Mathematical Physics Studies

Kasia Rejzner

Perturbative Algebraic Quantum Field Theory

An Introduction for Mathematicians

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York, UK Kasia Rejzner

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Chapter 1
Introduction

Quantum field theory is an important research area in theoretical physics, with a wide range of applications and an impressive agreement with experiment. Despite this success, the mathematical foundations of this theory are still under investigation and many fundamental questions remain open. The rapid development of the field makes it difficult to find textbooks which are up to date with all the recent advances, especially if one looks for a mathematically rigorous approach. It is a common misconception that working with QFT necessarily implies doing something "not-well defined", while in fact most of the formal manipulations presented in the physics literature can be made completely rigorous.

For me quantum field theory is a beautiful bizarre world full of wonders suspended somewhere in-between mathematics and physics. It charms physicists by providing results that agree with experiments with incredible precision. It lures mathematicians seeking to explore the land of QFT and get a closer look at the beautiful mathematical structures that inhabit it. And yet, after more than 50 years of research, we do not fully understand what QFT really is and what wonders it is hiding from us deep in its conceptual roots.

As both a physicist and a mathematician, I am fascinated by the richness of structures that one can encounter in QFT land, and from my first visit I have decided that I do not want to leave it ever again. So what is this book about? Well, maybe first I should explain what it isn't about…It is far from being a complete account of what has been done in QFT research (this would have taken multiple volumes!). It also doesn't touch the problem of non-perturbative construction of models of interacting quantum field theories, which at the moment remains open.

You can think of this book as a mathematician's diary from a journey into an exotic land. As opposed to some other textbooks on the subject, I will not use the excuse that "physicists often do something that is not well defined", so as mathematicians we don't need to bother and just turn around for a while, until it's over. Instead, I will jump straight into the lion's den and will try to make mathematical sense of perturbative QFT all the way from the initial definition of the model to the interpretation of the

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results. This is not always easy and sometimes I will have to bring into the story results from several fields of mathematics at once. I hope this will not discourage you from exploration of the QFT wonderland. After all, its beauty lies in the fact that it is so diverse and full of surprises… So, come along! Our journey starts here.

Chapter 2
Algebraic Approach to Quantum Theory

2.1 Algebraic Quantum Mechanics

Before entering the realm of the quantum theory of fields, let's have a look at something simpler and better understood, namely *quantum mechanics* (QM). To prepare the ground for what follows, we will present an abstract formulation of QM and discuss how it relates to the more standard Dirac–von Neumann axioms [Dir30, vN32]. The exposition presented in this chapter is based on [BF09b, Mor13, Fre13, Str08].

2.1.1 Functional Analytic Preliminaries

Let us start by recalling some basic definitions from functional analysis. For more information see [Rud91, RS80, BR87, BR97, Kad83]. Readers familiar with basic functional analysis can skip this subsection.

Definition 2.1 An *algebra* $\mathfrak{A}$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a $\mathbb{K}$-vector space with an operation $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ called the *product* with the following properties:

1. $(A \cdot B) \cdot C = A \cdot (B \cdot C), \quad \forall A, B, C \in \mathfrak{A}$ (associativity),
2. $A \cdot (B + C) = A \cdot B + A \cdot C$, $(B + C) \cdot A = B \cdot A + C \cdot A$,
   $\alpha(A \cdot B) = (\alpha A) \cdot B = A \cdot (\alpha B)$, for all $A, B, C \in \mathfrak{A}$, $\alpha \in \mathbb{K}$ (distributivity).

We will usually denote the algebra product $\cdot$ simply by juxtaposition, i.e. $A \cdot B \equiv AB$.

Definition 2.2 An algebra $\mathfrak{A}$ is said to have a unit (i.e. $\mathfrak{A}$ is unital) if there exists an element $\mathbb{1} \in \mathfrak{A}$ such that $\mathbb{1}A = A\mathbb{1} = A$, for all $A \in \mathfrak{A}$.

Definition 2.3 An *involutive complex algebra* (a $*$-*algebra*) $\mathfrak{A}$ is an algebra over the field of complex numbers, together with a map, $^* : \mathfrak{A} \to \mathfrak{A}$, called an *involution*. The image of an element A of $\mathfrak{A}$ under the involution is written A^*. Involution is required to have the following properties:

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1. for all $A, B \in \mathfrak{A}$: $(A + B)^* = A^* + B^*$, $(AB)^* = B^* A^*$,
2. for every $\lambda \in \mathbb{C}$ and every $A \in \mathfrak{A}$: $(\lambda A)^* = \overline{\lambda} A^*$,
3. for all $A \in \mathfrak{A}$: $(A^*)^* = A$.

Definition 2.4 A $*$-morphism is a map $\varphi : \mathfrak{A} \to \mathfrak{B}$ between $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$, which is an algebra morphism compatible with the involution, i.e.:

1. $\varphi(AB) = \varphi(A)\varphi(B)$, for all $A, B \in \mathfrak{A}$,
2. $\varphi(\lambda A + B) = \lambda\varphi(A) + \varphi(B)$, for all $A, B \in \mathfrak{A}$, $\lambda \in \mathbb{C}$,
3. $\varphi(A^*) = \varphi(A)^*$ for every $A \in \mathfrak{A}$.

Up to now all the properties we have considered are purely algebraic. In order to quantify the notion of distance between the elements of the algebra we need some topology.

Let us start with some basic definitions and notation.

Definition 2.5 *A topological space* $\mathcal{X}$ is a pair (X, τ), where X is a set X and τ is a collection of subsets of X (called open sets), with the following properties:

- $X \in \tau$
- $\varnothing \in \tau$
- the intersection of any two open sets is open: $U \cap V \in \tau$ for $U, V \in \tau$
- the union of every collection of open sets is open: $\bigcup_{\alpha \in A} U_\alpha \in \tau$ for $U_\alpha \in \tau$ $\forall \alpha \in A$, where A is some index set.

Consider mappings between topological spaces. A topology tells us something about the regularity of those mappings, since it contains already a notion of "being close to something" and we can ask ourselves to what extend a given map preserves this notion.

Definition 2.6 A function $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces, is *continuous* if and only if for every open set $V \subseteq Y$, the inverse image:

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \tag{2.1}$$

is open.

Given a collection of topological spaces, one can define a new topological space by taking their *Cartesian product*. This is a very commonly used operation, so we recall here the definition of a natural topology on such product.

Definition 2.7 Let X be a set such that

$$X = \prod_{i \in I} X_i$$

is the *Cartesian product of topological spaces* X_i, indexed by i in some set I. Let $p_i : X \to X_i$ be the canonical projections. The product topology on X is defined as the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections p_i are continuous.

In our applications the topology will not be enough to capture all the structure we need. In the physics context it is common that we want to add certain quantities and scale them. This leads in a natural way to a vector space structure. We want this structure to be compatible also with the topology.

Definition 2.8 A *Topological vector space* (TVS) over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (with their standard topologies) is a pair $(X, \tau) \equiv \mathcal{X}$, where τ is a topology such that:

- every point of X is a closed set (i.e. its complement is an open set),
- vector addition $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and scalar multiplication $\mathbb{K} \times \mathcal{X} \to \mathcal{X}$ are continuous functions with respect to the product topology on the respective domains.

Definition 2.9 Let $\mathcal{X}, \mathcal{Y}$ be topological vector spaces over the field $\mathbb{K}$. We denote by $L(\mathcal{X}, \mathcal{Y})$ the space of continuous linear maps from $\mathcal{X}$ to $\mathcal{Y}$ and by $\mathcal{X}'$ the topological dual of $\mathcal{X}$, i.e. the space of continuous linear maps from $\mathcal{X}$ to $\mathbb{K}$.

A topology can be introduced for example by means of a norm. This leads to the concept of a normed space.

Definition 2.10 A complex normed space is a vector space $\mathcal{X}$ over $\mathbb{C}$, equipped with a map $\|.\| : \mathcal{X} \to \mathbb{R}$, which satisfies:

1. $\|\lambda A\| = |\lambda|\|A\|$ (scaling),
2. $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality also called subadditivity),
3. If $\|A\| = 0$, then A is the zero vector (separates points).

One of the nice features of normed spaces is that the continuity of maps between such spaces can be probed by convergent sequences. Recall that in general:

Definition 2.11 A point x of the topological space $\mathcal{X}$ is the limit of the sequence (x_n) in $\mathcal{X}$ if, for every neighbourhood U of x, there is an N such that, for every $n \geq N$, $x_n \in U$.

In particular, for normed spaces:

Definition 2.12 A point x of a normed space $(X, \|.\|)$ is the limit of the sequence (x_n) if, for all $\varepsilon > 0$, there is an N such that, for every $n \geq N$, $\|x_n - x\| < \varepsilon$. A sequence that has a limit is called convergent.

Definition 2.13 Let $\mathcal{X}, \mathcal{Y}$ be topological spaces. Then a function $f : \mathcal{X} \to \mathcal{Y}$ is said to be *sequentially continuous* if for every convergent sequence (x_n) in $\mathcal{X}$ with the limit x we have $f(x_n) \to f(x)$ in $\mathcal{Y}$.

An elementary result from analysis states that if $\mathcal{X}, \mathcal{Y}$ are normed spaces equipped with topologies induced by the respective norms then $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if it is sequentially continuous. However, in Sect. 2.4.1 we will consider spaces where these two notions do not coincide.

Having defined the notion of convergence of sequences, we are now ready to introduce the notion of *completeness*. First we define:

Definition 2.14 A sequence (x_n) in a normed space $\mathcal{X}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all integers m, n such that $m, n > N$ we have $\|x_n - x_m\| < \varepsilon$.

Definition 2.15 A normed space $\mathcal{X}$ in which every Cauchy sequence converges to an element of $\mathcal{X}$ is called *complete*.

Given a normed space $\mathcal{X}$ that is not complete one can always construct its *completion*,[1] i.e. a complete normed space that contains $\mathcal{X}$ as a dense subspace.

Let us now come back to our algebras. If an algebra $\mathfrak{A}$ is equipped with a norm, we can ask for the continuity of the algebraic relations with respect to the norm topology and for some notion of completeness. This leads to the following definitions.

Definition 2.16 A *normed algebra* $\mathfrak{A}$ is a normed vector space whose norm $\|.\|$ satisfies

$$\|AB\| \leq \|A\|\|B\|.$$

If $\mathfrak{A}$ is unital, then it is a normed unital algebra if in addition $\|\mathbb{1}\| = 1$.

Definition 2.17 A *Banach space* is a normed vector space equipped with the norm-induced topology that is complete with respect to this topology. A Banach (unital) algebra is a Banach space and a normed (unital) algebra with respect to the same norm.

A particularly important class of Banach algebras with involution is distinguished by the C^*-property. We will see in this chapter that such algebras can be used to describe spaces of observables in quantum systems.

Definition 2.18 A *C^*-algebra* is a Banach involutive algebra (Banach algebra with involution satisfying $\|A^*\| = \|A\|$), such that the norm has the C^*-property:

$$\|A^*A\| = \|A\|\|A^*\|, \quad \forall A \in \mathfrak{A}.$$

2.1.2 Observables and States

In this section we will see how the structures introduced in the previous section are used in quantum physics. First note that in order to describe a physical system we need to specify a collection of physical quantities, which we want to measure (we call them *observables*) and a collection of *states* in which the system can be prepared. Now we want to deduce what kind of mathematical structure is suitable to describe observable and states. Operationally, each observable corresponds to some measurement apparatus, which measures given properties of the system. An example of such an apparatus is a particle detector localized in some region of space.

[1] The completion of $\mathcal{X}$ can be constructed as a set of equivalence classes of Cauchy sequences in $\mathcal{X}$.

Next, one considers operations that can be performed on observables. Scaling of the measurement apparatus means multiplying the corresponding observable A by a real number. One can also consider other functions of the observables, which can be operationally realized as "repainting the scale". The simplest examples are monomials A^n, interpreted as measuring the observable A and taking the nth power of the result.

Now we discuss the notion of states. We need to assume that we are able to repeat experiments, so that we can measure a given observable repeatedly in the same state (i.e. for the same preparation of the system). This statistical interpretation presupposes that each experiment comes with a protocol that allows us to obtain the same initial condition each time it is repeated. Under this assumption, a state ω associates to an observable A a real number $\omega(A)$ obtained by averaging the results of measurements of A for the system prepared to be in the state ω. It is natural to assume that $\omega(\lambda A) = \lambda\omega(A)$ for $\lambda \in \mathbb{R}_+$ (scaling). Let $\mathbb{1}$ be the observable, which always takes value 1. For this observable we require that $\omega(\mathbb{1}) = 1$. One can also deduce the positivity of states from the fact that the average of positive numbers is positive, so $\omega(A^2) \geq 0$.

If we assume that physical properties of observables can be measured only by looking at expectation values in various states of the system, it is natural to identify the observables that give the same expectation values in all the states. Now let $\mathcal{A}$ be the space of equivalence classes of observables, where $A \sim B$ if $\omega(A) = \omega(B)$ for all states ω of the system. A notion of a norm can be introduced by assigning to each observable $A \in \mathcal{A}$ a finite positive number defined by

$$\|A\| \doteq \sup_{\omega} |\omega(A)|$$

The operational properties of states imply that $\|\lambda A\| = |\lambda|\|A\|$ for $\lambda \in \mathbb{R}$ and $\|A\| = 0$ implies that $A = 0$ (states separate observables). What is still missing is the linear structure on $\mathcal{A}$ and the product. Let us start with the linear structure. We want to be able to construct measuring devices that measure the sum of any two observables A and B, i.e. we need the operation "$A + B$". This operation has to satisfy

$$\omega(A + B) = \omega(A) + \omega(B),$$

for all states of the system. It is, however, not clear if an element "$A + B$" exists in $\mathcal{A}$, so one needs to embed the initial space of observables in a larger structure in such a way that states will remain positive linear functionals on this enlarged space. Further considerations (see for example [Str08]) lead to the notion of Jordan algebras [Jor33, JvNW34] and finally, by bringing in a complex structure, to C^*-algebras, introduced in [Gel43] and discussed in [Seg47a, Seg47b] in the context of quantum mechanics.

We can summarize the basic axioms in the algebraic approach to QM as follows:

1. A physical system is defined by its unital C^*-algebra $\mathfrak{A}$.
2. States are identified with positive, normalized linear functionals on $\mathfrak{A}$, i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and $\omega(\mathbb{1}) = 1$.

Note that on a unital C^*-algebra a positive, normalized linear functional is automatically continuous with respect to the topology induced by the C^*-norm. More generally, we can define states also on involutive topological algebras.

Definition 2.19 A *state* on an involutive algebra $\mathfrak{A}$ is a linear functional ω, such that:

$$\omega(A^*A) \geq 0, \ \omega(\mathbb{1}) = 1.$$

Observables are self-adjoint elements of $\mathfrak{A}$ and possible measurement results for an observable A are characterized by its spectrum $\sigma(A)$. Recall that an element A of a C^*-algebra is called self-adjoint if $A^* = A$.

Definition 2.20 The *spectrum* $\mathrm{spec}(A)$ of $A \in \mathfrak{A}$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda\mathbb{1}$ has no inverse in $\mathfrak{A}$.

A standard result from functional analysis states that a spectrum of self-adjoint element is a subset of the real line and this agrees with the physical intuition, as outcomes of measurements have to be real.

2.1.3 Hilbert Space Representations

Having defined the abstract setup we can proceed to a more concrete description that provides a way to recover the Dirac–von Neumann axioms. The crucial observation is that abstract elements of an involutive algebra $\mathfrak{A}$ can be realized as operators on some Hilbert space by a choice of a *representation*. Definitions introduced in this section follow closely [Mor13, RS80]. First let us recall the definition of a Hilbert space.

Definition 2.21 Let $\mathcal{H}$ be a complex vector space. A map $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a *Hermitian inner product* if

1. $\langle u, v \rangle = \overline{\langle u, v \rangle}, \forall u, v \in \mathcal{H}$,
2. $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$ (linear in the second argument),
3. $\langle v, v \rangle \geq 0$ where the case of equality holds precisely when $v = 0$ (positive definite).

Properties 1 and 2 imply that $\langle ., . \rangle$ is antilinear in the first argument. One can define a norm on $\mathcal{H}$ by setting

$$\|v\| \doteq \sqrt{\langle v, v \rangle}.$$

Definition 2.22 A *Hilbert space* $\mathcal{H}$ is a complex vector space with a Hermitian inner product $\langle ., . \rangle$ such that the norm induced by this product makes $\mathcal{H}$ into a Banach space.

In physics *separable Hilbert spaces* play an important role.

Definition 2.23 A Hilbert space $\mathcal{H}$ is called *separable* if it admits a countable subset whose linear span is dense in $\mathcal{H}$. In fact a Hilbert space is separable if it is either finite dimensional or has a countable basis.

We are ready to define the notion of linear operators on Hilbert spaces, which is important in the context of C^*-algebras and physical observables.

Definition 2.24 An *operator* A on a Hilbert space $\mathcal{H}$ is a linear map from a subspace $D \subset \mathcal{H}$ into $\mathcal{H}$. In particular, if $D = \mathcal{H}$ and A satisfies $||A|| \doteq \sup_{||x||=1}\{||Ax||\} < \infty$, it is called *bounded*.

We will always assume that D is dense in $\mathcal{H}$ (i.e. A is *denesly defined*).

Definition 2.25 Let A be a densly defined linear operator on a Hilbert space $\mathcal{H}$. Let $D(A^*)$ be the set of all $v \in \mathcal{H}$ such that there exists $u \in \mathcal{H}$ with

$$\langle Aw, v \rangle = \langle w, u \rangle, \quad \forall w \in D(A).$$

For each such $v \in D(A^*)$ we define $A^*v = u$. A^* is called the adjoint of A.

An important class of bounded operators is provided by the unitary ones.

Definition 2.26 A bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called a *unitary operator* if it satisfies $U^*U = UU^* = \mathbb{1}$.

Note that the space $\mathscr{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ forms a C^*-algebra. We will see later on that one can argue the other way and realize any abstract C^*-algebra as the algebra of bounded operators on some $\mathcal{H}$. If A is a bounded operator on a Hilbert space then the self-adjointness is the same as hermiticity, i.e. is the condition that $A^* = A$. In general this is not sufficient.

Definition 2.27 An operator A on a Hilbert space $\mathcal{H}$ with a dense domain $D(A) \subset \mathcal{H}$ is called *symmetric* if for any vectors $u, v \in D(A)$ we have $\langle u, Av \rangle = \langle Au, v \rangle$. This implies that $D(A) \subseteq D(A^*)$. A symmetric operator A is *self-adjoint* if in addition $D(A^*) \subset D(A)$.

Definition 2.28 Let A be an operator on a Hilbert space $\mathcal{H}$ with a dense domain $D(A) \subset \mathcal{H}$. A self-adjoint operator A' is called a *self-adjoint extension* of A if $D(A) \subseteq D(A')$ and if $A'v = Av$ for any $v \in D(A)$.

A is called *essentially self-adjoint* if it admits a unique self-adjoint extension.

Abstract elements of an involutive algebra $\mathfrak{A}$ are realized as operators on some Hilbert space by a choice of a *representation*.

Definition 2.29 A *representation* of an involutive unital algebra $\mathfrak{A}$ is a unital *-homomorphism π into the algebra of linear operators on a dense subspace $\mathcal{K}$ of a Hilbert space $\mathcal{H}$. In particular, a representation of a C^*-algebra $\mathfrak{A}$ is a unital *-homomorphism $\pi : \mathfrak{A} \to \mathscr{B}(\mathcal{H})$.

A representation π is called *faithful* if $\mathrm{Ker}\,\pi = \{0\}$. It is called *irreducible* if there are no non-trivial subspaces of $\mathcal{H}$ invariant under $\pi(\mathfrak{A})$.

Definition 2.30 Two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of a C^*-algebra $\mathfrak{A}$ are called *unitarily equivalent*, if $U\pi_1(A) = \pi_2(A)U$ holds for all $A \in \mathfrak{A}$ with some unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$.

In the Dirac–von Neumann axiomatic framework, one postulates that physical observables are self-adjoint operators acting on some Hilbert space. The connection between the algebraic formulation and the Hilbert space picture is provided by means of the famous GNS (Gelfand–Naimark–Segal) theorem.

Theorem 2.1 *Let ω be a state on an involutive unital algebra $\mathfrak{A}$. Then there exists a representation π of the algebra by linear operators on a dense subspace $\mathcal{K}$ of some Hilbert space $\mathcal{H}$ and a unit vector $\Omega \in \mathcal{K}$, such that*

$$\omega(A) = (\Omega, \pi(A)\Omega),$$

and $\mathcal{K} = \{\pi(A)\Omega,\, A \in \mathfrak{A}\}$.

Proof First we introduce a scalar product on the algebra $\mathfrak{A}$ using the state ω:

$$\langle A, B\rangle \doteq \omega(A^*B).$$

Linearity for the right and antilinearity for the left argument are easy to prove. Hermiticity $\langle A, B\rangle = \overline{\langle B, A\rangle}$ follows from the positivity of ω and the fact that we can write A^*B and B^*A as linear combinations of positive elements:

$$2(A^*B + B^*A) = (A + B)^*(A + B) - (A - B)^*(A - B),$$
$$2(A^*B - B^*A) = -i(A + iB)^*(A + iB) + i(A - iB)^*(A - iB).$$

From the positivity of ω, it also follows that the scalar product is positive semidefinite, i.e. $\langle A, A\rangle \geq 0$ for all $A \in \mathfrak{A}$. We now study the set

$$\mathfrak{N} \doteq \{A \in \mathfrak{A} | \omega(A^*A) = 0\}.$$

We show that $\mathfrak{N}$ is a left ideal of $\mathfrak{A}$. Because of the Cauchy–Schwarz inequality $\mathfrak{N}$ is a subspace of $\mathfrak{A}$. The same inequality implies that for $A \in \mathfrak{N}$ and $B \in \mathfrak{A}$ we obtain

$$\omega((BA)^*BA) = \omega(A^*B^*BA) = \left\langle B^*BA, A\right\rangle \leq \sqrt{\langle B^*BA, B^*BA\rangle}\sqrt{\langle A, A\rangle} = 0,$$

hence $BA \in \mathfrak{N}$. Let us define $\mathcal{K}$ as the quotient $\mathfrak{A}/\mathfrak{N}$. Clearly, the scalar product is positive definite on $\mathcal{K}$ and we complete it to obtain a Hilbert space $\mathcal{H}$. The representation π is induced by the operation of left multiplication on $\mathfrak{A}$, i.e.

$$\pi(A)(B + \mathfrak{N}) \doteq AB + \mathfrak{N},$$

and we set $\Omega = \mathbb{1} + \mathfrak{N}$. If $\mathfrak{A}$ is a C*-algebra, one can show that the operators $\pi(A)$ are bounded, hence admitting unique continuous extensions to bounded operators on $\mathcal{H}$.

We now show that the construction is unique up to unitary equivalence. Let $(\pi', \mathcal{K}', \mathcal{H}', \Omega')$ be another quadruple satisfying the conditions of the theorem. Then we define an operator $U : \mathcal{K} \to \mathcal{K}'$ by

$$U\pi(A)\Omega \doteq \pi'(A)\Omega'.$$

U is well defined, since $\pi(A)\Omega = 0$ if and only if $\omega(A^*A) = 0$, but then also $\pi'(A)\Omega' = 0$. Moreover U preserves the scalar product and is invertible, hence it has a unique extension to a unitary operator from $\mathcal{H}$ to $\mathcal{H}'$. It follows that π and π' are unitarily equivalent. □

The representation π is in general not irreducible, i.e. there may exist a nontrivial closed invariant subspace. In this case, the state ω is not pure, which means that it is a convex combination of other states,

$$\omega = \lambda\omega_1 + (1 - \lambda)\omega_2 \ , \ 0 < \lambda < 1 \ , \ \omega_1 \neq \omega_2.$$

We have seen that the algebraic formulation of quantum mechanics (QM) allows us to characterize a physical system purely in terms of its observable C^*-algebra $\mathfrak{A}$ and states on it. The Hilbert space representations can then be obtained from states by means of the GNS theorem. One can also obtain the probabilistic interpretation of QM as follows. Given an observable A and a state ω on a C*-algebra $\mathfrak{A}$ we reconstruct the full probability distribution $\mu_{A,\omega}$ of measured values of A in the state ω from its moments, i.e. the expectation values of powers of A,

$$\int \lambda^n d\mu_{A,\omega}(\lambda) = \omega(A^n).$$

We can now apply these methods to some simple physical situations. The first example is related to the *canonical commutation relations*.

Example 2.1 Let L be a real vector space with a symplectic form σ, i.e. a bilinear form σ on L which is antisymmetric,

$$\sigma(x, y) = -\sigma(y, x),$$

and nondegenerate,

$$\sigma(x, y) = 0 \; \forall \; y \in L \text{ implies } x = 0.$$

We consider the unital *-algebra $\mathcal{W}(L, \sigma)$ over $\mathbb{C}$ generated by abstract symbols $W(x)$ (the Weyl generators), satisfying the relation

$$W(x)W(y) = e^{i\sigma(x,y)} W(x + y).$$

The involution is defined by

$$W(x)^* = W(-x)$$

and the unit is $\mathbb{1} = W(0)$.

We define a norm on $\mathcal{W}(L, \sigma)$ by

$$\left\| \sum_{i=1}^{n} \lambda_i W(x_i) \right\|_1 = \sum_{i=1}^{n} |\lambda_i|.$$

This norm satisfies the condition $\|AB\|_1 \leq \|A\|_1 \, \|B\|_1$ of an algebra norm. Moreover, the involution is isometric, $\|A^*\|_1 = \|A\|_1$ and we obtain an involutive normed algebra $\mathcal{W}(L, \sigma)$.

After [Mor13] we recall known facts about the existence of the unique C^*-norm on $\mathcal{W}(L, \sigma)$.

Proposition 2.1 *The following hold true:*

1. *There exists a norm $\|.\|_0$ on $\mathcal{W}(L, \sigma)$ satisfying the C^*-property,*
2. *In any C^*-norm Weyl generators have norm 1.*
3. *If we set*

$$\|A\|_c \doteq \sup\{\|A\|_0, \;\; \textit{such that} \;\; \|.\|_0 : \mathcal{W}(L, \sigma) \to [0, \infty) \textit{ is a } C^*\textit{-norm}\},$$

   then $\|.\|_c$ is a C^*-norm.
4. *Let $\mathfrak{W}(L, \sigma)$ be the completion of $\mathcal{W}(L, \sigma)$ with respect to $\|.\|_c$, then $\mathfrak{W}(L, \sigma)$ is a C^*-algebra, associated to (L, σ) uniquely up to isomorphism.*
5. *$\mathfrak{W}(L, \sigma)$ is simple, i.e. there are no non-trivial closed, *-invariant two-sided ideals.*

Proof For proof see [BGP07] as well as [Mor13]. To see that the supremum defining $\|.\|_c$ is finite, note that generators $W(x)$ are of norm 1 with respect to every C^*-norm, so if $A = \sum_i a_i W(x_i)$, then $\|A\| \leq \sum_i |a_i| = \|A\|_1$, which provides the upper bound for the supremum. □

Let's consider a particular example of a symplectic space (L, σ), which realizes canonical commutation relations for a free quantum particle in d dimensional space. In this case $L = \mathbb{R}^{2d}$ and we write elements of L in the form $\boldsymbol{X} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$. We define

$$\sigma\left((\boldsymbol{\alpha},\boldsymbol{\beta}),(\boldsymbol{\alpha}',\boldsymbol{\beta}')\right)=-\frac{1}{2}\hbar(\boldsymbol{\alpha}\cdot\boldsymbol{\beta}'-\boldsymbol{\alpha}'\cdot\boldsymbol{\beta}),$$

where $\cdot$ is the scalar product on $\mathbb{R}^d$. If the generators of the resulting Weyl C^*-algebra $\mathfrak{W}(L,\sigma)$ are represented by operators on a Hilbert space in such a way that they depend strongly continuously[2] on the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, then such a representation is called regular. It was proven by von Neumann that all the regular reducible representations of the resulting Weyl algebra are unitary equivalent. Another theorem important in this context is due to Stone [Sto30]:

Theorem 2.2 *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a unique (not necessarily bounded) self-adjoint operator A such that*

$$U_t=e^{itA},\quad \forall t\in\mathbb{R}.$$

Conversely, if A is a (not necessarily bounded) self-adjoint operator on a Hilbert space $\mathcal{H}$, then the one-parameter family $(U_t)_{t\in\mathbb{R}}$ of unitary operators defined by means of the Spectral Theorem for Self-Adjoint Operators (see for example Chap. 9 [Mor13]) as

$$t\mapsto U_t:=e^{itA}$$

is strongly continuous.

For $\mathfrak{W}(L,\sigma)$ this implies that there exist self-adjoint generators $q^1,\ldots,q^d$, $p^1,\ldots,p^d$ of 1-parameter groups of unitary operators

$$W(0,\ldots,\alpha_k,\ldots,0)=e^{i\alpha_k p^k},\quad W(0,\ldots,\beta_k,\ldots,0)=e^{i\beta_k q^k},$$

We denote $\boldsymbol{p}\doteq(p^1,\ldots,p^d)$, $\boldsymbol{q}\doteq(q^1,\ldots,q^d)$. Generators $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the canonical commutation relations

$$[q^k,p^j]=\delta_{kj},\quad [q^k,q^j]=0,\quad [p^k,p^j]=0$$

and one can write an arbitrary generator $W(\boldsymbol{\alpha},\boldsymbol{\beta})$ in the form

$$W(\boldsymbol{\alpha},\boldsymbol{\beta})=e^{-\frac{i\hbar\alpha\cdot\beta}{2}}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{p}}e^{i\boldsymbol{\beta}\cdot\boldsymbol{q}}=e^{\frac{i\hbar\alpha\cdot\beta}{2}}e^{i\boldsymbol{\beta}\cdot\boldsymbol{q}}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{p}}.$$

The Schrödinger representation of this Weyl algebra is defined on the Hilbert space of square integrable functions $\mathcal{L}_2(\mathbb{R}^d)$ with

$$\left(\pi(W(\boldsymbol{\alpha},\boldsymbol{\beta}))\Phi\right)(\boldsymbol{X})=e^{\frac{i\hbar\alpha\cdot\beta}{2}}e^{i\boldsymbol{\beta}\cdot\boldsymbol{X}}\Phi(\boldsymbol{X}+\hbar\boldsymbol{\alpha})\,,\tag{2.2}$$

[2] A net $\{T_\alpha\}$ of operators on a Hilbert space $\mathcal{H}$ converges strongly to an operator T if and only if $||T_\alpha x-Tx||\to 0$ for all $x\in\mathcal{H}$. The definition of a net is at p. 22 in Footnote 5.

for $\Phi \in \mathcal{L}_2(\mathbb{R}^d)$. As mentioned before, all the regular irreducible representations are unitary equivalent to this one. If one does not require continuity there are many more representations. In quantum field theory this uniqueness result does not apply, and one has to deal with a huge class of inequivalent representations. Note, however, that the construction of the Weyl algebra makes sense also for L infinite dimensional, so it can be applied in the context of field theory.

A particularly interesting class of states on $\mathfrak{W}(L, \sigma)$ is provided by *quasi-free states*.

Definition 2.31 A state ω on $\mathfrak{W}(L, \sigma)$ is called *quasi-free* if there exists $\eta : L \times L \to \mathbb{R}$, a symmetric form such that

$$\omega\left(W(x)\right) = e^{-\frac{1}{2}\eta(x,x)}.$$

The form η is then called the covariance of the quasi-free state ω.

The following theorem provides a way to easily find quasi-free states.

Theorem 2.3 *Let $\eta : L \times L \to \mathbb{R}$ be a symmetric form. The following are equivalent:*

1. *$\eta_{\mathbb{C}} + \frac{i}{2}\sigma_{\mathbb{C}} \geq 0$ on $L^{\mathbb{C}}$, the complexification of L, where $\eta_{\mathbb{C}}, \sigma_{\mathbb{C}} : L^{\mathbb{C}} \times L^{\mathbb{C}} \to \mathbb{C}$ are canonical sesquilinear extensions of η, σ.*
2. *$|\sigma(x_1, x_2)| \leq 2\sqrt{\eta(x_1, x_1)}\sqrt{\eta(x_2, x_2)}$, for all $x_1, x_2 \in L$.*
3. *There exists a quasi-free state ω_η on $\mathfrak{W}(L, \sigma)$ with covariance η.*

Proof For proof see for example [AS71, DG13a]. □

This result holds also if L is infinite dimensional and will be used later in Sect. 5.3. We define a complex scalar product on the complex vector space $L^{\mathbb{C}}$ by

$$\langle x, y\rangle = \eta_{\mathbb{C}}(x, y) + \frac{i}{2}\sigma_{\mathbb{C}}(x, y). \tag{2.3}$$

The GNS Hilbert space representation corresponding to ω_η turns out to be the bosonic Fock space:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} (\mathcal{H}_1^{\otimes n})_{\text{symm}} \; ; \mathcal{H}_1 = \overline{L^{\mathbb{C}}/\mathrm{Ker}(\langle ., .\rangle)}$$

The state ω_η is pure (i.e. the associated GNS representation is irreducible) if and only if the map $L \to L^{\mathbb{C}}/\mathrm{Ker}(\langle ., .\rangle)$ is surjective. The latter holds if and only if the pair $(2\eta, \sigma)$ is Kähler.

Definition 2.32 A pair $(2\eta, \sigma)$ consisting of a symmetric form 2η and symplectic form σ on L is called *Kähler* if the ranges of the two coincide $\mathrm{Ran}(2\eta) = \mathrm{Ran}(2\sigma)$, 2η is positive definite and $J \doteq \sigma^{-1}2\eta$ satisfies $J^2 = -\mathbb{1}$ (i.e. J is an anti-involution).

If $(2\eta, \sigma)$ is Kähler, then the quadruple $(L, 2\eta, \sigma, J)$ is a *Kähler space*. We will come back to this structure in the context of QFT in Sect. 5.3.

2.1.4 Dynamics and the Interaction Picture

If we want to model a physical system that evolves with time, we need to introduce the notion of dynamics. A very detailed discussion of quantum dynamics can be found in [BR87, BR97]. Here we only sketch the main ideas. Let $\mathfrak{A}$ be a C^*-algebra of observables and let $A_t \in \mathfrak{A}$ be some observable corresponding to the measurement apparatus A at time t.[3] We postulate that the algebra of observables $\mathfrak{A}$ doesn't change with time, so the assignment $t \mapsto A_t$ can be described by a 1-parameter group of automorphisms α_t, such that $A_t = \alpha_t(A)$ and we assume that α_t is strongly continuous.

For a given state ω we consider the family of states that are related to it by time-translations and it is natural to require some stability properties from the GNS-associated representation π_ω. If π_ω is irreducible, this stability requirement is realized as the condition that α_t has to be implemented by some unitary operator $U(t)$, i.e.

$$\pi_\omega(\alpha_t(A)) = U(t)^{-1}\pi_\omega(A)U(t), \quad \forall A \in \mathfrak{A}. \tag{2.4}$$

Now we apply Stone's Theorem 2.2 to deduce the existence of a self-adjoint generator H, called the *Hamiltonian* and we write

$$U(t) = e^{-itH} \quad \forall t \in \mathbb{R}.$$

By differentiating (2.4) we obtain the known evolution equation in the Heisenberg picture,

$$\frac{d}{dt}A(t) = i[H, A(t)], \tag{2.5}$$

where we have put $A(t) = U(t)^* A U(t)$ and we have omitted the symbol π_ω. To get the Schrödinger picture, we consider $\psi \in D(H)$ a Hilbert space vector in the domain of essential selfadjointness (see Definition 2.28) of H, and define $\psi_S(t) \doteq U(t)\psi$. We can now rewrite (2.5) in the form

$$i\frac{d}{dt}\psi_S(t) = H\psi_S(t). \tag{2.6}$$

This is time-evolution in the Schrödinger picture. If we want to construct a model of a quantum dynamical system, we usually start with a Hamiltonian H which is an operator on $\mathcal{H}_\pi$ that solves (2.6) for some initial data $\psi_S(0)$, within the domain $D(H)$. A solution to the initial value problem then defines the *propagator* $U(t, 0)$, i.e.

$$\psi_S(t) = U(t, 0)\psi_S(0).$$

[3] As sharp localization is physically impossible, operationally we can think of A_t as the average over some interval $[t - \epsilon, t + \epsilon]$ centered at t, for a fixed value of $\epsilon > 0$.

Note that the main difference between (2.5) and (2.6) is that in the Heisenberg picture states remain stationary and operators evolve with time, while in the Schrödinger picture it is the other way round. Often, solving the initial value problem of the form (2.6) is very difficult and it is convenient to split the Hamiltonian into two terms

$$H = H_0 + H_{int},$$

where the propagator for H_0 can be found relatively easily and then we try to solve the problem perturbatively. This point of view is something in-between the Heisenberg and Schrödinger pictures and we call it the *interaction picture*. H_{int} is called the *interaction Hamiltonian*. Let $U_0 \doteq e^{-itH_0}$. In the interaction picture the states are represented by

$$\psi_I(t) = U_0^* \psi_S(t) = e^{itH_0}\psi_S(t) = e^{itH_0}e^{-itH}\psi,$$

where ψ_S is a state in the Schrödinger picture and ψ is a state in the Heisenberg picture. Observables of the interaction picture evolve according to

$$A(t) = U_0(t)^* A_S U_0(t),$$

where A_S denotes the Schrödinger picture observable. In particular

$$H_{int} = U_0(t)^* H_{int} U_0(t)$$

for the interaction Hamiltonian H_{int}. Now the evolution Eq. (2.6) implies that

$$i\frac{d}{dt}\psi_I = H_{int}\,\psi_I. \tag{2.7}$$

Given initial data $\psi_I(t_0)$, we want to find the solution to this equation in terms of a propagator $U_I(t, t_0)$, so that

$$\psi_I(t) = U_I(t, t_0)\psi_I(t_0).$$

By definition we have

$$U_I(t, t_0) = e^{itH_0}e^{-i(t-t_0)H}e^{-it_0H_0},$$

and from (2.7) it follows that the propagator has to satisfy

$$i\frac{d}{dt}U_I(t, t_0) = H_{int}(t)U_I(t, t_0), \qquad U_I(t_0, t_0) = \mathbb{1}. \tag{2.8}$$

A formal solution to the above equation is then given by the Dyson series

$$\begin{aligned} U_I(t,t_0) &= \mathbb{1} - i\int_{t_0}^{t} H_{int}(t_1)dt_1 - \int_{t_0}^{t}\int_{t_0}^{t_2} H_{int}(t_2)H_{int}(t_1)dt_1dt_2 + \dots \\ &= \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_{t_0<t_1<\dots<t_n<t} \cdots \int H_{int}(t_n)\dots H_{int}(t_1)dt_1\dots dt_n. \end{aligned} \tag{2.9}$$

We can simplify the notation by introducing the *time-ordering operator* T defined on operators $A(t)$ and $B(t)$ by

$$T(A(t)B(t')) = \begin{cases} A(t)B(t'), \text{ if } t < t' \\ B(t')A(t), \text{ if } t' < t \end{cases}. \tag{2.10}$$

We can now rewrite the formula (2.9) as a time-ordered exponential, i.e.

$$\begin{aligned} U_I(t,t_0) &= \mathbb{1} + T\left[\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\left(\int_{t_0}^{t} H_{int}(t')dt'\right)^n\right] \\ &= T\left[\exp\left(-i\int_{t_0}^{t} H_{int}(t')dt'\right)\right]. \end{aligned} \tag{2.11}$$

We define the Møller operators $\mathcal{S}^{\pm}$ as the strong limits of $U_I(t,t_0)$ as t_0 approaches $\pm\infty$, as long as these limits exist.

$$\mathcal{S}_{\pm} \doteq \underset{t\to\pm\infty}{\text{s-lim}}\, U_I(0,t).$$

The scattering operator $\mathcal{S}$ (the S-matrix) is then defined by

$$\mathcal{S} \doteq \mathcal{S}_+^*\mathcal{S}_-. \tag{2.12}$$

We will use these ideas later on, in Sect. 6.1 to perturbatively construct QFT models.

2.2 Causality

After introducing basic notions from quantum mechanics, the next step towards quantum field theory leads through *spacetime geometry*. Historically, QFT was conceived as a framework that allows us to combine quantum mechanics with special relativity. The latter is based on concepts such as *Minkowski spacetime* and *causality*. In fact, the algebraic approach to QFT can be generalized beyond Minkowski spacetime and one can apply it to construct models on a wide class of Lorenzian manifolds. In this section we will review some basic concepts from Lorentzian geometry that are relevant for our framework.

Definition 2.33 *A spacetime* is a pair $\mathcal{M} = (M, g)$, where M is a smooth (4-dimensional) manifold (we assume it to be Hausdorff, paracompact, connected) and g is a smooth Lorentzian metric, i.e. a smooth tensor field $g \in \Gamma(T^*M \otimes T^*M)$, s.t. for every $p \in M$, g_p is a symmetric non-degenerate bilinear form. We require the metric g to be of the Lorentz signature $(+, -, -, -)$.

Remark 2.1 Let us make a few remarks concerning the above definition:

1. The assumption for a manifold to be *Hausdorff* means that points can be separated (for every pair of points x, y, there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \varnothing$)). In general topology one can drop this assumption and an example of a non-Hausdorff manifold is a line with two origins, i.e. the quotient space of two copies of the real line $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$, with the equivalence relation $(x, a) \sim (x, b)$ if $x \neq 0$.
2. The *paracompactness* is needed as a sufficient condition for the existence of partitions of unity. It means that for every open cover $(U_\alpha)_{\alpha \in A}$, there exists a refinement[4] $(V_\beta)_{\beta \in B}$ that is locally finite, i.e. each $x \in M$ has a neighborhood that intersects only finitely many sets of $(V_\beta)_{\beta \in B}$.
3. We assumed also that M is connected, i.e. it cannot be represented as a disjoint union of two or more non-empty sets. Later on we will see that in a more general context one has to drop this assumption and consider manifolds that are not connected.

Definition 2.34 A spacetime $\mathcal{M}$ is said to be *oriented* if it is equipped with a differential form of maximal degree (a volume form) that does not vanish anywhere. We say that $\mathcal{M}$ is *time-oriented* if it is equipped with a smooth vector field u on M such that for every $p \in M$, $g(u, u) > 0$ holds.

We will always assume that our spacetimes are orientable and time-orientable. We fix the orientation and choose the time-orientation by selecting a specific vector field u with the above property.

Example 2.2 A standard example is 4 dimensional Minkowski spacetime $\mathbb{M}$, which is $\mathbb{R}^4$ with the diagonal metric $\eta = \text{diag}(1, -1, -1, -1)$.

An important feature of the Lorentzian signature, which distinguishes it from the Euclidean signature, is that it allows to introduce some important classes of smooth curves.

Definition 2.35 Let $\gamma : \mathbb{R} \supset I \to M$ be a smooth curve in M, for I an interval in $\mathbb{R}$ and let $\dot{\gamma}$ be the vector tangent to the curve. We say that γ is

[4]An open cover $(V_\beta)_{\beta \in B}$ is a refinement of an open cover $(U_\alpha)_{\alpha \in A}$, if $\forall \beta \in B$, $\exists \alpha$ such that $V_\beta \subseteq U_\alpha$.

- timelike, if $g(\dot{\gamma}, \dot{\gamma}) > 0$,
- spacelike, if $g(\dot{\gamma}, \dot{\gamma}) < 0$,
- lighlike (null), if $g(\dot{\gamma}, \dot{\gamma}) = 0$,
- causal, if $g(\dot{\gamma}, \dot{\gamma}) \geq 0$.

The classification of curves defined above is referred to as the *causal structure*. The presence of time orientation allows for a further refinement of this classification.

Definition 2.36 Given the global timelike vector field u (the time orientation) on M, a causal curve γ is called future-directed if $g(u, \dot{\gamma}) > 0$ all along γ. It is past-directed if $g(u, \dot{\gamma}) < 0$.

Using the causal structure one can distinguish points of spacetime that are in the future or in the past of a given point $p \in \mathcal{M}$.

Definition 2.37 Let $p \in \mathcal{M}$ be a point in a time-oriented spacetime.

(i) $J^{\pm}(p)$ is defined to be the set of all points in M which can be connected to p by a future(+)/past(−)-directed causal curve $\gamma : I \to M$ so that $x = \gamma(\inf\ I)$.
(ii) The set $J^{+}(p)$ is called the causal future and $J^{-}(p)$ the causal past of p. The boundaries $\partial J^{\pm}(p)$ of these regions are called respectively: the *future/past lightcone*.
(iii) The future (past) of a subset $B \subset M$ is defined by

$$J^{\pm}(B) = \bigcup_{p \in B} J^{\pm}(p) \, .$$

The physical importance of the structures presented above becomes clear in the context of general relativity (GR). One of the postulates of GR states that *massive particles can move only on time-like curves and light travels following null curves*, i.e. *nothing travels faster than light*. Consequently, one of the fundamental principles of physics, *the principle of causality*, states that an event happening at a point p can be influenced only by events in $J^{-}(p)$ and that the consequences of this event can influence only the events in $J^{+}(p)$.

Definition 2.38 A subset $A \subset M$ is called *past-(future-) compact* if $A \cap J^{\mp}(p)$ is compact for all $p \in M$.

Definition 2.39 Two subsets O_1 and O_2 in M are called *causally separated* (or spacetime separated) if they cannot be connected by a causal curve, i.e. if for all $x \in \overline{O_1}$, $J^{\pm}(x)$ has empty intersection with $\overline{O_2}$.

Another important definition is that of the *causal complement* of a given region O.

Definition 2.40 The *causal complement* $O^{\perp}$ is defined as the largest open set in M that is causally separated from O.

It follows from the principle of causality that events happening at spacelike separated points cannot influence each other. In classical physics this property is realized as a consequence of some properties of normally hyperbolic partial differential equations. In Sect. 2.3 we will see how these ideas can be implemented into the framework of quantum theory.

Example 2.3 Consider Minkowski spacetime $\mathbb{M} = (\mathbb{R}^4, \eta)$. The set of points that are causally separated from a given point $P \in \mathbb{M}$ is called the *lighcone* with apex P. It is easy to verify that a point $Q \in \mathbb{M}$

- lies on the lightcone with apex P if and only if the vector $\overrightarrow{PQ}$ is lightlike,
- is in the future (past) of P if and only if the vector $\overrightarrow{PQ}$ is time-like and its 0th component is positive (negative),
- is spacelike to P if and only if $\overrightarrow{PQ}$ is spacelike.

These concepts are illustrated in Fig. 2.1.

Definition 2.41 Motivated by Example 2.3 we introduce the following notation:

- $\overline{V}_+ \doteq \{v \in \mathbb{R}^4 | \eta(v, v) \geq 0,\ v_0 > 0\}$ is called the *closed future lightcone*.
- $\overline{V}_- \doteq \{v \in \mathbb{R}^4 | \eta(v, v) \geq 0,\ v_0 \leq 0\}$ is called the *closed past lightcone*.

These definitions can also be applied to subsets of tangent and cotangent spaces $T_x M$ and $T^*_x M$, as these spaces can be mapped to $\mathbb{R}^4$ with the use of appropriate charts.

Not all Lorentzian spacetimes are equally convenient for constructing quantum field theory models. For example, several conceptual and technical problems appear when we consider spacetimes with closed time-like curves. To exclude such situations, we will restrict ourselves to spacetimes that are *globally hyperbolic*.

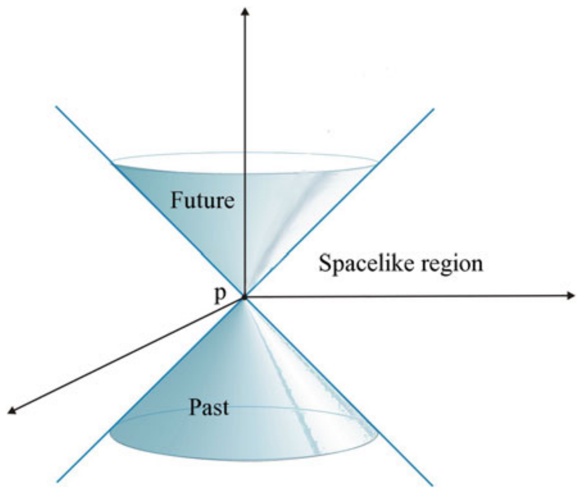

Fig. 2.1 A lightcone in Minkowski spacetime

Definition 2.42 (*after* [BS03]) A spacetime is called *globally hyperbolic* if it does not contain closed causal curves and if for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.

It was shown in [BS03] that globally hyperbolic spacetimes have many important features. To understand them better we need to introduce some further definitions.

Definition 2.43 A causal curve is *future inextendible* if there is no $p \in M$ such that:

$$\forall U \subset M \text{ open neighborhoods of } p,\ \exists t' \text{ s.t. } \gamma(t) \in U\ \forall t > t'.$$

Definition 2.44 *A Cauchy hypersurface* in M is a smooth subspace of M such that every inextendible causal curve intersects it exactly once.

The significance of Cauchy hypersurfaces lies in the fact that one can use them to formulate the initial value problem for partial differential equations and for some classes of such equations this problem has a unique solution. The fundamental theorem relating different equivalent notions of global hyperbolicity has been proven in [BS03].

Theorem 2.4 (after [BS03]) *The following definitions of global hyperbolicty of a Lorentzian manifold $\mathcal{M}$ are equivalent:*

- $\mathcal{M}$ *does not contain closed causal curves and for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.*
- $\mathcal{M}$ *contains a Cauchy surface.*
- $\mathcal{M}$ *admits a foliation by Cauchy surfaces.*

2.3 Haag–Kastler Axioms

In Sect. 2.1 we introduced such fundamental notions of quantum theory as states and observables. Now we want to make these compatible with the ideas of special and general relativity, reviewed in Sect. 2.2, where the causal structure plays an important role. The main conceptual difficulty is to find a way to implement the idea that "nothing travels faster than light" in such a way that it doesn't contradict the existence of quantum correlations in the theory. The groundbreaking idea of Rudolf Haag was to combine these notions using the principle of *locality* (*Nahwirkungsprincip*). In this framework, locality is the feature of observables, while states might exhibit correlations, i.e. they carry global information. One defines a QFT model by assigning to each bounded region $\mathcal{O} \subset \mathbb{M}$ of Minkowski spacetime the C^*-algebra of observables $\mathfrak{A}(\mathcal{O})$ that can be measured in this region. The notion of subsystem is realized by the requirement that if $\mathcal{O} \subset \mathcal{O}'$, then $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$. This condition is called *isotony* and it guarantees that one doesn't lose observables when considering a larger region of spacetime. The complete set of axioms for algebraic quantum field theory (AQFT) can be found in [HK64, Haa93, Ara99]; we will recall them briefly in this section.

The Haag–Kastler axioms (also called Araki-Haag–Kastler axioms) for a net[5] of C^*-algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ are:

- **Isotony**. For $\mathcal{O} \subset \mathcal{O}'$ we have $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$, see Fig. 2.2.
- **Locality (Einstein causality)**. Algebras associated to spacelike separated regions commute: if $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$, then $[A, B] = 0$, $\forall A \in \mathfrak{A}(\mathcal{O}_1)$, $B \in \mathfrak{A}(\mathcal{O}_2)$, where the commutator is taken in the sense of the inductive limit algebra $\mathfrak{A}$ (see the Definition 2.45 below). This expresses the "independence" of physical systems associated to regions $\mathcal{O}_1$ and $\mathcal{O}_2$.
- **Covariance**. Minkowski spacetime has a large group of isometries, namely the connected component of the Poincaré group $\mathcal{P}$. We require that for each $L \in \mathcal{P}$ there exists an isomorphism $\alpha_L^{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(L\mathcal{O})$, and that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_L^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_L^{\mathcal{O}_1}$ and $\alpha_{L'}^{L\mathcal{O}} \circ \alpha_L^{\mathcal{O}} = \alpha_{L'L}^{\mathcal{O}}$.
- **Time slice axiom**: The algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region. Physically this correspond to the well-posedness of an initial value problem, i.e. we only need to determine our observables in some small time interval $(t_0 - \epsilon, t_0 + \epsilon)$ to reconstruct the full algebra.
- **Spectrum condition**. Physically this condition is interpreted as the positivity of energy. One assumes that there exists a compatible family of faithful representations $\pi_{\mathcal{O}}$ of $\mathfrak{A}(\mathcal{O})$ on a fixed Hilbert space (i.e. the restriction of $\pi_{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\pi_{\mathcal{O}_1}$ for $\mathcal{O}_1 \subset \mathcal{O}_2$) such that translations are unitarily implemented, i.e. there is a unitary representation U of the translation group satisfying

$$U(a)\pi_{\mathcal{O}}(A)U(a)^{-1} = \pi_{\mathcal{O}+a}(\alpha_a(A)) \ , \ A \in \mathfrak{A}(\mathcal{O}),$$

  and such that the joint spectrum of the generators P_μ of translations $e^{a\cdot P} = U(a)$, $a \cdot P = \sum_{\mu=0}^{3} a^\mu P_\mu$, is contained in the closed future lightcone: $\sigma(P) \subset \overline{V}_+$.

Definition 2.45 The inductive limit of local algebras $\mathfrak{A}(\mathcal{O})$ defines the *quasilocal algebra* $\mathfrak{A} \doteq \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}$ (the bar means taking the completion in the norm topology).

All these axioms, apart from the **Spectrum condition**, can be generalized to QFT's on general globally hyperbolic spacetimes. We will discuss this in more detail in the next section. There are many important conceptual results that have been proven in the AQFT framework. The first major success was the development of the Haag–Ruelle scattering theory [Haa58, Rue62], which provided an explanation why quantum field theory yields a theory of interacting particles. It is, however, an open question, whether all states in the vacuum representation admit a particle interpretation (the problem of asymptotic completeness). For recent works on that topic see [DT11, DG14b, DG14a]. Another remarkable result of AQFT is the Reeh–Schlieder

[5] A net in a topological space $\mathcal{X}$ is a function from some directed set (nonempty set with a reflexive and transitive binary relation) A to $\mathcal{X}$.

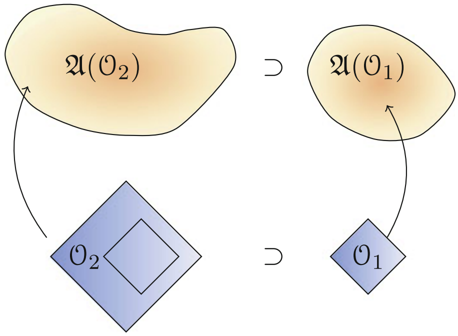

Fig. 2.2 Diagram representing inclusion of spacetime regions and corresponding C^*-algebras

Theorem [Haa93, RS61], see [BS14] for a recent discussion. Another known result achieved with the AQFT methods is the analysis of the superselection structure of QFT models [DHR71, DHR74]. Despite all this insight into the general structure of QFT, there remains the difficulty of constructing 4 dimensional interacting models that fulfil the Haag–Kastler axioms. For models in 2 dimensions see [Lec08, Tan12, BT13, Ala13, BC13] and references therein.

2.4 pAQFT Axioms

In this book we explore the possibility of dropping some of the assumptions of the Haag–Kastler framework, in order to allow for models that exist only in the formal, perturbative sense. The resulting framework is called *perturbative Algebraic Quantum Field Theory* (pAQFT). The generalization of the HK axioms to the perturbative context has been developed in a series of papers [DF01a, DF02, DF04, DF07, DF01b, BD08, Boa00, DB01, BDF09, Rej11b].

The generalization of the HK framework to curved spacetime has been for a long time an independent development. Some important early contributions include [Kay78, Dim80, KW91, Dim92]. Later these two generalizations met as the pAQFT on curved spacetimes after a seminal series of papers [BFK96, BF97, BF00, BFV03, HW01, HW02a, HW02b, HW05].

Abelian gauge theories were later treated in [DF98], while the Yang–Mills theories are the subject of [Hol08]. At the same time the mathematical foundations of pAQFT became better understood, mainly with the use of the functional approach, which is also the approach we take in this book. In [FR12b, FR12a, Rej11a] this framework was used to add the Batalin–Vilkovisky (BV) formalism to the pAQFT toolbox, which allows us to treat very general theories possessing local symmetries, including the bosonic string [BRZ14] and effective quantum gravity [BFR13].

2.4.1 More Functional Analysis

On the functional analytic side, we leave the realm of Banach spaces and allow for structures that have more general topologies. This involves some technical complications, but gives more flexibility in terms of model building. The most general class of topological vector spaces that we will use is the class of locally convex ones.

Definition 2.46 A topological vector space $\mathcal{X} \equiv (X, \tau)$ is called a *locally convex topological vector space* (LCVS) if there is a local base $\mathscr{T}$ whose members are convex.

Here by a *local base* we mean a collection $\mathscr{T}$, of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathscr{T}$. The open sets of $\mathcal{X}$ are then precisely those that are unions of translates of members of $\mathscr{T}$.

There is another way to characterize locally convex vector spaces, which allows us to make a connection with normed spaces, introduced in Definition 2.10. Instead of having one norm that characterizes the topology, we have a family of *seminorms*. A seminorm differs from a norm by not fulfilling property 3 in Definition 2.10. More precisely:

Definition 2.47 A *seminorm* on a vector space X is a real-valued function p on X such that:

1. $p(x + y) < p(x) + p(y)$ for all $x, y \in X$.
2. $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and all scalars $\lambda \in \mathbb{K}$.

We see that a seminorm already provides us with a lot of information, but it doesn't separate points. However, it is possible that a certain family of seminorms is separating.

Definition 2.48 A family $\mathscr{P}$ of seminorms on X is said to be *separating* if for each $x \neq 0$ there exists at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

Note that a separating family of seminorms already allows us to distinguish two elements of X.

Theorem 2.5 *To each separating family of seminorms on X we can associate a locally convex topology τ on X and vice versa: every locally convex topology is generated by some family of separating seminorms.*

Proof See [Rud91]. □

In the pAQFT framework a LCVS is usually the best that one can expect. Unfortunately it doesn't share many of the nice properties of a Banach space, but there are some distinguished classes of LCVS that are relatively well behaved and good for defining calculus on them. The "nicest" ones are *Fréchet* spaces. They are distinguished by the fact that their topology can be described in terms of a *metric*.

Theorem 2.6 *A locally convex topological vector space $\mathfrak{X} = (X, \tau)$ is metrizable if and only if τ can be defined by $\mathscr{P} = \{p_n : n \in \mathbb{N}\}$ a countable separating family of seminorms on X. One can equip X with a metric which is compatible with τ and which provides a family of convex balls.*

Proof See [Köt69, Rud91]. □

Usually a Fréchet space topology is defined explicitly by providing a countable separating family of seminorms. A LCVS from Theorem 2.6 can be equipped with the metric:

$$d(x, y) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)} \tag{2.13}$$

This metric is compatible with τ but in general it doesn't provide convex balls (see the discussion in [Rud91] after Theorem 1.24 and Exercise 18). Nevertheless it is good to know that you have a metric that can actually be written down in a closed form.

Definition 2.49 If X is complete with respect to the metric from Theorem 2.6, it is a *Fréchet space*.

In locally convex topological vector spaces which are not Fréchet, using sequences to probe continuity of maps is not enough and some important properties like for example completeness have to be formulated in terms of nets (for definition of a net, see Footnote 5 in Sect. 2.3).

Definition 2.50 A Cauchy net in a locally convex space is a net $\{x_\alpha\}_\alpha$ such that for every $\epsilon > 0$ and every seminorm p, there exists an α such that for all $\lambda, \mu > \alpha$, $p(x_\lambda - x_\mu) < \epsilon$. A locally convex space is complete if and only if every Cauchy net converges.

Compare this with Definitions 2.14 and 2.15 that are valid in normed space. For a locally convex topological vector space that is not complete, one can always construct a completion.

To end this section we remark on one more important aspect of LCVS, namely the definition of tensor products. In quantum theory tensor products are used to model systems that consist of independent subsystems. This is closely related to the notion of causality and we will come back to this issue in Sect. 2.5.

Definition 2.51 Let E and F be locally convex topological vector spaces and let $\otimes : E \times F \to E \otimes F$ be the canonical map into the corresponding tensor product. The finest topology on $E \otimes F$ that makes $\otimes$ continuous is called *the projective tensor topology* or the π-topology. The space $E \otimes F$ equipped with this topology is denoted by $E \otimes_\pi F$ and its completion by $E \hat{\otimes}_\pi F$.

It can be shown that the topology π is locally convex. Another possible topology on $E \otimes F$ is the so called *injective tensor topology*. Its definition is a little bit more involved. In some sense it is the weakest well behaved topology one can put on $E \otimes F$, while the projective tensor topology is the strongest.

