

Mathematics for Industry 31

Nikos I. Kavallaris
Takashi Suzuki

Non-Local Partial Differential Equations for Engineering and Biology

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Aims & Scope

The meaning of “Mathematics for Industry” (sometimes abbreviated as MI or MfI) is different from that of “Mathematics in Industry” (or of “Industrial Mathematics”). The latter is restrictive: it tends to be identified with the actual mathematics that specifically arises in the daily management and operation of manufacturing. The former, however, denotes a new research field in mathematics that may serve as a foundation for creating future technologies. This concept was born from the integration and reorganization of pure and applied mathematics in the present day into a fluid and versatile form capable of stimulating awareness of the importance of mathematics in industry, as well as responding to the needs of industrial technologies. The history of this integration and reorganization indicates that this basic idea will someday find increasing utility. Mathematics can be a key technology in modern society.

The series aims to promote this trend by (1) providing comprehensive content on applications of mathematics, especially to industry technologies via various types of scientific research, (2) introducing basic, useful, necessary and crucial knowledge for several applications through concrete subjects, and (3) introducing new research results and developments for applications of mathematics in the real world. These points may provide the basis for opening a new mathematics oriented technological world and even new research fields of mathematics.

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Non-Local Partial Differential Equations for Engineering and Biology

Mathematical Modeling and Analysis

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*I dedicate this monograph to my beloved wife
Yota and to my dear parents.*

Nikos I. Kavallaris

Preface

Many classical partial differential equations, arising in modeling of various physical and biological phenomena, are only taking into consideration local spatial or time variables, ignoring any possible spatial dependence on the neighboring points as well as neglecting any feasible memory effects. For instance, the classical (local) nonlinear heat equation provides the rate of change of temperature at every (localized) point and for any time where the source term overlooking any possible effects from any nearby points and ignoring the history of the heating process. In contrast, *non-local* models consider any possible reliance of the involved physical quantities on the evolution of the inspected process all over the nearby spatial points or any dependence on preceding times. Such a *non-local dependence* usually stems from a distance interaction or from several *conservation laws*. Some of the first *non-local equations* appeared in the literature are encountered in the field of phase transition and are related to theories due to van der Waals, Ginzburg & Landau and Cahn & Hilliard [5]. Lately, a wide variety of *non-local equations* emerged in the literature with significant applications in engineering, astrophysics, and biology. For instance such models with *non-local* spatial terms are encountered in the Ohmic heating production [29, 30], in the shear banding formation in metals being deformed under high strain rates [1, 2], in the theory of gravitational equilibrium of polytropic stars [27], in the investigation of the fully turbulent behavior of real flows, using invariant measures for the Euler equation [3], in population dynamics [8], in modeling aggregation of cells via interaction with a chemical substance (chemotaxis) [32], just to mention a few of them. Some more applications of *non-local equations* with memory (integral) terms can be found in [31]. Therefore, *non-locality* is not a technical obstruction to scientific research but it actually provides an essence of what happens in reality, and for that purpose, its mathematical study can provide very useful predictions in many areas of applications. In general, *non-local models* provide more accurate predictions compared to their local counterparts since they actually use all the available information regarding the evolution of the inspected process. On the other hand, the presence of the *non-local* terms might be responsible for the lack of some fundamental features, that share the local analogue problems,

like the maximum principle [30, 31]. Additionally, most of the *non-local* problems exhibit quite rich dynamics, which is usually more complicated to the dynamics of their local counterparts. In particular, the long-time behavior of *non-local* parabolic equations might be more complex than the one of their local complements, see for example [4, 6, 7, 8]. Another very intriguing phenomenon which is basically due to the presence of the *nonlinearity*, but whose impact grows when a *non-local term* is also present, is the occurrence of *finite-time blow-up* or *finite-time quenching*, when solutions of nonlinear equations cannot be extended after a finite time. Notably, blow-up and quenching fight against the *well-posedness* of nonlinear evolution equations, since both, under some circumstances, they rule out the possibility of existence of global in time solutions. Detailed profiles of the *blowing up* and *quenching* solutions, however, are heavily associated with the form of the *nonlinearity*, and the whole mechanisms of these phenomena have not yet been clarified in many equations as it happens with the *blow-up* of the solutions for the famous example of the *Navier–Stokes equation*. Still a lot of efforts have been paid since the pioneering work of Fujita [9, 10] on the *blow-up* of semilinear parabolic equations, and many important outcomes have been produced. At the earliest stage, studies on *blow-up* and *quenching* were thought to be related only within the field of pure mathematics where several toy models were investigated. However, recently it has been recognized that many realistic models are reduced to semilinear equations with *non-local terms* and many of them exhibit the phenomena of finite *blow-up* and *quenching*. This monograph is devoted to this type of *nonlinear* partial differential equations, investigating both their mathematical modeling and their mathematical analysis.

Part I of the current monograph is devoted to the investigation of some non-local models linked with applications from engineering. Chapter 1 focuses on the study of the following *non-local* model associated with electrostatic MEMS control

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda}{(1-u)^2 \left(1 + \alpha \int_{\Omega} \frac{1}{1-u}\right)^2} \quad \text{in } \Omega \times (0, T) \quad (1)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T) \quad (2)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \quad (3)$$

where $u(x, t)$ denotes the deformation of an elastic membrane which is part of the MEMS device. Here and henceforth, λ stands for a positive parameter. In the first place, the construction of the above model is presented. To this end, we describe the two main physical problems which build up the operation of an idealized MEMS device: the elastic and the electric problem. In the second place, we proceed with its mathematical analysis. First, the structure of the set of radially symmetric steady-state solutions is investigated together with their stability. Then, the circumstances under which *finite-time quenching*, or otherwise called *touching down*

in the MEMS context, occurs are investigated following the approach developed in [11, 16]. Finally Chap. 1 closes up with the investigation of a hyperbolic variation of Eqs. (1)–(3) along the lines of the approach introduced in [14].

Chapter 2 discusses some *non-local models* describing Ohmic heat production in various industrial processes. In the first part of the chapter, the process of food sterilization by using Ohmic heating is considered and the following one-dimensional *non-local model* is formulated

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\lambda f(u)}{\left(\int_{-1}^1 f(u) dx\right)^2} \quad \text{in } (0, 1) \times (0, T), \quad (4)$$

$$u(0, t) = u_x(1, t) = 0 \quad \text{in } (0, T), \quad (5)$$

$$u(x, 0) = u_0(x) \quad \text{in } (0, 1), \quad (6)$$

along the lines of [24, 29]. Here, $u(x, t)$ stands for the temperature of the sterilized food, while the nonlinearity $f(u)$ represents either electrical conductivity or resistivity and is taken to be positive. Next, and under different circumstances, a hyperbolic variation of Eqs. (4)–(6) with a *non-local* convection velocity is built up following the approach introduced in [25]. Both of these *non-local* models are inspected in terms of their stability and the occurrence of finite-time *blow-up*, where the latter in the current context means food burning. Different approaches should be followed though depending on the monotonicity of the nonlinearity f , since no maximum principle is available for the *non-local* parabolic problem (Eqs. (4)–(6)) when f is increasing. In the case where f is decreasing, some useful estimates of the blow-up (burning) time are given via the method developed in [19, 28]. Finally, the hyperbolic problem is treated via the method of characteristics. The second part of Chap. 2 deals with another application of Ohmic heating process in the thermistor device which is modeled by the following

$$\frac{\partial u}{\partial t} = \nabla(k(u)\nabla u) + \frac{\lambda f(u)}{\left(\int_{\Omega} f(u)\right)^2} \quad \text{in } \Omega \times (0, T), \quad (7)$$

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (8)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \quad (9)$$

where $u(x, t)$ is the temperature across the thermistor device, while $\frac{\partial}{\partial \nu}$ stands for the normal outward derivative to $\partial\Omega$ and $0 \leq \beta \leq \infty$, $k(u) > 0$. The finite-time blow-up, which suggests either the destruction of the thermistor device or the failure of the model, is investigated via the methods developed in [18].

Chapter 3 debates an application arising in the process of linear friction welding applied in metallurgy. Initially, the following one-dimensional *non-local* model is constructed

$$u_t = u_{xx} + \frac{e^u}{\left(\int_0^\infty e^u dx\right)^{1+\alpha}}, \quad \text{in } (0, \infty) \times (0, T), \quad \alpha > 0, \quad (10)$$

$$u_x(0, t) = 0, \quad \lim_{x \rightarrow \infty} u_x(x, t) = -1, \quad \text{in } (0, T), \quad (11)$$

$$u(x, 0) = u_0(x), \quad \text{in } (0, \infty), \quad (12)$$

for the soft-material case, where $u(x, t)$ serves as the temperature across the welding region. Next, a similar *non-local* model is derived for the hard-material case where the exponential nonlinearity is replaced by $f(u) = (-u)^{-p}$ for $p = \frac{1}{\alpha}$. The stability of Eqs. (10)–(12) is investigated using the analytical approach developed in [13], which actually proves convergence to the unique steady state. On the other hand, the stability in the case of the power-law nonlinearity is investigated by means of a numerical approach [13].

The following degenerate *non-local* model

$$u_t = \Delta \beta(u) + \frac{\lambda f(\beta(u))}{\left(\int_\Omega f(\beta(u)) dx\right)^2}, \quad \text{in } \Omega \times (0, T), \quad T > 0, \quad (13)$$

$$\beta(u) + k(x) \frac{\partial \beta(u)}{\partial \nu} = 0, \quad \text{on } \partial \Omega \times (0, T), \quad (14)$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \quad (15)$$

is produced in Chap. 4, where $0 \leq k(x) \leq \infty$ and $\beta(u) \geq 0$ are continuous functions with $\beta(0) = 0$. *Non-local* model (Eqs. (13)–(15)) is associated with the industrial process of resistance spot welding and $u(x, t)$ serves as the temperature in the welding area. By using a numerical scheme developed in [26], the occurrence of an emerging interface (free boundary), stemming from the degeneracy due to the condition $\beta(0) = 0$, is revealed.

Part II is handling some applications of *non-local* models coming for the field of biology. In Chap. 5, the following *non-local* problem is derived

$$u_t = \Delta u - u + \frac{u^p}{\left(\int_\Omega u^r dx\right)^{\gamma}}, \quad \text{in } \Omega \times (0, T), \quad (16)$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega \times (0, T), \quad (17)$$

$$u(x, 0) = u_0(x) > 0, \quad \text{in } \Omega, \quad (18)$$

by the shadow system of Gierer–Meinhardt system, an inhibitor–activator system arising in cell biology, when the inhibitor diffuses much faster than the activator does. The longtime behavior of $u(x, t)$, representing the concentration of the activator, is examined according to the values of parameters p, r and γ . Among other interesting results, and following the approach in [23], a diffusion-driven blow-up (a sort of *Turing instability* result) for the solution of Eqs. (16)–(18) is proven under the *Turing instability* condition $p - 1 < r\gamma$.

In Chap. 6, we deal with an application arising in evolutionary game dynamics and in particular in its subarea known as replicator dynamics. Considering an infinite continuous strategy space, corresponding, for example, to a continuously varying trait of a biological population, as well as payoff functions of Gaussian type, we end up with the following *non-local* degenerate model

$$\frac{\partial u}{\partial t} = u(\Delta u + \int_{\Omega} |\nabla u|^2 dx), \quad \text{in } \Omega \times (0, T), \quad (19)$$

$$u(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (20)$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \quad (21)$$

where $u(x, t)$ denotes the probability density of the probability measure providing the state of the biological population (population of players). As it is appropriate for degenerate problems, a regularized approximation of Eqs. (19)–(21) is used and then some a priori estimates for its solutions are derived. Afterward, by adopting the arguments introduced in [17], global-in-time existence and blow-up results are obtained according to the value of the initial mass $\|u_0\|_{L^1(\Omega)}$.

Chapter 7 debates the biological phenomenon of chemotaxis. Initially, a version of Keller–Segel system is considered which describes the movement of some cell population toward a chemo-attractant produced by the population itself. Next, it is shown, using the approach of [20, 32], that in the case where the chemo-attractant diffuses much faster than the cell population, then Keller–Segel reduces to the following *non-local* problem

$$u_t = \Delta u + \frac{\lambda e^u}{\int_{\Omega} e^u} \quad \text{in } \Omega \times (0, T), \quad (22)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (23)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \quad (24)$$

where $u(x, t)$ stands for the concentration of the chemo-attractant. For $\lambda < 8\pi$ global-in-time existence is derived whereas for $\lambda > 8\pi$ and for radially symmetric solutions, by using the approach developed in [20], the occurrence of blow-up is proven.

Chapter 8 introduces the following *non-local* reaction–diffusion system

$$u_t = d_1 \Delta u - ku \cdot \oint_{B(\cdot, R) \cap \Omega} v \quad \text{in } \Omega \times (0, T), \quad (25)$$

$$v_t = d_2 \Delta v - kv \cdot \oint_{B(\cdot, R) \cap \Omega} u \quad \text{in } \Omega \times (0, T), \quad (26)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (27)$$

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega, \quad (28)$$

where

$$\oint = \frac{1}{\omega_N R^N} \int.$$

In the first place, system (Eqs. (25)–(28)) is built up as a mathematical model in cell biology to describe the evolution of protein dimers within human cells. Indeed, system (Eqs. (25)–(28)) inspects the situation when chemical reactions occur only when two chemicals within cells and with concentrations u and v are in distance R , called the reaction radius [12, 21]. Next, the longtime behavior of the solutions of Eqs. (25)–(28) is investigated as well as the phase separation phenomenon develops on extremely fast reaction rates ($k \rightarrow +\infty$) is also examined [21, 22].

In the Appendix, some more *non-local* models are presented arising in further applications ranged from point vortices theory to differential geometry.

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References

1. Bebernes, J.W., Talaga, P.: Non-local problems modelling shear banding. *Comm. Appl. Nonlin. Anal.* **3**, 79–103 (1996)
2. Bebernes, J.W., Li, C., Talaga, P.: Single-point blow-up for non-local parabolic problems. *Phys. D* **134**, 48–60 (1999)
3. Caglioti, E., Lions, P.-L., Marchioro, C., Pulvirenti, M.: A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description. *Comm. Math. Phys.* **143**, 501–525 (1992)
4. Chafee, N.: The electric ballast resistor : homogeneous and nonhomogeneous equilibria. In: de Mottoni, P., Salvadori, L.: *Nonlinear Differential Equations : Invariance Stability and Bifurcations* pp. 97–127. Academic Press, New York (1981)
5. Chen, C.-K., Fife, P.C.: *Nonlocal* models of phase transitions in solids. *Adv. Math. Sci. Appl.* **10**, 821–849 (2000)
6. Freitas, P.: Bifurcation and stability of stationary solutions of nonlocal reaction-diffusion equations in m -dimensional case. *J. Dyn. Differ. Equ.* **6**, 613–629 (1994)
7. Freitas, P., Vishnevskii, M.P.: Stability of stationary solutions of nonlocal scalar reaction-diffusion equations. *Diff. Int. Equ.* **13**, 265–288 (2000)
8. Furter, J., Grinfeld, M.: Local vs. nonlocal interactions in population dynamics. *J. Math. Biol.* **27**(1), 65–80 (1989)
9. Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sec. IA* **13**, 109–124 (1966)
10. Fujita, H.: On the nonlinear equations $\Delta u + e^u = 0$ and $\partial v / \partial t = \Delta v + e^v$. *Bull. Amer. Math. Soc.* **75**, 132–135 (1969)
11. Guo, J.-S., Kavallaris, N.I.: On a non-local parabolic problem arising in electrostatic MEMS control. *Disc. Cont. Dyn. Systems-A* **32**, 1723–1764 (2012)
12. Ichikawa, K., Rouzaimaimaiti, M., Suzuki, T.: Reaction diffusion equation with non-local term arises as a mean field limit of the master equation. *Disc. Cont. Dyn. Systems-S* **5**(1), 115–126 (2012)
13. Kavallaris, N.I., Lacey, A.A., Nikolopoulos, C.V., Voong, C.: Behaviour of a non-local equation modelling linear friction welding. *IMA J. Appl. Math.* **75** (2), 597–616 (2007)
14. Kavallaris, N.I., Lacey, A.A., Nikolopoulos, Tzanetis, D.E.: A hyperbolic non-local problem modelling MEMS technology. *Rocky Mountain J. Math.* **41**, 505–534 (2011)
15. Kavallaris, N.I., Lacey, A.A., Nikolopoulos, C.V.: *On the quenching of a nonlocal parabolic problem arising in electrostatic MEMS control*, *Nonl. Analysis (TMA)* **138** 189–206, (2016), *Nonlinear Partial Differential Equations, in honor of Juan Luis Vázquez for his 70th birthday*

16. Kavallaris, N.I., Lankeit, J., Winkler, M.: *On a degenerate non-local parabolic problem describing infinite dimensional replicator dynamics*, SIAM J. Math. Anal. **49**(2), 954–983 (2017)
17. Kavallaris, N.I., Nadzieja, T.: *On the blow-up of the non-local thermistor problem*, Proc. Edinbrugh Math. Soc. **50**, 389–409 (2007)
18. Kavallaris, N.I., Nikolopoulos, C.V., Tzanetis, D.E.: Estimates of blowup time for a non-local problem modelling an Ohmic heating process. Euro. J. Appl. Math. **13**, 337–351 (2002)
19. Kavallaris, N.I., Suzuki, T.: On the finite-time blowup of a parabolic equation describing chemotaxis. Diff. Int. Equ. **20**, 293–308 (2007)
20. Kavallaris, N.I., Suzuki, T.: Nonlocal reaction-diffusion system involved by reaction radius I. IMA J. Appl. Math. **78** (3), 614–632 (2013)
21. Kavallaris, N.I., Suzuki, T.: Nonlocal reaction-diffusion system involved by reaction radius II: rate of convergence. IMA J. Appl. Math. **79** (1), 1–21 (2014)
22. Kavallaris, N.I., Suzuki, T.: On the dynamics of a non-local parabolic equation arising from the Gierer-Meinhardt system. Nonlinearity **30**(5), 1734–1761 (2017)
23. Kavallaris, N.I., Tzanetis, D.E.: Blow-up and stability of a non-local diffusion-convection problem arising in Ohmic heating of foods. Diff. Int. Equ. **15**(3), 271–288 (2002)
24. Kavallaris, N.I., Tzanetis, D.E.: Behaviour of a non-local reactive-convective problem with variable velocity in ohmic heating of food. Nonlocal elliptic and parabolic problems, Banach Center Publ.**66**, Polish Acad. Sci., Warsaw, 189–198 (2004)
25. Kavallaris, N.I., Yan, Y.: A Time Discretization Scheme for a Nonlocal Degenerate Problem Modelling Resistance Spot Welding. Math. Model. Nat. Phenom. **10** (6), 90–112 (2015)
26. Krzywicki, A., Nadzieja, T.: Some results concerning the Poisson–Boltzmann equation. Zastosowania Mat. (Appl. Math. (Warsaw)) **21**, 265–272 (1991)
27. Lacey, A.A.: Mathematical analysis of thermal runaway for spatially inhomogeneous reactions. SIAM J. Appl. Math. **43**, 1350–1366 (1983)
28. Lacey, A.A.: Thermal runaway in a non-local problem modelling Ohmic heating. Part I: Model derivation and some special cases. Euro. J. Appl. Math. **6**, 127–144 (1995)
29. Lacey, A.A.: Thermal runaway in a non-local problem modelling Ohmic heating. Part II: General proof of blow-up and asymptotics of runaway. Euro. J. Appl. Math. **6**, 201–224, (1995)
30. Quittner, P., Souplet, Ph.: Superlinear Parabolic Equations, Blow-up, Global Existence and Steady States. Birkhäuser, Basel, (2007)
31. Wolansky, G.: A critical parabolic estimate and application to nonlocal equations arising in chemotaxis. Appl. Anal. **66**, 291–321 (1997)

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Part I
Applications in Engineering

Chapter 1

Micro-Electro-Mechanical-Systems (MEMS)

Abstract In the current chapter we first present the construction of some non-local models describing the operation of an idealized MEMS (Micro-electro-mechanical system). In particular, the MEMS device is considered to be part of an electrical circuit and using elastic and electric theories two different non-local models are derived: a parabolic and a hyperbolic one. In the first place, the investigation of the structure of the corresponding non-local elliptic steady state problem is undertaken and some estimates of the pull-in voltage are obtained. Next, we focus on the mathematical analysis of the derived evolutionary non-local equations. Notably, the circumstances under which finite-time quenching occurs for both of evolutionary problems are investigated, so then some useful conclusions regarding the possible destruction of the MEMS device or the invalidity of the used models can be derived. Since, maximum principle is not available for both of the inspected non-local models, and thus comparison methods are not applicable, finally energy methods are called forth to investigate their long-time behavior.

1.1 Derivation of the Basic Model and Its Variations

The term “MEMS” more precisely refers to precision devices which combine mechanical processes with electrical circuits. In particular, electrostatic actuation is a popular application of MEMS. MEMS devices range in size from millimetres down to microns, and involve precision mechanical components that can be constructed using semiconductor manufacturing technologies. Various electrostatic actuated MEMS have been developed and used in a wide variety of devices applied as sensors and have fluid-mechanical, optical, radio frequency (RF), data-storage, and biotechnology applications. Examples of microdevices of this kind include microphones, temperature sensors, RF switches, resonators, accelerometers, micromirrors, micropumps, microvalves, data-storage devices etc., [9, 36, 46].

The principal part of such an electrostatic actuated MEMS device usually consists of an elastic plate suspended above a rigid ground plate whose length is equal to L and which are in distance d . Typically the elastic plate (or membrane) is held fixed at two ends while the other two edges remain free to move and additionally both plates are

considered to be perfect conductors, see Fig. 1.1. An alternative configuration could entail the plate or membrane being held fixed around its entire edge. When a potential difference V_s is applied between the membrane and the plate, the membrane deflects towards the ground plate. We first make the realistic assumption that the width of the gap, between the membrane and the bottom plate, is small compared to the device length, and the configuration of the two parallel plates is connected in series with a fixed voltage source and a fixed capacitor. Then our main purpose in the current section is to derive a mathematical model describing the deformation of the elastic membrane by following the approach presented in [36, 37]. We actually separate the composed procedure into two different problems: the elastic and the electric.

1.1.1 The Elastic Problem

To describe the elastic problem we use variational calculus. In particular, we apply Hamilton's principle (or the principle of least action) which is equivalent to the stationary action $\mathcal{S}$ of the system from time t_1 to t_2 given as

$$\mathcal{S} = \int_{t_1}^{t_2} \int_{\Omega} \mathcal{L} dx' dy' dt,$$

where $\Omega \subset \mathbb{R}^2$ is the domain occupied by the elastic membrane (plate) considering that the plate is very thin.

Here $\mathcal{L}$ is the Lagrangian of the system defined at any point of the system as

$$\mathcal{L} = \text{Kinetic Energy} + \text{Damping Energy} - \text{Potential Energy}.$$

Let $v = v(x', y', t')$ be the dynamic deflection of the membrane, then the kinetic energy of the membrane can be written as

$$E_k = \frac{\rho h}{2} \int_{\Omega} v_{t'}^2 dx' dy',$$

where ρ, h stand for the mass density per unit volume of the membrane and the thickness of the membrane respectively.

Furthermore, the damping energy E_d has the form

$$E_d = \frac{a}{2} \int_{\Omega} v^2 dx' dy',$$

where a represents the damping constant.

The potential energy is $E_p = E_s + E_b$, where E_s, E_b are the stretching and bending energy respectively. The former one is provided by

$$E_s = \mu \left(\int_{\Omega} \sqrt{1 + |\nabla_{\perp} v|^2} dx' dy' - |\Omega| \right),$$

where we are using the notation $\nabla_{\perp}$ to indicate the differentiation only with respect to x' and y' , while μ stands for the tension of the membrane and $|\Omega|$ represents the area of the domain Ω . It should be noted that for small deflections we may linearize to derive

$$E_s = \mu \int_{\Omega} |\nabla_{\perp} v|^2 dx' dy'.$$

On the other hand, the bending energy is defined by

$$E_b = \frac{D}{2} \int_{\Omega} \text{Curvature}^2 dx' dy'$$

and by again linearizing we deduce

$$E_b = \frac{D}{2} \int_{\Omega} (\Delta_{\perp} v)^2 dx' dy',$$

where D is the flexural rigidity of the membrane.

We thus obtain

$$E_p = \int_{\Omega} \left(\frac{\mu}{2} |\nabla_{\perp} v|^2 + \frac{D}{2} (\Delta_{\perp} v)^2 \right) dx' dy',$$

and the Lagrangian is provided by

$$\mathcal{L} = \frac{\rho h}{2} v_{t'}^2 + \frac{a}{2} v^2 - \frac{\mu}{2} |\nabla_{\perp} v|^2 - \frac{D}{2} (\Delta_{\perp} v)^2.$$

Therefore, the functional to be stationary is

$$\mathcal{S} = \int_{t_1}^{t_2} \int_{\Omega} \left(\frac{\rho h}{2} v_{t'}^2 + \frac{a}{2} v^2 - \frac{\mu}{2} |\nabla_{\perp} v|^2 - \frac{D}{2} (\Delta_{\perp} v)^2 \right) dx' dy' dt'.$$

Consequently, the Euler–Lagrange equation for v is

$$\rho h v_{t't'} + a v_{t'} - \mu \Delta_{\perp} v + D \Delta_{\perp}^2 v = 0, \quad (x', y', z') \in \Omega \times (0, d),$$

where $\Delta_{\perp}^2$ represents the bi-laplacian operator again with respect to x' and y' . If in addition the membrane is subject to a force F_e due to an electric field then

$$\rho h v_{t't'} + a v_{t'} - \mu \Delta_{\perp} v + D \Delta_{\perp}^2 v = F_e, \quad (x', y', z') \in \Omega \times (0, d).$$

1.1.2 The Electric Problem

It is assumed that the elastic (plate) membrane is held at the potential V_s , while the ground fixed plate is held at zero potential. Therefore, if ϕ is the electrostatic potential then due to the conservation of electric charge we have the Laplace equation

$$\Delta\phi = 0, \quad (1.1)$$

holding everywhere in the region between the elastic (plate) membrane and the ground plate as well as in the region surrounding the MEMS device. Moreover the following boundary conditions are imposed

$$\phi = V_s \quad \text{on the elastic membrane,} \quad (1.2)$$

and

$$\phi = 0 \quad \text{on the ground plate.} \quad (1.3)$$

Although, at the first glance the electric problem seems to be uncoupled to the elastic problem, however the two problems are coupled through the boundary condition (1.3) since V_s depends on v , see also below.

Furthermore, due to Ohm's law the electric field can be expressed as

$$F_e = -\frac{\varepsilon_0}{2} |\nabla\phi|^2,$$

where ε_0 is the permittivity of the free space and thus v satisfies

$$\rho h v_{t't'} + a v_{t'} - \mu \Delta_\perp v + D \Delta_\perp^2 v = -\frac{\varepsilon_0}{2} |\nabla\phi|^2, \quad (1.4)$$

for $(x', y', z') \in \Omega \times (0, d)$ and $t' > 0$.

Next we rescale our system (1.1)–(1.4) in order to express our governing equation into a dimensionless formulation. In particular, we consider the new variables

$$\theta = \frac{v}{d}, \quad \psi = \frac{\phi}{V_s}, \quad x = \frac{x'}{L}, \quad y = \frac{y'}{L}, \quad z = \frac{z'}{d}, \quad t = \frac{\mu t'}{aL^2}, \quad (1.5)$$

and substituting to (1.1)–(1.4) we derive the dimensionless system

$$\varepsilon^2 \Delta_\perp \psi + \psi_{zz} = 0, \quad (1.6)$$

$$\psi = 1 \quad \text{on the elastic membrane,} \quad (1.7)$$

$$\psi = 0 \quad \text{on the ground plate,} \quad (1.8)$$

$$\beta^2 \theta_{tt} + \theta_t - \Delta_\perp \theta + \delta \Delta_\perp^2 \theta = -\lambda \left[\varepsilon^2 |\nabla_\perp \psi|^2 + (\psi_{zz})^2 \right], \quad (1.9)$$

where the parameter β is defined as

$$\beta = \frac{\sqrt{\rho h \mu}}{aL}. \quad (1.10)$$

The parameter δ defined by

$$\delta = \frac{D}{L^2 \mu}, \quad (1.11)$$

is a measure of the relative importance of tension and rigidity, whereas the parameter

$$\varepsilon = \frac{d}{L}, \quad (1.12)$$

is the aspect ratio of the system.

Finally, the parameter

$$\lambda = \frac{\varepsilon_0 V_s^2 L^2}{2d^3 \mu}, \quad (1.13)$$

represents a ratio of a reference electrostatic force to a reference elastic force. Observe, that since λ is proportional to the square of the applied voltage V_s it could be actually used as a *tuning* parameter of the MEMS device.

1.1.3 An Uncoupled Local Model

In many applications, including the Grating Light Valves (GLV), the lateral dimensions of the MEMS device in Fig. 1.1 are much larger compared to the size of the gap between the elastic plate and the ground fixed plate, that is $\varepsilon = \frac{d}{L} \ll 1$. In that case, as we will see in the following the coupled system of the elastic and the electric problems can be reduced to a single equation.

Indeed, under the aspect ratio limit $\varepsilon \rightarrow 0$ the electrostatic problem reduces to

$$\psi_{zz} = 0, \quad \text{for } (x, y, z) \in \Omega \times (0, 1), \quad t > 0,$$

which can be solved exactly to

$$\psi = Cz + D.$$

Now using the boundary conditions

$$\psi(x, y, \theta, t) = 1,$$

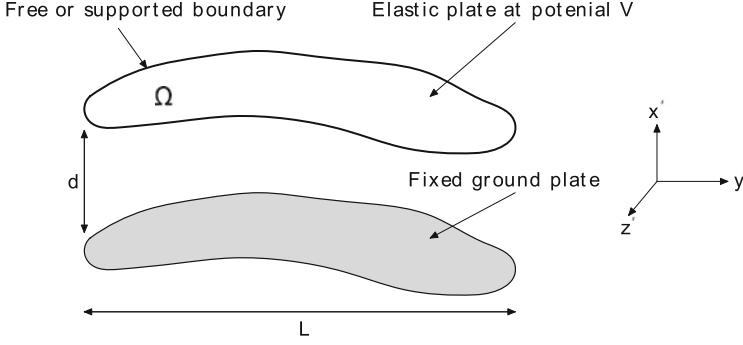


Fig. 1.1 Sketch of the capacitive control circuit (controlled-voltage operation)

and

$$\psi(x, y, 0, t) = 0,$$

we finally deduce

$$\psi(\theta) = \frac{z}{\theta}, \quad \text{for } z \in (0, 1). \quad (1.14)$$

We should note here that the electric potential given by (1.14) actually corresponds to the case where the fringing fields have been ignored.

Now considering the limit $\varepsilon \rightarrow 0$ into (1.9) and taking into account (1.14) we finally derive the uncoupled equation

$$\beta^2 \theta_{tt} + \theta_t - \Delta \theta + \delta \Delta^2 \theta = -\frac{\lambda}{\theta^2},$$

for $(x, y) \in \Omega$, and $t > 0$. Note that we have removed the index $\perp$ from the differentiations since now θ depends only on x and y .

In various applications, e.g., certain MEMS micro-pumps, GLV, e.t.c., the elastic plate is of very low rigidity and thus we can take $\delta = 0$ into (1.15) to derive

$$\beta^2 \theta_{tt} + \theta_t - \Delta \theta = -\frac{\lambda}{\theta^2}, \quad (1.15)$$

for $(x, y) \in \Omega$, and $t > 0$. Moreover, since it is assumed that the edges of elastic plate are kept fixed then (1.15) is also combined with boundary conditions of the form

$$\theta = 1 \quad \text{on } \partial\Omega, \quad (1.16)$$

as well as with initial conditions

$$\theta(x, 0) = \theta_0(x) \quad \text{in } \Omega. \quad (1.17)$$

From many experiments it is evident that the applied voltage V_s controls the operation of the MEMS device. It is observed that when V_s exceeds a critical threshold V_{cr} , called the *pull-in voltage*, then the phenomenon of *touch-down* (or *pull-in instability* as it is also known in MEMS literature) occurs when the elastic membrane touches the rigid ground plate.

For the mathematical Eqs. (1.15), (1.16), this actually means that there is some critical value λ_{cr} of the parameter λ above which singular behavior should be anticipated. Focusing on the nonlinear term of Eqs. (1.15), (1.16) one can notice that such singular behavior is possible only when u takes the value 0, a phenomenon known in the mathematical literature as *quenching*.

Additionally, it has been experimentally observed, see [46], that there is a significant uncertainty regarding the values of V_s . In particular, V_s fluctuates around an average value V_0 , hence this implies that $\lambda = \lambda_0 + \alpha \eta(x, t)$ where $\alpha > 0$ is a coefficient measuring the intensity of the noise term $\eta(x, t)$. The coefficient α might depend on the deformation θ as well, whereas the noise $\eta(x, t)$ could be taken to be a space-time *white noise* i.e., $\eta(x, t) = \partial_t W(x, t)$, where $W(x, t)$ is a Wiener process and thus the following stochastic variation of (1.15)–(1.17) is derived

$$\beta^2 \theta_{tt} + \theta_t - \Delta \theta = -\frac{\lambda}{\theta^2} + \sigma(1 - \theta) \partial_t W(x, t), \quad \text{in } \Omega \times (0, T), \quad T > 0, \quad (1.18)$$

$$\theta = 1, \quad \text{on } \partial\Omega \times (0, T), \quad (1.19)$$

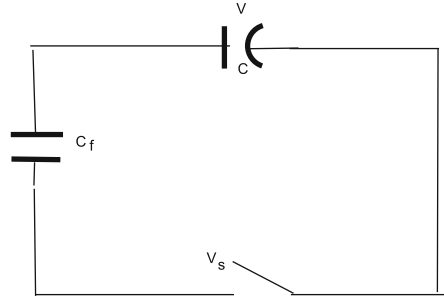
$$\theta(x, 0) = \theta_0(x), \quad x \in \Omega, \quad (1.20)$$

where σ is a positive function of the quantity $1 - \theta$, in the current monograph we mainly focus on the mathematical analysis of the deterministic model (1.15)–(1.17) and its non-local variation. Whilst regarding the mathematical analysis of the *touching down* behavior of the stochastic local MEMS (1.18)–(1.20) the reader can see [25].

1.1.4 An Uncoupled Non-local Model

As it has been pointed out above, the pull-in instability is a ubiquitous feature of electrostatically actuated systems. Many researchers have focused on extending the stable operation of electrostatically actuated systems beyond the pull-in regime. In particular, in [42, 43] the basic capacitive control scheme was first proposed by Seeger and Crary to elaborate this kind of stabilization, see also [5]. More precisely, this scheme provides control of the voltage by the addition of a series capacitance to the circuit containing the MEMS device, see Fig. 1.2, since the added capacitance

Fig. 1.2 The basic capacitive control circuit



acts as a voltage divider. The novelty to the produced mathematical model is that now the induced nonlinearity is of a non-local form.

Indeed, a straightforward application of Kirchoffs laws to the circuit gives that the voltage across the electrical circuit is provided by

$$V = \frac{V_s}{1 + \frac{C}{C_f}}, \quad (1.21)$$

where V_s stands for the source voltage while C , C_f represent the capacitance of the device and the capacitance of the fixed series capacitor respectively.

Now the stabilization effect of the new configuration is justified as follows: the capacitance C depends on the deflection of the system and it actually increases as the gap between the components of the system decreases. This, in turn, produces a drop in the value of V and thus reduces the electrostatic force stabilizing the device. This will be depicted later on by the mathematical analysis of the produced model. In that case the electric problem is modified as follows

$$\Delta\phi = 0 \quad \text{on the space between the two plates,} \quad (1.22)$$

$$\phi = V_s f\left(\frac{v}{d}\right) \quad \text{on the elastic membrane,} \quad (1.23)$$

$$\phi = 0 \quad \text{on the ground plate,} \quad (1.24)$$

where the dimensionless function f exhibits the fact that the voltage drop across our device when embedded in a circuit may depend upon the deflection of the elastic plate v .

Now implementing again the scaling (1.5) system (1.22)–(1.24) is transformed to

$$\varepsilon^2 \Delta_{\perp} \psi + \psi_{zz} = 0, \quad (1.25)$$

$$\psi = f(\theta) \quad \text{on the elastic membrane,} \quad (1.26)$$

$$\psi = 0 \quad \text{on the ground plate,} \quad (1.27)$$

$$\beta^2 \theta_{tt} + \theta_t - \Delta_{\perp} \theta + \delta \Delta_{\perp}^2 \theta = -\lambda [\varepsilon^2 |\nabla_{\perp} \psi|^2 + (\psi_{zz})^2], \quad (1.28)$$

where the parameters β , δ , ε and λ are defined again by (1.10), (1.11), (1.12) and (1.13) respectively.

Considering now the small aspect ratio limit $\varepsilon \rightarrow 0$ into (1.25)–(1.28) and following the same steps as in the Sect. 1.1.3 we derive

$$\psi(\theta) = \frac{zf(\theta)}{\theta}, \quad (1.29)$$

and thus the uncoupled equation

$$\beta^2 \theta_{tt} + \theta_t - \Delta \theta + \delta \Delta^2 \theta = -\lambda \frac{f^2(\theta)}{\theta^2}, \quad (1.30)$$

where we have again dropped the subscript $\perp$ since θ is independent of z .

It then remains to compute the term $f(\theta)$ which is actually given as

$$f(\theta) = \frac{V}{V_s} = \frac{1}{1 + \frac{C}{C_f}} = \frac{1}{1 + \frac{C_0}{C_f} \int_{\Omega} \psi_z(x, y, 0) dx dy} \quad (1.31)$$

by virtue of (1.21) and taking also into account that the capacity C is provided by

$$C = C_0 \int_{\Omega} \psi_z(x, y, 0) dx dy,$$

where C_0 the capacitance of the undeflected device.

Utilizing (1.30) as well as that $\psi_z(x, y, 0) = 1$, due to the fact that $V = V_s$ at $z = 0$, then (1.31) yields

$$f(\theta) = \frac{1}{1 + \alpha \int_{\Omega} \frac{1}{\theta(x, y)} dx dy} \quad (1.32)$$

where $\alpha = \frac{C_0}{C_f}$.

Now plugging (1.32) into (1.30) we end up with the following

$$\beta^2 \theta_{tt} + \theta_t - \Delta \theta + \delta \Delta^2 \theta = - \frac{\lambda}{\theta^2 \left(1 + \alpha \int_{\Omega} \frac{1}{\theta(x, y)} dx dy \right)^2}, \quad (1.33)$$

for $(x, y) \in \Omega$ and $t > 0$. An interesting feature of Eq.(1.33) is that it is non-local due to the presence of the spatial integral which is actually an output of the model reduction and coupled domain structure of the initial problem. Its occurrence also indicates that the deformation of each point of the elastic plate also depends on the deformation of the neighboring points as it is anticipated.

If we consider again the case where the elastic plate has low rigidity, i.e., $\delta \rightarrow 0$, then (1.33) is reduced to

$$\beta^2 \theta_{tt} + \theta_t - \Delta \theta = - \frac{\lambda}{\theta^2 \left(1 + \alpha \int_{\Omega} \frac{1}{\theta(x, y)} dx dy \right)^2}, \quad (1.34)$$

for $(x, y) \in \Omega$ and $t > 0$, which is also associated with the boundary condition

$$\theta = 1 \quad \text{on} \quad \partial \Omega, \quad (1.35)$$

elaborating the assumption that the edges of the elastic plate are kept fixed as well as with the initial condition.

$$\theta(x, 0) = \theta_0(x) \quad \text{in} \quad \Omega. \quad (1.36)$$

1.2 Mathematical Analysis

In the current section we investigate the mathematical behavior of the non-local models derived in Sect. 1.1. We begin with the investigation of the parabolic problem, when $\beta \ll 1$, and, we continue with study of the hyperbolic problem, when $\beta \gg 1$.

1.2.1 A Non-local Parabolic Problem

We now consider the non-local parabolic problem in $Q_T := \Omega \times (0, T)$

$$u_t - \Delta u = \frac{\lambda}{(1 - u)^2 \left(1 + \alpha \int_{\Omega} \frac{1}{1 - u} dx dy \right)^2}, \quad 0 \leq u < 1 \quad \text{in} \quad Q_T, \quad (1.37)$$

$$u = 0 \quad \text{on} \quad \partial \Omega \times (0, T), \quad (1.38)$$

$$0 \leq u(x, 0) = u_0(x) < 1 \quad \text{in} \quad \Omega, \quad (1.39)$$

which describes the operation of the MEMS device when it is a part capacitive control circuit. Henceforth, without any loss of generality we consider $\alpha = 1$.

1.2.1.1 Local Existence and Uniqueness

We first begin with the local existence and uniqueness of classical solutions to problem (1.37)–(1.39). Note that since the integrand in the non-local term is an increasing

function, the usual comparison principle for parabolic problems is not applicable for problem (cf. [30]) and thus lower and upper solutions of (1.37)–(1.39) are not necessarily ordered. Therefore, in the following we define the notion of lower-upper solution pairs which will be applied for comparison purposes.

Definition 1.2.1 A pair of functions $0 \leq v(x, t), z(x, t) < 1$ with $v, z \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is called a lower-upper solution pair of (1.37)–(1.39), if $v(x, t) \leq z(x, t)$ for $(x, t) \in Q_T$, $v(x, 0) \leq u_0(x) \leq z(x, 0)$ in $\overline{\Omega}$, $v(x, t) \leq 0 \leq z(x, t)$ for $(x, t) \in \partial\Omega \times [0, T]$, and

$$\begin{aligned} v_t &\leq \Delta v + \frac{\lambda}{(1-v)^2 \left(1 + \int_{\Omega} \frac{dx}{1-v}\right)^2} \quad \text{in } Q_T, \\ z_t &\geq \Delta z + \frac{\lambda}{(1-z)^2 \left(1 + \int_{\Omega} \frac{dx}{1-v}\right)^2} \quad \text{in } Q_T. \end{aligned}$$

If the above inequalities are strict, then (v, z) is called a strict lower-upper solution pair. Using now the concept of lower-upper solution pair introduced in Definition 1.2.1 we show the following local in time existence result.

Proposition 1.2.2 Let (v, z) be a lower-upper solution pair to (1.37)–(1.39) in Q_T for some $T > 0$. Then there exists a unique (classical) solution u to (1.37)–(1.39) such that $v \leq u \leq z$ in Q_T .

Proof First, we define the iteration scheme starting with $\bar{u}_0 = z$ and $\underline{u}_0 = v$, and proceeding according to

$$\begin{aligned} \underline{u}_{nt} &= \Delta \underline{u}_n + \frac{\lambda}{(1 - \underline{u}_{n-1})^2 \left(1 + \int_{\Omega} \frac{dx}{1 - \bar{u}_{n-1}}\right)^2} \quad \text{in } Q_T, \\ \bar{u}_{nt} &= \Delta \bar{u}_n + \frac{\lambda}{(1 - \bar{u}_{n-1})^2 \left(1 + \int_{\Omega} \frac{dx}{1 - \underline{u}_{n-1}}\right)^2} \quad \text{in } Q_T, \\ \underline{u}_n(x, t) &= \bar{u}_n(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T), \\ \underline{u}_n(x, 0) &= \bar{u}_n(x, 0) = u_0(x), \quad \text{for } x \in \overline{\Omega}, \end{aligned}$$

for $n = 1, 2, \dots$. The above problems are local, therefore, using the standard comparison arguments for parabolic problems and Definition 1.2.1, we easily see that the sequences $\{\underline{u}_n\}_{n=1}^{\infty}, \{\bar{u}_n\}_{n=1}^{\infty} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ are strictly positive and satisfy

$$v \leq \underline{u}_{n-1} \leq \underline{u}_n \leq \dots \leq \bar{u}_n \leq \bar{u}_{n-1} \leq z.$$

Hence, by the parabolic regularity theory and Dini's theorem, the sequences $\{\underline{u}_n\}_{n=1}^{\infty}, \{\bar{u}_n\}_{n=1}^{\infty}$ converge as $n \rightarrow \infty$ uniformly to $u_1, u_2 \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, respectively, such that $v \leq u_1 \leq u_2 \leq z$.

We claim that $u_1 = u_2$. Indeed, u_1, u_2 satisfy

$$\begin{aligned} u_{1t} &= \Delta u_1 + \frac{\lambda}{(1 - u_1)^2 \left(1 + \int_{\Omega} \frac{dx}{1 - u_2}\right)^2} \quad \text{in } Q_T, \\ u_{2t} &= \Delta u_2 + \frac{\lambda}{(1 - u_2)^2 \left(1 + \int_{\Omega} \frac{dx}{1 - u_1}\right)^2} \quad \text{in } Q_T, \\ u_1(x, t) &= u_2(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T), \\ u_1(x, 0) &= u_2(x, 0) = u_0(x), \quad \text{for } x \in \overline{\Omega}. \end{aligned}$$

Denote $\psi(x, t) = u_1(x, t) - u_2(x, t)$ then $\psi(x, t)$ satisfies

$$\begin{aligned} \psi_t &= \Delta \psi + A(x, t)\psi + B(x, t) \int_{\Omega} \int_0^1 \frac{d\theta}{\left[1 - \theta u_1 - (1 - \theta)u_2\right]^2} \psi dx, \quad \text{in } Q_T, \\ \psi(x, t) &= 0, \quad \text{for } (x, t) \in (\partial\Omega \times (0, T)) \cap (\overline{\Omega} \times \{0\}), \end{aligned}$$

where

$$A(x, t) := 2\lambda \frac{\int_0^1 \frac{d\theta}{\left[1 - \theta u_1 - (1 - \theta)u_2\right]^3}}{\left(1 + \int_{\Omega} \frac{dx}{1 - u_2}\right)^2} > 0,$$

and

$$B(x, t) := \frac{\lambda}{(1 - u_2)^2} \frac{2 + \int_{\Omega} \frac{dx}{1 - u_1} + \int_{\Omega} \frac{dx}{1 - u_2}}{\left(1 + \int_{\Omega} \frac{dx}{1 - u_1}\right)^2 \left(1 + \int_{\Omega} \frac{dx}{1 - u_2}\right)^2} > 0.$$

Applying Proposition 52.24 in [39] (which is actually a maximum principle for non-local problems) we easily obtain that $\psi(x, t) = 0$ in $\overline{Q}_T$. Hence $u_1 = u_2 := u$ in $\overline{Q}_T$.

Finally, suppose that there is a second solution U satisfying $v \leq U \leq z$. Then by the preceding iterative scheme we derive that $\underline{u}_n \leq U \leq \overline{u}_n$ for every $n = 1, 2, \dots$. By sending $n \rightarrow \infty$, we obtain that $U = u$. This proves the proposition. $\square$

Remark 1.2.3 By the above analysis we derive that the solution of problem (1.37)–(1.39) continues to exist as long as it remains less than or equal to b for some $b < 1$. This argument implies that u ceases to exist only by “quenching”, i.e., if there exists a sequence $(x_n, t_n) \rightarrow (x^*, t^*)$ as $n \rightarrow \infty$ with $t^* \leq \infty$ such that $u(x_n, t_n) \rightarrow 1$ as $n \rightarrow \infty$.

Let $z(x, t)$ be the unique solution of the following problem:

$$z_t - \Delta z = \frac{\lambda}{(1-z)^2 (1+|\Omega|)^2} \quad \text{in } Q_T, \quad (1.40)$$

$$z = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1.41)$$

$$z(x, 0) = z_0(x) \quad \text{for } x \in \overline{\Omega}. \quad (1.42)$$

Then we have the following.

Corollary 1.2.4 *If $z_0(x) \geq u_0(x)$ for every $x \in \Omega$, then problem (1.37)–(1.39) has a unique (classical) solution u on $\Omega \times [0, T)$, where $[0, T)$ is the maximal existence time interval for the solution $z(x, t)$ of problem (1.40)–(1.42), and $u(x, t) \leq z(x, t)$ on $\Omega \times [0, T)$.*

Proof Set $v(x, t) = 0$, then we have

$$z_t - \Delta z = \frac{\lambda}{(1-z)^2 (1+|\Omega|)^2} = \frac{\lambda}{(1-z)^2 \left(1 + \int_{\Omega} \frac{dx}{1-v}\right)^2} \quad \text{in } Q_T,$$

$$z = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$z(x, 0) = z_0(x) \quad \text{for } x \in \overline{\Omega},$$

while $v(x, t)$ satisfies

$$v_t - \Delta v = 0 \leq \frac{\lambda}{(1-v)^2 \left(1 + \int_{\Omega} \frac{dx}{1-v}\right)^2} \quad \text{in } Q_T,$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$v(x, 0) = 0 \quad \text{for } x \in \overline{\Omega}.$$

Therefore, (v, z) is a lower-upper solution pair for problem (1.37)–(1.39) and the result immediately follows by Proposition 1.2.2. $\square$

Remark 1.2.5 All the results of this section could be carried out without any change for equation

$$u_t - \Delta u = \frac{\lambda f(x)}{(1-u)^2 \left(1 + \int_{\Omega} \frac{1}{1-u} dx dy\right)^2}, \quad (1.43)$$

as well, when $f(x) > 0$.

1.2.1.2 Estimates of Pull-In Voltage

The corresponding steady-state problem to (1.37)–(1.39) has the form

$$\Delta w + \frac{\lambda}{K(1-w)^2} = 0, \quad x \in \Omega, \quad w = 0, \quad x \in \partial\Omega, \quad (1.44)$$

where

$$K = K(w) := \left(1 + \int_{\Omega} \frac{dx}{1-w}\right)^2. \quad (1.45)$$

Note that we always have $0 < w < 1$ in Ω for a (classical) solution of (1.44).

By setting

$$\mu = \frac{\lambda}{K} = \frac{\lambda}{\left(1 + \int_{\Omega} \frac{dx}{1-w}\right)^2}, \quad (1.46)$$

then (1.44) is transformed to

$$\Delta w + \mu(1-w)^{-2} = 0 \quad x \in \Omega, \quad w = 0 \quad x \in \partial\Omega. \quad (1.47)$$

Problems (1.44) and (1.47) are equivalent through (1.46), i.e., w is a solution of (1.44) corresponding to λ if and only if w satisfies (1.47) for $\mu = \lambda/K$. Thus, some features of the solution set

$$\mathcal{S} = \{(\lambda, w) | w = w_{\lambda}(x) \text{ is a classical solution to (1.44) for } \lambda > 0\},$$

resemble those of the solution set

$$\mathcal{C} = \{(\mu, w) | w = w_{\mu}(x) \text{ is a classical solution to (1.47) for } \mu > 0\}.$$

The structure of $\mathcal{C}$ is well known. Indeed, for any N , we recall from [15] that there exists a positive constant μ^* such that a solution of (1.47) exists, if $\mu < \mu^*$, and no solutions of (1.47) exist if $\mu > \mu^*$. Moreover, for each $\mu \in (0, \mu^*)$ the minimal solution of (1.47), denoted by w_{μ} , satisfies

$$0 < w_{\mu_1}(x) < w_{\mu_2}(x) < 1 \quad \text{for } x \in \Omega, \quad \text{if } 0 < \mu_1 < \mu_2 < \mu^*, \quad (1.48)$$

i.e., the minimal solutions are ordered with respect to the parameter μ .

Remark 1.1 We should point out here that a solution of (1.44) can be constructed for small values of λ by using monotone iteration techniques and the concept of lower-upper pairs of solutions, see [1].

In the radial symmetric case, i.e., when $\Omega = B_1 \equiv B_1(0) = \{x \in \mathbb{R}^N \mid 0 \leq |x| < 1\}$ is the unit ball of $\mathbb{R}^N$, $N \geq 2$, it is proved in [21] (see also the proof of Theorem 1.2.10 below) that $\mathcal{S}$ is homeomorphic to $\mathbb{R}$ and has end points $(0, 0)$ and $(\hat{\mu}, 1 - |x|^{2/3})$ with $\hat{\mu} = [4 + 6(N - 2)]/9$. Moreover, when $2 \leq N < 7$ the solution curve $(\mu(s), w(\cdot, s))$, $s \in \mathbb{R}$, of (1.47) bends infinitely many times around the singular point $(\hat{\mu}, 1 - |x|^{2/3})$. Whereas for $N \geq 7$ the solution curve terminates at $(\hat{\mu}, 1 - |x|^{2/3})$ and no bendings occur. For a detailed analysis on the structure of problem (1.47) when the term $\mu(1-w)^{-2}$ is also multiplied by the function $f(x) = |x|^p$, $p > 0$, see [6, 8, 13, 27].

Recall that for $N = 1$ there is $\lambda^* > 0$ such that a steady state of (1.37)–(1.39) exists if and only if $\lambda \leq \lambda^*$. In fact, by Theorem 2.1 of [18], there are exactly two solutions of (1.44) if $\lambda \in (0, \lambda^*)$, while (1.44) has a unique solution if $\lambda = \lambda^*$ and no solution if $\lambda > \lambda^*$. This indicates that the solution structure of the local problem (cf. [31]) is preserved by the non-local one when $N = 1$. However, it is not obvious that the aforementioned features of the local problem (1.47) are preserved by the non-local problem (1.44) for the case $N \geq 2$.

In the case where the Eq. (1.37) of problem (1.37)–(1.39) is substituted by (1.43), the related stationary problem is given by

$$\Delta w + \frac{\lambda f(x)}{K(1-w)^2} = 0, \quad x \in \Omega, \quad w = 0, \quad x \in \partial\Omega, \quad (1.49)$$

where K is given by (1.45) and $f(x) \geq 0$.

In the following, we will focus on the case $N \geq 2$. We first prove some results regarding the case of a general smooth domain Ω and then we shift to the radial symmetric case when $\Omega = B_1(0) = \{x \in \mathbb{R}^N \mid 0 \leq |x| < 1\}$.

Using the monotonicity property (1.48) we can prove the following existence theorem, which actually provides a lower bound of the pull-in voltage λ^* .

Theorem 1.2.6 *There exists a (classical) steady-state solution of (1.37)–(1.39) if $\lambda \in (0, (1 + |\Omega|)^2 \mu^*)$.*

Proof We look for $w = w_\mu$ for some μ . Note that, by (1.48), we have

$$(1 + |\Omega|)^2 = K(0) < K(w_{\mu_1}) < K(w_{\mu_2}), \quad \text{if } 0 < \mu_1 < \mu_2 < \mu^*.$$

Hence there exists a unique $\mu \in (0, \mu^*)$ such that

$$\mu K(w_\mu) = \lambda, \quad (1.50)$$

if $\lambda \in (0, (1 + |\Omega|)^2 \mu^*)$. This proves the theorem, since problems (1.44) and (1.47) are equivalent through (1.46). $\square$

Let Ω be a strictly star-shaped domain of $\mathbb{R}^N$, $N \geq 2$. Then there exists a constant $\beta > 0$ such that

$$s \cdot v \geq \beta \int_{\partial\Omega} ds \quad \text{for any } s \in \partial\Omega,$$

where v is the unit outward normal at s . Let v be a solution of the problem

$$\Delta v + \mu f(v) = 0 \quad x \in \Omega, \quad v = 0 \quad x \in \partial\Omega.$$

Then we have the following Pohozaev's identity (see [38]):

$$\mu N \int_{\Omega} F(v) dx - \frac{\mu(N-2)}{2} \int_{\Omega} v f(v) dx = \frac{1}{2} \int_{\partial\Omega} (s \cdot v) \left(\frac{\partial v}{\partial \nu} \right)^2 ds, \quad (1.51)$$

where $F(v) := \int_0^v f(s)ds$.

Next we make use of Pohozaev's identity to obtain an upper bound of the pull-in voltage λ^* .

Theorem 1.2.7 *Let Ω be a strictly star-shaped domain of $\mathbb{R}^N$, $N \geq 2$. Then problem (1.44) has no solution if*

$$\lambda > \lambda_0 := \frac{N(1 + |\Omega|)^4}{2\beta|\Omega|^2},$$

where $|\Omega|$ is the N -dimensional Lebesgue measure of Ω . Moreover, for $N \geq 2$ there exists $\lambda^* > 0$ such that problem (1.44) has a unique solution for $0 < \lambda < \lambda^* < \lambda_0$.

Proof We first prove the second statement of the theorem. For $N \geq 3$, by Theorem 3.1 in [7] there exists a positive constant μ_* such that problem (1.47) has a unique solution for $0 < \mu < \mu_*$. A similar result holds true for $N = 2$, see Theorem 4.1 in [20]. On the other hand, any solution $w = w(x)$ to (1.44) solves (1.47) with

$$\mu = \frac{\lambda}{(1 + \int_{\Omega} \frac{dx}{1-w})^2} \leq \frac{\lambda}{(1 + |\Omega|)^2},$$

due to the fact that $\frac{1}{1-w} \geq 1$, when w is a regular solution i.e., $\|w\|_{\infty} < 1$. We claim that problem (1.44) has a unique solution for $0 < \lambda < \lambda_* := \mu_*(1 + |\Omega|)^2$. Indeed, let us assume that problem (1.44) has two distinct solutions w_1, w_2 for some $0 < \lambda < \lambda_*$. We claim that

$$\int_{\Omega} \frac{dx}{1-w_1} = \int_{\Omega} \frac{dx}{1-w_2}. \quad (1.52)$$

Otherwise, if, for example,

$$\int_{\Omega} \frac{dx}{1-w_1} < \int_{\Omega} \frac{dx}{1-w_2},$$

then we have

$$\frac{\lambda}{(1 + |\Omega|)^2} \geq \mu_1 := \frac{\lambda}{(1 + \int_{\Omega} \frac{dx}{1-w_1})^2} > \mu_2 := \frac{\lambda}{(1 + \int_{\Omega} \frac{dx}{1-w_2})^2}. \quad (1.53)$$

Since $0 < \lambda < \lambda_* = \mu_*(1 + |\Omega|)^2$, we obtain by (1.53) that $0 < \mu_2 < \mu_1 < \mu_*$. Hence $w_1(x) = w(x; \mu_1)$ and $w_2(x) = w(x; \mu_2)$ are minimal solutions of the local problem (1.47). It follows by the monotonicity property (1.48) for the minimal solution branch of (1.47) that $w_1(x) > w_2(x)$ for $x \in \Omega$. This implies that

$$\int_{\Omega} \frac{dx}{1-w_1} > \int_{\Omega} \frac{dx}{1-w_2},$$

which is a contradiction. Using the same arguments we can exclude the possibility of

$$\int_{\Omega} \frac{dx}{1-w_1} > \int_{\Omega} \frac{dx}{1-w_2}.$$

Hence (1.52) is established and so local problem (1.47) has two distinct solutions w_1, w_2 corresponding to the same μ , which contradicts to [7, Theorem 3.1] and [20, Theorem 4.1]. Therefore, the second statement of the theorem is proved.

For the first statement, we suppose that problem (1.44) has a solution w for some $\lambda > 0$. Then applying Pohozaev's identity (1.51) to problem (1.44) with

$$f(w) = \frac{1}{K(1-w)^2}, \quad F(w) = \frac{w}{K(1-w)},$$

we obtain

$$\frac{\lambda N}{K} \int_{\Omega} \frac{w}{1-w} dx - \frac{\lambda(N-2)}{2K} \int_{\Omega} \frac{w}{(1-w)^2} dx = \frac{1}{2} \int_{\partial\Omega} (s \cdot \nu) \left(\frac{\partial w}{\partial \nu} \right)^2 ds. \quad (1.54)$$

By Hölder's inequality, we have

$$0 \leq - \int_{\partial\Omega} \frac{\partial w}{\partial \nu} ds \leq \left(\int_{\partial\Omega} \left(\frac{\partial w}{\partial \nu} \right)^2 ds \right)^{1/2} \left(\int_{\partial\Omega} ds \right)^{1/2},$$

hence due to the divergence theorem we derive

$$\begin{aligned} \frac{1}{2} \int_{\partial\Omega} (s \cdot \nu) \left(\frac{\partial w}{\partial \nu} \right)^2 ds &\geq \frac{\beta}{2} \left(\int_{\partial\Omega} ds \right) \int_{\partial\Omega} \left(\frac{\partial w}{\partial \nu} \right)^2 ds \geq \frac{\beta}{2} \left(\int_{\partial\Omega} -\frac{\partial w}{\partial \nu} ds \right)^2 \\ &= \frac{\beta}{2} \left(\int_{\Omega} -\Delta w dx \right)^2 = \frac{\lambda^2 \beta}{2K^2} \left(\int_{\Omega} \frac{1}{(1-w)^2} dx \right)^2. \end{aligned} \quad (1.55)$$

By dropping out the negative terms in (1.54) and using again Hölder's inequality, it follows from (1.54) that

$$\begin{aligned} \frac{\lambda N \int_{\Omega} \frac{dx}{1-w}}{\left(1 + \int_{\Omega} \frac{dx}{1-w} \right)^2} &\geq \frac{\lambda N}{K} \int_{\Omega} \frac{w}{1-w} dx \geq \frac{\lambda^2 \beta}{2} \left[\frac{\int_{\Omega} \frac{dx}{(1-w)^2}}{\left(1 + \int_{\Omega} \frac{dx}{1-w} \right)^2} \right]^2 \\ &\geq \frac{\lambda^2 \beta}{2|\Omega|^2} \left[\frac{\int_{\Omega} \frac{dx}{1-w}}{\left(1 + \int_{\Omega} \frac{dx}{1-w} \right)} \right]^4 \geq \frac{\lambda^2 \beta}{2|\Omega|^2} \left(\frac{|\Omega|}{1 + |\Omega|} \right)^4 \\ &= \frac{\lambda^2 \beta}{2} \left(\frac{|\Omega|^{1/2}}{1 + |\Omega|} \right)^4, \end{aligned} \quad (1.56)$$

where the last inequality in (1.56) comes out from the fact that

$$\int_{\Omega} \frac{dx}{1-w} \geq |\Omega|,$$

and since the function $g(s) = s/(1+s)$ is increasing in $s > 0$. Note also that

$$\frac{s}{(1+s)^2} \leq \frac{1}{4} \quad \text{for all } s \geq 0. \quad (1.57)$$

Combining (1.56) and (1.57), we get

$$\frac{\lambda N}{4} \geq \frac{\lambda^2 \beta}{2} \left(\frac{|\Omega|^{1/2}}{1+|\Omega|} \right)^4,$$

or equivalently

$$\lambda \leq \frac{N(1+|\Omega|)^4}{2\beta|\Omega|^2}.$$

This completes the proof of the theorem. $\square$

Remark 1.2.8 The second statement of Theorem 1.2.7 is still true for a general domain Ω . Indeed, for $N = 2$ Theorem 4.1 in [21] guarantees the existence of μ_* such that the local problem (1.47) has a unique solution for $0 < \mu < \mu_*$, see also Theorem 3.3 in [7]. On the other hand in higher dimensions an analogous result can be found in Sect. 5 of [44]. Furthermore, the second statement of Theorem 1.2.7 holds for problem (1.49) when $f(x) = |x|^p$, $p > 0$, and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, since in this case Theorem 3.2 in [7] guarantees the existence of $\mu_* > 0$ such that the local problem (1.47) has only the minimal solution for $0 < \mu < \mu_*$.

Remark 1.2.9 Let ω_N denote the volume of the N -dimensional unit ball $B_1(0)$. Then Theorem 1.2.7 implies that no steady states exist if $\lambda > \frac{N^2(1+\omega_N)^4}{2\omega_N}$, when $\Omega = B_1(0)$.

1.2.1.3 Structure of Radial Symmetric Solutions

When $\Omega = B_1 \equiv B_1(0)$, then the solution of (1.44) is radial symmetric, [14], i.e., $w(x) = w(r)$, where $r = |x|$ and (1.44) is reduced to

$$(r^{N-1}w_r)_r + \frac{\lambda r^{N-1}}{K(1-w)^2} = 0 \quad \text{for } r \in (0, 1), \quad (1.58)$$

$$w(1) = 0, \quad w_r(0) = 0, \quad (1.59)$$

where now K has the form

$$K = \left(1 + N\omega_N \int_0^1 \frac{r^{N-1}}{1-w} dr\right)^2,$$

or equivalently to the local problem

$$(r^{N-1}w_r)_r + \frac{\mu r^{N-1}}{(1-w)^2} = 0 \quad \text{for } r \in (0, 1), \quad (1.60)$$

$$w(1) = 0, \quad w_r(0) = 0, \quad (1.61)$$

where $\mu = \frac{\lambda}{K}$.

We then denote $\mathcal{C}_r, \mathcal{S}_r$ instead of $\mathcal{C}, \mathcal{S}$ respectively. Also, set

$$\mathcal{S}_r^\lambda = \{w \in C^2(B_1) \cap C^0(\overline{B_1}) \mid w \text{ solves (1.58)–(1.59)}\},$$

the section of $\mathcal{S}_r$ cut by $\lambda > 0$. Then we have the following characterization of $\mathcal{S}_r$.

Theorem 1.2.10 *If $N \geq 2$, then $\mathcal{S}_r$ is homeomorphic to $\mathbb{R}$ and has end points $(0, 0)$ and $(\widehat{\lambda}, 1 - |x|^{2/3})$, where*

$$\widehat{\lambda} = \Lambda \left(1 + \frac{N\omega_N}{\gamma}\right)^2, \quad \Lambda := \frac{2}{3}(N - 4/3) = \widehat{\mu}, \quad \gamma = N - 2/3.$$

Moreover, if $2 \leq N < 7$, then $\mathcal{S}_r$ bends infinitely many times with respect to λ around $\widehat{\lambda}$ and there exist two positive constants λ^* and λ_* with $0 < \lambda_* < \widehat{\lambda} < \lambda^*$ such that problem (1.58)–(1.59) has

- a unique solution for $0 < \lambda < \lambda_*$ and for $\lambda = \lambda^*$,
- a finite number of solutions for $\lambda \in (\lambda_*, \lambda^*)$ and $\lambda \neq \widehat{\lambda}$,
- infinite number of solutions for $\lambda = \widehat{\lambda}$,
- no solutions for $\lambda > \lambda^*$,

see Fig. 1.4.

Whereas no bending occurs in the case of $N \geq 7$, i.e., problem (1.58)–(1.59) has a unique solution for any $0 < \lambda < \widehat{\lambda} = \lambda^*$ and no solution for $\lambda \geq \widehat{\lambda} = \lambda^*$, see Fig. 1.5.

Proof We proceed as in [21] using phase-plane analysis. First we note that every positive solution of problem (1.60)–(1.61) can be obtained as a solution of the following initial-value problem

$$(r^{N-1}w_r)_r + \frac{\mu r^{N-1}}{(1-w)^2} = 0, \quad r > 0, \quad (1.62)$$

$$w(0) = A, \quad w_r(0) = 0, \quad (1.63)$$

with a certain positive constant $A \in (0, 1)$. We always require that $0 < w(r) < 1$ for $r \in (0, 1)$.

Putting

$$k = \sqrt{\frac{\Lambda(1-A)^3}{\mu}} = \sqrt{\frac{K\Lambda(1-A)^3}{\lambda}},$$

we apply the Emden transformation

$$w(r; t) = 1 - (1 - A) e^{2t/3} z(t), \quad r = k e^t, \quad (1.64)$$

for $\Lambda = \frac{[6(N-2)+4]}{9}$. Then problem (1.62)–(1.63) is transformed to

$$\ddot{z} + \gamma \dot{z} + \Lambda \left(z - \frac{1}{z^2} \right) = 0, \quad t > -\infty, \quad (1.65)$$

$$\lim_{t \rightarrow -\infty} e^{2t/3} z(t) = 1, \quad \lim_{t \rightarrow -\infty} e^{2t/3} \dot{z}(t) = -\frac{2}{3}, \quad (1.66)$$

where we require that $0 < z(t) < e^{-2t/3}/(1 - A)$. Next if we set

$$w(r; t) = 1 - (1 - A) e^{-2t/3} \tilde{z}(t), \quad r = k e^{-t},$$

then $\tilde{z}(t)$ satisfies

$$\ddot{\tilde{z}} - \gamma \dot{\tilde{z}} + \Lambda \left(\tilde{z} - \frac{1}{\tilde{z}^2} \right) = 0, \quad t > -\infty,$$

$$\lim_{t \rightarrow +\infty} e^{-2t/3} \tilde{z}(t) = 1, \quad \lim_{t \rightarrow +\infty} e^{-2t/3} \dot{\tilde{z}}(t) = \frac{2}{3}.$$

It has been proved in [21], that problem (1.65)–(1.66) has a global-in-time solution with the orbit $\mathcal{O} = \{(z(t), \dot{z}(t)), t \in \mathbb{R}\}$ starting at $t = -\infty$ from the point $(+\infty, -\infty)$ above its tangent line $\dot{z} + 2z/3 = 0$ and approaching the point $(1, 0)$ as $t \rightarrow +\infty$, see Fig. 1.3. On the other hand the orbit $\tilde{\mathcal{O}} = \{(\tilde{z}(t), \dot{\tilde{z}}(t)), t \in \mathbb{R}\}$ starts at $t = -\infty$ from the point $(+\infty, +\infty)$ below its tangent line $\dot{z} - 2z/3 = 0$ and approaches the point $(1, 0)$ as $t \rightarrow +\infty$. Moreover, the orbits $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are symmetric with respect to z -axis in the phase plane, see [21]. Hence we obtain

$$\left| \frac{\dot{z}(t)}{z(t)} \right| < \frac{2}{3}, \quad (1.67)$$

for any $t \in \mathbb{R}$.

Through the transformation (1.64), the boundary condition $w(1) = 0$ in (1.59) is converted to

$$z(\tau) = [\Lambda/\mu(\tau)]^{1/3}, \quad \tau := -\frac{1}{2} \ln \frac{[6(N-2)+4](1-A)^3}{9\mu} = -\ln k.$$

In other words, for any $\tau \in \mathbb{R}$, $(A(\tau), \mu(\tau), w(\cdot, \tau))$ defined by

$$w(r; \tau) = 1 - \frac{z(\tau + \ln r)}{z(\tau)} r^{2/3}, \quad (1.68)$$

$$\mu(\tau) = \frac{\Lambda}{z^3(\tau)}, \quad (1.69)$$

$$A(\tau) = 1 - \frac{1}{e^{2\tau/3} z(\tau)}, \quad (1.70)$$

satisfies the initial-value problem (1.62)–(1.63) or equivalently the local problem (1.60)–(1.61). Let

$$K(\tau) := \left(1 + N\omega_N \int_0^1 \frac{r^{N-1}}{1 - w(r; \tau)} dr\right)^2,$$

then $(\lambda(\tau), w(\cdot, \tau))$ with $\lambda(\tau) = \mu(\tau)K(\tau)$ satisfies the non-local steady-state problem (1.58)–(1.59). Conversely, every solution of (1.58)–(1.59) corresponds, through the Emden transformation (1.64) (where μ and A are given by (1.69) and (1.70) respectively) and $\lambda = \mu K$, to an element of the orbit $\mathcal{O}$ for a certain $\tau \in \mathbb{R}$. Therefore, there is an one-to-one and onto correspondence between the solutions to (1.58)–(1.59) and the elements of the orbit $\mathcal{O}$, hence $\mathcal{S}_r$ is homeomorphic to $\mathcal{O}$ and so to $\mathbb{R}$.

Now, taking the limit as $\tau \rightarrow -\infty$ in (1.69), we obtain that $\mu(\tau) \rightarrow 0$, which implies as well that $w(\cdot; \tau) \rightarrow 0$ in B_1 as $\tau \rightarrow -\infty$, hence $\lambda(\tau) \rightarrow 0$ as $\tau \rightarrow -\infty$. On the other hand, taking the limit as $\tau \rightarrow +\infty$ we obtain that $\mu(\tau) \rightarrow \Lambda$ and $w(\cdot; \tau) \rightarrow w^*(\cdot)$ in B_1 , where $w^*(r) := 1 - r^{2/3}$ is actually a singular, $\|w^*\|_\infty = 1$, solution to (1.58)–(1.59). This implies that

$$\lambda(\tau) \rightarrow \widehat{\lambda} := \Lambda \left(1 + N\omega_N \int_0^1 r^{N-5/3} dr\right)^2 = \Lambda \left(1 + \frac{N\omega_N}{\gamma}\right)^2 \text{ as } \tau \rightarrow +\infty. \quad (1.71)$$

Hence the terminating point of the solution curve $\mathcal{S}_r$ is $(\widehat{\lambda}, w^*)$. Besides, the singular points $(+\infty, -\infty)$ and $(0, 1)$ of (1.65)–(1.66) correspond to the end points $(1, 0)$ and $(\widehat{\lambda}, w^*)$ of the solution curve $\mathcal{S}_r$. This proves the first part of the theorem.

Moreover, by Lemma 6 in [21], for $2 \leq N < 7$ the orbit $\mathcal{O}$ of (1.65)–(1.66) starts from $(+\infty, -\infty)$ above the tangent line $\dot{z} + 2z/3 = 0$ and terminates to $(1, 0)$ crossing clockwise infinitely many times the positive part of the z -axis, say for $t_1 < t_2 < \dots < t_k < \dots$, $k \in \mathbb{N}$, as well as the curve Γ defined by

$$\dot{z} = -\frac{\Lambda}{\gamma} \left(z - \frac{1}{z^2}\right) := f(z), \quad (1.72)$$

for $s_1 < s_2 < \dots < s_k < \dots$, $k \in \mathbb{N}$, without crossing itself.

