explicitly reflecting the quantitative dependence on θ. For the reason of simplicity of representation of the results in this paper we restrict ourselves to the infinite time horizon case.

In other words the optimal economic growth problem requires finding an optimal control on the pre-perturbed period, $u^0(\cdot)$, the switching time τ, and the optimal control on the perturbed period, $u^1(\cdot)$.

3 Optimal Production and Optimal Cleaning

In this section and in what follows we simplify the dynamics of the production and cleaning technology stocks by ignoring technology depreciation, i.e., in (1), (2) $\mu = 0$. Due to the bilinear structure of the dynamic equations (1), (2) in case $\mu > 0$ all qualitative conclusions made along the paper remain whereas quantitatively the depreciation decreases the GDP growth and thus leads to later entering the "unsafe" mode.

3.1 Perturbed Period

Let us analyze the behavior of system (1)–(6) after $E(t)$ has exceeded the critical level E_*. It turns out that regardless what is happening on the pre-perturbed period and the time moment when the system's dynamics switches from deterministic to stochastic, one finds the optimal control on the perturbed period. Theorem 1 comprises this result.

Theorem 1 *Let $u(\cdot)$ be a control optimal in problem (10) of form (11). Let $\bar{u} \leq u_*$ be the maximum control admissible for the perturbed period.*[3] *Then on the perturbed period the optimal control takes its maximum admissible value*, *i.e.*,

$$u^1(t) = \bar{u} \quad (t \in (\tau, \infty)). \tag{13}$$

A formal proof of Theorem 1 is given in Appendix B.

The fact that the optimal control in the perturbed period does not depend on the current value of aggregated emissions, $E(t)$, is a consequence of features of Poisson process and the assumption that all hazards are equal in strength. Moreover we see that as soon as the world economy abandons a "safe" zone where no natural hazards driven by industrial GHG occur, there is no economic profit any more (in our model) to prevent further increase in the atmospheric GHG. In other words we assume the environmental impact to be insensitive to the level of the aggregated GHG emissions in the "unsafe" area. This rather extreme assumption nevertheless can be accepted on rather middle time perspective.[4]

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[3]One can choose $\bar{u}$ imposing risks constraints in "unsafe" zone. It is often that a dispersion of a random variable acts as a measure of risks in a stochastic dynamics. Limiting the dispersion of the utility J after the system enters "unsafe" zone one can get additional constraints on the optimal investment policy.

[4]But see suggestions on overcoming this effect in Discussion section.

Because of Theorem 1 we are now aimed at finding an optimal control on the pre-perturbed period and the switching time τ. Let us remind that τ is a time moment when aggregated GHG emissions $E(t)$ hits the level E_*. Namely we have

Problem A Supposing that $u^1(\cdot)$ has form (13) and the economy's dynamics is given by (1)–(6) with $\mu = 0$, find a couple $(\tau, u^0(\cdot))$ such that

$$W_1[\tau, u^0] = W[\tau, u^0, u^1] \to \max_{\tau, u^0(\cdot)}$$

where W is defined by (12).

The following lemma gives an alternative formula for the utility W_1 and will allow to simplify Problem A.

Lemma 1 *Problem A is equivalent to the next optimal control problem*

$$\begin{aligned}
&W_1[\tau, u^0] \to \max_{\tau \geq 0, u^0},\\
&\dot{T}(t) = Au(t)T(t), \qquad T(0) = T_0,\\
&\dot{C}(t) = A(u_* - u^0(t))T(t), \qquad C(0) = C_0,\\
&\dot{E}(t) = \beta T(t) - \gamma C(t), \qquad E(0) = E_0, \qquad E(\tau) = E_*,\\
&u^0(t) \in [0, u_*],\\
&(t \in [0, \tau]),
\end{aligned}$$

where

$$W_1[\tau, u^0] = \int_0^\tau e^{-\rho t} \ln AT(t)dt + \frac{e^{-\rho\tau}}{\rho}\left[\ln(AT(\tau)) + \frac{A\bar{u} + \lambda \ln a}{\rho}\right]. \quad (14)$$

We provide a proof to this lemma in Appendix C.

3.2 Pre-perturbed Period

In this section we solve Problem A, construct an optimal control on the pre-perturbed period $u^0(\cdot)$ and define the optimal switching time τ which completes the process of solving problem (10).

First let us specify the deterministic dynamics of system (1)–(6). In this section for technical reason we eliminate the upper index of a control on a pre-perturbed interval, i.e., instead of u^0 we will simply write u. Since for $t \in [0, \tau]$ we have

$\Omega(t) = 1$, by (1)–(6) for an arbitrary control $u(t)$ we get

$$
\begin{aligned}
&T(t) = T_0 e^{Ap(t)},\\
&C(t) = C_0 + u_* A T_0 \int_0^t e^{Ap(s)} ds - T_0(e^{Ap(s)} - 1),\\
&E(t) = E_0 - \gamma(T_0 + C_0)t + T_0(\beta + \gamma)\int_0^t e^{Ap(s)} ds \\
&\qquad - \gamma u_* A T_0 \int_0^t \int_0^r e^{Ap(s)} ds\, dr\\
&\quad (t \in [0, \tau])
\end{aligned}
\tag{15}
$$

where

$$p(t) = \int_0^t u(s) ds. \tag{16}$$

For simplicity we normalize the production technology stock $T(t)$ by its initial value T_0 and put

$$x(t) = \frac{T(t)}{T_0} = e^{Ap(t)} \quad (t \in [0, \tau]); \tag{17}$$

introduce an auxiliary variable $y(\cdot)$:

$$y(t) = \int_0^t e^{Ap(s)} ds \quad (t \in [0, \tau]). \tag{18}$$

In terms of variables x, y and based on Lemma 1, we represent Problem A as the following optimal control Problem B:

Problem B Find a couple $(\tau, u(\cdot))$ such that

$$
\begin{aligned}
W_1[\tau, u] = &\int_0^\tau e^{-\rho t} \ln x(t) dt + \frac{e^{-\rho\tau}}{\rho}\left[\ln x(\tau) + \frac{A\bar{u} + \lambda \ln a}{\rho}\right]\\
&\to \max_{\tau \geq 0, u(\cdot)},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\dot{x}(t) = Au(t)x(t), \qquad x(0) = 1,\\
&u(t) \in [0, u_*],\\
&\dot{y}(t) = x(t), \qquad y(0) = 0,\\
&E_0 - \gamma_0\tau + \gamma_2 y(\tau) - \gamma_1 \int_0^\tau y(s) ds = E_*,\\
&(t \in [0, \tau]),
\end{aligned}
\tag{20}
$$

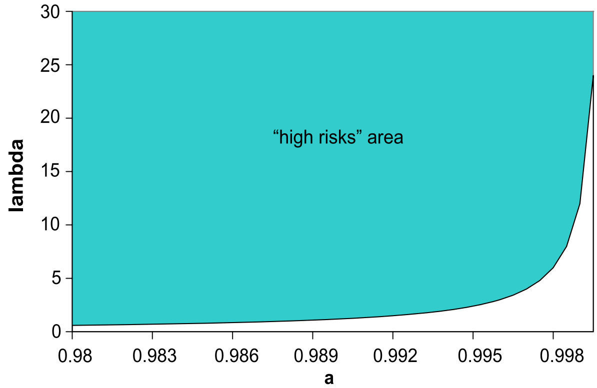

Fig. 2 "High risks" area in terms of a and λ

where

$$\gamma_0 = \gamma(C_0 + T_0),$$
$$\gamma_1 = \gamma u_* A T_0,$$
$$\gamma_2 = T_0(\beta + \gamma).$$

In what follows we assume that the probability of a single hazard, λ, as well as its percentage loss $(1 - a)$ are large enough. Namely we introduce the following

High Risks Assumption: $A\bar{u} + \lambda \ln a < 0$.

Figure 2 illustrates the area of admissible values of parameters a and λ under High Risks Assumption.

Lemma 2 *Let High Risks Assumption be satisfied. Let* $(\tau, u(\cdot))$ *be an arbitrary couple in which* $u(t)$ $(t \in [0, \tau])$ *is a control in Problem* B, $\tau \geq 0$. *Let* $y(t)$ $(t \in [0, \tau])$ *be a corresponding solution of* (20) *and*

$$E_0 - \gamma_0 \tau + \gamma_2 y(\tau) - \gamma_1 \int_0^{\tau} y(s) ds < E_*. \tag{21}$$

Then there exist a $\hat{\tau} \geq \tau$ *and a control* $\hat{u}(t)$ *extending* $u(t)$ *to* $t \in [0, \hat{\tau}]$, *that*

(1)

$$E_0 - \gamma_0 \hat{\tau} + \gamma_2 \hat{y}(\hat{\tau}) - \gamma_1 \int_0^{\hat{\tau}} \hat{y}(s) ds = E_* \tag{22}$$

where $(\hat{x}(t), \hat{y}(t))$ *is the solution of* (20) *corresponding to* $\hat{u}(t)$ $(t \in [0, \hat{\tau}])$; *and*
(2)

$$W_1[\hat{\tau}, \hat{u}] \geq W_1[\tau, u]. \tag{23}$$

Proof Let $\hat{u}(t) = 0$ for $t > \tau$. Let us show that $W_1[t, u]$ grows as t grows, starting at $t = \tau$. Indeed, taking the derivative we get

$$\frac{\partial W_1[t, u]}{\partial t} = \frac{e^{-\rho t}}{\rho}[A\bar{u} + \lambda \ln a]$$

which is positive by High Risks Assumption. Taking into account the fact that $E(t)$ grows (see Remark 1) we come to the conclusion of the lemma. □

The following theorem provides the main result of this section.

Theorem 2 *If a couple* $(\tilde{\tau}, \tilde{u}(\cdot))$ *is a solution of Problem B then control* $\tilde{u}(\cdot)$ *has necessarily a single switching point of the max-min type*, *i.e.*,

$$\tilde{u}(t) = \begin{cases} u_*, & t \in [0, \xi), \\ 0, & t \in [\xi, \tilde{\tau}], \end{cases} \tag{24}$$

for some $\xi \in [0, \tilde{\tau}]$.

Proof 1. Suppose the contrary: let an optimal couple $(\bar{\tau}, \bar{u}(\cdot))$ be not of the max-min type. $\bar{y}(\cdot)$ the solution of (20) corresponding to $\bar{u}(t)$. We assume that a trivial control $u(t) = 1$ $(t \in [0, \bar{\tau}])$ is not optimal in Problem B, i.e., $\bar{y} > \bar{\tau}$. Let us fix $\bar{y} = \bar{y}(\bar{\tau})$.

2. Let us fix $\bar{\tau}$ and consider the following optimal control problem:

Problem C find a control $u(\cdot)$ such that

$$\begin{aligned} &W_2[u] = \int_0^{\bar{\tau}} y(s)ds \to \max_{u(\cdot)}, \\ &\dot{x}(t) = Au(t)x(t), \qquad x(0) = 1, \\ &u(t) \in [0, u_*], \\ &\dot{y}(t) = x(t), \qquad y(0) = 0, \qquad y(\bar{\tau}) = \bar{y}, \\ &\quad (t \in [0, \bar{\tau}]). \end{aligned}$$

Since $\bar{\tau}$ is fixed we set equivalent $W_2[u] = W_2[\bar{\tau}, u]$. By Lemma 3 (see Appendix D) the single optimal control in Problem C is

$$\tilde{u}_C(t) = \begin{cases} u_*, & t \in [0, \xi), \\ 0, & t \in [\xi, \bar{\tau}], \end{cases} \tag{25}$$

where ξ is a single root of the equation

$$e^{Au_*\xi}\left(\frac{1}{Au_*} + \bar{\tau} - \xi\right) - \frac{1}{Au_*} = \bar{y}.$$

3. Now let us consider the following optimal control problem:

Problem D Find a control $u(\cdot)$ such that

$$W_3[u] = \int_0^{\bar{\tau}} e^{-\rho t} \ln x(t) dt + \frac{e^{-\rho \bar{\tau}}}{\rho} \left[\ln x(\tau) + \frac{A\bar{u} + \lambda \ln a}{\rho} \right] \to \max_{u(\cdot)},$$

$$\dot{x}(t) = Ax(t)u(t), \qquad x(0) = 1,$$

$$u(t) \in [0, u_*],$$

$$\dot{y}(t) = x(t), \qquad y(0) = 0, \qquad y(\bar{\tau}) = \bar{y},$$

$$(t \in [0, \bar{\tau}]).$$

Similar to the case of Problem C we set $W_3[u] = W_3[\bar{\tau}, u]$. By Lemma 4 (see Appendix D) $\tilde{u}_C(t)$ (25) is the single optimal control in Problem D.

4. Now let us consider the couple $(\bar{\tau}, \tilde{u}_C(\cdot))$. Let $(\tilde{x}(t), \tilde{y}(t))$ be the solution of (20) corresponding to $\tilde{u}_C(t)$.

The fact that a control $\tilde{u}_C(\cdot)$ (25) is optimal in Problem C leads to

$$E_0 - \gamma_0 \bar{\tau} + \gamma_2 \tilde{y}(\bar{\tau}) - \gamma_1 \int_0^{\bar{\tau}} \tilde{y}(s) ds < E_0 - \gamma_0 \bar{\tau} + \gamma_2 \bar{y}(\bar{\tau})$$
$$- \gamma_1 \int_0^{\bar{\tau}} \bar{y}(s) ds = E_*. \tag{26}$$

The fact that a control $\tilde{u}_C(\cdot)$ (25) is optimal in Problem D and equality $\tilde{y}(\bar{\tau}) = \bar{y}(\bar{\tau})$ lead to

$$W_3[\tilde{u}_C] = W_1[\bar{\tau}, \tilde{u}] > W_1[\bar{\tau}, \bar{u}] \tag{27}$$

(see the form of W_1 in (19)).

5. In a view of inequalities (26) and (27), by Lemma 2 there exist a $\hat{\tau} > \bar{\tau}$ and a control $\hat{u}(t)$ extending $\bar{u}(t)$ to $[0, \hat{t}]$ such that (22) and (23) hold. Now (23) and (27) show that the pair $(\bar{\tau}, \bar{u}(\cdot))$ is not optimal in Problem B. Thus we have arrived to the contradiction which proves the statement of the theorem. □

From Theorems 1 and 2 follows

Theorem 3 *If* $(\tau, u(\cdot))$ *is an optimal couple in the optimal economic growth problem* (10) *then*

$$u(t) = \begin{cases} u_*, & t \in [0, \xi), \\ 0, & t \in [\xi, \tau], \\ \bar{u}, & t \in (\tau, \infty), \end{cases} \tag{28}$$

where $\xi \in [0, \tau]$.

4 Global Calibration

For practical simulation we provide a calibrated version of model (1)–(10) in the global scale and run our model for 100 years time horizon starting from the year 2000. Table 1 provides calibrated values for the model's parameters. Some values are rather standard. For example, Barro and Sala-i-Martin (1995) and Nordhaus (1994) estimate the production technology intensity and the discount factor as $\rho = 0.03$ year^{-1} and $A = 4$ year^{-1}, respectively.

Here we restrict GHG to the main contributor in global warming—carbon dioxide CO_2. In the year 2000 GDP value $Y_0 = 26.7$ tril. US dollars and CO_2 atmospheric concentration $E_0 = 262$ ppm can easily be found in economic-environmental databases—see, e.g., IPCC (2007a); the size of the production technology stock in the year 2000, $T_0 = 6.6$ tril. US dollars, is calculated via (6).

The initial size of the cleaning technology stock can hardly be well estimated; we assume its value in the year 2000 to be negligibly small and put $C_0 = 0$.

We estimate the maximum resource for investment, u_* assuming that in the period preceding 2000 business as usual (BAU) strategy of investment in production has been implemented. In other words, investment in cleaning has been insufficient for substantial growth of its stock which has led to the exponential GDP growth $Y(t) = Y_0 e^{Au_*(t-2000)}$ (1), (6) for $t \leq 2000$, where Y_0 refers to the production technology stock in the year 2000. Evolving the past century world GDP statistics, available, e.g., in Maddison (1995) we regress Y on t and get u_* equal 0.3%.

When calibrating the CO_2 growth function $\beta T - \gamma C$ (4) we identify $\dot{E}(t)$ with emissions $e(t)$ ignoring natural adaptation effects. Under the assumption on BAU strategy of investment in production implemented in the period preceding 2000, i.e., for $t < 2000$ we put emission function as $e(t) = \beta T(t) = \beta T_0 e^{Au_*(t-2000)}$. We take data on global CO_2 emissions from Marland (2007) (http://cdiac.ornl.gov/trends/emis/em_cont.htm) provide regression e on t and get value for β as 0.8 Gt/(year*tril. US dollars).

Table 1 Calibrated values for the model's parameters

2000 year	Production technology intensity with respect to GDP
$\theta = 2100$ year	GDP elasticity with respect to production technology
$A = 4$ year^{-1}	Production technology intensity
$u_* = 0.003$	GDP fraction to be invested for technology development
$\rho = 0.03$ year^{-1}	Discount factor
$\beta = 0.8$ Gt/(year*tril. US dollars)	Production technology intensity with respect to emissions
$\lambda = 3.5$ year^{-1}	Expected (mean) value for annual number of catastrophes
$a = 0.9996$	Not damaged fraction of GDP as a result of each catastrophe
$Y_0 = 26.7$ tril. US dollars	Initial GDP
$T_0 = 6.7$ tril. US dollars	Initial production technology
$C_0 = 0$	Initial cleaning technology
$E_0 = 262$ ppm	Initial CO_2 atmospheric concentration

In trying to assess the Poisson distribution's (8) parameters, λ and a, we come across with a serious difficulty which appear rather often when calibrating continuous effects as discrete ones. Namely, the nature of the impact of the increase in atmospheric GHG on natural hazards is not very well explored. However, both the frequency of and the damage from them are expected to increase gradually while GHG concentration increases. Instead, in this work we simplify this complex gradual dependence into a switch between two extreme modes: a "safe" mode when no hazards occur (if $E(t) < E_*$) and a "dangerous" mode when hazards occur with a constant frequency λ and damage $1-a$ (if $E(t) > E_*$). We assume that initially (in the year 2000) the system is in the "safe" mode. The latter assumption implies that the frequency and loss parameter values, λ and a, are to be calibrated for a period in which the impact of hazards is ignorably small. Nevertheless we find average loss from each catastrophe to be 0.02% and average number of catastrophes to be 9 per year for the "safe" mode (IPCC 2007b). For simulations we will vary the values of λ and a around these estimates carrying out the sensitivity analysis of the model's outcome to the input uncertainty in Poisson process parameters.

On the same reason, it turns out to be not possible to calibrate the critical level E_* and cleaning technology intensity γ since the society has not come across with a "dangerous" mode of the environmental behavior so far. This fact adds E_* and γ to the number of uncertain parameters in the model.

5 Optimal GDP and Optimal GHG

Let us first calculate the optimal utility W (12). Substituting the form of the optimal control u (24) in the utility W_1 (19) we find that the optimal utility value has the form

$$W = W[\xi, \tau] = \frac{e^{-\rho\tau}}{\rho^2}[A\bar{u} + \lambda \ln a] + \frac{Au_*}{\rho^2} - \frac{e^{-\rho\xi}}{\rho^2} + \frac{\ln Y_0}{\rho}, \tag{29}$$

here $\tau > 0$ is the point in time at which the accumulated emission hits the critical level E_* and $\xi \in [0, \tau]$ is the switching time for the optimal control in Problem B. The optimal couple (ξ, τ), determining the optimal control (24) maximizes $W[\xi, \tau]$ under the constraints $\xi \geq 0$, $\tau \geq \xi$, $E(\tau) = E_*$. Given a $\xi \geq 0$ we find $\tau = \tau(\xi)$ from

$$\begin{aligned} &\frac{\beta T_0}{Au_*}[e^{Au_*\xi} - 1] - \gamma C_0\xi + (\beta T_0 e^{Au_*\xi} - \gamma C_0)(\tau - \xi) - \gamma u_* Y_0 e^{Au_*\xi}\frac{(\tau-\xi)^2}{2} \\ &\quad = E_* - E_0. \end{aligned} \tag{30}$$

Hence, in the optimal couple (ξ, τ) we have $\tau = \tau(\xi)$ and ξ is found as the solution to the one-dimensional optimization problem

$$W[\xi, \tau(\xi)] \to \max_{\xi \geq 0}.$$

Now let us give and analyze the optimal paths in Problem A. From (24) we see that the pre-perturbed period $[0, \tau]$ is split into two sub-periods: a period of intense production $[0, \xi]$ and a subsequent period $(\xi, \tau]$, at which special abatement measures on reducing GHG emissions are implemented. In period $[0, \xi]$ the optimal production technology stock exponentially grows and after $t = \xi$ it stabilizes:

$$T(t) = \begin{cases} T_0 e^{Au_* t}, & t \in [0, \xi], \\ T_0 e^{Au_* \xi}, & t \in (\xi, \tau]. \end{cases} \tag{31}$$

The optimal GDP is developing proportionally to the production technology stock with a coefficient A.

In period $[0, \xi]$ cleaning technology develops according to BAU strategy. After $t = \xi$ it grows linearly:

$$C(t) = \begin{cases} C_0, & t \in [0, \xi], \\ C_0 + u_* Y_0 e^{Au_* \xi}(t - \xi), & t \in (\xi, \tau]. \end{cases} \tag{32}$$

In period $[0, \xi]$, because of exponentially increasing production and BAU cleaning, the atmospheric GHG stock grows exponentially with the rate $\beta T_0 e^{Au_* t} - \gamma C_0$; in the subsequent period $(\xi, \tau]$ in spite of the fact that intense production is not being developed any more and all resources are invested in cleaning, the atmospheric GHG continue growing with the linearly decreasing rate $\beta T_0 e^{Au_* \xi} - \gamma C_0 - \gamma u_* Y_0 e^{Au_* \xi}(t - \xi)$ until they reach the critical level E_* at $t = \tau$:

$$E(t) = \begin{cases} E_0 - \gamma C_0 t + \frac{\beta T_0}{Au_*}[e^{Au_* t} - 1], \quad t \in [0, \xi], \\ E(\xi) + [\beta T_0 e^{Au_* \xi} - \gamma C_0](t - \xi) - \gamma u_* Y_0 e^{Au_* \xi} \frac{(t-\xi)^2}{2}, \\ \quad t \in (\xi, \tau]. \end{cases} \tag{33}$$

As the society enters the "dangerous" zone, natural hazards start to occur randomly. The optimal trajectories for the production technology stock, GDP, cleaning technology stock and GHG stock become stochastic. Due to the structure of the Poisson distribution describing the occurrence of natural hazards, we assess the values for these variables in nodes of a time grid only. We choose a time grid with a step δ (say, one year)

$$\{t_k\}_{k=0,1,\ldots}: \quad t_0 = \tau, \quad t_k = t_0 + k\delta. \tag{34}$$

In accordance with (1), (6) for every realization $(w_0, \ldots, w_k)$ we get

$$Y_{k+1} = w_k Y_k + A\bar{u} w_k Y_k \delta = w_k Y_k [1 + A\bar{u}\delta],$$

hence

$$Y_k = Y_\tau (1 + A\bar{u}\delta)^k \prod_{i=0}^{k-1} w_i = Y_\tau (1 + A\bar{u}\delta)^k a^{\eta_0 + \cdots + \eta_k}, \tag{35}$$

where $Y_\tau = Y_0 e^{Au_* \xi}$, is the value of the optimal GDP at time ξ the point of leaving a "safe" zone, $\eta_0, \ldots, \eta_k$ are numbers of catastrophes which occur in each year up to the year t_k. The latter formula holds because due to (8) the damage ratio each year does not depend on the number of the year i and is given by $1 - w_i = 1 - a^{\eta_i}$. Let us notice that the randomly damaged GDP in year t_k (35) can be written as

$$Y_k = Y_\tau e^{Au_* t_k} a^{\eta_0 + \cdots + \eta_k} + O(\delta), \tag{36}$$

where $O(\delta) \to 0$.

For $\eta_0 + \cdots + \eta_k = j$ $(j = 0, 1, \ldots)$ formula (35) defines a spectrum of the optimal GDP at year t_k

$$Y_{kj} = Y_\tau (1 + A\bar{u}\delta)^k a^j \quad (j = 0, 1, \ldots) \tag{37}$$

with corresponding probabilities

$$P_{kj} = P[Y_k = Y_{kj}] = \frac{(\lambda\delta k)^j}{j!} e^{-\lambda\delta k} \quad (j = 0, 1, \ldots). \tag{38}$$

From (36) we see that in the year t_k natural hazards reduce the deterministic annual GDP $Y_\tau e^{A\bar{u}t_k}$ for the fraction $1 - a^{\eta_0 + \cdots + \eta_k}$.

The expected optimal GDP is then

$$\begin{aligned} \mathbf{E}[Y_k] &= Y_\tau (1 + A\bar{u}\delta)^k \mathbf{E}[a^{\eta_0 + \cdots + \eta_k}] \\ &= Y_\tau (1 + A\bar{u}\delta)^k \sum_{i=0}^{\infty} a^i \frac{(\lambda\delta k)^i}{i!} e^{-\lambda\delta k} \\ &= Y_\tau (1 + A\bar{u}\delta)^k e^{-\lambda\delta k(1-a)}. \end{aligned} \tag{39}$$

Let us notice that the production technology stock obeys to the same probabilistic distribution as the GDP Y, i.e.,

$$\begin{aligned} T_{kj} &= T_\tau (1 + A\bar{u}\delta)^k a^j \quad \text{with the probability given by (38)}, \\ \mathbf{E}[T_k] &= T_\tau (1 + A\bar{u}\delta)^k e^{-\lambda\delta k(1-a)}. \end{aligned}$$

Since $u(t) = u_*$ in the "unsafe" mode, from (2) we get that the cleaning technology stock remains constant, i.e., $C_k = C_\tau$.

The GHG stock in year t_k and its expectation are given by

$$E_k = E_\tau - \gamma k\delta C_\tau + \beta\delta T_\tau \sum_{i=0}^{k} (1 + A\bar{u}\delta)^i a^{\eta_0 + \cdots + \eta_i}, \tag{40}$$

$$\mathbf{E}[E_k] = E_\tau - \gamma k\delta C_\tau + \beta\delta T_\tau \frac{(1 + A\bar{u}\delta)^{k+1} e^{-\lambda\delta(1-a)(k+1)} - 1}{(1 + A\bar{u}\delta) e^{-\lambda\delta(1-a)} - 1}. \tag{41}$$

In section Sensitivity analysis one can find a calibrated version of model (1)–(10), numerically calculated optimal trajectories for the state variables and sensitivity analysis of the model's output to the uncertain parameters.

6 Sensitivity Analysis

We us note that the life of the modeled system has two important time points, ξ and τ, switching time from intense production to intense cleaning, and starting time of catastrophes, respectively. Let us analyze them in terms of input uncertainties in β, E_*, λ, and a.

First, let us specify the procedure of maximizing W (29). We find $\tau(\xi)$ from (30). Notice that the longer the period of intense production $[0, \xi]$, the shorter the period of intense cleaning $[\xi, \tau]$, and, moreover, the shorter the whole "safe" mode period $[0, \tau]$.

Let us notice also that the switching ξ is constrainted both from above and from below. On one hand, the duration of intense production period, ξ, can not be too long—it is limited by the condition $\xi \le \tau$. On the other hand, ξ should be long enough to guarantee that by the time τ GHG concentration $E(t)$ approaches the critical level E_*. Therefore,

$$\xi \in [\xi_{\min}, \xi_{\max}],$$

where

$$\xi_{\min} = \max\left\{0, \frac{1}{Au_*} \ln \frac{2\gamma Au_*(E_* - E_0 + \frac{\beta T_0}{Au_*})}{\beta T_0(\beta + 2\gamma)}\right\}, \tag{42}$$

$$\xi_{\max} = \frac{1}{Au_*} \ln\left[1 + (E_* - E_0)\frac{Au_*}{\beta T_0}\right]. \tag{43}$$

Notice then, that assumption (3) implies $\xi_{\min} = 0$. Finally, we maximize W (29) as $\xi \in [0, \xi_{\max}]$. From (43) we see that the area of admissible ξ depends neither on a and λ, nor on γ, logariphmically expanding with respect to E_* (see (43)). Thus, if E_* equals to 393 ppm (1.5 times of E_0) one will choose the optimal switching time between the year 2000 and the year 2022, whereas if E_* equals to 524 ppm (doubled value of E_0) the upper limit for the interval for choosing the optimal switching time becomes 2039.

Now let us discuss the optimal choice of investment policy in the "safe" mode, i.e., the optimal time moments ξ and τ. The simulations show that there may be two principally different situations.

(i) *Low damages from catastrophes and high cost of cleaning*.

The aggregated damage of natural catastrophes over the whole perturbed period is less than the aggregated loss in the GDP due to special investment in cleaning. It means that it is not profitable to develop cleaning technology and the optimal investment strategy prescribes to allocate all resources in developing production. Then optimal time of starting catastrophes is given by $\tau = \xi = \xi_{\max}$. As we already mentioned, τ depends neither on a and λ, nor on γ, logariphmically growing with respect to E_* (see Fig. 4).

Both the optimal GDP and the optimal GHG concentration grow exponentially if $t \in [0, \tau]$ (see (31) and (33))[5] independently on E_*, γ, λ and a. The expected GDP and the expected GHG concentration values for $t > \tau$ exponentially depend on $-\lambda(1-a)$ (see (39) and (41)). Figure 3 illustrates.

(ii) *High damages from catastrophes and high cleaning efficiency.*

The cleaning technology stock slows down the exponential growth of GHG concentration in the atmosphere and postpones the time when the system enters the "unsafe" mode and catastrophes start to damage the fraction of the GDP. Abridgement of the perturbed period leads to decrease of the aggregated GDP losses and hence cleaning becomes profitable. Then the optimal investment strategy prescribes to start cleaning right in beginning of the modeling period. Then and $\xi = 0$,

$$\tau = \frac{\beta T_0 - \sqrt{\beta^2 T_0^2 - 2\gamma u_* Y_0 (E_* - E_0)}}{\gamma u_* Y_0}.$$

From the latter formula we see that the optimal time for starting natural catastrophes τ depends sensitively on E_*: the higher critical level of GHG concentration (i.e., the bigger the "safe" mode), the later catastrophes start to occur with increasing return to scale. The optimal time for starting natural catastrophes τ is rather insensitive to γ, but the increase the cost of cleaning technology leads to a slight increase in τ. Comparison with the case (i) shows that catastrophes start later in case (ii) then in case (i) for all values of critical GHG level E_* (see Fig. 4).

Since all the investment is allocated in developing cleaning, the optimal GDP does not grow keeping its initial value Y_0 during the whole "safe" mode (31). The optimal cleaning technology stock grows linearly independently on E_*, γ, λ and

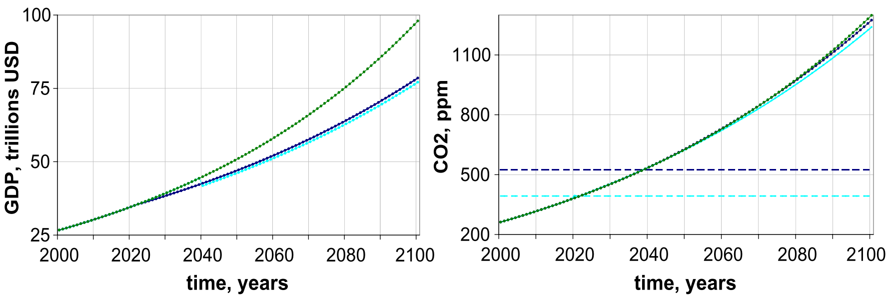

Fig. 3 (Color online) Case (i): optimal GDP trajectories (*left plot*) and optimal trajectories for GHG concentration in the atmosphere (*right plot*). *The upper, dark-green curves* correspond to $Y(t)$ and $E(t)$ trajectories in case of BAU production. *The dark-blue* and *light-blue curves* correspond to optimal $Y(t)$ and $E(t)$ for $E_* = 1.5E_0 = 393$ ppm and $E_* = 2E_0 = 524$ ppm, respectively. Parameters' values: $\gamma = 0.1$ ppm/year*tril. US dollars, $a = 0.9996$, $\lambda = 3.5$

[5]Let us remind that in our simulations $C_0 = 0$.

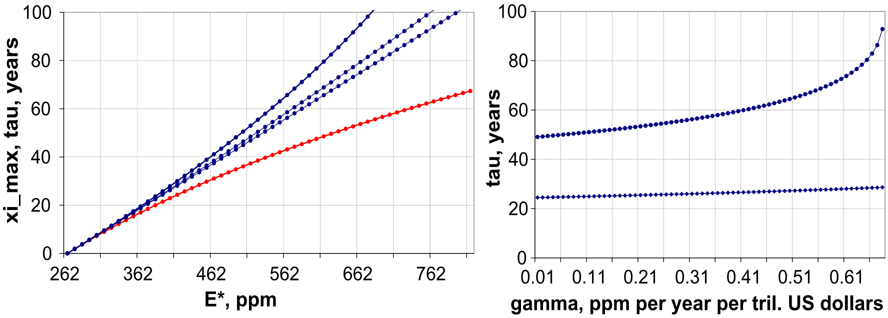

Fig. 4 (Color online) Optimal time moments of starting catastrophes. *The left plot* gives values for τ with respect to uncertain E_* for the case (i)—*the lower red curve*, and for the case (ii)—three *dark-blue curves*, *the upper* of which corresponds to $\gamma = 0.3$ ppm/year*tril. US dollars, *the middle curve* corresponds to $\gamma = 0.1$ ppm/year*tril. US dollars, *the lower curve* corresponds to $\gamma = 0.01$ ppm/year*tril. US dollars. *The right plot* gives values for τ in case (ii) with respect to uncertain γ for $E_* = 1.5E_0 = 526$ ppm (*the lower curve*) and $E_* = 2E_0 = 526$ ppm (*the upper curve*)

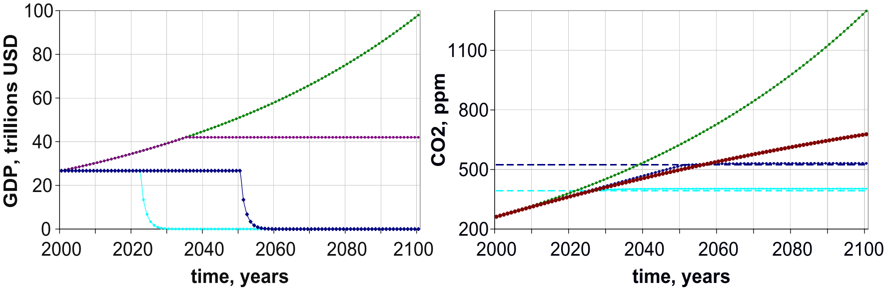

Fig. 5 (Color online) Case (ii): optimal GDP trajectories (*left plot*) and optimal trajectories for GHG concentration in the atmosphere (*right plot*). *The upper, dark-green curves* correspond to $Y(t)$ and $E(t)$ trajectories in case of BAU production. *The dark-blue* and *light-blue curves* correspond to optimal $Y(t)$ and $E(t)$ for $E_* = 1.5E_0 = 393$ ppm and $E_* = 2E_0 = 524$ ppm, respectively. *The dark-red curve* on the right plot illustrates $E(t)$ trajectory in case of intense cleaning. Parameters' values: $\gamma = 0.1$ ppm/year*tril. US dollars, $a = 0.95$, $\lambda = 14$

a (32). Because of that, the optimal GHG concentration grows, and its rate is negatively affected by γ (see (33)).

The expected GDP dramatically falls down damaged by catastrophes (39). Thanks to that the expected GHG concentration values stabilizes as soon as the system passes $t = \tau$ (41). Figure 5 illustrates.

7 Discussion

Let us start this section with discussion of what in fact new do we learn from explicit modeling of random hazards and their damages. Why not restrict ourselves to expected (mean) damages and cancel complicated stochastic dynamics and optimization in this problem? Generally speaking, these two ways are about an order of taking an expectation and non-linear instantaneous utility $f(Y)$. Namely, if case we consider stochastic dynamics, the utility to be maximized have a form $E[\int_0^\infty e^{-\rho t} f(Y(t))dt]$ whereas in case of equivalent deterministic dynamics the utility takes a formula $\int_0^\infty e^{-\rho t} f(E[Y(t)])dt]$. Owing to non-linearity, not only values of these two functionals are be different for the same control which determines $Y(t)$, but also their properties related to optimization may not provide the same optimizer for both functionals.[6]

Second concern with respect to the results presented in this paper is a choice of Poisson distribution for the number of catastrophes in "unsafe" zone. Poisson distribution is often used for modeling events which occur with a known average rate, and which are independent in time since the last event. A classic example is the nuclear decay. At the same time, it leads to a rather strong assumption on independence of the number catastrophes on the current level of atmospheric GHG provided the system exceeds the critical level E_*. On such a middle time horizon as 20–50 years we might accept that as a zero approximation. A way to overcome this problem might be in introducing a number of critical levels $E_*^1, E_*^2, \dots, E_*^n$, and corresponding "unsafe" zones. In each zone the random number of natural hazards is distributed according to Poisson distribution which should have specific parameters' values (λ and a): both the damage and the mean annual number of catastrophes should increase with an increase of E_*^i.

Let us summarize results which are obtained in this work:

(1) We considered a one-sector economic growth model with production technology as a key driver of the economy and cleaning technology which is used for retraining greenhouse gases from the atmosphere. The important factor in the model—random natural hazards damaging the current GDP provided the atmospheric GHG level is high. We choose the utility as an expectation of the integrated discounted consumption. We formulated a problem of optimization of the economic growth on the infinite time horizon with respect to the utility. The optimal investment policy in production and cleaning is to be found.

(2) We found the analytic structure of the optimal investment. It turns out that one should switch an optimal control mode twice. One should start from intense developing of production providing zero investment in cleaning. GHG concentration is growing exponentially at that stage. The first switching point, ξ, opens a period of intense cleaning when the rate of increase of GHG in the atmosphere slows down. However at a time moment τ the system enters the "unsafe" mode, and catastrophes start to occur. In this period one invests all admissible resources in production.

[6]For precise quantitative estimate of the difference in consequences of optimal decisions made by means of these two different functionals, one should specially consider the corresponding deterministic optimization problem which is out of the goals of this paper.

(3) We calibrated the model based on data available. Since the "unsafe" mode is only assumed to happen in the future, we reveal uncertainty in values of the critical level E_* of GHG concentrations in the atmosphere above which catastrophes affect the GDP significantly, of the cleaning efficiency, as well as of distribution parameters of random hazards.
(4) It turns out that depending on the correlation between parameters of random catastrophes, a and λ and costs and efficiency of cleaning γ, two extreme case may hold. The first corresponds to the case of law damages from catastrophes and high cost of cleaning. Then the damage from catastrophes is less significant than investment in cleaning. Thus it is optimal to allocate all admissible resources for developing production, providing zero investment in cleaning. The system approaches the critical level E_* fast, and, hence, catastrophes start to harm early which is nevertheless compensated by a relatively law damages.

The second case corresponds to a completely opposite case—high damages from catastrophes and high cleaning efficiency. In this case one should start cleaning as soon as possible, i.e., with the initial time moment of modeling. Because of high eventual damages from catastrophes it is optimal to postpone the time of starting catastrophes as long as possible. This idea implies zero investment in production and intensive developing of cleaning. Zero economic growth in the beginning is expected to be compensated by saving the GDP from catastrophes. Nevertheless, calculations show dramatic decrease of the GDP due to catastrophes in the "dangerous" mode.

The presented work acts as a step toward understanding how random natural hazards impact the technological development. Even under rather significant simplifications and strong assumptions made in this research, it reveals the eventual bifurcation of optimal dealing with economic growth harmed by natural hazards. Further quantitative and qualitative analysis of alternative hypothesis on both economic model and catastrophes regularities, as well as data analysis are needed to specify or refute them.

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Appendix A: Assumption (3)

Let us consider the best-case scenario for emissions, i.e., a control

$$u(t) = 0 \quad (t \geq 0).$$

Then

$$T(t) = T_0 \quad \text{and} \quad C(t) = C_0 + u_* Y_0 t$$

for $t \geq 0$. Emissions become

$$E(t) = E_0 + (\beta T_0 - \gamma C_0)t - \frac{\gamma u_* Y_0 t^2}{2}.$$

In this case $E(\cdot)$ is a quadratic function which increases from $t = 0$, approaches its maximum value

$$E_{\max} = E_0 + \frac{(\beta T_0 - \gamma C_0)^2}{2\gamma u_* Y_0}$$

at

$$t = \frac{\beta T_0 - \gamma C_0}{\gamma u_* Y_0}$$

and then decreases. Thus letting E_* be less then $E_{\max}$ we get increasing trajectories of emissions $E(t)$ for all admissible controls.

Appendix B: Proof of Theorem 1

In this section we provide a proof for Theorem 1. First we discretesize model (1)–(7) on the perturbed period $t \geq \tau$. We introduce a discrete time grid

$$\{t_k\}_{k=0,1,\ldots}: \quad t_0 = \tau, \quad t_k = t_0 + k\delta \tag{B.1}$$

with a small positive time step δ.

According to (7) a random fraction of the production loss in each period $[t_i, t_{i+1}]$ becomes

$$\omega_i = a^{\eta_i} \tag{B.2}$$

where η_i is a random number of natural hazards which occur over a time interval $[t_i, t_{i+1}]$. Let $(u_0, u_1, \ldots)$ be an approximation of a control $u^1(t)$ $(t \in [\tau, \infty))$ (11). Model's dynamics (1)–(4) becomes

$$T_{i+1} = w_i T_i + A u_i \omega_i T_i \delta, \qquad T_0 = T_\tau, \tag{B.3}$$

$$C_{i+1} = C_i + A(u_* - u_i)\omega_i T_i \delta, \qquad C_0 = C_\tau, \tag{B.4}$$

$$E_{i+1} = E_i + (\beta T_i - \gamma C_i)\delta, \qquad E_0 = E_\tau, \tag{B.5}$$

$$Y_i = A w_i T_i \tag{B.6}$$

$$(i = 0, 1, \ldots)$$

where $T_\tau = T(\tau)$, $C_\tau = C(\tau)$ and $E_\tau = E(\tau)$ are non-perturbed values of the production technology stock, cleaning technology stock and GHG stock at the moment $t = \tau$ at which the system leaves a non-perturbed zone.

Correspondingly taking into account (B.6), (B.3) the utility on the perturbed period J^1 (see (12)) becomes

$$\begin{aligned} J^1 &= \sum_{i=0}^{\infty} e^{-\rho(\tau+t_i)} \ln Y_i \delta \\ &= \sum_{i=0}^{\infty} e^{-\rho(\tau+t_i)} \ln(A w_i T_i) \delta \\ &= c_*^1(\delta) + e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i \delta} [\ln \omega_i + \ln T_i] \delta \end{aligned}$$

where

$$\begin{aligned} c_*^1(\delta) &= \ln A e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i \delta} \delta \\ &= \ln A e^{-\rho\tau} \frac{\delta}{1 - e^{-\rho\delta}}. \end{aligned} \tag{B.7}$$

Note that

$$c_*^1(\delta) \to \ln A \frac{e^{-\rho\tau}}{\rho} \quad \text{as } \delta \to 0.$$

Clearly,

$$\mathbf{E} J^1 = c_*^1(\delta) + e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i \delta} \mathbf{E}[\ln \omega_i + \ln T_i] \delta. \tag{B.8}$$

Let us specify the latter formula. From (B.3) we get

$$\begin{aligned} T_i &= T_\tau \prod_{k=0}^{i-1} (1 + A u_k \delta) \prod_{k=0}^{i-1} \omega_k \\ &= T_\tau \prod_{k=0}^{i-1} e^{A u_k \delta} \prod_{k=0}^{i-1} \omega_k + O(\delta) \\ &= T_\tau e^{A p_i} \prod_{k=0}^{i-1} \omega_k + O(\delta) \end{aligned}$$

where

$$p_i = \sum_{k=0}^{i-1} u_k \delta \tag{B.9}$$

and

$$O(\delta) \to 0 \quad \text{as } \delta \to 0.$$

Hence,

$$\ln T_i = \ln T_\tau + Ap_i + \sum_{k=0}^{i-1} \ln \omega_k + O(\delta)$$

and

$$\mathbf{E} \ln T_i = \ln T_\tau + Ap_i + \sum_{k=0}^{i-1} \mathbf{E} \ln \omega_k + O(\delta). \tag{B.10}$$

By (B.2) and (8)

$$\begin{aligned}
\mathbf{E} \ln \omega_k &= \sum_{j=0}^{K} \frac{(\lambda\delta)^j}{j!} e^{-\lambda\delta} \ln a_j \\
&= \sum_{j=0}^{K} \frac{(\lambda\delta)^j}{j!} j e^{-\lambda\delta} \ln a \\
&= \sum_{j=1}^{K} \frac{(\lambda\delta)^j}{(j-1)!} e^{-\lambda\delta} \ln a \\
&= \lambda\delta \ln a + O(\delta).
\end{aligned}$$

Substituting in (B.10) we get

$$\mathbf{E} \ln T_i = \ln T_\tau + Ap_i + i\lambda\delta \ln a + O(\delta). \tag{B.11}$$

Coming back to (B.8) we find that

$$\begin{aligned}
\mathbf{E} J^1 &= c_*^1(\delta) + e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i\delta} \mathbf{E}[\ln \omega_i + \ln T_i]\delta \\
&= c_*^1(\delta) + e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i\delta} [\lambda\delta \ln a + \ln T_\tau + Ap_i + (i-1)\delta\lambda \ln a + O(\delta)]\delta \\
&= c_*^1(\delta) + e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i\delta} [\ln T_\tau + Ap_i + i\delta\lambda \ln a]\delta + O(\delta) \\
&= C_*^1(\delta) + e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i\delta} (Ap_i + i\delta\lambda \ln a)\delta + O(\delta)
\end{aligned} \tag{B.12}$$

where

$$C^1_*(\delta) = c^1_*(\delta) + e^{-\rho\tau} \sum_{i=0}^{\infty} e^{-\rho i \delta} \ln T_\tau \delta$$
$$= c^1_*(\delta) + \ln T_\tau e^{-\rho\tau} \frac{\delta}{1 - e^{-\rho\delta}}.$$

Note that

$$C^1_*(\delta) \to \ln Y_\tau \frac{e^{-\rho\tau}}{\rho} \quad \text{as } \delta \to 0.$$

From (B.12) we see that regardless the value of τ and the grid step δ, $\mathbf{E}J^1$ approaches its maximum if each p_i $(i = 0, 1, \ldots)$ takes the maximum value. In other words $u_k = \bar{u}$ $(k = 0, 1, \ldots)$ brings the maximum value to $\mathbf{E}J^1$. Passage to a limit as $\delta \to 0$ finishes proving of the statement of the Theorem.

Appendix C: Proof of Lemma 1

In this section we provide a proof for Lemma 1. In other words we are aimed at specification of a form of the utility $W = J^0 + \mathbf{E}J^1$ (12).

Let us specify $\mathbf{E}J^1$. From (B.12) (see Appendix B) by passage to a limit as $\delta \to 0$ we get

$$\mathbf{E}J^1 = \int_\tau^\infty e^{-\rho t} \mathbf{E} \ln Y(t) dt \tag{C.1}$$

where

$$\mathbf{E}Y(t) = Y_\tau e^{Ap^1(t) + \lambda \ln a (t-\tau)} \quad (t \in [\tau, \infty)).$$

According to the Theorem 1 for $t \geq \tau$ $u^1(t) = \bar{u}$ and

$$p^1(t) = \bar{u}(t - \tau) \quad (t \in [\tau, \infty)).$$

Hence

$$\mathbf{E}J^1 = \int_\tau^\infty e^{-\rho t}[\ln Y_\tau + (A\bar{u} + \lambda \ln a)(t - \tau)]dt$$
$$= \frac{e^{-\rho\tau}}{\rho}\left(\ln Y_\tau + \frac{A\bar{u} + \lambda \ln a}{\rho}\right). \tag{C.2}$$

Finally we obtain

$$W = J^0 + \mathbf{E}J^1$$
$$= \int_0^\tau e^{-\rho t} \ln Y(t) dt + \frac{e^{-\rho\tau}}{\rho}\left(\ln Y_\tau + \frac{A\bar{u} + \lambda \ln a}{\rho}\right). \tag{C.3}$$

Appendix D: Lemmas 3 and 4

Lemma 3 *The control optimal in the Problem C is*

$$\tilde{u}_C(t) = \begin{cases} u_*, & t \in [0, \xi], \\ 0, & t \in (\xi, \bar{\tau}], \end{cases} \tag{D.1}$$

where ξ is a root of an equation

$$e^{Au_*\xi}\left(\frac{1}{Au_*} + \bar{\tau} - \xi\right) - \frac{1}{Au_*} = \bar{y}. \tag{D.2}$$

Proof We apply standard Pontryagin maximum principle to find a control optimal in Problem C. Let $\psi_1(\cdot)$, $\psi_2(\cdot)$ be adjoint variables. The Hamiltonian becomes

$$H(x, y, \psi_1, \psi_2) = y + \psi_1 Aux + \psi_2 x$$

and the Hamilton system supplying a solution of Problem C is

$$\dot{\psi}_1 = -Au\psi_1 - \psi_2, \qquad \psi_1(\bar{\tau}) = \bar{y}, \tag{D.3}$$

$$\dot{\psi}_2 = -1, \tag{D.4}$$

$$\dot{x} = Aux, \qquad x(0) = 1, \tag{D.5}$$

$$\dot{y} = x, \qquad y(0) = 0, \qquad y(\bar{\tau}) = \bar{y}. \tag{D.6}$$

The maximum condition becomes

$$u(t) = \begin{cases} u_*, & \text{if } \psi_1(t) > 0, \\ \in [0, u_*], & \text{if } \psi_1(t) = 0, \\ 0, & \text{if } \psi_1(t) < 0. \end{cases}$$

From (D.4) we have

$$\psi_2(t) = \psi_2^0 - t$$

with unknown initial value ψ_2^0. Then (D.3) becomes

$$\dot{\psi}_1 = -Au\psi_1 + (t - \psi_2^0)$$

and hence

$$\psi_1(t) = \frac{1}{x(t)}\left[\psi_1^0 + \int_0^t x(s)(s - \psi_2^0)ds\right].$$

Let us analyze the behavior of $\psi_1(t)$ in terms of its positiveness/negativeness. Since $x(t) > 0$ for all admissible controls $u(\cdot)$ and all $t \in [0, \bar{\tau}]$ we focus on the expression in the square brackets only. Consider a function

$$t \mapsto \phi_1(t) = \psi_1^0 + \int_0^t x(s)(s - \psi_2^0)ds$$

whose derivative is

$$\dot{\phi}_1(t) = x(t)(t - \psi_2^0) \tag{D.7}$$

and necessarily

$$\phi_1(\bar{\tau}) = 0.$$

From (D.7) it follows that in Problem C there is no special modes (controls for which $\psi_1(t) = 0$ more then in one point). In other words an optimal control takes only its extreme values. Next, from (D.7) we see that the derivative $\dot{\phi}_1(t)$ changes its sign not more then in one point on $[0, \bar{\tau}]$. It means that necessarily the following cases satisfy to the optimality conditions provided by Pontryagin maximum principle:

(i) $\psi_1^0 > 0$ and $\psi_1(t) > 0$ for all $t < \bar{\tau}$ and $\psi_1(\bar{\tau}) = 0$; then

$$u(t) = u_* \quad (t \in [0, \bar{\tau}]);$$

(ii) $\psi_1^0 > 0$ and $\psi_1(t)$ changes its sign at some $t \in (0, \bar{\tau})$, namely, $\psi_1(t) > 0$ for $t \in [0, \xi)$ and $\psi_1(t) < 0$ for $t \in (\xi, \tau]$, and $\psi_1(\bar{\tau}) = 0$; then

$$u(t) = \begin{cases} u_*, & t \in [0, \xi), \\ 0, & t \in [\xi, \bar{\tau}]; \end{cases}$$

(iii) $\psi_1^0 \le 0$ and $\psi_1(t) < 0$ for all $t < \bar{\tau}$ and $\psi_1(\bar{\tau}) = 0$; then

$$u(t) = 0 \quad (t \in [0, \bar{\tau}]).$$

Control (iii), clearly, does not satisfy the condition $\bar{y} > \bar{\tau}$; depending on the value of $\bar{y}$ extreme control (i) may be not admissible in Problem C. Otherwise it become a particular case of control (ii). Thus the two-stair control (ii) generalize the structure of an optimal control in Problem C. The switching time ξ is determined in such a way that the edge condition $y(\bar{\tau}) = \bar{y}$ holds which leads to (D.2). The lemma is proved. □

Lemma 4 *The control optimal in Problem D is $\tilde{u}_C(t)$ $(t \in [0, \bar{\tau}])$ (D.1) (see Lemma 3).*

Proof First, let us notice that since the final time $\bar{\tau}$ is fixed, the goal function W_3 can be modified. Namely, we have

$$\ln x(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{\dot{x}(s)}{x(s)} ds + \ln x_0 = \int_0^{\bar{\tau}} Au(s)ds + \ln x_0$$

and hence Problem D is equivalent to the optimal control problem of maximization of the goal function

$$W_3'[u] = \int_0^{\bar{\tau}} [e^{-\rho t} \ln x(t) + Au(t)]dt$$

under the same dynamics.

Similar to the proof of the Lemma 3 we apply standard Pontryagin maximum principle to find a control optimal in the Problem D. Let $\psi_1(\cdot)$, $\psi_2(\cdot)$ be adjoint variables. The Hamiltonian becomes

$$H(t, x, y, \psi_1, \psi_2) = e^{-\rho t} \ln x + Au + \psi_1 Aux + \psi_2 x$$

and the Hamiltonian system supplying a solution of the Problem C is

$$\dot{\psi}_1 = -Au\psi_1 - \psi_2 + \frac{R}{x^2} - \frac{e^{-\rho t}}{x}, \qquad \psi_1(\bar{\tau}) = \bar{y}, \tag{D.8}$$

$$\dot{\psi}_2 = 0, \tag{D.9}$$

$$\dot{x} = Aux, \qquad x(0) = 1, \tag{D.10}$$

$$\dot{y} = x, \qquad y(0) = 0, \qquad y(\bar{\tau}) = \bar{y}. \tag{D.11}$$

The maximum condition becomes

$$u(t) = \begin{cases} u_*, & \text{if } \psi_1(t) > 0, \\ \in [0, u_*], & \text{if } \psi_1(t) = 0, \\ 0, & \text{if } \psi_1(t) < 0. \end{cases}$$

From (D.9) we have

$$\psi_2(t) = \psi_2^0$$

with unknown initial value ψ_2^0. Then (D.8) becomes

$$\dot{\psi}_1 = -Au\psi_1 + \left(\psi_2^0 + \frac{e^{-\rho t}}{x}\right)$$

and hence

$$\begin{aligned} \psi_1(t) &= \frac{1}{x(t)}\left(\psi_1^0 + \int_0^t \left(e^{-\rho s} + \psi_2^0 x(s)\right) ds\right) \\ &= \frac{1}{x(t)}\left(\psi_1^0 - \frac{1 - e^{-\rho t}}{\rho} + \psi_2^0 y(t)\right). \end{aligned}$$

Let us analyze the behavior of $\psi_2(t)$ in terms of its positiveness/negativeness. Since $x(t) > 0$ for all admissible controls $u(\cdot)$ and all $t \in [0, \bar{\tau}]$ we focus on the expression in the square brackets only. Consider a function

$$t \mapsto \phi_2(t) = \psi_1^0 - \frac{1 - e^{-\rho t}}{\rho} + \psi_2^0 y(t)$$

whose derivative is

$$\dot{\phi}_2(t) = -e^{-\rho t} + \psi_2^0 x(t). \tag{D.12}$$

and necessarily

$$\phi_2(\bar{\tau}) = 0.$$

Since $e^{-\rho t}$ decreases and $x(t)$ increases, from (D.12) we see that there may not be special modes in Problem D. Next, from (D.12) we see that the derivative $\dot{\phi}_2(t)$ changes its sign not more then in one point on $[0, \bar{\tau}]$. It means that necessarily the following cases satisfy to the optimality conditions provided by Pontryagin maximum principle:

(i) $\psi_2^0 \le 0$ and $\psi_1^0 > 0$ for all $t < \bar{\tau}$ and $\psi_1(\bar{\tau}) = \bar{y}$; then

$$u(t) = u_* \quad (t \in [0, \bar{\tau}]);$$

(ii) $0 < \psi_2^0 < 1$ and either $\psi_1(t) > 0$ for all $t < \bar{\tau}$ and $\psi_1(\bar{\tau}) = \bar{y}$, or $\psi_1(t)$ changes its sign at some $t \in (0, \bar{\tau})$, namely, $\psi_1(t) > 0$ for $t \in [0, \xi)$ and $\psi_1(t) < 0$ for $t \in (\xi, \bar{\tau}]$, and $\psi_1(\bar{\tau}) = 0$; then either

$$u(t) = u_* \quad (t \in [0, \bar{\tau}]);$$

or

$$u(t) = \begin{cases} u_*, & t \in [0, \xi), \\ 0, & t \in [\xi, \bar{\tau}]; \end{cases}$$

(iii) $\psi_1^0 > 1$ and $\psi_1(t) < 0$ for all $t < \bar{\tau}$, and hence

$$u(t) = 0 \quad (t \in [0, \bar{\tau}]).$$

Similar to the case of Lemma 3 control (iii) does not satisfy to the condition $\bar{y} > \bar{\tau}$; depending on the value of $\bar{y}$ extreme control (i) may be not admissible in Problem C. Otherwise it become a particular case of control (ii). Thus the two-stair control (ii) generalize the structure of an optimal control in Problem D. The switching time ξ is determined in such a way that the edge condition $y(\bar{\tau}) = \bar{y}$ holds which leads to the equation (D.2). The Lemma is proved. □

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Optimal Economic Growth with a Random Environmental Shock

Sergey Aseev, Konstantin Besov, Simon-Erik Ollus, and Tapio Palokangas

Abstract The government in a small open economy uses both an old "dirty," or "polluting," technology and a new "clean" technology simultaneously. However, because of climate change, it should take into account that at some stage in the future it will be penalized for production based on the old technology. In this paper, pollution is alleviated through international agreements that restrict polluting activities. The government's incentives to invest in cleaner technologies are based on productivity of the technology and randomly increasing abatement costs for pollution in future. In contrast to the Schumpeterian model of creative destruction, both technologies can be used simultaneously. The technologies are subject to AK production functions. Assuming that the exogenous environmental shock follows a Poisson process, we use Pontryagin's maximum principle to find the optimal investment policy. We find conditions under which a rational government should invest all its resources in one technology, while the other is moderately run down, as well as conditions under which it should divide the investments between the technologies in a certain ratio.

1 Introduction

Facing the possibility of climate change and global sanctions, the government in a small open economy attempts to reduce pollution and develop new cleaner production technologies. In many circumstances, the old "polluting" technology is not immediately replaced by a modern efficient "environment-saving" technology, instead they coexist. Why is not the former one abandoned immediately, or why not try to benefit from both technologies? The answer is that "clean" technologies are expensive to develop, less productive (at least in the initial stage) and often more expensive in use.

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S. Aseev (✉)
Steklov Mathematical Institute, Gubkina str. 8, Moscow 119991, Russia
e-mail: aseev@mi.ras.ru

S. Aseev
International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1, Laxenburg, 2361, Austria

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The government knows that it is only a matter of time when old polluting technologies will be internationally penalized (through, e.g., quotas, carbon trade, taxes or standards), but there is a lot of uncertainty when these sanctions will actually take place. This is supported by the fact that even today there are no global binding agreements on reducing greenhouse gas emissions. The Kyoto process is a first step towards such sanctions, but it is applied only to a part of the world countries. More binding emission targets and stricter sanctions are, however, being negotiated. Given that the international negotiation process and climate change are ongoing, a rational government takes into account that at some stage in the future it will be penalized for an old "polluting" technology. Under these circumstances, it is instructive to study how a rational government should adjust to an expected exogenous environmental change that will increase the abatement costs of emissions some time in future.

Many endogenous growth models of environmental change assume that international pollution is an externality for a household. In that case, the level of global pollution is incorporated into a household's utility function as a public good. In this paper, we do not adopt this approach. We rather assume that the governments of different countries are still too small to internalize the externality of pollution. It is likely that global pollution generates international sanctions that restrict polluting activities in each economy. We model such sanctions in the form of abatement costs. Thus, the incentive for a single government's investment in cleaner technologies is based on randomly increasing abatement costs in the future.[1]

Traditional growth models (e.g., Aghion and Howitt 1998; Barro and Sala-i-Martin 1995; Wälde 2002, 2007) with random technological change are built on a Schumpeterian process of creative destruction. The old technology and the capital bound into it cannot be recycled and the new technology takes immediately over. Thus a technology jump occurs and the old technology is destroyed. However, in contrast to this, it is empirically evident that both technologies are bound to coexist for a while, and a rational government does not abandon the old technology at once. We assume that the sector that does not receive new investments is moderately run down.

In this paper, we consider two alternative technologies that produce the same composite good (or perfect substitutes); this good can be both consumed and invested in capital. The first technology is "clean," or "non-polluting," while the second one is "dirty," or "polluting." Both technologies are used simultaneously and are characterized by AK production functions. At some stage of development, the sanctions for the polluting technology will become stricter due to worsening of environment (cf. climate crisis), increased international awareness and the need to develop new technologies. We model this as an "exogenous environmental shock" and assume that it follows a Poisson process. The government knows that this shock

[1] E.g., Chinese economic growth seen in recent years is much based on focusing only on the productivity, neglecting environmental concerns. However, at some stage in the future Chinese decision makers will need to take environmental concerns into account, as global pressure for sanctions increases and the domestic environmental problems grow. Simply put, they need to optimize the relation between environmental concerns and national welfare in monetary terms.

is coming, but does not know exactly when, and the probability of its occurrence is proportional to the length of time. The shock increases the consumer cost for the second technology through sanctions and higher abatement costs.

In this paper, we show how a rational government in a small economy adjusts to an expected environmental change. The results are based on the technical assumption that the level of consumption is fixed as compared with the total income. We are at the moment working on extensions of our two-sector growth model with uncertainty of random shock.

It should be noted that the potential application of the model constructed below is wider than that considered in the present paper. The developed mathematical methodology can also be applied to other economic growth problems, with random exogenous shock not necessarily of "environmental" character.

2 The Model

Let us consider an economy with two economic sectors, one based on "clean" technology and the other, on "dirty" technology. The productivity of the dirty technology is assumed to be higher in the initial stage, and the externality of pollution in the initial stage is taken into account as abatement costs by the social planner (government). Let a state variable $K_1(t) > 0$ ($K_2(t) > 0$) represent capital (capital stock) in the clean (dirty) sector at time $t \geq 0$. At each moment $t \geq 0$, the output of the clean (dirty) sector, $Q_1(t)$ ($Q_2(t)$), is a linear function of its capital:

$$Q_1(t) = A_1 K_1(t) \quad \text{and} \quad Q_2(t) = A_2 K_2(t),$$

where the parameter $A_1 > 0$ ($A_2 > 0$) is the level of technology in that sector. The outputs $Q_1(t)$ and $Q_2(t)$ are perfect substitutes as a private consumption good, but the dirty sector produces more emissions as a by-product in proportion to its output $Q_2(t)$. Let T be the time of the expected environmental shock that changes the abatement costs for these emissions from $(1-q)Q_2(t)$ to $(1-p)Q_2(t)$ units of the final good, where $1 > q > p > 0$.[2] In the mathematical part of this paper we also included the opposite case where $q < p$, due to the mathematical interest of the model, but this case is not interesting for this particular problem with an exogenous environmental shock.[3]

National income in terms of "money" is equal to the total output $Q_1(t) + Q_2(t)$ minus the abatement costs $(1-q)Q_2(t)$, that is,

$$Y_q(t) = Q_1(t) + Q_2(t) - (1-q)Q_2(t) = A_1 K_1(t) + q A_2 K_2(t) \tag{1}$$

[2]The mathematical solution of the optimal control problem described below is valid for all $p, q \in (0, \infty)$; that is, the assumption $p, q < 1$ is never used in what follows. However, if $p, q \geq 1$, the abatement costs turn around and become a support for the dirty technology. Thus, we excluded this case from the final results.