The idea behind the injective topology is to define it via the topology on the space of continuous linear mappings $L(E'_\gamma, F)$.

Definition 2.52 We equip E' with the finest locally convex topology γ that coincides with the weak one on equicontinuous[6] sets. One can identify $E \otimes F$ with a subspace of $L(E'_\gamma, F)$. Next we equip $L(E'_\gamma, F)$ with the topology of uniform convergence on equicontinuous compact sets in E'. We denote the resulting topological space by $E\varepsilon F$. It is called *the ε-product* of E and F. The corresponding topology induced on $E \otimes F$ is called the ε-topology and $E \otimes F$ equipped with it is the injective tensor product $E \otimes_\varepsilon F$. The corresponding completion is denoted by $E\hat{\otimes}_\varepsilon F$

This topology is better behaved if we want to consider vector-valued distributions and was used (in a slightly modified version) by L. Schwartz in [Sch57, Sch58]. Inequivalent notions of tensor products on LCVS can create problems, but there is a large class of spaces where these notions coincide. These are *nuclear* locally convex topological vector spaces, studied by A. Grothendieck in [Gro55].

2.4.2 Axioms

In this section we introduce the generalization of the Haag–Kastler axioms which is the foundation of pAQFT. It is in fact convenient to extend the pAQFT framework also to classical field theory, to keep a uniform language.

Definition 2.53 A classical field theory model on a spacetime $\mathcal{M}$ is a net of locally convex topological Poisson $*$-algebras $\mathfrak{P}(\mathcal{O})$, each with sequentially continuous product and Poisson bracket $\lfloor.,.\rfloor$;

$$\mathcal{O} \mapsto \mathfrak{P}(\mathcal{O}),$$

where $\mathcal{O} \subset \mathcal{M}$ are bounded, simply-connected regions. The global algebra is obtained as the inductive limit

$$\mathfrak{P}(\mathcal{M}) \doteq \lim_{\mathcal{O}\subset\mathcal{M}} \mathfrak{P}(\mathcal{O}).$$

We require that **Locality** holds, i.e. if $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$, then

$$\lfloor A, B \rfloor = 0,$$

$\forall A \in \mathfrak{P}(\mathcal{O}_1)$, $B \in \mathfrak{P}(\mathcal{O}_2)$, where the Poisson bracket $\lfloor, \rfloor$ is taken in $\mathfrak{P}(\mathcal{M})$.

[6] A set A of continuous functions between two topological spaces E and F is equicontinuous at the points $x_0 \in E$ and $y_0 \in F$ if for any open set $\mathcal{O}$ around y_0, there are neighborhoods U of x_0 and V of y_0 such that for every $f \in A$, if the intersection of $f(U)$ and V is nonempty, then $f(U) \subseteq \mathcal{O}$. One says that A is equicontinuous if it is equicontinuous for all points $x_0 \in E$, $y_0 \in F$. The notion of equicontinuity becomes more intuitive, if we choose E and F to be metric spaces. The family A is equicontinuous at a point x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \epsilon$ for all $f \in A$ and all x such that $d(x_0, x) < \delta$. In other words we require all member of the family A to be continuous and to have equal variation over a given neighbourhood.

In Chap. 4 we show how to construct models of classical field theories in agreement with the above definition. In Chaps. 5–8 we will show how to quantize such classical models perturbatively. The resulting structure is not a net of C^*-algebras, due to the perturbative character of the construction. Nevertheless, many of the features of a Haag–Kastler net are still present.

Definition 2.54 A perturbative algebraic quantum field theory (pAQFT) model on a spacetime $\mathcal{M}$ is a net of topological $*$-algebras with sequentially continuous product

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}),$$

where $\mathcal{O} \subset \mathcal{M}$ are bounded, simply-connected regions and we require **Locality**. The global algebra is obtained as the inductive limit

$$\mathfrak{A}(\mathcal{M}) \doteq \lim_{\mathcal{O}\subset\mathcal{M}} \mathfrak{A}(\mathcal{O}).$$

The remaining Haag–Kastler axioms from Sect. 2.3, apart from the **Spectrum condition**, can be easily translated to a pAQFT context.

Definition 2.55 Further axioms:

1. A classical/quantum field theory model on a globally hyperbolic spacetime $\mathcal{M}$ satisfies the **Time-slice axiom** if the algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region.
2. If the underlying spacetime $\mathcal{M}$ has a non-trivial group of symmetries $\mathcal{G}$, we say that a model is **Covariant** on $\mathcal{M}$, if for $\beta \in \mathcal{G}$ there exists a family of isomorphisms $\alpha_\beta^{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(\beta\mathcal{O})$, such that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_\beta^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_\beta^{\mathcal{O}_1}$ and $\alpha_{\beta'}^{g\mathcal{O}} \circ \alpha_\beta^{\mathcal{O}} = \alpha_{\beta'\beta}^{\mathcal{O}}$.

The spectrum condition cannot be meaningfully defined on an arbitrary globally hyperbolic spacetime, as it relies on the action of translations, which is a special feature of $\mathbb{M}$. We will replace this condition with a requirement we impose on preferred states on our net of algebras. These preferred states are called *Hadamard states* and they realize the idea of positivity of energy. We discuss them in detail in Sect. 5.1.

2.5 Locally Covariant Quantum Field Theory

In the previous section we recalled the Haag–Kastler axioms and reviewed the generalization of these axioms to the situation where we drop some of the regularity conditions on the topology of local algebras and we drop the restriction to Minkowski spacetime, allowing for general globally hyperbolic backgrounds. We can go a step further and see what happens if we replace the embeddings of bounded regions $\mathcal{O}$ into a fixed spacetime $\mathcal{M}$ with arbitrary embeddings between pairs of globally hyperbolic spacetimes $\mathcal{N}$ and $\mathcal{M}$. We formalize this idea by introducing the notion of an *admissible embedding*.

Definition 2.56 We call an embedding $\chi : \mathcal{M} \to \mathcal{N}$ of a globally hyperbolic manifold $\mathcal{M}$ into another one $\mathcal{N}$ *admissible* if it is an isometry and it preserves orientations and the causal structure. The property of *preserving the causal structure* is defined as follows: for any causal curve $\gamma : [a, b] \to N$, if $\gamma(a), \gamma(b) \in \chi(M)$ then for all $t \in]a, b[$ we have: $\gamma(t) \in \chi(M)$.

The generalization of AQFT which we discuss in this section is called *Locally Covariant Quantum Field Theory* (LCQFT). For a recent extensive review of the area, see [FV15].

As in the original AQFT framework, we assign algebras of observables to globally hyperbolic spacetimes and we also want to require that for each such admissible embedding there exists an injective homomorphism

$$\alpha_\chi : \mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N}) \tag{2.14}$$

of the corresponding algebras of observables assigned to them, moreover if $\chi_1 : \mathcal{M} \to \mathcal{N}$ and $\chi_2 : \mathcal{N} \to \mathcal{L}$ are embeddings as above then we require the covariance relation

$$\alpha_{\chi_2 \circ \chi_1} = \alpha_{\chi_2} \circ \alpha_{\chi_1} \ . \tag{2.15}$$

Such an assignment $\mathfrak{A}$ of algebras to spacetimes and algebra-morphisms to embeddings can be interpreted in the language of category theory as a *covariant functor* between two categories: the category **Loc** which is an appropriate sub-category of the category whose objects are globally hyperbolic spacetimes and arrows are the admissible embeddings; and the category **Obs** of topological $*$-algebras. The precise choice of the category **Loc** depends on the kind of objects we want to study. If the physical theory we consider is sensitive to some topological (hence non-local) features of the underlying manifold, one first restricts the class of objects considered and then studies possible extensions. The detailed analysis of such topological effects has been provided in [BSS14]. In this section we will present the framework in the simplest version, suitable for the study of scalar fields, as introduced in [BFV03]. First we recall some basic notions of category theory, which are relevant for LCQFT.

Definition 2.57 A *category* **C** consists of:

- a class of objects $\text{Obj}(\mathbf{C})$,
- a class of morphisms (arrows) $\text{Hom}(\mathbf{C})$, such that each $f \in \text{Hom}(\mathbf{C})$ has a unique *source object* and *target object* (both are elements of $\text{Obj}(\mathbf{C})$). For a fixed $a, b \in \text{Obj}(\mathbf{C})$, we denote by $\text{Hom}(a, b)$ the set of morphisms with a as a source and b as a target,
- a binary associative operation $\circ : \text{Hom}(a, b) \times \text{Hom}(b, c) \to \text{Hom}(a, c)$, $f, g \mapsto f \circ g$, called composition of morphisms,
- the identity morphism id_c for each $c \in \text{Obj}(\mathbf{C})$.

Definition 2.58 Let **C**, **D** be categories. A *covariant functor* $\mathfrak{F}$ assigns to each object $c \in \mathbf{C}$ and object $\mathfrak{F}(c)$ of **D** and to each morphism $f \in \mathrm{Hom}(\mathbf{C})$, a morphism $\mathfrak{F}(f) \in \mathrm{Hom}(\mathbf{D})$ in such a way that the following two conditions hold:

- $\mathfrak{F}(\mathrm{id}_c) = \mathrm{id}_{\mathfrak{F}(c)}$ for every object $c \in \mathbf{C}$.
- $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$ for all morphisms $f : a \to b$ and $g : b \to c$.

Definition 2.59 Let $\mathfrak{F}$ and $\mathfrak{G}$ be functors between categories **C** and **D**, then a *natural transformation* η from $\mathfrak{F}$ to $\mathfrak{G}$ associates to every object $a \in \mathbf{C}$ a morphism $\mathrm{Hom}(\mathbf{D}) \ni \eta_a : \mathfrak{F}(a) \to \mathfrak{G}(a)$, such that for every morphism $\mathrm{Hom}(\mathbf{C}) \ni f : a \to b$ we have:

$$\eta_b \circ \mathfrak{F}(f) = \mathfrak{G}(f) \circ \eta_a .$$

We denote the family of natural transformations between $\mathfrak{F}$ and $\mathfrak{G}$ by $\mathrm{Nat}(\mathfrak{F}, \mathcal{G})$.

For more details on categories and functors, see [ML78]. In LCQFT applied to scalar fields we adopt the following definitions of categories **Loc** and **Obs**.

Definition 2.60 The category **Loc** is a category where objects are globally hyperbolic, oriented time-oriented spacetimes and morphisms are admissible embeddings (see Definition 2.56).

Remark 2.2 Note that **Loc** is a large category, i.e. its class of objects Obj(**Loc**) is not a small set. It was shown in [Few07] that one can improve the situation with the use of the Whitney embedding theorem, which states that every smooth manifold of dimension d may be embedded as a smooth submanifold of $\mathbb{R}^{2d+1}$. Hence the collection of isomorphism equivalence classes in Obj(**Loc**) may be identified with a subset of the power set of $\mathbb{R}^{2d+1}$, so it is a small set. This makes **Loc** essentially small.

Definition 2.61 Depending on the context, we have the following choices for the category of observables.

(i) In the non-perturbative setting: **Obs** is the category with unital C^*-algebras as objects and injective unit-preserving $*$-homomorphisms as arrows.
(ii) In classical theory: $\mathbf{Obs}_c$ is the category with locally convex topological Poisson algebras as objects and injective Poisson homomorphism as arrows.
(iii) In the perturbative setting: $\mathbf{Obs}_p$ is the category with locally convex topological unital $*$-algebras as objects and injective unit-preserving $*$-homomorphisms as arrows.

We are now ready to give a definition of a classical/quantum field theory model in the LCQFT setting.

Definition 2.62 In the LCQFT framework, a model is a functor $\mathfrak{A}$ from **Loc** to …

(i) …**Obs** for a non-perturbative locally covariant QFT model,
(ii) …$\mathbf{Obs}_c$ for a locally covariant classical field theory model,
(iii) …$\mathbf{Obs}_p$ for a perturbative locally covariant QFT model.

If we don't want to specify the context, we write $\mathbf{Obs}_*$. Moreover, we often use the notation $\alpha_\chi \equiv \mathfrak{A}\chi$, where $\chi \in \mathrm{Hom}(\mathbf{Loc})$.

Another useful category is the category of locally convex topological vector spaces.

Definition 2.63 Define **Vec** to be the category whose objects are locally convex topological vector spaces (LCVS) and whose morphisms are injective homomorphisms of LCVS.

The requirement that $\mathfrak{A}$ is a covariant functor already generalizes the Haag–Kastler axioms of *Isotony* and *Covariance*. We can impose further requirements:

- **Einstein causality**: let $\chi_i : \mathcal{M}_i \to \mathcal{M}$, $i = 1, 2$ be morphisms of **Loc** such that $\chi_1(M_1)$ is causally disjoint from $\chi_2(M_2)$, then we require that:

$$[\alpha_{\chi_1}(\mathfrak{A}(\mathcal{M}_1)), \alpha_{\chi_2}(\mathfrak{A}(\mathcal{M}_2))] = \{0\},$$

- **Time-slice axiom**: let $\chi : \mathcal{N} \to \mathcal{M}$, if $\chi(\mathcal{N})$ contains a neighborhood of a Cauchy surface $\Sigma \subset \mathcal{M}$, then α_χ is an isomorphism.

The **Einstein causality** requirement reflects the commutativity of observables localized in spacelike separated regions. From the point of view of category theory, this property is encoded in the tensor structure of the functor $\mathfrak{A}$. In order to make this statement precise, we need to equip our categories **Loc** and $\mathbf{Obs}_*$ with tensor structures (for a precise definition of a tensor category, see [ML78]).

Definition 2.64 We call a category **C** *strictly monoidal (tensor category)* if there exists a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is associative, i.e. $\otimes(\otimes \times 1) = \otimes(1 \times \otimes)$ and there exists an object e which is a left and right unit for $\otimes$. If $\otimes$ is associative up to a natural isomorphism, then **C** is called *monoidal*.

The category of globally hyperbolic manifolds **Loc** can be extended to a monoidal category $\mathbf{Loc}^\otimes$, if we extend the class of objects with finite disjoint unions of elements of $\mathrm{Obj}(\mathbf{Loc})$,

$$\mathcal{M} = \mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_k,$$

where $\mathcal{M}_i \in \mathrm{Obj}(\mathbf{Loc})$. Morphisms of $\mathbf{Loc}^\otimes$ are isometric embeddings, preserving orientations and causality. More precisely, they are maps $\chi : \mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_k \to \mathcal{M}$ such that each component satisfies the requirements for a morphism of **Loc** and additionally all images are spacelike to each other, i.e., $\chi(M_i) \perp \chi(M_j)$, for $i \neq j$. $\mathbf{Loc}^\otimes$ has the disjoint union as a tensor product, and the empty set as unit object. It is a monoidal category and, using the results of [JS93], it is tensor equivalent to a strict monoidal category, which we will denote by the same symbol $\mathbf{Loc}^\otimes$.

On the level of C*-algebras the choice of a tensor structure is less obvious, since, in general, the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ of two C^*-algebras can be completed to a C^*-algebra with respect to many non-equivalent tensor norms. The choice of an appropriate norm has to be based on some further physical indications. This problem was discussed in [BFIR14], where it is shown that a physically justified tensor norm is the minimal C^*-norm $\|.\|_{\min}$ defined by

$$\|A\|_{\min} \doteq \sup\{\|(\pi_1 \otimes \pi_2)(A)\|_{\mathscr{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}\}\ , \quad A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2,$$

where π_1 and π_2 run through all representations of $\mathfrak{A}_1$ and of $\mathfrak{A}_2$ on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. $\mathscr{B}$ denotes the algebra of bounded operators. If we choose π_1 and π_2 to be faithful, then the supremum is achieved, i.e. $\|A\|_{\min} = \|(\pi_1 \otimes \pi_2)(A)\|_{\mathscr{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$. The completion of the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ with respect to the minimal norm $\|A\|_{\min}$ is denoted by $\mathfrak{A}_1 \underset{\min}{\otimes} \mathfrak{A}_2$. It was shown in [BFIR14] that, under some technical assumptions, a functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Obs}$ satisfies the axiom of **Einstein causality** if and only if it can be extended to a tensor functor $\mathfrak{A}^{\otimes} : \mathbf{Loc}^{\otimes} \to \mathbf{Obs}^{\otimes}$, which means that

$$\mathfrak{A}^{\otimes}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \mathfrak{A}^{\otimes}(\mathcal{M}_1) \otimes_{\min} \mathfrak{A}^{\otimes}(\mathcal{M}_2), \tag{2.16}$$

$$\mathfrak{A}^{\otimes}(\chi \otimes \chi') = \mathfrak{A}^{\otimes}(\chi) \otimes \mathfrak{A}^{\otimes}(\chi'), \tag{2.17}$$

$$\mathfrak{A}^{\otimes}(\varnothing) = \mathbb{C}. \tag{2.18}$$

In the perturbative setting, we face a similar problem with extending $\mathbf{Obs}_p$ to a tensor category, as there are many possibilities to chose a tensor product. The most natural choices are the injective tensor product (Definition 2.52) and the projective tensor product (Definition 2.51). A way out is to restrict $\mathbf{Obs}_p$ to the category of *nuclear* topological algebras, where these two notions coincide.

Let us now discuss the **Time slice axiom**. We use it to describe the evolution between different Cauchy surfaces. We fix a spacetime $\mathcal{M} = (M, g)$. Let N, K be subsets of M. We denote by ι_{KN} the inclusion of $\mathcal{N} \doteq (N, g\restriction_N)$ into $\mathcal{K} \doteq (K, g\restriction_K)$ and by $\alpha_{KN} \doteq \mathfrak{A}\iota_{KN}$, the corresponding morphism in Hom($\mathbf{Obs}$). These morphisms allow us to associate to each Cauchy surface Σ the inverse limit

$$\mathfrak{A}(\Sigma) = \lim_{\mathcal{N} \supset \Sigma} \mathfrak{A}(\mathcal{N}), \tag{2.19}$$

which comes with natural projections $\alpha_{M\Sigma}$ from the algebra $\mathfrak{A}(\Sigma)$ into $\mathfrak{A}(\mathcal{M})$.

From the time slice axiom it follows that each homomorphism α_{KN} is an isomorphism. Hence $\alpha_{M\Sigma}$ is also an isomorphism, and we obtain the "propagator" between two Cauchy surfaces Σ_1 and Σ_2 by

$$\alpha^M_{\Sigma_1\Sigma_2} = \alpha^{-1}_{M\Sigma_1} \circ \alpha_{M\Sigma_2}. \tag{2.20}$$

This construction resembles constructions in topological field theory [Seg].

Another important notion in LCQFT is that of a *local quantum field*. In the Haag–Kastler framework on Minkowski spacetime an essential ingredient was the translation symmetry. This symmetry allowed the comparison of observables in different regions of spacetime. This is not possible in the generally covariant framework we describe here, because on a generic spacetime the isometry group might be trivial. It follows that there is a priori no natural way to say what it means to have the same observable in a different region. We need to introduce some extra labels for the observables, which make such a comparison possible. This is where locally covariant quantum fields come into the game. We can think of them as operator-valued distributions assigned to all the objects of **Loc** in a coherent way. Before we give the precise definition, we need to make clear what we mean by test function spaces.

Definition 2.65 Let $\mathfrak{D}$ denote the functor from **Loc** to **Vec** that associates to every spacetime M its space of compactly supported $\mathcal{C}^\infty$-functions,

$$\mathfrak{D}(\mathcal{M}) = \mathscr{D}(M) \doteq \mathcal{C}^\infty_c(M, \mathbb{R}) \,, \tag{2.21}$$

and to every embedding $\chi : M \to N$ of spacetimes the pushforward of test functions $f \in \mathfrak{D}(\mathcal{M})$

$$\mathfrak{D}\chi \equiv \chi_* \; \chi_* f(x) = \begin{cases} f(\chi^{-1}(x)) & x \in \chi(M) \\ 0 & \text{else} \end{cases} \,. \tag{2.22}$$

Note that $\mathfrak{D}$ is a covariant functor. We are now ready to state the definition of a locally covariant quantum field.

Definition 2.66 A locally covariant quantum/classical field Φ is defined as a natural transformation from the functor $\mathfrak{D}$ of test function spaces to the functor $\mathfrak{A}$ of field theory composed with the forgetful functor from $\mathbf{Obs}_*$ to **Vec**.

More concretely, Φ is defined by a family of morphisms $\Phi_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$, $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$ such that

$$\mathfrak{A}\chi \circ \Phi_{\mathcal{M}} = \Phi_{\mathcal{N}} \circ \mathfrak{D}\chi \tag{2.23}$$

The category theory language, which is used to formulate the axioms of LCQFT, is not only a convenient way to phrase known results, but also leads to new insights. For example, one can use it to formulate what it means to have the same physics in all spacetimes. This property, called SPASS, is a property of the QFT functor and it has been extensively studied in [FV11a, FV11b]. Further study of structures appearing in LCQFT led recently to construction of new theories by using symmetries of the QFT functor [Few13].

The structure of **Loc** introduced above is suitable for scalar fields, but things get more complicated if we want to consider Dirac fields or 1-forms (like in electrodynamics). A convenient and operationally motivated way to do this is to extend the LCQFT framework to the situation where **Loc** is replaced by the category of framed manifolds. This idea has been proposed in [FV15] to prove the locally covariant version of the spin-statistics theorem and presented in more detail in [Few15]. In this book, we apply these concepts in Sect. 6.5.1 to describe the construction of time-ordered products of local functionals that involve derivatives of field configurations. Let us recall after [Few15] some basic definitions.

Definition 2.67 Define the objects of the category **FLoc** to be pairs $\mathcal{M} \doteq (M, \boldsymbol{e})$, where M is a smooth manifold of a fixed dimension (in our context equal to 4), $\boldsymbol{e} = (e^a)_{a=0,\ldots,3}$ is a co-tetrad, (a collection of four smooth linearly independent 1-forms on M) and M, equipped with the metric, orientation and time-orientation induced by $\boldsymbol{e}$ is an object in **Loc**.

The metric induced by $\boldsymbol{e}$ is defined by

$$g = \sum_{a,b=0}^{3} \eta_{ab} e^a e^b, \tag{2.24}$$

where η is the Minkowski metric in four dimensions. The existence of orientation and time-orientation is guaranteed if we require that $e^0 \wedge \cdots \wedge e^3$ is everywhere positive and that e^0 is future-directed.

Definition 2.68 Given $(\mathcal{M}, \boldsymbol{e}), (\mathcal{M}', \boldsymbol{e}') \in \mathrm{Obj}(\mathbf{FLoc})$, a morphism ψ in $\mathrm{Hom}((M, \boldsymbol{e}), (M', \boldsymbol{e}'))$ is a smooth map between the underlying manifolds inducing a **Loc**-morphism $(\mathcal{M}, e) \to (\mathcal{M}', e')$ and obeying $\psi^* \boldsymbol{e}' = \boldsymbol{e}$, where $\mathcal{M} = (M, g)$, $\mathcal{M}' = (M', g')$ and g, g' are defined by (2.24).

Given a co-tetrad we obtain its dual tetrad (a set of four independent vector fields) $(e_a)_{a=0,\ldots,3}$ by requiring that

$$e^a(e_b) = e^a_\mu e^\mu_b = \delta^a_b,$$

where δ^a_b is the Kronecker delta.

Geometrically, the four vector fields $(e_a)_{a=0,\ldots,3}$ define a global section of the frame bundle (a parallelization of M), i.e. they provide an isomorphism $TM \cong M \times \mathbb{R}^4$.

The operational interpretation for elements of Obj(**FLoc**) is provided in terms of *rods and clocks*, but this description is redundant. This corresponds to the freedom to make global frame rotations by elements of the proper orthochronous Lorentz group $\Lambda \in \mathcal{L}^\uparrow_+$ (the group of isometries of Minkowski spacetime that leave the origin fixed). There is a representation of this group as automorphisms of **FLoc** and given a locally covariant QFT functor $\mathfrak{A}$ one obtains a family of theories by applying such frame rotations. More precisely, to each $\Lambda \in \mathcal{L}^\uparrow_+$, there is a functor $\mathscr{T}(\Lambda) : \mathbf{FLoc} \to \mathbf{FLoc}$

$$\mathscr{T}(\Lambda)(M, \boldsymbol{e}) = (M, \Lambda \boldsymbol{e}), \qquad \text{where} \qquad (\Lambda e)^a = \Lambda^a{}_b e^b, \qquad \Lambda \in \mathcal{L}^\uparrow_+. \tag{2.25}$$

Physically, theories defined by $\mathfrak{A} \circ \mathscr{T}(\Lambda)$ for different $\Lambda \in \mathcal{L}^\uparrow_+$ have to be equivalent, so one needs to impose an additional condition on $\mathfrak{A}$ that guarantees that this is indeed the case.

- **Independence of global frame rotations** To each $\Lambda \in \mathcal{L}^\uparrow_+$, there exists a natural transformation $\eta(\Lambda) : \mathfrak{A} \to \mathfrak{A} \circ \mathscr{T}(\Lambda)$, such that

$$\eta(\Lambda)_{(M,\boldsymbol{e})} \alpha_{(M,\boldsymbol{e})} = \alpha_{(M,\Lambda\boldsymbol{e})} \eta(\Lambda)_{(M,\boldsymbol{e})}, \qquad \forall \alpha \in \mathrm{Aut}(\mathfrak{A}), \tag{2.26}$$

where Aut($\mathfrak{A}$) is the group of natural transformations that are automorphisms of the functor $\mathfrak{A}$, see [Few13] for more detail.

The next step in LCQFT research is the proper understanding of the structures of gauge theories, where the topological features lead to new difficulties [DL12, SDH14, BSS14]. It would be desirable to obtain for local symmetries a framework similar to the DHR analysis done for global symmetries [DHR71, DHR74].

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Chapter 3
Kinematical Structure

In the framework of perturbative algebraic quantum field theory (pAQFT) we start with the classical theory, which is subsequently quantized. We work in the Lagrangian framework, but there are some modifications that we need to make to deal with the infinite dimensional character of field theory. In this chapter we give an overview of mathematical structures that will be needed later on to construct models of classical and quantum field theories. Since we do not fix the dynamics yet, the content of this chapter describes the kinematical structure of our model. Readers familiar with some of the concepts we introduce here can skip corresponding sections.

3.1 The Space of Field Configurations

We start with a globally hyperbolic spacetime $\mathcal{M} = (M, g)$ (see Definition 2.33). The next step is to define the space of field configurations. This specifies what kind of objects our model describes (e.g. scalar fields, Dirac fields, gauge fields, etc.). In the simplest situation the configuration space $\mathcal{E}$ is a vector space.

Definition 3.1 The *configuration space* $\mathcal{E}$ on the fixed spacetime $\mathcal{M} = (M, g)$ is realized as the space of smooth sections $\Gamma(E \to M)$ of some vector bundle $E \xrightarrow{\pi} M$ over M. Let V be the finite dimensional vector space which constitutes the fibre of E. We assume that there exists a bilinear pairing $\langle ., . \rangle_E : V \times V \to \mathbb{R}$. This pairing also defines an isomorphism between V and its dual V^*.

Example 3.1 Examples of configuration spaces for commonly used theories:

- for the real scalar field: $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R})$,
- for Yang-Mills theories (see Chap. 7) with the trivial principal bundle: $\mathcal{E} = \Omega_1(M, \mathfrak{k})$, where $\mathfrak{k}$ is some Lie algebra of some compact Lie group K,
- for effective quantum gravity (see Chap. 8): $\mathcal{E} = \Gamma((T^*M)^{\otimes 2})$.

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Let us now comment on the differentiable structure on $\mathcal{E}$. A natural way to introduce a smooth structure on $\mathcal{E}$ is to equip it first with the standard Fréchet topology (as defined below) and use this topology to define functional derivatives on $\mathcal{E}$.

Definition 3.2 Let $\Omega \subset \mathbb{R}^n$ be an open subset and $\mathcal{C}^\infty(\Omega, \mathbb{R}) \equiv \mathscr{E}(\Omega)$ the space of smooth functions on it. We equip this space with a Fréchet topology generated by the family of seminorms:

$$p_{K,m}(\varphi) = \sup_{\substack{x \in K \\ |\alpha| \leq m}} |\partial^\alpha \varphi(x)|, \tag{3.1}$$

where $\alpha \in \mathbb{N}^N$ is a multiindex, $N \in \mathbb{N}$, and $K \subset \Omega$ is a compact set. This is just the topology of uniform convergence of all the derivatives on compact sets.

The definition above can be applied to define a Fréchet topology on $\mathcal{C}^\infty(M, \mathbb{R})$ with the use of coordinate charts, as M is locally $\mathbb{R}^4$. It also generalizes easily to the vector-valued case $\mathcal{E} = \Gamma(E \to M)$. We will always assume that $\mathcal{E}$ is equipped with this Fréchet topology.

Next we introduce a natural topology on the space of compactly supported functions $\mathscr{D}(M) \doteq \mathcal{C}_c^\infty(M, \mathbb{R})$. This topology is locally convex, but is not Fréchet.

Definition 3.3 Let $\mathscr{D}(\Omega) \doteq \mathcal{C}_c^\infty(\Omega, \mathbb{R})$, $\Omega \subset \mathbb{R}$. The fundamental system of seminorms on $\mathscr{D}(\Omega)$ is given by:

$$p_{\{m\},\{\epsilon\},a}(\varphi) = \sup_\nu \Big(\sup_{\substack{|x| \geq \nu, \\ |p| \leq m_\nu}} \big| D^p \varphi^a(x) \big| / \epsilon_\nu \Big), \tag{3.2}$$

where $\{m\}$ is an increasing sequence of positive numbers going to $+\infty$ and $\{\epsilon\}$ is a decreasing one tending to 0.

The generalization of the above topology to the vector-valued case $\mathcal{E}_c \doteq \Gamma_c(E \to M)$ is straightforward.

Remark 3.1 We can also consider situations in which the configuration space $\mathcal{E}$ is not a vector space, but still can be made into an infinite dimensional affine manifold in the sense of [Mic84]. This happens for example with the space of all Lorentzian metrics or the space of gauge connections, but this is beyond the scope of this book.

3.2 Functionals on the Configuration Space

We model classical and quantum observables as smooth functions on $\mathcal{E}$. Intuitively, a classical measurement assigns a real number to a field configuration. The smoothness condition is a regularity requirement which we need in our formalism in order to introduce various algebraic structures on the space of functionals. The main feature of the functional approach, which we advocate in this book, is that both the classical and quantum theory are defined in terms of the *same set* of functionals and differ by algebraic structures on this set. The classical theory is defined in terms of a Poisson bracket while the quantum theory in terms of a non-commutative product. We will construct these structures for the scalar field in Chaps. 4, 5 and 6, but first we need to define the underlying space of functionals, which is the main subject of the present section. Having the quantization in mind, we work from the start with complex-valued functionals.

For functions on $\mathcal{E}$, the smoothness is understood in the sense of Bastiani calculus [Bas64, Ham82, Mil84, Nee06], i.e.

Definition 3.4 (*after* [Nee06]) Let $\mathcal{X}$ and $\mathcal{Y}$ be topological vector spaces, $U \subseteq \mathcal{X}$ an open set and $f : U \to \mathcal{Y}$ a map. The derivative of f at $x \in U$ in the direction of $h \in \mathcal{X}$ is defined as

$$\left\langle f^{(1)}(x), h \right\rangle \doteq \lim_{t \to 0} \frac{1}{t} \left(f(x + th) - f(x) \right) \tag{3.3}$$

whenever the limit exists. The function f is called differentiable at x if $\left\langle f^{(1)}(x), h \right\rangle$ exists for all $h \in \mathcal{X}$. It is called continuously differentiable if it is differentiable at all points of U and $f^{(1)} : U \times \mathcal{X} \to \mathcal{Y}$, $(x, h) \mapsto \left\langle f^{(1)}(x), h \right\rangle$ is a continuous map. It is called a $\mathcal{C}^1$-map if it is continuous and continuously differentiable. Higher derivatives are defined by

$$\left\langle f^{(k)}(x), v_1 \otimes \cdots \otimes v_k \right\rangle \doteq \frac{\partial^k}{\partial t_1 \ldots \partial t_k} f(x + t_1 v_1 + \cdots + t_k v_k) \Big|_{t_1 = \cdots = t_k = 0}, \tag{3.4}$$

and f is $\mathcal{C}^k$ if $f^{(k)}$ is jointly continuous as a map $U \times \mathcal{X}^k \to \mathcal{Y}$. We say that f is smooth if it is $\mathcal{C}^k$ for all $k \in \mathbb{N}$.

This notion of differentiability is also referred to as Michal-Bastiani differentiability, since the definition of differentiability at a point is equivalent to the one proposed by Michal [Mic38, Mic40]. However, Bastiani differentiability on an open set is a stronger notion.

We apply this to define derivatives of $\mathbb{C}$-valued functionals F on $\mathcal{E}$. By definition $F^{(1)}(\varphi)$, if it exists, is an element of the complexified dual space $\mathcal{E}'^{\mathbb{C}} \doteq \mathcal{E}' \otimes \mathbb{C}$. More generally, we have the following result.

Proposition 3.1 *Let $F : \mathcal{E} \to \mathbb{C}$ be a Bastiani smooth functional, then*

(i) *$F^{(n)}(\varphi)$ is a linear continuous map from $\mathcal{E}^n$ to $\mathbb{C}$,*
(ii) *$F^{(n)}(\varphi)$ induces a continuous map on the completed projective tensor product $\mathcal{E}^{\hat{\otimes}_\pi k} \cong \Gamma(E^{\boxtimes n} \to M^n)$, where $\boxtimes$ is the exterior tensor products of vector bundles, defined below in Definition 3.6. Here we denote the map on $\mathcal{E}^{\hat{\otimes}_\pi k}$ by the same symbol as the original differential, i.e. $F^{(n)}(\varphi)$.*

Proof Property (i) follows directly from the definition of continuous differentiability. For (ii) it is crucial that $\mathcal{E}$ is a Fréchet space. For the proof of the claim, see for example [Tre06]. □

Definition 3.5 (*after* [BGP07]) Let $E_1 \xrightarrow{\pi_1} M_1$, $E_2 \xrightarrow{\pi_2} M_2$ be two vector bundles over M_1 and M_2 with fibers V_1, V_2 respectively. The *exterior tensor product* $E_1 \boxtimes E_2$ is defined as the vector bundle over $M_1 \times M_2$, whose fiber over $(x, y) \in M_1 \times M_2$ is $E_{1x} \otimes E_{2y}$, where E_{1x} is the fibre of E_1 over x and E_{2y} is the fibre of E_2 over y.

This definition has to be contrasted with the definition of the ordinary tensor product of vector bundles.

Definition 3.6 The *tensor product of vector bundles* $E_1 \xrightarrow{\pi_1} M$, $E_2 \xrightarrow{\pi_2} M$ is a vector bundle over M, denoted by $E_1 \otimes E_2$ whose fiber over a point $x \in M$ is the tensor product of vector spaces $E_{1x} \otimes E_{2x}$.

It is clear from the discussion above that in order to better understand the behavior of functional derivatives of smooth functionals on $\mathcal{E}$ we need to bring some notions from the theory of distributions into our framework.

Definition 3.7 The space of *distributions* on $\Omega \subset \mathbb{R}^n$ is defined to be the dual $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ with respect to the topology given in the Definition 3.3.

Equivalently, given a linear map L on $\mathscr{D}(\Omega)$ we can decide if it is a distribution by checking one of the equivalent conditions given in the theorem below [Tre06, Rud91, Hor03].

Theorem 3.1 *A linear map u on $\mathscr{D}(\Omega)$ is a distribution if it satisfies the following equivalent conditions:*

1. *To every compact subset K of Ω there exists an integer k and a constant $C > 0$ such that for all $\varphi \in \mathscr{D}(\Omega)$ with support contained in K*

$$|u(\varphi)| \leq C \max_{p \leq k} \sup_{x \in \Omega} |\partial^p \varphi(x)|.$$

   We call $||u||_{\mathcal{C}^k(\Omega)} \doteq \max_{p\leq k} \sup_{x\in\Omega} |\partial^p \varphi(x)|$ the $\mathcal{C}^k$-norm and if the same integer k can be used in all K for a given distribution u, then we say that u is of order k.
2. *If a sequence of test functions $\{\varphi_l\}$, as well as all their derivatives converge uniformly to 0 and if all the test functions φ_l have their supports contained in a compact subset $K \subset \Omega$ independent of the index l, then $u(\varphi_l) \to 0$.*

Proof See for example [Hor03]. □

The localization of a distribution is characterised by its support.

Definition 3.8 Let $u \in \mathscr{D}'(\Omega)$. The *support* supp u of a distribution $u \in \mathscr{D}'(\Omega)$ is the smallest closed set $\mathcal{O}$ such that $u|_{\Omega \setminus \mathcal{O}} = 0$. In other words:

$$\text{supp } u \doteq \{x \in \Omega | \forall x \in U \subset \Omega \text{ open } \exists \varphi \in \mathscr{D}(\Omega),\ \text{supp } \varphi \subset U,\ u(\varphi) \neq 0\}.$$

Definition 3.9 Let $\mathscr{E}(\Omega)$ denote the space of smooth functions on Ω with its standard Fréchet topology (see Definition 3.2).

Proposition 3.2 (after [Hor03, Hor12]) *$\mathscr{E}'(\Omega)$, the topological dual of $\mathscr{E}(\Omega)$, is the space of compactly supported distributions.*

The regularity of a distribution can be characterized in terms of the falloff conditions for its Fourier transform.

Theorem 3.2 *A distribution $u \in \mathscr{E}'(\Omega)$ is smooth if and only if for every N there is a constant C_N such that:*

$$|\hat{u}(k)| \leq C_N(1+|k|)^{-N},$$

where $\hat{u}$ denotes the Fourier transform of u.

Proof See for example [Hor03]. □

Definition 3.10 The *singular support* of a distribution u is the complement of the largest open set on which u is smooth.

If a distribution has a nonempty singular support we can give a further characterization of its singularity structure by specifying the direction in which it is singular. This is exactly the purpose of the definition of a wave front set.

Definition 3.11 For a distribution $u \in \mathscr{D}'(\Omega)$ the *wavefront set* $\mathrm{WF}(u)$ is the complement in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ of the set of points $(x, k_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ such that there exist

- a function $f \in \mathscr{D}(\Omega)$ with $f(x) = 1$,
- an open conic neighborhood C of k_0, with

$$\sup_{k \in C}(1+|k|)^N |\widehat{f \cdot u}(k)| < \infty \qquad \forall N \in \mathbb{N}_0.$$

On a manifold M the definition of the Fourier transform depends on the choice of a chart, but the property of strong decay in some direction (characterized now by a point (x, k), $k \neq 0$ of the cotangent bundle T^*M) turns out to be independent of this choice. Therefore the wave front (WF) set of a distribution on a manifold M is a well defined closed conical subset of $\dot{T}^*M$, the cotangent bundle (with the zero section removed).

Wavefront sets provide us with a simple criterion for the existence of point-wise products distributions. Before we give it, we prove a more general result concerning the pullback. We will present all the proofs in the context of distributions on $\mathbb{R}^n$. The generalization can be found in [BGP07] and is mentioned also in [Hor03].

We follow closely [BF09a, Hor03]. Let $\alpha : \Omega \to \widetilde{\Omega}$ be a smooth map between $\Omega \subset \mathbb{R}^m$ and $\widetilde{\Omega} \subset \mathbb{R}^n$. We define the normal set N_α of the map α as:

$$N_\alpha \doteq \{(\alpha(x), \eta) \in \widetilde{\Omega} \times \mathbb{R}^n | (d\alpha_x)^T(\eta) = 0\},$$

where $(d\alpha_x)^T$ is the transposition of the differential of α at x.

Theorem 3.3 *Let Γ be a closed cone*[1] $\widetilde{\Omega}\times(\mathbb{R}^n\setminus\{0\})$ *and* $\alpha:\Omega\to\widetilde{\Omega}$ *as above, such that* $N_\alpha\cap\Gamma=\varnothing$. *Then the pullback* $\alpha^*:\mathscr{E}(\Omega)\to\mathscr{E}(\widetilde{\Omega})$ *has a unique, sequentially continuous extension to a sequentially continuous map* $\mathscr{D}'_\Gamma(\widetilde{\Omega})\to\mathscr{D}'(\Omega)$, *where* $\mathscr{D}'_\Gamma(\widetilde{\Omega})$ *denotes the space of distributions with WF sets contained in* Γ.

Proof For proof see [BF09a, Hor03]. □

Using this theorem we can define the pointwise product of two distributions t, s on an n-dimensional manifold M as a pullback by the diagonal map $D: M\to M\times M$ if the pointwise sum of their wave front sets

$$\mathrm{WF}(t)+\mathrm{WF}(s)=\{(x,k+k')|(x,k)\in\mathrm{WF}(t),(x,k')\in\mathrm{WF}(s)\},$$

does not intersect the zero section of T^*M (see Theorem 8.2.10 of [Hor03]). To see that this is the right criterion, note that the set of normals of the diagonal map $D: x\mapsto(x,x)$ is given by $N_D=\{(x,x,k,-k)|x\in M, k\in T^*M\}$. The product ts is then defined by: $ts=D^*(t\otimes s)$ and if one of t, s is compactly supported, then so is ts and we define the pointwise product by $\langle t,s\rangle\doteq\widehat{ts}(0)$. Another way of seeing that the construction works is to look at the Fourier transformed version. For $t,s\in\mathscr{E}(\Omega)$ we have

$$\langle ts, fg\rangle=\frac{1}{(2\pi)^n}\int\widehat{tf}(k)\widehat{sg}(-k)dk, \tag{3.5}$$

where $f,g\in\mathscr{D}(\Omega)$ are chosen with sufficiently small support. We will now give a brief argument for why the integral above converges. Note that if $k\neq 0$, then either $\widehat{tf}$ is fast-decaying in a conical neighborhood around k or $\widehat{sg}$ is fast-decaying in a conical neighborhood around $-k$, while the other factor is polynomially bounded.

Motivated by the criterion above we can distinguish certain important classes of functionals by analyzing the WF set properties of their derivatives. Before we move to this task, there is one more concept we need to introduce first. Following the spirit of AQFT we would like to define some notion of localization (on spacetime) of the functionals we consider. Later on we will construct algebraic structures associated to bounded regions of M, so we need to be able to decide if a given observable (modeled as a smooth functional) belongs to a given region or not. We achieve this by introducing the notion of the *spacetime support*.

Definition 3.12 Let F be a map from $\mathfrak{X}=\Gamma(E\to M)$ to $\mathfrak{Y}$, where E is a vector bundle over M and $\mathfrak{Y}$ is a set. The *spacetime support* of F is defined by

$$\begin{aligned}\mathrm{supp}\ F\doteq\{x\in M|&\forall\text{ neighborhoods }U\text{ of }x\ \exists\varphi,\psi\in\mathfrak{X}, \mathrm{supp}\ \psi\subset U,\\ &\text{such that }F(\varphi+\psi)\neq F(\varphi)\}.\end{aligned} \tag{3.6}$$

[1]We say that a subset Γ of $\widetilde{\Omega}\times(\mathbb{R}^n\setminus\{0\})$ is a *cone* if $(x,\lambda k)\in\Gamma$ whenever $(x,k)\in\Gamma$, $\lambda>0$. A cone is said to be closed (open) if it is closed (open) in $\widetilde{\Omega}\times(\mathbb{R}^n\setminus\{0\})$.

Here we rely on the fact that $\Gamma(E \to M)$ is equipped with a linear structure, but the concept of spacetime support generalizes to the case where $\mathcal{X}$ is a space of sections of an arbitrary bundle over M [BFR13]. Note that if F is linear and $\mathcal{X} = \mathcal{C}^\infty_c(\Omega, \mathbb{R})$, where $\Omega \subset \mathbb{R}^n$, then Definition 3.12 reduces to the Definition 3.8.

An even closer relation between the distributional support and the spacetime support of a functional can be seen in the following result [BDLGR16]:

$$\text{supp } F = \overline{\bigcup_{\varphi \in \mathcal{E}} \text{supp}\,(F^{(1)}(\varphi))}.$$

The concept of spacetime support also applies to functions of several variables.

Definition 3.13 Let F be a map on the product $\mathcal{X} \equiv \Gamma(E_1 \to M) \times \cdots \times \Gamma(E_k \to M)$ taking values in a set $\mathcal{Y}$. Then the spacetime support of F is defined as

$$\begin{aligned} \text{supp } \; F \doteq \{ & x \in M | \forall \text{ neighborhoods } U \text{ of } x \; \exists \varphi = (\varphi_1, \ldots \varphi_k) \in \mathcal{X} \\ & \text{and } \psi \in \Gamma(E_i \to M) \text{ for some } i \in \{1, \ldots, k\}, \text{supp}\, \psi \subset U, \\ & \text{such that } F(\varphi_1, \ldots, \varphi_i + \psi, \ldots, \varphi_k) \neq F(\varphi)\}. \end{aligned} \tag{3.7}$$

We have already seen that functional derivatives of Bastiani smooth functionals on $\mathcal{E}$ are compactly supported distributions. Further restrictions on regularity and support of distributions appearing as functional derivatives are obtained for local functionals.

Definition 3.14 A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ is called *local* (an element of $\mathcal{F}_{\text{loc}}$) if for each $\varphi_0 \in \mathcal{E}$ there exists an open neighbourhood V in $\mathcal{E}$ and $k \in \mathbb{N}$ such that for all $\varphi \in V$ we have

$$F(\varphi) = \int_M \alpha(j^k_x(\varphi)), \tag{3.8}$$

where $j^k_x(\varphi)$ is the kth jet prolongation of φ and α is a density-valued function on the jet bundle.

Remark 3.2 If F is local then $F^{(n)}(\varphi)$ is a distribution supported on the thin diagonal

$$\text{Diag}_n \doteq \{(x_1, \ldots, x_n) \in M^n, x_1 = \cdots = x_n\}.$$

We equip the space $\mathcal{F}_{\text{loc}}$ of local functionals on the configuration space with the pointwise product using the prescription

$$(F \cdot G)(\varphi) \doteq F(\varphi)G(\varphi), \tag{3.9}$$

where $\varphi \in \mathcal{E}$. $\mathcal{F}_{\text{loc}}$ is not closed under this product, but we can consider instead the space $\mathcal{F}$ of *multilocal functionals*, which is defined as the algebraic closure of $\mathcal{F}_{\text{loc}}$ under the product (3.9). We introduce the involution operator $*$ on $\mathcal{F}$ using complex conjugation, i.e.

$$F^*(\varphi) \doteq \overline{F(\varphi)}.$$

In this way we obtain a commutative $*$-algebra.

Local and multilocal functionals satisfy some important regularity properties. Firstly, for local functionals the wavefront set of $F^{(n)}(\varphi)$ is orthogonal to $T\mathrm{Diag}_n$, the tangent bundle of the thin diagonal. In particular, $F^{(1)}(\varphi)$ has empty wavefront set, and so is smooth for each fixed $\varphi \in \mathcal{E}$. The latter is true also for multilocal functionals, i.e. for $F \in \mathcal{F}$. Note that using the invariant volume form μ_g we can therefore identify $F^{(1)}(\varphi)$ with an element of $\mathcal{E}_c^{*\mathbb{C}} \doteq \Gamma_c^{\mathbb{C}}(E^* \to M)$, where E^* is the dual bundle of E and we denote its fiber by V^* (algebraic dual of V).

Actually, one can characterize locality in an abstract way, using Bastiani smoothness and WF set properties as well as support properties of the first and the second derivative. The following result will be proven in [BDLGR16], based on the ideas of [BFR12].

Theorem 3.4 *Let U be an open subset of $\mathscr{E}(M) \doteq C^\infty(M, \mathbb{R})$ and $F : U \to \mathbb{R}$ be smooth in the sense of Bastiani. Assume that*

1. *F is additive.*
2. *For every $\varphi \in U$, the differential $F^{(1)}(\varphi)$ of F at φ has empty WF set and the induced map $F^{(1)} : U \to \mathscr{D}(M)$ is Bastiani smooth.*

Then, for every $\varphi \in U$, there is a neighborhood V of the origin, an integer k and a smooth real-valued function f on the k-jet bundle $J^k(M)$ such that $F(\varphi + \psi) = \int_M f(j_x^k\psi)d\mu_g(x)$ for every $\psi \in V$, where $j_x^k\psi$ is the k-jet of ψ at x.

Another class of functionals with nice properties is the space of regular functionals $\mathcal{F}_{\mathrm{reg}}$.

Definition 3.15 A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ is called regular if for all $\varphi \in \mathcal{E}$ and $n \in \mathbb{N}$ the WF set of $F^{(n)}(\varphi)$ (seen as a distribution in $\Gamma'^{\mathbb{C}}(E^{\boxtimes n} \to M^n)$ by means of the Proposition 3.1) is empty. Equivalently this means that we can identify

$$F^{(n)}(\varphi) \in \Gamma_c^{\mathbb{C}}((E^*)^{\boxtimes n} \to M^n)\,.$$

The space of regular functionals is denoted by $\mathcal{F}_{\mathrm{reg}}$.

Note that a local functional can be regular if and only if it is at most linear in φ.

3.3 Fermionic Field Configurations

Up to now we have treated configuration spaces that are ordinary infinite dimensional manifolds. Now we will consider a situation where the configuration space is graded. Physically this becomes relevant when we want to describe fermionic field configurations, like for example matter fields in QED (Dirac fields). Here we will use the term "fermionic" in the sense of "anticommuting", rather than "non-integer spin". The later context is related to the spin-statistic theorem (see [SW00]), which, however, doesn't apply to auxiliary, non-physical field variables like ghosts and antighosts that will be introduced in Chap. 7.

There are several ways to define graded manifolds geometrically, and in the infinite dimensional context the approach proposed by [Sac08] seems to be the most appropriate. Here, however, we are not interested in the structure of graded manifolds themselves, but we take the algebraic point of view and focus on their rings of polynomial functions. As in earlier chapters we denote by $\mathcal{E}$ the space of sections of some vector bundle $E \xrightarrow{\pi} M$ (this could be for example the Dirac bundle describing the electrons in QED). Unless stated otherwise, $\mathcal{E}'$ is understood as the *strong dual* (i.e. topological dual space equipped with the topology of uniform convergence on bounded sets of $\mathcal{E}$).

We want to give meaning to the notion of odd $\mathcal{E}$, i.e. $\mathcal{E}[1]$, where the number in square brackets denotes the degree shift. We characterize this space in terms of its ring of functions $\mathcal{O}(\mathcal{E}[1])$ and call the latter *the space of antisymmetric functionals on* $\mathcal{E}$. Before we can make this notion precise we quote a known result from [Buc72].

Theorem 3.5 *Let $\mathcal{X}$, $\mathcal{Y}$ be two Fréchet spaces, one of which has the approximation property (see the Definition 3.16 below). Then*

(i) $\mathcal{X}'\hat{\otimes}_\pi\mathcal{Y}' \cong (\mathcal{X}\hat{\otimes}_\epsilon\mathcal{Y})'$,
(ii) $\mathcal{X}'\hat{\otimes}_\epsilon\mathcal{Y}' \cong (\mathcal{X}\hat{\otimes}_\pi\mathcal{Y})'$,

where all the duals are meant as the strong duals.

Proof See [Buc72]. □

The approximation property is essentially reflecting how well the space of continuous linear maps from the given space is approximated by the tensor product with its dual. More precisely:

Definition 3.16 Let $\mathcal{X}$, $\mathcal{Y}$ be Hausdorff locally convex vector spaces and $\mathcal{B}$ be the family of bounded sets of the completion of E (i.e. a bornology, see Definition 3.17). Let $\tau_\mathcal{B}$ be the topology of uniform convergence on bounded sets it induces on $L(\mathcal{X}, \mathcal{Y})$. We say that $\mathcal{X}$ has the *(sequential) approximation property* if one of the following equivalent conditions holds:

1. $\mathcal{X}' \otimes \mathcal{Y}$ is (sequentially) dense in $(L(\mathcal{X}, \mathcal{Y}), \tau_\mathcal{B})$ for every $\mathcal{Y}$,
2. $\mathcal{X}' \otimes \mathcal{X}$ is (sequentially) dense in $(L(\mathcal{X}, \mathcal{X}), \tau_\mathcal{B})$,
3. $\mathrm{id}_\mathcal{X}$ is the $\tau_\mathcal{B}$-limit of some (sequence) net in $\mathcal{X}' \otimes \mathcal{X}$.

In the definition above we have used the notion of *bornology*, which will become useful later on, so here we spell out the precise definition.

Definition 3.17 A *bornology* on a set X is a family $\mathcal{B}$ of subsets of X (called the **bounded (sub)sets** of X) such that:

(i) every one-element subset of X belongs to $\mathcal{B}$,
(ii) if $A \in \mathcal{B}$ and $B \subset A$ then $B \in \mathcal{B}$,
(iii) if A and $\mathcal{B}$ are in $\mathcal{B}$ then $A \cup B \in \mathcal{B}$.

We can prove the following useful result concerning tensor products of $\mathcal{E}$ and its strong dual.

Proposition 3.3 *$\Gamma'(E^{\boxtimes n} \to M^n)$, the strong dual of $\Gamma(E^{\boxtimes n} \to M^n)$ is isomorphic to $\Gamma'(E)^{\hat{\otimes}_\pi n}$.*

Proof Since $\Gamma(E)$ is Fréchet and has the approximation property (see [Jar12]), we have $(\Gamma(E)^{\hat{\otimes}_\pi n})' \cong \Gamma'(E)^{\hat{\otimes}_\epsilon n}$. From the nuclearity of $\Gamma'(E)$ follows that the later can be identified with $\Gamma'(E)^{\hat{\otimes}_\pi n}$. As $\Gamma(E)^{\hat{\otimes}_\pi n} \cong \Gamma(E^{\boxtimes n} \to M^n)$, we obtain

$$\Gamma'(E^{\boxtimes n} \to M^n) \cong \Gamma'(E)^{\hat{\otimes}_\pi n}.$$

□

Next we want to make precise the notion of antisymmetric functionals. Recall that $\bigwedge^n \mathcal{X}$, the nth exterior tensor power of a complex vector space $\mathcal{X}$, can be identified[2] with the image of the anti-symmetrization operator $A : \mathcal{X}^{\otimes n} \to \mathcal{X}^{\otimes n}$ induced by the multi-linear map

$$\mathcal{X}^n \to \mathcal{X}^{\otimes n},$$
$$(v_1, \dots, v_n) \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

This identification allows us to treat $\bigwedge^n \mathcal{X}$ as a subspace of $\mathcal{X}^{\otimes n}$ and hence equip it with the induced projective tensor product topology. The same can be done for the symmetric tensor product. We use this fact in the following definition.

Definition 3.18 Let $E \xrightarrow{\pi} M$ be a vector bundle with fiber V.

(i) We define $\Gamma'_a(E^{\boxtimes n} \to M^n)$ as the completion of $\Gamma'(E)^{\wedge n}$ with respect to the topology of $\Gamma'(E)^{\hat{\otimes}_\pi n}$. The subscript "$a$" stands for antisymmetry.
(ii) Analogously, $\Gamma'_s(E^{\boxtimes n} \to M^n)$ is the completion of the symmetric tensor product $\Gamma'(E)^{\otimes_s n}$.
(iii) This generalizes. The completion of $\Gamma'(E_1)^{\wedge p} \otimes \Gamma'(E_2)^{\otimes_s q}$ will be denoted by $\Gamma'_{\boldsymbol{p}|q}(E_1^{\boxtimes p} \boxtimes E_2^{\boxtimes q} \to M^{p+q})$. In general we consider labels which are arbitrary finite sequences of the type $p_1|\boldsymbol{p_2}|\boldsymbol{p_3}|\dots|p_k$, where integers in boldface indicate the antisymmetric factors and the remaining indices correspond to totally symmetric factors.

[2]This holds true, because $\mathbb{C}$ is a field of characteristic 0.

Definition 3.19 We define $\mathcal{O}(\mathcal{E}[1])$, the space of *odd (antisymmetric) functionals* on $\mathcal{E}$ as

$$\mathcal{A} \doteq \prod_{k=0}^{\infty} \mathcal{A}^k \doteq \prod_{k=0}^{\infty} \Gamma'_a(E^{\boxtimes k} \to M^k) \otimes \mathbb{C},$$

where the notation Γ'_a is clarified in Definition 3.18.

The elements of $\mathcal{A}$ can be written as (possibly infinite) sequences: $T = (T_k)_{k \in \mathbb{N}}$, where the components $T_k \in \mathcal{A}^k$ are referred to as homogeneous functionals. It is convenient to introduce a notation for such functionals, which is commonly used in physics

Definition 3.20 Let $\mathcal{E} = \Gamma(E \to M)$ with fiber V. Choose a basis on the fiber labelled by the index set I. For $\alpha \in I$, $x \in M$, define $\Phi^\alpha_x \in \mathcal{E}'$ as the evaluation functional

$$\Phi^\alpha_x(\varphi) \doteq \varphi^\alpha(x).$$

Formally, we can write elements of $T \in \mathcal{A}^k$ in terms of integral kernels

$$\begin{aligned} &T(u_1 \otimes \cdots \otimes u_k) \\ &\quad = \sum_{\alpha_1,\dots,\alpha_k} \int T(x_1, \dots, x_k)_{\alpha_1,\dots,\alpha_k} u_1(x_1)^{\alpha_1} \dots u_k(x_k)^{\alpha_k} d\mu(x_1) \dots d\mu(x_k). \end{aligned}$$

We can also write this in terms of evaluation functionals, defined in Definition 3.20, as

$$T = \sum_{\alpha_1,\dots,\alpha_k} \int T(x_1, \dots, x_k)_{\alpha_1,\dots,\alpha_k} \Phi^{\alpha_1}_{x_1} \dots \Phi^{\alpha_k}_{x_k} \, d\mu(x_1) \dots d\mu(x_k).$$

Using this notation we are able to translate some formal expressions used in the physics literature into our framework. The key point is that in physics one usually identifies Φ^α_x's with some abstract Grassmann-valued functions. We refrain from that interpretation, since we want to avoid dealing with infinitely many Grassmann parameters. Instead we treat Φ^α_x's as honest, real-valued *functionals*. This viewpoint on functionals of Fermionic fields has been proposed in [Rej11b].

The generalization of the Definition 3.19 to the graded case is straightforward.

Definition 3.21 Define $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1] \oplus \mathcal{E}_2[2])$ as $\mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A})$, where

$$\mathcal{A} \doteq \prod_{\substack{k=0 \\ p+q=k}}^{\infty} \Gamma'_{\boldsymbol{p}|q}(E_1^{\boxtimes p} \boxtimes E_2^{\boxtimes q} \to M^k) \otimes \mathbb{C}.$$

Adding further terms to $\mathcal{E}_0 \oplus \mathcal{E}_1[1] \oplus \mathcal{E}_2[2]$ is reflected by adding further factors in $\boldsymbol{p}|q$, where odd degrees contribute antisymmetric tensor powers and even degrees contribute symmetric tensor powers.

We also introduce the notation

$$\mathcal{O}^k(\mathcal{E}_0 \oplus \mathcal{E}_1[1] \oplus \mathcal{E}_2[2]) \doteq \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k),$$

where $\mathcal{A}^k \doteq \bigoplus_{p+q=k} \Gamma'_{\boldsymbol{p}|q}(E_1^{\boxtimes p} \boxtimes E_2^{\boxtimes q} \to M^k) \otimes \mathbb{C}$.

Remark 3.3 For the future reference, we make a distinction between $\mathcal{O}(\mathcal{E})$ and $\mathcal{O}(\mathcal{E}[0])$. The former is always understood as the space of smooth functionals, while the latter is the space of (potentially infinite) series in symmetric tensor products, without any notion of convergence. We make this distinction in Sect. 7, where we need formal objects of the type $\mathcal{O}(\mathcal{E}[0])$.

Next we introduce the notion of a derivative of a graded functional.

Definition 3.22 Let $F \in \mathcal{A}^k$, $u \in \mathcal{E}^{\otimes k-1}$, $h \in \mathcal{E}$.

(i) The left derivative of F at u in the direction of h is defined by

$$\left\langle \frac{\delta_l F}{\delta\varphi}(u), h \right\rangle \doteq F(h \wedge u), k > 0$$
$$\frac{\delta_l F}{\delta\varphi} = 0 \quad F \in \mathcal{A}^0.$$

We extend this definition to $\mathcal{A}$ by linearity.