Let $Q_k = (z(t_k), 0)$ and $P_k = (z(s_k), \dot{z}(s_k))$, $k \in \mathbb{N}$, be the intersection points of the orbit $\mathcal{O}$ with the z -axis and the curve Γ respectively, see Fig. 1.3. Then it suffices

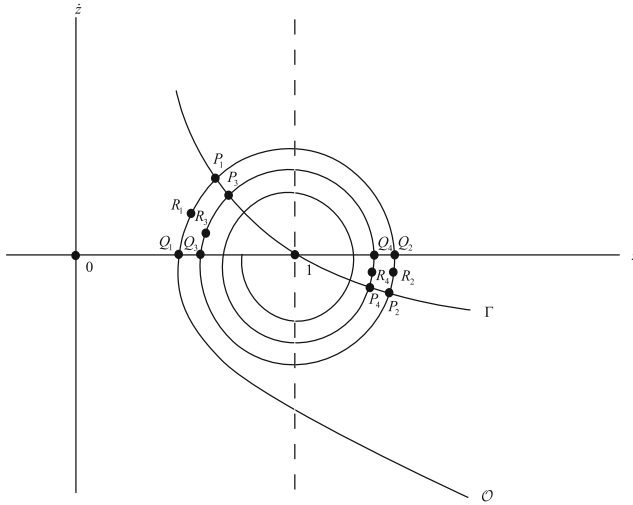


Fig. 1.3 The phase plane for $2 \leq N < 7$

to show that $\dot{\lambda}(\tau)$ changes sign across the arcs $Q_{2k-1}P_{2k-1}$ and $Q_{2k}P_{2k}$, see also [33].

We note

$$\lambda(\tau) = \mu(\tau)G^2(\tau) = \Lambda \frac{G^2(\tau)}{z^3(\tau)}, \quad G(\tau) := K^{1/2}(\tau). \quad (1.73)$$

Also, from (1.68) it follows that

$$\begin{aligned} G(\tau) &= 1 + N\omega_N z(\tau) \int_0^1 \frac{1}{z(\tau + \ln r)} r^{\gamma-1} dr \\ &= 1 + N\omega_N z(\tau) e^{-\gamma\tau} \int_{-\infty}^{\tau} \frac{1}{z(s)} e^{\gamma s} ds, \end{aligned} \quad (1.74)$$

and hence

$$\dot{G}(\tau) = \frac{\dot{z}(\tau)}{z(\tau)} [G(\tau) - 1] - \gamma [G(\tau) - 1] + N\omega_N. \quad (1.75)$$

Then

$$\begin{aligned} \dot{\lambda}(\tau) &= 2\Lambda \frac{G(\tau)\dot{G}(\tau)}{z^3(\tau)} - 3\Lambda \frac{G^2(\tau)}{z^4(\tau)} \dot{z}(\tau) \\ &= \frac{\Lambda G(\tau)}{z^4(\tau)} [2\dot{G}(\tau)z(\tau) - 3\dot{z}(\tau)G(\tau)]. \end{aligned} \quad (1.76)$$

Using (1.75), relation (1.76) reads

$$\dot{\lambda}(\tau) = \frac{\Lambda G(\tau)}{z^4(\tau)} \left\{ -\dot{z}(\tau)[G(\tau) + 2] + 2(N\omega_N - \gamma[G(\tau) - 1])z(\tau) \right\}, \quad (1.77)$$

which implies

$$\dot{\lambda}(t_{2k-1}) = \frac{2\Lambda G(t_{2k-1})}{z^3(t_{2k-1})} \left(\gamma + N\omega_N - \gamma G(t_{2k-1}) \right), \quad (1.78)$$

and

$$\begin{aligned} \dot{\lambda}(s_{2k-1}) &= \frac{\Lambda G(s_{2k-1})}{z^4(s_{2k-1})} \left[- (G(s_{2k-1}) + 2) f(z(s_{2k-1})) \right. \\ &\quad \left. + 2(\gamma + N\omega_N - \gamma G(s_{2k-1})) z(s_{2k-1}) \right], \end{aligned} \quad (1.79)$$

respectively, because Q_{2k-1} is on the z -axis.

By the mean value theorem equality (1.74) can be written in the form

$$G(\tau) = 1 + N\omega_N \frac{z(\tau)}{z(\tau + \ln r_0(\tau))} \int_0^1 r^{\gamma-1} dr = 1 + \frac{N\omega_N}{\gamma} \frac{z(\tau)}{z(\tau + \ln r_0(\tau))}, \quad (1.80)$$

where $r_0(\tau) \in (0, 1)$. In particular, we have

$$G(t_{2k-1}) = 1 + \frac{N\omega_N}{\gamma} \frac{z(t_{2k-1})}{z(t_{2k-1} + \ln r_0(t_{2k-1}))}, \quad (1.81)$$

for any k .

Moreover, due to (1.71) we have

$$G(\tau) \rightarrow 1 + \frac{N\omega_N}{\gamma} \quad \text{as } \tau \rightarrow \infty.$$

Therefore, by (1.80) and the continuity of $z(\tau)$ we derive

$$\ln r_0(\tau) \rightarrow 0 \quad \text{as } \tau \rightarrow \infty.$$

The later implies that at t_{2k-1} and for k sufficiently large we have $|\ln r_0(t_{2k-1})| \ll 1$. But, at t_{2k-1} a local minimum for $z(t)$ occurs and hence $z(t_{2k-1}) < z(t_{2k-1} + \ln r_0(t_{2k-1}))$ for k sufficiently large. It finally yields

$$G(t_{2k-1}) < 1 + \frac{N\omega_N}{\gamma} \quad \text{for } k \text{ sufficiently large}, \quad (1.82)$$

by (1.81). Then (1.78) and (1.82) imply

$$\dot{\lambda}(t_{2k-1}) > 0, \quad (1.83)$$

for k sufficiently large.

On the other hand, at the point P_{2k-1} we have $z(s_{2k-1}) < 1$, and hence the first term in the bracket in the RHS of (1.79) is strictly negative. Besides, there holds

$$G(s_{2k-1}) = 1 + \frac{N\omega_N}{\gamma} \frac{z(s_{2k-1})}{z(s_{2k-1} + \ln r_0(s_{2k-1}))}, \quad (1.84)$$

with $0 < -\ln r_0(s_{2k-1}) \ll 1$ whenever k is large enough. But, since z is strictly increasing along the arc $Q_{2k-1}P_{2k}$, equality (1.84) yields that

$$G(s_{2k-1}) > 1 + \frac{N\omega_N}{\gamma}, \quad (1.85)$$

for k sufficiently large. Therefore, we conclude from (1.79) that

$$\dot{\lambda}(s_{2k-1}) < 0, \quad (1.86)$$

for k sufficiently large.

Now, from (1.83) and (1.86), whenever k is large there exists a point on the orbit $\mathcal{O}$ denoted by $R_{2k-1} = (z(\tau_{2k-1}), \dot{z}(\tau_{2k-1}))$ with $t_{2k-1} < \tau_{2k-1} < s_{2k-1}$, located in the arc $Q_{2k-1}P_{2k-1}$ such that $\dot{\lambda}(\tau_{2k-1}) = 0$.

In a similar manner, it is derived that $\dot{\lambda}(t_{2k}) < 0$ and $\dot{\lambda}(s_{2k}) > 0$, because z achieves a local maximum at t_{2k} and $z(\tau)$ is strictly decreasing along the arc $Q_{2k}Q_{2k+1}$. Therefore, we can again find, for k sufficiently large, a point $R_{2k} = (z(\tau_{2k}), \dot{z}(\tau_{2k}))$, $t_{2k} < \tau_{2k} < s_{2k}$, lying on the arc $Q_{2k}P_{2k}$ such that $\dot{\lambda}(\tau_{2k}) = 0$. Also by (1.77) we obtain $\dot{\lambda}(\tau) < 0$ for any $s_{2k-1} < \tau < t_{2k}$, while $\dot{\lambda}(\tau) > 0$ for any $s_{2k} < \tau < t_{2k+1}$. Hence only the points $\{R_k\}$ correspond to the bendings of $\mathcal{S}_r$. For the distribution of points $\{R_k\}$ across the orbit $\mathcal{O}$ see Fig. 1.3. The above analysis implies that $\mathcal{S}_r$ bends infinitely many times around $\hat{\lambda}$, see Fig. 1.4.

On the other hand, when $N \geq 7$, the orbit starts from $(+\infty, -\infty)$ and stays inside the area $\{(z, \dot{z}) \mid z > 0, 0 > \dot{z} > -2z/3\}$ terminating to $(1, 0)$ without crossing neither the z -axis nor itself. Therefore, through the homeomorphism between $\mathcal{S}_r$ and $\mathcal{O}$, the curve $\mathcal{S}_r$ exhibits no bending, and hence the local problem (1.60)–(1.61) has a unique solution for $0 < \mu < \hat{\mu} = \Lambda$. This means that the non-local problem (1.58)–(1.59) has a unique solution for any $0 < \lambda < \hat{\lambda}$, see Fig. 1.5. The proof of the theorem is complete. $\square$

Remark 1.2.11 Using the same reasoning as above, an analogous result to Theorem 1.2.10 could be proven for problem (1.49), see also Theorem 1.2 in [27].

Remark 1.2.12 By Theorem 1.2.10, if $2 \leq N < 7$ there is a strictly increasing sequence $\{\tau_k\}_{k=1}^\infty \subset \mathbb{R}$ such that the functions $\tau \in [\tau_{2k-1}, \tau_{2k}] \mapsto \lambda(\tau)$ and

Fig. 1.4 The response diagram of (1.58)–(1.59) when $2 \leq N < 7$

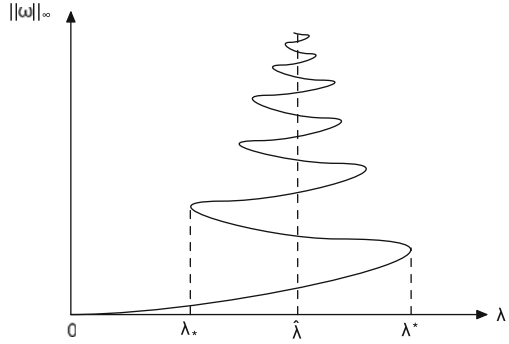
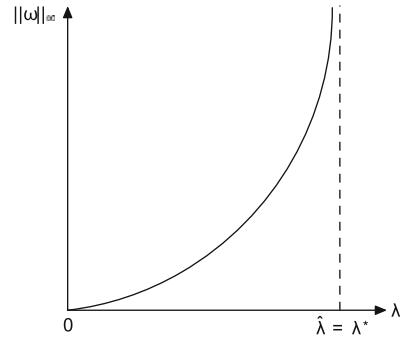


Fig. 1.5 The response diagram of (1.58)–(1.59) when $N \geq 7$



$\tau \in [\tau_{2k}, \tau_{2k+1}] \mapsto \lambda(\tau)$ are strictly decreasing and increasing respectively, and $\lambda(\tau_2) < \lambda(\tau_4) < \dots < \lambda(\tau_{2k}) < \lambda(\tau_{2k+2}) < \dots < \hat{\lambda} < \dots < \lambda(\tau_{2k+1}) < \lambda(\tau_{2k-1}) < \dots < \lambda(\tau_3) < \lambda(\tau_1)$. On the other hand, if $N \geq 7$ the function $\tau \in (-\infty, \infty) \mapsto \lambda(\tau)$ is strictly increasing and for each $\lambda \in (0, \hat{\lambda})$ the non-local steady-state problem (1.58)–(1.59) has a unique solution.

Given $w \in \mathcal{S}^\lambda$, the linearized eigenvalue problem around (λ, w) is given as follows

$$-\Delta\phi - \frac{2\lambda\phi}{(1-w)^3 \left(1 + \int_{\Omega} \frac{dx}{1-w}\right)^2} + \frac{2\lambda \int_{\Omega} \frac{\phi dx}{(1-w)^2}}{(1-w)^2 \left(1 + \int_{\Omega} \frac{dx}{1-w}\right)^3} = \rho\phi, \quad x \in \Omega, \quad (1.87)$$

$$\phi = 0, \quad x \in \partial\Omega. \quad (1.88)$$

The number of the negative eigenvalues ρ of problem (1.87)–(1.88), denoted by $i = i(\lambda, w)$, is called the Morse index at (λ, w) .

In the case where $\Omega = B_1(0)$, the number of the negative eigenvalues of the corresponding linearized eigenvalue problem in the space of radially symmetric functions around the radial symmetric solution $w(r) \in \mathcal{S}_r^\lambda$ is denoted by $i_R = i_R(\lambda, w)$ and is called the radial Morse index at (λ, w) . Regarding the radial Morse

index the following result holds. For an analogous result associated to the non-local Gelfand problem see [33, 34].

Theorem 1.2.13 *If $\Omega = B_1(0)$, then $i(\lambda, w) = i_R(\lambda, w)$ and this index increases by one at each turning (bending) point of the solution curve $\mathcal{S}_r$.*

Proof The first statement of the theorem is proven by using a similar argument to the proof of Proposition 3.3 in [32]. Indeed, when $\Omega = B_1(0)$ any solution w to (1.44) is radially symmetric, i.e., $w(x) = w(r)$, $r = |x|$. Hence, applying the separation of variables, any solution to (1.87)–(1.88) can be written in the form

$$\phi_k(x) = \psi_k(r)e_k(x/|x|), \quad x \in B_1(0), \quad k = 1, 2, \dots,$$

where $\{e_k\}_{k=1}^\infty$ is the sequence of eigenfunctions of the Laplace operator on the unit sphere $\partial B_1(0)$ with corresponding eigenvalues $0 = \nu_1 < N - 1 = \nu_2 \leq \nu_3 \leq \dots$. Owing to the radial symmetry of w and ψ_k we obtain

$$\int_{B_1(0)} \frac{\phi_k dx}{(1-w)^2} = 0 \quad \text{for } k \geq 2.$$

Then $\psi_k(r)$ satisfies the problem

$$-\psi_k'' - \frac{N-1}{r}\psi_k' + \frac{\nu_k}{r^2}\psi_k - \frac{2\lambda\psi_k}{(1-w)^3\left(1 + \int_B \frac{dx}{1-w}\right)^2} = \rho\psi_k, \quad 0 < r < 1, \quad (1.89)$$

$$\psi_k'(0) = \psi_k(1) = 0, \quad (1.90)$$

for any $k \geq 2$. Note that problem (1.89)–(1.90) has the same form with the problem obtained by (1.47) under linearization and separation of variables.

We claim that, under the assumption $\rho < 0$, we get $\psi_k = 0$ and so $\phi_k = 0$ for $k \geq 2$. Indeed, let assume that $\psi_k \neq 0$ and r_1 be the first positive zero of ψ_k such that $\psi_k > 0$ in $(0, r_1)$. Then using similar arguments as in [32] we derive for $0 < r < r_1$

$$\begin{aligned} \left[\frac{\partial \psi_k}{\partial r} \frac{\partial w}{\partial r} r^{N-1} - \psi_k \frac{\partial^2 w}{\partial r^2} r^{N-1} \right]_{r=0}^{r=r_1} + \int_0^{r_1} \frac{N-1-\nu_k}{r^2} \psi_k \frac{\partial w}{\partial r} r^{N-1} dr \\ = -\rho \int_0^{r_1} \psi_k \frac{\partial w}{\partial r} r^{N-1} dr. \end{aligned} \quad (1.91)$$

Taking into account that $w(r)$ is radial decreasing as well as that $\nu_k \geq N - 1$ for $k \geq 2$ we derive by (1.91) that $\rho > 0$, leading to a contradiction. Therefore, under the hypothesis $\rho < 0$ we have that the solution of (1.87)–(1.88) could be written in the form $\phi(x) = c\psi(r)$, for some constant c with $\psi(r)$ satisfying (1.89)–(1.90), which implies the desired result $i(\lambda, w) = i_R(\lambda, w)$.

The second part of the theorem is proven by the method of [34]; note that each $(\lambda, w) \in \mathcal{C}_r$ corresponds to $(z(\tau), \dot{z}(\tau)) \in \mathcal{O}$ and therefore, it is parameterized by

$\tau \in \mathbb{R}$: $(\lambda, w) = (\lambda(\tau), w(\tau))$. We denote by ρ_τ^ℓ , $\ell = 1, 2, \dots$, the ℓ -th eigenvalue of the linearized problem (1.87)–(1.88) around $(\lambda(\tau), w(\tau))$ which corresponds to a radially symmetric eigenfunction. Each ρ_τ^ℓ is simple. If $(\lambda(\tau), w(\tau))$ is on a turning point of $\mathcal{C}_r$, then, by the implicit function theorem, there is $\ell \geq 1$ such that $\rho_\tau^\ell = 0$. If $\rho_\tau^\ell = 0$ holds for some $\ell \geq 1$ with $(\lambda(\tau), w(\tau)) \in \mathcal{C}_r$ not on a turning point, then, by the bifurcation theory on the critical point of odd multiplicity, [40, 41], it is actually a bifurcation point of $\mathcal{C}_r$. But, this is impossible by the geometric features of $\mathcal{S}_r$ provided by Theorem 1.2.10, and therefore, $(\lambda(\tau), w(\tau))$ is on a turning point of $\mathcal{C}_r$ if and only if (1.87)–(1.88) has a zero eigenvalue $\rho_\tau^\ell = 0$ for some $\ell \geq 1$. Denoting the k -th turning point of $\mathcal{C}_r$ by $T_k = (\lambda(\tau_k), w(\cdot, \tau_k))$, $\tau_1 < \tau_2 < \dots < \tau_k < \dots$, then the assertion follows if we can prove that $\dot{\rho}_{\tau_k}^{\ell(k)} < 0$ for $k \geq 1$, where $\ell(k)$ is such that $\rho_{\tau_k}^{\ell(k)} = 0$.

By differentiating (1.44) with respect to τ , we deduce

$$\Delta \dot{w} + \frac{\dot{\lambda}(1-w)^{-2}}{\left(1 + \int_{B_1(0)} \frac{dx}{1-w}\right)^2} + \frac{2\lambda(1-w)^{-3}\dot{w}}{\left(1 + \int_{B_1(0)} \frac{dx}{1-w}\right)^2} - \frac{2\lambda(1-w)^{-2} \int_{B_1(0)} (1-w)^{-2} \dot{w} dx}{\left(1 + \int_{B_1(0)} \frac{dx}{1-w}\right)^3} = 0 \text{ in } B_1(0), \quad (1.92)$$

$$\dot{w} = 0 \text{ on } \partial B_1(0), \quad (1.93)$$

where $\dot{w} := \partial w / \partial \tau$ and $\dot{\lambda} := d\lambda/d\tau$. Hence $w_k := w(\cdot, \tau_k)$ satisfies

$$\Delta \dot{w}_k + \frac{2\lambda(1-w_k)^{-3}\dot{w}_k}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^2} - \frac{2\lambda(1-w_k)^{-2} \int_{B_1(0)} (1-w_k)^{-2} \dot{w}_k dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} = 0 \text{ in } B_1(0),$$

$$\dot{w}_k = 0 \text{ on } \partial B_1(0),$$

recalling that $\dot{\lambda}(\tau_k) = 0$.

We claim that

$$\dot{w}(r, \tau) = r^{2/3} \frac{z(\tau + \ln r) \dot{z}(\tau) - \dot{z}(\tau + \ln r) z(\tau)}{z^2(\tau)} \not\equiv 0, \quad \forall \tau \in \mathbb{R}.$$

Indeed, if there is a $\tau \in \mathbb{R}$ such that $\dot{w}(r, \tau) \equiv 0$, then we have $\dot{z}/z$ is identically equal to a constant. This constant must be $-2/3$ by (1.66). Then (1.65) implies that $\frac{1}{z^2} \equiv 0$, a contradiction. Hence the claim is proved and we obtain that $\dot{w}_k$ is an eigenfunction of the linearized problem (1.87)–(1.88) corresponding to $\rho = \rho_{\tau_k}^{\ell(k)} = 0$. Therefore, the standard perturbation theory ([24]) guarantees the existence of a function $\psi = \psi(\cdot, \tau)$ and $\rho = \rho(\tau)$ satisfying the linearized problem (1.87)–(1.88) such that $\psi(\cdot, \tau_k) = \dot{w}_k$ and $\rho(\tau_k) = \rho_{\tau_k}^{\ell(k)} = 0$.

Differentiating problems (1.92)–(1.93) and (1.87)–(1.88) with respect to τ and taking into account that $\dot{\lambda}(\tau_k) = \rho(\tau_k) = 0$, we obtain

$$\begin{aligned} \Delta \ddot{w}_k + \frac{\ddot{\lambda}(1-w_k)^{-2}}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^2} + \frac{6\lambda(1-w_k)^{-4}\dot{w}_k^2}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^2} + \frac{2\lambda(1-w_k)^{-3}\ddot{w}_k}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^2} \\ - \frac{8\lambda(1-w_k)^{-3}\dot{w}_k \int_B (1-w_k)^{-2}\dot{w}_k dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} - \frac{4\lambda(1-w_k)^{-2} \int_B (1-w_k)^{-3}\dot{w}_k^2 dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} \\ + \frac{6\lambda(1-w_k)^{-2} \left(\int_B (1-w_k)^{-2}\dot{w}_k dx\right)^2}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^4} - \frac{2\lambda(1-w_k)^{-2} \int_B (1-w_k)^{-2}\ddot{w}_k dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} = 0, \end{aligned} \quad (1.94)$$

for $x \in B_1(0)$ with $\ddot{w}_k = 0$ on $\partial B_1(0)$, and

$$\begin{aligned} \Delta \dot{\psi}_k + \frac{6\lambda(1-w_k)^{-4}\psi_k \dot{w}_k}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^2} + \frac{2\lambda(1-w_k)^{-3}\dot{\psi}_k}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^2} \\ - \frac{4\lambda(1-w_k)^{-3}\psi_k \int_B (1-w_k)^{-2}\dot{w}_k dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} - \frac{4\lambda(1-w_k)^{-3}\dot{w}_k \int_B (1-w_k)^{-2}\psi_k dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} \\ - \frac{4\lambda(1-w_k)^{-2} \int_B (1-w_k)^{-3}\psi_k^2 dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} + \frac{6\lambda(1-w_k)^{-2} \left(\int_B (1-w_k)^{-2}\psi_k dx\right)^2}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^4} \\ - \frac{2\lambda(1-w_k)^{-2} \int_B (1-w_k)^{-2}\dot{\psi}_k dx}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^3} = -\dot{\rho}\psi_k, \end{aligned} \quad (1.95)$$

for $x \in B_1(0)$ with $\dot{\psi}_k = 0$ on $\partial B_1(0)$. Using $\psi_k = \dot{w}_k$, it follows by (1.94) and (1.95) that

$$\frac{\ddot{\lambda}(\tau_k)(1-w_k)^{-2}}{\left(1 + \int_B \frac{dx}{1-w_k}\right)^2} = \dot{\rho}(\tau_k)\dot{w}_k. \quad (1.96)$$

Multiplying (1.96) by $\dot{w}_k$ and integrating over B_1 , we derive that

$$\frac{\ddot{\lambda}(\tau_k) \int_B (1-w_k)^{-2}\dot{w}_k dx}{\left(1 + \int_{B_1} \frac{dx}{1-w_k}\right)^2} = \dot{\rho}(\tau_k) \int_{B_1} \dot{w}_k^2 dx. \quad (1.97)$$

Thus, in order to obtain the desired result we need to prove that the left-hand side of (1.97) is negative.

Using that $\dot{\lambda}(\tau_k) = 0$ relation (1.76) implies

$$\dot{G}(\tau_k) = \frac{3}{2} \frac{\dot{z}(\tau_k)G(\tau_k)}{z(\tau_k)}, \quad (1.98)$$

while (1.77) also reads

$$\dot{z}(\tau_k) \left(G(\tau_k) + 2 \right) = 2 \left(N\omega_N - \gamma(G(\tau_k) - 1) \right) z(\tau_k), \quad (1.99)$$

for every $k = 1, 2, \dots$

Differentiating (1.77) with respect to τ , we obtain

$$\begin{aligned} \ddot{\lambda}(\tau) = & \Lambda \frac{\dot{G}(\tau)z^4(\tau) - 4G(\tau)z^3(\tau)\dot{z}(\tau)}{z^8(\tau)} \times \left[-\dot{z}(\tau)(G(\tau) + 2) + 2 \left(N\omega_N - \gamma(G(\tau) - 1) \right) z(\tau) \right] \\ & + \frac{\Lambda G(\tau)}{z^4(\tau)} \left[-\ddot{z}(\tau)(G(\tau) + 2) - \dot{z}(\tau)\dot{G}(\tau) - 2\gamma\dot{G}(\tau)z(\tau) + 2 \left(N\omega_N - \gamma(G(\tau) - 1) \right) \dot{z}(\tau) \right], \end{aligned}$$

and due to (1.99) yields

$$\begin{aligned} \ddot{\lambda}(\tau_k) = & \frac{\Lambda G(\tau_k)}{z^4(\tau_k)} \left[-\ddot{z}(\tau_k)(G(\tau_k) + 2) - \dot{z}(\tau_k)\dot{G}(\tau_k) - 2\gamma\dot{G}(\tau_k)z(\tau_k) \right. \\ & \left. + 2 \left(N\omega_N - \gamma(G(\tau_k) - 1) \right) \dot{z}(\tau_k) \right]. \end{aligned} \quad (1.100)$$

Now (1.100), via (1.98), yields

$$\begin{aligned} \ddot{\lambda}(\tau_k) = & \frac{2}{3} \frac{\Lambda \dot{G}(\tau_k)}{z^3(\tau_k)} \left[-\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)} (G(\tau_k) + 2) - \dot{G}(\tau_k) - 2\gamma\dot{G}(\tau_k) \frac{z(\tau_k)}{\dot{z}(\tau_k)} \right. \\ & \left. + 2 \left(N\omega_N - \gamma(G(\tau_k) - 1) \right) \right]. \end{aligned}$$

Using (1.75), (1.98) and (1.99), we finally end up with the following

$$\ddot{\lambda}(\tau_k) = \frac{2}{3} \frac{\Lambda \dot{G}(\tau_k)}{z^3(\tau_k)} \left[-\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)} (G(\tau_k) + 2) - \gamma(1 + 2G(\tau_k)) + 3 \frac{\dot{z}(\tau_k)}{z(\tau_k)} - N\omega_N \right].$$

Noting that

$$\dot{G}(\tau_k) = \int_{B_1(0)} (1 - w_k)^{-2} \dot{w}_k dx,$$

it follows by (1.97)

$$\frac{2\Lambda \dot{G}(\tau_k)^2}{3z^3(\tau_k)} \frac{C(\tau_k)}{\left(1 + \int_{B_1(0)} \frac{dx}{1-w_k} \right)^2} = \dot{\rho}(\tau_k) \int_{B_1(0)} \dot{w}_k^2 dx,$$

where

$$C(\tau_k) := -\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)}(G(\tau_k) + 2) - \gamma(1 + 2G(\tau_k)) + 3\frac{\dot{z}(\tau_k)}{z(\tau_k)} - N\omega_N.$$

Therefore, it is enough to prove that $C(\tau_k) < 0$ as well as that $\dot{G}(\tau_k) \neq 0$.

Note that for $\tau = \tau_{2k-1}$ there holds

$$\frac{\ddot{z}(\tau_{2k-1})}{\dot{z}(\tau_{2k-1})} > 0,$$

since $\dot{z}(\tau_{2k-1}) > 0$ and $\ddot{z}(\tau_{2k-1}) > 0$.

The latter actually holds because the point $R_{2k-1} = (z(\tau_{2k-1}), \dot{z}(\tau_{2k-1}))$ is below the curve Γ , i.e., $\dot{z}(\tau_{2k-1}) < f(z(\tau_{2k-1}))$ and due to the orbit equation (1.65) the strictly positivity of $\ddot{z}(\tau_{2k-1})$ is obtained, see also Fig. 1.2. Analogously, we have that $\dot{z}(\tau_{2k}) < 0$ and $\ddot{z}(\tau_{2k}) < 0$, since now $R_{2k} = (z(\tau_{2k}), \dot{z}(\tau_{2k}))$ is above the curve Γ . Hence at every turning point $R_k = (z(\tau_k), \dot{z}(\tau_k))$ we have

$$\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)} > 0. \quad (1.101)$$

Furthermore, by (1.67)

$$3\frac{\dot{z}(\tau_k)}{z(\tau_k)} - N\omega_N < 2 - N\omega_N < 0 \quad \text{for any } N \geq 2. \quad (1.102)$$

Hence $C(\tau_k)$ is strictly negative as an immediate consequence of (1.101) and (1.102) since also $G(\tau) > 0$.

Moreover, differentiating (1.73) with respect to τ we obtain

$$\dot{\mu}(\tau_k) = -\frac{2\lambda(\tau_k)\dot{G}(\tau_k)}{G^3(\tau_k)}. \quad (1.103)$$

On the other hand, differentiating (1.69) with respect to τ we have

$$\dot{\mu}(\tau_k) = -3\Lambda z^{-4}(\tau_k)\dot{z}(\tau_k) \neq 0, \quad (1.104)$$

since at $\tau = \tau_k$ the orbit $\mathcal{O}$ does not meet the curve z -axis. Therefore by (1.103) and (1.104) we derive that $\dot{G}(\tau_k) \neq 0$ and we finally conclude that $\dot{\rho}(\tau_k) < 0$ which proves the desired assertion. This completes the proof of the theorem. $\square$

Remark 1.2.14 The above theorem characterizes the level of stability of the steady-state solutions. In particular, for $2 \leq N < 7$ we have $i = i_R = k$ on the arc $T_k T_{k+1}$ of the curve $\mathcal{S}_r$, where $T_k = (\lambda(\tau_k), w(\tau_k))$, $k = 0, 1, 2, \dots$ with $\tau_0 = -\infty$. That means that moving from the minimal to the maximal branch of the steady-state problem, for $0 < \lambda < \hat{\lambda}$ the solutions become less stable. On the other hand,

there always holds $i = i_R = 0$ for $N \geq 7$, i.e., the unique steady-state solution for $0 < \lambda < \widehat{\lambda}$ is asymptotically stable.

Remark 1.2.15 Following the same arguments as above, the second statement of Theorem 1.2.13 could be also proved for problem (1.49) when $f(x) = |x|^p$, $p > 0$, see also Theorem 1.3 in [27]. On the other hand, the first statement of Theorem 1.2.13 does not hold for problem (1.49), since general theorems, see [14, 35], of radial symmetry for stationary problems are not valid when $f_r > 0$, $r = |x|$ (which is the case for $f(x) = |x|^p$, $p > 0$).

1.2.1.4 Global Existence Versus Quenching

Given $\lambda \in (0, (1 + |\Omega|)^2 \mu^*)$, by Theorem 1.2.6, there exists a minimal steady-state solution of (1.37)–(1.39), denoted by w_λ , such that $w_\lambda = w_\mu$ with μ satisfying (1.50). Then we have the following theorem on the global existence for small values of λ .

Theorem 1.2.16 *For any $\lambda \in (0, (1 + |\Omega|)^2 \mu^*)$, then problem (1.37)–(1.39) with $u_0 \leq w_\lambda$ has a solution which exists globally in time and converges to the minimal steady state w_λ as $t \rightarrow \infty$.*

Proof By Corollary 1.2.4, we obtain that $0 \leq u \leq z \leq w_\lambda$, where z is the unique solution of the local problem (1.40)–(1.42) with $z_0 = u_0 \leq w_\lambda$. Therefore we derive that $0 \leq u \leq z \leq w_\mu$ for $0 < \mu = \lambda / (1 + |\Omega|)^2 < \mu^*$. But it is known that the local problem (1.40)–(1.42) has a global-in-time solution for $0 < \mu < \mu^*$ and $z_0 \leq w_\mu$ hence the non-local problem (1.37)–(1.39) does so.

The proof of convergence result can be carried out by a similar argument as that in [18] and so we omit it here. $\square$

Next, we show that the solution of problem (1.37)–(1.39) quenches in finite time for certain initial data. It is easy to see that problem (1.37)–(1.39) admits an energy functional of the form

$$E[u](t) \equiv E(t) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{\lambda}{(1 + \int_{\Omega} (1 - u)^{-1} dx)}, \quad (1.105)$$

which decreases with respect to time across any solution of problem (1.37)–(1.39). More precisely there holds

$$\frac{dE}{dt} = - \int_{\Omega} u_t^2(x, t) dx < 0, \quad (1.106)$$

hence

$$0 \leq \int_0^T \int_{\Omega} u_t^2(x, t) dx dt = E(0) - E(T) \leq E(0) < \infty, \quad (1.107)$$

for any $0 < T < T_{max}$, where T_{max} is the maximal existence time of problem (1.37)–(1.39). The following quenching result reveals the fact that quenching can also be controlled via initial conditions too, see [16, 17].

Theorem 1.2.17 *For any fixed $\lambda > 0$, there exist initial data such that the solution of problem (1.37)–(1.39) quenches in finite time provided the corresponding initial energy $E(0)$ is chosen sufficiently small, i.e., if*

$$E(0) < \frac{\lambda q(|\Omega|)}{2}. \quad (1.108)$$

Proof Suppose that problem (1.37)–(1.39) has a global-in-time (classical) solution u . Set

$$Z(t) = \int_{\Omega} u^2(x, t) dx.$$

Multiplying Eq. (1.37) by u and integrating by parts over Ω , we derive

$$\begin{aligned} \frac{1}{2} \frac{dZ}{dt} &= \int_{\Omega} u \left[\Delta u + \frac{\frac{\lambda}{(1-u)^2}}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2} \right] dx \\ &= - \int_{\Omega} |\nabla u|^2 dx + \lambda \frac{\int_{\Omega} \frac{u}{(1-u)^2} dx}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2}. \end{aligned} \quad (1.109)$$

Using (1.105) and the energy dissipation formula (1.106), relation (1.109) reads

$$\begin{aligned} \frac{1}{2} \frac{dZ}{dt} &= -2E(t) + \frac{2\lambda}{1 + \int_{\Omega} (1-u)^{-1} dx} + \lambda \frac{\int_{\Omega} \frac{u}{(1-u)^2} dx}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2} \\ &\geq -2E(0) + \lambda \frac{2\left(1 + \int_{\Omega} \frac{dx}{1-u}\right) + \int_{\Omega} \frac{u}{(1-u)^2} dx}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2} \\ &= -2E(0) + \lambda \frac{2 + \int_{\Omega} \frac{2-u}{(1-u)^2} dx}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2}. \end{aligned} \quad (1.110)$$

On the other hand, using Hölder's and Young's inequalities, we can deduce that

$$\left(1 + \int_{\Omega} \frac{dx}{1-u}\right)^2 \leq 2 + 2|\Omega| \int_{\Omega} \frac{dx}{(1-u)^2}. \quad (1.111)$$

Combining (1.110) and (1.111) yields

$$\frac{1}{2} \frac{dZ}{dt} \geq -2E(0) + \lambda q(|\Omega|),$$

where

$$q(|\Omega|) := \begin{cases} 1, & |\Omega| \leq \frac{1}{2}, \\ \frac{1}{2|\Omega|}, & |\Omega| \geq \frac{1}{2}, \end{cases}$$

see [16, 17], and finally

$$Z(t) \geq 2[q(|\Omega|) \lambda - 2E(0)]t + Z(0).$$

The latter implies that $Z(t) \rightarrow \infty$ as $t \rightarrow \infty$ provided that $E(0)$ satisfies (1.108).

On the other hand, by our initial assumption we have that $Z(t) \leq |\Omega|$ for any $t > 0$, leading to a contradiction. Therefore the theorem follows. $\square$

Remark 1.2.18 Note that for $N = 1$ and $\Omega = (0, 1/2)$ the condition (1.108) holds if for example choose

$$u_0(x) = \begin{cases} \frac{1}{1-\delta}x, & 0 \leq x < \frac{\delta}{2}(1-\delta), \\ \frac{\delta}{2}, & \frac{\delta}{2}(1-\delta) \leq x \leq \frac{1}{2} - \frac{\delta}{2}(1-\delta), \\ \frac{1}{1-\delta}(x - \frac{1}{2}), & \frac{1}{2} - \frac{\delta}{2}(1-\delta) \leq x \leq \frac{1}{2}, \end{cases}$$

where $0 < \delta < 1$ and for λ sufficiently large.

Indeed for such initial data we have

$$E(0) := \frac{1}{2} \int_0^{1/2} (u'_0(x))^2 dx + \frac{\lambda}{1 + \int_0^{1/2} \frac{dx}{1-u_0(x)}} = \frac{\delta}{2(1-\delta)} + \frac{\lambda}{1 + \frac{1}{2-\delta} - 2(1-\delta) \ln(1-\frac{\delta}{2})}$$

and then condition (1.108) is satisfied for any $\lambda > 5.5$ by choosing for example $0.97 < \delta < 1$. Such initial data as above, leading to quenching, could be constructed in higher dimensions as well.

In the following we present a quenching result for problem (1.37)–(1.39) holding for any $\lambda > \lambda^*$, recalling that λ^* is given by Theorem 1.2.10, in the radial symmetric case $\Omega = B_1(0)$. Indeed the following holds see also [29].

Theorem 1.2.19 *Consider symmetric and radial decreasing initial data $u_0(r)$. Then for any $\lambda > \lambda^*$ the solution of problem (1.37)–(1.39) quenches in finite time $T_q < \infty$.*

The proof of Theorem 1.2.19 will be given in different steps and thus we need first to provide some auxiliary concepts and results.

For the purposes of the current section we will need the notion of a weak solution of problem (1.44). In particular we define the following form of weak solution for (1.44).

Definition 1.2.20 A function $w \in H_0^1(\Omega)$ is called a *weak finite-energy solution* of (1.44) if there exists a sequence $\{w_j\}_{j=1}^\infty \in C^2(\Omega) \cap C_0(\Omega)$ satisfying as $j \rightarrow \infty$

$$w_j \rightharpoonup w \text{ weakly in } H_0^1(\Omega), \quad (1.112)$$

$$w_j \rightarrow w \text{ a.e.}, \quad (1.113)$$

$$\frac{1}{(1-w_j)^2} \rightarrow \frac{1}{(1-w)^2} \text{ in } L^1(\Omega), \quad (1.114)$$

$$\frac{1}{(1-w_j)} \rightarrow \frac{1}{(1-w)} \text{ in } L^1(\Omega), \quad (1.115)$$

and

$$\Delta w_j + \frac{\lambda}{(1-w_j)^2 \left(1 + \int_\Omega \frac{dx}{1-w_j}\right)^2} \rightarrow 0 \text{ in } L^2(\Omega). \quad (1.116)$$

It follows that any weak finite-energy solution of (1.44) also satisfies

$$-\int_\Omega \nabla \phi \cdot \nabla w \, dx + \lambda \frac{\int_\Omega \frac{\phi}{(1-w)^2} \, dx}{\left(1 + \int_\Omega \frac{dx}{1-w}\right)^2} = 0 \text{ for all } \phi \in H_0^1(\Omega),$$

i.e., it is a weak $H_0^1(\Omega)$ -solution of (1.44) as well, see also [47].

Set

$$\widehat{\lambda} := \sup \{ \lambda > 0 : \text{problem (1.44) admits a weak finite-energy solution} \}.$$

The relation between λ^* and $\widehat{\lambda}$ is then provided by the following:

Proposition 1.2.21 *For radially symmetric problems, with radially decreasing solutions, the suprema of the spectra of the classical and weak problems are identical, that is $\lambda^* = \widehat{\lambda}$.*

Proof Since any classical solution of (1.44) is also a weak finite-energy solution, then we have $\lambda^* \leq \widehat{\lambda}$.

On the other hand, we can take λ_1 arbitrarily close to $\widehat{\lambda}$ so that there is a weak finite-energy solution w_1 for $\lambda = \lambda_1$. Since w_1 is decreasing with $0 \leq w \leq 1$, either $w < 1$ for $0 < r \leq 1$ or there is some $s > 0$ such that $w = 1$ for $0 \leq r < s$. In the latter case $\int_\Omega (1-w)^{-1} \, dx$ becomes infinite so that w is then a weak finite-energy solution of $\Delta w = 0$ satisfying $0 \leq w \leq 1$, as well as the boundary condition $w = 0$, giving $w \equiv 0$. We must then have $w_1(r) < 1$ for $r > 0$ and it follows that w_1 is regular for $r > 0$.

For $N = 1$, simple integration now gives that the solution is classical. For $N \geq 2$, following now [17] (see also [21]), the (classical) problem can be solved in $r > 0$ to find that there is precisely one limiting value of λ , say λ_* , for which $w(0) = 1$ and

w is then a weak finite-energy solution but not classical. Depending upon the value of N , $\lambda_* < \lambda^*$ or $\lambda_* = \lambda^*$. In either case, $\lambda^* = \widehat{\lambda}$. $\square$

A key estimate for proving our quenching result is given by the following:

Lemma 1.2.22 *Let u be a global-in-time solution of problem (1.37)–(1.39). Then there is a sequence $\{t_j\}_{j=1}^\infty \uparrow \infty$ as $j \rightarrow \infty$ such that*

$$\lambda \int_{\Omega} u_j (1 - u_j)^{-2} dx \leq C_1 H^2(u_j), \quad (1.117)$$

for some positive constant C_1 , where $u_j = u(\cdot, t_j)$ and

$$H(u_j) := 1 + \int_{\Omega} (1 - u_j)^{-1} dx > 0. \quad (1.118)$$

Proof Suppose that the problem (1.37)–(1.39) has a global-in-time solution $u(x, t) = u(x, t; \lambda)$. Then, multiplying Eq. (1.37) by u and integrating over Ω , we derive

$$\begin{aligned} \int_{\Omega} u u_t dx &= \int_{\Omega} u \left[\Delta u + \frac{\lambda(1-u)^{-2}}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2} \right] dx \\ &= - \int_{\Omega} |\nabla u|^2 dx + \frac{\lambda \int_{\Omega} u(1-u)^{-2} dx}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2} \\ &= -2E(t) + \frac{2\lambda}{1 + \int_{\Omega} (1-u)^{-1} dx} + \frac{\lambda \int_{\Omega} u(1-u)^{-2} dx}{\left(1 + \int_{\Omega} (1-u)^{-1} dx\right)^2}, \end{aligned} \quad (1.119)$$

using also integration by parts and relation (1.105).

By virtue of Hölder's inequality and (1.107) then (1.119) implies

$$\begin{aligned} \lambda \int_{\Omega} u(1-u)^{-2} dx &= 2E(t)H^2(u) - 2\lambda H(u) + H^2(u) \int_{\Omega} u u_t dx \\ &\leq 2E_0 H^2(u) + \|u(\cdot, t)\|_2 \|u_t(\cdot, t)\|_2 H^2(u) \\ &\leq 2E_0 H^2(u) + |\Omega|^{1/2} \|u_t(\cdot, t)\|_2 H^2(u). \end{aligned} \quad (1.120)$$

On the other hand, the energy dissipation formula (1.106) reads

$$0 \leq \int_{\tau}^t \int_{\Omega} u_t^2(x, s) dx ds = E(\tau) - E(t),$$

and thus from (1.107) we deduce that

$$\int_{\tau}^{\infty} \int_{\Omega} u_t^2(x, s) dx ds \leq C < \infty, \quad (1.121)$$

where the constant C is independent of τ .

Now (1.121) yields the existence of a sequence $\{t_j\}_{j=1}^\infty \uparrow \infty$ such that

$$\|u_t(\cdot, t_j)\|_2^2 = \int_{\Omega} u_t^2(x, t_j) dx \rightarrow 0 \quad \text{as } t_j \rightarrow \infty, \quad (1.122)$$

and thus by virtue of (1.120)

$$\lambda \int_{\Omega} u_j(1 - u_j)^{-2} dx \leq C_1 H^2(u_j), \quad (1.123)$$

for some $C_1 > 0$. □

The next step is to provide another key estimate for $H(u)$ which will allow us not only to prove finite-time quenching but also to characterize the form of the quenching set.

In the radial symmetric case, i.e., when $\Omega = B_1(0)$ problem (1.37)–(1.39) in N dimensions is written as

$$u_t - \Delta_r u = F(r, t), \quad (r, t) \in (0, 1) \times (0, T), \quad (1.124)$$

$$u_r(0, t) = u(1, t) = 0, \quad t \in (0, T), \quad (1.125)$$

$$0 \leq u(r, 0) = u_0(r) < 1, \quad 0 < r < 1, \quad (1.126)$$

where $\Delta_r u := u_{rr} + (N - 1)r^{-1}u_r$ for $N \geq 1$ and

$$F(r, t) = \lambda k(t)(1 - u(r, t))^{-2}, \quad (1.127)$$

for

$$k(t) = \left(1 + N\omega_N \int_0^1 r^{N-1} (1 - u(r, t))^{-1} dr \right)^{-2},$$

where $\omega_N = |B_1(0)| = \frac{\pi^{N/2}}{\Gamma(N/2)}$ stands for the volume of the N -dimensional unit sphere $B_1(0)$ in $\mathbb{R}^N$ and Γ is the gamma function.

Condition $u_r(0, t) = 0$, for $N \geq 1$ is imposed to guarantee the regularity of the solution u . If we consider radial decreasing initial data $u_0(r)$, i.e., $u'_0(r) \leq 0$, then it is a standard result that the monotonicity property is inherited by u so that $u_r(r, t) \leq 0$ for $r > 0$ and $t > 0$.

For the sake of simplicity we obtain the desired estimate for $H(1 - v)$ where v is defined as $v := 1 - u$. Then $v \rightarrow 0+$ if $u \rightarrow 1-$. Moreover v satisfies

$$v_t - v_{rr} - (N - 1)r^{-1}v_r = -fv^{-2}, \quad (r, t) \in (0, 1) \times (0, T), \quad (1.128)$$

$$v_r(0, t) = 0, \quad v(1, t) = 1, \quad t \in (0, T), \quad (1.129)$$

$$0 < v(r, 0) = v_0(r) \leq 1, \quad 0 < r < 1, \quad (1.130)$$

where

$$f = f(t) = \frac{\lambda}{(1 + N\omega_N \int_0^1 r^{N-1} v^{-1} dr)^2}. \quad (1.131)$$

Then we have the following:

Lemma 1.2.23 *Consider symmetric and radial increasing initial data $v_0(r)$. Then for any $k > \frac{2}{3}$ there exists a positive constant $C(k)$ such that*

$$1 - u(r, t) \geq C(k)r^k \quad \text{for } (r, t) \in (0, 1) \times (0, T_{\max}), \quad (1.132)$$

where $T_{\max}$ is the maximum existence time of solution u .

Furthermore, there exists C_2 uniform in λ and independent of time t such that

$$H(u) = H(1 - v) \leq C_2 \quad \text{for any } 0 < t < T_{\max}. \quad (1.133)$$

Proof Fixing some a , $1 < a < 2$, there are some $t_1 > 0$ and $\varepsilon_1 > 0$ such that

$$v_r > \varepsilon_1 r v^{-a} \quad \text{at } t = t_1 \text{ for } 0 < r < 1. \quad (1.134)$$

We define

$$z = r^{N-1} v_r, \quad (1.135)$$

and it is then easy to check, [11], by differentiating (1.128), that

$$z_t - z_{rr} + (N - 1)r^{-1}z_r = 2r^{N-1}f v^{-3}v_r. \quad (1.136)$$

We define

$$J = z - \varepsilon r^N v^{-a}, \quad (1.137)$$

where

$$0 < \varepsilon < \varepsilon_1.$$

Then

$$J_t = z_t + a\varepsilon r^N v^{-a-1}v_t, \quad (1.138)$$

$$J_r = z_r + a\varepsilon r^N v^{-a-1}v_r - N\varepsilon r^{N-1}v^{-a}, \quad (1.139)$$

and

$$J_{rr} = z_{rr} + a\varepsilon r^N v^{-a-1}v_{rr} + 2Na\varepsilon r^{N-1}v^{-a-1}v_r - a(a+1)\varepsilon r^N v^{-a-2}v_r^2 - N(N-1)\varepsilon r^{N-2}v^{-a}. \quad (1.140)$$

We define a function $G(\varepsilon)$ by

$$G(\varepsilon) = \frac{\varepsilon^{\frac{2}{a+1}}}{(\varepsilon^{\frac{1}{a+1}} + \frac{N\omega_N}{Na+N-2}(a+1)^{\frac{a}{a+1}}2^{\frac{1}{a+1}})^2}. \quad (1.141)$$

Our choice of ε is then, more precisely, given by

$$0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}, \quad (1.142)$$

where $\varepsilon_2 > 0$ is chosen to satisfy

$$\varepsilon_2 < \sup \left\{ \varepsilon : \varepsilon \leq \min \left\{ \frac{1}{N}, \left(\frac{2-a}{2a} \right) \right\} \lambda G(\varepsilon) \right\}; \quad (1.143)$$

a small ε_2 satisfying (1.143) can be found since $G(\varepsilon)$ is of order $\varepsilon^{\frac{2}{a+1}} \gg \varepsilon$ for ε small (recall that $a > 1$). Then

$$J > 0 \text{ for } 0 < r \leq 1 \text{ at } t = t_1. \quad (1.144)$$

As long as $J > 0$,

$$z > \varepsilon r^N v^{-a} \Rightarrow v_r > \varepsilon r v^{-a} \Rightarrow v > \left(\frac{(a+1)\varepsilon}{2} \right)^{\frac{1}{a+1}} r^{\frac{2}{a+1}}, \quad (1.145)$$

which leads to (1.132).

Then

$$\int_0^1 r^{N-1} v^{-1} dr < \left(\frac{2}{(a+1)\varepsilon} \right)^{\frac{1}{a+1}} \int_0^1 r^{\frac{Na+N-2}{a+1}-1} dr = \left(\frac{2}{(a+1)\varepsilon} \right)^{\frac{1}{a+1}} \left(\frac{a+1}{Na+N-2} \right)$$

and so

$$f(t) = \frac{\lambda}{(1 + N\omega_N \int_0^1 r^{N-1} v^{-1} dr)^2} > \lambda G(\varepsilon). \quad (1.146)$$

In particular, $f(t) > \lambda G(\varepsilon)$ in a neighbourhood of $t = t_1$.

We suppose for a contradiction that

$$\text{there is some } t_2 \in (t_1, T_{max}) \text{ such that } f(t_2) = \lambda G(\varepsilon) \text{ with } f(t) > \lambda G(\varepsilon) \text{ for } t_1 \leq t < t_2. \quad (1.147)$$

Now

$$J = 0 \text{ on } r = 0. \quad (1.148)$$

On the boundary $r = 1$ we have

$$J = v_r - \varepsilon \text{ and then,}$$

$$J_r = v_{rr} + (N-1)v_r + a\varepsilon v_r - N\varepsilon = (f - (N-1)v_r) + (N-1)v_r + a\varepsilon v_r - N\varepsilon = f + a\varepsilon v_r - N\varepsilon,$$

$$\text{so both } J_r - a\varepsilon J = f + a\varepsilon^2 - N\varepsilon > f - N\varepsilon \text{ on } r = 1, \quad (1.149)$$

$$\text{and, since } v_r \geq 0 \text{ on } r = 1, \quad J_r \geq f - N\varepsilon \text{ on } r = 1. \quad (1.150)$$

Provided that

$$\varepsilon < f/N, \quad (1.151)$$

either (1.149) or (1.150) gives a positive boundary condition on $r = 1$.

Now

$$\begin{aligned} J_t - J_{rr} + (N-1)r^{-1}J_r &= 2r^{N-1}fv^{-3}v_r + (a\varepsilon r^N v^{-a-1})(2(N-1)r^{-1}v_r - fv^{-2}) \\ &\quad - 2Na\varepsilon r^{N-1}v^{-a-1}v_r \\ &\quad + a(a+1)\varepsilon r^N v^{-a-2}v_r^2 + N(N-1)\varepsilon r^{N-2}v^{-a} \\ &\quad - N(N-1)\varepsilon r^{N-2}v^{-a} \\ &> (2r^{N-1}fv^{-3} + 2(N-1)a\varepsilon r^{N-1}v^{-a-1} - 2Na\varepsilon r^{N-1}v^{-a-1})v_r \\ &\quad - a\varepsilon f r^N v^{-a-3} \\ &= 2(fv^{-3} - a\varepsilon v^{-a-1})v_r - a\varepsilon f r^N v^{-a-3} \\ &= 2(fv^{-3} - a\varepsilon v^{-a-1})J + 2(fv^{-3} - a\varepsilon v^{-a-1})\varepsilon r^N v^{-a} \\ &\quad - a\varepsilon f r^N v^{-a-3} \\ &= 2(fv^{-3} - a\varepsilon v^{-a-1})J + \varepsilon(2-a)f r^N v^{-a-3} \\ &\quad - 2a\varepsilon^2 r^N v^{-2a-1}. \end{aligned}$$

Thus

$$J_t - J_{rr} + (N-1)r^{-1}J_r > 2(fv^{-3} - a\varepsilon v^{-a-1})J, \quad (1.152)$$

as long as

$$\varepsilon < \frac{(2-a)}{2a}f. \quad (1.153)$$

Following the standard arguments for the maximum principle, we now show that $J > 0$ for $0 < r \leq 1, t_1 \leq t \leq t_2$.

In $0 < r \leq 1, t_1 \leq t \leq t_2$, because $v > 0$, the coefficient of J in (1.152) is bounded. We can then define a new variable $\tilde{J} = e^{-D_1 t} J$ which then satisfies boundary condition (1.148), boundary inequality (1.149) and

$$\tilde{J}_t - \tilde{J}_{rr} + (N-1)r^{-1}\tilde{J}_r > -D_2 \tilde{J}, \quad (1.154)$$

where D_1 and D_2 are positive constants. Should $\tilde{J}$ be non-positive somewhere (with $r > 0$), it must take a non-positive minimum at some (r_3, t_3) with $0 < r_3 \leq 1$ and $t_1 < t_3 \leq t_2$.

For $r_3 = 1$, (1.150) gives $\tilde{J}_r > 0$ on $r = 1$, leading to a contradiction, so the supposed minimum must have $0 < r_3 < 1$, where $\tilde{J}_t \leq 0$, $\tilde{J}_r = 0$ and $\tilde{J}_{rr} \geq 0$. With $\tilde{J} \leq 0$ and $D_2 > 0$, (1.154) gives another contradiction. Hence both $\tilde{J}$ and J remain positive in $r > 0$ for $t_1 \leq t \leq t_2$.

This now gives that (1.146) holds at $t = t_2$, contradicting the assumption (1.147). Thus, as long as the solution exists, $f(t) > \lambda G(\varepsilon)$ for $t \geq t_1$.

It follows that $J > 0$, (1.132) holds and

$$\int_0^1 r^{N-1} v^{-1} dr < \frac{1}{Na + N - 2} (a + 1)^{\frac{a}{a+1}} \left(\frac{2}{\varepsilon} \right)^{\frac{1}{a+1}} \quad (1.155)$$

for all $t \geq t_1$, if (1.128)–(1.130) have a global solution, and up to and including the quenching time, if the solution quenches. Thanks to the definition of $H(u)$ (1.155) implies the desired estimate. $\square$

Remark 1.2.24 An estimate similar to (1.132) has been also obtained in [18] but only for the one-dimensional case. Furthermore, here we also prove that the exponent $\frac{2}{3}$ is optimal for the validity of (1.132), see below.

Proof of Theorem 1.2.19 Let $\lambda > \lambda^*$ and assume that problem (1.124)–(1.126) has a global-in-time solution. Then (1.117) in conjunction with (1.133) yields

$$\lambda N \omega_N \int_0^1 r^{N-1} u_j (1 - u_j)^{-2} dr \leq C_3, \quad \text{for any } t > 0, \quad (1.156)$$

where the constant C_3 is independent of j .

From this and (1.133) we have

$$\begin{aligned} N \omega_N \int_0^1 \frac{r^{N-1} dr}{(1 - u_j)^2} &= N \omega_N \int_0^1 \frac{r^{N-1} dr}{(1 - u_j)} + N \omega_N \int_0^1 \frac{r^{N-1} u_j dr}{(1 - u_j)^2} \\ &\leq (C_2 - 1) + \frac{C_3}{\lambda} := C_4, \end{aligned} \quad (1.157)$$

where C_4 is independent of j .

From the energy dissipation formula (1.107) we also have

$$\|\nabla u_j\|_{L^2(B_1)}^2 \leq C_5 < \infty, \quad (1.158)$$

with constant C_5 being independent of j as well. Passing to a subsequence, if necessary, relation (1.158) implies the existence of a function w such that

$$u_j \rightharpoonup w \quad \text{in } H_0^1(B_1(0)), \quad (1.159)$$

$$u_j \rightarrow w \quad \text{a.e. in } B_1(0). \quad (1.160)$$

For $N \geq 2$ by virtue of (1.132) we directly derive that $1/(1 - u_j)^2$ is uniformly integrable and since $1/(1 - u_j)^2 \rightarrow 1/(1 - w)^2$, a.e. in $B_1(0)$, due to (1.160), we deduce

$$\frac{1}{(1 - u_j)^2} \rightarrow \frac{1}{(1 - w)^2} \quad \text{as } j \rightarrow \infty \quad \text{in } L^1(B_1(0)), \quad (1.161)$$

applying the dominated convergence theorem. Similarly we also derive

$$H(u_j) \rightarrow H(w) \quad \text{as } j \rightarrow \infty \quad \text{in } L^1(B_1(0)). \quad (1.162)$$

Note that the weak formulation of (1.124) along the sequence $\{t_j\}$ is given by

$$\int_{B_1(0)} \frac{\partial u_j}{\partial t} \phi \, dx = - \int_{B_1(0)} \nabla u_j \cdot \nabla \phi \, dx + \lambda H^{-1}(u_j) \int_{B_1(0)} \phi (1 - u_j)^{-2} \, dx \quad \text{as } j \rightarrow \infty, \quad (1.163)$$

for any $\phi \in H_0^1(B_1(0))$.

Passing to the limit as $j \rightarrow \infty$ in (1.163), and in conjunction with (1.122), (1.159), (1.161) and (1.162), we derive

$$\Delta u_j + \frac{\lambda}{(1 - u_j)^2 \left(1 + \int_{\Omega} \frac{dx}{1 - u_j}\right)^2} \rightarrow 0 \quad \text{in } L^2(B_1(0)),$$

which implies that w is an weak finite-energy solution of problem (1.44) corresponding to $\lambda > \lambda^*$, contradicting the result of Proposition 1.2.21.

On the other hand, for $N = 1$ by using (1.122), (1.124), (1.127), (1.157) and (1.158) we deduce that $(u_j)_x$ is bounded in $W^{1,1}(-1, 1)$ and thus, by virtue of Sobolev's inequality,

$$(u_j)_x \quad \text{is bounded in } L^\infty(-1, 1), \quad (1.164)$$

Furthermore

$$[(1 - u_j)^{-1}] \quad \text{is bounded in } W^{1,1}(-1, 1),$$

since

$$[(1 - u_j)^{-1}]_x = \frac{(u_j)_x}{(1 - u_j)^2} \quad \text{is bounded in } L^1(-1, 1),$$

due to (1.157) and (1.164), and

$$(1 - u_j)^{-1} \text{ is bounded in } L^1(-1, 1),$$

by virtue of (1.133).

Therefore Sobolev's inequality guarantees that

$$[(1 - u_j)^{-1}] \text{ is bounded in } L^\infty(-1, 1),$$

and thus

$$[(1 - u_j)^{-2}] \text{ is bounded in } L^\infty(-1, 1).$$

Now by virtue of (1.160) we derive that

$$(1 - u_j)^{-1} \rightarrow (1 - w)^{-1} \text{ and } (1 - u_j)^{-2} \rightarrow (1 - w)^{-2} \text{ as } j \rightarrow \infty \text{ in } L^\infty(-1, 1).$$

Consequently,

$$\Delta w + \frac{\lambda}{(1 - w)^2 \left(1 + \int_{\Omega} \frac{dx}{1 - w}\right)^2} = 0 \text{ in } L^2(B_1(0)),$$

where the non-local term is bounded and hence elliptic regularity arguments entail that w classical steady solution, again contradicting Proposition 1.2.21. $\square$

Remark 1.2.25 Theorem 1.2.19 improves the results of Theorem 4.1 in [18] and Theorems 5.2, 5.3 in [21]. Indeed, the earlier results have provided finite-time quenching only for large values of the parameter λ , without giving a threshold for λ above which quenching occurs.

Another quenching result for big initial data, i.e., for $0 < u_0(x) < 1$ close to 1, can be obtained by employing the widely used classical technique of Kaplan, [23]. The estimate of Lemma 1.2.23, permitting us to treat the non-local problem (1.37)–(1.39) as a local one. In particular we have:

Theorem 1.2.26 *For any $\lambda > 0$ there exist symmetric initial data u_0 satisfying the assumptions of Theorem 1.2.19 that are close to 1 such that the solution u of (1.37)–(1.39) quenches in finite time $T_q < \infty$.*

Proof Set $\lambda_1 = \lambda_1(B_1(0)) > 0$ the principal eigenvalue of the following problem

$$-\Delta \phi = \lambda \phi, \quad x \in B_1(0), \quad \phi(x) = 0, \quad x \in \partial B_1(0),$$

with associated positive eigenfunction $\phi_1(x)$ normalized so that

$$\int_{B_1(0)} \phi_1(x) dx = 1.$$

Let us assume that problem (1.124)–(1.126) has a global-in-time solution, i.e., $T_{max} = \infty$ so that $0 < u(x, t) < 1$ for any $(x, t) \in B_1(0) \times (0, +\infty)$.

Multiplying (1.124) by ϕ_1 , integrating over $B_1(0)$ and using Green's second identity, we obtain, via Lemma 1.2.23,

$$\begin{aligned} \frac{dA}{dt} &= -\lambda_1 A(t) + \frac{\lambda \int_{B_1(0)} \phi_1 (1-u)^{-2} dx}{H^2(u)} \\ &\geq -\lambda_1 A(t) + \frac{\lambda \int_{B_1(0)} \phi_1 (1-u)^{-2} dx}{C_2^2}, \end{aligned} \quad (1.165)$$

where $A(t) = \int_{B_1(0)} u \phi_1 dx$. Applying Jensen's inequality to (1.165),

$$\frac{dA}{dt} \geq -\lambda_1 A(t) + \frac{\lambda}{C_2^2} (1 - A(t))^{-2} \quad \text{for any } t > 0. \quad (1.166)$$

Choosing $\gamma \in (0, 1)$ so that

$$\Psi(s) := \frac{\lambda}{C_2^2} (1-s)^{-2} - \lambda_1 s > 0 \quad \text{for all } s \in [\gamma, 1),$$

then by considering u_0 close enough to 1 such that $A(0) \geq \gamma$, relation (1.165) yields

$$\frac{dA}{dt} \geq \Psi(A(t)) > 0 \quad \text{for any } t > 0,$$

which then leads to

$$t \leq \int_{A(0)}^{A(t)} \frac{ds}{\Psi(s)} \leq \int_{A(0)}^1 \frac{ds}{\Psi(s)} < \infty,$$

contradicting the assumption $T_{max} = \infty$. This completes the proof of the theorem. $\square$

Below we address the question: where does quenching takes place? Namely for radially symmetric and monotonic decreasing initial data, quenching, when it occurs, takes place at a single point, the origin:

Theorem 1.2.27 *If we consider initial data as in Theorem 1.2.19 so that the solution of problem (1.124)–(1.126) quenches in finite time $T_q < \infty$, the quenching occurs only at the origin $r = 0$.*

Proof The proof follows immediately from relation (1.132). $\square$

Due to the non-locality, obtaining the sharp profile of the standard (local) problem (cf. [10]) for (1.124)–(1.126) might be hard. However it can be rigorously shown

that the exponent $\frac{2}{3}$ in (1.132) is optimal, at least in the sense that (1.132) cannot be true for any exponent $k < \frac{2}{3}$.

The optimality of the exponent $\frac{2}{3}$ is a consequence of the following result.

Proposition 1.2.28 *Let T_q be the quenching time of the solution u of (1.124)–(1.126) then*

$$\lim_{t \rightarrow T_q} \|(1-u)^{-1}\|_m = \infty \text{ for any } m > \frac{3N}{2} > 1. \quad (1.167)$$

Proof First note $\theta = (1-u)^{-1}$ satisfies $\theta_t - \Delta\theta \leq f(t)\theta^4 \leq \lambda\theta^4$ with $\theta = 1$ on $\partial\Omega$. Next fix any $\Lambda > 0$ and assume that $\|\theta(t_0)\|_m \leq \Lambda$ for some $m > \frac{3N}{2} > 1$ and $t_0 \in (0, T_q)$.

By virtue of [39, Theorem 15.2, Example 51.27], see also [2, Theorem 1] and [45, Theorem 1], we have that problem

$$\begin{aligned} z_t - \Delta z &= \lambda(1+z)^4 \quad \text{in } \Omega \times (t_0, T_q), \\ z &= 0 \quad \text{on } \partial\Omega \times (t_0, T_q), \\ z(x, t_0) &= \theta(x, t_0) \quad x \in \Omega, \end{aligned}$$

is well posed and in particular there exists $\tau > 0$ such that

$$\|z(t_0 + s)\|_\infty \leq Ks^{-N/2m}, \quad s \in (0, \tau], \quad (1.168)$$

where K, τ depend only on $\Lambda, m, \Omega, \lambda$. By comparison θ exists and satisfies $\theta \leq z + 1$ on $[t_0, t_0 + \tau]$. Since $\lim_{t \rightarrow T_q} \|\theta(t)\|_\infty = \infty$ it follows that $\|\theta(t)\|_m > \Lambda$ for all $t \in (\max(0, T_q - \tau), T_q)$ and thus (1.167). $\square$

Remark 1.2.29 It should be noted that in the critical case $m = m_c = \frac{3N}{2}$ the time τ in (1.168) depends on $\theta(t_0)$ and not just on $\|\theta(t_0)\|_m$ (see [2] and [39, Remarks 15.4 and 16.2(iv)]). Therefore, in this case, it is no longer certain that $T_q < \infty$ implies the finite-time blow-up of the norm $\|\theta(t_0)\|_m$.

Corollary 1.2.30 *Relation (1.132) is not valid for any $k < \frac{2}{3}$.*

Proof First note that since $T_q = T_{max} < \infty$ it is easily seen by the proof of Lemma 1.2.23 that (1.132) is valid up to T_{max} . Assume now that there is $k_0 < \frac{2}{3}$ such that

$$1 - u(r, t) \geq C(k_0)r^{k_0} \quad \text{for } (r, t) \in (0, 1) \times (0, T_{max}],$$

then

$$\lim_{t \rightarrow T_{max}} \|(1-u)^{-1}\|_m = \int_0^1 \frac{r^{N-1}}{(1-u(T_{max}))^m} dr \leq C^{-1}(k_0) \int_0^1 r^{N-1-mk_0} dr < \infty,$$

for $m > \frac{3N}{2}$ close to $\frac{3N}{2}$, contradicting Proposition 1.2.28. $\square$

We close the study of (1.124)–(1.126) by providing a lower estimate of its quenching rate. We recall that considering radial decreasing initial data u_0 then u inherits this property and hence

$$M(t) := \max_{x \in \bar{B}_1} u(x, t) = u(0, t).$$

The next result provides a lower estimate of the quenching rate:

Theorem 1.2.31 *The lower bound of the quenching rate of problem (1.124)–(1.126) is given by*

$$M(t) \geq 1 - \widehat{C}(T_q - t)^{1/3} \quad \text{for } 0 < t < T_q, \quad (1.169)$$

where $\widehat{C}$ is a positive constant independent of time t .

Proof It can be easily checked that the function $M(t)$ is Lipschitz continuous and hence, by Rademacher's theorem, is almost everywhere differentiable, see [11, 26]. Furthermore, since u is decreasing in r , $\Delta_r u(0, t) \leq 0$ for all $t \in (0, T_q)$. Therefore, for any t where dM/dt exists, we have

$$\frac{dM}{dt} \leq \lambda \frac{(1 - M(t))^{-2}}{\left(1 + \int_{B_1(0)} \frac{1}{1-u} dx\right)^2} \leq \lambda \frac{(1 - M(t))^{-2}}{(1 + N\omega_N)^2} \quad \text{for a.e. } t \in (0, T_q),$$

which yields

$$\int_{M(t)}^1 (1-s)^2 ds \leq \lambda C(T_q - t),$$

for $C = \frac{1}{(1+N\omega_N)^2}$, giving the desired estimate

$$M(t) \geq 1 - \widehat{C}(T_q - t)^{1/3} \quad \text{for } 0 < t < T_q,$$

where $\widehat{C} = (3\lambda C)^{1/3}$. □

1.2.2 A Non-local Hyperbolic Problem

Now we are investigating the following non-local problem

$$u_{tt} = u_{xx} + \frac{\lambda}{(1-u)^2 \left(1 + \int_0^1 \frac{1}{1-u} dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \quad (1.170)$$

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0, \quad (1.171)$$

$$0 < u(x, 0) = u_0(x) < 1, \quad u_t(x, 0) = u_1(x), \quad 0 < x < 1. \quad (1.172)$$

The steady-state problem corresponding to (1.170)–(1.172) is

$$w'' + \frac{\lambda}{(1-w)^2 \left(1 + \int_0^1 \frac{1}{1-w} dx\right)^2} = 0, \quad 0 < x < 1; \quad w(0) = 0, \quad w(1) = 0. \quad (1.173)$$

If we set $W = 1 - w$ then (1.173) becomes

$$W'' = \frac{\mu}{W^2}, \quad 0 < x < 1; \quad W(0) = 1, \quad W(1) = 1, \quad (1.174)$$

where

$$\mu = \frac{\lambda}{\left(1 + \int_0^1 \frac{1}{W} dx\right)^2}. \quad (1.175)$$

Then multiplying both sides by W' and integrating from $m = \min\{W(x), x \in [0, 1]\} = W\left(\frac{1}{2}\right)$ to W we derive

$$\int_0^{W'} W' dW' = \int_{\frac{1}{2}}^x W'' W' dx = \mu \int_{\frac{1}{2}}^x \frac{W'}{W^2} dx = \mu \int_m^W \frac{dW}{W^2},$$

hence

$$\frac{1}{2}(W')^2 = \mu \left(\frac{1}{m} - \frac{1}{W} \right).$$

This gives equivalently

$$\frac{dx}{dW} = \sqrt{\frac{m}{2\mu}} \sqrt{\frac{W}{W-m}},$$

which implies

$$x - \frac{1}{2} = \sqrt{\frac{m}{2\mu}} \left[\sqrt{W(W-m)} - \frac{1}{2}m \ln(m) + m \ln(\sqrt{W} + \sqrt{W-m}) \right].$$

This yields, on setting $x = 1$ so that $W = 1$,

$$\mu = 2m \left[\sqrt{1-m} - \frac{1}{2}m \ln(m) + m \ln(1 + \sqrt{1-m}) \right]^2.$$

Furthermore we have

$$\begin{aligned}
\int_0^1 \frac{1}{W} dx &= \int_0^1 \frac{dx}{dW} \frac{dW}{W} = \sqrt{\frac{2m}{\mu}} \int_m^1 \frac{1}{\sqrt{W(W-m)}} dW \\
&= \frac{1}{\sqrt{1-m} - \frac{1}{2}m \ln(m) + m \ln(1 + \sqrt{1-m})} \int_m^1 \frac{1}{\sqrt{W(W-m)}} dW \\
&= \frac{1}{\sqrt{1-m} - \frac{1}{2}m \ln(m) + m \ln(1 + \sqrt{1-m})} \ln\left(\frac{2-m+2\sqrt{1-m}}{m}\right).
\end{aligned}$$

On using (1.175), we finally establish the following relation between λ and m ,

$$\lambda = 2m \left[\sqrt{1-m} + m \ln\left(\frac{(1+\sqrt{1-m})}{\sqrt{m}}\right) + \ln\left(\frac{2-m+2\sqrt{1-m}}{m}\right) \right]^2. \quad (1.176)$$

From the above relation, we can obtain the response (bifurcation) diagram of problem (1.173) see Fig. 1.6. In particular, (1.176) implies that $\lambda \sim 2\alpha^2 m (\ln m)^2$ as $m \rightarrow 0$.

The presented mathematical analysis follows [28].

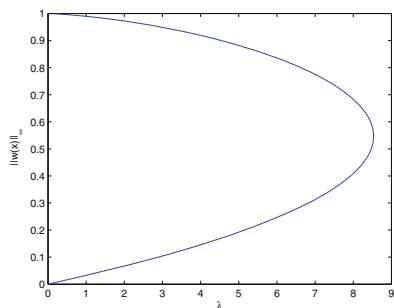
1.2.2.1 Local Existence

In this section we establish local existence of the solution of problem (1.170)–(1.172) where $u_0, u_1 \in C^1((0, 1))$ and $u_0(0) = u_0(1) = 0$, by modifying appropriately the proof given for the local problem in [4], see also [12].