[3]In this setup of the problem, $q < p$ would imply that the abatement costs fall at the shock at time T, i.e., the dirty technology is awarded in the future.

for $t \in [0, T)$ and

$$Y_p(t) = Q_1(t) + Q_2(t) - (1-p)Q_2(t) = A_1K_1(t) + pA_2K_2(t) \tag{2}$$

for $t \in [T, \infty)$. At the moment T, we have

$$Y_p(T) = \lim_{t \to T-0} Y_q(t) + (p-q)A_2K_2(T). \tag{3}$$

Note that the change in abatement costs decreases the monetary value of the national income in the economy, as $q > p$. We also assume that abatement costs are some exogenous costs that are paid to the world economy as a penalty fee for pollution and are not returned to the economy as any subvention. Thus, the model does not have any budget constraint.

The social planner (government) of the economy distributes all income $Y_q(t)$ (or $Y_p(t)$) between consumption $C(t)$,[4] investment in the first sector $I_1(t)$, and investment in the second sector $I_2(t)$[5] at each moment $t \geq 0$. This implies

$$\begin{aligned}
&Y_q(t) = C(t) + I_1(t) + I_2(t),\\
&C(t) = u(t)Y_q(t), \qquad I_1(t) = i_1(t)Y_q(t), \qquad I_2(t) = i_2(t)Y_q(t),\\
&u(t) + i_1(t) + i_2(t) = 1, \quad u(t) > 0,\ i_1(t) \geq 0 \text{ and } i_2(t) \geq 0,
\end{aligned}$$

for $t \in [0, T)$, and

$$\begin{aligned}
&Y_p(t) = C(t) + I_1(t) + I_2(t),\\
&C(t) = u(t)Y_p(t), \qquad I_1(t) = i_1(t)Y_p(t), \qquad I_2(t) = i_2(t)Y_p(t),\\
&u(t) + i_1(t) + i_2(t) = 1, \quad u(t) > 0,\ i_1(t) \geq 0 \text{ and } i_2(t) \geq 0,
\end{aligned}$$

for $t \in [T, \infty)$.

By introducing a new control parameter $v(t)$, $t \geq 0$, for the relation between the investments in the two sectors, we can decrease the number of control parameters as follows:

$$\begin{aligned}
0 \leq v(t) \leq 1, &\quad t \geq 0,\\
i_1(t) = v(t)(1-u(t)), &\quad t \geq 0,\\
i_2(t) = (1-v(t))(1-u(t)), &\quad t \geq 0.
\end{aligned}$$

The quantities $u(\cdot)$ and $v(\cdot)$ are treated as control parameters[6] (or, simply, controls); $u(\cdot)$ symbolizes the control for consumption, and $v(\cdot)$, for the ratio of in-

[4]Consumption is assumed to include both public and private consumption in the small open economy.

[5]Here both sectors are assumed to be representative producers, and thus represent all production in the economy.

[6]Here and below the symbol $\cdot$ is used as an argument to indicate that the listed quantities are considered as functions of an independent variable.

vestments in the first and second sectors. As usual, we assume that these control parameters are (Lebesgue) measurable functions defined on $[0,\infty)$ and satisfying the indicated constraints.

Now, the capital stocks $K_1(t)$ and $K_2(t)$, $t \geq 0$, accumulate according to

$$\dot{K}_1(t) = b_1 I_1(t) - \delta K_1(t),$$
$$\dot{K}_2(t) = b_2 I_2(t) - \delta K_2(t).$$

This is equivalent to (see (1))

$$\dot{K}_1(t) = v(t)(1-u(t))b_1\big[A_1K_1(t) + qA_2K_2(t)\big] - \delta K_1(t), \tag{4}$$

$$\dot{K}_2(t) = (1-v(t))(1-u(t))b_2\big[A_1K_1(t) + qA_2K_2(t)\big] - \delta K_2(t) \tag{5}$$

on the time interval $[0,T)$, and (see (2))

$$\dot{K}_1(t) = v(t)(1-u(t))b_1\big[A_1K_1(t) + pA_2K_2(t)\big] - \delta K_1(t), \tag{6}$$

$$\dot{K}_2(t) = (1-v(t))(1-u(t))b_2\big[A_1K_1(t) + pA_2K_2(t)\big] - \delta K_2(t) \tag{7}$$

on the rest infinite time interval $[T,\infty)$.

Here $b_1 > 0$, $b_2 > 0$ and $1/b_1$, $1/b_2$ are constant costs of units of capital in the first and second economic sectors, respectively; $\delta \geq 0$ is the depreciation rate of capital, which is assumed to be the same for both economic sectors.

Due to (1), (4) and (5), the instantaneous income $Y_q(\cdot)$ satisfies on $[0,T]$ (in the sense of Carathéodory) the differential equation

$$\begin{aligned}\dot{Y}_q(t) &= A_1\dot{K}_1(t) + qA_2\dot{K}_2(t)\\ &= \big(v(t)(1-u(t))b_1A_1 + q(1-v(t))(1-u(t))b_2A_2\big)Y_q(t) - \delta Y_q(t)\end{aligned} \tag{8}$$

with the initial condition $Y_q(0) = A_1K_1(0) + qA_2K_2(0)$.

Similarly, due to (2), (6) and (7), the instantaneous income $Y_p(\cdot)$ satisfies on $[T,\infty)$ the differential equation

$$\begin{aligned}\dot{Y}_p(t) &= A_1\dot{K}_1(t) + pA_2\dot{K}_2(t)\\ &= \big(v(t)(1-u(t))b_1A_1 + p(1-v(t))(1-u(t))b_2A_2\big)Y_p(t) - \delta Y_p(t)\end{aligned} \tag{9}$$

with the initial condition (3).

The utility function for the social planner (government) is assumed to be the Ramsey utility function in the Cobb–Douglas form. Thus the social planer evaluates the quality of the control pair $(u(\cdot), v(\cdot))$ on the time interval $[0,T]$, $T > 0$, with the following utility index:

$$\begin{aligned}J_T(u(\cdot), v(\cdot), Y_q(\cdot), K_1(T), K_2(T)) = &\int_0^T e^{-\rho t}\ln(u(t)Y_q(t))dt\\ &+ e^{-\rho T}V(T, K_1(T), K_2(T)),\end{aligned} \tag{10}$$

where $\rho > 0$ is a subjective discount rate (time preference in the utility function) and

$$V(T, K_1(T), K_2(T)) = \max_{u(\cdot), v(\cdot)} \int_T^\infty e^{-\rho(t-T)} \ln(u(t)Y_p(t))dt \qquad (11)$$

is the current value of the capital stocks $K_1(T)$ and $K_2(T)$ at instant T.

Note that the first term on the right-hand side of (10) represents the aggregated discounted logarithm of consumption (in terms of money) on the time interval $[0, T]$, while the second one (the discounted current value of the capital stocks $K_1(T)$ and $K_2(T)$) represents the maximal possible value of the aggregated discounted logarithm of consumption (again in terms of money) on the rest infinite interval $[T, \infty)$:

$$e^{-\rho T} V(T, K_1(T), K_2(T)) = \max_{u(\cdot), v(\cdot)} \int_T^\infty e^{-\rho t} \ln(u(t)Y_p(t))dt.$$

Consider the integral in (11). Since due to (9)

$$\begin{aligned}
&\int_T^\infty e^{-\rho t} \ln Y_p(t)dt \\
&= \frac{e^{-\rho T}}{\rho} \ln Y_p(T) \\
&\quad + \frac{1}{\rho} \int_T^\infty e^{-\rho t} \big[(v(t)b_1A_1 + p(1 - v(t))b_2A_2)(1 - u(t)) - \delta\big]dt,
\end{aligned}$$

we have (see (11))

$$\begin{aligned}
&\int_T^\infty e^{-\rho(t-T)} \ln(u(t)Y_p(t))dt \\
&= \frac{1}{\rho} \ln Y_p(T) - \frac{\delta}{\rho^2} \\
&\quad + e^{\rho T} \int_T^\infty e^{-\rho t} \left[\ln u(t) + \frac{1}{\rho}(v(t)b_1A_1 + p(1 - v(t))b_2A_2)(1 - u(t))\right]dt.
\end{aligned} \qquad (12)$$

The integral on the right-hand side of (12) does not depend on the state variables $K_1(\cdot)$ and $K_2(\cdot)$, while the first two terms are constants. Hence, the integral on the right-hand side of (12) can be maximized in the control parameters $v(\cdot)$ and $u(\cdot)$ independently.

Thus, two cases are possible: (i) $b_1A_1 - pb_2A_2 > 0$ and (ii) $b_1A_1 - pb_2A_2 \leq 0$.

The first case (i) indicates that the productivity of capital in the modern sector is higher than in the old sector after the shock. In this case, the increased abatement costs are high enough to reduce the polluting technology and direct investment towards the modern sector. The second case (ii) indicates that the productivity of capital is still higher for the old technology despite the increase in abatement costs for pollution after the shock. Case (ii) is not interesting for our specific problem, but we still include it for the sake of completeness.

Consider case (i): Maximizing the integral on the right-hand side of (12) in $v(\cdot)$ and $u(\cdot)$ gives the following optimal controls on $[T,\infty)$: $\hat{v}_*(t) \equiv 1$ (all investments are directed to the modern technology) for all $t \geq T$, while $\hat{u}_*(t) \equiv \rho/(b_1 A_1)$ if $\rho \leq b_1 A_1$ or $\hat{u}_*(t) \equiv 1$ (all income is consumed and not invested in the second period) if $\rho > b_1 A_1$ for all $t \geq T$. Thus, the size of the time preference ρ matters for consumption in the second period.

Substituting these optimal controls in (12), we get the following value of the value function $V(T, K_1(T), K_2(T))$:

$$V(T, K_1(T), K_2(T)) = \frac{1}{\rho} \ln Y_p(T) + M_1, \tag{13}$$

where either

$$M_1 = \frac{\ln \rho - \ln(b_1 A_1) - 1}{\rho} + \frac{b_1 A_1 - \delta}{\rho^2} \quad \text{if } \rho \leq b_1 A_1 \tag{14}$$

or

$$M_1 = -\frac{\delta}{\rho^2} \quad \text{if } \rho > b_1 A_1. \tag{15}$$

Consider case (ii). In this case, the maximization of the integral on the right-hand side of (12) gives $\hat{v}_*(t) \equiv 0$ (all investments are directed to the old technology[7]) for all $t \in [0, T]$, while $\hat{u}_*(t) \equiv \rho/(pb_2 A_2)$ if $\rho \leq pb_2 A_2$ or $\hat{u}_* \equiv 1$ if $\rho > pb_2 A_2$ for all $t \in [0, T]$. Hence, substituting these optimal controls in (12), we get the following value for $V(T, K_1(T), K_2(T))$ in this case:

$$V(T, K_1(T), K_2(T)) = \frac{1}{\rho} \ln Y_p(T) + M_2, \tag{16}$$

where

$$M_2 = \frac{\ln \rho - \ln(pb_2 A_2) - 1}{\rho} + \frac{pb_2 A_2 - \delta}{\rho^2} \quad \text{if } \rho \leq pb_2 A_2 \tag{17}$$

or

$$M_2 = -\frac{\delta}{\rho^2} \quad \text{if } \rho > pb_2 A_2. \tag{18}$$

Thus, due to (1) and (3), in both cases (i) and (ii) (see (13), (16) and (3)) we have

$$V(T, K_1(T), K_2(T)) = \frac{1}{\rho} \ln(A_1 K_1(T) + pA_2 K_2(T)) + M, \tag{19}$$

where the constant M is either M_1 (see (14), (15)) or M_2 (see (17), (18)) depending on the relations between the values of the parameters. Recall that here the state vari-

[7] A quite natural result as the abatement costs of the polluting technology are not large enough to compensate for the higher productivity in the polluting sector.

ables $K_1(\cdot)$ and $K_2(\cdot)$ satisfy (4) and (5), respectively, on the time interval $[0, T]$.

All previous constructions have been performed under the assumption that the instant of time $T > 0$ is fixed.

Assume now that the instant of time T at which the environmental shock happens is a Poisson random variable (see, for example, Gnedenko 1997). This means that on each small time interval $[t, t + \Delta t]$, $t \geq 0$, $\Delta t > 0$, the relative probability of the event that the abatement cost of the unit of production in the second sector jumps to a value of $1 - p$ under the condition that before t it equals $1 - q$ is proportional to the length Δt of this time interval. Analytically this property of the random variable T can be expressed as follows:

$$\mathsf{P}(T < t + \Delta t \mid T \geq t) = \nu \Delta t + o(\Delta t).$$

Here $\nu > 0$ is a proportionality coefficient of the distribution and $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$.

In this case the distribution $\Phi(t) = \mathsf{P}(T < t)$ and the density $\varphi(t) = \dot{\Phi}(t)$, $t > 0$, of the random variable T are

$$\Phi(t) = 1 - e^{-\nu t} \quad \text{and} \quad \varphi(t) = \nu e^{-\nu t}, \quad t \geq 0. \tag{20}$$

In this situation the social planer faces the problem of maximization (by choosing an appropriate control pair $(u_*(\cdot), v_*(\cdot))$ on $[0, \infty)$) of the expected value of the random variable $J_T(u(\cdot), v(\cdot), Y_q(\cdot), K_1(T), K_2(T))$ (see (10)) at an uncertain (random) instant T.

For an arbitrary admissible control pair $(u(\cdot), v(\cdot))$ on $[0, \infty)$, due to (10), (19) and (20), we have

$$\begin{aligned}
&\mathsf{E}(J_T(u(\cdot), v(\cdot), Y_q(\cdot), K_1(T), K_2(T)))\\
&\quad = \int_0^\infty \nu e^{-\nu t} J_t(u(\cdot), v(\cdot), Y_q(\cdot), K_1(t), K_2(t)) dt\\
&\quad = \int_0^\infty \left[\nu e^{-\nu t} \int_0^t e^{-\rho s} \ln(u(s) Y_q(s)) ds \right] dt\\
&\qquad + \nu \int_0^\infty e^{-(\nu+\rho)t} \left[\frac{1}{\rho} \ln(A_1 K_1(t) + p A_2 K_2(t)) + M \right] dt\\
&\quad = \frac{\nu M}{\nu + \rho} + \int_0^\infty e^{-(\nu+\rho)t} \ln(u(t) Y_q(t)) dt\\
&\qquad + \frac{\nu}{\rho} \int_0^\infty e^{-(\nu+\rho)t} \ln(A_1 K_1(t) + p A_2 K_2(t)) dt\\
&\quad = \frac{\nu M}{\nu + \rho} + \int_0^\infty e^{-(\nu+\rho)t} \ln u(t) dt + \frac{\nu + \rho}{\rho} \int_0^\infty e^{-(\nu+\rho)t} \ln Y_q(t) dt\\
&\qquad + \frac{\nu}{\rho} \int_0^\infty e^{-(\nu+\rho)t} \ln \frac{A_1 K_1(t) + p A_2 K_2(t)}{Y_q(t)} dt.
\end{aligned} \tag{21}$$

Consider the second integral on the right-hand side of (21). Due to (8) we have

$$
\begin{aligned}
&\frac{\nu+\rho}{\rho}\int_0^\infty e^{-(\nu+\rho)t}\ln Y_q(t)dt\\
&=\frac{1}{\rho}\ln Y_q(0)\\
&\quad+\frac{1}{\rho}\int_0^\infty e^{-(\nu+\rho)t}\big[(v(t)b_1A_1+q(1-v(t))b_2A_2)(1-u(t))-\delta\big]dt\\
&=\frac{1}{\rho}\ln Y_q(0)-\frac{\delta}{\rho(\nu+\rho)}\\
&\quad+\frac{1}{\rho}\int_0^\infty e^{-(\nu+\rho)t}(v(t)b_1A_1+q(1-v(t))b_2A_2)(1-u(t))dt.
\end{aligned}
$$

Therefore, we can rewrite formula (21) for the expected value of the random variable $J_T(u(\cdot), v(\cdot), Y_q(\cdot), K_1(T), K_2(T))$ as follows:

$$
\begin{aligned}
&\mathsf{E}(J_T(u(\cdot), v(\cdot), Y_q(\cdot), K_1(T), K_2(T)))\\
&=\frac{\nu M}{\nu+\rho}+\frac{1}{\rho}\ln Y_q(0)-\frac{\delta}{\rho(\nu+\rho)}\\
&\quad+\int_0^\infty e^{-(\nu+\rho)t}\left[\ln u(t)+\frac{1}{\rho}(1-u(t))(v(t)b_1A_1+q(1-v(t))b_2A_2)\right]dt\\
&\quad+\frac{\nu}{\rho}\int_0^\infty e^{-(\nu+\rho)t}\ln\frac{A_1K_1(t)+pA_2K_2(t)}{A_1K_1(t)+qA_2K_2(t)}dt.
\end{aligned}
\tag{22}
$$

Since the first three terms on the right-hand side of (22) are constants, they can be neglected when optimizing the expected value of $J_T(u(\cdot), v(\cdot), Y_q(\cdot), K_1(T), K_2(T))$. So we can formulate the social planner's optimal control problem as the following optimal control problem (P):

$$
\begin{aligned}
&\dot K_1(t)=v(t)(1-u(t))b_1[A_1K_1(t)+qA_2K_2(t)]-\delta K_1(t),\\
&\dot K_2(t)=(1-v(t))(1-u(t))b_2[A_1K_1(t)+qA_2K_2(t)\big]-\delta K_2(t),\\
&K_1(0)=K_{10},\qquad K_2(0)=K_{20},\qquad u(t)\in(0,1],\qquad v(t)\in[0,1],\\
&J(K_1(\cdot), K_2(\cdot), u(\cdot), v(\cdot))\\
&\quad=\int_0^\infty e^{-(\nu+\rho)t}\left[\ln u(t)+\frac{(1-u(t))(v(t)b_1A_1+q(1-v(t))b_2A_2)}{\rho}\right.\\
&\qquad\left.+\frac{\nu}{\rho}\ln\frac{A_1K_1(t)+pA_2K_2(t)}{A_1K_1(t)+qA_2K_2(t)}\right]dt\to\max.
\end{aligned}
$$

Note that for $p=q$ problem (P) is trivial (the jump of the abatement costs at the instant T disappears). In this case, the utility functional $J(K_1(\cdot), K_2(\cdot), u(\cdot), v(\cdot))$

does not depend on the state variables $K_1(\cdot)$ and $K_2(\cdot)$, and the solution (optimal control pair $(u(\cdot), v(\cdot))$) is obtained by maximizing the integrand

$$\ln u(t) + \frac{1}{\rho}(1 - u(t))\big(v(t)b_1A_1 + q(1 - v(t))b_2A_2\big)$$

in the variables $u(t) \in (0, 1]$ and $v(t) \in [0, 1]$ at each instant t independently. It is natural that this expression coincides with that in (12), and we obtain similar optimal controls in this case: $v_*(t) \equiv 1$ for all $t \geq 0$ if $b_1A_1 > qb_2A_2$ and $v_*(t) \equiv 0$ for all $t \geq 0$ if $b_1A_1 \leq qb_2A_2$, while $u_*(t) \equiv \rho/\max\{b_1A_1, qb_2A_2\}$ if $\rho \leq \max\{b_1A_1, qb_2A_2\}$ and $u_*(t) \equiv 1$ if $\rho > \max\{b_1A_1, qb_2A_2\}$.

So, in what follows, we consider only the most interesting case

$$p \neq q.$$

Under this condition we can simplify the problem by reducing the dimension of the state variable. Namely, we introduce a new state variable $x(\cdot)$ as follows:

$$x(t) = \frac{A_1K_1(t) + pA_2K_2(t)}{A_1K_1(t) + qA_2K_2(t)}, \quad t \geq 0. \tag{23}$$

Note that $x(t)$ is equal to $Y_p(t)/Y_q(t)$ if we formally define $Y_p(t)$ for $t \in [0, T]$ by the same formula (2). This can be interpreted as the ratio of the "fictitious" instantaneous income $Y_p(t)$ to the real instantaneous income $Y_q(t)$ at time t, where the term "fictitious" means that $Y_p(t)$ would be the instantaneous income if the shock happened at this current point of time.

Below we study both the case $p > q$ and the case $p < q$, although only the latter (where sanctions increase) is meaningful in our particular economic problem. And in this latter case the state variable x is a monotonically increasing function of the ratio K_1/K_2.

Differentiating (23), we obtain

$$\begin{aligned}
\dot{x}(t) = {} & \frac{1}{(A_1K_1(t) + qA_2K_2(t))^2}\big[(A_1\dot{K}_1(t) + pA_2\dot{K}_2(t))(A_1K_1(t) + qA_2K_2(t)) \\
& - (A_1\dot{K}_1(t) + qA_2\dot{K}_2(t))(A_1K_1(t) + pA_2K_2(t))\big] \\
= {} & A_1A_2(q - p)\frac{\dot{K}_1(t)K_2(t) - K_1(t)\dot{K}_2(t)}{(A_1K_1(t) + qA_2K_2(t))^2} \\
= {} & A_1A_2(q - p)(1 - u(t))\frac{b_1v(t)K_2(t) - b_2(1 - v(t))K_1(t)}{A_1K_1(t) + qA_2K_2(t)}.
\end{aligned}$$

Note that

$$\begin{aligned}
& A_1A_2(q - p)[b_1v(t)K_2(t) - b_2(1 - v(t))K_1(t)] \\
& \quad = [b_1A_1v(t) + pb_2A_2(1 - v(t))](A_1K_1(t) + qA_2K_2(t)) \\
& \qquad - [b_1A_1v(t) + qb_2A_2(1 - v(t))](A_1K_1(t) + pA_2K_2(t)).
\end{aligned}$$

Therefore,

$$\dot{x}(t) = (1 - u(t))\big[b_1 A_1 v(t) + p b_2 A_2 (1 - v(t)) - (b_1 A_1 v(t) \\ + q b_2 A_2 (1 - v(t)))x(t)\big]. \tag{24}$$

Since for any admissible trajectory $(K_1(\cdot), K_2(\cdot))$ of problem (P), we have $K_1(t) > 0$ and $K_2(t) > 0$ for all $t \geq 0$, it follows from (23) that

$$\min\left\{1, \frac{p}{q}\right\} < x(t) < \max\left\{1, \frac{p}{q}\right\} \quad \text{for all } t \geq 0. \tag{25}$$

It can also be verified directly that any admissible trajectory $x(\cdot)$ of (24) (under arbitrary admissible controls $u(\cdot)$ and $v(\cdot)$) with an initial condition $x(0) = x_0$ such that

$$\min\{1, p/q\} < x_0 < \max\{1, p/q\}$$

satisfies (25).

Thus, we have reduced problem (P) to the following equivalent optimal control problem (P1):

$$\begin{aligned}
&\dot{x}(t) = (1 - u(t))\big[b_1 A_1 v(t) + p b_2 A_2 (1 - v(t)) - (b_1 A_1 v(t) \\
&\qquad + q b_2 A_2 (1 - v(t)))x(t)\big], \\
&x(0) = x_0, \qquad u(t) \in (0, 1], \qquad v(t) \in [0, 1], \\
&J_1(x(\cdot), u(\cdot), v(\cdot)) \\
&\quad = \int_0^\infty e^{-(\nu+\rho)t}\left[\ln u(t) + \frac{(1 - u(t))(v(t) b_1 A_1 + q(1 - v(t)) b_2 A_2)}{\rho}\right. \\
&\quad \left. + \frac{\nu}{\rho} \ln x(t)\right] dt \to \max,
\end{aligned}$$

where

$$x_0 = \frac{A_1 K_1(0) + p A_2 K_2(0)}{A_1 K_1(0) + q A_2 K_2(0)}.$$

The equivalence of problems means, in particular, that a control pair $(u(\cdot), v(\cdot))$ is optimal in problem (P) if and only if it is optimal in problem (P1).

As we are interested in how the social planner allocates investments, we simplify the problem and fix consumption at a certain share of the total income. Thus, we consider from now on a particular situation when the control for consumption $u(\cdot)$ is a constant $0 < u_0 < 1$, i.e., we assume that

$$u(t) \equiv u_0, \quad t \in [0, \infty).$$

To simplify the notations, we set $a = b_1 A_1 (1 - u_0)$, $b = q b_2 A_2 (1 - u_0)$ and $c = p b_2 A_2 (1 - u_0)$. Then problem (P1) reduces to the following optimal control problem (P2):

$$\dot{x}(t) = av(t) + c(1 - v(t)) - \big(av(t) + b(1 - v(t))\big)x(t), \tag{26}$$

$$x(0) = x_0, \qquad v(t) \in [0, 1], \tag{27}$$

$$J_2(x(\cdot), v(\cdot)) = \int_0^\infty e^{-(\nu+\rho)t}\big[av(t) + b(1 - v(t)) + \nu \ln x(t)\big]dt \to \max, \tag{28}$$

where x_0 is a given number in the interval $(\min\{1, p/q\}, \max\{1, p/q\})$. Here we discarded the constant term $\ln u_0$ in the utility functional J_1 and multiplied it by the positive constant ρ (the time preference of the utility function).

Denote

$$f(x, v) = av + c(1 - v) - (av + b(1 - v))x, \quad x > 0,\ v \in [0, 1], \tag{29}$$

and

$$g(x, v) = av + b(1 - v) + \nu \ln x, \quad x > 0,\ v \in [0, 1], \tag{30}$$

so that (26) and (28) become

$$\dot{x}(t) = f(x(t), v(t)) \quad \text{and} \quad J_2(x(\cdot), v(\cdot)) = \int_0^\infty e^{-(\nu+\rho)t} g(x(t), v(t))dt. \tag{31}$$

The formulated problem (P2) is affine in the control $v(\cdot)$. Hence, due to the standard existence theorem (see, for example, Aseev and Kryazhimskii 2007), there is an optimal admissible pair $(x_*(\cdot), v_*(\cdot))$ in (P2). This problem (P2) is in the focus of all our analysis below. In the next section we characterize all optimal regimes in problem (P2) with the use of optimal control theory. In Sect. 4 we offer an economic interpretation of the solution. Section 5 contains some conclusions. In the Appendix we give an alternative, direct, solution of problem (P2).

3 Solution of the Problem

In the standard way, we define the current value Hamilton–Pontryagin function $\mathcal{M}(x, v, \phi)$ and the current value Hamiltonian $M(x, \phi)$ for problem (P2) in the normal form:

$$\begin{aligned}\mathcal{M}(x, v, \phi) &= f(x, v)\phi + g(x, v)\\ &= \big[av + c(1 - v) - (av + b(1 - v))x\big]\phi\\ &\quad + av + b(1 - v) + \nu \ln x,\end{aligned} \tag{32}$$

$$M(x, \phi) = \sup_{v\in[0,1]} \mathcal{M}(x, v, \phi). \tag{33}$$

Here $x > 0$, $v \in [0, 1]$ and $\phi \in \mathbb{R}^1$.

Applying Theorem 12.1 from (Aseev and Kryazhimskii 2007), we obtain the following version of the Pontryagin maximum principle for problem (P2):

Theorem 1 *Let a pair* $(x_*(\cdot), v_*(\cdot))$ *be an optimal process in problem* (P2). *Then there exists a current value adjoint variable* $\phi(\cdot)$ (*corresponding to the pair* $(x_*(\cdot), v_*(\cdot))$) *such that the following conditions hold*:

(i) *The admissible pair* $(x_*(\cdot), v_*(\cdot))$, *together with the current value adjoint variable* $\phi(\cdot)$, *satisfies the core relations of the Pontryagin maximum principle in the normal form on the infinite time interval* $[0, \infty)$:

$$\dot{\phi}(t) \stackrel{\text{a.e.}}{=} (\nu + \rho)\phi(t) - \frac{\partial \mathcal{M}(x_*(t), v_*(t), \phi(t))}{\partial x}, \tag{34}$$

$$\mathcal{M}(x_*(t), v_*(t), \phi(t)) \stackrel{\text{a.e.}}{=} M(x_*(t), \phi(t)). \tag{35}$$

(ii) *The admissible pair* $(x_*(\cdot), v_*(\cdot))$, *together with the current value adjoint variable* $\phi(\cdot)$, *satisfies the normal-form stationarity condition*:

$$M(x_*(t), \phi(t)) = (\nu + \rho)e^{(\nu+\rho)t} \int_t^\infty e^{-(\nu+\rho)s} g(x_*(s), v_*(s))ds$$
$$\textit{for all } t \geq 0. \tag{36}$$

(iii) *For any* $t \geq 0$

$$\phi(t) = e^{(\nu+\rho)t} e^{z(t)} \int_t^\infty e^{-(\nu+\rho)s} e^{-z(s)} \frac{\partial g(x_*(s), v_*(s))}{\partial x} ds, \tag{37}$$

where $z(t) = -\int_0^t \frac{\partial f(x_*(s), v_*(s))}{\partial x} ds \geq 0$.

Proof It suffices to verify that problem (P2) satisfies all hypotheses of Theorem 12.1 from (Aseev and Kryazhimskii 2007). Due to (25), (27), (29) and (30), we have

$$|x(t)| \leq \text{const}, \qquad |f(x(t), v(t))| \leq \text{const}, \qquad \frac{\partial f(x(t), v(t))}{\partial x} \leq 0,$$
$$|g(x(t), v(t))| \leq \text{const}, \qquad \left|\frac{\partial g(x(t), v(t))}{\partial x}\right| \leq \text{const}, \qquad t \geq 0,$$

for all admissible $x(\cdot)$ and $v(\cdot)$. Thus, it only remains to note that the functions $f(x, \cdot)$ and $g(x, \cdot)$ are affine in the control variable v for arbitrary fixed $x > 0$. □

Theorem 1 serves as a main tool in our construction of optimal regimes in this section. As we will see below, there is a unique admissible pair $(x_*(\cdot), v_*(\cdot))$ satisfying the conditions of Theorem 1. Due to Theorem 1 and the standard existence theorem (see Aseev and Kryazhimskii 2007), this pair $(x_*(\cdot), v_*(\cdot))$ is a unique optimal admissible pair in problem (P2). An alternative direct solution of problem (P2) is presented below in the Appendix as well.

Corollary 1 *The current value adjoint variable* $\phi(\cdot)$ *corresponding to an optimal process in problem* (P2) *is positive and bounded*:

$$0 < \phi(t) \leq C \quad \textit{for all } t \geq 0. \tag{38}$$

Proof Since $\frac{\partial g(x,v)}{\partial x} = \frac{\nu}{x} > 0$ for $x > 0$, it follows from (37) that $\phi(t) > 0$ for all $t \geq 0$. On the other hand, $\frac{\nu}{x} < \nu \max\{1, q/p\}$ for any x satisfying (25). Note that

$$z(t) = \int_0^t (av_*(s) + b(1 - v_*(s)))ds \quad \text{for } t \geq 0;$$

thus, $z(\cdot)$ is a nonnegative monotonically increasing function of t. Therefore, we can estimate $\phi(\cdot)$ from (37) as follows:

$$\phi(t) \leq e^{(\nu+\rho)t} e^{z(t)} \int_t^\infty e^{-(\nu+\rho)s} e^{-z(t)} \nu \max\{1, q/p\} ds = \frac{\nu}{\nu + \rho} \max\{1, q/p\}.$$

□

Note that since $x(t) > 0$, $\phi(t) > 0$ for all $t \geq 0$ and any admissible trajectory $x(\cdot)$ is bounded (see (25)), inequality (38) implies the validity of the standard transversality condition at infinity (see, for example, Aghion and Howitt 1998; Aseev and Kryazhimskii 2007; Barro and Sala-i-Martin 1995)

$$\lim_{t \to \infty} e^{-(\rho+\nu)t} x(t)\phi(t) = 0.$$

Now we analyze the maximum condition (35). Since the current value Hamilton–Pontryagin function $\mathcal{M}(x, \cdot, \phi)$ (see (32)) is affine in the control v, the maximum value of $\mathcal{M}(x, \cdot, \phi)$ over $v \in [0, 1]$ for arbitrary fixed $x > 0$ and $\phi > 0$ is reached either at $v = 0$, at $v = 1$, or at all points $v \in [0, 1]$ simultaneously.

Denote

$$\mathcal{M}_0(x, \phi) = \mathcal{M}(x, 0, \phi) = (c - bx)\phi + b + \nu \ln x, \quad x > 0, \ \phi > 0, \tag{39}$$

and

$$\mathcal{M}_1(x, \phi) = \mathcal{M}(x, 1, \phi) = a(1 - x)\phi + a + \nu \ln x, \quad x > 0, \ \phi > 0. \tag{40}$$

Thus, $\mathcal{M}_0(x, \phi) > \mathcal{M}_1(x, \phi)$ if and only if

$$(bx - c + a - ax)\phi < b - a,$$

and $\mathcal{M}_0(x, \phi) < \mathcal{M}_1(x, \phi)$ if and only if

$$(bx - c + a - ax)\phi > b - a.$$

Define a function ϕ_0: $(\min\{1, p/q\}, \max\{1, p/q\}) \to \mathbb{R}^1$ as follows:

$$\phi_0(x) = \frac{b - a}{bx - c + a - ax}, \quad \min\{1, p/q\} < x < \max\{1, p/q\}. \tag{41}$$

Note that if $p < q$, then the denominator in (41) is positive, while if $p > q$, this denominator is negative.

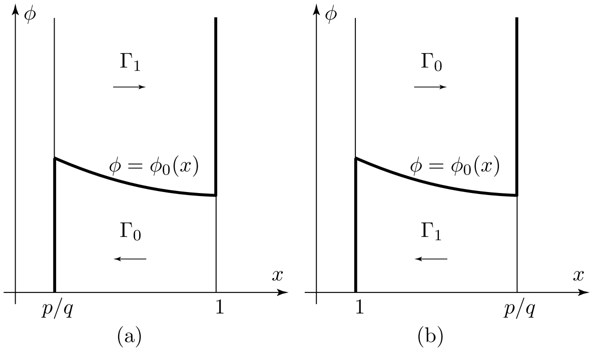

Fig. 1 The sets Γ_0 and Γ_1 (**a**) in the case $p < q$ and (**b**) in the case $p > q$. *The arrows* indicate the direction of the change of the x-coordinate for trajectories of the Hamiltonian system. An optimal trajectory tends to a limit point that lies on *the thick curve*

Introduce the set

$$\Gamma = \left\{(x, \phi) \in \mathbb{R}^2 \colon \min\{1, p/q\} < x < \max\{1, p/q\},\ \phi > 0\right\}$$

of admissible values of x and ϕ. The graph of $\phi_0(\cdot)$

$$\operatorname{gr}\phi_0 = \{(x, \phi) \in \Gamma \colon \phi = \phi_0(x)\}$$

(if it intersects Γ) divides Γ into two parts

$$\Gamma_0 = \{(x, \phi) \in \Gamma \colon \mathcal{M}_0(x, \phi) > \mathcal{M}_1(x, \phi)\}$$

and

$$\Gamma_1 = \{(x, \phi) \in \Gamma \colon \mathcal{M}_0(x, \phi) < \mathcal{M}_1(x, \phi)\}$$

(see Fig. 1). If $p < q$, then Γ_0 lies below the graph of $\phi_0(\cdot)$ (or is empty if $\operatorname{gr}\phi_0 = \emptyset$) and Γ_1 lies above the graph of $\phi_0(\cdot)$ (or coincides with Γ if $\operatorname{gr}\phi_0 = \emptyset$), while if $p > q$, Γ_0 lies above the graph of $\phi_0(\cdot)$ (or coincides with Γ if $\operatorname{gr}\phi_0 = \emptyset$) and Γ_1 lies below this graph (or is empty if $\operatorname{gr}\phi_0 = \emptyset$).

In the open set Γ_0, due to condition (35), the Hamiltonian system of the maximum principle for problem (P2) has the form

$$\dot{x}(t) = c - bx(t), \tag{42}$$

$$\dot{\phi}(t) = (\nu + \rho)\phi(t) + b\phi(t) - \frac{\nu}{x(t)}. \tag{43}$$

In the open set Γ_1, the Hamiltonian system of the maximum principle for problem (P2) has the form

$$\dot{x}(t) = a - ax(t), \tag{44}$$

$$\dot{\phi}(t) = (\nu + \rho)\phi(t) + a\phi(t) - \frac{\nu}{x(t)}. \tag{45}$$

Remark 1 If $p < q$, then for any trajectory $(x(\cdot), \phi(\cdot))$ of the Hamiltonian system (42)–(45) we have $\dot{x}(t) > 0$ if $(x(t), \phi(t)) \in \Gamma_1$ and $\dot{x}(t) < 0$ if $(x(t), \phi(t)) \in \Gamma_0$. If $p > q$, then for any trajectory $(x(\cdot), \phi(\cdot))$ of the Hamiltonian system (42)–(45) we have $\dot{x}(t) < 0$ if $(x(t), \phi(t)) \in \Gamma_1$ and $\dot{x}(t) > 0$ if $(x(t), \phi(t)) \in \Gamma_0$.

It follows from Remark 1 that fixed points of the Hamiltonian system of the maximum principle for problem (P2) may only be on the graph of the function $\phi_0(\cdot)$. Let us find them.

Let $\vec{V}_0(x,\phi) = (V_{01}(x,\phi), V_{02}(x,\phi))$ be the vector field generated by the right-hand side of system (42), (43) in Γ_0, and let $\vec{V}_1(x,\phi) = (V_{11}(x,\phi), V_{12}(x,\phi))$ be the vector field generated by the right-hand side of system (44), (45) in Γ_1. We can extend these fields by continuity to assume that they are also defined on $\operatorname{gr}\phi_0$. Then at any point of $\operatorname{gr}\phi_0$ we have a family of admissible vectors (velocities)

$$\vec{V}(x,\phi) = \lambda \vec{V}_1(x,\phi) + (1-\lambda)\vec{V}_2(x,\phi), \quad 0 \le \lambda \le 1. \tag{46}$$

This family contains a zero vector if and only if the vectors $\vec{V}_1(x,\phi)$ and $\vec{V}_2(x,\phi)$ have opposite directions (or one of them vanishes). Since the first coordinates of $\vec{V}_1(x,\phi)$ and $\vec{V}_2(x,\phi)$ are nonzero and have different signs (see Remark 1), this condition is equivalent to

$$V_{01}(x,\phi)V_{12}(x,\phi) - V_{11}(x,\phi)V_{02}(x,\phi) = 0, \tag{47}$$

or

$$(c-bx)\left[(\nu+\rho+a)\phi - \frac{\nu}{x}\right] - a(1-x)\left[(\nu+\rho+b)\phi - \frac{\nu}{x}\right] = 0. \tag{48}$$

Recall that $\phi = \phi_0(x)$ (since we are on the graph of $\phi_0(\cdot)$). Then we can rewrite this equation as

$$\phi_0(x)[(\nu+\rho+a)(c-bx) - (\nu+\rho+b)a(1-x)] - \frac{\nu}{x}(c-bx-a(1-x)) = 0. \tag{49}$$

Note that this automatically implies $\phi_0(x) > 0$ if x is a solution of (49) in the interval $(\min\{1, p/q\}, \max\{1, p/q\})$. Indeed, if $p < q$, then $bx > c$ and $x < 1$, while if $p > q$, then $bx < c$ and $x > 1$.

Substituting (41) into (49), we obtain

$$(b-a)x[(\nu+\rho+a)(c-bx) - (\nu+\rho+b)a(1-x)] + \nu(c-bx-a(1-x))^2 = 0,$$

or

$$(b-a)x[(\nu+\rho)(bx-c+a(1-x)) + ab - ac] - \nu(bx-c+a(1-x))^2 = 0,$$

or

$$P(x) := \rho(b-a)^2x^2 + (b-a)[a(b-c) + (\rho-\nu)(a-c)]x - \nu(a-c)^2 = 0. \tag{50}$$

If $a = b$, then this equation has no solutions (because $p \neq q$ and hence $a \neq c$ in this case). Otherwise, the only nonnegative solution is given by

$$\bar{x} = -\frac{a(b-c) + (\rho - \nu)(a-c)}{2\rho(b-a)} + \frac{\sqrt{[a(b-c) + (\rho-\nu)(a-c)]^2 + 4\nu\rho(a-c)^2}}{2\rho|b-a|}. \tag{51}$$

The corresponding point $(\bar{x}, \phi_0(\bar{x}))$ belongs to Γ (and hence is a fixed point of the Hamiltonian system of the maximum principle for problem (P2)) if and only if

$$P(\min\{1, p/q\}) < 0 \quad \text{and} \quad P(\max\{1, p/q\}) > 0. \tag{52}$$

Thus, the Hamiltonian system of the maximum principle for problem (P2) either has no fixed points or has one fixed point $(\bar{x}, \phi_0(\bar{x}))$.

Corollary 2 *If* $(x_*(\cdot), v_*(\cdot))$ *is an optimal process in problem* (P2), *then the corresponding trajectory* $(x_*(\cdot), \phi(\cdot))$ *cannot cross the curve* $\operatorname{gr}\phi_0$, *i.e., cannot lie partly in* Γ_0 *and partly in* Γ_1.

Proof Indeed, the x-coordinates of the vector fields in Γ_0 and Γ_1 have different signs (see Remark 1). Therefore, if a trajectory $(x_*(\cdot), \phi(\cdot))$ crosses $\operatorname{gr}\phi_0$, then there is an interval $[x_\alpha, x_\beta]$ such that $x_*(\cdot)$ moves from x_α to x_β on one time interval and from x_β to x_α on another time interval. Then, by (Aseev and Kryazhimskii 2007, Theorem 4.4) each point ξ of the interval $[x_\alpha, x_\beta]$ would be an optimal stationary trajectory in problem (P2) with the initial condition $x(0) = \xi$. However, this is impossible, because, as we have just shown, the Hamiltonian system of the maximum principle for problem (P2) has at most one fixed point. □

Now, we show that no part of any trajectory $(x(\cdot), \phi(\cdot))$ can go along the curve $\operatorname{gr}\phi_0$, except for staying at the fixed point.

Lemma 1 *If* $(x(\cdot), \phi(\cdot))$ *is a trajectory of the Hamiltonian system of the maximum principle for problem* (P2) *such that* $(x(t), \phi(t)) \in \operatorname{gr}\phi_0$ *for any* t *from some time interval* $[t_0, t_1]$, $t_1 > t_0$, *then* $(x(t), \phi(t)) \equiv (\bar{x}, \phi_0(\bar{x}))$ *for any* $t \in [t_0, t_1]$.

Proof The curve $\operatorname{gr}\phi_0$ is defined by the equation $\mathcal{M}_0(x, \phi) - \mathcal{M}_1(x, \phi) = 0$ (see (39), (40)). Therefore, a normal to this curve is given by

$$\vec{n} = \left(\frac{\partial \mathcal{M}_0}{\partial x} - \frac{\partial \mathcal{M}_1}{\partial x}, \frac{\partial \mathcal{M}_0}{\partial \phi} - \frac{\partial \mathcal{M}_1}{\partial \phi}\right).$$

Note that the second coordinate of $\vec{n}$ is always nonzero. The vector fields $\vec{V}_0(x, \phi)$ and $\vec{V}_1(x, \phi)$ in Γ_0 and Γ_1, respectively, are given by

$$\vec{V}_0 = (V_{01}, V_{02}) = \left(\frac{\partial \mathcal{M}_0}{\partial \phi}, (\nu + \rho)\phi - \frac{\partial \mathcal{M}_0}{\partial x}\right)$$

and

$$\vec{V}_1 = (V_{11}, V_{12}) = \left(\frac{\partial \mathcal{M}_1}{\partial \phi}, (\nu + \rho)\phi - \frac{\partial \mathcal{M}_1}{\partial x} \right).$$

Let us calculate the scalar products $\langle \vec{n}, \vec{V}_0 \rangle$ and $\langle \vec{n}, \vec{V}_1 \rangle$:

$$\langle \vec{n}, \vec{V}_0 \rangle = -\frac{\partial \mathcal{M}_1}{\partial x} \frac{\partial \mathcal{M}_0}{\partial \phi} + \left(\frac{\partial \mathcal{M}_0}{\partial \phi} - \frac{\partial \mathcal{M}_1}{\partial \phi} \right)(\nu + \rho)\phi$$
$$+ \frac{\partial \mathcal{M}_0}{\partial x} \frac{\partial \mathcal{M}_1}{\partial \phi} = \langle \vec{n}, \vec{V}_1 \rangle. \tag{53}$$

But then for any vector $\vec{V}$ from the family (46) we obtain the same value of the scalar product with $\vec{n}$:

$$\langle \vec{n}, \vec{V} \rangle = \langle \vec{n}, \vec{V}_0 \rangle = \langle \vec{n}, \vec{V}_1 \rangle.$$

If a trajectory goes along $\mathrm{gr}\,\phi_0$, then we must have $\langle \vec{n}, \vec{V} \rangle = 0$ at any point of this trajectory, i.e.,

$$\langle \vec{n}, \vec{V}_0 \rangle = \langle \vec{n}, \vec{V}_1 \rangle = 0.$$

This implies that the vector fields $\vec{V}_0$ and $\vec{V}_1$ have either the same direction or opposite directions, i.e., $V_{01}V_{12} - V_{02}V_{11} = 0$, which is nothing else but equation (47) of the fixed point. The lemma is proved. □

Remark 2 If a trajectory $(x(\cdot), \phi(\cdot))$ of the Hamiltonian system of the maximum principle reaches the curve $\mathrm{gr}\,\phi_0$ at a point other than $(\bar{x}, \phi_0(\bar{x}))$, then it must necessarily cross the curve $\mathrm{gr}\,\phi_0$ (due to the equality of scalar products (53)) and hence cannot be optimal by Corollary 2.

Now we are ready to describe all trajectories $(x_*(\cdot), \phi(\cdot))$ corresponding to optimal processes $(x_*(\cdot), v_*(\cdot))$ in problem (P2) by virtue of Theorem 1.

Such a trajectory $(x(\cdot), \phi(\cdot))$

(i) either tends to the fixed point $(\bar{x}, \phi_0(\bar{x})) \in \mathrm{gr}\,\phi_0$,
(ii) or lies in Γ_0 for all t starting from a certain $t_0 \geq 0$,
(iii) or lies in Γ_1 for all t starting from a certain $t_0 \geq 0$.

In case (i) the trajectory goes to the fixed point either from Γ_0 or from Γ_1 (but not along $\mathrm{gr}\,\phi_0$ by Lemma 1) and hence reaches the fixed point in some finite time $\tau > 0$ and then stays at this point for all $t \geq \tau$. The optimal control $v_*(\cdot)$ after reaching the fixed point can be obtained by equating the right-hand side of (26) to zero at $x(\tau) = \bar{x}$:

$$v_*(t) \equiv v_* = \frac{b\bar{x} - c}{b\bar{x} - c + a - a\bar{x}}, \quad t \geq \tau. \tag{54}$$

We always have $0 < v_* < 1$ if $\min\{1, p/q\} < \bar{x} < \max\{1, p/q\}$. Note that in this case $v_*(\cdot)$ is an optimal singular control on $[\tau, \infty)$ (see, for example, Gabasov and Kirillova 1982).

In case (ii) $x_*(t) \to \frac{p}{q}$ and in case (iii) $x_*(t) \to 1$ as $t \to \infty$ due to (42) and (44), respectively. By Corollary 1 the trajectory $(x_*(\cdot), \phi(\cdot))$ is bounded. Since the vector fields generated by (42), (43) in Γ_0 and (44), (45) in Γ_1 have continuous extensions to the boundaries $x = \frac{p}{q}$ and $x = 1$ of Γ_0 and Γ_1, respectively, the trajectory $(x_*(\cdot), \phi(\cdot))$ must tend to an "infinite" fixed point $(\bar{x}_0, \bar{\phi}_0) \in \partial\Gamma_0$ or $(\bar{x}_1, \bar{\phi}_1) \in \partial\Gamma_1$, where $\bar{x}_0 = \frac{p}{q}$ and $\bar{x}_1 = 1$. For $\bar{\phi}_0$ from (43) we obtain the equation

$$(\nu + \rho + b)\bar{\phi}_0 - \frac{\nu q}{p} = 0.$$

For $\bar{\phi}_1$ from (45) we obtain the equation

$$(\nu + \rho + a)\bar{\phi}_1 - \nu = 0.$$

Thus,

$$\bar{\phi}_0 = \frac{\nu q}{p(\nu + \rho + b)} \quad \text{and} \quad \bar{\phi}_1 = \frac{\nu}{\nu + \rho + a}.$$

Suppose that $p < q$. Then the condition $(\bar{x}_0, \bar{\phi}_0) \in \partial\Gamma_0$ is equivalent to $\bar{\phi}_0 \leq \phi_0(\frac{p}{q})$, or

$$\frac{\nu q}{p(\nu + \rho + b)} \leq \phi_0\left(\frac{p}{q}\right). \tag{55}$$

Note that this inequality is equivalent to the fact that the left-hand side of (48) is nonpositive at $x = \frac{p}{q} = \frac{c}{b}$, i.e., the left-hand side of (49) is nonpositive at $x = \frac{p}{q}$, i.e., $P(\frac{p}{q}) \geq 0$ (see (50)).

Similarly, the condition $(\bar{x}_1, \bar{\phi}_1) \in \partial\Gamma_1$ for $p < q$ is equivalent to $\bar{\phi}_1 \geq \phi_0(1)$, or

$$\frac{\nu}{\nu + \rho + a} \geq \phi_0(1). \tag{56}$$

This inequality is equivalent to the fact that the left-hand side of (48) is nonnegative at $x = 1$, i.e., the left-hand side of (49) is nonnegative at $x = 1$, i.e., $P(1) \leq 0$ (see (50)).

The conditions $P(\frac{p}{q}) \geq 0$, $P(1) \leq 0$, and the condition (52) of the existence of a fixed point in $\operatorname{gr}\phi_0$ are pairwise incompatible for $p < q$ and describe all possible situations (i.e., one and only one condition from these three holds for any relation between the parameters of the problem). Thus, one and only one fixed point $((\bar{x}, \phi_0(\bar{x}))$, $(\bar{x}_0, \bar{\phi}_0)$, or $(\bar{x}_1, \bar{\phi}_1))$ exists in the closure of Γ (this fixed point lies on the thick curve in Fig. 1a), and for any initial condition $\frac{p}{q} < x_0 < 1$ there is only one trajectory $(x(\cdot), \phi(\cdot))$ with $x(0) = x_0$ that tends to this fixed point. Indeed, suppose that there are two such trajectories $(x_1(\cdot), \phi_1(\cdot))$ and $(x_2(\cdot), \phi_2(\cdot))$ that tend to the same fixed point from the same set Γ_0 or Γ_1. Then $x_1(t) = x_2(t)$, $\phi_1(t) \neq \phi_2(t)$ for all $t \geq 0$ and, on the one hand, the difference of their ϕ-coordinates must tend to

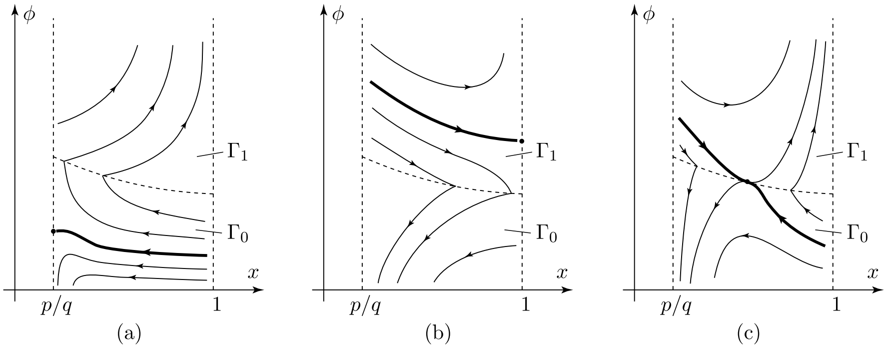

Fig. 2 Trajectories of the Hamiltonian system of the maximum principle for problem (P2) in the case $p < q$. The optimal trajectory is shown by *a thick line*. Cases (**a**), (**b**) and (**c**) correspond to cases I, II and III of Theorem 2. The picture in case (**c**) resembles a saddle; note, however, that in this case the optimal process, moving along *the thick curve*, always reaches the steady state in finite time

zero, but, on the other hand, it satisfies the differential equation

$$\frac{d}{dt}(\phi_1(t) - \phi_2(t)) = (\nu + \rho + b)(\phi_1(t) - \phi_2(t))$$

if the trajectories lie in Γ_0 or the differential equation

$$\frac{d}{dt}(\phi_1(t) - \phi_2(t)) = (\nu + \rho + a)(\phi_1(t) - \phi_2(t))$$

if the trajectories lie in Γ_1. This contradiction shows that there is only one trajectory $(x(\cdot), \phi(\cdot))$ with $x(0) = x_0$ that satisfies the relations of Theorem 1. This trajectory is shown by a thick line in Fig. 2; it gives the optimal solution in problem (P2). All other trajectories either are unbounded or intersect the line $\phi = 0$.

Suppose now that $p > q$. Then the condition $(\bar{x}_0, \bar{\phi}_0) \in \partial\Gamma_0$ is equivalent to $\bar{\phi}_0 \geq \phi_0(\frac{p}{q})$, or

$$\frac{\nu q}{p(\nu + \rho + b)} \geq \phi_0\left(\frac{p}{q}\right). \tag{57}$$

This inequality is equivalent to the fact that the left-hand side of (48) is nonpositive at $x = \frac{p}{q} = \frac{c}{b}$, i.e., the left-hand side of (49) is nonpositive at $x = \frac{p}{q}$, i.e., $P(\frac{p}{q}) \leq 0$ (the denominator of ϕ_0 is negative for $p > q$, see (41)). Similarly, the condition $(\bar{x}_1, \bar{\phi}_1) \in \partial\Gamma_1$ for $p > q$ is equivalent to $\bar{\phi}_1 \leq \phi_0(1)$, or

$$\frac{\nu}{\nu + \rho + a} \leq \phi_0(1). \tag{58}$$

This inequality is equivalent to the fact that the left-hand side of (48) is nonnegative at $x = 1$, i.e., the left-hand side of (49) is nonnegative at $x = 1$, i.e., $P(1) \geq 0$.

The conditions $P(\frac{p}{q}) \leq 0$, $P(1) \geq 0$, and the condition (52) of the existence of a fixed point in $\operatorname{gr}\phi_0$ are pairwise incompatible for $p > q$ and describe all possible situations. Thus, one and only one fixed point $((\bar{x}, \phi_0(\bar{x}))$, $(\bar{x}_0, \bar{\phi}_0)$, or $(\bar{x}_1, \bar{\phi}_1))$ exists in the closure of Γ, and for any initial condition $1 < x_0 < \frac{p}{q}$ there is only one trajectory $(x(\cdot), \phi(\cdot))$ with $x(0) = x_0$ that tends to this fixed point. This trajectory gives the optimal solution in problem (P2).

Note that (55) for $p < q$ and (57) for $p > q$ can be written in a unified way as

$$\nu a(q - p) \leq (b - a)p(\nu + \rho + b),$$

or

$$\nu a(b - c) \leq (b - a)c(\nu + \rho + b),$$

or

$$\nu b(a - c) \leq (b - a)c(\rho + b), \tag{59}$$

Similarly, (56) for $p < q$ and (58) for $p > q$ can be written in a unified way as

$$\nu(b - c) \geq (b - a)(\nu + \rho + a),$$

or

$$\nu(a - c) \geq (b - a)(a + \rho). \tag{60}$$

Remark 3 One can check directly that conditions (59) and (60) are incompatible (the case $a = b = c$ is impossible because $p \neq q$), which agrees well with the above analysis.

Summarizing the results of our analysis, we obtain the following optimal synthesis (see Pontryagin et al. 1964) in problem (P2):

Theorem 2

(I) *If condition* (59) *holds, then the optimal control* $v_*(\cdot)$ *in problem* (P2) *as a function of* x *is* $v_*(x) \equiv 0$.

(II) *If condition* (60) *holds, then the optimal control* $v_*(\cdot)$ *in problem* (P2) *as a function of* x *is* $v_*(x) \equiv 1$.

(III) *Suppose that both conditions* (59) *and* (60) *are violated. Then* $(\bar{x}, \phi_0(\bar{x})) \in \Gamma$.

(a) *If* $p < q$, *then the optimal control* $v_*(\cdot)$ *in problem* (P2) *as a function of* x *is given by* (*see* (54))

$$v_*(x) = \begin{cases} 1 & \text{if } x < \bar{x}, \\ \frac{b\bar{x} - c}{b\bar{x} - c + a - a\bar{x}} & \text{if } x = \bar{x}, \\ 0 & \text{if } x > \bar{x}, \end{cases}$$

where $\bar{x}$ *is defined in* (51).

(b) *If* $p > q$, *then the optimal control* $v_*(\cdot)$ *in problem* (P2) *as a function of* x *is given by*

$$v_*(x) = \begin{cases} 0 & \textit{if } x < \bar{x}, \\ \frac{b\bar{x}-c}{b\bar{x}-c+a-a\bar{x}} & \textit{if } x = \bar{x}, \\ 1 & \textit{if } x > \bar{x}. \end{cases}$$

For an arbitrary initial state $x_0 \in (\min\{1, p/q\}, \max\{1, p/q\})$, the optimal synthesis $v_*(x)$, $x \in (\min\{1, p/q\}, \max\{1, p/q\})$, uniquely defines the optimal trajectory $x_*(\cdot)$ in problem (P2) as the solution of the Cauchy problem

$$\dot{x}(t) = av_*(x(t)) + c(1 - v_*(x(t))) - \big(av_*(x(t)) + b(1 - v_*(x(t)))\big)x(t),$$
$$x(0) = x_0,$$

and the corresponding optimal control $v_*(\cdot)$ as the function

$$v_*(t) = v_*(x_*(t)), \quad t \in [0, \infty).$$

Note that in case III the optimal trajectory $x_*(\cdot)$ always reaches the steady state $\bar{x}$ in finite time and then the control $v_*(\cdot)$ switches to the value $v_*(\bar{x})$.

4 Economic Interpretation

Let us now pass on to the economic interpretation of the results obtained. As we mentioned earlier, our model expects an exogenous environmental shock at time $T > 0$ after which the abatement costs increase, i.e., $p < q$. In other words, the productivity of capital (including the abatement costs) of the old technology sector after the shock, $pb_2A_2 = c/(1 - u_0)$, is lower than its productivity before the shock, $qb_2A_2 = b/(1 - u_0)$. So we will assume throughout this section that $p < q$.

In our model any admissible state trajectory $x(\cdot)$ takes values

$$x(t) = \frac{A_1K_1(t) + pA_2K_2(t)}{A_1K_1(t) + qA_2K_2(t)}, \quad t \geq 0,$$

i.e., is the ratio of

$$Y_p(t) = A_1K_1(t) + pA_2K_2(t)$$

to

$$Y_q(t) = A_1K_1(t) + qA_2K_2(t),$$

where $Y_q(t)$ is the real instantaneous income at instant $t \geq 0$ (before the shock) and $Y_p(t)$ is the corresponding “fictitious” instantaneous income at the same instant t, which shows what the income would be if the shock happened right now.

Consider the variable

$$x = \frac{A_1 K_1 + pA_2 K_2}{A_1 K_1 + qA_2 K_2} = \frac{A_1 \frac{K_1}{K_2} + pA_2}{A_1 \frac{K_1}{K_2} + qA_2}$$

as a function of the ratio $y = K_1/K_2$ of the capital stocks of the modern and old technology sectors, i.e., put

$$x = \frac{A_1 y + pA_2}{A_1 y + qA_2}, \quad y \in (0, \infty).$$

It is easy to note that $x(\cdot)$ is a monotonically increasing function of the variable $y \in (0, \infty)$. Hence there is the inverse

$$y = \frac{A_2}{A_1} \frac{qx - p}{1 - x}, \quad x \in \left(\frac{p}{q}, 1\right) \tag{61}$$

which is a monotonically increasing function of the variable $x \in (\frac{p}{q}, 1)$ as well. This fact allows one to formulate the results obtained in terms of the ratio of the volumes of the modern and old technology sectors.

In particular, (61) implies that if the value $x_*(t)$ of an optimal trajectory $x_*(\cdot)$ approaches its lower bound p/q as $t \to \infty$, then the corresponding ratio $y(x_*(t))$ of the volumes of the modern and old technology sectors goes to zero, meaning that the old technology sector dominates the modern technology sector and develops more rapidly. If the value $x_*(t)$ of an optimal trajectory $x_*(\cdot)$ approaches its upper bound 1 as $t \to \infty$, then the corresponding ratio $y(x_*(t))$ goes to infinity, meaning that the market share of the old technology sector vanishes.

Denote by $\bar{y}$ the value of $y(\cdot)$ corresponding to the steady state $\bar{x}$, i.e.

$$\bar{y} = y(\bar{x}) = \frac{A_2}{A_1} \frac{q\bar{x} - p}{1 - \bar{x}}. \tag{62}$$

Let us now describe all possible optimal investment strategies:

(i) Both before and after the shock,[8] all income after consumption is invested in the old technology sector: $v_*(t) \equiv 0$, $t \in [0, T)$, and $\hat{v}_*(t) \equiv 0$, $t \in [T, \infty)$. This strategy applies only when the productivity of the modern technology sector is larger than the productivity of the old technology sector (after the abatement) both before and after the shock:

$$b_1 A_1 \leq pb_2 A_2, \qquad b_1 A_1 \leq qb_2 A_2.$$

[8]Here and below, under the optimal strategy before the shock, we mean the optimal control $v_*(\cdot)$ in problem (P2) on the time interval $[0, T)$; analogously, under the optimal strategy after the shock, we mean the strategy $\hat{v}_*(\cdot)$ that maximizes the utility functional (11) on the rest infinite time interval $[T, \infty)$.

Note that in this case condition (59) holds, which can be rewritten as

$$\nu q(b_1 A_1 - p b_2 A_2) \leq p(q b_2 A_2 - b_1 A_1)(\rho + (1 - u_0) q b_2 A_2).$$

This corresponds to case I of Theorem 2. From the economical point of view, this situation is not so interesting: the sanctions are not effective enough to reduce polluting technologies and the economy is in fact running down the modern technology sector.

(ii) In the first period (before the shock), all income after consumption is invested in the old technology sector, $v_*(t) = 0$ for $t \in [0, T)$, while in the second period (after the shock), all income is invested in the modern technology sector, $\hat{v}_*(t) = 1$ for $t \in [T, \infty)$. This strategy applies when the productivity of the modern technology sector is greater than the productivity of the old technology sector in the second period, but is "much" less than the productivity of the old technology sector in the first period:

$$p b_2 A_2 < b_1 A_1 < q b_2 A_2 \quad \text{and} \quad \nu \frac{b_1 A_1 - p b_2 A_2}{q b_2 A_2 - b_1 A_1} \leq \frac{p}{q} \rho + (1 - u_0) p b_2 A_2.$$

In this case condition (59) still holds; i.e., we are again under the case I of Theorem 2, but the sanctions are effective.