(ii) Analogously, the right derivative of F at u in the direction of h is defined by

$$\left\langle \frac{\delta_r F}{\delta\varphi}(u), h \right\rangle \doteq F(u \wedge h), k > 0$$
$$\frac{\delta_r F}{\delta\varphi} = 0 \quad F \in \mathcal{A}^0.$$

Clearly, for $F \in \mathcal{A}$, $\left\langle \frac{\delta_l F}{\delta\varphi}(.), h \right\rangle$ is an element of $\mathcal{A}$ and we can think of $\frac{\delta_l F}{\delta\varphi}$ as a distribution (i.e. continuous linear map) on $\mathcal{E}$ with values in the graded algebra $\mathcal{A}$. This point of view has been adapted in [Rej11b]. We equip the space of such distributions with the strong topology (the topology of uniform convergence on bounded sets) and use the notation $L_b(\mathcal{E}, \mathcal{A})$. The theory of distributions taking values in general locally convex vector spaces has been developed in [Sch57, Sch58]. One can define the notions of convolution, Fourier transform and WF set for such objects. We also have the analogue of Theorem 3.3. The following definitions allow us to distinguish important classes of graded functionals. For simplicity of notation we only explicitly spell out the case where the configuration space is $\mathcal{E}_0 \oplus \mathcal{E}_1[1]$. By definition, we can differentiate elements of $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ as functionals on $\mathcal{E}_0$. The n-th derivative of $F \in \mathcal{O}(\mathcal{E}_0, \mathcal{A})$ at a point $\varphi \in \mathcal{E}_0$ will be denoted by $F^{(n)}(\varphi)$ or $\frac{\delta F}{\delta\varphi_0}(\varphi)$, to distinguish it from the graded derivative $\frac{\delta_l F(\varphi)}{\delta\varphi_1}$ on $\mathcal{A}$.

Definition 3.23 The support of a graded functional $F \in \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k)$ is defined by

$$\overline{\bigcup_{\varphi\in\mathcal{E}_0, u\in\mathcal{E}_1^{\hat{\otimes}_\pi k}} \text{supp}\left(\frac{\delta F}{\delta\varphi_0}(\varphi; u)\right)} \cup \overline{\bigcup_{\varphi\in\mathcal{E}_0, u\in\mathcal{E}_1^{\hat{\otimes}_\pi k-1}} \text{supp}\left(\frac{\delta_l F(\varphi)}{\delta\varphi_1}(u)\right)},$$

where $\frac{\delta}{\delta\varphi_0}$ denotes the usual Bastiani derivative with respect to the variable in $\mathcal{E}_0$, while $\frac{\delta_l}{\delta\varphi_1}$ is the graded derivative on $\mathcal{A}$, defined in Definition 3.22. The definition of support generalizes to the case of $\mathcal{E}_0 \oplus \mathcal{E}_1[r_1] \oplus \ldots \mathcal{E}_N[r_N]$, $r_1, \ldots, r_N \in \mathbb{N}$ by adding further terms in the union.

Definition 3.24 A graded functional $F \in \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k)$ is called local (an element of $\mathcal{C}^\infty_{\text{loc}}(\mathcal{E}_0, \mathcal{A}^k)$) if it is compactly supported and for each $\varphi \in \mathcal{E}$ there exist $k \in \mathbb{N}$ and $i_0, \ldots, i_k$ such that

$$F(\varphi; h_1, \ldots, h_k) = \int_M \alpha(j_x^{i_0}(\varphi), j_x^{i_1}(h_1), \ldots, j_x^{i_k}(h_k)), \tag{3.10}$$

where $\varphi \in \mathcal{E}_0$, $h_1, \ldots, h_k \in \mathcal{E}_1$ and α is a density-valued function on the jet bundle. We denote

$$\mathcal{O}_{\text{loc}}(\mathcal{E}_0 \oplus \mathcal{E}_1[1]) \doteq \prod_{k=0}^{\infty} \mathcal{C}^\infty_{\text{loc}}(\mathcal{E}_0, \mathcal{A}^k).$$

Definition 3.25 We equip $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ with the antisymmetric product:

$$\begin{aligned}&\left\langle (F \wedge G)(\varphi); h_1, \ldots, h_{p+q}\right\rangle \\ &\quad\doteq \frac{1}{p!q!} \sum_{\sigma\in S_{p+q}} \text{sgn}(\sigma) \left\langle F(\varphi); h_{\sigma(1)}, \ldots, h_{\sigma(p)}\right\rangle \left\langle G(\varphi); h_{\sigma(p+1)}, \ldots, h_{\sigma(p+q)}\right\rangle,\end{aligned} \tag{3.11}$$

where F and G are of degree p and q respectively, $\varphi \in \mathcal{E}_0$ and $h_i \in \mathcal{E}_1$.

Definition 3.26 We define $\mathcal{O}_{\text{ml}}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])\mathcal{C}^\infty_{\text{ml}}(\mathcal{E}_0, \mathcal{A})$ as the algebraic completion of $\mathcal{O}_{\text{loc}}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ with respect to the graded product $\wedge$.

The following result allows us to characterize derivatives of compactly supported functionals.

Proposition 3.4 *Let $F \in \mathcal{O}^k(\mathcal{E}_0 \oplus \mathcal{E}_1[1]) \doteq \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k)$ be compactly supported, then the n-the derivative of F can be extended to an element of*

$$\Gamma'^{\mathbb{C}}(E_0^{\boxtimes n} \boxtimes E_1^{\boxtimes k} \to M^{k+n}) \cong (\mathcal{E}_0')^{\hat{\otimes}_\pi n} \hat{\otimes}_\pi (\mathcal{E}_1')^{\hat{\otimes}_\pi k} \otimes \mathbb{C},$$

for all $\varphi \in \mathcal{E}_0$ and we denote this extension also by $F^{(n)}(\varphi)$.

Proof We start with $F^{(1)}$. By definition $F^{(1)}(\varphi)$ is a continuous linear map from $\mathcal{E}_0$ to $(\mathcal{E}_1')^{\hat{\otimes}_\pi k} \otimes \mathbb{C}$, and so is a vector-valued distribution in $L_b(\mathcal{E}_0, (\mathcal{E}_1')^{\hat{\otimes}_\pi k}) \otimes \mathbb{C}$. Since $\mathcal{E}_0$ is a Montel space, the bounded sets are the same as equicontinuous sets, so $L_b(\mathcal{E}_0, (\mathcal{E}_1')^{\hat{\otimes}_\pi k})$ is identified with $L_\varepsilon(\mathcal{E}_0, (\mathcal{E}_1')^{\hat{\otimes}_\pi k})$, the space with the topology of uniform convergence on equicontinuous sets. The latter is then isomorphic to $\mathcal{E}_0' \hat{\otimes}_\varepsilon (\mathcal{E}_1')^{\hat{\otimes}_\pi k}$, since both arguments are complete and have the approximation property. Next we use the fact that $\mathcal{E}_0'$ and $(\mathcal{E}_1')^{\hat{\otimes}_\pi k}$ are nuclear to conclude that the injective product can be replaced with the projective product. Finally, we use Buchwalter's Theorem 3.5 to conclude that

$$F^{(1)}(\varphi) \in (\mathcal{E}_0')\hat{\otimes}_\pi(\mathcal{E}_1')^{\hat{\otimes}_\pi k} \otimes \mathbb{C} \cong \Gamma'^{\mathbb{C}}(E_0 \boxtimes E_1^{\boxtimes k} \to M^{k+1}),$$

for all $\varphi \in \mathcal{E}_0$. By iterating this procedure we obtain the result for all n. □

In analogy to the bosonic case, we can conclude that for a local functional $F \in \mathcal{O}^k_{\mathrm{loc}}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$, the WF set of $F^{(n)}(\varphi) \in \Gamma'(E_0^{\boxtimes n} \boxtimes E_1^{\boxtimes k} \to M^{k+n})$ is orthogonal to $T\mathrm{Diag}_{k+n}$, the tangent bundle of the thin diagonal.

3.4 Vector Fields

In this section we define some important geometrical structures on $\mathcal{E}$. Firstly, note that we can view $\mathcal{E}$ (in a trivial way) as an infinite dimensional manifold modeled on a locally convex topological vector space (a Fréchet space in this case). For a precise definition of infinite dimensional manifolds see [KM97, Nee06]. The tangent space to $\mathcal{E}$ is then given by

$$T\mathcal{E} = \mathcal{E} \times \mathcal{E},$$

and smooth vector fields are smooth sections

$$\Gamma(T\mathcal{E}) \cong \mathcal{C}^\infty(\mathcal{E}, \mathcal{E}).$$

As in the finite dimensional situation, vector fields on $\mathcal{E}$ form a Lie algebra, where the Lie bracket is given by the commutator $[.,.]$.

We can also define differential forms on $\mathcal{E}$, but here the definition is a bit more tricky than for vector fields. In [Nee06] one uses the following

Definition 3.27 Let X be a differentiable manifold (possibly infinite dimensional) and $\mathcal{Y}$ a locally convex topological vector space, then a $\mathcal{Y}$-valued p-form on X is a function that associates to each $x \in X$ a p-linear alternating map $\omega_x = \omega(x) : (T_xX)^p \to \mathcal{Y}$ such that in local coordinates the map $(x, v_1, \dots, v_p) \mapsto \omega_x(v_1, \dots, v_p)$ is smooth.

It follows from Proposition 3.1 that first derivatives of Bastiani smooth functionals are 1-forms in the above sense. However, later on we will need objects more general than these, so it is better to modify the concepts of vector fields and forms already at this point.

We are interested in the complexification of $\Gamma(T\mathcal{E})$, i.e. in $\Gamma^{\mathbb{C}}(T\mathcal{E})$. This space can be identified with $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}^{\mathbb{C}})$. As we did with the functionals, we want to require vector fields to be compactly supported in some sense. There are two possibilities here; we can view a vector field X as an element of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}^{\mathbb{C}})$ or as a derivation on $\mathcal{C}^\infty(\mathcal{E}, \mathbb{R})$. The notion of support that we invoke here takes both these aspects into account. First, we restrict ourselves to vector fields that are elements of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^{\mathbb{C}})$. We define the support as follows.

Definition 3.28 Let $X \in \mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^{\mathbb{C}})$ be a vector field. We define

$$\text{supp } X = \overline{\bigcup_{\varphi \in \mathcal{E}} \text{supp}\,(X^{(1)}(\varphi))} \cup \overline{\bigcup_{\varphi \in \mathcal{E}} \text{supp}\,(X(\varphi))} \tag{3.12}$$

From now on we will consider only vector fields that are compactly supported. This implies that in particular they need to induce elements of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^{\mathbb{C}})$. The first part of the formula (3.12) refers to the support of X, as a function on $\mathcal{E}$. The second term is the support of X seen as a derivation. Next we define locality.

Definition 3.29 Let $\mathcal{V}_{\text{loc}} \subset \Gamma^{\mathbb{C}}(T\mathcal{E})$ denote the space of compactly supported complexified vector fields X that can be written in the form

$$X(\varphi)(x) = \tilde{X}(j_x^k(\varphi)) \equiv X_x(\varphi),$$

where $k \in \mathbb{N}$ and $\tilde{X}$ is some $\mathcal{E}_c$-valued function on the jet bundle. Such vector fields are called *local*.

Note that elements of $\mathcal{V}_{\text{loc}}$ are derivations of $\mathcal{F}_{\text{loc}}$ and $\mathcal{V}_{\text{loc}}$ is a Lie subalgebra (over $\mathbb{C}$) of $\Gamma^{\mathbb{C}}(T\mathcal{E})$, where the Lie bracket is given by the commutator of vector fields. However, it is not an $\mathcal{F}$-submodule of $\Gamma^{\mathbb{C}}(T\mathcal{E})$. In the next step we define multivector fields. We use a definition that differs from the standard one used in the literature, but is more natural in our context. Let us start with some motivation. Firstly, we want to be able to insert a differential $F^{(1)}$ of a local functional into a local bi-vector field and the result should be an element of $\mathcal{V}_{\text{loc}}$. Secondly, we want antisymmetry, some smoothness conditions and the compact support requirement. Locality implies that we have to consider objects more general than just elements of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^{\mathbb{C}} \wedge \mathcal{E}_c^{\mathbb{C}})$, which would be the standard notion of the space of bi-vector fields on an infinite dimensional manifold. Instead, we use the framework introduced in Sect. 3.3 and view the complexified multivector fields as $\mathcal{O}(T^*[1]\mathcal{E})$, where $T^*[1]\mathcal{E}$ is the odd cotangent bundle of $\mathcal{E}$, i.e.

$$T^*[1]\mathcal{E} \doteq \mathcal{E} \oplus \mathcal{E}^*[1],$$

where $\mathcal{E}^* \doteq \Gamma(E^* \to M)$. The notion of functions on a graded space has been clarified in Definition 3.21. Among all elements of $\mathcal{O}(T^*[1]\mathcal{E})$ we distinguish the local ones.

Definition 3.30 The space of *local multivector fields* is defined as $\mathcal{O}_{\text{loc}}(\mathcal{E} \oplus \mathcal{E}^*[1])$ in the sense of Definition 3.24.

In particular, we identify $\mathcal{V}_{\text{loc}}$ with $\mathcal{O}^1_{\text{loc}}(\mathcal{E} \oplus \mathcal{E}^*[1])$. Note that the notion of the support of a vector field which we have introduced in Definition 3.12 is now just a special case of Definition 3.23. The space of local multivector fields defined this way is closed under insertion of differentials $F^{(1)}$ of local functionals, as required.

In the next step we introduce multilocal multivector fields.

Definition 3.31 The space of *multilocal multivector fields* $\bigwedge \mathcal{V}$ is defined as $\mathcal{O}_{\text{ml}}(\mathcal{E} \oplus \mathcal{E}^*[1])$, in the sense of Definition 3.26. We also denote $\mathcal{V} \doteq \mathcal{O}^1_{\text{ml}}(\mathcal{E} \oplus \mathcal{E}^*[1])$.

Remark 3.4 Note that $\mathcal{V}$ is just the algebraic completion of $\mathcal{V}_{\text{loc}}$ as a $\mathcal{F}$-module and a Lie subalgebra (over $\mathbb{C}$) of $\Gamma^{\mathbb{C}}(T\mathcal{E})$. Elements of $\mathcal{V}$ are derivations of $\mathcal{F}$.

At this point it is convenient to introduce some notation. The action of a vector field $X \in \mathcal{V}$ on a functional $F \in \mathcal{F}$ can be written as

$$(\partial_X F)(\varphi) = \left\langle F^{(1)}(\varphi), X(\varphi) \right\rangle .$$

As $F^{(1)}(\varphi)$ is represented be a certain integration measure formally written as $\frac{\delta F}{\delta\varphi(x)}$, we can use the notation

$$(\partial_X F)(\varphi) = \int_M X_x(\varphi) \frac{\delta F}{\delta\varphi(x)},$$

which motivates the following:

$$X = \int_M X_x \frac{\delta}{\delta\varphi(x)}.$$

This notation is analogous to the one adopted in the finite dimensional case, i.e. $v = \sum_{i=1}^N v^i \partial_i$, where $v \in \Gamma(T\mathbb{R}^N)$. In the physics literature this formal notation is commonly used, but one replaces $\frac{\delta}{\delta\varphi(x)}$ with a formal generator $\varphi^\ddagger(x)$ called the *antifield*, i.e.

$$X(\varphi, \varphi^\ddagger) = \int_M X_x(\varphi)\varphi^\ddagger(x)$$

This way, vector fields in our approach can be identified with functions of φ and $\varphi^\ddagger$ present in other approaches. Similarly, we write k-vector fields $Y \in \bigwedge^k \mathcal{V}$ in the form

$$Y(\varphi) = \int_M Y(\varphi)(x_1, \dots, x_k) \frac{\delta}{\delta\varphi(x_1)} \cdots \frac{\delta}{\delta\varphi(x_k)},$$

where all the indices of the fiber V^* have been suppressed and $Y(\varphi)(x_1, \dots, x_k)$ is a distributional kernel with antisymmetry properties reflecting Definition 3.18. If Y is local, then this distribution is supported on the thin diagonal. In the antifield notation we express Y as

$$Y(\varphi, \varphi^\ddagger) = \int_M Y(\varphi)(x_1, \dots, x_k)\varphi^\ddagger(x_1) \dots \varphi^\ddagger(x_k).$$

This notation is actually not as formal as it seems, if we interpret $\varphi^\ddagger(x)$ as an evaluation functional, i.e. $\varphi^\ddagger_\alpha(x)(v) \doteq v_\alpha(x)$, where $v \in \mathcal{E}^*$. We can then understand the above formula as a definition of an antisymmetric k-form on $\mathcal{E}^*$

$$\left\langle Y(\varphi, \varphi^\ddagger); v_1, \dots, v_k \right\rangle = \int_M Y(\varphi)(x_1, \dots, x_k)v(x_1) \dots v(x_k),$$

where we have suppressed all the indices of the fiber V^* and we do not need to antisymmetrize, as $Y(\varphi)(x_1, \dots, x_k)$ is already antisymmetric. This agrees with the formal notation introduced in Sect. 3.3.

3.5 Functorial Interpretation

All the constructions we have performed can be done covariantly across all spacetimes, so we can reformulate them in the category theory language.

Definition 3.32 The configuration space functor is a contravariant functor $\mathfrak{E}$ from **Loc** to **Vec**, such that for all $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$, $\mathfrak{E}(\mathcal{M})$ is a configuration space according to the Definition 3.1.

Let us consider some examples.

Example 3.2 For the theory of scalar fields, the *configuration space functor* $\mathfrak{E}$ is a *contravariant functor* (similar to a covariant functor, but reverses direction of the arrows) from **Loc** to **Vec**, defined by

$$\mathfrak{E}(\mathcal{M}) = \mathcal{C}^\infty(M, \mathbb{R}),$$
$$\mathfrak{E}\chi = \chi^*,$$

where $\mathcal{M} = (M, g) \in \mathrm{Obj}(\mathbf{Loc})$ and for a morphism $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ we have the pullback $\chi^* : \mathfrak{E}(\mathcal{N}) \to \mathfrak{E}(\mathcal{M})$ defined by

$$\chi^*\varphi \doteq \varphi \circ \chi,$$

where $\varphi \in \mathfrak{E}(\mathcal{N})$.

This definition generalizes in a straightforward way to the case where $\mathcal{E}(M)$ is defined as the space of k-forms on M, since the pullback is still well defined. For k-vectors we use the metric g to obtain k-forms, so we can define pullbacks of arbitrary tensor fields.

Example 3.3 For effective quantum gravity we set

$$\begin{aligned}\mathfrak{E}(\mathcal{M}) &= \Gamma((T^*M)^{\otimes 2}),\\ \mathfrak{E}\chi &= \chi^*,\end{aligned}$$

where

$$(\chi^* h)(u, v) \doteq h \circ \chi(T\chi(u), T\chi(v)),$$

where $\chi : \mathcal{M} \to \mathcal{N}$, $u, v \in \Gamma(TM)$, $h \in \mathfrak{E}(\mathcal{N})$ and $T\chi$ is the tangent map.

The generalization to forms taking values in a fixed vector space is also straightforward.

Example 3.4 For Yang-Mills theories with a trivial bundle we set

$$\begin{aligned}\mathfrak{E}(\mathcal{M}) &= \Omega_1(M, \mathfrak{k}),\\ \mathfrak{E}\chi &= \chi^*,\end{aligned}$$

where

$$(\chi^* A)(u) \doteq h \circ \chi(T\chi(u)),$$

where $\chi : \mathcal{M} \to \mathcal{N}$, $u \in \Gamma(TM)$, $A \in \mathfrak{E}(\mathcal{N})$.

Given $\mathfrak{E}$, the spaces of local and multilocal functionals are assigned to spacetimes in a functorial way.

Proposition 3.5 *The space of local functionals is a functor* $\mathfrak{F}_{\text{loc}} : \mathbf{Loc} \to \mathbf{Vec}$. *So is the space of multilocal functionals* $\mathfrak{F} : \mathbf{Loc} \to \mathbf{Vec}$.

Proof We set $\mathfrak{F}_{\text{loc}}(\mathcal{M}) \doteq \mathcal{F}_{\text{loc}}(M)$ and $\mathfrak{F}(\mathcal{M}) \doteq \mathcal{F}(M)$ for the objects and $\mathfrak{F}_{\text{loc}}\chi(F)(\varphi) \doteq F(\mathfrak{E}\chi\varphi)$, $\mathfrak{F}\chi(G)(\varphi) \doteq F(\mathfrak{E}\chi\varphi)$ for the morphisms, where $\chi \in \text{Hom}(\mathcal{M}, \mathcal{N})$, $F \in \mathfrak{F}_{\text{loc}}(\mathcal{M})$, $G \in \mathfrak{F}(\mathcal{M})$ and $\chi \in \mathfrak{E}(\mathcal{N})$. □

The space of configurations is in a natural way a contravariant functor, but the space of compactly supported configurations can be assigned to a spacetime in a covariant way. As an example, consider the space of test functions in Definition 2.65. This motivates the following definition

Definition 3.33 The space of compactly supported configurations $\mathfrak{E}_c$ is a covariant functor from **Loc** to **Vec**, which acts on the objects as $\mathfrak{E}_c(\mathcal{M}) = \mathcal{E}_c(\mathcal{M})$ and maps morphisms $\chi \in \text{Hom}(\mathbf{Loc})$ to appropriate pushforwards χ_*.

The vector fields on $\mathcal{E}$ that we consider are maps from $\mathcal{E}$ to $\mathcal{E}_c$, so they transform covariantly.

Proposition 3.6 *Given $\mathfrak{E}$ and $\mathfrak{E}_c$, the space of multi-local vector fields is a functor $\mathfrak{V}$: **Loc** $\to$ **Vec**, the same holds for local vector fields.*

Proof We set $\mathfrak{V}(\mathcal{M}) \doteq \mathcal{V}(\mathcal{M})$ and $\mathfrak{V}\chi(X) \doteq \mathfrak{E}_c\chi \circ X \circ \mathfrak{E}\chi$, where $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$, $X \in \mathfrak{V}(\mathcal{M})$. The same for local vector fields. □

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Chapter 4
Classical Theory

Having defined the essential kinematical structure we are now ready to introduce the dynamics. To this end we use a generalization of the Lagrange formalism. The precise relation to notions known from classical mechanics will be explained in Sect. 4.5.

4.1 Dynamics

We start with introducing some definitions.

Definition 4.1 A *generalized Lagrangian* on a fixed spacetime $\mathcal{M} = (M, g)$ is a map $L : \mathscr{D}(M) \to \mathcal{F}_{\text{loc}}$ such that

(i) $L(f + g + h) = L(f + g) - L(g) + L(g + h)$ for $f, g, h \in \mathscr{D}$ with $\text{supp}\ f \cap \text{supp}\ h = \varnothing$ (**Additivity**).
(ii) $\text{supp}\,(L(f)) \subseteq \text{supp}\,(f)$ (**Support**).
(iii) Let $\mathcal{G}$ be the isometry group of the spacetime $\mathcal{M}$ (for Minkowski spacetime we set $\mathcal{G}$ to be the proper orthochronous Poincaré group $\mathcal{P}_+^{\uparrow}$.). We require that $L(f)(\beta^*\varphi) = L(\beta_* f)(\varphi)$ for every $\beta \in \mathcal{G}$ (**Covariance**).

Proposition 4.1 *Let L be a generalized Lagrangian. The additivity of L in the test function implies that:*

(i) $L(f)$ is an additive functional on $\mathcal{E}$.
(ii) For any fixed test function $f \in \mathscr{D}(M)$, $L(f)$ can be written as a finite sum of additive functionals of arbitrarily small space-time support.

Proof See [BFR12]. □

Definition 4.1 formalizes the idea that the generalized Lagrangian associates to a test function f the local functional $L(f)$ obtained by integrating f with the Lagrangian density $\mathcal{L}(x)[\varphi]$ that depends locally on the field configuration φ. Introducing the cutoff function f is necessary because the manifold M, being globally

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hyperbolic, is non-compact. Moreover, it is not possible to restrict ourselves to compactly supported configurations φ, since later on we will have to impose equations of motion that are normally hyperbolic and non-trivial solutions to such equations cannot be compactly supported. Let us consider some examples.

Example 4.1 Examples of generalized Lagrangians:

(i) Free scalar field:

$$L_0(f)[\varphi] = \frac{1}{2}\int_M \left(\nabla_\nu \varphi \nabla^\nu \varphi - m^2\varphi^2\right) f d\mu_g, \tag{4.1}$$

where $d\mu_g \doteq \sqrt{-g}d^4x$ is the volume form on $\mathcal{M} = (M, g)$ induced by the metric, ∇ is the covariant derivative, and we use the Einstein summation convention for the indices, so $\nabla_\nu\varphi\nabla^\nu\varphi \equiv \sum_{\nu=0}^3 \nabla_\nu\varphi\nabla^\nu\varphi$.

(ii) Interaction term in the φ^4 theory:

$$L_I(f)[\varphi] = \int_M \frac{1}{4!}\varphi^4 f d\mu_g.$$

(iii) Yang-Mills Lagrangian:

$$L^{YM}(f)(A) = -\frac{1}{2}\int_M f \ \mathrm{tr}\,(F \wedge *F),$$

where $A \in \Omega_1(M, \mathfrak{k})$, $F = dA + \frac{1}{2}[A, A]$ and $*$ is the Hodge operator and tr is the trace in the adjoint representation given by the Killing-Cartan metric.

(iv) Einstein-Hilbert Lagrangian:

$$L^{EH}(f)[h] \doteq \int R[g+h]f \, d\mu_{g+h},$$

where $h \in \Gamma((T^*M)^{\otimes_s 2})$.

The cutoff function f is only an auxiliary tool, which allows us to formulate the problem in a mathematically rigorous way, but it has no physical meaning. Therefore, the crucial structures in our classical model cannot depend on the choice of f. This is achieved by means of the following definition:

Definition 4.2 The *Euler-Lagrange derivative* of L is a map $S' : \mathcal{E} \to \mathcal{E}_c'^{\mathbb{C}}$ defined by

$$\left\langle S'(\varphi), h\right\rangle \doteq \left\langle L(f)^{(1)}[\varphi], h\right\rangle,$$

where $h \in \mathcal{E}_c$ and $f \in \mathcal{D}(M)$ is chosen in such a way that $f \equiv 1$ on supp h.

Since $L(f)$ is a local functional, S' doesn't depend on the choice of f. Moreover, S' would not change if we added to L a generalized Lagrangian that is supported in the region where f is not constant. Therefore, the dynamical structure is not encoded in L's, but rather in equivalence classes of generalized Lagrangians. This leads to the following definition:

Definition 4.3 An *action* is an equivalence class of Lagangians under the following equivalence relation [BDF09]:

$$L_1 \sim L_2 \quad \text{iff} \quad \text{supp}\,((L_1 - L_2)(f)) \subset \text{supp}\ df. \tag{4.2}$$

Remark 4.1 We write S instead of $[L]$ to denote the equivalence class of L. For example, the action corresponding to a Lagrangian denoted by L_0 will be written as S_0 rather than $S[L_0]$.

The physical meaning of (4.2) is to identify Lagrangians that "differ by a total divergence". Note that two Lagrangians equivalent under the relation (4.2) induce the same Euler-Lagrange derivative, so dynamics is a structure coming from actions rather than Lagrangians. We are now ready to introduce the equations of motion (EOM's).

Definition 4.4 The *equation of motion* (EOM) corresponding to the action S is

$$S'(\varphi) \equiv 0, \tag{4.3}$$

understood as a condition on $\varphi \in \mathcal{E}$.

The space of solutions to (4.3) will be denoted by $\mathcal{E}_S$ and it is a submanifold of $\mathcal{E}$. Physically, classical observables should be modeled as multilocal functionals on $\mathcal{E}_S$. Let us denote the space of such functionals by $\mathcal{F}_S$. Note that it can be characterized as the quotient

$$\mathcal{F}_S = \mathcal{F}/\mathcal{I}_S,$$

where $\mathcal{I}_S$ is the ideal of $\mathcal{F}$ consisting of functionals that vanish on $\mathcal{E}_S$.

Definition 4.5 The second variational derivative S'' of the action S is defined by

$$\left\langle S''(\varphi), h_1 \otimes h_2 \right\rangle \doteq \left\langle L^{(2)}(f)(\varphi), h_1 \otimes h_2 \right\rangle,$$

where $f \equiv 1$ on supp h_1 and supp h_2.

By definition S'' is a linear map

$$S'' : \mathcal{E} \to L(\mathcal{E}_c \times \mathcal{E}_c, \mathbb{C}).$$

From the locality of the Lagrangian it follows that in fact $S''(\varphi)$ can be extended to a linear map on $\mathcal{E}_c \times \mathcal{E}$ and the Schwarz kernel theorem (see [Hör03, Chap. 5]) implies that this induces a continuous linear operator $P_S(\varphi) : \mathcal{E}^{\mathbb{C}} \to \mathcal{E}^{*\mathbb{C}}$, where $\mathcal{E}^* \doteq \Gamma(E^* \to M)$. For details concerning the proofs of these statements, see [BDLGR16]. Note that if S is quadratic then $P_S(\varphi) \equiv P$ is the same for all φ and $S'(\varphi) = P\varphi$. This is the case for the free scalar field described by the Lagrangian L_0 from Example 4.1, where $P = -(\Box + m^2)$.

The crucial assumption in the pAQFT approach is that $P_S(\varphi)$ is a normally hyperbolic operator. We recall that an operator on $\mathcal{E}$ is normally hyperbolic if its principal symbol is of the metric type, i.e. it is given by

$$\sigma_{P_S(\varphi)}(\xi, \xi) = g(\xi, \xi)\mathrm{id}_{E_x},$$

for all $\xi \in T^*_x M$ and all $x \in M$. Here id_{E_x} denotes the identity on the fiber. For more details on normally hyperbolic operators see [BGP07]. In the same reference it is also shown that for such operators there exist unique *retarded* and *advanced Green's functions* (fundamental solutions) $\Delta^{\mathrm{R}}_S(\varphi),\ \Delta^{\mathrm{A}}_S(\varphi) : \mathcal{E}^{*\mathbb{C}}_c \to \mathcal{E}^{\mathbb{C}}$ defined by the requirements

$$P_S(\varphi) \circ \Delta^{\mathrm{R/A}}_S = \mathrm{id}_{\mathcal{E}^{*\mathbb{C}}_c},$$
$$\Delta^{\mathrm{R/A}}_S \circ P_S(\varphi) \restriction_{\mathcal{E}_c} = \mathrm{id}_{\mathcal{E}^{\mathbb{C}}_c},$$

and the support properties

$$\mathrm{supp}\ \Delta_A(f) \subset J^+(\mathrm{supp}\ f),$$
$$\mathrm{supp}\ \Delta_R(f) \subset J^-(\mathrm{supp}\ f),$$

where $f \in \mathcal{E}^{*\mathbb{C}}_c$. Note that, by the Schwarz kernel theorem, these operators can be written in terms of their integral kernels, which then satisfy appropriate support properties and

$$\Delta^{\mathrm{R}}_S(y, x) = \Delta^{\mathrm{A}}_S(x, y). \tag{4.4}$$

Let us define the *causal propagator* as

$$\Delta_S(\varphi) \doteq \Delta^{\mathrm{R}}_S(\varphi) - \Delta^{\mathrm{A}}_S(\varphi). \tag{4.5}$$

Due to (4.4) the causal propagator is antisymmetric, i.e. its integral kernel satisfies

$$\Delta_S(\varphi)(y, x) = -\Delta_S(\varphi)(x, y).$$

Remark 4.2 Note that in the convention we use here (following for example [DF02]), the second derivative of the free action induces the differential operator P which is *minus* the usual Klein-Gordon operator. As a consequence, what we call $\Delta^{\mathrm{R}}_{S_0}$ is $-\Delta^{ret}_m$, where Δ^{ret}_m is the retarded Green's function for $\Box + m^2$. similarly for the advanced propagator. Consequently

$$\Delta_{S_0} = \Delta^{adv}_m - \Delta^{ret}_m,$$

which is in agreement with the convention used for example in [FV11a, FV11b].

Example 4.2 Consider the free scalar field (i.e. $P = -(\Box + m^2)$) on Minkowski spacetime. We look for solutions to the equation

$$(\Box + m^2)_x \Delta_{S_0}^{\mathrm{R/A}}(x, y) = -\delta(x, y). \tag{4.6}$$

with the appropriate support properties. The standard way to proceed (see for example [IZ06]) is to use the Fourier transform. Translation invariance implies that the integral kernels of $\Delta_{S_0}^{\mathrm{R/A}}$ depend only on $x - y$, so we set

$$\Delta_{S_0}^{\mathrm{R/A}}(x, y) = \frac{1}{(2\pi)^4} \int e^{-ip\cdot(x-y)} \widehat{\Delta_{S_0}^{\mathrm{R/A}}}(p) d^4 p, \tag{4.7}$$

where $p \cdot x \doteq p^0 x^0 - \boldsymbol{p}.\boldsymbol{x}$ and $\boldsymbol{p}.\boldsymbol{x} \doteq \sum_{i=1}^3 p^i x^i$. From (4.6) follows that

$$(-p^2 + m^2)\widehat{\Delta_{S_0}^{\mathrm{R/A}}}(p) = -1$$

To obtain the explicit formulas for the Green's functions we insert $\widehat{\Delta_{S_0}^{\mathrm{R/A}}}(p) = -\frac{1}{-p^2+m^2}$ back to equation (4.7) and choose the appropriate contour of integration for the integral over p^0. More precisely, we set

$$\widehat{\Delta_{S_0}^{\mathrm{R/A}}}(p) = \frac{1}{(p^0 \pm i\epsilon)^2 - \boldsymbol{p}^2 - m^2}.$$

Using Cauchy's theorem for the contour integrals obtained from the ϵ-prescription above, we conclude that $\Delta_{S_0}^{\mathrm{R/A}}$ have indeed correct support properties. Performing the integral over p^0 we obtain explicit formulas for the Green's functions on Minkowski spacetime:

$$\Delta_{S_0}^{\mathrm{R/A}}(x, y) = \mp \frac{i\theta(\pm(x^0 - y^0))}{(2\pi)^3} \int \left(e^{-i\omega_{\boldsymbol{p}}(x^0-y^0)+i\boldsymbol{p}.\boldsymbol{x}} - e^{i\omega_{\boldsymbol{p}}(x^0-y^0)+i\boldsymbol{p}.\boldsymbol{x}} \right) \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}},$$

where $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. Hence the causal propagator is given by the formula

$$\Delta_{S_0}(x, y) = -\frac{i}{(2\pi)^3} \int \left(e^{-i\omega_{\boldsymbol{p}}(x^0-y^0)+i\boldsymbol{p}.\boldsymbol{x}} - e^{i\omega_{\boldsymbol{p}}(x^0-y^0)+i\boldsymbol{p}.\boldsymbol{x}} \right) \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}. \tag{4.8}$$

4.2 Natural Lagrangians

There is an elegant way to describe generalized Lagrangians using the language of category theory.

Definition 4.6 A *natural Lagrangian L* is a natural transformations from the functor $\mathfrak{D}$ to $\mathfrak{F}_{\mathrm{loc}}$, such that for all $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$ we have:

(i) $L_{\mathcal{M}}(f+g+h) = L_{\mathcal{M}}(f+g) - L_{\mathcal{M}}(g) + L_{\mathcal{M}}(g+h)$ for $f, g, h \in \mathscr{D}$ with $\mathrm{supp}\ f \cap \mathrm{supp}\ h = \varnothing$ (**Additivity**).
(ii) $\mathrm{supp}\,(L_{\mathcal{M}}(f)) \subseteq \mathrm{supp}\,(f)$ (**Support**).

Note that L is fixed by a family of maps $L_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$ satisfying the covariance condition

$$L_{\mathcal{M}}(f)(\mathfrak{E}\chi(\varphi)) = L_{\mathcal{N}}(\mathfrak{D}\chi f)(\varphi), \tag{4.9}$$

where $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$, $f \in \mathfrak{D}(\mathcal{M})$, $\varphi \in \mathfrak{E}(\mathcal{N})$. The following result shows the relation between natural Lagrangians and generalized Lagrangians introduced earlier in this chapter.

Proposition 4.2 *Let L be a natural Lagrangian from Definition 4.6, then for each* $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$*,* $L_{\mathcal{M}}$ *is a generalized Lagrangian in the sense of Definition 4.1.*

Proof Since the additivity and support conditions are included in the definition, it suffices to show covariance under isometries of the spacetime. This, however, follows from the general local covariance of L, expressed by the condition (4.9). □

Example 4.3 All the Lagrangians from Example 4.1 give rise to natural Lagrangians.

In the following we will always assume that our generalized Lagrangians arise in this way.

4.3 Homological Characterization of the Solution Space

We have already remarked that the space of multilocal functionals on the space $\mathcal{E}_S$ of solutions to EOM's can be understood as the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{I}_S$. In this section we make this statement precise and we find a nice homological interpretation of this quotient. Since $\mathcal{E}_S$ is the zero locus of S', it is natural to use at this point the derived critical locus construction.

Note that if $X \in \mathcal{V}$ is a multilocal vector field, then the multilocal functional $\langle S', X \rangle$, obtained by contracting this vector field with the one-form S', vanishes on $\mathcal{E}_S$. Let us define a map $\delta_S : \mathcal{V} \to \mathcal{F}$ by

$$\delta_S \doteq -\langle S', . \rangle .$$

The minus sign is introduced for future convenience. Clearly, $\delta_S(\mathcal{V}) \subset \mathcal{I}_S$. In general the opposite inclusion can hold only locally, since the structure of the global solution space of nonlinear PDE's can be very complicated. Since our ultimate goal is the quantum theory, we will avoid these complications by *defining*, from now on, $\mathcal{I}_S$ as $\delta_S(\mathcal{V})$.

Definition 4.7 The ideal $\mathcal{I}_S \subset \mathcal{F}$ is defined as $\delta_S(\mathcal{V})$ and we call it "the ideal generated by the equations of the motion". The space of on-shell functionals is defined as

$$\mathcal{F}_S \doteq \mathcal{F}/\mathcal{I}_S.$$

Since QFT models are often constructed by means of some quantization procedure from classical field theory models, the space of solutions to classical EOM's is bound to appear. However, in the approach to pAQFT which we advocate in this book, we use $\mathcal{F}_S$ rather than $\mathcal{E}_S$, so it is natural to give up the traditional point of view on the space of solutions. This now allows for an algebraic interpretation of $\mathcal{F}_S$ as the 0th homology of the following complex:

$$\underset{}{\ldots} \rightarrow \underset{2}{\bigwedge^2 \mathcal{V}} \overset{\delta_S}{\rightarrow} \underset{1}{\mathcal{V}} \overset{\delta_S}{\rightarrow} \underset{0}{\mathcal{F}} \rightarrow 0\,, \tag{4.10}$$

where δ_S is extended to the exterior algebra $\bigwedge \mathcal{V}$ by requiring the graded Leibniz rule, acting from the right, with respect to the exterior product $\wedge$ and by continuity.

Let us now discuss $H_1\left(\bigwedge \mathcal{V}, \delta_S\right)$. The kernel of $\delta_S : \mathcal{V} \to \mathcal{F}$ (here denoted by $\ker \delta_S\big|_{1\to 0}$) consists of those vector fields X which satisfy

$$\partial_X S(\varphi) \doteq \left\langle S'(\varphi), X(\varphi)\right\rangle = 0, \quad \forall \varphi \in \mathcal{E}.$$

Geometrically, these vector fields correspond to directions in the configuration space in which the action S is constant. In this sense, we interpret them as *symmetries of the action*. Among all the symmetries we can distinguish those which are of the form $\delta_S Z$ for some $Z \in \bigwedge^2 \mathcal{V}$. In order to understand the meaning of such symmetries, let us consider a bivector field of the form $Z = X \wedge Y$ for some $X, Y \in \mathcal{V}$. We have

$$\delta_S\,(X \wedge Y) = -(\delta_S X)Y + (\delta_S Y)X.$$

Note that the vector field obtained this way vanishes identically on $\mathcal{E}_S$. For this reason, such symmetries are called in the physics literature *trivial symmetries*. As a result, one interprets

$$H_1\left(\bigwedge \mathcal{V}, \delta_S\right) \doteq \frac{\mathrm{Ker}\,\delta_S\big|_{1\to 1\to 0}}{\mathrm{Im}\,\delta_S\big|_{2\to 1}} \tag{4.11}$$

as the space of *non-trivial local symmetries*. Theories that don't possess non-trivial local symmetries include the scalar field theory with a polynomial interaction (e.g. φ^4). One of the simplest examples of a theory with non-trivial $H_1\left(\bigwedge \mathcal{V}, \delta_S\right)$ is QED. Yang-Mills theories and gravity also fall into this class. We will discuss these in more detail in Chaps. 7 and 8. A simple criterion to decide that the theory has no non-trivial local symmetries has been provided in [FR12b].

Proposition 4.3 *If the linearised equation of motion*

$$P_S(\varphi)\psi = 0$$

doesn't have any non-trivial compactly supported solutions ψ *for all* $\varphi \in \mathcal{E}$*, then the action* S *possesses no non-trivial symmetries.*

Proof See the discussion at the end of Sect. 2 in [FR12b]. □

To end this section, we will introduce one more algebraic structure on $\bigwedge \mathcal{V}$. As noted before, $\mathcal{V}$ is a Lie subalgebra of $\Gamma(T\mathcal{E})$, where the bracket is just minus the commutator of vector fields. Now, using the graded Leibniz rule and continuity, one can extend this structure to minus the Schouten bracket $\{.,.\}$ on $\bigwedge \mathcal{V}$, fixed uniquely by the following properties:

1. $\{X, F\} \doteq -\partial_X F$, for $F \in \mathcal{F}$ and $X \in \mathcal{V}$,
2. $\{X, Y\} \doteq -[X, Y]$, for $X, Y \in \mathcal{V}$,
3. $\{.,.\}$ fulfills the graded Leibniz rule in both arguments.

In the physics literature this structure is called the *antibracket*, where it is usually expressed with the use of the antifield notation introduced in Sect. 3.4. Recall that $X \in \bigwedge^k \mathcal{V}$ is an element of $\mathcal{O}^k_{\mathrm{ml}}(T^*[1]\mathcal{E})$, where $T^*[1]\mathcal{E} = \mathcal{E} \oplus \mathcal{E}^*[1]$ is the odd cotangent bundle of $\mathcal{E}$. We denote the nth derivative with respect to φ by $\frac{\delta^n X}{\delta \varphi^n}$ and the left derivative of X in the direction of $v \in \mathcal{E}^*$ is denoted by

$$\left\langle \frac{\delta_l X}{\delta \varphi^\ddagger}(\varphi), v \right\rangle.$$

Similarly, we denote the right derivative as

$$\left\langle \frac{\delta_r X}{\delta \varphi^\ddagger}(\varphi), v \right\rangle.$$

Using this notation we write

$$\{X, Y\}(\varphi) \doteq \left\langle \frac{\delta_r}{\delta \varphi} X(\varphi), \frac{\delta_l}{\delta \varphi^\ddagger} Y(\varphi) \right\rangle - \left\langle \frac{\delta_r}{\delta \varphi^\ddagger} X(\varphi), \frac{\delta_l}{\delta \varphi} Y(\varphi) \right\rangle. \tag{4.12}$$

This expression is well defined, since the first derivatives of a multilocal vector field with respect to both φ and $\varphi^\ddagger$ are, by definition, smooth compactly supported sections.

Note that δ_S is locally generated by the bracket in the sense that

$$\delta_S X = \{X, S\} \doteq \{X, L(f)\},$$

where $f \equiv 1$ on the support of X and L is the Lagrangian defining the theory.

4.4 The Net of Topological Poisson Algebras

In this section we show how to construct a causal net of topological Poisson algebras in the sense of Definition 2.53. First we introduce a Poisson bracket on an appropriate space of functionals and next we equip this space of functionals with a topology that makes this bracket sequentially continuous.

4.4.1 The Peierls Bracket and Microcausal Functionals

We equip $\mathcal{F}$ with a Poisson bracket called the *Peierls bracket* [Pei52]. It was shown in [FR14] that this bracket is equivalent to the canonical bracket commonly used in classical mechanics, if the latter exists. The advantage of using the Peierls bracket is that it can be defined completely within the Lagrangian formalism, without the need to pass to the Hamiltonian. This is important for example in cases when the Hamiltonian vanishes. Another advantage of the Peierls bracket is that it is defined in a completely covariant way, so it doesn't require to introduce a foliation of M with Cauchy surfaces.

Definition 4.8 Let $F, G \in \mathcal{F}$. The Peierls bracket is defined by

$$\lfloor F, G \rfloor_S(\varphi) \doteq \left\langle F^{(1)}(\varphi), \Delta_S^{\mathbb{C}}(\varphi) G^{(1)}(\varphi) \right\rangle, \tag{4.13}$$

The bracket defined in Definition 4.8 is antisymmetric (by the antisymmetry of $\Delta_S(\varphi)$), bilinear and satisfies the Jacobi identity [Jak09]. Since $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ are smooth, it is clear that $\lfloor ., . \rfloor$ is well defined on multilocal functionals. However, $\mathcal{F}$ is *not* closed under this bracket. The natural question to ask is how to extend the domain of definition of $\lfloor ., . \rfloor$, so that the resulting space is closed under this bracket. To answer this question, it is useful to look at the WF set of $\Delta_S(\varphi)$.

$$\mathrm{WF}(\Delta_S(\varphi)) = \{(x, k; x', -k') \in \dot{T}^* M^2 | (x, k) \sim (x', k')\},$$

where the equivalence relation $\sim$ means that there exists a null geodesic strip such that both (x, k) and (x', k') belong to it. Recall that a null geodesic strip is a curve in T^*M of the form $(\gamma(\lambda), k(\lambda))$, $\lambda \in I \subset \mathbb{R}$, where $\gamma(\lambda)$ is a null geodesic parametrized by λ and $k(\lambda)$ is given by $k(\lambda) = g(\dot{\gamma}(\lambda), \cdot)$. The form of the WF set of $\Delta_S(\varphi)$ follows from the theorem on the propagation of singularities together with the initial conditions and the antisymmetry of $\Delta_S(\varphi)$ (See [Rad96] for a details).

We can now use Hörmander's criterion Theorem 3.3 to determine a class of distributions that can be pointwise multiplied with $\Delta_S(\varphi)$ and hence the condition that has to be satisfied by the differentials $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ in (4.13). This leads to the following definition

Definition 4.9 A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{R})$ is called *microcausal* if it is compactly supported and satisfies

$$\mathrm{WF}(F^{(n)}(\varphi)) \subset \Xi_n, \quad \forall n \in \mathbb{N},\ \forall \varphi \in \mathcal{E}, \tag{4.14}$$

where Ξ_n is an open cone defined as

$$\Xi_n \doteq T^*M^n \setminus \left\{(x_1, \dots, x_n; k_1, \dots, k_n) | (k_1, \dots, k_n) \in (\overline{V}_+^n \cup \overline{V}_-^n)_{(x_1,\dots,x_n)}\right\}, \tag{4.15}$$

where $(\overline{V}_\pm)_x$ is the closed future/past lightcone understood as a conic subset of T_x^*M.

In [BFR13] it is additionally required that the first derivative $F^{(1)}(\varphi)$ is smooth for all $\varphi \in \mathcal{E}$ and $\varphi \mapsto F^{(1)}(\varphi)$ is smooth as a map $\mathcal{E} \to \mathcal{E}^{*\mathbb{C}}$. We will call functionals satisfying this additional property *strongly microcausal*. We denote the space of microcausal functionals by $\mathcal{F}_{\mu c}$ and the space of the strongly microcausal ones by $\mathcal{F}_{s\mu c}$.

At this point it is convenient to introduce a notation for spaces of distributions with WF sets contained in open and closed cones.

Definition 4.10 Let $\mathscr{D}'_\Gamma(M^n)$ denote the space of distributions whose WF sets are contained in a closed cone $\Gamma \subset \dot{T}^*M^n$. Similarly, $\mathscr{E}'_\Lambda(M^n)$ denotes the space of distributions with compact support whose WF sets are contained in an open cone $\Lambda \subset \dot{T}^*M^n$.

We can now rephrase Definition 4.9 by saying that nth derivatives of elements of $\mathcal{F}_{\mu c}$ are distributions belonging to the corresponding spaces $\mathscr{E}'_{\Xi_n}(M^n)$. The generalization of the Definition 4.9 to the graded case is straightforward. We spell out the definition for $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$.

Definition 4.11 Let $F \in \mathcal{O}^k(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$. We say that F is microcausal, i.e. an element of $\mathcal{O}_{\mu c}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ if it is compactly supported and

$$F^{(n)}(\varphi) \in \mathscr{E}'_{\Xi_{n+k}}(M^{n+k}),$$

for all $\varphi \in \mathcal{E}_0$, $n \in \mathbb{N}$.

F is said to be strongly microcausal if in addition

(i) $F(\varphi, u)$ has an empty WF set for all $\varphi \in \mathcal{E}_0$, $u \in \mathcal{E}_1^{\hat{\otimes}_\pi (k-1)}$ and the map $(\varphi, u) \mapsto F(\varphi, u)$ is smooth as a map from $\mathcal{E}_0 \times \mathcal{E}_1^{\hat{\otimes}_\pi (k-1)}$ to $\mathcal{E}_1^{*\mathbb{C}}$,

(ii) $F^{(1)}(\varphi, u)$ has an empty WF set for all $\varphi \in \mathcal{E}_0$, $u \in \mathcal{E}_1^{\hat{\otimes}_\pi k}$ and the map $(\varphi, u) \mapsto F^{(n)}(\varphi, u)$ is smooth as a map from $\mathcal{E}_0 \times \mathcal{E}_1^{\hat{\otimes}_\pi k}$ to $\mathcal{E}_0^{*\mathbb{C}}$.

The following proposition is crucial for the construction of the local net of Poisson algebras.

Proposition 4.4 $(\mathcal{F}_{s\mu c}, \lfloor ., . \rfloor)$ *is closed under the bracket and is a Poisson algebra. If* Δ_S *doesn't depend on* φ*, then* $(\mathcal{F}_{\mu c}, \lfloor ., . \rfloor)$ *is also a Poisson algebra.*

Proof See [BFR12, BFR13] □

Remark 4.3 The stronger version of microcausality is needed if Δ_S depends on φ, because otherwise the proof of the Jacobi identity given in [Jak09] would fail. Alternatively, following Dabrowski [Dab14b], one can use a more refined definition of microcausality that involves the notion of dual WF sets. We will give more details on that in Sect. 4.4.2.

If the discussion applies to all the notions of microcausality introduced above, we use the notation $\mathcal{F}_{*\mu c}$.

4.4.2 Topologies on the Space of Microcausal Functionals

We now come to the important problem of introducing on $\mathcal{F}_{*\mu c}$ a topology that will be appropriate for constructing models of classical and quantum theories in the sense of Definitions 2.53 and 2.54. We have already seen that the regularity of smooth functionals is governed by the regularity of their derivatives. The latter is measured using the notion of a WF set. Clearly we need a topology that controls all these regularity properties. In the light of the discussion from Sect. 2.5 it would be desirable to use a topology that is nuclear. Other useful properties are completeness and being bornological. At the moment there is no definite consensus in the literature as to which choice is the most natural, so we will review the proposals that are most relevant for the scope of this book.

The basic idea is to introduce a topology on the space $\mathscr{E}'_{\Xi_n}(M^n)$ of distributions with WF sets contained within the open cone Ξ_n defined by (4.15) and use this topology to control the regularity of derivatives of functionals. At this point there are several possibilities. The simplest one is to invoke the topology of pointwise convergence of all the functional derivatives [BDF09], but the resulting space is not complete, so it might be better to use some weak notion of uniform convergence instead (see [Dab14b]).

We start by reviewing the proposal made by [BDF09], where the authors equip $\mathscr{D}'_\Gamma(M^n)$ with the locally convex topology proposed by Duistermat [Dui96]. For simplicity we state the definition for $\mathbb{R}^n$, but it generalizes to manifolds in a straightforward way with the use of local charts.

Definition 4.12 We define τ_H on $\mathscr{D}'_\Gamma(\mathbb{R}^n)$ as the locally convex topology given by the following systems of seminorms:

(i) All the seminorms on $\mathscr{D}'(\mathbb{R}^n)$ for the weak topology: $||u||_f = |\langle u, f\rangle|$ for all $f \in \mathscr{D}(\mathbb{R}^n)$.

(ii) The seminorms of the form

$$||u||_{m,V,\chi} = \sup_{k\in V}(1+|k|)^m |\widehat{u\chi}(k)|,$$

where $m \geq 0$, $\chi \in \mathscr{D}(\mathbb{R}^n)$, and $V \in \mathbb{R}^n$ is a closed cone with $(\text{supp}\,\chi \times V) \cap \Gamma = \varnothing$.

This is also called the Hörmander topology, since the seminorms (ii) of the definition above were present in [Hör71], where they were used to define a pseudo-topology rather than topology. In [BDH14] it was shown that it in fact defines a *bornology* (i.e. a family of bounded sets, see Definition 3.16).

In [BDH14] it is proven that convergence in the bornological sense (Mackey convergence) in $\mathscr{D}'_\Gamma(\mathbb{R}^n)$ is the same as convergence in the sense of the Hörmander pseudotopology.

Proposition 4.5 (after [BDH14]) *A sequence u_j in $\mathscr{D}'_\Gamma$ converges to u in the sense of Hörmander iff it Mackey-converges to u for the bornology (see Definition 3.16) of $\mathscr{D}'_\Gamma$.*

Proof See Proposition 3.1 of [BDH14]. □

There are several indications that the notion of bornology is more natural than topology for the applications in physics. Another argument is that the family of smooth curves in a topological vector space is determined by the bornology rather than topology. In the *convenient setting* of global analysis [KM97] a map is smooth if it maps smooth curves into smooth curves. This is a very elegant and robust notion of smoothness and allows one to do geometry on infinite dimensional manifolds of the type that appear commonly in physics. In [Mey04] it is pointed out that bornological vector spaces are also very useful in non-commutative geometry and representation theory. At the moment the formalism of bornological vector spaces hasn't been fully embraced by mathematical physicists studying QFT, so we refrain from formulating everything in these terms as well. Nevertheless, it seems tempting to explore the consequences of taking a purely bornological viewpoint on the foundations of QFT in the future.

Let us come back to the main problem of the present section. For the definition of a topology on microcausal functionals we still need to make one more step. The topology defined above was meant for distributions with WF sets contained in a *closed* cone. However, in Definition 4.9 one uses *open cones* instead. It was proposed in [BDF09] to introduce a topology on $\mathscr{E}'_{\Xi_n}(M^n)$ as the inductive limit topology over a countable family of spaces $(\mathscr{D}'_C(M^n), \tau_H)$ with closed cones contained in Ξ_n. The proof has been spelled out in detail in [BFR12, Lemma 4.0.18]. By a slight abuse of notation, we denote the resulting topological spaces by $(\mathscr{E}'_{\Xi_n}(M^n), \tau_H)$.

Remark 4.4 It is crucial, that the inductive limit in the definition of $(\mathscr{E}'_{\Xi_n}(M^n), \tau_H)$ is countable, since this allows us to conclude that $(\mathscr{E}'_{\Xi_n}(M^n), \tau_H)$ is nuclear.

In [BDF09] $(\mathscr{E}'_{\Xi_n}(M^n), \tau_H)$, or rather its complexification, is used to define a topology on $\mathfrak{F}_{\mu c}$.

Definition 4.13 Equip $\mathcal{F}_{\mu c}$ with the topology τ_{BDF} defined as the initial topology with respect to all the maps

$$\begin{aligned} \mathcal{F}_{\mu c} &\to (\mathscr{E}'^{\mathbb{C}}_{\Xi_n}(M^n), \tau_H), \\ F &\mapsto F^{(n)}(\varphi), \end{aligned}$$

where $n \in \mathbb{N}$, $\varphi \in \mathcal{E}$.

Definition 4.14 To obtain a topology on $\mathcal{F}_{s\mu c}$, we replace in Definition 4.13 $\mathscr{E}'_{\Xi_1}(M)$ with $\mathscr{D}(M)$, equipped with its standard topology.

The topology τ_{BDF} is nuclear, but is not complete and it has been argued in [DB14] that from the functional analytic viewpoint this is not the most optimal choice. The authors of [DB14] point out that according to [Sch57] the spaces of distributions most optimal for applications are the *normal spaces of distributions*.

Definition 4.15 A Hausdorff locally convex vector space $\mathcal{X}$ is said to be a normal space of distributions if there are continuous injective linear maps $i : \mathscr{D}(\Omega) \hookrightarrow \mathcal{X}$ and $j : \mathcal{X} \hookrightarrow \mathscr{D}'(\Omega)$, where $\mathscr{D}'(\Omega)$ is equipped with its strong topology, such that:

(i) The image of i is dense in $\mathcal{X}$,
(ii) for any f and g in $\mathscr{D}(\Omega)$, $\langle j \circ i(f), g\rangle = \int_\Omega f(x)g(x)dx$.

The motivation for using normal spaces of distributions is that they have a better behaviour under duality, in particular, the dual space can also be equipped with a normal topology. In order to make $\mathscr{D}'_\Gamma(\mathbb{R}^n)$ into a normal space of distributions one needs to refine its topology. It was shown in [DB14] that this can be done simply by replacing in Definition 4.12 the seminorms of the weak topology with the seminorms of the strong topology.

Definition 4.16 (*after* [DB14]) We define τ_N on $\mathscr{D}'_\Gamma(\mathbb{R}^n)$ as the locally convex topology given by the following system of seminorms:

(i) All the seminorms on $\mathscr{D}'(\mathbb{R}^n)$ for the strong topology: $p_B(u) = \sup_{f\in B} |\langle u, f\rangle|$, where B runs over the bounded sets of $\mathscr{D}(\Omega)$.
(ii) The seminorms $||u||_{m,V,\chi}$, where $m \geq 0$, $\chi \in \mathscr{D}(\mathbb{R}^n)$, and $V \in \mathbb{R}^n$ is a closed cone with $(\text{supp}\,\chi \times V) \cap \Gamma = \varnothing$.

With this choice of topologies one obtains a duality between $\mathscr{D}'_\Gamma(M^n)$ and $\mathscr{E}'_\Lambda(M^n)$, where $\Lambda = -\Gamma^c$.

Proposition 4.6 (after [DB14]) *The dual of $\mathscr{D}'_\Gamma(M^n)$ for its normal topology τ_N is $\mathscr{E}'_\Lambda(M^n)$, where Γ is a closed cone and $\Lambda = -\Gamma^c$.*

Proof See Proposition 7 in [DB14]. □

The space $\mathscr{E}'_\Lambda(M^n)$ can be equipped with an inductive limit topology similar to the one proposed in [BDF09] and spelled out in [BFR12]. It was shown in [DB14] that this topology is equivalent to the strong topology coming from the duality with

$\mathscr{D}'_\Gamma(M^n)$. In the same reference it was also proven that both $\mathscr{E}'_\Lambda(M^n)$ and $\mathscr{D}'_\Gamma(M^n)$ are nuclear and that $\mathscr{D}'_\Gamma(M^n)$ is complete. The latter property unfortunately doesn't hold for $\mathscr{E}'_\Lambda(M^n)$, which was the motivation for further study, done in [Dab14a].

Definition 4.17 Let τ_{BDH} denote the topology on the space of (strongly) microcausal functionals given by replacing in Definition 4.13 the Hörmander topology τ_H with the normal topology τ_N.

This topology has much better functional properties than τ_{BDF}, but is not complete. In order to obtain a complete topological space one needs to modify the definition of the space of microcausal functionals as well. The idea in [Dab14a] is to control not only the WF set of distributions but also the dual WF set defined as follows.

Definition 4.18 (*after* [Dab14a]) Let $u \in \mathscr{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n$. The *dual WF set*[1] is defined as $\mathrm{DWF}(u) = \bigcup_{s>0}, WF_s(u)$, i.e. the union of Sobolev H^s-wave front sets.

A point $(x, k_0) \notin WF_s(u)$ if there is a neighborhood U of x and a conic neighborhood C of k_0 such that for any $f \in \mathscr{D}(U)$

$$(1 + |k|^2)^{s/2}\widehat{fu} \in L^2(C).$$

Note that if the H^s-wave front set of a distribution u is empty, it means that u is of Sobolev type s, just as a distribution with empty WF set is smooth.

The notion of the DWF set introduced above allows for an explicit characterisation of the completion of $\mathscr{E}'_\Lambda(M^n)$. It was shown in [Dab14a] that the completion of $(\mathscr{E}'_\Lambda(M^n), \tau_n)$ is the space of distributions in $\mathscr{E}'(M^n)$ with DWF sets contained in Λ.