Definition 1.2.32 We say that u is a weak solution of (1.170)–(1.172) in $Q_T \equiv (0, 1) \times (0, T)$ if:

- (i) u is continuous in $\bar{Q}_T$ and satisfies the initial and boundary conditions.
- (ii) There exists some $\delta > 0$ such that $|u| \leq 1 - \delta$ in $\bar{Q}_T$.
- (iii) u has weak derivatives u_x, u_t in $\bar{Q}_T$ and for all $t \in (0, T)$, $u_x, u_t \in L^2((0, 1))$.
- (iv) For any function $\zeta(x, t) \in C^2(\bar{Q}_T)$ satisfying the boundary conditions and for $0 \leq t \leq T$,

Fig. 1.6 Response diagram for problem (1.173) where $\lambda^* \approx 8.533$



$$\begin{aligned} \int_0^1 \zeta(x, t) u_t(x, t) dx &= \int_0^t \int_0^1 [\zeta_\tau(x, \tau) u_\tau(x, \tau) - \zeta_x(x, \tau) u_x(x, \tau)] dx d\tau \\ &\quad + \lambda \int_0^t \int_0^1 \frac{\zeta(x, \tau) dx d\tau}{(1 - u(x, \tau))^2 \left(1 + \int_0^1 \frac{1}{1-u(y, \tau)} dy\right)^2}. \end{aligned} \quad (1.177)$$

- (v) The total energy associated with (1.170)–(1.172) is preserved i.e.,

$$E_T(t) = \frac{1}{2} \int_0^1 (u_x^2 + u_t^2) dx + \frac{\lambda}{\left(1 + \int_0^1 \frac{1}{1-u} dx\right)} = E_T(0) := E_0. \quad (1.178)$$

We consider a fixed $\delta \in (0, 1)$ and assume that the initial data satisfy the condition

$$\|u_0\|_\infty + T\|u_1\|_\infty < 1 - 2\delta, \quad (1.179)$$

for a positive T . Define the odd periodic (with period two) extensions with respect to x on $\mathbb{R} \times [0, T]$ of u, u_0, v_0 which are denoted, without any confusion, again as u, u_0, v_0 .

We also define the function $G_p : \mathbb{R} \times [0, +\infty) \times (-1, 1) \rightarrow \mathbb{R}$, as

$$G_p(x, t, u) = \begin{cases} F(u), & x \in [2n, 2n+1), \\ F(-u), & x \in [2n-1, 2n), \end{cases} \quad (1.180)$$

for $n = 0, \pm 1, \pm 2, \dots$, where

$$F(u) = \frac{1}{(1-u)^2 \left(1 + \int_0^1 \frac{1}{1-u} dx\right)^2}.$$

Then by standard arguments applied to wave equations, we have that u is a solution of (1.170)–(1.172) if and only if u solves in $\mathbb{R} \times [0, T]$ the integral equation

$$u(x, t) = \tilde{u}(x, t) + \frac{\lambda}{2} \int_0^t \int_{x-t+\tau}^{x+t+\tau} G_p(y, \tau, u(y, \tau)) dy d\tau, \quad (1.181)$$

where

$$\tilde{u}(x, t) = \frac{1}{2} [u_0(x+t) + u_0(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} u_1(z) dz.$$

Note that (1.179) implies that

$$\|\tilde{u}\|_T = \sup_{t \in [0, T]} \|\tilde{u}(t)\|_\infty < 1 - 2\delta. \quad (1.182)$$

Let $\mathcal{B}_T$ be the Banach space of continuous odd periodic (with period two) functions with respect to x , defined in $\mathbb{R} \times [0, T]$, vanishing on $x = n$, $n \in \mathbb{Z}$, with the norm $\|\cdot\|_T$. Also let $\bar{B}(\tilde{u}, \delta)$ be the closed ball of radius δ centered on $\tilde{u}$ in $\mathcal{B}_T$. Consider the operator

$$\mathcal{T}[u(x, t)] = \tilde{u}(x, t) + \frac{\lambda}{2} \int_0^t \int_{x-t+\tau}^{x+t+\tau} G_p(y, \tau, u(y, \tau)) dy d\tau,$$

which, owing to the definition of the function G_p , is well-defined.

In order to show that (1.170)–(1.172), or equivalently (1.181), has a local-in-time solution, it is enough to show that the operator $\mathcal{T}$ is a contraction from $\bar{B}(\tilde{u}, \delta)$ to $\bar{B}(\tilde{u}, \delta)$. (Note that $\tilde{u} \in \mathcal{B}_T$.) In particular, we have to show that

$$\|\mathcal{T}u - \tilde{u}\|_T < \delta, \quad \|\mathcal{T}u - \mathcal{T}v\|_T < K\|u - v\|_T,$$

for $u, v \in \bar{B}(\tilde{u}, \delta)$ and $0 < K < 1$.

In the following, for convenience, we write $f(u) := \frac{1}{1-u}$ and $I(u) := \int_0^1 f(u) dx$, hence $F(u) := \frac{f^2(u)}{[1+I(u)]^2}$. Then we have

$$\|\mathcal{T}v(x, t) - \mathcal{T}u(x, t)\|_T = \frac{\lambda}{2} \left\| \int_0^t \int_{x-t+\tau}^{x+t+\tau} [G_p(y, \tau, v(y, \tau)) - G_p(y, \tau, u(y, \tau))] dy d\tau \right\|_T,$$

where for $x - t + \tau > 0$

$$\begin{aligned} [G_p(y, \tau, v(y, \tau)) - G_p(y, \tau, u(y, \tau))] &= \frac{f^2(v)}{[1+I(v)]^2} - \frac{f^2(u)}{[1+I(u)]^2} \\ &= f^2(v) \left(\frac{1}{[1+I(v)]^2} - \frac{1}{[1+I(u)]^2} \right) + \frac{1}{[1+I(u)]^2} (f^2(v) - f^2(u)). \end{aligned}$$

Note that $f^2(v) - f^2(u) = (f(v) + f(u))(f(u) - f(v))$ and $\|f(u) - f(v)\|_T \leq |f'(1 - \delta)| \|u - v\|_T$, by taking into account (1.182) and the fact that f is a convex function.

Moreover we have

$$\begin{aligned} \frac{1}{[1+I(v)]^2} - \frac{1}{[1+I(u)]^2} &= \frac{[1+I(u)]^2 - [1+I(v)]^2}{[1+I(v)]^2 [1+I(v)]^2} \\ &< [2 + (I(u) + I(v))][I(u) - I(v)], \end{aligned}$$

since $I(u), I(v) > 0$. Using again (1.182) and the fact that f is an increasing and convex function we derive $\|I(u) - I(v)\|_T \leq f'(1 - \delta) \|u - v\|_T$ and $\|2 + (I(u) + I(v))\|_T \leq 2[1 + f(1 - \delta)]$. Thus finally

$$\begin{aligned} |G_p(y, \tau, v(y, \tau)) - G_p(y, \tau, u(y, \tau))| &\leq 2f^2(1-\delta)f'(1-\delta)[1+f(1-\delta)]\|u-v\|_T \\ &\quad + 2f(1-\delta)f'(1-\delta)\|u-v\|_T, \end{aligned}$$

or

$$|G_p(y, \tau, v(y, \tau)) - G_p(y, \tau, u(y, \tau))| \leq C(f, \delta)\|u-v\|_T,$$

for $C(f, \delta) = 2f^2(1-\delta)f'(1-\delta)[1+f(1-\delta)] + 2f(1-\delta)f'(1-\delta)$. The same final estimate also holds when $x-t+\tau < 0$.

Therefore

$$\begin{aligned} \|\mathcal{T}v(x, t) - \mathcal{T}u(x, t)\|_T &= \frac{\lambda}{2} \left\| \int_0^t \int_{x-t+\tau}^{x+t+\tau} [G_p(y, \tau, v(y, \tau)) - G_p(y, \tau, u(y, \tau))] dy d\tau \right\|_T \\ &\leq \frac{\lambda}{2} t^2 C(f, \delta) \|u-v\|_T \leq \frac{\lambda}{2} \sigma^2 C(f, \delta) \|u-v\|_T, \end{aligned} \quad (1.183)$$

for $0 \leq t \leq \sigma$. Note that $K = \frac{\lambda}{2} \sigma^2 C(f, \delta) < 1$ if $\sigma < \sqrt{\frac{2}{\lambda} \frac{1}{\sqrt{C(f, \delta)}}}$.

It remains to show that $\|\mathcal{T}u - u_1\|_T < \delta$. We have again for $x-t+\tau > 0$,

$$\begin{aligned} \|\mathcal{T}u(x, t) - u_1\|_T &= \frac{\lambda}{2} \left\| \int_0^t \int_{x-t+\tau}^{x+t+\tau} G_p(y, \tau, u(y, \tau)) dy d\tau \right\|_T \\ &\leq \left\| \frac{\lambda}{2} \int_0^t \int_{x-t+\tau}^{x+t+\tau} \frac{f^2(u)}{[1+I(u)]^2} dy d\tau \right\|_T \\ &\leq \frac{\lambda}{2} f^2(1-\delta) \left\| \int_0^t \int_{x-t+\tau}^{x+t+\tau} dy d\tau \right\|_T \\ &\leq \frac{\lambda}{2} t^2 f^2(1-\delta) \leq \frac{\lambda}{2} \sigma^2 f^2(1-\delta). \end{aligned}$$

Since the same final estimate is obtained for $x-t+\tau < 0$ (when the first inequality in (1.183) becomes strict), we end up with $\|\mathcal{T}u(x, t) - u_1\|_T \leq \delta$ as long as $\frac{\lambda}{2} \sigma^2 f^2(1-\delta) \leq \delta$, i.e., if $\sigma \leq \sqrt{\frac{2}{\lambda} \sqrt{\frac{\delta}{f^2(1-\delta)}}}$.

Thus finally, if we choose σ such that

$$\sigma < \min \left\{ T, \sqrt{\frac{2}{\lambda}} \left(\frac{\delta}{f^2(1-\delta)} \right)^{\frac{1}{2}}, \sqrt{\frac{2}{\lambda}} \left(\frac{1}{C(f, \delta)} \right)^{\frac{1}{2}} \right\},$$

we conclude that the operator $\mathcal{T} : B(u_1, \delta) \rightarrow B(u_1, \delta)$ is a contraction and hence the Banach fixed point theorem guarantees the existence of a unique fixed point for $\mathcal{T}$.

We have thus established local existence of solution of problem (1.170)–(1.172) in an interval $[0, \sigma]$. In particular:

Theorem 1.2.33 *If the initial data $u_0(x), v_0(x) \in C^1((0, 1))$ satisfy condition (1.179), then, for any $\lambda > 0$, problem (1.170)–(1.172) has a unique weak C^1 –solution on $Q_T = (0, 1) \times [0, T]$ if T is sufficiently small.*

Remark 1.2.34 By the proof of the above theorem we also obtain that the solution u to (1.170)–(1.172) is piecewise C^2 in Q_T . Furthermore, the solution of (1.170)–(1.172) could be extended to any interval of the form $[0, T + \tau]$ for τ sufficiently small and positive as long as $|u| < 1$ on $\bar{Q}_T$.

Remark 1.2.35 In the case of zero initial datum we could obtain, by differentiating relation (1.181), that $u(x, t)$ is a regular solution to (1.170)–(1.172) except on the point set

$$\{(x, t) \in \mathbb{R} \times [0, T] | x, x - t, x + t \text{ are integers}\},$$

see also [4].

Definition 1.2.36 The solution $u(x, t)$ of problem (1.170)–(1.172) quenches at point $x^* \in (0, 1)$ in finite time $0 < T < \infty$ if there exist sequences $\{x_n\}_{n=1}^\infty \in (0, 1)$ and $\{t_n\}_{n=1}^\infty \in (0, \infty)$ with $x_n \rightarrow x^*$, $t_n \rightarrow T$ – and $u(x_n, t_n) \rightarrow 1$ – as $n \rightarrow \infty$. In the case where $T = \infty$ we say that $u(x, t)$ quenches in infinite time at x^* .

By Remark 1.2.34 we conclude that the solution $u(x, t)$ to problem (1.170)–(1.172) ceases to exist only by quenching. In MEMS terminology quenching is usually called touch-down since it describes the phenomenon when the elastic membrane touches the rigid plate on the bottom of the MEMS device.

1.2.2.2 Global Existence Versus Quenching

In order to prove global existence of problem (1.170)–(1.172), we need as a first step to determine the corresponding energy functional. Unless otherwise stated, $\|\cdot\|$ now denotes $\|\cdot\|_2$.

To find the energy, we multiply Eq. (1.170) by u_t and integrate over $[0, 1]$. Thus we obtain

$$\int_0^1 (u_t u_{tt} - u_t u_{xx}) dx = \lambda \int_0^1 \frac{u_t dx}{(1-u)^2} \frac{1}{(1 + \alpha \int_0^1 \frac{1}{1-u} dx)^2},$$

or

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_0^1 (u_t^2 + u_x^2) dx \right) &= \lambda \frac{d}{dt} \left(\int_0^1 \frac{1}{1-u} dx \right) \frac{1}{(1 + \int_0^1 \frac{1}{1-u} dx)^2} \\ &= -\lambda \frac{d}{dt} \left(\frac{1}{1 + \int_0^1 \frac{1}{1-u} dx} \right). \end{aligned}$$

Thus finally we get

$$\frac{1}{2} \int_0^1 (u_t^2 + u_x^2) dx + \lambda \left(\frac{1}{1 + \int_0^1 \frac{1}{1-u} dx} \right) = E_0, \quad (1.184)$$

where E_0 is a constant representing the initial energy of the system which is conserved and is given by

$$E_0 = \frac{1}{2} \int_0^1 (u_1^2 + u_{0x}^2) dx + \lambda \left(\frac{1}{1 + \int_0^1 \frac{1}{1-u_0} dx} \right). \quad (1.185)$$

From Eq. (1.184) we deduce that

$$\|u_t\|^2 + \|u_x\|^2 + 2\lambda \left(\frac{1}{1 + \int_0^1 \frac{1}{1-u} dx} \right) = 2E_0,$$

where $\|u_t\| = \|u_t(\cdot, t)\|$, $\|u_x\| = \|u_x(\cdot, t)\|$, and that

$$\|u_x\|^2 + 2\lambda \left(\frac{1}{1 + \alpha \int_0^1 \frac{1}{1-u} dx} \right) \leq 2E_0. \quad (1.186)$$

At this point we can use the one-dimensional Sobolev embedding and the Poincaré inequality to obtain

$$4u^2(x, t) \leq \|u_x\|^2, \quad \text{for } (x, t) \in [0, 1] \times [0, T]. \quad (1.187)$$

We also define $M := \max\{u(x, t), (x, t) \in [0, 1] \times [0, T]\}$. For cases where $M < 1$ irrespective of the value of T , the maximal time of existence of u is infinite.

Therefore we have that $u \leq M$, $1 - u \geq 1 - M$, $1 + \int_0^1 \frac{1}{1-u} dx \leq 1 + \frac{1}{1-M}$, and that

$$\frac{1}{1 + \int_0^1 \frac{1}{1-u} dx} \geq \frac{1}{1 + \frac{1}{1-M}}.$$

Thus we obtain by (1.186)

$$\|u_x\|^2 + 2\lambda \left(\frac{1}{1 + 1/(1-M)} \right) \leq 2E_0,$$

and due to (1.187) we conclude that

$$4u^2(x, t) + 2\lambda \left(\frac{1}{1 + 1/(1-M)} \right) \leq 2E_0 \quad \text{for } (x, t) \in [0, 1] \times [0, T]. \quad (1.188)$$

Considering the simplest case where $u_0 = 0$, $v_0 = 0$ we have that $E_0 = \lambda/2$.

Note that the solution u ceases to exist if $M = 1$, and the inequality (1.188) yields

$$h(M) = h(M; \lambda) := 2M^2 + \lambda(1 - M)/(2 - M) \leq E_0. \quad (1.189)$$

We have that $h(1) = 2$ and $h(0) = \frac{\lambda}{2}$, hence for $\lambda < 4$ we have that $h(1) > h(0)$. Also $h'(M) = 4M - \lambda/(2 - M)^2$ with $h'(1) = 4 - \lambda > 0$ for $\lambda < 4$.

Hence the following global-in-time existence result has been established.

Theorem 1.2.37 *Problem (1.170)–(1.172) with zero initial data has a global-in-time solution, i.e., $0 < u(x, t) < 1$ for all $(x, t) \in [0, 1] \times [0, \infty)$, provided that $\lambda < 4$.*

Let λ^* be the largest value of the parameter λ so that problem (1.170)–(1.172) has a global-in-time solution for $\lambda < \lambda^*$, then Theorem 1.2.37 implies that $\lambda^* \geq 4$.

Now consider the case of non-zero initial data, i.e., $u_0 \neq 0$ or $v_0 \neq 0$. In this case the initial energy is given by the expression

$$E_0 = E_0(\lambda, u_0, u_1) = \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|u_{0x}\|^2 + \lambda \left(\frac{1}{1 + \int_0^1 \frac{1}{1-u_0} dx} \right).$$

Following the above, for u to exist it would then be enough to show that

$$4M^2 + 2\lambda \left(\frac{1-M}{2-M} \right) \leq \|u_1\|^2 + \|u_{0x}\|^2 + 2\lambda F(u_0), \quad (1.190)$$

for $F(u_0) = \frac{1}{1 + \int_0^1 \frac{1}{1-u_0} dx}$, guarantees that $M < 1$. Clearly inequality (1.190) fails at $M = 1$ if

$$4 > \|u_1\|^2 + \|u_{0x}\|^2 + 2\lambda, \quad (1.191)$$

since $F(u_0) \leq 1$. Thus the solution exists for all time as long as

$$\lambda < \left(2 - \frac{1}{2} (\|u_1\|^2 + \|u_{0x}\|^2) \right).$$

(N.B. With $u_0 = 0$, $F(u_0) = 1/2$, and a stronger estimate, as of Theorem 1.2.37, is achieved.)

We first consider the case of zero initial conditions, i.e., $u_0(x) = u_1(x) \equiv 0$. Our purpose is to show that there exists a critical value $\lambda_+^* \geq \lambda_-^*$ of λ such that the solution of problem (1.170)–(1.172) quenches in finite time for any $\lambda \geq \lambda_+^*$. Thus λ_+^* is the supremum of those λ for which (1.170)–(1.172) has a global solution.¹

For convenience and due to the expected symmetry of the solution $u(x, t)$ at the point $x = \frac{1}{2}$ we will consider the equivalent problem

¹Because (1.170)–(1.172) lacks clear monotonicity properties, it is not obvious that $\lambda_+^* = \lambda_-^*$.

$$u_{tt} = u_{xx} + \frac{\lambda}{(1-u)^2 \left(1 + 2\alpha \int_0^{\frac{1}{2}} \frac{1}{1-u} dx\right)^2}, \quad 0 < x < \frac{1}{2}, \quad t > 0, \quad (1.192)$$

$$u(0, t) = 0, \quad u_x\left(\frac{1}{2}, t\right) = 0, \quad t > 0, \quad (1.193)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 < x < \frac{1}{2}. \quad (1.194)$$

We restrict our analysis for times $0 < t < 1/2$ so that the characteristic line $t = x$ remains in the strip $0 < x < 1/2$.

Initially we consider the general problem

$$u_{tt} - u_{xx} = g(t), \quad 0 < x < \frac{1}{2}, \quad t > 0, \quad (1.195)$$

$$u(0, t) = 0, \quad u_x\left(\frac{1}{2}, t\right) = 0, \quad t > 0, \quad (1.196)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 < x < \frac{1}{2}, \quad (1.197)$$

with $g(t) > 0$ and continuous. For $x \geq t$, the solution of (1.195)–(1.197) is given by

$$\begin{aligned} u(x, t) &= \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} g(s) d\xi ds = \frac{1}{2} \int_0^t g(t) [x + (t-s) - x + (t-s)] ds \\ &= \int_0^t (t-s) g(s) ds := G(t), \end{aligned} \quad (1.198)$$

since in that case the domain of dependence of the point (x, t) is the triangle $D = \{(\xi, s) \mid x - (t-s) < \xi < x + (t-s) \text{ and } 0 < s < t\}$. On the other hand, for $x \leq t$ we have to subtract the contribution coming from $x < 0$ so we get

$$u(x, t) = G(t) - G(t-x) = \int_0^t (t-s) g(s) ds - \int_0^{t-x} (t-x-s) g(s) ds. \quad (1.199)$$

Note that in this case the domain of dependence is no longer a triangular but is instead split into two parts $D_1 = \{(\xi, s) \mid x - (t-s) < \xi < x + (t-s) \text{ and } t-x < s < t\}$ and $D_2 = \{(\xi, s) \mid t-x-s < \xi < x + (t-s) \text{ and } 0 < s < t-x\}$.

From the above analysis we bear in mind that the solution of (1.195)–(1.197) in the area $x \geq t$ is only a function of time, $u(x, t) = G(t)$. In addition for $t \leq 1/2$ by the form of the solution we can easily deduce that $u_x \geq 0$ ($= 0$ for $x \geq t$, $= \int_0^{t-x} g(s) ds > 0$ for $x \leq t$). Finally for every $0 < x < 1/2$, $0 < t < 1/2$ we have $0 \leq u(x, t) \leq G(t) = \max_{x \in [0, 1/2]} u(x, t)$.

In the following we consider the more general problem

$$u_{tt} - u_{xx} = h(x, t), \quad 0 < x < \frac{1}{2}, \quad t > 0, \quad (1.200)$$

$$u(0, t) = 0, \quad u_x\left(\frac{1}{2}, t\right) = 0, \quad t > 0, \quad (1.201)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 < x < \frac{1}{2}, \quad (1.202)$$

with $h(x, t) > 0$, $h_x(x, t) > 0$ and continuous.

Using the same reasoning as above we easily obtain that the solution of problem (1.200)–(1.202) is given by

$$u(x, t) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} h(y, s) d\xi ds, \quad (1.203)$$

for $x \geq t$, and

$$u(x, t) = \frac{1}{2} \int_{t-x}^t \int_{x-(t-s)}^{x+(t-s)} h(y, s) d\xi ds + \frac{1}{2} \int_0^{t-x} \int_{t-s-x}^{x+(t-s)} h(y, s) d\xi ds,$$

for $x \leq t$. Due to the fact that $h(x, t) > 0$ and $h_x(x, t) > 0$ we have $u_x(x, t) \geq 0$ and hence $\max_{x \in [0, 1/2]} u(x, t)$ is given by (1.203).

In our case, going back to our original problem (1.192)–(1.194), the function h has the specific form $h(x, t) = f(u)g(t)$ where $f(u) = \frac{1}{(1-u)^2}$ and

$$g(t) = \frac{\lambda}{\left(1 + 2 \int_0^{1/2} \frac{1}{1-u} dx\right)^2}.$$

Note also that the function f is such that $f(0) = 1$ and $f'(s) > 0$ for $0 < s < 1$.

We shall show that for $x \geq t$ the solution of problem (1.192)–(1.194) is purely a function of time, i.e., $u(x, t) = U(t) = \max_{x \in [0, 1/2]} u(x, t)$, by applying a Picard iteration.

Initially we define u_0 to be the solution of the problem

$$u_{0tt} - u_{0xx} = g(t), \quad 0 < x < \frac{1}{2}, \quad 0 < t < \frac{1}{2},$$

with the boundary and initial conditions as defined above. As we have already stated, by the analysis of the problem (1.195)–(1.197), $u_0(x, t) = U_0(t)$ for $x \geq t$, $u_{0x} \geq 0$ and $0 \leq u_0(x, t) \leq U_0(t)$ where $U_0(t) = \max_{x \in [0, 1/2]} u_0(x, t)$ coincides with $G(t)$ given by (1.198).

We inductively define the function $u_n(x, t)$ to be the solution of the problem

$$u_{ntt} - u_{nxx} = f(u_{n-1})g(t), \quad 0 < x < \frac{1}{2}, \quad 0 < t < \frac{1}{2},$$

with the standard initial and boundary conditions. By the analysis of problem (1.200)–(1.202) we have, since $h(x, t) = f(u_{n-1})g(t) > 0$ and

$$h_x(x, t) = f'(u_{n-1})u_{(n-1)x}g(t) > 0,$$

($f' > 0$ and $(u_{n-1})_x > 0$ by the induction hypothesis), that u_n has the required property $u_{nx} > 0$. By the induction hypothesis we also have that $u_{n-1}(x, t)$ is purely a function of time in the area $x \geq t$ hence $f(u_{n-1})g(t)$ is also, implying that $u_n(x, t) = U_n(t) = \max_{x \in [0, 1/2]} u_n(x, t)$ for any $n \in \mathbb{N}$ and $x \geq t$ and $U_n(t)$ is given by

$$U_n(t) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(U_{n-1}(s))g(s) d\xi ds, \quad (1.204)$$

which implies that the sequence $\{U_n(t)\}_{n=1}^\infty$ is increasing, recalling that $f(s) > 1$ and $f'(s) > 0$ for $0 < s < 1$. Moreover, we have that $0 < U_n(t) < 1$ for every $0 < t < t_0$ for some $t_0 \leq 1/2$ hence $\{U_n(t)\}_{n=1}^\infty$ converges as $n \rightarrow \infty$ to a function $U(t)$ which is actually the maximum value of the unique solution $u(x, t)$ of (1.192)–(1.194) achieved for $x \geq t$, i.e., $u(x, t) = U(t) = \max_{x \in [0, 1/2]} u(x, t)$ for $x \geq t$.

The latter implies that the solution of problem (1.192)–(1.194) satisfies for $x \geq t$ the equation

$$U_{tt} = f(U)g(t),$$

and since

$$g(t) \geq \frac{\lambda}{\left[1 + \frac{1}{(1-U)}\right]^2},$$

$U(t)$ finally satisfies the differential inequality

$$U_{tt} \geq \frac{\lambda}{[2 - U]^2}, \quad (1.205)$$

with $U(0) = 0$ and $U_t(0) = 0$.

Therefore we obtain that the following inequality is satisfied

$$\sqrt{2\lambda}t \leq 2^{\frac{3}{2}} \left[\sin^{-1} \left(\sqrt{\frac{U}{2}} \right) + \frac{\sqrt{U(2-U)}}{2} \right],$$

and we deduce that $U(t)$ reaches 1 before time $t = 1/2$, provided that

$$\lambda \geq 4 \left[2 \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) + 1 \right]^2 = \lambda^+ = 26.43596. \quad (1.206)$$

Finally we have proved the following result:

Theorem 1.2.38 *If $\lambda \geq \lambda^+$, where λ^+ is given by (1.206), then the solution $u(x, t)$ to problem (1.192)–(1.194) quenches in finite time $T \leq 1/2$, i.e., $\|u(\cdot, t)\|_\infty \rightarrow 1$ as $t \rightarrow T \leq 1/2$.*

The least upper bound for global existence, λ_+^* , then satisfies $\lambda_+^* \leq \lambda^+$.

Remark 1.2.39 For the local problem, differential inequality (1.205) implies that $\|u_{tt}(\cdot, t)\|_\infty$ blows up at the quenching time, i.e., $\|u_{tt}(\cdot, t)\|_\infty \rightarrow +\infty$ as $t \rightarrow T$, which is in agreement with the result in [3].

A result similar to Theorem 1.2.38 can be deduced when the initial data are non-zero as it is shown below.

Indeed we take λ large enough, depending on the initial data u_0 and u_1 , and we prove finite-time quenching using some of the arguments developed previously.

We consider problem (1.170)–(1.172) and we assume that u_0 is bounded away from 1.

Let w be the solution of the problem

$$w_{tt} - w_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \quad (1.207)$$

$$w(0, t) = w(1, t) = 0, \quad t > 0, \quad (1.208)$$

$$w(x, 0) = u_0(x), \quad w_t(x, 0) = v_0(x), \quad 0 < x < 1. \quad (1.209)$$

We also define the function v as the difference $v = u - w$ hence $v(x, 0) = 0$ and $v_t(x, 0) = 0$; moreover v satisfies homogeneous Dirichlet boundary conditions $v(0, t) = 0 = v(1, t)$.

Now we assume that $u_0(x)$ and $v_0(x)$ are smooth enough so that $w_t(x, t)$ is bounded below for $x \in (1/4, 3/4)$ and $t \in (0, 1/4)$. Then for some $t_0 < 1/4$ there exists $\varepsilon > 0$ such that $\sup_{(1/4, 3/4) \times (0, t_0)} w(x, t) < 1 - 2\varepsilon$. Note that t_0 and ε can be taken independent of λ , and, having fixed t_0 , ε can be arbitrarily small. We define C_0 , independent of λ and of ε , by $C_0 = \inf_{[0, 1] \times [0, t_0]} \{w\} \leq 0$.

Now take $t_1(\varepsilon, \lambda) \leq t_0$ such that $\|v(\cdot, t)\|_\infty < \varepsilon$ for $0 \leq t < t_1$. Then

$$C_0 - \varepsilon \leq u = w + v \leq 1 - 2\varepsilon + \varepsilon \quad \text{for } 0 \leq t \leq t_1.$$

Hence, for $0 < t < t_1$,

$$\varepsilon < 1 - u < 1 + C_0 - \varepsilon, \quad 1 + \frac{1}{1 + C_0 - \varepsilon} < 1 + \int_0^1 (1 - u)^{-1} dx < 1 + \frac{1}{\varepsilon},$$

and

$$\frac{\lambda}{\left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon}\right)\right]^2} < v_{tt} - v_{xx} < \frac{\lambda}{\left(\varepsilon \left[1 + \frac{1}{1+C_0-\varepsilon}\right]\right)^2}.$$

In particular, following the previous subsection, since $t \leq x \leq 1 - t$ for $1/4 \leq x \leq 3/4$, $0 \leq t \leq 1/4$, for $0 < t < t_1 \leq t_0 \leq 1/4$,

$$v(x, t) = V(t) \text{ for } \frac{1}{4} \leq x \leq \frac{3}{4}, \quad \text{with } 0 \leq v(x, t) \leq V(t) \text{ for } x \leq \frac{1}{4} \text{ and for } x \geq \frac{3}{4},$$

where

$$0 < \frac{\lambda}{\left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon}\right)\right]^2} < \frac{d^2 V}{dt^2} < \frac{\lambda}{\left(\varepsilon \left[1 + \frac{1}{1+C_0-\varepsilon}\right]\right)^2}.$$

Then, at $t = t_1(\varepsilon, \lambda)$,

$$v_t = \frac{dV}{dt} \geq \frac{\lambda t_1}{((1 + C_0 - \varepsilon) (1 + 1/\varepsilon))^2} \quad \text{and}$$

$$\frac{\lambda t_1^2}{2 \left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon}\right)\right]^2} < v = V < \frac{\lambda t_1^2}{2 \left(\varepsilon \left[1 + \frac{1}{1+C_0-\varepsilon}\right]\right)^2},$$

for $1/4 < x < 3/4$.

Additionally, $v \leq V \leq \varepsilon$ is guaranteed in $0 \leq t \leq t_1$ by taking $\lambda t_1^2 \leq 2\varepsilon^3 \left[1 + \frac{1}{1+C_0-\varepsilon}\right]^2$, for example by

$$t_1 = \sqrt{\frac{2}{\lambda}} \left[1 + \frac{1}{1 + C_0 - \varepsilon}\right] \varepsilon^{\frac{3}{2}}. \quad (1.210)$$

In $t_1 < t < t_0$, $1/4 \leq x \leq 3/4$, we still have $v(x, t) = V(t)$ with $d^2 V/dt^2 > 0$, as long as the solutions u and v exist. It follows that

$$v\left(\frac{1}{2}, t\right) = V(t) > \frac{\lambda t_1^2}{2 \left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon}\right)\right]^2} + \frac{\lambda t_1(t - t_1)}{\left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon}\right)\right]^2}.$$

Assuming the solution exists up to t_0 , i.e., $T \geq t_0$,

$$u\left(\frac{1}{2}, t\right) = v\left(\frac{1}{2}, t\right) + w\left(\frac{1}{2}, t\right) > C_0 + \frac{\lambda t_1 \left(t_0 - \frac{t_1}{2}\right)}{\left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon}\right)\right]^2} \geq 1,$$

if

$$\lambda t_1 \left(t_0 - \frac{t_1}{2} \right) \geq \left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon} \right) \right]^2. \quad (1.211)$$

Therefore, taking t_1 to be given by (1.210), quenching before $t = t_0$ will take place on choosing λ large enough to satisfy

$$\sqrt{2\lambda t_1} \left(t_0 - \left[1 + \frac{1}{1 + C_0 - \varepsilon} \right] \varepsilon^{\frac{3}{2}} / \sqrt{2\lambda t_1} \right) \geq \left[(1 + C_0 - \varepsilon) \left(1 + \frac{1}{\varepsilon} \right) \right]^2. \quad (1.212)$$

Note that, for restricted regions of smoothness for the initial data, different intervals of x and smaller values of t_0 could be used.

References

1. Al-Refai, M., Kavallaris, N.I., Hajji, M.A.: Monotone iterative sequences for non-local elliptic problems. *Euro. J. Appl. Math.* **22**(6), 533–552 (2011)
2. Brezis, H., Cazenave, T.: A nonlinear heat equation with singular initial data. *J. Anal. Math.* **68**, 277–304 (1996)
3. Chan, C.Y., Nip, K.K.: On the blow-up of $|u_{tt}|$ at quenching for semilinear Euler-Poisson-Darboux equations. *Comp. Appl. Mat.* **14**, 185–190 (1995)
4. Chan, E.K., Dutton, R.W.: Effects of capacitors, resistors and residual change on the static and dynamic performance of electrostatically actuated devices. *Proc. SPIE* **3680**, 120–130 (1999)
5. Chang, P.H., Levine, H.A.: The quenching of solutions of semilinear hyperbolic equations. *SIAM J. Math. Analysis* **12**, 893–903 (1981)
6. Esposito, P.: Compactness of a nonlinear eigenvalue problem with a singular nonlinearity. *Comm. Contemp. Math.* **10**, 17–45 (2008)
7. Esposito, P., Ghoussoub, N.: Uniqueness of solutions for an elliptic equation modeling MEMS. *Methods Appl. Anal.* **15**, 341–354 (2008)
8. Esposito, P., Ghoussoub, N., Guo, Y.: Compactness along the first branch of unstable solutions for an elliptic problem with a singular nonlinearity. *Comm. Pure Appl. Math.* **60**, 1731–1768 (2007)
9. Esposito, P., Ghoussoub, N., Guo, Y.: *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*. Courant Lecture Notes in Mathematics, vol. 20. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence (2010)
10. Filippas, S., Guo, J.S.: Quenching profiles for one-dimensional semilinear heat equations. *Quart. Appl. Math.* **51**, 713–729 (1993)
11. Friedman, A., McLeod, B.: Blow-up of positive solutions of semilinear heat equations. *Ind. Univ. Math. J.* **34**, 425–447 (1985)
12. Garabedian, P.R.: *Partial Differential Equations*. Wiley, New York (1964)
13. Ghoussoub, N., Guo, Y.: On the partial differential equations of electrostatic MEMS devices: stationary case. *SIAM J. Math. Anal.* **38**, 1423–1449 (2007)
14. Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* **68**, 209–243 (1979)
15. Guo, J.-S.: Quenching problem in nonhomogeneous media. *Differ. Integral Equ.* **10**, 1065–1074 (1997)
16. Guo, J.-S.: Recent developments on a nonlocal problem arising in the micro-electro mechanical system. *Tamkang J. Math.* **45**, 229–241 (2014)

17. Guo, J.-S., Kavallaris, N.I.: On a nonlocal parabolic problem arising in electromechanical MEMS control. *Disc. Cont. Dyn. Systems* **32**, 1723–1746 (2012)
18. Guo, J.-S., Hu, B., Wang, C.-J.: A nonlocal quenching problem arising in micro-electro mechanical systems. *Quart. Appl. Math.* **67**, 725–734 (2009)
19. Guo, Y., Pan, Z., Ward, M.J.: Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties. *SIAM J. Appl. Math.* **166**, 309–338 (2006)
20. Guo, Z., Wei, J.: Asymptotic behavior of touch-down solutions and global bifurcations for an elliptic problem with a singular nonlinearity. *Comm. Pure Appl. Anal.* **7**, 765–786 (2008)
21. Hui, K.-M.: The existence and dynamic properties of a parabolic nonlocal MEMS equation. *Nonl. Anal. (TMA)* **74**, 298–316 (2011)
22. Joseph, D.D., Lundgren, T.S.: Quasilinear dirichlet problems driven by positive sources. *Arch. Ration. Mech. Anal.* **49**, 241–269 (1973)
23. Kaplan, S.: On the growth of solutions of quasilinear parabolic equations. *Comm. Pure Appl. Maths.* **16**, 327–330 (1963)
24. Kato, T.: *Perturbation Theory for Linear Operators*. Springer, Berlin (1966)
25. Kavallaris, N.I.: Quenching solutions of a stochastic parabolic problem arising in electrostatic MEMS control. *Math. Meth. Appl. Sci* (2016). <https://doi.org/10.1002/mma.4176>
26. Kavallaris, N.I., Nardziejca, T.: On the blow-up of the non-local thermistor problem. *Proc. Edin. Math. Soc.* **50**, 389–409 (2007)
27. Kavallaris, N.I., Miyasita, T., Suzuki, T.: Touchdown and related problems in electrostatic MEMS device equation. *Non. Diff. Eqns. Appl.* **15**, 363–385 (2008)
28. Kavallaris, N.I., Lacey, A.A., Nikolopoulos, C.V., Tzanetis, D.E.: A hyperbolic nonlocal problem modelling MEMS technology. *Rocky Mountain J. Math.* **41**, 505–534 (2011)
29. Kavallaris, N.I., Lacey, A.A., Nikolopoulos, C.V.: On the quenching of a nonlocal parabolic problem arising in electrostatic MEMS control. *Nonl. Anal.* **138**, 189–206 (2016). *Nonlinear Partial Differential Equations*, in honor of Juan Luis Vázquez for his 70th birthday
30. Lacey, A.A.: Thermal runaway in a non-local problem modelling Ohmic heating. Part I: model derivation and some specail cases. *Euro. J. Appl. Math.* **6**, 127–144 (1995)
31. Levine, H.A.: Quenching, nonquenching, and beyond quenching for solution of some parabolic equations. *Ann. Mat. Pura Appl.* **155**, 243–260 (1989)
32. Lin, C.-S., Ni, W.-M.: A counterexample to the nodal domain conjecture and a related semilinear equation. *Proc. Amer. Math. Soc.* **102**, 271–277 (1988)
33. Miyasita, T.: Non-local elliptic problem in higher dimension. *Osaka J. Math.* **44**, 159–172 (2007)
34. Nagasaki, K., Suzuki, T.: Spectral and related properties about the Emden-Fowler equation $-\Delta u = \lambda e^u$ on circular domains. *Math. Ann.* **299**, 1–15 (1994)
35. Naito, Y., Suzuki, T.: Radial symmetry of positive solutions for semilinear elliptic equations on the unit ball in $\mathbb{R}^n$. *Funkcial Ekvac.* **41**, 215–234 (1998)
36. Pelesko, J.A., Bernstein, D.H.: *Modeling MEMS and NEMS*. Chapman Hall and CRC Press, Boca Raton (2002)
37. Pelesko, J.A., Triolo, A.A.: Nonlocal problems in MEMS device control. *J. Eng. Math.* **41**, 345–366 (2001)
38. Pohozaev, S.I.: Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Doklady* **165**, 1408–1411 (1965)
39. Quittner, P., Souplet, Ph.: *Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States*. Birkhäuser Advanced Texts. Birkhäuser Verlag, Basel (2007)
40. Rabinowitz, P.H.: Some global results for nonlinear eigenvalue problems. *J. Func. Anal.* **7**, 487–513 (1971)
41. Rabinowitz, P.H.: Some aspects of nonlinear eigenvalue problems. *Rocky Mountain J. Math.* **3**, 161–202 (1973)
42. Seeger, J.J., Crary, S.B.: Stabilization of electrostatically actuated mechanical devices. In: *Proceedings of the 1997 International Conference on Solid-State Sensors and Actuators*, pp. 1133–1336 (1997)

43. Seeger, J.J., Crary, S.B.: Analysis and simulation of MOS capacitor feedback for stabilizing electrostatically actuated, mechanical devices. In: Second International Conference on the Simulation and Design of Microsystems and Microstructures-MICROSIM97, pp. 199–208 (1997)
44. Schaaf, R.: Uniqueness for semilinear elliptic problems supercritical growth and domain geometry. *Adv. Diff. Equ.* **5**, 1201–1220 (2000)
45. Weissler, F.B.: Local existence and nonexistence for semilinear parabolic equations in L^p . *Indiana Univ. Math. J.* **29**, 79–102 (1980)
46. Younis, M.: MEMS Linear and Nonlinear Statics and Dynamics. Springer, New York (2011)
47. Zheng, G.-F.: Some dichotomy results for the quenching problem. *Acta Math. Sinica* **28**, 1491–1506 (2012)

Chapter 2

Ohmic Heating Phenomena

Abstract The current chapter considers two main applications associated with Ohmic heating phenomena. Initially we deal with an application from food industry, building up two one-dimensional non-local problems illustrating the evolution of the temperature of the sterilized food. The former model consists of a diffusion-convection equation while the latter of a convection equation with non-local convection velocity. Both of these non-local models are investigated in terms of their stability and the occurrence of finite-time blow-up, where the latter in the current context indicates food burning. Different approaches should be followed though depending on the monotonicity of the nonlinearity appearing in the non-local term, since no maximum principle is available for the non-local parabolic problem when this nonlinearity is increasing. The second part of the chapter is devoted to the study of a non-local parabolic model illustrating the operation of the thermistor device. Notably, conditions under which finite-time blow-up, which here indicates the destruction of the thermistor device, occurs are investigated by using both energy and comparison methods.

2.1 Ohmic Heating of Foods

2.1.1 Derivation of the Basic Model and Its Variations

One of the methods developed in recent years for sterilizing food is to heat it rapidly by means of an electric current. The food is passed through a conduit, part of which lies between two electrodes. A high electric current flowing between the electrodes results in Ohmic heating of the food which quickly gets hot. We next present the derivation of the mathematical model describing the above process. A more detailed background on this type of process can be found in [8, 12, 16, 38, 41, 44].

The electric potential φ and the current density $\vec{j}$ are related by Ohm's law,

$$\vec{j} = -\sigma \nabla \varphi,$$

where σ is the electrical conductivity, which is assumed to vary with temperature. Then for conservation of charge, we have

$$\nabla \cdot (\sigma \nabla \phi) = 0, \quad (2.1)$$

where it is assumed that on electro-magnetic time scales the situation is only slowly varying and the change density is small.

For food (or other substance) with density ρ , specific heat c , and velocity $\vec{v}$, all assumed constant here, the temperature T satisfies

$$\rho c \left[\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right] = \nabla \cdot (k \nabla T) + \sigma |\nabla \phi|^2. \quad (2.2)$$

Here, k is the thermal conductivity and the last term on the right hand side of (2.2) represents Ohmic heating. This model can be additionally simplified on assuming the following:

- (i) the thermal conductivity is constant independent on the temperature T ;
- (ii) the food enters the heater with a temperature T_0 independent of its position along the channel;
- (iii) end effects for the problem can be neglected so that the potential is 0 at the start of the heater, $z = 0$, and $V(t)$ at the far (down-stream) end, $z = L$.

Here, z is the distance along the channel (i.e. in the direction of $\vec{v}$) which has parallel sides and owing assumptions (i) – (ii) we then derive that the potential and temperature only vary with z and time t and satisfy the system

$$\frac{\partial}{\partial z} \left(\sigma \frac{\partial \phi}{\partial z} \right) = 0, \quad 0 < z < L, \quad t > 0, \quad (2.3)$$

$$\rho c \left(\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial z} \right) = k \frac{\partial^2 T}{\partial z^2} + \sigma \left(\frac{\partial \phi}{\partial z} \right)^2, \quad 0 < z < L, \quad t > 0, \quad (2.4)$$

$$\phi = 0, \quad T = T_0, \quad z = 0; \quad \phi = V, \quad z = L. \quad (2.5)$$

Also, V is known if the potential difference across the device is specified (but has to be determined if the process is controlled in some other way). Equation (2.3) is integrated to give,

$$\sigma \frac{\partial \phi}{\partial z} = J(t), \quad \text{so} \quad V = J \int_0^L \frac{dz}{\sigma},$$

where J is the electric current density (along the channel) and finally (2.4) is transformed to

$$\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial z} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial z^2} + \frac{J^2}{\rho c \sigma} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial z^2} + \left(\frac{V^2}{\rho c} \right) \left(\frac{1}{\sigma} \right) \left(\int_0^L \frac{1}{\sigma} dz \right)^{-2}.$$

This is a variant of the non-local parabolic problem considered in [10, 29, 30]; heat transport is now happening both by convection and conduction.

It is convenient to scale the distance with the length of channel and the time to make the convective velocity 1. Additionally this scaling changes the temperature variable so that it becomes 0 at the inlet and its derivative becomes zero at the outlet, i.e. the right hand side of the device is thermally insulated. Then the dimensionless temperature u satisfies the following conduction-convection problem [22],

$$u_t + u_x = u_{xx} + \frac{\lambda f(u)}{\left(\int_0^1 f(u) dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \quad (2.6)$$

$$u(0, t) = u_x(1, t) = 0, \quad t > 0, \quad (2.7)$$

$$u(x; 0) = u_0(x), \quad 0 < x < 1, \quad (2.8)$$

with $f > 0$ being the dimensionless electrical reactivity ($\propto \frac{1}{\sigma}$). The parameter $\lambda > 0$ can be identified with (the square of) applied potential difference; on occasions we shall absorb λ into f .

If the heater is part of a circuit, so that it is connected in series with a constant resistance, and a fixed EMF (Electromotive Force) is applied across the two, then the scaled non-local equation is replaced by

$$u_t + u_x = u_{xx} + \frac{\lambda f(u)}{\left[a + b \int_0^1 f(u) dx\right]^2}, \quad 0 < x < 1, \quad t > 0, \quad (2.9)$$

where $a, b > 0$.

When the heating of the food is rapid, heat diffusion both in the direction of flow and normal to it can be neglected, i.e. $0 < k \ll 1$, as suggested in [36]. Following then the same steps as above, we end up with the model [31],

$$u_t + u_x = \frac{\lambda f(u)}{\left(\int_0^1 f(u) dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \quad (2.10)$$

$$u(0, t) = 0, \quad t > 0, \quad (2.11)$$

$$u(x; 0) = u_0(x), \quad 0 < x < 1, \quad (2.12)$$

which is a non-local hyperbolic model.

If it is assumed that the density ρ and the velocity $\vec{v}$ of the food vary significantly with temperature T , then one has to take the change of mass of the food into account as well, so additionally to system (2.1)–(2.2) we have the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (2.13)$$

expressing the conservation of mass.

Under assumptions (i) – (iii) and for a rapid food heating, system (2.1), (2.2) and (2.13) is then reduced to

$$\frac{\partial}{\partial x} \left(\sigma \frac{\partial \varphi}{\partial x} \right) = 0, \quad 0 < x < L, \quad t > 0, \quad (2.14)$$

$$\rho \frac{\partial T}{\partial t} + \rho v \frac{\partial T}{\partial x} = \sigma \left(\frac{\partial \varphi}{\partial x} \right)^2, \quad 0 < x < L, \quad t > 0, \quad (2.15)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \quad 0 < x < L, \quad t > 0. \quad (2.16)$$

We integrate (2.14) to derive

$$\rho \frac{\partial T}{\partial t} + \rho v \frac{\partial T}{\partial x} = \frac{I^2}{\sigma} = \frac{V^2}{\sigma} \left(\int_0^1 \frac{dx}{\sigma} \right)^{-2}, \quad (2.17)$$

and then integrating (2.16) and substituting ρ back to (2.17) we deduce after proper scaling that the dimensionless temperature u satisfies the following hyperbolic non-local problem

$$\rho(u)u_t + \left(1 - \int_0^x \rho'(u)u_t dy \right) u_x = \frac{\lambda f(u)}{\left(\int_0^1 f(u) dx \right)^2}, \quad 0 < x < 1, \quad t > 0, \quad (2.18)$$

$$u(0, t) = 0, \quad t > 0, \quad (2.19)$$

$$u(x; 0) = u_0(x), \quad 0 < x < 1, \quad (2.20)$$

where now $\lambda = V^2$, see also [24]. Obviously problem (2.18)–(2.20) reduces to (2.10)–(2.12) when ρ is constant.

Remarkably, non-local equations similar to (2.6) arise in various applications including the shear banding formation in metals being deformed under high strain rates, [1, 2], the investigation of the fully turbulent behavior of real flows, using invariant measures for the Euler equation, [3], to mention a few of them. Additionally, for the case of high strained metals and when there is lack of knowledge of certain physical parameters of the system, then we end up, [20], with the following stochastic system

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda e^u}{\left(\int_D e^u dx \right)^p} + \sigma(u) \partial_t W(x, t), \quad \text{in } D_T := D \times (0, T), \quad (2.21)$$

$$u(x, t) = 0, \quad \text{o } \partial D \times (0, T), \quad (2.22)$$

$$u(x, 0) = \xi(x), \quad \text{in } D, \quad (2.23)$$

for a bounded domain $D \in \mathbb{R}^N$, $N \geq 1$ and $0 < p \leq 1$. Here $W(x, t)$ is a Wiener process and the multiplicative noise $\sigma(u) \partial_t W(x, t)$ encapsulates the present uncertainty in the system. In the current manuscript, only the deterministic version,

i.e. when $\sigma(u) \equiv 0$, of (2.21)–(2.23) is investigated, however the interested reader is referred to [20] for the inspection of the finite-time blow-up (occurrence of shear banding formation) of the stochastic problem (2.21)–(2.23).

2.1.2 Local Existence and Monotonicity

2.1.2.1 Parabolic Case

The well-posedness of problem (2.6)–(2.8) can be obtained by using a Picard type argument. In particular, by $\{u^n\}_{n=1}^\infty$ satisfying

$$\begin{aligned} u_t^n + u_x^n &= u_{xx}^n + \frac{\lambda u^{n-1}}{\left(\int_0^1 f(u^{n-1})dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \\ u^n(0, t) &= u_x^n(1, t) = 0, \quad t > 0, \\ u^n(x; 0) &= u_0(x), \quad 0 < x < 1, \end{aligned}$$

we can easily prove the following

Theorem 2.1 Fix $\lambda > 0$ and take f strictly positive satisfying a Lipschitz condition in the interval (α, β) where

$$\alpha < \min \left\{ 0, \inf_{(0,1)} u_0(x) \right\} \quad \text{and} \quad \beta > \max \left\{ 0, \sup_{(0,1)} u_0(x) \right\}.$$

Assume also that $u_0 \in L^\infty(0, 1)$ then there exists $T > 0$ such that problem (2.6)–(2.8) has a unique (classical) solution in $[0, 1] \times [0, T]$.

For a proof of Theorem 2.1 see [18].

Remark 2.1.1 The (unique) solution provided by Theorem 2.1 continues to exist as long as it remains less than or equal to β . This argument implies that u ceases to exist only by blow-up; that is, there exists a sequence $(x_n, t_n) \rightarrow (x^*, t^*)$ with $t^* \leq \infty$ such that $u(x_n, t_n) \rightarrow \infty$ as $n \rightarrow \infty$.

For the study of the long-time behavior of the solutions to problem (2.6)–(2.8) we will need the concept of lower and upper solutions.

Definition 2.1.2 A function $\bar{u}$ is an upper solution to problem (2.6)–(2.8) if it satisfies

$$\begin{aligned}\bar{u}_t + \bar{u}_x &\geq \bar{u}_{xx} + \frac{\lambda f(\bar{u})}{\left(\int_0^1 f(\bar{u}) dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \\ \bar{u}(x, t) &\geq u(x, t), \quad x = 0, 1, \quad t > 0, \\ \bar{u}(x, 0) &\geq u(x, 0), \quad 0 < x < 1,\end{aligned}$$

whereas a lower solution $\underline{u}$ to (2.6)–(2.8) satisfies the above inequalities but reversed, i.e.

$$\begin{aligned}\underline{u}_t + \underline{u}_x &\leq \underline{u}_{xx} + \frac{\lambda f(\underline{u})}{\left(\int_0^1 f(\underline{u}) dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \\ \underline{u}(x, t) &\leq u(x, t), \quad x = 0, 1, \quad t > 0, \\ \underline{u}(x, 0) &\leq u(x, 0), \quad 0 < x < 1.\end{aligned}$$

An importance observation is that if f is a decreasing function then we may use comparison methods, i.e. upper and lower solutions of problem (2.6)–(2.8) are ordered.

Indeed, if we define $v(x, t) := \bar{u}(x, t) - \underline{u}(x, t)$, by continuity there exists a time $T > 0$ such that $v \geq 0$ for $0 \leq t < T$ and

$$\begin{aligned}v_t + v_x &\geq v_{xx} + I(s, \underline{u})v, \quad (x, t) \in Q_T := (0, 1) \times (0, T), \\ v(x, t) &\geq 0, \quad x = 0, 1, \quad 0 < t < T, \quad v(x, 0) \geq 0, \quad 0 < x < 1,\end{aligned}$$

where

$$I(s, \underline{u}) := \frac{\lambda f'(s)}{\left(\int_0^1 f(\underline{u}) dx\right)^2},$$

is bounded for any $s \in (\underline{u}, \bar{u})$. The latter, by the maximum principle, implies that $v = \bar{u} - \underline{u} \geq 0$ for $0 \leq t \leq T$, see also [29].

This gives an alternative proof for the existence of a local-in-time solution u of problem (2.6)–(2.8) which lies in $(\underline{u}, \bar{u})$ under the monotonicity of f , [35, 39], and the monotonicity of f as well as the ordering of lower and upper solutions.

When f is increasing, however, the maximum principle is not applicable [29, 37]. Then upper and lower solutions are not necessarily ordered. Therefore, some new type of comparison functions should be defined, see also Chap. 1.

Definition 2.1.3 A pair of functions $v, z \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is called a lower-upper solution pair of (2.6)–(2.8), if $v(x, t) \leq z(x, t)$ for $(x, t) \in Q_T$, $v(x, 0) \leq u_0(x) \leq z(x, 0)$ in $[0, 1]$, $v(x, t) \leq 0 \leq z(x, t)$ for $x = 0, 1$, $0 < t < T$, and

$$\begin{aligned}
v_t + v_x &\leq v_{xx} + \frac{\lambda f(v)}{\left(\int_0^1 f(z) dx\right)^2} & \text{in } Q_T, \\
z_t + z_x &\geq z_{xx} + \frac{\lambda f(z)}{\left(\int_0^1 f(v) dx\right)^2} & \text{in } Q_T.
\end{aligned}$$

If the above inequalities are strict, then (v, z) is called a strict lower-upper solution pair.

Then we can prove local-in-time existence of the solution by similar arguments to Proposition 1.2.2.

2.1.2.2 Hyperbolic Case

Notably local-in-time existence for the hyperbolic problems (2.10)–(2.12) and (2.18)–(2.20) can be established through the theory of characteristic curves and applying a Picard type approach. In the following we focus only to the more general problem (2.18)–(2.20).

The characteristics of (2.18)–(2.20) are given as a solution of the following system of ordinary differential equations

$$\frac{dt}{d\tau} = \rho(u), \quad (2.24)$$

$$\frac{dx}{d\tau} = 1 - \int_0^x \rho'(u) u_t dy, \quad (2.25)$$

$$\frac{du}{d\tau} = \frac{\lambda f(u)}{\left(\int_0^1 f(u) dx\right)^2}. \quad (2.26)$$

Although discontinuities of u_0 or a mismatch between u_0 and the boundary condition give rise to irregular behavior of u , these are simply propagated along the characteristics and allow the existence of a (local) weak solution. So, in the following we will generally be thinking of u_0 being continuous (and normally, but not always, differentiable) with $u_0(0) = 0$.

Now if f is a Lipschitz continuous function and $\rho \in C^1(0, \infty)$, then the iteration argument implies the existence of a solution to (2.24)–(2.26); also nonexistence can only come about through blow-up with u becoming infinite after some finite time t^* , see [24]. Especially, (2.24) and (2.26), together with the iteration argument imply, since $\rho \in C^1(0, \infty)$, that u_t is bounded as far as u is bounded. Using the same arguments it is proved that u_x becomes unbounded only when u becomes unbounded.

Although (2.18)–(2.20) is a hyperbolic problem in the case where f is a decreasing function, more information can be gained by a comparison result. In fact, if f is a

decreasing and Lipschitz continuous function, then $0 \leq f(\beta) - f(\alpha) \leq K(\alpha - \beta)$, where $\beta \leq \alpha \leq M$, $M > \sup_{(0,1)} u_0(x)$ for some positive constant $K \equiv K(M)$. Then a lower solution $\underline{u}$ and a solution u to problem (2.18)–(2.20) satisfy the inequality $\frac{dv}{d\tau} \leq \frac{\lambda f(0)}{f^2(M)}$ on a characteristic curve as long as they lie under M . So $\underline{u} \leq M$ and $u \leq M$ while

$$\tau \leq \frac{(M - \sup u_0) f^2(M)}{\lambda f(0)}.$$

Considering now $v_0 = \underline{u}$, it can be defined iteratively $\{v_n\}$ for $n \geq 1$ by

$$\frac{dv_n}{d\tau} + \frac{\lambda K}{f^2(M)} v_n = \frac{\lambda f(v_{n-1})}{\left(\int_0^1 f(v_{n-1}) dx\right)^2} + \frac{\lambda K}{f^2(M)} v_{n-1}, \quad (2.27)$$

with $v_n = u_0$ at $\tau = 0$ and $v_n = 0$ for $x = 0$.

Problem (2.27) has a unique solution since is linear and more precisely there holds $v_n \leq M$ for $\tau \leq T \equiv (M - \sup u_0) f^2(M) / \lambda(f(0) + KM) \leq T_1$; note also that

$$\frac{dv_1}{d\tau} + \frac{\lambda K}{f^2(M)} v_1 = \frac{\lambda f(v_0)}{\left(\int_0^1 f(v_0) dx\right)^2} + \frac{\lambda K}{f^2(M)} v_0 \geq \frac{dv_0}{d\tau} + \frac{\lambda K}{f^2(M)} v_0,$$

since $v_0 = \underline{u}$ is a lower solution to Eq. (2.26) and $v_1 \geq v_0$ for $\tau = 0$, $x = 0$; thus $v_1 \geq v_0$ for $0 \leq \tau \leq T_1$ ($0 \leq x \leq 1$) and some $T_1 > 0$.

Moreover,

$$\begin{aligned} \frac{dv_1}{d\tau} - \frac{\lambda f(v_1)}{\left(\int_0^1 f(v_1) dx\right)^2} &= \frac{\lambda f(v_1) \int_0^1 (f(v_1) + f(v_0)) dx \int_0^1 (f(v_1) - f(v_0)) dx}{\left(\int_0^1 f(v_1) dx\right)^2 \left(\int_0^1 f(v_0) dx\right)^2} \\ &\quad + \frac{\lambda(f(v_0) - f(v_1))}{\left(\int_0^1 f(v_0) dx\right)^2} + \frac{\lambda K}{f^2(M)} (v_0 - v_1) \\ &\leq \frac{\lambda f(v_1) \int_0^1 (f(v_1) + f(v_0)) dx \int_0^1 (f(v_1) - f(v_0)) dx}{\left(\int_0^1 f(v_1) dx\right)^2 \left(\int_0^1 f(v_0) dx\right)^2} \\ &\quad + \lambda K (v_0 - v_1) \left(\frac{1}{f^2(M)} - \frac{1}{\left(\int_0^1 f(v_0) dx\right)^2} \right) \leq 0, \end{aligned}$$

provided that f is Lipschitz continuous and decreasing. It follows, inductively, that $\underline{u} = v_0 \leq v_1 \leq v_2 \leq \dots \leq v_n \leq \dots \leq M$ and so $v_n \rightarrow u \geq \underline{u}$ for some solution $u \leq M$ and $0 \leq \tau \leq T$. The uniqueness of the solution for $\tau \in [0, T]$ is proved similarly. Supposing that there exist two solutions u_1, u_2 in $[0, T]$ then $0 \leq u_1, u_2 \leq M$ and using the Lipschitz continuity of f we get

$$\left| \frac{d}{d\tau}(u_1 - u_2) \right| \leq \frac{\lambda f(u_1) \int_0^1 (f(u_1) + f(u_2)) dx \int_0^1 |f(u_1) - f(u_2)| dx}{\left(\int_0^1 f(u_1) dx \right)^2 \left(\int_0^1 f(u_2) dx \right)^2} + \frac{\lambda |f(u_1) - f(u_2)|}{\left(\int_0^1 f(u_2) dx \right)^2} \leq \Lambda |u_1 - u_2|, \quad (2.28)$$

where $\Lambda = (2\lambda f^2(0) + \lambda f^2(M))K/f^4(M)$. Since for $0 \leq \tau \leq T$ there holds $|u_1 - u_2| \leq M$ due to (2.28) we get $|u_1 - u_2| \leq \Lambda M T$ and inductively we obtain $|u_1 - u_2| \leq \frac{M(\Lambda T)^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$ resulting in $u_1 \equiv u_2$.

Using the same arguments but now starting at $\tau = T$ we deduce that $u \geq \underline{u}$ as long as they both exist. The proof that $u \leq \bar{u}$, if $\bar{u}$ is an upper solution to (2.18)–(2.20), is similar.

2.1.3 Stationary Problem

The key point for the study of the long-time behavior of problems (2.6)–(2.8) and (2.18)–(2.20) is the study of the corresponding stationary problems. Henceforth, we assume that function f satisfies

$$f(s) > 0, \quad f'(s) < 0 \quad s \geq 0. \quad (2.29)$$

2.1.3.1 Parabolic Case

The corresponding steady-state problem to (2.6)–(2.8) is

$$w'' - w' + \mu f(w) = 0, \quad 0 < x < 1, \quad (2.30)$$

$$w(0) = w'(1) = 0, \quad (2.31)$$

where

$$\mu = \frac{\lambda}{\left(\int_0^1 f(w) dx \right)^2}, \quad (2.32)$$

is a positive local parameter (λ is the non-local one). Problem (2.30)–(2.31) has a unique solution for every $\mu > 0$, since f satisfies (2.29) (by using monotone type arguments), [5, 9, 34], and Lemma 2.1.5 holds, see below.

Here, due to the convection term, the steady problem is not symmetric, in contrast with the pure diffusion problem, [30]. In the sequel we shall investigate the spectrum of (2.30)–(2.31).

Integrating (2.30) over $(0, 1)$ we obtain

$$w'(0) + M = \mu \int_0^1 f(w) dx,$$

and by virtue of (2.32) we finally derive

$$\lambda(M) = \frac{(w'(0) + M)^2}{\mu}, \quad (2.33)$$

where $M = \|w\|_\infty = w(1)$.

Also multiplying (2.30) by w' and integrating we obtain

$$\frac{(w'(0))^2}{\mu} = 2 \left[\int_0^M f(s) ds - \frac{1}{\mu} \int_0^1 (w'(x))^2 dx \right] \leq 2 \int_0^M f(s) ds. \quad (2.34)$$

We now have:

Lemma 2.1.4 *If $\int_0^\infty f(s) ds < \infty$ then $\frac{(w'(0))^2}{\mu} \rightarrow 2I_\infty$ as $\mu \rightarrow \infty$ where $I_\infty = \int_0^\infty f(s) ds$.*

Proof We first consider the auxiliary problem:

$$z''(x) + \mu g(z(x)) = 0, \quad 0 < x < 1 - \delta, \quad (2.35)$$

$$z(x) = \sup_x w(x) = M, \quad z'(x) = 0 \quad \text{for } 1 - \delta \leq x \leq 1 \quad (2.36)$$

$$z(0) = 0, \quad (2.37)$$

where $0 < g(s) < f(s)$, and z, z' , are continuous at the point $x = 1 - \delta$.

Now, multiplying (2.35) by z' and integrating over the interval $(0, 1 - \delta)$ we obtain

$$(z'(x))^2 = 2\mu \int_{z(x)}^M g(s) ds = 2\mu[G(z) - G(M)], \quad (2.38)$$

where $G(z) = \int_z^\infty g(s) ds$. Since $z'(x) > 0$ in $[0, 1 - \delta)$, then (2.38) entails

$$\int_0^M [G(z) - G(M)]^{-1/2} dz = (1 - \delta)\sqrt{2\mu}. \quad (2.39)$$

We next prove that the solution of problem (2.35)–(2.37) is a lower solution to problem (2.30)–(2.31). Indeed,

$$z''(x) - z'(x) + \mu f(z) = \mu f(z) > 0 \quad \text{for } 1 - \delta \leq x \leq 1.$$

Also taking into account (2.35)–(2.39),

$$\begin{aligned} z'' - z' + \mu f(z) &= z' + \mu(f(z) - g(z)) \\ &= -\sqrt{2\mu} [G(z) - G(M)]^{1/2} + \mu(f(z) - g(z)) \quad \text{for } 0 < x < 1 - \delta. \end{aligned} \quad (2.40)$$

Now choosing μ such that

$$\mu \geq \mu_0 = \sup_{x \in (0, M)} \frac{2[G(z) - G(M)]}{[f(z) - g(z)]^2}, \quad (2.41)$$

and $\delta < 1$, relations (2.40), (2.41) imply

$$z'' - z' + \mu f(z) > 0 \quad \text{for } 0 < x < 1 - \delta,$$

In addition $z(0) = z'(1) = w(0) = w'(1) = 0$, and thus z is a lower solution problem (2.30)–(2.31). Therefore

$$z(x) \leq w(x) \quad \text{for } 0 < x \leq 1 \quad \text{and} \quad 0 < z'(0) \leq w'(0). \quad (2.42)$$

Now we choose:

- (a) g , such that $0 < g(s) < f(s)$ and $I_\infty - \epsilon \leq G(0) = \int_0^\infty g(s) ds < I_\infty$,
- (b) M such that $[G(0) - G(M)] > I_\infty - 2\epsilon$ for $\epsilon > 0$,
- (c) μ to satisfy (2.41).

Note that since $G'(z) = -g(z) < 0$ and we have $G(0) \leq I_\infty$ and then by virtue of (2.34), (2.38) and (2.42) we obtain

$$2I_\infty > \frac{(w'(0))^2}{\mu} \geq \frac{(z'(0))^2}{\mu} = 2[G(0) - G(M)] > 2(I_\infty - 2\epsilon), \quad \text{for any } \epsilon > 0,$$

which implies the desired result. $\square$

Also, by adapting some ideas from [9], we can prove following:

Lemma 2.1.5 *If w is the solution to (2.30)–(2.31) and $\int_0^\infty f(s) ds < \infty$ then $w(x; \mu) \rightarrow \infty$ as $\mu \rightarrow \infty$ for every x in $(0, 1]$.*

Proof We first prove that $\Phi(\mu) = \int_0^1 f(w(x; \mu)) dx \rightarrow 0$ as $\mu \rightarrow \infty$. To this end we construct a lower solution of (2.30)–(2.31) of the form $z = \beta \phi_1$ for some $\beta > 0$ where ϕ_1 is the principal eigenfunction of

$$-\phi'' + \phi' = \lambda\phi, \quad 0 < x < 1, \quad (2.43)$$

$$\phi(0) = \phi'(1) = 0. \quad (2.44)$$

It is known that $\lambda_1 > 0$ and $\phi_1 > 0$, also we normalize ϕ_1 by taking $\|\phi_1\|_\infty = 1$. Now on choosing β to satisfy $\frac{\lambda_1\beta}{f(\beta)} \leq \mu$, $\beta\phi_1$ becomes a lower solution of (2.30)–(2.31).

Thus, it is sufficient to choose $\frac{\beta}{f(\beta)} = \frac{\mu}{\lambda_1}$.

This choice of β is unique for each $\mu > 0$. Indeed $\mathcal{L}(\beta) := \frac{\lambda_1\beta}{f(\beta)}$ is one to one since $\mathcal{L}'(\beta) > 0$ and maps $\mathbb{R}_+$ onto $\mathbb{R}_+$ since $\mathcal{L}((0, \infty)) = (0, \infty)$. Finally $\mathcal{L}$ is a diffeomorphism, hence to each μ corresponds a unique $\beta(\mu)\phi_1$ which is a lower solution to (2.30)–(2.31).

Therefore we obtain that

$$\Phi(\mu) = \int_0^1 f(w(x; \mu)) dx \leq \int_0^1 f(\beta(\mu)\phi_1(x)) dx \rightarrow 0 \quad \text{as } \mu \rightarrow \infty.$$

The last limit implies that $w(x; \mu) \rightarrow \infty$ as $\mu \rightarrow \infty$ for $0 < x \leq 1$, otherwise we could find an x_0 so that $w(x; \mu) < \infty$ in $(0, x_0)$, but this would imply that $\lim_{\mu \rightarrow \infty} \Phi(\mu) > 0$, contradicting that $\Phi(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$. $\square$

An immediate consequence of Lemma 2.1.5 is that $M = w(1) \rightarrow \infty$ as $\mu \rightarrow \infty$. Furthermore, by using maximum principle to problem (2.30)–(2.31) we obtain the response local diagram, see Fig. 2.1d.

Now, multiplying (2.30) by $w' - w$, we obtain

$$\frac{(w'(0))^2}{\mu} = 2 \left[\int_0^M f(s) ds - \int_0^1 f(w)w dx \right] + \frac{M^2}{\mu}. \quad (2.45)$$

In addition we have

$$\int_0^1 f(w)w dx = f(w(\xi; \mu))w(\xi; \mu), \quad \xi \in (0, 1), \quad (2.46)$$

and hence we derive

$$\int_0^1 f(w)w dx \rightarrow 0 \quad \text{as } \mu \rightarrow \infty, \quad (2.47)$$

by virtue of Lemma 2.1.5, taking also into account that f is decreasing.

Also by virtue of (2.45) we obtain, by taking the limit as $\mu \rightarrow \infty$,

$$\frac{M^2}{\mu} \rightarrow 0. \quad (2.48)$$

Henceforth, for convenience we normalize the integral

$$\int_0^\infty f(s) ds = I_\infty = 1. \quad (2.49)$$

Now we have the following:

Proposition 2.1.6 *If (2.49) holds then $\lambda(M) \rightarrow 2$ as $M \rightarrow \infty$ (or equivalently as $\mu \rightarrow \infty$).*

The proof is an immediate consequence of (2.33), (2.48) and Lemma 2.1.4.

We now consider the complementary case where

$$\int_0^\infty f(s) ds = \infty. \quad (2.50)$$

Proposition 2.1.7 *Let f satisfy (2.50) and w be the solution of (2.30)–(2.31). Then*

$$\frac{(w'(0))^2}{\mu} \rightarrow \infty \text{ as } \mu \rightarrow \infty \text{ and } \lambda(M) \rightarrow \infty \text{ as } M \rightarrow \infty.$$

Proof Let z satisfy

$$z''(x) + \mu g(z(x)) = 0, \quad 0 < x < 1 - \delta, \quad (2.51)$$

$$z(x) = M, \quad z'(x) = 0, \quad 1 - \delta \leq x \leq 1, \quad z(0) = 0. \quad (2.52)$$

It is easily proved that z is a lower solution of (2.30)–(2.31) provided that

- (a) $0 < g(s) < f(s)$, $g'(s) < 0$ and $\int_0^\infty g(s) ds = \infty$, for instance g can be taken as $g(s) = \gamma f(s)$, $0 < \gamma < 1$,
 (b) $\mu \geq \mu_0 = \sup_{z \in (0, M)} \{ [2 \int_z^M g(s) ds] / [f(z) - g(z)]^2 \}$,

thus we obtain:

$$\frac{(w'(0))^2}{\mu} \geq \frac{(z'(0))^2}{\mu} = 2 \int_0^M g(s) ds \rightarrow \infty,$$

as $\mu \rightarrow \infty$.

Hence

$$\lambda(M) = \frac{(w'(0) + M)^2}{\mu} \geq \frac{(w'(0))^2}{\mu} \rightarrow \infty \text{ as } M \rightarrow \infty.$$

□

From the above analysis we can obtain the main possible response diagrams, see Fig. 2.1. It is possible, see Fig. 2.1b, to have more than one turning points. This can occur even in the cases of Fig. 2.1a, c, see also [34, 42].

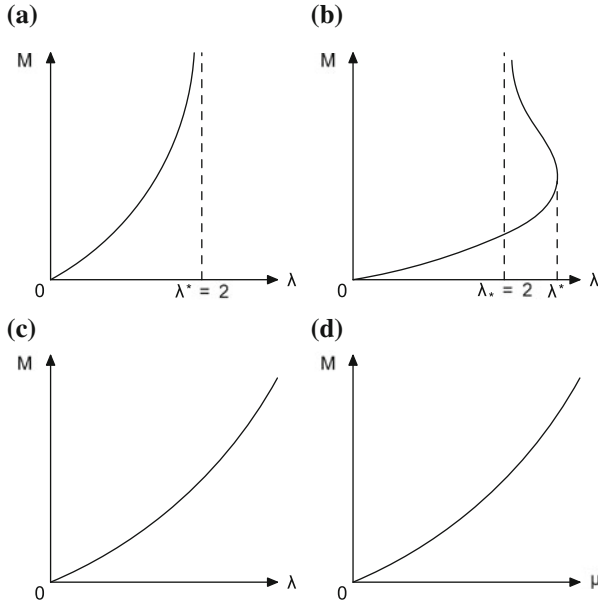


Fig. 2.1 λ , μ -diagrams represent the non-local, local response diagrams respectively of problem (2.30)–(2.31). (i) (a), (b) for the case $\int_0^\infty f(s) ds = 1$ and (ii) (c) for the case $\int_0^\infty f(s) ds = \infty$ where $M = w(1)$

Remark 2.1 Solutions of the steady-state problem (2.30)–(2.31) for small values of the parameter λ can be constructed via monotone iteration techniques for both decreasing and increasing nonlinear functions f , see [1].