(iii) In the first period all income after consumption is invested in the modern technology sector until the ratio of the capital stocks $K_1(t)/K_2(t)$ reaches the value $\bar{y}$ (see (62)) at some instant $\tau > 0$ (or until the shock if it happens earlier), i.e., either $v_*(t) = 1$ for $t \in [0, \tau)$ if $\tau < T$ and then the investments are divided between the two sectors in a certain ratio until the shock happens, $v_*(t) = \frac{b\bar{x} - c}{b\bar{x} - c + a - a\bar{x}}$ for $t \in [\tau, T)$, or $v_*(t) = 1$ for $t \in [0, T)$ if $\tau \geq T$; after the shock all income is again invested in the modern technology sector, $\hat{v}_*(t) = 1$ for $t \geq T$. This strategy applies when the ratio of the capital stocks is small at the initial moment,

$$\frac{K_1(0)}{K_2(0)} < \bar{y},$$

and when the productivity of the modern technology sector belongs to a certain intermediate interval between the productivity of the old technology sector in the second period and the productivity of the old technology sector in the first period:

$$\begin{gathered} p b_2 A_2 < b_1 A_1 < q b_2 A_2 \quad \text{and} \\ \frac{p}{q} \rho + (1 - u_0) p b_2 A_2 < \nu \frac{b_1 A_1 - p b_2 A_2}{q b_2 A_2 - b_1 A_1} < \rho + (1 - u_0) b_1 A_1. \end{gathered} \tag{63}$$

This situation falls into case III of Theorem 2. The sanctions are effective.

(iv) In the first period the investments are divided between the two sectors in a certain ratio, $v_*(t) = \frac{b\bar{x} - c}{b\bar{x} - c + a - a\bar{x}}$ for $t \in [0, T)$; after the shock all income is invested in the modern technology sector, $\hat{v}_*(t) = 1$ for $t \in [T, \infty)$. This strategy applies only when (63) holds and the ratio of the capital stocks at the initial moment is exactly $\bar{y}$,

$$\frac{K_1(0)}{K_2(0)} = \bar{y}.$$

This is again case III of Theorem 2. However, the situation is not very realistic, because it is hardly likely that the ratio of the capital stocks at the initial moment turns out to be exactly $\bar{y}$. This case may be considered as "exotic."

(v) The situation similar in a sense to (iii): In the first period all income after consumption is invested in the old technology sector until the ratio of the capital stocks $K_1(t)/K_2(t)$ reaches the value $\bar{y}$ at some instant $\tau > 0$ (or until the shock if it happens earlier), i.e. either $v_*(t) = 0$ for $t \in [0, \tau)$ if $\tau < T$ and then the investments are divided between the two sectors in a certain ratio until the shock happens, $v_*(t) = \frac{b\bar{x}-c}{b\bar{x}-c+a-a\bar{x}}$ for $t \in [\tau, T)$, or $v_*(t) = 0$ for $t \in [0, T)$ if $\tau \geq T$; and after the shock all income is invested in the modern technology sector, $\hat{v}_*(t) = 1$ for $t \in [T, \infty)$. This strategy applies when the ratio of the capital stocks is large at the initial moment,

$$\frac{K_1(0)}{K_2(0)} > \bar{y},$$

and when the productivity of the modern technology sector belongs to the intermediate interval between the productivity of the old technology sector in the second period and the productivity of the old technology sector in the first period, i.e., when (63) holds. This is again case III of Theorem 2.

(vi) Both before and after the shock, all income after consumption is invested in the modern technology sector: $v_*(t) \equiv 1$, $t \in [0, T)$, and $\hat{v}_*(t) \equiv 1$, $t \in [T, \infty)$. This strategy applies either when the productivity of the modern technology sector is higher than the productivity of the old technology sector from the very beginning,

$$b_1 A_1 \geq p b_2 A_2, \qquad b_1 A_1 \geq q b_2 A_2,$$

or when the productivity of the modern technology sector is less than the productivity of the old technology sector before the shock but is close to the latter,

$$p b_2 A_2 < b_1 A_1 < q b_2 A_2 \quad \text{and} \quad \nu \frac{b_1 A_1 - p b_2 A_2}{q b_2 A_2 - b_1 A_1} \geq \rho + (1 - u_0) b_1 A_1.$$

This situation falls into case II of Theorem 2.

Cases (i) and (vi) are bang-bang solutions. All resources are from the beginning invested in one of the sectors and the investment decision does not change in spite of the jump in the abatement costs at time $T > 0$. The other sector is then moderately run down, and this process is determined only by the value of the depreciation coefficient δ (if it is positive). Both technologies co-exist as long as they are productive, but the losing technology sector's capital stock depreciates. This point distinguishes our model from traditional models of creative destruction (see, e.g., Aghion and Howitt 1998; Wälde 2002, 2007).

In case (ii) the decision on the allocation of investments is based only on the current productivity of technologies. In other words, the government knows about expected sanctions but does not change its investment policy until sanctions come into effect.

In cases (iii)–(v) there exists a steady state $\bar{y}$, and at the initial stage in cases (iii) and (v) all resources are invested in one of the sectors so as to drive the ratio $K_1(t)/K_2(t)$ of the capital stocks to this state on some finite time interval $[0, \tau]$, $\tau > 0$, while in case (iv) the ratio $K_1(0)/K_2(0)$ already coincides with this steady state. The steady state is determined by the initial productivity of capital, the time preference ρ, the proportionality coefficient of the Poisson distribution ν and the level of the abatement costs.

Note that a higher proportionality coefficient of the Poisson distribution ν works in favor of the modern technology sector, while a higher time preference ρ works in favor of the old technology sector.

Indeed, let $c < a < b$, i.e., $pb_2A_2 < b_1A_1 < qb_2A_2$. If ν is sufficiently small, then condition (59) holds and all investments are attracted by the old technology sector. When ν increases, the left-hand side of (59) becomes greater than the right-hand side, i.e., a steady state $\bar{x}$ appears. One can check that the derivative $\frac{\partial \bar{x}}{\partial \nu}$ is positive. Hence, as ν increases, $\bar{x}$ increases, and $v(\cdot)$ as a function of the state variable x also increases; thus investments in the modern technology sector grow. When ν increases further, $\bar{x}$ reaches the point $x = 1$ and condition (60) becomes valid, i.e., all investments are attracted by the modern technology sector.

Similarly, if ρ is sufficiently large, then condition (59) holds and all investments are attracted by the old technology sector. When ρ decreases, a steady state $\bar{x}$ may appear. One can check that the derivative $\frac{\partial \bar{x}}{\partial \rho}$ is negative. Hence, as ρ decreases, $\bar{x}$ increases, and $v(\cdot)$ as a function of the state variable x also increases; thus investments in the modern technology sector grow. When ρ decreases further, $\bar{x}$ may reach the point $x = 1$ and condition (60) may become valid; in this case all investments are driven to the modern technology sector.

5 Conclusions

This paper shows how a rational government could adjust to an expected environmental change. The time of the change is assumed to be random, as the government does not exactly know when it will occur, but knows that the abatement costs for production with old polluting technology will rise in the future. The government needs to take into account the change in the abatement costs already today, when it decides on the allocation of resources between the two sectors.

In this paper the decision on the allocation of resources between the sectors depends only on the productivity of capital and the abatement costs. If the productivity of capital in the clean technology sector is higher than in the dirty technology sector, there is a bang-bang solution and all resources are from the beginning invested in the modern technology sector. A modernization of the economy occurs. On the other hand, if the abatement costs are too low to punish for the pollution, the old technology sector receives all the investments in both periods (before and after the shock).

If the relative productivity between the two sectors switches at time T in favor of the modern technology sector, there may exist an intermediate solution in which, first, the optimal ratio of the capital stocks of the two sectors is reached and then the resources are invested simultaneously in both sectors.

For this intermediate solution we show that when the time preference decreases, meaning that for consumers the difference between consumption today and in the future diminishes, the value of the optimal ratio of the capital stocks of the sectors increases in favor of the old technology sector. Similarly a decrease in the Poisson distribution coefficient favors investments in the old technology sector.

For the abatement costs to have an impact on the resource allocation, they must be large enough to compensate for the lower productivity in the modern technology sector. Thus, the international community, when setting up global sanctions for polluting technologies, needs to be aware of this fact. It is interesting that the value of production declines in the second period as sanctions on the old technology sector increase.

We also show that the transition to the more productive technology is slow and is determined by the depreciation rate of capital. If all resources are invested in the modern technology sector, the old technology sector is run down at the depreciation rate. Similarly, if the old technology sector after sanctions is still more productive, the modern technology sector is run down, and the economy focuses on the old technology all the time. In our model we assume that it is not rational to eliminate the less productive sector completely as long as it is productive, even if this is technologically fully possible. This distinguishes our results from traditional models of technology destruction, where the less productive sector is assumed to disappear immediately through a process of creative destruction.

In this paper, our model is limited to the solution where the proportion of consumption is fixed in relation to national income. This paper is a first attempt to use our two-sector setup with an exogenous random shock for economic modelling. We plan to complete this setup in subsequent works and to include more specific characteristics of the two sectors and the technological processes.

We plan to develop potential applications of our model to other economic growth problems. Note that in some applications (e.g., to the R&D sector), the case of a "favorable" random shock, when the productivity of capital (or price) after the shock increases, may also have good reasons to be considered. This case is described by the inequality $p > q$ in our model and is analyzed in the mathematical part of the present paper.

Appendix

Here we present another way of solving problem (P2), without using optimal control theory.

From (26) we can express $v(t)$ a.e. on $[0, \infty)$ as a function of $x(t)$ and $\dot{x}(t)$:

$$v(t) = \frac{\dot{x}(t) + bx(t) - c}{(b-a)x(t) + a - c}, \qquad 1 - v(t) = \frac{a - ax(t) - \dot{x}(t)}{(b-a)x(t) + a - c}.$$

The denominator is always nonzero, because $bx(t) > c$ and $ax(t) < a$ for $p < q$, while $bx(t) < c$ and $ax(t) > a$ for $p > q$.

Then for the instantaneous utility $g(x(t), v(t))$, we have

$$\begin{aligned} g(x(t), v(t)) &= av(t) + b(1 - v(t)) + \nu \ln x(t) \\ &= \frac{(a-b)\dot{x}(t) + a(b-c)}{(b-a)x(t) + a - c} + \nu \ln x(t) \\ &= -\frac{d}{dt} \ln\left|(b-a)x(t) + a - c\right| + \frac{a(b-c)}{(b-a)x(t) + a - c} + \nu \ln x(t). \end{aligned}$$

Integrating by parts the first term on the right-hand side, we obtain (see (31))

$$\begin{aligned} J_2(x(\cdot), v(\cdot)) = \ln|(b-a)x(0) + a - c| & \\ + \int_0^\infty e^{-(\nu+\rho)t} \Bigg[& -(\nu+\rho) \ln\left|(b-a)x(t) + a - c\right| \\ & + \frac{a(b-c)}{(b-a)x(t) + a - c} + \nu \ln x(t) \Bigg] dt. \end{aligned}$$

Let us find positive extremum points of the function

$$g_1(x) = -(\nu+\rho) \ln\left|(b-a)x + a - c\right| + \frac{a(b-c)}{(b-a)x + a - c} + \nu \ln x.$$

We have

$$\begin{aligned} g_1'(x) &= -\frac{(\nu+\rho)(b-a)}{(b-a)x + a - c} + \frac{\nu}{x} - \frac{a(b-c)(b-a)}{((b-a)x + a - c)^2} \\ &= \frac{-(\nu+\rho)(b-a)x(bx - c + a - ax) + \nu(bx - c + a - ax)^2 - a(b-c)(b-a)x}{((b-a)x + a - c)^2} \\ &= \frac{-P(x)}{((b-a)x + a - c)^2}, \end{aligned}$$

where $P(x)$ is defined by (50).

If $P(\min\{1, p/q\}) < 0$ and $P(\max\{1, p/q\}) > 0$, then $\bar{x}$ (see (51)) is a unique maximum point of $g_1(\cdot)$ on the interval $(\min\{1, p/q\}, \max\{1, p/q\})$, $g_1(\cdot)$ increases for $x < \bar{x}$, and $g_1(\cdot)$ decreases for $x > \bar{x}$. Thus, the optimal process $x_*(\cdot)$ must reach the point $\bar{x}$ with maximum possible velocity ($v_* \equiv 0$ or 1) and then stay at $\bar{x}$.

If $P(\min\{1, p/q\}) \geq 0$, then $g_1(\cdot)$ decreases on the whole interval $(\min\{1, p/q\}, \max\{1, p/q\})$, and hence the optimal process $x_*(\cdot)$ must decrease with maximum possible velocity (and tend to $\min\{1, p/q\}$) on the whole infinite time interval $[0, \infty)$.

If $P(\max\{1, p/q\}) \leq 0$, then $g_1(\cdot)$ increases on the whole interval $(\min\{1, p/q\}, \max\{1, p/q\})$, and hence the optimal process $x_*(\cdot)$ must increase with maximum possible velocity (and tend to $\max\{1, p/q\}$) on the whole infinite time interval $[0, \infty)$.

Obviously, this gives the same optimal synthesis as in Theorem 2.

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Prices Versus Quantities in a Vintage Capital Model

Thierry Bréchet, Tsvetomir Tsachev, and Vladimir M. Veliov

Abstract The heterogeneity of the available physical capital with respect to productivity and emission intensity is an important factor for policy design, especially in the presence of emission restrictions. In a vintage capital model, reducing pollution requires to change the capital structure through investment in cleaner machines and to scrap the more polluting ones. In such a setting we show that quantity-based or a price-based regulation may yield contrasting outcomes. We also show that some failures in the permits market may undermine its efficiency and that imposing the emission cap over longer periods plays a regularizing role in the market, that is, ensures a positive market price of permits and decreases its volatility.

1 Introduction

It is well-established in the economic literature that regulating pollution through prices (e.g. emission charges) or quantities (e.g. emission quotas) is equivalent. Both yield the same resource allocation and welfare level. In his seminal paper Weitzman (1974) showed that such equivalence does not hold anymore when information is imperfect, be it on pollution abatement costs or damages. Following Weitzman, many papers elaborated on the uncertainty issue and the choice of policy instrument (for some recent papers, see Zhao 2003; Krysiac 2008). A few authors introduced alternative motives for which this equivalence may not hold. As examples, Finkelshtain and Kislev (1997) found political motives, and Kelly (2005) stressed the role of general equilibrium effects. The contribution of our paper is to show that this equivalence may not hold even under perfect information, simply because the capital stock that generates pollution is not homogeneous. In a vintage capital model, reducing pollution requires to change the capital structure through investment in cleaner machines and scrapping the more polluting ones. In such a setting we show that emission tax and auction emission permits may yield contrasting outcomes.

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T. Bréchet (✉)
CORE, and Chair Lhoist Berghmans in Environmental Economics and Management, Université catholique de Louvain, Voie du Roman Pays, 34, 1348 Louvain-la-Neuve, Belgium
e-mail: thierry.brechet@uclouvain.be

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The heterogeneity of the available physical capital with respect to productivity and emission intensity is an important factor for policy design, especially in the presence of emission restrictions. To decide which machines to scrap and which machines to buy is an indispensable right of the firm's management. The vintage model of a firm that we employ is essentially a version of the one introduced in Barucci and Gozzi (1998, 2001) and investigated by several authors (see also Feichtinger et al. (2006, 2008) and the bibliography therein). In particular, the model allows for investing and scrapping in technologies of any vintage.[1] Polluting emissions are regulated either by a tax or with auctioned pollution permits.

When the price of emission is endogenized by auctioning tradable emission permits (and not set as a tax) it has to be determined by a market equilibrium equation that involves the emission paths resulting from the optimal behavior of the participating firms. The derivation and the investigation of this equation is the main goal of this paper. It turns out that the equation for the auction price of emissions (being a Fredholm integral equation of the first kind) is ill-posed, in the sense that (i) it may fail to have a solution; (ii) it may have multiple solutions; (iii) it may not have a positive solution (in which case the solution does not represent a market price); (iv) the solution, even it exists and is unique and positive, may be highly volatile and fluctuating. For this reason we introduce period-wise restrictions on the emission, which correspond to the *commitment periods* in the terminology of the Kyoto protocol. It is argued below that such a period-wise emission restriction plays a regularizing role on the auction price of emissions. Namely, a sufficiently large commitment period ensures existence, uniqueness (under somewhat restrictive conditions) and (according to numerical evidence) positiveness of the solution of the auction price equation. In addition it decreases the volatility of the auction market.

The paper is organized as follows. In Sect. 2 we present the basic model of a firm facing an exogenous cost for emissions (a tax) and we characterize its optimal behavior. The model is of vintage-type, that is, the physical capital is differentiated with respect to technologies of different dates. In Sect. 3 we present some properties and analytic expressions of the emission of the firm along its optimal path as a function of the exogenous costs. Based on this, in Sect. 4 we investigate the equilibrium equation for the auction of emission permits and introduce period-wise emission restrictions (commitment periods) as a tool for regularization of the auction price. Then in Sect. 5 we present some numerical results supporting the regularizing role of the period-wise emission restrictions.

2 The Firm's Problem with an Emission Tax

The model of the firm presented below is a version of the PDE-vintage models introduced in Barucci and Gozzi (1998, 2001) and further investigated and applied

[1]This is a substantial difference with the delay-differential equation models, see Boucekkine et al. (2004, 2005).

in a large number of contributions (see Feichtinger et al. 2006, 2008 and the literature therein). The formal difference of our model with the abovementioned ones is technical and not essential from a methodological point of view.

First we describe the basic model of a firm that is composed of machines of different vintages (technologies) τ: $K(\tau, s)$ will denote the capital stock of vintage τ and of age s. That is, $K(\tau, t-\tau)$ is the stock of vintage τ that exists at time $t \geq \tau$. The maximal life-time of machines of each technology will be denoted by ω, and the depreciation rate of each technology—by δ. Both are assumed independent of the vintage just for notational convenience. At any time $t > 0$ the firm may invest with intensity $I(\tau, s)$ in machines of vintage τ that are of age s at time t (so that $s = t - \tau$).

The planning horizon of the firm is $[0, \infty)$, therefore the stock of machines of vintage $\tau \in [-\omega, 0]$ which are present in the firm at time $t = 0$ is considered as exogenous and will be denoted by $K_0(\tau)$. These machines have age $-\tau$ at time $t = 0$ and may be in use until they reach age ω. Machines of vintage $\tau > 0$ may be in use for ages $s \in [0, \omega]$, and their stock at age zero equals zero. Therefore $K_0(\tau)$ will be defined as zero for $\tau > 0$. The ages of possible use of any vintage can be written as $[s_0(\tau), \omega]$, where $s_0(\tau) = \max\{0, -\tau\}$.

Summarizing, the dynamics of each vintage $\tau \in [-\omega, \infty)$ is given by the equation

$$\dot{K}(\tau, s) = -\delta K(\tau, s) + I(\tau, s), \quad K(\tau, s_0(\tau)) = K_0(\tau), \quad s \in [s_0(\tau), \omega], \tag{1}$$

where "dot" means everywhere the derivative with respect to s (the argument representing the age).

The productivity of machines of vintage τ is denoted by $f(\tau)$, while $g(\tau)$ denotes the emission per machine of vintage τ. The firm faces costs due to emissions at an exogenous price $v(t)$ per unit of emission. At this stage of the paper $v(t)$ represents a tax on pollution set up by a regulator. This price will be endogenized later on when auctioned emission permits will be considered. Due to this cost the firm may decide to (possibly temporarily) switch off a part of the machines. Let us denote by $W(\tau, s) \in [0, 1]$ the fraction of the machines of vintage τ that operate at age s.

The cost of investment I in s years old machines of any vintage will be denoted by $C(s, I)$.

The present value (at time $t = 0$) of the total production of machines of vintage τ, discounted with a rate r, is

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} f(\tau) K(\tau, s) W(\tau, s) ds,$$

the cost of emission is

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} v(\tau + s) g(\tau) K(\tau, s) W(\tau, s) ds,$$

and the investment costs are

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} C(s, I(\tau, s)) ds.$$

The firm maximizes the aggregated over time discounted net revenue, that is, solves the problem

$$\max_{I \geq 0, W \in [0,1]} \int_{-\omega}^{\infty} e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} [(f(\tau) - v(\tau + s) g(\tau)) K(\tau, s) W(\tau, s) \\ - C(s, I(\tau, s))] ds d\tau \tag{2}$$

subject to (1).

The emission of the firm at time $t > 0$ is given by the expression

$$E(t) = \int_{t-\omega}^{t} g(\tau) K(\tau, t - \tau) W(\tau, t - \tau) d\tau. \tag{3}$$

Remark 1 Due to the specific form of the problem there is an easy way to define optimality even though the optimal value in (2) may be infinite. Namely (I^*, W^*, K^*) is a solution of (2), (1) if for any $T > 0$ the restriction of these functions to $D_T = \{(\tau, s) : \tau \in [-\omega, T],\ s \in [s_0(\tau), \omega]\}$ is an optimal solution of the problem in which the integration in (2) is carried out on D_T.

Problem (2), (1) fits to the general framework of heterogeneous optimal control systems developed in Veliov (2008). However, the present problem can be treated also by the classical Pontryagin optimality conditions for ODEs since it decomposes along vintages: every technology vintage $\tau \in [-\omega, \infty)$ is managed separately by solving the problem

$$\max_{i \geq 0, w \in [0,1]} \int_{s_0(\tau)}^{\omega} e^{-rs} [(f(\tau) - v(\tau + s) g(\tau)) k(s) w(s) - C(s, i(s))] ds \tag{4}$$

subject to

$$\dot{k}(s) = -\delta k(s) + i(s), \quad k(s_0(\tau)) = K_0(\tau), \quad s \in [s_0(\tau), \omega]. \tag{5}$$

If for any fixed $\tau \in [-\omega, \infty)$ the triple $(i(\cdot), w(\cdot), k(\cdot)) = (I(\tau, \cdot), W(\tau, \cdot), K(\tau, \cdot))$ is a solution of this problem, then (I, W, K) is a solution of (2), (1) and vice versa.[2]

The following is assumed throughout the paper.

(A1) The exogenous data K_0, f, g are non-negative and continuous, f and g are continuously differentiable, $g > 0$, $f' > 0$, $g' \leq 0$, $r \geq 0$, $\delta \geq 0$;

[2] This is not a self-evident fact, but can easily be proven in natural space settings for the two problems and on the assumptions made below.

(A2) The cost function $C(s,i)$ is two times differentiable in i, the derivatives C_i' and C_{ii}'' are continuous in (s,i), $C(s,0)=0$, $C_i'(s,0)\geq 0$, $C_{ii}''\geq \varepsilon_C>0$;

(A3) The price of emission, $v(\cdot)$, is a measurable bounded function.[3]

Assumption (A1) means that newer machines are more productive and less polluting than older machines Under these conditions problem (2), (1) has a unique solution $(I^*[v], W^*[v], K^*[v])$. The corresponding emission given by (3) will be denoted by $E^*[v]$.

Since W enters only in the objective function, clearly we have

$$W^*[v](\tau,s)=\begin{cases}0 & \text{if } f(\tau)-v(\tau+s)g(\tau)\leq 0,\\ 1 & \text{if } f(\tau)-v(\tau+s)g(\tau)>0.\end{cases} \tag{6}$$

The optimal control i^* of problem (4), (5) for a fixed τ and $v(\cdot)$ is easy to obtain by applying the Pontryagin maximum principle (Pontryagin et al. 1962). Namely,

$$i^*(s)=\begin{cases}0 & \text{if } \xi^*(s)<C_i'(s,0),\\ (C_i')^{-1}(s,\xi^*(s)) & \text{if } \xi^*(s)\geq C_i'(s,0),\end{cases} \tag{7}$$

where $\xi\to(C_i')^{-1}(s,\xi)$ is the inverse of the function $i\to C_i'(s,i)$ and $\xi^*(s)$ is the unique solution of the *adjoint equation*

$$\dot{\xi}(s)=(r+\delta)\xi(s)-(f(\tau)-v(\tau+s)g(\tau))w^*(s), \qquad \xi(\omega)=0. \tag{8}$$

3 The Emission Mapping

An exogenously given tax on emission, that is a function $v(t)$ as in assumption (A3), determines the optimal emission of the firm

$$E^*[v](t)=\int_{t-\omega}^{t} g(\tau)K^*[v](\tau,t-\tau)W^*[v](\tau,t-\tau)d\tau, \tag{9}$$

where $(I^*[v], W^*[v], I^*[v])$ is the optimal solution at the firm level corresponding to v. In this section we investigate in some more details the mapping $v\to E^*[v]$.

Since the emission restriction takes effect at time $\hat{t}$, we consider v as equal to zero before $\hat{t}$. For a technical convenience we assume that $\hat{t}\geq\omega$, although this is not essential.

Denote by $\mathcal{V}$ the space of all measurable and locally bounded functions $v:[0,\infty)\mapsto\mathbf{R}$ that equal zero on $[0,\hat{t})$. The space $\mathcal{V}$ will be sometimes considered as a subspace of $L_\infty^{\text{loc}}(0,\infty)$, the latter consisting of all measurable function that are bounded on every bounded interval $[0,T]$.

[3] For reasons that will become clear later we formally allow for negative values of $v(t)$.

The next proposition claims a specific Lipschitz continuity property of E^*. Since we allow for negative values of v we denote $v_- = \min\{0, v\}$. In the text below we shall use also the notation

$$\sigma(\tau, v) = f(\tau) - vg(\tau).$$

Proposition 1 *Assume that* (A1)–(A3) *hold. There is a constant c and a non-increasing function* $[0, \infty) \ni t \to \gamma_t > 0$ *such that for every $v', v'' \in \mathcal{V}$ and $t > \hat{t}$*

$$|E^*[v'](t) - E^*[v''](t)| \leq c\Bigg[\|v' - v''\|_{L_\infty[t-\omega,t+\omega]} + \frac{f(t+\omega) + \|v'_- + v''_-\|_{L_\infty[t-\omega,t+\omega]}}{\gamma_t}|v'(t) - v''(t)|\Bigg].$$

Proof The function γ_t can be explicitly defined as $\gamma_t = \inf_{\tau \leq t} \gamma(\tau) > 0$, where $\gamma(\tau) = f'(\tau) - \frac{f(-\omega)}{g(-\omega)} g'(\tau)$, which is strictly positive according to (A1).

For arbitrarily fixed $v', v'' \in \mathcal{V}$ denote $\Delta K = K^*[v'] - K^*[v'']$, $\Delta I = I^*[v'] - I^*[v'']$, $\Delta W = W^*[v'] - W^*[v'']$, $\Delta\xi = \xi^*[v'] - \xi^*[v'']$. We have

$$|E^*[v'](t) - E^*[v''](t)| \leq \int_{t-\omega}^{t} g(\tau)|\Delta K(\tau, t-\tau)|W^*[v'](\tau, t-\tau)d\tau + \int_{t-\omega}^{t} g(\tau)K^*[v''](\tau, t-\tau)|\Delta W(\tau, t-\tau)|d\tau. \tag{10}$$

Using (6) we obtain from the adjoint equation (8), which is satisfied by $\xi[v_k](s)$ for (almost) every τ, that

$$\begin{aligned}|\Delta\xi(\tau, s)| &\leq \int_{s}^{\omega} e^{-(r+\delta)(\theta-s)}|\chi(f(\tau) - v'(\tau+\theta)g(\tau)) \\ &\quad - \chi(f(\tau) - v''(\tau+\theta)g(\tau))|d\theta \\ &\leq \omega\bar{g}\|v' - v''\|_{L_\infty[\tau,\tau+\omega]},\end{aligned}$$

where $\chi(x) = x$ for $x > 0$, $\chi(x) = 0$ for $x \leq 0$, $\bar{g} = g(-\omega)$ is an upper bound for g, and we use that χ is Lipschitz continuous with Lipschitz constant equal to one.

From assumption (A2) we easily obtain that $(C'_i)^{-1}(s, \cdot)$ is Lipschitz continuous with Lipschitz constant $1/\varepsilon_C$, thus according to (7)

$$|\Delta I(\tau, s)| \leq \frac{\omega\bar{g}}{\varepsilon_C}\|v' - v''\|_{L_\infty[\tau,\tau+\omega]}.$$

Then from (1) or (5) we obtain that

$$|\Delta K(\tau, s)| \leq \frac{\omega^2\bar{g}}{\varepsilon_C}\|v' - v''\|_{L_\infty[\tau,\tau+\omega]}, \quad s \in [s_0(\tau), \omega].$$

Thus the first term in (10) can be estimated by

$$\frac{\omega^3 \bar{g}^2}{\varepsilon_C} \|v' - v''\|_{L_\infty[t-\omega,t+\omega]}. \tag{11}$$

To estimate the second term in (10) we first note that by a similar argument as above we estimate

$$\xi[v''](\tau,s) \leq \int_s^\omega e^{-(r+\delta)(\theta-s)} \chi(f(\tau) - v''(\tau+\theta)g(\tau))d\theta$$
$$\leq \omega\big(f(\tau) + \bar{g}\|v''_-\|_{L_\infty[\tau,\tau+\omega]}\big).$$

Using that the function $(C_i')^{-1}(s,\cdot)$ is Lipschitz continuous with Lipschitz constant $\leq 1/\varepsilon_C$, which is implied in a standard way by (A2), and (7) we obtain that

$$I[v''](\tau,s) \leq \frac{\omega}{\varepsilon_C}\big(f(\tau) + \bar{g}\|v''_-\|_{L_\infty[\tau,\tau+\omega]}\big).$$

Hence, using that $t \geq \hat{t} \geq \omega$,

$$K[v''](\tau,s) \leq \frac{\omega^2}{\varepsilon_C}(f(\tau) + \bar{g}\|v''_-\|_{L_\infty[\tau,\tau+\omega]}). \tag{12}$$

It remains to estimate the term

$$|\Delta W(\tau, t-\tau)| = \begin{cases} 0 & \text{if } \sigma(\tau, v'(t)) \text{ and } \sigma(\tau, v''(t)) \text{ are both positive} \\ & \text{or both non-positive,} \\ 1 & \text{otherwise} \end{cases}$$

in (10). To do this we compare the functions $\sigma(\tau, v'(t))$ and $\sigma(\tau, v''(t))$, assuming without any restriction that $v'(t) \leq v''(t)$. Clearly $\Delta W(\tau, t-\tau) \neq 0$ (and equals one) for some $\tau \in [t-\omega, t]$ if and only if $\sigma(\tau, v''(t)) \leq 0 < \sigma(\tau, v'(t))$.

Note also that if $v < 0$, then $\sigma(\tau, v)$ and $\sigma(\tau, 0)$ have both positive signs, hence the sign of $\sigma(\tau, v)$ does not change if we replace v with $v_+ := \max\{0, v\}$. Since $|v'_+ - v''_+| \leq |v' - v''|$, the estimations below would not change if we replace $v'(t)$ and $v''(t)$ with $v'(t)_+$ and $v''(t)_+$, or equivalently, if we assume that $v'(t) \geq 0$.

Due to (A1) (implying that $\sigma(\tau, v)$ is strictly increasing for $v \geq 0$) and (A3) the set

$$\{\tau \in [t-\omega, t] : \sigma(\tau, v''(t)) \leq 0 < \sigma(\tau, v'(t))\} \tag{13}$$

(if nonempty) is an interval $(\tilde{\tau} - \nu, \tilde{\tau}]$, where $\sigma(\tilde{\tau}, v''(t)) = 0$ and $\sigma(\tilde{\tau} - \nu, v'(t)) \geq 0$. Then

$$\sigma(\tilde{\tau} - \nu, v''(t)) = \sigma(\tilde{\tau}, v''(t)) - \dot{\sigma}(\tilde{\tilde{\tau}}, v''(t))\nu = -\dot{\sigma}(\tilde{\tilde{\tau}}, v''(t))\nu,$$

where $\tilde{\tilde{\tau}} \in [\tilde{\tau} - \nu, \tilde{\tau}]$. Subtracting the above equality from $\sigma(\tilde{\tau} - \nu, v'(t)) \geq 0$ we obtain that

$$(v''(t) - v'(t))g(\tilde{\tau} - \nu) \geq \dot{\sigma}(\tilde{\tilde{\tau}}, v''(t))\nu = \big(f'(\tilde{\tilde{\tau}}) - v''(t)g'(\tilde{\tilde{\tau}})\big)\nu. \tag{14}$$

Since

$$0 \geq \sigma(\tilde{\tilde{\tau}}, v''(t)) = f(\tilde{\tilde{\tau}}) - v''(t)g(\tilde{\tilde{\tau}}),$$

we have according to (A1) that

$$v''(t) \geq \frac{f(\tilde{\tilde{\tau}})}{g(\tilde{\tilde{\tau}})} \geq \frac{f(-\omega)}{g(-\omega)}.$$

Then (14) and $g'(\tilde{\tilde{\tau}}) \leq 0$ imply

$$(v''(t) - v'(t))g(\tilde{\tau} - \nu) \geq \left(f'(\tilde{\tilde{\tau}}) - \frac{f(-\omega)}{g(-\omega)} g'(\tilde{\tilde{\tau}}) \right) \nu = \gamma(\tilde{\tilde{\tau}})\nu \geq \gamma_t \nu.$$

Hence

$$\nu \leq \frac{\bar{g}}{\gamma_t}(v''(t) - v'(t)).$$

Thus the measure of the set in (13) does not exceed $\frac{\bar{g}}{\gamma_t}|v''(t) - v'(t)|$. Combining this with (12), (11) and (10) we obtain the claim of the proposition. □

Lemma 1 *For every $v \in \mathcal{V}$ and $t \geq \hat{t}$ the value $E^*[v](t)$ depends only on the restriction of v to $[t - \omega, t + \omega]$.*

Proof The proof of this lemma is straightforward: $E^*[v](t)$ depends only on $K^*[v](\tau, t - \tau)$ for $\tau \in [t - \omega, t]$ (see (9)), $K^*[v](\tau, t - \tau)$ depends only on $I^*[v](\tau, s)$ with $s \in [0, \omega]$, the latter depends only on $\xi^*[v](\tau, s)$, which on its turn, depends on $v(\theta)$ with $\theta \in [\tau, \tau + \omega]$ (see (8)). □

The analysis of the market price of emissions in the next section involves a rather complicated functional equation for v. Together with the general case we consider also a particular cost function $C(s, I)$ for which the equation substantially simplifies.

(A4) $C(s, I) = 1/2c(s)I^2$, where $c(s) > 0$, $s \in [0, \omega]$, is a measurable non-increasing function.

The advantage of the above assumption, which is still economically meaningful, is that the optimal investment becomes a linear function of the "shadow price" of the capital stock. Namely, (7) becomes

$$I^*[v](\tau, s) = \frac{\xi^*(\tau, s)}{c(s)}, \quad s \in [0, \omega), \tag{15}$$

where $\xi(\tau, \cdot)$ is the solution of the adjoint equation (8), that is,

$$\dot{\xi}(\tau, s) = (r + \delta)\xi(\tau, s) - \sigma(\tau, v(\tau + s))W^*[v](\tau, s), \qquad \xi(\tau, \omega) = 0. \tag{16}$$

Note that due to (A1) the function $\tau \longrightarrow \sigma(\tau, v)$, restricted to an interval $[t-\omega, t]$, either has a definite sign or has a single zero in this interval, denoted further by $\theta(t, v)$. In addition, we set $\theta(t, v) = t - \omega$ or $\theta(t, v) = t$ if $\sigma(\tau, v) > 0$ or $\sigma(\tau, v) < 0$ in $[t-\omega, t]$, respectively. With this notation we have, according to (6), that $W^*[v](\tau, t-\tau) = 1$ for $\tau \in (t-\omega, t)$ if and only if $\tau \in (\theta(t, v(t)), t)$.

Lemma 2 *Let assumptions* (A1) *and* (A4) *hold and let* $v \in \mathcal{V}$. *Then for every* $t \geq \hat{t}$

$$
\begin{aligned}
E^*[v](t) = &\int_{\theta(t,v(t))}^{t} \int_{\theta(t,v(t))\vee\theta(\alpha,v(\alpha))}^{\alpha} g(\tau)\kappa(t,\tau,\alpha-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha \\
&+ \int_{t}^{\theta(t,v(t))+\omega} e^{-(r+\delta)(\alpha-t)} \int_{\theta(t,v(t))\vee\theta(\alpha,v(\alpha))}^{t} g(\tau)\kappa(t,\tau,t-\tau) \\
&\times \sigma(\tau, v(\alpha))d\tau\, d\alpha \\
&+ \int_{\theta(t,v(t))+\omega}^{t+\omega} e^{-(r+\delta)(\alpha-t)} \int_{\theta(\alpha,v(\alpha))}^{t} g(\tau)\kappa(t,\tau,t-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha,
\end{aligned}
$$

where $a \vee b := \max\{a, b\}$ *and*

$$
\kappa(t,\tau,\beta) = \int_0^{\beta} \frac{1}{c(s)} e^{-\delta(t-\tau-s)-(r+\delta)(\beta-s)} ds.
$$

The proof of this lemma uses the Cauchy formula for the solution ξ of the adjoint equation (16), the formula (15) for the optimal control, the Cauchy formula for the corresponding solution $K^*[v]$ of (1), and (9). This results in an expression for $E^*[v](t)$ in the form of a triple integral depending only on the data and v, from which one can derive the expression in the lemma after a sequence of changes of the order and the variables of integration. We skip the cumbersome calculations.

Definition We call the price function $v \in \mathcal{V}$ *regular* if $f(\tau) - v(t)g(\tau) > 0$ for all $\tau \geq 0$ and $t \in (\tau, \tau + \omega)$.

In other words, regularity means that the price $v(t)$ does not invoke switch-off of existing machines. In the context of the emission restrictions the existence of a regular auction price (see next section) would mean that the environmental goals can be achieved only by appropriate investment policies (without premature scrapping).

For a regular $v \in V$ the expression for $E^*[v]$ substantially simplifies, since $\theta(t, v(t)) = t - \omega$ for all $t \geq \hat{t}$. Hence,

$$
\begin{aligned}
E^*[v](t) = &\int_{t-\omega}^{t} \int_{t-\omega}^{\alpha} g(\tau)\kappa(t,\tau,\alpha-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha \\
&+ \int_{t}^{t+\omega} e^{-(r+\delta)(\alpha-t)} \int_{\alpha-\omega}^{t} g(\tau)\kappa(t,\tau,t-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha.
\end{aligned}
$$

Having in mind the definition of $\sigma(\tau, v)$ we split each of the above outer integrals into two parts: one term depending on v, the other—independent of v. The resulting expression that is independent of v is exactly the emission corresponding to $v(t) \equiv 0$. Having in mind also that $v(t) = 0$ for $t < \hat{t}$ we obtain the following lemma.

Lemma 3 *Let assumptions* (A1) *and* (A4) *hold and let* $v \in \mathcal{V}$ *be regular. Then for every* $t \geq \hat{t}$

$$E^*[v](t) = E^*[0](t) - \int_{\hat{t} \vee (t-\omega)}^{t+\omega} \varphi(t, \alpha) v(\alpha) d\alpha, \tag{17}$$

where

$$\varphi(t, \alpha) = \begin{cases} \int_{t-\omega}^{\alpha} g^2(\tau)\kappa(t, \tau, \alpha - \tau) d\tau & \text{if } \alpha \in [t - \omega, t], \\ e^{-(r+\delta)(\alpha - t)} \int_{\alpha-\omega}^{t} g^2(\tau)\kappa(t, \tau, t - \tau) d\tau & \text{if } \alpha \in [t, t + \omega] \end{cases}$$

and $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$.

The following properties of the kernel $\varphi(t, \cdot)$ play a role in the study of the auction price of emissions defined in the next section.

Lemma 4 *Let assumptions* (A1) *and* (A4) *hold. Then for every* $t > \hat{t}$ *the function* $\varphi(t, \cdot) : [t - \omega, t + \omega]$ *has the following properties*:

$$\varphi(t, t - \omega) = \varphi(t, t + \omega) = \varphi'_\alpha(t, t - \omega) = \varphi'_\alpha(t, t + \omega) = 0,$$

$\varphi'_\alpha(t, \alpha)$ *is strictly positive* (*negative*) *on* $(t - \omega, t)$ (*on* $(t, t + \omega)$, *respectively*).

That is, the "mass" of the kernel $\varphi(t, \cdot)$ is concentrated mostly around $\alpha = t$.

4 The Auction Price of Permits and Its Regularization

We consider an economy consisting of n identical firms described by the model considered in Sect. 2. Let $\hat{E}(t)$ be a cap for the emission permits of the economy for $t \geq \hat{t}$. As before we assume that the regulation (here, the emission cap) is known by the firms at time $t = 0$ for all the future. The cap takes effect at time $\hat{t} \geq \omega$.

The emission permits are auctioned at time $\hat{t}$. The main issue investigated in this section is: does the auction (the primary market) really determine the price $v(t)$? Since the answer is "NO", we analyze theoretically and numerically the primary market behavior and the appearance of market failure.

The equation for the auction price of emission permits, $v(t)$, is

$$E^*[v](t) = \hat{E}(t)/n, \quad t \geq \hat{t}. \tag{18}$$

The market would determine the price of permits if

(i) equation (18) has a solution $v(\cdot) \in \mathcal{V}$;
(ii) the solution is unique;
(iii) the solution is positive for all t.

In general, neither of the above requirement is fulfilled. The first needs assumptions for the data, and these assumptions are difficult to specify due to the complexity of (18) (even in the special case considered in Lemma 3). The second requirement is obviously not fulfilled if, for example, $\hat{E}(t) \equiv 0$, since if v is a solution of (18), then every $\tilde{v}$ with $\tilde{v}(t) > v(t)$ would also be a solution. Even if the above two possibilities could be classified as "academic", as we show below, a failure of the third requirement could be rather realistic.

Problems for which a meaningful solution may fail to exist (in the present case—a positive market price) or the solution is not unique (indeterminacy), or a unique solution exists but is arbitrarily sensitive to perturbations, are usually called *ill-posed*. A modification (approximation) of an ill-posed problem that turns it into a well posed one is called *regularization*. Regularization methods for equations of the kind we face in (18) are developed first by A. Tikhonov in the 60-th years of the last century. The regularization that we employ below is different and has the advantage that it has a clear economic meaning and is implementable in the real market (see the next two paragraphs for further explanations).

In the simplest case, in which the price function v solving (18) is regular, it solves also the equation

$$\int_{\hat{t}\vee(t-\omega)}^{t+\omega} \varphi(t,\alpha)v(\alpha)d\alpha = E^*[0](t) - \hat{E}(t) \tag{19}$$

(we skip the division by n considering $\hat{E}(t)$ as already divided by n). This is a Fredholm integral equation of the first kind on the infinite interval $[\hat{t}, +\infty)$. Such equations are inherently ill-posed. To see that it is enough to add to a solution v a highly fluctuating term, such as $\sin(nt)$. Then $v(t) + \sin(nt)$ would be a solution of the same equation with the right-hand side modified by the quantity $\int_{\hat{t}\vee(t-\omega)}^{t+\omega} \varphi(t,\alpha)\sin(n\alpha)d\alpha$, which is arbitrarily small when n is large enough. Thus an arbitrarily small disturbance in the right-hand side can lead to a finite (even arbitrarily large) change of the solution. This difficulty is caused by the smoothing effect on v of the integration with φ: high-frequency components of v are "smoothed out". Therefore one can expect that computing v would tend to amplify any high-frequency component or irregularity of the right-hand side. In effect, the right-hand side of (19) has to be somewhat "smoother" than the solution v in order to obtain satisfactory numerical approximation (Hansen 1992).

A good numerical method to solve a Fredholm integral equation of the first kind should be able to somehow filter out the high-frequency components in the singular value expansion of the solution (if such exists). Different methods of regularization aimed at finding reasonable numerical approximation to the solution are known (Tikhonov and Arsenin 1977; Delves and Mohamed, 1985; Kress 1989). However, we stress that our aim is not just to solve the price equation (19) or some of its more complicated nonlinear versions (say, that resulting from

Lemma 2). Our ultimate goal is to imitate the behavior of the auction market, therefore the regularization we apply for solving (18) should be implementable also in the real auction market. We argue below and in the next section that a relevant regularization mean is to formulate the emission constraint (18) period-wise, rather than at any time instant. This amounts to applying *regularization by time-aggregation*, related to the Nyström's method (Delves and Mohamed, 1985, Chap. 12). In contrast to the celebrated singular expansion/decomposition this method is directly applicable also in the general (nonlinear) case of (18) and has a clear policy implementation.

Namely, we introduce the discrete version of the emission mapping E^* as follows. Assume that an emission restriction is given period-wise: $\hat{E}_k = \frac{1}{t_k - t_{k-1}} \times \int_{t_{k-1}}^{t_k} \hat{E}(t)dt$ is the cap for the emission intensity in the time-period $[t_{k-1}, t_k]$. For simplicity we assume that all time periods have the same length $h > 0$, thus $t_k = kh$, and also that $\omega = mh$, $\hat{t} = \hat{k}h$ for appropriate natural numbers m and $\hat{k}$. Correspondingly, the price of emission will be constant, v_k, on each interval $[t_{k-1}, t_k)$ and zero for $k \leq \hat{k}$. Thus instead of the space $\mathcal{V}$ of price functions we consider

$$\mathcal{V}^h = \{(v_1, v_2, \dots) : v_k \in (-\infty, +\infty),\ v_k = 0 \text{ for } k \leq \hat{k}\},$$

which can be viewed as a subset of $\mathcal{V}$ by piece-wise constant embedding of $\mathcal{V}^h$ in $\mathcal{V}$. The emission resulting from $v \in \mathcal{V}^h$ becomes a vector $E^{*h}[v] = (E_1^{*h}[v], E_2^{*h}[v], \dots)$, where

$$E_k^{*h}[v] = \frac{1}{h} \int_{t_{k-1}}^{t_k} \int_{t-\omega}^{t} g(\tau) K^*[v](\tau, t-\tau) W^*[v](\tau, t-\tau) d\tau\, dt. \tag{20}$$

Lemma 1 can be directly translated to the discrete case. It claims that $E_k^{*h}[v]$ depends only on $v_{k-m}, \dots, v_{k+m}$.

Then instead of the price equation (18) we consider the equation (skipping again the division by n)

$$E_k^{*h}[v] = \hat{E}_k^h, \quad k \geq \hat{k} \tag{21}$$

for $v \in \mathcal{V}^h$.

Under the conditions of Lemma 3 the period-wise version of the emission as function of $v \in \mathcal{V}^h$ becomes (after a change of the order of integration):

$$\begin{aligned} E_k^{*h}[v] &= E_k^{*h}[0] - \sum_{i=\hat{k}\vee(k-m)}^{k+m} \frac{1}{h} \int_{B_{ki}} \varphi(t, \alpha) dt\, d\alpha\, v_i \\ &=: E_k^{*h}[0] - \sum_{i=\hat{k}\vee(k-m)}^{k+m} a_{ki}^h v_i, \quad k \geq \hat{k}, \end{aligned} \tag{22}$$

where B_{ki} is the square $[t_i, t_{i+1}] \times [t_{k-1}, t_k]$ in the (α, t)-plane for $i = k-m, \dots, k+m-2$, while $B_{k,k-m}$ is the triangle with vertices $\{(t_{k-m-1}, t_{k-1}), (t_{k-m}, t_{k-1}), (t_{k-m}, t_k)\}$, and $B_{k,k+m}$ is the triangle with vertices $\{(t_{k+m-1}, t_{k-1}), (t_{k+m-1}, t_k),$

$(t_{k+m}, t_k)\}$. However, the numerical analysis given in the next section, as well as the proposition below apply to the general case of an emission map $E_k^{*h}[v]$ defined in (20).

In the rest of this section we address the issue of existence of a solution to the (nonlinear non-smooth) equation (21) for $v \in \mathcal{V}^h$. First of all we claim a certain Lipschitz continuity property of the mapping E^{*h}, similarly as in Proposition 1.

Proposition 2 *Assume that* (A1)–(A3) *hold. There is a constant c and a function $\gamma_t > 0$* (*the same as in Proposition* 1) *such that for every $v', v'' \in \mathcal{V}^h$ and $k > \hat{k}$*

$$|E_k^{*h}[v'] - E_k^{*h}[v'']|$$
$$\leq c\left[\max_{k-m<i\leq k+m} |v_i' - v_i''| + \frac{f(t_{k+m}) + \max_{k-m<i\leq k+m} |v_{i-}' + v_{i-}''|}{\gamma_{t_k}} |v_k' - v_k''|\right].$$

The claim of this proposition follows directly from Proposition 1 and (20).

In the existence result presented below we consider the truncated version of (21). That is, given the aggregation time-step $h > 0$ we solve the finite system of equations

$$E_k^{*h}[v] = \hat{E}_k, \quad k = \hat{k} + 1, \ldots, M, \tag{23}$$

with respect to $v = (v_{\hat{k}+1}, \ldots, v_M)$ where M is presumably a large number, so that $T = Mh$ is also “very large”. According to the discrete version of Lemma 1, $E_k^{*h}[v]$ depends only on $v_{k-m}, \ldots, v_{k+m}$. The values v_i for $i = 1, \ldots, \hat{k}$ are fixed a priori equal to zero, the values $v_{M+1}, \ldots, v_{M+m}$ will be considered as fixed parameters. Thus (23) becomes a system of $M - \hat{k}$ equations for the $M - \hat{k}$ unknown variables $v_{\hat{k}+1}, \ldots, v_M$.

It as a rather difficult task to prove that the solutions of (23) would converge to a solution of (21) when $M \longrightarrow +\infty$ (this is not simple also in the linear case (22)). The truncation of the time horizon is based on the common belief that the (hypothetical) knowledge of the economic factors thousands of years from now would not essentially influence the behavior of the economic agents in the next 100 years. Still, the plausibility of the truncation is not evident due to the possibly unlimited technological progress (this is clearly exhibited by the requirement (24) in the proof of the proposition below).

Another support for the truncation of the time horizon is provided by our numerical experiments, where the auction price for the next 80 years (to which we restrict the numerical analysis in the next section) remains practically the same when (23) is solved for $T = Mh = 150$ or more years, and also when the parameters $v_{M+1}, \ldots, v_{M+m}$ vary in a reasonable range (taking all of them equal to zero is a relevant choice, since it means that no emission cap is posed after time $T = Mh$).

Proposition 3 *Let $M > \hat{k}$ and $\bar{e} > 0$ be arbitrarily fixed. Then there exists $\bar{v}$ such that for every $\hat{E}_k$, $k = \hat{k} + 1, \ldots, M$, with $\hat{E}_k \in [0, \bar{e}]$ and any choice of $v_i \in [0, \bar{v}]$ for $i = M + 1, \ldots, M + m$ system* (23) *has a solution $v_{\hat{k}+1}, \ldots, v_M \in [-\bar{v}, \bar{v}]$.*

Proof First we shall define the number $\bar{v}$ by the requirements

$$\begin{aligned} &\bar{v} \geq 2\frac{f(Mh)}{g(Mh)}, \qquad \bar{v} \geq 2\bar{e}, \\ &\frac{h}{2}e^{-h(r+\delta)}(f(\hat{t}) + 0.5\bar{v}g(Mh)) \geq \max_{s\in[0,\omega]} C_i'\left(s, \frac{16e^{\delta h}\bar{e}}{h(\omega-h)g(Mh)}\right). \end{aligned} \tag{24}$$

The above definition of $\bar{v}$ applies to the more interesting case $h < \omega$. The alternative case requires minor changes in the last inequality.

After having $\bar{v}$ fixed so that (24) are satisfied we denote

$$\mathcal{V}_h^M(\bar{v}) = \{v = (v_{\hat{k}+1}, \dots, v_M) : |v_k| \leq \bar{v}\}.$$

The proof utilizes the Brouwer fixed point theorem. Therefore we reformulate system (23) as a fixed point equation:

$$F_k(v) = v_k, \quad k = \hat{k}+1, \dots, M, \quad \text{where } F_k(v) = v_k + \beta_k(E_k^{*h}[v] - \hat{E}_k),$$

and $\beta_k > 0$ are chosen in such a way that

$$\beta_k \max_{v\in\mathcal{V}_h^M(\bar{v})} E_k^{*h}[v] \leq \frac{\bar{v}}{2}, \quad \beta_k \leq 1.$$

The above maximum exists due to the continuity of $E_k^{*h}[v]$ and the compactness of $\mathcal{V}_h^M(\bar{v})$.

We shall prove that $F(v) = (F_{\hat{k}+1}(v), \dots, F_M(v))) \in \mathcal{V}_h^M(\bar{v})$ for $v \in \mathcal{V}_h^M(\bar{v})$, which would imply the claim of the proposition due to the Brouwer fix point theorem, since F is continuous and $\mathcal{V}_h^M(\bar{v}) \subset \mathbf{R}^{M-\hat{k}}$ is convex and compact.

Take an arbitrary $v \in \mathcal{V}_h^M(\bar{v})$ and $k \in \{\hat{k}+1, \dots, M\}$. All we have to prove is that $-\bar{v} \leq F_k(v) \leq \bar{v}$.

To prove the second inequality we consider two cases:

1. Let $v_k \geq \bar{v}/2$. Then for $\tau \in [0, Mh]$ we have $f(\tau) - v_k g(\tau) \leq f(Mh) - 0.5\bar{v}g(Mh) \leq 0$ according the first inequality in (24). Hence (see (9)) $E_k^{*h}[v] = 0$ and $F_k(v) = v_k - \beta_k\hat{E}_k \leq v_k \leq \bar{v}$.

2. Let $v_k < \bar{v}/2$. Then $F_k(v) \leq v_k + \beta_k E_k^{*h}[v] \leq \bar{v}/2 + \bar{v}/2 = \bar{v}$ according to the choice of β_k.

To prove that $F_k(v) \geq -\bar{v}$ we consider the next two cases.

3. Let $v_k \geq -\bar{v}/2$. Then $F_k(v) \geq -\bar{v}/2 - \beta_k\hat{E}_k \geq -\bar{v}/2 - \bar{e} \geq -\bar{v}$ according to the second inequality in (24).

4. Let $v_k < -\bar{v}/2$. Clearly $W^*[v](\tau, s) = 1$ if $\tau + s \in [t_{k-1}, t_k]$. Then from the adjoint equation (8) we have that for $\tau \in [t_k - \omega, t_{k-1}]$

$$\xi[v](\tau, s) = \int_s^\omega e^{-(r+\delta)(\theta-s)}(f(\tau) - v(\tau+\theta)g(\tau))W^*[v](\tau, \theta)d\theta$$

and if $s \in [t_{k-1} - \tau, t_k - \tau]$ then

$$\xi[v](\tau, s) \geq \int_s^{t_k - \tau} e^{-(r+\delta)(\theta - s)}(f(\tau) - v_k g(\tau))d\theta,$$

and if $s \in [t_{k-1} - \tau, t_{k-1} - \tau + h/2]$ then

$$\xi[v](\tau, s) \geq e^{-(r+\delta)h} \frac{h}{2}\left(f(\hat{t}) + \frac{\bar{v}}{2} g(Mh)\right).$$

Thus for $\tau \in [t_k - \omega, t_{k-1}]$ and $s \in [t_{k-1} - \tau, t_{k-1} - \tau + h/2]$

$$I^*[v](\tau, s) \geq (C_i')^{-1}\left(s, e^{-(r+\delta)h} \frac{h}{2}\left(f(\hat{t}) + \frac{\bar{v}}{2} g(Mh)\right)\right) \geq \frac{16 e^{\delta h} \bar{e}}{h(\omega - h) g(Mh)} =: I^\#,$$

where we use the last inequality in (24). Then from (1) we obtain that for $\tau \in [t_k - \omega, t_{k-1}]$ and $s \in [t_{k-1} - \tau + h/4, t_{k-1} - \tau + h/2]$

$$K^*[v](\tau, s) \geq \frac{h}{4} e^{-\delta h} I^\# = \frac{4\bar{e}}{(\omega - h) g(Mh)}.$$

Finally we estimate

$$\begin{aligned}
E_k^{*h}[v] &= \frac{1}{h}\int_{t_{k-1}}^{t_k}\int_{t-\omega}^{t} g(\tau) K^*[v](\tau, t-\tau) W^*[v](\tau, t-\tau) d\tau\, dt \\
&\geq \frac{1}{h}\int_{t_{k-1}+h/4}^{t_{k-1}+h/2}\int_{t_k-\omega}^{t_{k-1}} g(\tau) K^*[v](\tau, t-\tau) W^*[v](\tau, t-\tau) d\tau\, dt \\
&= \frac{1}{h}\int_{t_k-\omega}^{t_{k-1}}\int_{t_{k-1}-\tau+h/4}^{t_{k-1}-\tau+h/2} g(\tau) K^*[v](\tau, s) ds\, d\tau \\
&\geq \frac{1}{h}(\omega - h)\frac{h}{4} g(Mh) \frac{4\bar{e}}{(\omega - h) g(Mh)} = \bar{e}.
\end{aligned}$$

Using this we obtain $F_k(v) \geq v_k + \beta_k(E_k^{*h}[v] - \hat{E}_k) \geq v_k \geq -\bar{v}$. This proves the invariance of $\mathcal{V}_h^M(\bar{v})$ with respect to F and the proposition. □

Thus Proposition 3 ensures at least a positive answer to the issue (i) at the beginning of the section at least for the truncated equation (23). Then in the special case where assumption (A4) holds and (23) has a regular solution v with the natural choice of the parameters $v_{M+1} = \cdots = v_{M+m} = 0$, this must be the unique regular solution (due to the freedom in the choice of the right-hand side $\hat{E}_k$). Issue (iii) from the beginning of this section remains unclear, and will be somewhat enlightened by the numerical experiments in the next section. Here we only mention that under (A4) and the assumption $\hat{E}(\hat{t}) = E^*[0](\hat{t})$ (that is if the cap starts from the unconstrained emission at $\hat{t}$) the market equation (18) cannot have a regular positive solution. Indeed, if it has a regular solution v, then v solves also (19). This equation applied for $t = \hat{t}$ shows that v cannot be positive on $(\hat{t}, \hat{t} + \omega)$ due to Lemma 4.

5 Numerical Analysis of the Market Price of Emission

In this section we present some experimental results for the auction price of permits assuming that the firms participating to the auction are identical (thus the price is determined by (18), if a positive solution exists). In particular we investigate the role of the aggregation step h. Following the terminology used under the Kyoto protocol, we shall call it a *commitment period*, and h will be its length.[4] The existence and the regularity of the auction price for emission permits will be scrutinized for different h.

The following data specifications are used in the experiments:

$T = 120$—time horizon;
$\omega = 20$—maximal age of capital;
$\delta = 0.1$—depreciation rate of each technology;
$r = 0.04$—discount rate;
$K_0(\tau)$ for vintages $\tau < 0$ is obtained by solving the firm's optimization problem with $v(\cdot) \equiv 0$ on $[-\omega, T]$;
$K_0(\tau) = 0$ for vintages $\tau > 0$;
$f(\tau) = 3 + 0.03\tau$ for $\tau \in [-\omega, T]$—the productivity of one unit of capital of vintage τ;
$g(\tau) = \frac{2.5}{1.7+ln(2+\omega+\tau)} + 0.0002$ for $\tau \in [-\omega, T]$—the emissions produced by one unit of capital of vintage τ;
$C(s,i) = 20(1 - s/\omega)i + 0.5i^2$—the cost of investment i in s-years-old machines of any vintage;

In the figures below the plotted time horizon is 80 years. However, the calculations are made for a time horizon of 120 years, in order to eliminate the truncation error. The results obtained with larger time horizons are visually indistinguishable from the ones given below.

In the first group of plots (Figs. 1 and 2) a constant emission cap $\hat{E}(t) = 300$ is imposed at time $\hat{t} = 30$. This cap is below the level of emission that would be produced without any emission restriction (represented by the dash-dotted line in Fig. 1). The solid line represents the emission of the firm obeying the cap. It is remarkable that the reduction of the emission of the firm begins much earlier than $\hat{t} = 30$—this is the so called anticipation effect. The left plot in Fig. 2 represents the price of the emission permits in the primary market (as auctioned). In fact, it is obtained by using a small commitment periods of $h = 1$ year. The price is highly oscillating close to the time at which the cap is imposed, as it is expected from the explanations given at the beginning of Sect. 4. The right plot corresponds to a commitment periods of $h = 5$ years, for which the fluctuations of the price are substantially lower. Although the primary market is efficient in the considered test example (that is, it determines a positive price of the emissions) a larger commitment period h clearly decreases the volatility of the market.

[4]Under the Kyoto protocol of the UN Framework Convention on Climate Change, the (first) commitment period is a five-year period covering 2008 to 2012.

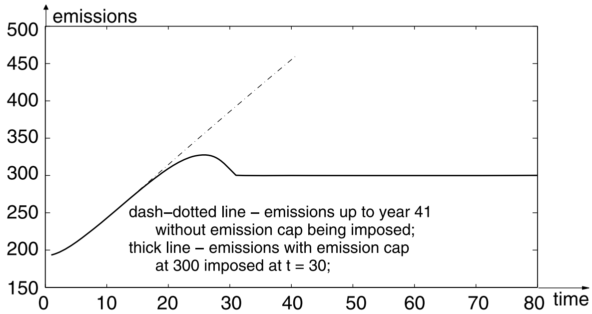

Fig. 1 Emission resulting from an emission restriction posed at time $t = 30$

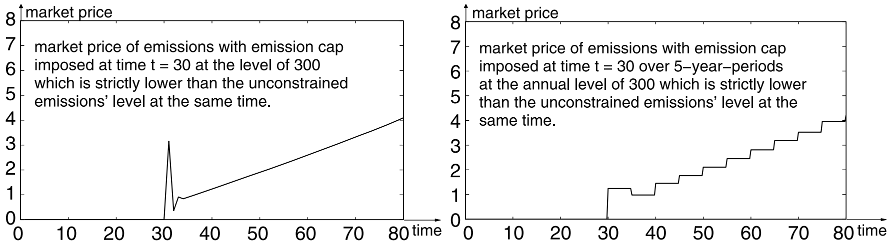

Fig. 2 Auction price of permits with commitment periods of 1 year (*left*) and auction price with commitment period of 5 years (*right*) resulting from an emission restriction posed at time $t = 30$

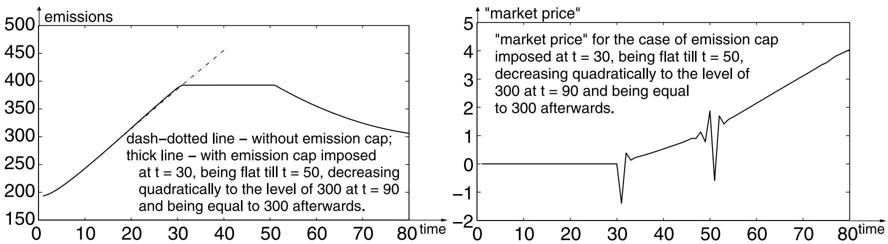

Fig. 3 Emissions (*left*) and a solution to the price equation (18) (*right*) resulting from an emission restriction posed at time $t = 30$

In Fig. 3 the emission cap imposed at time $t = 30$ is first constant, starting from the emission level before the beginning of the emission restriction (the dash-dotted line represents the unrestricted emissions), then it decreases quadratically. The solid line represents the emission of the firm obeying the cap. Here it is remarkable that (in contrast to the first scenario) there is no visible reduction of the emission level before time $t = 30$, i.e. no visible anticipation effect. This phenomenon will be discussed in more detail in the continuation of the paper.