In order to obtain a space of functionals with nice functionala analytic properties, in [Dab14b] the following modifications are made, with respect to the Ansatz of [BDF09]:

1. In Definition 4.9, view derivatives $F^{(n)}(\varphi)$ not as distributions in $\mathscr{D}'(M^n)$, but rather as multilinear maps between spaces of distributions with control on both the WF set and DWF set,
2. Use a topology on the space of distributions with control on both the WF set and the DWF set, which is complete, nuclear and bornological (the latter means essentialy that it is compatible with the bornology),
3. Replace the pointwise convergence of all the derivatives with the uniform convergence on images of compact subsets of $\mathbb{R}$ under smooth curves (this is a natural choice from the point of view of the convenient setting [KM97]).

we denote the resulting space of functionals by $\mathcal{F}_{D\mu c}$ and we will denote the topology on this space by τ_D. It has been shown in [Dab14b] that $(\mathcal{F}_{D\mu c}, \tau_D)$ is complete, nuclear and bornological.

[1]The name "dual" is meant as the indication that this notion behaves better under dualities of the type mentioned in Proposition 4.6 than the usual WF set.

4.4.3 The Classical Causal Net

We are now ready to construct a classical field theory model in the sense of Definition 2.53. The only missing ingredient is localization, but this can be easily introduced in our case, using the notion of the spacetime support of a functional. We obtain the following result

Proposition 4.7 *Given a spacetime* $\mathcal{M} = (M, g)$*, the configuration space* $\mathcal{E} = \Gamma(E \xrightarrow{\pi} M)$ *and the generalized Lagrangian L, consider the net*

$$\mathcal{O} \mapsto (\mathcal{F}_{\mathrm{s}\mu\mathrm{c}}, \tau_{BDH}, \lfloor ., . \rfloor_S, *),$$

where S is the action corresponding to L, $\lfloor ., . \rfloor_S$ *is given by the formula* (4.13) *and the involution* $*$ *is the pointwise complex conjugation of complex-valued functionals. This net is a classical field theory model in the sense of Definition 2.53.*

Proof The proof has been outlined in [BDF09] and [BFR13]. The main idea is to reduce the problem to the problem involving basic operations on distribution. The fact that $\lfloor ., . \rfloor_S$ is well defined on the space of strongly microcausal functionals and that $\mathcal{F}_{\mathrm{s}\mu\mathrm{c}}$ is stable under this bracket follow from the WF set properties of derivatives of microcausal functionals and of $\Delta_S^+(\varphi)$. The only missing step, not commented on in [BDF09] and [BFR13], is the (sequential) continuity. We will fill this gap here. For simplicity of notation, we spell out the proof for the example of the scalar field, i.e. $\mathcal{E} = \mathscr{E}(M)$. First we note that with the initial topology we are using it is sufficient to show the continuity pointwise in φ, so it reduces to the continuity of operations on distributions of the form

$$(f, g) \mapsto \langle f, \Delta_S g \rangle ,$$

where $f, g \in \mathscr{E}'_{\Xi_1}(M)$. This in turn is reduced to proving the sequential continuity of the tensor product and the sequential continuity of the distributional pullback. In the closed cone case these two results are well known (see [CP81, p. 511] and [Hör03, 8.2.4] respectively). For the open cone, we will use the strategy proposed by Dabrowski (private communication), based on the results of [BDH14, DB14, Dab14a, Dab14b].

We start with the sequential continuity of the tensor product. It was proven in [DB14, Proposition 28] that $\mathscr{E}'_{\Xi_n}(M^n)$ is a barrelled space, so, following [Trè06, Theorem 41.2] we conclude that the separate continuity implies hypocontninuity[2] thus sequential continuity. By the definition of the inductive limit, the separate continuity follows from the separate continuity in the closed cone case, which has been proven in [BDH14]. Another way to see it is to use the fact that for linear maps Hörmander's sequential continuity implies boundedness, hence continuity between

[2]Hypocontinuity of a bilinear map is a notion stronger than sequential continuity on the product space, but is weaker than the joint continuity.

the bornologifications. It was shown in [Dab14a, Proposition 33] that the inductive limit topology on $\mathscr{E}'_{\Xi_n}(M^n)$ is also an inductive limit of bornologifications, hence it is itself bornological.

As for the continuity of the pull-back, see [Dab14a, Proposition 36]. □

Remark 4.5 In fact the argument presented above proves something stronger than sequential continuity, namely the *hypocontinuity* of the product. This result would not be possible with τ_H replacing τ_N, as shown by counterexamples provided in [BDH14]. Therefore, to show the joint sequential continuity of the bracket with the topology τ_{BDF} one needs to use a different, indirect argument (work in progress).

A major improvement of the above result has been obtained recently in [Dab14b].

Proposition 4.8 (after [Dab14b]) $(\mathfrak{F}_{D\mu c}, \tau_D)$ *Equipped with the Peierls bracket* $\lfloor , \rfloor$ *is a complete, nuclear and bornological topological Poisson algebra with hypocontinuous operations.*

Proof See [Dab14b]. □

4.5 Analogy with Classical Mechanics

In this section we show how the formalism developed in this chapter relates to the more standard formulation of classical field theory in a simple example. We consider the free scalar field with the field equation

$$P\varphi = 0, \tag{4.16}$$

where $P = -(\Box + m^2)$ is minus the Klein-Gordon operator, and we assume the spacetime $\mathcal{M} = (M, g)$ to contain a compact Cauchy surface.

As mentioned before, the retarded and advanced Green's functions exist for this equation and for every $f \in \mathcal{E}$ whose support is past and future compact, Δf is a solution to (4.16). Conversely, every smooth solution of the Klein-Gordon equation is of the form Δf for some $f \in \mathcal{E}$ with future and past compact support.

Without the loss of generality $M = \mathbb{R} \times \Sigma$ with compact Cauchy surfaces $\{t\} \times \Sigma$, $t \in \mathbb{R}$. The space of Cauchy data $\Sigma \ni \boldsymbol{x} \mapsto (\varphi(t, \boldsymbol{x}), \dot{\varphi}(t, \boldsymbol{x}))$ on the surface $\{t\} \times \Sigma$ is

$$\mathcal{C} = \{(\phi, \psi) \in \mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma)\},$$

where $\mathscr{E}(\Sigma) \doteq \mathcal{C}^\infty(\Sigma, \mathbb{R})$. This space is isomorphic to $\mathcal{E}_S$, the of smooth solutions to (4.16).

As in classical mechanics, equations of motion can be derived from the least action principle. Elements of $\mathcal{C}$ play the role of generalized coordinates and generalized velocities, while a smooth trajectory $t \mapsto \phi(t)$, $t \in \mathbb{R}$ is a function which assigns to an instant of time t a function $\phi(t) \in \mathscr{E}(\Sigma)$ such that trajectories ϕ are in one to one correspondence with field configurations $\varphi : (t, \boldsymbol{x}) \to \phi(t)(\boldsymbol{x})$, i.e. elements of $\mathcal{E}$.

In this setting, the Lagrangian L is a functional on $\mathcal{C}$, typically given in terms of a Lagrangian density $\mathcal{L}$,

$$L(\phi, \psi) = \int_\Sigma \mathcal{L}(\phi(\boldsymbol{x}), \nabla\phi(\boldsymbol{x}), \psi(\boldsymbol{x}))d\sigma(\boldsymbol{x}),$$

and the action is, for every finite time interval I, a function on the space of trajectories defined by

$$S_{I\times\Sigma}(\phi) = \int_I L(\phi(t), \dot{\phi}(t))dt = \int_I \left(\int_\Sigma \mathcal{L}(\varphi(t, \boldsymbol{x}), \nabla_{\boldsymbol{x}}\varphi(t, \boldsymbol{x}), \dot{\varphi}(t, \boldsymbol{x}))d\sigma_t(\boldsymbol{x}) \right) dt\ . \tag{4.17}$$

Solutions are configurations for which, for all I, $S_{I\times\Sigma}$ is stationary under variations $\delta\phi$ with support in the interior of $I \times \Sigma$. If $\mathcal{L}$ is the Lagrangian density of the free scalar field, then the least action principle yields (4.16) as the equation of motion.

Now let F, G be two functions on the space of trajectories which depend only on the restriction of the trajectory to $[t_1, t_2] \times \Sigma$ and $t_1 < t_2$. Let $\mathcal{E}_S$ be the space of solutions for an action S, and let $r_{\lambda G} : \mathcal{E}_S \to \mathcal{E}_{S+\lambda G}$ be the map which associates to a solution for S a solution for $S + \lambda G$ such that both solutions coincide for $t < t_1$ ($r_{\lambda G}$ is called the retarded Møller map). Following the idea of Peierls, we consider the change of F under the change of the action and set, for a solution $\varphi \in \mathcal{E}_S$,

$$r_1(G, F)(\varphi) = \frac{d}{d\lambda} F(r_{\lambda G}(\varphi))\Big|_{\lambda=0}\ .$$

Similarly, we introduce the advanced Møller map $a_{\lambda F} : \mathcal{E}_S \to \mathcal{E}_{S+\lambda F}$ where the solutions coincide for $t > t_2$, and set

$$a_1(G, F)(\varphi) = \frac{d}{d\lambda} F(a_{\lambda G}(\varphi))\Big|_{\lambda=0}\ .$$

The Peierls bracket of G and F was originally defined as

$$\{G, F\}_{\mathrm{Pei}} \doteq r_1(G, F) - a_1(G, F). \tag{4.18}$$

Now we show that the formula of Peierls (4.18) is equivalent to (4.13), if $P_S(\varphi)$ is a normally hyperbolic operator. Let $G \in \mathcal{F}_{\mathrm{loc}}$ be a local functional. We are interested in the flow (Φ_λ) on $\mathcal{E}$ which deforms solutions of the original field equation $S'(\varphi) = 0$ to those of the perturbed equation $S'(\varphi) + \lambda G^{(1)}(\varphi) = 0$. Let $\Phi_0(\varphi) = \varphi$ and

$$\frac{d}{d\lambda} \left(S'(\Phi_\lambda(\varphi)) + \lambda G^{(1)}(\Phi_\lambda(\varphi)) \right) \Big|_{\lambda=0} = 0. \tag{4.19}$$

The vector field $\varphi \mapsto X(\varphi) = \frac{d}{d\lambda}\Phi_\lambda(\varphi)|_{\lambda=0}$ satisfies the equation

$$\langle P_S(\varphi), X(\varphi)\rangle + G^{(1)}(\varphi) = 0. \tag{4.20}$$

Let $\Delta_S^{\mathrm{R/A}}(\varphi)$ be the retarded/advanced Green's function of the normally hyperbolic operator $P_S(\varphi)$ and let $\Delta_S(\varphi) = \Delta_S^{\mathrm{R}}(\varphi) - \Delta_S^{\mathrm{A}}(\varphi)$ be the causal propagator. We obtain now two distinguished solutions to the equation (4.20),

$$X^{\mathrm{R/A}}(\varphi) = -\left\langle \Delta_S^{\mathrm{R/A}}(\varphi), G^{(1)}(\varphi)\right\rangle . \tag{4.21}$$

Note that $X^{\mathrm{R}}(\varphi) = r_{1,1}(G, \Phi)(\varphi)$, where Φ is the evaluation functional $\Phi_x(\varphi) \doteq \varphi(x)$, hence

$$r_1(G, F) = -\left\langle F^{(1)}, \Delta_S^{\mathrm{R/A}} G^{(1)}\right\rangle.$$

The difference $X = X^{\mathrm{R}} - X^{\mathrm{A}}$ defines a vector field $X \in \Gamma(T\mathcal{E})$ and it follows that

$$\begin{aligned}\{G, F\}_{\mathrm{Pei}}(\varphi) \doteq r_1(G, F)(\varphi) - a_1(G, F)(\varphi) &= \left\langle G^{(1)}(\varphi), \Delta_S(\varphi) F^{(1)}(\varphi)\right\rangle \\ &= \lfloor G, F\rfloor(\varphi).\end{aligned}$$

Next, following [FR15], we prove the equivalence between (4.13) and the canonical bracket. We fix a Cauchy surface $\{t\} \times \Sigma$. Note that, given Cauchy data $(\phi, \psi) \in \mathcal{C}$, we can write the unique solution φ corresponding to these Cauchy data as

$$\varphi(x) = \beta(\phi, \psi)(x) \equiv \int_\Sigma \left(\Delta_S(x; t, \mathbf{y})\psi(\mathbf{y}) - \frac{\partial}{\partial t}\Delta_S(x; t, \mathbf{y})\phi(\mathbf{y})\right) d\sigma_t(\mathbf{y}). \tag{4.22}$$

Canonical momenta are obtained as distributional densities by

$$\langle \pi(\phi, \psi), h\rangle \doteq \frac{d}{d\lambda}|_{\lambda=0} L(\phi, \psi + \lambda h)\,,\ h \in \mathcal{D}(\Sigma)\,.$$

We assume that for the Lagrangians of interest π is always smooth. The phase space is then

$$\mathcal{P} = \mathcal{E}(\Sigma) \times \mathcal{E}_d(\Sigma), \tag{4.23}$$

where $\mathcal{E}_d(\Sigma)$ is the space of smooth densities. The phase space has the canonical symplectic form

$$\sigma((f_1, f_2), (g_1, g_2)) = \int_\Sigma (f_1 g_2 - f_2 g_1).$$

Note that $\mathcal{E}(\Sigma) \times \mathcal{E}_d(\Sigma) \subset \mathcal{E}(\Sigma) \times \mathcal{D}'(\Sigma) \cong T^*(\mathcal{E}(\Sigma))$, so $(\mathcal{P}, \sigma)$ is indeed the analog of the phase space in classical mechanics.

For simplicity we consider an action S induced by a Lagrangian L which depends on $\dot\phi$ only through the kinetic term $\frac{1}{2}\dot\phi^2$, hence $\pi(\mathbf{y}) \doteq \dot\phi(\mathbf{y}) d\sigma_t(\mathbf{y})$. Let $\alpha : (\phi, \pi) \mapsto$

$(\phi, \dot{\phi})$ and $\tilde{\beta} \doteq \beta \circ \alpha : \mathcal{P} \to \mathcal{E}_S$. We can now prove the equivalence of the canonical and the Peierls bracket. Let $F, G \in \mathcal{F}$. Using (4.22) we obtain

$$\begin{aligned}
&\{F \circ \tilde{\beta}, G \circ \tilde{\beta}\}_{\text{can}} \\
&= \int_\Sigma \left(\left\langle \tfrac{\delta F}{\delta \varphi} \circ \tilde{\beta}, \tfrac{\delta \tilde{\beta}}{\delta \phi(\boldsymbol{x})} \right\rangle \left\langle \tfrac{\delta G}{\delta \varphi} \circ \tilde{\beta}, \tfrac{\delta \tilde{\beta}}{\delta \pi(\boldsymbol{x})} \right\rangle - \left\langle \tfrac{\delta F}{\delta \varphi} \circ \tilde{\beta}, \tfrac{\delta \tilde{\beta}}{\delta \pi(\boldsymbol{x})} \right\rangle \left\langle \tfrac{\delta G}{\delta \varphi} \circ \tilde{\beta}, \tfrac{\delta \tilde{\beta}}{\delta \phi(\boldsymbol{x})} \right\rangle \right) \\
&= \left\langle \Theta, F^{(1)} \circ \tilde{\beta} \otimes G^{(1)} \circ \tilde{\beta} \right\rangle,
\end{aligned}$$

where Θ is given by

$$\Theta(z', z) = \int_\Sigma \left(\dot{\Delta}_S(z'; t, \boldsymbol{x}) \Delta_S(z; t, \boldsymbol{x}) - \dot{\Delta}_S(z; t, \boldsymbol{x}) \Delta_S(z'; t, \boldsymbol{x}) \right) d\sigma(\boldsymbol{x}).$$

From general properties of the causal propagator Δ_S (the generalization of (4.22) to distributional Cauchy data) it follows that Θ is equal to Δ_S. Hence, on the solution space $\mathcal{E}_S$,

$$\{F \circ \tilde{\beta}, G \circ \tilde{\beta}\}_{\text{can}} = \lfloor F, G \rfloor \circ \tilde{\beta}.$$

For another argument, valid for the interacting theory, see Sect. 3.3 of [FR15].

4.6 Classical Møller Maps Off-Shell

In the previous section we introduced the notion of classical Møller operators as maps between solution spaces for equations of motion induced by different Lagrangians. Proving the existence of such maps in concrete models would require using some subtle properties of solution spaces of non-linear partial differential equations. Claims about a potential existence result have been made in [BFR12]. Here we take a different approach and following [DF02], we work off-shell and formulate the notion of classical Møller operators on the space of functionals on the configuration space.

To avoid the functional analytic difficulties we define all the structures only for regular functionals $\mathcal{F}_{\text{reg}}$ or local functionals $\mathcal{F}_{\text{loc}}$. Extension to more general classes of functionals requires a careful analysis of WF set conditions (see for example [Dab14a, Dab14b, BFR12]). To further simplify the matters, we work perturbatively, i.e. we use formal power series in λ, which plays a role of the coupling constant.

Let S_0 be the free action and let $\lambda V \in \mathcal{F}_{\text{reg}}$ be the perturbation. Formally, we can construct the classical Møller operators $r_{S_0+\lambda V, S_0}$ and $a_{S_0+\lambda V, S_0}$ between the corresponding solution spaces $\mathcal{E}_{S_0}$ and $\mathcal{E}_{S_0+\lambda V}$ by following [DF02]. We require that

$$r_{S_1, S_2} \circ r_{S_2, S_3} = r_{S_1, S_3}. \tag{4.24}$$

To simplify the notation we abbreviate $r_{S_0+\lambda V, S_0} \equiv r_{\lambda V}$, and $a_{S_0+\lambda V, S_0} \equiv a_{\lambda V}$.

Møller maps act on the on-shell functionals by the pullback.

$$r_{\lambda V} F(\varphi) \doteq F \circ r_{\lambda V}(\varphi), \text{ and} \tag{4.25}$$

$$a_{\lambda V} F(\varphi) \doteq F \circ a_{\lambda V}(\varphi), \tag{4.26}$$

where $F \in (\mathcal{F}_{\text{reg}})_{S_0+\lambda V}$, $\varphi \in \mathcal{E}_{S_0}$. By a slight abuse of notation we use the same symbol to denote a Møller map acting on functionals, as the symbol for a map between the solution spaces. In order to get a consistent off-shell definition we also require the off-shell extensions to satisfy

$$r_{\lambda V}(\mathcal{I}_{S_0+\lambda V}) = \mathcal{I}_{S_0},$$
$$a_{\lambda V}(\mathcal{I}_{S_0+\lambda V}) = \mathcal{I}_{S_0}.$$

Expanding the classical Møller maps in the powers of λ we get

$$r_{\lambda V} F = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} r_n(V^{\otimes n}; F), \tag{4.27}$$

$$a_{\lambda V} F = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a_n(V^{\otimes n}; F), \tag{4.28}$$

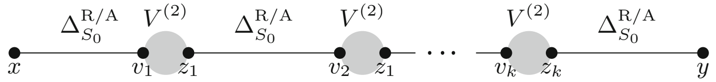
and call the coefficients of these expansions n-fold retarded and advanced products, respectively. In the previous section we have seen that if V is local, then the first order terms of this expansion can be expressed in terms of retarded and advanced Green's functions. Here we generalize to non-local interactions, but we have to work with formal power series. The retarded and advanced Green's functions $\Delta^{\text{R}}_{S_0+\lambda V}$ and $\Delta^{\text{A}}_{S_0+\lambda V}$ are now understood as formal power series with coefficients in distributions. The following proposition shows the existence of such objects for a general class of interactions, including the local and the regular ones.

Proposition 4.9 (after [DF02, BD08]) *Let $V \in \mathcal{F}_{\text{loc}}$ or $V \in \mathcal{F}_{\text{reg}}$. The retarded (advanced) Green's function corresponding to the linear operator $P + V^{(2)}$ induced by the second derivative of the action $S_0 + \lambda V$ is given by the formula*

$$\Delta^{\text{R/A}}_{S_0+\lambda V}(x, y) = \Delta^{\text{R/A}}_{S_0}(x, y) + \sum_{k=1}^{\infty} (-\lambda)^k \int \Delta^{\text{R/A}}_{S_0}(x, v_1) \tag{4.29}$$

$$\cdot \frac{\delta^2 V}{\delta\varphi(v_1)\delta\varphi(z_1)} \Delta^{\text{R/A}}_{S_0}(z_1, v_2) \dots \frac{\delta^2 V}{\delta\varphi(v_k)\delta\varphi(z_k)} \Delta^{\text{R/A}}_{S_0}(z_k, y) d\mu_g(v_1) \dots d\mu_g(z_k). \tag{4.30}$$

Graphically the integral is represented by:



The support of $\Delta^{\mathrm{R/A}}_{S_0+\lambda V}(x, y)$ *is contained in the set*

$$\left\{(x, y)|x \in \mathrm{supp}\left(\tfrac{\delta V}{\delta\varphi}\right) + \overline{V}_\pm \wedge y \in \mathrm{supp}\left(\tfrac{\delta V}{\delta\varphi}\right) + \overline{V}_\mp\right\} \cup \left\{(x, y)|x \in y + \overline{V}_\pm\right\}. \tag{4.31}$$

Proof The recursive formula is proven by induction in k. For the support property, first note that $\mathrm{supp}\big(\frac{\delta^2 V}{\delta\varphi^2}\big) \subset \mathrm{supp}\big(\frac{\delta V}{\delta\varphi}\big) \times \mathrm{supp}\big(\frac{\delta V}{\delta\varphi}\big)$. Therefore, in order to get a non-vanishing contribution from the integral in (4.29), x has to lie in the future of $u_1 \in \mathrm{supp}\big(\frac{\delta V}{\delta\varphi}\big)$ and y in the past of $v_n \in \mathrm{supp}\big(\frac{\delta V}{\delta\varphi}\big)$. □

Note that the uniqueness of $\Delta^{\mathrm{R}}_{S_0}$ and $\Delta^{\mathrm{A}}_{S_0}$ implies that $\Delta^{\mathrm{R}}_{S_0}(f, g) = \Delta^{\mathrm{A}}_{S_0}(g, f)$ and using the recursive formula (4.29) we can also verify that

$$\Delta^{\mathrm{R}}_{S_0+\lambda V}(f, g) = \Delta^{\mathrm{A}}_{S_0+\lambda V}(g, f).$$

The following proposition has been proven in [DF02] and it demonstrates the existence of classical Møller operators in the sense of formal power series.

Proposition 4.10 (after [DF02]) *Let* $V \in \mathfrak{F}_{\mathrm{loc}}$ *or* $V \in \mathfrak{F}_{\mathrm{reg}}$. *The retarded Møller operator* $r_{\lambda V}$ *exists as a map on* $\mathfrak{F}_{\mathrm{loc}} \cup \mathfrak{F}_{\mathrm{reg}}$ *valued in formal power series in* $\mathfrak{F}_{\mu c}$ *and its coefficients satisfy the recursion relation:*

$$\begin{aligned} &r_{n+1}(V^{\otimes(n+1)}, F) \\ &= -\sum_{l=0}^{n} \binom{n}{l} r_l \left(V^{\otimes l}, \int \frac{\delta V}{\delta\varphi(x)} \Delta^{\mathrm{A}\,(n-l)}_{S_0}[V](x, y) \frac{\delta F}{\delta\varphi(y)} d\mu_g(x) d\mu_g(y)\right), \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} \Delta^{\mathrm{A}\,(k)}_{S_0}[V](x, y) &\doteq \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} \Delta^{\mathrm{A}}_{S_0+\lambda V}(x, y) \\ &= (-1)^k k! \int \Delta^{\mathrm{A}}_{S_0}(x, v_1) \frac{\delta^2 V}{\delta\varphi(v_1)\delta\varphi(z_1)} \Delta^{\mathrm{A}}_{S_0}(z_1, v_2) \cdot \\ &\qquad \cdots \cdot \frac{\delta^2 V}{\delta\varphi(v_k)\delta\varphi(z_k)} \Delta^{\mathrm{A}}_{S_0}(z_k, y) d\mu_g(v_1) \ldots d\mu_g(z_k). \end{aligned}$$

Analogously for the advanced Møller operator.

Proof differentiating the property (4.24) we obtain

$$\frac{d}{d\lambda} r_{\lambda V}(F) = r_{\lambda V}\left(-\left\langle F^{(1)}, \Delta^{\mathrm{R}}_{S_0+\lambda V} V^{(1)}\right\rangle\right) = r_{\lambda V}\left(-\left\langle V^{(1)}, \Delta^{\mathrm{A}}_{S_0+\lambda V} F^{(1)}\right\rangle\right), \tag{4.33}$$

Comparing the terms with the same power of λ on both sides yields the required recursion relation. □

In particular, for $V \in \mathcal{F}_{\mathrm{reg}}$, $r_{\lambda V}$ and $a_{\lambda V}$ are well defined maps from $\mathcal{F}_{\mathrm{reg}}[[\lambda]]$ to itself. The next proposition shows that the classical Møller operators satisfy a natural intertwining property.

Proposition 4.11 *Let $F, G, V \in \mathcal{F}_{\mathrm{loc}}$ or $F, G, V \in \mathcal{F}_{\mathrm{reg}}$. The retarded Møller operator $r_{\lambda V}$ preserves the Peierls bracket, i.e.*

$$\lfloor r_{\lambda V} F, r_{\lambda V} G \rfloor_{S_0} = r_{\lambda V}(\lfloor F, G \rfloor_{S_0 + \lambda V})$$

The analogous statement holds also for $a_{\lambda V}$.

Proof See [DF02, Proposition 2]. □

If the interaction V is local, the recursion relation can be simplified and for $M = \mathbb{M}$ one obtains the following expression for the n-fold retarded product:

$$r_n(V^{\otimes n}, G) \overset{\text{o.s.}}{=} n! \int_{x_1^0 \leq \ldots \leq x_n^0} (\mathcal{R}_V(x_1) \cdots \mathcal{R}_V(x_n) G) d^4x_1 \ldots d^4x_n, \tag{4.34}$$

where

$$\mathcal{R}_V(x) := -\int \Big(\frac{\delta V}{\delta\varphi(x)} \Delta_S^{\mathrm{R}}(y, x) \frac{\delta(.)}{\delta\varphi(y)} \Big) d^4y, \tag{4.35}$$

and o.s. means on-shell, i.e. modulo the ideal $\mathcal{I}_{S_0}$ generated by the free equations of motion. For proof see Proposition 3 of [DF02].

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Chapter 5
Deformation Quantization

The reformulation of classical theory done in Chap. 3 served as a preparation for constructing QFT models. The framework that we are going to use is deformation quantization combined with causal perturbation theory. To quantize a given theory described by the action S we first need to split S into a free part S_0 (at most quadratic in field configurations) and the interaction term S_I. Then, we quantize the theory defined by S_0, using deformation quantization based on a Moyal-type formula, and in the final step we will re-introduce the interaction using causal perturbation theory. This last step will be discussed in Chap. 6, while the present chapter deals with deformation quantization.

The idea of deformation quantization goes back to Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [BFF+78a, BFF+78b] and the first attempt to use these structures in quantum field theory is due to Dito [Dit90]. Based on these ideas Brunetti, Dütsch, and Fredenhagen developed a formalism, which we present here [DF01a, DF01b, BDF09]. At the moment this quantization method is known to work only perturbatively, but the ultimate aim is to obtain some converegence results as well.

5.1 Star Products

In this chapter we will focus on quantizing theories where no local symmetries are present. We will also restrict ourselves to even (bosonic) field configurations, to avoid extra complication with the signs. Let $\mathcal{E} = \Gamma(E \to M)$ be the configuration space and S an action that doesn't possess non-trivial local symmetries and for which $S''(\varphi)$ induces a normally hyperbolic operator for every $\varphi \in \mathcal{E}$. In the first step we split $S = S_0 + S_I$, where S_0 is at most quadratic. This can be done by means of Taylor expansion around any field configuration φ_0, i.e.

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$$L(f)(\varphi_0+\varphi) = L(f)(\varphi_0) + \underbrace{\left\langle L(f)^{(1)}(\varphi_0), \varphi\right\rangle + \frac{1}{2}\left\langle L(f)^{(2)}(\varphi_0), \varphi\otimes\varphi\right\rangle}_{L_0(f)}\dots,$$

The constant term can be neglected, as it doesn't affect the dynamics. Note that if we choose φ_0 to be a solution to the equations of motion (i.e. $S'(\varphi_0)=0$), then

$$\left\langle L(f)^{(1)}(\varphi_0), \varphi+\psi\right\rangle = \left\langle L(f)^{(1)}(\varphi_0), \varphi\right\rangle,$$

if $\psi \in \mathcal{E}_c$ is supported in the region, where $f = const$. Hence

$$\text{supp}\left\langle L(f)^{(1)}(\varphi_0), .\right\rangle \subset \text{supp}\,(df)$$

and by means of the equivalence relation (4.2) we conclude that in such a situation S_0 contains only the quadratic term. Otherwise, S_0 has both a quadratic and a linear term. For simplicity, we will consider here only the situation where φ_0 is a solution, so S_0 is quadratic and we denote the linear operator induced by S_0'' by P. Moreover, we assume that P is formally self-adjoint, i.e. for all $f, g \in \mathcal{E}_c^{\mathbb{C}}$

$$\int_M \langle f, Pg\rangle_E\, d\mu_g = \int_M \langle Pf, g\rangle_E\, d\mu_g,$$

where $\langle ., .\rangle_E$ is the bilinear pairing introduced in Definition 3.1.

Starting from the Poisson algebra $(\mathcal{F}_{\mu c}, \lfloor ., .\rfloor_{S_0})$, formal deformation quantization means constructing an associative algebra $(\mathcal{F}_{\mu c}[[\hbar]], \star)$, where the product $\star$ is expressed as

$$F\star G = \sum_{n=0}^{\infty}\hbar^n B_n(F,G), \tag{5.1}$$

in terms of some differential (in the sense of calculus on $\mathcal{E}$) operators B_n such that

$$B_0(F,G) = F\cdot G,$$
$$B_1(F,G) - B_1(G,F) = i\hbar\lfloor F, G\rfloor_{S_0}.$$

Note that the second condition corresponds to Dirac's idea that in order to quantize a classical theory one should "replace canonical brackets with commutators". Including terms of higher order in $\hbar$ is necessary to avoid the Groenewald-van Hove no-go theorem, which states that (also in the finite dimensional case) a Dirac type quantization prescription is not possible in the strict sense [Gro46, VH51].

More precisely (see [Wal07]), let $\mathfrak{h}$ be the Lie algebra spanned by the canonical coordinate and momentum functions $q^1,\dots,q^N, p_1,\dots,p_N$ and 1, with the canonical Poisson bracket $\{.,.\}_{\text{can}}$. This algebra is a Lie subalgebra of $\mathfrak{g} \doteq (\text{Pol}(T^*\mathbb{R}^N), \{.,.\}_{\text{can}})$ (polynomials on the phase space). According to the Groenewald-van Hove Theorem, there exists no faithful irreducible representation of $\mathfrak{h}$ by operators on a dense domain of some Hilbert space which can be extended to a representation of $\mathfrak{g}$. As a result, one cannot have a Dirac quantization map $\mathfrak{Q}$ from $\mathfrak{g}$ to the space of operators on some Hilbert space $\mathcal{H}$, such that

$$[\mathfrak{Q}(f), \mathfrak{Q}(g)] = i\hbar\mathfrak{Q}(\{f, g\}). \tag{5.2}$$

There is, however a way out. Deformation quantization [BFF+78a, BFF+78b] allows one to avoid this no-go result, by weakening the condition (5.2) to

$$[\mathfrak{Q}(f), \mathfrak{Q}(g)] = \mathfrak{Q}([f, g]_\star) = i\hbar\mathfrak{Q}(\{f, g\}) + \mathcal{O}(\hbar^2).$$

A stronger notion than formal deformation quantization is *strict deformation quantization*. In this case, instead of constructing a space of formal power series, one aims at constructing a continuous field of C^*-algebras. This fits well with the algebraic framework for quantum theory described in Sect. 2.3. The notion of strict deformation quantization has been introduced in [Rie94]. For a review on the current status of the subject refer to [Rie98], see also [Haw08, BMS94].

As for formal deformation quantization, the most significant recent result is the one of Kontsevich, who has proven in [Kon03] the existence of formal deformation quantization for arbitrary finite dimensional Poisson manifolds. Unfortunately this cannot be applied directly in field theory, as the configuration space $\mathcal{E}$ is infinite dimensional. However, if S_0 is at most quadratic, there exists an explicit formula for the star product and we will focus on this construction for the rest of the present chapter.

To understand the algebraic structure, it is helpful to put the functional analytic aspects aside for the time being and work on $\mathcal{F}_{\text{reg}}$.

We define

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), \left(\tfrac{i}{2}\Delta_{S_0}\right)^{\otimes n} G^{(n)}(\varphi)\right\rangle, \tag{5.3}$$

for $F, G \in \mathcal{F}_{\text{reg}}$. It is convenient to introduce the notation

$$D_\Delta(F \otimes G) \doteq \left\langle F^{(1)}, \Delta_{S_0} G^{(1)}\right\rangle.$$

In this notation

$$F \star G = m \circ e^{\frac{i\hbar}{2} D_\Delta}(F \otimes G),$$

where m is the pointwise multiplication of functionals.

Example 5.1 (*Weyl algebra*) For the free scalar field on a spacetime $\mathcal{M} = (M, g)$ we have $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R})$ and the Lagrangian is given by Example 4.1(i). Consider regular functionals of the form

$$F_f(\varphi) = \int_M f(x)\varphi(x)d\mu_g(x) \equiv \int_M f\varphi d\mu_g, \quad \text{where } f \in \mathscr{D}(M),$$

We define $\mathcal{W}(f) \doteq \exp(iF_f)$ and verify that

$$\left\langle (\mathcal{W}(f))^{(1)}(\varphi), h \right\rangle = \frac{d}{d\lambda} e^{i\int f(\varphi+\lambda h)d\mu_g}\Big|_{\lambda=0} = \left(i\int f h \, d\mu_g\right)\mathcal{W}(f)(\varphi).$$

Hence

$$\left\langle (\mathcal{W}(f))^{(n)}(\varphi), h^{\otimes n} \right\rangle = \left(i\int f h \, d\mu_g\right)^n \mathcal{W}(f)(\varphi)$$

and we obtain the following formula for the star product:

$$\mathcal{W}(f) \star \mathcal{W}(\tilde{f}) = \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^n \frac{(-1)^n}{n!} \left(\int \Delta_{S_0}(x,y)\tilde{f}(y)f(x)d\mu_g(x)d\mu_g(y)\right)^n \mathcal{W}(f+\tilde{f})$$
$$= e^{-\frac{i\hbar}{2}\Delta_{S_0}(f,\tilde{f})}\mathcal{W}(f+\tilde{f}), \tag{5.4}$$

which reproduces the Weyl relations from Example 2.1, with the difference that now we are dealing with a bilinear form which is Poisson, but not symplectic (has a non-trivial kernel).

In the next step we extend the star product to the space of microcausal functionals. To prepare for this task we take another look at the singularity structure of the distribution Δ_{S_0}. Note that the WF set of this distribution is composed of two parts: one in $\overline{V}_+$ and another with in $\overline{V}_-$, where $\overline{V}_\pm$ is the closed future/past lightcone. As shown in [Rad96], one can decompose Δ_{S_0} into two distributions with WF sets corresponding to these two components

$$\tfrac{i}{2}\Delta_{S_0} = \Delta^+_{S_0} - H, \tag{5.5}$$

where the WF set of $\Delta^+_{S_0}$ is

$$\mathrm{WF}(\Delta^+_{S_0}) = \{(x,k;x,-k') \in \dot{T}^*M^2 | (x,k) \sim (x',k'), k \in (\overline{V}_+)_x\}, \tag{H 0}$$

and in addition the following properties hold:

(H 1) $\Delta_{S_0} = 2\,\mathrm{Im}(\Delta^+_{S_0})$
(H 2) $\Delta^+_{S_0}$ is a distributional bisolution to the field equation, i.e. $\langle \Delta^+_{S_0}, Pf \otimes g\rangle = 0$ and $\langle \Delta^+_{S_0}, f \otimes Pg\rangle = 0$ for all $f, g \in \mathcal{E}^{\mathbb{C}}_c$.
(H 3) $\Delta^+_{S_0}$ is of positive type, meaning that $\langle \Delta^+_{S_0}, \bar{f} \otimes f\rangle \geq 0$, where $\bar{f}$ is the complex conjugate of $f \in \mathcal{E}^{\mathbb{C}}_c$.

Example 5.2 On Minkowski spacetime it is natural to choose $\Delta^+_{S_0}$ as

$$\Delta^+_{S_0}(x,y) = \frac{1}{(2\pi)^3}\int \left(e^{-i\omega_{\boldsymbol{p}}(x^0-y^0)+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})}\right)\frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}. \tag{5.6}$$

with $\omega(\mathbf{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$. We can now verify explicitly that

$$\begin{aligned}\Delta^+_{S_0}(x,y) - \Delta^+_{S_0}(y,x) &= \frac{1}{(2\pi)^3}\int \left(e^{-i\omega_{\boldsymbol{p}}(x^0-y^0)+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} - e^{i\omega_{\boldsymbol{p}}(x^0-y^0)-i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})}\right)\frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\\ &= i\Delta_{S_0}(x,y),\end{aligned}$$

so $\frac{i}{2}\Delta_{S_0}$ is the antisymmetric part of $\Delta^+_{S_0}$. It is also easy to see that

$$\langle \Delta^+_{S_0}, \bar{f}\otimes f\rangle = \frac{1}{(2\pi)^3}\int \left|\int e^{-i\omega_{\boldsymbol{p}}x^0+i\boldsymbol{p}.\boldsymbol{x}} f(x)d^4x\right|^2 \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} \geq 0,$$

so $\Delta^+_{S_0}$ is of positive type. The symmetric part of $\Delta^+_{S_0}$ is given by

$$\Delta_1 = \frac{1}{(2\pi)^3}\int \cos\left(\omega_{\boldsymbol{p}}(x^0-y^0) + i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})\right)\frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}},$$

so we write $\Delta^+_{S_0} = \frac{i}{2}\Delta_{S_0} + \Delta_1$.

On general globally hyperbolic spacetimes a decomposition (5.5) with properties (**H** 0)–(**H** 3) always exists but is not unique. If H and H' correspond to two different choices of the split (5.5), then their difference $H - H'$ is a smooth symmetric bisolution to the field equations (a smooth symmetric function with $P_x(H-H')(x,y) = P_y(H-H')(x,y) = 0$). Physically, the split of the causal propagator into $\Delta^+_{S_0}$ and H is interpreted as "taking the positive frequency part" of the causal propagator. In flat spacetime this corresponds to the spectrum condition, which is one of the Haag-Kastler axioms listed in Sect. 2.3.

The WF set of Δ^+_S has better properties than the WF set of Δ_S. If we now replace $\frac{i}{2}\Delta_S$ with Δ^+_S in (5.3), then the new product, denoted by $\star_H$ can be extended from $\mathcal{F}_{\mathrm{reg}}$ to $\mathcal{F}_{\mu c}$ (see [BDF09]). On $\mathcal{F}_{\mathrm{reg}}[[\hbar]]$ the two star products $\star$ and $\star_H$ are isomorphic. To see this, consider the map $\alpha_H : \mathcal{F}_{\mathrm{reg}}[[\hbar]] \to \mathcal{F}_{\mathrm{reg}}[[\hbar]]$ given by

$$\alpha_H \doteq e^{\frac{\hbar}{2}\mathcal{D}_H}, \tag{5.7}$$

where $\mathcal{D}_H \doteq \langle H, \frac{\delta^2}{\delta\varphi^2}\rangle$.

It is easy to check that

$$F \star_H G = \alpha_H\left((\alpha_H^{-1}F)\star(\alpha_H^{-1}G)\right), \quad F, G \in \mathcal{F}_{\mathrm{reg}}. \tag{5.8}$$

We say that α_H provides a gauge transformation between $\star$ and $\star_H$.

Definition 5.1 A gauge transformation is a map $F \mapsto F + \sum_{\hbar\geq 1}\hbar^n B_n(F)$, where each B_n is a differential operator.

If there exists a gauge transformation relating two star products $\star$ and $\star'$, then they define the same deformation quantization. In the case of α_H, $B_n = \frac{1}{n!2^n}(\mathcal{D}_{H-H'})^n$. Physically, we can identify the transition between $\star$ and $\star_H$ with normal ordering, so passing to the $\star_H$-product is the algebraic version of Wick's theorem (more detail in Sect. 6.2.1). Note that the codomain of $\alpha_H : \mathfrak{F}_{\text{reg}} \to \mathfrak{F}_{\text{reg}}$ is sequentially dense in a larger space $\mathfrak{F}_{\mu c}$ (with respect to the topology τ_{BDH} described in Sect. 4.4.2, Definition 4.17) and we can also build a corresponding (sequential) completion $\overline{\mathfrak{F}_{\text{reg}}}$ of the domain. Next we extend $\mathfrak{F}_{\text{reg}}$ with all elements of the form $\lim_{n\to\infty} \alpha_H^{-1}(F_n)$, where (F_n) is a convergent sequence in $\mathfrak{F}_{\mu c}$. We denote this space by $\alpha_H^{-1}(\mathfrak{F}_{\mu c}) \subset \overline{\mathfrak{F}_{\text{reg}}}$. This motivates the following definition

Definition 5.2 The *quantum algebra of the free theory* $\mathfrak{A}$ is defined as the extension of $\mathfrak{F}_{\text{reg}}[[\hbar]]$ by limits $\lim_{n\to\infty} \alpha_H^{-1}(F_n)$, where (F_n) is a convergent sequence in $\mathfrak{F}_{\mu c}$. $\mathfrak{A}$ is equipped with the star product defined by

$$A \star B = \alpha_H^{-1}\left(\alpha_H(A) \star_H \alpha_H(B)\right)$$

and with the involution given by

$$A^* \doteq \alpha_H^{-1}(\overline{\alpha_H(A)}),$$

where bar denotes the complex conjugation of a functional. The support of $A \in \mathfrak{A}$ is defined by

$$\text{supp } A \doteq \text{supp } \alpha_H(A).$$

Remark 5.1 Note that $\mathfrak{A}$ as an abstract unital involutive topological algebra is unique up to an isomorphism, since different choices of H are related by

$$F \star_{H'} G = \alpha_{H-H'}^{-1}\left(\alpha_{H-H'}(F) \star_H \alpha_{H-H'}(G)\right),$$

and $H - H'$ is a smooth function, so $\alpha_{H-H'}$ is an isomorphism. This isomorphism doesn't change the support of a functional, so the notion of support introduced in Definition 5.2 is also independent of the choice of H.

Remark 5.2 We can also characterize $\mathfrak{A}$ as the space of families (F_H), labeled by possible choices of H, where $F_H \in \mathfrak{A}^H \doteq (\mathfrak{F}_{\mu c}[[\hbar]], \star_H)$ fulfill the relations

$$F_{H'} = \alpha_{H'-H} F_H,$$

and the product is defined by

$$(F \star G)_H = F_H \star_H G_H.$$

The map α_H^{-1} associates to a classical functional $F \in \mathfrak{F}_{\mu c}$ an element A of $\mathfrak{A}$ such that $A_H = F$. A different choice of the 2-point function $\Delta_{S_0}^+$ (and hence H) leads to a different identification of classical functionals with elements of $\mathfrak{A}$. If $H' \neq H$, then α_H^{-1} maps F to $B \in \mathfrak{A}$ such that $B_{H'} = F$ and $B_H = \alpha_{H-H'}(F)$, so clearly, $A \neq B$. We will come back to discussion of this ambiguity in Sect. 6.2.1.

We think of $\mathfrak{A}$ as the abstract algebra of observables, while the choice of α_H corresponds to a choice of realization of $\mathfrak{A}$ as an algebra of formal power series with coefficients in some space of functionals. Although we are primarily interested in the abstract structure, the concrete realization is important if we want to make some computations. The diagram below summarizes the algebraic structures introduced so far.

$$\begin{array}{ccc} (\mathcal{F}_{\mathrm{reg}}, \star) & \xrightarrow{\alpha_H} & (\mathcal{F}_{\mathrm{reg}}, \star_H) \\ {\scriptstyle \mathrm{dense}}\downarrow\cap & & {\scriptstyle \mathrm{dense}}\downarrow\cap \\ \mathfrak{A} & \xleftarrow{\alpha_H^{-1}} & (\mathcal{F}_{\mu c}, \star_H) \end{array}$$

We use the notion of support defined for the elements of $\mathfrak{A}$ to build a net of involutive topological algebras.

Proposition 5.1 *Given a spectime $\mathcal{M} = (M, g)$, a configuration space $\mathcal{E} = \Gamma(E \to M)$ and a quadratic Lagrangian L_0, which induces a formally selfadjoint differential operator P, consider the net*

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}),$$

*where $\mathfrak{A}(\mathcal{O})$ is generated by elements with support contained in $\mathcal{O}$. This net is a quantum field theory model in the sense of Definition 2.54 and it satisfies the axiom of **Covariance** and the **Time-slice axiom**.*

Proof The properties of covariance and causality are clear from the construction. The time-slice axiom has been proven in [CF08]. □

The construction is fully covariant, so the assignment of unital $*$-algebras to spacetime defined above induces a functor from **Loc** to $\mathbf{Obs}_p$. It acts on morphisms $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ by (see equation (13) in [Zah14]):

$$(\mathfrak{A}\chi(F))_H \doteq \mathfrak{F}_{\mu c}\chi(F_{\chi^* H}),$$

where $\chi^* H$ is the pullback of the restriction of H to $\chi(M) \times \chi(M)$ and $\mathfrak{F}_{\mu c}$ is the functor which associates to objects $\mathcal{M} \in \mathbf{Loc}$ the spaces of microcausal functionals $\mathcal{F}_{\mu c}(\mathcal{M})$ and acts on morphisms by

$$\mathfrak{F}_{\mu c}\chi F(\varphi) \doteq F(\chi^* \varphi).$$

Proposition 5.2 *A locally covariant QFT model in the sense of Definition 2.66 is obtained by assigning $\mathfrak{A}(\mathcal{M})$ to objects of* **Loc**. *For morphisms $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$, $\mathfrak{A}\chi$ is defined with the use of the pullback of functionals, as in Proposition 3.5.*

Let us now discuss the existence of states. One obtains a family of states on $\mathfrak{A}$ by setting

$$\omega_{H,\varphi}(F) \doteq \alpha_{_H}(F)(\varphi) = F_H(\varphi),$$

where $\varphi \in \mathcal{E}_S$. This is well defined, since F_H is a functional in $\mathcal{F}_{\mu c}$, hence the evaluation at a field configuration φ makes sense.

Example 5.3 We continue with the example of the free scalar field. We define $\mathfrak{A}_W$ as the subalgebra of $\mathfrak{A}$ generated by the Weyl generators $\mathcal{W}(f) \doteq \exp(iF_f)$, where $f \in \mathscr{D}(M)$. Note that

$$\mathcal{D}_H \left(i \int f \varphi d\mu_g \right)^n = -\frac{n!}{(n-2)!} H(f,f) \left(i \int f \varphi d\mu_g \right)^{n-2}.$$

Hence

$$\alpha_{_H}(\mathcal{W}(f)) = e^{\frac{\hbar}{2}\mathcal{D}_H}\mathcal{W}(f) = e^{-\frac{\hbar}{2}H(f,f)}\mathcal{W}(f).$$

It follows now that

$$\omega_{H,0}(\mathcal{W}(f)) = e^{-\frac{\hbar}{2}H(f,f)},$$

so $\omega_{H,0}$ is a quasi-free state with covariance H (compare with the Definition 2.31).

5.2 The Star Product on the Space of Multivector Fields

In the classical theory we have defined "going on-shell" as taking the quotient of the algebra of classical observables by the ideal $\mathcal{I}_S$ generated by the equations of motion. Now we want to do something similar in the quantum theory.

Definition 5.3 The equations of motion ideal $\mathfrak{I}^H_{S_0}$ of $\mathfrak{A}^H$ is defined as the $\star_H$-ideal of $\mathfrak{A}^H$ generated by elements of the form $\langle S_0', X\rangle$, where $X \in \mathcal{V}_{\text{loc}}$. The ideal $\mathfrak{I}_{S_0}$ is then defined by the requirement that $F \in \mathfrak{I}_{S_0}$ if and only if $F_{_H} \in \mathfrak{I}^H_{S_0}$ for one and hence for all H (note that $\alpha_{_{H-H'}}\langle S_0', X\rangle = \langle S_0', \alpha_{_{H-H'}}(X)\rangle$).

The on-shell quantum algebra of the free theory is then understood as the quotient

$$\mathfrak{A}_{S_0} \doteq \mathfrak{A}/\mathfrak{I}_{S_0}.$$

Remark 5.3 Note that $\mathfrak{A}_{S_0}$ can be characterized as the space of families $F = (F_H)$, where

$$F_H \in \mathfrak{A}^H_{S_0} \doteq \mathfrak{A}^H/\mathfrak{I}^H_{S_0}.$$

The definition used here suggests that we can express it also in terms of the differential δ_{S_0}, as defined in Sect. 4.3. The star product $\star$ can be easily extended to the space $\bigwedge \mathcal{V}$ of vector fields, if one replaces the pointwise product in (5.3) with the graded product $\wedge$. Before we give the explicit formula for $\star_H$ of two multivector fields, it is convenient to introduce some notation.

Definition 5.4 The space $\bigwedge^k \mathcal{V}_{\mu c}$ of microcausal k-vector fields is identified with $\mathcal{O}^k_{\mu c}(T^*[1]\mathcal{E})$, see Definition 4.11.

Using this definition, we can now introduce the star product of multivector fields:

$$\begin{aligned}&\left\langle (X \star_H Y)(\varphi); v_1, \ldots, v_{p+q}\right\rangle \\ &\doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!p!q!} \sum_{\sigma \in S_{p+q}} \left\langle\left\langle \tfrac{\delta^n X}{\delta\varphi^n}(\varphi); v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right\rangle, (\Delta^+_{S_0})^{\otimes n} \left\langle \tfrac{\delta^n Y}{\delta\varphi^n}(\varphi); v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}\right\rangle\right\rangle,\end{aligned}$$

where $X, Y \in \bigwedge \mathcal{V}_{\mu c}$ are of degree p and q respectively. In the 0th order, the star product gives just the wedge product of two multivector fields. In the first order, one obtains the extension of $\lfloor ., . \rfloor$ to $\bigwedge \mathcal{V}_{\mu c}$.

We can now characterize the ideal $\mathfrak{A}_0$ by means of the differential δ_{S_0}, as we did in classical theory. First we note that since $\Delta^+_{S_0}$ is a distributional bisolution for the operator P (see property (**H** 1)), we have

$$\left\langle S_0', f\right\rangle \star F = \left\langle S_0', f\right\rangle \cdot F + \left\langle S_0'', f \otimes \Delta^+_{S_0} F^{(1)}\right\rangle = \left\langle \frac{\delta S_0}{\delta\varphi}, f\right\rangle \cdot F,$$

where $f \in \mathcal{E}_c$, $F \in \mathcal{F}_{\mu c}$. It follows that

$$\delta_{S_0}(X \star_H Y) = (\delta_{S_0} X) \star_H Y + (-1)^{|X|} X \star_H (\delta_{S_0} Y),$$

for $X, Y \in \bigwedge \mathcal{V}_{\mu c}$, i.e. δ_{S_0} is a derivation with respect to the product $\star_H$. In particular we can write $\mathfrak{A}^H_{S_0}$ as

$$\mathfrak{A}^H_{S_0} = \frac{\mathrm{Ker}\ \delta_{S_0} \restriction_{\mathcal{F}_{\mu c}}}{\mathrm{Im}\ \delta_{S_0} \restriction_{\mathcal{V}_{\mu c}}}$$

Note that in our case $\mathrm{Ker}\ \delta_{S_0} \restriction_{\mathcal{F}_{\mu c}} = (\mathcal{F}_{\mu c}, \star_H)$, but we will see later on that this is no longer the case in gauge theories (Chap. 7).

Example 5.4 Consider the algebra $\mathfrak{A}_W$ from Example 5.3. Let $\mathfrak{I}_{W,S_0} \doteq \mathfrak{I}_{S_0} \cap \mathfrak{A}_W$. Note that $S_0'(\varphi) = P\varphi = -(\Box + m^2)\varphi$, and using integration by parts, we conclude that $F_{(\Box+m^2)f}(\varphi) = -\left\langle S_0', f\right\rangle$. It is then easy to verify that $\mathfrak{I}_{W,S_0}$ is generated by the elements

$$\mathcal{W}((\Box + m^2)f) - 1, \quad f \in \mathcal{D}(M) \tag{5.9}$$

We denote $\mathfrak{A}_W/\mathfrak{I}_{W,S_0}$ by $\mathfrak{A}_{W,S_0}$. This algebra is then (algebraically) isomorphic to the Weyl algebra $\mathcal{W}(L, \sigma)$, with $L = \mathcal{D}(M)/P\mathcal{D}(M)$ and $\sigma = \Delta_S$. States of the form $\omega_{H,0}$ are well defined on the quotient $\mathfrak{A}_{W,S_0}$, as H is a bisolution for the operator P.

5.3 Kähler Structure

Objects introduced in the previous section have an interpretation in terms of a Kähler structure, as in Definition 2.32. H is a symmetric, non-degenerate bilinear form on $L = \mathscr{D}(M)/P\mathscr{D}(M)$, Δ_S is the symplectic structure and $\Delta_S^+ = \frac{i}{2}\Delta_S + H$ is a Hermitian 2-form on $L^{\mathbb{C}}$, as in formula (2.3).

We have seen in Example 5.3 that H is the covariance of a quasi-free state $\omega_{H,0}$ on $\mathfrak{A}_{W,S_0}$. If this state is pure, then the pair $(\Delta_S, 2H)$ is Kähler. In general one can always use Δ_S and H to define an anti-involution J, but $\Delta_S \circ J = 2H$ holds only in the case of pure $\omega_{H,0}$.

Let us now recall briefly the construction of J. We follow [DG13]. First note that since H is the covariance of a quasi-free state, we know from Theorem 2.3 that

$$|\Delta_S(f_1, f_2)| \leq 2\sqrt{H(f_1, f_2)}\sqrt{H(f_1, f_2)}, \quad \forall f_1, f_2 \in L. \tag{5.10}$$

We complete L with the product $(.,.)_H \doteq \langle ., H.\rangle$ to a real Hilbert space $\mathcal{H}$ and the inequality (5.10) implies that Δ_S is a bilinear form on $\mathcal{H}$ with norm less than or equal 2. Therefore, there exists an operator $A \in \mathcal{B}(\mathcal{H})$ with $||A|| \leq 1$ such that

$$\langle f_1, \Delta_S f_2\rangle = 2(f_1, Af_2)_H$$

If the kernel of A is trivial, then we take the polar decomposition $A = -J|A|$ and J satisfies $J^2 = -1$. More generally, as in the proof of theorem 17.12 of [DG13], we define $L_{\text{sg}} \doteq \text{Ker } A$ and $L_{\text{reg}} \doteq L_{\text{sg}}^{\perp}$. Then we set $A_{\text{reg}} \doteq A\restriction_{L_{\text{reg}}}$ and construct the polar decomposition $A_{\text{reg}} = -J_{\text{reg}}|A_{\text{reg}}|$. If the dimension of L_{sg} is even or infinite (which is the case in the situation we are interested in), then there exist an orthogonal anti-involution J_{sg} on L_{sg} and we set $J = J_{\text{reg}} \oplus J_{\text{sg}}$. J constructed this way defines an almost-complex structure on L.

We define the holomorphic and anti-holomorphic subspaces of $L^{\mathbb{C}}$ as

$$\mathcal{Z} \doteq \{(f - iJf)|f \in L\},$$
$$\overline{\mathcal{Z}} \doteq \{(f + iJf)|f \in L\},$$

respectively. Projections onto these subspaces are defined as $\mathbb{1}_{\mathcal{Z}} = \frac{1}{2}(\text{id} - iJ)$ and $\mathbb{1}_{\overline{\mathcal{Z}}} = \frac{1}{2}(\text{id} + iJ)$

If $\omega_{H,0}$ is pure then the quadruple $(L, 2H, \Delta_S, J)$ is a Kähler structure. We decompose Δ_S^+ in the holomorphic basis to obtain

$$\left\langle \mathbb{1}_{\overline{\mathcal{Z}}} f_1, \Delta_S^+(\mathbb{1}_{\mathcal{Z}} f_2)\right\rangle = \left\langle f_1, \Delta_S^+ f_2\right\rangle,$$

where $f_1, f_2 \in L^{\mathbb{C}}$ and remaining components vanish. As a result, Δ_S^+ is represented in the holomorphic basis as

$$\begin{pmatrix} 0 & 0 \\ \Delta_S^+ & 0 \end{pmatrix},$$

so it acts only on the holomorphic part of the first argument and the anti-holomorphic part of the second argument.

We have seen in this section that it is natural to reformulate the construction of the quantum algebra of the free field in the language of Kähler geometry. There is also a hope that such concepts might allow us to go beyond the formal quantization. This will be addressed in our future works.

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Chapter 6
Interaction and Renormalization of the Scalar Field Theory

6.1 Outline of the Approach

In the previous chapter we have covered the quantization of free theories (quadratic actions); now is the time to introduce the interactions. This is where we have to start working perturbatively. The ultimate goal of AQFT is to be able to construct interacting models in 4 spacetime dimensions non-perturbatively, but at the moment no such models are known. The perturbative approach, on the other hand, has proven to be successful in describing many phenomena in particle physics, so it is worthwhile to try to understand its mathematical foundations. It turns out that a careful analysis of the problem and employing some tools from functional analysis allow us to avoid dealing with ill defined "divergent" expressions, as is often done in physics textbooks.

We will follow the ideas on renormalization developed by [BP57, BS59, Hep66, EG73, SR50, Ste71]. The approach is motivated by the interaction picture of quantum mechanics, as outlined in Sect. 2.1.4. We begin with a heuristic argument and then we will show how to make it rigorous. Let H_0 be the free Hamiltonian and let $H_{t,I} = -\int_K :\mathcal{L}_I(0,\mathbf{x}): d\sigma_t$ be the interaction Hamiltonian, where $:\mathcal{L}_I:$ is the normal-ordered Lagrangian density (the precise definition of normal-ordering will be introduced later in this chapter), constructed from the classical quantity $\mathcal{L}_I(x)$ and K is some compact subset of a Cauchy surface Σ.

We would like to use the Dyson formula (2.11) for the interacting time evolution operator $U_I(t,s)$, so formally we write

$$U_I(t,s) = 1 + \sum_{n=1}^{\infty} \frac{i^n \lambda^n}{n!} \int_{([s,t]\times\mathbb{R}^3)^n} T(:\mathcal{L}_I(x_1):\dots:\mathcal{L}_I(x_n):) d^4x_1 \dots d^4x_n,$$

where λ is the coupling constant, T denotes time-ordering and $:\mathcal{L}_I:$ is an operator-valued function given by

$$:\mathcal{L}_I(x): = e^{iH_0x^0} :\mathcal{L}_I(0,\mathbf{x}): e^{-iH_0x^0}.$$

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Heuristically, one would use the unitary map defined above to obtain interacting fields as

$$\varphi_I(x) = U(x^0, s)^{-1}\varphi(x)U(x^0, s) = U(t, s)^{-1}U(t, x^0)\varphi(x)U(x^0, s), \tag{6.1}$$

where $s < x^0 < t$.

There are, however, serious problems with this idea. The first obvious difficulty is the fact that typical Lagrangian densities like $\mathcal{L}_I(x) = \varphi(x)^4$ cannot be used to define operator-valued distributions on a Cauchy surface Σ_0 (they are too singular). This is the source of the so called UV problem. Moreover, having the sharp cutoff function in the Lagrangian and Hamiltonian (i.e. integrating with the characteristic function of $K \times [s, t]$) leads to additional divergences, called Stückelberg divergences. Finally there is a problem with taking the adiabatic limit, as the integral of the Lagrangian density over $\mathbf{x}$ does not exist if Σ is non-compact. Last but not least, the overall sum might not converge.

Fortunately all these problems, apart from the last one, can be easily dealt with by a slight modification of the above idea. First, to avoid the Stückelberg divergences, we replace the sharp cutoffs with smooth test functions. Next, we solve the UV problem by using causal perturbation theory in the sense of Epstein and Glaser [EG73], where the interaction is switched on only in a compact region of spacetime. Finally, we take the adiabatic limit algebraically, as a certain inductive limit at the level of interacting observable algebras.