2.1.3.2 Hyperbolic Case

The steady-state problem to (2.18)–(2.20), which actually coincides with the one of (2.10)–(2.12), is

$$w' = \mu f(w), \quad 0 < x < 1, \quad w(0) = 0, \quad (2.53)$$

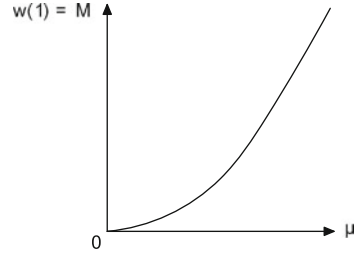
where again $\mu = \frac{\lambda}{\left(\int_0^1 f(w) dx\right)^2}$ is referred to as the local parameter while λ as the non-local one.

Equation (2.53) can be written

$$\frac{dw}{f(w)} = \mu dx, \quad 0 < x < 1, \quad (2.54)$$

from which by integration over $(0, 1)$ we obtain

Fig. 2.2 The local response diagram to (2.53), where $M(\mu) = w(1; \mu) = w(1)$



$$\mu = \mu(M) = \int_0^M \frac{ds}{f(s)}, \quad M = \|w\|_\infty = w(1). \quad (2.55)$$

The latter implies that $\mu'(M) = \frac{1}{f(M)} > 0$ leading to the response diagram appearing in Fig. 2.2. Also by integration of (2.53) over $(0, 1)$ we get $\lambda = M^2/\mu$ and so $\lambda = \lambda(M) = M^2 / \int_0^M \frac{ds}{f(s)}$. Since $\lim_{M \rightarrow \infty} \lambda(M) = 2 \lim_{M \rightarrow \infty} Mf(M)$, we distinguish two cases:

- (i) $\int_0^\infty f(s)ds < \infty$, then $Mf(M) \leq 2 \int_{M/2}^M f(s)ds \rightarrow 0$ as $M \rightarrow \infty$, and so there exists a λ^* such that for $0 < \lambda < \lambda^*$ problem (2.53) has at least two steady-state solutions while for $\lambda > \lambda^*$ there is no steady-state solution, see Fig. 2.3c.
- (ii) $\int_0^\infty f(s)ds = \infty$, if $\lim_{M \rightarrow \infty} Mf(M)$ exists then two things might happen. Either $Mf(M) \rightarrow c$, $0 < c < \infty$ as $M \rightarrow \infty$ and so the spectrum to (2.53) is bounded, see Fig. 2.3b, or $Mf(M) \rightarrow \infty$ as $M \rightarrow \infty$ and (2.53) has at least one steady state for any $\lambda > 0$ ($\lambda^* = \infty$), see Fig. 2.3a).

Moreover, if $\mu(M) = \int_0^M \frac{ds}{f(s)} > M/2f(M)$ for $M > 0$, then

$$\lambda'(M) = \frac{M}{\mu(M)} \left[2\mu(M) - \frac{M}{f(M)} \right] > 0,$$

and thus there is a unique steady state to each $0 < \lambda < \lambda^*$. From the above analysis we get the possible non-local response diagrams of Fig. 2.3. Each diagram may contain more turning points than shown (so that for some λ there are more solutions).

2.1.4 Stability

2.1.4.1 Parabolic Case

We now study the stability of the steady solutions, for $0 < \lambda \leq \lambda^* < \infty$ if $\int_0^\infty f(s)ds < \infty$ or for any $\lambda > 0$ if $\int_0^\infty f(s)ds = \infty$, by using comparison methods, see also [22].

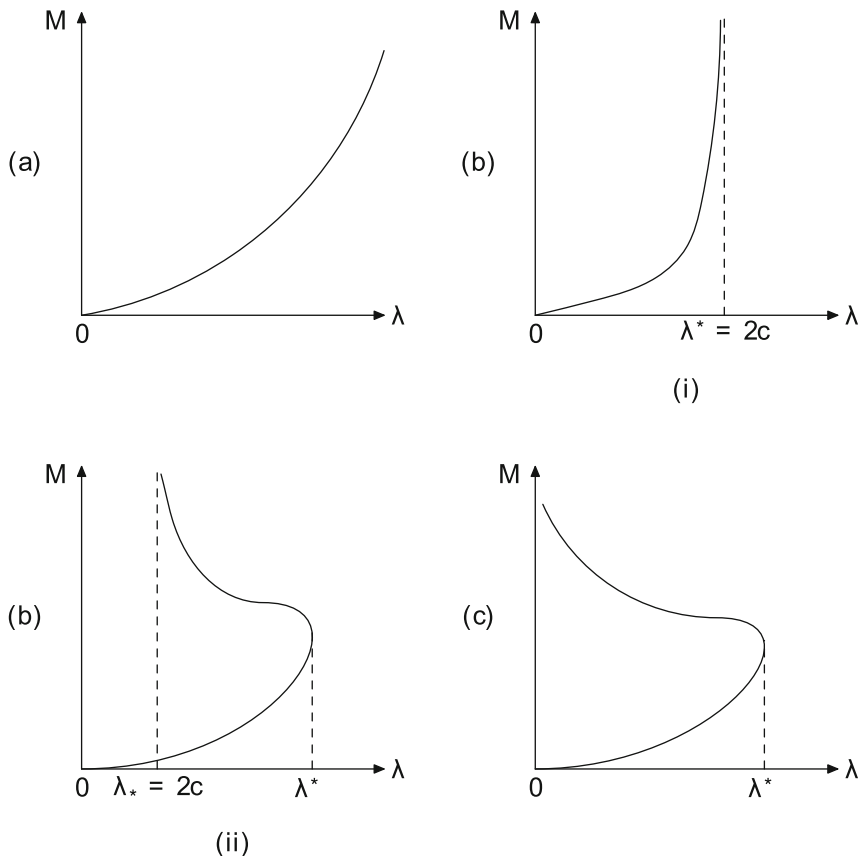


Fig. 2.3 Possible non-local response diagrams to (2.53). (a) $Mf(M) \rightarrow \infty$ as $M \rightarrow \infty$, (b) $Mf(M) \rightarrow c$, $0 < c < \infty$ as $M \rightarrow \infty$, (c) $Mf(M) \rightarrow 0$ as $M \rightarrow \infty$, where $M = w(1)$

In particular, we construct an upper solution $U(x, t) = w(x; \bar{\mu}(t))$ decreasing in time, and a lower solution $z(x, t) = w(x; \underline{\mu}(t))$ increasing in time, of problem (2.6)–(2.8). Both $\bar{\mu}$ and $\underline{\mu}$ satisfy the following initial value problem:

$$\dot{v} = h(v) \equiv \inf_{x \in (0,1)} \left\{ \frac{f(w)}{w_v} \right\} \frac{\lambda - \lambda(v)}{\left(\int_0^1 f(w) dx \right)^2}, \quad v(0) = v_0, \quad (2.56)$$

where $w = w(x; v(t))$ is the solution of (2.30)–(2.31) with $\mu = v(t)$. Furthermore, $w = w(x; \mu)$ is the unique solution of problem (2.30)–(2.31), where $(v, M) = (v(t), M(t))$ is a point on the graph indicated in Fig. 2.1d. The w_v –problem is derived by differentiating (2.30)–(2.31) with respect to μ and substituting v for μ .

The existence of $w = w(x; v(t))$ is a consequence of the fact that to each $t > 0$ there corresponds μ so that $\mu = v(t)$ is continuous and maps $\mathbb{R}_+$ onto $[v_0, \infty)$. Moreover, the continuity of the function $h(v)$ and the form of diagrams in Fig. 2.1 imply the existence and uniqueness of the solution $v = v(t)$ of (2.56), see also [11].

The function $v = v(t)$ is global-in-time since v is bounded: $\underline{\mu}(0) \leq v(t) \leq \bar{\mu}(0)$. We have also that v is strictly monotone. More precisely, $\bar{\mu}(t)$ and $\underline{\mu}(t)$ are decreasing and increasing, respectively (see below). Moreover, it holds that $w_v > 0$ by the maximum principle, recalling $f'(s) < 0$. Also w_v is finite; indeed for a fixed v , any sufficiently large constant is an upper solution of the w_v -problem. Hence $\inf_{x \in (0,1)} \{f(w)/w_v\}$ is always positive since also $f(s)$ is bounded away from zero for $s \leq \sup w < \infty$.

To construct upper and lower solutions, we choose v_0 satisfying $w(x; v(0)) \geq u_0(x)$ and $w(x; v(0)) \leq u_0(x)$, respectively. This is possible since $w_v > 0$. More precisely, this can be done by $u_0 \geq 0$ and $u_0 \in C^1([0, 1])$. Otherwise we choose $v(\epsilon)$ so that $w(\cdot; v(\epsilon)) \geq u(\cdot, \epsilon)$ for small $\epsilon > 0$.

If the response diagram is as in Fig. 2.1a, c, or b for $\lambda < \lambda_*$, a unique steady-state solution corresponds to each λ . Then we have a unique $M(\mu)$ and take either $v_0 > \mu$ or $v_0 < \mu$. In the complementary case, when two steady-states correspond to each λ denoted by $M_1 = M(\mu_1)$ and $M_2 = M(\mu_2)$, for example, we take $\mu(\lambda_*) = \mu_* < v_0 < \mu_1$, or $\mu_1 < v_0 < \mu_2$, or $v_0 > \mu_2$ for $\lambda_* < \lambda < \lambda^*$ as in Fig. 2.1b. The case with more than two turning points is similar, where each λ corresponds to more than two M 's.

The above analysis implies that $\lambda - \lambda(v) < 0$, $v = \bar{\mu}(t)$ and $\lambda - \lambda(v) > 0$, $v = \underline{\mu}(t)$ for the upper and lower solutions, respectively. These inequalities extend to a proper region of v , that is for the case of unique solution, $\lambda - \lambda(v) < 0$ and $\lambda - \lambda(v) > 0$ if $v > \mu$ and $v < \mu$, respectively, and for the case of two solutions, $\lambda - \lambda(v) < 0$ and $\lambda - \lambda(v) > 0$ if $\mu_1 < v < \mu_2$, $\mu_* < v < \mu_1$ and $v > \mu_2$, respectively.

These properties imply $U(x, t) = w(x; \bar{\mu}(t))$ and $z(x, t) = w(x; \underline{\mu}(t))$ are upper and lower solutions, respectively, and hence $\lambda - \lambda(\bar{\mu}) < 0 < \lambda^* - \lambda(\bar{\mu})$ and $\lambda - \lambda(\underline{\mu}) > 0 > \lambda_* - \lambda(\underline{\mu})$, respectively. Then it holds that $U_t = w_v \dot{v} < 0$ and $z_t = w_v \dot{v} > 0$, since $w_v > 0$, $\dot{\bar{\mu}}(t) < 0$, and $\dot{\underline{\mu}}(t) > 0$.

Returning to the case of a unique steady-state $w(x) \equiv w_1(x)$, the above construction implies that $z(x, t) \leq u(x, t) \leq U(x, t)$. Then $u(x, t)$ is a global-in-time solution and $z(x, t) \uparrow w(x)$ and $U(x, t) \downarrow w(x)$ as $t \rightarrow \infty$ uniformly in x , since $\bar{\mu}(t) \rightarrow \bar{l} = \mu+$, $\underline{\mu}(t) \rightarrow \underline{l} = \mu-$ as $t \rightarrow \infty$, which means that $\bar{\mu}(t) \rightarrow \mu$, $(\underline{\mu}(t) \rightarrow \mu)$ and $\bar{\mu}(t) > \mu$ ($\underline{\mu}(t) < \mu$). The latter is true i.e. $\bar{l} = \underline{l} = \mu$, since assuming that $\bar{\mu}(t) \rightarrow \hat{\mu}+ = \hat{\mu} > \mu$, as $t \rightarrow \infty$ problem (2.56)

would imply $\int_{\bar{\mu}(0)}^{\bar{\mu}(t)} \frac{ds}{h(s)} = t$, and by taking the limit as $t \rightarrow \infty$ we should have

$\int_{\bar{\mu}(0)}^{\hat{\mu}} \frac{ds}{h(s)} = \infty$. But this can occur if and only if $h(\hat{\mu}) = 0$ or equivalently $\lambda(\hat{\mu}) = \lambda$ which contradicts the uniqueness of the solution of the non-local steady problem, (otherwise we would have $\lambda(\hat{\mu}) = \lambda(\mu) = \lambda$ with $\hat{\mu} > \mu$, which is a contradiction,

see Fig. 2.1c, d). The same argument applies to all other cases with more steady-states, giving always an extra steady solution. Similarly it is shown that $\underline{\mu}(t) \rightarrow \mu-$, as $t \rightarrow \infty$.

Consequently we deduce that w is a globally asymptotically stable solution. Here, if

$$\int_0^\infty f(s)ds = \infty,$$

then we can also prove that $u(x, t)$ is a global in time solution. Indeed we have

$$\dot{M}(t) \leq \frac{\lambda f(M)}{\left(\int_0^1 f(u)dx\right)^2} < \frac{\lambda}{f(M)} \quad \text{or} \quad \int_{M(0)}^{M(t)} f(s) ds < \lambda t,$$

which implies that $u(x, t)$ is global in time.

In case that more than one stationary solutions exist we can apply a similar analysis to show the stability alternates starting from a stable minimal stationary solution, proceeding to greater one which is unstable and so on, see [22].

Now we examine the long time behavior of u when f is an increasing function inspired by [14, 22]. For simplicity we consider Dirichlet boundary conditions,

$$u(0, t) = u(1, t) = 0,$$

and expand $u(x, t)$ as

$$u(x, t) = \sum_{n=1}^{\infty} E_n(t) y_n(x), \quad (2.57)$$

where $y_n(x) = e^{x/2} \sin(n\pi x)$, $n = 1, 2, \dots$, are the eigenfunctions of the operator $-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}$, under Dirichlet boundary conditions.

Substituting (2.57) into Eq. (2.6) and taking into account the monotonicity of f , see also [22], we obtain the following estimate:

$$|E_n(t)| \leq |E_n(0)|e^{-\lambda_n t} + \frac{2\lambda}{n^2\pi^2 f(0)}[1 - e^{-\lambda_n t}],$$

hence

$$\limsup_{t \rightarrow \infty} |E_n(t)| \leq \frac{2\lambda}{n^2\pi^2 f(0)},$$

where $\lambda_n = \frac{1}{4} + n^2\pi^2$.

By virtue of (2.57) we have

$$\limsup_{t \rightarrow \infty} u(x, t) \leq e^{1/2} \limsup_{t \rightarrow \infty} \sum_{n=1}^{\infty} |E_n(t)| \leq \frac{2\lambda e^{1/2}}{f(0)} \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2} = \frac{\lambda e^{1/2}}{3f(0)},$$

since the series of $|E_n(t)|$ converges uniformly and hence in that case a global-in-time solution exists which is bounded for each time $t > 0$.

For the Neumann boundary condition

$$u_x(0, t) = u_x(1, t) = 0, \quad (2.58)$$

a similar analysis yields that the solution $u(x, t)$ of problem (2.6), (2.58) and (2.8) is global-in-time but not necessarily bounded, see [22].

2.1.4.2 Hyperbolic Case

Now we investigate the stability of steady states of (2.18)–(2.20), using the comparison result given in Sect. 2.1.2.

For this reason we consider comparison functions of the form $v(x, t) = w(x; \mu(t))$. Using $v_x = \mu f(v)$ and $\mu x = \int_0^w \frac{ds}{f(s)}$ obtained by (2.53) and (2.54), respectively, we get $v_t = \dot{\mu}(t)x f(v)$ with $v(0, t) = 0$. Thus by following the calculations in [22] we derive

$$\begin{aligned} \mathcal{H}(v) &:= \rho(v)v_t + \left(1 - \int_0^x \rho'(v)v_t dy\right)v_x - \frac{\lambda f(v)}{\left(\int_0^1 f(v)dx\right)^2} \\ &= f(v) \left[\dot{\mu}(t)x\rho(v) + \mu(t) - \dot{\mu}(t) \int_0^v y\rho'(s)ds - \lambda / \left(\int_0^1 f(v)dx\right)^2 \right]. \end{aligned}$$

However, since $\int_0^1 f(v)dx = \frac{1}{\mu(t)} \int_0^1 v_x dx = \frac{M(t)}{\mu(t)}$ for $M(t) = \sup_x v(x, t) = v(1, t)$, choosing

$$\dot{\mu}(t) = \dot{\mu} = h(\mu(t)) \equiv \frac{1}{\rho(0)} \left(\frac{\lambda \mu^2}{M^2(\mu)} - \mu \right), \quad \text{for } t > 0, \quad (2.59)$$

we obtain

$$\mathcal{H}(v) = f(v) \left[\frac{1}{\rho(0)} \left(\frac{\lambda \mu^2}{M^2(\mu)} - \mu \right) \left(\rho(v)x - \int_0^v y\rho'(s)ds \right) + \mu - \frac{\lambda \mu^2}{M^2(\mu)} \right].$$

Let $\lambda > \frac{M^2(\mu)}{\mu}$. Using the fact that $\rho(s)$ is a positive decreasing function, we get $\mathcal{H}(v) \leq 0$ for $0 < x < 1$. Thus in this case $v(x, t)$ is an increasing-in-time ($v_t = \dot{\mu}(t)xf(v) > 0$) lower solution to (2.18)–(2.20), provided that $v(x, 0) = w(x; \mu(0)) \leq u_0(x)$. Also for $\lambda < \frac{M^2(\mu)}{\mu}$ and $v(x, 0) = w(x; \mu(0)) \geq u_0(x)$ we obtain that $v(x, t)$ is a decreasing-in-time upper solution to (2.18)–(2.20).

We start with the case that a unique steady state w exists. Then to each $0 < \lambda < \lambda^*$ ($\lambda^* = \infty$ when $\int_0^\infty f(s)ds = \infty$) there exists $\mu > 0$ such that $\lambda = \lambda(\mu) := \mu \left(\int_0^1 f(w)dx \right)^2$ and the function $\lambda(\mu)$ is increasing. For the case $u_0(x) \leq w(x)$ we can choose $0 < \underline{\mu}(t) < \mu$, so $\lambda = \lambda(\mu) > \lambda(\underline{\mu}(t)) = \frac{M^2(t)}{\underline{\mu}(t)}$, satisfying the Eq. (2.59). Then $\underline{\mu}(t)$ satisfies the transcendental equation

$$\int_{\underline{\mu}(0)}^{\underline{\mu}(t)} \frac{ds}{h(s)} = t, \quad t > 0, \quad \text{where } h(s) = \frac{1}{\rho(0)} \left(\frac{\lambda s^2}{M^2(s)} - s \right). \quad (2.60)$$

Equation (2.60) has a unique solution in $[\underline{\mu}(0), \mu]$, for any $\underline{\mu}(0) \geq 0$. Hence in this case $\mathcal{G} : [\underline{\mu}(0), \mu] \rightarrow [0, \infty)$ with $\mathcal{G}(\xi) := \int_{\underline{\mu}(0)}^{\xi} \frac{ds}{h(s)}$ is a C^1 -diffeomorphism, [22]. Thus (2.59) has a unique solution $\underline{\mu}(t)$ and since $w_\mu = xf(w) \geq 0$ we can choose $\underline{\mu}(0) \geq 0$ such that $w(x; \underline{\mu}(0)) \leq u_0(x)$. Hence $v(x, t) = w(x; \underline{\mu}(t))$ is an increasing-in-time lower solution to (2.18)–(2.20), so $v(x, t) \leq u(x, t) \leq w(x)$ for $x \in [0, 1]$ and $t > 0$. Moreover $\underline{\mu}(t) \rightarrow \mu^-$ as $t \rightarrow \infty$, because, otherwise there would be another steady state. Therefore, $v(\cdot, t) \rightarrow w(\cdot)$ as $t \rightarrow \infty$ uniformly in x resulting in $u(\cdot, t) \rightarrow w(\cdot)$ as $t \rightarrow \infty$ uniformly in x .

When $u_0(x) \geq w(x)$, it is possible to choose $\bar{\mu}(t) > \mu$ (so $\lambda < \lambda(\bar{\mu}(t))$) to satisfy (2.59) and construct a decreasing-in-time upper solution $z(x, t)$ to (2.18)–(2.20), provided that $z(x, 0) = w(x; \bar{\mu}(0)) \geq u_0(x)$. The latter is achieved since $u_0(x), u'_0(x)$ are bounded and $w_{\bar{\mu}} > 0$. Thus we obtain $w(x) \leq u(x, t) \leq z(x, t)$ and finally $u(\cdot, t) \rightarrow w(\cdot)$ as $t \rightarrow \infty$ uniformly in x , with $\bar{\mu}(t) \rightarrow \mu^+$ as $t \rightarrow \infty$. Hence the unique steady state $w(x)$ is globally asymptotically stable and $u(x, t)$ is a global-in-time bounded solution.

We turn to the case where (2.53) has two steady states $w_1 = w(x; \mu_1)$ and $w_2 = w(x; \mu_2)$. Then for each $\lambda_* < \lambda < \lambda^*$ there exist μ_1 and μ_2 such that $\lambda = \lambda(\mu_1) = \lambda(\mu_2)$ and function $\lambda(\mu)$ is increasing for $0 < \mu < \mu^*$ and decreasing for $\mu > \mu^*$ with μ^* satisfying $\lambda'(\mu^*) = 0$. For $0 < u_0(x) < w_1(x)$, choosing $0 < \underline{\mu}(t) < \mu_1 < \mu^*$ such that (2.59), we get as above a lower solution $v(x, t) = w(x; \underline{\mu}(t))$ with $v(\cdot, t) \rightarrow w_1(\cdot)$ as $t \rightarrow \infty$ uniformly in x . Whereas for $w_1(x) < u_0(x) < w_2(x)$, on choosing $\mu_1 < \bar{\mu}(t) < \mu^*$, we construct an upper solution $z(x, t) = w(x; \bar{\mu}(t))$ such that $z(\cdot, t) \rightarrow w_1(\cdot)$ as $t \rightarrow \infty$ uniformly in x . Hence for $\lambda_* < \lambda < \lambda^*$ the minimal steady state w_1 is asymptotically stable with a region of attraction $[0, w_2]$, while for $0 < \lambda < \lambda_*$ w_1 is globally asymptotically stable. This implies that $u(x, t)$ is a global-in-time bounded solution.

If we consider $u_0(x) > w_2(x)$ and choose $\underline{\mu}(t) > \mu_2$ satisfying (2.59) then an unbounded lower solution $v(x, t) = w(x; \underline{\mu}(t))$ can be constructed. More precisely, $\underline{\mu}(t) \rightarrow \infty$ as $t \rightarrow T^* \leq \infty$. Otherwise there would be a third steady state which is a contradiction. Hence $\|u(\cdot, t)\|_\infty \rightarrow \infty$ as $t \rightarrow t^* \leq T^* \leq \infty$, which means that $u(x, t)$ is unbounded. The latter implies that the maximal steady state w_2 is unstable.

Moreover, $w^*(x) = w(x; \lambda^*)$ is unstable. In fact, w^* is stable from below, $0 < u_0(x) < w^*(x)$, and unstable from above, $u_0(x) > w^*(x)$. If for each $\lambda_* < \lambda < \lambda^*$ more than two steady states exist, then the above arguments imply that the minimal steady state is stable, that the greater one is unstable, and so on.

We note that problem (2.18)–(2.20) has unbounded solutions for $\lambda > \lambda^*$. Indeed, in this case there holds $\lambda > \lambda(\mu) = \frac{M^2}{\mu}$ for $\mu > 0$. Hence we can construct a lower solution of the form $w(x; \mu(t))$, which actually becomes unbounded since $\mu(t) \rightarrow \infty$ as $t \rightarrow T^* \leq \infty$ and due to the fact that for $\lambda > \lambda^*$ there is no steady state. Consequently $u(x, t)$ becomes unbounded at some $t^* \leq T^* \leq \infty$, i.e. $\|u(\cdot, t)\|_\infty \rightarrow \infty$ as $t \rightarrow t^* -$.

2.1.5 Finite-Time Blow-Up

The phenomenon of finite-time blow-up, apart from its mathematical interest, is very significant for the process of sterilization of food. Indeed, in this context finite-time blow-up is closely associated with thermal runaway which finally leads to food burning and it should be avoided, since, otherwise the whole process of the food sterilization is inefficient. Therefore, it is important from the application point of view to investigate under which circumstances finite-time blow-up occurs so we can optimize the whole process.

In the following we provide a thorough study of the phenomenon of finite-time blow-up for the solutions of both (2.6)–(2.8) and (2.18)–(2.20). Here by finite-time blow-up we mean that there exists a point $(x^*, t^*) \in (0, 1)$ and sequence $(x_n, t_n) \rightarrow (x^*, t^*)$ as $n \rightarrow +\infty$ such that $u(x_n, t_n) \rightarrow +\infty$ as $n \rightarrow +\infty$.

2.1.5.1 Parabolic Case

So far we have shown that when $\lambda \in (\lambda_*, \lambda^*]$ and $u_0(x) > w_2(x)$ or when $\lambda > \lambda^*$, the solution u of (2.6)–(2.8) is unbounded.

In the following, we actually prove that u blows up in finite time in these two cases. We first consider the case

$$\lambda > \lambda^* \quad \text{and} \quad \int_0^\infty f(s) ds < \infty, \quad (2.61)$$

and we construct a lower solution $z = z(x, t)$ to (2.6)–(2.8) of the form:

$$z_{xx} + \mu(t)f(z) = 0, \quad 0 < x < \delta(t), \quad t > 0, \quad (2.62)$$

$$z(0, t) = 0, \quad t > 0, \quad (2.63)$$

$$z(x, t) = M(t) = \sup_{x \in (0, \delta)} z(x, t), \quad z_x(x, t) = 0, \quad \delta(t) \leq x \leq 1, \quad t > 0. \quad (2.64)$$

Multiplying (2.62) by z_x and integrating we get

$$z_x(x, t) = \sqrt{2\mu} [F(z) - F(M)]^{1/2}, \quad (2.65)$$

where $F(\sigma) = \int_{\sigma}^{\infty} f(s) ds$, so $F'(\sigma) = -f(\sigma) < 0$.

The relation (2.65) implies

$$\int_0^M \frac{ds}{[F(s) - F(M)]^{1/2}} = \delta \sqrt{2\mu}. \quad (2.66)$$

Also

$$\begin{aligned} \int_0^1 f(z) dx &= - \int_0^{\delta} \frac{z_{xx}}{\mu} dx + (1 - \delta)f(M) = \left(\frac{2}{\mu} \int_0^M f(s) ds \right)^{1/2} + (1 - \delta)f(M) \\ &= \sqrt{\frac{2}{\mu}} + f(M) + o(1), \end{aligned} \quad (2.67)$$

as $\delta \rightarrow 0$ and $M \rightarrow +\infty$.

Let

$$\delta(M) = \frac{a}{2} f(M) \int_0^M [F(s) - F(M)]^{-1/2} ds, \quad (2.68)$$

where a is a suitable constant with $a > 1 / \left[\left(\frac{\lambda}{2} \right)^{1/2} - 1 \right]$. It is easily seen that such a choice of a entails $\Lambda = \frac{1}{2} \left[\frac{\lambda}{(1+a)^2} - \frac{2}{a^2} \right] > 0$. Then it holds that

$$\sqrt{\frac{2}{\mu}} = a f(M), \quad (2.69)$$

and thus, by using (2.29), (2.66) and (2.69), we obtain

$$\delta = \frac{1}{\sqrt{2\mu}} \int_0^M [F(s) - F(M)]^{-1/2} ds \leq a(M f(M))^{1/2},$$

which implies $\delta \rightarrow 0$ as $M \rightarrow \infty$ (or equivalently, as $\mu \rightarrow \infty$.)

Therefore we deduce that

$$\int_0^1 f(z) dx = (1+a)f(M) + o(1) \quad \text{as} \quad M \rightarrow +\infty. \quad (2.70)$$

Now we can prove the following finite-time blow-up result.

Theorem 2.1.8 *Under the condition (2.61) the solution u of problem (2.6)–(2.8) blows up globally in finite time.*

Proof As the first step we show that z defined above is a blowing up lower solution of problem (2.6)–(2.8). Indeed, for $\delta \leq x \leq 1$ we have

$$\begin{aligned} \mathcal{F}(z) &:= z_t - z_{xx} + z_x - \frac{\lambda f(z)}{\left(\int_0^1 f(z) dx\right)^2} = \dot{M} - \frac{\lambda f(M)}{\left(\int_0^1 f(z) dx\right)^2} \\ &= \dot{M} - \frac{\lambda}{(1+a)^2 f(M)} + o(1) < \dot{M} - \frac{\Lambda}{f(M)} + o(1), \quad \text{as} \quad M \rightarrow +\infty. \end{aligned} \quad (2.71)$$

Hence $\mathcal{F}(z) \leq 0$ provided that $0 < \dot{M} \leq \Lambda/f(M)$ and for M sufficiently large. Notably it can be proven that $M(t)$ is differentiable almost everywhere following the approach in [15, 21].

Now we integrate relation (2.65) with respect to $x \in (0, \delta)$ and then differentiate with respect to t , to obtain

$$\begin{aligned} z_t &= \frac{x\dot{\mu}}{\sqrt{2\mu}}[F(z) - F(M)]^{1/2} + \frac{1}{2}\dot{M}f(M)[F(z) - F(M)]^{1/2} \int_0^z [F(s) - F(M)]^{-3/2} ds \\ &:= A + B. \end{aligned} \quad (2.72)$$

Owing to (2.66) and (2.69) we have

$$\begin{aligned} A &= -\frac{f'(M)\dot{M}}{f(M)}[F(z) - F(M)]^{1/2} \int_0^z [F(s) - F(M)]^{-1/2} ds \\ &\leq -\frac{f'(M)\dot{M}f^{1/2}(z)M}{2f^{3/2}(M)} \\ &\leq \frac{\Lambda f(z)}{2f^2(M)}, \end{aligned} \quad (2.73)$$

since $s(M^{1/2} - s) \leq \frac{1}{4}M$, provided that $0 \leq \dot{M} \leq -\Lambda/Mf'(M)$.

Also we have

$$B \leq \frac{1}{2}\dot{M}f(M)f^{1/2}(z)(M-z)^{1/2}f^{-3/2}(M) \int_0^z (M-s)^{-3/2} ds$$

$$\leq \frac{\dot{M} f^{1/2}(z)}{f^{1/2}(M)} \leq \frac{\Lambda f(z)}{2f^2(M)}, \quad (2.74)$$

as long as

$$0 \leq \dot{M} \leq \frac{\Lambda}{2f(M)}.$$

Finally, if $0 \leq \dot{M}(t) \leq \min \left\{ \frac{\Lambda}{2f(M)}, \frac{-\Lambda}{M f'(M)} \right\}$ in $x \in (0, \delta]$, we obtain

$$\begin{aligned} \mathcal{F}(z) &\leq \frac{\Lambda f(z)}{f^2(M)} + \mu f(z) + \frac{2M^{1/2} f(z)}{af^{3/2}(M)} - \frac{\lambda f(z)}{(1+a)^2 f^2(M)} + o(1) \\ &\leq \frac{f(z)}{f^2(M)} \left[\Lambda + \frac{2}{a^2} + \Lambda - \frac{\lambda}{(1+a)^2} \right] + o(1) = \frac{f(z)}{f^2(M)} [2\Lambda - 2\Lambda] + o(1) = o(1), \end{aligned} \quad (2.75)$$

as $M \rightarrow +\infty$, taking also into account that $\frac{2(f(M)M)^{1/2}}{a} \leq \Lambda$ for M sufficiently large.

Therefore, by choosing

$$\dot{M} = \min \left\{ \frac{\Lambda}{2f(M)}, -\frac{\Lambda}{M f'(M)} \right\}, \quad (2.76)$$

there holds that $\mathcal{F}(z) \leq 0$ for $x \in (0, \delta) \cup (\delta, 1)$. Since z, z_x are continuous and $z(0, t) = z_x(1, t) = 0$, the function z is a lower solution of (2.6)–(2.8) for M large enough, after some time at which u is sufficiently large.

We shall show now that z blows up in finite time. Indeed relation (2.76) implies

$$\Lambda \frac{dt}{dM} = \max\{2f(M), -M f'(M)\} \leq 2f(M) - M f'(M) \quad \text{or}$$

$$\Lambda t \leq 3 \int_0^M f(s) ds - M f(M) < 3 \int_0^\infty f(s) ds < \infty,$$

since $M f(M) \rightarrow 0$ as $M \rightarrow \infty$. The latter implies that z blows up at $T^* = \frac{3}{\Lambda} \int_0^\infty f(s) ds < \infty$. Hence u must also blow up at some $t^* \leq T^* < \infty$.

Now, we claim that u blows up globally, which means that $u(x, t) \rightarrow \infty$ as $t \rightarrow t^*$ for all x in $(0, 1]$ and $u_x(0, t) \rightarrow \infty$ as $t \rightarrow t^*$. Indeed, noting that

$$M(t) = \sup_{[0,1]} u(\cdot, t) \quad \text{satisfies} \quad \dot{M} \leq \frac{\lambda f(0)}{\left(\int_0^1 f(u) dx \right)^2} = h(t),$$

we have $M(t) - M(0) \leq \int_0^t h(s) ds \rightarrow \infty$ as $t \rightarrow t^*-$, which implies $\int_0^1 f(u) dx \rightarrow 0$ as $t \rightarrow t^*-$. Thus for $\lambda > \lambda^*$ or for $\lambda_* < \lambda \leq \lambda^*$ and $u_0 > w_2$, u blows up globally and $u_x(0, t) \rightarrow \infty$ as $t \rightarrow t^*-$. $\square$

An analogous result to Theorem 2.1.8 could be proved in the case

$$\lambda_* < \lambda \leq \lambda^* \quad \text{and} \quad \int_0^\infty f(s) ds < \infty,$$

by a similar construction of a blowing lower solution of problem (2.6)–(2.8). For more details see [22].

Remark 2.1.9 All the results obtained in Sects. 2.1.2–2.1.5 can be easily derived when the Dirichlet boundary condition $u(0, t) = 0$ is replaced by $u_x(0, t) - a u(0, t) = 0$, $t > 0$ for $a > 0$. Similar results can be derived if Eq. (2.6) is replaced by (2.9).

A nonlinearity is encountered quite often in application is the Heaviside function:

$$H(s) = \begin{cases} 0, & s < 0, \\ 1, & s \geq 0. \end{cases}$$

In fact, it is a good approximation for the food resistivity since in many occasions food materials change their resistivity during the sterilization process [40].

In that case problem (2.6)–(2.8) takes the form, [23],

$$u_t - u_{xx} + u_x = \frac{\lambda H(1 - u)}{\left(\int_0^1 H(1 - u) dx \right)^2}, \quad 0 < x < 1, \quad t > 0, \quad (2.77)$$

$$u(0, t) = u_x(1, t) = 0, \quad t > 0, \quad (2.78)$$

$$u(x, 0) = u_0(x), \quad 0 < x < 1, \quad (2.79)$$

where now $u_0(x)$, $u'_0(x)$ are considered to be bounded with $u_0(x) \geq 0$ in $[0, 1]$ (the last requirement is a consequence of the fact that for any $u_0(x)$ the solution u becomes non-negative throughout $0 < x < 1$ at some time t).

From the mathematical point of view, the Heaviside function is neither Lipschitz nor strictly positive however it is decreasing and so the techniques used in the previous sections can be modified to derive analogous results regarding the long time behavior of u . It should be pointed out that in this case the finite-time blow-up cannot occurs but instead finite-time quenching takes place when $\lambda > \lambda^*$ or when $0 < \lambda < \lambda^*$ and for large enough initial data. For a rigorous investigation of the long time behavior of the solution to (2.77)–(2.79) see [23].

2.1.5.2 Hyperbolic Case

It has been noted in Sect. 2.1.4 that the unbounded solutions to problem (2.18)–(2.20) exist either for $\lambda > \lambda^*$ or for $u_0(x)$ sufficiently large and $\lambda \leq \lambda^*$. The exact behavior of such solutions to (2.18)–(2.20) depends upon the decreasing rate of $f(s)$. More precisely we have, [24],

Theorem 2.1.10 *If $\int_0^\infty f(s)ds < \infty$ and $\rho(s) \geq \gamma > 0$ for $s > 0$ then the unbounded solutions to (2.18)–(2.20) blow up globally in finite time, i.e. $u(x, t) \rightarrow \infty$ as $t \rightarrow t^* < \infty$ for any $x \in (0, 1]$ and $u_x(0, t) \rightarrow \infty$ as $t \rightarrow t^*$.*

Proof As in Sect. 2.1.4 we can construct a lower solution of the form $v(x, t) = w(x; v(t))$ with $v(t)$ satisfying (2.59). Moreover, recalling that $M(\mu)$ is defined implicitly by

$$\mu(M) = \int_0^M \frac{ds}{f(s)},$$

we note that $M(0) = 0$, and thus Hardy's inequality [17] entails

$$\int_0^v \left(\frac{M(\sigma)}{\sigma} \right)^2 d\sigma \leq 4 \int_0^v (M'(\sigma))^2 d\sigma. \quad (2.80)$$

Also, by virtue of $M'(v) = f(M(v))$, then (2.80) implies

$$\int_0^v \left(\frac{M(\sigma)}{\sigma} \right)^2 d\sigma \leq 4 \int_0^v (M'(\sigma))^2 d\sigma = 4 \int_0^{M(v)} f(s)ds < 4 \int_0^\infty f(s)ds < \infty. \quad (2.81)$$

Since $v(t)$ satisfies (2.59), we obtain

$$t = \rho(0) \int_0^{v(t)} \frac{M^2(\sigma)}{\sigma^2 \left(\lambda - \frac{M^2(\sigma)}{\sigma} \right)} d\sigma \quad \text{for any } t > 0. \quad (2.82)$$

Taking into account $\frac{M^2(\mu)}{\mu} = \frac{M^2(\mu)}{\int_0^M \frac{ds}{f(s)}} \rightarrow 0$ as $\mu \rightarrow \infty$ and $\int_0^\infty f(s) < \infty$, we deduce

$$\int_\beta^\infty \frac{M^2(\sigma)}{\sigma^2 \left(\lambda - \frac{M^2(\sigma)}{\sigma} \right)} d\sigma = \frac{1}{\lambda} \int_\beta^\infty \left(\frac{M(\sigma)}{\sigma} \right)^2 d\sigma + o(1) \quad \text{as } \beta \rightarrow +\infty. \quad (2.83)$$

Finally, combining (2.81), (2.82) and (2.83) we derive $v(t) \rightarrow \infty$ as $t \rightarrow T^* -$, where

$$T^* = \rho(0) \int_0^\infty \frac{M^2(\sigma)}{\sigma^2 \left(\lambda - \frac{M^2(\sigma)}{\sigma} \right)} d\sigma < \infty.$$

Hence $u(x, t)$ blows up in finite time, i.e. $\|u(\cdot, t)\|_\infty \rightarrow \infty$ as $t \rightarrow t^* - \leq T^* < \infty$.

To prove global blow-up, we first note that $N(t) = \max_{[0,1]} u(\cdot, t)$ satisfies

$$\frac{dN}{dt} = \frac{\lambda f(N)}{\rho(N) \left(\int_0^1 f(u) dx \right)^2} \leq \frac{\lambda f(0)}{\gamma \left(\int_0^1 f(u) dx \right)^2} = h(t),$$

and since u blows up we take $N(t) - N(0) \leq \int_0^t h(s) ds \rightarrow \infty$ as $t \rightarrow t^*$. The latter implies $h(t) \rightarrow \infty$ as $t \rightarrow t^*$ and so $\int_0^1 f(u) dx \rightarrow 0$ as $t \rightarrow t^*$, giving that $u(x, t) \rightarrow \infty$ as $t \rightarrow t^*$ for any $x \in (0, 1]$ and $u_x(0, t) \geq w_x(0, v(t)) = v(t) f(0) \rightarrow \infty$ as $t \rightarrow t^*$. $\square$

A complementary result to Theorem 2.1.10 is the following, [24],

Theorem 2.1.11 *If $\int_0^\infty f(s) ds = \infty$ and $\rho(s) \geq \gamma > 0$ for $s > 0$ then any unbounded solution $u(x, t)$ to (2.18)–(2.20) diverges or blows up in infinite time globally, i.e. $u(x, t) \rightarrow \infty$ as $t \rightarrow \infty$ for any $x \in (0, 1]$ and $u_x(0, t) \rightarrow \infty$ as $t \rightarrow \infty$.*

Proof We consider the function $v = v(t) > 0$ such that

$$\frac{dv}{dt} = \frac{\lambda}{\gamma f(v)}. \quad (2.84)$$

Then there holds that

$$\begin{aligned} \mathcal{H}(z) &:= \rho(z) z_t + \left(1 - \int_0^x \rho'(z) z_t dy \right) z_x - \frac{\lambda f(z)}{\left(\int_0^1 f(z) dx \right)^2} \\ &= \rho(v) \frac{dv}{dt} - \frac{\lambda f(v)}{\left(\int_0^1 f(v) dx \right)^2} \geq \frac{\lambda}{f(v)} - \frac{\lambda}{f(v)} = 0. \end{aligned} \quad (2.85)$$

Choosing $v(0)$ such that $u_0(x) \leq v(0)$, we see that $z(x, t) = v(t)$ is an upper solution to (2.18)–(2.20). Also (2.84) implies $\int_{v(0)}^{v(t)} f(s) ds = \frac{\lambda}{\gamma} t$, leading, due to the hypothesis $\int_0^\infty f(s) ds = \infty$, to $v(t) \rightarrow \infty$ as $t \rightarrow \infty$. Hence, $z(x, t)$ is a global-in-time unbounded upper solution to (2.18)–(2.20). This implies that $u(x, t)$ diverges or blows up in infinite time, i.e. $\|u(\cdot, t)\|_\infty \rightarrow \infty$ as $t \rightarrow \infty$. Using similar arguments as in Theorem 2.1.10, it is proved that $\int_0^1 f(u) dx \rightarrow 0$ as $t \rightarrow \infty$. Thus $u(x, t) \rightarrow \infty$, for any $x \in (0, 1]$ and $u_x(0, t) \rightarrow \infty$ as $t \rightarrow \infty$. This completes the proof. $\square$

2.2 A Non-local Thermistor Problem

The second part of the current chapter is devoted to the study of the thermistor, a device for regulating electric current in a circuit.

The operation of the thermistor is described by the following system

$$u_t = \nabla \cdot (\kappa(u) \nabla u) + \rho(u) |\nabla \phi|^2, \quad x \in \Omega, \quad t > 0, \quad (2.86)$$

$$\nabla \cdot (\rho(u) \nabla \phi) = 0, \quad x \in \Omega, \quad t > 0, \quad (2.87)$$

together with some boundary conditions for $u(x, t)$ and $\phi(x, t)$ on $\partial\Omega$, see for example [3, 4, 13]. Here, Ω is assumed to be a smooth, bounded open set of $\mathbb{R}^N$, $N \geq 1$, and stands for the spatial domain occupied by the conductor (the body of the thermistor); the physical situation corresponds to $N = 3$. Moreover, $\phi(x, t)$ is the electrical potential, $u(x, t)$ the temperature inside the conductor, $\kappa(u) > 0$ the thermal conductivity, and $\rho(u) > 0$ stands for the electrical conductivity. The parabolic equation (2.86) describes the heat flow in the system, while the elliptic equation (2.87) describes the conservation of charge in the system, provided that its variation in space and time is not too rapid.

Using a similar approach to Sect. 2.1.1 we can reduce system (2.86)–(2.87) to the following non-local equation

$$u_t = \nabla \cdot (\kappa(u) \nabla u) + \frac{\lambda f(u)}{(\int_{\Omega} f(u) dx)^2}, \quad x \in \Omega, \quad t > 0, \quad (2.88)$$

which is also associated with boundary and initial conditions. The form of the boundary conditions depending on the flux conditions are applied at the edge of the thermistor device. Below we will mainly deal with zero-flux (Neumann), mixed (Robin), and Dirichlet type boundary conditions.

Local-in-time existence of problems for Eq. (2.88) can be obtained using similar ideas as in Sect. 2.1.2 and therefore we focus on the blowing-up behavior of the associated problems. Again blow-up behavior is closely linked with the occurrence of thermal runaway which might cause destruction of thermistor device. Therefore its thorough investigation is interesting from applications point of view as well.

2.2.1 Neumann Problem

We start our study from the simplest case, when $u(x, t)$ satisfies Neumann boundary conditions, i.e. the boundary of the thermistor is thermally insulated, and so the (dimensionless) temperature $u(x, t)$ satisfies the problem

$$u_t = \nabla \cdot (\kappa(u) \nabla(u)) + \frac{\lambda f(u)}{(\int_{\Omega} f(u) dx)^2}, \quad x \in \Omega, \quad t > 0, \quad (2.89)$$

$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (2.90)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (2.91)$$

where $f(s) > 0$, $f'(s) < 0$ and $\kappa(u) \geq c > 0$. Here $\partial/\partial\nu$ denotes the normal outward derivative to the boundary $\partial\Omega$. In the following we assume that $u_0(x) \geq 0$.

In this case the associated steady-state problem

$$\nabla \cdot (\kappa(w) \nabla(w)) + \frac{\lambda f(w)}{(\int_{\Omega} f(w) dx)^2} = 0, \quad x \in \Omega, \quad \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad (2.92)$$

does not permit any kind of solution for every $\lambda > 0$.

Otherwise, if we integrate the equation of problem (2.92) we get

$$0 = \frac{\lambda}{\int_{\Omega} f(w) dx},$$

which is a contradiction. The lack of stationary solutions is an indication that time-dependent solutions should be unbounded. From the physical point of view this is justified since the source term is positive, the system is provided with heat. On the other hand the boundary condition $\partial u/\partial\nu = 0$ prevents any heat from escaping. Thus, in such a situation, one expects the solution (concentration of heat) to become unbounded. This physical situation is described by the following.

Theorem 2.2.1 *There exists $t^* \leq \infty$ such that $\|u(\cdot, t)\|_{\infty} \rightarrow \infty$ as $t \rightarrow t^*$. If $\int_0^{\infty} f(s) ds < \infty$ then $u(x, t)$ blows up in finite time, i.e. $t^* < \infty$, whereas if $\int_0^{\infty} f(s) ds = \infty$ then blow-up in infinite-time occurs, i.e. $t^* = \infty$ and $\|u(\cdot, t)\|_{\infty} \rightarrow \infty$ as $t \rightarrow t^*$. Moreover in each case blow-up is global and flat (uniform) i.e.*

$$u(x, t) = \|u(\cdot, t)\|_{\infty}(1 + o(1)) \quad \text{as } t \rightarrow t^* - \text{ for a.e. } x \in \Omega.$$

Proof We first assume $\int_0^{\infty} f(s) ds < \infty$. Under this assumption and the positivity of u (which is a consequence of the maximum principle) the functional

$$Y(t) = \int_{\Omega} \int_{u(x,t)}^{\infty} f(\sigma) d\sigma dx,$$

is well-defined and nonnegative [3, 21].

Taking the derivative of $Y(t)$ with respect to t and using Eq. (2.89) we obtain

$$Y'(t) = - \int_{\Omega} f(u) u_t dx = - \int_{\Omega} f(u) \nabla \cdot (\kappa(u) \nabla(u)) dx - \frac{\lambda \int_{\Omega} f^2(u) dx}{\left(\int_{\Omega} f(u) dx\right)^2}.$$

Using Jensen's inequality for $\phi(s) = s^2$ and integration by parts we have

$$Y'(t) \leq - \int_{\Omega} f(u) \nabla \cdot (\kappa(u) \nabla(u)) dx - \frac{\lambda}{|\Omega|} = \int_{\Omega} f'(u) \kappa(u) |\nabla u|^2 dx - \frac{\lambda}{|\Omega|}.$$

Now using the monotonicity of $f(s)$ as well the positivity of $\kappa(u)$ then

$$Y'(t) \leq - \frac{\lambda}{|\Omega|},$$

which finally yields

$$0 \leq Y(t) \leq Y(0) - \frac{\lambda}{|\Omega|} t.$$

The latter yields that $u(x, t)$ cannot exist beyond t^* , where

$$t^* \leq T_u^* = \frac{|\Omega| Y(0)}{\lambda} < \infty. \quad (2.93)$$

Since the solution $u(x, t)$ of (2.89)–(2.91) ceases to exist only when it becomes unbounded, we deduce that $\|u(\cdot, t)\|_{\infty} \rightarrow \infty$ as $t \rightarrow t^*$ – (finite-time blow-up). An immediate result of relation (2.93) is that as the L^1 –norm of the initial conditions increases then the bound T_u^* on the blow-up time, as it is expected, decreases. Moreover we can prove that $u(x, t)$ blows up globally, using similar ideas as in Theorem 2.1.8, and uniformly i.e. $u(x, t) \sim \|u(\cdot, t)\|_{\infty}$ as $t \rightarrow t^*$ for a.e. $x \in \Omega$ (global and flat blow-up).

Applying a similar approach as in [15, 21], we obtain that $n(t) = \min_{x \in \bar{\Omega}} u(x, t)$ and $N(t) = \max_{x \in \bar{\Omega}} u(x, t)$ are differentiable almost for almost every t and thus it satisfies

$$\frac{dn}{dt} \geq \frac{\lambda f(n)}{\left(\int_{\Omega} f(u) dx\right)^2} \geq \frac{\lambda f(N)}{\left(\int_{\Omega} f(u) dx\right)^2} \geq \frac{dN}{dt}, \quad \text{a.e. in } (0, t^*),$$

which implies $n(t) < N(t) \leq n(t) + C$, $C = N(0) - n(0)$, for every $t \in (0, t^*)$. Hence the blow-up is uniform and it holds that

$$u(x, t) = \|u(\cdot, t)\|_{\infty}(1 + o(1)) \quad \text{as } t \rightarrow t^* \text{ for a.e. } x \in \Omega.$$

We now consider the case where $\int_0^{\infty} f(s) ds = \infty$. Again $N(t)$ satisfies

$$\frac{dN}{dt} \leq \frac{\lambda f(N)}{(\int_{\Omega} f(u) dx)^2} \leq \frac{\lambda}{f(N) |\Omega|^2} \quad \text{a.e in } (0, t),$$

and integrating over $(0, t)$ yields

$$\int_{N(0)}^{N(t)} f(s) ds \leq \frac{\lambda}{|\Omega|^2} t,$$

for every $t > 0$. The latter implies that the solution cannot blow-up in finite time. By the inequality

$$\left(\int_{\Omega} u(x, t) dx \right)_t \geq \frac{\lambda}{|\Omega| f(0)} > 0,$$

we deduce $\lim_{t \rightarrow \infty} \int_{\Omega} u(x, t) dx = \infty$ and thus $N(t)$ diverges in infinite time. $\square$

2.2.2 Robin Problem

We now investigate a problem with Robin-type boundary conditions, i.e. non-Newtonian cooling,

$$u_t = \nabla \cdot (\kappa(u) \nabla(u)) + \frac{\lambda f(u)}{(\int_{\Omega} f(u) dx)^2}, \quad x \in \Omega, \quad t > 0, \quad (2.94)$$

$$\frac{\partial u}{\partial \nu} + \beta u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (2.95)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (2.96)$$

where $\beta = \beta(x) > 0$. We still assume that $f(s) > 0$, $f'(s) < 0$ and $\int_0^{\infty} f(s) ds < \infty$; actually for simplicity we take

$$\int_0^{\infty} f(s) ds = 1. \quad (2.97)$$

Then the following blow-up result holds:

Theorem 2.2.2 *Assume $(f(s)\kappa(s))' < 0$ then if $u(x, t)$ is a solution of (2.94)–(2.96) then for λ sufficiently large there exists $t^* < \infty$ such that $\|u(\cdot, t)\|_{\infty} \rightarrow \infty$ as $t \rightarrow t^*-$, i.e. $u(x, t)$ blows up in finite time. Moreover blow-up is global and flat.*

Proof In this case the technique used for the Neumann problem fails because when integrating by parts the term containing $\partial u / \partial \nu$ does not vanish any more on the boundary. In order to overcome this difficulty we modify the form of the functional

$Y(t)$. This can be achieved by using an auxiliary function in the definition of $Y(t)$. More precisely we consider

$$Y(t) = \int_{\Omega} \Psi(x) \left(\int_{u(x,t)}^{\infty} f(\sigma) d\sigma \right) dx,$$

where $\Psi(x)$ is the Robin eigenfunction corresponding to the principal eigenvalue of the Laplacian, i.e. $\Psi(x)$ satisfies the problem

$$-\Delta \Psi = \mu_1 \Psi, \quad x \in \Omega, \quad \frac{\partial \Psi}{\partial \nu} + \beta \Psi = 0, \quad x \in \partial \Omega. \quad (2.98)$$

It is known, see for example [2] Theorem 4.3, that μ_1 is positive and that $\Psi(x)$ does not change sign in $\overline{\Omega}$, so it can be chosen to be positive and for simplicity normalised so that

$$\int_{\Omega} \Psi(x) dx = 1. \quad (2.99)$$

Hence the functional $Y(t)$ is nonnegative and well-defined due to (2.97).

Differentiating $Y(t)$ and using (2.94) we obtain

$$\begin{aligned} Y'(t) &\leq - \int_{\partial \Omega} \Psi f(u) \kappa(u) \frac{\partial u}{\partial \nu} ds + \int_{\Omega} \nabla (\Psi(x) f(u)) \cdot \nabla u dx - \frac{\lambda m}{|\Omega|} \\ &= \beta \int_{\partial \Omega} \Psi f(u) \kappa(u) u ds + \int_{\Omega} \nabla \Psi(x) \cdot \nabla \left(\int_0^u f(\sigma) \kappa(\sigma) d\sigma \right) dx \\ &\quad + \int_{\Omega} \Psi(x) f'(u) \kappa(u) |\nabla u|^2 dx - \frac{\lambda m}{|\Omega|}, \end{aligned}$$

due to the boundary conditons. Then $f'(s) < 0$ implies

$$\begin{aligned} Y'(t) &\leq \beta \int_{\partial \Omega} \Psi f(u) \kappa(u) u ds - \beta \int_{\partial \Omega} \Psi \left(\int_0^u f(\sigma) \kappa(\sigma) d\sigma \right) ds \\ &\quad - \int_{\Omega} \Delta \Psi \left(\int_0^u f(\sigma) \kappa(\sigma) d\sigma \right) dx - \frac{\lambda m}{|\Omega|} \\ &= \mu_1 - \frac{\lambda m}{|\Omega|}, \end{aligned}$$

for $m = \min_{x \in \overline{\Omega}} \Psi(x) > 0$ by $(f(s)\kappa(s))' < 0$, (2.99), and (2.97).

The latter yields

$$0 < Y(t) \leq Y(0) - \left(\frac{\lambda m}{|\Omega|} - \mu_1 \right) t,$$

and finally, we conclude that the solution of (2.94)–(2.96) blows up globally in finite time by

$$T_u^* = \frac{Y(0)}{\frac{\lambda m}{|\Omega|} - \mu_1},$$

provided that $\lambda > \hat{\lambda} = \frac{\mu_1 |\Omega|}{m}$. It can also be proven that blow-up is global and uniform. $\square$

The above method, apart from the existence of blow-up, provides us with an upper estimate of λ^* , the supremum of the spectrum of the corresponding steady-state problem. Indeed, we claim that $\lambda^* \leq \hat{\lambda}$, because otherwise, for initial conditions $u(x, 0) < w_m(x; \lambda)$ where $w_m(x; \lambda)$ is the minimal stationary solution corresponding to a $\hat{\lambda} < \lambda < \lambda^*$, we could prove, using similar ideas as in [30], that $u(x, t) \rightarrow w_m(x)$ as $t \rightarrow \infty$, which is a contradiction since blow-up occurs for every initial conditions if $\lambda > \hat{\lambda}$. This estimate does not seem to be optimal, at least for $f(s) = e^{-s}$. In fact, as Bebernes and Lacey conjectured in [5] that λ^* in this case will be proportional to $2|\partial\Omega|^2$. However, for the one-dimensional case the above estimate $\hat{\lambda}$ improves the ones in [30, 42].

As it has been pointed out in the second part of the current chapter, the occurrence of blow-up is linked with thermal runaway which might be responsible for the destruction of thermistor device. Therefore, it is important from applications point of view to estimate when finite blow-up takes place. Thus, in the following we provide some useful estimates of the blow-up time when $\lambda > \lambda^*$.

For simplicity we take a nonlinearity f satisfying

$$f(s), f''(s) > 0, \quad f'(s) < 0, \quad s \geq 0, \quad (2.100)$$

$$f(s) \leq \frac{c}{s^2}, \quad c > 0 \text{ for } s \gg 1 \quad \text{and} \quad \int_0^\infty f(s) ds < \infty, \quad (2.101)$$

and we also deal with the one-dimensional case, e.g. $\Omega = (-1, 1)$, for $\kappa(u) = 1$, i.e.,

$$u_t = u_{xx} + \frac{\lambda f(u)}{\left(\int_{-1}^1 f(u) dx\right)^2}, \quad x \in (-1, 1), \quad t > 0, \quad (2.102)$$

$$u_x(x, t) \pm \beta u(x, t) = 0, \quad x = \pm 1, \quad t > 0, \quad (2.103)$$

$$u(x, 0) = u_0(x), \quad x \in (-1, 1). \quad (2.104)$$

In particular, in [26] is proven the following

Theorem 2.2.3 *Let f satisfy (2.100) and (2.101). Assume, further, that the steady-state problem of (2.102)–(2.104) has at least a classical solution, say w^* , corresponding to λ^* . Then for any λ sufficiently close to λ^* , the blow-up time T_b of the solution of (2.102)–(2.104) can be estimated as follows*

$$T_l (\lambda - \lambda^*)^{-\frac{1}{2}} \leq T_b \leq T_u (\lambda - \lambda^*)^{-\frac{1}{2}}, \quad (2.105)$$

where T_l, T_u are positive constants.

The proof of Theorem 2.2.3 is based on several ideas introduced in [28] and is lengthy. Thus it is omitted here. In [26], some estimates of the blow-up via asymptotic and numerical methods are also provided for $f(s) = e^{-s}$ which actually confirm the results of Theorem 2.2.3.

2.2.3 Dirichlet Problem

We finally close our study with the Dirichlet case where for simplicity we also consider $\kappa(u) = 1$ and thus we focus on the following

$$u_t = \Delta u + \frac{\lambda f(u)}{(\int_{\Omega} f(u) dx)^2}, \quad x \in \Omega, \quad t > 0, \quad (2.106)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (2.107)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega. \quad (2.108)$$

In this case we expect for blow-up to occur if $f(s)$ is decreasing and satisfies (2.97). Indeed, under those conditions there exists a critical parameter $\lambda^* \geq 2|\partial\Omega|^2$, see Theorem 2.2 in [5], such that the steady-state problem

$$\Delta w + \frac{\lambda f(w)}{(\int_{\Omega} f(w) dx)^2} = 0, \quad x \in \Omega, \quad w(x) = 0, \quad x \in \partial\Omega, \quad (2.109)$$

has no solutions for $\lambda > \lambda^*$. Actually, in the one-dimensional case a thorough study of finite-time blow-up for $\lambda > \lambda^*$ is provided in [29, 30]. Here we provide an extension of those blow-up results in the higher dimensional case however we should restrict the class of the considered nonlinear functions f .

Notably the approach used in the case of Neumann and Robin boundary conditions fails here, for more details see [21]. Therefore, we will proceed for the proof of blow-up by constructing a proper blowing up lower solution. We actually have the following

Theorem 2.2.4 *For $\lambda > \lambda^*$ and sufficiently large initial conditions the solution of the problem (2.106)–(2.108) blows up globally and uniformly in finite time provided that $f(s)$ satisfies*

$$\int_0^{\infty} [2s f(s) - s^2 f'(s)] ds < \infty. \quad (2.110)$$

Proof For the construction of this lower solution we will need first to define a lower solution of the problem for the steady-state problem (2.109), which was first

introduced in [5]. This lower solution has the form

$$v(x; \mu, \nu, \delta) = \begin{cases} V(\sqrt{\nu \delta} d(x, \partial\Omega); \mu), & d(x, \partial\Omega) < \frac{\mu}{\sqrt{\nu \delta}}, \\ V(\mu; \mu) = M, & d(x, \partial\Omega) \geq \frac{\mu}{\sqrt{\nu \delta}}, \end{cases} \quad (2.111)$$

where V is the solution of the problem

$$V'' + f(V) = 0, \quad 0 < y < \mu, \quad V(0; \mu) = 0, \quad V'(\mu; \mu) = 0, \quad (2.112)$$

while $\delta = \lambda / (\int_{\Omega} f(w) dx)^2$ is the so called local parameter of (2.109) and ν is a constant to be determined later. Here $d(x, \partial\Omega)$ stands for the distance between $x \in \Omega$ and the boundary $\partial\Omega$. Also, d is smooth and more precisely $|\Delta d| < K$, for some K , in a neighborhood of the boundary if $\partial\Omega$ is smooth. In particular, such a neighborhood consists of all $x \in \Omega$ such that $d(x, \partial\Omega) \leq \frac{\mu}{\sqrt{\nu \delta}} < \rho$, where ρ is smaller than the infimum of the radius of the largest interior ball touching the boundary at some $z \in \partial\Omega$. Hence in the following δ is chosen large enough to ensure that $\frac{\mu}{\sqrt{\nu \delta}} < \rho$.

Obviously ν satisfies the correct boundary condition and is C^1 . Moreover there holds

$$\begin{aligned} \Delta v + \delta f(v) &= \delta f(V(\mu)) > 0 \quad \text{for } d(x, \partial\Omega) \geq \frac{\mu}{\sqrt{\nu \delta}}, \\ \Delta v + \delta f(v) &= \nu \delta |\nabla d|^2 V'' + \sqrt{\nu \delta} V' \Delta d + \delta f(V) \\ &= (1 - \nu) \delta f(V) + \sqrt{\nu \delta} V' \Delta d \quad \text{where } d(x, \partial\Omega) < \frac{\mu}{\sqrt{\nu \delta}} \\ &> (1 - \nu) \delta f(V) - \sqrt{\delta} K V' \geq 0, \quad \text{for } \nu < 1, \end{aligned}$$

provided that $\nu \leq 1 - \frac{K G(\mu)}{\sqrt{\delta}}$, where $G(\mu) = \sup_{y \in (0, \mu)} \frac{V'}{f(V)}$.

Therefore, by choosing $\nu = 1 - K G(\mu)/\sqrt{\delta}$, ν is a lower solution of problem (2.109) for sufficiently large δ , i.e. for $\lambda < \lambda^*$ sufficiently close to λ^* .

Problem (2.112) also implies that V satisfies

$$V'^2 = 2 \int_V^M f(s) ds = 2 \int_V^\infty f(s) ds - 2 \int_M^\infty f(s) ds = 2(F(V) - F(M)),$$

where $F(\sigma) = \int_\sigma^\infty f(s) ds$ and thus the relation between μ and M is defined by

$$\mu = \mu(M) = \frac{1}{\sqrt{2}} \int_0^M [F(s) - F(M)]^{-1/2} ds. \quad (2.113)$$

We now consider as a candidate lower solution the function

$$z(x, t) = \begin{cases} V(\sqrt{v} \delta d(x, \partial\Omega); \mu), & d(x, \partial\Omega) < \frac{\mu}{\sqrt{v} \delta}, \\ V(\mu; \mu) = M(t), & d(x, \partial\Omega) \geq \frac{\mu}{\sqrt{v} \delta}, \end{cases} \quad (2.114)$$

where the functions $\mu = \mu(t)$, $M = M(t)$, $\delta = \delta(t)$ and the constant v will be determined below.

Indeed, due to (2.113) the relation between $M(t)$ and $\mu(t)$ is already determined, while we are free to choose the relation between $M(t)$ and $\delta(t)$. It is evident, from the definition of $z(x, t)$, that if the ratio $\mu/\sqrt{v}\delta$ decreases to 0 as t increases then the spatial independent behavior of $z(x, t)$ dominates the behavior near the boundary, i.e. the growth of $z(x, t)$ is uniform (flat) as t increases. In the following we will choose $M(t)$ and the dependence between M and δ such that, for large enough initial conditions, $z(x, t)$ is a lower solution of problem (2.106)–(2.108) which blows up in finite time.

We choose

$$\delta = \delta(M) = \frac{M^2}{f(M)}, \quad (2.115)$$

while we impose $M(t)$ to satisfy

$$\dot{M}(t) := \frac{dM}{dt} = \frac{\lambda - \lambda^*}{(\int_{\Omega} f(z) dx)^2} \inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} > 0 \quad \text{for } t > 0, \quad (2.116)$$

and $M(0)$ is chosen large enough so that $\delta_0 = \delta(M(0)) = M^2(0)/f(M(0))$ is also sufficiently large (note that $\delta'(M) > 0$).

Consequently, we have that

$$z_M = \frac{\partial z}{\partial M} = \frac{\partial V}{\partial y} y'(M) + \frac{\partial V}{\partial \mu} \mu'(M) > 0,$$

where $y = y(M) = \sqrt{v} \delta(M) d(x, \partial\Omega)$. In fact, differentiating problem (2.112) with respect to y and using the monotonicity of $f(s)$ together with maximum principle arguments we get $\partial V / \partial y > 0$. Besides, $y'(M) = \frac{1}{2} y(M) \delta'(M) / \delta(M) > 0$ and $\mu'(M)$, $V_{\mu}(y; \mu) > 0$.

Setting $Y = y/\mu$ then problem (2.112) can be written as

$$W'' + \mu^2 f(W) = 0, \quad 0 < Y < 1, \quad W(0; \mu) = 0, \quad W'(1; \mu) = 0,$$

where $W(Y) = V(y)$. Differentiating this problem with respect to μ we obtain

$$-W''_{\mu} - \mu^2 f'(W) W_{\mu} = 2\mu f(W) > 0, \quad 0 < Y < 1, \quad W_{\mu}(0; \mu) = 0, \quad W'_{\mu}(1; \mu) = 0,$$

which implies (due the maximum principle) that $W_{\mu}(Y; \mu) = V_{\mu}(y; \mu) > 0$ for $0 < y < \mu$ and $W_{\mu}(1; \mu) = M'(\mu) > 0$, hence $\mu'(M) > 0$.

Now if we choose δ_0 large enough then the fact that $\delta'(M) > 0$ and $M(t)$ satisfies (2.116) guarantee that $\delta(t)$ remains large enough for every $t > 0$.

Therefore

$$-\Delta z \leq \delta(t) f(z) \quad \text{for every } t > 0, \quad (2.117)$$

provided that v is chosen such that

$$v \leq \inf_{M > M(0)} \left(1 - \frac{K G(M)}{\sqrt{\delta(M)}} \right), \quad (2.118)$$

where $M(0)$ is also taken large enough. The latter choice is possible since

$$\lim_{M \rightarrow +\infty} \frac{G(M)}{\sqrt{\delta(M)}} = 0.$$

Therefore we derive

$$\begin{aligned} z_t - \Delta z - \frac{\lambda f(z)}{(\int_{\Omega} f(z) dx)^2} &= z_M \dot{M}(t) - \Delta z - \frac{\lambda f(z)}{(\int_{\Omega} f(z) dx)^2} \\ &\leq z_M \dot{M}(t) + \delta(t) f(z) - \frac{\lambda f(z)}{(\int_{\Omega} f(z) dx)^2} \quad (\text{due to (2.11.7)}) \\ &\leq z_M \dot{M}(t) + \frac{[\lambda^* - \lambda] f(z)}{(\int_{\Omega} f(z) dx)^2} + o(1) \\ &= z_M \left(\frac{\lambda - \lambda^*}{(\int_{\Omega} f(z) dx)^2} \inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} + \frac{[\lambda^* - \lambda]}{(\int_{\Omega} f(z) dx)^2} \frac{f(z)}{z_M} \right) + o(1) \\ &= \frac{z_M (\lambda - \lambda^*)}{(\int_{\Omega} f(z) dx)^2} \left[\inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} - \frac{f(z)}{z_M} \right] + o(1) \leq 0 \quad \text{as } M \rightarrow +\infty, \end{aligned}$$

since $\delta(t) \leq \lambda^*/(\int_{\Omega} f(z) dx)^2 + o(1)$ as $M \rightarrow +\infty$, and by choosing $M(0)$ sufficiently large.

Consequently, $z(x, t)$ is a lower solution of problem (2.106)–(2.108) provided that $M(t)$ satisfies (2.116) with sufficiently large initial conditions $M(0)$. Furthermore, we show that (2.116) guarantees the occurrence of finite-time blow-up for the lower solution $z(x, t)$.

Indeed, we first note that

$$\inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} = \min \left\{ f(M), \inf_{d(x, \partial\Omega) < \mu/\sqrt{v\delta}} \left\{ \frac{f(z)}{z_M} \right\} \right\},$$

where

$$\inf_{d(x, \partial\Omega) < \mu/\sqrt{v\delta}} \left\{ \frac{f(z)}{z_M} \right\} = \inf_{y \in (0, \mu)} \left\{ \frac{f(V(y; \mu))}{V_M(y; \mu)} \right\} = \frac{1}{\sup_{y \in (0, \mu)} \left\{ \frac{V_M(y; \mu)}{f(V(y; \mu))} \right\}}.$$

Since

$$V_M(y; \mu) = \frac{1}{2} y \frac{\delta'(M)}{\delta(M)} \frac{\partial V}{\partial y} + \frac{\partial V}{\partial \mu} \mu'(M),$$

then

$$\begin{aligned} \sup_{y \in (0, \mu)} \left\{ \frac{V_M(y; \mu)}{f(V(y; \mu))} \right\} &\leq \frac{1}{2} \frac{\delta'(M)}{\delta(M)} \sup_{y \in (0, \mu)} \left\{ \frac{V_y(y; \mu) y}{f(V(y; \mu))} \right\} + \sup_{y \in (0, \mu)} \left\{ \frac{V_\mu(y; \mu) \mu'(M)}{f(V(y; \mu))} \right\} \\ &\leq \frac{1}{2} \frac{\delta'(M)}{\delta(M)} G(\mu(M)) \mu(M) + \sup_{y \in (0, \mu)} \left\{ \frac{V_\mu(y; \mu)}{f(V(y; \mu))} \right\} \mu'(M). \end{aligned} \quad (2.119)$$

Note that

$$\lim_{M \rightarrow +\infty} \frac{\mu(M)}{\sqrt{\delta(M)}} = 0, \quad (2.120)$$

and

$$\lim_{M \rightarrow +\infty} \frac{G(\mu(M))}{\sqrt{\delta(M)}} = 0. \quad (2.121)$$

For the former by virtue of (2.113) we have

$$\begin{aligned} \mu(M) &= \frac{1}{\sqrt{2}} \int_0^M [F(s) - F(M)]^{-1/2} ds \leq \frac{1}{\sqrt{2}} \int_0^M [(M - s) f(M)]^{-1/2} ds \\ &= \sqrt{\frac{2M}{f(M)}}, \end{aligned}$$

and thus

$$0 < \lim_{M \rightarrow +\infty} \frac{\mu(M)}{\sqrt{\delta(M)}} \leq \lim_{M \rightarrow +\infty} \frac{\sqrt{\frac{2M}{f(M)}}}{\sqrt{\delta(M)}} = \lim_{M \rightarrow +\infty} \frac{\sqrt{\frac{2M}{f(M)}}}{\frac{M}{\sqrt{f(M)}}} = 0.$$

For the latter, taking into account the monotonicity of f we obtain that

$$f^2(M) (F(V) - F(M)) / f^2(V) \leq (M - V) f^2(M) / f(V) \leq (M - V) f(M) \leq M f(M),$$

which implies

$$G(\mu(M)) = \sup_{V \in (0, M)} \frac{\sqrt{2(F(V) - F(M))}}{f(V)} \leq \sqrt{\frac{2M}{f(M)}},$$

and thus

$$0 < \lim_{M \rightarrow +\infty} \frac{G(\mu(M))}{\sqrt{\delta(M)}} \leq \lim_{M \rightarrow +\infty} \frac{\sqrt{\frac{2M}{f(M)}}}{\sqrt{\delta(M)}} = \lim_{M \rightarrow +\infty} \frac{\sqrt{\frac{2M}{f(M)}}}{\frac{M}{\sqrt{f(M)}}} = 0.$$

Furthermore,

$$\sup_{y \in (0, \mu)} \left\{ \frac{V_\mu(y; \mu)}{f(V(y; \mu))} \right\} \leq \frac{V_\mu(\mu; \mu)}{f(V(\mu; \mu))} = \frac{M'(\mu)}{f(M)}, \quad (2.122)$$

since

$$\frac{\partial^2 V(y; \mu)}{\partial \mu \partial y} = - \int_0^y f'(V(s; \mu)) V_\mu(s; \mu) ds + \frac{\partial^2 V(0; \mu)}{\partial \mu \partial y} \geq 0.$$

Now by using (2.120)–(2.122) then relation (2.119) entails

$$\sup_{y \in (0, \mu)} \left\{ \frac{V_M(y)}{f(V(y))} \right\} < \frac{1}{2} \delta'(M) + \frac{1}{f(M)} = \frac{1}{2} \delta'(M) (1 + o(1)) \quad \text{as } M \rightarrow +\infty,$$

taking into account

$$\lim_{M \rightarrow \infty} \frac{\frac{1}{f(M)}}{\frac{\delta'(M)}{2}} = \lim_{M \rightarrow \infty} \frac{\frac{2}{\delta'(M)}}{f(M)} = \lim_{M \rightarrow \infty} \frac{2}{2M - \frac{M^2 f'(M)}{f(M)}} = 0,$$

and thus

$$\inf_{x \in \Omega} \left\{ \frac{f(z)}{z_M} \right\} \geq \min \left\{ f(M), \frac{2}{\delta'(M)} \right\} + o(M) = \frac{2}{\delta'(M)} + o(M) \quad \text{as } M \rightarrow +\infty.$$

Consequently in view of relation (2.116) we obtain

$$\begin{aligned} \dot{M}(t) &\geq \frac{\lambda - \lambda^*}{\left(\int_{\Omega} f(z) dx \right)^2} \frac{2}{\delta'(M)} + o(M) = \frac{2(\lambda - \lambda^*)}{|\Omega|^2 \delta'(M) f^2(M)} + o(M) \\ &= \frac{(\lambda - \lambda^*)}{|\Omega|^2 [2M f(M) - M^2 f'(M)]} + o(M), \quad \text{as } M \rightarrow +\infty, \end{aligned}$$

since $\int_{\Omega} f(z) dx = f(M) |\Omega| + o(M)$ as $M \rightarrow +\infty$ due to (2.120) and the definition of $z(x, t)$. The latter implies that $M(t) \rightarrow \infty$ as $t \rightarrow t_1^*$, where

$$t_1^* \leq \frac{|\Omega|^2}{(\lambda - \lambda^*)} \int_{M(0)}^{\infty} [2\sigma f(\sigma) - \sigma^2 f'(\sigma)] d\sigma + o(M) < \infty, \quad \text{as } M \rightarrow +\infty, \quad (2.123)$$

recalling that f satisfies (2.110).