The right plot in Fig. 3 represents the auctioned price of the emission permit. As before it is obtained by using small commitment periods of $h = 1$ year. In contrast to the first scenario, here the auction fails in two time-periods. First this happens immediately after the introduction of the emission cap at time $\hat{t} = 30$, then around

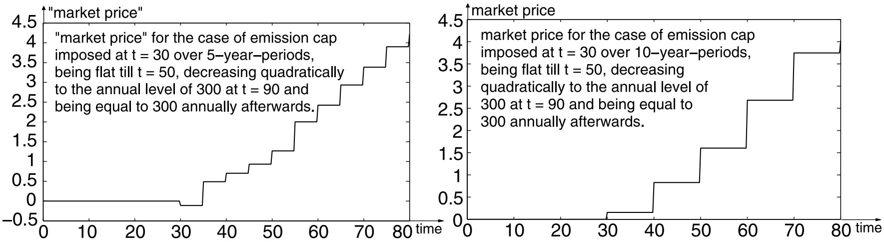

Fig. 4 Auction price of emission permits for commitment periods $h = 5$ years (*left*) and $h = 10$ years (*right*)

the change of the shape of the cap in year $t = 50$: the solution of the auction equation (18) takes negative values. In addition, the solution is highly oscillating around these times. The regularization of the market by posing the cap in commitment periods is seen on Fig. 4. The left plot corresponds to commitment periods of $h = 5$ years. The market still fails immediately after the introduction of the cap, but not around the change of the shape of the cap. The right plot in Fig. 4 represents the price function with commitment periods of 10 years. The market is now efficient and looks quite regular.

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International Emission Policy with Lobbying and Technological Change

Tapio Palokangas

Abstract I examine emission policy in a union of countries when production in any country incurs emissions that pollute all over the union, but efficiency in production is improved by research and development (R&D). I compare four cases: Laissez-faire, Pareto optimal policy, and the case of a self-interested central planner that decides on nontraded or traded emission quotas. I show that with nontraded quotas, the growth rate is socially optimal, but welfare sub-optimal. Trade in quotas speeds up growth from the initial position of laissez-faire, but slows down growth from the initial position of nontraded quotas.

1 Introduction

In this study, I examine the implementation of emission policy in a union of countries. The production of goods in any country incurs emissions that pollute all over the union, but efficiency in production in each country can be improved by research and development which has a random outcome. In every country, there is a local planner that maximizes welfare and has enough instrument to control the allocation of resources in the country.

In particular, I examine the following cases of exercising emission policy:

(i) *Laissez-faire.* All countries choose their optimal emissions ignoring the externality through pollution.
(ii) *Pareto optimum.* In the union, there is a benevolent central planner that is able to transfer resources between the countries.
(iii) *Lobbying over nontraded emission quotas.* In the union, there is a self-interested central planner that sets nontraded emission quotas for all countries. That planner is subject to lobbying and has no financial resources of its own.
(iv) *Lobbying over traded emission quotas.* In the union, there is a self-interested central planner that sets emission quotas for all countries, and a market through

T. Palokangas (✉)
Department of Economics, University of Helsinki and HECER, Arkadiankatu 7,
P.O. Box 17, 00014 Helsinki, Finland
e-mail: tapio.palokangas@helsinki.fi

T. Palokangas
International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1,
2361 Laxenburg, Austria

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which the countries can sell their quotas to each others. The central planner is subject to lobbying and has no financial resources of its own.

In this model, there are two sources of inefficiency. One is negative externality through pollution, for which a single country has too much production with emissions and too little investment in R&D. The second externality is waste due to lobbying. Given that the central planner consists at least partly of different households than the rest of the population, political contributions are waste from the viewpoint of the latter. The relative weight of these sources determine the outcome of the comparison between cases (i)–(iv).

The impact of any environmental policy depends crucially on the existence of uncertainty. The papers Corsetti (1997), Smith (1996), Turnovsky (1995, 1999) consider public policy by a growth model where productivity shocks follow a Wiener process. Soretz (2003) applies that approach to environmental policy. In one of my earlier publications (Palokangas 2008), I examine an economic union where member countries produce emissions in fixed proportion to labor in production and where uncertainty is embodied in technological change in the form of Poisson processes. As a result of this, I obtain Pareto-optimal emission taxes for the member countries. In this paper, I modify Palokangas' (2008) model so that (i) the central planner is self-interested and subject to lobbying, (ii) quotas are the main instrument for emission policy, and (iii) labor and emissions are different inputs.

This paper is organized as follows. Sections 2 and 3 present the general structure of the union and a single country. Sections 4, 5, 6 and 7 examine the cases (i)–(iv) above, respectively.

2 The Union

I consider a union of fixed number n of similar countries. Each country $j \in \{1,\dots,n\}$ has a fixed labor supply L, of which the amount l_j is used in production and the rest

$$z_j = L - l_j \tag{1}$$

in R&D. I assume that all countries $j \in \{1,\dots,n\}$ produce the same good, for simplicity.[1] The goods market is then balanced, if

$$C = \sum_{j=1}^{n} Y_j, \tag{2}$$

[1] With some complication, but with no significant effect on the results, it would be possible to assume the final consumption good as a CES function of the outputs of all countries.

where C is total consumption in the union and Y_j output in country $j \in \{1, \ldots, n\}$. Pollution P is determined by total emissions in the union:

$$P = \sum_{j=1}^{n} m_j, \tag{3}$$

where m_j is emissions in country j. Environmental policy is called the more *centralized* (*de-centralized*), the smaller (bigger) n. The bigger the number of countries, n, the bigger the negative externality through pollution.

I assume that environmental policy in the union is exercised by the central planner who collects political contributions from the local planners of countries $j = 1, \ldots, n$ and spends them in its personal consumption. I assume that this central planner lives in a different country $j = 0$ than the other households of the union, for simplicity.[2]

All households in countries $j = 0, 1, \ldots, n$ share the same preferences. Their utility increases with personal consumption and decreases with pollution P. A single household is small enough to take its income, the consumption price p, the interest rate r and the level of pollution, P, as given. With these assumptions, the households in countries $j = 0, 1, \ldots, n$ behave as if there were a representative household in these countries. I specify that this chooses its flow of consumption C to maximize its utility starting at time T,

$$\int_T^{\infty} (\log D) e^{-\rho(\theta - T)} d\theta, \quad D = CP^{-\delta}, \tag{4}$$

where θ is time, $\rho > 0$ the constant rate of time preference, $D = CP^{-\delta}$ the composite commodity of consumption C and pollution P and $\delta \in (0, 1)$ the parameter that characterizes the disutility of pollution. One can equivalently assume that the household chooses the flow of the quantities D of the composite good to maximize utility (4). This maximization leads to the Euler equation (cf. Grossman and Helpman 1994b)

$$\frac{\dot{\mathcal{E}}}{\mathcal{E}} = \frac{d\mathcal{E}}{dt} \frac{1}{\mathcal{E}} = r - \rho \quad \text{with } \mathcal{E} \doteq pD = pCP^{-\delta}, \tag{5}$$

where p the consumption price, $\mathcal{E}$ household spending and r the interest rate.

Because in the model there is no money that would pin down the nominal price level at any time, it is convenient to normalize the households' total expenditure on the composite commodity in the union, $\mathcal{E}$, at unity. From (5) it then follows that the interest rate r is constant in terms of the numeraire:

$$r = \rho > 0. \tag{6}$$

[2] This is the simplest way of modeling the social cost of political contributions. Alternatively, one could assume that the central planner spends political contributions in services that crowd out labor from production and R&D, but this would excessively complicate the model without any significant impact in the results.

3 The Countries

The development of a new technology in any "producing" country $j \in \{1, \ldots, n\}$ increases total factor productivity (TFP) in that country by constant $a > 1$. TFP in country j is then equal to a^{γ_j}, where γ_j is the serial number of technology in country j. In the advent of technological change in country j, TFP increases from a^{γ_j} to a^{γ_j+1}.

Given TFP, all "producing" countries $j \in \{1, \ldots, n\}$ have the same CES production function $f(l_j, m_j)$, where l_j is labor input and m_j emissions in the country. Country j then produces consumption good C according to $Y_j = a^{\gamma_j} f(l_j, m_j)$ (cf. (2)). I define, for convenience, country j's production function of the composite commodity $D = CP^{-\delta}$ as follows:

$$\begin{aligned}
&y_j = Y_j P^{-\delta} = a^{\gamma_j} f(l_j, m_j) P^{-\delta}, \qquad f_l \doteq \partial f/\partial l_j > 0,\\
&f_m \doteq \partial f/\partial m_j > 0,\\
&f_{ll} \doteq \frac{\partial^2 f}{\partial l_j^2} < 0, \qquad f_{lm} \doteq \frac{\partial^2 f}{\partial l_j \partial m_j} > 0, \qquad f_{mm} \doteq \frac{\partial^2 f}{\partial m_j^2} < 0,\\
&\frac{f_l f_m}{f_{lm} f} = \sigma \in (0,1) \cup (1,\infty),\\
&\frac{m_j f_m(l_j, m_j)}{f(l_j, m_j)} = \frac{m_j}{l_j}\frac{f_m(l_j/m_j, 1)}{f(l_j/m_j, 1)} \doteq \xi\left(\frac{m_j}{l_j}\right),\\
&\frac{l_j f_l(l_j, m_j)}{f(l_j, m_j)} = 1 - \frac{m_j f_m(l_j, m_j)}{f(l_j, m_j)}, \qquad \xi'\left(\frac{m_j}{l_j}\right)\begin{cases} >0 & \text{for } \sigma > 1,\\ <0 & \text{for } \sigma < 1,\end{cases}
\end{aligned} \tag{7}$$

where σ is the constant elasticity of substitution between labor and emissions, δ the constant elasticity of output y_j with respect to pollution P.

The local planner in country j (hereafter local planner j) pays political contributions R_j to the central planner of the union. Because all the households (including the central planner) consume the same composite commodity D, real political contributions can be defined in terms of that commodity. Real income in country j is therefore given by $y_j - R_j$, where y_j is output and R_j political contributions. Noting (7), I obtain local planner j's utility from an infinite stream of real income beginning at time T as follows:

$$E\int_T^\infty (y_j - R_j)e^{-r(t-T)}dt = E\int_T^\infty \left[a^{\gamma_j} f(l_j, m_j)P^{-\delta} - R_j\right]e^{-r(t-T)}dt, \tag{8}$$

where E is the expectation operator and $r > 0$ the interest rate (cf. (6)).

The improvement of technology in country j depends on labor devoted to R&D, z_j. In a small period of time dt, the probability that R&D leads to development of a new technology with a jump from γ_j to $\gamma_j + 1$ is given by $\lambda z_j dt$, while the probability that R&D remains without success is given by $1 - \lambda z_j dt$, where λ is productivity in R&D. Noting (1), this defines a Poisson process χ_j with

$$d\chi_j = \begin{cases} 1 & \text{with probability } \lambda z_j dt = \lambda(L - l_j)dt, \\ 0 & \text{with probability } 1 - \lambda z_j dt = 1 - \lambda(L - l_j)dt, \end{cases} \tag{9}$$

where $d\chi_j$ is the increment of the process χ_j. The expected growth rate of productivity a^{γ_j} in the production sector in the stationary state is given by

$$g_j \doteq E[\log a^{\gamma+1} - \log a^{\gamma}] = (\log a)\lambda z_j = (\log a)\lambda(L - l_j),$$

where E is the expectation operator (cf. Aghion and Howitt 1998, p. 59, and Wälde 1999). In other words:

Proposition 1 *The expected growth rate g_j of country j's output is in fixed proportion to labor devoted to R&D, $z_j = L - l_j$, in that country.*

Given this result, I can use labor devoted to R&D, z_j, as a proxy for the growth rate in each country j.

4 Laissez-faire

If there is laissez-faire, there is no lobbying and no political contributions either, $R_j = 0$ for all j. Local planner j then maximizes its utility (8) by emissions m_j and labor input l_j subject to pollution (3) and Poisson technological change (9), given emissions in the rest of the union,

$$m_{-j} \doteq \sum_{k \neq j} m_k. \tag{10}$$

The value of the optimal program for planner j starting at time T is then

$$\Omega^j(\gamma_j, m_{-j}, n, T) \doteq \max_{(m_j, l_j) \text{ s.t. } (9)} E \int_T^{\infty} a^{\gamma_j} f(l_j, m_j)(m_j + m_{-j})^{-\delta} \times e^{-r(t-T)} dt. \tag{11}$$

I denote $\Omega^j = \Omega^j(\gamma_j, m_{-j}, n, T)$ and $\widetilde{\Omega}^j = \Omega^j(\gamma_j + 1, m_{-j}, n, T)$. The Bellman equation corresponding to the optimal program (11) is

$$r\Omega^j = \max_{m_j, l_j} \Phi^j(m_j, l_j, \gamma_j, m_{-j}, n, T), \tag{12}$$

where

$$\Phi^j(m_j, l_j, \gamma_j, m_{-j}, n, T) = a^{\gamma_j} f(l_j, m_j)(m_j + m_{-j})^{-\delta} + \lambda(L - l_j)[\widetilde{\Omega}^j - \Omega^j]. \tag{13}$$

This leads to the first-order conditions

$$\frac{\partial \Phi^j}{\partial m_j} = \frac{a^{\gamma_j} f_m(l_j, m_j)}{(m_j + m_{-j})^\delta} - \frac{\delta a^{\gamma_j} f(l_j, m_j)}{(m_j + m_{-j})^{\delta+1}} = 0, \tag{14}$$

$$\frac{\partial \Phi^j}{\partial l_j} = \frac{a^{\gamma_j} f_l(l_j, m_j)}{(m_j + m_{-j})^\delta} - \lambda[\widetilde{\Omega}^j - \Omega^j] = 0. \tag{15}$$

To solve the dynamic program, I try the solution that the value of the program, Ω^j, is in fixed proportion $\varphi_j > 0$ to instantaneous utility:

$$\Omega^j(\gamma_j, m_{-j}, n, T) = \varphi_j a^{\gamma_j} f(l_j, m_j)(m_j + m_{-j})^{-\delta}. \tag{16}$$

This implies

$$(\widetilde{\Omega}^j - \Omega^j)/\Omega^j = a - 1. \tag{17}$$

Inserting (16) and (17) into the Bellman equation (12) and (13) yields

$$1/\varphi_j = r + (1-a)\lambda(L - l_j) > 0. \tag{18}$$

Inserting (16) and (17) into the first-order conditions (14) and (15) yields

$$\begin{aligned} \frac{m_j}{\Omega^j}\frac{\partial \Phi^j}{\partial m_j} &= \frac{1}{\varphi_j}\left[\frac{m_j f_m(l_j, m_j)}{f(l_j, m_j)} - \frac{\delta m_j}{m_j + m_{-j}}\right] = 0, \\ \frac{l_j}{\Omega^j}\frac{\partial \Phi^j}{\partial l_j} &= \frac{1}{\varphi_j}\frac{l_j f_l(l_j, m_j)}{f(l_j, m_j)} - (a-1)\lambda l_j = 0. \end{aligned} \tag{19}$$

Because there is symmetry throughout all countries $j = 1, \ldots, n$ in the model, conditions $l_j = l$ and $m_j = m$ hold in equilibrium. Inserting these conditions into (18) and (19) and noting (1), (7) and (10) yield

$$\xi\left(\frac{m}{l}\right) \doteq \frac{m f_m(l, m)}{f(l, m)} = \frac{\delta}{n} \in (0, 1), \tag{20}$$

$$\begin{aligned} (a-1)\lambda l &= [r + (1-a)\lambda(L-l)]\frac{l f_l}{f} = [r + (1-a)\lambda(L-l)]\left[1 - \frac{m f_m}{f}\right] \\ &= [r + (1-a)\lambda(L-l)](1 - \delta/n). \end{aligned} \tag{21}$$

Solving for m/l from (20) and l from (21) and noting (7) yield

$$\frac{m}{l} = \xi^{-1}\left(\frac{\delta}{n}\right) \doteq \varphi(n), \qquad \frac{d\varphi}{dn} = -\frac{\delta}{n^2\xi'} = \begin{cases} < 0 & \text{for } \sigma > 1, \\ > 0 & \text{for } 0 < \sigma < 1, \end{cases}$$

$$l(n) \doteq \underbrace{\frac{r + (1-a)\lambda L}{(a-1)\lambda}}_{+} \underbrace{\left(\frac{n}{\delta} - 1\right)}_{+} > 0, \quad r + (1-a)\lambda L > 0, \qquad l' > 0, \tag{22}$$

$$z(n) = L - l(n), \qquad z' < 0, \qquad m(n) \doteq \varphi(n) l(n),$$

$$\frac{dm}{dn} = l\varphi' + \varphi l' > 0 \quad \text{for } 0 < \sigma < 1.$$

These results can be rephrased as follows:

Proposition 2 *Centralization* (*i.e. a decrease in* n)

(a) *decreases the labor input in production* (*i.e.* $l' > 0$), *but increases the growth rate* z (*i.e.* $z' < 0$),
(b) *decreases emissions* m *unambiguously* (*i.e.* $dm/dn > 0$), *when labor and emissions are gross complements*, $0 < \sigma < 1$,
(c) *increases emissions per labor input*, m/l (*i.e.* $\varphi' < 0$), *when labor and emissions are gross substitutes*, $\sigma > 1$.

Because centralization helps to internalize the effect of pollution, the local planners alleviate pollution by transferring resources from production into R&D. This decreases output, but speeds up economic growth. When labor and emissions are gross complements, the decrease of output reduces both labor and emissions in production. When labor and emissions are gross substitutes, labor transferred from production into R&D is partly replaced by emissions. This increases the emissions-labor ratio in production.

5 Pareto Optimum

Assume a benevolent central planner which has enough instruments to transfer income between countries,[3] and which does not collect political contributions, $R_j = 0$ for all j. Because it can internalize the externality of pollution entirely, the outcome is the Pareto optimum where the economic union behaves as if there were one jurisdiction only, $n = 1$. Noting (22), labor input in production at the Pareto optimum, l^P, is given by

$$l^P \doteq l(1) = \frac{r + (1-a)\lambda L}{(a-1)\lambda}\left(\frac{1}{\delta} - 1\right). \tag{23}$$

[3] In the model, it would be sufficient if the central planner could tax consumption in all countries at any rate and then use the revenue for subsidizing R&D.

Furthermore, Proposition 2 has the following corollary:

Proposition 3 *Labor input in production*, l, *is at the lowest level and the growth rate* z *at the highest level at the Pareto optimum. When labor and emissions are gross complements in production*, $0 < \sigma < 1$, *emissions* m *and pollution* $P = nm$ *are at the lowest level at the Pareto optimum.*

For the remainder of this paper, I assume that the central planner is self-interested, not benevolent. In Sect. 6, I assume that the central planner taylors a specific emission quota m_j for each country, but the countries cannot trade with these quotas. In Sect. 7, I introduce trade in these quotas.

6 Lobbying over Nontraded Emission Quotas

Following Grossman and Helpman (1994a, 1994b), I assume that the central planner of the union has its own interests and collects political contributions. Local planner j in each country $j \in \{1, \ldots, n\}$ pays political contributions R_j to the central planner which decides on a specific emission quota m_j for each country $j \in \{1, \ldots, n\}$. The order of this common agency game is the following. First, the local planners set their political contributions $(R_1, \ldots, R_n)$ conditional on the central planner's prospective policy $(m_1, \ldots, m_n)$. Second, the central planners sets the quotas $(m_1, \ldots, m_n)$ and collect the contributions for its personal consumption. Third, the local planners maximize their utilities given the level of political contributions $(R_1, \ldots, R_n)$. This game is solved in reversed order as follows. Section 6.1 considers a local planner, Sect. 6.2 the central planner and Sect. 6.3 the political equilibrium.

6.1 The Local Planners

Local planner j maximizes its utility (8) by labor input l_j subject to Poisson technological change (9) on the assumption that the interest rate r, the quotas $m_1, \ldots, m_n$, pollution $P = \sum_j m_j$ (cf. (3)) and its political contributions R_j are kept constant. It is equivalent to maximize

$$E \int_T^{\infty} a^{\gamma_j} f(l_j, m_j) P^{-\delta} e^{-r(t-T)} dt$$

by l_j subject to (9), given r, m_j, P and R_j. The value of the optimal program for local planner j starting at time T can then be defined as follows:

$$\Gamma^j(\gamma_j, m_j, P, T) = \max_{l_j \text{ s.t. } (9)} E \int_T^{\infty} a^{\gamma_j} f(l_j, m_j) P^{-\delta} e^{-r(t-T)} dt. \tag{24}$$

I denote $\Gamma^j = \Gamma^j(\gamma_j, m_j, P, T)$ and $\widetilde{\Gamma}^j = \Gamma^j(\gamma_j + 1, m_j, P, T)$. The Bellman equation corresponding to the optimal program (24) is

$$r\Gamma^j = \max_{l_j} \Psi^j(l_j, \gamma_j, m_j, P, T), \tag{25}$$

where

$$\Psi^j(l_j, \gamma_j, m_j, P, T) = a^{\gamma_j} f(l_j, m_j) P^{-\delta} + \lambda(L - l_j)[\widetilde{\Gamma}^j - \Gamma^j]. \tag{26}$$

This leads to the first-order condition

$$\frac{\partial \Psi^j}{\partial l_j} = a^{\gamma_j} f_l(l_j, m_j) P^{-\delta} - \lambda[\widetilde{\Gamma}^j - \Gamma^j] = 0. \tag{27}$$

To solve the dynamic program, I try the solution that the value of the program, Γ^j, is in fixed proportion $\vartheta_j > 0$ to instantaneous utility:

$$\Gamma^j(\gamma_j, m_j, P, T) = \vartheta_j a^{\gamma_j} f(l_j, m_j) P^{-\delta}, \qquad \frac{\partial \Gamma^j}{\partial m_j} = \frac{f_m(l_j, m_j)}{f(l_j, m_j)} \Gamma_j, \\ \frac{\partial \Gamma^j}{\partial P} = -\delta \frac{\Gamma^j}{P}. \tag{28}$$

This implies

$$(\widetilde{\Gamma}^j - \Gamma^j)/\Gamma^j = a - 1. \tag{29}$$

Inserting (28) and (29) into the Bellman equation (25) and (26) yields

$$1/\vartheta_j = r + (1 - a)\lambda(L - l_j) > 0. \tag{30}$$

Inserting (28), (29) and (30) into the first-order condition (27) and noting (7), one obtains

$$\begin{aligned} \frac{l_j}{\Gamma^j} \frac{\partial \Psi^j}{\partial l_j} &= \frac{1}{\vartheta_j} \frac{l_j f_l(l_j, m_j)}{f(l_j, m_j)} - (a - 1)\lambda l_j = \frac{1}{\vartheta_j}\left[1 - \xi\left(\frac{m_j}{l_j}\right)\right] - (a - 1)\lambda l_j \\ &= [r + (1 - a)\lambda(L - l_j)]\left[1 - \xi\left(\frac{m_j}{l_j}\right)\right] - (a - 1)\lambda l_j = 0. \end{aligned} \tag{31}$$

Noting (3), (7), (24) and (28), local planner j's utility (8) becomes

$$
\begin{aligned}
\Upsilon^j(R_j, m_1, \ldots, m_n) &= \Gamma^j(\gamma_j, m_j, P, T) - \int_T^\infty R_j e^{-r(t-T)} dt \\
&= \Gamma^j(\gamma_j, m_j, P, T) - R_j/r, \\
\frac{\partial \Upsilon^j}{\partial m_j} &= \frac{\partial \Gamma^j}{\partial m_j} + \frac{\partial \Gamma^j}{\partial P}\frac{\partial P}{\partial m_j} = \Gamma^j\left[\frac{f_m(l_j, m_j)}{f(l_j, m_j)} - \frac{\delta}{P}\frac{\partial P}{\partial m_j}\right] \\
&= \frac{\Gamma^j}{m_j}\left[\xi\left(\frac{m_j}{l_j}\right) - \frac{\delta m_j}{P}\right], \\
\frac{\partial \Upsilon^j}{\partial m_k} &= \frac{\partial \Gamma^j}{\partial P}\frac{\partial P}{\partial m_k} = -\frac{\delta \Gamma^j}{P}\frac{\partial P}{\partial m_k} = -\frac{\delta \Gamma^j}{P} \quad \text{for } k \neq j, \qquad \frac{\partial \Upsilon^j}{\partial R_j} = -\frac{1}{r}.
\end{aligned}
\tag{32}
$$

6.2 The Self-interested Central Planner

The present value the expected flow of the real political contributions R_j from all countries j at time T is given by

$$
E\int_T^\infty \sum_{j=1}^n R_j e^{-r(\theta-T)} d\theta. \tag{33}
$$

Given this, (3) and (32), I specify the central planner's utility function as:

$$
\begin{aligned}
&G(m_1, \ldots, m_n, R_1, \ldots, R_n) \\
&\quad \doteq E\int_T^\infty \sum_{j=1}^n R_j e^{-r(\theta-T)} d\theta + \sum_{j=1}^n \zeta_j \Upsilon^j(R_j, m_1, \ldots, m_n) \\
&\quad = \frac{1}{r}\sum_{j=1}^n R_j + \sum_{j=1}^n \zeta_j \Upsilon^j(R_j, m_1, \ldots, m_n),
\end{aligned}
\tag{34}
$$

where constants $\zeta_j \geq 0$ are the weight of planner j's welfare in the central planner's preferences. Grossman and Helpman's (1994a) objective function (34) is widely used in models of common agency and it has been justified as follows. The politicians are mainly interested in their own income which consists of the contributions from the public, $\sum_j R_j$, but because they must defend their position in general elections, they must sometimes take the utilities of the interest groups Υ^j into account directly. The linearity of (34) in $\sum_j R_j$ is assumed, for simplicity.

6.3 The Political Equilibrium

Each local planner j tries to affect the central planner by its contributions R_j. The contribution schedules are therefore functions of the central planner's policy variables (= the emission quotas m_j):

$$R_j(m_1, \ldots, m_n), \quad j = 1, \ldots, n. \tag{35}$$

Following Proposition 1 of Dixit et al. (1997), a subgame perfect Nash equilibrium for this game is a set of contribution schedules $R_j(m_1, \ldots, m_n)$ and a policy $(m_1, \ldots, m_n)$ such that the following conditions (i)–(iv) hold:

(i) Contributions R_j are non-negative but no more than the contributor's income, $\Upsilon_j \geq 0$.

(ii) The policy $(m_1, \ldots, m_n)$ maximizes the central planner's welfare (34) taking the contribution schedules R_j as given,

$$\begin{aligned}&(m_1, \ldots, m_n)\\ &\quad \in \arg\max_{m_1,\ldots,m_n} G\big(m_1, \ldots, m_n, R_1(m_1, \ldots, m_n), \ldots, R_n(m_1, \ldots, m_n)\big);\end{aligned} \tag{36}$$

(iii) Local planner j cannot have a feasible strategy $R_j(m_1, \ldots, m_n)$ that yields it a higher level of utility than in equilibrium, given the central planner's anticipated decision rule,

$$(m_1, \ldots, m_n) = \arg\max_{m_1,\ldots,m_n} \Upsilon^j\big(R_j(m_1, \ldots, m_n), m_1, \ldots, m_n\big). \tag{37}$$

(iv) Local planner j provides the central planner at least with the level of utility than in the case it offers nothing ($R_j = 0$), and the central planner responds optimally given the other local planners contribution functions,

$$\begin{aligned}&G(m_1, \ldots, m_n, R_1(m_1, \ldots, m_n), \ldots, R_n(m_1, \ldots, m_n))\\ &\quad \geq \max_{m_1,\ldots,m_n} G\big(m_1, \ldots, m_n, R_1(m_1, \ldots, m_n), \ldots, R_{j-1}(m_1, \ldots, m_n), 0,\\ &\qquad\qquad R_{j+1}(m_1, \ldots, m_n), \ldots, R_n(m_1, \ldots, m_n)\big).\end{aligned}$$

Noting (32), the conditions (37) are equivalent to

$$0 = \frac{\partial \Upsilon^j}{\partial R_j}\frac{\partial R_j}{\partial m_k} + \frac{\partial \Upsilon^j}{\partial m_j} = -\frac{1}{r}\frac{\partial R_j}{\partial m_k} + \frac{\partial \Upsilon^j}{\partial m_k} \quad \text{for all } k,$$

and

$$\frac{\partial R_j}{\partial m_j} = r\frac{\partial \Upsilon^j}{\partial m_j} = r\frac{\Gamma^j}{m_j}\left[\xi\left(\frac{m_j}{l_j}\right) - \frac{\delta m_j}{P}\right], \qquad \frac{\partial R_j}{\partial m_k} = -\frac{r\delta\Gamma^j}{P} \quad \text{for } k \neq j.$$

Given these equations, one obtains

$$\frac{\partial}{\partial m_k}\sum_{j=1}^{n} R_j = \sum_{j=1}^{n}\frac{\partial R_j}{\partial m_k} = \frac{\partial R_k}{\partial m_k} + \sum_{j\neq k}\frac{\partial R_j}{\partial m_k}$$
$$= r\frac{\Gamma^k}{m_k}\left[\xi\left(\frac{m_k}{l_k}\right) - \frac{\delta m_k}{P}\right] - \sum_{j\neq k}\frac{r\delta\Gamma^j}{P}$$
$$= r\frac{\Gamma^k}{m_k}\left[\xi\left(\frac{m_k}{l_k}\right) - \frac{\delta m_k}{P}\frac{1}{\Gamma^k}\sum_{j=1}^{n}\Gamma^j\right]. \tag{38}$$

Noting (35) and (37), the central planner's utility function (34) becomes

$$\mathcal{G}(m_1,\ldots,m_n) \doteq G\big(m_1,\ldots,m_n, R_1(m_1,\ldots,m_n),\ldots,R_n(m_1,\ldots,m_n)\big)$$
$$= \frac{1}{r}\sum_{j=1}^{n} R_j(m_1,\ldots,m_n)$$
$$+ \sum_{j=1}^{n}\zeta_j \max_{m_1,\ldots,m_n} \Upsilon^j\big(R_j(m_1,\ldots,m_n), m_1,\ldots,m_n\big). \tag{39}$$

Noting (38) and (39), the equilibrium conditions (36) are equivalent to the first-order conditions

$$\frac{\partial\mathcal{G}}{\partial m_k} = \frac{1}{r}\frac{\partial}{\partial m_k}\sum_{j=1}^{n} R_j = \frac{\Gamma^k}{m_k}\left[\xi\left(\frac{m_k}{l_k}\right) - \frac{\delta m_k}{P}\frac{1}{\Gamma^k}\sum_{j=1}^{n}\Gamma^j\right] = 0 \quad \text{for all } k. \tag{40}$$

The political equilibrium is now specified by the equilibrium conditions (31) for all local planners $j = 1,\ldots,n$ plus those (40) for the central planner. In this system, there are $2n$ unknowns, (l_j, m_j) for $j = 1,\ldots,n$. I assume, for simplicity, uniform initial productivity in the union, $\gamma_k = \gamma_1$ for all $k \neq 1$. In the system, noting (28), this yields perfect symmetry $l_j = l$, $m_k = m$ and $\Gamma_j = \Gamma$ for the countries $j = 1,\ldots,n$ in equilibrium. Given this, (3) and (7), the equilibrium conditions (31) and (40) change into

$$\xi\left(\frac{m}{l}\right) = \xi\left(\frac{m_j}{l_j}\right) = \frac{\delta m_k}{P}\frac{1}{\Gamma^k}\sum_{j=1}^{n}\Gamma^j = \delta\frac{m_k}{P}n = \delta,$$
$$\frac{(a-1)\lambda l}{r + (1-a)\lambda(L-l)} = 1 - \frac{m f_m}{f} = 1 - \xi = 1 - \delta. \tag{41}$$

The results (41) are the same as the result (20) and (21) with $n = 1$. This shows that m, l and $z = L - l$ are the same as at the Pareto optimum (23):

Proposition 4 *In the case of lobbying over nontraded emission quotas, emissions m and the growth rate z are Pareto optimal.*

The introduction of the central planner as a decision maker for emissions eliminates the externality through pollution. This effect is the same for both a benevolent and a self-interested central planner.

In the case of lobbying, the countries pay political contributions, $R_j > 0$ for all j, while in the case of Pareto-optimal policy, there are no such contributions, $R_j = 0$ for all j. If the central planner consists of different households than the rest of the population (even partly), one can define political contributions are a waste from the viewpoint of the latter. Thus, there is the following corollary for Proposition 4:

Proposition 5 *In the case of lobbying over nontraded emission quotas, welfare is Pareto sub-optimal.*

7 Lobbying over Traded Emission Quotas

In this section, I assume that the central planner sets quotas for the countries' emissions, but that the countries can trade in these quotas among themselves. To enable a stationary state equilibrium in the model, I assume that the quotas are in fixed proportion to the level of productivity a^{γ_j} so that more advanced countries get tighter restrictions. Therefore, the quota for country j's productivity-adjusted emissions $m_j a^{\gamma_j}$ is given by q_j. When country j has excess quotas, $q_j > m_j a^{\gamma_j}$, it can sell the difference $q_j - m_j a^{\gamma_j}$ to the other members of the union at the price p. Correspondingly, when country j has excess emissions, $m_j a^{\gamma_j} - q_j$, it must buy the difference $m_j a^{\gamma_j} - q_j$ from other countries at the price p. At the level of the whole union, productivity-adjusted emissions $\sum_{j=1}^{n} m_j a^{\gamma_j}$ are equal to total quotas $\sum_{j=1}^{n} q_j$,

$$\sum_{j=1}^{n} m_j a^{\gamma_j} = \sum_{j=1}^{n} q_j. \tag{42}$$

Local planner j in each country $j \in \{1, \ldots, n\}$ pays political contributions R_j to the central planner. The order of this common agency game is the following. First, the local planners set their political contributions $(R_1, \ldots, R_n)$ conditional on the central planner's prospective policy $(q_1, \ldots, q_n)$. Second, the central planners sets the quotas $(q_1, \ldots, q_n)$ and collect the contributions for its personal consumption. Third, the local planners maximize their utilities given the level of political contributions $(R_1, \ldots, R_n)$. This game is solved in reversed order as follows. Section 7.1 considers a local planner, Sect. 7.2 the central planner and Sect. 7.3 the political equilibrium.

7.1 The Local Planners

Planner j's utility starting at time T, (8), can be extended into

$$\Upsilon^j \doteq E \int_T^\infty \left[a^{\gamma_j} f(l_j, m_j)(m_j + m_{-j})^{-\delta} + p(q_j - m_j a^{\gamma_j}) - R_j\right] e^{-r(t-T)} dt, \tag{43}$$

where $p(q_j - m_j a^{\gamma_j})$ is country j's net income from trade in quotas. Local planner j maximizes its utility (43) by labor input l_j and emissions m_j subject to Poisson technological change (9) on the assumption that the interest rate r, the quotas $q_1, \ldots, q_n$, the emission price p, emissions in the rest of the union, m_{-j}, and its political contributions R_j are kept constant. It is equivalent to maximize

$$E \int_T^{\infty} a^{\gamma_j} \big[f(l_j, m_j)(m_j + m_{-j})^{-\delta} - p m_j \big] e^{-r(t-T)} dt$$

by (l_j, m_j) subject to (9), given r, p and m_{-j}. The value of the optimal program for local planner j can then be defined as follows:

$$\begin{aligned} &\Gamma^j(\gamma_j, p, m_{-j}, T) \\ &\quad = \max_{(m_j, l_j) \text{ s.t. } (9)} E \int_T^{\infty} a^{\gamma_j} \big[f(l_j, m_j)(m_j + m_{-j})^{-\delta} - p m_j \big] e^{-r(t-T)} dt. \end{aligned} \tag{44}$$

I denote $\Gamma^j = \Gamma^j(\gamma_j, p, m_{-j}, T)$ and $\widetilde{\Gamma}^j = \Gamma^j(\gamma_j + 1, p, m_{-j}, T)$. The Bellman equation corresponding to the optimal program (44) is

$$r\Gamma^j = \max_{l_j, m_j} \Psi^j(l_j, \gamma_j, p, m_{-j}, T), \tag{45}$$

where

$$\begin{aligned} &\Psi^j(l_j, \gamma_j, p, m_{-j}, T) \\ &\quad = a^{\gamma_j} \big[f(l_j, m_j)(m_j + m_{-j})^{-\delta} - p m_j \big] + \lambda(L - l_j) \big[\widetilde{\Gamma}^j - \Gamma^j \big]. \end{aligned} \tag{46}$$

This leads to the first-order conditions

$$\frac{\partial \Psi^j}{\partial m_j} = a^{\gamma_j} \left[\frac{f_m(l_j, m_j)}{(m_j + m_{-j})^{\delta}} - \frac{\delta f(l_j, m_j)}{(m_j + m_{-j})^{\delta+1}} - p \right] = 0, \tag{47}$$

$$\frac{\partial \Psi^j}{\partial l_j} = \frac{a^{\gamma_j} f_l(l_j, m_j)}{(m_j + m_{-j})^{\delta}} - \lambda[\widetilde{\Gamma}^j - \Gamma^j] = 0. \tag{48}$$

I try the solution that the value of the program, Γ^j, is given by

$$\begin{aligned} &\Gamma^j(\gamma_j, p, m_{-j}, T) = \vartheta_j a^{\gamma_j} \left[\frac{f(l_j, m_j)}{(m_j + m_{-j})^{\delta}} - p m_j \right], \\ &\frac{\partial \Gamma^j}{\partial m_{-j}} = -\frac{\delta \vartheta_j a^{\gamma_j} f(l_j, m_j)}{(m_j + m_{-j})^{\delta+1}}, \qquad \frac{\partial \Gamma^j}{\partial p} = -\vartheta_j a^{\gamma_j} m_j, \end{aligned} \tag{49}$$

where $\vartheta_j > 0$ is independent of the control variables. This implies

$$(\widetilde{\Gamma}^j - \Gamma^j)/\Gamma^j = a - 1. \tag{50}$$

Inserting (49) and (50) into the Bellman equation (45) and (46) yields

$$1/\vartheta_j = r + (1-a)\lambda(L - l_j) > 0. \tag{51}$$

Given (49), (50) and (51) the first-order conditions (47) and (48) change into

$$p = \frac{f_m(l_j, m_j)}{(m_j + m_{-j})^\delta} - \frac{\delta f(l_j, m_j)}{(m_j + m_{-j})^{\delta+1}}, \tag{52}$$

$$\begin{aligned}\frac{1}{\Gamma^j}\frac{\partial \Psi^j}{\partial l_j} = {} & \frac{r + (1-a)\lambda(L - l_j)}{f(l_j, m_j)(m_j + m_{-j})^{-\delta} - pm_j}\frac{f_l(l_j, m_j)}{(m_j + m_{-j})^\delta} \\ & - (a-1)\lambda = 0. \end{aligned}\tag{53}$$

In the system (10), (51), (52) and (53), there is perfect symmetry $l_j = l$, $m_j = m$ and $\vartheta_j = \vartheta$ throughout $j = 1, \ldots, n$. Noting (10), this yields the following system of three equations:

$$1/\vartheta = r + (1-a)\lambda(L - l) > 0, \tag{54}$$

$$p = \frac{f_m(l, m)}{(nm)^\delta} - \frac{\delta f(l, m)}{(nm)^{\delta+1}}, \tag{55}$$

$$\frac{r + (1-a)\lambda(L - l)}{f(l, m)(nm)^{-\delta} - pm}\frac{f_l(l, m)}{(nm)^\delta} = (a-1)\lambda. \tag{56}$$

Because in the two equations (55) and (56) there are two unknown variables—labor input in production, l, and emissions m—and two given variables—the emission price p and the number of countries, n—one obtains

$$l_j = l(p, n), \qquad m_j = m(p, n). \tag{57}$$

By duality, a higher price for emissions decreases the input of emissions:

$$m_p(p, n) \doteq \partial m/\partial p < 0. \tag{58}$$

If the number n of similar countries is large, the sum $\sum_{\ell=1}^n a^{\gamma_\ell}$ can be taken as a deterministic variable. Inserting (57) into (42) yields

$$m(p, n)\sum_{\ell=1}^n a^{\gamma_\ell} = \sum_{j=1} q_j. \tag{59}$$

From this and (58) it follows that the emission price p is a decreasing function of the emission quotas q_j for all countries $j \in \{1, \ldots, n\}$:

$$p(q_1, \ldots, q_n, n), \qquad \frac{\partial p}{\partial q_j} = \frac{1}{m_p(p, n)\sum_{\ell=1}^n a^{\gamma_\ell}} < 0. \tag{60}$$

7.2 The Self-interested Central Planner

Given (10), (44), (57) and (60), local planner j's utility (43) changes into

$$
\begin{aligned}
&\Delta^j(R_j, q_1, \ldots, q_n, n) \\
&\quad = \Upsilon^j = \Gamma^j(\gamma_j, p, m_{-j}, T) + \int_T^\infty (pq_j - R_j)e^{-r(t-T)}dt \\
&\quad = \Gamma^j(\gamma_j, p, (n-1)m(p,n), T) + \int_T^\infty (pq_j - R_j)e^{-r(t-T)}dt, \\
&\frac{\partial \Delta^j}{\partial R_j} = -\frac{1}{r}.
\end{aligned}
\tag{61}
$$

From this, (49), (54), (55), (59) and (60) it follows that

$$
\begin{aligned}
&\sum_{j=1}^n \Delta^j(R_j, q_1, \ldots, q_n, n) \\
&\quad = \sum_{j=1}^n \Gamma^j + \int_T^\infty \left(p\sum_{j=1}^n q_j - \sum_{j=1}^n R_j\right)e^{-r(t-T)}dt \\
&\quad = \sum_{j=1}^n \Gamma^j + \int_T^\infty \left[pm(p,n)\sum_{\ell=1}^n a^{\gamma_\ell} - \sum_{j=1}^n R_j\right]e^{-r(t-T)}dt \\
&\quad = \sum_{j=1}^n \Gamma^j(\gamma_j, p, (n-1)m(p,n), T) + \frac{1}{r}\left[pm(p,n)\sum_{\ell=1}^n a^{\gamma_\ell} - \sum_{j=1}^n R_j\right]
\end{aligned}
$$

with

$$
\begin{aligned}
&\frac{\partial}{\partial q_k}\sum_{j=1}^n \Delta^j(R_j, q_1, \ldots, q_n, n) \\
&= \left\{\sum_{j=1}^n \frac{\partial \Gamma_j}{\partial p} + \sum_{j=1}^n \frac{\partial \Gamma_j}{\partial m_{-j}}(n-1)m_p + \frac{1}{r}[m + pm_p]\left(\sum_{\ell=1}^n a^{\gamma_\ell}\right)\right\}\frac{\partial p}{\partial q_j} \\
&= \left\{-m - \frac{\delta f(l,m)}{(nm)^{\delta+1}}(n-1)m_p + \frac{1}{r\vartheta}[m + pm_p]\right\}\left(\sum_{\ell=1}^n a^{\gamma_\ell}\right)\vartheta\frac{\partial p}{\partial q_j} \\
&= \left\{-m - \frac{\delta f(l,m)}{(nm)^{\delta+1}}(n-1)m_p + [1 + (1-a)\lambda(L-l)/r][m + pm_p]\right\}\frac{\vartheta}{m_p} \\
&= \left\{pm_p - \frac{\delta f(l,m)}{(nm)^{\delta+1}}(n-1)m_p + (1-a)\frac{\lambda}{r}(L-l)[m + pm_p]\right\}\frac{\vartheta}{m_p}
\end{aligned}
$$

$$= \left\{ p + (1-n) \frac{\delta f(l(p,n), m(p,n))}{[nm(p,n)]^{\delta+1}} + (1-a)\frac{\lambda}{r}[L - l(p,n)] \left[\frac{m(p,n)}{m_p(p,n)} + p \right] \right\} \vartheta \quad \text{for all } k. \tag{62}$$

The local planners $j = 1, \dots, n$ lobby the central planner which decides on the emission quotas $(q_1, \dots, q_n)$. Following Grossman and Helpman (1994a, 1994b), I assume that the central planner has its own interests and collects contributions $(R_1, \dots, R_n)$ from the local planners. Given this, I specify Grossman and Helpman's (1994a, 1994b) utility function for the central planner as follows:

$$\begin{aligned} &G(q_1, \dots, q_n, R_1, \dots, R_n, n) \\ &\doteq E \int_T^\infty \sum_{j=1}^n R_j e^{-r(\theta - T)} d\theta + \sum_{j=1}^n \zeta_j \Delta^j (R_j, q_1, \dots, q_n, n) \\ &= \frac{1}{r} \sum_{j=1}^n R_j + \sum_{j=1}^n \zeta_j \Delta^j (R_j, q_1, \dots, q_n, n), \end{aligned} \tag{63}$$

where constants $\zeta_j \geq 0$ are the weight of planner j's welfare.

7.3 The Political Equilibrium

Each local planner j tries to affect the central planner by its contributions R_j. The contribution schedules are therefore functions of the central planner's policy variables, the emission quotas q_j:

$$R_j(q_1, \dots, q_n), \quad j = 1, \dots, n. \tag{64}$$

The central planner maximizes its utility function (63) by $(q_1, \dots, q_n)$, given the contribution schedules (64). A subgame perfect Nash equilibrium for this game is a set of contribution schedules $R_j(q_1, \dots, q_n)$ and policy $(q_1, \dots, q_n)$ such that the conditions (i)–(iv) in Sect. 6.3 hold, with $(m_1, \dots, m_n)$ being replaced by $(q_1, \dots, q_n)$. Thus, it must be true that $\Delta^j \geq 0$ and

$$(q_1, \dots, q_n) \in \arg\max_{q_1, \dots, q_n} G\big(q_1, \dots, q_n, R_1(q_1, \dots, q_n), \dots, R_n(q_1, \dots, q_n), n\big), \tag{65}$$

$$(q_1, \dots, q_n) = \arg\max_{q_1, \dots, q_n} \Delta^j \big(R_j(q_1, \dots, q_n), q_1, \dots, q_n, n\big), \tag{66}$$

$$G\big(q_1,\dots,q_n,R_1(q_1,\dots,q_n),\dots,R_n(q_1,\dots,q_n),n\big)$$
$$\geq \max_{q_1,\dots,q_n} G(q_1,\dots,q_n,R_1(q_1,\dots,q_n),\dots,R_{j-1}(q_1,\dots,q_n),0,$$
$$R_{j+1}(q_1,\dots,q_n),\dots,R_n(q_1,\dots,q_n),n).$$

Noting (61), the conditions (66) are equivalent to

$$0=\frac{\partial\Delta^j}{\partial R_j}\frac{\partial R_j}{\partial q_k}+\frac{\partial\Delta^j}{\partial q_k}=-\frac{1}{r}\frac{\partial R_j}{\partial q_k}+\frac{\partial\Delta^j}{\partial q_k}\quad\text{for all } j \text{ and } k,$$

and

$$\frac{\partial R_j}{\partial q_k}=r\frac{\partial\Delta^j}{\partial q_k}\quad\text{for all } j \text{ and } k, \tag{67}$$

which suggests that in equilibrium the change in the lobby's contribution (R^j) due to a change in quota m_j is equal to the change in the lobby's rent Δ^j due to this same fact, holding the contribution R^j constant.

Noting (64) and (66), the central planner's utility function (63) becomes

$$\mathcal{G}(q_1,\dots,q_n,n)\doteq G\big(q_1,\dots,q_n,R_1(q_1,\dots,q_n),\dots,R_n(q_1,\dots,q_n),n\big)$$
$$=\frac{1}{r}\sum_{j=1}^{n}R_j(q_1,\dots,q_n)$$
$$+\sum_{j=1}^{n}\zeta_j\max_{q_1,\dots,q_n}\Delta^j\big(R_j(q_1,\dots,q_n),q_1,\dots,q_n,n\big). \tag{68}$$

Noting (62), (67) and (68), the equilibrium conditions (65) are equivalent to

$$\frac{\partial\mathcal{G}}{\partial q_k}=\frac{1}{r}\frac{\partial}{\partial q_k}\sum_{j=1}^{n}R_j=\frac{\partial}{\partial q_k}\sum_{j=1}^{n}\Delta_j$$
$$=\bigg\{p+(1-n)\frac{\delta f(l(p,n),m(p,n))}{[nm(p,n)]^{\delta+1}}$$
$$+(1-a)\frac{\lambda}{r}[L-l(p,n)]\bigg[\frac{m(p,n)}{m_p(p,n)}+p\bigg]\bigg\}\vartheta=0.$$

Thus, the equilibrium price p for emissions is determined by

$$\underset{+}{p}+\underbrace{(1-n)}_{-}\underbrace{\frac{\delta f(l(p,n),m(p,n))}{[nm(p,n)]^{\delta+1}}}_{+}+\underbrace{(1-a)}_{-}\underbrace{\frac{\lambda}{r}[L-l(p,n)]}_{+}\bigg[\underbrace{\frac{m(p,n)}{m_p(p.n)}}_{-}+\underbrace{p}_{+}\bigg]$$
$$=0.$$

Given (7), local planner j's first-order conditions (55) and (56) become

$$\xi\left(\frac{m}{l}\right) = \frac{mf_m(l,m)}{f(l,m)} = \frac{\delta}{n} + \frac{pm(nm)^\delta}{f(l,m)} > \frac{\delta}{n}, \tag{69}$$

$$\begin{aligned} 1-\xi\left(\frac{m}{l}\right) &= \frac{lf_l(l,m)}{f(l,m)} = \frac{(a-1)\lambda l}{r+(1-a)\lambda(L-l)}\left[1-\frac{pm(nm)^\delta}{f(l,m)}\right] \\ &= \frac{(a-1)\lambda l}{r+(1-a)\lambda(L-l)}\left[1-\xi\left(\frac{m}{l}\right)+\frac{\delta}{n}\right], \\ l = l^T &\doteq \frac{r+(1-a)\lambda L}{(a-1)\lambda}\frac{1-\xi(m/l)}{\delta/n} < l(n) \\ &\doteq \frac{r+(1-a)\lambda L}{(a-1)\lambda}\left(\frac{n}{\delta}-1\right), \end{aligned} \tag{70}$$

where l^T is the employment of labor in production with traded emission quotas. The comparison of the equilibrium in the case of laissez-faire, (20) and (21), to that in the case of traded emission quotas, (69) and (70), shows the following. First, $l = l(n)$ in the case of laissez-faire, but $l = l^T < l(n)$ in the case of traded emission quotas. Second, in the case of laissez-faire, the function $\xi(m/l)$ is equal to $\frac{\delta}{n}$, but in the case of traded emission quotas, it higher than $\frac{\delta}{n}$. Because $\xi' > 0$ (< 0) for $\sigma > 1$ ($0 < \sigma < 1$) by (7), it follows that m/l is bigger (smaller) with traded emission quotas than in with laissez-faire for $\sigma > 1$ ($0 < \sigma < 1$). These results can be rephrased as follows:

Proposition 6 *In the lobbying equilibrium with traded emission quotas,*

(a) *the level of employment in production, l, is lower, but the growth rate $z = L - l$ higher,*
(b) *the level of emissions, m, is lower when labor and emissions are gross complements (i.e. $0 < \sigma < 1$),*
(c) *the emissions-labor ratio m/l is higher when labor and emissions are gross substitutes (i.e. $\sigma > 1$),*

than with laissez-faire.

With traded emission quotas, one more unit of R&D costs less in terms of lost output. Thus, trade in emission quotas boosts R&D and decreases labor in production. When labor and emissions are gross complements, a smaller labor input in production leads to smaller emissions as well. When labor and emissions are gross substitutes, labor transferred from production into R&D is partly replaced by emissions and the emissions-labor ratio increases.

On the condition that the number of countries, n, is large enough, it holds true that $\xi \doteq mf_m/f < 1 - (1-\delta)/n$.[4] The comparison of (23) and (70) then leads to

[4]Because $\delta \in (0,1)$, it is enough that $\xi \doteq mf_m/f < 1 - 1/n = (n-1)/n$.

the results

$$l^T \doteq \frac{r+(1-a)\lambda L}{(a-1)\lambda}\frac{1-\xi}{\delta/n} > \frac{r+(1-a)\lambda L}{(a-1)\lambda}\left(\frac{1}{\delta}-1\right) \doteq l^P$$

and $z^T = L - l^T < L - l^P = z^P$. Thus, I conclude:

Proposition 7 *In the lobbying equilibrium with traded emission quotas, the growth rate z is Pareto sub-optimal (i.e. less than z^P).*

With nontraded emission quotas, the central planner determines the emissions at the level of the union. With traded emission quotas, however, the emissions are determined at the level of countries and the externality through pollution cannot be internalized. Due to the distortion through externality, the growth rate is smaller with traded than with nontraded emission quotas.

8 Conclusions

A higher level of centralization increases the growth rate, and decreases the level of emissions unambiguously when labor and emissions are gross complements. A higher level of centralization helps to internalize the effect of pollution. In that case, a local planner alleviates pollution by transferring resources from production into R&D. This speeds up economic growth. When labor and emissions are gross complements, the decrease of labor in production decreases emissions as well.

With a benevolent central planner, the union of countries behaves as if there were only one jurisdiction. Given the result above, the growth rate is then at the highest level, and emissions at the lowest level when labor and emissions are gross complements.

In the case of lobbying over nontraded emission quotas, the emissions-labor ratio, the growth rate and pollution are the same as in the Pareto optimal case where a benevolent central planner can transfer resources between countries. In either case, the introduction of the central planner as a decision maker for emissions eliminates the externality through pollution. On the other hand, in the case of lobbying, the countries pay political contributions, while in the case of a benevolent central planner, there are no such contributions. This means that lobbying decreases the welfare of countries, although the allocation of resources were the same.

In the case of lobbying over traded emission quotas, the growth rate is smaller than in the case of lobbying over nontraded quotas. With traded quotas, the emissions are determined at the level of countries and the externality through pollution cannot be internalized. With nontraded quotas, the central planner determines the emissions at the level of the union, and the externality through pollution can be internalized. Thus, with traded quotas, externality distorts the allocation of resources and decreases the growth rate.

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The Role of Product Differentiation in the Producer-targeted Promotion of Renewable Energy Technologies

Ina Meyer and Serguei Kaniovski

Abstract Carbon-based technologies continue to dominate the energy sector due to their high productivity and economies of scale. This creates an obligation for governments to provide incentives, such as taxes, subsidies and regulations, to encourage producers to implement cleaner technologies. We study a duopoly in which the incumbent is more efficient, has a higher propensity to invest and has a lower cost of capital. We derive the minimal subsidy (to the entrant) or tax (on the incumbent) sufficient to preserve the entrant in the market in the long run. The rate of the subsidy or tax depends on the underlying demand structure. The more differentiated the products and preferences of the consumers, the lower the subsidy or tax required to safeguard new entrants with innovative clean technologies.

1 Introduction

Carbon-based technologies continue to be prevalent in the energy sector due to their high productivity, economies of scale and low technological and economic uncertainty. The market share of low-carbon and renewable energy technologies remains below that required for the mitigation of anthropogenic climate change.[1]

[1]The stabilization "…of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system…" (UNFCCC, article 2) is the guiding principle of international efforts to deal with climate change. Deciding what level of climate change is dangerous is both a scientific question and a normative one, as it involves social and political judgments on acceptable risks. The latest IPCC assessment (IPCC 2007) and other studies find a level of warming of a maximum of 2 degrees Celsius (relative to pre-industrial levels) not to be dangerous. The EU has committed itself to this objective. Yet some scientists call for a more stringent stabilization of greenhouse gases, i.e. a stabilization of concentration in the atmosphere below present levels (350 ppm instead of present levels close to 390 ppm). This would still induce a warming of 1 degree Celsius in the long term. Ultimately, it appears to be impossible to define a decisive warming limit that ensures a safe climate. Even a stabilization at a maximum warming of plus 2 degrees Celsius poses a risk to natural and human systems as, from thermal expansion of sea water alone, the sea level could rise over 1 meter or more over centuries (e.g. Hare 2009; Schellnhuber 2008; German Advisory Council on Global Change 2007). Working out

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I. Meyer (✉)
Austrian Institute of Economic Research (WIFO), P.O. Box 91, 1103 Vienna, Austria
e-mail: ina.meyer@wifo.ac.at

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This situation is likely to prevail unless prices of fossil energy resources markedly rise. Reversing the current trend of rising greenhouse gas emissions creates an obligation for governments to provide producers with incentives to switch to low-carbon technologies. Producer-targeted policies include the provision of tax credits and subsidies to innovators and producers of renewable energies, taxing or limiting carbon emissions by emission trading, and regulating minimum quality standards of products.

The output of the energy sector such as electricity or heat is typically homogeneous. Indeed, whether produced by burning coal or harnessing wind, the physical characteristics of the final product are identical. If product differentiation instead pertained to the environmental impact of the production process, then raising consumer awareness towards environmentally cleaner energy products might be a viable policy for creating market niches for clean producers.

In this paper we study the effect of product differentiation on market structure using a simple dynamic model of a duopoly. The incumbent uses an established technology while the entrant is a clean producer. The incumbent is more efficient, has a higher propensity to invest in the production capacity and has a lower cost of capital. We derive the minimal subsidy or tax credit (to the entrant), or the minimal tax or cost-incurring environmental regulation (on the incumbent) sufficient to preserve the entrant in the market in the long run. The rate of the subsidy or tax depends on the underlying demand structure. The rate is higher when the customers do not differentiate the products. The more differentiated the products are, the more stable the duopoly and the less government intervention is necessary. This shows the importance of raising consumer awareness towards environmentally friendly products as a viable strategy against climate change.

Firms in real world industries rarely compete in a single, perfectly homogeneous product. The academic interest in the economic consequences of product differentiation is hence justified on empirical grounds by the sheer prevalence of the phenomenon. On theoretical grounds, product differentiation leads to remarkable market outcomes that are highly sensitive to assumed consumer behavior and the informational structure of economic models.

Existing models of environmental product differentiation are based on vertical product differentiation (e.g. Crampesa and Hollander 1995; Cremera and Thisse 1999; Eriksson 2004; Conrad 2005). Products are called vertically differentiated if they can be ordered according to their objective quality from the highest to the lowest.[2] Firms first choose the environmental quality of their products and then set prices. In three-stage models, firms first decide whether to enter the market then

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an emissions path that would achieve a defined warming limit is not only fraught with political and economic challenges on different governance levels, but also with uncertainties in the causal link from emissions to greenhouse gas concentration and radiative forcing, and ultimately to climate change. However, research has demonstrated that it is technically and economically feasible to reduce CO_2 emissions quickly enough to ensure a peak warming below 2 degrees Celsius (Hare 2009). Achieving this requires emissions abating to zero between 2050 and 2100.

[2]Horizontally differentiated products differ but no intrinsic quality ordering exists. For a survey of different models of product differentiation, see Beath and Katsoulacos (1991).

choose the quality and price. Varieties of a product find demand because their prices differ. The above models confirm the fundamental assertion that product differentiation implies softer competition than when the products are homogeneous.

We do not model environmental preferences of consumers directly. Instead, we express these preferences in terms of price-elasticities of demand. The higher the environmental awareness, the higher is the difference in the price-elasticity of the clean and dirty product. An environmentally aware consumer reacts sluggishly to a fall in the price of the dirty product relative to that of the clean product. Similarly, we do not explicitly model the environmental characteristics of the products. The consequences of these for consumer behavior are again conveyed by price-elasticity.

The existing models make very strong assumptions about rationality and perfect foresight. Firms know the consequences of their choices and the choices of their competitors. Perhaps, least realistically, equilibrium models do not model competition as a dynamic process that possibly, but not necessarily, tends to an equilibrium state. They only model what appears to be a conceivable result of such competition. The equilibrium concept is that of a subgame-perfect Nash equilibrium.

We do not impute the firms with rationality and perfect foresight. Instead, we assume that the firms invest a constant fraction of cash flow in production capacity. This behavioral assumption is typical of evolutionary models of firm dynamics in the tradition of Nelson and Winter (1982), and consistent with Cyert and March's (1963) managerial theory of the firm and the empirical phenomenon of X-inefficiency first discussed by Leibenstein (1966).

We model product differentiation using multi-product demand functions. An inverse demand function expresses the price of a product in terms of the quantities sold. In modeling product differentiation we follow the approach introduced by Bulow et al. (1985), which is based on the notion of strategic substitutes. Two products are called strategic substitutes if an increase in sales of the rival good has an adverse effect on a firm's own sales. The inverse demand function is embedded in a dynamic model of capital accumulation borrowed from Winter et al. (2003), less stochastic entry and exit. Exclusion of stochastic entry renders the model deterministic and thus permits an analytic inquiry into the properties of the selection process implied by the underlying dynamic system. This dynamic element is fairly representative for evolutionary models in the tradition of Nelson and Winter (1982).[3] In a dynamic model, the fundamental assertion that product differentiation implies softer competition than when the products are homogeneous should be reflected in slacker conditions for the emergence and stability of coexistence equilibria. By coexistence equilibria we mean those with strictly positive equilibrium outputs.

In the next section we formulate our model of duopoly and derive the main results. The final section offers concluding remarks.

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[3]Examples range from the early modeling efforts by Nelson and Winter (1982) and Winter (1984) to more complex models by Jonard and Yildizoglu (1998), Winter et al. (2000, 2003). Surveys of evolutionary modeling in economics are available in Silverberg (1997) and Kwasnicki (2002).

2 The Model

Let q_1, q_2 be the quantities produced by firms and p_1, p_2 the products' prices. The price of every product depends on the quantity of every other product supplied to a common market.

Let the firm-specific inverse demand functions $p_1(q_1,q_2)$ and $p_2(q_1,q_2)$ be bounded, differentiable, and strictly decreasing in own output in the domain where they are positive. The last two properties together imply $\partial p_1/\partial q_1$, $\partial p_2/q_2 < 0$. The products q_1 and q_2 are related as strategic substitutes if $\partial p_1/\partial q_2$, $\partial p_2/\partial q_1 \leq 0$. Since any two units of the same product are perfect substitutes, whereas any two units of differentiated products are not, the effect of own sales on price is dominant, i.e. $\partial p_1/\partial q_1 \geq |\partial p_1/\partial q_2|$ and $\partial p_2/\partial q_2 \geq |\partial p_2/\partial q_1|$.

In our model, we assume linear demand and hence also linear inverse demand functions,

$$p_1(q_1,q_2) = A_1 - B_1 q_1 - C_1 q_2,$$

$$p_2(q_1,q_2) = A_2 - B_2 q_2 - C_2 q_1,$$

where all parameters are strictly positive, $B_1 \geq C_1$, $B_2 \geq C_2$ and $B_1 \geq C_2$, $B_2 \geq C_1$. The degree of product differentiation is given by $B_1 - C_1$ and $B_2 - C_2$.

In the following we specify the model using the example of Firm 1, the incumbent. The model for Firm 2 is completely analogous. Let $q_1(t) > 0$ be the output of Firm 1 at time $t \in [0, \infty)$. The basic building blocks of the model include a production function with constant returns to scale, the perpetual inventory method that describes the dynamics of the net capital stock and a simple investment rule.

Output $q_1(t)$ is produced by employment $l_1(t)$ and capital $k_1(t)$ under constant returns to scale, or $q_1(t) = F_1(l_1(t), k_1(t))$, with F_1 homogeneous of degree one. Therefore,

$$a_1 = \frac{q_1(t)}{k_1(t)} = F_1\left(\frac{l_1(t)}{k_1(t)}, 1\right). \tag{1}$$

The parameter $a_1 > 0$ is the reciprocal of the capital coefficient. The relation $a_1 > a_2$ implies that the first firm employs capital more productively than the second firm. A constant labor to capital ratio is assumed, so that a_1 is constant. Having made this assumption, the dynamic counterpart of the above equation, $\dot{q}_1(t) = a_1 \dot{k}_1(t)$, can be used for output determination.[4]

Capital is accumulated according to a continuous time version of the perpetual inventory method. Given the gross investment $i_1(t)$, the net change in capital stock is $\dot{k}_1(t) = i_1(t) - \rho k_1(t)$, subject to some initial capital endowment $k_1(0) > 0$ and the common depreciation rate $\rho \in (0, 1]$. Substitution in $\dot{q}_1(t) = a_1 \dot{k}_1(t)$ produces

$$\dot{q}_1(t) = a_1[i_1(t) - \rho k_1(t)] = a_1 i_1(t) - \rho q_1(t). \tag{2}$$

[4]Both the Leibnitz and the raised dot notation will be used to denote the derivative with respect to time.