6.2 Scattering Matrix and Time Ordered Products

Modifications of the Dyson formula described at the end of the previous section lead to the definition of the *formal S-matrix* as

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n \lambda^n}{n!} \int g(x_1)\dots g(x_n) T(:\mathcal{L}_I(x_1):\dots:\mathcal{L}_I(x_n):) d^4x_1 \dots d^4x_n,$$

where g is a test function. Compare this formula with (2.12). In order to make this formula mathematically rigorous, first we need to make sense of the normal ordering operation $:\mathcal{L}_I(x):$, and then we have to define the time-ordered products of $:\mathcal{L}_I(x_i):$. Finally, in order to make sense of the formula (6.1) for the interacting field we interpret it as the definition of a *distribution*, rather than a function. Hence, for a test function f we obtain

$$
\begin{aligned}
&\int f(x)\varphi_I(x)d^4x \\
&\quad = S(g)^{-1}\sum_{n=0}^{\infty}\frac{i^n\lambda^n}{n!}\int f(x)g(x_1)\dots g(x_n)T\varphi(x)\mathcal{L}_I(x_1)\dots\mathcal{L}_I(x_n)d^4x_1\dots d^4x_n \\
&\quad = \frac{d}{d\lambda}\, S(g)^{-1}S(g,\lambda f)\big|_{\lambda=0}\,, \qquad (6.2)
\end{aligned}
$$

where $S(g, f)$ is the formal S-matrix with the Lagrangian density $\lambda g\mathcal{L}_I + f\varphi$. This is called *Bogoliubov's formula* [BS59].

6.2.1 Wick Products

Let us start our construction by defining the Wick-ordered quantities $:\mathcal{L}_I(x):$. In our framework normal (Wick) ordering is meant as a prescription for identifying classical quantities with their quantum counterparts. More precisely, it is a map $\mathfrak{F}_{\mathrm{loc}} \to \mathfrak{A}$, $F \mapsto :F:$. An example of such map is provided by $:F:_H \doteq \alpha_H^{-1} \circ \mathcal{T}^H_{(\mathrm{F})}$, where $\mathcal{T}^H : \mathfrak{F}_{\mathrm{loc}} \to \mathfrak{A}^H$. Note that both classical and quantum observables are understood as (formal power series in) microcausal functionals, so the easiest way is define the normal ordering is to set $\mathcal{T}^H = \mathrm{id}$ and hence

$$
:F:_H = \alpha_H^{-1}(F), \qquad F \in \mathfrak{F}_{\mathrm{loc}}. \qquad (6.3)
$$

Clearly, $:\,.\,:_H$ defined this way depends on the choice of H, so there is no distinguished choice of normal ordering on spacetimes with no distinguished states.

Example 6.1 (*Algebraic Wick's theorem*) Wick's theorem plays an important role in physics, so we will show here how it follows from our algebraic definition of normal ordering. To this end we will use the abstract definition of $\mathfrak{A}$, rather the one in terms of families of functionals labeled by H. Consider the free scalar field. Let

$$
F_n(\varphi) = \int \varphi(x)\varphi(y)g_n(y-x)f(x)d\mu_g(x)d\mu_g(y),
$$

where $f \in \mathcal{D}(M)$ and g_n is a sequence of smooth compactly supported functions that converges to the Dirac delta distribution $\delta(x-y)$ in the topology τ_N defined in Definition 4.16. By applying α_H^{-1} to the sequence (F_n), we obtain a sequence

$$
\alpha_H^{-1}F_n = \int (\varphi(x)\varphi(y)g_n(y-x)f(x) - \hbar H(x,y)g_n(y-x)f(x))d\mu_g(x)d\mu_g(y),
$$

The limit of this sequence is identified with the normally ordered expression $\int :\varphi(x)^2:_H f(x)d\mu_g(x)$, i.e.:

$$
\begin{aligned}
{:}F{:}_H &= \int {:}\varphi^2{:}_H \, f d\mu_g \\
&= \lim_{n\to\infty} \int (\varphi(x)\varphi(y) - \hbar H(x,y)) g_n(y-x) f(x) d\mu_g(x) d\mu_g(y).
\end{aligned}
$$

Note that the right-hand side cannot be interpreted as a functional on the configuration space, but is just a formal expression representing an abstract element of the extension of $\mathcal{F}_{\mathrm{reg}}$, as defined in Definition 5.2. We write this expression in a short-hand notation as a coinciding point limit:

$$
{:}\varphi(x)^2{:}_H = \lim_{y\to x} (\varphi(x)\varphi(y) - \hbar H(x,y)).
$$

This shows that transforming with α_H^{-1} corresponds formally to the subtraction of $\hbar H(x,y)$.

Now, to recover Wick's theorem, consider a product of two Wick squares ${:}\varphi(x)^2{:}_H \, {:}\varphi(y)^2{:}_H$. Note that in our setting this has to be understood as the $\star$-product of two elements of $\mathfrak{A}$. Using the α_H^{-1} prescription we identify this product as

$$
\begin{aligned}
&\left(\alpha_H^{-1}\int \varphi^2 f_1 d\mu_g\right) \star \left(\alpha_H^{-1}\int \varphi^2 f_2 d\mu_g\right) \\
&= \alpha_H^{-1}\left(\left(\int \varphi^2 f_1 d\mu_g\right) \star_H \left(\int \varphi^2 f_2 d\mu_g\right)\right) \\
&= \alpha_H^{-1}\Bigg(\int \varphi^2 f_1 d\mu_g \int \varphi^2 f_2 d\mu_g + 4\hbar\langle f_1\varphi, \Delta^+_{S_0}(f_2\varphi)\rangle \\
&\qquad + \frac{\hbar^2}{2}\int (\Delta^+_{S_0}(x,y))^2 f_1(x) f_2(y) d\mu_g(x) d\mu_g(y)\Bigg).
\end{aligned}
$$

Omitting the test functions we obtain

$$
{:}\varphi(x)^2{:}_H \, {:}\varphi(y)^2{:}_H = {:}\varphi(x)^2\varphi(y)^2{:}_H + 4{:}\varphi(x)\varphi(y){:}_H \hbar\Delta^+_{S_0}(x,y) + 2\left(\hbar\Delta^+_{S_0}(x,y)\right)^2,
$$

which is a familiar form of Wick's theorem applied to ${:}\varphi(x)^2{:}_H \, {:}\varphi(y)^2{:}_H$.

6.2.2 *Locally Covariant Wick Products*

The normal ordering prescription (6.3) is not the optimal one if we work on curved spacetime. This is because we would like to define the normal ordering on all globally hyperbolic spacetimes in a coherent way and the choice of H across different spacetimes cannot be made covariantly. More precisely, we want to lift ${:}\,.\,.\,{:}$ to the level of locally covariant fields in the sense of Definition 2.66.

Definition 6.1 Let $:..:_{\mathcal{M}}$ be a Wick-ordering prescription on spacetime $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$. Given a natural transformation $\Phi : \mathfrak{D} \to \mathfrak{F}_{\mathrm{loc}}$ we say that $(:..:_{\mathcal{M}})_{\mathcal{M}\in\mathrm{Obj}(\mathbf{Loc})}$ is covariant if the family $:\Phi:_{\mathcal{M}}$ defines a natural transformation from $\mathfrak{D}$ to $\mathfrak{A}$.

We will now discuss the existence of such a family of Wick-ordering prescriptions and the remaining renormalization freedom.

Even though Hadamard states on different spacetimes cannot be chosen in a coherent way, it turns out that this is possible for a family of Hadamard parametrics. The difference between $\Delta^{+}_{S_0}$, a 2-point function of a Hadamard state, and a parametrix is that the latter is a bi-solution of the linearized equations of motion only up to smooth terms. Due to the covariance condition, it is more appropriate to define the normal ordering by a prescription where only the singular part of H enters into α_H^{-1}. We realize this idea by setting

$$:F:_{\mathcal{M}} = \alpha_H^{-1}(\alpha_w F) = \alpha_{H-w}^{-1} F,$$

where $F \in \mathcal{F}_{\mathrm{loc}}$ and where w is the smooth part of the Hadamard 2-point function. In other words, we set $\mathcal{T}^H \equiv \alpha_w$. To understand what w is, we need to recall some facts about the singularity structure of 2-point functions of Hadamard states. We follow closely [KW91] and a recent review [FR14].

We begin by introducing some notation. Let $t : \mathcal{M} \to \mathbb{R}$ be a time function (smooth function with a timelike and future directed gradient field) and let

$$\sigma_\varepsilon(x, y) \doteq \sigma(x, y) + 2i\varepsilon(t(x) - t(y)) + \varepsilon^2,$$

where $\sigma(x, y)$ is half of the square geodesic distance between x and y, i.e.

$$\sigma(x, y) \doteq \frac{1}{2} g(\exp_x^{-1}(y), \exp_x^{-1}(y)).$$

Definition 6.2 We say that a bi-distribution W on $\mathcal{M}$ is of local Hadamard form if, for every $x_0 \in M$, there exists a geodesically convex neighbourhood V of x_0 such that, for every integer N, $W(x, y)$ on $V \times V$ can be written in the form

$$W = \lim_{\varepsilon \downarrow 0} \left(\frac{u}{\sigma_\varepsilon} + \sum_{n=0}^{N} \sigma^n v_n \log\left(\frac{\sigma_\varepsilon}{\lambda^2}\right) + w_N \right) = W_N^{sing} + w_N, \tag{6.4}$$

where $u, v_n \in \mathcal{C}^\infty(M^2, \mathbb{R})$, $n = 0, \ldots, N$ are solutions of the transport equations and are uniquely determined by the local geometry, λ is a free parameter with the dimension of inverse length and w_N is an $2N + 1$ times continuously differentiable real-valued function.

We now define

$$\alpha_w F \doteq \lim_{N\to\infty} \alpha_{w_N} F, \tag{6.5}$$

for $F \in \mathcal{F}_{\mathrm{loc}}$. This limit makes sense, because the series converges after finitely many steps. It is crucial that in this formula F is local, since the limit of w_N is well defined only in a geodesically convex neighborhood of the diagonal in $M \times M$. Since functional derivatives of a local functional are supported on the diagonal, $\alpha_{w_N} F$ has a well-defined limit. For details see [HW02a]. Note that in our definition of $:\,.\,.\,:_{\mathcal{M}}$ is fixed up to the choice of the energy scale λ in Eq. 6.4. One can, however, allow additional freedom, staying consistent with the requirements of locality, covariance and some regularity properties. This possibility has been investigated in [HW02a], where it was shown that for the scalar field $:\,.\,.\,:_{\mathcal{M}}$ is fixed up to two parameters. The choice of these parameters is then treated as additional renormalization freedom.

A slightly different point of view on the problem of finding locally covariant Wick products is presented in section 5 of [BFV03]. Here we only give a sketch of the argument for the Wick square. For each object $\mathcal{M} \in$ **Loc** we choose $H_{\mathcal{M}}$ and going through the definitions it is easy to see that, for an admissible embedding $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{M}')$ we obtain

$$\mathfrak{A}\chi\big({:}\Phi^2{:}_{H_{\mathcal{M}}}(x)\big) = {:}\Phi^2{:}_{H_{\mathcal{M}'}}(\chi(x)) + H_{\mathcal{M}'}(\chi(x), \chi(x)) - H_{\mathcal{M}}(x, x)\,.$$

It was shown in [BFV03] that redefining Wick powers to become covariant amounts to solving a cohomological problem. Note that given H and H', symmetric parts of 2-point functions of Hadamard states on a given spacetime $\mathcal{M}'$, we can define a smooth function $B_{H,H'}$ on $\mathcal{M}'$ by setting

$$B_{H,H'}(x') \doteq {:}\Phi^2{:}_{H}(x') - {:}\Phi^2{:}_{H'}(x')\,.$$

These functions are covariant under embeddings $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{M}')$ in the sense that

$$B_{\chi^* H, \chi^* H'}(x) = B_{H,H'}(\chi(x))\,,$$

where $\chi^* H$ denotes the pullback of the restriction of H to $\chi(M) \times \chi(M)$. Moreover, functions B fulfill a cocycle condition

$$B_{H,H'} + B_{H',H''} + B_{H'',H} = 0\,.$$

The problem of finding a covariant normal-ordering prescription is then reduced to the cohomological problem of trivialization of the above cocycle while preserving the covariance property. To see that this is indeed the case, note that if the cocycle trivializes, then there exists a family of functions f_H labeled by different choices of Hadamard states such that

$$B_{H,H'}(x) = f_H(x) - f_{H'}(x)$$

and we can set

$${:}\Phi^2{:}_{\mathcal{M}}(x) \doteq {:}\Phi^2{:}_{H_{\mathcal{M}}}(x) - f_H(x)\,.$$

In fact, the cohomological problem above is solved by using the smooth part of the Hadamard 2-point-function, i.e. one can take

$$f_H(x) = w(x, x)\,.$$

6.2.3 Time-Ordered Products

After giving sense to normally ordered expressions we now want to define their time-ordered products. In the first step we consider only regular functionals, to understand the algebraic structure. We want to define the time-ordered product $\cdot_{\mathcal{T}}$ as a binary operation on $\mathfrak{F}_{\mathrm{reg}}[[\hbar]]$ that satisfies the condition (2.10), i.e.

$$F \cdot_{\mathcal{T}} G = \begin{cases} F \star G & \text{if supp}\, G \prec \text{supp}\, F, \\ G \star F & \text{if supp}\, F \prec \text{supp}\, G, \end{cases} \tag{6.6}$$

where the relation "$\prec$" means "not later than" i.e. there exists a Cauchy surface that separates supp G and supp F and in the first case supp F is in the future of this surface and in the second case it is in the past. We postulate

$$F \cdot_{\mathcal{T}} G = \sum_{n=0}^{\infty} \hbar^n B_n(F, G), \tag{6.7}$$

for some functional differential operators B_n. Now let $f \in \Gamma_c^{\mathbb{C}}(E^* \to M)$ and define a linear functional $F_f(\varphi) = \int \langle f, \varphi \rangle \, d\mu_g$ where $\langle ., . \rangle$ is the duality between V^* and V, the fibers of E^* and E, respectively. Condition (6.6) implies

$$B_0(F_f, F_g) + \hbar B_1(F_f, F_g) = \begin{cases} F_f F_g + \frac{i}{2}\hbar \big\langle f, \Delta_{S_0} g \big\rangle & \text{if supp}\, g \prec \text{supp}\, f, \\ F_f F_g + \frac{i}{2}\hbar \big\langle g, \Delta_{S_0} f \big\rangle & \text{if supp}\, f \prec \text{supp}\, g, \end{cases}$$

where $f, g \in \Gamma_c^{\mathbb{C}}(E^* \to M)$. This suggests setting $B_0(F, G) = FG$ for all $F, G \in \mathfrak{F}_{\mathrm{reg}}$. To analyse the first order condition in more detail, we note that $\Delta_{S_0} = \Delta_{S_0}^{\mathrm{R}} - \Delta_{S_0}^{\mathrm{A}}$ and from the support properties of the advanced and retarded Green functions it follows that in the example above

$$B_1(F_f, F_g) = \begin{cases} \frac{i}{2} \big\langle f, \Delta_{S_0}^{\mathrm{R}} g \big\rangle & \text{if supp}\, g \prec \text{supp}\, f, \\ \frac{i}{2} \big\langle f, \Delta_{S_0}^{\mathrm{A}} g \big\rangle & \text{if supp}\, f \prec \text{supp}\, g, \end{cases}$$

Now, since B_1 has to be a differential operator, the condition above fixes its coefficients by the first order derivatives up to diagonal, i.e.

$$B_1 = m \circ \left\langle t, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle + \text{higher derivatives},$$

where m is the pointwise multiplication operator, i.e. $m(F_1, \ldots, F_n)(\varphi) \doteq F_1(\varphi) \cdot \ldots \cdot F_n(\varphi)$ and t is a distribution in $\Gamma_c'^{\mathbb{C}}((E^*)^{\boxtimes 2} \to M^2)$ with a kernel satisfying

$$t(x, y) = \begin{cases} \frac{i}{2}\Delta_{S_0}(x, y) = \frac{i}{2}\Delta^{\mathrm{R}}_{S_0}(x, y) & \text{if } y \prec x, \\ -\frac{i}{2}\Delta_{S_0}(x, y) = \frac{i}{2}\Delta^{\mathrm{A}}_{S_0}(x, y) & \text{if } x \prec y. \end{cases}$$

Note that the relations $x \prec y$ and $y \prec x$ require that there is a Cauchy surface separating x and y, so the case $x = y$ is not affected by the above and there is an ambiguity in choosing $t(x, x)$.

It turns out that a consistent choice is provided by $t = i\Delta^{\mathrm{D}}_{S_0}$, where $\Delta^{\mathrm{D}}_{S_0} \doteq \frac{1}{2}(\Delta^{\mathrm{R}}_{S_0} + \Delta^{\mathrm{A}}_{S_0})$ is the Dirac propagator. Let us introduce the notation

$$D_{\mathrm{D}} \doteq \left\langle \Delta^{\mathrm{D}}_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle.$$

Using the higher order conditions and requiring associativity leads to the following formula for the time-ordered product:

$$\begin{aligned} F \cdot_{\mathcal{T}} G &\doteq m \circ e^{i\hbar D_{\mathrm{D}}}(F \otimes G) \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}, \left(i\Delta^{\mathrm{D}}_{S_0}\right)^{\otimes n} G^{(n)} \right\rangle, \end{aligned} \tag{6.8}$$

where $F, G \in \mathcal{F}_{\mathrm{reg}}$. In contrast to $\star$, the product defined this way is commutative, since $\Delta^{\mathrm{D}}_{S_0}$ is symmetric. It is also equivalent to the pointwise product by means of

$$F \cdot_{\mathcal{T}} G = \mathcal{T}\left(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G\right), \tag{6.9}$$

where

$$\mathcal{T} = e^{\frac{i\hbar}{2}\mathcal{D}_{\mathrm{D}}}, \tag{6.10}$$

and

$$\mathcal{D}_{\mathrm{D}} \doteq \left\langle \Delta^{\mathrm{D}}_{S_0}, \frac{\delta^2}{\delta\varphi^2} \right\rangle,$$

or more precisely

$$(\mathcal{T}F)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle (i\Delta^{\mathrm{D}}_{S_0})^{\otimes n}, F^{(2n)}(\varphi) \right\rangle.$$

The linear operator $\mathcal{T}$ defined above plays a role of the quantization map; it goes from the classical world to the quantum world, i.e.

$$\begin{matrix} (\mathcal{F}_{\mathrm{reg}}, \cdot) & \xrightarrow{\mathcal{T}} & (\mathfrak{A}_{\mathrm{reg}}, \cdot_{\mathcal{T}}) \\ \text{classical} & & \text{quantum} \end{matrix},$$

where $\mathfrak{A}_{\mathrm{reg}} \doteq (\mathcal{F}_{\mathrm{reg}}[[\hbar]], \star)$. Compare this with the notion of the normal-ordering prescription discussed in Sects. 6.2.1 and 6.2.2, which plays the same role for local functionals, as $\mathcal{T}$ for the regular ones. Later on we will use this fact to extend $\mathcal{T}$ to local non-linear functionals.

Note that on the quantum side there are *two* products, the non-commutative $\star$ and commutative $\cdot_{\mathcal{T}}$.

Remark 6.1 There are several propagators and Green's functions relevant in the pAQFT framework and they all induce some distinguished differential operators. To simplify the notation, in this book we will always write

$$D_* \doteq \left\langle \Delta^*_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle,$$

where $* = \mathrm{A, R, F}$ for the advanced, retarded and Feynman Green's functions, $* = \mathrm{D}$ for the Dirac propagator and $* = +$ for the 2-point function; for the causal propagator we denote the corresponding functional differential operator by D_Δ. Similarly we denote

$$\mathcal{D}_{\mathrm{F}} \doteq \left\langle \Delta^{\mathrm{F}}_{S_0}, \frac{\delta^2}{\delta\varphi^2} \right\rangle,$$

and

$$\mathcal{D}_H \doteq \left\langle H, \frac{\delta^2}{\delta\varphi^2} \right\rangle.$$

6.2.4 The Formal S-Matrix and Møller Operators

Consider a theory where the interaction is given by λV, where $V \in \mathcal{F}_{\mathrm{reg}}$ and λ is the coupling constant, treated from now on as a formal parameter (similarly to $\hbar$).

We start with introducing some notation. Let $\mathfrak{A}^H((\hbar))[[\lambda]] \doteq (\mathcal{F}_{\mu c}((\hbar))[[\lambda]], \star_H)$, where $\mathcal{F}_{\mu c}((\hbar))[[\lambda]]$ means formal power series in λ with coefficients in Laurent series in $\hbar$. Similarly, Let $\mathfrak{A}^H[[\lambda]] \doteq (\mathcal{F}_{\mu c}[[\hbar, \lambda]], \star_H)$. By obvious modification of the definitions introduced up to now, we define $\mathfrak{A}((\hbar))[[\lambda]]$, $\mathfrak{A}[[\hbar, \lambda]]$, etc.

Using these objects is crucial for the definition of the S-matrix, since we hold on to the convention, commonly used in physics, in which the action that enters the expression for the S-matrix is divided by $\hbar$. Alternatively, one can absorb this negative power of $\hbar$ into the definition of the action, as was done in [BDF09].

Definition 6.3 Define the formal S-matrix as a map $\mathcal{S} : \mathfrak{A}_{\mathrm{reg}}[[\lambda]] \to \mathfrak{A}_{\mathrm{reg}}((\hbar))[[\lambda]]$ given by the time-ordered exponential

$$\mathcal{S}(F) \doteq e_{\mathcal{T}}^{iF/\hbar} = \mathcal{T}\big(e^{i(\mathcal{T}^{-1}F)/\hbar}\big), \tag{6.11}$$

where F is of order at least λ.

In what follows we choose $F = \lambda V$ for $V \in \mathfrak{A}_{\mathrm{reg}}$ and call the terms in the λ-expansion of (6.11) the n-fold time-ordered products. More precisely

$$\mathcal{S}(\lambda V) \doteq \sum_{n=0}^{\infty} \left(\tfrac{i\lambda}{\hbar}\right)^n \frac{1}{n!}\mathcal{T}_n(V^{\otimes n}),$$

and

$$\mathcal{T}_n(V_1, \ldots, V_n) \doteq V_1 \cdot_{\mathcal{T}} \ldots \cdot_{\mathcal{T}} V_n,$$

where $V_1, \ldots, V_n \in \mathfrak{A}_{\mathrm{reg}}$.

Remark 6.2 Note that the S-matrix defined by the formula (6.11) is not unitary (i.e. $\mathcal{S}(\lambda V^*)^* \star \mathcal{S}(\lambda V) \neq 1$), unless V is linear. This problem is related to the non-locality of regular non-linear interactions. If we use only local interaction, the unitarity problem is solved on the level of the renormalized theory (see Sect. 6.2.5) by a suitable re-definition of time-ordered products. Note that this is to be expected, since non-local interactions are known to cause problems in quantum field theory [Kir67, Efi67, BDFP02].

Interacting fields are obtained by means of the Bogoliubov formula, which reads

$$\begin{aligned} R_{\lambda V}(F) &= -i\hbar\frac{d}{dt}\left(\mathcal{S}(\lambda V)^{\star -1} \star \mathcal{S}(\lambda V + tF)\right)\Big|_{t=0} \\ &= \left(e_{\mathcal{T}}^{i\lambda V/\hbar}\right)^{\star -1} \star \left(e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F\right). \end{aligned} \tag{6.12}$$

The formal inverse of $R_{\lambda V}$ is defined as

$$R_{\lambda V}^{-1}(G) \doteq \left(e_{\mathcal{T}}^{i\lambda V/\hbar} \star G\right) \cdot_{\mathcal{T}} e_{\mathcal{T}}^{-i\lambda V/\hbar}.$$

Physically, we interpret $R_{\lambda V}(F)$ as the interacting observable corresponding to F, constructed from free fields. The map $R_{\lambda V}$ is often called the (formal) quantum Møller operator. Expanding in λ we obtain

$$R_{\lambda V}(F) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} R_n(V^{\otimes n}, F)$$

and call maps R_n *n-fold retarded products*.

In the physics literature one can find the Bogoliubov formula written in terms of *antichronological products*. To understand the connection with our formulation, first write the $\star$-inverse of $e_{\mathcal{T}}^{i\lambda V/\hbar}$ in the form

$$\left(e_{\mathcal{T}}^{i\lambda V/\hbar}\right)^{\star -1} = \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{\hbar^k} \sum_{P \in \mathcal{P}_k^{\mathrm{ord}}} (-1)^{|P|} \prod_{I \in P}^{\star} V^{\cdot_{\mathcal{T}} |I|} \frac{1}{|I|!}, \tag{6.13}$$

where $\mathcal{P}_k^{\mathrm{ord}}$ is the set of ordered partitions (partitions P with an order relation $<$ on the elements $I \in P$) of a set with k non-distinguishable elements, $\overset{\star}{\prod}$ is the product built with the use of $\star$ and

$$V^{\cdot_{\mathcal{T}}|I|} \doteq \underbrace{V \cdot_{\mathcal{T}} \ldots \cdot_{\mathcal{T}} V}_{|I|}.$$

Terms in the λ-expansion of the formula (6.13) are called *antichronological products* $\bar{\mathcal{T}}_k$. More precisely

$$\bar{\mathcal{T}}_k(F_1, \ldots, F_k) \doteq k! \sum_{P \in \mathcal{P}_k^{\mathrm{ord}}} (-1)^{|P|+k} \overset{\star}{\prod_{I \in P}} \frac{1}{|I|!} \mathcal{T}_{|I|}\left(\bigotimes_{i \in I} F_i\right), \tag{6.14}$$

so

$$\left(e_{\mathcal{T}}^{i\lambda V/\hbar}\right)^{\star -1} = \sum_{k=0}^{\infty} \frac{(-i\lambda)^k}{\hbar^k k!} \bar{\mathcal{T}}_k(V^{\otimes k}).$$

The notation $\mathcal{T}_{|I|}\left(\bigotimes_{i \in I} F_i\right)$ means taking the time-ordered product of functionals F_i, where $i \in I$. The order is irrelevant, as $\cdot_{\mathcal{T}}$ is commutative. We can now express the n-fold retarded product as:

$$R_n(V^{\otimes n}, F) = \frac{i^n}{\hbar^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \bar{\mathcal{T}}_k(V^{\otimes k}) \star \mathcal{T}_{n-k+1}(V^{\otimes (n-k)} \otimes F).$$

This almost reproduces the formula from [DF01a], the only difference being the factor of $\frac{i^n}{\hbar^n}$, which we prefer to include into the definition of R_n. The convention we chose is natural in the context of the classical limit. In order to be able to take this limit at all, we need to show that the terms with negative powers in $\hbar$ vanish in the expression for the retarded Møller operator. Before we turn to this task, we need to take one more step.

Working with time-ordered products and $\star$-products becomes easier if we express the $\star$-product in the following form[1]

$$F \star G = m_{\mathcal{T}} \circ e^{-i\hbar D_{\mathrm{A}}}(F \otimes G),$$

where $m_{\mathcal{T}} \doteq m \circ e^{i\hbar D_{\mathrm{D}}}$ and $F, G \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$. In this way, we can rewrite $R_{\lambda V}(F)$ as

$$R_{\lambda V}(F) = m_{\mathcal{T}} \circ e^{-i\hbar D_{\mathrm{A}}} \left(\left(e_{\mathcal{T}}^{i\lambda V/\hbar}\right)^{\star -1} \otimes e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F\right),$$

[1] Thanks to Eli Hawkins for making this crucial observation. This is related to a new way of understanding the algebraic structures in pAQFT, which will be described in detail in the upcoming publication [HR16].

The next proposition shows that the Møller operator $R_{\lambda V}$ contains no negative powers in $\hbar$. The original proof is due to [DF01a] and was presented for the case of local functionals. Here we are working only with regular functionals, so we present a simplified version of the proof for this case (due to E. Hawkins, to appear in [HR16]).

Proposition 6.1 *Let $F, G, V \in \mathcal{F}_{\mathrm{reg}}$, then the expression $R_{\lambda V}(F)$ contains no negative powers of $\hbar$.*

Proof It is easier to work with the inverse quantum Møller operator. Note that

$$e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} R_{\lambda V}^{-1}(G) = e_{\mathcal{T}}^{i\lambda V/\hbar} \star G = m_{\mathcal{T}} \circ e^{-i\hbar D_{\mathrm{A}}} \left(e_{\mathcal{T}}^{i\lambda V/\hbar} \otimes G\right).$$

Using the fact that D_{A} is a derivation in its left argument, we can "pull out" the exponential on the right-hand side and write

$$e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} R_{\lambda V}^{-1}(G) = e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} m_{\mathcal{T}} \left\langle \sum_{k=0}^{\infty} \frac{(-1)^n}{n!} P_n(V) \overset{\otimes}{,} (\Delta_{S_0}^{\mathrm{A}})^{\otimes n} G^{(n)} \right\rangle,$$

where the pairing above is the pairing of smooth compactly supported functions on M^k taking values in $\mathcal{F}_{\mathrm{reg}}$ with distributions on M^k taking values in $\mathcal{F}_{\mathrm{reg}}$ where the product on functionals is the tensor product $\otimes$, so the result of the pairing is a functional in two variables. After applying the multiplication operator $m_{\mathcal{T}}$ we arrive at a functional in one variable. We spell out $P_n(V)$ explicitly as $P_0(V) = 1$ and

$$P_n(V) = \sum_{P \in \mathcal{P}_n} i^{n+|P|} \hbar^{n-|P|} \lambda^{|P|} \prod_{I \in P}^{\cdot_{\mathcal{T}}} V^{(|I|)}, \quad n > 0,$$

so it is now clear that $P_n(V)$ does not contain negative powers in $\hbar$. The final formula for the inverse quantum Møller operator can be written as

$$R_{\lambda V}^{-1}(G) = m_{\mathcal{T}} \left\langle \sum_{k=0}^{\infty} \frac{(-1)^n}{n!} P_n(V) \overset{\otimes}{,} (\Delta_{S_0}^{\mathrm{A}})^{\otimes n} G^{(n)} \right\rangle. \tag{6.15}$$

Since $R_{\lambda V}^{-1}$ starts with identity and is well defined as a map on formal power series in λ and $\hbar$, so is $R_{\lambda V}$. □

From the proposition above it follows that $R_{\lambda V}$ is a well defined map on $\mathcal{F}_{\mathrm{reg}}[[\hbar, \lambda]]$. We introduce the interacting star product on $\mathcal{F}_{\mathrm{reg}}[[\hbar, \lambda]]$ by

$$F \star_V G \doteq R_V^{-1}(R_V F \star R_V G)$$

and the $\star_V$-commutator by

$$[F, G]_{\star_V} = F \star_V G - G \star_V F.$$

Let us now discuss the classical limit.

Proposition 6.2 *The 0th order in $\hbar$ part of the quantum Møller operator $R_{\lambda V}$ agrees with the classical Møller operator, i.e.*

$$R_{\lambda V}(F)\big|_{\hbar=0} = r_{\lambda V}(F).$$

Proof As in the previous proposition, it is easier to work with the inverse Møller operator. Taking the $\hbar = 0$ term in formula (6.15) results in

$$\tilde{r}_{\lambda V}^{-1}(G) \doteq R_{\lambda V}^{-1}(G)\big|_{\hbar=0} = m\left\langle \sum_{k=0}^{\infty} \frac{1}{n!}\left(\lambda V^{(1)}\Delta_{S_0}^{\mathrm{A}}\right)^n \overset{\otimes}{,} G^{(n)} \right\rangle.$$

We will now show that the operator $\tilde{r}_{\lambda V}^{-1}$ is equal to $r_{\lambda V}^{-1}$ introduced in Sect. 4.6. First note that $\tilde{r}_{\lambda V}^{-1}$ is in fact a pullback of the operator that acts on $\mathcal{E}[[\lambda]]$ as

$$\tilde{r}_{\lambda V}^{-1}(\varphi) = \varphi + \lambda\Delta_{S_0}^{\mathrm{R}} V^{(1)}(\varphi) \tag{6.16}$$

Clearly, if φ_λ is a formal solution for the interacting field equation $P(\varphi_\lambda) + \lambda V^{(1)}(\varphi_\lambda) = 0$, then $\tilde{r}_{\lambda V}^{-1}(\varphi_\lambda)$ is a solution to the free field equation. The equation for the inverse of $\tilde{r}_{\lambda V}^{-1}$ can be obtained by setting $\varphi_\lambda = \tilde{r}_{\lambda V}(\varphi)$. We arrive at

$$\tilde{r}_{\lambda V}(\varphi) = \varphi - \lambda\Delta_{S_0}^{\mathrm{R}} V^{(1)}(\tilde{r}_{\lambda V}(\varphi)),$$

which is the *Yang–Feldmann equation*. Differentiating with respect to λ results in

$$\frac{d}{d\lambda}\tilde{r}_{\lambda V}(\varphi) = -\left(\Delta_{S_0}^{\mathrm{R}} V^{(1)}(\tilde{r}_{\lambda V}(\varphi)) + \lambda\Delta_{S_0}^{\mathrm{R}} \circ V^{(2)}(\tilde{r}_{\lambda V}(\varphi))\frac{d}{d\lambda}\tilde{r}_{\lambda V}(\varphi)\right).$$

Now we bring the derivative of $\tilde{r}_{\lambda V}(\varphi)$ to the left-hand side.

$$\left(\mathrm{id} + \Delta_{S_0}^{\mathrm{R}} \circ V^{(2)}(\tilde{r}_{\lambda V}(\varphi))\right)\frac{d}{d\lambda}\tilde{r}_{\lambda V}(\varphi) = -\Delta_{S_0}^{\mathrm{R}} V^{(1)}(\tilde{r}_{\lambda V}(\varphi))$$

Applying P (the differential operator induced by S_0'') to both sides we obtain

$$\left(P + V^{(2)}(\tilde{r}_{\lambda V}(\varphi))\right)\frac{d}{d\lambda}\tilde{r}_{\lambda V}(\varphi) = -V^{(1)}(\tilde{r}_{\lambda V}(\varphi))$$

Now apply the inverse of $P + V^{(2)}$.

$$\frac{d}{d\lambda}\tilde{r}_{\lambda V}(\varphi) = -\Delta_{S_0+\lambda V}^{R} V^{(1)}(\tilde{r}_{\lambda V}(\varphi))$$

On the level of functionals this coincides with the relation (4.32), so $\tilde{r}_{\lambda V}$ satisfies the same recursion relation as $r_{\lambda V}$. Clearly, the 0th and the 1st orders of both maps coincide, hence $\tilde{r}_{\lambda V} = r_{\lambda V}$. □

It is now easy to see that the theory defined by $\star_V$ is indeed a quantization of the classical theory defined by the Poisson bracket $\lfloor .,. \rfloor_{S_0+\lambda V}$ given in Definition 4.8.

Proposition 6.3 *Let* $F, G, V \in \mathfrak{F}_{\text{reg}}$*, then*

$$\frac{1}{i\hbar}[F, G]_{\star_V}\Big|_{\hbar=0} = \lfloor F, G \rfloor_{S_0+V}.$$

Proof This is a straightforward consequence of Propositions 4.11 and 6.1. □

The interacting theory for regular functionals is the algebra $(\mathfrak{F}_{\text{reg}}[[\hbar, \lambda]], \star_V)$ and R_V acts as the intertwining map between the free quantum theory and the interacting quantum theory, i.e.

$$\underset{\text{classical}}{(\mathfrak{F}_{\text{reg}}, \cdot)} \xrightarrow{\mathcal{T}} \underset{\substack{\text{free}\\ \text{quantum}}}{(\mathfrak{A}_{\text{reg}}, \cdot_{\mathcal{T}})} \xrightarrow{R_V^{-1}} \underset{\substack{\text{interacting}\\ \text{quantum}}}{(\mathfrak{F}_{\text{reg}}[[\hbar, \lambda]], \star_V)}. \tag{6.17}$$

The formula (6.9) for the time-ordered product makes sense if we restrict ourselves to regular functionals. This is, however, not enough, since typical interactions appearing in particle physics are local and non-linear (hence not regular). In the first attempt to fix the problem we pass to a different star product, which amounts to replacing $\frac{i}{2}\Delta_{S_0}$ by $\Delta^+_{S_0}$ and $i\Delta^{\text{D}}_{S_0}$ by the Feynman propagator $\Delta^{\text{F}}_{S_0} = i\Delta^{\text{D}}_{S_0} + H$, so we obtain now

$$\underset{\text{classical}}{(\mathfrak{F}_{\text{reg}}, \cdot)} \xrightarrow{\mathcal{T}^H} (\mathfrak{A}^H_{\text{reg}}, \cdot_{\mathcal{T}^H}) \xrightarrow{\alpha_H^{-1}} \underset{\substack{\text{free}\\ \text{quantum}}}{(\mathfrak{A}_{\text{reg}}, \cdot_{\mathcal{T}})} \xrightarrow{R_V^{-1}} \underset{\substack{\text{interacting}\\ \text{quantum}}}{(\mathfrak{F}_{\text{reg}}[[\hbar, \lambda]], \star_V)},$$

where $\mathcal{T}^H \doteq e^{\frac{\hbar}{2}\mathcal{D}_{\text{F}}}$, and $\mathcal{D}_{\text{F}} \doteq \langle \Delta^{\text{F}}_{S_0}, \frac{\delta^2}{\delta\varphi^2} \rangle$, so $\mathcal{T} = \alpha_H^{-1} \circ \mathcal{T}^H$.

Remark 6.3 Note that here we use a convention where the factor of i is absorbed into the definition of the Feynman propagator. This means that $\Delta^{\text{F}}_{S_0}$ is i *times a Green's function* for the differential operator P induced by S_0''.

Unfortunately the modification we have made is not sufficient to extend the time-ordered products to arbitrary local functionals. The difficulty that we have to face is that the WF set of $\Delta^{\text{F}}_{S_0}$ on the diagonal behaves like the WF set of the Dirac delta (i.e. contains all non-zero covectors), hence the tensor powers of $\Delta^{\text{F}}_{S_0}$ cannot be contracted with derivatives of local functionals.

Example 6.2 Consider the example of Minkowski spacetime. Let $F_f(\varphi) = \int \varphi(x) f(x) d^4x$ be a smeared field and we want to find an explicit formula for $F_f \cdot_{\mathcal{T}} F_g$ for $f, g \in \mathscr{D}(\mathbb{M})$, starting from first principles. By the definition of time-ordering we have

$$F_f \cdot_{\mathcal{T}} F_g = \int \theta(\tau)\Delta^+_{S_0}(x,y) f(x) g(y) d^4x\, d^4y$$
$$+ \int \theta(-\tau)\Delta^+_{S_0}(y,x) f(x) g(y) d^4x\, d^4y,$$

where $\tau = x^0 - y^0$. Using formula (5.6) for the 2-point function we get

$$\begin{aligned}
&\theta(\tau)\Delta^+_{S_0}(x,y) + \theta(-\tau)\Delta^+_{S_0}(y,x) \\
&= \frac{1}{(2\pi)^3}\int \left(\theta(\tau)e^{-i\omega_{\boldsymbol{p}}\tau + i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} + \theta(-\tau)e^{i\omega_{\boldsymbol{p}}\tau - i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})}\right)\frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} \\
&= \frac{1}{(2\pi)^3}\int \left(\theta(\tau)e^{-i\omega_{\boldsymbol{p}}\tau} + \theta(-\tau)e^{i\omega_{\boldsymbol{p}}\tau}\right) e^{i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})}\frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}.
\end{aligned}$$

We can now use the known trick to rewrite the last integral as

$$\frac{1}{(2\pi)^4}\int \frac{i}{p^2 - m^2 + i\epsilon} e^{ip\cdot(x-y)} d^4p.$$

Comparing with $F_f \cdot_{\mathcal{T}} F_g = \langle f, \Delta^{\mathrm{F}}_{S_0} g\rangle$, we conclude that on Minkowski spacetime

$$\Delta^{\mathrm{F}}_{S_0}(x,y) = \frac{1}{(2\pi)^4}\int \frac{i}{p^2 - m^2 + i\epsilon} e^{ip\cdot(x-y)} d^4p,$$

as expected. For a final consistency check, we note that $\Delta^{\mathrm{F}}_{S_0}$ given by the formula above is indeed i times a Green's function for $P = -(\Box + m^2)$, i.e.

$$-(\Box_x + m^2)\Delta^{\mathrm{F}}_{S_0}(x,y) = i\delta(x-y).$$

Remark 6.4 On the Minkowski spacetime, the operator $\mathcal{T}^H$ formally corresponds to convolution with a Gaussian measure with covariance $i\hbar\Delta^{\mathrm{F}}_{S_0}$, i.e.

$$\mathcal{T}^H F(\varphi) \overset{\text{formal}}{=} \int F(\varphi - \phi) d\mu_{i\hbar\Delta^{\mathrm{F}}_{S_0}}(\phi). \tag{6.18}$$

The path integral above is not well defined as an integral, but the differential operator $\mathcal{T}^H$ that we use instead is much better behaved. Therefore, one can think of the functional formalism we present in this book as a way to make path integrals and other formulas used in perturbative QFT rigorous.

6.2.5 Epstein–Glaser Axioms

In the next step we want to extend time-ordered products defined in Sect. 6.2.3 to local non-linear functionals. This corresponds to what the physics literature calls the *renormalization problem*. Its mathematically rigorous solution has been proposed by Epstein and Glaser in the seminal paper [EG73] and it makes use of the causal structure of spacetime. Here we follow a similar approach.

Note that the quantization operator $\mathcal{T}^H$ acts on $\mathcal{F}_{\text{loc}}$ by $\mathcal{T}^H = \alpha_w$, where α_w is defined by (6.5). On Minkowski spacetime we can alternatively set $\mathcal{T}^H = \text{id}$. The subspace $\mathcal{F}_{\text{loc}}[[\hbar]] \subset \mathfrak{A}^H \equiv (\mathcal{F}_{\mu c}[[\hbar]], \star_H)$ is denoted by $\mathfrak{A}^H_{\text{loc}}$.

We define $\mathcal{F}_{\text{loc}}$ as the image of $\mathfrak{A}^H_{\text{loc}}$ under α_H^{-1}. The definition makes sense, since this subspace is the same vector space for all the choices of H.

Definition 6.4 Denote by $(\mathcal{F}_{\text{loc}})^{\otimes n}_{\text{pds}}$ the subspace of $(\mathcal{F}_{\text{loc}})^{\otimes n}$ spanned by $F_1 \otimes \cdots \otimes F_n$, where $F_1, \ldots, F_n \in \mathcal{F}_{\text{loc}}$ have pairwise disjoint supports.

Definition 6.5 Let $n \in \mathbb{N}$, $n > 1$. On $(\mathfrak{A}^H_{\text{loc}})^{\otimes n}_{\text{pds}}$ we define the n-fold time-ordered product as a map $\mathcal{T}^H_n : (\mathfrak{A}^H_{\text{loc}})^{\otimes n}_{\text{pds}} \to \mathfrak{A}^H$ given by

$$\mathcal{T}^H_n(F_1, \ldots, F_n) \doteq F_1 \cdot_{\mathcal{T}_H} \ldots \cdot_{\mathcal{T}_H} F_n .$$

Let $F_1 \otimes \cdots \otimes F_n \in (\mathfrak{A}^H_{\text{loc}})^{\otimes n}_{\text{pds}}$ such that the supports $\operatorname{supp} F_i$, $i = 1, \ldots, k$ of the first k entries do not intersect the past of the supports $\operatorname{supp} F_j$, $j = k+1, \ldots, n$ of the last $n-k$ entries. If follows from the definition of the time-ordered product that in this case

$$\mathcal{T}^H_n(F_1 \otimes \cdots \otimes F_n) = \mathcal{T}^H_k(F_1 \otimes \cdots \otimes F_k) \star_H \mathcal{T}^H_{n-k}(F_{k+1} \otimes \cdots \otimes F_n), \tag{6.19}$$

This is called the *causal factorisation property*. It is a crucial feature which we want to require also from the extended n-fold time-ordered products. This motivates the following, axiomatic definition.

Definition 6.6 Renormalized time-ordered products are multilinear maps $\mathcal{T}^H_n : \mathcal{F}^{\otimes n}_{\text{loc}} \to \mathfrak{A}^H = (\mathcal{F}_{\mu c}[[\hbar]], \star_H)$, $n \in \mathbb{N}$, satisfying:

(T 1) Causal factorisation property

$$\mathcal{T}^H_n(F_1 \otimes \cdots \otimes F_n) = \mathcal{T}^H_k(F_1 \otimes \cdots \otimes F_k) \star_H \mathcal{T}^H_{n-k}(F_{k+1} \otimes \cdots \otimes F_n)$$

if the supports $\operatorname{supp} F_i$, $i = 1, \ldots, k$ of the first k entries do not intersect the past of the supports $\operatorname{supp} F_j$, $j = k+1, \ldots, n$ of the last $n-k$ entries.

(T 2) Starting element: $\mathcal{T}^H_0 = 1$, $\mathcal{T}^H_1 = \text{id}$.[2]

(T 3) Symmetry: For a purely bosonic theory $\mathcal{T}^H_n$ is symmetric in its arguments. If fermions are present, $\mathcal{T}^H_n$ is graded-symmetric.

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[2] One can leave some freedom in the definition of $\mathcal{T}^H_1$ which can then be used to absorb the renormalization ambiguity in defining the normal ordering $:.:_{\mathcal{M}}$.

(T 4) **Field independence**: $\mathcal{T}^H_n(F_1, \ldots, F_n)$, as a functional on $\mathcal{E}$, depends on φ only via the functional derivatives of $F_1, \ldots, F_n$, i.e.

$$\frac{\delta}{\delta\varphi}\mathcal{T}^H_n(F_1, \ldots, F_n) = \sum_{i=1}^{n} \mathcal{T}^H_n\left(F_1, \ldots, \frac{\delta F_i}{\delta\varphi}, \ldots, F_n\right)$$

(T 5) **φ-Locality**: $\mathcal{T}^H_n(F_1, \ldots, F_n) = \mathcal{T}^H_n(F_1^{[N]}, \ldots, F_n^{[N]}) + \mathcal{O}(\hbar^{N+1})$, where $F_i^{[N]}$ is the Taylor series expansion of the functional F_i up to the Nth order.

Remark 6.5 Note that the property **(T 5)** allows one to reduce the problem of constructing the time-ordered products of local functionals to the construction of the time-ordered products of polynomials. Property **(T 4)** is crucial to show that the construction reduces to extensions of numerical distributions.

Given the family of maps $\{\mathcal{T}^H_n\}_{n\in\mathbb{N}}$, we can define time-ordered products of Wick-ordered quantities and then finally the S-matrix.

Definition 6.7 We define $\mathcal{T}_n : \mathfrak{A}_{\mathrm{loc}}^{\otimes n} \to \mathfrak{A}$ by requiring that $\alpha_H \circ \mathcal{T}_n = \mathcal{T}^H_n \circ \alpha_H^{\otimes n}$, so in a slightly formal way we write

$$\mathcal{T}_n \doteq \alpha_H^{-1} \circ \mathcal{T}^H_n \circ \alpha_H^{\otimes n},$$

and the renormalized S-matrix is a map $\mathcal{S} : \mathfrak{A}_{\mathrm{loc}}[[\lambda]] \to \mathfrak{A}((\hbar))[[\lambda]]$, given by

$$\mathcal{S}(F) \doteq \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!}\mathcal{T}_n(F^{\otimes n}),$$

where F is of order at least λ.

Analogously to the regular case, we consider $F = \lambda V$, where $V \in \mathfrak{A}_{\mathrm{loc}}$. We want the S-matrix defined this way to be unitary, i.e. we require that

$$\mathcal{S}(\lambda V^*)^* \star \mathcal{S}(\lambda V) = \mathbb{1},$$

This condition can be translated into an additional axiom for the time-ordered products:

(T 6) **Unitarity**: $\mathcal{T}^H_n(F_1^*, \ldots, F_n^*)^* = \bar{\mathcal{T}}^H_n(F_1, \ldots, F_n)$, where $\bar{\mathcal{T}}^H_n$ is the antichronological product defined in (6.14).

Remark 6.6 Let V be a real-valued local functional. The unitarity property implies that the renormalized quantum Møller operator satisfies the following relation:

$$\begin{aligned} R_{\lambda V}(F)^* &= -i\hbar \frac{d}{dt}\left(\mathcal{S}(\lambda V)^{-1}\star \mathcal{S}(\lambda V + tF)\right)^*\Big|_{t=0} \\ &= -i\hbar\left(\frac{d}{dt}\left(\mathcal{S}(\lambda V + tF^*)\right)^{-1}\Big|_{t=0}\star \mathcal{S}(\lambda V)\right) \\ &= i\hbar\, \mathcal{S}(\lambda V)^{-1}\star \frac{d}{dt}\left(\mathcal{S}(\lambda V + tF^*)\right)\Big|_{t=0} = R_{\lambda V}(F^*). \end{aligned}$$

The final consideration is to ensure that the time-ordered products are covariant, i.e. they are defined on all spacetimes in a coherent way. Denote by $\mathcal{T}_{n\mathcal{M}}$ the n-fold time-ordered product on spactime $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$.

Definition 6.8 Let $\mathcal{M}, \mathcal{N} \in \mathrm{Obj}(\mathbf{Loc})$ and $\chi : \mathcal{M} \to \mathcal{N}$ be an admissible embedding. We say that the family $(\mathcal{T}_{n\mathcal{M}})_{\mathcal{M}\in\mathrm{Obj}(\mathbf{Loc})}$ defines a covariant n-fold time-ordered product if

$$\mathfrak{A}_{\mu c}\chi \circ \mathcal{T}_{n\mathcal{M}}(F_1, \ldots, F_n) = \mathcal{T}_{n\mathcal{N}}(\mathfrak{A}_{\mathrm{loc}}\chi(F_1), \ldots, \mathfrak{A}_{\mathrm{loc}}\chi(F_n)),$$

where $F_1, \ldots, F_n \in \mathfrak{A}_{\mathrm{loc}}(\mathcal{M})$.

The following is an obvious consequence of our notion of covariance and shows the relation with covariant Wick products.

Proposition 6.4 *Let $\mathcal{T}_{n\mathcal{M}}$ be a covariant n-fold time-ordered product on spactime $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$. Given locally covariant quantum fields*

$$:\Phi_i:_{\mathcal{M}} \in \mathrm{Nat}(\mathfrak{D}, \mathfrak{A}_{\mathrm{loc}}),\ i = 1, \ldots n,$$

the family

$$\mathcal{T}_{n\mathcal{M}} \circ (:\Phi_1:_{\mathcal{M}}, \ldots :\Phi_n:_{\mathcal{M}})$$

defines a natural transformation in $\mathrm{Nat}(\mathfrak{D}^n, \mathfrak{A})$.

Finding the right notion of covariance for time-ordered products was a very important step in understanding quantum field theory on curved spacetime. The idea was developed by Brunetti, Fredenhagen, Hollands, Verch and Wald [HW01, Ver01, BFV03]. The requirement of covariance can be translated into another axiom for $\mathcal{T}_n^H$'s.

(T 7) Covariance: $\mathcal{T}_n \doteq \alpha_H^{-1} \circ \mathcal{T}_n^H \circ \alpha_H^{\otimes n}$ is covariant in the sense of Definition 6.8.

More explicitely the **Covariance** condition can be written as

$$\mathfrak{F}_{\mu c}\psi \circ \mathcal{T}_n^{\psi^* H} = \mathcal{T}_n^H \circ (\mathfrak{F}_{\mathrm{loc}}\psi)^{\otimes n}\,,$$

where $\psi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$, H is a distribution on $N \times N$ and $\psi^* H$ denotes the pullback of its restriction to $\psi(M) \times \psi(M)$, by ψ.

Using the inductive method of [EG73], generalized to curved spacetimes by [BF00, HW01, HW02a], one shows that a family of maps satisfying axioms **(T 1)–(T 7)** exists and the non-uniqueness of the construction is fully absorbed into adding multilinear maps $\mathcal{Z}^H_n : \mathcal{F}^{\otimes n}_{\text{loc}} \to \mathcal{F}_{\text{loc}}[[\hbar]]$, i.e.

$$\widetilde{\mathcal{T}^H}_n(F_1, \dots, F_n) = \mathcal{T}^H_n(F_1, \dots, F_n) + \mathcal{Z}^H_n(F_1, \dots, F_n), \tag{6.20}$$

where $\{\mathcal{T}^H_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{\mathcal{T}^H_n}\}_{n\in\mathbb{N}}$ are two choices of families of n-fold time-ordered products that coincide up to order $n-1$. A detailed argument for the existence of time-ordered products is given in Sect. 6.5.

The causal factorisation property for time-ordered products implies that the S-matrix satisfies Bogoliubov's factorization relation

$$\mathcal{S}(F_1 + F_2 + F_3) = \mathcal{S}(F_1 + F_2) \star \mathcal{S}(F_2)^{-1} \star \mathcal{S}(F_2 + F_3) \tag{6.21}$$

if the support of F_1 does not intersect the past of the support of F_3.

Definition 6.9 We define the renormalized map $\mathcal{T} : \mathcal{F} \to \mathfrak{A}$ by $\mathcal{T} \doteq \bigoplus_n \alpha_H^{-1} \circ \mathcal{T}^H_n \circ \alpha_w \circ m^{-1}$, where $m^{-1} : \mathcal{F} \to S^\bullet \mathcal{F}^{(0)}_{\text{loc}}$ is the inverse of the multiplication, as defined in [FR12a] and $\mathcal{F}^{(0)}_{\text{loc}}$ is the space of local functionals that vanish at 0.

Using $\mathcal{T}$ we conclude that the renormalized time-ordered product $\cdot_\mathcal{T}$ is a binary operation defined on the domain $D_\mathcal{T} \doteq \mathcal{T}(\mathcal{F}) \subset \mathfrak{A}$.

6.3 Renormalization Group

In this section we discuss in detail the ambiguity arising in defining $\mathcal{T}_n$'s. In physics this is known as the *renormalization ambiguity* and is related to the notion of the *renormalization group*. First note that the ambiguities $\mathcal{Z}^H_n$ appearing in formula (6.20) give rise to maps $\mathcal{Z}_n : \mathfrak{A}^{\otimes n}_{\text{loc}} \to \mathfrak{A}_{\text{loc}}$ defined by

$$\mathcal{Z}_n \doteq \alpha_H^{-1} \circ \mathcal{Z}^H_n \circ \alpha_H^{\otimes n}.$$

Next we define a map $\mathcal{Z} : \mathfrak{A}_{\text{loc}} \to \mathfrak{A}_{\text{loc}}$ by summing up all the $\mathcal{Z}_n$'s relating two chosen prescriptions to define the time-ordered products, i.e.

$$\mathcal{Z}(V) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{Z}_n(V^{\otimes n}), \quad V \in \mathfrak{A}_{\text{loc}}.$$

For any two choices of the families $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$, the corresponding map $\mathcal{Z}$, which relates them, has the following properties:

(Z 1) $\mathcal{Z}(0) = 0$,

(Z 2) $\mathcal{Z}^{(1)}(0) = \mathrm{id}$,

(Z 3) $\mathcal{Z} = \mathrm{id} + \mathcal{O}(\hbar)$,

(Z 4) $\mathcal{Z}(F + G + H) = \mathcal{Z}(F + G) + \mathcal{Z}(G + H) - \mathcal{Z}(G)$, if supp $F \cap$ supp G,

(Z 5) $\frac{\delta \mathcal{Z}}{\delta \varphi} = 0$.

The group of formal diffeomorphisms of $\mathfrak{A}_{\mathrm{loc}}$ that fulfill **(Z 1)**–**(Z 5)** is called the *Stückelberg–Petermann renormalization group* $\mathcal{R}$. The relation between the formal S-matrices and the elements of $\mathcal{R}$ is provided by the *main theorem of renormalization*. This theorem, originally formulated in an unpublished preprint by Stora and Popineau [PS82], was later generalized and improved, in particular by Pinter [Pin01]. Its final version, which relies heavily on a proof of Stora's "Action Ward Identity" [DF04, DF07], was obtained in [HW02b, DF04] and was then further analyzed in [BDF09].

Theorem 6.1 *Let* $\mathcal{S}$ *and* $\hat{\mathcal{S}}$ *be two formal S-matrices, built from time ordered products satisfying the axioms* **(T 1)**–**(T 7)**. *Then there exists* $\mathcal{Z} \in \mathcal{R}$ *such that*

$$\hat{\mathcal{S}} = \mathcal{S} \circ \mathcal{Z}, \tag{6.22}$$

where $\mathcal{Z} \in \mathcal{R}$. *Conversely, if* $\mathcal{S}$ *is an S-matrix satisfying the axioms* **(T 1)**–**(T 7)** *and* $\mathcal{Z} \in \mathcal{R}$ *then* $\hat{\mathcal{S}}$ *defined by* (6.22) *also fulfils the axioms.*

Proof For proof see [DF04, BDF09]. □

The abstract notion of the renormalization group can be made more concrete when we consider a specific model. This is done by fixing a natural Lagrangian (see Definition 4.6) $L_I \in \mathrm{Nat}(\mathfrak{D}, \mathfrak{F}_{\mathrm{loc}})$, which defines the interaction term of the theory. Since we defined the Wick-ordering in a covariant way (in the sense of Definition 6.1), the normally-ordered quantity $:L_I:$ is a natural transformation in $\mathrm{Nat}(\mathfrak{D}, \mathfrak{A}_{\mathrm{loc}})$. For a fixed spacetime $\mathcal{M}$ the map $:L_I:_{\mathcal{M}} : \mathscr{D} \to \mathfrak{A}_{\mathrm{loc}}$ satisfies **additivity**, the **support property** and **covariance** from Definition 4.1. This motivates the following definition.

Definition 6.10 A quantum Lagrangian on a fixed spacetime $\mathcal{M} = (M, g)$ is a map $L : \mathscr{D}(M) \to \mathfrak{A}_{\mathrm{loc}}$ such that

(i) $L(f + g + h) = L(f + g) - L(g) + L(g + h)$ for $f, g, h \in \mathscr{D}$ with supp $f \cap$ supp $h = \varnothing$ (**Additivity**).

(ii) $\mathrm{supp}(L(f)) \subseteq \mathrm{supp}(f)$ (**Support**).

(iii) Let $\mathcal{G}$ be the isometry group of the spacetime $\mathcal{M}$ (for the Minkowski spacetime we set $\mathcal{G}$ to be the proper orthochronous Poincaré group $\mathcal{P}_+^{\uparrow}$.). We require that $L(f)(\beta^* \varphi) = L(\beta_* f)(\varphi)$ for every $\beta \in \mathcal{G}$ (**Covariance**).

In a given pQFT model we are interested in computing expectation values of such quantum Lagrangians L_i, $i = 1, \ldots, N$ in a given state ω. The following result is crucial for our framework.

Theorem 6.2 (Proposition 6.2 of [BDF09]) *The space $\mathscr{L}$ of quantum Lagrangians is invariant under the renormalization group $\mathcal{R}$.*

Proof For the details of the proof see [BDF09]. Note that the additivity property for the transformed Lagrangians is a direct consequence of the property (**Z 4**) of the renormalization group. □

Remark 6.7 It is convenient to choose the state ω as a quasi-free state of the form $\omega_{H,0}$. On Minkowski spacetime there is a distinguished choice of H as Δ_1, so that $\Delta^+_{S_0} = \frac{i}{2}\Delta_{S_0} + \Delta_1$ is the Wightman 2-point function, as in Example 5.2. In this case $\omega_{\Delta_1,0}$ is just the usual *vacuum expectation value*, which can be compared with the standard physics literature.

We are now ready to give an abstract definition of a *theory*.

Definition 6.11 A theory in pQFT on a spacetime $\mathcal{M}$ is defined by fixing the space $\mathcal{E}$ of field configurations and a complex vector space $\mathbf{\Phi}$ of quantum Lagrangians that is closed under the action of the renormalization group $\mathcal{R}$.

From the point of view of physical interpretation, it is more appropriate to talk about *actions* than Lagrangians. Classically, we defined actions as equivalence classes under the equivalence relation (4.2) (see Definition 4.3). In the quantum theory the test function appearing in the Lagrangian has the interpretation of a point-dependent coupling constant. For example in the φ^4 theory the interaction Lagrangian takes the form

$$L(f) = \frac{1}{4}\int \lambda f(x)\varphi^4(x) d\mu_g(x),$$

so $\lambda(x) \equiv \lambda f(x)$ is seen as a point-dependent coupling constant.

The limit where $f \to 1$ is called the *adiabatic limit*. In the next section we will see that in pAQFT it is replaced by a weaker notion, the *algebraic adiabatic limit*. The presence of Lagrangians that are non-linear in the coupling constant is necessary for example in non-abelian gauge theories, where the coupling constant enters in different terms with different powers, in order to maintain the gauge symmetry. The necessity of non-linear dependence on the test function was stressed in [BDF09], where the *additivity* (in this book, property (i) in Definition 4.1) has been proposed as a weaker version of linearity.

The interpretation presented above motivates the following definition.