Consequently $z(x, t)$ is lower solution which blows up in finite time t_1^* , provided that $M(0)$ is chosen large enough. The latter implies that $u(x, t)$ blows up in finite

time $t^* \leq t_1^*$, i.e. $\|u(\cdot, t)\|_\infty \rightarrow \infty$ as $t \rightarrow t^*$, for large enough initial conditions. It can be proven, using the same arguments as in the Neumann problem, that blow-up is global.

Finally, in this case we can also prove that blow-up is uniform (flat). Actually, if we consider the problem

$$\begin{aligned} v_t &= \Delta v + h(t) f(N), \quad x \in \Omega, \quad t > 0, \\ v(x, t) &= 0, \quad x \in \partial\Omega, \quad t > 0, \\ v(x, 0) &= 0, \quad x \in \Omega, \end{aligned}$$

where $N = N(t) = \max_{x \in \Omega} u(x, t)$ and $h(t) = \lambda / (\int_\Omega f(u(x, t)) dx)^2$ then $v(x, t)$ is a lower solution of problem (2.106)–(2.108) since $f(s)$ is decreasing.

Set $v(x, t) = \theta(x, t) + V(t)$ where $V(t)$ is the solution of the problem

$$\frac{dV(t)}{dt} = h(t) f(N), \quad V(0) = 0,$$

and $\theta(x, t)$ satisfies

$$\begin{aligned} \theta_t &= \Delta \theta, \quad x \in \Omega, \quad t > 0, \\ \theta(x, t) &= -V(t), \quad x \in \partial\Omega, \quad t > 0, \\ \theta(x, 0) &= 0, \quad x \in \Omega. \end{aligned}$$

Thus θ has the integral representation

$$\theta(x, t) = \int_0^t V(\tau) \int_{\partial\Omega} \frac{\partial G(x, s, t - \tau)}{\partial \nu} ds d\tau,$$

where $G(x, y, t)$ is the Green's function for the heat equation in Ω with Dirichlet boundary conditions. Thus

$$v(x, t) = V(t) + \int_0^t V(\tau) \int_{\partial\Omega} \frac{\partial G(x, s, t - \tau)}{\partial \nu} ds d\tau, \quad (2.124)$$

and for any fixed $x \in \Omega$ the second term on the right-hand side of (2.124) is much smaller than the first as $t \rightarrow t^*$, due to the contribution of the Green's function term, and so $v(x, t) = V(t)(1 + o(1))$ as $t \rightarrow t^*$ for any $x \in \Omega$. Hence $N(t) \geq u(x, t) \geq V(t)(1 + o(1))$ as $t \rightarrow t^*$ for any $x \in \Omega$.

Now using the fact

$$\frac{dN(t)}{dt} \leq h(t) f(N) = \frac{dV(t)}{dt},$$

we obtain $N(t) \leq V(t) \leq N(t)(1 + o(1))$ as $t \rightarrow t^*$ and thus it follows that $V(t) = N(t)(1 + o(1))$ as $t \rightarrow t^*$ and consequently we deduce $u(x, t) = N(t)(1 + o(1))$ as $t \rightarrow t^*-$ for any $x \in \Omega$ (uniform blow-up). $\square$

Remark 2.2.5 Relation (2.110) is satisfied by $f(s) = e^{-s}$ as well as by $f(s) = 1/(1+s)^{1+k}$, for $k > 1$.

Remark 2.2.6 It should be pointed out that the upper bound of blow-up time given by relation (2.123) is of the same form with the upper estimate obtained in Theorem 2.2.3, although the method used to derive (2.123) is independent of the spatial dimension.

In the case of a non-constant thermal conductivity $\kappa(u)$ some blow-up results not only for the Dirichlet case can be found in [19, 32].

The case $\lambda = \lambda^*$ seems to be critical since an infinite-time blow-up result holds for the one-dimensional case, e.g. $\Omega = (-1, 1)$. Indeed, we can construct an upper solution V to problem (2.106)–(2.108) which is global in time and unbounded.

Such an upper solution has the form

$$\begin{aligned} V(x, t) &= w(y(x, t); \mu(t)), \quad \delta(t) \leq |x| \leq 1, \quad t > 0, \\ V(x, t) &= M(t) = \max_{\delta(t) \leq |x| \leq 1} w(y(x, t); \mu(t)), \quad 0 \leq |x| < \delta(t), \quad t > 0, \end{aligned}$$

where $0 \leq y(x, t) = \frac{|x| - \delta(t)}{1 - \delta(t)} \leq 1$, and $0 < \delta(t) < 1$ is a function can be chosen properly whereas $w(y; \mu(t))$ satisfies the following quasi steady-state problem

$$w_{yy} + \mu(t)f(w) = 0, \quad 0 < y < 1, \quad t > 0, \quad w'(0) = w(1) = 0,$$

with $\varepsilon(t) = 1 - \delta(t)$.

More precisely the following holds

Theorem 2.2.7 *Let $f(s)$ be a decreasing function satisfying relation (2.97) as well as the one of the following conditions*

$$\liminf_{s \rightarrow \infty} g(s) > c > 0$$

or

$$\liminf_{s \rightarrow \infty} g(s) = 0 \quad \text{and} \quad g(s) \geq \frac{\sqrt{2}(1+\alpha)}{\sqrt{\ln s}}, \quad \alpha > 0 \quad \text{for } s \gg 1,$$

where

$$g(s) = \frac{f(s)\sqrt{\mu(s)}}{\int_s^\infty f(\sigma) d\sigma}.$$

Then the solution $u^*(x, t)$ of problem (2.106)–(2.108) for $\lambda = \lambda^*$ is global-in-time. Moreover,

$$\lim_{t \rightarrow \infty} u^*(x, t) = \infty \quad \text{for all } x \in (-1, 1),$$

and $u_x^*(\pm 1, t) \rightarrow \mp \infty$ as $t \rightarrow \infty$, i.e. u^* blows in infinite time (diverges) globally in x .

The proof of Theorem 2.2.7 is very lengthy and thus we omit it here, however it can be found in [27], where also a divergence result for the radial case is proven.

An analogous divergence result to Theorem 2.2.7 for the case of a non-constant thermal conductivity $\kappa(u)$ is given in [33].

Closing the current chapter we would like to specify that the following non-local parabolic equation

$$u_t = \nabla \cdot (\kappa(u) \nabla(u)) + \frac{\lambda f(u)}{\left(\int_{\Omega} f(u) dx\right)^p}, \quad x \in \Omega, \quad t > 0, \quad (2.125)$$

where $p \neq 2$ can serve as a mathematical model for a variety of industrial processes. In particular, for $p \leq 1$ (2.125) can be used to describe the torsion test in metallurgy where finite-time blow-up is now associated with the phenomenon of shear banding formation, [6, 7].

For some finite-time blow-up results for Eq. (2.125) associated with different boundary conditions see [5, 7, 19, 25, 43] where a variety of methods are employed.

References

1. Al-Refai, M., Kavallaris, N.I., Hajji, M.A.: Monotone iterative sequences for non-local elliptic problems. *Eur. J. Appl. Math.* **22**(6), 533–552 (2011)
2. Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18**, 620–709 (1976)
3. Antontsev, S.N., Chipot, M.: The thermistor problem: existence, smoothness, uniqueness, blow-up. *SIAM J. Math. Anal.* **25**(4), 1128–1156 (1994)
4. Antontsev, S.N., Chipot, M.: The analysis of blow-up for the thermistor problem. *Sib. Math. J.* **38**, 827–841 (1997)
5. Bebernes, J.W., Lacey, A.A.: Global existence and finite-time blow-up for a class of nonlocal parabolic problems. *Adv. Differ. Equ.* **2**, 927–953 (1997)
6. Bebernes, J.W., Talaga, P.: Nonlocal problems modelling shear banding. *Commun. Appl. Nonlinear Anal.* **3**, 79–103 (1996)
7. Bebernes, J.W., Li, C., Talaga, P.: Single-point blowup for nonlocal parabolic problems. *Phys. D* **134**, 48–60 (1999)
8. Biss, C.H., Coombers, S.A., Skudder, P.J.: The development and applications of Ohmic heating for continuous processing of particulate foodstuffs. In: Field, R.W., Howell, J.A. (eds.) *Process Engineering in the Food Industry*, pp. 17–27. Elsevier, London (1989)
9. Carillo, J.A.: On a non-local elliptic equation with decreasing nonlinearity arising in plasma physics and heat conduction. *Nonlinear Anal.* **32**, 97–115 (1998)

10. Chafee, N.: The electric ballast resistor: homogeneous and nonhomogeneous equilibria. In: de Mottoni, P., Salvadori, L. (eds.) *Nonlinear Differential Equations: Invariance Stability and Bifurcations*, pp. 97–127. Academic Press, New York (1981)
11. Chipot, M., Lovat, B.: Some remarks on non local elliptic and parabolic problems. *Nonlinear Anal. TMA*, **30**, 4619–4627 (1997)
12. De Alwis, A.A.P., Fryer, P.J.: Operability of the Ohmic heating process: electrical conductivity effects. *J. Food Eng.*, **15**, 21–48 (1992)
13. Fowler, A.C., Frigaard, I., Howison, S.D.: Temperature surges in current - limiting circuit devices. *SIAM J. Appl. Math.*, **52**, 998–1011 (1992)
14. Freitas, P.: A nonlocal Sturm–Liouville eigenvalue problem. *Proc. Rog. Soc. Ed* **124A**, 169–188 (1994)
15. Friedman, A., McLeod, B.: Blow-up of positive solutions of semilinear heat equations. *Indiana Univ. Math. J.*, **34**, 425–447 (1985)
16. Fryer, P.J., De Alwis, A.A.P., Koury, E., Stapley, A.G.F., Zhang, L.: Ohmic processing of solid-liquid mixtures: heat generation and convection effects. *J. Food Eng.*, **18**, 101–125 (1993)
17. Hardy, G.H., Littlewood, J.E., Polya, E.: *Inequalities*. Cambridge University Press, Cambridge (1967)
18. Kavallaris, N.I.: Blow-up and global existence of solutions of some non-local problems arising in Ohmic heating process. Ph.D. Thesis, National Technical University of Athens (2000) (in Greek)
19. Kavallaris, N.I.: Asymptotic behaviour and blow-up for a nonlinear diffusion problem with a non-local source term. *Proc. Edinb. Math. Soc.*, **47**, 375–395 (2004)
20. Kavallaris, N.I.: Explosive solutions of a stochastic non-local reaction-diffusion equation arising in shear band formation. *Math. Methods Appl. Sci.*, **38**(16), 3564–3574 (2015)
21. Kavallaris, N.I., Nadzieja, T.: On the blow-up of the non-local thermistor problem. *Proc. Edinb. Math. Soc.*, **50**, 389–409 (2007)
22. Kavallaris, N.I., Tzanetis, D.E.: Blow-up and stability of a nonlocal diffusion-convection problem arising in Ohmic heating of foods. *Differ. Integral Equ.*, **15**, 271–288 (2002)
23. Kavallaris, N.I., Tzanetis, D.E.: An Ohmic heating non-local diffusion-convection problem for the Heaviside function. *ANZIAM J.*, **44** (E), E114–E142 (2002)
24. Kavallaris, N.I., Tzanetis, D.E.: Behaviour of a non-local reactive-convective problem with variable velocity in Ohmic heating of food, Nonlocal elliptic and parabolic problems, Banach Center Publ. **66**, Polish Acad. Sci. Warsaw, 189–198 (2004)
25. Kavallaris, N.I., Tzanetis, D.E.: On the blow-up of a non-local parabolic problem. *Appl. Math. Lett.*, **19**, 921–925 (2006)
26. Kavallaris, N.I., Nikolopoulos, C.V., Tzanetis, D.E.: Estimates of blow-up time for a non-local problem modelling an Ohmic heating process. *Eur. J. Appl. Math.*, **13**, 337–351 (2002)
27. Kavallaris, N.I., Lacey, A.A., Tzanetis, D.E.: Global existence and divergence of critical solutions of a non-local parabolic problem in Ohmic heating process. *Nonlinear Anal. TMA*, **58**, 787–812 (2004)
28. Lacey, A.A.: Mathematical analysis of thermal runaway for spatially inhomogeneous reactions. *SIAM J. Appl. Math.*, **43**, 1350–1366 (1983)
29. Lacey, A.A.: Thermal runaway in a non-local problem modeling Ohmic heating: part I: model derivation and some special cases. *Eur. J. Appl. Math.*, **6**, 127–144 (1995)
30. Lacey, A.A.: Thermal runaway in a non-local problem modelling Ohmic heating: part II: general proof of blow-up and asymptotics of runaway. *Eur. J. Appl. Math.*, **6**, 201–224 (1995)
31. Lacey, A.A., Tzanetis, D.E., Vlamos, P.M.: Behaviour of a non-local reactive convective problem modelling Ohmic heating of foods. *Q. J. Mech. Appl. Math.*, **52**, 623–644 (1999)
32. Latos, E.A., Tzanetis, D.E.: Existence and blow-up of solutions for a non-local filtration and porous medium problem. *Proc. Edinb. Math. Soc.*, **53**, 195–209 (2010)
33. Latos, E.A., Tzanetis, D.E.: Grow-up of critical solutions for a non-local porous medium problem with Ohmic heating source. *Nonlinear Differ. Equ. Appl.*, **17**, 137–151 (2010)
34. Lopez-Gomez, J.: On the structure and stability of the set of solutions of a non-local problem modelling Ohmic heating. *J. Dyn. Differ. Equ.*, **10**, 537–559 (1998)

35. Pao, C.V.: Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York (1992)
36. Please, C.P., Schwendeman, D.W., Hagan, P.S.: Ohmic heating of foods during aseptic processing. *IMA J. Math. Bus. Ind.* **2**, 26–28 (1994)
37. Quittner, P., Souplet, P.: Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States. Birkhäuser Advanced Texts. Birkhäuser, Basel (2007)
38. Skudder, P., Biss, S.: Aseptic processing of food products using Ohmic heating. *Chem. Eng.* **2**, 26–28 (1987)
39. Smoller, J.: Shock Waves and Reaction-Diffusion Equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 2nd edn. Springer, New York (1994)
40. Stakgold, I.: Boundary Value Problems of Mathematical Physics, vol. I. The Macmillan Company, Collier-Macmillan, London (1970)
41. Stirling, R.: Ohmic heating - a new process for the food industry. *Power Eng. J.* **6**, 365 (1987)
42. Tzanetis, D.E.: Blow-up of radially symmetric solutions of a non-local problem modelling Ohmic heating. *Electron. J. Differ. Equ.* **11**, 1–26 (2002)
43. Tzanetis, D.E., Vlamos, P.M.: A nonlocal problem modelling Ohmic heating with variable thermal conductivity. *Nonlinear Anal. RWA* **2**, 443–454 (2001)
44. Zang, L., Fryer, P.J.: Models for the electrical heating of solid-liquid food mixtures. *Chem. Eng. Sci.* **48**, 633–642 (1993)
45. Zheng, G.-F.: On finite-time blow-up for a nonlocal parabolic problem arising from shear bands in metals. *Proc. Amer. Math. Soc.* **135**, 1487–1494 (2007)

Chapter 3

Linear Friction Welding

Abstract The current chapter discusses an application arising in the process of linear friction welding applied in metallurgy. In the first place a one-dimensional non-local model defined in the half-line is constructed in order to describe the evolution of the temperature within the welding region. In this study we mainly consider two cases: the soft-material which is modeled by an exponential nonlinearity and the hard-material case when a power-law nonlinearity is regarded. In the former case the non-local problem has variational structure, and so can be treated as a gradient flow, which is used to derive appropriate a priori estimates for the solution. Thus parabolic regularity theory can be used to prove global-in-time existence and finally prove the convergence of its solution towards the unique steady state. On the other hand, the power-law case lacks such a variational structure and thus we have to appeal to a numerical scheme of Crank–Nicolson type in order to presume the long-time behavior in this case as well as to confirm the analytical results derived in the exponential case.

3.1 Derivation of the Model

Problems of thermo-viscous flow and of thermo-viscoelastic flow in channels have previously been seen to lead to non-local parabolic equations [1]. These models have held in bounded domains, corresponding to cross-sections of the channels. Conditions under which such problems can exhibit some form of blow-up, possibly corresponding to shear-band formation, have been investigated in a number of papers [1–5, 16, 17].

A related problem of thermo-plastic flow arises from the consideration of *linear friction welding*. In this process two metal workpieces are forced to slide against each other, in a rubbing motion, while also being pressed together. The rubbing leads to heat generation, consequent softening near the workpieces' adjoining surfaces, and plastic flow. Viscous dissipation continues heat generation within a thin softened layer where the two workpieces merge. The forcing together of the workpieces results in an additional, but rather smaller, squeezing motion, with the workpieces moving slowly towards each other, and some material – including impurities on the original

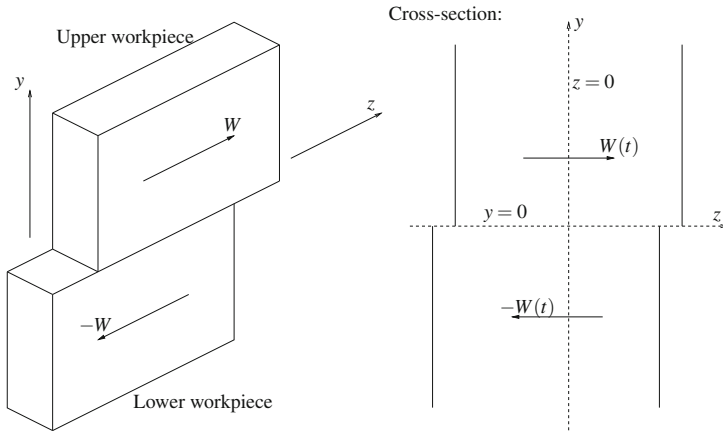


Fig. 3.1 A schematic description of the procedure of linear friction welding

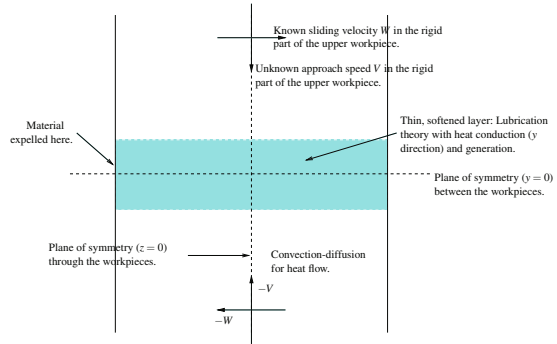
surfaces – being expelled from the sides as “flash”. In a time the process is halted and cooling of previously warmer regions results in a weld between the workpieces. For a schematic description of the above process see Fig. 3.1.

In [6, 14], the process has been modelled by assuming that the forced sliding velocities of the workpieces, $\pm W(t)$, are square waves with $W(t) = \pm W_e$. This means that the sliding speed is always W_e . This, combined with an assumption that in the soft layer the corresponding (oscillating) sliding velocity dominates the non-oscillatory squeezing velocity, means that the rate of heat generation within the layer is, to leading order, independent of time. The latter assumption will hold provided, roughly speaking, that the squeezing force is small enough in comparison with other operating parameters and material constants.

Another key part of the model in [6, 14], mentioned also in [10], is the thinness of the soft layer, which can then be looked at using lubrication theory, subject to matching conditions with outer problems applying to the cooler parts of the workpieces which move as simple rigid bodies. Figure 3.2 indicates some of the key features of the model. If the sides of the workpieces have low Biot number, that is, the heat transfer between them and their surroundings is poor, and the oscillations have small amplitude, temperature will be, to leading order, independent of position, z , along the weld.

All this allows the process to be represented by an essentially one-dimensional mathematical model. The squeezing velocity along the weld, the z direction, $\bar{w}$, is proportional to z , measured from the centre of the weld, but on writing $\bar{w} = zw^*$, the new variable w^* is independent of z . The flow in the soft layer is dominated by one-dimensional, plastic, sliding flow, and is modelled in the same way as the shear-band models of [1, 2]. The energy equation for temperature, however, requires a matching condition, with y , the distance from the plane of symmetry along the weld, large compared with h , the width of the soft layer. (More accurately, h is the length scale

Fig. 3.2 Some features of the model for linear friction welding



across this thin, soft layer.) The matching is with an outer problem for temperature, in the rigid parts of the workpieces, which gives a value for the temperature gradient in terms of the approach speed V . Thus, looking at the inner problem which applies over a length scale of h in the thin, soft zone, a temperature gradient is applied at infinity – more accurately, for $y/h \gg 1$. This contrasts with the shear-band models, which have boundary conditions imposed for temperature at the surfaces which are a finite distance apart.

A clear problem with the model as outlined above, and discussed more fully in [10, 14], is the case that $W(t)$ is not of constant size; variation something like $W(t) = W_e \cos \Omega t$ is much more reasonable. A preliminary investigation has suggested possible bad behavior as $W(t)$ passes through 0, when the relative sliding is stopped for an instant, with certain quantities becoming infinite [14]. The modelling was based upon a quasi-steady theory for temperature in the inner region; this approximation will fail for t sufficiently close to the zeros of $W(t)$. For example, if $|t - (n + \frac{1}{2})\pi/\Omega| = O(h^2/D)$, where D is the thermal diffusivity, for any integer n the time derivative of temperature, T_t , contributes a lot and is not neglectable.

In this chapter we discuss a model problem for a guidance on the behavior of the solution to the inner problem for such time regimes. A key sub-problem for this purpose is the energy equation for the temperature T :

$$\rho c T_t = k T_{yy} + \tau w_y, \quad (3.1)$$

where ρ , c , and k are the density, the specific heat, and the thermal conductivity, respectively, all regarded as being constant. Furthermore, τ is the shear stress which is a-priori unknown and w is the velocity along the weld, i.e. along the soft layer, denoted by the z direction. It should be noted that even if there were significant temperature variation with z , thinness of the layer, $0 < h \ll L$, with L being the width of a workpiece, ensures that the T_{zz} term is negligible in comparison with T_{yy} . Also, the convective term involving $v T_y$, with v standing for the velocity in the y direction, is similar from (3.1). The last term in (3.1), τw_y , represents the dissipation and acts as a body heat source. Equation (3.1) is taken to hold for $0 < y < \infty$, with an assumed symmetry condition holding along the central plane,

$$T_y = 0 \quad \text{on} \quad y = 0,$$

and a matching condition at infinity,

$$T_y \rightarrow -A \quad \text{as} \quad y \rightarrow \infty.$$

Matching with the outer region should be expected to provide a rather stronger condition, $T \sim -Ay + T^*$ for $y \rightarrow \infty$, and some results appearing in the following section will require such a hypothesis.

The flow turns out to be “slow”, in the sense that inertial terms can be neglected from momentum equations, as is usual in lubrication theory. One of these equations then becomes simply $\tau_y = 0$ and hence

$$\tau = \tau(t).$$

The material is, in the lubricating layer, non-Newtonian and undergoes both shear thinning and thermal softening so that a relevant constitutive equation is of the form

$$\tau = F(T)|w_y|^{a-1}w_y = F(T)|w_y|^a \text{sign}(w_y), \quad (3.2)$$

with $F(T)$ a decreasing function of temperature and a is typically $1/4$. In fact, many metals have exponent a close to this value, although other types of material have different power laws [6, 11, 13]. Symmetry is again applied so that

$$w = 0 \quad \text{on} \quad y = 0,$$

while the imposed sliding motion gives

$$w \rightarrow W(t) \quad \text{as} \quad y \rightarrow \infty.$$

Equation (3.2) can be rewritten as $w_y = F(T)^{-1/a}|\tau|^{1/a} \text{sign} \tau$ which gives

$$W = \int_0^\infty w_y dy = (\text{sign}(\tau))|\tau|^{1/a} \int_0^\infty F(T)^{-1/a} dy,$$

and hence

$$\tau = \frac{|W(t)|^a \text{sign}(W(t))}{\left(\int_0^\infty F(T)^{-1/a} dy\right)^a}.$$

Substitution of this into (3.1) leads to

$$\rho c T_t = k T_{yy} + \frac{|W(t)|^{a+1} F(T)^{-1/a}}{\left(\int_0^\infty F(T)^{-1/a} dy\right)^{(1+a)}},$$

or

$$T_t = DT_{yy} + \frac{g(t)f(T)}{\left(\int_0^\infty f(T)dy\right)^{(1+a)}}, \quad (3.3)$$

where $g(t)$ is regarded as specified and $f(T) = F(T)^{-1/a}$ is an increasing function of temperature.

One further aspect of the problem must be mentioned. Since we are looking at a thin zone, the temperature T varies little in comparison with the outer region. Under various circumstances it is then possible to approximate $F(T)$ by a simple function. In [11], the temperature-dependent factor is often taken in the Arrhenius type, $\text{const.} \times \exp(T_a/T)$ for some activation temperature T_a . But in [6], a product of an Arrhenius term and a linear term vanishing at T_m was taken: $F(T) = \text{const.} \times (T - T_m) \exp(T_a/T)$. It is more appropriate if T approaches the melting temperature denoted by T_m .

The report [6] is concentrated on a typical case of a “hard” material. There, the operating conditions and material properties are such that T lies extremely close to T_m where $F(T)$ can be approximated by a linear function. A partially scaled version of (3.3) then becomes

$$u_t = u_{xx} - \frac{G(t)u^{-p}}{\left(\int_0^\infty u^{-p}dy\right)^{1+1/p}} \quad \text{for } 0 < x < \infty,$$

with

$$u_x(0, t) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} u_x(x, t) = 1,$$

where $p = 1/a \approx 4$ and $T = T_m - \text{const.} \times u$.

To justify this model, we shall investigate the corresponding autonomous problem where G is replaced by a constant numerically in Sect. 3.3.

In [10, 14], both “hard” and “soft” materials were considered. The latter case is subject to suitable material constants and operating conditions, where T does not get close to T_m in contrast to the former. For T_a sufficiently large, it is the Arrhenius term that varies significantly in the soft layer. Then F can be approximated by a simple exponential, as in high-activation-energy asymptotics, for instance in thin-flame theory. Then the simplified problem with suitable scaling is

$$u_t = u_{xx} + \frac{G(t)e^u}{\left(\int_0^\infty e^u dy\right)^{1+a}} \quad \text{for } 0 < x < \infty,$$

with

$$u_x(0, t) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} u_x(x, t) = -1.$$

Again we investigate the autonomous problem. In this particular equation we can, by using suitable rescaling, replace G by 1. Numerical experiments are again carried out in Sect. 3.3. Because of the structure of this problem, see Sect. 3.2, it admits a suitable

energy. Then we are able to prove that the solution exists for all time and tends, uniformly on compact intervals, to the unique steady state. This is strongly indicative of the non-autonomous model being a well-posed and sensible mathematical problem and of being a good model for the local behavior of temperature and flow in these thin, short-time regimes occurring in linear friction welding.

3.2 The Exponential Case

In the exponential case the time-dependent problem is written in the form

$$u_t = u_{xx} + \frac{e^u}{\left(\int_0^\infty e^u dx\right)^{1+a}}, \quad 0 < x < \infty, \quad t > 0, \quad a > 0, \quad (3.4)$$

$$u_x(0, t) = 0, \quad u_x(x, t) \rightarrow -1 \text{ as } x \rightarrow \infty, \quad t > 0, \quad (3.5)$$

$$u(x, 0) = u_0(x), \quad 0 < x < \infty. \quad (3.6)$$

We consider $u_0(x)$ which satisfies the compatibility conditions

$$u'_0(0) = 0, \quad \text{and} \quad u'_0(x) \rightarrow -1 \text{ as } x \rightarrow \infty, \quad (3.7)$$

and the monotonicity condition

$$u'_0(x) < 0, \quad 0 < x < \infty.$$

This condition implies, via the maximum principle, that $u_x(x, t) < 0$ for $0 < x < \infty$ and $t > 0$. For the results below we need the stronger condition that

$$u_0(x) + x \quad \text{is bounded.} \quad (3.8)$$

Theorem 3.2.1 *Problem (3.4)–(3.6) has a global-in-time solution.*

Proof We consider a self-similar solution of the problem

$$\begin{aligned} v_t &= v_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ v_x(0, t) &= 0, \quad v_x(x, t) \rightarrow -1 \text{ as } x \rightarrow \infty, \quad t > 0, \end{aligned}$$

that is, $v(x, t) = -t^{1/2}V(\eta)$ for $\eta = xt^{-1/2}$, where $V(\eta)$ satisfies

$$V'' + \frac{\eta}{2} V' - \frac{1}{2} V = 0, \quad V'(0) = 0, \quad V(\eta) \sim \eta, \quad \eta \rightarrow \infty.$$

Then for c sufficiently large negative, $c + v$ is lower solution of problem (3.4)–(3.6) and so

$$u \geq c - t^{1/2}V(Rt^{-1/2}) \quad \text{for every } 0 \leq x \leq R. \quad (3.9)$$

Moreover the function $\theta(x, t)$ satisfying the problem

$$\begin{aligned}\theta_t &= \theta_{xx} + \frac{e^u}{\left(\int_0^R e^u dx\right)^{1+a}}, \quad 0 < x < R, \quad t > 0, \quad a > 0, \\ \theta_x(0, t) &= 0, \quad \theta_x(R, t) = 0, \quad t > 0, \\ \theta(x, 0) &= u_0(x), \quad 0 < x < R,\end{aligned}$$

is an upper solution of problem (3.4)–(3.6) restricted to the interval $[0, R]$ for every $R > 0$, and hence $u(x, t) \leq \theta(x, t)$ for $0 \leq x \leq R$.

Setting $h(x, t) = e^u / \left(\int_0^R e^u dx\right)^{1+a}$, we have

$$\int_0^R h(x, t) dx = \frac{1}{\left(\int_0^R e^u dx\right)^a} \leq \frac{1}{\left(e^c \int_0^R e^{-t^{1/2}} V(Rt^{-1/2}) dx\right)^a} \leq R^{-a} e^{-ac} e^{at^{1/2} V(Rt^{-1/2})},$$

and so

$$u(x, t) \leq \theta(x, t) \leq B_1(R) + B_2(R) t^{1/2} e^{a V(Rt^{-1/2}) t^{1/2}}, \quad 0 \leq x \leq R,$$

for every $R > 0$. This, along with the fact that $u(x, t)$ is decreasing in x , excludes blow-up in finite time. $\square$

The associated steady-state problem of (3.4)–(3.6) is

$$w'' + \frac{e^w}{\left(\int_0^\infty e^w dx\right)^{1+a}} = 0, \quad 0 < x < \infty, \quad w'(0) = 0, \quad w'(x) \rightarrow -1 \quad \text{as } x \rightarrow \infty, \quad (3.10)$$

and can be solved exactly to find $w(x) = \ln\left(\frac{1}{2} \operatorname{sech}^2 \frac{x}{2}\right)$. Note also that $\int_0^\infty e^w dx = 1$.

Setting $z(x, t) = u(x, t) - w(x)$, we arrive at

$$z_t = z_{xx} + \frac{e^u}{\left(\int_0^\infty e^u dx\right)^{1+a}} - \frac{e^w}{\left(\int_0^\infty e^w dx\right)^{1+a}}, \quad 0 < x < \infty, \quad t > 0, \quad a > 0, \quad (3.11)$$

$$z_x(0, t) = 0, \quad z_x(x, t) \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad t > 0, \quad (3.12)$$

$$z(x, 0) = z_0(x) = u_0(x) - w(x). \quad (3.13)$$

To obtain a priori estimates we consider the functional

$$J(t) = \frac{1}{2} \int_0^\infty z_x^2 dx + \frac{1}{a} \left(\left(\int_0^\infty e^w e^z dx \right)^{-a} + a \int_0^\infty e^w z dx - 1 \right)$$

which satisfies, due to the boundary conditions,

$$\begin{aligned}
\frac{dJ(t)}{dt} &= \int_0^\infty z_x z_{xt} dx - \frac{\int_0^\infty e^u u_t dx}{\left(\int_0^\infty e^u dx\right)^{1+a}} + \int_0^\infty e^w z_t dx \\
&= - \int_0^\infty (u_{xx} - w'') u_t dx - \frac{\int_0^\infty e^u u_t dx}{\left(\int_0^\infty e^u dx\right)^{1+a}} + \int_0^\infty e^w u_t dx \\
&= - \int_0^\infty u_t^2 dx \leq 0.
\end{aligned}$$

Thus we obtain

$$J(t) \leq J(0) \equiv J_0 = \frac{1}{2} \int_0^\infty (z'_0)^2 dx + \frac{1}{a} \left(\left(\int_0^\infty e^{u_0} dx \right)^{-a} + a \int_0^\infty e^w z_0 dx - 1 \right),$$

or equivalently,

$$\frac{1}{2} \int_0^\infty z_x^2 dx + \int_0^\infty e^w z dx \leq J_0 + \frac{1}{a} - \frac{1}{a} \left(\int_0^\infty e^u dx \right)^{-a} \leq J_0 + \frac{1}{a} = K,$$

which implies

$$0 \leq \frac{1}{2} \int_0^\infty z_x^2 dx = K_1 \leq K - \int_0^\infty e^w z dx. \quad (3.14)$$

Using the Cauchy–Schwarz inequality, we derive

$$\left(\int_0^r |z_x| dx \right)^2 \leq r \int_0^r z_x^2 dx \leq 2 K_1 r,$$

which yields that

$$|z(r, t) - z(0, t)| \leq \int_0^r |z_x| dx \leq \sqrt{2 K_1 r}.$$

Setting $m(t) = z(0, t)$, we obtain

$$m - \sqrt{2 K_1 r} \leq z(r, t) \leq m + \sqrt{2 K_1 r} \quad \text{for every } r > 0, \quad (3.15)$$

and so

$$m - \sqrt{2 K_1 C_0^2} \leq \int_0^\infty e^w z dx \leq m + \sqrt{2 K_1 C_0^2}, \quad (3.16)$$

writing $\int_0^\infty \sqrt{x} e^w dx = C_0 \approx 1.072$ and recalling that $\int_0^\infty e^w dx = 1$.

From (3.14) we get

$$0 \leq K_1 = \frac{1}{2} \int_0^\infty z_x^2 dx \leq K - m + \sqrt{2 K_1 C_0^2},$$

so

$$(\sqrt{K_1} - \sqrt{2} C_0) \sqrt{K_1} \leq K - m,$$

or

$$\left(\sqrt{K_1} - \frac{C_0}{\sqrt{2}} \right)^2 \leq K - m + \frac{C_0^2}{2}. \quad (3.17)$$

Hence $m(t)$ is bounded above. Furthermore, the next lemma clarifies the long-time behavior of $m(t)$.

Lemma 3.2.2 *It holds that $\lim_{t \rightarrow \infty} m(t) = 0$.*

Proof Since $m(t)$ is bounded above, there are three possible types of long-time behavior of $m(t)$: (i) $m(t) \rightarrow -\infty$ as $t \rightarrow \infty$, (ii) $m(t)$ oscillates as $t \rightarrow \infty$ or (iii) $m(t) \rightarrow m^* > -\infty$.

In the first case, using $\frac{\partial z}{\partial x} \leq 1$, we obtain $z(x, t) - m(t) \leq x$. Then inequality (3.17) implies

$$z(x, t) - m(t) \leq \min\{x, \sqrt{2 K_1 x}\} = \begin{cases} x, & 0 \leq x \leq 2 K_1 \\ \sqrt{2 K_1 x}, & x \geq 2 K_1 \end{cases},$$

and hence

$$\int_0^\infty e^u dx = \int_0^\infty e^{w+z} dx \leq e^m \left(\int_0^{2K_1} e^{w+x} dx + \int_{2K_1}^\infty e^{w+\sqrt{2 K_1 x}} dx \right).$$

Let $M(t) = -u(0, t) = \ln 2 - m(t)$. Relation (3.17) gives

$$K_1 \leq M + C\sqrt{M} \quad \text{for } M \text{ sufficiently large,}$$

where C is a positive constant. Then

$$\int_0^{2K_1} e^{w+x} dx \leq \int_0^{2K_1} e^{C_1} dx \leq C_2(M + C\sqrt{M}),$$

and

$$\int_{2K_1}^\infty e^{w+\sqrt{2 K_1 x}} dx \leq \int_{2K_1}^\infty e^{C_1 - x + \sqrt{2 K_1 x}} dx = C_3 \int_0^\infty e^{2K_1 \sqrt{1+x_1/2K_1} - 2K_1 - x_1} dx_1 < C_4,$$

for C_1 being a constant such that $w(x) + x \leq C_1$, and C_2, C_3 , and C_4 are other positive constants. Finally we get

$$\int_0^\infty e^u dx < 2e^{-M} \left(C_2(M + C\sqrt{M}) + C_4 \right) < C_5 M e^{-M}, \quad (3.18)$$

for M sufficiently large.

Setting $g(t) = \left(\int_0^\infty e^u dx \right)^{-(1+a)}$, in view of (3.18) we obtain

$$g(t) \geq C_6 M^{-(1+a)} e^{M(1+a)} \quad \text{for } M \gg 1,$$

for some positive constant C_6 . Now we take $M_2 > M_1 > 0$ to be chosen suitably later. First, for some time t_1 we assume

$$u(0, t_1) = -M_1, \quad u(0, t) < -M_1 \quad \text{for } t_1 < t < t_1 + \delta, \quad (3.19)$$

and

$$u(0, t) > -M_2 \quad \text{for } 0 \leq t \leq t_1 + \delta, \quad (3.20)$$

where $\delta > 0$. If (3.19) does not hold then $u(0, t)$ is certainly bounded below and so is $m(t)$, leading to contradiction.

While $M(t) = u(0, t) \geq -M_2$, $-M_2 - x$ is a lower solution for u , so that $u(x, t) > -M_2 - x$ for $0 \leq t \leq t_1 + \delta$ and $x > 0$. On the other hand, relation (3.15) implies

$$u(x, t) \geq M(t) + \ln 2 + w(x) - \sqrt{2K_1(M(t))x} \quad \text{for } x > 0, \quad t > 0,$$

with $K_1(M)$ denoting the largest value of K_1 such that (3.17). In particular, at $t = t_1$ we have

$$u(x, t) \geq \max \left\{ -M_2 - x, -M_1 + \ln 2 + w(x) - \sqrt{2K_1(M_1)x} \right\}.$$

Also, as long as $M(t) \geq M_1$, i.e. $u(0, t) \leq -M_1$, which is certainly true for $t_1 \leq t \leq t_1 + \delta$, we get $g(t) \geq C_6 e^{(1+a)M} M^{-(1+a)} \geq C_6 e^{(1+a)M_1} M_1^{-(1+a)}$ for M_1 large enough. Writing $u = U - M_2$, we have

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + g(t)e^{-M_2}e^U, \quad 0 < x < \infty, \quad t > t_1, \quad (3.21)$$

$$\frac{\partial U}{\partial x} = 0 \quad \text{at } x = 0, \quad U_x \rightarrow -1 \quad \text{as } x \rightarrow \infty, \quad t > t_1. \quad (3.22)$$

Then it holds that

$$\begin{aligned} U(x, t_1) &\geq \max \left\{ -x, M_2 - M_1 + \ln 2 + w(x) - \sqrt{2K_1(M_1)x} \right\} \\ &\geq \max \left\{ -x, C_7 \sqrt{M_1}(1 - \sqrt{x}) + w(x) \right\} \\ &\geq \begin{cases} C_7 \sqrt{M_1}(1 - \sqrt{x}) - w(x), & 0 \leq x < 1 \\ -x, & x \geq 1 \end{cases}, \end{aligned} \quad (3.23)$$

for C_7 being a positive constant and $M_2 = M_1 + C_7\sqrt{M_1} - w(1)$ for M_1 sufficiently large. Also, it holds that $g(t)e^{-M_2} \geq C_6e^{(aM_1 - C_7\sqrt{M_1})}M_1^{-(1+a)} \geq C_8e^{C_9M_1}$ for M_1 sufficiently large.

It is easily seen that

$$W(x) \equiv \max\{-x, k(1 - x^2) - 1\} = \begin{cases} k(1 - x^2) - 1, & 0 \leq x < 1 \\ -x & x \geq 1 \end{cases},$$

is a lower solution of problem (3.21)–(3.23) for $t \geq t_1$, while $U(0, t) \leq C_7\sqrt{M_1}$, provided that

- (i) $k > 1$ to ensure $U(0, t) \geq W(0) > 0$, i.e. $u(0, t) > -M_2$, and that $W(x)$ is well-defined, i.e. $k(1 - x^2) - 1 + x \geq 0$ for $0 \leq x < 1$,
- (ii) $k \leq \frac{C_7\sqrt{M_1}}{4} - w(x)$ to ensure that $W(x) \leq U(x, t)$ at $t = t_1$, i.e. $k(1 - x^2) - 1 \leq C_7\sqrt{M_1}(1 - \sqrt{x})$ for $0 \leq x < 1$,
- (iii) $k \leq \frac{C_8}{2}e^{C_9M_1-1}$ (to ensure $W'' + ge^{-M_2}e^W \geq 0$).

Taking $k = \frac{C_7}{4}\sqrt{M_1}$, and M_1 large, we have (i)–(iii) all, and W is a lower solution to (3.21)–(3.23) for $t \geq t_1$. Thus $U(0, t) > W(0, t) > 0$, i.e. $u(0, t) > -M_2$ as long as $u(0, t) \leq -M_1$. Consequently, we have $u(0, t) > -M_2$ for every $t > 0$ which contradicts the hypothesis (i), $m(t) \rightarrow -\infty$ as $t \rightarrow \infty$.

In the second case that $m(t)$ oscillates, there exist $m_a < m_b$ and sequences of time $t_1 \leq \hat{t}_1 \leq t_2 \leq \hat{t}_2 \leq \dots$ such that $m(t_n) = m_a$, $m(\hat{t}_n) = m_b$ and $m_a < m(t) < m_b$ for $t_n < t < \hat{t}_n$. Then for $t_n < t < \hat{t}_n$ we obtain

$$\begin{aligned} \int_0^\infty e^u dx &\geq \frac{1}{2} e^m \int_0^\infty \operatorname{sech}^2 \frac{x}{2} e^{-\sqrt{2K_1}x} dx \geq \frac{1}{2} e^m \int_0^\infty e^{-x-\sqrt{2K_1}x} dx \\ &\geq \frac{1}{2} e^{m_a} \int_0^\infty e^{-x-\sqrt{2\widehat{C}_1}x} dx = A > 0, \end{aligned}$$

where $\widehat{C}_1 = \left(\sqrt{K - m_a + \frac{C_0^2}{2}} + \frac{C_0}{\sqrt{2}} \right)^2$ and A is a constant.

In such time intervals, $m(t)$ satisfies

$$\frac{dm}{dt} = \frac{d}{dt}u(0, t) \leq \frac{e^{u(0, t)}}{(\int_0^\infty e^u dx)^{1+a}} = \frac{e^m}{2(\int_0^\infty e^u dx)^{1+a}} \leq \frac{1}{2} e^{m_b} A^{-(1+a)} = \widehat{C}_2,$$

which, by integration, implies $\hat{t}_n - t_n \geq \delta = \frac{m_b - m_a}{\widehat{C}_2} > 0$.

Using the estimate (3.15) for $z(x, t)$, we obtain a lower bound for the functional $J(t)$. Indeed, taking also into account of (3.16) and (3.17), we derive

$$J(t) \geq \int_0^\infty e^w z dx - \frac{1}{a} \geq m - \sqrt{2K_1 C_0^2} - \frac{1}{a} \geq m_a - \sqrt{2K_1 C_0^2} - \frac{1}{a}.$$

Thus, taking a subsequence if necessary, there exists a sequence $\tilde{t}_n \in (t_n, \hat{t}_n)$ s.t. $\dot{J}(\tilde{t}_n) = \max_{[t_n, \hat{t}_n]} \dot{J}(t)$ with $\dot{J}(\tilde{t}_n) \rightarrow 0$ as $n \rightarrow \infty$, where $\dot{\cdot} = \frac{d}{dt}$, and $m(\tilde{t}_n) \rightarrow m^*$ as $n \rightarrow \infty$. Hence the sequence $u(\cdot, \tilde{t}_n)$ is uniformly bounded in $C([0, R])$ for every $R > 0$. Using Schauder-type estimates we obtain that $u(\cdot, \tilde{t}_n)$ is uniformly bounded in $C^1([0, R]) \cap C^2((0, R))$ as well. Therefore, there is a subsequence denoted again by $u(x, \tilde{t}_n)$ and a function $\widehat{w}(x)$ such that

$$u(\cdot, \tilde{t}_n) \rightarrow \widehat{w}(\cdot) \quad \text{as } n \rightarrow \infty \quad \text{in } C^1([0, R]) \cap C^2((0, R)) \quad \text{for every } R > 0.$$

Taking now the inner product in $L^2([0, R])$ of (3.4) with an arbitrary function $\xi \in H_0^1([0, R])$ for $t = \tilde{t}_n$, we derive

$$\int_0^R u_t(x, \tilde{t}_n) \xi(x) dx = - \int_0^R u_x(x, \tilde{t}_n) \xi'(x) dx + \int_0^R \frac{e^{u(x, \tilde{t}_n)}}{(\int_0^\infty e^{u(y, \tilde{t}_n)} dy)^{1+a}} \xi dx, \quad (3.24)$$

where ξ' is the distributional derivative of ξ .

Passing to the limit as $\tilde{t}_n \rightarrow \infty$ in (3.24) and taking also into account of

$$\int_0^R u_t^2(x, \tilde{t}_n) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for every } R > 0,$$

with Lebesgue's dominated convergence theorem, we deduce

$$- \int_0^R \widehat{w}'(x) \xi'(x) dx + \int_0^R \frac{e^{\widehat{w}(x)}}{(\int_0^\infty e^{\widehat{w}(y)} dy)^{1+a}} \xi dx = 0.$$

Since $\widehat{w}(x) \in C^1([0, R]) \cap C^2((0, R))$ for every $R > 0$, thus $\widehat{w}(x)$ coincides with the unique classical steady-state solution, hence $m^* = 0$. It is a contradiction because under the hypothesis that $m(t)$ oscillates we can always find $m_a < m_b$ with $0 \notin (m_a, m_b)$. Using the same arguments, we can also rule out case (iii) that $m(t) \rightarrow m^* \neq 0$. Thus we finally deduce that $m(t) \rightarrow 0$ as $t \rightarrow \infty$. $\square$

We claim that $u(x, t) \rightarrow w(x)$ as $t \rightarrow \infty$ for every $0 < x < \infty$. To this end we need the following

Lemma 3.2.3 *The difference between the solution u and equilibrium w , $z(x, t) = u(x, t) - w(x)$, is uniformly bounded.*

Proof Using the fact that $u(0, t) \rightarrow w(0) = -\ln 2$ as $t \rightarrow \infty$, we get that $z(x, t)$ is bounded from below, i.e. $u(x, t) \geq M_l - x$ for

$$M_l = \min\{\inf_{t>0} u(0, t), \inf_{x>0} (u_0(x) + x)\} < 0.$$

Note that, by (3.8) and Lemma 3.2.2, M_l is finite.

To obtain an upper bound for $u(x, t)$ we apply the intersection comparison arguments. Consider a related steady-state solution $w(x; \lambda, R)$ satisfying the equation $w'' + \lambda e^w = 0$, centred on $R \geq 0$, i.e. $w(x; \lambda, R) = \ln\left(\frac{1}{2\lambda} \operatorname{sech}^2\left(\frac{x-R}{2}\right)\right)$, for some $\lambda > 0$. For any λ we can take R large enough so that $u_0(x)$ crosses $w(x; \lambda, R)$ exactly once, using the fact that $u'_0 < 0$. With such an R , $w(\cdot; \lambda, R) > u(\cdot)$ at infinity and, because $u_x < 0$, if a second crossing occurs at some time it must appear at some time t_a with $u(x, t_a)$ tangent to $w(x, \lambda, R)$ at some point $x_a > R$.

There remain two possibilities:

- (a) $u(x, t_a)$ crosses $w(x, \lambda, R)$ at some $x < R$ and then $u(x, t_a)$ touches $w(x, \lambda, R)$ from below at $x_a > R$;
- (b) at some time $t = t_b$, $u(\cdot)$ crosses $w(\cdot; \lambda, R)$ at $x = R$.

In the former case we should have $\lambda \leq \left(\int_0^\infty e^u dx\right)^{-(1+a)}$ at $t = t_a$. On the other hand with $u(x, t)$ bounded below, e.g. $u(x, t) \geq M_l - x$, a choice of

$$\lambda \geq \lambda_1 = \left(\int_0^\infty e^{M_l - x} dx\right)^{-(1+a)} \geq \left(\int_0^\infty e^u dx\right)^{-(1+a)},$$

ensures that this cannot happen. Notice that λ_1 is independent of R .

In case (b) we proceed as following. From the definition of $J(t)$ we deduce that

$$J(t) > \frac{1}{2} \int_0^R z_x^2 dx + \int_0^\infty e^w z dx - \frac{1}{a} \quad \text{for every } R > 0. \quad (3.25)$$

From the lower bound on $u(x, t)$, we derive

$$\int_0^\infty e^{w(x)} z(x) dx \geq \int_0^\infty e^{w(x)} (M_l - x - w(x)) dx = K_2.$$

Furthermore, at $t = t_b$ we have, writing $M = \sup_{t>0} m(t)$,

$$\begin{aligned} \int_0^R z_x^2 dx &\geq \frac{[z(R, t_b) - z(0, t_b)]^2}{R} \\ &= \frac{[u(R, t_b) - w(R) - m(t_b)]^2}{R} \\ &= \frac{[-\ln 2\lambda + \ln(2 \cosh^2 \frac{R}{2}) - m(t_b)]^2}{R} \\ &\geq [\ln e^R - \ln 4\lambda - M]^2 / R \\ &\geq R - 2(\ln 4\lambda + M), \end{aligned}$$

for R sufficiently large. Then, by (3.25) we obtain, taking $R \geq 2[J(0) - K_2 + \frac{1}{a} + \ln 4\lambda + M]$,

$$J(t_b) > \frac{R}{2} - (\ln 4\lambda + M) + K_2 - \frac{1}{a} \geq J(0),$$

which cannot happen.

Thus $u(x, t)$ intersects $w(x; \lambda, R)$ at most once for every $t > 0$ and so there must be some constant M_u such that $u(x, t) \leq M_u - x$ for $0 < x < \infty$ and $t > 0$. Therefore, z is uniformly bounded. Before proceeding to the proof of the convergence towards the unique steady-state solution we need the following auxiliary result. $\square$

Lemma 3.2.4 *There exists a constant $C > 0$ such that $|\ddot{J}(t)| < C$ for every $t > 0$.*

Proof Since

$$|\ddot{J}(t)| = 2 \left| \int_0^\infty z_t z_{tt} dx \right| \leq 2 \int_0^\infty |z_t| |z_{tt}| dx,$$

it suffices to show that

$$\int_0^\infty |z_t| |z_{tt}| dx \leq C < \infty.$$

Since $M_l - x \leq u(x, t) \leq M_u - x$ for $0 < x < \infty$, we have that $H(x, t) = e^u / (\int_0^\infty e^u dx)^{1+a}$ is uniformly integrable in $(0, \infty)$ for every $t > 0$. Here,

$$\begin{aligned} u_{xx}(x, t) &= \int_0^\infty G_{xx}(x, \zeta, t) u_0(\zeta) d\zeta \\ &+ \int_0^t \int_0^\infty G_{xx}(x, \zeta, t - \tau) (H(\zeta, \tau) - H(x, \tau)) d\zeta d\tau, \end{aligned} \quad (3.26)$$

where $G(x, t)$ is the Green's function of the heat equation in $(0, \infty)$ satisfying the boundary conditions $G_x(0, t) = 0$ and $G_x(x, t) \rightarrow 0$ as $x \rightarrow \infty$. Using the smoothing effect of the Green's function and the uniform integrability of $H(x, t)$ in $(0, \infty)$, we obtain from (3.26) that $u_{xx}(x, t)$ is uniformly integrable in the half line for every $t > 0$. Coming back to the Eq. (3.4), we deduce that the same holds for $u_t(x, t)$. Since $z_t = u_t$ we derive that z_t is bounded for $t \geq \varepsilon > 0$, as well as its summability in $(0, \infty)$.

Using now a bootstrap argument, we can prove that $\int_0^\infty |H_t(x, t)| dx < K_4 < \infty$. Via relation

$$\begin{aligned} u_{txx}(x, t) &= \int_0^\infty G_{xx}(x, \zeta, t) u_t(\zeta, 0) d\zeta \\ &+ \int_0^t \int_0^\infty G_{xx}(x, \zeta, t - \tau) (H_t(\zeta, \tau) - H_t(x, \tau)) d\zeta d\tau, \end{aligned}$$

we obtain $\int_0^\infty |u_{txx}(x, t)| dx < K_5 < \infty$. Therefore, from

$$u_{tt}(x, t) = u_{txx}(x, t) + H_t(x, t),$$

we deduce that $u_{tt}(x, t)$ is uniformly integrable in the half line for every $t > 0$ and so is $z_{tt}(x, t)$. Hence it holds that

$$|\ddot{J}(t)| \leq 2 \int_0^\infty |z_t| |z_{tt}| dx < C < \infty, \quad t > 0. \quad (3.27)$$

□

Theorem 3.2.5 *The solution $u(x, t)$ of problem (3.4)–(3.6) converges as $t \rightarrow \infty$ to the unique steady-state solution $w(x)$.*

Proof First, we show

$$\lim_{t \rightarrow \infty} \dot{J}(t) = 0. \quad (3.28)$$

We assume that there exists a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \dot{J}(t_n) = - \lim_{n \rightarrow \infty} \int_0^\infty u_t^2(x, t_n) dx = -c < 0, \quad (3.29)$$

and derive a contradiction.

Due to (3.29) there exists N such that $\dot{J}(t_n) < -2c/3$, for $n \geq N$. Using $\ddot{J}(t) < K$, we obtain $\dot{J}(t) < -c/3$ for $t_n \leq t \leq t_n + c/3K$ and $n \geq N$. The latter yields $J(t_n) \rightarrow -\infty$ as $n \rightarrow \infty$ leading to a contradiction.

Now since $z(x, t)$ is uniformly bounded in every interval $[0, R]$, $R > 0$, we obtain via a Schauder-type estimate that $z(x, t)$ belongs to $C^{2+q, 1+q/2}([0, R] \times (0, \infty))$, for some $q > 0$, and for every $R > 0$.

The latter implies that there exists a sequence $t_n \rightarrow \infty$ and a function $\psi(x)$ such that

$$\|z(\cdot, t_n) - \psi(\cdot)\|_{C^1([0, R]) \cap C^2((0, R))} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{for every } R > 0,$$

or equivalently,

$$\begin{aligned} \omega(u_0) = \{ \phi \in L^\infty((0, \infty)) : \text{there exists } t_n \rightarrow \infty : \|u(\cdot, t_n; u_0) \\ - \phi(\cdot)\|_{C^1([0, \infty)) \cap C^2((0, \infty))} \rightarrow 0 \} \neq \emptyset. \end{aligned}$$

Now we claim that $\omega(u_0) \subseteq S$, for S the set of the steady states, and hence $\omega(u_0) = \{w\}$, since $S = \{w\}$. Indeed, considering $\phi \in \omega(u_0)$, there is a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $\|u(\cdot, t_n) - \phi(\cdot)\|_{C^1([0, R]) \cap C^2((0, R))} \rightarrow 0$, or equivalently $\|z(\cdot, t_n) - \psi(\cdot)\|_{C^1([0, R]) \cap C^2((0, R))} \rightarrow 0$ as $n \rightarrow \infty$, for $\psi = \phi - w$. By similar arguments as in Lemma 3.2.2, taking also into account (3.28), we derive $\psi \equiv 0$, hence $\omega(u_0) = \{w\}$. The latter yields the desired result, otherwise, due to the uniform boundedness of z , there must be a subsequence $t_m \rightarrow \infty$ such that $\|z(\cdot, t_m) - \psi(\cdot)\|_{C^1([0, \infty)) \cap C^2((0, \infty))} \rightarrow 0$ with $\psi \neq 0$ or, equivalently, $\|u(\cdot, t_m) - \phi(\cdot)\|_{C^1([0, \infty)) \cap C^2((0, \infty))} \rightarrow 0$ as $m \rightarrow \infty$ with $\phi \neq w$, contradicting the above. □

3.3 Numerical Results

In this section we present a numerical approach for both cases: the “soft” and the “hard” material case.

3.3.1 The Soft Material Case

Since the problem is defined in an infinite domain, $[0, \infty)$, an approximated problem is solved numerically in the domain, $[0, b]$, for b large enough, i.e.

$$u_t = u_{xx} + \frac{e^u}{I^{1+a}}, \quad 0 < x < b, \quad t > 0, \quad (3.30)$$

$$u_x(0, t) = 0, \quad u_x(b, t) = -1, \quad (3.31)$$

$$u(x, t) = u_0(x). \quad (3.32)$$

Regarding the term, I , we have

$$I = \int_0^\infty e^u dx = \int_0^b e^u dx + \int_b^\infty e^u dx \sim \int_0^b e^u dx + 2e^{-b}, \text{ taking } b \gg 1,$$

assuming that for x large $u \sim w$.

In order to solve problem (3.30)–(3.32) numerically, a three-step Crank–Nicolson scheme (which is unconditionally stable) is used. Taking a partition of $M + 1$ points in $[0, b]$, $0 = x_0, x_0 + \delta x = x_1, \dots, x_M = b$ and using a time step δt for a partition in the time interval $[0, T]$, we have

$$\begin{aligned} \frac{u_j^{i+1} - u_j^{i-1}}{2\delta t} &= \frac{1}{2} \frac{u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1}}{\delta x^2} + \frac{1}{2} \frac{u_{j+1}^{i-1} - 2u_j^{i-1} + u_{j-1}^{i-1}}{\delta x^2} \\ &\quad + \frac{e^{u_j^i}}{\left(\int_0^b e^{u^i} dx + e^{-b}\right)^{1+a}}. \end{aligned} \quad (3.33)$$

The integral $\int_0^b e^{u^i} dx$, where $u^i = (u_1^i, u_2^i, \dots, u_M^i)^T$, is evaluated in each time step by Simpson’s rule. Taking into account the boundary conditions the numerical scheme takes the form

$$Au^{i+1} = Bu^{i-1} + bu^i, \quad (3.34)$$

or

$$u^{i+1} = A^{-1}Bu^{i-1} + A^{-1}bu^i,$$

where A, B are $M \times M$ matrices and b a $1 \times M$ vector.

Fig. 3.3 The numerical solution of problem (3.30)–(3.32), for various values of b and $a = 1/2$. In each case the plot of $u(x, t)$ is at $t = 25$

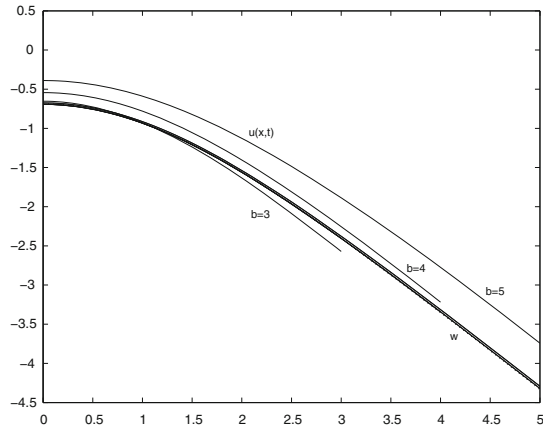
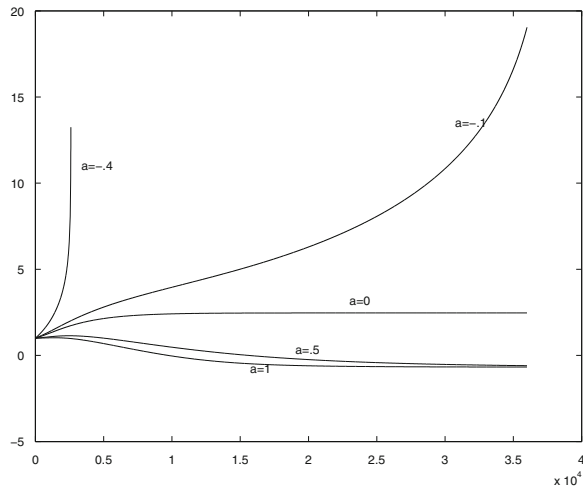


Fig. 3.4 The numerical solution of problem (3.30)–(3.32) at $x = 0$, $u(0, t)$, is plotted for the values of a , $-0.4, -0.1, 0, 0.5, 1$, taking $b = 5$



The first attempt is to investigate if by changing b we notice any difference in the solution.

In Fig. 3.3, $u(x, t)$ is plotted, at $t = 25$, against $x \in [0, b]$ for various values of b together with the steady-state solution w . It is found that $u(x, t)$ converges to the steady state: running the program for a time interval $[0, T]$ and in a space interval $[0, b]$ for increasing values of both T and b , starting with initial condition $u_0(x) = 0$, the plot of $u(x, T)$ approaches with that of w (the thick line). The conclusion is that the numerical solution approximates the solution of (3.4)–(3.6) if b is large enough, e.g. $b \geq 5$. However, in Fig. 3.3, the relatively poor agreement for $b = 5$ arises from the slower convergence, with respect to time, associated with the larger interval.

In the next plot, Fig. 3.4, we plot $u(t, 0)$ against time starting with initial condition $u_0(x) = 0$ for various values of a . For $a > 0$ the solution converges to the steady state while for $a < 0$ the solution blows up.

3.3.2 The Hard Material Case

For the problem with $f(u) = u^{-p}$ no analytical results are yet available. For this case we must therefore employ a numerical approximation in order to get an indication of the behavior of the solution.

The problem in this case takes the form

$$u_t = u_{xx} + \frac{(-u)^{-p}}{\left(\int_0^\infty (-u)^{-p} dx\right)^{1+a}}, \quad 0 < x < \infty, \quad t > 0, \quad \text{with } a = 1/p > 0, \quad (3.35)$$

$$u_x(0, t) = 0, \quad u_x(x, t) \rightarrow -1 \text{ as } x \rightarrow \infty, \quad t > 0, \quad (3.36)$$

$$u(x, 0) = u_0(x), \quad 0 < x < \infty. \quad (3.37)$$

An equivalent form of the problem can be given by replacing u by $-u$ and setting $u_x(x, t) \rightarrow 1$. Then the problem has the form

$$u_t = u_{xx} - \frac{u^{-p}}{\left(\int_0^\infty u^{-p} dx\right)^{1+a}}, \quad 0 < x < \infty, \quad t > 0, \quad a > 0, \quad (3.38)$$

$$u_x(0, t) = 0, \quad u_x(x, t) \rightarrow 1 \text{ as } x \rightarrow \infty, \quad t > 0, \quad (3.39)$$

$$u(x, 0) = u_0(x), \quad 0 < x < \infty. \quad (3.40)$$

As for the exponential case we will apply a finite difference scheme to the approximate problem in the domain $0 < x < b$ for b being large enough. The approximate problem is

$$u_t = u_{xx} - \frac{u^{-p}}{\left(\int_0^b u^{-p} dx\right)^{1+a}}, \quad 0 < x < b, \quad t > 0, \quad a > 0, \quad (3.41)$$

$$u_x(0, t) = 0, \quad u(b, t) = b \quad t > 0, \quad (3.42)$$

$$u(x, 0) = u_0(x), \quad 0 < x < \infty. \quad (3.43)$$

The finite difference scheme in this case takes the form

$$\begin{aligned} \frac{u_j^{i+1} - u_j^{i-1}}{2\delta t} &= \frac{1}{2} \frac{u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1}}{\delta x^2} + \frac{1}{2} \frac{u_{j+1}^{i-1} - 2u_j^{i-1} + u_{j-1}^{i-1}}{\delta x^2} \\ &\quad - \frac{(u_j^i)^{-p}}{\left(\int_0^b (u^i)^{-p} dx + \frac{b^{-p+1}}{-p+1}\right)^{1+a}}. \end{aligned} \quad (3.44)$$

Again the integral $\int_0^b (u^i)^{-p} dx$, where $u^i = (u_1^i, u_2^i, \dots, u_M^i)^T$, is evaluated in each time step by Simpson's rule. Taking into account the boundary conditions the numerical scheme takes the form of Eq. (3.34).

In addition in this case, in order to test the convergence of the scheme (3.44), we have to approximate the steady-state solution of the problem,

$$w_{xx} = \frac{w^{-p}}{\left(\int_0^b w^{-p} dx\right)^{1+a}}, \quad 0 < x < b, \quad a > 0, \quad (3.45)$$

$$w_x(0, t) = 0, \quad w(b, t) = b. \quad (3.46)$$

Again using a simple finite difference scheme we have that

$$\frac{w_{j+1} - 2w_j + w_{j-1}}{\delta x^2} = \frac{w^{-p}}{\left(\int_0^b (w)^{-p} dx\right)^{1+a}}. \quad (3.47)$$

This results in a nonlinear algebraic system of the form $Mw = b(w) + c$ which is solved by using a Newton–Raphson iterative scheme.

In Fig. 3.5 the numerical solution of the steady-state problem is plotted, for $p = 4$ and for different values of the parameter b . The result indicates that as b increases the numerical solution converges.

In Fig. 3.6 the problem (3.41)–(3.43) is solved numerically in a time interval $[0, T]$ and the solution at time t is plotted against space for different values of the parameter p (solid line). In each case the solution approaches the relevant steady state (dotted line). In each case the initial condition is $u_0(x) = 1$ for $x < 1$, $u_0(x) = x$ for $x \geq 1$; the solution is plotted at $t = T = 25$. Finally in Fig. 3.7 the problem (3.41)–(3.43), with $u(x, 0) = 1$, is solved numerically and the minimum of the solution $u(0, t)$ is

Fig. 3.5 The numerical solution of problem (3.45)–(3.46), $w(x)$ is plotted for b values of 5, 10, 20, 30, with $p = 4$

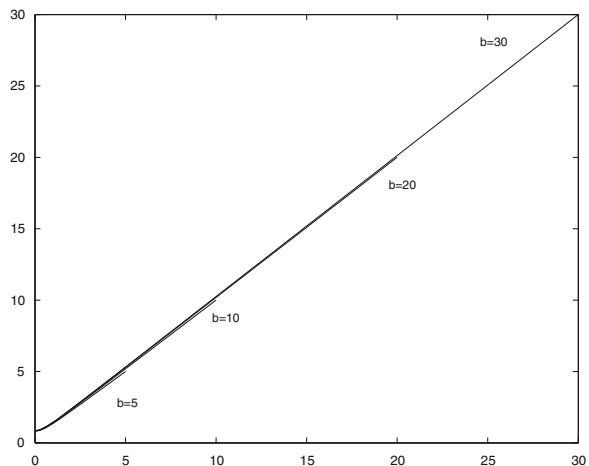


Fig. 3.6 The numerical solution of problem (3.41)–(3.43), $u(x, t)$ at $t = 25$ (solid curves) compared with steady states $w(x)$ (dotted curves) for $p = 1.1, 2, 4, 6$, all with $b = 5$

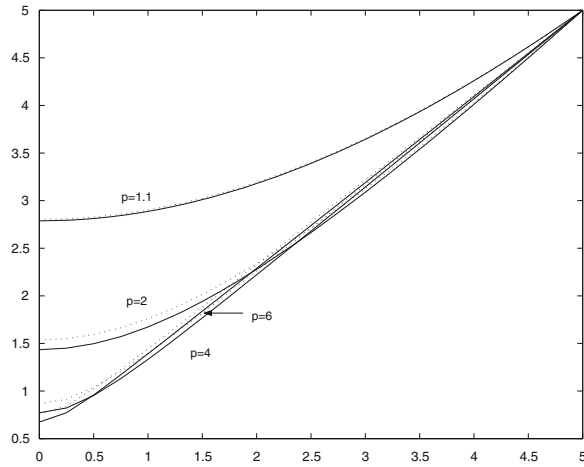
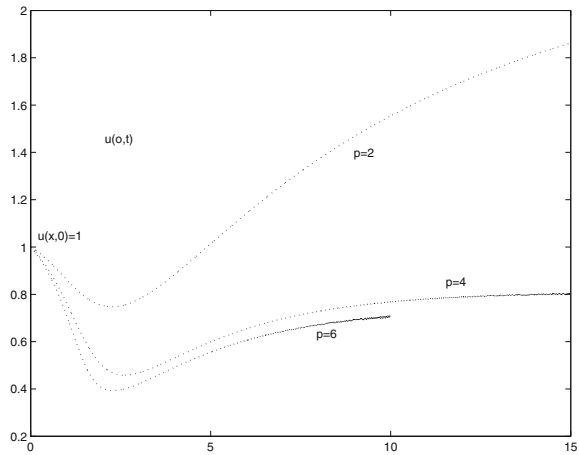


Fig. 3.7 The numerical solution of problem (3.41)–(3.43) at $x = 0$, $u(0, t)$, is plotted against time for p values of 2, 4, 6, taking $b = 5$ and $u(x, 0) = 1$



plotted against time for different value of the parameter p . In all of the simulations $u(0, t)$ initially decreases but it never reaches zero.

In conclusion, these numerical results indicate that the solution u of the problem (3.38)–(3.40) does not quench and that instead it converges to the steady state.