The assumption of a constant labor to capital ratio implies the constancy of the labor coefficient $l_1(t)/q_1(t)$ and that a variable production cost per unit of output $w_1 > 0$.

In the absence of fixed production costs, the firm generates a cash flow

$$c_1(t) = [h_1(q_1(t), q_2(t)) - w_1]q_1(t).$$

We assume that the firm invests a fixed portion of the cash flow if the latter is positive. If there are no other sources of funding available to the firm, then the investment rule implies $v_1 i_1(t) = \lambda_1[c_1(t)]^+$, where $v_1 > 0$ is the cost per unit of capital, $\lambda_1 \in (0, 1)$ is the propensity to invest, and $[\cdot]^+$ is a short-hand notation for $\max[\cdot, 0]$. The difference between cash flow and investment outlays is the firm's current profit $\pi_1(t) = (1 - \lambda_1)[c_1(t)]^+$. As $q_1(t) > 0$, we have $i_1(t) = \lambda_1 a_1 v_1^{-1}[h_1(q_1(t), q_2(t)) - w_1]^+ q_1(t)$. Since $\lambda_1 < 1$, the firm earns a positive profit when its cash flow is positive.

Substitution of this expression into (2) yields the reaction function

$$\dot{q}_1(t) = \left\{\frac{\lambda_1 a_1}{v_1}[h_1(q_1(t), q_2(t)) - w_1]^+ - \rho\right\} q_1(t).$$

We study the stability of a system defined by a pair of growth equations:

$$\dot{q}_1(t) = \left\{\frac{\lambda_1 a_1}{v_1}[A_1 - B_1 q_1(t) - C_1 q_2(t) - w_1]^+ - \rho\right\} q_1(t),$$

$$\dot{q}_2(t) = \left\{\frac{\lambda_2 a_2}{v_2}[A_2 - B_2 q_2(t) - C_2 q_1(t) - w_2]^+ - \rho\right\} q_2(t),$$

where $q_1(0) > 0$ and $q_2(0) > 0$.

We model a subsidy or a production tax credit to Firm 2 as a reduction in the variable production cost per unit of output w_2.[5] Specifically, if the second firm receives a subsidy or production tax credit $s \in [0, w_2]$, then its variable cost becomes $w_2 - s$. Other types of fiscal measures such as those aimed at reducing the cost of capital v_2 can also be studied using the above model. In what follows we assume that Firm 2 is the clean entrant (Sects. 3 and 4).

2.1 Competitive Dynamical System

The definition of strategic substitutes stated in terms of a reaction function corresponds to the notion of a competitive dynamical system. A dynamical system is said

[5] Production and other types of tax credits are used to foster innovation and adoption of low-carbon technologies. For instance, the US economic stimulus packages formulated in The American Recovery and Reinvestment Act of 2009 foresee production and research tax credits as an incentive for a post-carbon transition. Similar measures have been implemented in other countries, e.g. France.

to be competitive if an increase in fitness of one entity adversely affects other entities. Conversely, a system is cooperative if entities interact in mutually supportive ways.[6] Formally, an autonomous two-dimensional dynamical system with a differentiable right-hand side

$$\dot{q}_1(t) = \Phi_1(q_1, q_2),$$
$$\dot{q}_2(t) = \Phi_2(q_1, q_2),$$

is called competitive if $\partial\Phi_1/\partial q_2$, $\partial\Phi_2/\partial q_1 < 0$ and cooperative if $\partial\Phi_1/\partial q_2$, $\partial\Phi_2/\partial q_1 > 0$. One fundamental feature of such systems can be inferred despite the generality of the right-hand side. Competitive and cooperative dynamical systems fulfil the Bendixson condition for the non-existence of periodic solutions. Consequently, the outputs of a two-dimensional system of either type are monotone functions of time and the orbits of the system (output trajectories) converge either to infinity or to a rest point. A rest point represents market equilibrium. This result does not extend to three or more dimensional systems, in which periodic solutions remain a possibility even in the case of linear reaction functions.

3 The Degree of Product Differentiation and Stability of the Duopoly

Next, we show that the degree of product differentiation influences the stability of the duopoly. A firm earns sufficient cash flow to grow as long as the market price of its product exceeds the firm's break-even price. Isoclines are the loci of all pairs (q_1, q_2) that support a firm's break-even price.

$$L_1 = \left\{ (q_1, q_2) \in \mathbb{R}^2_+ \text{ such that } \frac{\lambda_1 a_1}{v_1}(A_1 - B_1 q_1 - C_1 q_2 - w_1) = \rho \right\},$$
$$L_2 = \left\{ (q_1, q_2) \in \mathbb{R}^2_+ \text{ such that } \frac{\lambda_2 a_2}{v_2}(A_2 - B_2 q_2 - C_2 q_1 - w_2) = \rho \right\}$$

or, setting $V_1 = [A_1 - w_1 - \frac{\rho v_1}{\lambda_1 a_1}]$ and $V_2 = [A_2 - w_2 - \frac{\rho v_2}{\lambda_2 a_2}]$,

$$q_2 = \frac{V_1}{C_1} - \frac{B_1}{C_1} q_1,$$
$$q_2 = \frac{V_2}{B_2} - \frac{C_2}{B_2} q_1.$$

In $\mathbb{R}^2_+$, the isoclines either intersect, do not intersect, or coincide. Their relative position defines the number, arrangement and stability properties of the equilibria.

[6]See Hofbauer and Sigmund (1998, Chap. 3.4) for a discussion and further references.

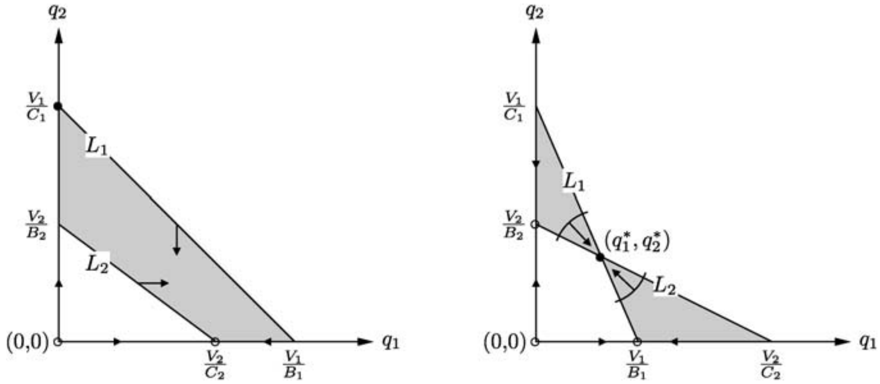

Fig. 1 Imperfect substitutes. Here, $V_1 = [A_1 - w_1 - \frac{\rho v_1}{\lambda_1 a_1}]$ and $V_2 = [A_2 - w_2 - \frac{\rho v_2}{\lambda_2 a_2}]$. *The encircled points* are stationary for $B_1 > C_1$, $B_2 > C_2$. *The filled point* is globally asymptotically stable

For all combinations of individual outputs lying above the isoclines, market prices are such that both firms contract. This situation occurs only when both firms start with outputs that cannot be sustained at market entry. In our analysis we focus on the situations in which both firms grow at the onset. Coinciding isoclines imply that both firms are identical in all respects, and hence there is no product differentiation. Figure 1 illustrates the relevant cases.

The slope of L_1 is given by B_1/C_1, while the slope of L_2 is given by C_2/B_2. Since $B_1 \geq C_1$ and $B_2 \geq C_2$, L_1 is steeper than L_2, as is shown in the right panel of Fig. 1. In the left panel, the market price is such that Firm 2, the entrant and clean energy producer, is unable to sustain a constant capacity. It is forced to gradually reduce its output and eventually exits the market. The incumbent keeps growing until its cash flow dwindles down to the point where its output stagnates. The equilibrium output of the duopoly equals $q_1 = V_1/C_1$, as the entrant exists the market. It can be shown that this solution is a locally asymptotically stable equilibrium.[7] It is also globally stable (GAS) when $q_1(0) > 0$ and $q_2(0) > 0$, or when the firms start with strictly positive production capacities. In the situation portrayed above, the entrant's product is not sufficiently differentiated to compensate for its inefficiency, or customer awareness towards the environmental-friendliness of clean-energy products has yet not been developed adequately.

Let the isoclines intersect in $\mathbb{R}^2_+$. Again, for combinations of individual outputs located below the isoclines, each firm earns sufficient cash flow to grow. The shaded region to the left represents all combinations of individual outputs that yield a market price of Product 1 below the break-even price of Firm 1, and a market price of Product 2 above the break-even price of Firm 2. In this region, Firm 1 grows, while Firm 2 contracts. The opposite holds for all combinations of individual out-

[7]This is accomplished using the standard linearization procedure, which involves an analysis of the eigenvalues of the Jacobian matrix of the model evaluated at the singular point.

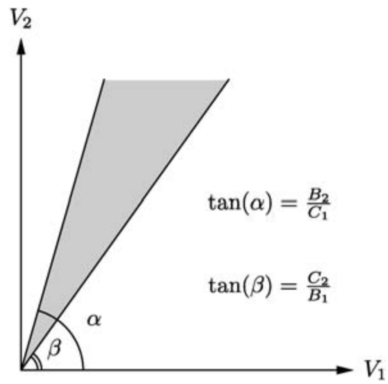

Fig. 2 Stability with imperfect substitutes. All other things being equal, the larger $B_1 - C_1$ and $B_2 - C_2$ are, the larger is the set of parameters that supports stable duopoly

puts belonging to the shaded region to the right. There are four stationary points, but only the intersection is GAS. Thus, independently of initial (positive) capital endowments, individual outputs converge to

$$q_1^* = \frac{V_1 B_2 - V_2 C_1}{B_1 B_2 - C_1 C_2}, \tag{3}$$

$$q_2^* = \frac{V_2 B_1 - V_1 C_2}{B_1 B_2 - C_1 C_2}. \tag{4}$$

The degree of product differentiation is given by $B_1 - C_1$ and $B_2 - C_2$. It can be shown that the more differentiated the products are, the more stable the duopoly. In this sense, product differentiation indeed leads to softer competition. Since the denominator in (3) is positive, an intersection in the interior of $\mathbb{R}^2_+$ is only feasible if

$$V_1 B_2 - V_2 C_1 > 0, \tag{5}$$

$$V_2 B_1 - V_1 C_2 > 0, \tag{6}$$

subject to the mild condition $V_1, V_2 > 0$ that supposes a sufficient carrying capacity in the market (i.e. sufficiently high A_1, A_2). In the space spanned by V_1 and V_2, the solution to (5)–(6) is given by the shaded area delimited by two lines with slopes given by

$$\frac{B_1}{C_2} > \frac{C_1}{B_2} > 0. \tag{7}$$

The geometry of Fig. 2 reveals that the larger $B_1 - C_1$ and $B_2 - C_2$ are, the more likely the isoclines are to intersect in the interior of $\mathbb{R}^2_+$, and the more stable the duopoly.

3.1 Profits

The market value of a firm is commonly given by the present value of the future profit flow. In a dynamic setting, the comparison of future profit flows is complicated by the fact that firms vary in their lifetimes. As a consequence, it is possible for a short-lived firm to earn more profit than an infinitely long-lived firm would, provided it has a sufficiently high discount rate. Discounting makes profits to be earned in the distant future of little value today.

The current profit of Firm 1 is the difference between cash flow and investment outlays, or $\pi_1(t) = (1-\lambda_1)[p_1(q_1(t), q_2(t)) - w_1]^+ q_1(t)$. The dynamics of the current profit of Firm 2 described by a similar equation. Let $r > 0$ is a discount rate, for example, a risk-free rate of return. The sign of the derivative $\dot{\Pi}_1(t) = (1-\lambda_1)e^{-rt}\pi_1(t) \geq 0$ tells us that, even if the current profit eventually declines, the discounted future profit is non-decreasing in time. Since $q_1(0) > 0$ and p_1 is bounded away from zero for any small q_1 and q_2, so is $\Pi_1(t)$. Both properties ensure that $\lim_{t\to\infty} \Pi_1(t) = \Pi_1^*$ exists.

The reaction function $\dot{q}_1(t) = \{\lambda_1 a_1 v_1^{-1}[p_1(q_1(t), q_2(t)) - w_1]^+ - \rho\} q_1(t)$ is used to obtain the present value of all future profits, Π_1^*. To do so, multiply both sides of the reaction function by e^{-rt} and integrate

$$\begin{aligned}
\int_0^\infty e^{-rt}\dot{q}_1(t)dt &= \frac{\lambda_1 a_1}{v_1}\int_0^\infty e^{-rt}[p_1(q_1(t), q_2(t)) - w_1]^+ q_1(t)dt \\
&\quad - \rho\int_0^\infty e^{-rt} q_1(t)dt \\
&= \frac{\lambda_1 a_1}{(1-\lambda_1)v_1}\Pi_1^* - \rho L_1(r),
\end{aligned}$$

where $L_1(r) = \int_0^\infty e^{-rt} q_1(t)dt$. Integrating the left-hand side (by parts) and rearranging the terms yields

$$\Pi_1^* = \frac{(1-\lambda_1)v_1}{\lambda_1 a_1}\big[(\rho + r)L_1(r) - q_1(0)\big].$$

Taken in its full generality this expression can only be evaluated numerically. Nevertheless, it is possible to study the special cases of a small discount rate. Using the fact that Π_1^* is $O(r^{-1})$ as $r \to 0^+$ and the Final Value Theorem[8] $\lim_{t\to\infty} q_1(t) = \lim_{r\to 0^+} rL_1(r)$, one obtains

$$\lim_{r\to 0^+} r\Pi_1^* = \frac{(1-\lambda_1)v_1}{\lambda_1 a_1}\Big\{\rho q_1^* + [q_1^* - q_1(0)]\lim_{r\to 0^+} r\Big\} = \frac{(1-\lambda_1)v_1}{\lambda_1 a_1}\,\rho\, q_1^*.$$

Taking the ratio of the value of Firm 1 to the value of Firm 2, we obtain

$$\lim_{r\to 0^+}\frac{\Pi_1^*}{\Pi_2^*} = \frac{1-\lambda_1}{1-\lambda_2}\cdot\frac{\lambda_2 a_2 v_1}{\lambda_1 a_1 v_2}\cdot\frac{q_1^*}{q_2^*}.$$

Note as the suggested asymptotic analysis involves equilibrium output levels, the above result can only be used to compare the equilibrium market values of the firms.

[8]See Doetsch (1974, p. 233).

Moreover, the above expression is defined if Firm 2 remains in the equilibrium, as q_2^* vanish otherwise.

3.2 A Numerical Example

Consider a numerical example that illustrates the dynamics of the key quantities described above. Suppose that Firm 1 (the incumbent) employs capital more efficiently than Firm 2 (the clean energy producer entrant). For simplicity the firms are assumed to be identical in every other respect except the initial size. In model terms, the above assumption translates into a larger reciprocal of the capital coefficient a_n. Specifically, let $a_1 = 0.3$ and $a_2 = 0.2$, so that a unit of capital employed by the first (second) firm yields 0.3 (0.2) units of output.[9] The firm-specific multiproduct linear inverse demand functions are given by

$$p_1(q_1, q_2) = 10 - 0.6q_1 - 0.4q_2,$$
$$p_2(q_1, q_2) = 10 - 0.6q_2 - 0.4q_1.$$

The choice of parameters is fairly unambiguous and consistent with the assumption of the two products being strategic substitutes, as is characterized by a larger price effect of own sales.

In this example, the choice of parameters leads to an equilibrium devoid of firm exit. The evolution of outputs is shown in Fig. 3. While small, the firms expand depressing the market prices. At some point the firms arrive at levels of individual outputs such that one product is priced below the threshold of the producing firm, whereas the price of the substitute is still above the threshold of the rival firm. In this example it is Firm 1 that will eventually contract. This fact is also reflected in instantaneous growth rates plotted in the right panel of Fig. 3. The growth rate of Firm 1 becomes negative.

The evolution of current profits shown in the left panel in Fig. 4 conveys the same story. Both firms' current profits trace a bell-shaped curve as the generated cash flow eventually declines. The area under the current profit curve gives the present value of future profits. For illustrative purpose, the present value has been approximated by

$$\Pi_i = \sum_{t=0}^{T} \frac{\pi_i(t)}{(1+r)^t} \quad \text{for } T = 40, \ i = 1, 2.$$

The evolution of this magnitude is shown in the right panel in Fig. 4. Note that although current profits eventually vanish, the present value of future profits is positive. In principle, a finitely long-lived firm can generate more profit over lifespan than an infinitely long-lived firm.

[9] The remaining parameters are: depreciation rates $\rho_1 = \rho_2 = 0.08$, propensities to invest $\lambda_1 = \lambda_2 = 0.4$, capital costs in efficiency units $v_1/a_1 = 2$ and $v_2/a_2 = 3$, variable production cost per unit of output $w_1 = w_2 = 2$, initial outputs $q_1(0) = 0.4$, $q_2(0) = 0.6$, discount rate $r = 0.1$.

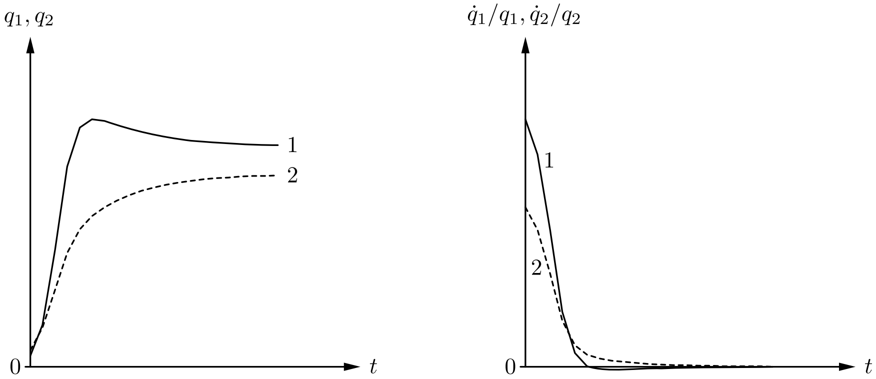

Fig. 3 Example 1: Output and growth

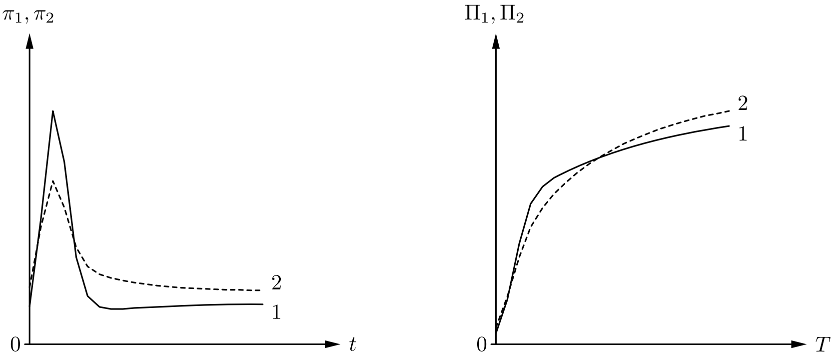

Fig. 4 Example 1: Current profit and the present value

4 The Minimal Sufficient Subsidy or Tax

The stability analysis of Sect. 3 allows us to derive the minimal subsidy to Firm 2 (the clean energy entrant) or, equivalently, the minimal tax on Firm 1 (the polluting incumbent) sufficient for Firm 2 to remain in the market in the long run (i.e. to have a positive equilibrium market share). In other words, starting with a situation in which the isoclines do not intersect (left panel of Fig. 1), we wish to shift L_2 to the right until it intersects L_1 on the boundary (right panel of Fig. 1). The required minimal subsidy is given by

$$\min[s] = \frac{V_1}{B_1} - \frac{V_2}{C_2}.$$

At any higher level of subsidy, the equilibrium market share of Firm 2 will be positive. Remarkably, the required minimal subsidy does not depend on the cross-derivatives B_2 and C_1. It is easy to see from (3) that the market share of Firm 2 increases in s, and that, given a sufficiently high subsidy, Firm 2 will squeeze the incumbent out of the market. Such a subsidy must exceed $\frac{1}{C_2}(\frac{B_2 V_1}{C_1} - V_2)$. The higher B_1 and the lower C_2, the lower is the minimal subsidy sufficient to preserve Firm 2 in the long run. All other things being equal, the higher the degree of product differentiation, $B_1 - C_1$ and $B_2 - C_2$, the lower is the subsidy.

5 Summary and Conclusions

Using a simple dynamic model of a duopoly, we have shown that the minimal tax credit (to the clean energy entrant) or tax (on the incumbent) sufficient to preserve the entrant in the market in the long run depends on the underlying demand structure. The rate is higher when customers do not differentiate products. The more differentiated the products, the more stable is the duopoly and the less government intervention is necessary.

Given that physical characteristics of electricity or heat do not depend on their means of production (or source), inducing product differentiation is particularly challenging. In this context, product differentiation needs to address preferences of consumers that go beyond immediate use, and encompass their broader and long term implications such as climate change. Ecological and energy labeling indicating the energy efficiency and environmental impact of consumer goods is an example in case.[10] This and similar measures are essentially the same as those firms use to differentiate their products. Firms have a strong incentive to induce customers to differentiate their products from those of their competitors and thus to soften the competition. Softer competition increases the chances for a prolonged coexistence of firms, and invites market entry. While product differentiation can considerably soften competition, it will not completely negate it. A monopoly is still feasible if the incumbent is vastly superior to the entrant in terms of efficiency or production costs, or if the degree of product differentiation is insufficient. In this case, saving the clean entrant may require an excessively high rate of subsidy. However, given the absence of a basis for product differentiation with respect to physical qualities, an additional price incentive through a reduced VAT rate on clean products may be required.

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Dynamic Oligopoly with Capital Accumulation and Environmental Externality

Davide Dragone, Luca Lambertini, and Arsen Palestini

Abstract We model the interplay between capital accumulation for production and environmental externalities in a differential oligopoly game with Ramsey dynamics. The external effect is determined, alternatively, by sales or production. While the externality does not affect the behaviour of profit-seeking firms, it may induce a benevolent planner to shrink sales as compared to the Cournot-Nash equilibrium because of a tradeoff between consumer surplus and the externality, if the latter is driven by sales. If instead it is determined by production, there emerges that the Ramsey golden rule is no longer socially optimal.

1 Introduction

The control of polluting emissions damaging the environment is a hot issue and is receiving an increasing amount of attention in the current literature in the field of environmental economics. Most of the existing contributions investigate the design of optimal Pigouvian taxation aimed at inducing firms to reduce damaging emissions, both in monopoly and oligopoly settings.[1] The established approach to this problem consists in taking the social optimum, where a benevolent planner chooses a production plan for the firms in the industry so as to maximise social welfare, as a benchmark against which the performance of the profit-seeking firms has to be assessed. This produces corrective policy measures which, ideally, should take the form of tax schemes able to reproduce the same social welfare level associated with the first best.

Another stream of literature analyses the feasibility of tradeable pollution permits, which, however, may lead to the monopolization of the industry.[2]

[1]See Bergstrom et al. (1987), Karp and Livernois (1992, 1994) and Benchekroun and Long (1998, 2002), inter alia.

[2]To this regard, see Newbery (1990) and von der Fehr (1993), inter alia.

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D. Dragone (✉)
Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy
e-mail: davide.dragone@unibo.it

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To the best of our knowledge, the interplay between environmental externalities and capital accumulation under oligopoly or imperfect competition has received scanty, if any, attention thus far.[3] Indeed, this is a relevant facet of the general matter, in particular in view of the current debate on globalization and the ambiguous attitude adopted in this respect by new major actors, like China and India, but also Brazil, in shaping the current look of the international economic system for the new millennium.

We illustrate a dynamic oligopoly model where firms accumulate capacity *à la* Ramsey (as in Cellini and Lambertini 1998, 2008) to produce the final good and either sales or production cause a negative environmental externality (pollution). Given the assumption that firms do not internalise the externality, the latter does not affect their optimal behaviour, yielding either the Cournot-Nash solution or the Ramsey golden rule as a saddle point equilibrium, depending upon the relative size of parameters. Clearly, the opposite holds at the social optimum, where the maximization of social welfare also accounts for the external effect. If the externality depends on sales, then a benevolent social planner may find it convenient to produce less than the profit-seeking firms if the weight attached to the externality is sufficiently high, in view of the tradeoff between the externality itself and consumer surplus. When instead the external effect is determined by production, the picture of the profit-seeking behaviour remains the same while the social optimum changes drastically, with the Ramsey golden rule disappearing as a stand-alone equilibrium.

The basic model is in Sect. 2. Section 3 contains the oligopoly game among profit-seeking firms, while the analysis of the social optimum in the case where the externality is determined by sales is in Sect. 4. The comparative analysis of the two regimes is carried out in Sect. 5. The alternative model where the externality depends on production is laid out in Sect. 6. Concluding remarks are in Sect. 7.

2 The Set Up

The present set up is a simplified version of the dynamic game presented in Cellini and Lambertini (1998). Consider a market where N identical firms produce and sell a homogeneous product under Cournot competition. The inverse demand function for the good is

$$p(t) = a - \sum_{i=1}^{N} q_i(t) \tag{1}$$

[3] Instead, uncountably many contributions studying the interplay between pollution and growth or technical change do exist. See Bovenberg and de Mooij (1997), Hartman and Kwon (2005), Jouvet et al. (2005), Dutta and Radner (2006), Greiner (2007), Ricci (2007), Bartz and Kelly (2008), Itaya (2008). for an overview, see Dockner et al. (2000).

where $q_i(t) \in [0, \overline{q})$ is the quantity produced and sold by firm i at time t and $a > 0$ is the reservation price. Production costs are linear in quantities

$$C_i(t) = cq_i(t)$$

with $c \geq 0$ being exogenously given and identical for all firms.

Production requires physical capital $k_i(t)$ that accumulates over time to create capacity. At any instant of time t, the output level of each firm is

$$y_i(t) = f(k_i(t)) \tag{2}$$

where $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f'' \equiv \partial^2 f(k_i(t))/\partial k_i^2(t) < 0$. We assume that, at any time t, $q_i(t) \leq y_i(t)$, so that the level of sales cannot exceed the quantity produced. Output that is not sold is used to build up productive capacity according to

$$\dot{k}_i(t) = f(k_i(t)) - q_i(t) - \delta k_i(t) \tag{3}$$

where $\delta > 0$ is the depreciation rate of capital.

Under the above assumptions, the instantaneous profit of each firm is $\pi_i(t) = (p(t) - c)q_i(t)$. Given a common intertemporal discount rate $\rho > 0$, the goal of each firm is to maximize the discounted value of its flow of profits

$$\Pi_i = \int_0^{\infty} e^{-\rho t} \pi_i(t) dt \tag{4}$$

under the dynamic constraint (3).

With respect to Cellini and Lambertini (1998), we now introduce the assumption that producing the good is polluting and that this externality is not taken into account by the single firm (which is myopic or simply not interested in this aspect of its activities), but it enters the social welfare evaluation made by a benevolent social planner. Assuming that the social cost of pollution at any time t is quadratic in the total amount of output sold, the social welfare function of the social planner is

$$\begin{aligned} sw(t) &= \sum_{i=1}^{N} \pi_i(t) + CS(t) - EXT(t), \\ EXT(t) &= \beta \left[\sum_{i=1}^{N} q_i(t) \right]^2 \end{aligned} \tag{5}$$

where the first term represents the profits of the N firms, $CS(t) = (a - p(t)) \times Q(t)/2 = Q(t)^2/2$ is consumer surplus and the last term represents the social cost of pollution, with $\beta > 0$. Observe that here the environmental externality $EXT(t)$ depends on actual sales (or equivalently, consumption) and not on production or installed capacity. In the remainder we will also discuss the alternative cases where either $EXT(t) = \beta[\sum_{i=1}^{N} f(k_i(t))]^2$.

In next section we determine the open-loop Nash equilibrium of the game played by N firms neglecting the social cost of pollution. Then we compare this equilibrium with the solution that would be implemented by a benevolent social planner that takes into consideration also the social cost of pollution.

3 Cournot Competition

Given that the model is not built in linear-quadratic form, we will focus our attention on the open-loop solution. The current-value Hamiltonian function of each firm i is

$$H_i(t) = \pi_i(t) + \lambda_{ii}(t)\dot{k}_i(t) + \sum_{j \neq i} \lambda_{ij}(t)\dot{k}_j(t) \tag{6}$$

$$= \left[a - c - q_i(t) - \sum_{j \neq i} q_j(t)\right] q_i(t) + \lambda_{ii}(t)[f(k_i(t)) - q_i(t) - \delta k_i(t)]$$
$$+ \sum_{j \neq i} \lambda_{ij}(t)[f(k_j(t)) - q_j(t) - \delta k_j(t)] \tag{7}$$

where $\lambda_{ii}(t)$ is the costate variable associated to $k_i(t)$ and $\lambda_{ij}(t)$ is the costate variable associated to $k_j(t)$ by firm i. The initial condition for firm i is $k_i(0) = k_{i0}$.

Under the requirement that the following set of transversality conditions

$$\lim_{t \to \infty} \lambda_{ij}(t) k_j(t) = 0 \tag{8}$$

is satisfied for all i and j, the necessary conditions for a path to be optimal are:

$$\frac{\partial H_i}{\partial q_i} = a - c - 2q_i - \sum_{j \neq i} q_j - \lambda_{ii} = 0, \tag{9}$$

$$-\frac{\partial H_i}{\partial k_i} = -\lambda_{ii}[f' - \delta] = \dot{\lambda}_{ii} - \rho\lambda_{ii} \quad \Longrightarrow \quad \dot{\lambda}_{ii} = \lambda_{ii}[\rho + \delta - f'], \tag{10}$$

$$-\frac{\partial H_i}{\partial k_j} = -\lambda_{ij}\left[\frac{\partial f(k_j)}{\partial k_j} - \delta\right] = \dot{\lambda}_{ij} - \rho\lambda_{ij}, \tag{11}$$

where the time arguments are omitted for brevity. From (9) we obtain, for all t,

$$q_i = \frac{1}{2}\left[a - c - \sum_{j \neq i} q_j - \lambda_{ii}\right]. \tag{12}$$

As the costate variables λ_{ij} are irrelevant for the optimal choice of sales q_i (indeed any costate equation (11) is in separable variables and admits the solution $\lambda_{ij} = 0$ at all times), we proceed by setting $\lambda_{ij}(t) = 0$ for all $i \neq j$ and all t. Differentiating

(12) with respect to time and using (9)–(10), we get

$$\dot{q}_i = -\frac{1}{2}\left[\sum_{j\neq i}\dot{q}_j + \dot{\lambda}_{ii}\right] = -\frac{1}{2}\left[\sum_{j\neq i}\dot{q}_j + \left(a - c - 2q_i - \sum_{j\neq i} q_j\right)(\delta + \rho - f')\right]. \quad (13)$$

Given the ex-ante symmetry, we impose that the choices made by the firms are symmetrical, i.e.:

$$q_i = q_j = q \quad \forall j \neq i, \ \forall t. \quad (14)$$

Under the above assumption, (13) simplifies to

$$(N+1)\dot{q} = [a - c - (N+1)q](f' - \delta - \rho). \quad (15)$$

The state-control dynamic system of the model is the following one:

$$\begin{cases} \dot{k} = f(k) - q - \delta k, \\ \dot{q} = \frac{1}{N+1}[a - c - (N+1)q](f'(k) - \delta - \rho). \end{cases} \quad (16)$$

The steady state pair (k, q) solves one of the following systems

$$\begin{cases} q^C = \frac{a-c}{N+1}, \\ q^C = f(k^C) - \delta k^C, \end{cases} \quad (17)$$

$$\begin{cases} f'(k^R) = \delta + \rho, \\ q^R = f(k^R) - \delta k^R, \end{cases} \quad (18)$$

where the first solution is the familiar Ramsey golden rule solution and the second one represents the static solution that emerges from the static version of the Cournot game. For further reference, the total output produced under the Cournot solution is

$$Q^C = Nq^C = \frac{N(a-c)}{N+1}. \quad (19)$$

To visualize one possible solution of the game, consider Fig. 1.

The locus $\dot{k} \equiv dk/dt = 0$, as well as the dynamics of k (depicted by the horizontal arrows), derives from the first equation of system (16). The locus $\dot{q} \equiv dq/dt = 0$ is given by the solutions of the second equation of (16) and consists of a horizontal branch (corresponding to the Cournot solution q^C) and of a vertical branch (corresponding to the Ramsey solution where $f'(k^R) = \delta + \rho$). The dynamics of q is summarised by the vertical arrows. Steady state equilibria, denoted by $E1$, $E3$ along the horizontal branch, and $E2$ along the vertical one, are identified by the intersections between loci. Notice that, as $E1$ and $E3$ entail the same levels of sales, point $E3$ is inefficient in that it requires a higher amount of capital.

Figure 1 describes only one out of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho + \delta$ is independent of demand parameters, while the locus $q^C = (a - c)/(N + 1)$ shifts upwards (resp., downwards) as a (resp., c) increases. Therefore, we obtain one out of five possible regimes:

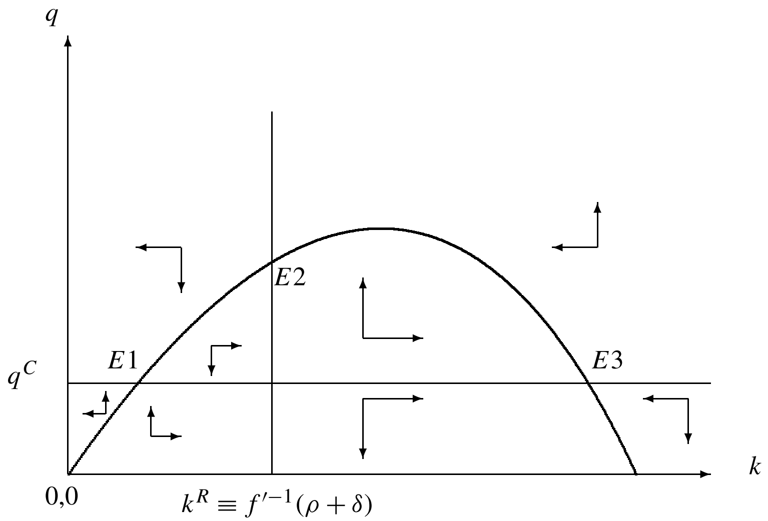

Fig. 1 The phase diagram under Cournot competition

1. There exist three steady state points, with $k_{E1} < k_{E2} < k_{E3}$ (this is the specific case portrayed in Fig. 1).
2. There exist two steady state points, with $k_{E1} = k_{E2} < k_{E3}$.
3. There exist three steady state points, with $k_{E2} < k_{E1} < k_{E3}$.
4. There exist two steady state points, with $k_{E2} < k_{E1} = k_{E3}$.
5. There exists a unique steady state equilibrium point, corresponding to $E2$.

To assess the stability properties of the steady state(s), consider the Jacobian matrix associated to (16):

$$J(k,q) = \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial q} \\ \frac{\partial \dot{q}}{\partial k} & \frac{\partial \dot{q}}{\partial q} \end{bmatrix} = \begin{bmatrix} f'(k) - \delta & -1 \\ \frac{a-c-(N+1)q}{N+1} f''(k) & \delta + \rho - f'(k) \end{bmatrix}.$$

Evaluating J in the Ramsey solution yields.

$$J(k^R, q^R) = \begin{bmatrix} \rho & -1 \\ \frac{a-c-(N+1)[f(k^R)-\delta k^R]}{N+1} f''(k^R) & 0 \end{bmatrix}.$$

(k^R, q^R) is a saddle point if $a - c - (N+1)[f(k^R) - \delta k^R] > 0$. Otherwise, taking the size of the market $a - c = \sigma$ as a bifurcation parameter, we can easily remark that:

- If $\sigma > \max\{0, (N+1)[\frac{\rho^2}{4f''(k^R)} + f(k^R) - \delta k^R]\}$, (k^R, q^R) is an unstable node.
- If $\sigma < (N+1)[\frac{\rho^2}{4f''(k^R)} + f(k^R) - \delta k^R]$, (k^R, q^R) is an unstable focus provided that $f(k^R) > \delta k^R - \frac{\rho^2}{4f''(k^R)}$.

Evaluating J in the Cournot-Nash equilibrium, we have:

$$J(k^C, q^C) = \begin{bmatrix} f'(k^C) - \delta & -1 \\ 0 & \delta + \rho - f'(k^C) \end{bmatrix},$$

whose determinant is negative if $f'(k^C) > \delta + \rho$. This implies that (k^C, q^C) is a saddle point whenever $q^C < q^R$, while it is an unstable node otherwise.

The discussion carried out so far can be intuitively summarised by noting that the sign of the determinant of the Jacobian matrix is the sign of $a - c - (N + 1) \times [f(k^R) - \delta k^R] = (N + 1)(q^C - q^R)$ and therefore, if $q^R > q^C$, the saddle point is identified by the intersection of the Cournot-Nash quantity with the locus $\dot{k} = 0$; conversely, if $q^R < q^C$, the saddle point coincides with the Ramsey golden rule. Residually, the dynamics illustrated in Fig. 1 intuitively reveals that the origin (point $(0, 0)$) is unstable.

The stability analysis reveals that:

Regime 1 $E1$ is a saddle point, while $E2$ is an unstable focus. $E3$ is again a saddle point, with the horizontal line as the stable manifold.

Regime 2 $E1$ coincides with $E2$, so that we have only two steady states which are both saddle points. In $E1 = E2$, the saddle path approaches the saddle point from the left only, while in $E3$ the stable manifold is again the horizontal line.

Regime 3 $E2$ is a saddle, $E1$ is an unstable focus. $E3$ is a saddle point, as in regimes 1 and 2.

Regime 4 Here, $E1$ and $E3$ coincide. $E3$ remains a saddle, while $E1 = E3$ is a saddle whose converging manifold proceeds from the right along the horizontal line.

Regime 5 Here, there exists a unique steady state point, $E2$, which is a saddle point.

We can sum up the above discussion as follows. The unique efficient steady state with saddle point stability is $E2$ if $k_{E2} < k_{E1}$, while it is $E1$ if the opposite inequality holds. Individual equilibrium output is q^C if the equilibrium is in $E1$, or $q^R = f(k^R) - \delta k^R$ (i.e., the output level corresponding to the optimal capital constraint k^R) if the equilibrium is point $E2$. The reason is that, if the capacity at which marginal instantaneous profit is nil is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and firms' decisions in each period are driven by the unconstrained maximisation of single-period profits only. Hence, we can state

Proposition 1 *The efficient steady state Nash equilibrium of the open-loop oligopoly game has saddle point stability and is associated to the following individual level of sales*

$$q^N_{OL} = \min\{q^C, q^R\}.$$

Some additional remarks are in order concerning the inefficient Cournot solution $E3$, whenever such a solution is a saddle point (as in Fig. 1). As shown in

Cellini and Lambertini (2008), this is a strongly time consistent equilibrium under the open-loop information structure, involving $\lambda_{ii} = 0$, provided the initial capital endowment $k_i(0)$ be large enough to allow the firm to produce q^C in every instant and let the capacity depreciate at the rate δ. If the externality depends on sales, as in this version of the model, adopting this solution has no effect on the amount of pollution. Yet, as we shall see in the remainder, this is no longer true if polluting emissions depend on production or installed capacity.

4 The Social Optimum

The open-loop Nash solution of the game clearly does not depend on pollution, because of the myopic attitude of firms. In this section we want to establish the conditions under which a social planner that trades-off the negative social externality due to pollution with the profits of the industry and consumer surplus would recommend a lower level of production. Introducing the symmetry assumption, so that $q_i = q$ for all i, the instantaneous social welfare (5) of the social planner is

$$\begin{aligned} sw(t) &= N\pi(t) + \frac{N^2q^2(t)}{2} - \beta N^2q^2(t) \\ &= N(a - c - Nq(t))q(t) + \frac{N^2q^2(t)}{2} - \beta N^2q^2(t). \end{aligned}$$

The social planner aims at maximizing the discounted value of social welfare

$$SW = \int_0^\infty e^{-\rho t} sw(t)dt$$

under the dynamic constraint

$$\dot{k}(t) = f(k(t)) - q(t) - \delta k(t). \tag{20}$$

The current value Hamiltonian for the social planner is (omitting the time argument for brevity)

$$H_{SP} = N(a - Nq - c)q + \frac{N^2q^2}{2} - \beta N^2q^2 + \mu[f(k) - q - \delta k]$$

where μ is the costate variable. The necessary conditions are

$$\frac{\partial H_{SP}}{\partial q} = N[a - c - (1 + 2\beta)Nq] - \mu = 0, \tag{21}$$

$$-\frac{\partial H_{SP}}{\partial k} = \mu[\delta - f'(k)] = \dot{\mu} - \rho\mu \quad \Longrightarrow \quad \dot{\mu} = \mu[\rho + \delta - f'(k))] \tag{22}$$

and the transversality condition $\lim_{t\to\infty} \mu(t)k(t) = 0$ applies.

From (21) one obtains

$$q = \frac{1}{N^2(1+2\beta)}(aN - cN - \mu) \tag{23}$$

and, differentiating w.r.t. t, we get

$$\dot{q} = -\frac{1}{N^2(1+2\beta)}\dot{\mu}.$$

Using (22) and (21), the latter expression simplifies as

$$\begin{aligned}\dot{q} &= -\frac{1}{N^2(1+2\beta)}(\rho + \delta - f'(k))\mu \\ &= -\frac{1}{N(1+2\beta)}(\rho + \delta - f'(k))[a - c - (1+2\beta)Nq]. \end{aligned} \tag{24}$$

The steady states must satisfy one of the following systems

$$\begin{cases} f'(k^R) = \delta + \rho, \\ q^R = f(k^R) - \delta k^R, \end{cases}$$

$$\begin{cases} q^{SP} = \frac{a-c}{(1+2\beta)N}, \\ q^{SP} = f(k^{SP}) - \delta k^{SP}. \end{cases}$$

The first solution coincides with the Ramsey golden rule already found in the previous section, while the second solution coincides with that chosen by a social planner in the static case where the market parameters and the sensitivity to pollution are taken into account (to see it, just maximize the instantaneous social welfare function with respect to output q). The two alternative steady states are portrayed in Fig. 2.

Considering first the market-driven solution, total output is

$$Q^{SP} = Nq^{SP} = \frac{a-c}{1+2\beta}.$$

Comparing the level of total output reached under social planning with total output under Cournot competition, one obtains:

$$Q^{SP} > Q^C \quad \Longleftrightarrow \quad \beta < \frac{1}{2N} \equiv \widehat{\beta}.$$

Clearly, in the limit case where $\beta = 0$, one obtains that the level of total output under social planning is necessarily larger than the sales level reached by the industry under Cournot competition. Nevertheless, if the social planner is sensitive to pollution, there is an incentive to reduce the total amount of output. In other words, there is an incentive for the social planner to reduce consumer surplus by decreasing the total amount of output (which corresponds to an increase in prices), as this is more than compensated by the reduction in the amount of pollution and by the increase in total

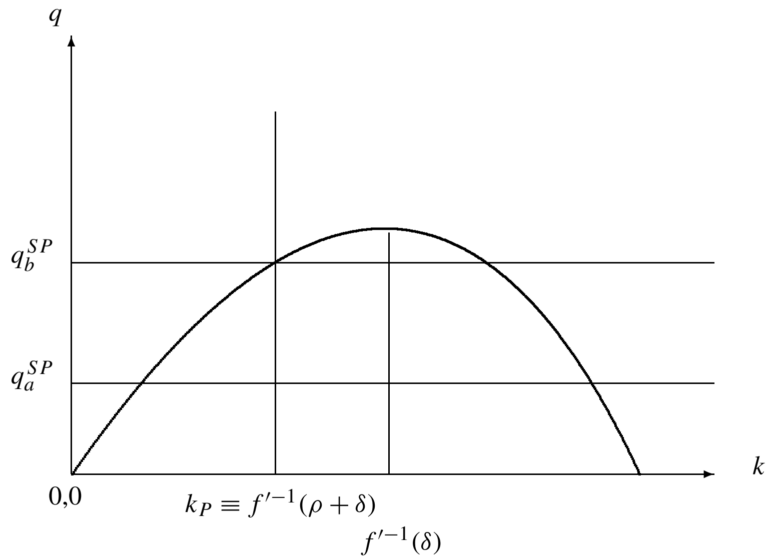

Fig. 2 The phase diagram under social planning

profits. This argument can be reinforced by observing that the industry output under social planning is smaller than under perfect competition $(a-c)$ for all $\beta > 0$. This discussion can be summarised by:

Proposition 2 *Suppose the industry produces* $Q^C = N(a-c)/(N+1)$ *at the Nash equilibrium of the open-loop game, and* $Q^{SP} = (a-c)/(1+2\beta)$. *If so, there exists a threshold value* $\widehat{\beta}$ *above which* $Q^{SP} < Q^C$. *Such a threshold level of* β *is decreasing in* N, *with* $\lim_{N\to\infty}\widehat{\beta} = 0$.

The last remark in the above Proposition entails that, if the market-driven solution prevails under both regimes, an increase in the intensity of market competition is not necessarily welcome from the standpoint of a social planner as it brings about an increase in the total amount of polluting emissions.[4] In the limit, as the Cournot-Nash equilibrium collapses onto perfect competition, any $\beta > 0$ implies that, from the standpoint of the planner, the external effect matters more than the price effect, and therefore the planner produces less than the industry output at the perfectly competitive equilibrium. That is, perfect competition *per se* is not efficient as firms do not internalise the externality.

Now we turn to the alternative case where social planning ends up in the Ramsey equilibrium, which happens when $Q^{SP} > Q^R$ and the latter is a saddle point solution. In this situation, the features of intertemporal capacity accumulation (i.e.,

[4] In this respect, a wave of horizontal mergers, or alternatively allowing for some degree of collusion among firms, could be a way of indirectly preserving the environment. To this regard, see Lambertini and Mantovani (2008).

parameters ρ and δ and the marginal productivity of capital) matter more than the environmental concern:

Proposition 3 *If $Q^{SP} > Q^R$, and therefore the Ramsey golden rule obtains as the socially optimal saddle point equilibrium, the benevolent planner neglects the environmental aspects of the industry and focuses upon optimal intertemporal growth only.*

A thorough assessment of the profit-driven equilibrium vs. the socially optimal allocation is carried out in next section.

5 Cournot Oligopoly vs. Social Planning

To begin with, consider the case where $q^{SP} \geq q^R > q^C$ (as in Fig. 3). If so, then the socially optimal allocation reflects the golden rule and the planner neglects the presence of environmental externalities.

This is indeed a case where no agent cares about it, as of course profit-seeking firms do not attach any weight to pollution. Hence, this situation is observationally equivalent (at least in terms of the phase diagram and the vector of optimal sales and capital endowment at the steady state(s)) to the case depicted in Cellini and Lambertini (1998, 2008) where environmental effects were ruled out by assumption.

The second case is that where exactly the opposite chain of inequalities applies: $q^C \geq q^R > q^{SP}$ (as in Fig. 4). In such a situation, at the Cournot equilibrium the industry produces and sells too much as compared to the social optimum. This may

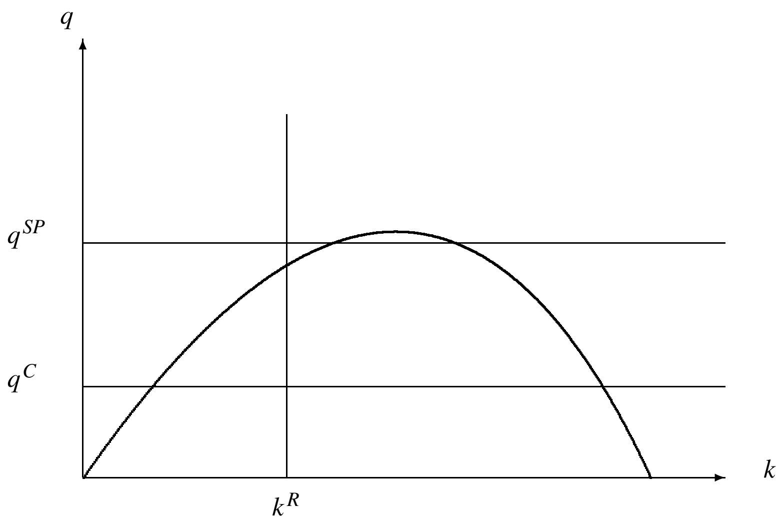

Fig. 3 $q^{SP} > q^R > q^C$

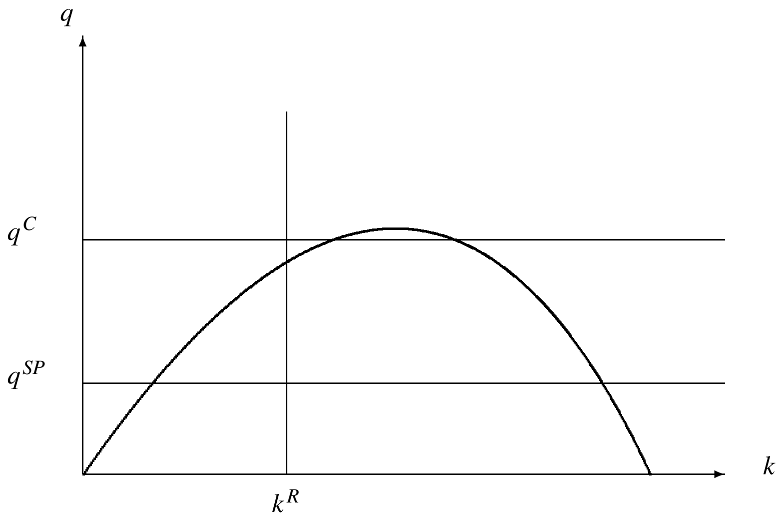

Fig. 4 $q^C > q^R > q^{SP}$

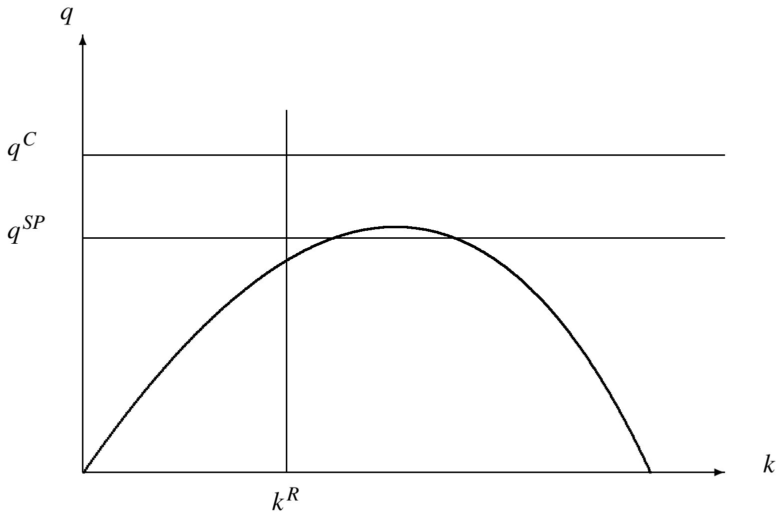

Fig. 5 $q^{SP}, q^C > q^R$

happen if (i) cost and demand parameters are such that $q^C \geq q^R$, and (ii) β is high enough that $q^R > q^{SP}$.

In the third case (see Fig. 5), $q^C, q^R \geq q^{SP}$ and the Ramsey golden Rule prevails irrespective of the market regime, and once again the steady state allocation is observationally equivalent to the one we would observe without environmental externalities.

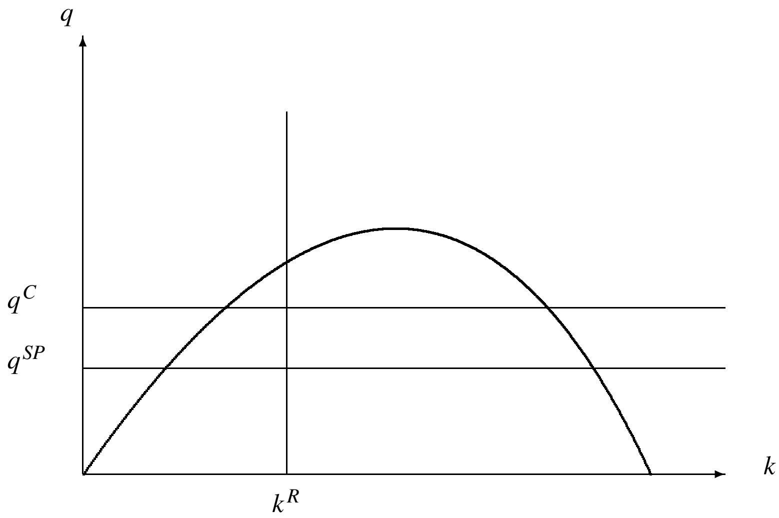

Fig. 6 $q^R > q^{SP}, q^C$

Last, there remains the case in which $q^R > q^{SP}, q^C$ (see Fig. 6). This is the situation described in Proposition 2, where what matters is the dimension of parameter β.

6 Extension: Pollution as a Function of Production

Here we abandon the assumption that pollution depends quadratically on sales (or consumption), to adopt the alternative view that it is induced by production itself, so that

$$EXT = \beta \left[\sum_{i=1}^{N} f(k_i) \right]^2 . \tag{25}$$

Of course this has no consequences on the behaviour of firms, as they neglect the externality, but it does affect the behaviour of a social planner interested in maximising the discounted flow of social welfare. The planner's Hamiltonian is now:

$$H_{SP} = N(a - Nq - c)q + \frac{N^2 q^2}{2} - \beta N^2 [f(k)]^2 + \mu[f(k) - q - \delta k].$$

The necessary conditions are

$$\frac{\partial H_{SP}}{\partial q} = N[a - c - Nq] - \mu = 0, \tag{26}$$

$$-\frac{\partial H_{SP}}{\partial k} = \mu[\delta - f'(k)] - 2\beta N^2 f(k) f'(k) = \dot{\mu} - \rho\mu$$

$$\implies \quad \dot{\mu} = \mu[\rho + \delta - f'(k))] - 2\beta N^2 f(k) f'(k) \tag{27}$$

together with the transversality condition $\lim_{t\to\infty} \mu(t)k(t) = 0$.

From (26) one obtains

$$\mu = N[a - c - Nq] \tag{28}$$

as well as the control equation:

$$\dot{q} = -\frac{\dot{\mu}}{N^2}. \tag{29}$$

Using (26) and (27), the latter expression simplifies as

$$\begin{aligned}\dot{q} &= -\frac{\mu[\rho + \delta - f'(k))] - 2\beta N^2 f(k) f'(k)}{N^2} \\ &= -\frac{(\rho + \delta - f'(k))[a - c - Nq] - 2\beta N f(k) f'(k)}{N}.\end{aligned} \tag{30}$$

On the basis of (30), we may state the following:

Lemma 1 *If the environmental externality is determined by the amount of production, then under social planning the industry cannot converge to the Ramsey golden rule for all* $\beta > 0$.

The proof of this claim is intuitive. It suffices to observe that (30) indeed coincides with (24) only in the limit case where $\beta = 0$, but this clearly would imply that the environmental externality is altogether absent.

The stationarity condition $\dot{q} = 0$ admits a unique steady state solution w.r.t. q:

$$q^{SP}(k) = \frac{(a - c)[f'(k) - \rho - \delta] + 2\beta N f(k) f'(k)}{N(f'(k) - \rho - \delta)}. \tag{31}$$

A sufficient condition for $q^{SP}(k) > 0$ is that $f'(k) > \rho + \delta$.

The Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial q} \\ \frac{\partial \dot{q}}{\partial k} & \frac{\partial \dot{q}}{\partial q} \end{bmatrix} = \begin{bmatrix} f'(k) - \delta & -1 \\ \frac{2\beta N[f'(k)]^2 + [a - c - Nq + 2\beta N f(k)] f''(k)}{N} & \delta + \rho - f'(k) \end{bmatrix},$$

whose trace and determinant are, respectively, $Tr(J) = \rho > 0$ and

$$\begin{aligned}\Delta(J) &= [f'(k) - \delta][\delta + \rho - f'(k)] \\ &\quad + \frac{2\beta N[f'(k)]^2 + [a - c - Nq + 2\beta N f(k)] f''(k)}{N}.\end{aligned} \tag{32}$$

In correspondence of (31), $\Delta(J)$ simplifies as follows:

$$\Delta(J) = \frac{[\delta - f'(k)][f'(k) - \delta - \rho]^2 - 2\beta\{[f'(k) - \delta - \rho][f'(k)]^2 - f(k)(\delta + \rho) f''(k)\}}{f'(k) - \delta - \rho}. \tag{33}$$

If $f'(k) > \rho + \delta$, $\Delta(J) < 0$ for all β such that:

$$\beta < \frac{[f'(k) - \delta][f'(k) - \delta - \rho]^2}{2\{[f'(k) - \delta - \rho][f'(k)]^2 - f(k)(\delta + \rho) f''(k)\}}. \tag{34}$$

In such a parameter region, the steady state is stable in the saddle point sense. On the basis of the foregoing discussion, we can formulate

Proposition 4 *If $f'(k) > \rho + \delta$, the steady state solution is a saddle point if the weight attached to pollution in the social welfare function is small enough.*

Alternatively, consider the region where $f'(k) \in (\delta, \rho + \delta)$. Here, $q^{SP} > 0$ if

$$\beta < -\frac{(a - c)[f'(k) - \delta - \rho]}{2Nf(k)f'(k)}. \tag{35}$$

Concerning the sign of $\Delta(J)$, one may easily establish that the sufficient condition for $\Delta(J) > 0$ is that $f(k)(\delta + \rho) f''(k) > [f'(k) - \delta - \rho][f'(k)]^2$. Otherwise, if

$$f(k)(\delta + \rho) f''(k) < [f'(k) - \delta - \rho][f'(k)]^2, \tag{36}$$

then $\Delta(J) < 0$ for all

$$\beta > \frac{[f'(k) - \delta][f'(k) - \delta - \rho]^2}{2\{[f'(k) - \delta - \rho][f'(k)]^2 - f(k)(\delta + \rho) f''(k)\}}. \tag{37}$$

Hence, whenever $f(k)(\delta + \rho) f''(k) < [f'(k) - \delta - \rho][f'(k)]^2$, any

$$\beta \in \left(\frac{[f'(k) - \delta][f'(k) - \delta - \rho]^2}{2\{[f'(k) - \delta - \rho][f'(k)]^2 - f(k)(\delta + \rho) f''(k)\}}, -\frac{(a - c)[f'(k) - \delta - \rho]}{2Nf(k)f'(k)} \right) \tag{38}$$

ensures that $q^{SP} > 0$ and also entails the saddle point stability. The interval specified in (38) exists if the market size is large enough:

$$a - c > -\frac{Nf(k)[f'(k) - \delta][f'(k) - \delta - \rho]f'(k)}{[f'(k) - \delta - \rho][f'(k)]^2 - f(k)(\delta + \rho) f''(k)} > 0. \tag{39}$$

Note that, if β is to the left of the inf of the interval (38), the equilibrium becomes unstable due to the following mechanism. Suppose the system is in the unstable steady state. The planner knows that a sales expansion induces an increase in consumer surplus and, as a side effect, also a decumulation of capacity and therefore also in production, as $\dot{k}$ becomes negative. This implies a reduction in the externality, which is also desirable. However, in the long run, this deviation is unsustainable as it implies that the size of firms decreases progressively. The phase diagram of this case is illustrated in Fig. 7.

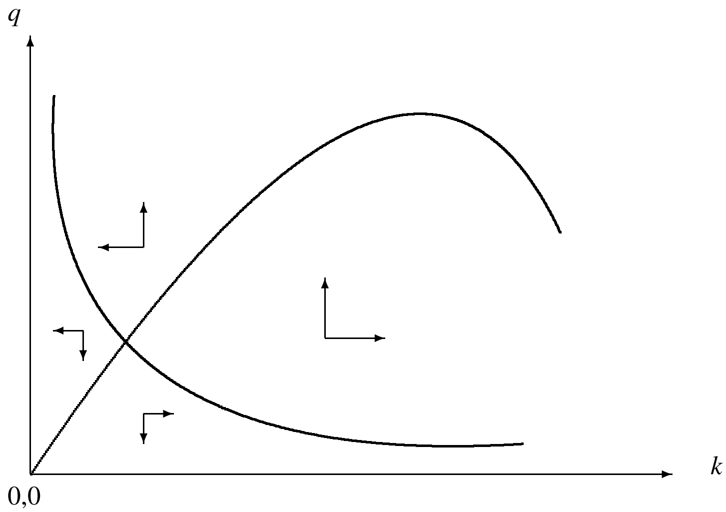

Fig. 7 The unstable case under planning, with $f'(k) \in (\delta, \rho + \delta)$

Example As an illustration, assume $f(k) = \alpha\sqrt{k}$, and take

$$a = 2, \qquad c = 0, \qquad N = 10, \qquad \alpha = 1, \qquad \delta = 1/20, \qquad \rho = 1/18$$

and consider the range where $f'(k) - \delta - \rho > 0$, which entails $k \in (0, 22.438)$. Also, set

$$\begin{aligned}\beta &= \frac{[f'(k) - \delta][f'(k) - \delta - \rho]^2}{2\{[f'(k) - \delta - \rho][f'(k)]^2 - f(k)(\delta + \rho)f''(k)\}} - \frac{1}{50}\\ &= \frac{(90 - 19\sqrt{k})^2(10 - \sqrt{k})}{162000} - \frac{1}{50}\end{aligned}$$

to satisfy (34). Then, impose $\dot{k} = 0$ to obtain $q(k) = f(k) - \delta k = \sqrt{k} - k/20$. The equation

$$q^{SP} - q(k) = 0$$

yields the following solutions:

$$k = 0.659, \qquad k = 24.93 \quad \text{and} \quad k = 148.996.$$

Only the first is acceptable, in view of the above assumption concerning the marginal productivity of capital. In correspondence of $k = 0.659$, the numerical value of the determinant of the Jacobian matrix is $\Delta(J) = -0.018 < 0$, and therefore this qualifies as a saddle point equilibrium. The corresponding optimal steady state quantity is $q^{SP} = 0.779$.

As a last remark, we would like to stress the following. If one keeps in mind that firms disregard the externality and by this very reason are able to converge to the golden rule (under appropriate conditions, which we already know from Sect. 3), what is likely to appear as the most striking feature of the present version of the model is that the golden rule doesn't look like a socially efficient rule any more because of the fact that pollution is determined by production instead of sales. Consequently, unlike the first version of the model, this one does not allow for any alignment of social and private incentives, except in the very special case in which $\beta = 0$, of course less than interesting as it amounts to assuming that pollution is not there at all.

7 Concluding Remarks

We have investigated a dynamic model where an environmental externality interacts with firms' capital accumulation, to show that (i) at the social optimum it may be optimal to trade off some amount of consumer surplus in order to reduce the externality, and (ii) if the external effect is proportional to the industry production, then the Ramsey golden rule just disappears as a stand-alone equilibrium.

The above analysis has been carried out under the open-loop information structure. The desirable extension to feedback models is left for future research.

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On a Decentralized Boundedly Rational Emission Reduction Strategy

Arkady Kryazhimskiy

Abstract We consider the emission reduction process involving several countries, in which the countries negotiate, in steps, frequently enough, on small, local emission reductions and implement their decisions right away. In every step, the countries either find a mutually acceptable local emission reduction vector and use it as a local emission reduction plan, or terminate the emission reduction process. We prove that the process necessarily terminates in some step and the final total emission reduction vector lies in a small neighborhood of a certain Pareto maximum point in the underlying emission reduction game. We use examples to illustrate some features of the proposed decision making scheme and discuss a way to organize negotiations in every step of the emission reduction process.