Definition 6.12

$$L_1 \overset{q}{\sim} L_2, \quad \text{iff } \operatorname{supp}(L_1 - L_2)(f) \subset (f^{-1}(1))^c, \tag{6.23}$$

where $L_1, L_2 \in \mathscr{L}$ and the superscript "c" denotes the complement of a set.

In other words, we identify Lagrangians that would coincide in the limit where the test function goes to 1.

Definition 6.13 We define the set of actions $\widetilde{\mathscr{L}}$ as $\mathscr{L}/\overset{q}{\sim}$, i.e. the set of quantum Lagrangians $\mathscr{L}$ modulo the equivalence relation (6.23).

We are now ready to formulate the definition of power-counting renormalizability in our framework.

Definition 6.14 A theory defined by $(\mathcal{E}, \mathbf{\Phi})$ is called *power-counting renormalizable* if $\mathbf{\Phi}/\overset{q}{\sim}$ is finite-dimensional (as a vector space).

6.4 Interacting Local Nets

In this section we show how to construct a perturbative model of an interacting quantum theory, in the sense of Definition 2.54. We follow closely the construction proposed in [FR15]. Let $\boldsymbol{L} = (L_1, \dots, L_N)$ be an N-tuple of Lagrangians relevant for the physical theory we are interested in. Assume $L_1, \dots, L_N$ to vanish at $\varphi = 0$, i.e. $L_i(f)[0] = 0\ \forall f \in \mathscr{D}(M)$. We choose $L_1 = L_I$ to be the interaction term and the remaining Lagrangians are some chosen observables, for example conserved currents, charges, etc. For each Lagrangian we introduce a formal parameter λ_i, $i = 1, \dots, N$ and we denote the N-tuple of these parameters by $\boldsymbol{\lambda}$. Define $\mathscr{D}^N(M)$ to be the space of compactly supported functions on M with values in $\mathbb{R}^N$.

For a fixed $f = (f_1, \dots, f_N) \in \mathscr{D}^N(M)$ we construct the S-matrices corresponding to $L_i(\lambda_i f_i)$, $i = 1, \dots, N$ and their linear combinations, using Epstein–Glaser renormalization. This allows us to define a map from $\mathscr{D}^N(M)$ to $\mathfrak{A}$ by means of

$$f \mapsto \mathcal{S}(f) \doteq \mathcal{S}\left(\sum_{i=1}^{N} L_i(\lambda_i f_i)\right) \tag{6.24}$$

In this way we obtain a family of unitaries (see condition **(T 6)**) $\{\mathcal{S}(f) | f \in \mathscr{D}^N(M)\}$ with $\mathcal{S}(0) = 0$, which generate a *-subalgebra $\mathfrak{A}_{\boldsymbol{L}}$ of $\mathfrak{A}((\hbar))[[\boldsymbol{\lambda}]]$ and satisfy for $f, g, h \in \mathscr{D}$ Bogoliubov's factorization relation

$$\mathcal{S}(f + g + h) = \mathcal{S}(f + g) \star \mathcal{S}(g)^{-1} \star \mathcal{S}(g + h) \tag{6.25}$$

if the past J_- of $\operatorname{supp} h$ does not intersect $\operatorname{supp} f$ (or, equivalently, if the future J_+ of $\operatorname{supp} f$ does not intersect $\operatorname{supp} h$).

Proposition 6.5 *Assigning to bounded simply-connected regions $\mathcal{O} \subset \mathcal{M}$ subalgebras $\mathfrak{A}_{\boldsymbol{L}}(\mathcal{O}) \subset \mathfrak{A}_{\boldsymbol{L}}$ generated by $\{\mathcal{S}(f) | \operatorname{supp} f \subset \mathcal{O}\}$ and the unit defines a local, covariant net in the sense of Definition 2.54 (understood in terms of formal power series in $\boldsymbol{\lambda}$ valued in Laurent series in $\hbar$).*

Proof The isotony condition, needed for the assignment $\mathcal{O} \mapsto \mathfrak{A}_{\boldsymbol{L}}(\mathcal{O})$ to be a net, is straightforward to verify. All the structures used in constructing this net are covariant, so $\mathcal{O} \mapsto \mathfrak{A}_{\boldsymbol{L}}(\mathcal{O})$ is covariant in the sense of Definition 2.54. The **Locality** axiom follows from the fact that for functions f, g with spacelike separated supports

$$\operatorname{supp} f \cap J_\pm(\operatorname{supp} g) = \varnothing \tag{6.26}$$

and hence

$$\mathcal{S}(f) \star \mathcal{S}(g) = \mathcal{S}(f+g) = \mathcal{S}(g) \star \mathcal{S}(f). \tag{6.27}$$

□

The net $\{\mathfrak{A}_{\boldsymbol{L}}(\mathcal{O})\}$ is not yet the interacting net we are looking for. However, it turns out that there is another way to assign subalgebras of $\mathfrak{A}_{\boldsymbol{L}}$ to bounded regions of spacetime, which takes the interaction into account. Let us come back to the heuristic Bogoliubov formula (6.2) and its mathematically rigorous version (6.12). Note that the formal Møller operator R_V, which relates the free and interacting theory, is written as the derivative of the *relative S-matrix* $\mathcal{S}(V)^{\star -1} \star \mathcal{S}(V+F)$. Here $V \equiv L_1(\lambda_1 g_1) = L_I(g_1)$ for a test function $g_1 \in \mathscr{D}(M)$. We choose $F = L_i(\lambda_i f_i)$, for some $i \in \{2, \dots, N\}$ and $f_i \in \mathscr{D}(M)$. This way we expressed the relative S-matrix using the elements of $\mathfrak{A}_{\boldsymbol{L}}$. We obtain a map

$$f \mapsto \mathcal{S}_g(f) \doteq \mathcal{S}(g)^{-1} \star \mathcal{S}(g+f) \tag{6.28}$$

where $f, g \in \mathscr{D}^N$ and $g = (g_1, 0, \dots, 0)$, $f = (0, f_2, \dots, f_N)$. $\mathcal{S}(g)$ as well as $\mathcal{S}(f+g)$ are defined by means of (6.24). We can now prove a crucial result concerning the relative S-matrices.

Proposition 6.6 (after [FR15]) *All the maps $\mathcal{S}_g$ labelled by different choices of g (or more precisely g^0), satisfy Bogoliubov's factorisation relation (6.25).*

Proof We follow the proof given in [FR15]. Let $f, h \in \mathscr{D}^N(M)$ such that $\operatorname{supp} f$ does not intersect $J_-(\operatorname{supp} h)$. Let $g, g' \in \mathscr{D}^N(M)$. Then

$$\begin{aligned}
\mathcal{S}_g(f+g'+h) &= \mathcal{S}(g)^{-1} \star \mathcal{S}(f+(g+g')+h) \\
&= \mathcal{S}(g)^{-1} \star \mathcal{S}(f+(g+g')) \star \mathcal{S}(g+g')^{-1} \star \mathcal{S}((g+g')+h) \\
&= \mathcal{S}_g(f+g') \star \mathcal{S}_g(g')^{-1} \star \underbrace{\mathcal{S}(g)^{-1} \star \mathcal{S}(g)}_{=1} \star \mathcal{S}_g(g'+h).
\end{aligned}$$

□

Example 6.3 For the case of $N=2$, $g=(g_1,0)$ and $f=(0,f_2)$ we have $S(g+f) = \mathcal{S}(L_I(\lambda_1 g_1) + L_2(\lambda_2 f_2))$. Hence the derivative of $S_g(f)$ with respect to λ_2, at $\lambda_2 = 0$ is just the retarded field

$$R_{L_I(g_1)}\left(\frac{d}{d\lambda_2} L_2(\lambda_2 f_2)\Big|_{\lambda_2=0}\right),$$

and if the quantum Lagrangian L_2 is linear in the test function, this reduces to

$$R_{L_I(g_1)}\left(L_2(f_2)\right).$$

We interpret $\mathcal{S}_g(f)$ as the generating function for retarded observable described by $\sum_{i=2}^N L_i(\lambda_i f_i)$, under the influence of the interaction $L_I(g_1)$. We are now ready to define the interacting net of observables.

Definition 6.15 The interacting quantum net $\mathfrak{A}_{\boldsymbol{L},g}$ corresponding to observables $\boldsymbol{L} = (L_1, \dots, L_N)$ is defined by assigning to bounded simply-connected regions $\mathcal{O}$ the local algebras $\mathfrak{A}_{\boldsymbol{L},g}(\mathcal{O}) \subset \mathfrak{A}((\hbar))[[\boldsymbol{\lambda}]]$ that are generated by the relative S-matrices $\mathcal{S}_g(f)$ with $\operatorname{supp} f \subset \mathcal{O}$, $g = (g_1, 0, \dots, 0)$ and $f = (0, f_2, \dots, f_N)$.

Note that the algebras $\mathfrak{A}_{\boldsymbol{L},g}$ are subalgebras of $\mathfrak{A}((\hbar))[[\boldsymbol{\lambda}]]$ but the net defined by Definition 6.15 differs from the net in Proposition 6.5. This is a very subtle point, since one could suspect that building interacting fields out of free fields should imply that the resulting theories are identical. Note, however, that in the algebraic approach physical information is contained not in the global algebra, but in the net structure, i.e. in the way we assign the elements of this algebra to bounded regions of spacetime. The next proposition shows that elements of $\mathfrak{A}_{\boldsymbol{L},g}(\mathcal{O})$ can be indeed interpreted as retarded observables, as indicated below Example 6.3.

Proposition 6.7 (after [FR15]) *The relative S-matrix has the following properties:*

1. $\mathcal{S}_g(f)$ *depends only on the behavior of* g *in the past of* $\operatorname{supp} f$.
2. $\mathcal{S}_g(f)$ *depends on the behavior of* g *outside of the future of* $\operatorname{supp} f$ *via a (formal) unitary transformation which does not depend on* f.

Proof (i) Let g and g' be such that $\operatorname{supp}(g - g') \cap J_-(\operatorname{supp} f) = \varnothing$, i.e. they only differ in regions outside the past of $\operatorname{supp} f$. This implies

$$\begin{aligned}\mathcal{S}_g(f) &= \mathcal{S}(g)^{-1} \star \mathcal{S}((g - g') + g' + f)\\ &= \mathcal{S}(g)^{-1} \star \mathcal{S}((g - g') + g') \star \mathcal{S}(g')^{-1} \star \mathcal{S}(g' + f) = \mathcal{S}_{g'}(f).\end{aligned}$$

(ii) For $\operatorname{supp}(g - g') \cap J_+(\operatorname{supp} f) = \varnothing$ we have

$$\begin{aligned}\mathcal{S}_g(f) &= \mathcal{S}(g)^{-1} \star \mathcal{S}(f + g' + (g - g'))\\ &= \mathcal{S}(g)^{-1} \star \mathcal{S}(f + g') \star \mathcal{S}(g')^{-1} \star \mathcal{S}(g' + (g - g'))\\ &= \mathcal{S}(g)^{-1} \star \mathcal{S}(g') \star \mathcal{S}(g')^{-1} \star \mathcal{S}(f + g') \star \mathcal{S}_{g'}(g - g')\\ &= \operatorname{Ad} \mathcal{S}_{g'}(g - g')^{-1}(\mathcal{S}_{g'}(f)),\end{aligned}$$

where we use the notation $\operatorname{Ad} B(A) \doteq B^{-1} \star A \star B$ for $A, B \in \mathfrak{A}((\hbar))[[\boldsymbol{\lambda}]]$. □

Remark 6.8 We can think of $\mathfrak{A}_{\boldsymbol{L},g}(\mathcal{O})$ as the algebra of generating functions for retarded fields localized in $\mathcal{O}$. Note that, although elements of $\mathfrak{A}_{\boldsymbol{L},g}(\mathcal{O})$ contain negative powers of $\hbar$, their derivatives

$$\left(\frac{i}{\hbar}\right)^{-n} \frac{d^n}{d\lambda_{i_1} \dots d\lambda_{i_n}} \mathcal{S}_g(f)$$

are at least of order 0 in $\hbar$ (see the proof in [DF01a] and the discussion in Sect. 6.2.4). In the simple case of Example 6.2, with L_I and L_2 linear in test functions, we obtain the expansion

$$\mathcal{S}_g(f) = \sum_{n,m} \frac{(g_1\lambda_1)^n (f_2\lambda_2)^m}{n!m!} \frac{d^{n+m}}{d\lambda_1^n d\lambda_2^m} \mathcal{S}_g(f)\,.$$

Denote

$$\left(\frac{i}{\hbar}\right)^{-m} \frac{d^{n+m}}{d\lambda_1^n d\lambda_2^m} \mathcal{S}_g(f) \equiv R_{n,m}(f_1, g_2)\,.$$

This reproduces the expansion of the relative S-matrix into retarded products obtained in [DF01a]. In particular, we have

$$R_{n,1}(g_1, f_2) \equiv R_n\big(L_I(g_1)^{\otimes n}, L_2(f_2)\big)\,.$$

Proposition 6.7 shows that the structure of local algebras depends only locally on the interaction. This allows us to perform the adiabatic limit directly on the level of local algebras as a certain inductive limit (*the algebraic adiabatic limit*). In the first step we remove the restriction to interactions with compact support. Let $G \in \mathcal{C}^\infty(M, \mathbb{R}^N)$ and let $\mathcal{O}$ be a bounded simply connected region in M. We define

$$[G]_{\mathcal{O}} = \{g \in \mathscr{D}^N(M) | g \equiv G \text{ on a neighborhood of } J_+(\mathcal{O}) \cap J_-(\mathcal{O})\}.$$

Next we consider the $\mathfrak{A}_{\boldsymbol{L}}$-valued maps

$$\mathcal{S}_{G,\mathcal{O}}(f) : [G]_{\mathcal{O}} \ni g \mapsto \mathcal{S}_g(f) \in \mathfrak{A}_{\boldsymbol{L}}.$$

We are now ready to extend the Definition 6.15 to functions with arbitrary support.

Definition 6.16 The local net of algebras $\mathcal{O} \mapsto \mathfrak{A}_{\boldsymbol{L},G}(\mathcal{O})$ is defined by assigning to $\mathcal{O}$ the algebra generated by $\mathcal{S}_{G,\mathcal{O}}(f)$, where supp $f \subset \mathcal{O}$. For $\mathcal{O}_1 \subset \mathcal{O}_2$ the embedding $i_{\mathcal{O}_2\mathcal{O}_1}$ of the corresponding algebras is defined on generators by

$$i_{\mathcal{O}_2\mathcal{O}_1} : \mathcal{S}_{G,\mathcal{O}_1}(f) \mapsto \mathcal{S}_{G,\mathcal{O}_2}(f)$$

for $f \in \mathscr{D}^N(M)$ with supp $f \subset \mathcal{O}_1$, $G = (G_1, 0, \ldots, 0)$, $f = (0, f_2, \ldots, f_N)$.

Let $\mathfrak{A}_{\boldsymbol{L},G}$ be the inductive limit of algebras $\mathfrak{A}_{\boldsymbol{L},G}(\mathcal{O})$ constituting the net defined in Definition 6.16. Let us denote the corresponding canonical embedding into $\mathfrak{A}_{\boldsymbol{L},G}$ by

$$i_{\mathcal{O}} : \mathfrak{A}_{\boldsymbol{L},G}(\mathcal{O}) \to \mathfrak{A}_{\boldsymbol{L},G}$$

We set

$$\mathcal{S}_G(f) = i_{\mathcal{O}}(\mathcal{S}_{G,\mathcal{O}}(f)).$$

The following result shows that the net defined in Definition 6.16 is covariant in the sense of Definition 2.55.

Theorem 6.3 *Let*

$$\alpha_\beta^{G,\mathcal{O}}\left(\mathcal{S}_G(f)\right) \doteq \mathcal{S}_G(\beta^* f), \tag{6.29}$$

where $\mathcal{S}_G(f) \in \mathfrak{A}_{L,G}(\mathcal{O})$, $\beta \in \mathcal{G}$ *is an element of the isometry group of* $\mathcal{M}$ *and* $\beta^* f \doteq f \circ \beta$. *Then* $\alpha_\beta^{G,\mathcal{O}}$, $\beta \in \mathcal{G}$ *extends to a family of isomorphisms* $\alpha_\beta^{G,\mathcal{O}} : \mathfrak{A}_{L,G}(\mathcal{O}) \to \mathfrak{A}_{L,G}(\beta\mathcal{O})$ *satisfying the conditions of Definition 2.55.*

Before we prove the theorem, we need a lemma that allows us to reduce the problem to the problem that involves the net $\{\mathfrak{A}_{L,g}(\mathcal{O})\}$.

Lemma 6.1 *The evaluation maps*

$$\gamma_{g,G} : \mathcal{S}_{G,\mathcal{O}}(f) \to \mathcal{S}_g(f)$$

extend to isomorphisms of $\mathfrak{A}_{L,G}(\mathcal{O})$ *and* $\mathfrak{A}_{L,g}(\mathcal{O})$ *for every* $g \in [G]_\mathcal{O}$.

Proof The only thing that we need to check is that the map $\gamma_{g,G}$ is injective. Assume that $\gamma_{g,G}(\mathcal{S}_{G,\mathcal{O}}(f)) = \gamma_{g,G}(\mathcal{S}_{G,\mathcal{O}}(f'))$ for some f, f' with supports contained in $\mathcal{O}$. Then $\mathcal{S}_{g,\mathcal{O}}(f) = \mathcal{S}_{G,\mathcal{O}}(f')$ and using Proposition 6.7 we conclude that $\mathcal{S}_{g,\mathcal{O}}(f) = \mathcal{S}_{G,\mathcal{O}}(f')$ for every $g' \in [G]_\mathcal{O}$, so $\mathcal{S}_{G,\mathcal{O}}(f) = \mathcal{S}_{G,\mathcal{O}}(f')$. □

Proof of the theorem We have to prove that the map $\alpha_\beta^{G,\mathcal{O}}$ given by (6.29) extends to an isomorphism $\mathfrak{A}_{L,G}(\mathcal{O}) \to \mathfrak{A}_{L,G}(\beta\mathcal{O})$. Let $\mathcal{O}_1 \supset \mathcal{O} \cup \beta\mathcal{O}$ and $g \in [G]_{\mathcal{O}_1}$. Then $g, \beta^* g \in [G]_\mathcal{O}$ and $\beta^* g = g + h_+^\beta + h_-^\beta$ with $\operatorname{supp} h_\pm^\beta \cap J_\mp(\mathcal{O}) = \varnothing$. By the causal factorization property (6.25) we obtain

$$\alpha_\beta^{G\mathcal{O}} = \gamma_{g,G}^{-1} \circ \operatorname{Ad} U_g(\beta) \circ \alpha_\beta^{\mathcal{O}} \circ \gamma_{g,G}$$

where $U_g(\beta) = \mathcal{S}_g(h_-^\beta)$ and

$$\alpha_\beta^{\mathcal{O}}(\mathcal{S}(f)) = \mathcal{S}(\beta^* f).$$

It is now apparent that (6.29) extends to an isomorphism. From the definition (6.29) it is also clear that if $\mathcal{O}_1 \subset \mathcal{O}_2$, then the restriction of $\alpha_\beta^{G\mathcal{O}_2}$ to $\mathfrak{A}_{L,G}(\mathcal{O}_1)$ coincides with $\alpha_\beta^{G\mathcal{O}_1}$ and for any $\beta, \beta' \in \mathcal{G}$, we have $\alpha_{\beta'}^{G\mathcal{O}} \circ \alpha_\beta^{G\mathcal{O}} = \alpha_{\beta'\circ\beta}^{G\mathcal{O}}$. □

6.5 Construction of Time-Ordered Products

In this section we review the abstract Epstein–Glaser construction of time-ordered products on Minkowski spacetime following [EG73, BF00, HW02a]. We will also show how to obtain more explicit formulas using regularization prescriptions and combinatorics involving Feynman diagrams.

Time-ordered products $\mathcal{T}_n^H$ are maps from $\mathcal{F}_{\mathrm{loc}}^{\otimes n}$ to $\mathcal{F}_{\mu c}[[\hbar]]$ and, as indicated in Sect. 6.2.5, they are obtained by extending non-renormalized expressions that are originally defined only on $(\mathcal{F}_{\mathrm{loc}})_{\mathrm{pds}}^{\otimes n}$. Let us consider $F \equiv F_1 \otimes \cdots \otimes F_n \in (\mathcal{F}_{\mathrm{loc}})_{\mathrm{pds}}^{\otimes n}$. Note that F induces a map from $\mathcal{E}^n$ to $\mathbb{C}$ by $F(\varphi_1, \ldots, \varphi_2) = F_1(\varphi_1) \cdots F_n(\varphi_n)$. When we talk about functionals on $\mathcal{E}$ we will denote the variable by φ and for functionals on $\mathcal{E}^n$, the variable is an n-tuple $(\varphi_1, \ldots, \varphi_n)$. For a multiindex $\beta \in \mathbb{N}_0^n$ we denote

$$\delta^\beta F(\varphi_1, \ldots, \varphi_n) \doteq \frac{\delta^{|\beta|} F}{\delta\varphi_1^{\beta_1} \ldots \delta\varphi_n^{\beta_n}}(\varphi_1, \ldots, \varphi_n),$$

where F is a smooth functional on $\mathcal{E}^n$ and $(\varphi_1, \ldots, \varphi_n) \in \mathcal{E}^n$. Clearly $\delta^\beta F(\varphi_1, \ldots, \varphi_n) \in L(\mathcal{E}^{|\beta|}; \mathbb{C})$ and by Proposition 3.1 it induces an element of $\Gamma'^{\mathbb{C}}(E^{\boxtimes|\beta|} \to M^{|\beta|})$.

6.5.1 *Existence of Time-Ordered Products (Abstract Proof)*

We have seen in Sect. 6.2.5 that the renormalization problem amounts to extending the maps $\mathcal{T}_n^H$ to local functionals with arbitrary supports. In fact, we will see that it reduces to extending numerical distributions defined everywhere outside certain subdiagonals in M^n (see [BF00]). The construction proceeds recursively; having constructed the time-ordered products of order $k < n$, at order n one is left with the problem of extending a distribution defined everywhere outside the thin diagonal of M^n. One way of explicitly constructing such distributional extensions relies on the splitting method (see for example [Sch95]). Here we take a different approach, based on the notion of Steinmann's scaling degree [Ste71].

Definition 6.17 Let $U \subset \mathbb{R}^d$ be a scale invariant open subset (i.e. $\lambda U = U$ for $\lambda > 0$), and let $t \in \mathscr{D}'(U)$ be a distribution on U. Let $t_\lambda(x) = t(\lambda x)$ be the scaled distribution. The scaling degree sd of t is

$$\mathrm{sd}\, t = \inf\{\delta \in \mathbb{R} | \lim_{\lambda \to 0} \lambda^\delta t_\lambda = 0\}. \tag{6.30}$$

The degree of divergence, another important concept used often in the literature, is defined as:

$$\mathrm{div}(t) \doteq \mathrm{sd}(t) - d.$$

The crucial result that allows us to construct time-ordered products is stated in the following theorem:

Theorem 6.4 *Let $t \in \mathscr{D}(\mathbb{R}^n \setminus \{0\})$ with scaling degree $\mathrm{sd}\, t < \infty$. Then there exists an extension of t to an everywhere defined distribution with the same scaling degree. The extension is unique up to the addition of a derivative $P(\partial)\delta$ of the delta function,* *where P is a polynomial with degree smaller or equal to* $\mathrm{div}(t)$ *(hence P vanishes for* $\mathrm{sd}\, t < n$*).*

In order to apply Theorem 6.4, we need to reduce the problem of constructing the time-ordered products to the problem of extending numerical distributions defined everywhere outside the thin diagonal. In this section we work on the level of quantum Lagrangians in the sense of Definition 6.10. A construction performed on the level of functionals will be presented in Sect. 6.5.2.

An n-fold time-ordered product of Lagrangians $L_1, \ldots, L_n$ induces a map

$$(f_1, \ldots, f_n) \mapsto \mathcal{T}^H_n(L_1(f_1), \ldots, L_n(f_n)), \tag{6.31}$$

We denote $\boldsymbol{L} \doteq (L_1, \ldots, L_n)$ and the map (6.31) by $\mathcal{T}^H_n(L_1, \ldots, L_n)$ or by $\mathcal{T}^H_{n,\boldsymbol{L}}$.

Similarly, let $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. We introduce the notation $\mathcal{T}^H_{I,\boldsymbol{L}}$ for the map

$$(f_1, \ldots, f_k) \mapsto \mathcal{T}^H_k(L_{i_1}(f_1), \ldots, L_{i_k}(f_k)).$$

Ultimately we need maps $\mathcal{T}^H_{n,\boldsymbol{L}}$ to construct the S-matrix and interacting local nets, as outlined in Sect. 6.4. For this purpose, it is convenient to formulate the problem of constructing the time-ordered products as the problem of constructing maps $\mathcal{T}^H_{n,\boldsymbol{L}}$. A method formulated on the level of local functionals, rather than such maps will be presented in the next subsection.

Let us start by formulating the axioms that the maps $\mathcal{T}^H_{n,\boldsymbol{L}}$ have to fulfil.

(TL 1) Causal factorisation property

$$\mathcal{T}^H_{n,\mathbf{L}}(f_1 \otimes \cdots \otimes f_n) = \mathcal{T}^H_{I,\mathbf{L}}(f_1 \otimes \cdots \otimes f_k) \star_H \mathcal{T}^H_{I^c,\mathbf{L}}(f_{k+1} \otimes \cdots \otimes f_n)$$

where $I = \{1, \ldots, k\}$ and the supports $\mathrm{supp}\, f_i$, $i = 1, \ldots, k$ of the first k arguments do not intersect the past of the supports $\mathrm{supp}\, f_j$, $j = k+1, \ldots, n$ of the last $n-k$ entries.

(TL 2) Starting element: $\mathcal{T}^H_0 = 1$, $\mathcal{T}^H_{1,L}(f) = L(f)$.

(TL 3) Symmetry: For a bosonic theory $\mathcal{T}^H_{I,\mathbf{L}} = \mathcal{T}^H_{I,\sigma(\mathbf{L})}$, where σ is a permutation of n elements and we define $\sigma(L_1, \ldots, L_n) \doteq (L_{\sigma(1)}, \ldots, L_{\sigma(n)})$. In the presence of fermions, each permutation of two elements introduces an extra minus sign.

(TL 4) Field independence: $\frac{\delta}{\delta\varphi}\mathcal{T}^H_{n,\boldsymbol{L}} = \mathcal{T}^H_n\left(L_1, \ldots, \frac{\delta L_i}{\delta\varphi}, \ldots, L_n\right)$.

(TL 5) φ-Locality: $\mathcal{T}^H_{n,\boldsymbol{L}} = \mathcal{T}^H_n\left(L_1^{[N]}, \ldots, L_n^{[N]}\right) + \mathcal{O}(\hbar^{N+1})$.

(TL 6) Unitarity: $\mathcal{T}^H_{n,\boldsymbol{L}^*} = \bar{\mathcal{T}}^H_{n,\boldsymbol{L}}$, where $\boldsymbol{L}^* \doteq (L_1^*, \ldots, L_n^*)$ and $L_i^*(f) \doteq \overline{L_i(f)}$.

(TL 7) Covariance: $\mathfrak{F}_{\mu c}\psi\left(\mathcal{T}^{\psi^*H}_{n,\boldsymbol{L}}(f_1, \ldots, f_n)\right) = \mathcal{T}^H_{n,\boldsymbol{L}}(\psi_* f_1, \ldots, \psi_* f_n)$, where $f_i \in \mathscr{D}(M)$, and $\psi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$.

In the first step we reduce the problem of defining the time-ordered products $\mathcal{T}^H_{n,\boldsymbol{L}}$ to the problem involving only $\mathbb{C}$-valued distributions. To this end, we use properties

(TL 4) and **(TL 5)** and write the n-fold time-ordered product $\mathcal{T}^H_{n,\boldsymbol{L}}$ in terms of its Taylor expansion around 0.

$$\begin{aligned}\mathcal{T}^H_{n,\boldsymbol{L}}(f_1 \otimes \cdots \otimes f_n)[\varphi] &= \mathcal{T}^H_{n,\boldsymbol{L}}(f_1 \otimes \cdots \otimes f_n)[0] \\ &+ \sum_{|\beta|>0} \left\langle \mathcal{T}^H_n(\delta^{\beta_1} L_1(f_1), \ldots, \delta^{\beta_n} L_n(f_n))[0], \varphi^{\otimes\beta_1} \otimes \cdots \otimes \varphi^{\otimes\beta_n} \right\rangle,\end{aligned} \tag{6.32}$$

where $\beta \in \mathbb{N}_0^n$. In the physics literature this is called the *Wick expansion*.

Remark 6.9 Note that this expansion converges at each order in $\hbar$, because due to **(TL 5)**, at each order in $\hbar$ the sum contains only finitely many terms. Therefore it is sufficient to know the time-ordered products of polynomials.

For the simplicity of notation now we restrict ourselves to the scalar field, i.e. $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R})$.

Recall that Φ_x denotes the evaluation functional on $\mathcal{E}$, i.e.

$$\Phi_x(\varphi) \doteq \varphi(x).$$

Denote by f_x the evaluation functional on $\mathscr{D}(M)$. Let $\langle ., . \rangle$ be the natural pairing between $\mathscr{D}(M)$ and $\mathcal{E}$ given explicitly by the integration with the integration measure $d\mu_g$.

In this notation

$$\int \varphi(x)^k f(x) d\mu_g(x) \equiv \int (\Phi_x^k \, \mathrm{f}_x)(\varphi, f) d\mu_g(x) \equiv \left\langle \Phi^k, \mathrm{f} \right\rangle (\varphi, f),$$

More generally, we can consider local functionals that involve derivatives of both the field configuration and the test function. To this end, we equip M with a co-tetrad $\boldsymbol{e}$ and its dual tetrad. This is always possible, as all oriented globally hyperbolic spacetimes in dimension 4 are parallelizable. Introducing a (co-)tetrad means that effectively we work in **FLoc**, the category of framed spacetimes, as given in Definitions 2.67 and 2.68.

Using the tetrad allows us to introduce functionals of the form

$$e^{\mu_1}_{(a_1} \ldots e^{\mu_n}_{a_n)} \nabla_{\mu_1} \ldots \nabla_{\mu_n} \Phi_x(\varphi) \doteq e^{\mu_1}_{(a_1} \ldots e^{\mu_n}_{a_n)} \nabla_{\mu_1} \ldots \nabla_{\mu_n} \varphi(x),$$

where we have used the Einstein summation convention for the indices $\mu_1, \ldots, \mu_n$ and the round brackets around indices mean symmetrization (antisymmetric derivatives will be introduced by means of the Riemann tensor). To make the notation more compact we will write the above contraction of tetrad components with covariant derivatives as

$$(\nabla)_{(a_1,\ldots,a_n)} \Phi_x \tag{6.33}$$

Similarly, we introduce the following functionals on $\mathscr{D}(M)$:

$$(\nabla)_{b_1,\dots,b_k}\mathrm{f}_x \tag{6.34}$$

Denote by $\mathscr{F}_x$ the algebra generated (with respect to the pointwise product) by functionals on $\mathscr{D}(M)$ that are of the form (6.34).

The generalized Lagrangians we consider depend locally and covariantly on the metric g. Let C be a local and covariant curvature tensor of type $(0, k)$. It is effectively a map from the space of globally hyperbolic metrics on the manifold M to $\Gamma((T^*M)^{\otimes k})$. We assume that C depend smoothly on the metric in the sense that it maps smooth curves to smooth curves. Denote by

$$C_{a_1,\dots,a_k}(x) \doteq e^{\mu_1}_{a_1}\dots e^{\mu_k}_{a_k} C_{\mu_1,\dots,\mu_k}(x) \tag{6.35}$$

the induced functional on $\Gamma((T^*M)^{\otimes_s 2})$.

We are now ready to introduce the notion of *Wick monomials* (compare with [HW05, DF07]).

Definition 6.18 Define a natural Wick monomial to be a natural transformation L from $\mathfrak{D}$ to $\mathfrak{F}_{\mathrm{loc}}$ (both functors seen as functors on **FLoc**) that is of the form

$$L_{\mathcal{M}}(f)[\varphi] = \left\langle \mathcal{L}_{a_1,\dots,a_l}(g;\varphi),\, p_{b_1,\dots,b_k}(f)\right\rangle, \tag{6.36}$$

where $\mathcal{M} = (M, \boldsymbol{e}) \in \mathrm{Obj}(\mathbf{FLoc})$, $\mathcal{L}_{a_1,\dots,a_l}(x)$ is a monomial in expressions (6.33) and (6.35), $p_{b_1,\dots,b_k}(x) \in \mathscr{F}_x$, $f \in \mathscr{D}(M)$, $\varphi \in \mathcal{E}$ and g is the metric induced by the co-tetrad $\boldsymbol{e}$ (see formula (2.24)). The pairing $\langle ., .\rangle$ is the integration over M.

Definition 6.19 Let $\mathscr{W}$ denote the vector space spanned by the natural Wick monomials.

Since our goal is to define the time-ordered products as a family of covariant maps, in the sense of Definition 6.8, it is convenient to work on the level of natural transformations. As indicated above, we change the underlying category from **Loc** to **FLoc** throughout this section and construct $\mathcal{T}_{n\mathcal{M}}$, $\mathcal{M} \in \mathrm{Obj}(\mathbf{FLoc})$, $n \in \mathbb{N}$.

Remark 6.10 The renormalization group $\mathcal{R}$ has a well defined action on $\mathscr{W}$, as well as $\mathscr{W}/\sim$, where $\sim$ is given by (6.23).

The combinatorics appearing in our formulas can be significantly simplified in the algebraic language [Bro06, Dan13, BFK06]. We now show how to construct some natural bi-algebras using natural Wick monomials.

Let $\mathcal{H}_x$ denote the algebra of functionals of the field configuration $\varphi \in \mathcal{E}$ and the metric g generated (with respect to the pointwise product) by functionals of the form $\mathcal{L}_{a_1,\dots,a_l}(x)$, with the notation of Definition 6.18.

Since the space of field configurations $\mathcal{E}$ is a vector space, it is in particular a group with addition. We can consider the Hopf algebra dual to this group. Restricting to $\mathcal{H}_x$, the induced co-product $\boldsymbol{\Delta}_T : \mathcal{H}_x \to \mathcal{H}_x \otimes \mathcal{H}_x$[3] is defined by

[3]We use the boldface notation $\boldsymbol{\Delta}_T$ for the co-product in order to distinguish it from Δ, which denotes the causal propagator at other places in the book.

$$\mathbf{\Delta}_T \mathcal{L}_x(g; \varphi, \psi) \doteq \mathcal{L}_x(g; \varphi + \psi)$$

and using the Taylor expansion we can express this co-product on generators as

$$\mathbf{\Delta}_T \left(C_{b_1,\dots,b_m}(x) \prod_{i=1}^{k} (\nabla)_{(a_{i1},\dots,a_{iN_i})} \Phi_x \right) \doteq \sum_{I \subset \{1,\dots,k\}} C_{b_1,\dots,b_m}(x) \prod_{i \in I} (\nabla)_{(a_{i1},\dots,a_{iN_i})} \Phi_x \otimes \prod_{j \in I^c} (\nabla)_{(a_{j1},\dots,a_{jN_j})} \Phi_x \tag{6.37}$$

where each $N_i \in \mathbb{N}$ and $a_{il} \in \{0, 1, 2, 3\}$.

Remark 6.11 By definition, an element of $\mathcal{H}_x \otimes \mathcal{H}_x$ is a functional on $\Gamma((T^*M)^{\otimes_s 2}) \times \mathcal{E} \times \mathcal{E}$. In order to obtain a functional on $\Gamma((T^*M)^{\otimes_s 2}) \times \mathcal{E}$ (i.e. a functional of a single field configuration and the metric) we can apply the pointwise multiplication map m. If we want to consider functionals depending on the choice of n points of spacetime M, we can use the exterior tensor product defined by

$$(\mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_n)_{x_1,\dots,x_n}(g; \varphi_1, \dots, \varphi_n) \doteq \mathcal{L}_{x_1}(g; \varphi_1) \dots \mathcal{L}_{x_n}(g; \varphi_n).$$

The unit e of $\mathcal{H}_x$ is identified with the constant functional $\Phi_x^0 \equiv 1$. Define the counit

$$\varepsilon(\mathcal{L}_x) = \begin{cases} 1 & \text{if } \mathcal{L}_x = e, \\ 0 & \text{else.} \end{cases} \tag{6.38}$$

where $\mathcal{L}_x \in \mathcal{H}_x$. The bi-algebra $\mathcal{H}_x$ can be equipped with two gradings. Let rk denote the polynomial rank, i.e.

$$\mathrm{rk}\left(\prod_{i=1}^{k} (\nabla)_{a_{i1},\dots,a_{iN_i}} \Phi_x \right) = k.$$

The second grading is the total degree of all the derivatives, i.e.

$$\deg\left(\prod_{i=1}^{k} (\nabla)_{a_{i1},\dots,a_{iN_i}} \Phi_x \right) = \sum_{i=1}^{k} N_i.$$

On Minkowski spacetime the $\deg = 0$ and $\mathrm{rk} = 0$ subspace of $\mathcal{H}_x$ consists of multiples of e, so $\mathcal{H}_x$ is connected, the antipode can be constructed and we obtain a Hopf algebra. In general, this is not the case, since $\deg = 0$ and $\mathrm{rk} = 0$ subspace contains elements that differ by curvature tensors $C_{b_1,\dots,b_m}(x)$.

Example 6.4 Consider the special case of Wick monomials of the form $\langle \Phi^k, \mathrm{f} \rangle$ and call these objects *Wick powers*. Note that

$$\left\langle \frac{\delta_l}{\delta\varphi^l}\Phi_x^k(\varphi), \psi^{\otimes l} \right\rangle = \frac{k!}{(k-l)!}\Phi_x^{k-l}(\varphi)\Phi_x^l(\psi) = \frac{k!}{(k-l)!}\Phi_x^{k-l} \otimes \Phi_x^l(\varphi,\psi).$$

Hence

$$\frac{\delta_l}{\delta\varphi^l}\langle \Phi^k, \mathrm{f} \rangle = \frac{k!}{(k-l)!}\langle \Phi^{k-l} \otimes \Phi^l, \mathrm{f} \rangle,$$

Let $\mathcal{H}_x^{pow}$ denote the algebra generated by evaluation functionals Φ_x^k, where the product is the pointwise product of functionals, as in the general case, i.e.

$$\Phi_x^k \Phi_x^l \doteq \Phi_x^{k+l}. \tag{6.39}$$

The coproduct $\mathbf{\Delta}_T$ acts on $\mathcal{H}_x^{pow}$ as

$$\mathbf{\Delta}_T(\Phi_x^k) = \sum_{l=0}^{k} \binom{k}{l} \Phi_x^{k-l} \otimes \Phi_x^l, \tag{6.40}$$

and the counit ε is as in (6.38). The bi-algebra $\mathcal{H}_x^{pow}$ is graded by k and the degree 0 component contains just e, hence the antipode exists and $\mathcal{H}_x^{pow}$ becomes a Hopf algebra, which is a subalgebra of $\mathcal{H}_x$.

We write the Taylor expansion of Φ_x^k as

$$\Phi_x^k(\varphi + \psi) = \mathbf{\Delta}_T(\Phi_x^k)(\varphi,\psi).$$

Hence

$$\mathbf{\Delta}_T \langle \Phi^k, \mathrm{f} \rangle = \sum_{l=0}^{k} \binom{k}{l} \langle \Phi^{k-l} \otimes \Phi^l, \mathrm{f} \rangle, \tag{6.41}$$

and we obtain

$$\langle \Phi^k, \mathrm{f} \rangle (\varphi + \psi; f) = \langle \mathbf{\Delta}_T(\Phi^k), \mathrm{f} \rangle (\varphi, \psi; f).$$

We come back to the general case. Note that the representation of a Wick monomial in terms of $\mathcal{L}_{a_1,\dots,a_l}$ and $p_{b_1,\dots,b_k}$ (see formula (6.36)) is not unique, since we can perform partial integration. Nevertheless, it is worth to put up with this redundancy for the moment in order to get a simple characterisation of $\mathscr{W}$.

Replace the evaluation functionals $(\nabla)_{a_1,\dots,a_n}\Phi_x$, $C_{a_1,\dots,a_l}(x)$ with abstract generators $(\nabla)_{a_1,\dots,a_n}\Phi$, $C_{a_1,\dots,a_l}$. Consider the free algebra generated by these symbols and define the coproduct $\mathbf{\Delta}_T$ by adapting formula (6.37). Unit and co-unit are introduced as above. The resulting bi-algebra is denoted by $\mathcal{H}$ and by definition is isomorphic to $\mathcal{H}_x$, so by choosing a spacetime M and fixing $x \in M$ we can identify abstract elements of $\mathcal{H}$ with concrete functionals.

Now take the free algebra generated by the symbols $(\nabla)_{b_1,\dots,b_k}\mathrm{f}$ and denote it by $\mathscr{F}$. By definition, every natural Wick monomial $L \in \mathscr{W}$ can be represented (non-uniquely!) as a pair $(\mathcal{L}, p)$, where $\mathcal{L} \in \mathcal{H}$ and $p \in \mathscr{F}$. Take $\boldsymbol{L} = (L_1, \dots, L_n)$, where $L_1, \dots, L_n \in \mathscr{W}$ and choose their representations $(\mathcal{L}_1, p_1), \dots, (\mathcal{L}_n, p_n)$ in terms of elements of $\mathcal{H}$ and $\mathscr{F}$. The Wick expansion (6.32) can now be written in the form

$$(\mathcal{T}_{n,\boldsymbol{L}})^H_{\mathcal{M}} = \sum_{(\mathcal{L}_1,\dots,\mathcal{L}_n)} \left\langle t^H_n(\mathcal{L}_{1(1)}, \dots, \mathcal{L}_{n(1)}) \cdot m\left(\mathcal{L}_{1(2)} \boxtimes \dots \boxtimes \mathcal{L}_{n(2)}\right), p_1 \boxtimes \dots \boxtimes p_n \right\rangle, \tag{6.42}$$

where we use the Sweedler notation $\mathcal{L} = \sum_{(\mathcal{L})} \mathcal{L}_{(1)} \otimes \mathcal{L}_{(2)}$, m is the pointwise multiplication of functionals, the notation with $\boxtimes$ is clarified in Remark 6.11, and the product $\cdot$ is the pointwise multiplication of a distribution on M^n and a functional-valued map on M^n.

It is clear from (6.32) and (6.42) that constructing the time-ordered products of Wick monomials reduces to extending some numerical distributions

$$t^H_n(\mathcal{L}_1, \dots, \mathcal{L}_n) \equiv t^H_{n,\mathcal{L}}.$$

Using the axioms **(TL 1)–(TL 7)** we can formulate the corresponding axioms for $t^H_{n,\mathcal{L}}$'s (see [HW01]) for details. We label these axioms by **(t 1)–(t 7)**. The non-uniqueness in representing Wick monomials in terms of pairs $(\mathcal{L}, p)$ implies some extra conditions on t's. For example, if we consider just the scalar field in the presence of field derivatives in Lagrangians, the non-uniquness arises due to the possibility of integration by parts. For consistency, distributions $t^H_{n,\mathcal{L}}$ have to satisfy the following condition:

(t 8) Action Ward Identity

$$e^\mu_a \nabla^{x_i}_\mu\, t^H_n(\mathcal{L}_1, \dots, \mathcal{L}_n) = t^H_n(\mathcal{L}_1, \dots, e^\mu_a \nabla_\mu \mathcal{L}_i, \dots \mathcal{L}_n), \tag{6.43}$$

where $\nabla^{x_i}_\mu$ is the covariant derivative with respect to the variable x_i.

It was shown in [DF07] that on Minkowski spacetime this condition can be fulfilled also for the extensions of t's to the diagonal. Before AWI has been proven, the construction of time-ordered products was known only on the level of formal generators (here formalized in terms of $\mathcal{H}$ and $\mathscr{F}$), not as functionals on the configuration space. It was then suggested by Stora [Sto02] to impose the additional requirement (6.43). The proof of this requirement achieved in [DF07] was a major breakthrough in mathematical pQFT since it opened the way for the functional formalism. In [HW05] this condition is called *the Leibniz rule*, and its proof is generalized to curved spacetimes.

The construction of time-ordered products $(\mathcal{T}_{k,\boldsymbol{L}})^H_{\mathcal{M}}$ proceeds inductively. To simplify the notation we keep the index $\mathcal{M}$ implicit. We assume that the maps $\mathcal{T}^H_{k,\boldsymbol{L}'}$ with $k < n$ are constructed for all possible k-tuples $\boldsymbol{L}'$ of generalized Lagrangians of the theory. For a power-counting renormalizable theory this is achieved through extending finitely many distributions $t^{H}_{n,\mathfrak{L}}$ (compare with Definition 6.11). Using the causal factorisation property **(TL 1)** we fix the values of $\mathcal{T}^{H}_{n,\boldsymbol{L}}$ outside the thin diagonal Diag_n.

Maps constructed this way are called *non-renormalized time-ordered products* and we denote them by $\mathring{\mathcal{T}}^{H}_{n,\boldsymbol{L}}$.

Definition 6.20 Let I be a proper subset of $\{1,\dots,n\}$ and we denote by C_I the subset of M^n defined by

$$C_I = \{(x_1,\dots,x_n)|x_i \notin J^+(x_j) \text{ for all } i \in I,\ \ j \in I^c\},$$

where I^c is the complement of I.

The sets C_I are open and they constitute an open covering of $M^n \setminus \mathrm{Diag}_n$.

Definition 6.21 Let I be a proper subset of $\{1,\dots,n\}$. Let $(f_1,\dots,f_n)$ be test functions such that $\mathrm{supp}\, f_i \prec \mathrm{supp}\, f_j$ for all $i \in I$, $j \in I^c$. Define

$$\mathcal{T}^{H}_{I^c|I,\boldsymbol{L}}(f_1,\dots,f_n) \doteq \mathcal{T}^{H}_{I^c,\boldsymbol{L}}(f_{I^c}) \star_H \mathcal{T}^{H}_{I,\boldsymbol{L}}(f_I),$$

where for $I = \{i_1,\dots,i_k\}$ we define $f_I \doteq (f_{i_1},\dots,f_{i_k})$.

For each $\mathcal{T}^{H}_{I^c|I,\boldsymbol{L}}$ choose a representation as

$$\mathcal{T}^{H}_{I^c|I,\boldsymbol{L}} = \left\langle \omega^{H}_{I^c|I,\boldsymbol{L}}, p_1 \boxtimes \cdots \boxtimes p_n \right\rangle,$$

where $\omega^{H}_{I^c|I,\boldsymbol{L}}$ is a functional-valued distribution on M^n and $p_1,\dots,p_n \in \mathscr{F}$.

Let $\{g_I\}$ be a partition of unity subordinate to the open covering introduced in Definition 6.20. We define the non-renormalized time-ordered products $\mathring{\mathcal{T}}^{H}_{n,\boldsymbol{L}}$ as maps on test functions such that $\mathrm{supp}(f_1 \cdot \ldots \cdot f_n)$ doesn't intersect Diag_n. We set

$$\mathring{\mathcal{T}}^{H}_{n,\boldsymbol{L}} \doteq \sum_{\substack{I \subsetneq \{1,\dots,n\} \\ I \neq \varnothing}} \left\langle g_I \cdot \omega^{H}_{I|I^c,\boldsymbol{L}}, p_1 \boxtimes \cdots \boxtimes p_n \right\rangle,$$

The construction presented here depends on the choice of the partition of unity and distributions $\omega^{H}_{I|I^c,\boldsymbol{L}}$, but since we are only interested in showing existence, we are not concerned about this issue. The following proposition has been proven in [HW01].

Proposition 6.8 *Assuming that time ordered products* $\mathring{\mathcal{T}}^{H}_{k,\boldsymbol{L}'}$ *with up to* $n-1$ *arguments have been defined in such a way that they satisfy properties* **(TL 1)–(TL 7)** *as maps on* $\mathscr{D}^{n-1}$. *Then the maps* $\mathring{\mathcal{T}}^{H}_{n,\boldsymbol{L}}$ *automatically satisfy the restrictions of properties* **(TL 1)–(TL 7)** *to maps on test functions functions such that* $\mathrm{supp}(f_1 \cdot \ldots \cdot f_n)$ *doesn't intersect* Diag_n.

Proof See the proof in [HW01]. □

Combining the result above with the Wick expansion, we can conclude that the renormalization problem reduces to extending numerical distributions $\mathring{t}^H_{n,\mathfrak{L}}$ to Diag_n. It is convenient to impose some additional, more technical requirements, which control the regularity of distributions $t^H_{n,\mathfrak{L}}$. These are:

(t 9) Almost homogeneous scaling,
(t 10) Microlocal spectrum condition,
(t 11) Smoothness,
(t 12) Analyticity.

The final two axioms concern the dependence on the background metric. The precise statement of these axioms can be found in Sect. 3.3 of [HW01].

Theorem 6.5 (after [HW01]) *There exists a family of distributions $t^H_{n,\mathfrak{L}}$, $n \in \mathbb{N}$ satisfying the axioms* **(t 1)–(t 12)**.

Proof See [BF00, HW01, HW05]. A more robust and mathematically cleaner method for constructing such distributional extension has been provided in [Dan13]. □

Having constructed the numerical distributions $t^H_{n,\mathfrak{L}}$ satisfying conditions **(t 1)–(t 12)** we use the formula (6.42) to construct maps $\mathcal{T}^H_n(L_1, \dots, L_n)$ satisfying properties **(TL 1)–(TL 7)**. Using linearity and contracting with appropriate $p \in \mathscr{F}$, this allows to construct time-ordered products of arbitrary Lagrangians in $\mathbf{\Phi}$. Note that all the way throughout this section we have worked purely on the level of topological algebras of functionals, without the need to refer to a concrete Hilbert space representation. This functional viewpoint is very useful in QFT on curved spacetimes, where there is no unique vacuum state and hence no distinguished vacuum representation.

6.5.2 Explicit Construction and Feynman Graphs

In Sect. 6.5.1 we have reviewed the existence proof for time ordered products, reformulating it in the language of the functional formalism. The drawback of the method presented there was that it relies on a (potentially non-unique) parametrisation of local functionals in terms of Lagrangians and also, that the existence proof is rather abstract and doesn't provide an explicit expression for time-ordered products. There is a more direct method to construct $\mathcal{T}^H_n$, which is formulated on the level of functionals, rather than the natural Lagrangians. For simplicity we will present it only in the context of Minkowski spacetime. The generalization to arbitrary globally hyperbolic spacetimes is not difficult and can be done using the methods of [BF00, Dan13].

In this section we work on Minkowski spacetime, so we set $H = \Delta_1$ and $\Delta^+_{S_0} = \frac{i}{2}\Delta_{S_0} + \Delta_1$ is the Wightman 2-point function. In this case $\Delta^{\mathrm{F}}_{S_0}$ is the "standard" Feynman propagator.

Theorem 6.4 allows us in principle to extend all the numerical distributions we need for the construction of time-ordered products. Although it is possible to argue on the level of $\mathcal{T}^H_n$s, here we use a perturbative expansion of time-ordered products in terms of Feynman graphs, to make the relation to other approaches to renormalization more apparent. Note, however, that in the pAQFT framework, Feynman graphs are not fundamental objects, but instead they are derived (together with the corresponding Feynman rules) from the definition of time-ordered products.

Let us denote $D^{ij}_{\mathrm{F}} \doteq \langle \Delta^{\mathrm{F}}_{S_0}, \frac{\delta^2}{\delta\varphi_i \delta\varphi_j} \rangle$ and recall that $\mathcal{D}_{\mathrm{F}} \doteq \langle \Delta^{\mathrm{F}}_{S_0}, \frac{\delta^2}{\delta\varphi^2} \rangle$. The Leibniz rule for differentiation can be formulated as

$$\frac{\delta}{\delta\varphi} \circ m = m \circ \left(\sum_{i=1}^{n} \frac{\delta}{\delta\varphi_i} \right), \tag{6.44}$$

where m is the pointwise multiplication (in this case of n arguments), or in other words, the pullback through the diagonal map $\mathcal{E} \to \mathcal{E}^n$, $\varphi \mapsto (\varphi, \ldots, \varphi)$.

Proposition 6.9 *The unrenormalized n-fold time ordered product* $\mathcal{T}^H_n : (\mathcal{F}_{\mathrm{loc}})^{\otimes n}_{\mathrm{pds}} \to \mathcal{F}_{\mu c}[[\hbar]]$ *can be expressed as*

$$\mathcal{T}^H_n(F_1, \ldots, F_n) = m \circ e^{\hbar \sum_{i<j} D^{ij}_{\mathrm{F}}} (F_1 \otimes \cdots \otimes F_n).$$

Proof By definition we have

$$\mathcal{T}^H_n(F_1, \ldots, F_n) = F_1 \cdot_{\mathcal{T}^H} \cdots \cdot_{\mathcal{T}^H} F_n = e^{\frac{\hbar}{2}\mathcal{D}_{\mathrm{F}}} \circ m(e^{-\frac{\hbar}{2} D^{11}_{\mathrm{F}}} F_1, \ldots, e^{-\frac{\hbar}{2} D^{nn}_{\mathrm{F}}} F_n).$$

The Leibniz rule implies that

$$e^{\frac{\hbar}{2}\mathcal{D}_{\mathrm{F}}} \circ m = m \circ e^{\hbar \sum_{i<j} D^{ij}_{\mathrm{F}} + \frac{\hbar}{2} \sum_i D^{ii}_{\mathrm{F}}},$$

hence

$$\mathcal{T}^H_n(F_1, \ldots, F_n) = m \circ e^{\hbar \sum_{i<j} D^{ij}_{\mathrm{F}} + \frac{\hbar}{2} \sum_i D^{ii}_{\mathrm{F}}} (e^{-\frac{\hbar}{2} D^{11}_{\mathrm{F}}} F_1, \ldots, e^{-\frac{\hbar}{2} D^{nn}_{\mathrm{F}}} F_n).$$

It is now clear that all the factors $e^{-\frac{\hbar}{2} D^{ii}_{\mathrm{F}}}$ cancel, which proves the result. □

Let us denote

$$T_n \doteq e^{\hbar \sum_{i<j} D^{ij}_{\mathrm{F}}},$$

so T_n is a map from $(\mathcal{F}_{\mathrm{loc}})^{\otimes n}_{\mathrm{pds}}$ to $\mathcal{F}^n_{\mu c}$. The identity

$$e^{\hbar \sum_{i<j} D^{ij}_{\mathrm{F}}} = \prod_{i<j} \sum_{l_{ij}=0}^{\infty} \frac{\left(\hbar D^{ij}_{\mathrm{F}}\right)^{l_{ij}}}{l_{ij}!} \tag{6.45}$$

allows us to express time ordered products in terms of graphs. Let $\mathcal{G}_n$ be the set of all graphs with vertex set $V(\Gamma) = \{1, \dots n\}$ and l_{ij} be the number of lines $e \in E(\Gamma)$ connecting the vertices i and j. We set $l_{ij} = l_{ji}$ for $i > j$ and $l_{ii} = 0$. If e connects i and j we set $\partial e := \{i, j\}$. Then

$$T_n = \sum_{\Gamma \in \mathcal{G}_n} T_\Gamma, \tag{6.46}$$

where

$$T_\Gamma = \frac{1}{\mathrm{Sym}(\Gamma)} \langle t_\Gamma, \delta_\Gamma \rangle, \tag{6.47}$$

with

$$\delta_\Gamma = \frac{\delta^{2|E(\Gamma)|}}{\prod_{i \in V(\Gamma)} \prod_{e: i \in \partial e} \delta\varphi_i(x_{e,i})}$$

and

$$t_\Gamma = \prod_{e \in E(\Gamma)} \hbar \Delta^{\mathrm{F}}_{S_0}(x_{e,i}, i \in \partial e) \tag{6.48}$$

The symmetry factor $\mathrm{Sym}(\Gamma)$ is the number of possible permutations of lines joining the same two vertices, $\mathrm{Sym}(\Gamma) = \prod_{i<j} l_{ij}!$. Note that T_Γ is a map from $(\mathcal{F}_{\mathrm{loc}})^{\otimes V(\Gamma)}_{\mathrm{pds}}$ to $\mathcal{F}^{|V(\Gamma)|}[[\hbar]]$, where $\otimes V(\Gamma)$ means that the factors in the tensor product are numbered by vertices and to a vertex $v \in V(\Gamma)$ we assign the variable φ_v. The renormalization problem to extend T_n's to maps on $(\mathcal{F}_{\mathrm{loc}})^{\otimes n}$ is now reduced to extending all the maps T_Γ. For the latter we can use methods relying on the combinatorics of Feynman graphs, as is done in other approaches to pQFT. In particular, one can establish a relation with the Connes–Kreimer approach [CK00, CK01] (see [Pin00, GBL00, DFKR14]).

Note that functional derivatives of local functionals are of the form

$$F^{(l)}(\varphi)(x_1, \dots, x_l) = \int \sum_{j=1}^{N} g_j[\varphi](y) p_j(\partial_{x_1}, \dots, \partial_{x_l}) \prod_{i=1}^{l} \delta(y - x_i) d\mu(y), \tag{6.49}$$

where $N \in \mathbb{N}$, p_j's are polynomials in partial derivatives and $g_j[\varphi]$ are φ-dependent test functions. The representation above is not unique, since some of the partial derivatives ∂_{x_i} can be replaced with ∂_y and applied to $g_j[\varphi]$. Another representation of $F^{(l)}(\varphi)$ is obtained by performing the integral above and using the centre of mass and relative coordinates:

$$F^{(l)}(\varphi)(x_1, \dots, x_l) = \sum_\beta f_\beta[\varphi](z) \partial^\beta \delta(x^{\mathrm{rel}}) \tag{6.50}$$

where $\beta \in \mathbb{N}_0^{4(l-1)}$, test functions $f_\beta[\varphi](x) \in \mathcal{D}(\mathbb{M})$ are now φ-dependent functions of the center of mass coordinate $z = (x_1 + \dots + x_k)/k$ and $x^{\mathrm{rel}} = (x_1 - z, \dots, x_k - z)$ denotes the relative coordinates.

Using (6.49) we see that the functional differential operator δ_Γ applied to $F \in \mathcal{F}_{\text{loc}}^{\otimes n}$ yields, at any n-tuple of field configurations $(\varphi_1, \dots, \varphi_n)$, a compactly supported distribution in the variables $x_{e,i}$, $i \in \partial e$, $e \in E(\Gamma)$ with support on the partial diagonal $\text{Diag}_\Gamma = \{x_{e,i} = x_{f,i}, i \in \partial e \cap \partial f, e, f \in E(\Gamma)\} \subset \mathbb{M}^{2|E(\Gamma)|}$ and with a wavefront set perpendicular to $T\text{Diag}_\Gamma$. Note that the partial diagonal Diag_Γ can be parametrized using the center of mass coordinates

$$z \doteq \frac{1}{\text{valence}(v)} \sum_{e:v\in\partial e} x_{e,v},$$

assigned to each vertex. The remaining relative coordinates are $x_{e,v}^{\text{rel}} = x_{e,v} - z_v$, where $v \in V(\Gamma)$, $e \in E(\Gamma)$ and $v \in \partial e$. Obviously, we have $\sum_{e|v\in\partial e} x_{e,v}^{\text{rel}} = 0$ for all $v \in V(\Gamma)$. In this parametrization $\delta_\Gamma F$ can be written as a finite sum

$$\delta_\Gamma F = \sum_\beta f^\beta \partial_\beta \delta_{\text{rel}},$$

where $\beta \in \mathbb{N}_0^{4|V(\Gamma)|}$, each $f^\beta(\varphi_1, \dots, \varphi_n)$ is a test function on Diag_Γ and δ_{rel} is the Dirac delta distribution in relative coordinates, i.e. $\delta_{\text{rel}}(g) = g(0, \dots, 0)$, where g is a function of $(x_{e,v}^{\text{rel}}, v \in V(\Gamma), e \in E(\Gamma))$.