References

1. Bebernes, J.W., Lacey, A.A.: Global existence and finite-time blow-up for a class of nonlocal parabolic problems. *Adv. Diff. Equ.* **2**, 927–953 (1997)
2. Bebernes, J.W., Lacey, A.A.: Shear Band Formation for a Non-Local Model of Thermo-Viscoelastic Flows. *Adv. Math. Sci. Appl.* **15**, 265–282 (2005)

3. Burns, T.J.: A mechanism for Shear Band Formation in the High Strain-Rate Torsion Test. *J. Appl. Mech.* **57**, 836–844 (1990)
4. Burns, T.J.: Does a shear band result from a thermal explosion? *Mech. Mater.* **17**, 261–271 (1994)
5. Flemming, R., Olmstead, W., Davis, S.: Shear localization with arrhenius flow law. *SIAM J. Appl. Math.* **60**, 1867–1886 (2000)
6. Hinch, J., Spence, J., Howison, S., Ockendon, J., Chapman, J., Cowley, S., Duursma, G., Guneratne, J., Gillies, B., Llewellyn-Smith, S., Parker, D.: Friction Welding, European Study Group with Industry Report (1998) (Unpublished)
7. Kavallaris, N.I., Lacey, A.A., Nikolopoulos, C.V., Voong, C.: Behavior of a non-local equation modelling linear friction welding. *IMA J. Appl. Math.* **75**(2), 597–616 (2007)
8. Lacey, A.A.: Thermal runaway in a non-local problem modelling Ohmic heating: Model derivation and some special cases. I. *Eu. J. Appl. Math.* **6**, 127–144 (1995)
9. Lacey, A.A.: Thermal runaway in a non-local problem modelling Ohmic heating: general proof of blow-up and asymptotics of runaway II. *Eur. J. Appl. Math.* **6**, 201–224 (1995)
10. Lacey, A.A., Voong, C.: Steady-state mathematical models of linear friction welding. *Quart. J. Mech. Appl. Math.* **65**(2), 211–237 (2012). preprint
11. Lenk, R.S.: *Polymer Rheology*. Applied Science Publisher Ltd. (1978)
12. Velázquez, J.J.L.: Global existence and quenching for a model of shear band formation. *Nonlin. Anal. Theor. Meth. Appl.* **5**, 337–348 (1993)
13. Tanner, R.I.: *Engineering Rheology*, 2nd edn. University Press, Oxford (2000)
14. Voong, C.: *The Mathematical Modelling of Linear Friction Welding*, Ph.D. Thesis, Heriot-Watt University (2006)
15. Wolansky, G.: A critical parabolic estimate and application to non-local equations arising in chemotaxis. *Appl. Anal.* **66**, 291–321 (1997)
16. Wright, T.: *The Physics and Mathematics of Adiabatic Shear Bands*. Cambridge University Press, Cambridge (2000)
17. Wright, T., Batra, R.: The initiation and growth of adiabatic shear bands. *Int. J. Plast.* **1**, 205–212 (1985)

Chapter 4

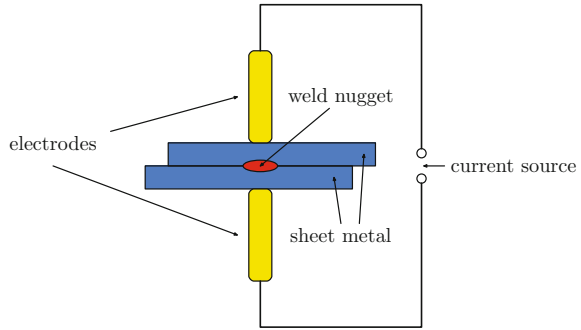
Resistance Spot Welding

Abstract In the current chapter we debate a joining process called resistance spot welding which finds many applications ranging from the automobile industry to robotics. In the first part of the chapter we present the construction of a non-local mathematical model illustrating the phase transition occurs during this joining process. Since the derived model consists of a degenerate non-local parabolic equation its analytical study is rather hard due to many arising technicalities and so we appeal to a numerical approach. We then consider a time discretization scheme for solving the resulting non-local moving boundary problem. The scheme consists of solving at each time step a linear elliptic partial differential equation and then making a correction which takes into account the nonlinearity. The stability and error estimates of the developed scheme are also investigated. Finally, some numerical experiments are presented which verify the efficiency of the developed numerical algorithm, as well as demonstrate the emergent interfaces due to the degeneracy of the problem.

4.1 Derivation of the Non-local Model

In the current chapter we focus on another application associated with the industrial process of welding. In particular we are dealing with the so called *resistance spot welding* process. Resistance welding is a thermo-electric process in which Joule heating is generated at the interface of the parts to be joined by passing an electrical current through the parts for a precisely controlled time and under a controlled pressure. The term *resistance* welding stems from the fact that the resistance of the workpieces and electrodes are used in combination or contrast to generate the heat at their interface. A common configuration of the resistance spot welding process can be seen in Fig. 4.1. Key advantages of the resistance welding process include: very short process time, no consumables, operator safety because of low voltage,

Fig. 4.1 Schematic diagram of resistance spot welding system



clean and environmentally friendly and finally a reliable electro-mechanical joint is formed. Resistance spot welding is the most popular joining process in automobile body assembly production lines; other applications include robotics, orthodontist's clinic and batteries manufacturing.

When an electric current, with local current density $\vec{j}$, flows through the electrodes and the sheet metals, then owing to a significantly higher resistivity in the contact area a rapid heating up of this area occurs, which is caused by the Joule effect. The rapid heating leads to the development of a weld nugget (a mushy region where the solid and liquid phases coexist) which actually grows quite fast. Once, the electrical current is switched off the weld nugget solidifies, leading to a lasting weld joint (known also as weldment) between the metal sheets. The main physical quantities are involved in the description of the configuration of Fig. 4.1 are: the temperature u , a parameter ψ standing for the proportion of the two phases and the applied electrical potential ϕ . In the melting-solidification process $\psi = 0$ in the solid phase while in the liquid phase we have $\psi = 1$.

The temperature evolution of the above system is governed by the following internal energy balance equation

$$\frac{\partial u}{\partial t} + \ell \frac{\partial \psi}{\partial t} = \nabla \cdot (k(u, \psi) \nabla u) + \rho(u, \psi) |\vec{j}|^2 \quad \text{in} \quad S_T := D \times (0, T), \quad T > 0, \quad (4.1)$$

where $k(u, \psi)$ and $\rho(u, \psi)$ represent the thermal conductivity and the electrical resistance of the metal sheets respectively whilst ℓ stands for the lateral heat of the phase change process. Here D denotes the joined part of the two metal sheets indicated with blue color in Fig. 4.1. In case the lateral heat ℓ is small, a rather realistic assumption, then (4.1) reduces to

$$\frac{\partial u}{\partial t} = \nabla \cdot (k(u, \psi) \nabla u) + \rho(u, \psi) |\vec{j}|^2 \quad \text{in} \quad S_T. \quad (4.2)$$

Owing to Ohm's law the current density satisfies

$$\vec{j} = -\tau(u, \psi)\nabla\phi, \quad (4.3)$$

where $\tau(u, \psi) = 1/\rho(u, \psi)$ is the electrical conductivity of the metal sheets and thus (4.2) reads

$$\frac{\partial u}{\partial t} = \nabla \cdot (k(u, \psi)\nabla u) + \tau(u, \psi)|\nabla\phi|^2, \quad \text{in } S_T. \quad (4.4)$$

Also due to conservation of charge $\nabla \cdot \vec{j} = 0$ and hence we derive the second governing equation of our system

$$\nabla \cdot (\tau(u, \psi)\nabla\phi) = 0, \quad \text{in } S_T. \quad (4.5)$$

Next we describe the derivation of the third mastering equation of our system which describes the phase transition. It is usually called the *phase equation* and describes the time-evolution of $0 \leq \psi \leq 1$. First we note that the *free energy* of the system can be chosen having the Landau–Ginzburg form

$$F(u, \psi) = c_V u(1 - \log u) + u \left(\hat{g}(\psi) + \mu \frac{|\nabla\psi|^2}{2} \right) + \ell\psi, \quad (4.6)$$

where c_V is the specific heat, taking henceforth equal to 1 for simplicity, and μ is a positive constant [24]; $\hat{g}$ has usually the form of a double well potential, e.g.

$$\hat{g}(\psi) = \frac{1}{2}(1 - \psi^2)^2,$$

see [4, 6].

Assuming that our isothermal system moves towards local minima of the *total free energy*

$$\mathcal{F}(u, \psi) = \int_D F(u, \psi) dx,$$

we can impose that the order-parameter dynamics is given by

$$u \frac{\partial \psi}{\partial t} = -\delta_\psi \mathcal{F}(u, \psi), \quad \text{in } S_T, \quad (4.7)$$

where $\delta_\psi \mathcal{F}(u, \psi)$ represents the variational derivative of $\mathcal{F}$ with respect to ψ . Taking into account (4.6) as well as the fact that $0 < \ell \ll 1$ then (6.10) yields

$$\frac{\partial \psi}{\partial t} = \mu \Delta \psi + g(\psi), \quad \text{in } S_T, \quad (4.8)$$

where g is the derivative of $\hat{g}$, [6].

From experimental data we can observe that usually both thermal and electrical conductivities have a discontinuity in the melting point, [16]. Actually we can assume the mixture ansatz $k(u, \psi) = (1 - \psi)\tilde{k}_1(u) + \psi\tilde{k}_2(u)$ and $\tau(u, \psi) = (1 - \psi)\tilde{\tau}_1(u) + \psi\tilde{\tau}_2(u)$ with possibly different $\tilde{k}_1, \tilde{k}_2$ and $\tilde{\tau}_1, \tilde{\tau}_2$, see [6, 18]. Or alternatively we have

$$k(u, \psi) = \tilde{k}(u) = \begin{cases} k_1(u), & \text{if } u < u_m, \\ k_2(u), & \text{if } u > u_m, \end{cases}$$

and

$$\tau(u, \psi) = \tilde{\tau}(u) = \begin{cases} \tau_1(u), & \text{if } u < u_m, \\ \tau_2(u), & \text{if } u > u_m, \end{cases}$$

for $k_1(s) \neq k_2(s)$ and $\tau_1(s) \neq \tau_2(s)$. Here u_m stands for the melting temperature of the metal workpieces.

Then the melting-solidification process is described by

$$\frac{\partial u}{\partial t} = \nabla \cdot (\tilde{k}(u) \nabla u) + \tilde{\tau}(u) |\nabla \phi|^2 \quad \text{in } S_T, \quad (4.9)$$

$$\nabla \cdot (\tilde{\tau}(u) \nabla \phi) = 0, \quad \text{in } S_T. \quad (4.10)$$

Consider now the case where the contact area D is long and thin of length L with axis parallel to z -direction and its ends at $z = 0, L$ where $\phi = 0, V$ respectively. Let Ω be the cross section of the contact area D , then we consider that its dimensions are much smaller compared L . For the case of a cylindrical contact area with diameter R much less than the length L , i.e. $R \ll L$, see Fig. 4.2. We also assume that the conduct area D is electrically insulated, $\frac{\partial \phi}{\partial n} = 0$ on $\Gamma_1 := \partial D$, and the temperature is kept fixed on its curved surface, $u = 0$ on Γ_1 . Neglecting the end effects at $z = 0, L$ we can actually take that the potential ϕ depends only on the z -variable and thus (4.10) yields $(\tilde{\tau}(u)\phi_z)_z = 0$ which finally implies that $(\tilde{\tau}(u)\phi_z)$ depends only on time t . Under integration over the cross section Ω we take

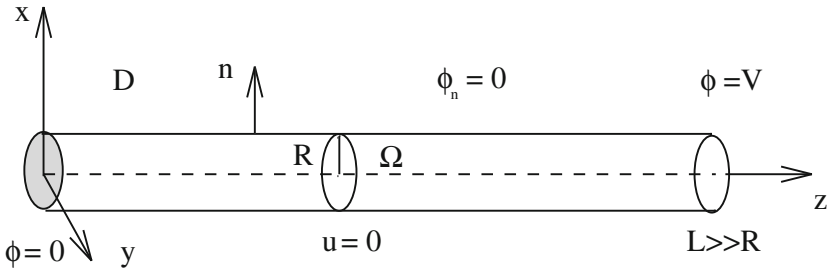


Fig. 4.2 A long and thin cylindrical conduct area

$$\int_{\Omega} \tilde{\tau}(u) \phi_z dx dy = \frac{I(t)}{A},$$

where $I(t)$ is the total current flowing through each cross section of the conductor and $A = |\Omega|$, thus

$$\phi_z = \frac{I(t)}{A \int_{\Omega} \tilde{\tau}(u) dx dy}. \quad (4.11)$$

Combining (4.9) and (4.11) and letting x denotes the position in the cross-section and Δ the two-dimensional Laplacian then system (4.9) and (4.10) is reduced to the single non-local equation

$$\frac{\partial u}{\partial t} = \Delta K(u) + \frac{\lambda \tilde{\tau}(u)}{(\int_{\Omega} \tilde{\tau}(u) dx)^2}, \quad \text{in } \Omega \times (0, T), \quad (4.12)$$

associated with boundary and initial conditions

$$K(u) = 0 \quad \text{on } x \in \partial\Omega,$$

and

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

provided also that the total current $I(t) = I$ is constant. Here $\lambda = \frac{I^2}{A^2}$ is the parameter that actually controls the welding process. Also $K(u)$ is defined as

$$K(u) = \int_0^u \tilde{k}(s) ds.$$

Alternatively, by applying Fourier's law for diffusion and heat conduction as well as conservation of energy, [12, 24], we can take u satisfying a mixed-type boundary condition

$$K(u) + k(x) \frac{\partial K(u)}{\partial \mathbf{n}} = 0 \quad \text{on } x \in \partial\Omega.$$

To our knowledge the mathematical model (4.12) and (4.13) represents the first coupling of Joule effect and phase-transition under a single non-local equation, see also [9]. It should be note that similar non-local models, describing Ohmic heating production as well as linear friction welding are derived and considered in Chaps. 2 and 3, see also [1, 2, 5, 8, 10, 11, 19].

4.2 The Mathematical Problem

Motivated by Sect. 4.1, in the rest of the current chapter we investigate the behavior of the following non-local degenerate problem

$$u_t = \Delta\beta(u) + \frac{\lambda f(\beta(u))}{(\int_{\Omega} f(\beta(u)) dx)^2}, \quad \text{in } Q_T := \Omega \times (0, T), \quad T > 0, \quad (4.13)$$

$$\beta(u) + k(x) \frac{\partial \beta(u)}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (4.14)$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \quad (4.15)$$

where λ is positive constant whilst as usual $\frac{\partial}{\partial \nu} = \nabla \cdot \nu$ denotes the normal derivative on the boundary $\Gamma = \partial\Omega$.

Moreover, $\beta(u)$ stands for a continuous function defined on $\mathbb{R}$ satisfying $\beta(0) = 0$ and Ω is a polyhedral and convex domain in $\mathbb{R}^d$ ($d \geq 1$). A common case, see [24], for function β is the power-law function $\beta(u) = u^m$, $m > 0$, and then (4.13) is a non-local porous medium equation. Nevertheless, under the general condition $\beta'(u) \geq 0$, which is actually necessary for making Eq. (4.13) formally parabolic, equation (4.13) is called a non-local filtration equation. The nonlinear function $f(s)$ can be considered, for some applications, as monotonic though in the current paper we only assume to be positive and Lipschitz continuous. Also $k(x)$ is considered in $C^{1+\delta}(\Gamma)$, $\delta > 0$, and $k(x) = 0$, $k(x) = \infty$ and $0 < k(x) < \infty$ correspond to Dirichlet, Neumann and Robin (mixed) type boundary conditions respectively.

Since $f(s) > 0$, whenever $u_0(x) > 0$ in Ω , we obtain via the comparison principle for the (local) porous medium operator that $u(x, t) > 0$ in $\Omega \times (0, T)$ and hence problem (4.13)–(4.15) is non-degenerate, [22]. Thus local-in-time existence and uniqueness of a classical solution to (4.13)–(4.15) is guaranteed and can be obtained by classical arguments, [7, 12]. Furthermore long-time behavior was studied by using comparison and energy methods for different nonlinear functions $f(s)$, [7, 12, 20, 21]. Even finite-time and infinite blow-up of the solution of (4.13)–(4.15) is proven to occur under some circumstances in the non-degenerate case, see [7, 8, 12, 13, 20, 21].

On the other hand, when $u_0(x)$ has compact support i.e. there exists a bounded closed set $\Omega_0 \subsetneq \Omega$ such that $u_0(x) = 0$, for $x \notin \Omega_0$ then $u(x, t)$ will remain compactly supported for all later times t , thus problem (4.13)–(4.15) becomes degenerate. In that case only a weak solution of (4.13)–(4.15) can be defined which is a function $u \in L^2(Q_T)$ with $\beta(u) \in L^\infty((0, T); L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega))$ and

$$\frac{\lambda f(\beta(u))}{(\int_{\Omega} f(\beta(u)) dx)^p} \in L^2(Q_T)$$

which satisfies the equation

$$\begin{aligned} & \int_{\Omega} [u(x, t_1)v(x, t_1) - u(x, 0)v(x, 0)] dx \\ &= \int_0^{t_1} \int_{\Omega} \left(uv_t - \nabla \beta(u) \cdot \nabla v + \frac{\lambda f(\beta(u))v}{(\int_{\Omega} f(\beta(u)) dx)^p} \right) dx dt, \end{aligned}$$

for any $t_1 \in [0, T]$ and $v \in \dot{H}^{1,1}(Q_{t_1})$ (where $\dot{H}^{1,1}(Q_{t_1})$ is the closure of $H^{1,1}(Q_{t_1})$ with respect to its norm) with $v = 0$ on $\partial\Omega$, [7].

This degeneracy results into the interesting phenomenon of the *finite speed of propagation*. Indeed, a *moving boundary* is formed, called also *interface*, separating the regions $P_u(t) = \{x \in \Omega : \beta(u(x, t)) > 0\}$ and $N_u(t) = \{x \in \Omega : \beta(u(x, t)) < 0\}$. It is defined as $\Gamma_u(t) = \partial P_u \cap Q_T$ and propagates with finite speed see also [22]. A intriguing problem both from mathematical and application point of view is the determination of the evolution in time of the emerged *interface*. In the current chapter we develop a numerical scheme which supplies an approximation of the *interface* (*moving boundary*) and also provides a uniform approach for both non-degenerate and degenerate cases.

4.3 The Numerical Scheme

In the current section we propose a numerical scheme for investigating the long-time behavior of the non-local problem (4.13)–(4.15). We are actually inspired by a numerical scheme developed in [15].

In particular, Nochetto and Verdi [15] studied the following local nonlinear parabolic problem

$$\frac{\partial u}{\partial t} - \Delta \beta(u) = f(\beta(u)), \quad \text{in } \Omega \times (0, T), \quad (4.16)$$

$$\beta(u) = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (4.17)$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \quad (4.18)$$

where $\beta(s)$ satisfies the assumption (H_{β}) below. They introduced a finite element method for solving (4.16)–(4.18) and the error estimates in both semidiscrete and fully discrete cases were derived. The usual technique to approximate (4.16)–(4.18) amounts to discretizing a nonlinear elliptic partial differential equations at each time step. The success of such a procedure relies on the smoothness of the solution u and $\theta = \beta(u)$. It is not a priori obvious that standard techniques for mildly nonlinear parabolic equations apply on low-regularity (degenerate) problem (4.16)–(4.18). Based on this observation, Nochetto and Verdi [15] tried the following numerical scheme.

Let $0 = t_0 < t_1 < \dots < t_N = T$ be a partition of $[0, T]$. Let $U^n \approx u(t_n)$ and $\Theta^n \approx f(\beta(u(t_n)))$ be the approximate solutions of $u(t_n)$ and $f(\beta(u(t_n)))$. Nochetto and

Verdi [15] introduced the following time discretization scheme which is a nonlinear Chernoff formula, find $\Theta^n \in H_0^1(\Omega)$, $U^n \in L^2(\Omega)$, such that, for any $\xi \in H_0^1(\Omega)$,

$$U^0 := u_0, \quad (4.19)$$

$$(\Theta^n, \xi) - \frac{\tau}{\mu} (\nabla \Theta^n, \nabla \xi) = (\beta(U^{n-1}), \xi) + \frac{\tau}{\mu} \left(f(\beta(U^{n-1})), \xi \right), \quad (4.20)$$

$$U^n := U^{n-1} + \mu(\Theta^n - \beta(U^{n-1})), \quad 1 \leq n < N := \frac{T}{\tau}. \quad (4.21)$$

Actually the above scheme was first introduced in [3] for tackling the filtration equation $u_t = \Delta \beta(u)$. Here $\tau > 0$ is the time step and $\mu > 0$ is the relaxation parameter which satisfies the stability constraint $\mu < L_\beta^{-1}$ ($L_\beta =$ Lipschitz constant of β). The numerical scheme consists of solving at each time step a linear elliptic partial differential equation and then making a correction to account for the nonlinearity.

One of the first numerical studies on non-local problems of the form of (4.13)–(4.15) was done in [17]. The authors, in [17], considered the following non-local filtration problem

$$\frac{\partial u}{\partial t} - \Delta \beta(u) = \frac{\lambda f(u)}{(\int_\Omega f(u) dx)^2}, \quad \text{in } \Omega \times (0, T), \quad (4.22)$$

$$\beta(u) = 0, \quad \text{on } \Gamma \times (0, T), \quad (4.23)$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega. \quad (4.24)$$

They introduced a time discretization scheme and proved the error estimates. However, no numerical results were presented in [17] for testing the validity and the efficiency of the produced numerical scheme.

Motivated by the numerical scheme considered in [15] we introduce a similar to (4.19)–(4.21) time discretization scheme which captures all the key features of the possible degenerate problem (4.13)–(4.15) considering for simplicity only homogeneous Dirichlet boundary conditions. So we look for a $\Theta^n \in H_0^1(\Omega)$, $U^n \in L^2(\Omega)$ such that, for any $\xi \in H_0^1(\Omega)$

$$U^0 := u_0, \quad (4.25)$$

$$(\Theta^n, \xi) - \frac{\tau}{\mu} (\nabla \Theta^n, \nabla \xi) = (\beta(U^{n-1}), \xi) + \frac{\lambda \tau}{\mu} \frac{(f(\beta(U^{n-1})), \xi)}{\left(\int_\Omega f(\beta(U^{n-1})) dx \right)^2}, \quad (4.26)$$

$$U^n := U^{n-1} + \mu(\Theta^n - \beta(U^{n-1})), \quad 1 \leq n < N := \frac{T}{\tau}. \quad (4.27)$$

Denote $\theta(t) = \beta(u(t))$, $e_\theta = \theta(t) - \Theta^n$ and $e_u = u(t) - U^n$ for any $t \in (t_{n-1}, t_n)$, then we will prove the following error estimates

$$\|e_\theta\|_{L^2(\Omega \times (0, T))} + \left\| \int_0^t e_\theta \right\|_{L^\infty(0, T; H^1(\Omega))} + \|e_u\|_{L^\infty(0, T; H^{-1}(\Omega))} = O(\tau^{\frac{1}{2}}).$$

We only deal with Dirichlet boundary conditions, though with some small modifications, the introduced numerical scheme could easily apply to the mixed type boundary conditions of the form (4.14), by following the technique developed in [14].

4.4 Stability

In this section, we consider the stability of the time discretization scheme (4.25)–(4.27). To that end, we first introduce some fundamental assumptions on f and β appearing in (4.13)–(4.15), see [15].

Assumption (H_β): $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and Lipschitz continuous function; more precisely

$$0 \leq \beta'(s) \leq L_\beta < +\infty, \quad \text{for a.e. } s \in \mathbb{R}. \quad (4.28)$$

Moreover, β satisfies $\beta(0) = 0$ and grows at least linearly at infinity, that is, there exist $c_1, c_2 > 0$ such that

$$|s| \leq c_1 + c_2|\beta(s)|, \quad \text{for any } s \in \mathbb{R}.$$

Assumption (H_f): $f : \mathbb{R} \rightarrow \mathbb{R}$ is a uniformly Lipschitz continuous function; that is,

$$|f(s_1) - f(s_2)| \leq L_f |s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbb{R};$$

moreover

$$f(s) \geq \sigma > 0, \quad \forall s \in \mathbb{R}.$$

Assumption (H_{u_0}): $u_0 \in L^\infty(\Omega)$.

We shall choose the relaxation parameter μ in (4.25)–(4.27) such that $0 < \mu \leq L_\beta^{-1}$, then the following property holds

$$\alpha := I - \mu\beta \quad \text{satisfies} \quad 0 \leq \alpha'(s) \leq 1, \quad \text{for a.e. } s \in \mathbb{R}. \quad (4.29)$$

Lemma 4.4.1 *Let (U^n, Θ^n) be the solution of (4.25)–(4.27). Assume that (H_β) , (H_f) and (H_{u_0}) hold. Then there exists a constant $C > 0$ such that*

$$\max_{1 \leq n \leq N} \|\beta(U^n)\|_{L^2(\Omega)} + \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \tau \|\Theta^n\|_{H^1(\Omega)}^2 \leq C.$$

Proof Let us denote $F(\beta(U^{n-1})) = \frac{(f(\beta(U^{n-1})), v)}{(\int_\Omega f(\beta(U^{n-1})) dx)^2}$. Then Eqs. (4.25)–(4.27) can be rewritten as, with $\partial U^n = (U^n - U^{n-1})/\tau$,

$$(\partial U^n, \xi) + (\nabla \Theta^n, \nabla \xi) = (F(\beta(U^{n-1})), \xi), \quad \text{for any } \xi \in H_0^1(\Omega). \quad (4.30)$$

Choose $v = \tau \Theta^n$ in (4.30) and sum up from 1 to N , then

$$\sum_{n=1}^N (\partial U^n, \tau \Theta^n) + \sum_{n=1}^N (\nabla \Theta^n, \tau \Theta^n) = \sum_{n=1}^N (F(\beta(U^{n-1})), \tau \Theta^n).$$

Note that, by virtue of (4.29),

$$\begin{aligned} \Theta^n &= \frac{1}{\mu}(U^n - U^{n-1}) + \beta(U^{n-1}) = \frac{1}{\mu}(U^n - U^{n-1}) + \frac{1}{\mu}(U^{n-1} - \alpha(U^{n-1})) \\ &= \frac{1}{\mu}U^n - \frac{1}{\mu}\alpha(U^{n-1}) \\ &= \frac{1}{2}\beta(U^n) + \frac{1}{2\mu}U^n + \frac{1}{2\mu}(\alpha(U^n) - \alpha(U^{n-1})) - \frac{1}{2\mu}\alpha(U^{n-1}), \end{aligned}$$

and thus we have

$$\begin{aligned} 2 \sum_{n=1}^N (\partial U^n, \tau \Theta^n) &= 2 \sum_{n=1}^N (U^n - U^{n-1}, \Theta^n) \\ &= 2 \sum_{n=1}^N \left(U^n - U^{n-1}, \frac{1}{2}\beta(U^n) \right) + 2 \sum_{n=1}^N \left(U^n - U^{n-1}, -\frac{1}{2\mu}\alpha(U^{n-1}) \right) \\ &\quad + 2 \sum_{n=1}^N \left(U^n - U^{n-1}, \frac{1}{2\mu}U^n \right) + 2 \sum_{n=1}^N \left(U^n - U^{n-1}, \frac{1}{2\mu}(\alpha(U^n) - \alpha(U^{n-1})) \right). \end{aligned}$$

Given a function $\pi : \mathbb{R} \rightarrow \mathbb{R}$, Φ_π stands for the convex function defined by

$$\Phi_\pi(s) = \int_0^s \pi(z) dz, \quad \text{for } s \in \mathbb{R}.$$

Since $\beta'(s) > 0$ and $\alpha'(s) > 0$, then we take

$$\begin{aligned} \Phi_\beta(U^n) - \Phi_\beta(U^{n-1}) &= \Phi'_\beta(c)(U^n - U^{n-1}) = \beta(c)(U^n - U^{n-1}) \leq \beta(U^n)(U^n - U^{n-1}), \\ \Phi_\alpha(U^n) - \Phi_\alpha(U^{n-1}) &= \Phi'_\alpha(c)(U^n - U^{n-1}) = \alpha(c)(U^n - U^{n-1}) \geq \alpha(U^{n-1})(U^n - U^{n-1}), \\ (U^n - U^{n-1}, \alpha(U^n) - \alpha(U^{n-1})) &\geq 0. \end{aligned}$$

Further, using the equality $2a(a - b) = a^2 - b^2 + (a - b)^2$, we derive

$$\begin{aligned}
2 \sum_{n=1}^N (\partial U^n, \tau \Theta^n) &\geq \int_{\Omega} \sum_{n=1}^N \{ [\Phi_{\beta}(U^n) - \Phi_{\beta}(U^{n-1})] \\
&\quad + \frac{1}{\mu} [\Phi_{\alpha}(U^{n-1}) - \Phi_{\alpha}(U^n)] \} dx + \frac{1}{\mu} \sum_{n=1}^N (U^n - U^{n-1}, U^n) \\
&= \int_{\Omega} \sum_{n=1}^N \left\{ [\Phi_{\beta}(U^n) - \Phi_{\beta}(U^{n-1})] + \frac{1}{\mu} [\Phi_{\alpha}(U^{n-1}) - \Phi_{\alpha}(U^n)] \right\} dx \\
&\quad + \frac{1}{\mu} \left[\|U^n\|_{L^2(\Omega)}^2 - \|U^0\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 \right].
\end{aligned}$$

Since $\frac{1}{2L_{\beta}}\beta^2(s) \leq \Phi_{\beta}(s) \leq \frac{L_{\beta}}{2}s^2$ and $0 \leq \beta'(s) \leq L_{\beta}$, we have

$$\begin{aligned}
\int_{\Omega} \sum_{n=1}^N [\Phi_{\beta}(U^n) - \Phi_{\beta}(U^{n-1})] dx &= \int_{\Omega} [\Phi_{\beta}(U^N) - \Phi_{\beta}(U^0)] \\
&\geq \frac{1}{2L_{\beta}} \|\beta(U^N)\|_{L^2(\Omega)}^2 - \frac{L_{\beta}}{2} \|U^0\|_{L^2(\Omega)}^2.
\end{aligned}$$

Furthermore since $\frac{1}{2L_{\alpha}}\alpha^2(s) \leq \Phi_{\alpha}(s) \leq \frac{L_{\alpha}}{2}s^2$ and $0 \leq \beta'(s) \leq 1$, we obtain

$$\begin{aligned}
\frac{1}{\mu} \int_{\Omega} \sum_{n=1}^N [\Phi_{\alpha}(U^{n-1}) - \Phi_{\alpha}(U^n)] dx &= \frac{1}{\mu} \int_{\Omega} [\Phi_{\alpha}(U^0) - \Phi_{\alpha}(U^N)] \\
&\geq \frac{1}{2\mu} \|\alpha(U^0)\|_{L^2(\Omega)}^2 - \frac{1}{2\mu} \|U^N\|_{L^2(\Omega)}^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&2 \sum_{n=1}^N (U^n - U^{n-1}, \Theta^n) \\
&\geq \frac{1}{2L_{\beta}} \|\beta(U^N)\|_{L^2(\Omega)}^2 - \frac{L_{\beta}}{2} \|U^0\|_{L^2(\Omega)}^2 + \frac{1}{2\mu} \|\alpha(U^0)\|_{L^2(\Omega)}^2 - \frac{1}{2\mu} \|U^N\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2\mu} \left[\|U^N\|_{L^2(\Omega)}^2 - \|U^0\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 \right] \\
&\geq -C + C \|\beta(U^N)\|_{L^2(\Omega)}^2 + \frac{1}{2\mu} \|U^n - U^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

We next consider the non-local term $\sum_{n=1}^N (F(\beta(U^{n-1})), \tau \Theta^n)$. Using the fact that f satisfies Lipschitz condition (H_f) as well as that $f(s) \geq \sigma > 0$ for $s \in \mathbb{R}$, we derive

$$|F(\beta(U^{n-1}))| = \left| \frac{\lambda f(\beta(U^{n-1}))}{(\int_{\Omega} f(\beta(U^{n-1})) dx)^2} \right| \leq C |f(\beta(U^{n-1}))| \leq C(1 + |\beta(U^{n-1})|).$$

Hence, noting that $\Theta^n = \frac{1}{\mu}(U^n - U^{n-1}) + \beta(U^{n-1})$, we have

$$\begin{aligned} \sum_{n=1}^N (F(\beta(U^{n-1})), \tau \Theta^n) &\leq \sum_{n=1}^N \tau \left(1 + \|\beta(U^{n-1})\|_{L^2(\Omega)}\right) \|\Theta^n\|_{L^2(\Omega)} \\ &\leq \sum_{n=1}^N \tau \left(1 + \|\beta(U^{n-1})\|_{L^2(\Omega)}\right) \left(\frac{1}{\mu} \|U^n - U^{n-1}\|_{L^2(\Omega)} + \|\beta(U^{n-1})\|_{L^2(\Omega)}\right) \\ &= \sum_{n=1}^N \tau \|\beta(U^{n-1})\|_{L^2(\Omega)} + \sum_{n=1}^N \tau \|\beta(U^{n-1})\|_{L^2(\Omega)}^2 \\ &\quad + \sum_{n=1}^N \tau \frac{1}{\mu} \|U^n - U^{n-1}\|_{L^2(\Omega)} + \sum_{n=1}^N \tau \|\beta(U^{n-1})\|_{L^2(\Omega)} \left(\frac{1}{\mu} \|U^n - U^{n-1}\|_{L^2(\Omega)}\right) \\ &\leq \sum_{n=1}^N \left(\tau + \tau \|\beta(U^{n-1})\|_{L^2(\Omega)}^2\right) + \sum_{n=1}^N \tau \|\beta(U^{n-1})\|_{L^2(\Omega)}^2 \\ &\quad + \sum_{n=1}^N \left(C_\varepsilon \tau^2 + \varepsilon \|U^n - U^{n-1}\|_{L^2(\Omega)}^2\right) + \sum_{n=1}^N \left(\|\tau \beta(U^{n-1})\|_{L^2(\Omega)}^2 + \varepsilon \|U^n - U^{n-1}\|_{L^2(\Omega)}^2\right) \\ &\leq C + \sum_{n=1}^N \tau \|\beta(U^{n-1})\|_{L^2(\Omega)}^2 + 2\varepsilon \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Combining all these estimates, and for $2\varepsilon = \frac{1}{4\mu}$, we obtain

$$\begin{aligned} &-C + C \|\beta(U^N)\|_{L^2(\Omega)}^2 + \frac{1}{2\mu} \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 + \tau C \sum_{n=1}^N \|\Theta^n\|_{H_0^1(\Omega)}^2 \\ &\leq C + C \sum_{n=1}^N \tau \|\beta(U^{n-1})\|_{L^2(\Omega)}^2 + \frac{1}{4\mu} \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2, \end{aligned}$$

which implies that

$$\begin{aligned} &C \|\beta(U^N)\|_{L^2(\Omega)}^2 + \frac{1}{4\mu} \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 \\ &\quad + \tau C \sum_{n=1}^N \|\beta(U^n)\|_{L^2(\Omega)}^2 \leq C + C \sum_{n=1}^N \tau \|\beta(U^{n-1})\|_{L^2(\Omega)}^2. \end{aligned}$$

Finally by virtue of Gronwall lemma, we derive

$$\max_{1 \leq n \leq N} \|\beta(U^n)\|_{L^2(\Omega)} + \sum_{n=1}^N \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \tau \|\Theta^n\|_{H^1(\Omega)}^2 \leq C,$$

and the proof of Lemma 4.4.1 is now complete. $\square$

4.5 Error Estimates

The main purpose of the current section is the derivation of some error estimates of the time discretization scheme (4.25)–(4.27). We actually have

Theorem 4.5.1 *Let (U^n, Θ^n) be the solution of (4.25)–(4.27). Let u be the solution of (4.13)–(4.15). Assume that (H_β) , (H_f) and (H_{u_0}) hold. Assume also that $u_0 \in L^\infty(\Omega)$, $\Delta\beta(u_0) \in L^1(\Omega)$ and in addition that*

$$\max_{1 \leq n \leq N} \|U^n\|_{L^\infty(\Omega)} \leq C.$$

Then we have

$$\begin{aligned} & \|e_\theta\|_{L^2(\Omega \times (0, T))} + \left\| \int_0^t e_\theta \right\|_{L^\infty(0, T; H^1(\Omega))} \\ & + \|e_u\|_{L^\infty(0, T; H^{-1}(\Omega))} = O(\tau^{\frac{1}{2}}), \quad \text{for any } t \in (t_{n-1}, t_n), \end{aligned}$$

where $\theta(t) = \beta(u(t))$, $e_\theta = \theta(t) - \Theta^n$ and $e_u = u(t) - U^n$.

In order to prove Theorem 4.5.1, we need the following auxiliary result.

Lemma 4.5.2 *If $u_0 \in L^\infty(\Omega)$, $\Delta\beta(u_0) \in L^1(\Omega)$, we have*

$$\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (u(t), U^i - U^{i-1}) dt \leq C\tau.$$

Proof The proof is similar to the proof of (4.24) in [15] and so it is omitted.

Proof (Theorem 4.5.1) We first denote

$$F(\beta(u)) = \frac{\lambda f(\beta(u))}{(\int_\Omega f(\beta(u)) dx)^2}.$$

Then (4.13)–(4.15) can be rewritten as

$$\frac{\partial u}{\partial t} - \Delta \beta(u) = F(\beta(u)), \quad \text{in } \Omega \times (0, T), \quad (4.31)$$

$$\beta(u) = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (4.32)$$

$$u(0) = u_0, \quad \text{in } \Omega. \quad (4.33)$$

The variational form of (4.31)–(4.33) is given as

$$(u_t, \xi) + (\nabla \theta, \nabla \xi) = (F(\beta(u)), \xi), \quad \text{for any } \xi \in H_0^1(\Omega), \quad (4.34)$$

where $u \in H_0^1(\Omega)$ and $\theta = \beta(u)$. Now (4.48) under integration over (t_{n-1}, t_n) , and with $u^n = u(t_n)$ implies

$$(u^n - u^{n-1}, \xi) + \tau(\nabla \bar{\theta}^n, \nabla \xi) = \tau \left(\overline{F(\beta(u))}^n, \xi \right), \quad \text{for any } \xi \in H_0^1(\Omega), \quad (4.35)$$

where

$$\bar{\theta}^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \theta(t) dt, \quad \overline{F(\beta(u))}^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} F(\beta(u(t))) dt.$$

The time discretization scheme (4.25)–(4.27) can be written as

$$(U^n - U^{n-1}, \xi) + \tau(\nabla \Theta^n, \nabla \xi) = \tau(F(U^{n-1}), \xi), \quad \forall \xi \in H_0^1(\Omega), \quad (4.36)$$

$$U^n - U^{n-1} = \mu(\Theta^n - \beta(U^{n-1})). \quad (4.37)$$

Subtracting (4.36) from (4.35) and summing up from 1 to i , we obtain

$$\begin{aligned} & \sum_{n=1}^i \left((u^n - U^n) - (u^{n-1} - U^{n-1}), \xi \right) + \tau \sum_{n=1}^i (\nabla(\bar{\theta}^n - \Theta^n), \nabla \xi) \\ &= \tau \sum_{n=1}^i \left(\overline{F(\beta(u))}^n - F(U^{n-1}), \xi \right), \quad \text{for any } \xi \in H_0^1(\Omega), \end{aligned}$$

or equivalently

$$\begin{aligned} & (u^i - U^i, \xi) + \tau \sum_{n=1}^i (\nabla(\bar{\theta}^n - \Theta^n), \nabla \xi) \\ &= \tau \sum_{n=1}^i \left(\overline{F(\beta(u))}^n - F(U^{n-1}), \xi \right), \quad \text{for any } \xi \in H_0^1(\Omega), \end{aligned} \quad (4.38)$$

by noting that $U^0 = u_0 = u(0)$.

Choosing $\xi = \tau(\bar{\theta}^i - \Theta^i)$ in (4.38) and summing up from 1 to N ,

$$\begin{aligned} & \sum_{i=1}^N (u^i - U^i, \tau(\bar{\theta}^i - \Theta^i)) + \tau^2 \sum_{i=1}^N \sum_{n=1}^i (\nabla(\bar{\theta}^n - \Theta^n), \nabla(\bar{\theta}^i - \Theta^i)) \\ &= \tau \sum_{n=1}^i (\overline{F(\beta(u))}^n - F(U^{n-1}), \tau(\bar{\theta}^i - \Theta^i)). \end{aligned} \quad (4.39)$$

Noting that

$$\begin{aligned} (u^i - U^i, \tau(\bar{\theta}^i - \Theta^i)) &= \int_{t_{i-1}}^{t_i} (u^i - U^i, \theta(t) - \Theta^i) dt \\ &= \int_{t_{i-1}}^{t_i} (u^i - U^i, e_\theta(t)) dt = \int_{t_{i-1}}^{t_i} (u^i - u(t) + u(t) - U^i, e_\theta(t)) dt \\ &= \int_{t_{i-1}}^{t_i} (e_u(t), e_\theta(t)) dt - \int_{t_{i-1}}^{t_i} (u(t) - u^i, e_\theta(t)) dt, \end{aligned}$$

we obtain

$$\begin{aligned} I + II &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (e_u(t), e_\theta(t)) dt + \tau^2 \sum_{i=1}^N \sum_{n=1}^i (\nabla(\bar{\theta}^n - \Theta^n), \nabla(\bar{\theta}^i - \Theta^i)) \\ &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (u(t) - u^i, e_\theta(t)) dt + \tau^2 \sum_{n=1}^i (\overline{F(\beta(u))}^n - F(U^{n-1}), \bar{\theta}^i - \Theta^i) \\ &= III + IV. \end{aligned}$$

For $t \in (t_{i-1}, t_i)$, we have

$$e_u(t) = u(t) - U^i = \mu\theta(t) + \alpha(u) - U^i = \mu e_\theta(t) + \alpha(u) - \alpha(U^{i-1}),$$

and

$$e_\theta(t) = \beta(u(t)) - \Theta^i = \beta(u(t)) - \beta(U^{i-1}) - \frac{1}{\mu}(U^i - U^{i-1}).$$

Hence

$$\begin{aligned} I &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (e_u(t), e_\theta(t)) dt \\ &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \mu(e_\theta(t), e_\theta(t)) dt + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\alpha(u) - \alpha(U^{i-1}), e_\theta(t)) dt \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \mu \|e_\theta(t)\|_{L^2(\Omega)}^2 dt + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\alpha(u) - \alpha(U^{i-1}), \beta(u) - \beta(U^{i-1})) dt \\
&\quad + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\alpha(u) - \alpha(U^{i-1}), -\frac{1}{\mu}(U^i - U^{i-1})) dt.
\end{aligned}$$

Furthermore, we have

$$\alpha(u) - \alpha(U^{i-1}) = u(t) - \mu\theta(t) - U^{i-1} + \mu\beta(U^{i-1}) = u(t) - U^i - \mu e_\theta(t),$$

and thus

$$\begin{aligned}
I &= \mu \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|e_\theta(t)\|_{L^2(\Omega)}^2 dt + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\alpha(u) - \alpha(U^{i-1}), \beta(u) - \beta(U^{i-1})) dt \\
&\quad + \frac{\tau}{\mu} \sum_{i=1}^N (U^i, (U^i - U^{i-1})) + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (e_\theta(t), (U^i - U^{i-1})) dt \\
&\quad - \frac{1}{\mu} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (u(t), (U^i - U^{i-1})) dt = I_1^N + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

For I_2 , we have the estimate

$$I_2 = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\alpha(u) - \alpha(U^{i-1}), \beta(u) - \beta(U^{i-1})) dt \geq 0,$$

taking into account that α, β are increasing functions.

On the other hand the term I_3 is estimated below as follows

$$\begin{aligned}
I_3 &= \frac{\tau}{\mu} \sum_{i=1}^N (U^i, U^i - U^{i-1}) \\
&= \frac{\tau}{\mu} \sum_{i=1}^N \left(\|U^i\|_{L^2(\Omega)}^2 - \|U^{i-1}\|_{L^2(\Omega)}^2 + \|U^i - U^{i-1}\|_{L^2(\Omega)}^2 \right) \\
&\geq \frac{\tau}{2\mu} \sum_{i=1}^N (\|U^i\|_{L^2(\Omega)}^2 - \|U^{i-1}\|_{L^2(\Omega)}^2) \\
&\geq \frac{\tau}{2\mu} (\|U^N\|_{L^2(\Omega)}^2 - \|U^0\|_{L^2(\Omega)}^2) \geq -\frac{\tau}{2\mu} \|U^0\|_{L^2(\Omega)}^2,
\end{aligned}$$

by using the equality $2a(a - b) = a^2 - b^2 + (a - b)^2$.

For term I_4 by virtue of Young's inequality $ab \leq \varepsilon a^2 + C_\varepsilon b^2$, we obtain the upper estimate

$$\begin{aligned}
 |I_4| &= \left| \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (e_\theta(t), U^i - U^{i-1}) dt \right| \leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|e_\theta(t)\|_{L^2(\Omega)} \|U^i - U^{i-1}\|_{L^2(\Omega)} dt \\
 &\leq \varepsilon \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|e_\theta\|_{L^2(\Omega)}^2 dt + C_\varepsilon \sum_{i=1}^N \tau \|U^i - U^{i-1}\|_{L^2(\Omega)}^2 \\
 &\leq \frac{1}{4} I_1^N + C\tau,
 \end{aligned}$$

where the last inequality follows from Lemma 4.4.1.

Thus we finally derive the estimate

$$I \geq I_1^N - \frac{\tau}{2\mu} \|U^0\|_{L^2(\Omega)}^2 - |I_4| + I_5.$$

Now regarding term II we have the following estimate from below

$$\begin{aligned}
 II &= \tau^2 \sum_{i=1}^N \sum_{n=1}^i (\nabla(\bar{\theta}^i - \Theta^i), \nabla(\bar{\theta}^n - \Theta^n)) \geq \frac{1}{2} \tau^2 \left\| \sum_{i=1}^N \nabla(\bar{\theta}^n - \Theta^n) \right\|_{L^2(\Omega)}^2 \\
 &= \frac{1}{2} \left\| \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \nabla(\theta(t) - \Theta^i) dt \right\|_{L^2(\Omega)}^2 = \frac{1}{2} \left\| \nabla \int_{t_0}^{t_N} e_\theta(t) dt \right\|_{L^2(\Omega)}^2,
 \end{aligned}$$

where we also made use of the equality $2 \sum_{i=1}^N a_i (\sum_{n=1}^i a_n) = (\sum_{i=1}^N a_i)^2 + \sum_{i=1}^N a_i^2$.

Next term III is estimated as

$$\begin{aligned}
 III &= \left| \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (u(t) - u^i, e_\theta(t)) dt \right| = \left| \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left(- \int_t^{t_i} \frac{\partial u}{\partial s} ds, e_\theta(t) \right) dt \right| \\
 &\leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{t_i} \left\| \frac{\partial u}{\partial s} \right\|_{H^{-1}} ds \|e_\theta(t)\|_{H^1(\Omega)} \right) dt \\
 &\leq \left[\sum_{i=1}^N \left(\int_{t_{i-1}}^{t_i} \left\| \frac{\partial u}{\partial s} \right\|_{H^{-1}(\Omega)}^2 ds \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[\sum_{i=1}^N \left(\int_{t_{i-1}}^{t_i} \|e_\theta(t)\|_{H^1(\Omega)}^2 dt \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\
 &\leq \tau \left(\int_0^{t_N} \left\| \frac{\partial u}{\partial s} \right\|_{H^{-1}(\Omega)}^2 ds \right)^{\frac{1}{2}} \left(\int_0^{t_N} \|e_\theta(t)\|_{H^1(\Omega)}^2 dt \right)^{\frac{1}{2}}.
 \end{aligned}$$

Finally for term IV, we derive

$$\begin{aligned}
IV &= \left| \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left(\sum_{n=1}^i \tau \left[\overline{F(\beta(u))^n} - F(\beta(U^{n-1})) \right], \bar{\theta}^i - \Theta^i \right) dt \right| \\
&= \left| \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left(\sum_{n=1}^i \tau \left[\overline{F(\beta(u))^n} - F(\beta(U^{n-1})) \right], e_\theta(t) \right) dt \right| \\
&\leq \varepsilon \int_0^{t_N} \|e_\theta(t)\|_{L^2(\Omega)}^2 dt + C_\varepsilon \sum_{i=1}^N \tau \left\| \sum_{n=1}^i \tau \left[\overline{F(\beta(u))^n} - F(\beta(U^{n-1})) \right] \right\|_{L^2(\Omega)}^2 \\
&\leq \varepsilon \int_0^{t_N} \|e_\theta(t)\|_{L^2(\Omega)}^2 dt + C_\varepsilon \sum_{i=1}^N \tau \left(\sum_{n=1}^i \tau \left\| \overline{F(\beta(u))^n} - F(\beta(U^{n-1})) \right\|_{L^2(\Omega)}^2 \right) \\
&= \varepsilon \int_0^{t_N} \|e_\theta(t)\|_{L^2(\Omega)}^2 dt + C_\varepsilon \sum_{i=1}^N \tau \left(\sum_{n=1}^i \tau \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} [F(\beta(u)) - F(\beta(U^{n-1}))] ds \right\|_{L^2(\Omega)}^2 \right) \\
&= \varepsilon \int_0^{t_N} \|e_\theta(t)\|_{L^2(\Omega)}^2 dt + C_\varepsilon \sum_{i=1}^N \tau \left(\sum_{n=1}^i \int_{t_{n-1}}^{t_n} \|F(\beta(u)) - F(\beta(U^{n-1}))\|_{L^2(\Omega)}^2 ds \right).
\end{aligned}$$

Since β and f satisfy Lipschitz conditions, we have

$$\begin{aligned}
|F(\beta(u)) - F(\beta(U^{n-1}))| &= \left| \frac{\lambda f(\beta(u))}{\left(\int_\Omega f(\beta(u)) dx \right)^2} - \frac{\lambda f(\beta(U^{n-1}))}{\left(\int_\Omega f(\beta(U^{n-1})) dx \right)^2} \right| \\
&\leq C \left| \frac{f(\beta(u)) - f(\beta(U^{n-1}))}{\left(\int_\Omega f(\beta(u)) dx \right)^2} \right| \\
&\quad + \left| \frac{f(\beta(U^{n-1})) \left[\int_\Omega f(\beta(u)) - f(\beta(U^{n-1})) dx \right] \left[\int_\Omega f(\beta(u)) + f(\beta(U^{n-1})) dx \right]}{\left(\int_\Omega f(\beta(u)) dx \right)^2 \left(\int_\Omega f(\beta(U^{n-1})) dx \right)^2} \right| \\
&\leq C |\beta(u) - \beta(U^{n-1})| + C \int_\Omega |f(\beta(u)) - f(\beta(U^{n-1}))| dx \\
&\leq C |\beta(u) - \beta(U^{n-1})| + C \left(\int_\Omega |f(\beta(u)) - f(\beta(U^{n-1}))|^2 dx \right)^{\frac{1}{2}} \\
&\leq C |\beta(u) - \beta(U^{n-1})| + C \|\beta(u) - \beta(U^{n-1})\|_{L^2(\Omega)},
\end{aligned}$$

noting that $\|U^n\|_{L^2(\Omega)}, \|f(\beta(U^{n-1}))\|_{L^2(\Omega)}, \|f(\beta(u))\|_{L^2(\Omega)} \leq C$ and $\beta(u) \geq \sigma > 0$.

Further we have

$$\beta(u) - \beta(U^{n-1}) = e_\theta(t) + \frac{1}{\mu}(U^n - U^{n-1}),$$

and thus

$$\begin{aligned}
& \|F(\beta(u)) - F(\beta(U^{n-1}))\|_{L^2(\Omega)}^2 = \int_{\Omega} |F(\beta(u)) - F(\beta(U^{n-1}))|^2 dx \\
& \leq C \int_{\Omega} |\beta(u) - \beta(U^{n-1})|^2 dx + C \|\beta(u) - \beta(U^{n-1})\|_{L^2(\Omega)}^2 \\
& \leq C \|\beta(u) - \beta(U^{n-1})\|_{L^2(\Omega)}^2 \leq C \|e_{\theta}(t)\|_{L^2(\Omega)}^2 + C \|U^n - U^{n-1}\|_{L^2(\Omega)}^2, \quad (4.40)
\end{aligned}$$

which implies the following estimate

$$\begin{aligned}
IV & \leq \varepsilon \int_0^{T_N} \|e_{\theta}(t)\|_{L^2(\Omega)}^2 dt \\
& \quad + C_{\varepsilon} \sum_{i=1}^N \tau \left[\sum_{n=1}^i \int_{t_{n-1}}^{t_n} \left(\|e_{\theta}(t)\|_{L^2(\Omega)}^2 + \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 \right) dt \right] \\
& = \frac{1}{4} \int_0^{T_N} \|e_{\theta}(t)\|_{L^2(\Omega)}^2 dt + C_{\varepsilon} \sum_{i=1}^N \tau \left[\sum_{n=1}^i \int_{t_{n-1}}^{t_n} \|e_{\theta}(t)\|_{L^2(\Omega)}^2 dt \right] \\
& \quad + C_{\varepsilon} \sum_{i=1}^N \tau \left[\sum_{n=1}^i \int_{t_{n-1}}^{t_n} \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 dt \right] \\
& \leq \frac{1}{4} I_1^N + C \sum_{i=1}^N \tau I_1^i + C \sum_{i=1}^N \tau \left[\sum_{n=1}^i \tau \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 \right] \\
& = \frac{1}{4} I_1^N + C \sum_{i=1}^N \tau I_1^i + C \sum_{i=1}^N \tau \|U^i - U^{i-1}\|_{L^2(\Omega)}^2 \\
& \leq \frac{1}{4} I_1^N + C \sum_{i=1}^N \tau I_1^i + C \tau.
\end{aligned}$$

Combining all the above estimates we finally derive

$$\begin{aligned}
& I_1^N - C\tau - |I_4| + I_5 + \frac{1}{4} \left\| \nabla \int_0^{T_N} e_{\theta}(t) dt \right\|_{L^2(\Omega)}^2 \\
& \leq C\tau + \frac{1}{4} I_1^N + C \sum_{i=1}^N \tau I_1^i + C\tau.
\end{aligned}$$

Besides we note that $|I_4| \leq \frac{1}{4} I_1^N$, and thus

$$\frac{\mu}{4} \|e_{\theta}\|_{L^2(\Omega \times (0, T))}^2 + \frac{1}{4} \left\| \nabla \int_0^{T_N} e_{\theta}(t) dt \right\|_{L^2(\Omega)}^2$$

$$\leq C \sum_{i=1}^N \tau \|e_\theta\|_{L^2(0,t_i;L^2(\Omega))}^2 + C\tau + \frac{1}{\mu} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (u(t), U^i - U^{i-1}) dt.$$

By virtue of Lemma 4.5.2 and Gronwall lemma, we have

$$\|e_\theta\|_{L^2(\Omega \times (0,T))} + \left\| \int_0^t e_\theta \right\|_{L^\infty(0,T;H^1(\Omega))} = O(\tau^{\frac{1}{2}}), \quad (4.41)$$

and thus the following $H^{-1}(\Omega)$ -error bound for the unknown u is derived,

$$\|e_u\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq C\tau^{\frac{1}{2}}. \quad (4.42)$$

Combining now estimates (4.41) and (4.42) we derive the desired error estimate and this completes the proof of Theorem 4.5.1. $\square$

4.6 Numerical Experiments

In this section, we present two numerical examples in order to test the efficiency of the numerical scheme introduced in Sect. 4.3. For each example, we consider the approximate solutions both for the local and non-local problems. We actually use finite element method for spatial discretization and backward Euler method for time discretization.

Consider the following non-local problem, with $p > 0$,

$$\frac{\partial u}{\partial t} - \Delta \beta(u) = \frac{\lambda f(\beta(u))}{(\int_\Omega f(\beta(u)) dx)^p}, \quad \text{in } \Omega \times (0, T), \quad (4.43)$$

$$- \frac{\partial \beta(u)}{\partial \nu} = k(x)(\beta(u) - q(x)), \quad \text{on } \partial\Omega \times (0, T), \quad (4.44)$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega. \quad (4.45)$$

For the functions involved in the mixed-type boundary condition (4.44) we take $k, q \in L^2(\partial\Omega)$ while $u_0 \in L^\infty(\Omega)$.

Denote $\theta = \beta(u)$. The variational form of (4.43)–(4.45) is to find $u(t) \in L^2(\Omega)$, $\theta(t) \in H^1(\Omega)$, such that

$$\begin{aligned} \left(\frac{\partial u}{\partial t}, \xi \right)_{L^2(\Omega)} + (\nabla \theta, \nabla \xi)_{L^2(\Omega)} + (k\theta, \xi)_{L^2(\partial\Omega)} \\ = (kq, \xi)_{L^2(\partial\Omega)} + \frac{\lambda(f(\theta), \xi)_{L^2(\Omega)}}{(\int_\Omega f(\theta) dx)^p}, \quad \forall \xi \in H^1(\Omega). \end{aligned} \quad (4.46)$$

For any $h > 0$, let τ_h be a decomposition of Ω into triangles $\tau_h : \{T_k\}_{k=1}^{N_h}$ with diameters bounded by h , which stands for the mesh size. Let $\{\tau_h\}_h$ be a regular family of decompositions. We do not require either the quasi-uniformity or the acuteness of the family $\{\tau_h\}_h$. We have

$$\Omega = \Omega_h = \bigcup_{k=1}^K T_k.$$

Let

$$\begin{aligned} M_h &= \{\psi : \Omega_h \rightarrow \mathbb{R} : \psi|_{T_k} \text{ is constant } \forall k = 1, 2, \dots, K\}, \\ S_h &= \{\chi \in C^0(\Omega_h) : \chi|_{T_k} \text{ is linear } \forall k = 1, 2, \dots, K\}. \end{aligned}$$

Let $\Pi_h : C(\Omega) \rightarrow S_h$ denote the local linear interpolation operator. We define

$$\langle \chi, \Phi \rangle_h = \sum_{k=1}^K \int_{T_k} \Pi_h(\chi \Phi) dx.$$

We also introduce the $L^2(\Omega)$ -projection operator P_h^0 onto M_h which, for any $\xi \in L^2(\Omega)$, is defined by

$$(P_h^0 \xi, \psi) = (\xi, \psi), \quad \forall \psi \in M_h.$$

Let $\tau = T/N$ be the time step (N is a positive integer) and $t_n = n\tau$. We define the following finite element method for solving (4.43)–(4.45). Find $U^n \in M_h$, $\Theta^n \in S_h$, where $U^n \approx u(t_n)$, $\Theta^n \approx \beta(u(t_n)) = \theta(t_n)$, such that, for any $\chi \in S_h$,

$$U^0 = P_h^0 u_0, \tag{4.47}$$

$$\begin{aligned} \langle \Theta^n, \chi \rangle_h + \frac{\tau}{\mu} (\nabla \Theta^n, \nabla \chi)_{L^2(\Omega)} + \frac{\tau}{\mu} (k \Theta^n, \chi)_{L^2(\partial\Omega)} \\ = \frac{\tau}{\mu} (k q, \chi)_{L^2(\partial\Omega)} + (\beta(U^{n-1}), \chi)_{L^2(\Omega)} + \frac{\tau}{\mu} \frac{\lambda(f(\beta(U^{n-1})), \chi)_{L^2(\Omega)}}{(\int_{\Omega} f(\beta(U^{n-1})) dx)^p}, \end{aligned} \tag{4.48}$$

$$U^n = U^{n-1} + \mu [P_h^0 \Theta^n - \beta(U^{n-1})], \tag{4.49}$$

where $\mu : 0 < \mu \leq L_{\beta}^{-1}$ is a fixed number (the so-called relaxation parameter).

Let φ_j , $j = 1, 2, \dots, N_h$ be the basis functions on the nodes P_j , $j = 1, 2, \dots, N_h$ on S_h . Assume that

$$\Theta^n = \sum_{j=1}^{N_h} \theta_j^n \varphi_j,$$

and choose $\chi = \varphi_l$, $l = 1, 2, \dots, N_h$ in (4.48) then we get

$$\begin{aligned} & \left\langle \sum_{j=1}^{N_h} \theta_j^n \varphi_j, \varphi_l \right\rangle_h + \frac{\tau}{\mu} \left(\nabla \sum_{j=1}^{N_h} \theta_j^n \varphi_j, \nabla \varphi_l \right)_{L^2(\Omega)} + \frac{\tau}{\mu} \left(k \sum_{j=1}^{N_h} \theta_j^n \varphi_j, \varphi_l \right)_{L^2(\partial\Omega)} \\ &= \frac{\tau}{\mu} (kg, \varphi_l)_{L^2(\partial\Omega)} + (\beta(U^{n-1}), \varphi_l)_{L^2(\Omega)} + \frac{\tau}{\mu} \frac{\lambda(f(\beta(U^{n-1})), \varphi_l)_{L^2(\Omega)}}{(\int_{\Omega} f(\beta(U^{n-1})) dx)^p}. \end{aligned} \quad (4.50)$$

Denote

$$\mathbf{M} = (\varphi_j, \varphi_l)_{L^2(\Omega)} \quad \mathbf{K} = (k\varphi_j, \varphi_l)_{L^2(\partial\Omega)}, \quad \mathbf{G} = (kq, \varphi_l)_{L^2(\partial\Omega)},$$

$$\mathbf{B} = (\beta(U^{n-1}), \varphi_l)_{L^2(\Omega)}, \quad \mathbf{F} = (f(\Theta^{n-1}), \varphi_l)_{L^2(\Omega)}, \quad \Theta^n = (\theta_j^n),$$

and $Q = \left(\int_{\Omega} f(\Theta^{n-1}) dx \right)^p$, then we have the matrix form

$$\left(\mathbf{M} + \frac{\tau}{\mu} \mathbf{S} + \frac{\tau}{\mu} \mathbf{K} \right) \Theta^n = \frac{\tau}{\mu} \mathbf{G} + \mathbf{B} + \frac{\tau}{\mu} \frac{\mathbf{F}}{Q}.$$

Once we obtain Θ^n , we can calculate U^n from (4.49).

We use the following steps to calculate $\mathbf{B}$, $\mathbf{F}$ and Q .

For the calculation of $\mathbf{B}$, we note that

$$(\beta(U^{n-1}), \varphi_l)_{L^2(\Omega)} = \int_{\Omega} \beta(U^{n-1}) \varphi_l dx = \sum_{k=1}^{N_h} \int_{T_k} \beta(U^{n-1}) \varphi_l dx,$$

where $\beta(U^{n-1})$ is a piecewise constant function and

$$\int_{T_k} \varphi_l dx = \begin{cases} |T_k|/3, & \text{if } P_l \text{ is a node of } T_k, \\ 0, & \text{otherwise,} \end{cases}$$

for $|T_k|$ denoting the area of T_k .

To calculate $\mathbf{F}$, we note that

$$\begin{aligned} (f(\Theta^{n-1}), \varphi_l)_{L^2(\Omega)} &= \left(f \left(\sum_{j=1}^{N_h} \theta_j^{n-1} \varphi_j \right), \varphi_l \right)_{L^2(\Omega)} \\ &\approx \sum_{j=1}^{N_h} f(\theta_j^{n-1}) (\varphi_j, \varphi_l)_{L^2(\Omega)} = (\mathbf{M} * f(\Theta^{n-1}))_l. \end{aligned}$$

Finally, we have

$$\begin{aligned} Q &= \left(\int_{\Omega} f(\Theta^{n-1}) dx \right)^p = \left(\int_{\Omega} f \left(\sum_{j=1}^{N_h} \theta_j^{n-1} \varphi_j \right) dx \right)^p \\ &= \left(\sum_{j=1}^{N_h} f(\theta_j^{n-1}) \cdot \int_{\Omega} \varphi_j dx \right)^p, \end{aligned}$$

where $\int_{\Omega} \varphi_j dx = \sum_{k=1}^{N_h} \int_{T_k} \varphi_j dx$. Therefore we have built the following algorithm ($\mathcal{A}$):

Step 1: Find $U^0 = P_h^0 u_0$, u_0 is the initial value.

Step 2: Find $\beta(U^0)$.

Step 3: Find Θ^1 by (4.48).

Step 4: Find U^1 by (4.49).

Step 5: Go to Steps 1–4 to find next Θ^n , U^n , $n = 2, 3, \dots$.

In the following we present two examples focusing on the two-dimensional case, which according to Sect. 4.1, is very interesting from the application point of view.

Example 4.1 ([3, 14, 15, 23]) Let $\Omega = (0, 0.5) \times (0, 0.25)$, $0 < t < T = 0.25$ and

$$\beta(u) = \begin{cases} u, & \text{if } u < 0, \\ 0, & \text{if } 0 \leq u \leq 1, \\ u - 1, & \text{if } u > 1. \end{cases}$$

When $f = 0$, the exact solution of (4.43)–(4.45) is

$$u(x, y, t) = \begin{cases} 2[e^{\Phi(x, y, t)} - 1] + 1, & \text{if } \Phi(x, y, t) \geq 0, \\ e^{\Phi(x, y, t)} - 1, & \text{if } \Phi(x, y, t) < 0, \end{cases}$$

where

$$\Phi(x, y, t) = -x - y + 2t + 0.1 = 0,$$

is the interface (moving boundary). Dirichlet data are prescribed on the boundary Γ .

Let M_1 and M_2 be any positive integers. Let $0 = x_0 < x_1 < x_2 < \dots < x_{M_1} = 0.5$ be the partition of the interval $[0, 0.5]$ and $h_x = 0.5/M_1$ the step-size. Similarly $0 = y_0 < y_1 < y_2 < \dots < y_{M_2} = 0.25$ is a partition of the interval $[0, 0.25]$ and $h_y = 0.25/M_2$ the step-size. We divide the rectangle $\Omega = [0, 0.5] \times [0, 0.25]$ into the triangles T_k , $k = 1, 2, \dots, N_h$ with the same size. The length of the longest side of each triangle is then $h = \sqrt{h_x^2 + h_y^2}$. Let now $0 = t_0 < t_1 < t_2 < \dots < t_N = T$

Table 4.1 The estimated order of convergence for Example 4.1 at $T = 0.25$

$M_1 (= M_2)$	Nel	E_u	E_θ	p_u	p_θ
5	50	0.062	0.00635		
10	200	0.0547	0.0042	0.18	0.60
15	450	0.0482	0.0033	0.31	0.57
20	800	0.0447	0.0028	0.26	0.55

be the time partition of $[0, T]$ and τ the time step-size. We will choose $\tau = C^*h$ for any constant C^* in our numerical simulations below.

The authors in [15] provided the error estimates both for time and space discretizations for local problem (4.16)–(4.18) and their numerical experiments show that the numerical results are consistent with the theoretical results. In the current chapter we only provided the error estimates for the time discretization for the non-local problem (4.43)–(4.45). However we believe that one can obtain similar error estimates for the space discretization as for the local problem in [15]. We will evaluate the actual order of convergence of both variables u and θ in our numerical experiments.

Denote

$$E_\theta^h := \left\{ \tau \sum_{n=1}^N \sum_{k=1}^{N_h} \int_{T_k} \Pi_h([\Theta^n - \theta^n]^2) dx \right\}^{\frac{1}{2}},$$

and analogously for E_u^h . Assume that $E_u^h = Ch^{p_u}$ and $E_\theta^h = Ch^{p_\theta}$, then we have

$$p_u = \frac{\log(E_u^{h_1}/E_u^{h_2})}{\log(h_1/h_2)}, \quad p_\theta = \frac{\log(E_\theta^{h_1}/E_\theta^{h_2})}{\log(h_1/h_2)}.$$

Since we do not know the exact solution θ^n and u^n for our non-local problem, in order to check the estimated convergence orders p_u and p_θ , we assume that the exact solutions θ^n and u^n are obtained by using a very small mesh size h with $M_1 = M_2 = 40$ which implies that the number of elements is $2 \times M_1 \times M_2 = 3200$. We then choose $M_1 = M_2 = 5, 10, 15, 20$ respectively and calculate the approximate solutions Θ^n and U^n . The estimated convergence orders p_θ and p_u are listed in Table 4.1. We note that $p_\theta \approx \frac{1}{2}$ and $p_u \approx \frac{1}{4}$ which are consistent with the error estimates for the local problem in [15]. According to the general theory, see [22], the formation of a moving boundary is expected for problem (4.43)–(4.45). As mentioned in Sect. 4.1 the moving boundary describes the mushy region formulated between the two phases (liquid and solid) in the contact area during the welding process. In Fig. 4.3, the discrete interfaces at different times with $f = 0$ for the function $\beta(u)$ considered in Example 4.1 are shown where we choose $\tau = 0.25/70$ and $M_1 = M_2 = 20$. Figure 4.4 depicts the formation of the interface at different times for $f(s) = s^2 + 1$ when $\lambda = 1$, $p = 2$ for $\tau = 0.25/70$ and $M_1 = M_2 = 20$. It is worth mentioning that

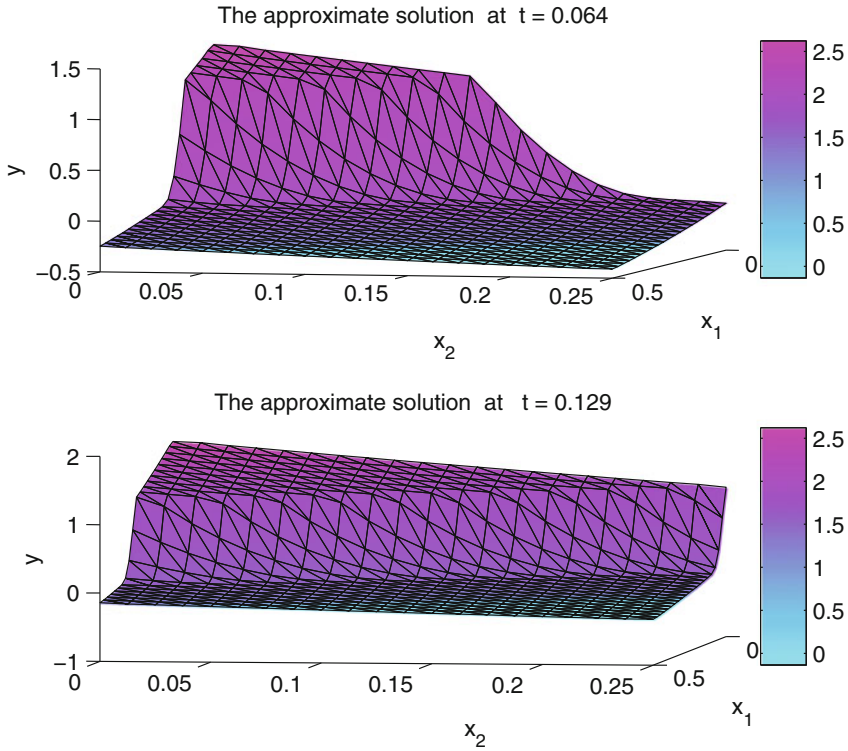


Fig. 4.3 Formation of interface for the function $\beta(u)$ considered in Example 4.1 when $f(s) = 0$ at various times

due to the external disturbance $f \neq 0$ the interface moves faster compared to the case $f = 0$. Furthermore, as λ , or equivalently the applied voltage, increases then again the evolution of the moving boundary in time becomes faster.

Example 4.2 ([14, 15, 23]) Let $\Omega = (-0.22, 0.18) \times (0, 0.2)$, $0 < t < T = 0.4$ and

$$\beta(u) = \begin{cases} \frac{1}{2}u, & \text{if } u < 0, \\ 0, & \text{if } 0 \leq u \leq 1, \\ \frac{1}{3}(u - 1), & \text{if } u > 1. \end{cases}$$

When $f = 0$, the exact solution of (4.43)–(4.45) is

$$u(x, y, t) = \begin{cases} 6\Phi(x, y, t) + 1, & \text{if } \Phi(x, y, t) \geq 0, \\ 2\Phi(x, y, t), & \text{if } \Phi(x, y, t) < 0, \end{cases}$$

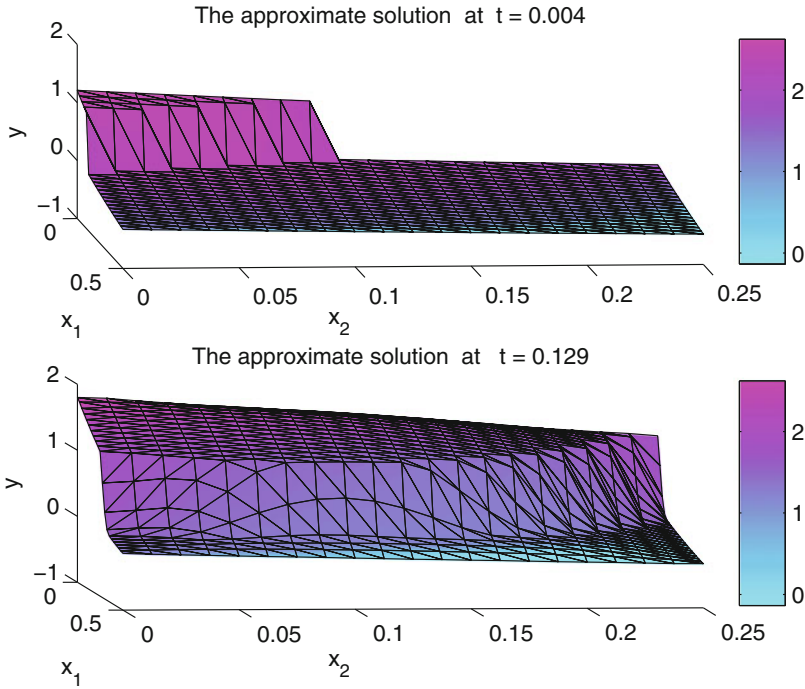


Fig. 4.4 Formation of interface for the function $\beta(u)$ considered in Example 4.1 when $f(s) = s^2 + 1$ and for $\lambda = 1$, $p = 2$ at various times

where

$$\Phi(x, y, t) = (x^2 + y^2 - e^{-4t})/4e^{2.4} = 0,$$

is the moving boundary. Dirichlet data are prescribed on the boundaries.

We use the same notation as in Example 4.1. In Fig. 4.5, the discrete interfaces at different times with $f = 0$ for the function $\beta(u)$ considered in Example 4.2 are shown where we choose $\tau = 0.25/70$ and $M_1 = M_2 = 20$.