1 Introduction

It has been recognized that emission reduction has been a common problem for all countries in a region. A country's industrial pollutants travel across borders and make neighboring countries suffer from contamination. The understanding that the emission reduction process involves multiple decision makers whose interests are interconnected but not identical has initiated a series of game-theoretic studies.

Today's practice in planning and controlling emission reductions is based on international agreements; accordingly, a significant part of research focuses on countries' incentives to participate in conventions, and on issues of formation and stability of coalitions (see Barrett 1994, 2003; Finus 2001). A considerable research effort concentrates on developing procedures that may lead the parties to an equilibrium solution and, in result, to a specification of emission reduction commitments. Part of the procedures proposed assumes that the parties use money transfers to compensate for cleaning up (see Maeler 1990; Chandler and Tulkens, 1992). Another approach suggests that the international agreements could be formed based on reciprocal emission reduction trade (see Hoel 1991; Nentjes 1993, 1994; Pethig 1982); an analogous theoretical framework has been developed in Ehtamo and Hamalainen (1993). Kryazhimskiy et al. (2001) interpret environmental negotiations as a "trade" between the governments, in which emission reductions act as the "goods" traded.

A. Kryazhimskiy (✉)
International Institute for Applied Systems Analysis (IIASA), Schlossplatz 1, Laxenburg, 2361, Austria
e-mail: kryazhim@iiasa.ac.at

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Martin et al. (1993) analyze a multi-agent dynamic game whose equilibrium solution may justify the countries' emission reduction plans.

The majority of the game-theoretic studies addressing the issue of emission reduction assume that every party has good knowledge on its own utility function—its overall gain due to emission reduction—and uses that knowledge in the negotiations leading to an international environmental agreement. That assumption natural from the standpoint of game theory, can however be criticized as an unrealistic one. Indeed, a country's utility has two components, the cost for national emission reduction (a negative component) and the ecological benefit from the emission reduction performed by all countries (a positive component). Even if we assume that a country's government is able to construct its cost function, based on economic considerations,[1] we should admit that it can hardly estimate in advance, with an acceptable precision, the sizes of the country's ecological benefits for all future emission reduction values. This uncertainty makes one view negotiation patterns, in which the countries use full information on their global utility functions, as useful but rather theoretical constructions.

In this paper, we study decisions on reducing emission in the situation where each country has limited information on its global utility function. Namely, we assume that given the actual state of the countries in the emission reduction process, i.e., the actual values of the countries' total emission reductions, every country is able to reconstruct its marginal cost and benefit functions, i.e., the growth rates for its global cost and benefit functions in small neighborhoods of the actual state. Moreover, each country has no information on the utility functions of the other countries.

In this situation, it is hardly possible to provide a classical game-theoretic basis for shaping, today, a long-term agreement on substantial emission reduction.[2] A realistic operational mode is "myopic" planning and "myopic" implementation. In the "myopic" mode, instead of fixing a long-term agreement, the countries negotiate, in steps, frequently enough, on small, local emission reductions and implement their decisions right away. In every negotiation step, each country uses its current marginal utility to understand if a proposed local emission reduction vector meets the country's local utility growth criterion, i.e., increases, locally, the value of the country's global utility function. The countries' goal is to identify an acceptable local emission reduction vector satisfying all local utility growth criteria. The identified acceptable emission reduction vector defines the countries' cooperative local emission reduction plan. If the countries fail to find an acceptable emission reduction vector, the negations are terminated and the latest total emission reduction vector is agreed to be the outcome of the emission reduction process. The described decision making scheme follows the approach of theory of repeated games (see, e.g., Brown 1951; Robinson 1951; Axelrod 1984; Smale 1980; Fudenberg and Kreps 1993;

[1] This assumption is however not so obvious; one can argue against it by saying that future changes in prices, unforeseeable today, will ruin today's cost estimates for high emission reduction values unreachable in the short run.

[2] This does not mean that the agreement is not reachable in principle; a reasonable decision can be found using, for example, political and general environmental considerations.

Weibull 1995; for examples of economic applications see, e.g., Friedman 1991; Kryazhimskiy et al. 2001, 2002).

In Sect. 2 we introduce technical assumptions and describe the emission reduction process. In Sect. 3 we prove that the process necessarily terminates in some step and its outcome lies in a small neighborhood of a certain Pareto maximum point in the emission reduction game; the radius of the neighborhood tends to zero together with the length of the time period between the points of decision making. In other words, we state that the proposed "myopic" decision making scheme allows the countries to find an equilibrium solution with an arbitrarily high precision. In Sect. 4 we discuss our solvability statement using two examples. One example shows that the statement may fail to hold if the countries' network is not fully connected in the sense that there are at least two countries such that pollution produced by one country is not transported to the other one. The other example shows that the set of all Pareto maximum points, which are reachable via the proposed emission reduction process, can be considerably smaller than the set of all Pareto maximum points in the emission reduction game. In Sect. 5 we discuss a possible way to organize negotiations bringing the countries to a common decision in each step of the emission reduction process.

2 Emission Reduction Process

We consider an emission reduction process involving n countries, numbered $1, \dots, n$, in which each country, i, controls its emission reduction value, $x_i \geq 0$, gradually increasing it over time. The process starts at time 0 with the zero emission reduction values and consists of successive periods of a fixed small length δ. In each period the countries negotiate on small reductions of their emissions so that the total reduction—as measured by the norm of the vector formed by the countries' emission reductions—is in a fixed proportion p to the length of the period, δ. We examine whether the process will terminante in some period and whether it will lead the countries to a Pareto optimum.

The utility function of each country i, w_i, is assumed to have the form

$$w_i(x) = -c_i(x_i) + b_i\left(\sum_{j=1}^{n} a_{ji}x_j\right). \tag{1}$$

Here $x = (x_1, \dots, x_n)$ is the full emission reduction vector; $c_i(x_i)$ is the cost paid by country i for the emission reduction x_i; $b_i(y)$ is the ecological benefit gained by country i thanks to the reduction of the total pollution load to its territory, $y = \sum_{j=1}^{n} a_{ji}x_j$; and a_{ji} is a proportion of emission from country j, which is transported to country i (a transport coefficient). Clearly, $\sum_{i=1}^{n} a_{ji} = 1$ $(j = 1, \dots, n)$. We assume that the countries' network is fully interconnected in the sense that each country pollutes itself and every other country, implying

$$a_{ji} > 0 \quad (j, i = 1, \dots, n). \tag{2}$$

We call a vector $x = (x_1, \dots, x_n)$ *positive* if $x_i > 0$ $(i = 1, \dots, n)$.

Our technical assumptions are the following.

(A1) The cost functions, c_i $(i = 1, \dots, n)$, defined on $[0, \infty)$ are continuously differentiable, convex, strictly monotonically increasing, positive-valued at all points except 0, and vanish at 0.

(A2) The benefit functions, b_i $(i = 1, \dots, n)$, defined on $[0, \infty)$ are continuously differentiable, strictly concave, strictly monotonically increasing, positive-valued at all points except 0, and vanish at 0; moreover, the benefit functions are bounded from above, implying, in particular, that

$$b_i'(y) \to 0 \quad \text{as } y \to \infty \ (i = 1, \dots, n). \tag{3}$$

(A3) The utility functions, w_i $(i = 1, \dots, n)$, take positive values at all positive emission vectors belonging to a certain neighborhood of the origin (in this manner we exclude a trivial situation, in which some of the countries are not interested in emission reduction, since their utilities are maximized at the zero emission reduction vector).

The emission reduction process develops in steps. A step k is performed over a time interval $[t_k, t_{k+1}]$ where $t_k = k\delta$ with a given small $\delta > 0$ $(k = 0, 1, \dots)$. For every country, i, we denote by $x_i(t_k)$ its total emission reduction value at the starting time of each step k, t_k. In step 0 the countries start with the zero emission reductions:

$$x_i(t_0) = x_i(0) = 0 \quad (i = 1, \dots, n). \tag{4}$$

In each step, k, every country, i, plans an extra local emission reduction, $\Delta x_i(t_k) \geq 0$; at time t_{k+1} the country completes the planned local emission reduction process bringing its total emission reduction value to a new state, $x_i(t_{k+1}) = x_i(t_k) + \Delta x_i(t_k)$. Introducing notations for the initial emission reduction vector in step k, $x(t_k) = (x_1(t_k), \dots, x_n(t_k))$, and for the local emission reduction vector in step k,

$$\Delta x(t_k) = (\Delta x_1(t_k), \dots, \Delta x_n(t_{k+1})), \tag{5}$$

we represent the transformation of the emission reduction vector in step k as

$$x(t_{k+1}) = x(t_k) + \Delta x(t_k). \tag{6}$$

Prior to considering the rules for choosing $\Delta x_i(t_k)$, we assume that information available for each country, i, a priori is the collection of the transport coefficients a_{ji} $(j = 1, \dots, n)$ only. Therefore, a priori each country may have no knowledge on the cost and benefit functions of the other countries and no knowledge on its own cost and benefit functions.

In each step, k, country i chooses $\Delta x_i(t_k)$ using the following additional information: the country's current emission reduction value, $x_i(t_k)$; the current value of the total reduction of the pollution load to its territory,

$$y_i(t_k) = \sum_{j=1}^{n} a_{ji} x_j(t_k); \tag{7}$$

and its marginal cost and benefit functions at points $x_i(t_k)$ and $y_i(t_k)$, respectively. The country constructs its marginal cost function at point $x_i(t_k)$ as a linear approximation to the virtual increment in its cost value, $c_i(x_i(t_k)+h)-c_i(x_i(t_k))$, corresponding to every small virtual positive increment in the emission reduction value, h; that linear approximation can be represented as $c_i'(x_i(t_k))h$. Similarly, the country constructs its marginal cost function at point $y_i(t_k)$ as a linear approximation to the virtual increment in its benefit value, $b_i(y_i(t_k)+h)-b_i(y_i(t_k))$, corresponding to a small virtual positive increment in the total emission reduction value, h; that linear approximation can be represented as $b_i'(y_i(t_k))h$.

Thus, we assume that in each step, k, the country is able to reconstruct, in linear approximation, the local structure of its cost and benefit functions in small neighborhoods of the actual emission reduction value, $x_i(t_k)$, and actual total pollution reduction value, $y_i(t_k)$, respectively. In more formal terms, we assume that in each step, k, every country, i, is able to reconstruct the derivatives $c_i'(x_i(t_k))$ and $b_i'(y_i(t_k))$.

While choosing a positive $\Delta x_i(t_k)$, country i negotiates with the other countries. In the negotiations, country i trades on exchanging its local emission reduction value, $\Delta x_i(t_k)$, to the local reduction of the total pollution load to its territory, which is due to the current efforts of the other countries, $\Delta y_i^0(t_k)$. Clearly, $\Delta y_i^0(t_k)$ is the sum of the local emission reductions of all the countries, except of country i, weighted with the corresponding transportation coefficients:

$$\Delta y_i^0(t_k) = \sum_{j=1,\ldots,n,\ j\neq i} a_{ji}\Delta x_j(t_k). \tag{8}$$

To each value of $\Delta y_i^0(t_k)$ emerging in the negotiations, country i responds with an emission reduction value $\Delta x_i(t_k)$ that can be exchanged to $\Delta y_i^0(t_k)$. The country's goal in the negotiations is to form a set of the local emission reduction values, $\Delta x_j(t_k)$ $(j=1,\ldots,n)$, that would locally increase the country's utility, i.e., ensure

$$w_i(x(t_k)+\Delta x(t_k)) > w_i(x(t_k)). \tag{9}$$

Thus, in each round the country acts as a boundedly rational agent (see, e.g., Rubinstein 1998).

Recall that in step k the country's knowledge about its cost and benefit functions, c_i and b_i, is restricted to the values $c_i'(x_i(t_k))$ and $b_i'(y_i(t_k))$. Using these values and referring to (1) and (7), country i reconstructs the partial derivatives

$$\frac{\partial w_i(x(t_k))}{\partial x_j} = a_{ji}b_i'(y_i(t_k)) \quad (j=1,\ldots,n,\ j\neq i), \tag{10}$$

$$\frac{\partial w_i(x(t_k))}{\partial x_i} = a_{ii}b_i'(y_i(t_k)) - c_i'(x_i(t_k)), \tag{11}$$

which give it its marginal utility at point $x(t_k)$, i.e., a linear approximation to the increment $w_i(x(t_k)+h)-w_i(x(t_k))$ as a function of h. The necessity to use the marginal utility at point $x(t_k)$ instead of w_i makes the country consider a linear approximation to the original criterion (9):

$$\sum_{j=1,\dots,n,\ j\neq i} \frac{\partial w_i(x(t_k))}{\partial x_j}\Delta x_j(t_k) + \frac{\partial w_i(x(t_k))}{\partial x_i}\Delta x_i(t_k) > 0. \tag{12}$$

The substitution of (11) and use of (8) transform (12) into

$$b_i'(y_i(t_k))\Delta y_i^0(t_k) + [a_{ii}b_i'(y_i(t_k)) - c_i'(x_i(t_k))]\Delta x_i(t_k) > 0$$

or

$$\Delta y_i^0(t_k) > \lambda_i(t_k)\Delta x_i(t_k) \tag{13}$$

where

$$\lambda_i(t_k) = \frac{c_i'(x_i(t_k))}{b_i'(y_i(t_k))} - a_{ii}. \tag{14}$$

We call (13) the *local utility growth criterion* for country i in step k.

Let us give several definitions. We call a positive emission reduction vector $\Delta x(t_k)$ (5) *acceptable* in step k if for every country, i, the values $\Delta y_i^0(t_k)$ given by (8) and $\Delta x_i(t_k)$ satisfy the country's local utility growth criterion (13) in step k. Every step k, in which there exists an acceptable emission reduction vector, will be said to be *nondegenerate*; every step that is not nondegenerate will be called *degenerate*.

Recall that $p > 0$ is a fixed proportionality coefficient that relates the norm of the emission reduction vector formed in each nondegenerate step of the emission reduction process to the length of the time period, during which the step is performed, δ. In what follows, $|\cdot|$ is a given norm in the n-dimensional linear space. The next assumption characterized the countries' abilities and outcomes in each step of the emission reduction process.

(A4) In the negotiations taking place in a nondegenerate step k, the countries find a positive emission reduction vector, $\Delta x(t_k)$ (5), acceptable in step k and such that $|\Delta x(t_k)| = p\delta$. In the negotiations taking place in a degenerate step k, the countries identify that step k is degenerate. (A possible negotiation pattern is presented Sect. 5.)

Our next assumption, (A5), suggest a rule for the termination of the emission reduction process.

(A5) In a first degenerate step, s, whose degeneracy is identified by the countries through negotiations (see (A4)), the countries terminate the emission reduction process and view $x(t_s)$ as its *outcome*.

Our final assumption, (A6) summarizes the rules for the countries' operation in the emission reduction process.

(A6) In each (nondegenerate) step k preceding the first degenerate step, s, the countries work out a local positive emission reduction vector $\Delta x(t_k)$ (5) through negotiations as described in (A4) and update the total emission reduction vector using (6). If all steps are nondegenerate, then in each step, k, the countries work out a local positive emission reduction vector $\Delta x(t_k)$ (5) through negotiations and update the total emission reduction vector using (6); in this situation the emission reduction process has no outcome.

3 Outcome of the Emission Reduction Process

Holding a game-theoretic viewpoint, we assume that a priori a goal of the countries' community is to bring the full emission reduction vector to a Pareto maximum point for the countries' utilities. A nonnegative emission reduction vector x^* is said to be a *Pareto maximum point* in the emission reduction game if switching from x^* to any nonnegative emission reduction vector $x \neq x^*$ either does not change the countries' utility values, i.e., $w_i(x) = w_i(x^*)$ for all $i = 1, \ldots, n$, or makes at least one country lose in utility, i.e., $w_i(x) < w_i(x^*)$ for some $i \in \{1, \ldots, n\}$. In view of the strict concavity of the utility functions $w_i, \ldots, w_n$ (see (A1) and (A2)), for every positive $z_1, \ldots, z_n$ the maximizer of the sum $z_1 w_1(x) + \cdots + z_n w_n(x)$ over all nonnegative emission reduction vectors x is a Pareto maximum point. Note that by (A3) the origin is not a Pareto maximum point. Thanks to the strict concavity of the utility functions (see (A1) and (A2)) a positive emission reduction vector x^* maximizes $z_1 w_1(x) + \cdots + z_n w_n(x)$ if and only if

$$z_1 \frac{\partial w_1(x^*)}{\partial x_i} + \cdots + z_n \frac{\partial w_n(x^*)}{\partial x_i} = 0 \quad (i = 1, \ldots, n) \tag{15}$$

(see, e.g., Germeyer 1976). Thus, every positive emission reduction vector x^* satisfying (15) for some positive $z_1, \ldots, z_n$ is a Pareto maximum point, which can be viewed as a target point in the emission reduction process. We call $z_1, \ldots, z_n$ a family of *Pareto multipliers* for the Pareto maximum point x^*.

Our goal in this section is to show that the decentralized boundedly rational emission reduction process described in the previous section brings the total emission reduction vector to a small neighborhood of some Pareto maximum point in a finite number of steps.

First, we state that the emission reduction process terminates in some step.

Proposition 1 *There is a degenerate step, in which the emission reduction process terminates* (*see* (A5)).

Proof Assume, to the contrary, that the emission reduction process never terminates, i.e., all the steps are nondegenerate. By (A5) in each step, k, the local emission reduction vector, $\Delta x(t_k)$, is positive and has the norm $p\delta$; hence, the norms of the total emission reduction vectors, $|x(t_k)|$ (see (6)), tend to infinity as $k \to \infty$. Then for each country, i, the total reduction of the pollution load to its territory, $y_i(t_k)$ (7), tends to infinity as $k \to \infty$ (here we take into account (2)). Therefore, by (3)

$$b_i'(y_i(t_k)) \to 0 \quad \text{as } k \to \infty \ (i = 1, \ldots, n). \tag{16}$$

By (A1) for each country, i, the cost function c_i, is strictly monotonically increasing and convex, implying that $c_i'(x_i(t_k)) \geq c^0 > 0$ uniformly for all steps k. Combining with (16), we find that for every country, i,

$$\lambda_i(t_k) \to \infty \quad \text{as } k \to \infty \ (i = 1, \ldots, n), \tag{17}$$

where $\lambda_i(t_k)$ is given by (14). For every step, k, let $i_k \in \{1, \ldots, n\}$ be such that $\Delta x_{i_k}(t_k) = \max\{\Delta x_1(t_k), \ldots, \Delta x_{i_k}(t_k)\}$. In view of (8), for every step, k, we have

$$\begin{aligned}\Delta y_{i_k}^0(t_k) - \lambda_{i_k}(t_k)\Delta x_{i_k}(t_k) &= \sum_{j=1,\ldots,n,\ j\neq i_k} a_{ji}\Delta x_j(t_k) - \lambda_i(t_k)\Delta x_i(t_k)\\ &\leq [(n-1) - \lambda_i(t_k)]\Delta x_{i_k}(t_k).\end{aligned}$$

By (17) the right hand side is negative for all k sufficiently large. Thus, for a large k the local utility growth criterion (13) is violated for country i_k; consequently, the local emission reduction vector $\Delta x(t_k)$ is not acceptable in step k. We get a contradiction with our initial assumption and finalize the proof. □

As we see from (4) and (A3), step 0 is nondegenerate. Therefore, for the first degenerate step, s (see Proposition 1), we have $s \geq 1$.

Consider the time interval $[t_{s-1}, t_s]$. For every $t \in [t_{s-1}, t_s]$ we set (see (6) and (5))

$$x(t) = (x_1(t), \ldots, x_n(t)) = x(t_{s-1}) + \frac{t - t_{s-1}}{\delta}\Delta x(t_{s-1}) \tag{18}$$

and extend notations (7) and (14) by setting

$$y_i(t) = \sum_{j=1}^{n} a_{ji}x_j(t), \quad \lambda_i(t) = \frac{c_i'(x_i(t))}{b_i'(y_i(t))} - a_{ii} \quad (i = 1, \ldots, n). \tag{19}$$

For every $t \in [t_{s-1}, t_s]$ let

$$h_i(t, z) = \sum_{j=1,\ldots,n,\ j\neq i} a_{ji}z_j - \lambda_i(t)z_i \quad (z = (z_1, \ldots, z_n),\ i = 1, \ldots, n), \tag{20}$$

$$H(t) = \{z > 0 : |z| = p\delta,\ h_i(t, z) > 0\ (i = 1, \ldots, n)\}; \tag{21}$$

here and below $z > 0$ marks that a vector z is positive.

The fact that the local emission reduction vector $\Delta x_{s-1}(t_{s-1})$ has the norm $p\delta$ and is acceptable in the nondegenerate step $s-1$, i.e., satisfies the local utility growth criterion for every country in step $s-1$ is equivalent to

$$\Delta x_{s-1}(t_{s-1}) \in H(t_{s-1}) \tag{22}$$

(see (A5), (13) and (8)). Similarly, we see that if $H(t_s)$ is nonempty, then for every $z \in H(t_s)$ the emission reduction vector $\Delta x_s(t_s) = z$ is acceptable in step s; consequently, step s is nondegenerate. Since step s is degenerate, we have

$$H(t_s) = \emptyset. \tag{23}$$

Let

$$T = \{t \in [t_{s-1}, t_s] : H(t) \neq \emptyset\}. \tag{24}$$

By (22) T is nonempty. Denote

$$\tau = \sup T. \tag{25}$$

Prior to formulating our main technical statement—Lemma 1, we make a few simple observations. In view of the continuity of the functions h_i (20) the set T is open in $[t_{s-1}, t_s]$. Therefore, if $\tau < t_s$, then $\tau \notin T$, i.e.,

$$H(\tau) = \emptyset; \tag{26}$$

note that if $\tau = t_s$, then (26) holds by (23). By the definition of τ, (25), there exist a sequence (τ_m) in $[t_{s-1}, \tau)$ such that $\tau_m \to \tau$ and $H(\tau_m) \neq \emptyset$ $(m = 1, 2, \ldots)$. Every sequence (z_m) such that $z_m \in H(\tau_m)$ $(m = 1, 2, \ldots)$ is bounded and has a limit point.

Lemma 1 *The following statements hold true.*

(1) *The emission reduction vector $x^* = x(\tau)$ is a Pareto maximum point.*
(2) *Let (τ_m) be a sequence in $[t_{s-1}, \tau)$ such that $\tau_m \to \tau$ and $H(\tau_m) \neq \emptyset$ $(m = 1, 2, \ldots)$, $z_m \in H(\tau_m)$ $(m = 1, 2, \ldots)$, and $z = (z_1, \ldots, z_n)$ be a limit point for the sequence (z_m). Then $z_1, \ldots, z_n$ is a family of Pareto multipliers for the Pareto maximum point x^*.*

Proof Let (τ_m) and (z_m) be the sequences defined above and $z = (z_1, \ldots, z_n)$ be a limit point for (z_m). Selecting, without renumeration, an appropriate subsequence, we assume that $z_m \to z$. Taking into account that $z_m > 0$ and $|z_m| = p\delta$ (see (21)), we get

$$z_i \geq 0 \quad (i = 1, \ldots, n), \tag{27}$$

$$|z| = p\delta. \tag{28}$$

Since $\tau_m \in T$ and $z_m \in H(\tau_m)$, we have $h_i(\tau_m, z_m) > 0$ $(i = 1, \ldots, n,\ m = 1, 2, \ldots)$. Due to the continuity of h_i $(i = 1, \ldots, n)$ it holds that $h_i(\tau, z) \geq 0$ $(i = 1, \ldots, n)$, or, more specifically (see (20)),

$$h_i(\tau, z) = \sum_{j=1,\ldots,n,\ j \neq i} a_{ji} z_j - \lambda_i(\tau) z_i \geq 0 \quad (i = 1, \ldots, n). \tag{29}$$

Suppose

$$h_{i_0}(\tau, z) = \sum_{j=1,\ldots,n,\ j \neq i_0} a_{ji_0} z_j - \lambda_{i_0}(\tau) z_{i_0} > 0 \tag{30}$$

for some $i_0 \in \{1, \dots, n\}$. Then

$$\sum_{j=1,\dots,n,\ j \neq i_0} a_{ji_0} z_j - \lambda_{i_0}(\tau)(z_{i_0} + \varepsilon_0) > 0 \tag{31}$$

for a sufficiently small $\varepsilon_0 > 0$. Let

$$\bar{z} = (\bar{z}_1, \dots, \bar{z}_n) = (z_1, \dots, z_{i_0 - 1}, z_{i_0} + \varepsilon_0, z_{i_0+1}, \dots, z_n).$$

Using (29), (2) and (31), we get

$$h_i(\tau, \bar{z}) = \sum_{j=1,\dots,n,\ j \neq i} a_{ji} \bar{z}_j - \lambda_i(\tau)\bar{z}_i > 0 \quad (i = 1, \dots, n).$$

Then

$$h_i(\tau, \bar{z}^*) = \sum_{j=1,\dots,n,\ j \neq i} a_{ji} \bar{z}_j^* - \lambda_i(\tau)\bar{z}_i^* > 0 \quad (i = 1, \dots, n)$$

where

$$\bar{z}^* = (\bar{z}_1^*, \dots, \bar{z}_n^*) = (\bar{z}_1 + \varepsilon_1, \dots, \bar{z}_n + \varepsilon_1)$$

with a sufficiently small $\varepsilon_1 > 0$. In view of (27) $\bar{z}^* > 0$. For $z^* = p\delta\bar{z}^*/|\bar{z}^*|$ we have $|z^*| = p\delta$ and

$$h_i(\tau, z^*) = \sum_{j=1,\dots,n,\ j \neq i} a_{ji} z_j^* - \lambda_i(\tau) z_i^* > 0 \quad (i = 1, \dots, n).$$

Thus, $z^* \in H(\tau)$. The latter contradicts (26). The contradiction shows that (30) is not possible for any $i_0 \in \{1, \dots, n\}$. Hence, in view of (29) we get

$$h_i(\tau, z) = \sum_{j=1,\dots,n,\ j \neq i} a_{ji} z_j - \lambda_i(\tau) z_i = 0 \quad (i = 1, \dots, n). \tag{32}$$

As seen from (28), there is an $i_* \in \{1, \dots, n\}$ such that $z_{i_*} > 0$. Then for every $i \in \{1, \dots, n\}$, $i \neq i_*$,

$$\sum_{j=1,\dots,n,\ j \neq i} a_{ji} z_j \geq a_{i_* i} z_{i_*} > 0$$

(here we use (2)). Now (32) shows that $\lambda_i(\tau) > 0$ and $z_i > 0$ for every $i \in \{1, \dots, n\}$, $i \neq i_*$. Thus, $z > 0$. Multiplying (32) by $b_i'(y_i(\tau))$ and using (19), we get

$$\sum_{j=1,\dots,n,\ j \neq i} a_{ji} b_i'(y_i(\tau)) z_j + [a_{ii} b_i'(y_i(\tau)) - c_i'(x_i(\tau))] z_i = 0$$
$$(i = 1, \dots, n), \tag{33}$$

or

$$z_1 \frac{\partial w_1(x(\tau))}{\partial x_i} + \cdots + z_n \frac{\partial w_n(x(\tau))}{\partial x_i} = 0 \quad (i = 1, \ldots, n)$$

(see the form of w_i (1)). Thus, the emission reduction vector $x^* = x(\tau)$ is a Pareto maximum point and $z_1, \ldots, z_n$ is a family of Pareto multipliers for x^*. The lemma is proved. □

Recall that the emission reduction process terminates in step s (see Proposition 1). By (A5) the emission reduction vector $x(t_s)$ is the outcome of the emission reduction process. Our principal statement is the following.

Proposition 2 *The outcome of the emission reduction process, $x(t_s)$, lies in the closed $p\delta$-neighborhood of the Pareto maximum point x^* described in Lemma 1.*

Proof By (6) and (18)

$$x(t_s) - x^* = x(t_s) - x(\tau) = \frac{t_s - \tau}{\delta} \Delta x(t_{s-1}). \tag{34}$$

By (22) and (21) $|\Delta x(t_{s-1})| = p\delta$ and by (25) $0 \leq t_s - \tau \leq \delta$. Hence, the norm of the right hand side in (34) is not bigger than $p\delta$. Therefore, $|x(t_s) - x^*| \leq p\delta$. The proposition is proved. □

Proposition 2 tells us that the distance between a Pareto maximum point, x^*, and the output of the emission reduction process, $x(t_s)$, goes to zero if the product $p\delta$ does. In other words, the smaller is the time duration of one step in the emission reduction process, δ, or the smaller is the total emission reduction size negotiated in each step, p, the better the process mimics a Pareto-optimal trade. Note that the decrease in each of the two values, δ and p, increases the countries' flexibility in the emission reduction process: the decrease in δ raises the frequency of negotiations, and the decrease in p reduces risk of unacceptable decisions in each step.

Let us also note that in every nondegenerate step of the emission reduction process, k, the local emission reduction vector, $\Delta x(t_k)$, being a result of the negotiations in step k (see (A6) and (A5)), is not defined uniquely. Therefore, the Pareto maximum point, x^*, that is approached, approximately, in the end of the emission reduction process is not pre-determined and can vary depending on the outcomes of the preceding negotiations. To summarize, we can say that Proposition 2 captures a robust qualitative property of the proposed decentralized boundedly rational emission reduction strategy: in the beginning of the emission reduction process the countries can be sure that the process will bring them close to a solution of the emission reduction game in a finite number of steps; however the countries should also realize that specific features of that solution will be seen after the termination of the process only.

4 Examples

The next example shows that the positivity of the transport coefficients (see (2)) is essential for the validity of Proposition 2.

Example 1 Let the emission reduction process involve two countries, country 1 and country 2 ($n = 2$). Let country 1 pollute itself only ($a_{11} = 1, a_{12} = 0$), country 2 pollute itself and country 1 in equal proportions ($a_{21} = a_{22} = 1/2$), and the countries' utility functions be given by

$$w_1(x) = 1 - \frac{1}{x_1 + x_2/2 + 1} - \frac{x_1}{2}, \qquad w_2(x) = 1 - \frac{1}{x_2/2 + 1} - \frac{x_2}{4};$$

here, in the right hand sides, the first terms and second terms represent the countries' benefit and cost functions, respectively. One can easily state that (A1)–(A3) are satisfied. We see that in contrast with the earlier assumptions, one of the transport coefficients, a_{12}, is zero. Let us show that Proposition 2 is no longer true.

Using expressions for the partial derivatives for the countries utility functions, w_1 and w_2,

$$\frac{\partial w_1(x)}{\partial x_1} = \frac{1}{(x_1 + x_2/2 + 1)^2} - \frac{1}{2}, \qquad \frac{\partial w_1(x)}{\partial x_2} = \frac{1}{2(x_1 + x_2/2 + 1)^2}, \tag{35}$$

$$\frac{\partial w_2(x)}{\partial x_1} = 0, \qquad \frac{\partial w_2(x)}{\partial x_2} = \frac{1}{2(x_2/2 + 1)^2} - \frac{1}{4}, \tag{36}$$

and the concavity of w_1 and w_2, we easily find that a nonnegative emission reduction vector x is a Pareto maximum point if and only if $x_1 = 0$ and $x_2 \geq r$.

Consider the emission reduction process. The fact that the total emission reduction for country 1, $x_1(t_k)$, grows in each nondegenerate step, k, whereas all the Pareto maximum points, x, have the zero first coordinates, $x_1 = 0$, tells us that the total emission reduction vector, $x(t_k)$, may never approach any Pareto maximum point. To support this intuitive observation, we argue as follows.

Take a step k such that

$$(x_1(t_k) + x_2/2 + 1)^2 < 3/2. \tag{37}$$

Using (35) and (36), we find that for every positive emission reduction vector, $\Delta x(t_k)$, it holds that

$$\frac{\partial w_1(x(t_k))}{\partial x_1} \Delta x_1(t_k) + \frac{\partial w_1(x(t_k))}{\partial x_2} \Delta x_2(t_k) \geq \alpha_{11} \Delta x_1(t_k) + \alpha_{12} \Delta x_2(t_k)$$

where $\alpha_{11} = 2/3 - 1/2 > 0$, $\alpha_{12} = 2/3 > 0$, and

$$\frac{\partial w_2(x(t_k))}{\partial x_1} \Delta x_1(t_k) + \frac{\partial w_2(x(t_k))}{\partial x_2} \Delta x_2(t_k) \geq \alpha_{22} \Delta x_2(t_k)$$

where $\alpha_{22} = 1/3 - 1/4 > 0$. Therefore, every step, k, such that (37) holds is nondegenerate and every positive emission reduction vector is acceptable in that step.

For $k = 0$ (37) holds since $x(0) = 0$ (see (4)). Suppose in every nondegenerate step, k, satisfying (37), the countries choose an acceptable local emission vector $\Delta x(t_k)$ such that $\Delta x_1(t_k) = \Delta x_2(t_k) = p\delta$ (we assume that the norm in the two-dimensional space is such that $|\Delta x(t_k)| = \max\{|\Delta x_1(t_k)|, |\Delta x_2(t_k)|\}$). Let k_* be the maximum of all such k. For every $k \le k_*$ we have

$$x_1(t_k) = x_2(t_k) = pk\delta; \tag{38}$$

hence, k_* is the maximum of all $k = 0, 1, \ldots$ such that $(3pk\delta/2 + 1)^2 < 3/2$ or $pk\delta < q$ where $q = 2[(3/2)^{1/2} - 1]/3 > 0$. Clearly, $pk_*\delta \ge q - p\delta$, or, in view of (38), $x_1(t_{k_*}) \ge q - p\delta$. Let δ be so small that $q - p\delta > q/2$. Since $x_1(t_k)$ grows, $x_1(t_k) > q/2$ in all nondegenerate steps $k \ge k_*$. Thus, the emission reduction process either never terminates or terminates with an $x_1(t_s) > q/2$ in some step $s > k_*$; in the latter case the final emission reduction vector, $x(t_s)$, is at a distance higher than $q/2$ from any Pareto maximum point. The statement of Proposition 2 is violated.

As noted in the previous section, the emission reduction process has multiple outcomes. By Proposition 2 each of those outcomes approximates a certain Pareto maximum point with accuracy $p\delta$. Let us call a Pareto maximum point *$p\delta$-reachable* if it is approximated by some outcome of the emission reduction process with accuracy $p\delta$. Let us ask ourselves if all the Pareto maximum points are $p\delta$-reachable. The next example shows that there can be a solid gap between the set of all Pareto maximum points and the set of all $p\delta$-reachable ones.

Example 2 Let two countries, country 1 and country 2, involved in the emission reduction process ($n = 2$) pollute each other in equal proportions ($a_{ji} = 1/2$, $j, i = 1, 2$), and the countries' utility functions be identical:

$$w_1(x) = 1 - \frac{1}{x_1/2 + x_2/2 + 1} - \frac{x_1}{2}, \qquad w_2(x) = 1 - \frac{1}{x_1/2 + x_2/2 + 1} - \frac{x_2}{2};$$

here, in the right hand sides, the first terms and second terms represent the countries' benefit and cost functions, respectively. One can easily state that (A1)–(A3) are satisfied.

We find the Pareto maximum points as nonnegative vectors x satisfying

$$z_1 \frac{\partial w_1(x)}{\partial x_1} + z_2 \frac{\partial w_2(x)}{\partial x_1} = 0, \qquad z_1 \frac{\partial w_1(x)}{\partial x_2} + z_2 \frac{\partial w_2(x)}{\partial x_2} = 0$$

with some $z_1, z_2 > 0$. Simple calculations result in the following: the set of all Pareto maximum points consists of all nonnegative x such that

$$x_1/2 + x_2/2 = \beta = 2^{1/2} - 1. \tag{39}$$

Geometrically, the latter set is the interval, I, with the end points $x^{(1)} = (2\beta, 0)$ and $x^{(2)} = (0, 2\beta)$. At the end point $x^{(1)}$ the utilities of countries 1 and 2 reach, respectively, their minimum and maximum values, $1 - 1/(\beta + 1) - \beta$ and $1 - 1/(\beta + 1)$, in I; at the end point $x^{(0)} = (0, 2\beta)$ the situation is symmetric. At the middle point of I, $x^{(0)}$, the countries have the same utility value, $1 - 1/(\beta + 1) - \beta/2$. One can view the "middle" Pareto maximum point, $x^{(0)}$, as the "most fair" one and the end points, $x^{(1)}$ and $x^{(2)}$, as the "most unfair" ones. Given a Pareto maximum point, x, the distance from x to the "most unfair" Pareto maximum point closest to x can be treated as "the degree of fairness" of x.

Let us consider the emission reduction process described earlier. Using Proposition 2, we find that in every nondegenerate step, k, the total emission reduction vector, $x(t_k)$, lies in the triangle bordered by the x_1-axis, x_2-axis and interval I, in particular,

$$y(t_k) = x_1(t_k)/2 + x_2(t_k)/2 < \beta. \tag{40}$$

In the first degenerate step, s, vector $x(t_s)$ constituting the outcome of the emission reduction process lies necessarily beyond the interior of the triangle, implying

$$y(t_s) \geq \beta. \tag{41}$$

Take a nondegenerate step k. We have $x(t_{k+1}) = x(t_k) + \Delta x(t_k)$ where $\Delta x(t_k)$ is a positive emission reduction vector acceptable in step k, i.e., satisfying

$$\frac{\partial w_1(x(t_k))}{\partial x_1}\Delta x_1(t_k) + \frac{\partial w_1(x(t_k))}{\partial x_2}\Delta x_2(t_k) > 0,$$
$$\frac{\partial w_2(x(t_k))}{\partial x_1}\Delta x_1(t_k) + \frac{\partial w_2(x(t_k))}{\partial x_2}\Delta x_2(t_k) > 0,$$

or

$$\left(\frac{1}{2(y(t_k)+1)^2} - \frac{1}{2}\right) + \frac{1}{2(y(t_k)+1))^2}\frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} > 0,$$
$$\frac{1}{2(y(t_k)+1)^2} + \left(\frac{1}{2(y(t_k)+1)^2} - \frac{1}{2}\right)\frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} > 0$$

(here we use explicit forms of the partial derivatives). After an elementary transformation, we get

$$\frac{1}{(y(t_k)+1)^2 - 1} > \frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} > (y(t_k)+1)^2 - 1.$$

The latter inequality implies

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta x_1(t_k)} = \frac{1}{2}\left(\frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} + 1\right) > \frac{(y(t_k)+1)^2}{2}. \tag{42}$$

We assume δ to be sufficiently small and view (42) as a difference approximation to the differential inequality

$$\frac{d\bar{y}(x_1)}{dx_1} > \frac{(\bar{y}(x_1)+1)^2}{2} \tag{43}$$

for a function $\bar{y}(x_1)$ at the point $(x_1(t_k), y(x_1(t_k)))$. One can prove that for an arbitrary $\varepsilon >$ and δ sufficiently small, there is a solution to the differential inequality (43), $\bar{y}$, defined on $[0,\infty)$, satisfying $\bar{y}(0)=0$ and such that $|y(t_k)-\bar{y}(x(t_k))| < \varepsilon$ for all nondegenerate steps, k. Clearly, $\bar{y}(x_1) \geq \bar{y}_*(x_1)$ $(x_1 \geq 0)$ where $\bar{y}_*$ is the solution to the differential equation

$$\frac{d\bar{y}_*(x_1)}{dx_1} = \frac{(\bar{y}_*(x_1)+1)^2}{2}, \tag{44}$$

defined on $[0,\infty)$ and satisfying $\bar{y}_*(0)=0$. Therefore, for the last nondegenerate step, $s-1$, it holds that

$$y(t_{s-1}) - \bar{y}_*(x(t_{s-1})) > -\varepsilon. \tag{45}$$

By (40) with $k=s-1$ and by (45) we have $\bar{y}_*(x(t_{s-1})) < \beta+\varepsilon$. Let $\bar{x}_1 > 0$ be such that $\bar{y}_*(x_1)=\beta$. If $\bar{y}_*(x(t_{s-1})) < \beta$, then $x(t_{s-1}) < \bar{x}_1$. If $\bar{y}_*(x(t_{s-1})) \geq \beta$, then $x(t_{s-1}) \geq \bar{x}_1$ and, due to (44),

$$x_1(t_{s-1}) - \bar{x}_1 \leq 2\frac{\bar{y}_*(x(t_{s-1})) - \bar{y}_*(\bar{x}_1)}{(\bar{y}_*(\bar{x}_1)+1)^2} \leq 2(\bar{y}_*(x_1(t_{s-1}) - \bar{y}_*(\bar{x}_1))) < 2\varepsilon.$$

Hence, for $x_1(t_s)$, the final emission reduction value for country 1, we have

$$x_1(t_s) \leq x_1(t_{s-1}) + p\delta < \bar{x}_1 + 2\varepsilon + p\delta. \tag{46}$$

Let us find $\bar{x}_1$. The integration of the differential equation (44) under the initial condition $\bar{y}_*(0)=0$ yields

$$\bar{y}_*(x_1) = \frac{2}{2-x_1} - 1.$$

Combining with $\bar{y}_*(\bar{x}_1)=\beta$ and resolving with respect to $\bar{x}_1$, we get

$$\bar{x}_1 = \frac{2\beta}{\beta+1}.$$

Then by (46)

$$x_1(t_s) \leq \frac{2\beta}{\beta+1} + 2\varepsilon + p\delta. \tag{47}$$

Using (41) or, equivalently, $x_1(t_s)+x_2(t_s) \geq 2\beta$, we find that

$$x_2(t_s) \geq 2\beta - x_1(t_s) = 2\beta\left(1 - \frac{1}{\beta+1}\right) - 2\varepsilon - p\delta. \tag{48}$$

Let the norm of a vector x in the two-dimensional space be defined as $|x| = \max\{|x_1|, |x_2|\}$. Consider the distance from the outcome vector, $x(t_s)$, to the “most unfair” Pareto maximum point $x^{(1)} = (2\beta, 0)$. From (47) and (48) we get

$$|x(t_s) - x^{(1)}| \geq 2\beta\left(1 - \frac{1}{\beta + 1}\right) - 2\varepsilon - p\delta.$$

Note that for every $p\delta$-reachable Pareto maximum point, x^*, it holds that $|x(t_s) - x^*| \leq p\delta$. Thus, for every such x^*, we have

$$|x^* - x^{(1)}| \geq |x(t_s) - x^{(1)}| - |x(t_s) - x^*| \geq 2\beta\left(1 - \frac{1}{\beta + 1}\right) - 2\varepsilon - 2p\delta.$$

We see that for an arbitrarily small $\gamma > 0$ one can choose ε and δ so small that all Pareto maximum points lying in the $(2\beta[1 - 1/(\beta + 1)] - \gamma)$-neighborhood of the “most unfair” Pareto maximum point $x^{(1)}$ are not $p\delta$-reachable.

A similar argument leads us to a symmetric statement: for an arbitrary $\gamma > 0$ one can choose ε and δ so small that all Pareto maximum points lying in the $(2\beta[1 - 1/(\beta + 1)] - \gamma)$-neighborhood of the “most unfair” Pareto maximum point $x^{(2)} = (2\beta, 0)$ are not $p\delta$-reachable.

Let us note in conclusion that a “converse” statement holds true as well: for an arbitrary $\gamma > 0$ one can choose ε and δ so small that all Pareto maximum points lying beyond the $(2\beta[1 - 1/(\beta + 1)] + \gamma)$-neighborhoods of the “most unfair” Pareto maximum points $x^{(1)}$ and $x^{(2)}$ are $p\delta$-reachable; for brevity, we omit a proof.

5 Negotiation Pattern

Here, we discuss a negotiation pattern satisfying assumption (A4), i.e., allowing the countries in each step to either find an acceptable positive emission reduction vector if the step is nondegenerate, or identify the fact that the step is degenerate.

Take an arbitrary step of the emission reduction process, k, which is either initial ($k = 0$) or such that all the preceding steps are nondegenerate and consider negotiations in step k. The goal of the negotiations is to either find a positive emission reduction vector acceptable in that step, or identify that the step is degenerate and terminate the process.

Recall that a positive emission reduction vector $\Delta x(t_k)$ (5) is acceptable in step k if for every country, i, its local utility growth criterion (13) is satisfied. Substituting (8) in (13), we represent the set of the countries’ local utility growth criteria in step k as a system of inequalities:

$$\lambda_i(t_k)\Delta x_i(t_k) < \sum_{j=1,\dots,n,\ j\neq i} a_{ji}\Delta x_j(t_k) \quad (i = 1, \dots, n). \tag{49}$$

We see that if $\lambda_i(t_k) \leq 0$, country i satisfies its local utility growth criterion in step k with any $\Delta x_i(t_k) > 0$; we call such a country, i, a *free negotiator* (in step k). Note that the strict inequality, $\lambda_i(t_k) < 0$, or, equivalently, $b_i'(y_i(t_k))a_{ii} - c_i'(x_i(t_k)) > 0$ (see (14)), implies that in step k the country's marginal cost is low enough and the country can gain in utility even by slightly reducing its emission solely. The opposite inequality, $\lambda_i(t_k) > 0$, implies that in step k the marginal cost for country i is high enough and a local growth in the country's utility is possible provided other countries reduce emission; we call such a country, i, a *constrained negotiator* (in step k).

Our negotiation pattern suggests that the negotiations in step k go in two phases, phase 1 and phase 2, the latter having two variants, phase 2a and phase 2b.

In phase 1 each country, i, reveals $\lambda_i(t_k)$. Based on that, the countries' community identifies the free negotiators and constrained negotiators. If there are free negotiators, the countries go to phase 2a. Otherwise the countries go to phase 2b.

Phase 2a is organized as follows. Based on some pre-defined rule, one free negotiator, i_*, is selected. The other countries, $i \neq i_*$, propose some $\Delta x_i(t_k) > 0$. The free negotiator i_* responds with a sufficiently large $\Delta x_{i_*}(t_k) > 0$ such that the utility growth criteria (49) are satisfied for all $i \neq i_*$; the latter is guaranteed, if, for example,

$$\Delta x_{i_*}(t_k) > \max_{i=1,\dots,n,\ i\neq i_*} \frac{\lambda_i(t_k)\Delta x_i(t_k)}{a_{i_* i}}.$$

For $i = i_*$ (49) is satisfied automatically. The vector $\Delta x(t_k)$ (5) resulting from the negotiations is acceptable in step k.

Let us give two comments to phase 2a. First, we see that if there exist free negotiators in step k, then step k is nondegenerate. Second, if there are several free negotiators in step k, the proposed simple decision making scheme in phase 2a "discriminates" the selected free negotiator, i_*, which is obliged to compensate for arbitrary choices of all the other negotiators, including the free ones. There are obviously a number of ways to modify the scheme and make it more cooperative; for the sake of brevity, we do not discuss such modifications here.

Phase 2b assuming that there are no free negotiators is organized as follows. In the beginning, the countries represent their local utility growth criteria (49) as

$$\Delta x_i(t_k) = \sum_{j=1,\dots,n,\ j\neq i} \frac{\beta_{ji}}{\gamma_i}\Delta x_j(t_k) \quad (i = 1, \dots, n) \tag{50}$$

where

$$\beta_{ji} = \frac{a_{ji}}{\lambda_i(t_k)} \quad (i = 1, \dots, n)$$

and

$$\gamma_i > 1 \quad (i = 1, \dots, n). \tag{51}$$

For each country, i, (50) is a formula for its *individual response*, $\Delta x_i(t_k) > 0$, to the proposals of the other countries, $\Delta x_j(t_k) > 0$, $j \neq i$.

Next, the countries switch to negotiations. The negotiations go through an *exploration stage* and a *decision making stage*. In the exploration stage the countries identify if step k is nondegenerate. If step k is degenerate, the countries cancel the decision making stage and terminate the emission reduction process (see (A5)). Otherwise, the countries switch to the decision making stage and find a local emission reduction vector acceptable in step k of the emission reduction process.

In the exploration stage the negotiations proceed in rounds. Round 1 is organized as follows. Country 1 communicates its individual response formula,

$$\Delta x_1(t_k) = \sum_{j=2}^{n} \frac{\beta_{j1}}{\gamma_1} \Delta x_j(t_k), \tag{52}$$

to country 2. Country 2 substitutes (52) in its individual response formula,

$$\Delta x_2(t_k) = \frac{\beta_{12}}{\gamma_2} \Delta x_1(t_k) + \sum_{j=3}^{n} \frac{\beta_{j2}}{\gamma_2} \Delta x_j(t_k),$$

transforming the latter into

$$\Delta x_2(t_k) = \sum_{j=3}^{n} \frac{\beta_{j2}^{(2)}(\gamma_1, \gamma_2)}{\gamma^{(2)}(\gamma_1, \gamma_2)} \Delta x_j(t_k) \tag{53}$$

where

$$\beta_{j2}^{(2)}(\gamma_1, \gamma_2) = \frac{\beta_{j1}\beta_{12}}{\gamma_1\gamma_2} + \frac{\beta_{j2}}{\gamma_2} \quad (j = 3, \ldots, n), \tag{54}$$

$$\gamma^{(2)}(\gamma_1, \gamma_2) = 1 - \frac{\beta_{21}\beta_{12}}{\gamma_1\gamma_2}; \tag{55}$$

here an in what follows we omit elementary transformations. The *updated individual response formula* for country 2, (53), takes into account the local utility growth criterion for country 1. The requirement that both sides in (53) are positive imposes a *positivity constraint* on $\gamma^{(2)}(\gamma_1, \gamma_2)$:

$$\gamma^{(2)}(\gamma_1, \gamma_2) > 0. \tag{56}$$

Country 2 communicates its updated individual response formula, (53), to country 1, and the latter substitutes (53) in its individual response formula (52) resulting in

$$\Delta x_1(t_k) = \sum_{j=3}^{n} \beta_{j1}^{(2)}(\gamma_1, \gamma_2) \Delta x_j(t_k) \tag{57}$$

where

$$\beta_{j1}^{(2)}(\gamma_1, \gamma_2) = \frac{\beta_{j2}^{(2)}(\gamma_1, \gamma_2)\beta_{21}}{\gamma_1\gamma^{(2)}(\gamma_1, \gamma_2)} + \frac{\beta_{j1}}{\gamma_1} \quad (j = 3, \ldots, n). \tag{58}$$

Two formulas, (57) and (53), represent the formula for a collective response of countries 1 and 2 to any proposed local emission reduction values of countries $3, \ldots, n$. The *collective response formula* (57), (53) and positivity constraint (56) constitute the result of round 1.

Round $m-1$ where $2 \leq m < n$ starts with the situation, in which countries $1, \ldots, m$ have generated their *collective response formula* in the form

$$\Delta x_i(t_k) = \sum_{j=m+1}^{n} \beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m)\Delta x_j(t_k) \quad (i = 1, \ldots, m-1), \tag{59}$$

$$\Delta x_m(t_k) = \sum_{j=m+1}^{n} \frac{\beta_{jm}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)}\Delta x_j(t_k), \tag{60}$$

where $\beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m)$ $(j = m+1, \ldots, n)$ are positive automatically, and a set of *positivity constraints*:

$$\gamma^{(i)}(\gamma_1, \ldots, \gamma_i) > 0 \quad (i = 1, \ldots, m). \tag{61}$$

Countries $1, \ldots, m$ communicate the collective response formula, (59), (60), and constraints (61) to country $m+1$.

Let $m+1 < n$. Country $m+1$ substitutes (59), (60) in its individual response formula and gets

$$\Delta x_{m+1}(t_k) = \sum_{j=m+2}^{n} \frac{\beta_{j\ m+1}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})}{\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})}\Delta x_j(t_k) \tag{62}$$

where

$$\begin{aligned}\beta_{j\ m+1}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) &= \sum_{i=1}^{m-1} \frac{\beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{i\ m+1}}{\gamma_{m+1}} \\ &\quad + \frac{\beta_{jm}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{m\ m+1}}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)\gamma_{m+1}} + \frac{\beta_{j\ m+1}}{\gamma_{m+1}},\end{aligned} \tag{63}$$

$$\begin{aligned}\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) &= 1 - \sum_{i=1}^{m-1} \frac{\beta_{m+1\ i}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{i\ m+1}}{\gamma_{m+1}} \\ &\quad - \frac{\beta_{m+1\ m}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{m\ m+1}}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)\gamma_{m+1}}.\end{aligned} \tag{64}$$

The fact that both sides in (62) are positive leads to the constraint

$$\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) > 0. \tag{65}$$

Equality (62) represents an updated individual response formula for country $m+1$, in which the local utility growth criteria for countries $1,\dots,m$ are taken into account.

Country $m+1$ communicates its updated individual response formula, (62), to countries $1,\dots,m$. Countries $1,\dots,m$ substitute (62) in their collective response formula (59), (60), transforming the latter into

$$\Delta x_i(t_k) = \sum_{j=m+2}^{n} \beta_{ji}^{(m+1)}(\gamma_1,\dots,\gamma_m)\Delta x_j(t_k) \quad (i=1,\dots,m) \tag{66}$$

where

$$\beta_{ji}^{(m+1)}(\gamma_1,\dots,\gamma_{m+1}) = \frac{\beta_{j\ m+1}^{(m+1)}(\gamma_1,\dots,\gamma_{m+1})\beta_{m+1\ i}^{(m)}(\gamma_1,\dots,\gamma_m)}{\gamma^{(m+1)}(\gamma_1,\dots,\gamma_{m+1})} + \beta_{ji}^{(m)}(\gamma_1,\dots,\gamma_m) \quad (i=1,\dots,m-1), \tag{67}$$

$$\beta_{jm}^{(m+1)}(\gamma_1,\dots,\gamma_{m+1}) = \frac{\beta_{j\ m+1}^{(m+1)}(\gamma_1,\dots,\gamma_{m+1})\beta_{m+1\ m}^{(m)}(\gamma_1,\dots,\gamma_m)}{\gamma^{(m)}(\gamma_1,\dots,\gamma_m)\gamma^{(m+1)}(\gamma_1,\dots,\gamma_{m+1})} + \frac{\beta_{jm}^{(m)}(\gamma_1,\dots,\gamma_m)}{\gamma^{(m)}(\gamma_1,\dots,\gamma_m)}. \tag{68}$$

Equalities (66) and (62) give a collective response formula for countries $1,\dots,m+1$. The collective response formula (66), (62) and positivity constraints (61), (65) form the result in round $m+1$. Equalities (67), (68), (63), (64) show how the collective response formula for countries $1,\dots,m+1$, (66), (62), are formed based on the collective response formula for countries $1,\dots,m$, (59), (60).

Let $m+1=n$. Country n substitutes (59), (60), where $m=n-1$, in its individual response formula,

$$\Delta x_n(t_k) = \sum_{i=1}^{n-1} \frac{\beta_{in}}{\gamma_n}\Delta x_i(t_k),$$

and gets in result a simplified analogue of (62):

$$\Delta x_n(t_k) = \frac{\varphi^{(n)}(\gamma_1,\dots,\gamma_{n-1})}{\gamma_n}\Delta x_n(t_k)$$

where

$$\varphi^{(n)}(\gamma_1,\dots,\gamma_{n-1}) = \sum_{i=1}^{n-2}\beta_{ni}^{(n-1)}(\gamma_1,\dots,\gamma_{n-1})\beta_{in} + \frac{\beta_{n\ n-1}^{(n-1)}(\gamma_1,\dots,\gamma_{n-1})\beta_{n-1\ n}}{\gamma^{(n-1)}(\gamma_1,\dots,\gamma_{n-1})}.$$

The inequality $\Delta x_n(t_k) > 0$ implies $\varphi^{(n)}(\gamma_1,\dots,\gamma_{n-1}) = \gamma_n$ and, in view of $\gamma_n > 1$ (see (51)),

$$\varphi^{(n)}(\gamma_1,\dots,\gamma_{n-1}) > 1. \tag{69}$$

Obviously, (69) is a necessary condition for the existence of a positive emission reduction vector $\Delta x(t_k)$ acceptable in step k, i.e., satisfying the countries' utility growth criteria (50). Country n communicates the criterion (69) to the other countries and finalizes round n.

In the final round of the exploration stage the countries verify if (69) is feasible under the constraints imposed on $\gamma_1, \dots, \gamma_{n-1}$ earlier:

$$\gamma_i > 1, \quad \gamma^{(i)}(\gamma_1, \dots, \gamma_i) > 0 \quad (i = 1, \dots, n-1) \tag{70}$$

(see (51), (61)). If the countries find that the system of inequalities (69), (70) is incompatible, they qualify step k as degenerate, stop the negotiations and terminate the emission reduction process. Otherwise the countries switch to the decision making stage.

Note that in order to tell if the system of inequalities (69), (70) is compatible, it is sufficient to find $\varphi_*^{(n)} = \sup \varphi^{(n)}(\gamma_1, \dots, \gamma_{n-1})$ under the constraints (70). This constrained optimization problem can be solved numerically using standard optimization techniques; for small n the problem can be treated analytically (for the sake of brevity we do not provide examples). Obviously, the system of inequalities (69), (70), is compatible if and only if $\varphi_*^{(n)} > 1$.

Let the system of inequalities (69), (70) be compatible and $\gamma_1, \dots, \gamma_{n-1}$ satisfy (69), (70). Consider the decision making stage in the negotiations in phase 2b. The proposed negotiation scheme implies that the compatibility of the system of inequalities (69), (70) is sufficient for the existence of a positive emission reduction vector, $\Delta x(t_k)$, satisfying the system of the countries' local utility growth criteria (50).

In round 1 country n chooses a positive emission reduction value $\Delta x_n(t_k)$ and communicates this value to the other countries. In round 2 countries $1, \dots, n-1$ compute their emission reduction values, $\Delta x_1(t_k), \dots, \Delta x_{n-1}(t_k)$, using their collective response formula (59), (60) designed in round $m = n-2$ of the exploration stage:

$$\Delta x_i(t_k) = \beta_{ni}^{(n-1)}(\gamma_1, \dots, \gamma_{n-1}) \Delta x_n(t_k) \quad (i = 1, \dots, n-2),$$

$$\Delta x_{n-1}(t_k) = \frac{\beta_{n\,n-1}^{(n-1)}(\gamma_1, \dots, \gamma_{n-1})}{\gamma^{(n-1)}(\gamma_1, \dots, \gamma_{n-1})} \Delta x_n(t_k).$$

A straightforward argument shows that the resulting emission reduction vector, $\Delta x(t_k)$, satisfies the countries' local utility growth criteria (50), constituting the desired outcome of the negotiations in step k of the emission reduction process.

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Environmental Mortality and Long-Run Growth

Ulla Lehmijoki and Elena Rovenskaya

Abstract There is emerging evidence that environmental degradation increases human mortality. This paper provides a long-run consumer optimization model in which mortality is endogenous to emissions generated by production. Emissions are assumed to follow the *EKC* path, first rising and then falling along with output. In the optimum, some deaths are accepted as an exchange for consumption. The model is estimated for the European outdoor air pollution data, showing that Europe has reached the downward sloping segment of the *EKC*. Economic growth will thus decrease rather than increase pollution in the future. Nevertheless, continuous population growth may increase the number of deaths in some countries.

1 Introduction

As several branches of science provide new findings concerning air pollution, climate change, salination of ground water, and pollution of the ocean, there is emerging evidence that environmental degradation harms human health. Therefore, the fear that economic growth increases this degradation is common.

This paper offers a long-run economic growth model in which population growth, through mortality, is endogenous to environmental degradation. In this framework, health—and even life—is one of the competing goals of utility-maximizing agents. Nevertheless, the association between economic degradation and economic growth is not linear. On the contrary, according to the Environmental Kuznets Curve hypothesis, *EKC*, pollution first increases but then decreases along with output (Selden and Song 1994; Arrow et al. 1995; Grossman and Krueger 1995).

Our model is derived from several building blocks and several simplifications are needed to keep it tractable. Mortality increases as a response to emissions generated as by-products of production. Since emissions mostly have their roots in energy combustion, it has been argued that the capital-intensive sector is "dirtier" than the labor-intensive one (Cole and Elliot 2003). We take this argument to the extreme by assuming that there is only one capital-intensive sector, where capital is the sole factor of production. Hence, environmental mortality causes no negative effect on

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U. Lehmijoki (✉)
Department of Economics, University of Helsinki, HECER, and IZA, P.O. Box 17, 00014, Helsinki, Finland
e-mail: ulla.lehmijoki@helsinki.fi
Fax: +358-9-19128736

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production and is important only because of welfare losses, modeled through the Benthamian utility function depending on per capita consumption and the population size. We assume that pollution is a public good consumed by all in equal amounts, so that only its overall extent is important. The *EKC* hypothesis is, therefore, considered in total rather than in per capita terms. Since our emphasis is on the basic trade-off between output and deaths, many important elements such as emission-limiting policies and health-promoting medical efforts are left out of the model (Stokey 1998).

There seems to be a consensus that of the several environmental hazards, outdoor air pollution currently causes the greatest risk to human health. Air pollution mortality was first reported in the Meuse Valley, Belgium (1930) and London (1953), where smog took the lives of 60 and 4000 people respectively (Logan 1953; Nemery et al. 2001). Air pollution raises mortality mainly through increases in respiratory and cardiovascular diseases and lung cancer, but an increase in skin cancer has also been reported (Samet et al. 2000; Brunekreef and Holgate 2002). All age groups are affected, but the unborn, young children, and the elderly are the most vulnerable. *CAFE*, the Clean Air for Europe program and *WHO*, the World Health Organization, have provided the first-ever estimates of environmental mortality in Europe, showing that nowadays there are more than 300000 premature deaths annually (WHO 2004). Hence, the relevant question arises how economic growth will change this number in the future.

To answer this question, we estimate the parameters of the model to accommodate it to the European economic and air pollution data. The critical question is whether the European countries have already passed the *EKC* peak. Since the derived results suggest that this is the case, our conclusion is that economic growth will decrease rather than increase air pollution in the future, the decrease being most rapid in countries with highest economic growth. Nevertheless, in some countries continuous population growth will increase the number of air pollution deaths because the number of exposed people increases. In the sub-sample of fourteen European countries for which a complete analysis is possible, the total number of deaths from 2000 to 2020 accumulates to more than four million, showing that even though Europe is one of the cleanest places in the world, its environmental deaths are numerous enough to be taken seriously.

The paper is organized as follows: Sect. 2 introduces the model and Sect. 3 works out its solution. Section 4 provides its application to the European air pollution data with country-specific results and Sect. 5 discusses the findings and closes the paper. The appendixes contain technical details and the data.

2 The Model

Consider an economy in which capital K is the only input, implying that the role of labor L as input is negligible. The production function takes the Cobb-Douglas formula

$$Y = AK^{\alpha}, \tag{1}$$

where $A > 0$ is the technology level and $0 < \alpha < 1$ is the elasticity of the output with respect to capital.