We can simplify our notation even further. Let Y_Γ denote the vector space spanned by derivatives of the Dirac delta distributions $\partial_\beta \delta_{\text{rel}}$, where $\beta \in \mathbb{N}_0^{4|V(\Gamma)|}$. Obviously, Y_Γ is graded by $|\beta|$. Let $\mathcal{D}(\text{Diag}_\Gamma, Y_\Gamma)$ denote the graded space of test functions on Diag_Γ with values in Y_Γ. With this notation we have $\delta_\Gamma F \in \mathcal{D}(\text{Diag}_\Gamma, Y_\Gamma)$ and if $F \in (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$, then $\delta_\Gamma F$ is supported on $\text{Diag}_\Gamma \setminus \text{DIAG}$, where DIAG is the large diagonal:

$$\text{DIAG} = \left\{z \in \text{Diag}_\Gamma |\, \exists v, w \in V(\Gamma), v \neq w : z_v = z_w\right\}.$$

We can now write (6.47) in the form

$$\frac{1}{\text{Sym}(\Gamma)} \langle t_\Gamma, \delta_\Gamma \rangle = \sum_\beta \left\langle f^\beta \partial_\beta \delta_{\text{rel}}, t_\Gamma \right\rangle$$

where the sum contains only finitely many terms and t_Γ is now written in terms of centre of mass and relative coordinates. To see that this expression is well defined, note that we can move all the partial derivatives ∂_β to t_Γ by formal partial integration. Then the contraction with δ_{rel} is just the pullback through the diagonal map ρ_Γ : $\text{Diag}_\Gamma \to \mathbb{M}^{2|E(\Gamma)|}$ given by

$$(\rho_\Gamma(z))_{e,v} = z_v \quad \text{if } v \in \partial e.$$

From the wavefront set properties of $\Delta_{S_0}^{\text{F}}$, we deduce that the pullback ρ_Γ^* of each $t_\Gamma^\beta \doteq \partial_\beta t_\Gamma$ is a well defined distribution on $\text{Diag}_\Gamma \setminus \text{DIAG}$, so (6.47) makes sense if $F \in (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$, as expected. We conclude that $t_\Gamma \in \mathcal{D}(\text{Diag}_\Gamma \setminus \text{DIAG}, Y_\Gamma)$, where the duality between t_Γ and a "test function" $f = \sum_\beta f^\beta \partial_\beta \delta$ is given by

$$\langle t_\Gamma, f \rangle \doteq \sum_\beta \langle t_\Gamma^\beta, f_\beta \rangle.$$

The renormalization problem now reduces to finding the extensions of all the distributions t_Γ^β appearing in the above expression, so that t_Γ gets extended to an element of $\mathscr{D}(\mathrm{Diag}_\Gamma, Y_\Gamma)$. The solution to this problem is obtained by using the inductive procedure of Epstein and Glaser. The induction step works as follows: if $t_{\Gamma'}$ is known for all graphs Γ' with fewer vertices than Γ, then t_Γ can be uniquely defined for all *disconnected*, all *connected one particle reducible* and all *one particle irreducible one vertex reducible graphs*. Graphs that are irreducible and do not contain any non-trivial irreducible subgraphs are called *EG-primitive*. For the remaining graphs, called *EG-irreducible*, t_Γ is defined uniquely on all $f \in \mathscr{D}(\mathrm{Diag}_\Gamma, Y_\Gamma)$, where f_β vanishes together with all its derivatives of order $\leq \omega_\Gamma + |\beta|$ on the thin diagonal of Diag_Γ. Here

$$\omega_\Gamma = (d-2)|E(\Gamma)| - d(|V(\Gamma)| - 1)$$

is the degree of divergence of the graph Γ and d denotes the dimension of the Minkowski spacetime $\mathbb{M}$ (in our case $d = 4$). We denote this subspace by $\mathscr{D}_{\omega_\Gamma}(\mathrm{Diag}_\Gamma, Y_\Gamma)$. Renormalization amounts to projecting a generic f to this subspace by a translation invariant projection $W_\Gamma : \mathscr{D}(\mathrm{Diag}_\Gamma, Y_\Gamma) \to \mathscr{D}_{\omega_\Gamma}(\mathrm{Diag}_\Gamma, Y_\Gamma)$. Different renormalization schemes differ by different choices of the projections W_Γ (see [DFKR14] for details).

By exploiting the translation invariance in Minkowski spacetime we find that, at each step of the recursive construction of time-ordered products, the renormalization problem reduces to the problem of extension of some distribution defined everywhere outside the origin, so this is what we will focus on now.

6.5.3 Regularization of Distributions

There is an important conceptual difference between the Epstein Glaser framework and other approaches to renormalization; namely from the EG point of view one constructs objects (e.g. time-ordered products, the S-matrix) which are already renormalized and can be physically interpreted. In other approaches, one first introduces some *regularization*, which renders the Feynman graphs well defined, and then in the next steps performs renormalization, which is some procedure that allows one to recover physically relevant information after the regularization parameters are removed by some limiting process.

In this section we show how introducing an explicit regularization procedure is related to the problem of extension of distributions.

Definition 6.22 We define

$$\mathscr{D}_\lambda(\mathbb{R}^n) := \{ f \in \mathscr{D}(\mathbb{R}^n) \,|\, (\partial^\alpha f)(0) = 0 \ \ \forall |\alpha| \leq \lambda \} \tag{6.51}$$

to be the space of functions with derivatives vanishing up to order λ and $\mathscr{D}'_\lambda(\mathbb{R}^n)$ is the corresponding space of distributions.

Theorem 6.6 ([Ste71, BF00]) *A distribution $t \in \mathscr{D}'(\mathbb{R}^n \setminus \{0\})$ with scaling degree* $\mathrm{sd}(t)$ *has a unique extension $\bar{t} \in \mathscr{D}'_\lambda(\mathbb{R}^n)$, $\lambda = \mathrm{sd}(t) - n$ that satisfies the condition* $\mathrm{sd}(\bar{t}) = \mathrm{sd}(t)$.

Definition 6.23 ([DFKR14]) Let $t \in \mathscr{D}'(\mathbb{R}^n \setminus \{0\})$ be a distribution with degree of divergence λ, and let $\bar{t} \in \mathscr{D}'_\lambda(\mathbb{R}^n)$ be the unique extension of t with the same degree of divergence. A family of distributions $\{t^\zeta\}_{\zeta \in \Omega \setminus \{0\}}$, $t^\zeta \in \mathscr{D}'(\mathbb{R}^n)$, with $\Omega \subset \mathbb{C}$ a neighborhood of the origin, is called a regularization of t, if

$$\forall g \in \mathscr{D}_\lambda(\mathbb{R}^n) : \quad \lim_{\zeta \to 0} \langle t^\zeta, g \rangle = \langle \bar{t}, g \rangle. \tag{6.52}$$

The regularization $\{t^\zeta\}$ is called analytic, if for all functions $f \in \mathscr{D}(\mathbb{R}^n)$ the map

$$\Omega \setminus \{0\} \ni \zeta \mapsto \langle t^\zeta, f \rangle \tag{6.53}$$

is analytic with a pole of finite order at the origin. The regularization $\{t^\zeta\}$ is called finite, if the limit $\lim_{\zeta \to 0} \langle t^\zeta, f \rangle \in \mathbb{C}$ exists $\forall f \in \mathscr{D}(\mathbb{R}^n)$; in this case $\lim_{\zeta \to 0} t^\zeta \in \mathscr{D}'(\mathbb{R}^n)$ is called an extension or renormalization of t.

Given the unique extension $\bar{t}$ one can define an (in general not unique) extension of t to a distribution in $\mathscr{D}(\mathbb{R}^n)$ by the choice of a projection $W : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}_\lambda(\mathbb{R}^n)$ defined as

$$Wf \doteq f - \sum_{|\gamma| \leq \mathrm{sd}(t) - n} w_\gamma \, \partial^\gamma f(0), \tag{6.54}$$

given in terms of functions $w_\gamma \in \mathscr{D}(\mathbb{R}^n)$, $|\beta| \leq \mathrm{sd}(t) - n$, fulfilling

$$\partial^\gamma w_\beta(0) = \delta^\gamma_\beta \qquad \forall \gamma \in \mathbb{N}^n_0 \tag{6.55}$$

In particular, it was shown in [Kel10, DFKR14] that for an analytic regularization $\{t^\zeta\}$ we can write (6.52) in the form

$$\langle \bar{t}, Wf \rangle = \lim_{\zeta \to 0} \left[\langle t^\zeta, f \rangle - \sum_{|\gamma| \leq \mathrm{sd}(t) - n} \langle t^\zeta, w_\gamma \rangle \, \partial^\gamma f(0) \right]. \tag{6.56}$$

and each of the two expressions appearing inside the square brackets can be expanded in a Laurent series around $\zeta = 0$. This allows us to perform Minimal Subtraction, as done in

Corollary 6.1 (Minimal Subtraction [DFKR14]) *The regular part* ($\mathrm{rp} \doteq \mathrm{id} - \mathrm{pp}$, *where* pp *denotes the principal part) of any analytic regularization* $\{t^\zeta\}$ *of a distribution* $t \in \mathscr{D}'(\mathbb{R}^n \setminus \{0\})$ *defines by*

$$\langle t^{\mathrm{MS}}, f\rangle := \lim_{\zeta\to 0} \mathrm{rp}(\langle t^{\zeta}, f\rangle) \tag{6.57}$$

an extension t^{MS} *of* t *with the same scaling degree,* $\mathrm{sd}(t^{\mathrm{MS}}) = \mathrm{sd}(t)$*, which we call the Minimal Subtraction.*

Let us now come back to the construction of the S-matrix. To relate this to the method of divergent counter terms, we need to find a family of renormalization group elements $Z^{\zeta} \in \mathscr{R}$ such that

$$\mathcal{S} = \lim_{\zeta\to\zeta_0} \mathcal{S}^{\zeta} \circ \mathcal{Z}^{\zeta}. \tag{6.58}$$

It was shown in [DFKR14, BDF09] that for a given $\mathcal{S}$, such a family $(\mathcal{Z}^{\zeta})$ exists and is uniquely determined up to a sequence which converges to the identity.

Remark 6.12 Note that, by construction, the maps $\mathcal{Z}^{\zeta}$ obtained in this way are local, so the construction provided in [DFKR14] automatically yields local counter terms.

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Chapter 7
Gauge Theories

In Sect. 4.3 we saw that the space of multilocal on-shell functionals $\mathcal{F}_S(\mathcal{M})$ can be characterized as the 0th homology of the differential complex $(\Lambda\mathcal{V}, \delta_S)$ (see 4.10). The 1st homology of this complex is interpreted as the space of non-trivial local symmetries. Now we discuss the quantization of theories where this homology group is non-trivial, using the BV framework, in the version proposed in [FR12b, FR12a].

7.1 Classical Gauge Theory

Recall from Sect. 4.3 that $\bigwedge \mathcal{V}$, as the space of multivector fields, is equipped with the natural structure of the Schouten bracket $\{.,.\}$ and the BV differential δ_S is locally generated by the bracket in the sense that

$$\delta_S X = \{X, S\}\,.$$

The triple $(\bigwedge \mathcal{V}, \{.,.\}, \delta_S)$ is an algebraic structure called a *differential Gerstenhaber algebra*. In the quantized theory this structure, together with a certain grade 1 operator gives rise to a *BV algebra*.

In this chapter we will use a slightly formal notation $X(\varphi) = \int X_x(\varphi)\frac{\delta}{\delta\varphi(x)}$ introduced in Sect. 3.4 for a vector field $X \in \mathcal{V}$. This notation allows us to make contact with the standard physics literature on the BV formalism, if one identifies $\frac{\delta}{\delta\varphi}$ with a formal generator $\varphi^\ddagger$, called the *antifield*.

The structure described above also appears in theories where local symmetries are present, but there the space of multivector fields on an infinite dimensional manifold $\mathcal{E}$ has to be replaced by the space of multivector fields on a certain *graded* infinite dimensional manifold, which we denote by $\overline{\mathcal{E}}$. We show how this space is constructed on the example of Yang–Mills theories and the free electromagnetic field.

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7.1.1 Dynamics and Symmetries

Let K be a finite dimensional semisimple compact Lie group and $\mathfrak{k} \doteq Lie(K)$ its Lie algebra. Consider the trivial principal bundle $P = M \times K$ over M. We take the point of view that the QFT model on a given spacetime should be first constructed from "simple building blocks", i.e. algebras associated to regions that are topologically trivial, and the global structure should be recovered from the properties of the extension of the local net to a structure where arbitrary regions are allowed [Rob76, Rob77]. Therefore, it is sufficient for our purposes to restrict ourselves to trivial principal bundles.

Globally the configuration space $\mathcal{E}$ for a Yang–Mills theory should is the space of connection 1-forms. After fixing a background connection A_0 we can characterize all the connection 1-forms in terms of one forms on M with values in the associated bundle $\mathfrak{k}_P \doteq P \times_K \mathfrak{k}$. We denote this space by $\Omega^1(M, \mathfrak{k}_P)$ and use it as our configuration space. As the bundle we consider is trivial, we choose A_0 is the trivial connection. For the discussion of the general case see [Zah13].

Definition 7.1 Define the configuration space $\mathcal{E}$ for a Yang–Mills theory as the space $\Omega^1(M, \mathfrak{k}_P)$ of one forms on M with values in the associated bundle $\mathfrak{k}_P \doteq P \times_K \mathfrak{k}$. Since P is assumed to be trivial, $\mathcal{E} = \Omega^1(M, \mathfrak{k})$.

Let $\mathcal{F}$ denote the space of multilocal compactly supported functionals on $\mathcal{E}$ and $\mathcal{V}$ the space of multilocal vector fields. The generalized Lagrangian of the Yang–Mills theory is given by

$$L_{YM}(f)(A) = -\frac{1}{2}\int_M f \ \mathrm{tr}(F \wedge *F), \tag{7.1}$$

where $F = dA + \frac{1}{2}[A, A]$, $*$ is the Hodge operator and tr is the trace in the adjoint representation, given by the Killing–Cartan metric. The equation of motion reads:

$$S'_{YM}(A) = D_A{*}F = 0,$$

where D_A is the covariant derivative induced by the connection A. To analyse $H^1(\bigwedge \mathcal{V}, \delta_{S_{YM}})$, we will explicitly construct non-trivial symmetries of the action corresponding to the Lagrangian (7.1). Let us define the gauge group as the space of vertical K-equivariant compactly supported diffeomorphisms of P:

$$\mathcal{G} := \{\alpha \in \mathrm{Diff}_c(P) | \alpha(p \cdot g) = \alpha(p) \cdot g, \pi(\alpha(p)) = \pi(p), \quad \forall g \in K, p \in P\}.$$

We can also characterize $\mathcal{G}$ as $\Gamma_c(M, P \times_K K)$ and for a trivial bundle P this reduces to $\mathcal{C}^\infty_c(M, K)$. It is known ([Nee04, Glö02, KM97], see also [Nee06, Woc06]) that $\mathcal{C}^\infty_c(M, K)$ can be equipped with the structure of an infinite dimensional Lie group modelled on its Lie algebra $\mathfrak{g}_c = \mathcal{C}^\infty_c(M, \mathfrak{k})$. Since the gauge group is just a subgroup of $\mathrm{Diff}(P)$, it has a natural action on $\Omega^1(P, \mathfrak{k})^K$ by the pullback. This induces the action of $\mathcal{G}$ on $\mathcal{E}$, and the corresponding derived action σ of $\mathfrak{g}_c$ is given by

$$\sigma(c)(A) = dc + [A, c] = D_A c, \quad c \in \mathfrak{g}_c \tag{7.2}$$

The Yang–Mills action is invariant under the transformation (7.2), in the sense that

$$\left\langle S'_{YM}(A), \sigma(c)(A)\right\rangle = 0, \quad \forall A \in \mathcal{E}, \quad \forall c \in \mathfrak{g}_c ,$$

so σ induces a map from $\mathfrak{g}_c$ to $\mathcal{V}$, whose image is contained in the kernel of $\delta_{S_{YM}}$. More generally, we consider $\mathfrak{G} \doteq \mathcal{C}^\infty_{\mathrm{ml}}(\mathcal{E}, \mathfrak{g}_c^{\mathbb{C}})$, the space of multilocal functionals on the configuration space with values in the (complexified) gauge algebra, and a map $\rho : \mathfrak{G} \to \mathcal{V}$ defined by $\rho(\Xi)(A) \doteq \sigma(\Xi(A))A$, i.e.

$$\partial_{\rho(\Xi)} F(A) \doteq \left\langle F^{(1)}(A), \sigma(\Xi(A))A\right\rangle .$$

Remark 7.1 The assignment of $\mathfrak{G}(\mathcal{M})$ to $\mathcal{M}$ induces a functor from **Loc** to **Vec**, which we denote by the same letter, i.e. $\mathfrak{G}$. The fact that the action of $\mathfrak{g}_c$ on $\mathcal{E}$ is local implies that ρ is a natural transformation between $\mathfrak{G}$ and $\mathfrak{V}$, both treated as functors from **Loc** to **Vec**.

Remark 7.2 To see a more geometrical interpretation of the map ρ, note that $\mathfrak{G} \subset \Gamma(\mathcal{E} \times \mathfrak{g}_c)$ (the space of sections of a trivial bundle over $\mathcal{E}$), and we have a morphism of vector bundles $\mathcal{E} \times \mathfrak{g}_c(\mathcal{M}) \to T\mathcal{E}$ given by $(A, c) \mapsto (A, \rho(c)A)$. In this way $\mathcal{E} \times \mathfrak{g}_c$ is made into a Lie algebroid.

7.1.2 The Koszul–Tate Complex

The invariance of the Yang–Mills action under σ implies that $\rho(\mathfrak{G}) \subset \mathrm{Ker}(\delta_{S_{YM}})$. In fact, one can characterize all non-trivial local symmetries in this way, in the sense that for each $X \in \mathrm{Ker}\delta_{S_{YM}}$ there exists an element $\Xi \in \mathfrak{G}$ and a trivial symmetry $I \in \delta_{S_{YM}}(\Lambda^2\mathcal{V})$ such that

$$X = I + \rho(\Xi).$$

We can use this fact to kill the homology in degree one of the differential complex (4.10). We extend the complex by adding $\mathfrak{G}$ in degree 2 and symmetric powers of $\mathfrak{G}$ in higher degrees. This idea is made precise in the following definition.

Definition 7.2 The underlying algebra of the Koszul–Tate complex is

$$\mathcal{KT} \doteq \mathcal{O}_{\mathrm{ml}}(\mathcal{E} \oplus \mathcal{E}^*[1] \oplus \mathfrak{g}^*[2])$$

where $\mathfrak{g}^* \equiv \mathcal{C}^\infty(M, \mathfrak{k}^*)$ and the notation $\mathcal{O}_{\mathrm{ml}}$ is explained in Definitions 3.24, 3.25 and 3.26.

Remark 7.3 Note that $\mathcal{KT}$ contains $\bigwedge \mathcal{V}$ and $\mathfrak{G}$ as subspaces. To see that $\mathcal{KT}$ contains $\mathfrak{G}$, note that

$$\mathcal{O}_{\mathrm{ml}}(\mathfrak{g}^*[2]) \subset \prod_{k=0}^{\infty} \Gamma'_s((\mathfrak{g}^*)^{\boxtimes k} \to M^k) \otimes \mathbb{C}$$

with appropriate WF set conditions. For $\mathcal{O}^1_{\mathrm{ml}}(\mathfrak{g}^*[2])$ the WF set has to be empty, hence $\mathcal{O}^1_{\mathrm{ml}}(\mathfrak{g}^*[2]) = \mathfrak{g}_c \otimes \mathbb{C}$ and therefore

$$\mathcal{O}_{\mathrm{ml}}(\mathcal{KT})\big|_{\#\mathrm{gh}=2} = \mathcal{C}^{\infty}_{\mathrm{ml}}(\mathcal{E}, \mathfrak{g}_c) \otimes \mathbb{C} \oplus \bigwedge^2 \mathcal{V} = \mathfrak{G} \oplus \bigwedge^2 \mathcal{V}\,.$$

For the precise definition of $\mathcal{C}^{\infty}_{\mathrm{ml}}$, see Definition 3.26.

We equip $\mathcal{KT}$ with a differential δ_{KT} which acts on $\bigwedge \mathcal{V}$ as $\delta_{S_{YM}}$, on $\mathfrak{G}$ is given by ρ and we extend to the whole space $\mathcal{KT}$ by means of the graded Leibniz rule. The resulting differential complex is called the Koszul–Tate complex.

$$\ldots \to \bigwedge^2 \mathcal{V} \oplus \mathfrak{G} \xrightarrow{\delta=\delta_{S_{YM}} \oplus \rho} \mathcal{V} \xrightarrow{\delta=\delta_{S_{YM}}} \mathcal{F} \to 0 \tag{7.3}$$

The 0th homology of this complex is $\mathcal{F}_{S_{YM}}$ and higher homologies are trivial, so $(\mathcal{KT}, \delta)$ provides a resolution of $\mathcal{F}_{S_{YM}}$.

7.1.3 The Chevalley–Eilenberg Complex

We have already seen how to characterize the space of on-shell functionals in Yang–Mills theory; now we want to find a homological interpretation for the space of gauge invariant ones. This can be done with the use of the Chevalley–Eilenberg complex.

Definition 7.3 The underlying algebra of the Chevalley–Eilenberg complex is $\mathcal{CE} = \mathcal{O}_{\mathrm{ml}}(\mathcal{E} \oplus \mathfrak{g}[1])$, where $\mathfrak{g} \doteq \mathcal{C}^{\infty}(M, \mathfrak{k})$.

Remark 7.4 The graded manifold $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{g}[1]$ is called the *extended configuration space*.

The Chevalley–Eilenberg differential γ is constructed in such a way that it encodes the action σ of the gauge algebra $\mathfrak{g}$ on $\mathcal{F}$, induced by (7.2). For $F \in \mathcal{F}$ we define $\gamma F \in \mathcal{O}^1_{\mathrm{ml}}\left(\overline{\mathcal{E}}\right) = \mathcal{C}^{\infty}_{\mathrm{ml}}(\mathcal{E}, \mathfrak{g}'^{\mathbb{C}})$ as

$$(\gamma F)(A, c) \doteq (\sigma(c)F)(A) = \left\langle F^{(1)}(A), D_A c \right\rangle, \tag{7.4}$$

where $c \in \mathfrak{g}$. Note that now we have dropped the restriction on the support of gauge parameters. For a form $\omega \in \mathfrak{g}'^{\mathbb{C}}$, which doesn't depend on A we set

$$\gamma\omega(c_1, c_2) \doteq \omega([c_1, c_2])\,.$$

Since γ is required to be nilpotent of order 2 and has to satisfy the graded Leibniz rule, for a general $F \in \mathcal{O}^q_{\mathrm{ml}}\left(\overline{\mathcal{E}}\right)$ we define

$$\begin{aligned}(\gamma F)(A; c_0, \ldots, c_q) \doteq &\sum_{i=0}^{q}(-1)^i \partial_{\sigma(c_i)}(\iota_{(c_0,\ldots,\hat{c}_i,\ldots,c_q)} F)(A)\\ &+\sum_{i<j}(-1)^{i+j} F(A, [c_i, c_j], \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots, c_q),\end{aligned}$$

where the hat over a variable means that this variable is omitted and ι denotes the insertion of n-vector fields into an n-form. The differential complex looks as follows:

$$0 \to \mathcal{F} \xrightarrow{\gamma} \mathcal{O}^1_{\mathrm{ml}}\left(\overline{\mathcal{E}}\right) \xrightarrow{\gamma} \mathcal{O}^2_{\mathrm{ml}}\left(\overline{\mathcal{E}}\right) \to \ldots \tag{7.5}$$

Note that from (7.4) it follows that the kernel of γ in degree 0 consists of all the multilocal functionals invariant under the action σ. Hence $H^0(\mathcal{CE}, \gamma) = \mathcal{F}^{\mathrm{inv}}$, the space of invariants.

Remark 7.5 Note that the assignment of $\mathcal{CE}(\mathcal{M})$ to a spacetime $\mathcal{M}$ induces a covariant functor $\mathfrak{CE}$ from **Loc** to **Vec** and the differential γ can be lifted to a natural transformation.

Remark 7.6 Formally we write elements of $\mathcal{CE}$ as sums of functionals of the form

$$\begin{aligned}&F(A)(c_1, \ldots, c_n)\\ &\quad= \sum_{a_1,\ldots,a_n} \int f(A)(x_1, \ldots, x_n)_{a_1,\ldots,a_n} c_1(x_1)^{a_1} \ldots c_n(X_n)^{a_n} d\mu(x_1) \ldots d\mu(x_n),\end{aligned}$$

where $f(A) \in \Gamma'_a(M^n, \mathfrak{k}^{\boxtimes n})$. Let us denote by C^a_x the evaluation functional $\mathrm{C}^a_x(c) \doteq c^a(x)$ (compare with Sect. 3.3). Clearly $\mathrm{C}^a(x) \in \mathfrak{g}'$. We call these evaluation functionals *ghosts* and we write

$$\begin{aligned}F(A, C) = \sum_{a_1<\ldots<a_n} \int &f(A)(x_1, \ldots, x_n)_{a_1,\ldots,a_n}\\ &\cdot C(x_1)^{a_1} \ldots C(x_n)^{a_n} d\mu(x_1) \ldots d\mu(x_n).\end{aligned}$$

Following Definition 3.21 from Sect. 3.3 we introduce $\frac{\delta_l}{\delta c}$, the graded left derivative on $\mathcal{O}_{\mathrm{ml}}\left(\overline{\mathcal{E}}\right)$.

7.1.4 The BV Complex

The Chevalley–Eilenberg complex and the Koszul–Tate complex fit together into one structure called the *BV complex*, which encodes information about both the equations of motion and the invariants. To see how it arises in a natural way it is worth looking back at the example of scalar fields, which we recalled at the beginning of this section. There, in order to characterize the space of on-shell functionals we needed to consider the space of multilocal vector fields on the configuration space. Now, to take the gauge symmetries into account, we have extended the configuration space into a graded manifold $\overline{\mathcal{E}}$. The space of multivector fields on $\overline{\mathcal{E}}$ is formally given as the algebra of functions on

$$T^*[-1]\overline{\mathcal{E}} = \mathcal{E} \oplus \mathfrak{g}[1] \oplus \mathcal{E}^*[-1] \oplus \mathfrak{g}^*[-2], \tag{7.6}$$

the odd cotangent bundle of $\overline{\mathcal{E}}$, with the *negative grading on the fiber*. Here $\mathfrak{g}^* \doteq \mathcal{C}^\infty(M, \mathfrak{k}^*)$ and $\mathfrak{k}^*$ is the algebraic dual of $\mathfrak{k}$. To give the odd cotangent bundle a meaning in the infinite dimensional situation we again apply Definitions 3.24, 3.25 and 3.26 from Sect. 3.3 and define

$$\mathcal{BV} \doteq \mathcal{O}_{\mathrm{ml}}(T^*[-1]\overline{\mathcal{E}}). \tag{7.7}$$

to be the underlying algebra of the BV complex. This geometric interpretation fits very well with the spirit of the functional approach; we are still working with multilocal functionals but we have to pass from infinite dimensional manifolds to graded infinite dimensional manifolds. Clearly, both $\mathcal{CE}$ and $\mathcal{KT}$ with inverted grading[1] are subalgebras of $\mathcal{BV}$.

On $\mathcal{BV}$ we introduce two gradings:

- the *pure ghost number* #pg, which is inherited from the grading of the Chevalley–Eilenberg complex, i.e. assigns grade $+1$ to elements of $\mathfrak{g}'$.
- the *antifield number* #af, which is inherited from the grading of the Koszul–Tate complex, i.e. assigns grade $+1$ to elements of $(\mathcal{E}^*)'$ and $+2k$ to elements of $(\mathfrak{g}^*)'$.

The total grading of the BV complex is called the *ghost number* #gh and is defined by

$$\#\mathrm{gh} = \#\mathrm{pg} - \#\mathrm{af}$$

and it reflects the grading of the graded manifold (7.6).

We will now extend the differentials δ and γ to the whole of $\mathcal{BV}$. As in the case of the scalar field, $\mathcal{BV}$ can be equipped with the graded Schouten bracket $\{.,.\}$ defined by formula 4.12. It fulfills the following properties:

[1]The grading assigned to the vector fields in the bicomplex $\mathcal{BV}$ is minus the grading of the $\mathcal{KT}$ complex. In the mathematical literature, the resulting structure is called a Beilinson–Drinfeld algebra [BD04], rather than a BV algebra.

$$\{X, Y\} = -(-1)^{(\epsilon_X+1)(\epsilon_Y+1)}\{Y, X\}\,,$$
$$0 = \{\{X, Y\}, Z\} + (-1)^{(\epsilon_X+1)(\epsilon_Y+\epsilon_Z)}\{\{Y, Z\}, X\}$$
$$+ (-1)^{(\epsilon_Z+1)(\epsilon_X+\epsilon_Y)}\{\{Z, X\}, Y\}\,.$$

where $\epsilon_X = \#\mathrm{gh}(X) \mod 2$, similarly for ϵ_Y and ϵ_Z. We use this structure to extend the Koszul–Tate differential to the whole algebra $\mathcal{BV}$ by setting

$$\delta X \doteq \{X, S_{YM}\}, \qquad X \in \mathcal{BV}.$$

The Chevalley–Eilenberg differential can also be written in terms of the bracket in a similar manner. To this end we need to find a natural Lagrangian that implements γ. Firstly, note that the assignment of $\mathcal{BV}(\mathcal{M})$ to spacetimes $\mathcal{M}$ induces a functor $\mathfrak{BV} : \mathbf{Loc} \to \mathbf{Vec}$. We denote this functor by $\mathfrak{BV}$.

Proposition 7.1 *There exists a natural transformation $\theta : \mathfrak{D} \to \mathfrak{BV}$ such that*

$$\gamma X = \{X, \theta_{\mathcal{M}}(f)\},$$

where $f \equiv 1$ on supp f *and* $X \in \mathfrak{CE}(\mathcal{M})$.

Proof γ is a derivation of $\mathfrak{CE}(\mathcal{M})$, so by definition a vector field on $\overline{\mathcal{E}}(\mathcal{M})$. It can be written as

$$\gamma(A, C) = \left\langle \frac{\delta}{\delta A}, \sigma(C)A \right\rangle + \frac{1}{2}\left\langle \frac{\delta_r}{\delta C}, [C, C] \right\rangle,$$

or in the antifield notation

$$\gamma(A, C, A^\ddagger, C^\ddagger) = \left\langle A^\ddagger, \sigma(C)A \right\rangle + \frac{1}{2}\left\langle C^\ddagger, [C, C] \right\rangle,$$

where we use the identification $C^\ddagger = \frac{\delta_r}{\delta C}$. In order to obtain an element of $\mathcal{BV}$, we modify γ by multiplying with a cutoff function. Define

$$\theta_{\mathcal{M}}(f)[A, C, A^\ddagger, C^\ddagger] \doteq \left\langle A^\ddagger, \sigma(fC)A \right\rangle + \frac{1}{2}\left\langle C^\ddagger, f[C, C] \right\rangle.$$

This definition is local and covariant and it can easily be checked that $\theta \in \mathrm{Nat}(\mathfrak{D}, \mathfrak{BV})$. □

Note that Proposition 4.2 easily generalizes to natural Lagrangians in $\mathrm{Nat}(\mathfrak{D}, \mathfrak{BV})$, so we can conclude that for a fixed spacetime $\mathcal{M}$, $\theta_{\mathcal{M}}$ is a generalized Lagrangian. We will denote the action corresponding to the Lagrangian $\theta_{\mathcal{M}}$ by $\gamma_{\mathcal{M}}$, or simply γ, if we work on a fixed spacetime. We can now write

$$\gamma X \doteq \{X, \theta_{\mathcal{M}}(f)\}, \qquad X \in \mathfrak{BV}(\mathcal{M}),$$

where $f \equiv 1$ on the support of X. On a fixed spacetime we will just use the shorthand notation

$$\gamma X = \{X, \gamma\},$$

since γ can be understood both as a differential on $\mathcal{CE}$ and a generalized action. We define the BV differential as the sum

$$s_{BV} = \delta + \gamma = \{., S_{YM} + \gamma\},$$

and call $S^{\mathrm{ext}} \doteq S + \gamma$ the *extended action* of the Yang–Mills theory. In this case $s_{BV}^2 = 0$, since both δ and γ are nilpotent and they anti-commute. The latter is a consequence of the gauge invariance of the equations of motion. For a detailed discussion see [Rej13]. In general, s_{BV} defined above would not be automatically nilpotent and one would need to add to it further terms. To do it systematically, it is convenient to formulate the problem in terms of natural Lagrangians. First we need some notation. Extend the equivalence relation (4.2) to natural Lagrangians depending on several test functions.

Definition 7.4 For $L_1, L_2 \in \mathrm{Nat}(\mathfrak{D}^k, \mathfrak{BV}_{\mathrm{loc}})$, we say that $L_1 \sim L_2$, if:

$$\mathrm{supp}((L_1 - L_2)_{\mathcal{M}}(f_1, \dots, f_k)) \subset \mathrm{supp}(df_1) \cup \dots \cup \mathrm{supp}(df_k), \qquad \forall f_1, \dots, f_k \in \mathfrak{D}^k(\mathcal{M}) \tag{7.8}$$

Next we lift the antibracket to the level of natural transformations.

Definition 7.5 Let $L_1 \in \mathrm{Nat}(\mathfrak{D}^p, \mathfrak{BV}_{\mathrm{loc}})$, $L_2 \in \mathrm{Nat}(\mathfrak{D}^q, \mathfrak{BV}_{\mathrm{loc}})$ be natural Lagrangians. We define

$$\{L_1, L_2\}_{\mathcal{M}}(f_1, \dots, f_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} \{L_{1\mathcal{M}}(f_{\pi(1)}, \dots, f_{\pi(p)}), L_{2\mathcal{M}}(f_{\pi(p+1)}, \dots, f_{\pi(p+q)})\}, \tag{7.9}$$

where P_{p+q} denotes the permutation group.

It was shown in [FR12b] that the nilpotency of s_{BV} is equivalent to the condition that

$$\{L^{\mathrm{ext}}, L^{\mathrm{ext}}\} \sim 0, \tag{7.10}$$

formulated for the natural Lagrangian L^{ext}. The equivalence relation $\sim$ was introduced in Definition 4.3. Condition (7.10) is called the *classical master equation*. One can now formulate the problem of finding the BV differential that extends $\delta + \gamma$ to the problem of finding a natural Lagrangian that satisfies (7.10), with fixed initial terms in degree $\#\mathrm{af} = -1$ and $\#\mathrm{af} = 0$.

Using the two gradings of $\mathcal{BV}$ we construct a bicomplex, whose columns are numbered by #af and rows by #gh. We obtain

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\delta} & \Lambda^2\mathcal{V}\oplus\mathfrak{G} & \xrightarrow{\delta} & \mathcal{V} & \xrightarrow{\delta} & \mathcal{F} & \xrightarrow{\delta} & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \\
\dots & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \mathcal{C}^\infty_{\mathrm{ml}}(\mathcal{E},\mathcal{E}_c\widehat{\otimes}_\pi\mathfrak{g}'^{\mathbb{C}}) & \xrightarrow{\delta} & \mathcal{C}^\infty_{\mathrm{ml}}(\mathcal{E},\mathfrak{g}'^{\mathbb{C}}) & \xrightarrow{\delta} & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & &
\end{array}$$

The grading of the total complex is #gh. Note that the first row is just the Koszul–Tate complex $\mathcal{KT}$ with inverted grading. The higher order rows are obtained by tensoring $\mathcal{KT}$ with powers of $\mathfrak{g}'$ and taking appropriate topological completions, so they are resolutions as well. A standard result in homological algebra tells us that the cohomology of the total complex is given by

$$H^k(\mathcal{BV}, s_{BV}) = H^k(H_0(\mathcal{BV}, \delta), \gamma).$$

Note that taking the 0th homology of δ amounts to going on-shell, while taking the 0th cohomology of γ characterizes gauge invariants. Hence,

$$H^0(\mathcal{BV}, s_{BV}) = \mathcal{F}^{\mathrm{inv}}_S,$$

is the space of gauge invariant on-shell functionals.

7.2 Gauge-Fixing

In the next step we use the differential complex $(\mathcal{BV}, s_{BV})$ to implement the gauge fixing by modifying the extended action. Note that the $\#\mathrm{af} = 0$ term of S^{ext} is still the original Yang–Mills action, which doesn't induce normally hyperbolic equations of motion, so we cannot construct retarded and advanced Green's operators. The idea now is to find an automorphism α of $(\mathcal{BV}, \{.,.\})$ such that $\tilde{S}^{\mathrm{ext}} \doteq \alpha(S^{\mathrm{ext}})$ at $\#\mathrm{af} = 0$ induces normally hyperbolic equations of motion. Such an automorphism can be defined by means of a *gauge fixing fermion*.

Definition 7.6 Let $\Psi \in \mathrm{Nat}(\mathfrak{D}, \mathfrak{BV})$ be a natural Lagrangian of degree $\#\mathrm{gh} = -1$. We call it the gauge fixing fermion and define

$$\alpha_\Psi(X) := \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\{\Psi_{\mathcal{M}}(f), \dots, \{\Psi_{\mathcal{M}}(f)}_{n}, X\}\dots\}, \tag{7.11}$$

where $X \in \mathfrak{BV}(\mathcal{M})$ and $f \equiv 1$ on supp X.

A concrete form of Ψ depends on the choice of gauge fixing and for particular choices one might need to extend $\mathcal{BV}$ with some further generators. This is the case for the Lorentz gauge, which is commonly used in the context of Yang–Mills theory.

We extend the BV-complex by adding to the configuration space the *non-minimal sector*. It consists of the Nakanishi–Lautrup fields $b \in \mathfrak{g}^*[0]$, which we add in degree 0 and the antighosts $\bar{c} \in \mathfrak{g}^*[-1]$, added in degree -1. The new extended configuration space is written explicitly as

$$\overline{\mathcal{E}} = \mathcal{E} \oplus \mathfrak{g}[1] \oplus \mathfrak{g}^*[0] \oplus \mathfrak{g}^*[-1].$$

The underlying algebra of the BV complex is defined as the space of multilocal functionals on $T^*[-1]\overline{\mathcal{E}}$, analogously to (7.7).

The BV differential is extended to the non-minimal sector is such a way that the cohomology of the BV complex remains unchanged. We define: $sF = 0$, and $sG = i\Pi G$ for $F \in S^1\mathfrak{g}'^{\mathbb{C}}$, $G \in \bigwedge^1 \mathfrak{g}'^{\mathbb{C}}$, where $i\Pi$ denotes the grade shift by $+1$ and multiplication of the argument by i. The last operation is just a convention used in physics to make antighosts hermitian. We adopt it to stay consistent with the literature. The extended Lagrangian is now:

$$L^{\text{ext}}(f)[A] = -\frac{1}{2}\int_M f\,\text{tr}(F \wedge *F) + \left\langle \frac{\delta}{\delta A}, \sigma(fC)A \right\rangle + \frac{1}{2}\left\langle \frac{\delta_r}{\delta C}, f[C, C] \right\rangle + \\ -i\left\langle \frac{\delta_r}{\delta \bar{C}}, fB \right\rangle . \tag{7.12}$$

The last term corresponds to the action of s on the non-minimal sector and we used the traditional notation B for evaluation functionals on the space of the Nakanishi–Lautrup fields and $\bar{C}$ for the evaluation functional in the antighosts.

Let us define $\tilde{S}^{\text{ext}} \doteq \alpha_\Psi(S^{\text{ext}})$ and $s \doteq \alpha_\Psi \circ s_{BV} \circ \alpha_\Psi^{-1} = \{., \tilde{S}^{\text{ext}}\}$. Clearly, we have $H^0(\mathcal{BV}, s) = H^0(\mathcal{BV}, s_{BV}) = \mathcal{F}_S^{\text{inv}}(\mathcal{M})$, so we didn't lose the information about the gauge invariant on-shell observables. We want to choose the gauge fixing fermion in such a way that the #af $= 0$ term of the new extended action $\tilde{S}$ induces normally hyperbolic EOM's. This is achieved in the Lorentz gauge, which is implemented by

$$\Psi_{\mathcal{M}}(f)[A] = i\int\limits_M f\left(\frac{\alpha}{2}\kappa(\bar{C}, B) + \langle \bar{C}, *d*A\rangle_{\mathfrak{k}}\right) d\mu, \tag{7.13}$$

where κ is the pairing on $\mathfrak{k}^*$ induced by the Killing–Cartan form on the Lie algebra $\mathfrak{k}$; $\langle ., . \rangle_{\mathfrak{k}}$ is the pairing between $\mathfrak{k}$ and its dual $\mathfrak{k}^*$. The transformed extended Lagrangian takes the form

$$\tilde{L}^{\text{ext}}(f) = -\frac{1}{2}\int\limits_M f\,\text{tr}(F \wedge *F)$$
$$+ i \int\limits_M f \left\langle d\bar{C}, *DC \right\rangle_{\mathfrak{k}}$$
$$- \int\limits_M f \left(\frac{\alpha}{2}\kappa(B, B) + \left\langle B, *^{-1} d * A \right\rangle_{\mathfrak{k}}\right) d\mu + L_{AF}(f)\,, \tag{7.14}$$

where $L_{AF}(f)$ is the term with #af > 0.

Now it is convenient to redefine the gradings again. Let #ta denote the *total antifield number*, which is 1 for all the vector fields on $\overline{\mathcal{E}}$ and 0 for functions. We decompose s with respect to this grading and obtain two terms

$$s = \tilde{\delta} + \tilde{\gamma},$$

where the first term has #ta $= -1$ and the second #ta $= 0$. Both $\tilde{\delta}$ and $\tilde{\gamma}$ are nilpotent and they anti-commute with each other.

$\tilde{\delta}$ is the Koszul differential for the Lagrangian L defined as the #ta $= 0$ term in (7.14). L is a graded functional depending on the multiplet of variables $\varphi = (A, c, \bar{c}, b)$ and we label the components in this multiplet by φ^α. We define S'' as a map from the extended configuration space to the space of vector-valued distributions (see Sect. 3.3 and [Rej11b]) given by

$$\left\langle (S'')_{\beta\alpha}, \psi_1^\alpha \otimes \psi_2^\beta \right\rangle \doteq \left\langle \frac{\delta_r}{\delta\varphi^\beta}\frac{\delta_l}{\delta\varphi^\alpha} L(f), \psi_1^\alpha \otimes \psi_2^\beta \right\rangle,$$

where $\psi_1 \in \overline{\mathcal{E}}$, $\psi_2 \in \overline{\mathcal{E}}_c$ are field configuration multiplets and $f \equiv 1$ on the support of ψ_2. Note that S'' induces normally hyperbolic equations of motion, so using (4.3) we conclude that there are no non-trivial local symmetries and hence $(\mathcal{BV}, \tilde{\delta})$ is a resolution. Explicitly the gauge-fixed equations of motion take the form

$$*^{-1} D{*}DA = -dB - i[d\bar{C}^{*_\kappa}, C]^{*_\kappa}\,, \tag{7.15}$$
$$*^{-1} d{*}A + \alpha B^{*_\kappa} = 0\,, \tag{7.16}$$
$$*^{-1} d{*}DC = 0\,,$$
$$*^{-1} D{*}d\bar{C} = 0\,,$$

where $D\omega = D + [A, \omega]$ denotes the covariant derivative and for $\xi \in \mathfrak{k}^*$, we denote $\xi^{*_\kappa} \doteq \kappa(\xi, .) \in \mathfrak{k}$. Acting with $*^{-1} D*$ on equation (7.15) we obtain:

$$*^{-1} D * dB = -i * [d\bar{C}^{*_\kappa}, *DC]^{*_\kappa}\,.$$

Equation (7.16) is the gauge-fixing condition.

The differential $\tilde{\gamma}$ is called the gauge-fixed BV differential, or just the BRST differential. The action of $\tilde{\gamma}$ on the elements of $\mathcal{BV}$ is summarized in the table below.

	$\tilde{\gamma}$
$F \in \mathcal{F}$	$\langle F^{(1)}, dC + [., C]\rangle$
C	$+\frac{1}{2}[C, C]$
B	0
$\bar{C}$	iB

Because $(\mathcal{BV}, \tilde{\delta})$ is a resolution, we can characterize the space of gauge invariant on-shell functionals as

$$\mathcal{F}^{\text{inv}}_{S_{YM}} = H^0(\mathcal{BV}, s) = H^0(H_0(\mathcal{BV}, \tilde{\delta}), \tilde{\gamma}).$$

The advantage of this reformulation is that now we are working with field equations that are normally hyperbolic and the abstract homological argument tells us that the we still recover the correct space of functionals at the end.

For the action S we can find Δ^{A}_S and Δ^{R}_S and introduce the Peierls bracket $\{., .\}_S$ on $\mathcal{BV}$, analogously to (4.8):

$$\lfloor F, G\rfloor(g, b_\mu, c, \overline{c}_\mu) \doteq \sum_{\alpha,\beta}\left\langle \frac{\delta_l F}{\delta\varphi^\alpha}, \Delta^{\alpha\beta}_S \frac{\delta_r G}{\delta\varphi^\beta}\right\rangle(g, b_\mu, c, \overline{c}_\mu), \qquad \Delta_S = \Delta^{\text{A}}_S - \Delta^{\text{R}}_S.$$

Note that L can be expressed as

$$L(f) = L_{YM}(f) + \tilde{\gamma}(\Psi(f)),$$

and S_{YM} is invariant under $\tilde{\gamma}$, so S is also $\tilde{\gamma}$-invariant. The latter can be expressed as

$$\{L, \tilde{\theta}\} \sim 0, \tag{7.17}$$

where $\tilde{\theta}$ is the natural Lagrangian implementing $\tilde{\gamma}$. This identity allows one to prove the following result, stated in [FR12a] generalized in [BFR13].

Proposition 7.2 (after [BFR13]) *The BRST differential $\tilde{\gamma}$ is a derivation with respect to the Peierls bracket induced by the gauge-fixed action S, modulo the image of $\tilde{\delta}$.*

Proof To prove the result we need to show that

$$m \circ (\tilde{\gamma} \otimes 1 + 1 \otimes \tilde{\gamma}) \circ D_{\Delta_S} = m \circ D_{\Delta_S} \circ (\tilde{\gamma} \otimes 1 + 1 \otimes \tilde{\gamma}),$$

modulo the image of $\tilde{\delta}$. Where

$$D_{\Delta_S} \doteq \sum_{\alpha,\beta} \left\langle {\Delta_S}^{\alpha\beta}, \frac{\delta_l}{\delta\varphi^\alpha} \otimes \frac{\delta_r}{\varphi^\beta} \right\rangle.$$

Note that, in contrast to differential operator D_Δ introduced in Sect. 5.1, D_{Δ_S} depends on φ, since the action S is non-linear. After a short calculation, we obtain the following condition (compare with Prop. 2.3. of [Rej13]):

$$(-1)^{|\sigma|} {K_\varphi}^\sigma{}_\beta(x)\Delta_S(\varphi)^{\beta\alpha}(x, y) + {K_\varphi}^\alpha{}_\beta(y)\Delta_S(\varphi)^{\sigma\beta}(x, y) = \tilde{\gamma}(\Delta_S(\varphi)^{\sigma\alpha}(x, y)), \tag{7.18}$$

where $|\sigma|$ denotes $\#\mathrm{gh}(\varphi^\sigma)$, while K_φ is defined with the use of the evaluation funsctionals Φ^α_x as

$$\gamma_{0\varphi}\Phi^\alpha_x = \sum_\sigma {K_\varphi}^\alpha{}_\sigma(x)\Phi^\sigma_x \equiv (K_\varphi\Phi)^\alpha,$$

and $\gamma_{0\varphi}$ is the linearization of $\tilde{\gamma}$ around φ. In a more compact notation we can write this condition as

$$(-1)^{|\sigma|}(K_\varphi \circ \Delta_S(\varphi))^{\sigma\alpha} + (\Delta_S(\varphi) \circ K^\dagger_\varphi)^{\sigma\alpha} = \tilde{\gamma}(\Delta_S(\varphi)^{\sigma\alpha}),$$

where $K^\dagger_\varphi$ means taking the transpose of the operator-valued matrix and adjoints of its entries. Now we use (7.17) to conclude that we can write

$$\left\langle \frac{\delta_r L(f')}{\delta\varphi^\alpha}, \tilde{\theta}^\alpha(f) \right\rangle = F(f, f'),$$

where $\tilde{\theta}^\alpha(f)$ is the term in $\tilde{\theta}(f)$ which is contracted with $\frac{\delta_l}{\delta\varphi^\alpha}$ and F is a generalized Lagrangian, with the support contained in $\mathrm{supp}(f) \cup \mathrm{supp}(f')$. We can now apply on the both sides the differential operator $\left\langle f_1 \Delta^{\mathrm{R}}_S(\varphi)^{\mu\beta} \circ \frac{\delta_r}{\delta\varphi^\beta}\frac{\delta_l}{\delta\varphi^\kappa}, \Delta^{\mathrm{R}}_S(\varphi)^{\kappa\nu} f_2 \right\rangle$, where $f_1, f_2 \in \mathscr{D}(\mathcal{M})$. We obtain

$$\begin{aligned}
&\left\langle f_1 \Delta^{\mathrm{R}}_S(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta_r}{\delta\varphi^\beta}\frac{\delta_r}{\delta\varphi^\alpha}\frac{\delta_l}{\delta\varphi^\kappa} L(f'), \tilde{\theta}^\alpha(f) \right\rangle, \Delta^{\mathrm{R}}_S(\varphi)^{\kappa\nu} f_2 \right\rangle \\
&\quad + \left\langle f_1 \Delta^{\mathrm{R}}_S(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta_r}{\delta\varphi^\beta}\frac{\delta_r}{\delta\varphi^\alpha} L(f'), \frac{\delta_l\tilde{\theta}^\alpha(f)}{\delta\varphi^\kappa} \right\rangle, \Delta^{\mathrm{R}}_S(\varphi)^{\kappa\nu} f_2 \right\rangle \\
&\quad + \left\langle f_1 \Delta^{\mathrm{R}}_S(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta_r}{\delta\varphi^\alpha}\frac{\delta_l}{\delta\varphi^\kappa} L(f'), \frac{\delta_r\tilde{\theta}^\alpha(f)}{\delta\varphi^\beta} \right\rangle \Delta^{\mathrm{R}}_S(\varphi)^{\kappa\nu} f_2 \right\rangle \\
&\quad + \left\langle f_1 \Delta^{\mathrm{R}}_S(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta_r L(f')}{\delta\varphi^\alpha}, \frac{\delta_l}{\delta\varphi^\kappa}\frac{\delta_r}{\delta\varphi^\beta}\tilde{\theta}^\alpha(f) \right\rangle, \Delta^{\mathrm{R}}_S(\varphi)^{\kappa\nu} f_2 \right\rangle \\
&= \left\langle \frac{\delta_r}{\delta\varphi^\beta}\frac{\delta_l}{\delta\varphi^\kappa} F(f, f'), \Delta^{\mathrm{A}}_S(\varphi)^{\beta\mu} f_1 \otimes (\Delta^{\mathrm{R}}_g)^{\kappa\nu} f_2 \right\rangle
\end{aligned}$$

Setting $f' \equiv 1$ on the support of f we see that the last term is proportional to the equations of motion, so we can ignore it. In the remaining terms on the left-hand side we can make use of the fact that $\Delta_S^{\mathrm{R}}(\varphi)$ is the Green's function for S''. As for the right-hand side, we choose f and f' such that they are equal to 1 on $J^-(\mathrm{supp}\ f_2) \cap J^+(\mathrm{supp}\ f_1)$. Now use the fact that F is local and depends locally on both f and f', and the support of $F(f, f')$ is contained within $\mathrm{supp}\ df \cup \mathrm{supp}\ df'$. It follows that the term on the right-hand side vanishes and because f_1, f_2 were chosen arbitrarily, we obtain the identity

$$\gamma(\Delta_S^{\mathrm{R}}) \overset{\text{o.s.}}{=} (-1)^{|\sigma|}(K_\varphi \circ \Delta_S^R(\varphi))^{\sigma\alpha} + (\Delta_S^R(\varphi) \circ K_\varphi^\dagger)^{\sigma\alpha}.$$

The same argument can be applied to $\Delta_S^{\mathrm{A}}(\varphi)$, so the identity (7.18) follows. This concludes the proof. □

Since $\tilde{\gamma}$ is a derivation with respect to the Peierls bracket modulo the image of $\tilde{\delta}$, $\lfloor .,. \rfloor_{\tilde{S}}$ is well defined on $H_0(\mathcal{BV}, \tilde{\delta})$ and on $\mathcal{F}^{\mathrm{inv}}_{S_{YM}}$. In order to obtain a space which is closed under the Poisson bracket, we extend $\mathcal{BV}$ to the space $\mathcal{BV}_{\mu c}$ of microcausal functionals on $T^*[-1]\overline{\mathcal{E}}$.

The classical net of local algebras on a spacetimes $\mathcal{M}$ is then defined by assignments

$$\mathcal{O} \mapsto (\mathcal{F}^{\mathrm{inv}}_{S,\mu c}(\mathcal{O}), \lfloor .,. \rfloor),$$

where $\mathcal{O} \subset \mathcal{M}$.

We define a functor $\mathfrak{BV}_{\mu c} : \mathbf{Loc} \to \mathbf{Vec}$ by setting $\mathfrak{BV}_{\mu c}(\mathcal{M}) \doteq \mathcal{BV}_{\mu c}(\mathcal{M})$ for the objects. As for the morphisms, let $\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ $F \in \mathcal{BV}_{\mu c}(\mathcal{M}) = \mathcal{O}_{\mu c}(T^*[-1]\overline{\mathcal{E}})$, then

$$\mathfrak{BV}_{\mu c}\chi F(\varphi) \doteq F(\chi^*\varphi),$$

where $\chi^* : \overline{\mathcal{E}}(\mathcal{N}) \to \overline{\mathcal{E}}(\mathcal{M})$ is the natural pull-back map.

Example 7.1 (*Electromagnetic field*) Let us illustrate the general construction described above on the example of the electromagnetic field. The gauge group is $K = U(1)$, so $\mathfrak{k} = \mathbb{R}$ and the Lagrangian takes the form

$$L_{\mathcal{M}}(f)(A) = -\frac{1}{2}\int_M f\,(F \wedge *F).$$

$\mathcal{E}$ is the space of principal connections on $M \times U(1)$ and it is an affine space modeled on $\mathfrak{E}_c(\mathcal{M}) = \Omega^1_c(M)$. As in the case of the free field we can consider the space $\mathcal{F}_{\mathrm{lin}}$ of linear functionals on $\mathcal{E}$. They are of the form

$$F_\beta(A) = \int_M A \wedge *\beta,$$

We can now apply to $\mathcal{F}_{\mathrm{lin}}$ the general BV formalism and compare with the construction of [Dim92]. The equation of motion is given by

$$\boldsymbol{\delta} dA = 0,$$

where $\boldsymbol{\delta} \doteq *^{-1} d*$ ia the codifferential.[2] It follows that the image of δ_S consists of functionals F_β, where $\beta = \boldsymbol{\delta} d\eta$ for some $\eta \in \Omega^1_c(M)$. We can realize $\mathcal{F}_{\mathrm{lin},S}$ as the space of equivalence classes of forms

$$\mathcal{F}_{\mathrm{lin},S} \cong \frac{\Omega^1_c(M)}{\delta d \Omega^1_c(M)}.$$

Now we have to characterize the kernel of γ. It consists of linear functionals that satisfy

$$0 = (\gamma F_\beta)(c) = \int_M dc \wedge *\beta = \int_M c \wedge *\boldsymbol{\delta}\beta.$$

It follows that $\boldsymbol{\delta}\beta = 0$. Let us denote $\Omega^1_{c,\boldsymbol{\delta}}(M) \doteq \{\omega \in \Omega^1_c(M) | \boldsymbol{\delta}\omega = 0\}$. The space of gauge invariant on-shell linear functionals is isomorphic to

$$\mathcal{F}^{\mathrm{inv}}_{\mathrm{lin},S} \cong \frac{\Omega^1_{c,\boldsymbol{\delta}}(M)}{\boldsymbol{\delta} d \Omega^1_c(M)}.$$

This is in agreement with the approach of [Dim92, SDH14, DS13, DL12]. Among these functionals we can distinguish the ones which are constructed from the field strength, i.e. those of the form

$$\int_M dA \wedge *\eta = \int_M A \wedge *\boldsymbol{\delta}\eta = F_{\boldsymbol{\delta}\eta}(A).$$

If $H^3(M)$ is trivial, then all elements of $\mathcal{F}^{\mathrm{inv}}_{\mathrm{lin},S}$ arise from field strength functionals, since all co-closed forms are also co-exact.

7.3 Quantization in the Batalin–Vilkoviski Formalism

In this section we discuss quantization along the lines of [FR12a]. We start with a discussion of the free scalar field. We consider the deformation of δ_{S_0} under the time-ordering operator $\mathcal{T}$. This deformation corresponds to the difference between the ideal generated by EOM's in the classical theory (i.e. with respect to "$\cdot$") and the ideal generated by EOM's with respect to $\cdot_{\mathcal{T}}$. We define

$$\delta^{\mathcal{T}}_{S_0} = \mathcal{T}^{-1} \circ \delta_{S_0} \circ \mathcal{T}, \tag{7.19}$$

[2] Here we use the boldface letter to denote the codifferential, in order to clearly distinguish it from the Koszul–Tate operator δ_S.

Let us first consider regular functionals. Explicit computation shows that, on $\bigwedge \mathcal{V}$,

$$\delta_{S_0}^{\mathcal{T}} := \mathcal{T}^{-1} \circ \delta_{S_0} \circ \mathcal{T} = \delta_{S_0} - i\hbar\triangle,$$

where $\triangle$ acts on regular vector fields $X \in \mathcal{V}_{\text{reg}}$ as

$$\triangle X(\varphi) = -\int \frac{\delta X_x}{\delta\varphi(x)}(\varphi), \qquad \text{where } X(\varphi) = \int X_x(\varphi)\frac{\delta}{\delta\varphi(x)},$$

For general $X \in \bigwedge\mathcal{V}$ we define

$$\triangle X \doteq -\sum_\alpha \left\langle \frac{\delta_r}{\delta\varphi}\frac{\delta_r}{\delta\varphi^\ddagger}X, 1\right\rangle.$$

It's remarkable that the operator $\triangle$ is "almost" a right derivation of $\bigwedge \mathcal{V}_{\text{reg}}$ and the failure is characterized by $\{.,.\}$, i.e.:

$$\triangle(X \wedge Y) - (-1)^{|Y|} \triangle(X) \wedge Y - X \wedge \triangle(Y) = (-1)^{|Y|}\{X, Y\},$$

The triple $(\bigwedge \mathcal{V}_{\text{reg}}, \{.,.\}, \triangle)$ is an example of a *BV algebra*.

Remark 7.7 Physically the relation between $\delta_{S_0}^{\mathcal{T}}$ and δ_{S_0} corresponds to the *Schwinger–Dyson equation*. Let $X(\varphi) = \int X_x(\varphi)\frac{\delta}{\delta\varphi(x)}$. We obtain

$$-(\delta_{S_0}^{\mathcal{T}}X)(\varphi) = \mathcal{T}^{-1}\int\left(\mathcal{T}X_x \cdot \frac{\delta S_0(\varphi)}{\delta\varphi(x)}\right)(\varphi) = \int X_x(\varphi)\frac{\delta S_0(\varphi)}{\delta\varphi(x)} - i\hbar\int\frac{\delta X_x}{\delta\varphi(x)}(\varphi),$$

where $\frac{\delta S_0(\varphi)}{\delta\varphi(x)}$ is a shorthand notation for $\frac{\delta L_0(f)(\varphi)}{\delta\varphi(x)}$, where we take the limit $f \to 1$. Note that

$$\int\mathcal{T}X_x(\varphi)\frac{\delta S_0(\varphi)}{\delta\varphi(x)} = \int\left(\mathcal{T}X_x \star \frac{\delta S_0}{\delta\varphi(x)}\right)(\varphi).$$

Hence,

$$\mathcal{T}\int\left(X_x \cdot \frac{\delta S_0(\varphi)}{\delta\varphi(x)}\right)(\varphi) = i\hbar\mathcal{T}\int\frac{\delta X_x}{\delta\varphi(x)}(\varphi),$$

modulo the $\star$-ideal generated by the EOM's. This is exactly the algebraic Schwinger–Dyson equation.

In the quantization of gauge theories, one simply replaces $\Lambda\mathcal{V}_{\text{reg}}$ with $\mathcal{BV}_{\text{reg}}$ and S_0 is the #af $= 0$ quadratic term of the extended action. The nilpotent operator $\triangle$ is now defined by

$$\triangle X \doteq \sum_\alpha (-1)^{\epsilon_\alpha+1}\left\langle \frac{\delta_r}{\delta\varphi^\alpha}\frac{\delta_r}{\delta\varphi_\alpha^\ddagger}X, 1\right\rangle,$$

where ϵ_α is the grade corresponding to the variable φ^α, modulo 2.