Experimental data show that the electrical conductivity, denoted by $f(s)$, usually has a discontinuous profile, see also Sect. 4.1. But a discontinuous function does not satisfy Assumption (H_f) and hence the developed algorithm in Sect. 4.3 is not applicable in this case. However, algorithm $(\mathcal{A})$ can still provide reliable numerical experiments in this delicate case. Therefore in the current example, in order to be consistent with applications, we consider the following discontinuous function

$$f_e(s) = \begin{cases} e^{s^2} + 1, & \text{if } s \geq 0, \\ e^{s^2} + 2, & \text{if } s < 0. \end{cases}$$

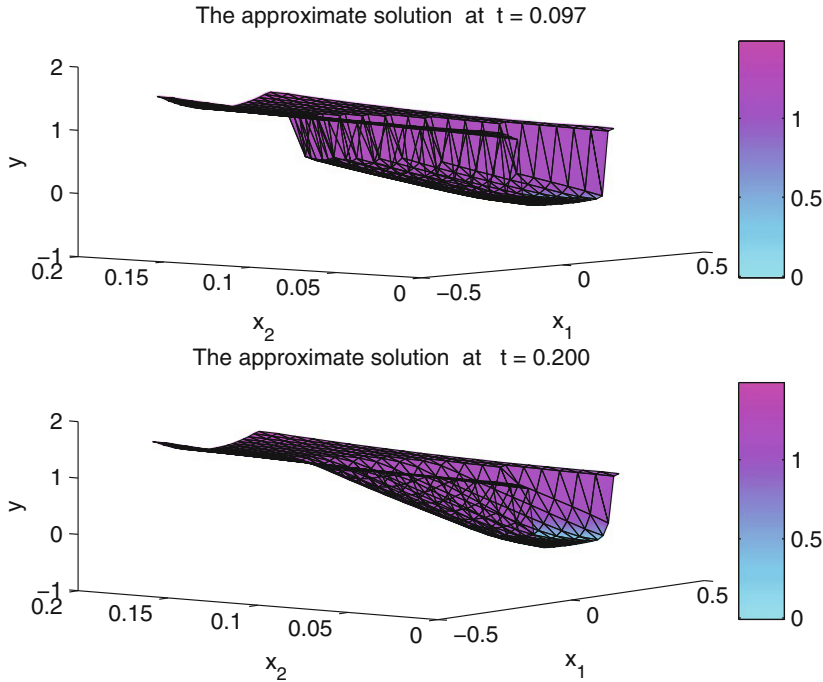


Fig. 4.5 Formation of interface for the function $\beta(u)$ considered in Example 4.2 when $f(s) = 0$ at various times

Table 4.2 The estimated order of convergence in Example 4.2 at $T = 0.4$

$M_1 (= M_2)$	Nel	E_u	E_θ	p_u	p_θ
5	50	0.0622	0.00135		
10	200	0.0516	0.0010	0.27	0.43
15	450	0.0478	8.1319e-004	0.19	0.51
20	800	0.0441	7.1035e-004	0.28	0.47

Following the same approach to estimate p_u and p_θ as in Example 4.1 we finally derive Table 4.2. We also note that $p_\theta \approx \frac{1}{2}$ and $p_u \approx \frac{1}{4}$ which are consistent with the error estimates obtained for the local problem [15].

Also Fig. 4.6 depicts the formation of this interface at various times for the function $\beta(u)$ considered in Example 4.2 and for $f(s) = s^2 + 1$ when $\lambda = 1$, $p = 2$ for $\tau = 0.25/70$ and $M_1 = M_2 = 20$. It is worth mentioning that due to the external disturbance $f \neq 0$ the interface moves faster compared to the case $f = 0$. Furthermore, as λ , or equivalently the applied voltage, increases then again the moving boundary evolves faster.

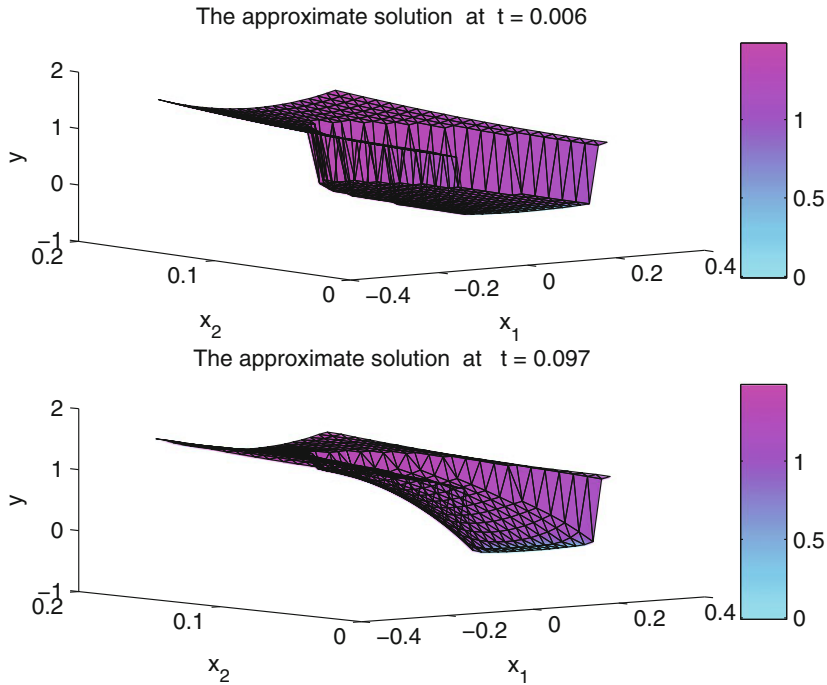


Fig. 4.6 Formation of interface for the function $\beta(u)$ considered in Example 4.2 when $f(s) = f_e(s)$ and $\lambda = 0.2$, $p = 1$ at various times

References

1. Antontsev, S.N., Chipot, M.: The analysis of blow-up for the thermistor problem. *Sib. Math. J.* **38**, 827–841 (1997)
2. Bebernes, J., Talaga, P.: Nonlocal problems modelling shear banding. *Commun. Appl. Non-linear Anal.* **3**, 79–103 (1996)
3. Berger, A.E., Brezis, H., Rogers, J.C.W.: A numerical method for solving the problem $u_t - \Delta f(u) = 0$. *RAIRO Anal. Numer.* **13**, 297–312 (1979)
4. Brokate, M., Sprekels, J.: *Hysteresis and Phase Transitions*. Applied Mathematical Sciences. Springer, New York (1996)
5. Hewitt, I.J., Lacey, A.A., Todd, R.I.: A mathematical model for flash sintering. *Math. Model. Nat. Phenom.* **10**(6), 77–89 (2015)
6. Hömberg, D., Rocca, E.: A model for resistance welding including phase transitions and Joule heating. *Math. Methods Appl. Sci.* **34**, 2077–2088 (2011)
7. Kavallaris, N.I.: Asymptotic behaviour and blow-up for a nonlinear diffusion problem with a non-local source term. *Proc. Edinb. Math. Soc.* **47**, 375–395 (2004)
8. Kavallaris, N.I., Nadzieja, T.: On the blow-up of the non-local thermistor problem. *Proc. Edinb. Math. Soc.* **50**, 389–409 (2007)
9. Kavallaris, N.I., Yan, Y.: A time discretization scheme for a nonlocal degenerate problem modelling resistance spot welding. *Math. Model. Nat. Phenom.* **10**(6), 90–112 (2015)
10. Kavallaris, N.I., Lacey, A.A., Nikolopoulos, C.V., Voong, C.: Behaviour of a non-local equation modelling linear friction welding. *IMA J. Appl. Math.* **72**, 597–616 (2007)

11. Lacey, A.A.: Thermal runaway in a nonlocal problem modelling Ohmic heating, part I, model derivation and some special cases. *Eur. J. Appl. Math.* **6**, 127–144 (1995)
12. Latos, E.A., Tzanetis, D.E.: Existence and blow-up of solutions for a non-local filtration and porous medium problem. *Proc. Edinb. Math. Soc.* **53**, 195–209 (2010)
13. Latos, E.A., Tzanetis, D.E.: Grow-up of critical solutions for a non-local porous medium problem with Ohmic heating source. *Nonlinear Differ. Equ. Appl.* **17**, 137–151 (2010)
14. Nochetto, R.H., Verdi, C.: Approximation of degenerate parabolic problems using numerical integration. *SIAM J. Numer. Anal.* **25**, 784–814 (1988)
15. Nochetto, R.H., Verdi, C.: An efficient linear scheme to approximate parabolic free boundary problems: error estimates and implementation. *Math. Comput.* **51**, 27–53 (1988)
16. Saunders, N., Li, X., Miodownik, A.P., Schillé, J-Ph.: Modelling of the thermo-physical and physical properties relevant to solidification. In: Stefanescu, D., Warren, J.A., Jolly, M.R., Krane, M.J.M. (eds.) *Advanced Solidification Processes X*, p. 669. TMS, Warrendale (2003)
17. Sidi Ammi, M.R., Mul, O.: Error estimates for the Chernoff scheme to approximate a nonlocal problem. *Proc. Est. Acad. Sci. Phys. Math.* **56**, 359–372 (2007)
18. Steinbach, I., Apel, M.: Multi phase field model for solid state transformation with elastic strain. *Phys. D* **217**, 153–160 (2006)
19. Tzanetis, D.E.: Blow-up of radially symmetric solutions of a non-local problem modelling Ohmic heating. *Electron. J. Differ. Equ.* **11**, 1–26 (2002)
20. Tzanetis, D.E., Vlamos, P.M.: A nonlocal problem modelling Ohmic heating with variable thermal conductivity. *Nonlinear Anal. RWA* **2**, 443–454 (2001)
21. Tzanetis, D.E., Vlamos, P.M.: Some interesting special cases of a non-local problem modelling Ohmic heating with variable thermal conductivity. *Proc. Edinb. Math. Soc.* **44**, 585–595 (2001)
22. Vazquez, J.L.: *The Porous Medium Equation: Mathematical Theory*. Oxford Science Publications, Oxford (2007)
23. Verdi, C.: On the numerical approach to a two-phase Stefan problem with non-linear flux. *Calcolo* **22**, 351–381 (1985)
24. Zel'dovich, Ya.B., Raizer, Yu.P.: *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, vol. II. Academic Press, New York (1966)

Part II

Applications in Biology

Chapter 5

Gierer–Meinhardt System

Abstract The purpose of the current chapter is to contribute to the comprehension of the dynamics of the shadow system of an activator-inhibitor system known as a Gierer–Meinhardt model. Shadow systems are intended to work as an intermediate step between single equations and reaction-diffusion systems. In the case where the inhibitor’s response to the activator’s growth is rather weak, then the shadow system of the Gierer–Meinhardt model is reduced to a single though non-local equation whose dynamics is investigated. We mainly focus on the derivation of blow-up results for this non-local equation which can be seen as instability patterns of the shadow system. In particular, a *diffusion driven instability (DDI)*, or *Turing instability*, in the neighbourhood of a constant stationary solution, which is destabilised via diffusion-driven blow-up, is obtained. The latter actually indicates the formation of some unstable patterns, whilst some stability results of global-in-time solutions towards non-constant steady states guarantee the occurrence of some stable patterns.

5.1 Derivation of the Non-local Model

In as early as 1952, A. Turing in his seminal paper [28] attempted, by using reaction-diffusion systems, to model the phenomenon of *morphogenesis*, the regeneration of tissue structures in hydra, an animal of a few millimeters in length made up of approximately 100,000 cells. Further observations on the morphogenesis in hydra led to the assumption of the existence of two chemical substances (morphogens), a slowly diffusing (short-range) activator and a rapidly diffusing (long-range) inhibitor. A. Turing was the first to indicate that although diffusion has a smoothing and trivializing effect on a single chemical, for the case of the interaction of two or more chemicals with different diffusion rates could force the uniform steady states of the corresponding reaction-diffusion systems to become unstable and to lead to nonhomogeneous distributions of such reactants. Such a phenomenon is now known as *diffusion driven instability (DDI)*, or *Turing instability*.

Exploring Turing’s idea further, A. Gierer and H. Meinhardt, [2], proposed in 1972 the following activator-inhibitor system, known since then as a Gierer–Meinhardt system, to model the regeneration phenomenon of hydra located in a domain $\Omega \subset$

$\mathbb{R}^N$, $N \geq 1$

$$u_t = \varepsilon^2 \Delta u - u + \frac{u^p}{v^q}, \quad \text{in } \Omega \times (0, T), \quad (5.1)$$

$$\tau v_t = D \Delta v - v + \frac{u^r}{v^s}, \quad \text{in } \Omega \times (0, T), \quad (5.2)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \partial \Omega \times (0, T), \quad (5.3)$$

$$u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) > 0, \quad \text{in } \Omega, \quad (5.4)$$

where Ω is the domain of interaction of the two reactants and ν denotes the unit normal vector to $\partial \Omega$, whilst u and v stand for the concentrations of the activator and the inhibitor respectively. System (5.1)–(5.4) intends to provide a thorough explanation of symmetry breaking as well as of de novo pattern formation by virtue of the coupling of a local activation and a long-range inhibition process. The inserted nonlinearities describe the fact that the activator promotes the differentiation process and it stimulates its own production, whereas the inhibitor acts as a suppressant against the self-enhancing activator to prevent the unlimited growth.

Here, ε^2 , D represent the diffusing coefficients whereas the exponents satisfying the conditions:

$$p > 1, \quad q, r, > 0, \quad \text{and } s > -1,$$

measure the morphogens interactions. In particular, the dynamics of system (5.1)–(5.4) can be characterised by two numbers: the *net self-activation index* $\rho \equiv (p-1)/r$ and the *net cross-inhibition index* $\gamma \equiv q/(s+1)$. Indeed, ρ correlates the strength of self-activation of the activator with the cross-activation of the inhibitor. So, if ρ is large, then the net growth of the activator is large no matter the inhibitor's growth. On the other hand, γ measures how strongly the inhibitor suppresses the production of the activator and that of itself. Now if γ is large then the production of the activator is strongly suppressed by the inhibitor. Finally, the parameter τ quantifies the inhibitor's response against the activator's growth.

Guided by biological interpretation as well as by mathematical reasons, it is usually assumed that the parameters p, q, r, s satisfy the following condition

$$\rho \equiv \frac{p-1}{r} < \frac{q}{s+1} \equiv \gamma,$$

or equivalently

$$p - r\gamma < 1. \quad (5.5)$$

Condition (5.5) is called a *Turing condition* whilst the reverse inequality

$$p - r\gamma > 1, \quad (5.6)$$

will be referred to as an *anti-Turing condition*, see also [7].

The *Turing condition* guarantees, [22], that the spatially homogeneous equilibrium $(u, v) = (1, 1)$ of the corresponding kinetic (ODE) system

$$\frac{du}{dt} = -u + \frac{u^p}{v^q}, \quad \tau \frac{dv}{dt} = -v + \frac{u^r}{v^s}, \quad (5.7)$$

is stable if $\tau < \frac{s+1}{p-1}$. Nevertheless, once diffusion terms are introduced, with $\varepsilon^2 \ll D$, and under (5.5) then $(u, v) = (1, 1)$ becomes unstable and bifurcation occurs, see also [22, 23]. Therefore, *diffusion driven instability (DDI)* takes place which leads to pattern formation and then explains the phenomenon of morphogenesis.

Apart from its vital biological importance the system (5.1)–(5.4) has also interesting mathematical features and emerging singularities. As such, it has attracted a lot of attention from the field of mathematical analysis. Subjects of interest include the existence of global-in-time solutions, which was first investigated in [25] and then studied more thoroughly in [17, 20]. The author in [6] proved that under the condition $\frac{p-1}{r} < 1$, a global-in-time solution exists, which is an almost optimal result, also taking into consideration the results in [22]. Furthermore, [8] contains an investigation of the asymptotic behavior of the solution of (5.1)–(5.4). In particular the authors showed that if $\tau = \frac{s+1}{p-1}$, $s > 0$, and

$$\frac{2\sqrt{d_1 d_2}}{d_1 + d_2} \geq \sqrt{\frac{(s+1)(p-1)}{sp}}, \quad d_1 = \varepsilon^2, \quad d_2 = \tau^{-1}D,$$

then the global-in-time solution of (5.1)–(5.4) is approaching uniformly a spatially homogeneous solution, which is always periodic-in-time unless it is a constant one. The occurrence of finite-time blow-up, which actually means unlimited growth for the activator, was first established in [17] and later in [9, 15, 30], whereas the case of nondiffusing activator finite-time blow-up is also investigated in [9]. The existence and stability of spiky stationary solutions is thoroughly studied in the survey paper [29].

As specified above, in the case of the Gierer–Meinhardt system, the inhibitor diffuses much faster compared to the activator, i.e. $\varepsilon^2 \ll D$, and thus the system (5.1)–(5.4) can be fairly approximated by its *shadow system* when $D \gg 1$. The concept of a *shadow system* was introduced by Keener, [11], to describe the qualitative behavior of reaction-diffusion systems when one of the diffusing coefficients is very large. Such a system is formed by a reaction-diffusion equation coupled with an ordinary differential equation (ODE) with non-local effects and it actually contains all the essential dynamics of the original reaction-diffusion system. In particular, if there is a compact attractor for the *shadow system* the original reaction-diffusion system has a compact attractor too, see also [3].

In the following we provide a formal derivation of the *shadow system* of the Gierer–Meinhardt system (5.1)–(5.4). A rigorous proof can be found in [18, 19] where it is also shown that the convergence of the original reaction-diffusion system towards its *shadow system* is valid locally in time except for an initial layer. Now,

dividing (5.2) by D and letting $D \uparrow +\infty$ for any fixed $t \in (0, T)$, then due to the boundary condition (5.3) v becomes spatial homogeneous, i.e. $v(x, t) = \xi(t)$. Next, integrating the resulting equation over Ω we finally derive that $u(x, t)$, $\xi(t)$ satisfy the *shadow system*:

$$u_t = \varepsilon^2 \Delta u - u + \frac{u^p}{\xi^q}, \quad \text{in } \Omega \times (0, T), \quad (5.8)$$

$$\tau \xi_t = -\xi + \frac{1}{\xi^s} \int_{\Omega} u^r dx \quad \text{in } \Omega \times (0, T), \quad (5.9)$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (5.10)$$

$$u(x, 0) = u_0(x) > 0, \quad \xi(0) = \bar{v}_0 = \frac{1}{|\Omega|} \int_{\Omega} v_0, \quad \text{in } \Omega, \quad (5.11)$$

where

$$\int_{\Omega} u^r dx \equiv \frac{1}{|\Omega|} \int_{\Omega} u^r dx.$$

Notably problem (5.8)–(5.11) is non-local due to the presence of the integral term in (5.9).

Since the convergence towards (5.8)–(5.11) holds only locally in time there might be discrepancies between the global-in-time dynamics of system (5.1)–(5.4) and those of (5.8)–(5.11) for some range of the involved parameters p, q, r, s ; this has been also indicated in [7, 13, 14]. On the other hand, there are ranges of the involved parameters, where the two systems have exactly the same long-time behavior [13, Theorem 1] and thus it is worth investigating the *shadow system* (5.8)–(5.11), which is simpler compared to the full system (5.1)–(5.4), so we can capture some of the features of (5.1)–(5.4).

Henceforth, we focus on the case where $\tau = 0$; i.e. when the inhibitor's response rate is quite small against the inhibitor's growth. For $\tau = 0$ the second equation (5.9) is solved as

$$\xi(t) = \left(\int_{\Omega} u^r(x, t) dx \right)^{\frac{1}{s+1}},$$

and thus the *shadow system* reduces to the following non-local problem

$$u_t = \Delta u - u + \frac{u^p}{\left(\int_{\Omega} u^r dx \right)^{\gamma}}, \quad \text{in } \Omega \times (0, T), \quad (5.12)$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (5.13)$$

$$u(x, 0) = u_0(x) > 0, \quad \text{in } \Omega, \quad (5.14)$$

where for simplicity it has been considered $\varepsilon = 1$.

The rest of the current chapter is devoted to the study of problem the (5.12)–(5.14) whose mathematical structure is intriguing. In particular, owing to the presence of the non-local term and the monotonicity of its nonlinearity, then problem (5.12)–(5.14) does not admit a maximum principle, [24], and so alternatives to comparison techniques should be employed to investigate its long-time behavior. Some global-in-time existence and blow-up results for problem (5.12)–(5.14) were presented in [14], whereas some slow moving spike solutions were constructed in [5]. Here, we provide novel global-in-time and blow-up results, extending further the mathematical analysis provided in [14], as well as describing the form of the destabilization patterns developed due to the phenomenon of *DDI*.

In addition, the investigation of the non-local problem (5.12)–(5.14) is also attractive from the biological point of view. Specifically, it will reveal under which circumstances the dynamics of the interaction of the two morphogens (activator and inhibitor) can be controlled by governing only the dynamics of the activator itself.

Note that under condition (5.5) the solution of the spatially homogeneous part

$$\frac{du}{dt} = -u + u^{p-r\gamma}, \quad u(0) = \bar{u}_0 > 0, \quad (5.15)$$

never exhibits blow-up, since the non-linearity is sublinear, and its unique stationary state $u = 1$ is asymptotically stable. Below, by using stability analysis we show that condition (5.5) implies linear instability.

Indeed, the linearized problem of (5.12)–(5.14) around $u = 1$ is given by

$$\begin{aligned} \phi_t &= \Delta\phi + (p-1)\phi - r\gamma \int_{\Omega} \phi, \quad \text{in } \Omega \times (0, T), \\ \frac{\partial\phi}{\partial\nu} &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

and can be written in the form of an evolution equation in $X = L^2(\Omega)$ as

$$\frac{d\phi}{dt} = -A\phi.$$

Here the generator A is a self-adjoint operator associated with the bi-linear form (see Kato [10])

$$a(\phi, w) = \int_{\Omega} (\nabla\phi \cdot \nabla w + (1-p)\phi w) \, dx + r\gamma \int_{\Omega} \phi \cdot \int_{\Omega} w, \quad \phi, w \in V = H^1(\Omega).$$

Now for $\phi = w$ we derive

$$\begin{aligned}
 a(\phi, \phi) &= \|\nabla \phi\|_2^2 + (1-p) \int_{\Omega} \phi^2 + r\gamma \left(\int_{\Omega} \phi \right)^2 \\
 &= \sum_{j=1}^{\infty} \mu_j^2 |(\phi, \varphi_j)|^2 + \sum_{j=1}^{\infty} (1-p) |(\phi, \varphi_j)|^2 + r\gamma |(\phi, \varphi_1)|^2 \\
 &= (1-p+r\gamma) |(\phi, \varphi_1)|^2 + \sum_{j=2}^{\infty} (\mu_j^2 + 1-p) |(\phi, \varphi_j)|^2,
 \end{aligned}$$

where $0 = \mu_1 < \mu_2 \leq \dots \rightarrow \infty$ denote the eigenvalues of $-\Delta$ associated with the Neumann boundary condition, and φ_j is the corresponding j -th eigenfunction normalized by $\|\varphi_j\|_2 = 1$. Note that under the Turing condition (5.5), the linearized instability of the steady-state solution $u = 1$ arises if and only if $\mu_2^2 < p - 1$. The latter suggests that under condition (5.5) a Turing instability phenomenon should be anticipated, which is actually shown in Theorem 5.7, this Turing instability is exhibited in the form of a finite-time blow-up.

5.2 Mathematical Analysis

Next we focus on the mathematical analysis of the dynamics of the non-local problem (5.12)–(5.14) which it will reveal intriguing features including the diffusion-driven finite-time blow-up mentioned in the previous section, see also [7].

5.2.1 Global-in-time Existence

We first point out that the solution of problem (5.12)–(5.14) could cease to exist either by quenching (when u reaches 0 at some finite or infinite time), i.e. there exists

$$T \leq +\infty \quad \text{and} \quad \liminf_{t \uparrow T} \|u(\cdot, t)\|_r = 0 \quad \text{for some } r > 1, \quad (5.16)$$

or via finite-time or infinite-time blow-up when there exists $T \leq +\infty$ such that

$$\limsup_{t \uparrow T} \|u(\cdot, t)\|_r = +\infty \quad \text{for some } r > 1.$$

Note that by the parabolic regularity there actually holds

$$T < +\infty \Rightarrow \lim_{t \uparrow T} \|u(\cdot, t)\|_{\infty} = +\infty. \quad (5.17)$$

Our first observation is that quenching in finite time cannot occur for u . Indeed, there holds

Proposition 5.1 *Each $T > 0$ admits $C_T > 0$ such that the solution of (5.12) and (5.13) satisfies*

$$u \geq C_T \text{ in } \Omega \times [0, T). \quad (5.18)$$

Proof By the maximum principle and comparison theorems there holds that $u = u(x, t) > 0$ and $u(x, t) \geq \tilde{u}(t)$, where $\tilde{u} = \tilde{u}(t)$ is the solution to the initial value problem

$$\frac{d\tilde{u}}{dt} = -\tilde{u} \text{ in } (0, T), \quad \tilde{u}(0) = \tilde{u}_0 \equiv \inf_{\Omega} u_0(x) > 0,$$

which implies (5.18) with $C_T = \tilde{u}_0 e^{-T}$. $\square$

Due to Proposition 5.1 the following alternatives are left; blow-up in finite time indicated by $T < +\infty$, blow-up in infinite time, quenching in infinite time, and global-in-time compact orbit in $C(\overline{\Omega})$. However, as the following Theorem shows infinite time quenching cannot occur.

Theorem 5.1 *There is $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ the solution of (5.12) and (5.13) admits the estimate*

$$\int_{\Omega} u^{-\delta} \leq C. \quad (5.19)$$

Proof Set $\chi = u^{\frac{1}{\alpha}}$, $\alpha \neq 0$, then χ satisfies

$$\alpha \chi_t = \alpha \left(\Delta \chi + 4(\alpha - 1) |\nabla \chi^{\frac{1}{2}}|^2 \right) - \chi + f \text{ in } \Omega \times (0, T), \quad (5.20)$$

$$\frac{\partial \chi}{\partial \nu} = 0, \text{ on } \partial \Omega \times (0, T), \quad (5.21)$$

$$\chi(x, 0) = u_0^{\frac{1}{\alpha}}(x), \text{ in } \Omega, \quad (5.22)$$

with

$$f = \frac{u^{p-1+\frac{1}{\alpha}}}{(\int_{\Omega} u^r)^{\gamma}}. \quad (5.23)$$

Averaging (5.20) over Ω , we obtain

$$\alpha \frac{d}{dt} \int_{\Omega} \chi + 4\alpha(1 - \alpha) \int_{\Omega} |\nabla \chi^{\frac{1}{2}}|^2 + \int_{\Omega} \chi = \int_{\Omega} f, \quad (5.24)$$

and hence

$$\frac{d}{dt} \int_{\Omega} \chi + 4(1 - \alpha) \int_{\Omega} |\nabla \chi^{\frac{1}{2}}|^2 + \frac{1}{\alpha} \int_{\Omega} \chi \leq 0, \quad (5.25)$$

for $\alpha < 0$ since also $f > 0$. Letting now $\delta = -\frac{1}{\alpha}$ we have

$$\frac{d}{dt} \int_{\Omega} \chi + 4(1 + \delta^{-1}) \int_{\Omega} |\nabla \chi^{\frac{1}{2}}|^2 \leq \delta \int_{\Omega} \chi.$$

Since Poincaré–Wirtinger’s inequality reads

$$\|\nabla w\|_2^2 \geq \mu_2 \|w\|_2^2, \quad \text{for any } w \in H^1(\Omega),$$

where μ_2 is the second eigenvalue of the Laplace operator associated with Neumann boundary conditions, then application of (5.25) for $w = \chi^{1/2}$ entails

$$\frac{d}{dt} \int_{\Omega} \chi + c \int_{\Omega} \chi \leq 0, \quad (5.26)$$

for $0 < \delta \ll 1$. Differential inequality (5.26) implies that $\chi(t) \leq C < \infty$ for any $t > 0$ and thus (5.19) follows by the fact that $\chi = u^{-\delta}$. $\square$

Remark 5.2.1 We note that this property does not arise neither for the original system (5.1)–(5.4) nor for the shadow system (5.8) and (5.9). The former can be easily concluded by the classification of the homogeneous orbits given in [23], while the latter is an immediate consequence of the Theorem 5.1.

In fact, inequality (5.19) implies $\int_{\Omega} u^{\delta} \geq c = C^{-1}$ and then

$$\int_{\Omega} u^r \geq \left(\int_{\Omega} u^{\delta} \right)^{r/\delta} \geq c^{r/\delta} > 0 \quad \text{for any } t > 0, \quad (5.27)$$

follows by Jensen’s inequality taking $\delta \leq r$, where again c is independent of time t . Consequently, relation (5.27) guarantees that the nonlinear term of non-local equation (5.12) stays away from zero and therefore the solution u is bounded away from zero as well.

Remark 5.2.2 Remark 5.2.1 is interpreted in biological context as follows: the activator can never be diminished.

The second global-in-time result is concerned with the special case $r = p + 1$ where a problem (5.12)–(5.14) admits a variational structure, which is not the case for the initial system (5.1)–(5.4). In particular we have

$$\frac{d}{dt} J(u) = -\|u_t\|_2^2 \leq 0, \quad (5.28)$$

for

$$J(u) = \frac{1}{2} (\|\nabla u\|_2^2 + \|u\|_2^2) - \frac{1}{(p+1)(1-\gamma)} \left(\int_{\Omega} u^{p+1} dx \right)^{1-\gamma},$$

which entails the following result by the method of double well potential, see [12, 26].

Theorem 5.2 *Let $N \geq 3$ and $r = p + 1$. If $\frac{p-1}{p+1} < \gamma < 1$ and $1 < p < \frac{N+2}{N-2}$ then a global-in-time solution exists, i.e.*

$$T = +\infty \quad \text{and} \quad \sup_{(0,T)} \|u(\cdot, t)\|_\infty \leq C. \quad (5.29)$$

Proof In this case we have $0 < \gamma < 1$ and $(p+1)(1-\gamma) < 2$. By $J(u) \leq J(u_0)$ it holds that

$$\frac{1}{2}(\|\nabla u\|_2^2 + \|u\|_2^2) \leq J(u_0) + \frac{|\Omega|^\gamma}{(p+1)(1-\gamma)} \left(\int_\Omega u^{p+1} \right)^{1-\gamma}.$$

In addition Sobolev's and Young's inequalities imply

$$\left(\int_\Omega u^{p+1} \right)^{1-\gamma} = \|u\|_{p+1}^{(p+1)(1-\gamma)} \leq \frac{1}{4} \|u\|_{H^1}^2 + C,$$

by $(p+1)(1-\gamma) < 2$ and thus

$$\|u(\cdot, t)\|_{H^1} \leq C. \quad (5.30)$$

Therefore u satisfies

$$u_t = \Delta u - u + a(t)u^p, \quad \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} = 0, \quad u \Big|_{t=0} = u_0(x) > 0,$$

with $0 \leq a(t) \leq C$ due to (5.27) and $1 < p < \frac{N+2}{N-2}$. Then (5.30) implies (5.29) by a standard bootstrap argument (see Lemma 8.1 of [27]). $\square$

Remark 5.2.3 Theorem 5.2 indicates that in case where $r = p + 1$ then there is a discrepancy between the behavior of the full system (5.1)–(5.4) and those of the non-local problem (5.12)–(5.14). Indeed, under the anti-Turing condition the full system does not exhibit any instability, whilst an instability occurs only when Turing condition (5.31) holds.

Remark 5.2.4 In the case where $\gamma = \frac{p-1}{p+1}$ and $1 < p < \frac{N+2}{N-2}$, we have always $T = +\infty$, whilst infinite-time blow-up, i.e. $\lim_{t \uparrow +\infty} \|u(\cdot, t)\|_\infty = +\infty$, may occur. In fact, by the proof of Theorem 5.2 we have

$$\|u(\cdot, t)\|_{H^1} \leq C(1+t) \quad \text{for } 0 < t < T,$$

and then by virtue of Sobolev's imbedding we obtain $\|u(\cdot, t)\|_\infty \leq C_T$, which entails $T = +\infty$ by the parabolic regularity. Furthermore, in case where $J(u_0) < 0$

we derive

$$\frac{d}{dt} \|u\|_2^2 \geq -2J(u_0) > 0,$$

and then it follows that $\lim_{t \uparrow \infty} \|u(\cdot, t)\|_2 = +\infty$. The latter implies that $\lim_{t \uparrow \infty} \|u(\cdot, t)\|_\infty = +\infty$ and thus infinite-time blow-up occurs in that case.

Note that for $r = p + 1$ Turing condition (5.5) is reduced to

$$\gamma > \frac{p-1}{p+1}. \quad (5.31)$$

Turing condition (5.31) further implies that the solution orbit for problem (5.12)–(5.14) is compact in $C(\overline{\Omega})$ and the ω -limit set

$$\omega(u_0) = \{u_* \in C(\overline{\Omega}) \mid \exists t_k \uparrow +\infty \text{ s.t. } \lim_{k \rightarrow \infty} \|u(\cdot, t_k) - u_*\|_\infty = 0\},$$

of this orbit is nonempty, connected, compact, and lies in the set of stationary solutions, which are defined as the solutions of the following problem

$$-\Delta u_* + u_* = \frac{u_*^p}{\left(\int_\Omega u_*^{p+1}\right)^\gamma}, \quad u_* > 0 \quad \text{in } \Omega, \quad \frac{\partial u_*}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \quad (5.32)$$

Concerning (5.32), existence of stable spiky stationary solutions is known (see the survey paper by Wei [29]) and thus formation of Turing patterns converging to these spiky solutions is guaranteed as long as (5.31) holds.

In the following, global-in-time existence of the solution is obtained via a priori estimates of some L^ℓ -norms of solution $u(x, t)$. Those a priori estimates hold in a parameter range which implies condition $\frac{p-1}{r} < 1$ which, as mentioned earlier, guarantees the global-in-time existence of the solution to the original model (5.1)–(5.4).

Theorem 5.3 *If $\frac{p-1}{r} < \min\{1, \frac{2}{N}, \frac{1}{2}(1 - \frac{1}{r})\}$ and $0 < \gamma < 1$, then problem (5.12)–(5.13) has again a global-in-time solution.*

Proof We assume $\frac{p-1}{r} < \min\{1, \frac{2}{N}, \frac{1}{2}(1 - \frac{1}{r})\}$ with $0 < \gamma < 1$ and we only focus on the case $N \geq 2$ since the complementary case $N = 1$ is simpler.

Since $p > 1$, the above assumption implies $\frac{p-1}{r} < \frac{2}{N}$ and $r > p$. Then there holds that

$$0 < \frac{1}{r-p+1} < \min \left\{ 1, \frac{1}{p-1} \cdot \frac{2}{N-2}, \frac{1}{1-p+r\gamma} \right\},$$

since also $0 < \gamma < 1$.

Choosing $\frac{1}{r-p+1} < \alpha < \min\{1, \frac{1}{p-1} \cdot \frac{2}{N-2}, \frac{1}{1-p+r\gamma}\}$, we have

$$\max \left\{ \frac{N-2}{N}, \frac{1}{\alpha r} \right\} < \frac{1}{-\alpha + 1 + \alpha p},$$

and hence there is $\beta > 0$ such that

$$\max \left\{ \frac{N-2}{N}, \frac{1}{\alpha r} \right\} < \frac{1}{\beta} < \frac{1}{-\alpha + 1 + \alpha p} < 2, \quad (5.33)$$

which also satisfies

$$\frac{\beta}{\alpha r} < 1 < \frac{\beta}{-\alpha + 1 + \alpha p}. \quad (5.34)$$

Note that for f defined by (5.23) holds

$$\int_{\Omega} f = \frac{\int_{\Omega} u^{p-1+\frac{1}{\alpha}}}{(\int_{\Omega} u^r)^{\gamma}} = \frac{\int_{\Omega} \chi^{-\alpha+1+\alpha p}}{(\int_{\Omega} \chi^{\alpha r})^{\gamma}}.$$

Also by virtue of (5.34)

$$\int_{\Omega} \chi^{-\alpha+1+\alpha p} \leq \left(\int_{\Omega} \chi^{\beta} \right)^{\frac{-\alpha+1+\alpha p}{\beta}} \quad \text{and} \quad \left(\int_{\Omega} \chi^{\alpha r} \right)^{\gamma} \geq \left(\int_{\Omega} \chi^{\beta} \right)^{\frac{\alpha r}{\beta} \cdot \gamma},$$

and thus

$$\int_{\Omega} f \leq \left(\int_{\Omega} \chi^{\beta} \right)^{\frac{-\alpha+1+\alpha p - \alpha r \gamma}{\beta}} = \|\chi^{\frac{1}{2}}\|_{2\beta}^{2(1-\sigma)}, \quad (5.35)$$

with $0 < \sigma = \alpha\{1 - p + r\gamma\} < 1$, recalling $\frac{p-1}{r} < \gamma$ and $\alpha < \frac{1}{1-p+r\gamma}$.

Now since $1 < 2\beta < \frac{2N}{N-2}$ holds due to (5.33), then Sobolev's and Young's inequalities entail

$$\frac{d}{dt} \int_{\Omega} \chi + c \|\chi^{\frac{1}{2}}\|_{H^1}^2 \leq C, \quad 0 < t < T,$$

using also $0 < \alpha < 1$, and in particular,

$$\int_{\Omega} \chi \leq C, \quad \text{for any } 0 < t < T.$$

Since $\frac{1}{\alpha}$ can be chosen to be close to $r - p + 1$, we have

$$\|u(\cdot, t)\|_q \leq C_q, \quad 0 < t < T, \quad \text{for any } 1 \leq q < r - p + 1, \quad (5.36)$$

taking into account that $\chi = u^{\frac{1}{\alpha}}$. Since $\frac{p-1}{r} < \frac{1}{2}(1 - \frac{1}{r})$ implies $\frac{r-p+1}{p} > 1$, then if there is $a > 1$ such that

$$\|u(\cdot, t)\|_q \leq C_q, \quad 0 < t < T, \quad \text{for any } 1 \leq q < a(r - p + 1), \quad (5.37)$$

by virtue of the classical semigroup estimate, see [25], then inequality (5.37) can be extended for any $q \geq 1$ as long as $\frac{N}{2}(\frac{1}{\ell} - \frac{1}{q}) < 1$. Therefore, we obtain

$$\|u(\cdot, t)\|_q \leq C_q, \quad 0 < t < T, \quad \text{for any } 1 \leq q < a_1(r - p + 1), \quad (5.38)$$

and for $a_1 > 0$ defined by

$$\frac{1}{a_1} = \frac{1}{a} - \frac{2}{N} \cdot \frac{r - p + 1}{p}, \quad (5.39)$$

as long as the right-hand side of (5.39) is positive, otherwise $q = \infty$ into relation (5.38). We eventually obtain (5.29), and the proof is complete. $\square$

Remark 5.2.5 The result of Theorem 5.3 is in agreement with the global-in-time existence result obtained in [6] for the full system, and so in that case (5.1)–(5.4) and (5.12)–(5.14) share the same dynamics.

Now we consider the following L^ℓ -norms, $\ell > 0$, of solution $u(x, t)$

$$\zeta(t) = \int_{\Omega} u^r dx, \quad z(t) = \int_{\Omega} u^{-p+1+r} dx, \quad w(t) = \int_{\Omega} u^{p-1+r} dx. \quad (5.40)$$

By choosing proper initial data and using phase plane analysis, we can actually derive estimates of $\zeta(t)$, $z(t)$ and $w(t)$, identifying also some invariant regions in the plane. In particular, the following result is satisfied.

Theorem 5.4 *Let $\gamma > 1$, $r \geq 1$ and $\frac{p-1}{r} < 1$. Assume also that $w(0) < \zeta(0)^{1-\gamma}$ and $\zeta(0)^{1+\gamma} > z(0)$. Then problem (5.12)–(5.13) has a global-in-time solution, i.e. (5.29) holds.*

Proof We first note that by virtue of Hölder's inequality we have

$$wz \geq \zeta^2. \quad (5.41)$$

Under the assumption $r \geq 1$ relation (5.48) entails

$$\frac{1}{r} \frac{d\zeta}{dt} \leq -\zeta + \frac{z}{\zeta^\gamma}, \quad \text{for } 0 \leq t < T. \quad (5.42)$$

Furthermore, since $\alpha = \frac{1}{p-1-r} < 0$ results from $\frac{p-1}{r} < 1$, then (6.10) implies

$$\alpha \frac{dw}{dt} \geq -w + \zeta^{1-\gamma}, \quad \text{for } 0 \leq t < T,$$

or equivalently

$$\frac{1}{-p+1+r} \frac{dw}{dt} \leq w - \zeta^{1-\gamma}, \quad \text{for } 0 \leq t < T. \quad (5.43)$$

We claim that the assumption $\zeta(0)^{1+\gamma} > z(0)$ yields that $\zeta(t)^{1+\gamma} > z(t)$ for any $0 \leq t < T$. Indeed, let us assume there exists $t_0 > 0$ such that

$$\zeta(t)^{1+\gamma} > z(t), \quad 0 \leq t < t_0, \quad \text{and} \quad \zeta(t_0) = z(t_0).$$

Then we obtain

$$\frac{d\zeta}{dt} < 0, \quad \text{for } 0 \leq t < t_0 \quad \text{and} \quad w(t_0) \geq \zeta(t_0)^{1-\gamma}, \quad (5.44)$$

by virtue of (5.42) and (5.41).

On the other hand, $w(0) < \zeta(0)^{1-\gamma}$, due to (5.43), entails

$$\frac{dw}{dt} < 0, \quad \text{for } 0 \leq t < t_0.$$

Consequently, since also $\gamma > 1$ then the curve $(w(t), \zeta(t))$ for $0 \leq t \leq t_0$, remains in the region $w < \zeta^{1-\gamma}$ and hence $w(t_0) < \zeta(t_0)^{1-\gamma}$, which contradicts the second inequality of (5.44).

Thus it follows that

$$\frac{d\zeta}{dt} < 0, \quad \frac{dw}{dt} < 0, \quad \text{for } 0 \leq t < T,$$

and in particular, we have

$$\|u(\cdot, t)\|_{p-1+r} \leq C, \quad \text{for } 0 \leq t < T.$$

Since $r \geq 1$ implies $\frac{p-1+r}{p} \geq 1$, we obtain (5.29) by the same bootstrap argument used in Theorem 5.3. $\square$

Remark 5.2.6 Theorem 5.4, on the contrary, deals with the case of global-in-time existence under Turing condition and it is also in agreement with Jiang's result in [6]. Consequently under assumptions of Theorem 5.4 both the full system (5.1)–(5.4) as well as the non-local problem (5.12)–(5.14) ensemble the same long-time behavior.

5.2.2 ODE Type Blow-Up

In this subsection we present some blow-up results for the non-local problem (5.12)–(5.14). Here, finite-time blow-up is actually induced by the presence of the non-local reaction term.

Our first blow-up result is based on the following

Proposition 5.2 *Assume that $p \geq r$ and condition (5.5) hold then*

$$\bar{u} \geq \max\{1, \bar{u}_0\}. \quad (5.45)$$

Whereas if anti-Turing condition (5.6) holds and $\bar{u}_0 > 1$, then finite-time blow-up occurs, i.e. $T < +\infty$.

Proof Since $p > 1$ and $p \geq r$, there is $r \leq \mu \leq p$ satisfying $s \geq 1$. Then we obtain

$$\int_{\Omega} u^p \geq \left(\int_{\Omega} u^{\mu} \right)^{\frac{p}{\mu}}, \quad \left(\int_{\Omega} u^r \right)^{\gamma} \leq \left(\int_{\Omega} u^{\mu} \right)^{\frac{r}{\mu}\gamma},$$

via Hölder's inequality and hence

$$\frac{d\bar{u}}{dt} = -\bar{u} + \frac{\int_{\Omega} u^p}{\left(\int_{\Omega} u^r \right)^{\gamma}} \geq -\bar{u} + \left(\int_{\Omega} u^{\mu} \right)^{\frac{p-r\gamma}{\mu}} \geq -\bar{u} + \bar{u}^{p-r\gamma}. \quad (5.46)$$

In case $p - r\gamma < 1$ then the differential inequality (5.46) implies (5.45). Whilst, in the complementary case $p - r\gamma > 1$ again by virtue of (5.46) we derive that $\bar{u}$ blows up in finite time provided $\bar{u}_0 > 1$, and hence u does so. $\square$

Remark 5.2.7 Proposition 5.2 illustrates that under the assumptions $p \geq r$ and (5.6) then the qualitative behavior of the full system (5.1)–(5.4) and those of the non-local problem (5.12)–(5.14) is quite different. It should be pointed out that under the anti-Turing condition (5.6) the full system does not exhibit any instability, whilst an instability emerges when Turing condition (5.5) is imposed.

Now we consider the case $r = p + 1$ where, as it has already pointed out in the previous section, problem (5.12)–(5.14) has a variational structure which is used to prove the following

Theorem 5.5 *Let $r = p + 1$ and $\gamma < \min\{1, \frac{p-1}{p+1}\}$. If $J(u_0) \leq 0$ then finite-time blow-up occurs, i.e. $T < +\infty$.*

Proof Since $J(u_0) \leq 0$, then via the dissipation relation (5.28) we derive $J(u(t)) \leq 0$ for any $0 < t < T$.

We also have

$$\frac{d}{dt} \|u\|_2^2 = -2I(u),$$

where

$$\begin{aligned}
 I(u) &= \|\nabla u\|_2^2 + \|u\|_2^2 - \frac{\int_{\Omega} u^{p+1}}{(\int_{\Omega} u^r)^{\gamma}} = \|\nabla u\|_2^2 + \|u\|_2^2 - \left(\int_{\Omega} u^{p+1}\right)^{1-\gamma} \\
 &= 2J(u) + \left(\frac{2}{(p+1)(1-\gamma)} - 1\right) \left(\int_{\Omega} u^{p+1}\right)^{1-\gamma} \\
 &\leq -\left(1 - \frac{2}{(p+1)(1-\gamma)}\right) \|u\|_{\frac{p+1}{p+1}}^{(p+1)(1-\gamma)}.
 \end{aligned}$$

Since $0 < \gamma < \min\{1, \frac{p-1}{p+1}\}$, there holds that $(p+1)(1-\gamma) > 2$, and thus by virtue of Hölder's inequality we can find $\alpha > 0$ such that

$$\frac{d}{dt} \|u\|_2^2 \geq c \|u\|_2^{2+\alpha}, \quad (5.47)$$

since also $p > 1$. Now (5.47) entails that $\|u\|_2^2$ blows up in finite time since $u_0(x) > 0$ and thus u exhibits a finite-time blow-up as well. $\square$

Remark 5.2.8 Theorem 5.5 can be interpreted in the biological context as follows: if the activator's initial concentration is large and its suppression by the inhibitor is rather small (since $0 < \gamma < 1$) then naturally the activator's growth becomes unlimited. Moreover, Theorem 5.5 similarly with Theorem 5.2 indicates that when $r = p + 1$ then there is a discrepancy between the behavior of the full system (5.1)–(5.4) and those of the non-local problem (5.12)–(5.14).

We close the current subsection with the following blow-up result

Theorem 5.6 *Let $0 < \gamma < 1$, $r \leq 1$, and $\frac{p-1}{r} > 1$. Assume, furthermore that either (1) $w(0) < \zeta(0)^{1-\gamma}$, or (2) $\frac{p-1}{r} \geq 2$ and $w(0) < 1$. Then finite-time blow-up occurs for problem (5.12)–(5.14), i.e. $T < +\infty$.*

Proof We first consider $r \leq 1 < \frac{p-1}{r}$ and $0 < \gamma < 1$. Since $r \leq 1$ then (5.24) for $\alpha = \frac{1}{r}$ yields

$$\frac{1}{r} \frac{d\zeta}{dt} = \frac{4}{r} \left(\frac{1}{r} - 1\right) \int_{\Omega} |\nabla u^{\frac{1}{r}}|^2 - \zeta + \frac{\zeta}{\zeta^{\gamma}} \quad \text{for } 0 < t < T, \quad (5.48)$$

and taking (5.41) into account we derive

$$\frac{1}{r} \frac{d\zeta}{dt} \geq -\zeta + \frac{\zeta^{2-\gamma}}{w} = \frac{\zeta}{w} (-w + \zeta^{1-\gamma}) \quad \text{for } 0 < t < T. \quad (5.49)$$

Furthermore, since $\frac{p-1}{r} > 1$ then (5.24) for $\alpha = \frac{1}{-p+1+r}$ reads

$$\alpha \frac{dw}{dt} = 4\alpha(\alpha - 1) \int_{\Omega} |\nabla u^{\frac{1}{2\alpha}}|^2 - w + \zeta^{1-\gamma}, \quad \text{for } 0 < t < T, \quad (5.50)$$

which, since $\alpha = \frac{1}{-p+1+r} < 0$, implies

$$\alpha \frac{dw}{dt} \geq -w + \zeta^{1-\gamma}, \quad \text{for } 0 < t < T,$$

or equivalently

$$\frac{1}{p-1-r} \frac{dw}{dt} \leq w - \zeta^{1-\gamma}, \quad \text{for } 0 < t < T. \quad (5.51)$$

The condition $0 < \gamma < 1$, entails that the curve

$$\Gamma_1 : w = \zeta^{1-\gamma}, \quad \zeta > 0, \quad (5.52)$$

is concave in the $w\zeta$ -plane, with its endpoint at the origin $(0, 0)$. Relations (5.49) and (5.51) imply that the region $\mathcal{R} = \{(\zeta, w) \mid w < \zeta^{1-\gamma}\}$ is invariant for the system (5.48), (6.10), i.e. if $(\zeta(0), w(0)) \in \mathcal{R}$ then $(\zeta(t), w(t)) \in \mathcal{R}$ for any $t > 0$. Furthermore, $\zeta = \zeta(t)$ and $w = w(t)$ are increasing and decreasing on $\mathcal{R}$, respectively.

In case $w(0) < \zeta(0)^{1-\gamma}$, then

$$\frac{dw}{dt} < 0, \quad \frac{d\zeta}{dt} > 0, \quad \text{for } 0 \leq t < T,$$

and thus,

$$\frac{1}{w} - \frac{1}{\zeta^{1-\gamma}} \geq \frac{1}{w(0)} - \frac{1}{\zeta(0)^{1-\gamma}} \equiv c_0 > 0, \quad \text{for } 0 \leq t < T.$$

Therefore by virtue of (5.49) we have

$$\frac{1}{r} \frac{d\zeta}{dt} \geq -\zeta + \frac{\zeta^{2-\gamma}}{w} = \zeta^{2-\gamma} \left(\frac{1}{w} - \frac{1}{\zeta^{1-\gamma}} \right) \geq c_0 \zeta^{2-\gamma}, \quad 0 \leq t < T. \quad (5.53)$$

Since $2 - \gamma > 1$ then (5.53) implies that $\zeta(t)$ blows up in finite time

$$t_1 \leq \hat{t}_1 \equiv \frac{(\zeta(0))^{\gamma-1}}{(1-\gamma)c_0 r},$$

and using the inequality

$$\zeta(t) = \int_{\Omega} u^r dx \leq \|u(\cdot, t)\|_{\infty}^r,$$

we conclude that $u(x, t)$ blows up in finite time $T \leq t_1$ as well.

We consider now the second case when $\frac{p-1}{r} \geq 2$ and thus $q = \frac{p-1-r}{r} \geq 1$. Then by virtue of Jensen's inequality

$$\int_{\Omega} u^r \cdot \left(\int_{\Omega} (u^{-r})^q \right)^{\frac{1}{q}} \geq \int_{\Omega} u^r \cdot \int_{\Omega} u^{-r} \geq 1,$$

and thus $\zeta^{\frac{1}{r}} \geq w^{-\frac{1}{p-1-r}}$ which entails

$$w \geq \zeta^{-\frac{p-1-r}{r}} = \zeta^{1-\frac{p-1}{r}}. \quad (5.54)$$

In addition, inequality $\frac{p-1}{r} \geq 2$ implies that the curve

$$\Gamma_2 : w = \zeta^{1-\frac{p-1}{r}}, \quad \zeta > 0,$$

is convex and approaches $+\infty$ and 0 as $\zeta \downarrow 0$ and $\zeta \uparrow +\infty$, respectively. The crossing of Γ_1 and Γ_2 is the point $(\zeta, w) = (1, 1)$, and therefore $w(0) < 1$ combined with (5.54) imply $w(0) < \zeta(0)^{1-\gamma}$. Consequently, the second case is reduced to the first one and again the occurrence of finite-time blow-up is established. $\square$

Remark 5.2.9 Since $\frac{p-1}{r} > 1 > \gamma$ is assumed, Theorem 5.6 is associated with the finite-time blow-up under anti-Turing condition and it is in agreement with the blow-up result [17, Theorem 2]. This is actually an indication, that under condition $\frac{p-1}{r} > 1 > \gamma$, the qualitative behavior of the non-local problem (5.12)–(5.14) resembles the one of the full system (5.1)–(5.4).

Remark 5.2.10 The biological interpretation of Theorem 5.6 is as follows: a large initial concentration for the activator combined with small *net cross-inhibition index* can lead to its unlimited growth.

Remark 5.2.11 The existence of the invariant region $\mathcal{R} = \{(\zeta, w) \mid w < \zeta^{1-\gamma}\}$ for the system (5.48), (5.50) entails that if the consumption of the activator cannot be suppressed initially then this can lead to its unlimited growth.

5.2.3 Diffusion Driven Blow-Up

In the current section we restrict ourselves to the radial case $\Omega = B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$ and we also consider $N \geq 3$. Then the solution of (5.12)–(5.14) is radial symmetric, that is $u(x, t) = u(\rho, t)$ for $0 \leq \rho = |x| < 1$.

We regard, as in [4], spiky initial data of the form

$$u_0(\rho) = \lambda \varphi_{\delta}(\rho), \quad (5.55)$$

with $0 < \lambda \ll 1$ and

$$\varphi_\delta(\rho) = \begin{cases} \rho^{-a}, & \delta \leq \rho \leq 1 \\ \delta^{-a} \left(1 + \frac{a}{2}\right) - \frac{a}{2} \delta^{-(a+2)} \rho^2, & 0 \leq \rho < \delta \end{cases}, \quad (5.56)$$

for $a = \frac{2}{p-1}$ and $0 < \delta < 1$.

It can be easily checked that $u_0(\rho)$ is decreasing, i.e. $u'_0(\rho) < 0$, and thus $\max_{\rho \in [0,1]} u_0(\rho) = u_0(0)$. Furthermore, due to the maximum principle we have that $u(\rho, t)$ is radial decreasing too, i.e. $u_\rho(\rho, t) < 0$. Now having specified the form of the considered initial data, we have the following

Theorem 5.7 *Let $N \geq 3$, $1 \leq r \leq p$, $p > \frac{N}{N-2}$ and $\frac{2}{N} < \frac{p-1}{r} < \gamma$. Then there is $\lambda_0 > 0$ with the following property: any $0 < \lambda \leq \lambda_0$ admits $0 < \delta_0 = \delta_0(\lambda) < 1$ such that any solution of problem (5.12)–(5.14) with initial data of the form (5.55) and $0 < \delta \leq \delta_0$ blows up in finite time, i.e. $T < +\infty$.*

We perceive that Theorem 5.7 for $r = 1$ is nothing but Proposition 3.3 in [13], which was proven using a series of auxiliary results and inspired by an approach introduced in [1, 4]. Therefore, in order to prove Theorem 5.7 we are following in short the arguments presented in [13], and we provide any modifications where are necessary.

The next lemma is elementary and so its proof is omitted.

Lemma 5.1 *The function ϕ_δ defined by (5.56) satisfies the following:*

(i) *There holds that*

$$\Delta \phi_\delta \geq -N a \phi_\delta^p, \quad (5.57)$$

in the weak sense for any $0 < \delta < 1$.

(ii) *If $m > 0$ and $N > ma$, we have*

$$\int_{\Omega} \phi_\delta^m = \frac{N}{N - ma} + O(\delta^{N-ma}), \quad \delta \downarrow 0. \quad (5.58)$$

Lemma 5.1 can be used to obtain some further useful estimates. Indeed, if we consider

$$\mu > 1 + r\gamma, \quad (5.59)$$

and set

$$\alpha_1 = \sup_{0 < \delta < 1} \frac{1}{\bar{\varphi}_\delta^\mu} \int_{\Omega} \phi_\delta^p, \quad \text{and} \quad \alpha_2 = \inf_{0 < \delta < 1} \frac{1}{\bar{\varphi}_\delta^\mu} \int_{\Omega} \phi_\delta^p, \quad (5.60)$$

then since $p > \frac{N}{N-2}$, relation (5.58) is applicable for $m = p$ and $m = 1$, and thus due to (5.59) we obtain

$$0 < \alpha_1, \alpha_2 < \infty. \quad (5.61)$$

Furthermore, there holds that

$$d \equiv \inf_{0 < \delta < 1} \left(\frac{1}{2\alpha_1} \right)^{\frac{r\gamma}{p}} \left(\frac{1}{2\varphi_\delta} \right)^{\frac{r\gamma}{p}\mu} > 0. \quad (5.62)$$

The following auxiliary result provides a key inequality satisfied by the initial data $u_0 = u_0(|x|)$ defined by (5.55). Indeed, we have

Lemma 5.2 *If $p > \frac{N}{N-2}$ and $\frac{p-1}{r} < \gamma$, there exists $\lambda_0 = \lambda_0(d) > 0$ such that for any $0 < \lambda \leq \lambda_0$ there holds*

$$\Delta u_0 + d\lambda^{-r\gamma} u_0^p \geq 2u_0^p. \quad (5.63)$$

Proof Note that inequality (5.63) is equivalent to

$$\Delta \varphi_\delta + d\lambda^{-r\gamma+p-1} \varphi_\delta^p \geq 2\lambda^{p-1} \varphi_\delta^p,$$

which is reduced to

$$d\lambda^{-r\gamma+p-1} \geq Na + 2\lambda^{p-1},$$

due to (5.57). Then the desired result follows since $\frac{p-1}{r} < \gamma$. $\square$

Henceforth we fix $0 < \lambda \leq \lambda_0 = \lambda_0(d)$ so that (5.63) is satisfied. Given $0 < \delta < 1$, let $T_\delta > 0$ be the maximal existence time of the solution to (5.12)–(5.14) with initial data of the form (5.55).

In order, to get rid off the linear dissipative term $-u$ we introduce the new variable $z = e^t u$, which then satisfies

$$z_t = \Delta z + K(t)z^p, \quad \text{in } Q \equiv \Omega \times (0, T_\delta), \quad (5.64)$$

$$\frac{\partial z}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, T_\delta), \quad (5.65)$$

$$z(x, 0) = u_0(|x|), \quad \text{in } \Omega, \quad (5.66)$$

where

$$K(t) = \frac{e^{(1+r\gamma-p)t}}{\left(\int_{\Omega} z^r \right)^\gamma}. \quad (5.67)$$

It is clear that u blows up in finite time if and only if z does so.

Due to (5.27), we have

$$0 < K(t) = \frac{e^{(1-p)t}}{\left(\int_{\Omega} u^r \right)^\gamma} \leq C < \infty, \quad (5.68)$$

thus (5.64) entails

$$\frac{d\bar{z}}{dt} = K(t) \int_{\Omega} z^p, \quad (5.69)$$

and we finally derive the following estimate

$$\bar{z}(t) \geq \bar{z}(0) = \int_{\Omega} u_0. \quad (5.70)$$

Another helpful estimate of z is given by the following lemma

Lemma 5.3 *There holds that*

$$\rho^N z(\rho, t) \leq \bar{z}(t) \quad \text{in } (0, 1) \times (0, T_{\delta}), \quad (5.71)$$

and

$$z_{\rho} \left(\frac{3}{4}, t \right) \leq -c, \quad 0 \leq t < T_{\delta}, \quad (5.72)$$

for any $0 < \delta < 1$.

Proof Set $w = \rho^{N-1} z_{\rho}$, then w satisfies

$$\begin{aligned} \mathcal{H}[w] &= 0, \quad \text{in } (0, 1) \times (0, T_{\delta}), \\ w(0, t) &= w(1, t) = 0, \quad \text{for } t \in (0, T_{\delta}), \\ w(\rho, 0) &< 0, \quad \text{for } 0 < \rho < 1, \end{aligned}$$

where

$$\mathcal{H}[w] \equiv w_t - w_{\rho\rho} + \frac{N-1}{\rho} w_{\rho} - pK(t)z^{p-1}w.$$

The maximum principle now implies $w \leq 0$, and hence $z_{\rho} \leq 0$ in $(0, 1) \times (0, T_{\delta})$. Then inequality (5.71) follows since

$$\begin{aligned} \rho^N z(\rho, t) &= z(\rho, t) \int_0^{\rho} N s^{N-1} ds \leq \int_0^{\rho} N z(s, t) s^{N-1} ds \\ &\leq \int_0^1 N z(s, t) s^{N-1} ds = \int_{\Omega} z = \bar{z}(t). \end{aligned}$$

Once $w \leq 0$ is proven, we have

$$\begin{aligned} w_t - w_{\rho\rho} + \frac{N-1}{\rho} w_{\rho} &= pK(t)z^{p-1}w \leq 0 \quad \text{in } (0, 1) \times (0, T_{\delta}), \\ w \left(\frac{1}{2}, t \right) &\leq 0, \quad w(1, t) \leq 0, \quad \text{for } t \in (0, T_{\delta}), \\ w(\rho, 0) &= \rho^{N-1} u'_0(\rho) \leq -c, \quad \text{for } \frac{1}{2} < \rho < 1, \end{aligned}$$

which entails $w \leq -c$ in $(\frac{1}{2}, 1) \times (0, T_\delta)$, and finally (5.72) holds. $\square$

The following lemma will ultimately provide another effective estimate of z

Lemma 5.4 *Given $\varepsilon > 0$ and $1 < q < p$ then ψ defined as*

$$\psi := \rho^{N-1} z_\rho + \varepsilon \cdot \frac{\rho^N z^q}{\bar{z}^{\gamma+1}}, \quad (5.73)$$

satisfies

$$\mathcal{H}[\psi] \leq -\frac{2q\varepsilon}{\bar{z}^{\gamma+1}} z^{q-1} \psi + \frac{\varepsilon \rho^N z^q}{\bar{z}^{2(\gamma+1)}} \left\{ 2q\varepsilon z^{q-1} - (\gamma+1)\bar{z}^{\gamma-r\gamma} \int_{\Omega} z^p - (p-q)z^{p-1}\bar{z}^{\gamma+1-r\gamma} \right\} \quad (5.74)$$

in $(0, 1) \times (0, T_\delta)$.

The proof of Lemma 5.4 follows the same steps as the proof of inequality (28) in [13], which holds for $r = 1$, and thus it is omitted.

Observe that when $p > \frac{N}{N-2}$ there is $1 < q < p$ such that $N > \frac{2p}{q-1}$ and thus the following quantities

$$A_1 \equiv \sup_{0 < \delta < 1} \frac{1}{\bar{u}_0^\mu} \int_{\Omega} u_0^p = \lambda^{\mu-p} \alpha_1, \quad A_2 \equiv \inf_{0 < \delta < 1} \frac{1}{\bar{u}_0^\mu} \int_{\Omega} u_0^p = \lambda^{\mu-p} \alpha_2, \quad (5.75)$$

are bounded due to (5.61). The following result, which is a modification of Lemma 3.3 in [13] for $r = 1$, provides a key estimate of the L^p -norm of z in terms of A_1 and A_2 and, since it is a core result for the proof of Theorem 5.7, we will provide a short proof for it.

Proposition 5.3 *There exist $0 < \delta_0 < 1$ and $0 < t_0 \leq 1$ independent of any $0 < \delta \leq \delta_0$, such that the following estimate is satisfied*

$$\frac{1}{2} A_2 \bar{z}^\mu \leq \int_{\Omega} z^p dx \leq 2 A_1 \bar{z}^\mu, \quad (5.76)$$

for any $0 < t < \min\{t_0, T_\delta\}$.

The proof of the above proposition requires some auxiliary results proven below.

We first take $0 < t_0(\delta) < T_\delta$ to be the maximal time for which inequality (5.76) holds true in $0 < t < t_0(\delta)$, then we have

$$\frac{1}{2} A_2 \bar{z}^\mu \leq \int_{\Omega} z^p \leq 2 A_1 \bar{z}^\mu, \quad \text{for } 0 < t < t_0(\delta). \quad (5.77)$$

We only consider the case $t_0(\delta) \leq 1$, since otherwise there is nothing to prove.

Now the first auxiliary result reads as follows

Lemma 5.5 *There exists $0 < t_1 < 1$ such that*

$$\bar{z}(t) \leq 2\bar{u}_0, \quad 0 < t < \min\{t_1, t_0(\delta)\}, \quad (5.78)$$

for any $0 < \delta < 1$.

Proof Since $r \geq 1$ and $t_0(\delta) \leq 1$, it follows that

$$\frac{d\bar{z}}{dt} \leq 2A_1 e^{1+r\gamma-p} \bar{z}^{\mu-r\gamma}, \quad 0 < t < t_0(\delta),$$

taking also into account relations (5.67) and (5.69).

Setting $C_1 = 2A_1 e^{1+r\gamma-p}$, we obtain

$$\bar{z}(t) \leq \left[\bar{u}_0^{1+r\gamma-\mu} - C_1(\mu - r\gamma - 1)t \right]^{-\frac{1}{\mu-r\gamma-1}},$$

by (5.59). Therefore, (5.78) holds for any $0 < t < \min\{t_1, t_0(\delta)\}$ where t_1 is estimated as

$$t_1 \leq \min \left\{ \frac{1 - 2^{1+r\gamma-\mu}}{C_1(\mu - r\gamma - 1)} \bar{u}_0^{1+r\gamma-\mu}, 1 \right\},$$

and it is independent of any $0 < \delta < 1$. □

Another fruitful estimate is provided by the next auxiliary result

Lemma 5.6 *There exist $0 < \delta_0 < 1$ and $0 < \rho_0 < \frac{3}{4}$ such that for any $0 < \delta \leq \delta_0$ the following estimate holds*

$$\frac{1}{|\Omega|} \int_{B(\rho_0, 0)} z^p \leq \frac{A_2}{8} \bar{z}^\mu, \quad \text{for } 0 < t < \min\{t_1, t_0(\delta)\}. \quad (5.79)$$

Proof First observe that

$$\bar{u}_0 \leq \bar{z}(t) \leq 2\bar{u}_0, \quad \text{for } 0 < t < \min\{t_1, t_0(\delta)\}, \quad (5.80)$$

follows from (5.70) and (5.78). Then, note that the growth of $\int_{\Omega} z^p$ is controlled by (5.76) for $0 < \min\{t_1, t_0(\delta)\}$. Since $p > q$ then Young's inequality guarantees that the second term of the right-hand side in (5.74) is negative for $0 < t < \min\{t_1, t_0(\delta)\}$, uniformly in $0 < \delta < 1$, provided that $0 < \varepsilon \leq \varepsilon_0$ for some $0 < \varepsilon_0 \ll 1$. Thus

$$\mathcal{H}[\psi] \leq -\frac{2q\varepsilon z^{q-1}}{\bar{z}^{\gamma+1}} \psi \quad \text{in } (0, 1) \times (0, \min\{t_1, t_0(\delta)\}). \quad (5.81)$$

Due to (5.71) and (5.80), we also have

$$\begin{aligned}\psi &= \rho^{N-1} z_\rho + \varepsilon \cdot \frac{\rho^N z^q}{\bar{z}^{\gamma+1}} \leq \rho^{N-1} z_\rho + \varepsilon \cdot \rho^{N(1-q)} \bar{z}^{q-\gamma-1} \\ &\leq \rho^{N-1} z_\rho + C \cdot \varepsilon \rho^{N(1-q)} \quad \text{in } (0, 1) \times (0, \min\{t_1, t_0(\delta)\}),\end{aligned}$$

which, for $0 < \varepsilon \leq \varepsilon_0$, entails

$$\psi\left(\frac{3}{4}, t\right) < 0, \quad 0 < t < \min\{t_1, t_0(\delta)\}, \quad (5.82)$$

by (5.72) and provided that $0 < \varepsilon_0 \ll 1$.

Additionally (5.73) for $t = 0$ gives

$$\psi(\rho, 0) = \rho^{N-1} \left(\lambda \varphi'_\delta(\rho) + \varepsilon \lambda^{q-\gamma-1} \rho \cdot \frac{\varphi_\delta^q}{\bar{\varphi}_\delta^{\gamma+1}} \right). \quad (5.83)$$

Now if $0 \leq \rho < \delta$ and ε are chosen small enough and independent of $0 < \delta < \delta_0$, then the right-hand side of (5.83) is estimated as follows:

$$\rho^N \lambda \left(-a \delta^{-a-2} + \varepsilon \lambda^{q-\gamma-2} \cdot \frac{\varphi_\delta^q}{\bar{\varphi}_\delta^{\gamma+1}} \right) \lesssim \rho^N \lambda \left(-a \delta^{-a-2} + \varepsilon \lambda^{q-\gamma-2} \cdot \delta^{-aq} \right) \lesssim 0,$$

since also

$$\frac{\varphi_\delta^q}{\bar{\varphi}_\delta^{\gamma+1}} \lesssim \delta^{-aq}, \quad \delta \downarrow 0, \quad \text{uniformly in } 0 \leq \rho < \delta,$$

holds by (5.56) and (5.58) for $m = 1$, taking also into account that $a+2 = ap > ak$.

On the other hand, if $\delta \leq \rho \leq 1$ then we obtain

$$\psi(\rho, 0) = \rho^N \lambda \left(-a \rho^{-a-1} + \varepsilon \lambda^{q-\gamma-1} \frac{\rho^{-aq+1}}{\bar{\varphi}_\rho^{\gamma+1}} \right), \quad (5.84)$$

by using again (5.58) for $m = 1$. Since $a+2 = ap > aq$ implies $-a-1 < -aq+1$, we derive

$$\psi(\rho, 0) < 0, \quad \delta \leq \rho \leq \frac{3}{4},$$

for any $0 < \delta \leq \delta_0$ and $0 < \varepsilon \leq \varepsilon_0$, provided ε_0 is chosen sufficiently small.

Consequently we deduce

$$\psi(\rho, 0) < 0, \quad 0 \leq \rho \leq \frac{3}{4}, \quad (5.85)$$

for any $0 < \delta \leq \delta_0$ and $0 < \varepsilon \leq \varepsilon_0$, provided $0 < \varepsilon_0 \ll 1$.

Combining (5.81), (5.82) and (5.85) we end up with

$$\psi = \rho^{N-1} z_\rho + \varepsilon \cdot \frac{\rho^N z^q}{\bar{z}^{\gamma+1}} \leq 0 \quad \text{in} \quad \left(0, \frac{3}{4}\right) \times (0, \min\{t_1, t_0(\delta)\}),$$

which implies

$$z(\rho, t) \leq \left(\frac{\varepsilon}{2}(q-1)\right)^{-\frac{1}{q-1}} \cdot \rho^{-\frac{2}{q-1}} \cdot \bar{z}^{\frac{\gamma-1}{q-1}}(t) \quad \text{in} \quad \left(0, \frac{3}{4}\right) \times (0, \min\{t_1, t_0(\delta)\}). \quad (5.86)$$

Since $-\frac{2}{q-1} \cdot p + N - 1 > -1$ due to $N > \frac{2p}{q-1}$, we finally obtain (5.79) for some $0 < \rho_0 < \frac{3}{4}$. $\square$

Remark 5.2.12 It is worth noting that relation (5.86) implies that if $z(\rho, t)$ blows up then this can only happen in the origin $\rho = 0$; that is, only a single-point blow-up is possible. In particular if we define

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \rightarrow x_0, \exists t_k \uparrow T_\delta, \lim_{k \rightarrow \infty} z(x_k, t_k) = +\infty\},$$

to be the blow-up set of z then $\mathcal{S} = \{0\}$ in the case z blows up in finite time.

Next we prove the key estimate (5.76) using essentially Lemmas 5.5 and 5.6.

Proof of Proposition 5.3 By virtue of (5.59) and since $\frac{p-1}{r} < \delta$, there holds that $\ell = \frac{\mu}{p} > 1$. We can easily see that $\theta = \frac{z}{\bar{z}^\ell}$ satisfies

$$\begin{aligned} \theta_t &= \Delta \theta + e^{(r\gamma+1-p)t} \left[\frac{z^p}{\bar{z}^\ell (f_\Omega z^r)^\gamma} - \frac{\ell z f_\Omega z^p}{\bar{z}^{\ell+1} (f_\Omega z^r)^\gamma} \right], \\ &\quad \text{in } Q_0 := \Omega \times (0, \min\{t_0, T_\delta\}), \end{aligned}$$

$$\frac{\partial \theta}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, \min\{t_0, T_\delta\}),$$

$$\theta(x, 0) = \frac{z(x, 0)}{\bar{z}_0^\ell} \quad \text{in } \Omega.$$

Now due to (5.27), (5.70), (5.71), (5.77), and (5.78), there holds that

$$\left\| \theta, \frac{z^p}{\bar{z}^\ell (f_\Omega z^r)^\gamma}, \frac{\ell z f_\Omega z^p}{\bar{z}^{\ell+1} (f_\Omega z^r)^\gamma} \right\|_{L^\infty((\Omega \setminus B(0, \rho_0)) \times \min\{t_1, t_0(\delta)\})} \leq C,$$

uniformly in $0 < \delta \leq \delta_0$.