All emissions E are generated as by-products of production. Hence, for all $Y \geq 0$ emissions become

$$E = g(Y) = g(AK^{\alpha}).$$

We assume that the Environmental Kuznets Curve hypothesis *EKC* holds, i.e., emissions first rise and then fall along with output. *EKC* arises when the emission-decreasing technology and composition effects dominate the emission-increasing scale effect, consisting of an increase in per capita incomes and population size. Therefore, the emission function $g(\cdot)$ assumes

$$\begin{aligned} &g'(Y) > 0 \quad \text{for } Y < \mu, \qquad g'(Y) = 0 \quad \text{for } Y = \mu, \\ &g'(Y) < 0 \quad \text{for } Y > \mu, \end{aligned} \tag{2}$$

and

$$\lim_{Y \to 0} g'(Y) < \infty, \qquad \lim_{Y \to \infty} g'(Y) = 0, \tag{3}$$

where μ refers to the *EKC* peak. The limit conditions (3) imply that emissions step in slowly and ultimately level off. The emission function also satisfies

$$g(0) = 0, \qquad \lim_{Y \to \infty} g(Y) \geq 0.$$

Emissions cause unwanted health consequences ranging from eye irritation to severe illness and death. We concentrate on deaths. Hence, we assume that the population growth rate $\dot{L}/L = n$ consists of two components, an autonomous component and a component describing the environmental deaths. Concentrating on the latter, we assume that the autonomous component ν is constant. Hence, for all $E \geq 0$, the population growth rate $n = n(E)$ satisfies

$$n(0) = \nu > 0, \qquad n'(E) < 0, \tag{4}$$

indicating that population growth is positive for zero emissions but decreases as emissions increase. Several additional specifications are possible, but since most epidemiological studies indicate that the association between mortality and pollution is linear (Samet et al. 2000; Brunekreef and Holgate 2002; Pope et al. 2002), we assume $n'' = 0$. Note also that because the production function (1) is highly stylized, environmental mortality induces no feedback on the output. This is justified if most environmental victims are children and elderly adults, as is the case of the air pollution deaths analyzed here.

Noting (4) and normalizing the initial population to unity, the population size at time t becomes

$$L(t) = \exp \int_0^t n[E(\tau)]d\tau. \tag{5}$$

Since the output can be either consumed or saved, the capital stock accumulates according to

$$\dot{K} = AK^{\alpha} - C - \delta K, \qquad K(0) = K_0, \tag{6}$$

where $\delta > 0$ refers to depreciation and K_0 to a positive initial value of the capital stock.

Consider a benevolent central planner facing the Benthamian societal utility $u(C/L) \cdot L$, which depends on individual utility from per capita consumption C/L and the number of people L, implying that environmental deaths cause disutility to the planner. To keep the model simple, environmental amenities are not included. Let the utility function adopt the *CIES* formula

$$u(C/L) = \frac{(C/L)^{1-\theta}}{1-\theta} \quad (\theta \neq 1).$$[1]

The central planner chooses consumption $C(\cdot)$ to maximize the utility index

$$\begin{aligned} U &= \int_0^{\infty} u[C(t)/L(t)]L(t)e^{-\rho t}dt \\ &= \int_0^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\int_0^t \{\rho - \theta n[E(\tau)]\}d\tau} dt \end{aligned} \tag{7}$$

subject to (2), (3), (4), (6). To keep (7) bounded, we assume

$$\rho - \theta \nu > 0. \tag{8}$$

Due to (4), assumption (8) is sufficient for positiveness of $\rho - \theta n(E)$ for all $E \geq 0$. Since we assume that emissions are commonly "consumed" by all, i.e., emissions are a public good, only their total amount is important. Hence, we keep the model at aggregative level, without reducing it to per capita terms.

3 Optimal Consumption and Investment

The fact that the discount factor $\Delta(t) = \int_0^t \{\rho - \theta n[E(\tau)]\}d\tau$ in (7) is not constant provides difficulties for the analysis. To eliminate them, we apply the virtual time technique suggested by Uzawa (1968). Given (8), the factor $\Delta(t)$ has the following properties:

(i) $\Delta(0) = 0$,
(ii) $\Delta(\infty) = \infty$,
(iii) $\Delta(t)$ is monotonically increasing with $\dot{\Delta}(t) = \rho - \theta n[E(t)] > 0$.

[1] The alternative specification, $u(C/L) = \frac{(C/L)^{1-\theta}}{1-\theta} - 1$, has the convenient property $\lim_{\theta \to 1} u(C/L) = \ln C/L$, but this fails in the shorter expression above. Hence the requirement $\theta \neq 1$. Both formulas lead to the same result.

Since $\Delta(t)$ thus satisfies the regularity conditions suggested by Uzawa (1968), it can be used as an alternative independent time variable and we set $C = C(\Delta)$, $K = K(\Delta)$, $E = E(\Delta)$. Furthermore,

$$dt = \frac{d\Delta(t)}{\rho - \theta n[E(t)]}. \tag{9}$$

Applying (9) to (2)–(7) turns the problem into[2]

$$\begin{aligned} &\text{maximize}_{C(\cdot)} \quad U = \int_0^\infty \frac{C^{1-\theta}}{(1-\theta)(\rho - \theta n(E))} e^{-\Delta} d\Delta, \\ &\text{subject to:} \qquad \mathring{K} = \frac{dK}{d\Delta} = \frac{AK^\alpha - C - \delta K}{\rho - \theta n(E)}, \qquad K(0) = K_0, \\ &\qquad\qquad\qquad\quad E = g(AK^\alpha), \end{aligned} \tag{10}$$

where the notation $\mathring{K}$ refers to differentiation in terms of virtual time. Problem (10) can be solved in virtual time by the Pontryagin maximum principle for optimal control problems on the infinite time horizon (as in Aseev and Kryazhimskiy 2007). Letting λ be the adjoint variable, the Hamiltonian and necessary conditions become:

$$H(K, C, \lambda) = \frac{1}{\rho - \theta n(E)} \left\{ \frac{C^{1-\theta}}{1-\theta} + \lambda[AK^\alpha - C - \delta K] \right\}, \tag{11}$$

$$\frac{\partial H}{\partial C} = 0 \quad \Longleftrightarrow \quad C^{-\theta} = \lambda, \tag{12}$$

$$\mathring{\lambda} = \frac{d\lambda}{d\Delta} = -\frac{\partial H}{\partial K} + \lambda. \tag{13}$$

Since

$$\dot{\lambda} = (d\lambda(\Delta)/d\Delta) \cdot (d\Delta/dt) = \mathring{\lambda}(\rho - \theta n(E)),$$

multiplying (13) by $(\rho - \theta n(E))$ transforms it back to the natural time and, after some algebra, the equation for the adjoint variable becomes

$$\dot{\lambda}/\lambda = -\left\{ \frac{\theta n' g' \alpha A K^{\alpha-1}}{\lambda} H + \alpha A K^{\alpha-1} - \delta - \rho + \theta n \right\}. \tag{14}$$

To eliminate λ, take the time derivative of (12) and insert into (14) to get

$$\frac{\dot{C}}{C} = \frac{1}{\theta} \left\{ \frac{\theta n' g' \alpha A K^{\alpha-1}}{\lambda} H + \alpha A K^{\alpha-1} - \delta - \rho + \theta n \right\}$$

[2] To make the formulas shorter, we leave out the arguments of the functions if possible.

$$= \frac{1}{\theta}\left\{\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho - \theta n}\left(\frac{\theta C}{1-\theta} + AK^{\alpha} - \delta K\right)\right.$$
$$\left. + \alpha A K^{\alpha-1} - \delta - \rho + \theta n\right\}. \tag{15}$$

Equations (6) and (15) supply the solution of the model. The phase lines are:

$$\frac{\dot{C}}{C} = 0 \quad \Longleftrightarrow \quad C = \frac{\theta - 1}{\theta}\left\{AK^{\alpha} - \delta K + \frac{\rho - \theta n}{\theta n' g' \alpha A K^{\alpha-1}}\right.$$
$$\left. \times \left[\alpha A K^{\alpha-1} - \delta - \rho + \theta n\right]\right\},$$
$$\dot{K} = 0 \quad \Longleftrightarrow \quad C = AK^{\alpha} - \delta K.$$

The phase line $\dot{K} = 0$ with slope $\alpha A K^{\alpha} - \delta$ is strictly concave, reaches its maximum at $\bar{K} = (\delta/\alpha A)^{1/(\alpha-1)}$, and hits the horizontal axis at the origin and $\tilde{K} = (\delta/A)^{1/(\alpha-1)}$ as is illustrated in Fig. 2.

The phase line $\dot{C}/C = 0$ can adopt several shapes depending upon the value of the parameter θ. Since Hall (1988) has argued that empirical elasticities tend to be large, we assume $\theta > 1$, implying that the fraction $(\theta - 1)/\theta$ is positive.[3] The limits of $\dot{C}/C = 0$ for $K \to 0$ and $K \to \tilde{K}$ are

$$\lim_{K \to 0} C = \frac{\theta - 1}{\theta}\frac{\rho - \theta n(0)}{\theta n'(0) g'(0)} < 0,$$
$$\lim_{K \to \tilde{K}} C = \frac{\theta - 1}{\theta}\frac{\rho - \theta n[g(A\tilde{K}^{\alpha})]}{\theta n'[g(A\tilde{K}^{\alpha})] g'(A\tilde{K}^{\alpha}) \alpha A \tilde{K}^{\alpha-1}}$$
$$\times \left[\alpha A \tilde{K}^{\alpha-1} - \delta - \rho + \theta n[g(A\tilde{K}^{\alpha})]\right] < 0.$$

Because g' changes its sign at the *EKC* peak, the line $\dot{C}/C = 0$ has a point of discontinuity at $K = \mu$. Since

$$0 \leq AK^{\alpha} - \delta K < \infty \tag{16}$$

for all $K < \tilde{K}$, the limit behavior of $\dot{C} = 0$ depends on its rightmost expression

$$\frac{\rho - \theta n}{\theta n' g' \alpha A K^{\alpha-1}}\left[\alpha A K^{\alpha-1} - \delta - \rho - \theta n\right].$$

Thus, noting (2), (3), (4) and (8), it holds

$$\lim_{K \uparrow \mu} C = -\infty,$$
$$\lim_{K \downarrow \mu} C = +\infty.$$

[3] The case where $\theta < 1$ can, however, be considered analogously.

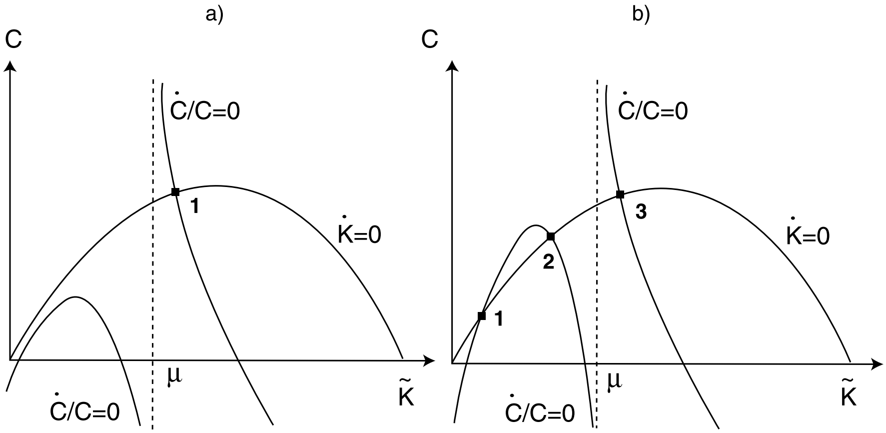


Fig. 1 The phase lines

All steady states, if they exist, are allocated on the upward sloping segment of $\dot{K} = 0$. For the *EKC* to be of economic interest, it should thus peak at lower values of K. Hence, we assume $\mu < \bar{K}$.

Finally, consider the slope of $\dot{C} = 0$ given by

$$\begin{aligned}\frac{dC}{dK} = \frac{\theta - 1}{\theta}\Bigg\{&\alpha A K^{\alpha-1} - \delta + \frac{d((\rho - \theta n)/\theta n' g' \alpha A K^{\alpha-1})}{dK} \\ &\times [\alpha A K^{\alpha-1} - \delta - \rho - \theta n] \\ &+ \frac{\rho - \theta n}{\theta n' g' \alpha A K^{\alpha-1}}[\alpha(\alpha - 1)A K^{\alpha-2} + \theta n' g' \alpha A K^{\alpha-1}]\Bigg\} \\ = \frac{\theta - 1}{\theta}\Bigg\{&2(\rho - \theta n) + \frac{\rho - \theta n}{\theta n' g'}(\alpha - 1)K^{-1}\Bigg\},\end{aligned}$$

implying that lim $K \to 0$ $dC/cK = +\infty$. Therefore, $\dot{C}/C = 0$ adopts an inverted-U shaped graph for $K < \mu$ but swings from ∞ to negative values when $K > \mu$. Inequality (16) affects the shape of $\dot{C}/C = 0$ in the vicinity of the K-axis. Figure 2 illustrates this.

Figure 1 shows that two cases are possible. The inverted-U part of the phase line $\dot{C}/C = 0$ can lie low enough to avoid the intersection with $\dot{K} = 0$. In this case, the number of interior steady states is one. Alternatively, the inverted-U part of $\dot{C} = 0$ may lie so high that it intersects $\dot{K} = 0$. In this case, the number of interior steady states is three.[4] The former is given in panel *a* and the latter in panel *b* in Fig. 1.

[4]The non-generic case in which $\dot{C}/C = 0$ is a tangent to $\dot{K} = 0$ is not analyzed. Because of the discontinuity and non-concavities in the phase lines, additional intersections can not be excluded a priori. The emission function may also exhibit several peaks, giving rise to several points of discontinuity without violating the basic structure of the model.

A standard local stability analysis in Appendix A shows that steady states 1 and 3 are saddles with stable branches running from the south-west and north-east while steady state 2 is an unstable focus or node. These results are summarized as follows:

Proposition *The problem* (2)–(7) *has at least one steady state* (K^*, C^*), *in which* $K^* > 0$, $C^* > 0$ *and* $K^* < \tilde{K} = (\delta/A)^{1/(\alpha-1)}$.

One can comprehend the model by comparing it with the standard model of optimal growth in which the central planner only faces the trade-off between current consumption and future consumption streams, whereas in the present model she also faces the trade-off between the future consumption streams and environmental deaths. Since high future consumption calls for emission-increasing investment and production, the planner evaluates current consumption against the future consumption net of induced deaths. In the optimum, she accepts some deaths as an exchange for sufficient consumption.

4 Air Pollution Mortality in Europe

CAFE, the Clean Air for Europe program and *WHO* have recently provided summary estimates of mortality caused by short-term exposure in Europe by collecting 629 time-series and 160 individual or panel studies that regress daily mortality against daily changes in outdoor air pollution (WHO 2004). These summary estimates show that there is a significant response in mortality to particulate matter (*PM*) and ozone.[5] Pope et al. (2002) have analyzed the effects of long-term *PM* exposures in the United States in a study in which questionnaires monitored individuals from 1982 onwards, making control for other risk sources possible. Their estimates were applied to the European data to derive the effects of long-term exposure; the short-term and long-term exposures together induced more than 300000 premature deaths in 2000 in Europe (WHO 2004).[6]

Although the available mortality numbers in WHO (2004) refer to several pollutants, most deaths are caused by particulate matter. Furthermore, as particulates are closely associated with other pollutants, they can be used as an *indicator* of outdoor air pollution (Cohen et al. 2004). Thus, we concentrate on particulate matter here. The data comes from Amann et al. (2007), who report the $PM_{2.5}$ emissions for 25

[5]Particulate matter, *PM*, consists of solid airborne particles of varying size, chemical composition, mainly generated by energy combustion (mobile or fixed site), often also from long-distance sources. Particulate matter is further classified according to its maximum diameter size, the main groups being $PM_{2.5}$ and PM_{10} with maximal diameters of 2.5 and 10 μm respectively.

[6]For methodological issues in epidemiological studies, see Chay et al. (2003). For studies on infant mortality, see Chay and Greenstone (2003) and Currie and Neidell (2005). For techniques for deriving country-level mortality estimates, see Ostro (2004).

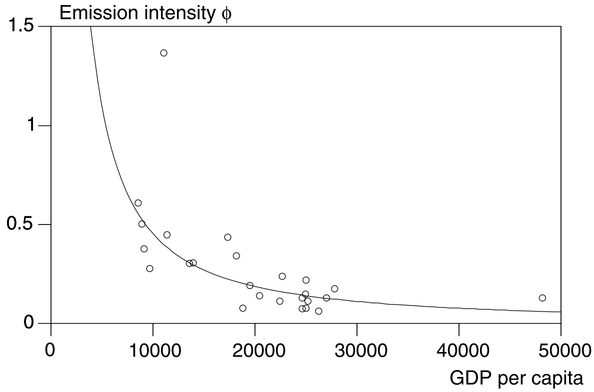


Fig. 2 Emission intensity and per capita *GDP* in 2000 in EU_{25}

European countries (EU_{25}) for the year 2000. Appendix B gives the list of countries together with mortality and emission data.

4.1 Generating the Missing Emission Data

The theoretical model claims that the emission function $E = g(Y) = g(GDP)$ should adopt an *EKC* consistent path, i.e., emissions should first rise and then fall as a function of output Y measured as the real gross domestic product *GDP*. To estimate such a function, several observations of E and *GDP* are necessary but, unfortunately, the $PM_{2.5}$ data above is available only for the year 2000. Therefore, our estimation strategy is to derive the country-specific formulas for $E = g(GDP)$ in two steps, both of which utilize the *EKC* approach. Namely, we first estimate the emission-output association from a cross-section of countries in 2000 and then we generalize this association to time series in individual countries.

For the cross-section, we calculate the emission intensities of outputs $\phi_{i,2000} = E_{i,2000}/GDP_{i,2000}$ for country i in 2000 to regress them against the values of the *per capita* domestic products ($GDPpc_{i,2000}$). Figure 2 shows that ϕ decreases as a function of *GDPpc*, implying that cleaner production methods are applied in richer countries. A suitable formula for the association seems to be $\phi = \gamma \cdot GDPpc^{\vartheta}$. Alternatively, one can try a hump-shaped curve, implying that the emission intensity should initially increase, even though such observations are not present in the current data consisting industrial countries alone.[7] The formula $\phi = \gamma \cdot GDPpc^{\vartheta}$ has the best fit with the data. Hence, by taking logs, we fit

$$\ln \phi_{i,2000} = \ln \gamma + \vartheta \cdot \ln GDPpc_{i,2000} + \varepsilon_i \tag{17}$$

by *OLS*, to derive the estimates $\gamma = 56298.77$ and $\vartheta = -1.27$. Model (17) explains 55% of the cross-country variation in ϕ.

[7] We also explored several other formulas, among them expressions for per capita emissions.

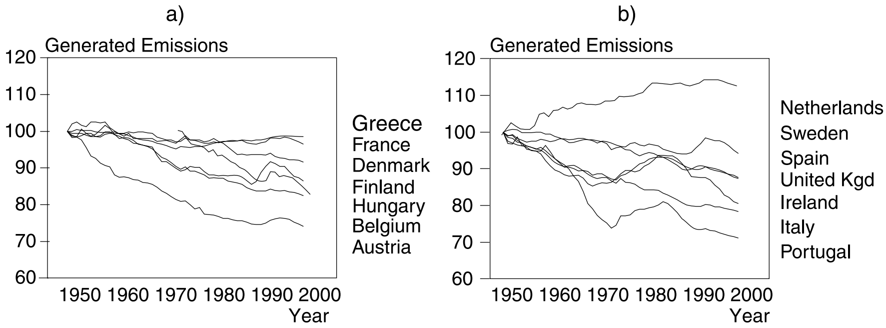


Fig. 3 The generated emissions data for EU_{14}; $1950 = 100$

To derive the country-specific time series from (17), note that $\phi = E/GDP$ implies

$$E = \phi \cdot GDP = \phi \cdot GDPpc \cdot L = 56298.77 \cdot GDPpc^{-0.27} \cdot L,$$

showing that the elasticity of emissions in terms of *GDPpc* (population) is negative (positive). Hence, knowing *GDPpc* and L one can calculate the time series

$$E_{i,t} = l_i \cdot 56298.77 \cdot GDPpc_{i,t}^{-0.27} \cdot L_{i,t}, \tag{18}$$

for each country with a multiplicative country-specific fixed factor $l_i = \varepsilon_i / \phi_{i,2000}$, derived from the residual error in (17). Equation (18) shows that, in spite of decreasing emission intensities, emissions themselves may increase or decrease, depending upon the growth rate of *GDPpc* and population L.

For most countries in EU_{25}, the *GDPpc* and population data are given from 1950 onwards in Heston et al. (2006), but for the former Soviet satellites, the *GDPpc* data only start from 1970 (or from 1993 in some cases). Furthermore, these data are markedly volatile, leading to violation of the parameter constraints $0 < \alpha < 1$ and $\rho - \theta\nu > 0$ (Hungary being an exception), while only Luxembourg violates these constraints among the group of the old EU members. For these reasons, we are able to perform a complete time series analysis only for 14 countries (EU_{14}), presenting 67% of the population in EU_{25}.

The generated $PM_{2.5}$ time series, indexed to $1950 = 100$, are shown in Fig. 3. Some of the series seem to have a peak soon after World War II but most show decreasing trends, interrupted by short booms in some cases. In The Netherlands, emissions are increasing exhibiting, however, a recent peak. Thus, the generated data does not conflict the *EKC*.

4.2 Estimating the Country-Specific Parameters

To apply cross-country results to a single-country model, some intermediate steps are usually needed and this holds in our case as well. Given that the cross-sectional comparison is meaningful only in terms of *per capita GDP*, the expression for emissions in (18) depends on $GDPpc$ and L. For mathematical reasons, however, our model takes emissions directly as a function of GDP, claiming that the function $E = g(GDP)$ satisfies (2) and (3), i.e., increases slowly, then peaks, and ultimately levels off. To estimate such a function, we take the generated time series for E as plain data, and regress them against the time series for GDP from each country. Several functional formulas satisfy the requirements of the theoretical model, maybe the simplest of them is

$$E(t) = \eta \cdot \mathrm{Exp}\left\{ -\left(\frac{GDP(t) - \mu}{\sigma} \right)^2 \right\},$$

where μ and η refer to the GDP and emissions at the EKC peak, respectively. In countries where the trend is downward sloping, the actual peak of the EKC remains unknown. In these cases, the peak is allocated to the earliest available year, but the observed peak is applied if available. For The Netherlands, the peak is allocated at 1996. By calculating $x_t = (GDP_t - \mu)^2$ for each year t and by taking logs, we estimate

$$\ln E_t = \ln \eta + s x_t + \varepsilon_t, \tag{19}$$

for each country separately to derive the country-specific estimates for η and $\sigma = -s^{0.5}$. All countries exhibit statistically highly significant values for σ. The estimates and the values for R^2 are reported in Table 1.

To evaluate A and α in the production function $Y = GDP = AK^{\alpha}$, we first apply the standard perpetual inventory method (Caselli 2004) to generate the capital stocks from 1950 to 2000 (from 1970 for Hungary) by accumulating investments (data from Heston et al. 2006). We assume that the depreciation rate $\delta = 0.05$ is the same in all countries. By taking logs, we can fit

$$\ln GDP_t = \ln A + \alpha \cdot \ln K_t + \varepsilon_t \tag{20}$$

for each country separately to derive the country-specific estimates for α and A (Table 1).

Consider next the demographic function $n = n(E)$ defined in (4). The linear function can be written as $n = n(E) = \nu - \beta E$. For the autonomous population growth ν, we adopt the country-specific average annual population growth rate from 1950 to 2000 (Table 1). Air pollution naturally has some effect on this number since urban air pollution used to be considerable in some cities, but given the long time-span and large population included, this effect seems negligible because environmental mortality constitutes only a small fraction of total mortality. Since βE is the death rate from air pollution, we calculate

$$\beta = \frac{\text{air pollution deaths}_{2000}}{\text{population}_{2000}} : E_{2000}$$

for each country to derive the country-specific estimates for β (Table 1).

Table 1 The parameters

Country	η	σ	R^2	A	α	R^2	ν	β	ρ	θ	R^2
Austria	34.77	364.35	0.66	1.32	0.78	0.99	0.0031	2.41E–05	0.055	6.04	0.86
Belgium	39.84	425.03	0.85	0.7	0.89	1.00	0.0035	3.81E–05	0.073	7.63	0.73
Denmark	29.37	313.99	0.84	1.21	0.78	0.99	0.0045	2.36E–05	0.08	7.64	0.77
Finland	32.57	222.69	0.64	0.76	0.82	0.99	0.0051	8.70E–06	0.046	4.62	0.62
France	347.59	7854.77	0.42	2.26	0.77	1.00	0.007	2.17E–06	0.059	6.8	0.90
Greece	48.7	546.28	0.08	1.6	0.73	0.99	0.0074	1.39E–05	0.079	4.89	0.64
Hungary	80.88	130.62	0.90	2.86	0.64	0.92	0.0018	2.41E–05	0.04	8.16	0.23
Ireland	16.62	209.32	0.40	1.1	0.8	0.97	0.0049	2.18E–05	0.048	8.54	0.59
Italy	184.97	2370.85	0.87	0.89	0.88	1.00	0.0041	5.85E–06	0.075	5.41	0.85
Netherlands	27.92	810.2	0.93	0.43	0.96	0.99	0.009	3.66E–05	0.066	6.86	0.77
Portugal	98.24	291.53	0.59	1.7	0.76	1.00	0.0039	6.42E–06	0.063	8.3	0.65
Spain	166.08	2832.71	0.25	2.74	0.73	1.00	0.0075	3.25E–06	0.051	6.2	0.65
Sweden	27.54	717.85	0.50	0.97	0.83	0.99	0.0047	1.46E–05	0.069	7.92	0.69
United Kd	126.22	2773.21	0.90	8.9	0.61	0.98	0.0033	6.16E–06	0.045	13.37	0.64

The country-specific average real interest from the post oil-crisis period (1983–2000) is chosen as the proxy for the time preference factor ρ (data from World Bank 2008). To estimate θ, consider equation (15). Unfortunately, (15) cannot be solved for θ, but we simplify it by setting $n' = 0$ to get $\dot{C}/C = (1/\theta)(\alpha A K^{\alpha-1} - \delta - \rho) + n$.[8] Hence, given the values for A, α, ρ, and δ and the data from Heston et al. (2006) we estimate $1/\theta$ from

$$\frac{\dot{C}_t}{C_t} - n_t = \frac{1}{\theta}(\alpha A K_t^{\alpha-1} - \delta - \rho) + \varepsilon_t \tag{21}$$

to derive the country-specific estimates for θ (Table 1). The F and t tests for the estimates in Table 1 show that all models and parameters are statistically significant.

4.3 Results

Given the estimated parameters, one can calculate the solution of the model and build projections of emissions and air pollution deaths for each country in EU_{14}.[9]

[8] We evaluated the average maximal error in EU_{14} from omitting the element $\frac{1}{\theta}\{\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n}(\frac{\theta C}{1-\theta} + AK^{\alpha} - \delta K)\}$ in (16) by noting that this term is an increasing function of θ and that the calibrated values for θ never exceed 15. Thus, by assuming θ=15 and calculating the omitted element for all countries we see that the average maximal error in the right hand side of (21) is 0.48%.

[9] All results are derived by the time-elimination method, in which the stable saddle path is calculated from the steady state (K^*, C^*) backwards to the origin (Mulligan and Sala-i-Martin 1991).

Table 2 The results

Country	$PM_{2.5}$ 2020	Death rate 2020	Deaths 2020	Deaths 2000–2020	Growth % 2000–2020
Austria	16.33	0.00039	3580	96869	2.05
Belgium	20.30	0.00077	8415	229168	2.19
Denmark	22.27	0.00053	2966	66104	1.42
Finland	21.03	0.00018	1003	24435	2.10
France	321.38	0.00070	45393	922930	1.85
Greece	42.11	0.00059	7535	155919	1.61
Hungary	42.74	0.00103	7052	211619	1.58
Ireland	12.79	0.00028	1058	23790	2.12
Italy	79.51	0.00047	28971	863269	2.58
Netherlands	21.09	0.00077	13786	325084	2.58
Portugal	38.13	0.00024	3061	86445	2.02
Spain	135.52	0.00044	19433	413681	2.06
Sweden	24.20	0.00035	3280	69350	1.64
United Kingdom	102.24	0.00063	37325	809758	0.98
Total/average	899.66	0.00053	182857	4298419	1.91

The time horizon chosen extends from 2000 to 2020. Only a single steady state arises in every EU_{14} country (panel a in Fig. 1). Simulated experiences show, however, that the empirical results derived below can be extended to multi-equilibrium cases (panel b in Fig. 1). The main reason is that the low and high-capital steady states are both saddles (Appendix A). The main results are shown in Table 2.

The first column in Table 2 shows the projected $PM_{2.5}$ emissions in 2020, its last row indicating that the total annual emissions in EU_{14} will decrease to 899.66 kilotons from 1097.34 kilotons in 2000 (Appendix B). To compare the country-specific values, we construct an index by normalizing the values for 2000 to 100. Figure 4 panel a indicates that emissions will decrease everywhere, the largest decrease taking place in Portugal and the smallest in France. Table 2 and Appendix B also show that the average death rate from air pollution will decrease to 0.00053 from 0.00069 in 2000. The country-specific indexed death rates are illustrated in Fig. 4, panel b.

The third column in Table 2 reports the number of air pollution deaths in 2020. A comparison with the data shows that the annual total in EU_{14} will decrease to 182857 from 220225 in 2000. Expressed as an index, this decrease is to 83.03. Figure 5 shows the indexed time paths for deaths for each country, indicating that most marked gains will be achieved by Italy, Portugal, and Hungary. On the other

The accuracy of the model, measured by its ability to meet the actual data point of air pollution deaths in 2000 is satisfactory, the average error in deaths being −3.84%. Mathematica 5.2 programs are available from the authors on request.

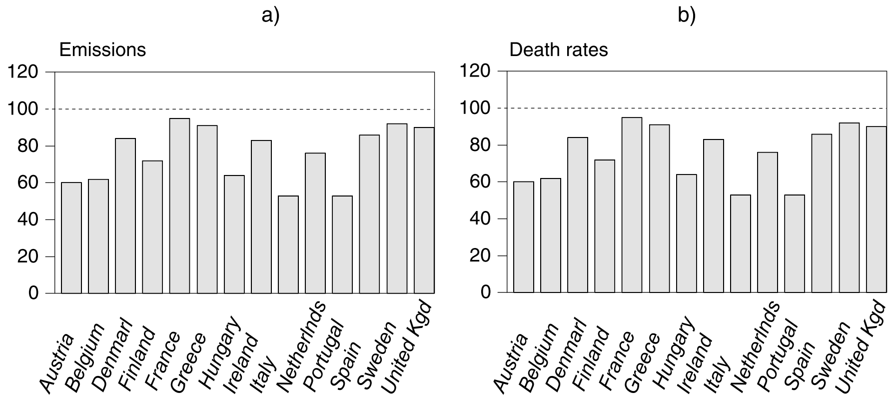


Fig. 4 The index of emissions and death rates in 2020. The value for 2000 is 100 for all countries

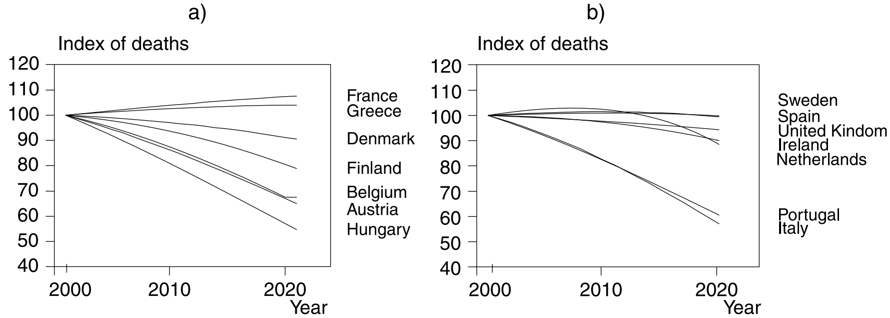


Fig. 5 Trends in air pollution deaths. The value for 2000 is 100 for all countries

hand, deaths will increase in France and Greece, where the decrease in emissions and deaths rates will be more than off-set by an increase in the population size, implying that the number of exposed individuals increases. In spite of the generally decreasing trends in deaths, the last row of the fourth column in Table 2 shows that the total number of air pollution deaths from 2000 to 2020 will accumulate to 4298419 persons in EU_{14}.

In general, the importance of population growth is considerable, even in an area like EU_{14}, where it is already relatively low. Figure 6 panel a plots the index of air pollution deaths in 2020 against the autonomous population growth rate (parameter ν), showing a strong positive association. One can see that the autonomous population growth rate is much higher in Greece (GRC), Spain (ESP) and France (FRC) than in Hungary (HUN), Italy (ITA), and Portugal (PRT), the former group exhibiting at most a marginal decrease in deaths, while the decrease in the latter group is large (Fig. 5, Table 2).

On the other hand, economic growth is also important because all economies show a downward-sloping emission trend, suggesting that the fast-growers should leap ahead in their EKC path. This is indeed confirmed by Fig. 6, panel b, which

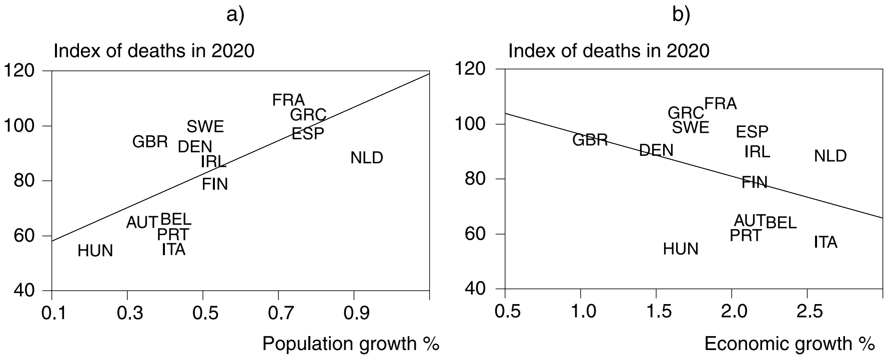


Fig. 6 The index of air pollution deaths in 2020 as a function of demographic and economic growth

plots the death index in 2020 against the projected average economic growth rate from 2000 to 2020 (Table 2, last column), revealing a negative association between these two. Comparison of panels a and b also shows several interesting cases. Italy (ITA), for example, gains a double advantage since its population growth is low and the projected economic growth rate is high. On the other hand, high economic growth in The Netherlands (NLD) will approximately off set the high demographic growth, although there will be a delay in the decrease in deaths, caused by a slow take-off after the recent peak (Fig. 5). The high death index in The United Kingdom (GBR), in turn, seems to be caused by the slow economic growth rate projected by the model.

In general, the projected economic growth rates will be smaller than in the past, a result which is expected because of the decreasing productivity of capital ($\alpha < 1$). Table 2 (last column, las row) shows that from 2000 to 2020 the average annual economic growth rate will be 1.91%, while it was 2.87% from 1950 to 2000 (from 1970 to 2000 for Hungary). Hence, the projected deceleration is considerable, implying that the here-calculated decreases in emissions and death rates are small rather than large.

All results here are based on the assumption that $\delta = 0.05$, so that they are subject to some uncertainty. To evaluate the magnitude of this uncertainty, we re-ran all estimates for $\delta = 0.04$ and $\delta = 0.06$. Figure 7 shows the time paths of the total annual deaths in EU_{14} for these alternative values, both as numbers and as an index, the latter showing that in 2020 the totals in EU_{14} are 87.13 (83.03) 78.65 for $\delta = 0.04$ ($\delta = 0.05$) $\delta = 0.06$ respectively. Hence, deaths decrease as δ increases because higher depreciation decreases the calculated capital stock for 1950–2000 (1970–2000), increasing the estimates for the productivity parameters in (20) which, in turn, implies higher growth for the period 2000–2020. One can also evaluate the sensitivity of the results in terms of other parameters. We give θ as an example, showing that the 95% confidence limits for total deaths in 2020 are 171904...194715 with a mean of 182857; i.e., the sensitivity of the results is not very considerable for θ.

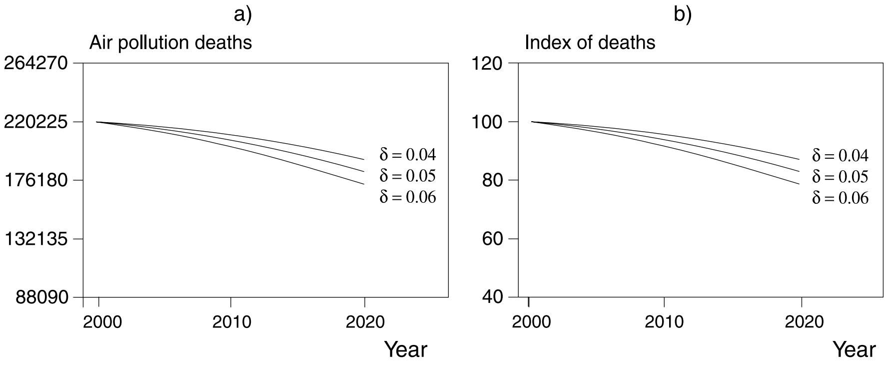
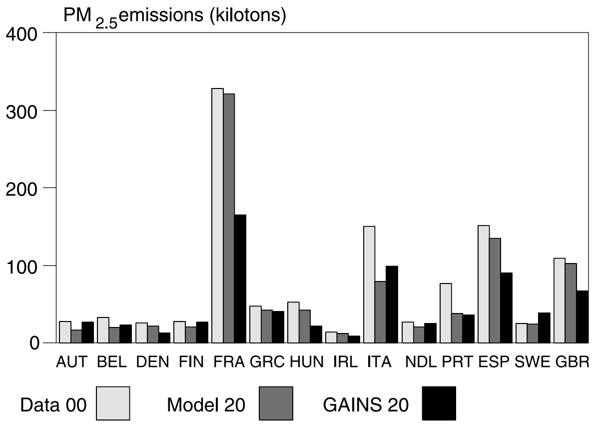


Fig. 7 Sensitivity analysis in terms of δ; total number and index of deaths

Fig. 8 Comparison of emissions



To put our analysis into a more general framework, it is necessary to compare it with other studies. One suitable source is the Regional Air pollution the INFormation and Simulation model RAINS (currently called GAINS once amended for greenhouse cases), which is a large-scale simulation model constructed to validate the emission data from the EU Member States (Amann et al. 2007). Depending on the scenario details, GAINS gives several projections for $PM_{2.5}$, among which Fig. 8, which also shows the data from 2000, compares the *Current Legislation* scenario with our results for 2020. The total emissions in EU_{14} in 2020 in our model (899.65 kilotons) will be larger than in GAINS (686.83 kilotons), the difference having its source in three countries, France (FRA), Spain (ESP) and The United Kingdom (GBR), while the other countries show almost similar numbers. One of the future challenges thus is to pay special attention to those countries where the difference between these two approaches is most significant.

5 Conclusions

There is emerging evidence that environmental degradation adds human mortality and a common fear is that economic growth exacerbates the number of environmental deaths by increasing degradation. To evaluate whether this fear is justified, we provide a long-run consumer optimization model where mortality is endogenous to emissions, which are assumed to follow the *EKC* path, first rising but then falling along with output.

The parameters of the model are estimated to accommodate European economic and air pollution data, showing that countries in Europe have reached the downward sloping segment of *EKC*. Economic growth will, therefore, decrease air pollution and the associated death rates in all fourteen countries for which the analysis is possible. In some countries, however, population growth is so high that the number of deaths will increase. The total number of air pollution deaths from 2000 to 2020 accounts to more than four million in these countries.

Several improvements to the current model are possible, but its simplicity is also an advantage. The simplicity of the production function, for example, reminds us why environmental deaths have been discussed so little: if no considerable productive feedbacks arise, as may be the case in air pollution deaths, then these deaths are human and welfare problems alone, indicating that sufficient attention may not be paid on them. On the other hand, the simplicity of the Benthamian utility function puts things very bluntly: only the total utility matters, and a situation in which some people suffer and die but others go on happily consuming ever more may well be optimal. Policy measures, such as emission limits and international agreements, are thus needed to decrease emissions faster than this study implies, and to prevent the total number of environmental deaths from growing to the kinds of numbers calculated in this paper.

Appendix A: Local Stability of the Steady States

Consider the system of (6) and (15). To simplify the notations, write $\dot{K} = \varphi(K, C)$ and $\dot{C}/C = \psi(K, C)$. The Jacobian of the system is

$$J = \begin{bmatrix} \varphi_K & \varphi_C \\ \psi_K & \psi_C \end{bmatrix}.$$

As evaluated around the steady state, its elements become

$$\begin{aligned}
\varphi_K &= \alpha A K^{\alpha-1} - \delta, \\
\varphi_C &= -1, \\
\psi_K &= \frac{1}{\theta}\left\{ \frac{d[\theta n' g' \alpha A K^{\alpha-1}/(\rho - \theta n)]}{dK} \left(\frac{\theta C}{1-\theta} + A K^{\alpha} - \delta K \right) \right. \\
&\qquad \left. + \frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho - \theta n} (\alpha A K^{\alpha-1} - \delta) + \alpha(\alpha - 1) A K^{\alpha-2} - \theta n' \right\}
\end{aligned}$$

$$= \frac{1}{\theta}\Bigg\{\frac{(\theta n' g' \alpha A K^{\alpha-1})^2}{(\rho-\theta n)^2}\left(\frac{\theta C}{1-\theta} + A K^{\alpha} - \delta K\right)$$
$$+ \frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n}(\alpha A K^{\alpha-1} - \delta) + \alpha(\alpha-1) A K^{\alpha-2} - \theta n'\Bigg\},$$
$$\psi_C = \frac{1}{1-\theta}\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n}.$$

Because ψ_K can be of either sign, we write

$$\begin{aligned}\mathrm{DET}\, J &= \varphi_K \cdot \psi_C - \psi_K \cdot \varphi_C \\ &= \left[\left(-\frac{\varphi_K}{\varphi_C}\right) - \left(-\frac{\psi_K}{\psi_C}\right)\right](-\varphi_C)\cdot \psi_C,\end{aligned}$$

where the square brackets give the difference in the slopes of $\dot{K}=0$ and $\dot{C}/C=0$. Consider the single steady states depicted in Fig. 1, panel a. Since this steady state is allocated after the *EKC* peak, g' is negative so that the expression $(-\varphi_C)\cdot\psi_C = \frac{1}{1-\theta}\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n}$ is negative. Since $\dot{C}/C=0$ hits $\dot{K}=0$ from above, the expression in the square brackets is positive, implying $\mathrm{DET}\, J < 0$. Hence, the single steady state is a saddle. By analogous reasoning, steady state 3 in panel b is also a saddle. In steady state 1 shown in panel b, $\dot{C}/C=0$ hits $\dot{K}=0$ from below, making the expression in the square brackets negative. But since $\frac{1}{1-\theta}\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n} > 0$ $(g' > 0)$, this steady state is again a saddle.

In steady state 2 in panel b, $\dot{C}=0$ hits $\dot{K}=0$ from above (square brackets positive) and $\frac{1}{1-\theta}\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n} > 0$ so that $\mathrm{DET}\, J > 0$. Since all steady states are allocated on the upward sloping segment of $\dot{K}=0$ with $\alpha A K^{\alpha-1} - \delta > 0$, and since $g' > 0$ for steady state 2, the trace of J, given by

$$\begin{aligned}\mathrm{TR}\, J &= \varphi_K + \psi_C \\ &= \alpha A K^{\alpha-1} - \delta + \frac{1}{1-\theta}\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n}\end{aligned}$$

is positive, implying that steady state 2 is unstable. Since the sign of $(\mathrm{TR}\, J)^2 - 4\cdot \mathrm{DET}\, J$ is unknown, steady state 2 can be either a focus or a node.

The dynamics outside the steady state are: because $\varphi_C = -1$, the capital stock increases (decreases) below (above) phase line $\dot{K}=0$. The behavior of consumption is given by $\psi_C = \frac{1}{1-\theta}\frac{\theta n' g' \alpha A K^{\alpha-1}}{\rho-\theta n}$. Consumption thus increases (decreases) above (below) the phase line $\dot{C}/C=0$ for $K<\mu$, whereas for $K>\mu$ this behavior is reversed. Hence, the stable saddle path which starts from the origin approaches the saddle-stable steady state (steady states) from the south-west whereas the other approaches it from the north-east.

Appendix B: Countries and Variables

Country	Isocode	$PM_{2.5}$	Death rate	Deaths
Austria	AUT	28.18	0.00068	5508
Belgium	BEL	32.86	0.00125	12904
Denmark	DNK	25.97	0.00061	3274
Finland	FIN	28.26	0.00025	1272
France	FRA	328.23	0.00071	42202
Greece	GRC	47.32	0.00066	7242
Hungary	HUN	52.38	0.00126	12895
Ireland	IRL	14.16	0.00031	1174
Italy	ITA	150.27	0.00088	50766
Netherlands	NLD	26.78	0.00098	15573
Portugal	PRT	76.99	0.00049	5053
Spain	ESP	151.14	0.00049	19976
Sweden	SWE	25.40	0.00037	3284
United Kingdom	GBR	109.40	0.00068	39543
EU_{14} (total/average)		1097.34	0.00069	220225
Cyprus	CYP	2.18	0.00030	231
Czech Rep.	CZE	42.69	0.00088	9086
Estonia	EST	21.69	0.00044	631
Germany	GER	159.86	0.00091	75150
Latvia	LVA	10.93	0.00055	1334
Lithuania	LTU	12.50	0.00061	2197
Luxembourg	LUX	2.73	0.00074	321
Malta	MLT	0.59	0.00049	193
Poland	POL	202.70	0.00085	32944
Slovak Rep.	SVK	14.50	0.00079	4265
Slovenia	SVN	12.08	0.00082	1582
EU_{25} (total/average)		1579.79	0.00068	348600

All numbers refer to year 2000. $PM_{2.5}$ emissions in kilotons (Amann et al. 2007), Deaths refer to air pollution induced deaths (WHO 2004). In addition, annual series from 1950 (from 1970 for Hungary) for population, GDPpc, and investments (Heston et al. 2006), and annual series from 1983 for real interest rates (World Bank 2008)

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Development of Transportation Infrastructure in the Context of Economic Growth

Manuel Benjamin Ortiz-Moctezuma, Denis Pivovarchuk, Jana Szolgayova, and Sabine Fuss

Abstract Developed road infrastructure is one of the main ingredients to economic growth. At the same time, economic growth enables further expansion of infrastructure. The co-evolutionary aspects of the growth of economic output and road infrastructure are thus apparent and represent the main motivation for the study presented in this chapter. We develop a model analyzing the interdependence between a country's economic growth and the development of transportation infrastructure in this country, explicitly taking into account the mutual influence of the rate of economic growth and the transportation capacity. Formulating an optimal control problem, the optimal investment rate can be determined. This model forms a comprehensive framework for understanding the underlying dynamics and the patterns of economic growth in relation to transport infrastructure. An analytical solution for the infinite horizon problem is derived and the steady state is shown to depend crucially on the rate of physical decay of roads. Testing the model for the data of two countries illustrates the usefulness of such an approach to real world problems and possibly policy recommendations, even though the model would have to be adapted to the specific characteristics of each country or region to make precise statements.

1 Introduction

Developed road infrastructure is an essential factor facilitating and accelerating economic growth, which will in turn enable the addition of more roads. At the same time, the marginal benefit of adding roads to a large stock of existing capacity might be diminishing. It is thus evident that the co-evolution of economic output and road infrastructure is rather intricate and deserves special attention. The model developed in this chapter therefore investigates the interdependency between a country's economic growth and the development of transportation infrastructure in this country. To this end, a co-evolutionary perspective is developed, where the mutual influence of the rate of economic growth and the capacity of transportation infrastructure are

M.B. Ortiz-Moctezuma (✉)
Dynamic Systems Program (DYN), IIASA, Schlossplatz 1, 2361 Laxenburg, Austria
e-mail: moctez@iiasa.ac.at

M.B. Ortiz-Moctezuma
Ciudad Victoria, Tamaulipas, Mexico

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explicitly taken into account. This approach enables us to set up an optimal control problem, where the optimal investment rate is determined considering the co-evolutionary dynamics of GDP growth and capacity expansion. This model forms a comprehensive framework for understanding the underlying dynamics and the patterns of economic growth in relation to transport infrastructure.

Following the seminal work by Aschauer (1989), interest in the relationship between economic growth and infrastructure had been rekindled and, as a consequence, a large body of mainly empirical studies emerged from the effort of the research community—also in response to demand for better insights from the side of policy-makers. Gramlich (1994) and later Button (1998) provide in-depth reviews of this work and also some valid criticism with respect to both methodology and—more importantly—the underlying mechanism of the dynamics.[1] One debate in the latter context is, for example, the question of causality involved in these processes, i.e. whether economic growth is accelerated by increases in the stock of infrastructure or whether additions to existing infrastructure are caused by enhanced economic growth. Methodologically, a point of criticism is that cross-country analysis is barely possible given the differences in measurement practices and infrastructure definitions in official accounting data. To this, add the differences in maintenance and utilization of infrastructure. More specifically, Gramlich (1994) claims that a sectorial view has to be taken, rather than an aggregate perspective, which would only give blurred results, as different *types* of infrastructure affect growth to varying extents and in different ways. Finally, it is not even clear what the best approach to such empirical estimates is: it is questionable whether it is even admissible to employ a production function as many authors do, for instance. Compared to what might reasonably be expected (also according to evidence at the micro-level) in terms of rates of returns, many studies' estimates are too high (see Button, p. 153).

Even though these points are all well taken and should be kept in mind as caveats, we are still convinced that it is of importance to pursue the topic, since if there is any conclusion to be drawn from the existing literature it is that there is a significant relationship between GDP and infrastructure (no matter what the precise nature is) and that this is highly policy-relevant from a development perspective and also in terms of the efficiency of ongoing production processes (i.e. infrastructure is not only an independent input into the production process, but also has an indirect effect on economic growth by enhancing the marginal products of other forms of capital, labor, energy and materials). Let us thus first begin with a definition of what infrastructure means in the context of our work and how we like to place our contribution within the range of the existing literature.

Button (1998) lists in his review a number of definitions of infrastructure, which range from very specific to highly aggregate and thus also quite vague notions sometimes. The one that appears most inclusive, yet precise, to us is the one by Hirschman (1958): "[...] it includes all public services from law and order through education and public health to transportation, communications, power and water supply as

[1] Gramlich (1994) discusses most lines of criticism and Button (1998) summarizes and extends this list.

well as agricultural overhead capital as irrigation and drainage systems. The core of the concept can probably be restricted to transportation and power" (Button, p. 150). We agree largely with this definition and have decided to focus on the first type of these "core" infrastructures, which is transportation, for our application. In particular, we concentrate our analysis on road infrastructures, which is of course an arbitrary choice. However, our main purpose is not to conduct another empirical study, but rather to introduce a new modeling approach, so our contribution is more on the theoretical side and all empirical implementations have only been conducted for the sake of demonstration and verification of applicability and usefulness for practical analysis. It is thus possible to use the method for any other type of infrastructure as well, given that the relationships between the variables in the model are adapted to the new infrastructure context.

Button (1998) distinguishes in his review of the infrastructure and growth literature between two "schools", where the first one is referred to as the Keynesian approach, which starts from the notion that any income or infrastructure can only be generated by economic growth itself in the first place. The other school is that of the neo-classical approach, which treats infrastructure as a production factor in the same style as labor and capital and which belong mainly to the literature of endogenous growth modeling. Fedderke et al. (2006), for example, carry out a time-series analysis for investment into road infrastructure and economic growth in South Africa and find that the former does indeed lead to enhanced economic growth, both by boosting GDP directly and by raising the marginal products of other production factors. They also test for the other direction of causality (i.e. from GDP growth to infrastructure expansion), but the evidence is significantly weaker in this case.

It is not entirely clear to which "camp" our approach developed here belongs: on the one hand, we develop a model, where the amount of newly added infrastructure is the control, which we optimize to foster growth and eventually reach a steady state. On the other hand, we adopt a co-evolutionary perspective taking on the view that the level of GDP and the stock of infrastructure develop simultaneously, thereby enhancing each other. In any case, we abstract from major secondary effects, such as pointed out by Button (1998), Sharp (1980), who claims that road infrastructure will not benefit regions, where the new roads simply serve transit traffic, or where producers do not have a comparative advantage over their newly accessible trade partners. Since our study takes the point-of-view of the social planner at the aggregate level, such considerations, which matter for the distribution of gains from infrastructure expansion *within* the country, are first neglected, even though we do not want to downplay their importance for further research.

Coming back to the issue of policy relevance, many of the empirical studies conducted so far have been motivated by the need to form policy recommendations targeted at exploiting the potential of infrastructure to foster economic growth. Liberini (2006), for example, employs an econometric approach to estimate the so-called "infrastructure gap", which is defined as the difference between infrastructure demand based on potential GDP[2] and the level of infrastructure that is actually

[2]Potential GDP is that level of output that could be produced if all production factors could be used to their fullest extent.

provided. The aim of Liberini's (2006) study is to determine the impact that government retrenchment on public funding of infrastructure had in the Latin American countries that were also affected by the debt crisis and to compare these results with infrastructure gaps computed for other regions like the OECD and East Asia, as far as data availability permits. Her reasoning with respect to the infrastructure gap provides a justification for our approach to set up a co-evolutionary model: in Liberini's (2006) framework, public infrastructure investment[3] exceeding the optimum evokes a negative correlation between economic growth and infrastructure expansion and vice versa, so that the idea that GDP and infrastructure co-evolve and are drawn towards a steady state (either from below or from above) appears quite intuitive. Liberini's (2006) findings indicate a positive elasticity of per capita GDP with respect to telecommunications and road capacity. Furthermore, she tests for the significance of infrastructure quality indices, which are found to be negative, implying that the existence of higher quality infrastructure will reduce the need for further expansions in the short term. In our study we also investigate the importance of quality differences modeled through variations in the rate of physical decay of roads. As regards the infrastructure gap, Liberini (2006) concludes that it has been increasing during the period of the debt crisis in many Latin American countries (and in most sectors), while East Asia seems to have maintained their gap at a stable level (and if better data were available, this would improve the results for East Asia even more, as some well-performing countries are not accounted for in the available data set used in the study). OECD countries can generally be reported with constant or even shrinking gaps. These results hold for telecommunications and power; however, in the case of road infrastructure a decrease in infrastructure gaps can be observed across all regions, which might point to the fundamentality of road infrastructure compared to other types of infrastructure and thus further justifies our choice to focus on roads in this chapter.

Our contribution is more of a theoretical nature, even though we also apply the developed framework to data from some OECD countries[4] in order to show that the approach can also be useful to gain insight into practical situations or in order to derive policy recommendations given specific conditions. Our results show that we can derive an analytical solution to the problem of optimal infrastructure expansion, for some pre-specified functional relationships between GDP, maintenance and investment costs and existing stocks and changes in stocks of infrastructure.[5] We find an analytical solution for the infinite horizon problem, where the control turns out to be a constant. The steady state is shown to depend crucially on the rate

[3]Liberini (2006) also mentions that private investment in infrastructure was not sufficient to counterbalance the retrenchment of public funds in Latin American countries. Since we take an aggregate view of the problem, we refrain from an explicit distinction between private and public investment as well.

[4]We present the cases of Finland and France here for illustrative purposes.

[5]These functional relationships can of course be changed, should the particular circumstances and characteristics of a country require so. We have here tried to come up with the most basic and intuitive reasoning to illustrate the usefulness of the co-evolutionary and optimal control approach.

of physical decay of roads, which we think can be interpreted as an index of quality, and the speed of adjustment, at which the economy moves along a trajectory. Testing the model for the data of two countries, France and Finland, illustrates the usefulness of such an approach to practical problems and possibly policy recommendations, where the model would have to be adapted to the peculiarities of each country or region to make precise statements. For the (more impressionistic) country studies presented here, both France and Finland are below their steady states, although France is rather close to it, while Finland is relatively farther removed. An increase in quality modeled through lower depreciation of the existing infrastructure stock is shown to lead to a higher steady state, which implies that a higher level of GDP can be reached in the long run. In this context, another important insight is the dependence of the results on the parametrization, in particular the tradeoff between the speed of adjustment, with which GDP approaches the asymptote, and the rate of decay of the existing road stock, as mentioned before.[6]

The chapter is organized as follows. In Sect. 2 the optimal control model is developed, motivated by a co-evolutionary perspective on the interactive development of infrastructure and GDP. We manage to solve the model analytically for the long run, i.e. when the planning horizon is infinite.[7] The results for two exemplary countries are presented and briefly discussed in Sect. 3. Section 4 summarizes the findings of this study, discusses their relevance and gives an outlook to future research in this area.

2 Optimal Control Approach to Infrastructure Investment & Economic Growth

2.1 Model

The model presented here is essentially based on the assumption that there is a strong interdependency between the capacity of transportation infrastructure and economic growth. Adopting this assumption we introduce a model of co-evolutionary dynamics that qualitatively describes how the development of transportation infrastructure affects the rate of economic growth and vice versa. The qualitative co-evolutionary model is in turn used to construct a control model of development of transportation infrastructure in the context of economic growth.

We assume that the capacity of the country's road infrastructure, z, creates a basis for the country's GDP growth and introduce the threshold function $f(z)$ that

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[6]Finally, it is important to note that the role of other economic factors is not the focus of this study, but this should not be mistaken to imply that they are considered to be constant. On the contrary, all production factors rather evolve *proportionally*.

[7]The Appendix gives an overview of the methods used to calibrate the core equations presented in Sect. 2.

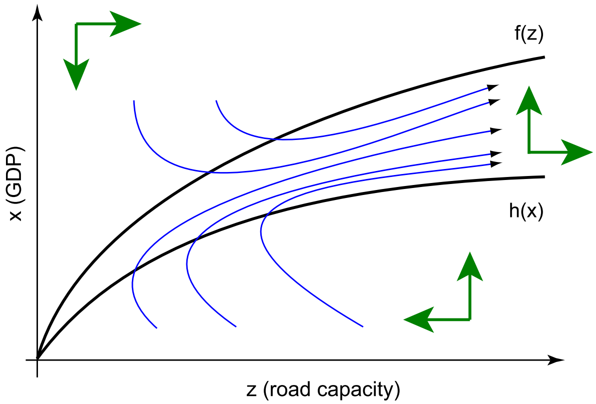

Fig. 1 Co-evolutionary dynamics

characterizes the maximal level of GDP possible at a given road capacity, z.[8] If the current level of GDP, x, is below the baseline, $x < f(z)$, then GDP grows. If the level of GDP is above the baseline, it decreases. Symmetrically, we assume that the level of a country's GDP, x, determines the development of the country's road infrastructure and introduce the threshold function $h(x)$ characterizing the size of the road capacity that can be supported by a given level of GDP, x. If the current level of GDP is too low for the existing road capacity, $z > h(x)$, then the size of road capacity decreases due to physical decay, as there is not sufficient investment to support the stock of road infrastructure. Conversely, if the current level of GDP can support a larger size of road capacity, then the capacity increases. Obviously, $f(z)$ and $h(x)$ are monotonically increasing functions.

Figure 1 shows how the phase diagram corresponding to the co-evolutionary model looks like. The threshold functions split the diagram into three regions: above the baseline $f(z)$, between the baselines $f(z)$, $h(x)$, and below the baseline $h(x)$. For each region, the directions of change for road capacity and GDP are indicated.

Based on the co-evolutionary model described, we construct a control-theoretic model of the development of road infrastructure. Assuming road capacity, z, to change over time, we get

$$\dot{z}(t) = u(t) - \delta z(t). \tag{1}$$

Here $z(t)$ is the road capacity at time t, $u(t)$ is its growth rate at time t and δ is the depreciation rate. We set

$$z(0) = z^0, \tag{2}$$

$$0 \leq u(t) \leq \bar{u}, \tag{3}$$

[8]As noted in the previous section, other economic factors are not constant, but rather evolve proportionally. This is consistent with the possible interpretation of the threshold function as a Cobb-Douglas production function.

where z^0 is road capacity at the initial time, 0, and $\bar{u}$ is the maximal possible growth rate of the road capacity. Assuming the level of a country's annual GDP, x, to depend on road capacity and assuming that the maximal possible level of GDP provided by existing road capacity, z, is determined by the threshold function $f(z)$, we arrive at

$$\dot{x}(t) = \gamma(f(z(t)) - x(t)), \tag{4}$$

where $x(t)$ is the level of GDP at time t and γ is a coefficient of the speed of adjustment. We set

$$x(0) = x_0, \tag{5}$$

where x^0 is the initial level of GDP.

Let $c(z)$ be the annual cost of maintaining road capacity z and $r(u)$ be the cost of increasing road capacity by an amount u in one year. Obviously, $c(z)$ and $r(u)$ are again monotonically increasing functions. It is reasonable to assume that c and r go to infinity as z and u do. The country's annual benefit is given by

$$b(z, x, u) = \mu x - c(z) - r(u), \tag{6}$$

where μ is the portion of GDP composed of road infrastructure. In Sect. 2.3 there will be further explanations on the way to calibrate μ and on the range in which we can expect this parameter to lie. Assuming an integrated benefit discounted at rate ρ,

$$J = \int_0^\infty e^{-\rho t} b(z(t), x(t), u(t)) dt, \tag{7}$$

to be the country's utility, we end up with an optimal control problem.

$$\begin{aligned} &\text{maximize} \quad J = \int_0^\infty e^{-\rho t} b(z(t), x(t), u(t)) dt \\ &\text{subject to} \quad (1)\text{–}(5). \end{aligned} \tag{8}$$

2.2 *Specifying Functions*

The most coherent data set with the longest cross-country time series for road length and other indicators for road infrastructure was compiled by Canning (1998, 1999). Figures 2 to 4 below display these data plotted against GDP, normalized with respect to the value in some given year. It is evident that the relationship is positive and in most cases close to linear. The other two figures show normalized road traffic and road energy consumption against GDP, respectively. These relationships confirm the previous observations.

Here we specify the functions needed for the control model: the threshold function, $f(z)$, for the country's GDP depending on the existing capacity of road infrastructure; the cost of expanding the infrastructure, $r(u)$, depending on the level of investment into new infrastructure; and the cost of maintaining the infrastructure, $c(z)$.