For Yang-Mills theories the quadratic linearized Lagrangian (i.e. the #ta $= 0$ term in L^{ext}) is (compare with [Hol08]):

$$L_0(f) = \frac{1}{2}\int_M f\,\mathrm{tr}(dA \wedge *dA) + i\int_M f\,\langle d\bar{C}, *dC\rangle_{\mathfrak{k}} + \\ - \int_M f\left(\frac{\alpha}{2}\kappa(B,B) + \kappa\langle B, *^{-1}d * A\rangle_{\mathfrak{k}}\right) d\mu_g\,,$$

where the linearizion has been done around the trivial connection $A_0 = 0$ and the basis is $(A, b, c, \bar{c})$. Choose $\alpha = 1$. The equations of motion are

$$S''(z,x) = \delta(z,x)\begin{pmatrix} \square_H + d\boldsymbol{\delta} & -d & 0 & 0 \\ \boldsymbol{\delta} & -1 & 0 & 0 \\ 0 & 0 & 0 & i\square_H \\ 0 & 0 & -i\square_H & 0 \end{pmatrix}(x)\,, \tag{7.20}$$

where $\square_H \doteq -(\boldsymbol{\delta} d + d\boldsymbol{\delta})$.

The retarded and advanced Green's functions for S_0'' are given by

$$\Delta^{\mathrm{A/R}}(x,y) = \begin{pmatrix} \Delta_v^{\mathrm{A/R}} & -d\Delta_v^{\mathrm{A/R}} & 0 & 0 \\ \boldsymbol{\delta}\Delta_v^{\mathrm{A/R}} & d\boldsymbol{\delta}\Delta_v^{\mathrm{A/R}} & 0 & 0 \\ 0 & 0 & 0 & i\Delta_s^{\mathrm{A/R}} \\ 0 & 0 & -i\Delta_s^{\mathrm{A/R}} & 0 \end{pmatrix},$$

where $\Delta_v^{\mathrm{A/R}}$ denotes Green's functions for the Hodge Laplacian on 1-forms and $\Delta_s^{\mathrm{A/R}}$ denotes Green's functions for the Hodge Laplacian on scalar functions. Construction of the free quantum algebra $\mathfrak{A}$ proceeds now the same as in Chap. 5.

Let us now consider a deformation of δ_{S_0} which corresponds to introducting the interaction. Here we treat the scalar field and gauge theories together (they differ by the choice of S_0 and V, but all the relevant equations have the same form), but still restrict to regular functionals, i.e. $V \in \mathcal{BV}_{\text{reg}}$ is a regular interaction term. We define the *quantum BV operator* $\hat{s}$ as

$$\hat{s} \doteq R_V^{-1} \circ \delta_{S_0} \circ R_V\,. \tag{7.21}$$

Analogously to Sect. 6.2.3, R_V is a map from the interacting quantum theory to the free quantum theory, so δ_{S_0} acts on $\mathfrak{A}_0$, as explained in detail in Sect. 5.2. Note that $F \in \mathcal{BV}$ is physical (i.e. and observable) if $R_V(F)$ is in $\ker(\delta_{S_0})$ and is identified with elements that differ by the image of $\hat{s}$, i.e. $R_V(F)$ is in in the same equivalence class as $R_V(F) + R_V(I)$, where $I \in \mathrm{Im}(\delta_0)$. This is the same equivalence relation as the one used in Sect. 5.2 to define the space of on-shell observables. The difference is that now the kernel of δ_{S_0} is not the full space of functionals, but only a subspace, since δ_{S_0} has non-trivial action on antifields.

Let us assume that

$$\delta_{S_0}\left(e_{\mathcal{T}}^{iV/\hbar}\right) = 0. \tag{7.22}$$

This condition reduces to the known *quantum master equation* QME, since

$$-i\hbar\delta_{S_0}\left(e_{\mathcal{T}}^{iV/\hbar}\right) = \left(\frac{1}{2}\{S_0 + V, S_0 + V\} - i\hbar \bigtriangleup V\right) \cdot_{\mathcal{T}} e_{\mathcal{T}}^{iV/\hbar}, \tag{7.23}$$

where we set $\{S_0, S_0\} \equiv 0$, since S_0 doesn't contain antifields. If (7.22) holds, then

$$\hat{s}F = \{F, S_0 + V\} - i\hbar \bigtriangleup F, \tag{7.24}$$

for $F \in \mathcal{F}_{\text{reg}}$. If QME is fulfilled, then the cohomology of $\hat{s}$ characterizes the space of *quantum* gauge invariant on-shell observables.

Now we want to extend the QME and $\hat{s}$ to local functionals. This is done by renormalizing the time-ordered products present in (7.21) and (7.22) using the Epstein–Glaser framework. Clearly, formulas (7.23) and (7.24) are not well defined for local arguments, since $\bigtriangleup$ is singular. Nevertheless, very similar results can be obtained using the anomalous Master Ward Identity ([BD08, Hol08]), which states that there exists a family of maps

$$\widetilde{\bigtriangleup}^n : \mathcal{T}(\mathcal{BV}_{\text{loc}})^{n+1} \to \mathfrak{A}_{\text{loc}}(M), \tag{7.25}$$

which depend locally on their arguments, and the formal power series

$$\widetilde{\bigtriangleup}(V) \doteq \sum_{n=0}^{\infty} \widetilde{\bigtriangleup}^n(V^{\otimes n}; V)$$

fulfills the identity

$$\int \left(e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \frac{\delta V}{\delta\varphi(x)}\right) \star \frac{\delta S_0}{\delta\varphi(x)} = e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \left(\tfrac{1}{2}\{V + S_0, V + S_0\}_{\mathcal{T}} - \widetilde{\bigtriangleup}(V)\right), \tag{7.26}$$

The maps $\widetilde{\bigtriangleup}^n$ can be determined recursively. For an explicit formula, see [BD08, Rej13]. We can now see that the renormalized QME reduces to

$$\delta_{S_0}\left(e_{\mathcal{T}}^{iV/\hbar}\right) = \frac{1}{2}\{S_0 + V, S_0 + V\} - i\hbar\widetilde{\bigtriangleup}(V). \tag{7.27}$$

and the renormalized quantum BV operator takes the form

$$\hat{s}F = \{F, S_0 + V\} - i\hbar \bigtriangleup_V F, \tag{7.28}$$

where $\triangle_V(F) \doteq \frac{d}{d\lambda}\Big|_{\lambda=0} \widetilde{\triangle}(V+\lambda F)$. Note that the renormalized operator $\triangle_V$ depends on V, in contrast to the non-renormalized one $\triangle$. It no longer has the interpretation of a graded Laplacian but is still a functional differential operator.

The construction outlined above allows us to choose V which is a local functional. In the Yang–Mills theory example, which we are focusing on, we can define the local interacting net corresponding to the region $\mathcal{O} \subset \mathcal{M}$ by taking $V = \tilde{L}^{\text{ext}}_{\mathcal{M}}(f) - L_{0\mathcal{M}}(f)$ and choosing $f \equiv 1$ on $\mathcal{O}$. The details of the construction, in particular the algebraic adiabatic limit, can be found in [FR12a].

Remark 7.8 The remarkable aspect of the approach presented above is that using the BV formalism in the sense of [FR12a] we can construct Møller maps R_V that intertwine between theories with *different gauge symmetries*. Indeed, the symmetries are encoded in the Chevalley–Eilenberg differential γ, which then enters into the extended action S^{ext} and consequently into V.

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Chapter 8
Effective Quantum Gravity

The functional approach to pQFT together with the BV framework introduced in Chap. 7 has been successfully applied to gauge theories [FR12a, FR12b] and can also be used in quantization of theories where the local symmetries involve transformation of spacetime points. The first model, where this has been achieved was the quantization of a bosonic string in arbitrary dimension, where the local symmetries are compactly supported diffeomorphisms of the string world-sheet. Recently these methods have also been applied to construct the effective theory of quantum gravity [BFR13]. By this we mean a quantum theory understood in terms of formal power series in both $\hbar$ and the coupling constant, where at each order of the perturbative construction new types of contributions appear, but they are always finite. The theory is not UV complete, so the relevant formal power series do not converge, but one can nevertheless apply it to model physical situations where the quantum gravity (QG) effects are not very strong. This seems to be a sensible Ansatz for the start, as the QG effects which we expect to observe in the near future should be relatively small. Example physical applications include cosmology and black hole physics.

8.1 From LCQFT to Quantum Gravity

The road to quantum gravity is paved with numerous technical and conceptual problems. In contrast to QFT on curved spacetimes, in QG the spacetime structure is dynamical. This means that we cannot treat the metric as a fixed structure, but it interacts with the matter fields. One can partially model this situation using the framework involving *backreaction*. In this formalism one treats matter fields as quantum objects and studies their effect of the metric by inserting the expectation value of the quantum stress-energy tensor in a given state ω into Einstein's equations:

$$\langle T_{\mu\nu}\rangle_\omega = G_{\mu\nu},$$

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where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. In the pAQFT framework this approach has been applied in cosmology and in the study of QFT in black-hole spacetimes, see for example [Hac10, Hac14, DMP09a, DMP09b, DMP11] and a recent book [Hac15].

On the next level of approximation one can split the metric g into the background metric g_0 and a perturbation h and quantize the perturbation as a quantum field on the background g_0. This is the approach which has been taken in [BFR13]. Since this tentative split into background and perturbation is not physical, one needs to show that the predictions of the theory do not depend on the way g is being split. This consistency condition is called *background independence* and we will come back to it later in this chapter. In the pAQFT approach, the background independence of effective QG has been proven in [BFR13] in the sense that a localized change in the background which formally yields an automorphism on the algebra of observables (called relative Cauchy evolution in [BFV03]) is actually trivial, in agreement with the proposal made in [BF07] (see also [FR12c]).

Another conceptual difficulty in quantizing gravity is that the Einstein–Hilbert action is reparametrization invariant, hence the theory has a huge symmetry group, the diffeomorphism group. This means that labeling of spacetime points doesn't have a physical meaning. As a consequence, physical observables have to be diffeomorphism invariant. In the framework of [BFR13] the characterization of diffeomorphism invariant observables is given by means of the BV formalism. The abstract setting looks very similar to the one discussed in Chap. 7, but there remains the difficulty in finding non-trivial elements of the 0th cohomology of the BV differential. In [FR12b, Rej11a] it has been proposed to use locally covariant quantum fields in the sense of Definition 2.66 as diffeomorphism invariant physical quantities in effective classical and quantum gravity. In [BFR13] this idea has been refined and a more explicit characterisation of these objects has been given in terms of *relational observables*. The latter are conceptually similar to the notion of observables introduced by Rovelli in the framework of loop quantum gravity [Rov02] and later used and further developed in [Dit06, Thi06].

Finally, there is a known difficulty that quantum gravity, as a QFT, is power counting non-renormalizable. We deal with this problem by using the Epstein–Glaser renormalization scheme, which allows us to calculate finite contributions to renormalized time-ordered products to every order in $\hbar$ and the coupling constant. The theory is then interpreted as an effective theory with the property that only finitely many parameters have to be considered below a fixed energy scale [GW96]. Another possible direction would be to make contact with the *asymptotic safety* approach. A theory is called asymptotically safe if there exists an ultraviolet fixed point of the renormalisation group flow[1] with only finitely many relevant directions [Wei79]. Results supporting this perspective have been obtained by Reuter et al. [Reu98, RS02].

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[1]Reuter et al. [Reu98, RS02] define the renormalisation group flow in terms of Wetterich equations [Wet93]. We expect that this notion is related to the Stückelberg-Petermann renormalization group we have introduced in Sect. 6.3 A result connecting the later to the Wilsonian flow has been already obtained in [BDF09].

8.2 Dynamics and Symmetries

We consider a formal metric g on a M given by $g = g_0 + \lambda h$, where λ is a formal parameter. In this setting $\mathcal{M} \equiv (M, g_0)$ is the background manifold and λh is the metric perturbation. For the effective theory of gravity the configuration space is $\mathcal{E}(\mathcal{M}) = \Gamma((T^*M)^{\otimes_s 2})$.

Definition 8.1 In this section $\mathcal{F}(\mathcal{M})$ denotes the space of multilocal functionals on $\mathcal{E}(\mathcal{M})$ that are Laurent series in λ.

Note that for the physical interpretation to make sense, the observables we obtain at the end cannot depend on negative powers of λ.

Definition 8.2 A functional derivative of $F \in \mathcal{F}(\mathcal{M})$ is defined by

$$\left\langle F^{(1)}(h), h_1 \right\rangle \doteq \frac{1}{\lambda} \left\langle F^{(1)}(h), h_1 \right\rangle,$$

where $F^{(1)}(h)$ means that we take the functional derivative of coefficients of F at each order in λ separately.

It is convenient to use natural units, where λ (identified with the square of the coupling constant) has a dimension of length, so h has a dimension of 1/length. The action used in quantization must be dimensionless, so we use L^{EH}/λ^2, where L^{EH} is the Einstein–Hilbert Lagrangian

$$L^{EH}_{(M,g)}(f)[h] \doteq \int R[g] f \, d\mu_g, \quad g = g_0 + \lambda h, \ h \in \mathcal{E}(\mathcal{M}), \tag{8.1}$$

The Euler-Lagrange derivative of L^{EH} is defined as in Definition 4.2, i.e.

$$\langle S^{EH\prime}_{\mathcal{M}}(h), h_1 \rangle \doteq \langle L^{EH}_{\mathcal{M}}(f)^{(1)}(h), h_1 \rangle,$$

where $f \equiv 1$ on supp h_1. For general relativity, local symmetries arise from infinitesimal diffeomorphism symmetries. The compactly supported diffeomorphism group acts on $\mathcal{E}(\mathcal{M})$ via the pullback: $\rho_{\mathcal{M}}(\alpha)h = (\alpha^{-1})^* h$, where $\alpha \in \mathrm{Diff}_c(\mathcal{M})$, $t \in \mathcal{E}(\mathcal{M})$. This induces the action of $\mathfrak{X}(\mathcal{M}) \equiv \Gamma(TM)$ via the Lie derivative:

$$\rho_{\mathcal{M}}(\xi)\lambda h \doteq \frac{d}{dt}\Big|_{t=0} (\exp(-t\xi))^* g = \pounds_\xi (g_0 + \lambda h),$$

where $\xi \in \mathfrak{X}(\mathcal{M})$. On the level of functionals $F \in \mathcal{F}$ we obtain

$$\rho_{\mathcal{M}}(\xi) F[h] \doteq \left\langle F^{(1)}(h), \pounds_\xi (g + \lambda h) \right\rangle.$$

Definition 8.3 $\mathcal{CE}(\mathcal{M})$, the underlying algebra of the Chevalley–Eilenberg complex, is defined as the space of Laurent series in λ with coefficients in multilocal functionals on the extended configuration space $\overline{\mathcal{E}}(\mathcal{M}) \doteq \mathcal{E}(\mathcal{M}) \oplus \mathfrak{X}[1](\mathcal{M})$. The Chevalley–Eilenberg differential γ_{CE} is defined as in Sect. 7.1.3.

Definition 8.4 $\mathcal{BV}(\mathcal{M})$, the underlying algebra of the BV complex is defined as $\mathcal{O}_{\mathrm{ml}}(T^*[-1]\overline{\mathcal{E}}(\mathcal{M}))$; compare with (7.7).

As in gauge theories, $\mathcal{BV}(\mathcal{M})$ is equipped with the natural structure of the Schouten bracket $\{.,.\}$ and the BV differential s_{BV} can be expressed as

$$s_{BV} = \{., S^{EH} + \gamma\},$$

where we choose the natural Lagrangian θ, which represents γ as

$$\theta^{CE}_{(M,g_0)}(f)[h, C, h^\ddagger, C^\ddagger] = \langle h^\ddagger, \pounds_{fC}g\rangle - \frac{1}{2}\langle C^\ddagger, f[C, C]\rangle, \tag{8.2}$$

The space of "gauge-invariant" on-shell observables, $\mathcal{F}^{\mathrm{inv}}_S$ is characterized by

$$\mathcal{F}^{\mathrm{inv}}_S(\mathcal{M}) = H^0(\mathcal{BV}(\mathcal{M}), s_{BV}).$$

In the next step we perform the gauge-fixing. In general relativity fixing the gauge means essentially fixing the coordinate system. For the specific choice of gauge we need, we have to extend the BV complex by adding auxiliary scalar fields: 4 scalar antighosts $\bar{c}_\mu$ in degree -1 and 4 scalar Nakanishi-Lautrup fields b_μ, $\mu = 0, \ldots, 3$ in degree 0. The new extended configuration space is again denoted by $\overline{\mathcal{E}}(\mathcal{M})$. We define s on functionals of antighosts and Nakanishi-Lautrup fields by fixing the action of s on evaluation functionals $\overline{C}_\mu(x)$, $B_\mu(x)$:

$$\begin{aligned} s(\overline{C}_\mu) &= iB_\mu - \pounds_C \overline{C}_\mu, \\ s(B_\mu) &= \pounds_C B_\mu. \end{aligned}$$

To implement these new transformation laws we add to the Lagrangian a term

$$\left\langle \overline{C}^\ddagger_\mu, ifB_\mu - \pounds_{fC}\overline{C}_\mu, \right\rangle + \left\langle B^\ddagger_\mu, \pounds_{fC}B_\mu, \right\rangle.$$

Next, we perform an automorphism α_Ψ of $(\mathcal{BV}(\mathcal{M}), \{.,.\})$ such that the part of the transformed action which doesn't contain antifields has a well posed Cauchy problem. We choose the gauge-fixing Fermion as

$$\Psi_{(M,g_0)}(f)[h, c, \bar{c}_\mu, b_\mu] = i\sum_{\mu,\nu}\int(\partial_\mu\bar{c}_\nu g^{\mu\nu} - \tfrac{1}{2}b_\mu\bar{c}_\nu\boldsymbol{\kappa}^{\mu\nu})f d\mu_g, \tag{8.3}$$

where $g = g_0 + \lambda h$ and $\boldsymbol{\kappa}$ is a non-degenerate 2-form on $\mathbb{R}^4$. The explicit appearance of this form in the gauge fixing Fermion is related to the choice of a dual pairing for Nakanishi-Lautrup fields. Note that expression (8.3) is explicitly coordinate-dependent. This is necessary because we need to break the reparametrization invariance of the action. The new terms appearing in the α_Ψ-transformed action arise from

$$\{\Psi(f'), L^{\text{ext}}(f)\}(h, c, \overline{c}_\mu, b_\mu) = -\int(\partial_\mu(fb_\nu)g^{\mu\nu} - \tfrac{1}{2}f_2 b_\mu b_\nu \boldsymbol{\kappa}^{\mu\nu})\sqrt{-\det g}d^4x \\ + i\int(\partial_\mu\overline{c}_\nu\sqrt{-\det g}\, g^{\mu\alpha}\partial_\alpha(fc^\nu))d^4x,$$

where $f' \equiv 1$ on supp f. We rewrite the above expression as

$$-\int\partial_\mu(fb_\nu)g^{\mu\nu}d\mu_g + \int\left(\tfrac{1}{2}fb_\mu b_\nu\right)\boldsymbol{\kappa}^{\mu\nu}d\mu_g + i\int c^\nu f\Box_g\overline{c}_\nu d\mu_g,$$

where $\Box_g$ is the d'Alembertian constructed from the formal metric $g = g_0 + \lambda h$. We denote the first term in the above formula by $L^{GF}(f)$ and the second by $L^{FP}(f)$ (gauge-fixing and Fadeev-Popov terms, respectively). The full transformed Lagrangian is given by:

$$L^{\text{ext}} = L^{EH} + L^{GF} + L^{FP} + L^{AF}, \tag{8.4}$$

where L^{AF} is the term containing antifields.

The variables of the theory (i.e. the components φ^α of the multiplet $\varphi \in \overline{\mathcal{E}}(\mathcal{M})$ are now: the metric $h \in \mathcal{E}(\mathcal{M})$, the Nakanishi-Lautrup fields b_μ, the antighosts $\overline{c}_\mu$, $\mu = 0, \ldots, 3$ (scalar fields) and the ghosts $c \in \mathfrak{X}(\mathcal{M})$. As in Sect. 7.2 we introduce a new grading #ta, which is equal to 0 for functions on $\overline{\mathcal{E}}(\mathcal{M})$ and equal to 1 for all the vector fields on $\overline{\mathcal{E}}(\mathcal{M})$.

The new field equations are now equations for the full multiplet $\varphi = (h, b_\mu, c, \overline{c}_\mu)$, $\mu = 0, \ldots, 3$ and are derived from the #ta = 0 term of L^{ext}, denoted by L. The α_Ψ-transformed BV differential $s = \alpha_\Psi \circ s_{BV} \circ \alpha_\Psi^{-1}$ is given by:

$$s = \{., S^{\text{ext}}\} = \gamma + \delta.$$

The action of γ on $\mathcal{F}(\mathcal{M})$ and the evaluation functionals $B_\mu, C, \overline{C}_\mu$ is summarized in the table below:

	γ
$F \in \mathcal{F}$	$\langle F^{(1)}, \pounds_C g\rangle$
C	$-\frac{1}{2}[C, C]$
B_μ	$\pounds_C B_\mu$
$\overline{C}_\mu$	$iB - \pounds_C\overline{C}_\mu$

The equations of motion are:

$$R_{\lambda\nu}[g] = -i\partial_\lambda \overline{C}_\alpha \, \partial_\nu C^\alpha - \partial_{(\lambda} B_{\nu)} \tag{8.5}$$

$$\Box_g C^\mu = 0 \tag{8.6}$$

$$\Box_g \overline{C}_\mu = 0 \tag{8.7}$$

$$\frac{1}{\sqrt{-\det g}}\partial_\mu(g^{\mu\nu}\sqrt{-\det g}) = B_\mu \boldsymbol{\kappa}^{\mu\nu} \tag{8.8}$$

where B_μ, C^μ, $\overline{C}_\mu$ are evaluation functionals. The equation for B^μ is obtained by using the Bianchi identity satisfied by $R_{\lambda\nu}[g]$ in Eq. (8.5) and takes the form

$$\Box_g B_\mu = 0. \tag{8.9}$$

8.3 Linearized Theory

Definition 8.5 The linearized Lagrangian L_0 is defined as

$$L_0 \doteq \frac{\lambda^2}{2} L^{(2)}{}_{(M,g_0)}(g_0, 0, 0, 0),$$

where L is the #af $= 0$ term in L^{ext}. We introduce the notation $S^{\text{ext}} = \lambda^2(S_0 + S_I)$.

Remark 8.1 The dimensionless action we use in the quantization is S^{ext}/λ^2, so its quadratic part is of order 0 in λ. If g_0 is not a solution to Einstein's equations, the λ-linear term in S^{ext} doesn't vanish and negative powers of λ appear in the interatction S_I. Formally, we solve this problem by introducing another parameter μ, so that $\frac{1}{\lambda} L^{(1)}_{(M,g_0)}(g_0, 0, 0, 0) \equiv \mu J_{g_0}$, where J_{g_0} is the source term, linear in h. Our observables are now formal power series in both λ and μ. For the physical interpretation we restrict ourselves to spacetimes where g_0 is a solution and put $\mu = 0$, but algebraically we can perform our construction of quantum theory on arbitrary backgrounds.

We choose from now on the gauge where the pairing $\boldsymbol{\kappa}$ is obtained from g_0 expressed in a fixed coordinate system. Let us introduce some notation.

Definition 8.6 The divergence operator div : $\Gamma((T^*M)^{\otimes_s 2}) \to \Gamma(T^*M)$ is defined by

$$(\text{div } t)_\alpha \doteq \frac{1}{\sqrt{-\det g_0}} g_0^{\beta\mu} \partial_\mu (t_{\beta\alpha}\sqrt{-\det g_0}).$$

Definition 8.7 We define a product

$$\langle u, v\rangle_{g_0} = \int_M \langle u^\#, v\rangle d\mu_{g_0},$$

where u, v are tensors of the same rank and # is the isomorphism between T^*M and TM induced by g_0.

Definition 8.8 The formal adjoint of div with respect to the product $\langle ., .\rangle_{g_0}$ is denoted by div^* : $\Gamma(T^*M) \to \Gamma((T^*M)^{\otimes_s 2})$.

Definition 8.9 The trace reversal operator $G : (TM)^{\otimes 2} \to (TM)^{\otimes 2}$ is defined by

$$Gt = t - \frac{1}{2}(\mathrm{tr}\ t) g_0. \tag{8.10}$$

We have $\mathrm{tr}(Gt) = -\mathrm{tr}\ t$ and $G^2 = \mathrm{id}$. Using the notation above we write $L_{0\mathcal{M}}$ in the form:

$$\begin{aligned} &L_{0\mathcal{M}}(f)[h, c, \overline{c}_\mu, b_\mu] \\ &= \int_M \frac{\delta}{\delta g}(Rf d\mu)\Big|_{g_0}(h) + 2i \sum_{\nu=0}^{3} \left\langle d\overline{c}_\nu, d(fc^\nu)\right\rangle_{g_0} + \left\langle fb, \mathrm{div}(Gh) - \tfrac{1}{2}b\right\rangle_{g_0}, \end{aligned}$$

where $\frac{\delta}{\delta g}(Rd\mu)\Big|_{g_0}(h)$ denotes the linearization of the Einstein–Hilbert Lagrangian density around the background g_0 and b is a 1-form on M constructed from b_ν's in the fixed coordinate system. The linearized EOM's are

$$S''_{\mathcal{M}}(z, x) = \delta(z, x) \begin{pmatrix} -\frac{1}{2}\left(\Box_L G + 2G\ \mathrm{div}^* \circ \mathrm{div} \circ G\right) & G \circ \mathrm{div}^* & 0 & 0 \\ \mathrm{div} \circ G & -1 & 0 & 0 \\ 0 & 0 & 0 & -i\Box_H \\ 0 & 0 & i\Box_H & 0 \end{pmatrix}(x), \tag{8.11}$$

where the variables are $(h, b, c^0, \ldots, c^3, \overline{c}_0, \ldots, \overline{c}_3)$; $\Box_H = \boldsymbol{\delta} d$ is the Hodge Laplacian, $\boldsymbol{\delta} \doteq *^{-1} d *$ is the codifferential and $\Box_L$ is given in local coordinates by

$$(\Box_L h)_{\alpha\beta} = \nabla^\mu \nabla_\mu h_{\alpha\beta} - 2(R_{(\alpha}{}^{\mu} h_{\beta)\mu} + R_{(\alpha}{}^{\mu\nu}{}_{\beta)} h_{\mu\nu}). \tag{8.12}$$

The retarded and advanced propagators for S_0 are given by:

$$\Delta^{\mathrm{R/A}}_{S_0}(x, y) = -2 \begin{pmatrix} G\Delta^{\mathrm{R/A}}_t & G\Delta^{\mathrm{A/R}}_t G \circ \mathrm{div}^*{}_y & 0 & 0 \\ \mathrm{div}_x \circ \Delta^{\mathrm{R/A}}_t & \mathrm{div}_x \circ \Delta^{\mathrm{R/A}}_t G \circ \mathrm{div}^*{}_y + \frac{1}{2}\delta_4 & 0 & 0 \\ 0 & 0 & 0 & -i\Delta^{\mathrm{R/A}}_s \\ 0 & 0 & i\Delta^{\mathrm{R/A}}_s & 0 \end{pmatrix},$$

where $\Delta^{\mathrm{R/A}}_t$ is the retarded/advanced Green's function for $\Box_L$ on symmetric covariant 2-tensors, $\Delta^{\mathrm{R/A}}_s$ is the retarded/advanced Green's function for $\Box_H$ on scalars, δ_4 denotes the Dirac delta in 4 dimensions and subscript $_y$ in div^*_y means that the operator should be applied on the second variable. We introduce the Peierls bracket on $\mathcal{BV}(\mathcal{M})$:

$$\lfloor F, G \rfloor_{g_0} = \sum_{\alpha,\beta} \left\langle \frac{\delta_l F}{\delta \varphi^\alpha} \Delta^{\alpha\beta}_{S_0} \frac{\delta_r G}{\delta \varphi^\beta} \right\rangle,$$

where $\Delta_{S_0} = \Delta^{\mathrm{R}}_{S_0} - \Delta^{\mathrm{A}}_{S_0}$. As in Chap. 7 we extend $\mathcal{BV}(\mathcal{M})$ to the space $\mathcal{BV}_{\mu c}(\mathcal{M})$ of microcausal functionals, which is closed under the Peierls bracket.

8.4 Quantization

For the definition of the $\star$-product we need a 2-point function $\Delta^{+}_{S_0}$. Assume that $\Delta^{+}_{S_0}$ is of the form:

$$\Delta^{+}_{S_0} = -2\begin{pmatrix} G\omega_t & \omega_t^T \ \mathrm{div}^*{}_y & 0 & 0 \\ \mathrm{div}_x \ \omega_t \ \mathrm{div}_x & G\,\omega^T \ \mathrm{div}^*{}_y & 0 & 0 \\ 0 & 0 & 0 & -i\omega_s \\ 0 & 0 & i\omega_s & 0 \end{pmatrix}, \tag{8.13}$$

In this case, the conditions for $\Delta^{+}_{S_0}$ to be a Hadamard 2-point function reduce to:

$$\omega_{s/t}(x, y) - \omega_{s/t}(y, x) = i\Delta_{s/t}(x, y), \tag{8.13a}$$

$$\Box_L\,\omega_t = 0,\ \Box_H\,\omega_s = 0, \tag{8.13b}$$

$$\mathrm{WF}(\omega_{s/t}) \subset C_+, \tag{8.13c}$$

$$\overline{\omega_{s/t}(x, y)} = \omega_{s/t}(y, x). \tag{8.13d}$$

We choose arbitrary parametrices ω_t, ω_s of $\Box_L$ and $\Box_H$ respectively. Their existence was already proven in [SV00] (the paper actually discusses general wave operators acting on vector-valued field configurations). Now, from a parametrix, one can construct a bisolution using a following argument: let ω be a Hadamard parametrix for the hyperbolic operator $O = \Box_L$ or$= \Box_H$. By definition, $O_x\omega = h,\ O_y\omega = k$, hold for some smooth functions h and k. Let χ be a smooth function such that supp χ is past-compact and supp$(1-\chi)$ is future-compact (see Definition 2.38). Define

$$G_\chi \doteq \Delta^{\mathrm{R}}\chi + \Delta^{\mathrm{A}}(1-\chi).$$

Clearly G_χ is a right inverse for O. A Hadamard bisolution ω_χ can now be obtained as

$$\omega_\chi \doteq (1 - G_\chi O) \circ \omega \circ (1 - OG_\chi^T).$$

From Hadamard solutions for $\Box_L$ and $\Box_H$ we can then construct $\Delta^{+}_{S_0}$ using (8.13).

We define $\mathfrak{A}(\mathcal{M})$ as in Sect. 5.1 and introduce the interaction using the Epstein–Glaser renormalization. The rest of the construction follows exactly the scheme described in Chaps. 6 and 7, so the abstract net of algebras can be defined without problems on arbitrary backgrounds. There are two questions that remain. First is the existence of a non-trivial gauge invariant observable and the other is the background independence of the resulting theory. We will address these problems in the following two sections, referring to the results of [BFR13].

8.5 Relational Observables

Abstractly we have characterized the classical gauge-invariant observables as

$$\mathcal{F}_S^{\text{inv}}(\mathcal{M}) = H^0(\mathcal{BV}(\mathcal{M}), s),$$

but there is no a priori reason for this space to be non-empty. To prove that non-trivial observables exist, we will construct some explicitly. We start with heuristic reasoning. If we think about an experiment that locally probes the geometric structure of spacetime, we can associate to our setup a causally convex spacetime region $\mathcal{O}$ of spacetime $\mathcal{M}$ and an observable Φ localised in $\mathcal{O}$, which we measure. Since the experiment has a finite resolution, we don't really measure values of the geometric data at a point. There is always some smearing involved. For example, we can model the measurement of the Ricci curvature R by defining our observable quantity as $\Phi(f) = \int f(x)R(x)$, where f is a smearing function with $\text{supp}(f) \subset \mathcal{O}$. In certain situations, we can think of the measured observable as a perturbation of the fixed background metric. This is for example the case if we want to observe gravitational waves. We make a tentative split: $g = g_0 + \lambda h$. The situation is pictured on the Fig. 8.1. To formulate what diffeomorphism invariance means, we first have to answer the question: What happens if we move our experimental setup to a different region $\mathcal{O}'$? Now to compare $\Phi_{(\mathcal{O},g)}(f)$ and $\Phi_{(\mathcal{O}',\alpha_* g)}(\alpha_* f)$ we need to know what it means to have "the same observable in a different region". We can give sense to this statement using the notion of locally covariant quantum fields, as defined in Definition 2.66.

Recall that the condition for Φ to be a locally covariant field reads

$$\Phi_{\mathcal{O}}(f)(\chi^* h) = \Phi_{\mathcal{M}}(\chi_* f)(h). \tag{8.14}$$

For a fixed spacetime $\mathcal{M}$ we define the action of the infinitesimal diffeomorphism algebra $\mathfrak{X}(\mathcal{M})$ on maps $\Phi_{\mathcal{M}}$ as

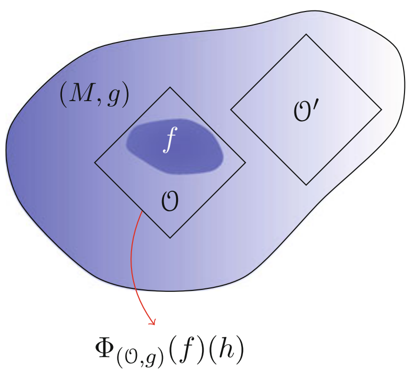

Fig. 8.1 Experimental situation while probing the spacetime geometry

$$(\rho(\xi)\Phi_{(M,g_0)})(f)[h] \doteq \left\langle (\Phi_{(M,g_0)}(f))^{(1)}(h), \pounds_\xi g \right\rangle + \Phi_{(M,g_0)}(\pounds_\xi f)[h],$$

where $\xi \in \mathfrak{X}(\mathcal{M})$.

Definition 8.10 We say that a locally covariant quantum field $\Phi_{\mathcal{M}}$ is diffeomorphism invariant if

$$\rho(\xi)\Phi_{\mathcal{M}} \equiv 0\,, \quad \forall\, \mathcal{M} \in \mathrm{Obj}(\mathbf{Loc}), \xi \in \mathfrak{X}(\mathcal{M}).$$

Example 8.1 As an example of a diffeomorphism invariant field we can take

$$\Phi_{1(M,g_0)}(f)[h] = \int R[g] f\, d\mu_g\,, \quad \text{where } g = g_0 + \lambda h.$$

Note that both the scalar curvature and the volume form depend on the full metric g. However if we take a field defined as

$$\Phi_{2(M,g_0)}(f)[h] = \int R[g] f\, d\mu_{g_0}\,, \quad \text{where } g = g_0 + \lambda h,$$

it is still a locally covariant quantum field, but it is no longer diffeomorphism invariant.

The reasoning presented above suggests that locally covariant quantum fields are good candidates for diffeomorphism invariant quantities. The question remains, how to relate these with non-trivial elements of $H^0(\mathcal{BV}(\mathcal{M}), s)$.

For a fixed spacetime $\mathcal{M}$ and a locally covariant quantum field Φ, a test function specifies the geometrical setup for an experiment, and the concrete choice of $f \in \mathfrak{D}(\mathcal{M})$ can be made only if we fix a coordinate system. In our framework, following [BFR13], we realize the choice of a coordinate system by introducing four scalar fields X^μ, which will parametrize points of spacetime. We can write any test function $f \in \mathfrak{D}(\mathcal{M})$ in the coordinate basis induced by $X \doteq (X^\mu | \mu = 0, \ldots, 3)$, so if we fix $\boldsymbol{f} : \mathbb{R}^4 \to \mathbb{R}$, then the change of $f = X^* \boldsymbol{f}$ due to the change of the coordinate system is realized through the change of scalar fields X^μ.

Definition 8.11 For a natural transformation $\Phi \in \mathrm{Nat}(\mathfrak{D}, \mathfrak{F})$ we obtain a map

$$\Phi_{\mathcal{M},\boldsymbol{f}}(h, X) \doteq \Phi_{\mathcal{M}}(X^* \boldsymbol{f})(h).$$

As long as we keep $\mathcal{M}$ fixed, we drop $\mathcal{M}$ in $\Phi_{\mathcal{M},\boldsymbol{f}}$ and use the notation $\Phi_{\boldsymbol{f}}$ instead. $\Phi_{\boldsymbol{f}}$ is a function of the metric and the coordinate system and transforms under infinitesimal diffeomorphisms according to

$$(\rho(\xi)\Phi_{\boldsymbol{f}}) = \left\langle \frac{\delta \Phi_{\boldsymbol{f}}}{\delta g}\Big|_X, \pounds_\xi g \right\rangle + \sum_{\mu=0}^{3} \left\langle \frac{\delta \Phi_{\boldsymbol{f}}}{\delta X^\mu}\Big|_g, \pounds_\xi X^\mu \right\rangle. \tag{8.15}$$

This is still not satisfactory, since the X^μ's are not dynamical variables, so there are no vector fields in $\mathcal{BV}(\mathcal{M})$ that would implement the second term in the above transformation. To solve this problem, we can replace X^μ with some scalars X^μ_g, $\mu = 0, \dots, 3$ that depend locally on the metric. They could be, for example, scalars constructed from the Riemann curvature tensor and its covariant derivatives. The caveat is that some particularly symmetric spacetimes do not admit such metric dependent coordinates, since in such cases the curvature might vanish (for a detailed discussion see [CHP09, HC10]). If matter field are present, one can construct X^μ's using the matter fields. A known example is the Brown–Kuchař model [BK95], which uses dust fields.

Let us denote by β the map $g \mapsto (X^0_g, \dots, X^3_g)$ and define

$$\Phi^\beta_{\boldsymbol{f}}(h) \doteq \Phi_{\boldsymbol{f}}(g, X_g), \tag{8.16}$$

where $g = g_0 + \lambda h$. Note that for (8.16) to be well defined we need to choose $\boldsymbol{f}$ and β in such a way that the support of $\boldsymbol{f}$ is contained in the interior of the image of M inside $\mathbb{M}$ under the quadruple of maps $X^\mu_{g_0}$. If this can be done, then a functional of the form (8.16) is an element of $H^0(\mathcal{BV}(\mathcal{M}), s)$ if and only if Φ is a diffeomorphism invariant locally covariant quantum field. Observables of this type are interpreted as relational observables, since they capture the relations between different quantities constructed from the metric (and possibly also matter fields) and they do not rely on absolute labeling of spacetime points. Instead, the map β provides relative labels X_g, which change with g.

Example 8.2 Assume that for a fixed background $\mathcal{M} = (M, g_0)$ we can choose $\boldsymbol{f}$ and β in such a way that $f \doteq X^*_{g_0}\boldsymbol{f}$ is compactly supported. Then an example $\mathfrak{X}(\mathcal{M})$-invariant functional can be obtained from the scalar curvature

$$\begin{aligned}\Phi^\beta_{\boldsymbol{f}}(h) &= \int_M R[g_0]\boldsymbol{f}(X_{g_0})d\mu_{g_0} \\ &+ \lambda\left(\int_M \boldsymbol{f}(X_{g_0})\,\frac{\delta}{\delta g}(Rd\mu)\bigg|_{g_0}(h) + \int_M R[g_0]\partial_\mu \boldsymbol{f}(X_{g_0})\,\frac{\delta X^\mu_g}{\delta g}\bigg|_{g_0}(h)\right) + \mathcal{O}(\lambda^2).\end{aligned}$$

To summarize, we have three ways to realize diffeomorphism invariant quantities in classical gravity:

- as locally covariant fields $\Phi_\mathcal{M} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{F}(\mathcal{M})$,
- as functionals of the metric and the coordinates $\Phi_{\boldsymbol{f}}(h, X)$,
- as relational observables $\Phi_{\boldsymbol{f}}(., X_g)$.

8.6 Background Independence

The last issue which we have to discuss is the background independence. We have made a tentative split into the free and interacting Lagrangian, relying on the Taylor expansion around the background metric g_0. Now we want to see what will happen if we slightly perturb the background. If the theory is background independent then physical quantities do not change under such a perturbation. Following [BFR13] we sketch the argument that this is true in effective QG constructed by the methods of pAQFT.

In [BF07] it was proposed that a condition of background independence can be formulated by means of relative Cauchy evolution. Let us fix a spacetime $\mathcal{M}_1 = (M, g_1) \in \text{Obj}(\textbf{Loc})$ and choose Σ_- and Σ_+, two Cauchy surfaces in $\mathcal{M}_1$, such that Σ_+ is in the future of Σ_-. Consider another globally hyperbolic metric g_2 on M, such that $k \doteq g_2 - g_1$ is compactly supported and its support K lies between Σ_- and Σ_+. Let us take $\mathcal{N}_\pm \in \text{Obj}(\textbf{Loc})$ that embed into $\mathcal{M}_1, \mathcal{M}_2$, via $\chi_{1\pm}, \chi_{2\pm}$ and $\chi_{i\pm}(\mathcal{N}_\pm), i = 1, 2$ are causally convex neighborhoods of $\Sigma_\pm$ in $\mathcal{M}_i$. For a visual representation of this construction, see Fig. 8.2. We use the time-slice axiom to define isomorphisms $\alpha_{\chi_{i\pm}} \doteq \mathfrak{A}\chi_{i\pm}$ and the free relative Cauchy evolution is an automorphism of $\mathfrak{A}(\mathcal{M}_1)$ given by $\beta_{0k} = \alpha_{0\chi_{1-}} \circ \alpha^{-1}_{0\chi_{2-}} \circ \alpha_{0\chi_{2+}} \circ \alpha^{-1}_{0\chi_{1+}}$. It was shown in [BFV03] that the functional derivative of β_{0k} with respect to k is the commutator with the free stress-energy tensor. A different proof of this result has been given in [BFR13] with the use of the *principle of perturbative agreement*, which is a condition introduced by Hollands and Wald in [HW05] and recently proven in a more general context in [DHP15]. Following these ideas, we introduce a map $\tau^{\text{ret}} : \mathfrak{A}(\mathcal{M}_2) \to \mathfrak{A}(\mathcal{M}_1)$, such that τ^{ret} maps $\Phi_{\mathcal{M}_2}(f)$ to $\Phi_{\mathcal{M}_1}(f)$ (modulo the image of δ_0), if the support of f lies outside the causal future of K. Physically it means that free algebras $\mathfrak{A}(\mathcal{M}_1)$ and $\mathfrak{A}(\mathcal{M}_2)$ are identified in the past of K. Analogously, we introduce a map τ^{adv},

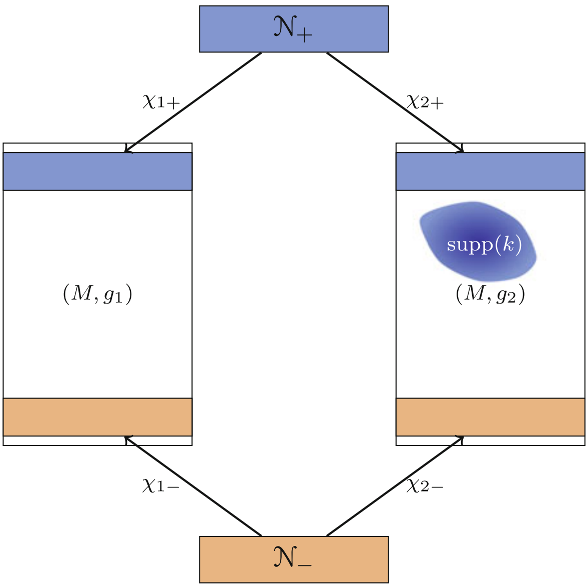

Fig. 8.2 Embeddings of neighborhoods of Cauchy surfaces into spacetimes $\mathcal{M}_1 = (M, g_1)$ and $\mathcal{M}_2 = (M, g_2)$

which identifies the free algebras in the future. The free relative Cauchy evolution is then expressed as

$$\beta_{0k} \doteq \tau^{\text{ret}}_{g_1 g_2} \circ (\tau^{\text{adv}}_{g_1 g_2})^{-1}, \tag{8.17}$$

As we choose to work off-shell, we define τ^{ret} as the classical retarded Møller operator constructed in [DF02]. The perturbative agreement is a condition that, on shell,

$$\tau^{\text{ret}}_{g_1 g_2} \circ \mathcal{S}_2 = \mathcal{S}_{S_{0\mathcal{M}_2} - S_{0\mathcal{M}_1}}. \tag{8.18}$$

Here $\mathcal{S}_{S_{0\mathcal{M}_1} - S_{0\mathcal{M}_2}}$ denotes the relative S-matrix constructed with the interaction $S_{0\mathcal{M}_1} - S_{0\mathcal{M}_2}$ and the background metric g_1, while $\mathcal{S}_2$ is the S-matrix constructed on $\mathcal{M}_2$ with the $\mathcal{T}_{\mathcal{M}_2}$ product. The perturbative agreement condition for $\tau^{\text{adv}}_{g_1 g_2}$ is analogous to (8.18). A straightforward calculation shows that The functional derivative of β_{0k} with respect to $k \doteq g_2 - g_1$ can now be easily calculated, yielding

$$\frac{\delta}{\delta k_{\mu\nu}} \beta_{0k} \left(e^{i\Phi_{\mathcal{M}_1 f'}/\hbar}_{\mathcal{T}_{\mathcal{M}_1}} \right) \Big|_{g_1} = -\frac{i}{\hbar} \left[T_0^{\mu\nu}, e^{i\Phi_{\mathcal{M}_1 f'}/\hbar}_{\mathcal{T}_{\mathcal{M}_1}} \right]_\star ,$$

where $T_0^{\mu\nu}$ is the stress-energy tensor of the linearized theory.

To obtain the relative Cauchy evolution for the full interacting theory, we use the quantum Møller maps introduced in (6.12). The following theorem has been proven in [BFR13]

Theorem 8.1 *The functional derivative $\Theta^{\mu\nu}$ of the relative Cauchy evolution can be expressed, on-shell, as*

$$\Theta^{\mu\nu}(\Phi_{\mathcal{M}_1}(f)) \overset{o.s.}{=} \frac{i}{\hbar}[R_{V_1}(\Phi_{\mathcal{M}_1}(f)), R_{V_1}(T^{\mu\nu})]_\star,$$

where $T^{\mu\nu}$ is the stress-energy tensor of the extended action and one can define the time-ordered products in such a way that $T^{\mu\nu} = 0$ holds, so the interacting theory is background independent.

Proof We write the interacting relative Cauchy evolution as:

$$\beta = R_{V_1}^{-1} \circ \tau^{\text{ret}}_{g_1 g_2} \circ R_{V_2} \circ A_{V_2}^{-1} \circ (\tau^{\text{adv}}_{g_1 g_2})^{-1} \circ A_{V_1}.$$

The condition of background independence is

$$R_{V_1}^{-1} \circ \tau^{\text{ret}}_{g_1 g_2} \circ R_{V_2} = A_{V_1}^{-1} \circ \tau^{\text{adv}}_{g_1 g_2} \circ A_{V_2}.$$

Differentiating with respect to $k_{\mu\nu}$ yields a condition

$$[R_{V_1}(\Phi_{\mathcal{M}_1 f'}), R_{V_1}(T(\eta))]_\star \overset{\text{o.s.}}{=} 0,$$

where

$$T(\zeta) \doteq \langle T^{\mu\nu}{}_f, \zeta_{\mu\nu} \rangle = \left\langle \frac{\delta L^{\text{ext}}_{\mathcal{M}_2 f}}{\delta k_{\mu\nu}}\Big|_0, \zeta_{\mu\nu} \right\rangle$$

To prove that the infinitesimal background independence is fulfilled, we have to show that $T(\eta) = 0$ in the cohomology of $\hat{s}$. This is easily done, as

$$T(\zeta) = \left\langle \frac{\delta S^{\text{ext}}_{\mathcal{M}_2}}{\delta k_{\mu\nu}}\Big|_0, \zeta^{\mu\nu} \right\rangle = \left\langle \frac{\delta S^{\text{ext}}_{\mathcal{M}_1}}{\delta h_{\mu\nu}}\Big|_0, \zeta^{\mu\nu} \right\rangle = s\left\langle h^{\ddagger}, \zeta \right\rangle = \hat{s}\left\langle h^{\ddagger}, \zeta \right\rangle,$$

where h is the perturbation metric. The last equality follows from the fact that the anomaly can always be removed for linear functionals [BD08]. This concludes the argument, so the theory is perturbatively background independent. □

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Glossary

Table A.1 contains a list of frequently used symbols, together with clarifications.

Table A.1 Glossary of frequently used notation

Symbol	Clarification
$\mathcal{X}$	A topological vector space, $\mathcal{X} = (X, \tau)$
ω	A state
$\Vert.\Vert$	A norm
τ	Topology (family of open sets)
$\mathcal{B}$	Bornology (family of bounded sets)
$\mathcal{E}$	Configuration space of the theory, $\mathcal{E} = \Gamma(E \to M)$
$\overline{\mathcal{E}}$	Extended configuration space used in the presence of symmetries, typically $\overline{\mathcal{E}} = \mathcal{E} \oplus \mathfrak{g}[1]$
E^*	Dual bundle of E
$\mathcal{E}^*$	$\Gamma(E^* \to M)$
$\mathcal{E}'$	Topological dual of $\mathcal{E}$, i.e. $\Gamma'(E \to M)$
$\mathscr{E}(\Omega)$	$\mathcal{C}^\infty(\Omega, \mathbb{R})$
$\mathscr{D}(\Omega)$	$\mathcal{C}^\infty_c(\Omega, \mathbb{R})$
$\mathcal{H}$	Typically a Hilbert space, in Sect. 6.5.1 a Hopf algebra
$\mathscr{B}(\mathcal{H})$	The space of bounded operators on the Hilbert space $\mathcal{H}$
$F^{(n)}(\varphi_0) \equiv \frac{\delta^n F}{\delta \varphi^n}(\varphi_0)$	nth functional derivative of the functional F at point φ_0
$T^*[-1](.)$	Odd cotangent bundle of …(cotangent bundle with the degree shift on the fibre)

(continued)

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Table A.1 (continued)

Symbol	Clarification
$\mathcal{C}^\infty(.)$	Smooth functions on …
$\mathcal{O}(.)$	Functions on a graded manifold …
$\mathcal{O}_{\mathrm{loc}}(.)$	Local functions on a graded manifold …
$\mathcal{O}_{\mathrm{ml}}(.)$	Multilocal functions on a graded manifold …
$\mathcal{V}_{\mathrm{loc}}$	Multilocal vector fields on $\mathcal{E}$
$\mathcal{V}$	Multilocal vector fields on $\mathcal{E}$
$\bigwedge\mathcal{V}$	Multilocal multi-vector fields on $\mathcal{E}$
$\mathcal{KT}$	Underlying algebra of the Koszul–Tate complex, $\mathcal{KT} \doteq \mathcal{O}_{\mathrm{ml}}(\mathcal{E} \oplus \mathcal{E}^*[1] \oplus \mathfrak{g}^*[2]) = \mathcal{C}^\infty_{\mathrm{ml}}(\mathcal{E}, \Lambda\mathcal{E}^{*\prime}\widehat{\otimes}_\pi S^\bullet\mathfrak{g}^{*\prime} \otimes \mathbb{C})$
$\mathcal{CE}$	Underlying algebra of the Chevalley–Eilenberg complex, $\mathcal{CE} \doteq \mathcal{O}_{\mathrm{ml}}(\mathcal{E} \oplus \mathfrak{X}[1]) = \mathcal{C}^\infty_{\mathrm{ml}}(\mathcal{E}, \Lambda\mathfrak{g}')$
$\mathcal{BV}$	Underlying algebra of the BV complex, $\mathcal{BV} \doteq \mathcal{O}_{\mathrm{ml}}(T^*[-1]\overline{\mathcal{E}}) = \mathcal{C}^\infty_{\mathrm{ml}}(\mathcal{E}, S^\bullet\mathfrak{g}_c\widehat{\otimes}_\pi\Lambda\mathcal{E}_c\widehat{\otimes}_\pi\Lambda\mathfrak{g}' \otimes \mathbb{C})$
$r_{\lambda V}$	Classical Møller operator for the interaction V, see Sect. 4.6
$R_{\lambda V}$	Quantum Møller operator for the interaction V, see Sect. 6.2.4
$\mathcal{S}(.)$ on $\mathfrak{A}_{\mathrm{reg}}[[\lambda]]$	Non-renormalized S-matrix, see Definition 6.3
$\mathcal{S}(.)$ on $\mathfrak{A}_{\mathrm{loc}}[[\lambda]]$	Renormalized S-matrix, see Definition 6.7
Spaces of functionals on $\mathcal{E}$	
$\mathcal{F}_{\mathrm{loc}}$	The space of local functionals on, see Definition 3.14
$\mathcal{F}$	The space of multilocal functionals, i.e. the algebraic completion of $\mathcal{F}_{\mathrm{loc}}$ with respect to the pointwise product
$\mathcal{F}_S$	On-shell multilocal functionals for the action S
$\mathcal{F}^{\mathrm{inv}}$	Gauge-invariant multilocal functionals
$\mathcal{F}^{\mathrm{inv}}_S$	Gauge invariant on-shell multilocal functionals for the action S
$\mathcal{F}_{\mu c}$	The space of microcausal local functionals, see Definition 4.9
$\mathcal{F}_{s\mu c}$	The space of strongly microcausal local functionals, see Definition 4.9
$(\mathcal{F}_{\mathrm{loc}})^{\otimes n}_{\mathrm{pds}}$	The subspace of $(\mathcal{F}_{\mathrm{loc}})^{\otimes n}$ spanned by $F_1 \otimes \cdots \otimes F_n$, where $F_1, \dots, F_n \in \mathcal{F}_{\mathrm{loc}}$ have pairwise disjoint supports

(continued)

Table A.1 (continued)

Symbol	Clarification
Products of functionals	
$\cdot$	Pointwise product, $(F \cdot G)(\varphi) = F(\varphi)G(\varphi)$
$\star$	Star product, $(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), \left(\frac{i}{2}\Delta_{S_0}\right)^{\otimes n} G^{(n)}(\varphi)\right\rangle$
$\star_H$	Star product, $(F \star_H G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), (\Delta^+_{S_0})^{\otimes n} G^{(n)}(\varphi)\right\rangle$
$\cdot_{\mathcal{T}}$ on $\mathcal{F}_{\mathrm{reg}}$	Non-renormalized time-ordered product, $F \cdot_{\mathcal{T}} G \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}, \left(i\Delta^{\mathrm{D}}_{S_0}\right)^{\otimes n} G^{(n)}\right\rangle = \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$
$\cdot_{\mathcal{T}}$ on $\mathcal{T}(\mathcal{F})$	Renormalized time-ordered product, $F \cdot_{\mathcal{T}} G = \mathcal{T}(\mathcal{T}^{-1} \cdot \mathcal{T}^{-1}G)$
$\cdot_{\mathcal{T}_H}$ on $\mathcal{F}_{\mathrm{reg}}$	Non-renormalized time-ordered product, $F \cdot_{\mathcal{T}_H} G \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}, \left(\Delta^{\mathrm{F}}_{S_0}\right)^{\otimes n} G^{(n)}\right\rangle = \mathcal{T}^H(\mathcal{T}^{H-1}F \cdot \mathcal{T}^{H-1}G)$
$\star_V$ on $\mathcal{F}_{\mathrm{reg}}$	Interacting star product for the interaction V, $F \star_V G \doteq R_V^{-1}(R_V F \star R_V G)$
Algebras	
$\mathfrak{A}$ in Chap. 2	An algebra
$\mathfrak{A}$ in Chaps. 5–8	The quantum algebra of the free theory, see Definition 5.2
$\mathfrak{A}^H$	$(\mathcal{F}_{\mu c}[[\hbar]], \star_H)$
$\mathfrak{A}_{\mathrm{reg}}$	$(\mathcal{F}_{\mathrm{reg}}[[\hbar]], \star)$
$\mathfrak{A}_{\mathrm{loc}}$	$\mathcal{T}(\mathcal{F}_{\mathrm{loc}}) \subset \mathfrak{A}$, where $\mathcal{T} \doteq \alpha_H^{-1} \circ \mathcal{T}^H$
$\mathfrak{A}^H_{\mathrm{loc}}$	$\mathcal{T}^H(\mathcal{F}_{\mathrm{loc}}) \subset \mathfrak{A}^H$
Categories	
Loc	Category of spacetimes
FLoc	Category of framed spacetimes
Obs	Category of unital C^*-algebras
$\mathbf{Obs}_c$	The category of locally convex topological Poisson algebras
$\mathbf{Obs}_p$	The category of locally convex topological unital $*$-algebras
Vec	Category of locally convex topological vector spaces

(continued)

Table A.1 (continued)

Symbol	Clarification
Functors	
$\mathfrak{F}_{\mathrm{loc}}$	Covariant functor of local functionals
$\mathfrak{F}$	Covariant functor of multilocal functionals
$\mathfrak{D}$	Covariant functor of test function spaces
$\mathfrak{E}$	Contravariant functor of configuration spaces
$\mathfrak{E}_c$	Covariant functor of compactly supported configurations spaces
Propagators	
$\Delta^{\mathrm{A}}_{S_0}$	Advanced Green's function
$\Delta^{\mathrm{R}}_{S_0}$	Retarded Green's function
Δ_{S_0}	Causal propagator $\Delta^{\mathrm{R}}_{S_0} - \Delta^{\mathrm{A}}_{S_0}$
$\Delta^{\mathrm{D}}_{S_0}$	Dirac propagator $\frac{1}{2}(\Delta^{\mathrm{R}}_{S_0} + \Delta^{\mathrm{A}}_{S_0})$
$\Delta^{+}_{S_0}$	2-point function $\frac{i}{2}(\Delta^{\mathrm{R}}_{S_0} - \Delta^{\mathrm{A}}_{S_0}) + H$
$\Delta^{\mathrm{F}}_{S_0}$	Feynman propagator $\frac{i}{2}(\Delta^{\mathrm{R}}_{S_0} + \Delta^{\mathrm{A}}_{S_0}) + H$
Functional differential operators	
D_{A}	$\left\langle \Delta^{\mathrm{A}}_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle$, e.g. $D_{\mathrm{A}}(F \otimes G)(\varphi_1, \varphi_2) = \left\langle F^{(1)}(\varphi_1), \Delta^{\mathrm{A}}_{S_0} G^{(1)}(\varphi_2) \right\rangle$
D_{R}	$\left\langle \Delta^{\mathrm{R}}_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle$
D_{D}	$\left\langle \Delta^{\mathrm{D}}_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle$
D_{Δ}	$\left\langle \Delta_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle$
D_{+}	$\left\langle \Delta^{+}_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle$
D_{F}	$\left\langle \Delta^{\mathrm{F}}_{S_0}, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle$
D^{ij}_{F}	$\langle \Delta^{\mathrm{F}}_{S_0}, \frac{\delta^2}{\delta\varphi_i \delta\varphi_j} \rangle$
$\mathcal{D}_{\mathrm{D}}$	$\left\langle \Delta^{\mathrm{D}}_{S_0}, \frac{\delta^2}{\delta\varphi^2} \right\rangle$, e.g. $\mathcal{D}_{\mathrm{D}}(F)(\varphi) = \left\langle \Delta^{\mathrm{D}}_{S_0}, F^{(2)}(\varphi) \right\rangle$
$\mathcal{D}_{\mathrm{F}}$	$\left\langle \Delta^{\mathrm{F}}_{S_0}, \frac{\delta^2}{\delta\varphi^2} \right\rangle$
$\mathcal{D}_{H}$	$\left\langle H, \frac{\delta^2}{\delta\varphi^2} \right\rangle$
α_H	$e^{\frac{\hbar}{2}\mathcal{D}_H}$
α_H^{-1}	$e^{-\frac{\hbar}{2}\mathcal{D}_H}$
α_w	$\alpha_w F \doteq \lim_{N\to\infty} \alpha_{w_N} F$, see Eq. (6.5) and the discussion above it
$\mathcal{T}$ on $\mathcal{F}_{\mathrm{reg}}$	$\mathcal{T} = e^{\frac{i\hbar}{2}\mathcal{D}_{\mathrm{D}}}$
$\mathcal{T}$ on $\mathcal{F}$	$\bigoplus_n \alpha_H^{-1} \circ \mathcal{T}^H_n \circ \alpha_w \circ m^{-1}$, see Definition 6.9
$\mathcal{T}^H$ on $\mathcal{F}_{\mathrm{reg}}$	$\mathcal{T} = e^{\frac{i\hbar}{2}\mathcal{D}_{\mathrm{F}}}$
$\mathcal{T}^H$ on $\mathcal{F}_{\mathrm{loc}}$	α_w

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