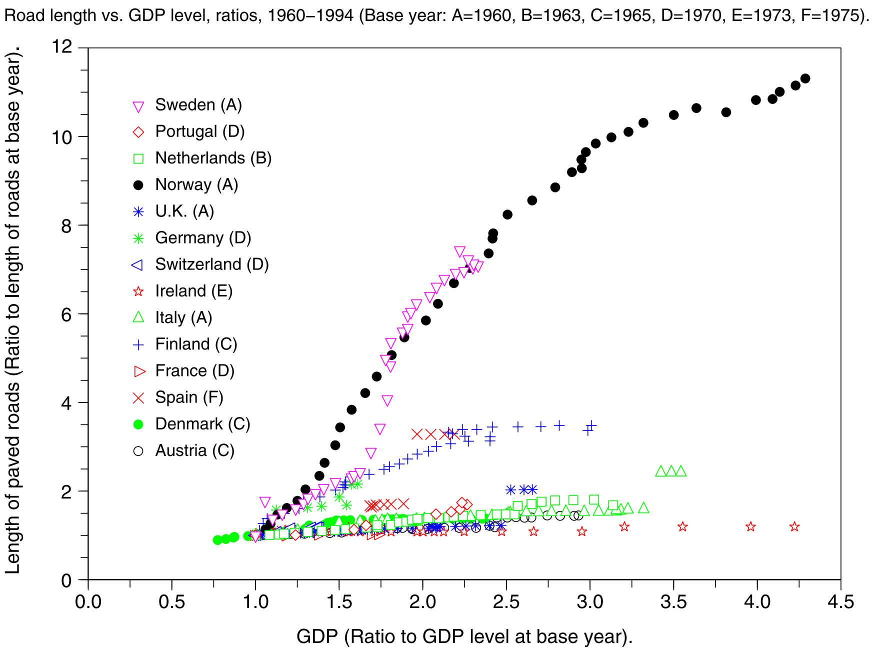

Fig. 2 Road length against the level of total GDP, ratios with respect to reference year

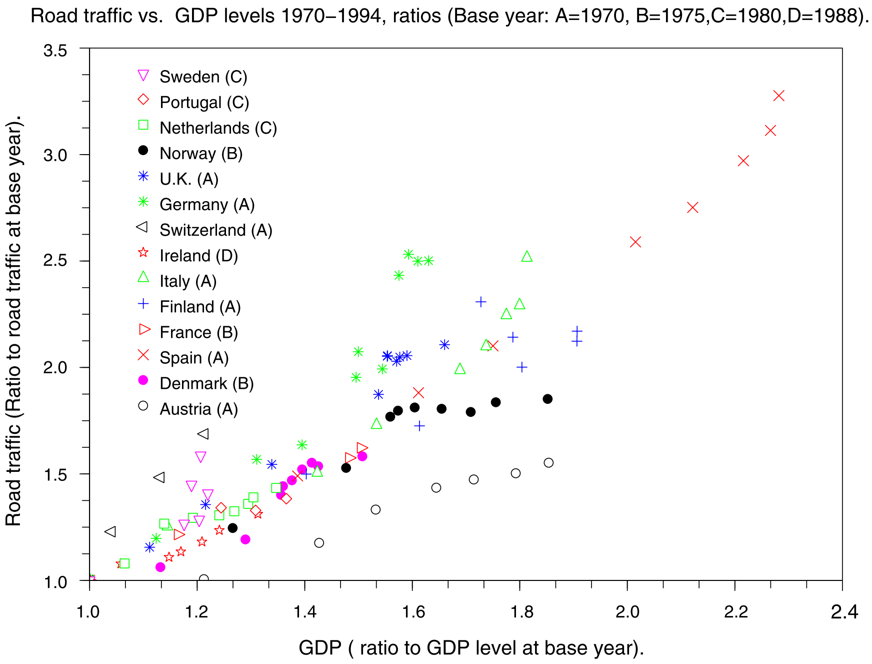

Fig. 3 Road traffic against the level of total GDP, ratios with respect to base year

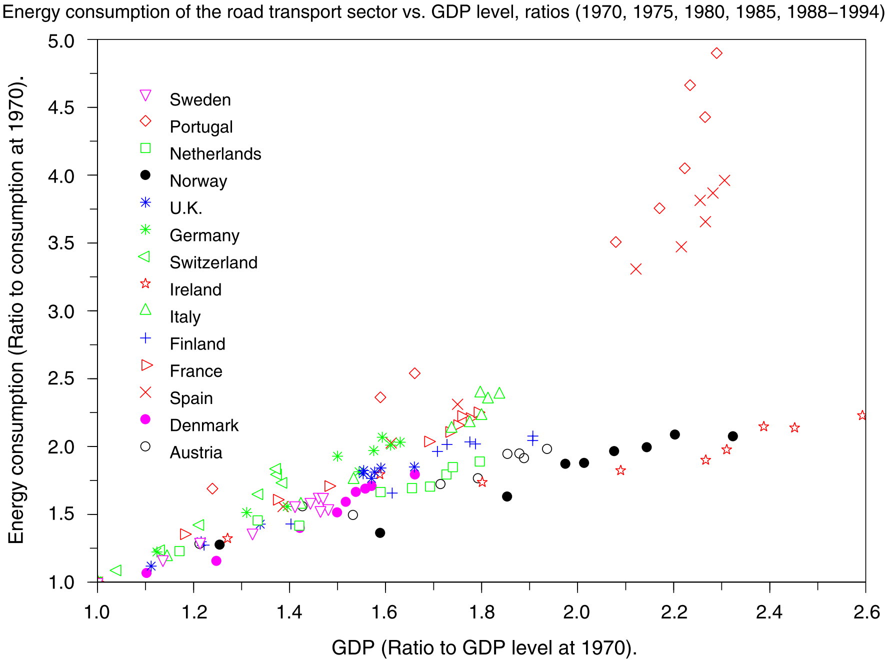

Fig. 4 Road transport energy consumption against the level of total GDP, ratios with respect 1970

1. Denote by $g(z)$ the function for the country's GDP, depending on the existing capacity of road infrastructure. This function is introduced to reflect the dependency between the country's GDP and the capacity of infrastructure based on statistical data. Looking at Figs. 2–4, we assume that it is a linear function, so

$$g(z) = \alpha z + \beta. \tag{9}$$

   The coefficients α and β will be calibrated for every country separately using statistical methods.
2. The threshold $f(z)$ is assumed to be a linear function

$$f(z) = Az + B. \tag{10}$$

3. In order to specify the function of the cost of investing into infrastructure, $r(u)$, we make the reasonable assumption that small amendments to existing infrastructure are relatively inexpensive, while setting up a major, new capacity item or a whole infrastructure system in the first place is much more costly. Moreover, we impose a restriction that no investment must also imply no cost, i.e. $r(0) = 0$. Hence, we infer that $r(u)$ is an exponential function:

$$r(u) = Le^{\theta u} - L, \tag{11}$$

   where L and θ are positive constants.

4. Considering an example of maintaining (e.g. through repairing) roads, we assume that the maintenance cost as a function of existing infrastructure is a linear function

$$c(z) = Dz + E, \tag{12}$$

where D and E are constants.

2.3 The Share of Road Infrastructure in Economic Output

The parameter μ is the portion of the GDP, which can be attributed to road infrastructure and so it can be interpreted as the importance of the role that road infrastructure plays in total economic output, the other contributing factors being labor, resources, other types of physical capital, energy, human capital and so forth. In the country case studies presented in the later sections, we have used a value of 5% as an—admittedly cautious—benchmark, since we did not want to overstate the effect of road infrastructure on total GDP in the face of relatively little constraints on that relationship. Table 1 shows, however, that μ could potentially be higher than that.

The data on total energy consumption were provided by BP p.l.c. (Statistical Review of World Energy, 2008), while the data on energy consumption in the road sector was taken from Madison (2001). It is necessary to estimate the ratio μ of GDP output, which is attributable to the activity in the road transportation sector. To this end, we refer to the plot in Fig. 4, which shows a linear relation between

Table 1 Ratio of energy used in road sector to total energy consumption

COUNTRY	1980	1985	1989	1990	1991	1992	1993	1994
Austria	0.154	0.154	0.168	0.168	0.177	0.180	0.174	0.182
Switzerland	0.131	0.142	0.168	0.168	0.170	0.171	–	–
Germany	0.113	0.115	0.136	0.147	0.154	0.159	0.165	0.163
Denmark	0.117	0.151	0.182	0.187	0.169	0.187	0.176	0.172
Spain	0.137	0.153	0.190	0.198	0.204	0.211	0.210	0.212
France	0.147	0.152	0.167	0.168	0.163	0.165	0.166	0.171
Finland	0.120	0.133	0.155	0.159	0.150	0.154	0.151	0.149
Italy	0.147	0.171	0.187	0.189	0.190	0.200	0.208	0.207
Ireland	0.197	0.199	0.181	0.180	0.177	0.192	0.185	0.184
Norway	0.065	0.067	0.069	0.067	0.076	0.073	0.074	0.076
Netherlands	0.083	0.086	0.095	0.094	0.091	0.097	0.098	0.100
Portugal	0.176	0.174	0.191	0.193	0.202	0.214	0.221	0.225
Sweden	0.113	0.107	0.130	0.121	0.122	0.126	0.125	0.131
UK	0.131	0.144	0.171	0.175	0.170	0.173	0.173	0.177

countries' total GDP and energy consumption of the road sector as well as Table 1, which shows the ratio of energy used in the road sector to total energy consumption. This ratio displays an increasing trend; the last available values, corresponding to the year 1994, are between 7.6% for Norway to 22.5% for Portugal. Some estimates say that transport industry is responsible for producing 6–8% of GDP in most countries, e.g. Weidlich et al. (1999). In the case of France a more precise evaluation states that the transport industry share of GDP is around 14%, according to the French Road Federation (2006). Since these estimates display a large range of diverse numbers, we have decided to keep μ low at around 5% for the beginning, as we want to avoid overstating the effects of a larger stock of road infrastructure on steady state GDP in the absence of strict constraints on that relationship. In the case studies presented in Sect. 3 the sensitivity of the results with respect to higher values of μ will be tested.

2.4 Solution of Optimal Control Problem

In this section, we approach the problem from an optimal control point-of-view (e.g. Pontryagin et al. 1962; Lee and Markus 1967; see Dorfman 1969 for a more economic exposition of optimal control problems).

We consider the following optimal control problem with infinite time horizon

$$
\begin{aligned}
&\text{maximize} && J = \int_0^\infty (\mu x(t) - Dz(t) - E - Le^{\theta u(t)} + L)dt \\
&\text{subject to} && \dot{z}(t) = u(t) - \delta z(t), \\
& && \dot{x}(t) = \gamma(Az(t) + B - x(t)), \\
& && u(t) \in [0,\ \bar{u}], \\
& && z(0) = z_0, \\
& && x(0) = x_0, \\
& && t \in [0,\ \infty).
\end{aligned}
\tag{13}
$$

The solution approach is based on the Pontryagin Maximum Principle for a case of infinite time horizon. More precisely, we use *Corollary 7* proven in Aseev and Kryazhimskiy (2005).[9] First, let us check that the problem satisfies a number of assumptions in order to prove the applicability of the method to the problem.

Assumption 1 [A3][10] For each z and each x, the function $b(z, x, u)$ is a concave function in u.

That follows from the convexity of the exponential function $e^{\theta u}$ and the condition $L > 0$.

[9] See also Aseev and Kryazhimskiy (2007).

[10] Numbers in square brackets refer to the assumptions in Aseev and Kryazhimskiy (2005).

Assumption 2 [A4] There exist positive-valued functions μ and ω on $[0, \infty)$ such that $\mu(t) \to 0$ as $t \to \infty$, and for any admissible pair (u, z, x),

$$e^{-\rho t} \max_{u \in [0, \bar{u}]} |b(z(t), x(t), u)| \leq \mu(t) \quad \text{for all } t > 0;$$

$$\int_T^\infty e^{-\rho t} |b(z(t), x(t), u(t))| dt \leq \omega(T) \quad \text{for all } T > 0.$$

That follows from the linearity of the function $b(z, x, u)$ in (z, x), and the restrictions on the control function.

Assumption 3 [A6] There exist a $k \geq 0$ and a $r \geq 0$ such that

$$\sqrt{\left(\frac{\partial b(z, x, u)}{\partial z}\right)^2 + \left(\frac{\partial b(z, x, u)}{\partial x}\right)^2} \leq k(1 + \sqrt{z^2 + x^2})^r,$$

for all x and for all $u \in [0, \bar{u}]$.

Taking into account the linearity of the function $b(z, x, u)$ in (z, x), we get that $k = \sqrt{\mu^2 + D^2}$ and $r = 0$.

Assumption 4 [Dominating discount case]

$$\rho > (r + 1)\lambda,$$

where λ is the maximal of the real parts of the eigenvalues of the dynamic system.

Taking into account that $r = 0$ and all eigenvalues of the dynamic system are negative, we get that it is sufficient that $\rho > 0$.

Now we can start solving the problem using the Maximum Principle. Let us compose the Hamilton–Pontryagin function

$$\begin{aligned} \mathcal{H}(t, z, x, u, \psi) = e^{-\rho t}(\mu x - Dz - E - Le^{\theta u} + L) + \psi_1(u - \delta z) \\ + \psi_2 \gamma (Az + B - x) \end{aligned} \tag{14}$$

and the adjoint equation

$$\begin{cases} \dot{\psi}_1 = -\dfrac{\partial \mathcal{H}}{\partial z} = \delta \psi_1 - \gamma A \psi_2 + De^{-\rho t}, \\ \dot{\psi}_2 = -\dfrac{\partial \mathcal{H}}{\partial x} = \gamma \psi_2 - \mu e^{-\rho t}. \end{cases} \tag{15}$$

Using *Corollary 7* (Aseev and Kryazhimskiy 2005), we get the following transversality condition

$$\lim_{t \to \infty} \psi_1(t) = 0, \tag{16}$$

$$\lim_{t \to \infty} \psi_2(t) = 0. \tag{17}$$

Let us consider the differential equation describing the adjoint variable ψ_2 separately

$$\dot{\psi}_2 = \gamma\psi_2 - \mu e^{-\rho t}.$$

A general integral of this equation has the following form

$$\psi_2(t) = \frac{\mu}{\rho+\gamma}e^{-\rho t} + C_1 e^{\gamma t},$$

where C_1 is a constant. Taking into account transversality condition (16), we get

$$C_1 = 0.$$

Hence, we have

$$\psi_2(t) = \frac{\mu}{\rho+\gamma}e^{-\rho t}. \tag{18}$$

Substitute the variable ψ_2 in the differential equation for the variable ψ_1 using the equality (18). We get the following differential equation for the variable ψ_1

$$\dot{\psi}_1 = \delta\psi_1 - \left(\frac{\gamma A\mu}{\rho+\gamma} - D\right)e^{-\rho t}.$$

This equation has the following general integral

$$\psi_1(t) = \frac{\gamma A\mu - D\gamma - D\rho}{(\gamma+\rho)(\delta+\rho)}e^{-\rho t} + C_2 e^{\delta t}.$$

Taking into account the transversality condition (17), we get

$$\psi_1(t) = \frac{\gamma A\mu - D\gamma - D\rho}{(\gamma+\rho)(\delta+\rho)}e^{-\rho t}. \tag{19}$$

The extremal control satisfies the following maximum condition

$$u(t,z,x,\psi) = \arg\max_{u\in[0,\bar{u}]}\mathcal{H}(t,z,x,u,\psi) = \arg\max_{u\in[0,\bar{u}]}\{\psi_1 u - Le^{-\rho t}e^{\theta u}\}.$$

Note that the function

$$M(u) = \psi_1 u - Le^{-\rho t}e^{\theta u}$$

is a concave function as $L > 0$. Therefore,

$$\arg\max_{u\in[0,\bar{u}]} M(u) = \begin{cases} 0, & \hat{u}(t) \le 0, \\ \hat{u}(t), & \hat{u}(t) \in (0,\bar{u}), \\ \bar{u}, & \hat{u}(t) \ge \bar{u}, \end{cases}$$

where $\hat{u}$ is a solution of the equation

$$\frac{\partial M(u)}{\partial u} = 0.$$

We get

$$\hat{u}(t) = \frac{1}{\theta} \ln\left(\frac{e^{\rho t}\psi_1(t)}{L\theta}\right).$$

Finally, the optimal control is

$$u^*(t) = \begin{cases} 0, & \hat{u} \le 0, \\ \hat{u}, & \hat{u} \in (0, \bar{u}), \\ \bar{u}, & \hat{u} \ge \bar{u}, \end{cases} \tag{20}$$

where

$$\hat{u} = \frac{1}{\theta} \ln\left(\frac{\gamma A\mu - D\gamma - D\rho}{L\theta(\gamma+\rho)(\delta+\rho)}\right). \tag{21}$$

An important conclusion is that the optimal control $u^*(t)$ is a constant function over the time interval $t \in [0, \infty)$. Figure 5 shows an example of a phase diagram that consists of a number of optimal trajectories starting at various initial states. The threshold curve corresponding to the function $f(z)$ has a lighter shade (green in color version). Trajectories starting above the threshold line decline, that is GDP decreases until a trajectory intersects the threshold line, whereupon GDP increases.

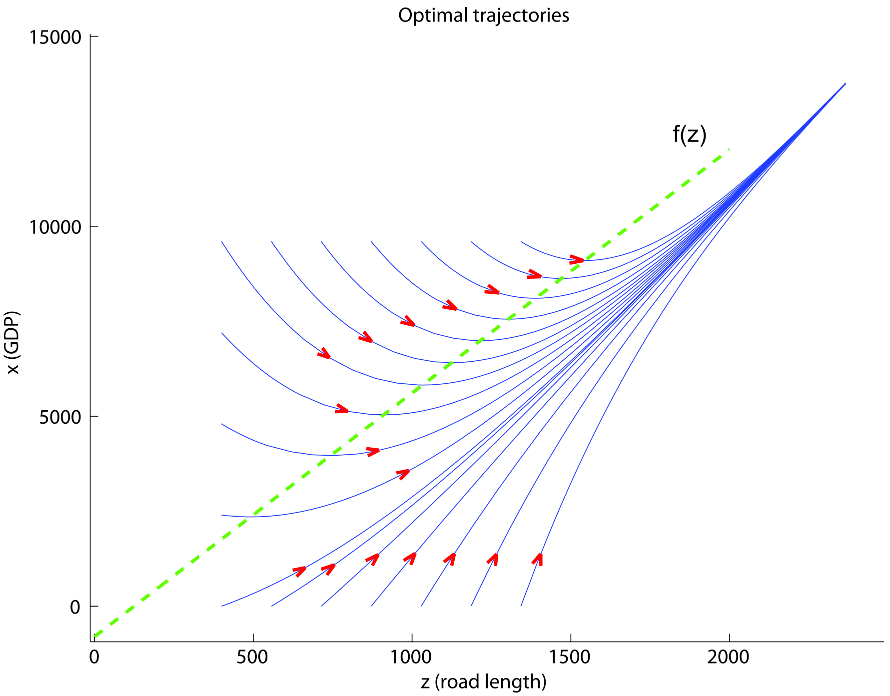

Fig. 5 Optimal trajectories starting at various initial states and the threshold curve $f(z)$ (*dashed line*, *green* in online version), GDP in bill. 2005 US$, road length in 1,000 km

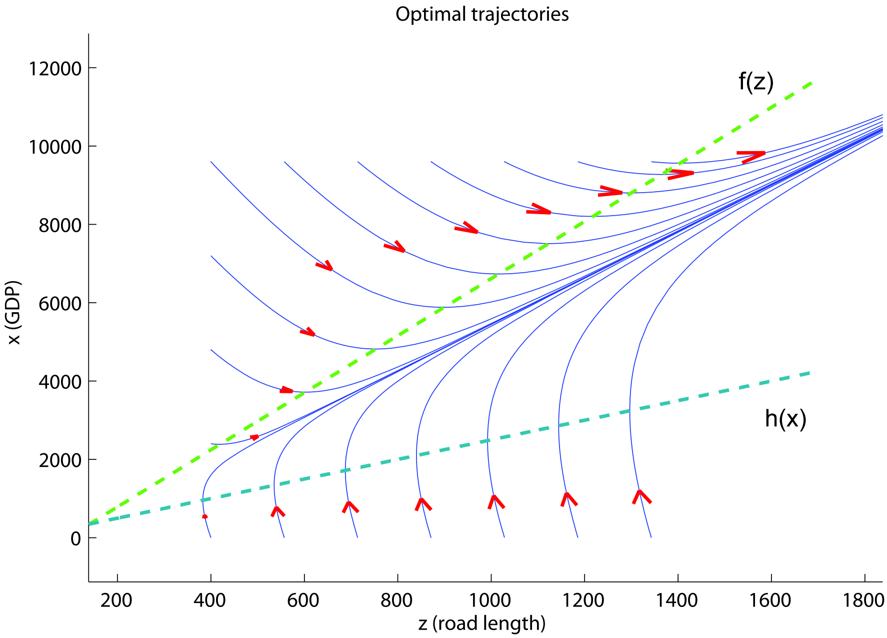

Fig. 6 Optimal trajectories and the threshold curves $f(z)$ (*light dashed line*, *green* in online version) and $h(x)$ (*dark dashed line*, *blue* in online version)

Later, we will show that the trajectories' behavior does not only depend on the threshold line but also on the steady state, which uniquely exists for every optimal trajectory in the model.

Let us describe how the second threshold function, $h(x)$, affects the phase diagram. We introduced a constant restriction on the control, $0 \le u \le \bar{u}$. However, the upper restriction actually depends on the current level of GDP because little money can be invested if the level of GDP is low, so the upper restriction is $u \le \bar{u}(x)$. Therefore, there is a possibility that the size of road capacity can decrease due to physical decay, while the level of GDP is low. Consequently, the function $\bar{u}(x)$ determines a threshold curve, denoted by $h(x)$, that separates areas of decreasing and increasing road capacity size. Figure 6 presents a phase diagram with an upper restriction on the control $0 \le u \le \bar{u}(x)$. In the present study, we will not be considering such kinds of restrictions on the control, as we have no information to calibrate the functions $\bar{u}(x)$ or $h(x)$.

Let us substitute the constant control u^* into the equations describing the dynamical system. The equations take the form

$$\begin{cases} \dot{z}(t) = u^* - \delta z(t), & z(0) = z_0, \\ \dot{x}(t) = \gamma(Az(t) + B - x(t)), & x(0) = x_0. \end{cases}$$

That means that, assuming $u = u^*$, the trajectory of the system can be computed as the solution of an affine system

$$\dot{y} = Fy + G,$$

where

$$y = \begin{pmatrix} z \\ x \end{pmatrix}, \qquad F = \begin{pmatrix} -\delta & 0 \\ \gamma A & -\gamma \end{pmatrix}, \qquad G = \begin{pmatrix} u^* \\ \gamma B \end{pmatrix}$$

satisfying the initial condition

$$y(0) = \begin{pmatrix} z_0 \\ x_0 \end{pmatrix}.$$

The solution of the system can be calculated analytically as

$$\begin{cases} z(t) = -\dfrac{u^* - \delta z_0}{\delta} e^{-\delta t} + \dfrac{u^*}{\delta}, \\ x(t) = \gamma A \dfrac{u^* - \delta z_0}{\delta(\delta - \gamma)} e^{-\delta t} + \left(x_0 - B + \delta A \dfrac{\gamma z_0 - u^*}{\delta(\delta - \gamma)} \right) e^{-\gamma t} + \dfrac{B\delta + A u^*}{\delta} \end{cases}$$

for $\delta \neq \gamma$ and

$$\begin{cases} z(t) = -\dfrac{u^* - \delta z_0}{\delta} e^{-\delta t} + \dfrac{u^*}{\delta}, \\ x(t) = A(\delta z_0 - u^*) t e^{-\delta t} + \left(x_0 - A\dfrac{u^*}{\delta} - B \right) e^{-\delta t} + \left(A\dfrac{u^*}{\delta} + B \right) \end{cases}$$

for $\delta = \gamma$. Since the eigenvalues of F are $-\gamma < 0$ and $-\delta < 0$, the unique stationary solution of the system

$$\hat{y} = \begin{pmatrix} \hat{z} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} \frac{u^*}{\delta} \\ A\hat{z} + B \end{pmatrix}$$

is a stable node and the trajectories converge to it along the eigenvector belonging to the eigenvalue closer to zero for $\delta \neq \gamma$. That means that for $\delta < \gamma$ the trajectories converge along the line

$$x = \frac{\gamma A}{\gamma - \delta} z - A \frac{u^*}{\gamma - \delta} + B$$

and for $\delta > \gamma$ along the line

$$z = \frac{u^*}{\delta}.$$

Let us assume that the developed countries already behave optimally (that means their observed real trajectory follows the asymptote whose eigenvector has a smaller modulus). Therefore, the asymptotic line computed for a developed country must coincide with the function $g(z)$ calibrated for the same country. We use this assumption to calibrate the threshold function $f(z)$. We get two equations to compute A and B

$$\begin{cases} \alpha = \dfrac{\gamma A}{\gamma - \delta}, \\ \beta = -A\dfrac{u^*}{\gamma - \delta} + B. \end{cases}$$

Solving the latter equations, we get

$$\begin{cases} A = \dfrac{\alpha(\gamma - \delta)}{\gamma}, \\ B = \beta + \dfrac{\alpha}{\gamma}u^*. \end{cases} \tag{22}$$

Figure 7 presents an example of two asymptotic lines with a darker shade (highlighted in magenta in the color version). All optimal trajectories converge along the inclined asymptotic line in the case of $\gamma > \delta$ and along the vertical asymptotic line in the case of $\gamma < \delta$ and terminate in a unique steady state.

The steady state is an essential element of the optimal behavior of the described control model. It gives the maximal possible level of GDP and the maximal possible road capacity to support that level of GDP. It is worth to note that the steady state depends on the parameters δ and γ and that, in the context of the model, the

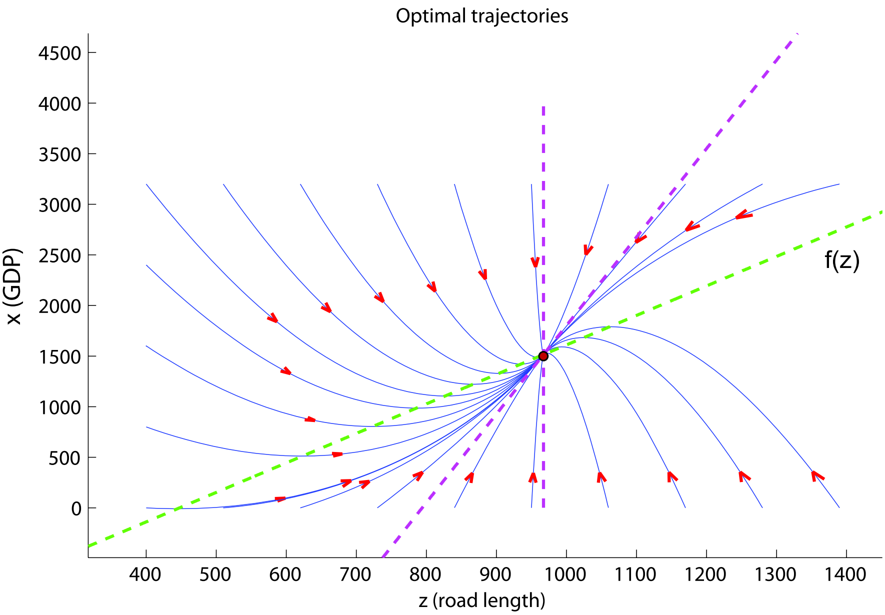

Fig. 7 Optimal trajectories, asymptotic lines (*darker dashed lines, magenta* in online version) and steady state (*circle*), GDP in bill. 2005 US$, road length in 1,000 km

parameter δ can be interpreted as the quality of the existing infrastructure. Since the country's GDP level is restricted by the steady state, the only way to accelerate economic growth (taking into account the dependency on road infrastructure only) is to improve the quality of road infrastructure or, in other words, to reduce δ.[11] This will be investigated in more detail in the following section.

3 Optimal Control Results: Country Case Studies

In Sect. 2 we have derived the analytical solutions for the long-run behavior of the economy and its convergence to a steady state,[12] denoting the maximum attainable GDP with the required stock of road infrastructure, which is — inter alia — determined by the rate of physical decay or the quality of the roads and the speed, at which the economy adjusts, i.e. the speed at which it moves along its trajectory. While this might seem rather technical to the reader, we also want to emphasize the usefulness of the type of approach we have taken for practical problems and the associated policy agenda. In this section we therefore derive some results for the cases of France and Finland with the help of the data and calibration presented in the Appendix.

Figures 8 and 9 display the phase diagrams for France and Finland respectively. The light dotted line (green in color version) is the threshold curve. The darker, dashed line (pink in color version) is the asymptote. The arrows of motion on the trajectories point to the steady state. The transparent dots correspond to the real data. For France, the results show that the country is currently below its long-run steady state and the same is true for Finland in Fig. 9.

Figure 10 demonstrates the sensitivity of the results with respect to the parameter δ. If we interpret δ as an indicator of quality, which means that a lower value implies better quality, then the long-run steady state will indeed shift (remember our discussion at the end of Sect. 2) upwards and to the right and the economy will move along the asymptote to a higher long-run GDP level supported by a larger stock of higher-quality infrastructure. This will be further investigated in the following subsection.

3.1 Infrastructure Quality & Steady State GDP

As mentioned in Sect. 2, it has been suggested that there is a relationship between the quality of existing infrastructure and steady state economic output. In the pre-

[11]However, this result has to be seen with caution, since we should not forget that u also depends on δ and so to find the "optimal" δ is not as straightforward as it seems because also the dependence of costs on the same would need to be considered in detail.

[12]Note that the phase diagrams do *not* imply constant GDP in the long run: the steady state should be considered as a restriction on growth, but it does not mean that this restriction will be reached at some finite time moment. GDP will grow (slowing down) and will not exceed the steady state level, but it will not be a constant in finite time.

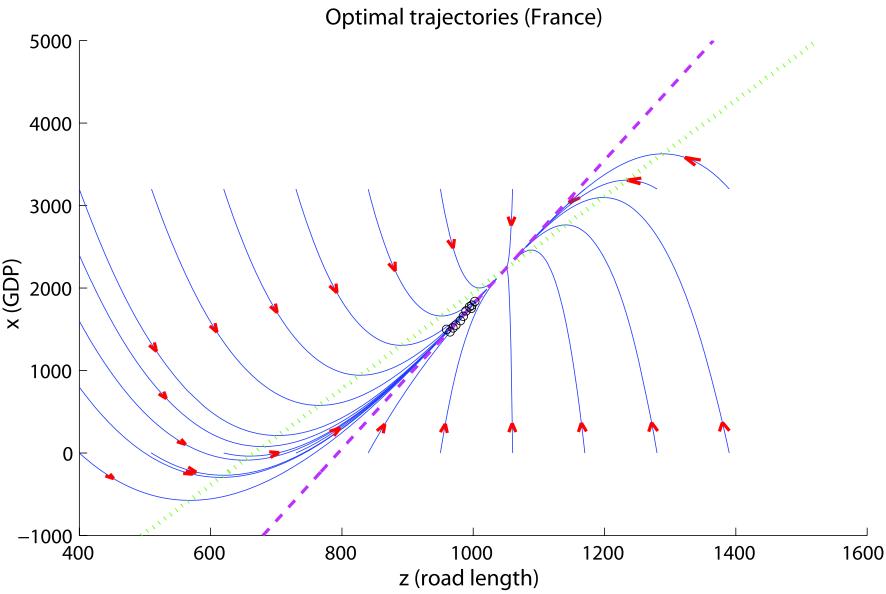

Fig. 8 Phase diagram for France with trajectories, GDP in bill. 2005 US$, road length in 1,000 km

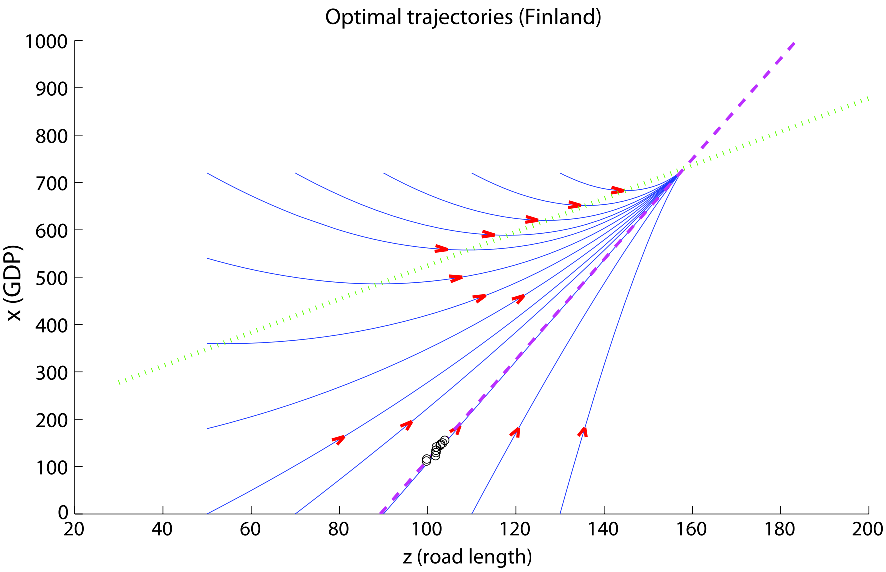

Fig. 9 Phase diagram for Finland with trajectories, GDP in bill. 2005 US$, road length in 1,000 km

vious section it has been indicated that this relationship is positive (see Fig. 10). Analytically, it is difficult to find the value for δ, which is "optimal" in the sense that it supports the maximally attainable steady state GDP. The reason is that one

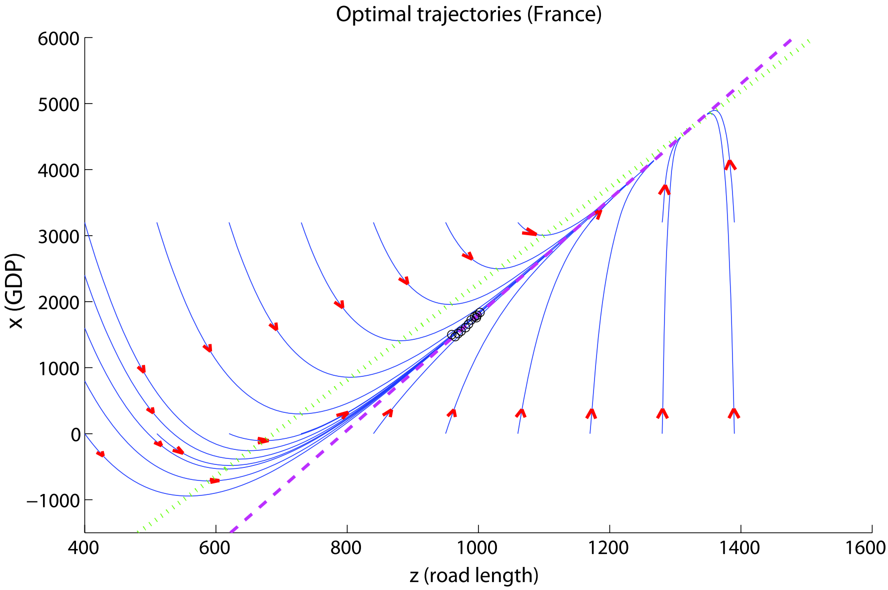

Fig. 10 Phase diagram for France with better quality road infrastructure ($\delta = 5\%$), GDP in bill. 2005 US$, road length in 1,000 km

would have to specify the exact dependence of costs on δ, which has not been done here. Empirically—not knowing the precise value of δ—we can use the available data to calibrate the model for a given δ and find the optimal solution corresponding to that value. Plotting these optimal solutions for increasing values of δ, we can then draw some conclusions about the relationship between infrastructure quality and steady state economic output.

Figures 11 and 12 show that for *decreasing* given δ—representing *increasing* infrastructure quality according to our interpretation—a more than proportionately higher steady state GDP level can be attained in both France and Finland. Both graphs display similar properties.

The numerical results from this sensitivity exercise indicate that for a relatively small improvement in quality (a small decrease in δ), a relatively large gain in terms of optimal GDP can be achieved. For relatively low levels of infrastructure quality (high δ), the results should be looked at with scrutiny, since the model does not have a constraint with respect to the maximum impact of δ on steady state GDP and so the reader should not be misled to think that long run economic output could drop to zero or even negative levels if existing infrastructure deteriorates at a relatively fast pace.

3.2 The Speed of Adjustment & Steady State GDP

Another parameter, which merits special attention in our model is γ. Looking back at Sect. 2.1 and, in particular, equation (4), we remember that γ is the coefficient of

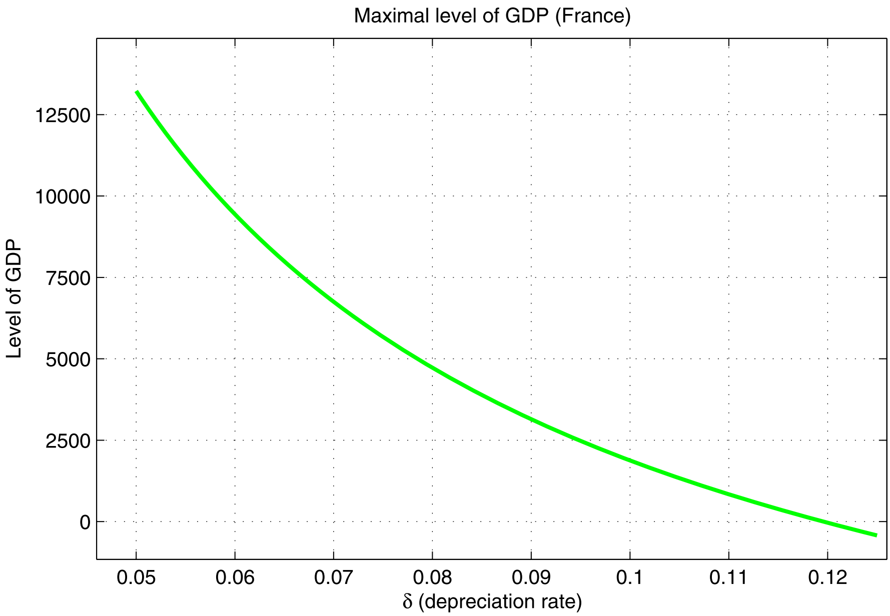

Fig. 11 Steady state GDP for France against decreasing infrastructure quality (modeled as increasing δ), GDP in bill. 2005 US$

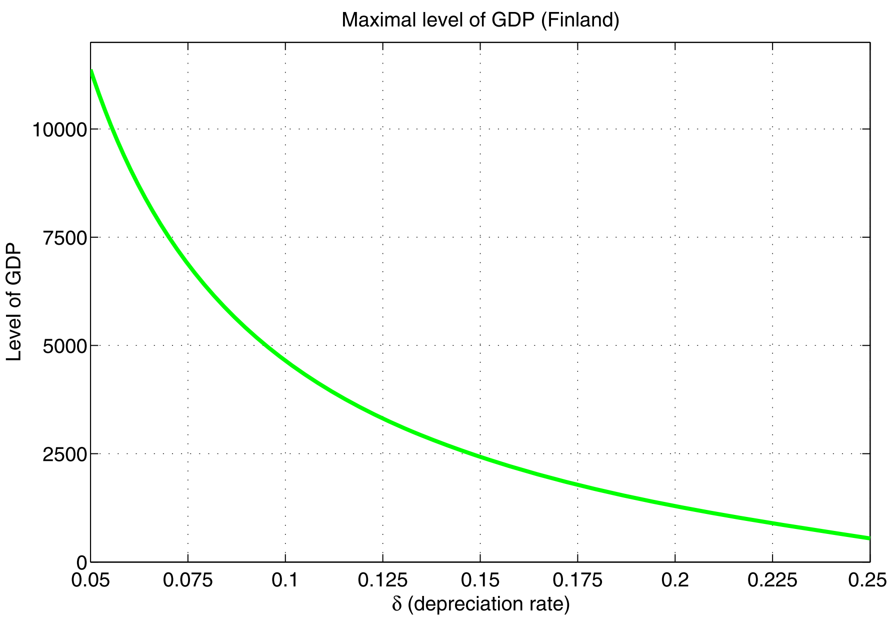

Fig. 12 Steady state GDP for Finland against decreasing infrastructure quality (modeled as increasing δ), GDP in bill. 2005 US$

the speed of adjustment. It therefore represents the rate at which GDP approaches its long-run, optimal level when the economy is on one of the trajectories traced out in the previous sections. In this section we will test the relationship between different levels of γ and steady state GDP for the calibrated cases of Finland and France. This sensitivity analysis will reveal how the ability of a country to adapt to its steady state influences the level that this steady state will have.

Remember from Fig. 7 in Sect. 2 that there are two asymptotic lines in the model and that the trajectories converge along the inclined asymptotic line in the case of $\gamma > \delta$ and along the vertical asymptotic line in the case of $\gamma < \delta$ until the steady state is reached. In this section we focus on the first case, since in the other case the economy would adjust more slowly than its infrastructure deteriorates and without adding more constraints this could easily lead to negative growth and a contraction of the long-run economic output below zero.

With this caveat in mind, let us turn to Figs. 13 and 14 displaying the results of the exercise for France and Finland respectively. In both cases it can be observed that the more quickly GDP approaches its long-run steady state level, the response of this level is initially huge and levels off afterwards, i.e. there is a level of maximal GDP that cannot be surpassed, no matter how large γ is. In other words, the sensitivity

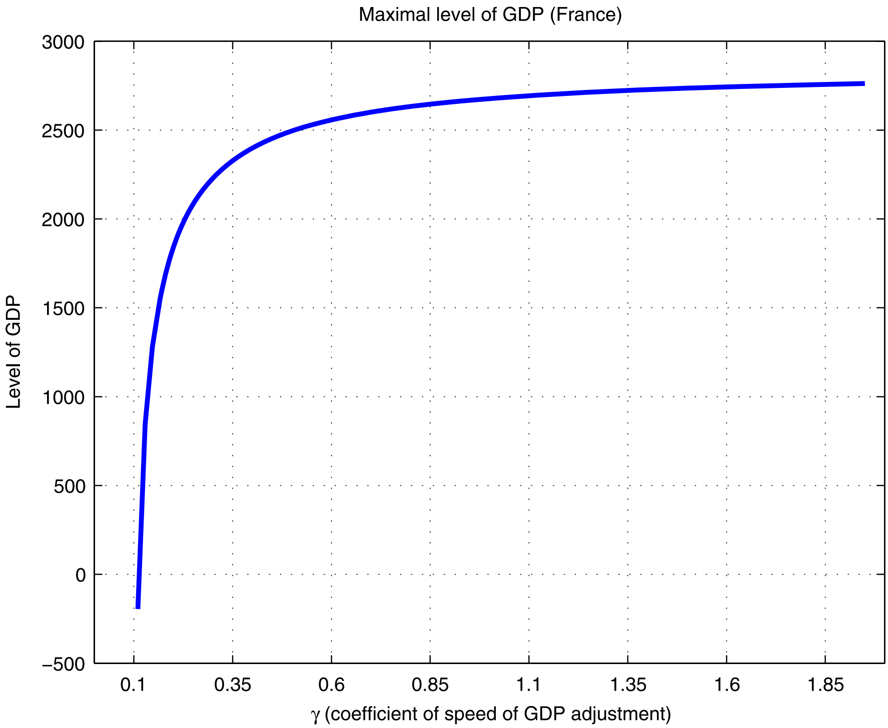

Fig. 13 Steady state GDP for France against speed of adjustment (modeled as increasing γ), GDP in bill. 2005 US$

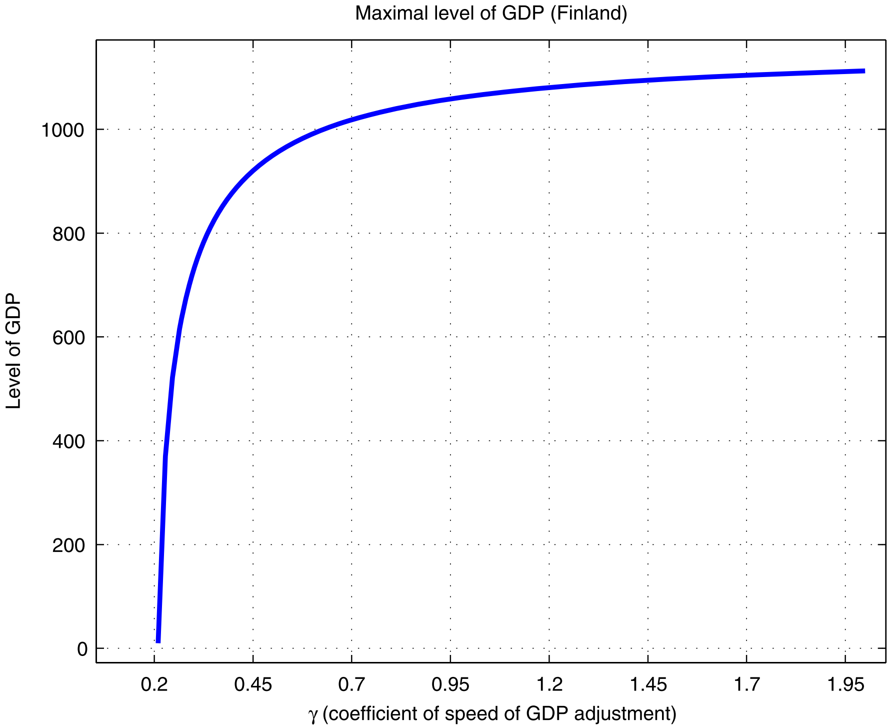

Fig. 14 Steady state GDP for Finland against speed of adjustment (modeled as increasing γ), GDP in bill. 2005 US$

analysis shows a positive but diminishing effect of the speed of adjustment on steady state GDP.

3.3 The Share of Road Infrastructure in GDP & Steady State GDP

Remember that in Sect. 2.3, Table 1 shows a proxy of the portion of GDP, which is composed of road infrastructure, based on the amounts of energy used in road transport and the total energy consumed in the economy. Even though we opted for a rather cautious value of μ (5%), which does not differ significantly from other studies' estimates (see Sect. 2.3) in our case studies, Table 1 indicates that μ might be higher than that and, in addition, the parameter varies across countries. The values estimated for Finland grow from 12% to almost 15% in the period from 1980 to 1994; France starts out at 15% ending up at about 17%.

The diversity of these figures raises the question how—in our case studies—the level of steady state GDP would be affected if we used a different μ in our framework with all other calibrated parameters unchanged. Therefore, we also present the corresponding sensitivity analysis for both France and Finland. Figures 15 and 16

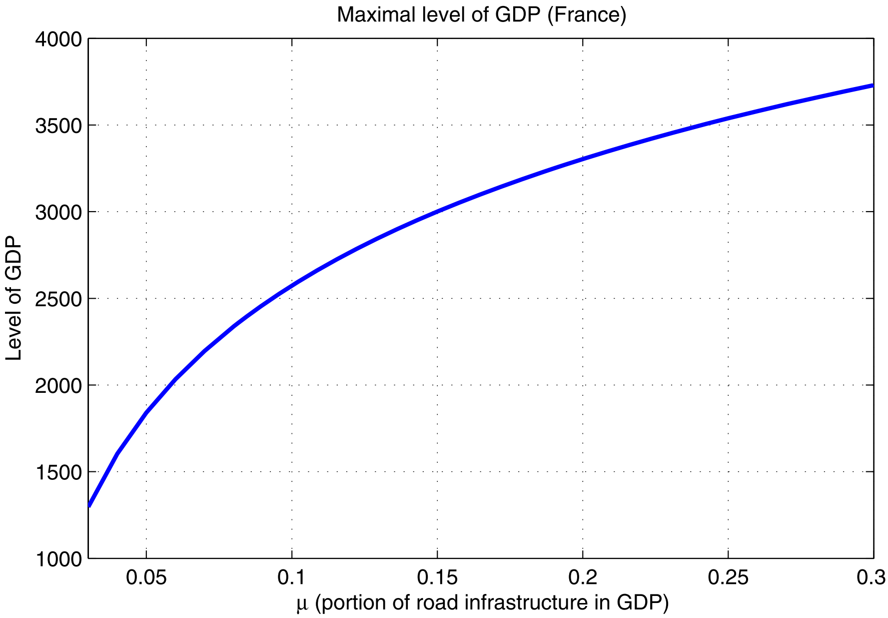

Fig. 15 Steady state GDP for France against the share of road infrastructure in output (modeled as increasing μ), GDP in bill. 2005 US$

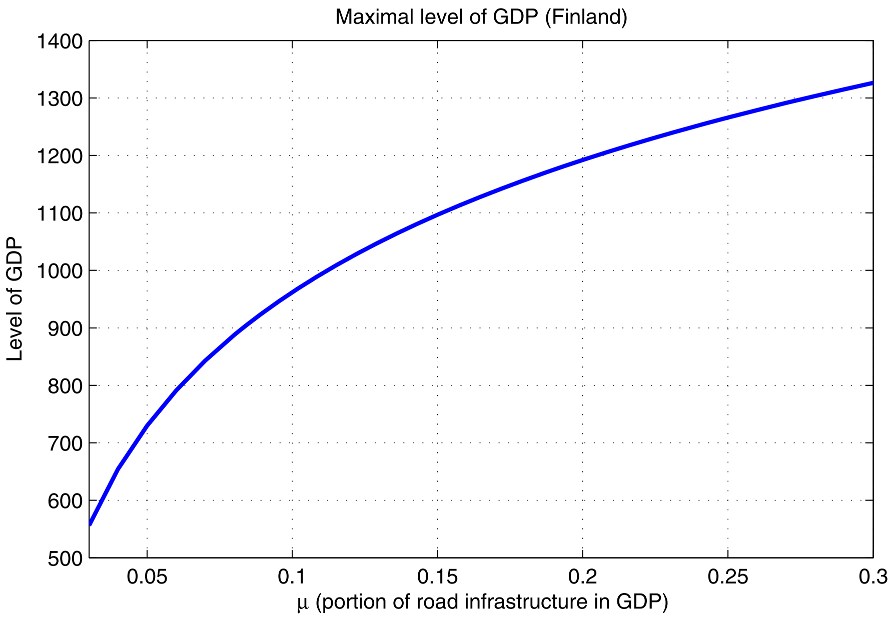

Fig. 16 Steady state GDP for Finland against the share of road infrastructure in output (modeled as increasing μ), GDP in bill. 2005 US$

display the same shape of relationship, where changing μ from a very small value to a slightly higher one has a very large impact on steady state economic output, but beyond 20% this effect levels off: while the graph continues to slope upwards, it is still slightly concave.

4 Summary and Conclusion

In this paper we have applied optimal control theory to a co-evolutionary framework, where the co-evolving variables are economic output and road infrastructure. The control in our problem is the expansion of existing road capacity—in other words investment. Our goal was to show that useful insights can be derived from developing such an approach and when the underlying (cost) functions and constraints are adapted to a specific country or region, then policy makers can make better informed decisions about public investment into roads or about providing incentives for private road investment. We believe that a sound theoretical framework should be the foundation for further empirical work and have therefore embarked on demonstrating how this can be approached and shown that it can be calibrated and implemented if appropriate data are available.

We find an analytical solution for the infinite horizon problem, where the control turns out to be a constant. The steady state is shown to depend crucially on the rate of physical decay of roads, which we think can be interpreted as an index of quality, and the speed of adjustment, at which the economy moves along a trajectory. Another parameter, which merits special attention is the degree to which GDP is composed of road infrastructure as an input factor, the other factors being labor, resources, human capital, other types of physical capital, and so forth.

Testing the model for the data of two countries, France and Finland, illustrates the usefulness of such an approach to practical problems and possibly policy recommendation, where the model would have to be adapted to the specific characteristics of each country or region to obtain a detailed and clear picture.

Sensitivity analyses with respect to the above-mentioned parameters show that better quality of road infrastructure implies that a higher level of steady state GDP can be reached if the other calibrated relationships are unchanged. While this relationship is exponential, a larger speed of adjustment to steady state GDP and a larger portion of GDP being composed of road infrastructure are shown to have a concave relationship with long-run economic output. In other words, a marginal change at low levels of γ and μ has a large impact on the output attainable in the long run, but this effect diminishes for larger values of these parameters.

We think that our study contributes to the existing literature by applying an existing modeling approach in combination with co-evolutionary features to a problem, which has previously mainly been the focus of empirical research and where there has been much debate about causality issues and other problems when estimating the underlying relationships. Our model is admittedly simple, but it has been implemented like this on purpose, so as to illustrate the usefulness of our approach in a transparent and straightforward way. The empirical part (i.e. the country case studies) suffers from a lack of appropriate data to give robust estimates of the parameters used, but serves the goal of demonstrating that the framework can be adapted to practical applications when the underlying relationships are adequately adapted to the situation and the context.

The current framework obviously offers several points of departure for further research. Most importantly, there will be an expansion of the model with respect to a multi-sector dimension. This will serve to take into account competing uses for available resources and to put conclusions better into perspective with respect to other economic factors and their mutual influence on each other. In addition, the current work should be developed further by extending it in a spatial dimension in order to explicitly consider network effects and spatial evolution. Finally, an effort will be made to collect a more comprehensive data set, as there is much scope for improvement on the empirical side.

Appendix: Data & Calibration

Calibration Methods

In this section we propose an approach to the calibration of the model given statistical data for a certain country. The model includes the following functions to be identified: $g(z)$, $f(z)$, $c(z)$, $r(u)$. We assume that the parameters δ, γ, μ and ρ have been specified. Moreover, we have chosen forms for the functions $g(z)$, $f(z)$, $c(z)$ and $r(u)$ (see (9), (10), (11), (12)). So we need to identify the parameters α, β, A, B, D, E, L, θ.

Let us assume that data are arranged as follows. All data are specified on a time grid that covers the time period $[t_0, T]$

$$G_T = \{t_0, t_1, \ldots, t_{N_T}\}, \quad t_{N_T} = T. \tag{A.1}$$

Assume that for every time moment on the grid G_T we have the following data (for a certain country):

x_i—GDP value at time t_i, $i = 0, \ldots, T_N$;
z_i—capacity of road infrastructure at time t_i, $i = 0, \ldots, T_N$;
c_i—maintenance cost at time t_i, $i = 0, \ldots, T_N - 1$;
r_i—building cost at time t_i, $i = 0, \ldots, T_N - 1$.

We split all parameters to be identified into three groups. The first group contains the parameters α and β relating to the function $g(z)$. As the function $g(z)$ has been chosen linear (based on statistical data for various countries), linear regression can be used to calibrate α and β. The second group is composed of the parameters A and B relating to the threshold function. It is not possible to calibrate the threshold line using statistical data for a single country. Therefore, we make the assumption that developed countries develop in the optimal way (in the sense of the described model). From this assumption follows that the optimal asymptote constructed for a given country has to coincide with the function $g(z)$ calibrated for the same country. That gives us equations to find A and B (see Sect. 2.4).

The third group contains the parameters D, E, L, θ. When calibrating these parameters we take into account that the trajectory $z(t)$ has to satisfy (1), consequently, we need to identify the control function $u(t)$ producing a given trajectory $\{z(t_i)\}_{i=0,\dots,T_N}$ as well. The approach to calibrating these parameters is based on the least–squares method. Let us introduce new variables u_i that corresponds to the control $u(t_i)$, $i = 0, \dots, N_T - 1$, at time moment t_i. Equation (1) imposes the following constraints

$$z_{i+1} - z_i = (u_i - \delta z_i)(t_{i+1} - t_i), \quad i = 0, \dots, N_T - 1. \tag{A.2}$$

We need to minimize the function

$$\sum_{i=0}^{N_T} [w_c(c_i - c(z_i))^2] + \sum_{i=0}^{N_T-1} [w_r(r_i - r(u_i))^2] \tag{A.3}$$

under the constraints (A.2) by choosing $\{u_i\}_{i=0,\dots,N_T-1}$, D, E, L, θ. The coefficients w_c, w_r should be chosen such that all items in the function (A.3) have the same scale.

Note that the constraints in (A.2) enable us to compute the variables u_i directly

$$u_i = \frac{z_{i+1} - z_i}{t_{i+1} - t_i} + \delta z_i, \quad i = 0, \dots, N_T - 1.$$

Therefore, minimization of (A.3) can be carried out for the first and second items independently. Taking into account that the function $c(z)$ is linear, we can apply linear regression to identify D and E. The last step is to calibrate $r(u)$. Having plotted statistical data for the function $r(u)$, we obtain rather a grouped set of points located on a relatively small part of the (u, r)-plane rather than a curve. We assumed that the function $r(u)$ has the form

$$r(u) = L(e^{\theta u} - 1).$$

Therefore, we choose such values for the coefficients L and θ that the exponential curve passes trough the set of points. In this case we arrive at an approximation of the extrapolation at least, since the exponential curve satisfies the condition $r(0) = 0$ (that must be imposed) and more or less approximates the group of points with a curve. To implement this approach, we can indicate (at least manually) a point located inside of the group of points and a slope at this point so that the exponential curves passes through the point and satisfies a previously specified slope.

Let $(\bar{u}, \bar{r})$ be a point, through which the curve has to pass with the slope in this point being k. We get the following equations to find L and θ:

$$\begin{cases} L(e^{\theta \bar{u}} - 1) = \bar{r}, \\ L\theta e^{\theta \bar{u}} = k. \end{cases}$$

Solving these equations, we get a nonlinear equation to find θ:

$$1 - \frac{\theta \bar{r}}{k} = \frac{1}{e^{\theta \bar{r}}} \tag{A.4}$$

and an equality to find L:

$$L = \frac{k}{\theta e^{\theta \bar{u}}}. \tag{A.5}$$

Calibration Results

Based on the data provided by Canning (1998, 1999) that we have been using above to motivate the functional forms of the relationships in the optimal control problem at hand, we have chosen to focus on two case studies: the two countries are Finland and France. We have chosen France as an example of one of the more mature economies with a relatively high income featuring in the upper right region in Figs. 2 to 4. Finland, on the other hand, is one the countries in the lower left corner of Figs. 2 to 4. If this is a matter of scale or whether it implies that these countries are farther removed from their steady state remains to be seen. GDP data are taken from the UNECE Statistical Division Database, compiled from national and international official sources such as EUROSTAT and the OECD. Road length is from the UNECE Transport Division Database. Investment and maintenance cost series are from the International Transport Forum, issued in May 2008.[13]

Before proceeding with the calibration results, a word of caution should be mentioned: the purpose of this exercise is not to obtain a probabilistic estimation of the model functions. Instead, the calibration aims to find such model parameters that make observed real (time-series) data for a country coincide with the trajectories generated by our model. The other trajectories are the extrapolations according to the co-evolutionary assumption that is the basis of the model.[14]

Starting with GDP as a function of road infrastructure, we perform a linear regression of GDP and road length, where A is the slope and B the constant. The calibration results for both France and Finland show that this provides a very good fit compared to the actual data, judging from the high values we find for R^2 (see Figs. 17 and 18).

[13] We have chosen to use the UNECE data for road length instead of Canning's (1998, 1999) data because there is a larger overlap in time with the other series and our goal was to maximize the number of data points, since empirical applications are already subject to many points of criticism, so at least the data set should be as complete as possible.

[14] Taking into account the form of model equations, it turns out that the model functions can be calibrated independently. So we fit curves and use the R^2 criteria in order to show the quality of these fittings, even though we know that R^2 is a misleading measure of fit when applied to non-stationary series to statistically estimate functions.

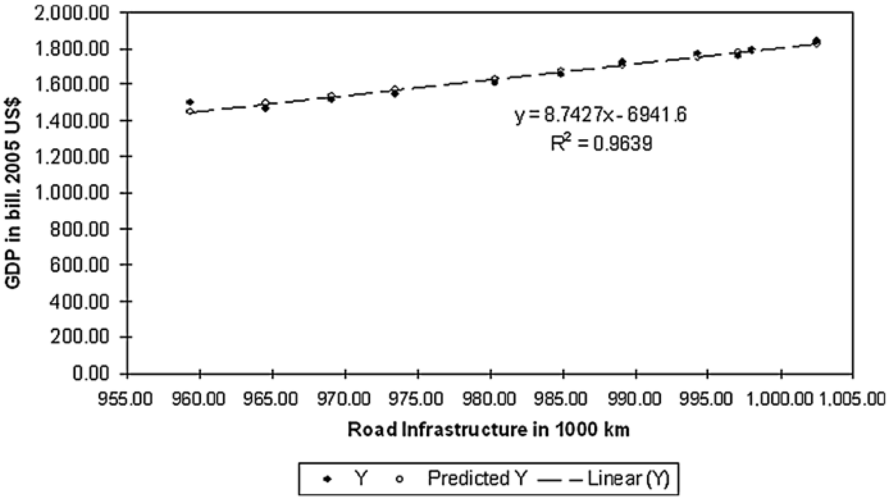

Fig. 17 GDP as a function of road length, linear fit for France

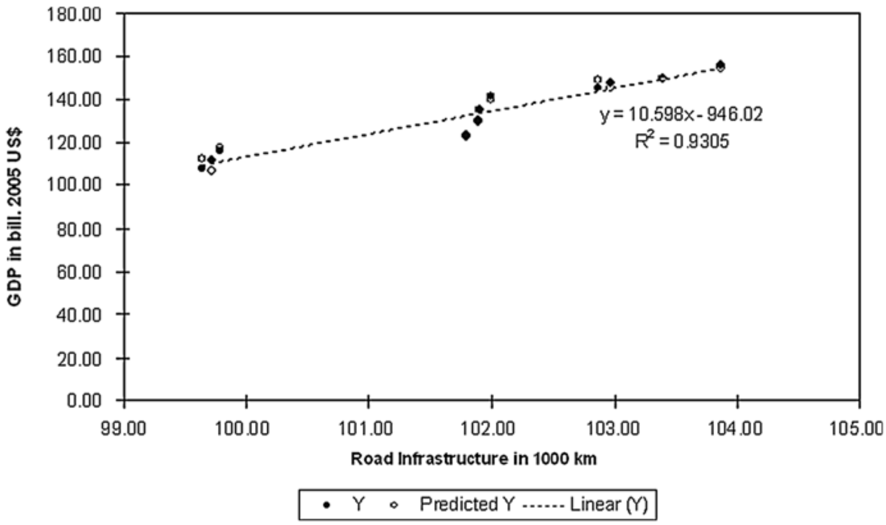

Fig. 18 GDP as a function of road length, linear fit for Finland

The depreciation rates used in the calibration of building cost as a function of the growth in road length is 10 and 20% respectively for France and Finland. In order to smooth the series for the latter variable, we take the average of the difference in u over the current and the coming year, where u is computed as the difference between z in the two years plus depreciation rate times current infrastructure. The exponential fit is then obtained by the method proposed in the previous section, i.e. we (manually) indicate a point located inside the group of points in the (r, u)-plane

Table 2 Building cost as a function of the growth in infrastructure capacity (the "Fit" is computed as the correspondence between the output (i.e. the predicted values) and the actual, observed data)

Country	France	Finland
Specified point (r, u)	(12,100)	(0.6, 21)
Specified slope	1.00	0.10
θ	0.0833	0.1610
L	0.0029	0.0211
Fit	21.79%	10.81%

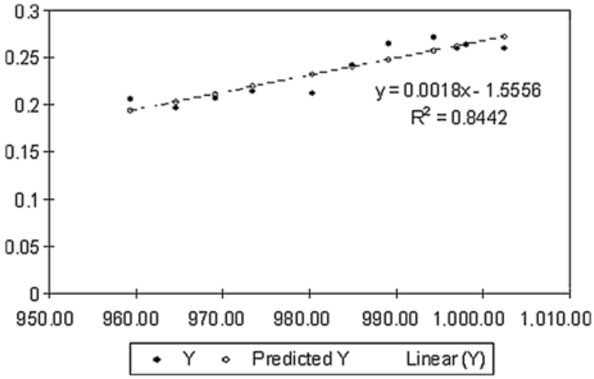

Fig. 19 Maintenance cost as a function of road length, linear fit for France

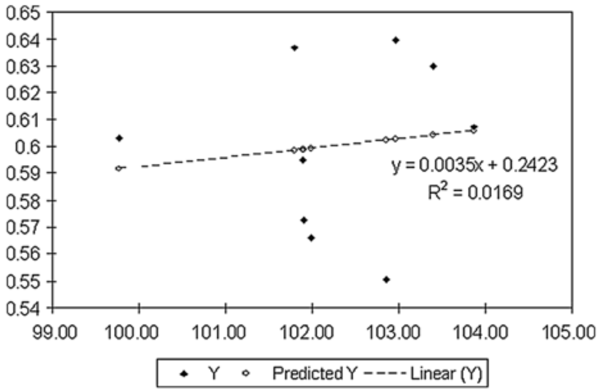

Fig. 20 Maintenance cost as a function of road length, linear fit for Finland

and a slope at this point, so that the exponential curves passes through the point and has a specified slope. The results of this are displayed in Table 2.

Even though the fit is far from perfect, we think that given the relative shortness of our time series and the justifications on the basis of the data set by Canning (1998, 1999) allow us to make use of the coefficients thus obtained, especially against the background that our results are not supposed to be numerically indicative of real developments, but rather illustrative of the new approach and the usefulness of applying optimal control theory to the problem of developing road infrastructure in a context of economic growth.

Finally, the relationship between maintenance cost and the existing road infrastructure stock is calibrated through linear regression again. Figure 19 above shows that this provides a very good fit in terms of R^2 for the case of France, while Finland (Fig. 20) has an R^2 of less than 2%. Still, the linear fit seems the closest we can get to the behavior of the actual data. It is of course admissible to specify a different function for $c(z)$ for Finland, but this would require the re-computation of the analytical solution, which would not add to the illustrative character of this exposition and is thus beyond the scope of this paper.

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