Therefore, by the standard parabolic regularity, see DeGiorgi–Nash–Moser estimates in [16, pp. 144–145], there is $0 < t_2 \leq t_1$ independent of $0 < \delta \leq \delta_0$ such

that

$$\sup_{0 < t < \min\{t_2, t_0(\delta)\}} \|\theta^p(\cdot, t) - \theta^p(\cdot, 0)\|_{L^1(\Omega \setminus B(0, \rho_0))} \leq \frac{A_2}{8} |\Omega|,$$

which implies

$$\left| \frac{1}{|\Omega|} \int_{\Omega \setminus B(0, \rho_0)} \frac{z^p}{\bar{z}^\mu} - \frac{1}{|\Omega|} \int_{\Omega \setminus B(0, \rho_0)} \frac{z_0^p}{\bar{z}_0^\mu} \right| \leq \frac{A_2}{8}, \quad 0 < t < \min\{t_2, t_0(\delta)\}, \quad (5.87)$$

for any $0 < \delta \leq \delta_0$. Inequalities (5.79) and (5.87) entail

$$\left| \oint_{\Omega} \frac{z^p}{\bar{z}^\mu} - \oint_{\Omega} \frac{z_0^p}{\bar{z}_0^\mu} \right| \leq \frac{3A_2}{8}, \quad \text{for } 0 < t < \min\{t_2, t_0(\delta)\} \text{ and } 0 < \delta \leq \delta_0,$$

and hence

$$\frac{5A_2}{8} \leq \oint_{\Omega} \frac{z^p}{\bar{z}^\mu} \leq \frac{11A_1}{8}, \quad 0 < t < \min\{t_2, t_0(\delta)\}, \quad 0 < \delta \leq \delta_0, \quad (5.88)$$

taking into account that

$$A_2 \leq \oint_{\Omega} \frac{z_0^p}{\bar{z}_0^\mu} \leq A_1.$$

Therefore, if we take $t_0(\delta) \leq t_2$ then it follows that

$$\frac{1}{2} A_2 \bar{z}^\mu < \frac{5}{8} A_2 \bar{z}^\mu \leq \oint_{\Omega} z^p \leq \frac{11}{8} A_1 \bar{z}^\mu < 2A_1 \bar{z}^\mu, \quad 0 < t < t_0(\delta),$$

and by a continuity argument we deduce that

$$\frac{1}{2} A_2 \bar{z}^\mu \leq \oint_{\Omega} z^p \leq 2A_1 \bar{z}^\mu, \quad 0 < t < t_0(\delta) + \eta,$$

for some $\eta > 0$, which contradicts the definition of $t_0(\delta)$.

Consequently, we obtain $t_2 < t_0(\delta)$ for any $0 < \delta \leq \delta_0$, and the proof is complete for $t_0 \leq t_2$. $\square$

Now we have all the ingredients to proceed to the proof of the main result of this section.

Proof of Theorem 5.7 Since $t_0 \leq t_1$ in (5.78), we have

$$\begin{aligned} K(t) &\geq \frac{1}{(f_\Omega z^r)^\gamma} \geq \frac{1}{(f_\Omega z^p)^{\frac{r\gamma}{p}}} \geq \left(\frac{1}{2A_1 \bar{z}^\mu} \right)^{\frac{r\gamma}{p}} = \left(\frac{1}{2A_1} \right)^{\frac{r\gamma}{p}} \cdot \left(\frac{1}{\bar{z}} \right)^{\frac{r\gamma}{p}\mu} \\ &= \left(\frac{1}{2\alpha_1} \right)^{\frac{r\gamma}{p}} \cdot \left(\frac{1}{2\bar{\varphi}_\delta} \right)^{\frac{r\gamma}{p}\mu} \lambda^{-r\gamma} \geq d\lambda^{-r\gamma} \equiv D, \quad 0 < t < \min\{t_0, T_\delta\}, \end{aligned} \quad (5.89)$$

by virtue of (5.62) and (5.75). Since $0 < \lambda \leq \lambda_0(d)$, then inequality (5.63) applies to derive

$$\Delta u_0 + Du_0^p \geq 2u_0^p, \quad (5.90)$$

for any $0 < \delta \leq \delta_0$.

By using (5.89) and (5.90) the comparison principle yields that the solution z of (5.64)–(5.66) satisfies

$$z \geq \tilde{z} \quad \text{in } Q_0 \equiv \Omega \times (0, \min\{t_0, T_\delta\}), \quad (5.91)$$

where $\tilde{z} = \tilde{z}(x, t)$ solves the following

$$\tilde{z}_t = \Delta \tilde{z} + D\tilde{z}^p, \quad \text{in } Q_0, \quad (5.92)$$

$$\frac{\partial \tilde{z}}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, \min\{t_0, T_\delta\}), \quad (5.93)$$

$$\tilde{z}(|x|, t) = u_0(|x|) \quad \text{in } \Omega. \quad (5.94)$$

Let us now introduce

$$h(x, t) := \tilde{z}_t(x, t) - \tilde{z}^p(x, t),$$

then due to (5.90) and (5.92) h satisfies

$$h_t = \Delta h + p(p-1)\tilde{z}^{p-2}|\nabla \tilde{z}|^2 + Dp\tilde{z}^{p-1}h \geq \Delta h + Dp\tilde{z}^{p-1}h \quad \text{in } Q_0,$$

and

$$h(x, 0) = \Delta \tilde{z}(x, 0) + D\tilde{z}^p(x, 0) - \tilde{z}^p(x, 0) = \Delta u_0 + (D-1)u_0^p \geq u_0^p > 0, \quad \text{in } \Omega,$$

with boundary condition

$$\frac{\partial h}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \min\{t_0, T_\delta\}).$$

Then the maximum principle entails that $h > 0$ in Q_0 , that is,

$$\tilde{z}_t > \tilde{z}^p \quad \text{in } Q_0. \quad (5.95)$$

Inequality (5.95) implies

$$\tilde{z}(0, t) \geq \left(\frac{1}{z_0^{p-1}(0)} - (p-1)t \right)^{-\frac{1}{p-1}} = \left\{ \left(\frac{\delta^a}{\lambda(1 + \frac{a}{2})} \right)^{p-1} - (p-1)t \right\}^{-\frac{1}{p-1}},$$

for $0 < t < \min\{t_0, T_\delta\}$, and therefore,

$$\min\{t_0, T_\delta\} < \frac{1}{p-1} \cdot \left(\frac{\delta^a}{\lambda(1 + \frac{a}{2})} \right)^{p-1}. \quad (5.96)$$

For $0 < \delta \ll 1$, the right-hand side on (5.96) is less than t_0 , and then $T_\delta < +\infty$ follows. Furthermore, by (5.96) $T_\delta \rightarrow 0$ as $\delta \rightarrow 0$ and the proof is complete. $\square$

Remark 5.2.13 The blowing up solution u obtained in Theorem 5.7 exhibits a single-point blow-up at the origin $\rho = 0$. Recalling that $z = e^t u$ we obtain the occurrence of single-point blow-up for u in view of Remark 5.2.12.

An alternative way to prove single-point blow-up is by virtue of the following estimate

$$\oint_{\Omega} z^p dx = N\omega_N \int_0^1 \rho^{N-1} z^p d\rho \leq C, \quad \text{for } 0 < t \leq T_\delta, \quad (5.97)$$

which holds due to (5.76) and (5.78), taking $0 < \delta \ll 1$ small enough such that $T_\delta \leq t_0$. Using the fact that $z = z(\rho, t)$ is radially decreasing, then (5.97) implies that $\mathcal{S} = \{0\}$.

5.2.4 Blow-Up Rate and Blow-Up Pattern

One of our purposes in the current section is to determine the blow-up rate of the diffusion-driven blowing up solution provided by Theorem 5.7. We also intend to identify its blow-up pattern (profile) and thus reveal the formed patterns anticipated under this DDI event.

Theorem 5.8 *Let $N \geq 3$, $\max\{r, \frac{N}{N-2}\} < p < \frac{N+2}{N-2}$ and $\frac{2}{N} < \frac{p-1}{r} < \gamma$. Then the blow-up rate of the diffusion-induced blowing-up solution of Theorem 5.7 is determined as follows*

$$\|u(\cdot, t)\|_\infty \approx (T_{\max} - t)^{-\frac{1}{p-1}}, \quad t \uparrow T_{\max}, \quad (5.98)$$

where $T_{\max}$ stands for the blow-up time.

Proof We first note that

$$0 < K(t) = \frac{e^{(1+r\gamma-p)t}}{\left(\int_{\Omega} z^r\right)^{\gamma}} \leq C < \infty, \quad (5.99)$$

by virtue of (5.97) and in view of Hölder's inequality since $p > r$.

Consider now Φ satisfying

$$\begin{aligned} \Phi_t &= \Delta \Phi + C \Phi^p, \quad \text{in } \Omega \times (0, T_{\max}), \\ \frac{\partial \Phi}{\partial \nu} &= 0, \quad \text{on } \partial \Omega \times (0, T_{\max}), \\ \Phi(x, 0) &= z_0(x), \quad \text{in } \Omega, \end{aligned}$$

then via comparison $z \leq \Phi$ in $\Omega \times (0, T_{\max})$.

Yet it is known, see [24, Theorem 44.6], that

$$|\Phi(x, t)| \leq C_{\eta} |x|^{-\frac{2}{p-1}-\eta} \quad \text{for } \eta > 0,$$

when $x \in \Omega$, $0 < t < T_{\max}$, and thus

$$|z(x, t)| \leq C_{\eta} |x|^{-\frac{2}{p-1}-\eta} \quad \text{for } (x, t) \in \Omega \times (0, T_{\max}), \quad (5.100)$$

which by virtue of (5.99), (5.100) and using also standard parabolic estimates entails that

$$z \in \mathcal{BUC}^{\sigma} \left(\{\rho_0 < |x| < 1 - \rho_0\} \times \left(\frac{T_{\max}}{2}, T_{\max} \right) \right), \quad (5.101)$$

for some $\sigma \in (0, 1)$ and each $0 < \rho_0 < 1$, where $\mathcal{BUC}^{\sigma}(M)$ denotes the Banach space of all bounded and uniform σ -Hölder continuous functions $h : M \subset \mathbb{R}^N \rightarrow \mathbb{R}$; see also [24].

Consequently (5.101) implies that $\lim_{t \rightarrow T_{\max}} z(x, t)$ exists and it is finite for all $x \in B_1(0) \setminus \{0\}$.

Recalling that $\frac{2p}{p-1} < N$ (or equivalently $p > \frac{N}{N-2}$, $N > 2$) then by using (5.99), (5.100) and in view of the dominated convergence theorem we derive

$$\lim_{t \rightarrow T_{\max}} K(t) = \omega \in (0, +\infty). \quad (5.102)$$

Applying now Theorem 44.3(ii) in [24], taking also into account (5.102), we can find a constant $C_u > 0$ such that

$$\|z(\cdot, t)\|_{\infty} \leq C_u (T_{\max} - t)^{-\frac{1}{(p-1)}} \quad \text{in } (0, T_{\max}). \quad (5.103)$$

On the other hand, setting $N(t) := \|z(\cdot, t)\|_\infty = z(0, t)$ then $N(t)$ is differentiable for almost every $t \in (0, T_{\max})$, in view of [1], and it also satisfies

$$\frac{dN}{dt} \leq K(t)N^p(t).$$

Now since $K(t) \in C([0, T_{\max}))$ is bounded in any time interval $[0, t]$, $t < T_{\max}$, then upon integration we obtain

$$\|z(\cdot, t)\|_\infty \geq C_l (T_\delta - t)^{-\frac{1}{(p-1)}} \quad \text{in } (0, T_{\max}), \quad (5.104)$$

for some positive constant C_l .

Since $z(x, t) = e^t u(x, t)$ then by virtue of (5.103) and (5.104) we obtain

$$\tilde{C}_l (T_{\max} - t)^{-\frac{1}{(p-1)}} \leq \|u(\cdot, t)\|_\infty \leq \tilde{C}_u (T_{\max} - t)^{-\frac{1}{(p-1)}} \quad \text{for } t \in (0, T_{\max}),$$

where now $\tilde{C}_l, \tilde{C}_u$ depend on $T_{\max}$, which actually leads to (5.98). $\square$

Remark 5.2.14 Condition (5.98) implies that the diffusion-induced blow-up provided by Theorem 5.7 is of type I, i.e. the blow-up mechanism is controlled by the ODE part of (5.12).

In contrast, for the finite-time blow-up furnished by Proposition 5.2 and Theorems 5.5 and 5.6 we cannot derive a blow-up rate as in (5.98) since the blow-up of some L^ℓ -norm, $\ell \geq 1$, in each of these cases entails that

$$K(t) = \frac{e^{(1-p)t}}{\left(\int_\Omega u^r\right)^\gamma} \rightarrow 0 \quad \text{as } t \rightarrow T_{\max},$$

and thus the approach of Theorem 5.8 fails. This might be an indication that in the preceding cases finite-time blow-up is rather of type II.

Remark 5.2.15 First observe that (5.100) provides a rough form of the blow-up pattern for z and thus for those of u as well. Nonetheless, owing to (5.99) then the non-local problem (5.64)–(5.66) can be treated as the corresponding local one for which the following more accurate asymptotic blow-up profile, [21], is available

$$\lim_{t \rightarrow T_{\max}} z(|x|, t) \sim C \left[\frac{|\log |x||}{|x|^2} \right] \quad \text{for } |x| \ll 1.$$

Therefore using again that $z = e^t u$ we derive a similar asymptotic blow-up profile for the driven-induced blowing up solution u . This actually reveals the form of the developed patterns which are induced as a result of the DDI.

References

1. Friedman, A., McLeod, J.B.: Blow-up of positive solutions of semilinear heat equations. *Indiana Univ. Math. J.* **34**, 425–447 (1985)
2. Gierer, A., Meinhardt, H.: A theory of biological pattern formation. *Kybernetik (Berlin)* **12**, 30–39 (1972)
3. Hale, J.K., Sakamoto, K.: Shadow systems and attractors in reaction-diffusion equations. *Appl. Anal.* **32**, 287–303 (1989)
4. Hu, B., Yin, H.-M.: Semilinear parabolic equations with prescribed energy. *Rend. Circ. Mat. Palermo* **44**, 479–505 (1995)
5. Iron, D., Ward, M.: A metastable spike solution for a nonlocal reaction-diffusion model. *SIAM J. Appl. Math.* **60**(3), 778–802 (2000)
6. Jiang, H.: Global existence of solutions of an activator-inhibitor system. *Discret. Contin. Dyn. Syst.* **14**, 737–751 (2006)
7. Kavallaris, N.I., Suzuki, T.: On the dynamics of a non-local parabolic equation arising from the Gierer–Meinhardt system. *Nonlinearity* **30**(5), 1734–1761 (2017)
8. Karali, G., Suzuki, T., Yamada, Y.: Global-in-time behavior of the solution to a Gierer–Meinhardt system disc. *Contin. Dyn. Syst.* **33**, 2885–2900 (2013)
9. Karch, G., Suzuki, K., Zienkiewicz, J.: Finite-time blowup of solutions to some activator-inhibitor systems. *Discret. Contin. Dyn. Syst.* **36**(9), 4997–5010 (2016)
10. Kato, T.: *Perturbation Theory for Linear Operators*. Springer, Berlin (1966)
11. Keener, J.: Activators and inhibitors in pattern formation. *Stud. Appl. Math.* **59**, 1–23 (1978)
12. Levine, H.A.: Some nonexistence and instability theorems for formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.* **51**, 371–386 (1973)
13. Li, F., Ni, W.-M.: On the global existence and finite time blow-up of shadow systems. *J. Differ. Equ.* **247**, 1762–1776 (2009)
14. Li, F., Yip, N.K.: Finite time blow-up of parabolic systems with nonlocal terms. *Indiana Univ. Math. J.* **63**(3), 783–829 (2014)
15. Li, F., Peng, R., Song, X.: Global existence and finite time blow-up of solutions of a Gierer–Meinhardt system. *J. Diff. Equ.* **262**(1), 559–589 (2017)
16. Li, M., Chen, S., Qin, Y.: Boundedness and blow up for the general activator-inhibitor model. *Acta Math. Appl. Sin.* **11**, 59–68 (1995)
17. Lieberman, G.M.: *Second Order Parabolic Differential Equations*. World Scientific Publishing Co., Inc., River Edge (1996)
18. Marciniak-Czochra, A., Mikelić, A.: Shadow limit using renormalization group method and center manifold method. *Vietnam J. Math.* **45**, 103–125 (2017)
19. Marciniak-Czochra, A., Härtling, S., Karch, G., Suzuki, K.: Dynamical spike solutions in a nonlocal model of pattern formation. [arXiv:1307.6236](https://arxiv.org/abs/1307.6236) (2013)
20. Masuda, K., Takahashi, K.: Reaction-diffusion systems in the Gierer–Meinhardt theory of biological pattern formation. *Jpn. J. Appl. Math.* **4**, 47–58 (1987)
21. Merle, F., Zaag, H.: Refined uniform estimates at blow-up and applications for nonlinear heat equations. *Geom. Funct. Anal.* **8**(6), 1043–1085 (1998)
22. Ni, W.-M.: *The Mathematics of Diffusion*. CBMS-NSF Series. SIAM (2011)
23. Ni, W.-M., Suzuki, K., Takagi, I.: The dynamics of a kinetic activator-inhibitor system. *J. Differ. Equ.* **229**, 426–465 (2006)
24. Quittner, P., Souplet, Ph: *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*. Birkhäuser, Basel (2007)
25. Rothe, F.: *Global Solution of Reaction-Diffusion Systems*. Lecture Notes in Mathematics, vol. 1072. Springer, Berlin (1984)
26. Sattinger, D.: On global solution of nonlinear hyperbolic equations. *Arch. Rational Mech. Anal.* **30**, 148–172 (1968)
27. Suzuki, T., Senba, T.: *Applied Analysis, Mathematical Methods in Natural Science*. Imperial College Press, London (2012)

28. Turing, A.M.: The chemical basis of morphogenesis. *Philos. Trans. R. Soc. B* **237**, 37–72 (1952)
29. Wei, J.: Existence and stability of spikes for the Gierer–Meinhardt system. *Handbook of Differential Equations: Stationary Partial Differential Equations*, pp. 487–585. Elsevier/North-Holland, Amsterdam (2008)
30. Zou, H.: Finite-time blow-up and blow-up rates for the Gierer–Meinhardt system. *Appl. Anal.* **94**(10), 2110–2132 (2015)

Chapter 6

A Non-local Model Illustrating Replicator Dynamics

Abstract The current chapter discusses a utilization from the field of evolutionary game dynamics and in particular from its subarea called replicator dynamics. Considering an infinite continuous strategy space, which for example might be considered as the sampling space of a continuously varying trait of a biological population, as well as payoff functions of Gaussian type we build up a non-local degenerate parabolic problem. As it is appropriate for degenerate problems, a regularized approximation is constructed and then some a priori estimates for its solutions are obtained. Using the derived estimates, we prove that solutions converge to the trivial solution if the initial population is small, whereas they undergo a blow-up in finite time if the initial population is large. In particular, in the latter case, it is shown that the blow-up set coincides with the whole strategy space, i.e. the finite-time blow-up is global.

6.1 Derivation of the Non-local Model

Evolutionary game dynamics is a major part of modern game theory. It was appropriately fostered by evolutionary biologists such as W. D. Hamilton and J. Maynard Smith (see [11] for a collection of survey papers and [42] for a popularized account) and it actually brought a conceptual revolution to the game theory analogous with the one of population dynamics in biology. The resulting population-based approach has also found many applications in non-biological fields like economics or learning theory and introduces a significant enrichment of *classical* game theory which focuses on the concept of a rational individual.

The main subject of evolutionary game dynamics is to explain how a population of players update their strategies in the course of a game according to the strategies' success. This contrasts with classical noncooperative game theory that analyzes how rational players will behave through static solution concepts such as the Nash Equilibrium (NE) (i.e., a strategy choice for each player whereby no individual has a unilateral incentive to change his or her behavior).

As Hofbauer and Sigmund [20] pointed out, strategies with high pay-off will spread within the population through learning, imitation or inheriting processes or even by infection. The pay-offs depend on the actions of the co-players, i.e. the

frequencies in which the various strategies appear, and since these frequencies change according to the pay-offs, a feedback loop appears. The dynamics of this feedback loop will determine the long time progress of the game and its investigation is exactly the course of evolutionary game theory.

According to the extensive survey paper [20] there is a variety of different dynamics in evolutionary game theory: replicator dynamics, imitation dynamics, best response dynamics, Brown-von Neumann-Nash dynamics e.t.c.. However, the dynamics most widely used and studied in the literature on evolutionary game theory are the replicator dynamics which were introduced in [49] and baptised in [40]. Such kind of dynamics illustrates the idea that in a dynamic process of evolution a strategy should increase in frequency if it is a successful strategy in the sense that individuals playing this strategy obtain a higher than average payoff.

Let us consider a game with m discrete pure strategies, forming the strategy space $S = \{1, 2, \dots, m\}$, and corresponding frequencies $p_i(t)$, $i = 1, 2, \dots, m$, for any $t \geq 0$. (Alternatively S could be considered as the set of different states (genetic programmes) of a biological population). The frequency (probability) vector $p(t) = (p_1(t), p_2(t), \dots, p_m(t))^T$ belongs to the invariant simplex

$$S(m) = \left\{ y = (y_1, y_2, \dots, y_m)^T \in \mathbb{R}^m : y_i \geq 0, i = 1, 2, \dots, m \text{ and } \sum_{i=1}^m y_i = 1 \right\}.$$

The game is actually determined by the pay-off matrix $A = (a_{ij})$, which is a real $m \times m$ symmetric matrix. Pay-off means expected gain, and if an individual plays strategy i against another individual following strategy j , then the pay-off to i is defined to be a_{ij} while the pay-off to j is a_{ji} . For symmetric games matrix A is considered to be symmetric. (In the case of a biological population pay-off represents fitness, or reproductive success.)

Then the expected pay-off for an individual playing strategy i can be expressed as

$$(A \cdot p(t))_i = \sum_{j=1}^m a_{ij} p_j(t),$$

whereas the average pay-off over the whole population is given by

$$(p(t)^T \cdot A \cdot p(t)) = \sum_{i=1}^m \sum_{j=1}^m a_{ij} p_i(t) p_j(t).$$

Particularly if our game is symmetric with infinitely many players (or if the biological population is infinitely big and its generations blend continuously to each other) then we obtain that $p_i(t)$ evolve as differentiable functions. Note that the rate of increase of the per capita rate of growth $\dot{p}_i/p_i$ of strategy (type) i is a measure of its evolutionary success; here $\dot{p}_i$ stands for the time derivative of p_i . A reasonable assumption, which is also in agreement with the basic tenet of Darwinism, is that the per capita rate of

growth (i.e. the logarithmic derivative) $\dot{p}_i / p_i$ is given by the difference between the pay-off for strategy (type) i and the average pay-off. This yields the *the replicator dynamical system*,

$$\frac{dp_i}{dt} = \left(\sum_{j=1}^m a_{ij} p_j(t) - \sum_{i=1}^m \sum_{j=1}^m a_{ij} p_i(t) p_j(t) \right) p_i(t), \quad i = 1, 2, \dots, m, \quad t > 0. \quad (6.1)$$

The dynamical system (6.1) actually describes the mechanism that individuals tend to switch to strategies that are doing well, or that individuals bear offspring who tend to use the same strategies as their parents, and the fitter the individual, the more numerous his offspring.

Most of the work on replicator dynamics has focused on games that have a finite strategy space, thus leading to a dynamical system for the frequencies of the population which is finite dimensional. However, interesting applications arise either in biology or economics where the strategy space is not finite or, even, not discrete, see [8, 33–35]. In case the strategy space S is discrete but consists of an infinite number of strategies, e.g. $S = \mathbb{Z}$, then the replicator dynamics describing the evolution of the infinite dimensional vector $p(t) = (\dots, p_1(t), p_2(t), \dots)$ is described by the following

$$\frac{dp_i}{dt} = \left(\sum_{j \in \mathbb{Z}} a_{ij} p_j(t) - \sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} a_{ij} p_i(t) p_j(t) \right) p_i(t), \quad t > 0,$$

which is a infinite dynamical system with $p_i(t) \geq 0$ for $i \in \mathbb{Z}$ and $\|p(t)\|_{\ell^1(\mathbb{Z})} = 1$ for any $t > 0$.

In the current chapter we are concentrating on games whose pure strategies belong to a continuum. For instance, this could be the aspiration level of a player or the size of an investment in economics or it might arise in situations where the pure strategies correspond to geographical points as in economic geography, [25]. On the other hand, in biology such strategies correspond to some continuously varying trait such as the sex ratio in a litter or the virulence of an infection, [20]. There are different ways of modelling the evolutionary dynamics in this case. However, in the current work we adapt the approach introduced in [8]. In that case the strategy set Ω is an arbitrary, not necessarily bounded, Borel set of $\mathbb{R}^N$, $N \geq 2$. Hence strategies can be identified by $x \in \Omega$. For the case of symmetric two-player games, the pay-off can be given by a Borel measurable function $f : \Omega \times \Omega \rightarrow \mathbb{R}$, where $f(x, y)$ is the pay-off for player 1 when she follows strategy x and player 2 plays strategy y . A population is now characterized by its state, a probability measure $\mathcal{P}$ in the measure space $(\Omega, \mathcal{A})$ where $\mathcal{A}$ is the Borel algebra of subsets of Ω . The average (mean) pay-off of a sub-population in state $\mathcal{P}$ against the overall population in state $\mathcal{Q}$ is given by the form

$$E(\mathcal{P}, \mathcal{Q}) := \int_{\Omega} \int_{\Omega} f(x, y) \mathcal{Q}(dy) \mathcal{P}(dx).$$

Then, the success (or lack of success) of a strategy x followed by population $\mathcal{Q}$ is provided by the difference

$$\sigma(x, \mathcal{Q}) := \int_{\Omega} f(x, y) \mathcal{Q}(dy) - \int_{\Omega} \int_{\Omega} f(x, y) \mathcal{Q}(dy) \mathcal{Q}(dx) = E(\delta_x, \mathcal{Q}) - E(\mathcal{Q}, \mathcal{Q}),$$

where δ_x is the unit mass concentrated on the strategy x .

The evolution in time of the population state $\mathcal{Q}(t)$ is given by the replicator dynamics equation

$$\frac{d\mathcal{Q}}{dt}(A) = \int_A \sigma(x, \mathcal{Q}(t)) \mathcal{Q}(t)(dx), \quad t > 0, \quad \mathcal{Q}(0) = \mathcal{P}, \quad (6.2)$$

for any $A \in \mathcal{A}$, where the time derivative should be understood with respect to the variational norm of a subspace of the linear span $\mathcal{M}$ of $\mathcal{A}$. The well-posedness of (6.2) as well as relating stability issues were investigated in [34, 35] under the assumption that the pay-off function $f(x, y)$ is bounded.

The abstract form of Eq. (6.2) does not actually allow us to obtain insight on the form of its solutions and thus a better understanding of the evolutionary dynamics of the corresponding game. In order to have a better overview of the evolutionary game, following the approach in [23, 24], we restrict our attention to measures $\mathcal{Q}(t)$ which, for each $t > 0$, are absolutely continuous with respect to the Lebesgue measure, with probability density $u(x, t)$. Then the replicator dynamics equation (6.2) can be reduced to the following integro-differential equation for $x \in \Omega$ and $t > 0$,

$$\frac{\partial u}{\partial t} = \left(\int_{\Omega} f(x, y) u(y, t) dy - \int_{\Omega} \int_{\Omega} f(z, y) u(y, t) u(z, t) dy dz \right) u(x, t), \quad (6.3)$$

taking also into account that the probability density u satisfies

$$\int_{\Omega} \int_{\Omega} u(y, t) u(z, t) dy dz = 1, \quad (6.4)$$

hence we can skip the denominator from the average pay-off term into (6.3).

There are applications in biology where the pay-off kernel has the form $f(x, y) = G(x - y)$ with G being a steep function of Gaussian type, see [18, 19]. This case, in general, models games where the pay-off is measured as the distance from some reference strategy and finally under some proper scaling leads to

$$\int_{\Omega} f(x, y) u(y, t) dy \approx \Delta u(x, t), \quad (6.5)$$

(see also [22]) which by virtue of (6.2) yields

$$\frac{\partial u}{\partial t} \approx \left(\Delta u - \int_{\Omega} u \Delta u \, dx \right) u. \quad (6.6)$$

Another alternative towards getting pay-offs of this type is to consider a game with a discrete strategy space and take the appropriate scaling limit. In that case a Taylor expansion and a proper scaling gives a similar approximation to (6.5), see also [23, 24].

Therefore, in case that Ω is a bounded and smooth domain in $\mathbb{R}^N$, it is easily seen that via integration by parts the non-local integro-differential dynamics equation (6.3) is approximated by the degenerate non-local parabolic equation

$$\frac{\partial u}{\partial t} = u \left(\Delta u + \int_{\Omega} |\nabla u|^2 \, dx \right), \quad x \in \Omega, \, t > 0. \quad (6.7)$$

The non-local equation (6.7) is associated with initial data

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (6.8)$$

and homogeneous Dirichlet boundary conditions

$$u(x, t) = 0, \quad x \in \partial\Omega, \, t > 0, \quad (6.9)$$

when the individuals of the biological population do not interact once they are close to the spatial boundary where probably the “food” is less. We remark that if on the boundary of the strategy space the population individuals do not really distinguish between nearby strategies and hence populate them equally, then the non-local Eq. (6.7) should rather be complemented with homogeneous Neumann boundary conditions not explicitly considered here, see [24].

Our analysis will reveal that initial unit-population is preserved and guarantees that

$$\int_{\Omega} u(x, t) \, dx = 1, \quad (6.10)$$

when

$$\int_{\Omega} u_0(x) \, dx = 1,$$

see also Theorem 6.2.14 (ii), which in this case provides an a-posteriori justification for (6.4).

From a mathematical perspective, the evolution in (6.7) is governed by two characteristic mechanisms, each of which already gives rise to considerable challenges on its own. First, diffusion in (6.7) is strongly degenerate at small densities in the sense that at near points where $u = 0$, typical diffusive effects are substantially inhibited.

Additionally in (6.7), this degenerate diffusion process interacts with a spatially non-local source which is such that unlike in large bodies of the literature on related non-local parabolic equations [39], even basic questions concerning local solvability appear to be far from trivial. Indeed, in light of an expected loss of appropriate solution regularity due to strongly degenerate diffusion, even for smooth initial data it seems a priori unclear whether solutions can be constructed which allow for a meaningful definition of the Dirichlet integral $\int_{\Omega} |\nabla u|^2$ for positive times. This is in stark contrast to most non-local parabolic problems previously studied, in which diffusion is non-degenerate and hence such first-order profiles are controllable by L^∞ – bounds of solutions. In the semi-linear problem

$$u_t = \Delta u + u^m \left(\int_{\Omega} |\nabla u|^2 dx \right)^r,$$

studied for $m \geq 1$, $r > 0$ in [45], e.g., allows local theories based on extensibility criteria in $L^\infty(\Omega)$ only (see [9, 43] and also the book [39]).

6.2 Mathematical Analysis

Previous mathematical studies on the non-local PDE in (6.7) have concentrated on analyzing self-similar solutions only. In [23], the authors constructed self-similar solutions in the case $\Omega = \mathbb{R}$, and in [36] the same was achieved in the multi-dimensional case $\Omega = \mathbb{R}^N$ with $N \geq 2$.

More recently, the authors in [37] investigated the existence of self-similar solutions in the one-dimensional case for a closely related problem in which the Laplacian is perturbed by a time-dependent term containing the first derivative as well; all these self-similar solutions are shown to be regular and to approach Dirac-type distributions as $t \searrow 0^+$. An analogous study in higher dimensions is provided in [38].

In the current chapter we develop a fundamental theory of local solvability for problem (6.7)–(6.9), and also provide the first step towards an understanding of the qualitative solution behavior following an approach developed in [21]. In order to formulate our results, we determine the specific setting within which (6.7)–(6.9) is investigated by requiring that Ω denotes a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary and by introducing the solution concept as follows.

Definition 6.2.1 Let $T \in (0, \infty]$. By a *weak solution* of (6.7)–(6.9) in $\Omega \times (0, T)$ we mean a nonnegative function

$$u \in L_{loc}^\infty(\bar{\Omega} \times [0, T)) \cap L_{loc}^2([0, T); W_0^{1,2}(\Omega)) \quad \text{with} \quad u_t \in L_{loc}^2(\bar{\Omega} \times [0, T)),$$

which satisfies

$$\begin{aligned}
& - \int_0^T \int_{\Omega} u \varphi_t \, dx \, dt + \int_0^T \int_{\Omega} \nabla u \cdot \nabla (u \varphi) \, dx \, dt = \int_{\Omega} u_0 \varphi(\cdot, 0) \, dx \\
& + \int_0^T \left(\int_{\Omega} u \varphi \, dx \right) \cdot \left(\int_{\Omega} |\nabla u|^2 \, dx \right) dt,
\end{aligned} \tag{6.11}$$

for all $\varphi \in C_0^\infty(\Omega \times [0, T])$. A weak solution u of (6.7)–(6.9) in $\Omega \times (0, T)$ is called *locally positive* if $\frac{1}{u} \in L_{loc}^\infty(\Omega \times [0, T])$.

Remark 6.2.2 Since $u \in L_{loc}^2([0, T]; W_0^{1,2}(\Omega))$ and $u_t \in L_{loc}^2(\bar{\Omega} \times [0, T])$ imply

$$u \in C^0([0, T]; L^2(\Omega)).$$

Equation (6.11) is equivalent to $u(\cdot, 0) = u_0$, and

$$\int_0^T \int_{\Omega} u_t \varphi \, dx \, dt + \int_0^T \int_{\Omega} \nabla u \cdot \nabla (u \varphi) \, dx \, dt = \int_0^T \left(\int_{\Omega} u \varphi \, dx \right) \cdot \left(\int_{\Omega} |\nabla u|^2 \, dx \right) dt, \tag{6.12}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times (0, T))$.

We assume that the initial data satisfy

- (H1) $u_0 \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$,
- (H2) $u_0 \geq 0$ and $\frac{1}{u_0} \in L_{loc}^\infty(\Omega)$,
- (H3) there exists $L > 0$ such that $\|u_0\|_{\Phi, \infty} \leq L$.

For a measurable function $v: \Omega \rightarrow \mathbb{R}$ we set

$$\|v\|_{\Phi, \infty} := \operatorname{ess\,sup}_{x \in \Omega} \left| \frac{v}{\Phi} \right|,$$

where $\Phi \in C^2(\bar{\Omega})$ denotes the solution to

$$-\Delta \Phi = 1 \quad \text{in } \Omega, \quad \Phi|_{\partial\Omega} = 0. \tag{6.13}$$

From the Hopf boundary point lemma, requiring $\|u_0\|_{\Phi, \infty} < \infty$ is equivalent to ask $|u_0| \leq \|u_0\|_{\Phi, \infty} \Phi$ on $\partial\Omega$.

6.2.1 Local Existence and Extendability of Weak Solutions

The first result asserts the local existence of positive weak solutions, along with a favorable extensibility criterion only involving the norm of the solution in $L^\infty(\Omega)$.

Theorem 6.2.3 *Let u_0 satisfy (H1)–(H3). Then there exist $T_{max} \in (0, \infty]$ and a locally positive weak solution u to (6.7)–(6.9) in $\Omega \times (0, T_{max})$ which satisfies*

$$\text{either } T_{max} = \infty \quad \text{or} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (6.14)$$

For each smooth bounded subdomain $\Omega' \subset \subset \Omega$ there exists $C_{\Omega'} > 0$ such that

$$\begin{aligned} \int_{\Omega} |\nabla u(\cdot, t)|^2 &\leq \int_{\Omega} |\nabla u_0|^2 \cdot \exp \left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (0, t)} \int_{\Omega} u(\cdot, \tau) \right) \right. \\ &\quad \cdot \left. \left(\int_{\Omega'} \phi \ln u(\cdot, \tau) \Big|_{\tau=0}^t + \int_0^t \int_{\Omega'} u \right) \right]. \end{aligned} \quad (6.15)$$

It holds also that

$$\|u(\cdot, t)\|_{\phi, \infty} \leq \max \left\{ \|u_0\|_{\phi, \infty}, \sup_{\tau \in (0, t)} \int_{\Omega} |\nabla u(x, \tau)|^2 \right\}, \quad (6.16)$$

for a.e. $t \in (0, T_{max})$.

If the second condition of (6.14) is fulfilled then we say that the solution u of (6.7)–(6.9) blows in finite time. When finite-time blow-up occurs, understanding the solution behavior near the blow-up time necessarily requires to describe the set of all points where the solution becomes unbounded. Accordingly, we shall next be concerned with the blow-up set $\mathcal{B}$ by

$$\mathcal{B}^c = \left\{ x \in \overline{\Omega} \mid \text{there exists an open set } U \text{ containing } x \text{ such that } \limsup_{t \uparrow T} \|u(\cdot, t)\|_{L^\infty(\Omega \cap U)} < +\infty \right\},$$

of exploding solutions.

We emphasize that the extensibility criterion (6.14) particularly excludes any *gradient blow-up* phenomenon in the sense of finite-time blow-up of ∇u despite boundedness of u itself. Indeed, the occurrence of unbounded gradients of bounded solutions appears to be a characteristic qualitative implication of various types of interplay between diffusion, possibly degenerate, and gradient-dependent nonlinearities [2, 4, 30, 48].

In order to prove Theorem 6.2.3 we will follow an approach well-established in the context of degenerate parabolic equations and thus some necessary auxiliary results will be provided first. In particular, we aim at constructing a solution u to (6.7)–(6.9) as the limit of solutions to certain regularized problems. For this purpose, let us fix a sequence $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ such that $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$, and a sequence $(u_{0\varepsilon})_{\varepsilon=\varepsilon_j} \subset C^3(\Omega)$ with the properties

$$u_{0\varepsilon} \geq \varepsilon \text{ in } \Omega, \quad u_{0\varepsilon} = \varepsilon \text{ on } \partial\Omega, \quad \Delta u_{0\varepsilon} = - \int_{\Omega} |\nabla u_{0\varepsilon}|^2 \text{ on } \partial\Omega, \quad (6.17)$$

for all $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ and

$$\limsup_{\varepsilon = \varepsilon_j \searrow 0} \|u_{0\varepsilon} - \varepsilon\|_{\Phi, \infty} \leq L, \quad (6.18)$$

with $L > \max \left\{ \int_{\Omega} |\nabla u_0|^2, \|u_0\|_{\Phi, \infty} \right\}$, cf. (H3), as well as for any compact set $K \subset \Omega$ there is $C_K > 0$ such that

$$\liminf_{\varepsilon = \varepsilon_j \searrow 0} \inf_K u_{0\varepsilon} \geq C_K, \quad (6.19)$$

with

$$u_{0\varepsilon} \rightarrow u_0 \quad \text{in } W^{1,2}(\Omega) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \quad (6.20)$$

and

$$\int_{\Omega} u_{0\varepsilon} = \int_{\Omega} u_0 \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}. \quad (6.21)$$

The first observation is that such an approximation is possible.

Lemma 6.2.4 *Let u_0 satisfy (H1)–(H3). Then there is a sequence $(u_{0\varepsilon})_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}} \subset C^3(\bar{\Omega})$ having the properties (6.17)–(6.21).*

Proof Here we restrict ourselves to giving an outline; a more detailed proof can be found in [21]. By modification of the usual mollification procedure (cf. [57, Sect. I 3]) commonly employed to obtain (6.20), it is possible to obtain the other properties as well. More precisely, we set

$$u_{0\varepsilon} = \varepsilon + C(1 - \rho)\Phi + \rho(\varphi + \alpha\vartheta),$$

where $\varphi \in C_0^\infty(\Omega)$ is a mollified version of u_0 after “locally shifting u_0 towards the interior of the domain”, $\rho \in C_0^\infty(\Omega)$, $0 \leq \rho \leq 1$, such that the supports of $\nabla \rho$ and φ are disjoint, $0 \leq \vartheta \in C_0^\infty(\Omega)$ with $\int_{\Omega} \vartheta = 1$ (in order to adjust (6.21)), Φ is the solution to $-\Delta \Phi = 1$ in Ω , $\Phi = 0$ on $\partial\Omega$ (for achieving the third property in (6.17)), and C and α are appropriately adjusted constants, depending on ε as well as several different integrals containing the functions Φ , ρ , θ , their gradients, and u_0 . $\square$

For $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$, we consider the regularized problem

$$u_{\varepsilon t} = u_{\varepsilon} \Delta u_{\varepsilon} + u_{\varepsilon} \cdot \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right), \quad x \in \Omega, \quad t > 0, \quad (6.22)$$

$$u_{\varepsilon}(x, t) = \varepsilon, \quad x \in \partial\Omega, \quad t > 0, \quad (6.23)$$

$$u_{\varepsilon}(x, 0) = u_{0\varepsilon}(x), \quad x \in \Omega, \quad (6.24)$$

where

$$\rho_\varepsilon(z) := \min \left\{ z, \frac{1}{\varepsilon} \right\} \quad \text{for } z \geq 0.$$

The first result regarding the regularized problem (6.22)–(6.24) can be seen below.

Lemma 6.2.5 *For all sufficiently small $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$, problem (6.22)–(6.24) has a unique classical global-in-time solution $u_\varepsilon \in C^{2,1}(\bar{\Omega} \times [0, \infty))$.*

Proof To prove the uniqueness statement for all ε , we assume that both u_1 and u_2 are classical solutions of (6.22)–(6.24) from the indicated class in $\Omega \times (0, T)$ for some $T > 0$. Then $w := u_1 - u_2$ satisfies $w = 0$ on $\partial\Omega$ and at $t = 0$, and

$$w_t = u_1 \Delta w + \Delta u_2 \cdot w + \rho_\varepsilon \left(\int_{\Omega} |\nabla u_2|^2 \right) \cdot w + u_1 \cdot \left[\rho_\varepsilon \left(\int_{\Omega} |\nabla u_1|^2 \right) - \rho_\varepsilon \left(\int_{\Omega} |\nabla u_2|^2 \right) \right], \quad (6.25)$$

for $t \in (0, T)$. Now given $T' \in (0, T)$, we can find a constant $M > 0$ such that u_1 , $|\nabla u_1|$, u_2 and $|\nabla u_2|$ are bounded above by M in $\Omega \times (0, T')$, since u_1, u_2 are classical solutions. Thus, by Hölder's inequality and the pointwise estimate

$$\left| |\nabla u_1| - |\nabla u_2| \right| \leq |\nabla(u_1 - u_2)|,$$

we obtain

$$\begin{aligned} \left| \rho_\varepsilon \left(\int_{\Omega} |\nabla u_1|^2 \right) - \rho_\varepsilon \left(\int_{\Omega} |\nabla u_2|^2 \right) \right| &\leq \|\rho'_\varepsilon\|_{L^\infty((0, \infty))} \cdot \left| \int_{\Omega} (|\nabla u_1|^2 - |\nabla u_2|^2) \right| \\ &\leq \int_{\Omega} \left| |\nabla u_1| - |\nabla u_2| \right| \cdot (|\nabla u_1| + |\nabla u_2|) \\ &\leq 2M \int_{\Omega} |\nabla w| \\ &\leq 2M |\Omega|^{\frac{1}{2}} \cdot \left(\int_{\Omega} |\nabla w|^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (6.26)$$

for all $t \in (0, T')$ by $\|\rho'_\varepsilon\|_{L^\infty((0, \infty))} \leq 1$. Upon multiplying (6.25) by w and integrating over Ω , we see that for $t \in (0, T')$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 &= \int_{\Omega} u_1 \Delta w w + \int_{\Omega} w^2 \Delta u_2 + \int_{\Omega} w^2 \rho_\varepsilon \left(\int_{\Omega} |\nabla u_2|^2 \right) \\ &\quad + \int_{\Omega} w u_1 \left[\rho_\varepsilon \left(\int_{\Omega} |\nabla u_1|^2 \right) - \rho_\varepsilon \left(\int_{\Omega} |\nabla u_2|^2 \right) \right] \\ &\leq - \int_{\Omega} u_1 |\nabla w|^2 - \int_{\Omega} \nabla u_1 \nabla w w - 2 \int_{\Omega} w \nabla w \nabla u_2 \\ &\quad + \int_{\Omega} w^2 \rho_\varepsilon \left(\int_{\Omega} |\nabla u_2|^2 \right) + \int_{\Omega} |w| u_1 \left| \rho_\varepsilon \left(\int_{\Omega} |\nabla u_1|^2 \right) - \rho_\varepsilon \left(\int_{\Omega} |\nabla u_2|^2 \right) \right|. \end{aligned} \quad (6.27)$$

Together with Young's inequality, (6.26) and the facts that $u_1 \geq \varepsilon$ (which, thanks to the actual non-degeneracy of problem (6.22)–(6.24) for positive ε , is an immediate consequence of the maximum principle) and $\rho_\varepsilon(s) \leq \frac{1}{\varepsilon}$ for all $s > 0$, this entails

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 &\leq -\varepsilon \int_{\Omega} |\nabla w|^2 + \frac{\varepsilon}{4} \int_{\Omega} |\nabla w|^2 + \frac{1}{\varepsilon} \int_{\Omega} w^2 |\nabla u_1|^2 + \frac{\varepsilon}{2} \int_{\Omega} |\nabla w|^2 + \frac{8}{\varepsilon} \int_{\Omega} w^2 |\nabla u_2|^2 \\ &\quad + \frac{1}{\varepsilon} \int_{\Omega} w^2 + 2M|\Omega|^{\frac{1}{2}} \left(\int_{\Omega} |\nabla w|^2 \right)^{\frac{1}{2}} \int_{\Omega} |w|u_1, \end{aligned}$$

for $t \in (0, T')$. The choice of M now ensures that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 &\leq -\frac{\varepsilon}{4} \int_{\Omega} |\nabla w|^2 + \frac{M^2}{\varepsilon} \int_{\Omega} w^2 + \frac{8M^2}{\varepsilon} \int_{\Omega} w^2 + \frac{1}{\varepsilon} \int_{\Omega} w^2 \\ &\quad + 2M|\Omega|^{\frac{1}{2}} \left(\int_{\Omega} |\nabla w|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |w|^2 \int_{\Omega} u_1^2 \right)^{\frac{1}{2}} \\ &\leq -\frac{\varepsilon}{4} \int_{\Omega} |\nabla w|^2 + \frac{9M^2 + 1}{\varepsilon} \int_{\Omega} w^2 + \frac{\varepsilon}{4} \int_{\Omega} |\nabla w|^2 + \frac{4M^4 |\Omega|^2}{\varepsilon} \int_{\Omega} |w|^2, \end{aligned} \tag{6.28}$$

for $t \in (0, T')$, so that (6.28) finally turns into

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 \leq \left(\frac{3M^2 + 1}{\varepsilon} + \frac{4M^4 |\Omega|^2}{\varepsilon} \right) \cdot \int_{\Omega} w^2,$$

for all $t \in (0, T')$.

Integration of this yields that $w \equiv 0$ in $\Omega \times (0, T')$ and hence also in $\Omega \times (0, T)$, because $T' < T$ was arbitrary. It remains to show that for all $T > 0$, (6.22)–(6.24) is classically solvable in $\Omega \times (0, T)$, provided that ε is sufficiently small. To this end, fix $T > 0$ and let $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ be so small that $\int_{\Omega} |\nabla u_{0\varepsilon}|^2 < \frac{1}{\varepsilon}$, which is possible due to (6.20). By [27, Theorem V.1.1], there are $K_1 > 0$ and $\theta > 0$ such that any classical solution w to the problem

$$w_t = w\Delta w + c(x, t) \text{ in } \Omega \times [0, T], \quad w|_{\partial\Omega} = \varepsilon, \quad w(\cdot, 0) = u_{0\varepsilon},$$

with $c \in L^\infty(\Omega \times (0, T))$ satisfies

$$\|w\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [0, T])} \leq K_1, \tag{6.29}$$

by $0 \leq c \leq \frac{1}{\varepsilon} \|u_{0\varepsilon}\|_{L^\infty(\Omega)} e^{\frac{T}{\varepsilon}}$ and $\varepsilon \leq w \leq \|u_{0\varepsilon}\|_\infty e^{\frac{T}{\varepsilon}}$.

Fix $\delta > 0$. By an application of [13, Theorem 7.4] to $w - u_{0\varepsilon} - tb_0$, corresponding to θ , K_1 and δ , there is K_2 such that any solution w to

$$w_t = a(x, t)\Delta w + b(x, t) \text{ in } \Omega \times [0, T], \quad w|_{\partial\Omega} = \varepsilon, \quad w(\cdot, 0) = u_{0\varepsilon},$$

fulfils

$$\|w\|_{C^{1+\delta, \frac{\delta}{2}}(\overline{\Omega} \times [0, T])} \leq K_2, \quad (6.30)$$

if $a \in C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [0, T])$, $a(x, t) = \varepsilon$ for $(x, t) \in \partial\Omega \times [0, T]$, $\varepsilon \leq a \leq \|u_{0\varepsilon}\|_{L^\infty} e^{\frac{T}{\varepsilon}}$, $\|a\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [0, T])} \leq K_1$, and $b \in C(\overline{\Omega} \times [0, T])$ with $b(x, 0) = b_0 \in \mathbb{R}$, $\|b\|_\infty \leq \frac{K_1}{\varepsilon}$.

With this in mind, in the space $X = C^{1+\frac{\delta}{2}, \frac{\delta}{4}}(\overline{\Omega} \times [0, T])$ we consider the set

$$S := \left\{ v \in X \mid v \geq \varepsilon \text{ in } \Omega \times (0, T), v(\cdot, 0) = u_{0\varepsilon} \text{ and } \|v\|_{C^{1+\delta, \frac{\delta}{2}}(\overline{\Omega} \times [0, T])} \leq K_2 \right\},$$

which is evidently closed, bounded, convex, and compact in X . For each $v \in S$, the definition of ρ_ε implies that

$$f(t) := \rho_\varepsilon \left(\int_\Omega |\nabla v(\cdot, t)|^2 \right), \quad t \in [0, T], \quad (6.31)$$

defines a non-negative $\frac{\delta}{2}$ -Hölder continuous function f on $[0, T]$. The choices of f , S and ε show that $f(0) = \int_\Omega |\nabla u_{0\varepsilon}|^2$ and thus (6.17) ensures that the compatibility condition of first order is satisfied. Therefore, the quasilinear, actually non-degenerate parabolic problem

$$u_{\varepsilon t} = u_\varepsilon \Delta u_\varepsilon + f(t)u_\varepsilon, \quad x \in \Omega, \quad t > 0, \quad (6.32)$$

$$u_\varepsilon(x, t) = \varepsilon, \quad x \in \partial\Omega, \quad t > 0, \quad (6.33)$$

$$u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad x \in \Omega, \quad (6.34)$$

possesses a classical solution $u_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0, T])$ by [27, Theorem V.6.1], which, by comparison, satisfies

$$\varepsilon \leq u_\varepsilon \leq \|u_{0\varepsilon}\|_{L^\infty(\Omega)} \cdot e^{\frac{T}{\varepsilon}} \quad \text{in } \Omega \times (0, T), \quad (6.35)$$

because $\underline{u}(x, t) := \varepsilon$ and $\overline{u}(x, t) := \|u_{0\varepsilon}\|_{L^\infty(\Omega)} \cdot e^{\frac{T}{\varepsilon}}$ are easily seen to define a sub- and a supersolution of (6.32)–(6.34), respectively.

We now introduce a mapping $F: S \rightarrow X$ by setting $Fv := u_\varepsilon$, where u_ε solves (6.32)–(6.34) with (6.31).

Then defining $c(x, t) := u_\varepsilon(x, t)f(t)$, $x \in \Omega$, $t \in [0, T]$, this function satisfies $\|c\|_\infty \leq \frac{1}{\varepsilon} \|u_{0\varepsilon}\|_\infty e^{\frac{T}{\varepsilon}}$ and accordingly, as stated in (6.29) above, $\|Fv\|_{C^{\theta, \frac{\theta}{2}}} \leq K_1$ for any $v \in S$.

Using $a(x, t) := (Fv)(x, t)$ and $b(x, t) := (Fv)(x, t) \cdot f(t)$, we see that, again, the above considerations are applicable and $\|Fv\|_{C^{1+\delta, \frac{\delta}{2}}(\overline{\Omega} \times [0, T])} \leq K_2$ for any $v \in S$ by (6.30). In particular, we observe that $FS \subset S$.

Furthermore invoking [27, IV.5.2], we can conclude the existence of $k > 0$ and $K_3 > 0$ such that

$$\|Fv\|_{C^{2+\delta, 1+\frac{\delta}{2}}(\bar{\Omega} \times [0, T])} \leq k \left(\|Fv \cdot f\|_{C^{\delta, \frac{\delta}{2}}(\bar{\Omega} \times [0, T])} + \|u_{0\varepsilon}\|_{C^{2+\delta}(\bar{\Omega} \times [0, T])} + \varepsilon \right) \leq K_3, \quad (6.36)$$

for all $v \in S$. To see that F is continuous, we suppose that $(v_k)_{k \in \mathbb{N}} \subset S$ and $v \in S$ are such that $v_k \rightarrow v$ in X . Then $f_k(t) := \rho_\varepsilon(\int_\Omega |\nabla v_k(\cdot, t)|^2)$ satisfies

$$f_k \rightarrow f \quad \text{in } C^0([0, T]), \quad (6.37)$$

as $k \rightarrow \infty$, with f as given by (6.31). By (6.36) and the theorem of Arzelà-Ascoli, $(Fv_k)_{k \in \mathbb{N}}$ is relatively compact in $C^{2,1}(\bar{\Omega} \times [0, T])$, and if $k_i \rightarrow \infty$ is any sequence such that $u_{k_i} := Fv_{k_i}$ converges in $C^{2,1}(\bar{\Omega} \times [0, T])$ to some w as $i \rightarrow \infty$, then in

$$\partial_t u_{k_i} = u_{k_i} \Delta u_{k_i} + f_{k_i}(t) u_{k_i}, \quad x \in \Omega, \quad t \in (0, T),$$

we may let $k_i \rightarrow \infty$ and use (6.37) to obtain that w is a classical solution of (6.32)–(6.34). Since classical solutions of (6.32)–(6.34) are unique due to the comparison principle, we must have $w = Fv$. We thereby derive that the whole sequence $(Fv_k)_{k \in \mathbb{N}}$ converges to Fv and hence conclude that F is continuous. Therefore the Schauder fixed point theorem asserts the existence of at least one $u_\varepsilon \in S$ for which $u_\varepsilon = Fu_\varepsilon$ holds. Since such a fixed point obviously solves (6.22)–(6.24), and the proof is complete. $\square$

The basis of both our existence proof and our boundedness result is formed by the next two lemmata which provide useful a priori estimates for u_ε in terms of certain presupposed bounds. The first lemma essentially derives a uniform pointwise bound for u_ε from a space-time integral estimate for $|\nabla u_\varepsilon|^2$.

Lemma 6.2.6 *For all $M > 0$ and $B > 0$ there exists $C(M, B) > 0$ with the following property: If*

$$u_{0\varepsilon} \leq M \quad \text{in } \Omega \quad \text{and} \quad \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq B, \quad (6.38)$$

holds for some $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ and $T \in (0, \infty]$ then we have

$$u_\varepsilon \leq C(M, B) \quad \text{in } \Omega \times [0, T]. \quad (6.39)$$

Proof Our plan is to use a separated function of the form

$$\bar{u}(x, t) := z(t) \cdot (M + \Phi(x)), \quad x \in \bar{\Omega}, \quad t \in [0, T], \quad (6.40)$$

as a comparison function, where M is as in the hypothesis of the lemma, $\Phi \in C^2(\bar{\Omega})$ is the solution of (6.13), and z denotes the solution of

$$z' = -z^2 + (f(t) + 1) \cdot \theta, \quad t \in (0, T), \quad z(0) = 1, \quad (6.41)$$

with $f(t) := \int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2$. In fact, it follows from (6.41) that $\zeta := \frac{1}{z}$ is a solution of $\zeta' = 1 - (f(t) + 1)\zeta$, $\zeta(0) = 1$, and hence given by

$$\zeta(t) = e^{-\int_0^t f(s)ds-t} + \int_0^t e^{-\int_s^t f(\sigma)d\sigma-(t-s)} ds, \quad t \in [0, T].$$

We claim that

$$1 \leq z(t) \leq e^{B+1} \quad \text{for all } t \in (0, T). \quad (6.42)$$

To see this, we note that if $t \in (0, T)$ satisfies $t < 1$, then (6.38) implies

$$\zeta(t) \geq e^{-\int_0^t f(s)ds-t} \geq e^{-B-t} \geq e^{-B-1},$$

whereas if $t \in [1, T)$ then again (6.38) shows

$$\begin{aligned} \zeta(t) &\geq \int_{t-1}^t e^{-\int_s^t f(\sigma)d\sigma-(t-s)} ds \geq \int_{t-1}^t e^{-B-(t-s)} ds \\ &\geq \int_{t-1}^t e^{-B-1} ds = e^{-B-1}. \end{aligned}$$

This yields the right inequality in (6.42), while the left immediately results from an ODE comparison of z with $\underline{z}(t) \equiv 1$, because $\underline{z}' + \underline{z}^2 - (f(t) + 1)\underline{z} = -f(t) \leq 0$. Consequently, since $\Phi \geq 0$ in Ω , the function $\bar{u}$ defined by (6.40) satisfies

$$\bar{u}(x, 0) = M + \Phi(x) \geq M \geq u_{\varepsilon}(x, 0) \quad \text{for all } x \in \Omega,$$

due to (6.38), and on the lateral boundary we have

$$\bar{u}(x, t) = z(t) \cdot M \geq M \geq \varepsilon \quad \text{for all } x \in \partial\Omega \text{ and } t \in (0, T).$$

Moreover,

$$\begin{aligned} \bar{u}_t - \bar{u}\Delta\bar{u} - f(t) \cdot \bar{u} &= z' \cdot (M + \Phi) + z^2 \cdot (M + \Phi) - f(t) \cdot \theta \cdot (M + \Phi) \\ &= z \cdot (M + \Phi) \geq 0 \quad \text{for all } x \in \Omega \text{ and } t \in (0, T), \end{aligned}$$

whence the comparison principle ensures that $u_\varepsilon \leq \bar{u}$ in $\Omega \times (0, T)$. In view of (6.42), this entails that

$$u_\varepsilon(x, t) \leq e^{B+1} \cdot (M + \|\Phi\|_{L^\infty(\Omega)}) \quad \text{for all } x \in \Omega \text{ and } t \in (0, t),$$

so that (6.39) is valid upon an obvious choice of $C = C(M, B)$. $\square$

Next, the fact that solutions of (6.22)–(6.24) cannot blow up immediately can be turned into a quantitative local-in-time boundedness estimate in terms of the norm of the initial data in $L^\infty(\Omega) \cap W^{1,2}(\Omega)$. Moreover, our technique at the same time yields an estimate involving integrals of $u_{\varepsilon t}$ and ∇u_ε , as long as u_ε is appropriately bounded.

Lemma 6.2.7 (i) *For all $M > 0$ there exist $T_1(M) > 0$ and $C_1(M) > 0$ such that if*

$$u_{0\varepsilon} \leq M \quad \text{in } \Omega \quad \text{and} \quad \int_{\Omega} |\nabla u_{0\varepsilon}|^2 \leq M, \quad (6.43)$$

hold for some $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$, then

$$u_\varepsilon \leq C_1(M) \quad \text{in } \Omega \times [0, T_1(M)). \quad (6.44)$$

(ii) *For each $M > 0$ and $T > 0$ there exist $T_2(M) \in (0, T]$ and $C_2(M) > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ is such that*

$$u_\varepsilon \leq M \quad \text{in } \Omega \times (0, T) \quad \text{and} \quad \int_{\Omega} |\nabla u_{0\varepsilon}|^2 \leq M, \quad (6.45)$$

are satisfied, then

$$\int_0^{T_2(M)} \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \sup_{t \in (0, T_2(M))} \int_{\Omega} |\nabla u_\varepsilon(\cdot, t)|^2 \leq C_2(M). \quad (6.46)$$

Proof (i) We multiply (6.22) by $\frac{u_{\varepsilon t}}{u_\varepsilon}$ and integrate by parts, use that $u_{\varepsilon t} = 0$ on $\partial\Omega$, and apply Hölder's together with Young's inequality to see that

$$\begin{aligned} \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u_\varepsilon|^2 &= \left(\int_{\Omega} u_{\varepsilon t} \right) \cdot \rho_\varepsilon \left(\int_{\Omega} |\nabla u_\varepsilon|^2 \right) \\ &\leq \left(\int_{\Omega} \frac{u_{\varepsilon t}^2}{u_\varepsilon} \right)^{\frac{1}{2}} \left(\int_{\Omega} u_\varepsilon \right)^{\frac{1}{2}} \int_{\Omega} |\nabla u_\varepsilon|^2 \\ &\leq \frac{1}{2} \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \frac{1}{2} \left(\int_{\Omega} u_\varepsilon \right) \left(\int_{\Omega} |\nabla u_\varepsilon|^2 \right)^2, \end{aligned} \quad (6.47)$$

for all $t > 0$, because $\rho_\varepsilon(\xi) \leq \xi$ for all $\xi \geq 0$. Hence,

$$\int_{\Omega} \frac{u_{\varepsilon t}^2}{u_{\varepsilon}} + \frac{d}{dt} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq \left(\int_{\Omega} u_{\varepsilon} \right) \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right)^2. \quad (6.48)$$

Using the Poincaré inequality, we obtain

$$\int_{\Omega} u_{\varepsilon}(\cdot, t) \leq c_1 \cdot \left(\left(\int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2 \right)^{\frac{1}{2}} + 1 \right),$$

with a positive constant c_1 independent of $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \in (0, 1)$ and $t > 0$. Therefore, (6.48) yields

$$\int_{\Omega} \frac{u_{\varepsilon t}^2}{u_{\varepsilon}} + \frac{d}{dt} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq c_1 \cdot \left(\left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right)^{\frac{1}{2}} + 1 \right) \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right)^2, \quad (6.49)$$

which in particular implies that $z(t) := \int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2$ satisfies

$$z'(t) \leq c(\sqrt{z} + 1)z^2 \quad \text{for all } t > 0, \quad \text{and } z(0) \leq M.$$

Hence, if we let ζ denote the local-in-time solution of

$$\begin{aligned} \zeta'(t) &= c(\sqrt{\zeta} + 1)\zeta^2, & t > 0, \\ \zeta(0) &= M, \end{aligned}$$

with maximal existence time $T_{\zeta} > 0$, then due to (6.43) and an ODE comparison we have $z \leq \zeta$ in $(0, T_{\zeta})$. Defining $T_1(M) := \frac{1}{2}T_{\zeta}$, for instance, we obtain from this that $\int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2 \leq \zeta(T_1(M))$ for all $t \in [0, T_1(M))$, whereupon (6.44) now results from Lemma 6.2.6.

(ii) If the first inequality in (6.45) holds then (6.48) entails that z as defined above even satisfies the nonlinear ODI

$$z'(t) \leq M|\Omega|z^2 \quad \text{for all } t > 0,$$

whence we have $\int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2 \leq \frac{1}{M^{-1} - M|\Omega|t}$ for all $t \in (0, T_2)$ with

$$T_2 := \min\{T, 1/(M^2|\Omega|)\},$$

by the second inequality in (6.45). Inserting this into (6.49) again and integrating over $(0, T_2)$ proves (6.46). $\square$

When constructing the solution u of (6.7)–(6.9) as the limit of solutions u_{ε} of (6.22)–(6.24), it will be comparatively easy to obtain the approximation property $\nabla u_{\varepsilon} \rightarrow \nabla u$ in the sense of $L_{loc}^2(\Omega \times [0, T))$ -convergence. For handling the non-local term in the equation, however, it seems appropriate to make sure that also $\int_{\Omega} |\nabla u_{\varepsilon}|^2 \rightarrow \int_{\Omega} |\nabla u|^2$ in $L_{loc}^1([0, T))$.

In order to achieve the latter we exclude certain boundary concentration phenomena of ∇u_ε in the following sense.

Lemma 6.2.8 *For any $T > 0$, $C > 0$, $M > 0$ and $\delta > 0$, there is $K = K(M, C, T, \delta) \subset \subset \Omega$ and $\eta > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ is such that $\varepsilon < \eta$ and*

$$\sup_{t \in [0, T]} \int_{\Omega} |\nabla u_\varepsilon(t)|^2 \leq C \quad \text{and} \quad u_\varepsilon \leq M, \quad (6.50)$$

we have

$$\int_0^T \int_{\Omega \setminus K} |\nabla u_\varepsilon|^2 < \delta.$$

Proof For $q \in (0, 1)$, we multiply (6.22) by u_ε^{q-1} and integrate by parts to obtain

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega} u_\varepsilon^q = \int_{\partial\Omega} u_\varepsilon^q \partial_\nu u_\varepsilon - \int_{\Omega} q u_\varepsilon^{q-1} |\nabla u_\varepsilon|^2 + \int_{\Omega} u_\varepsilon^q \rho_\varepsilon \left(\int_{\Omega} |\nabla u_\varepsilon|^2 \right),$$

where we can use $\partial_\nu u_\varepsilon \leq 0$ on $\partial\Omega$ and integrate with respect to time to derive

$$q \int_0^T \int_{\Omega} u_\varepsilon^{q-1} |\nabla u_\varepsilon|^2 \leq -\frac{1}{q} \int_{\Omega} u_\varepsilon^q(T) + \frac{1}{q} \int_{\Omega} u_{0\varepsilon}^q + \int_0^T \left(\int_{\Omega} u_\varepsilon^q \int_{\Omega} |\nabla u_\varepsilon|^2 \right) =: C(T), \quad (6.51)$$

for all $\varepsilon > 0$ satisfying (6.50), which gives control on $|\nabla u_\varepsilon|^2$ where ever u_ε is small – which is the case near the boundary, as we ensure next: In order to lay the groundwork for the corresponding comparison argument, note that by (6.50),

$$u_{\varepsilon t} = u_\varepsilon \Delta u_\varepsilon + u_\varepsilon \rho_\varepsilon \left(\int_{\Omega} |\nabla u_\varepsilon|^2 \right) \leq u_\varepsilon \Delta u_\varepsilon + C u_\varepsilon, \quad u_\varepsilon|_{\partial\Omega} = \varepsilon, \quad u_\varepsilon(0) = u_{0\varepsilon}.$$

Fix $\eta > 0$ such that $\frac{(2\eta)^{1-q} C(T)}{q} < \delta$. Let Φ solve (6.13). Choose $A > C$ such that $A\Phi + \eta > u_{0\varepsilon}$ for all $0 < \varepsilon < \eta$, which is possible due to condition (8.92). Then $\bar{u} := A\Phi + \eta$ satisfies

$$\bar{u}_t = 0 \geq -(A\Phi + \eta)A + (A\Phi + \eta)C = \bar{u}A\Delta\Phi + C\bar{u} = \bar{u}\Delta\bar{u} + C\bar{u}. \quad (6.52)$$

As long as $\varepsilon < \eta$, also $\bar{u}|_{\partial\Omega} \geq u_\varepsilon|_{\partial\Omega}$ holds and furthermore

$$\bar{u}(0) \geq u_{0\varepsilon}.$$

Therefore, by the comparison principle, we obtain $\bar{u} \geq u_\varepsilon$.

Now choose $K \subset \subset \Omega$ in such a way that

$$A\Phi \leq \eta \quad \text{in } \Omega \setminus K.$$

This entails $u_\varepsilon \leq \bar{u} = A\Phi + \eta \leq 2\eta$ in $\Omega \setminus K$. Then

$$\begin{aligned} \int_0^T \int_{\Omega \setminus K} |\nabla u_\varepsilon|^2 &= \int_0^T \int_{\Omega \setminus K} u_\varepsilon^{q-1} |\nabla u_\varepsilon|^2 u_\varepsilon^{1-q} \\ &\leq (2\eta)^{1-q} \int_0^T \int_{\Omega \setminus K} u_\varepsilon^{q-1} |\nabla u_\varepsilon|^2 \\ &\leq (2\eta)^{1-q} \int_0^T \int_{\Omega} u_\varepsilon^{q-1} |\nabla u_\varepsilon|^2 \leq \frac{(2\eta)^{1-q} C(T)}{q}, \end{aligned}$$

by virtue of (6.51). $\square$

We are now ready to prove that u_ε in fact approach a weak solution of (6.7)–(6.9) that is locally positive in the sense of Definition 6.2.1. Before we do so, however, we prepare the following estimate for u_ε that will be useful in proving assertions about the blow-up behavior of u .

Lemma 6.2.9 *Let $\Omega' \subset \subset \Omega$ be a domain with smooth boundary. Assume also that ϕ denotes the solution to $-\Delta\phi = 1$ in Ω' , $\phi|_{\partial\Omega'} = 0$. Then there exists $C_{\Omega'} > 0$ such that for each $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ and any $t > 0$ the solution u_ε of (6.22)–(6.24) satisfies*

$$\begin{aligned} \int_{\Omega} |\nabla u_\varepsilon(\cdot, t)|^2 &\leq \\ \int_{\Omega} |\nabla u_{0\varepsilon}|^2 \exp &\left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (0, t)} \int_{\Omega} u_\varepsilon(\tau) \right) \left(\int_{\Omega'} \phi \ln u_\varepsilon(\cdot, t) - \int_{\Omega'} \phi \ln u_{0\varepsilon} + \int_0^t \int_{\Omega'} u_\varepsilon \right) \right]. \end{aligned} \quad (6.53)$$

Proof As $u_{\varepsilon t} = 0$ on $\partial\Omega$, similarly to (6.47), multiplying (6.22) by $\frac{u_{\varepsilon t}}{u_\varepsilon}$ and integrating over Ω yields

$$\begin{aligned} \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_\varepsilon} &= \int_{\Omega} u_{\varepsilon t} \Delta u_\varepsilon + \int_{\Omega} u_{\varepsilon t} \rho_\varepsilon \left(\int_{\Omega} |\nabla u_\varepsilon|^2 \right) \\ &= -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u_\varepsilon|^2 + \int_{\Omega} u_{\varepsilon t} \rho_\varepsilon \left(\int_{\Omega} |\nabla u_\varepsilon|^2 \right). \end{aligned}$$

After rearranging, by Hölder's and Young's inequalities and the definition of ρ_ε this entails

$$\frac{d}{dt} \int_{\Omega} |\nabla u_\varepsilon|^2 \leq -2 \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_\varepsilon} + 2 \left[\left(\int_{\Omega} \left(\frac{u_{\varepsilon t}}{\sqrt{u_\varepsilon}} \right)^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} \sqrt{u_\varepsilon}^{-2} \right)^{\frac{1}{2}} \right] \rho_\varepsilon \left(\int_{\Omega} |\nabla u_\varepsilon|^2 \right),$$

$$\begin{aligned}
&\leq -2 \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_{\varepsilon}} + 2 \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_{\varepsilon}} + \frac{1}{2} \int_{\Omega} u_{\varepsilon} \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right)^2 \\
&\leq \frac{1}{2} \int_{\Omega} u_{\varepsilon} \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \int_{\Omega} |\nabla u_{\varepsilon}|^2 \quad \text{in } (0, \infty).
\end{aligned}$$

This looks like a quadratic differential inequality for $z(t) := \int_{\Omega} |\nabla u_{\varepsilon}|^2$ and at first does not seem helpful for obtaining an estimate for this quantity. Therefore we shall split the respective quadratic term and apply Gronwall's lemma to $z'(t) \leq g(t)z(t)$, where

$$g(t) = \frac{1}{2} \int_{\Omega} u_{\varepsilon}(t) \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}(t)|^2 \right),$$

which leads to

$$z(t) \leq z(0) \exp \int_0^t g(\tau) d\tau \quad \text{for all } t > 0. \quad (6.54)$$

In this situation, however, we are left with a term $\int_0^t \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right)$ in the exponent and we prepare an estimate for this in the following way: With ϕ as specified in the hypothesis, we let $C_{\Omega'} = \int_{\Omega'} \phi > 0$. Multiplication of (6.22) by $\frac{\phi}{u_{\varepsilon}}$ and integrating over Ω' then gives

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega'} \ln u_{\varepsilon} \phi &= \int_{\Omega'} \Delta u_{\varepsilon} \phi + \int_{\Omega'} \phi \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \\
&= \int_{\Omega'} u_{\varepsilon} \Delta \phi + \int_{\partial \Omega'} \partial_{\nu} u_{\varepsilon} \phi - \int_{\partial \Omega'} u_{\varepsilon} \partial_{\nu} \phi + C_{\Omega'} \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \quad \text{on } (0, \infty).
\end{aligned}$$

Taking into account the definition of ϕ and the following consequence of Hopf's boundary value lemma $\partial_{\nu} \phi|_{\partial \Omega'} \leq 0 = \phi|_{\partial \Omega'}$, we infer that

$$\frac{d}{dt} \int_{\Omega'} \phi \ln u_{\varepsilon} \geq - \int_{\Omega'} u_{\varepsilon} + C_{\Omega'} \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \quad \text{in } (0, \infty).$$

Therefore

$$\int_0^t \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \leq \frac{1}{C_{\Omega'}} \left[\int_0^t \int_{\Omega'} u_{\varepsilon} + \int_{\Omega'} \phi \ln u_{\varepsilon}(t) - \int_{\Omega'} \phi \ln u_{0\varepsilon} \right],$$

for any $t > 0$, and we can conclude from (6.54) that

$$\int_{\Omega} |\nabla u_{\varepsilon}(t)|^2 \leq \int_{\Omega} |\nabla u_{0\varepsilon}|^2 \exp \left[\frac{1}{2C_{\Omega'}} \sup_{\tau \in (0, t)} \int_{\Omega} u(\tau) \left(\int_0^t \int_{\Omega'} u_{\varepsilon} + \int_{\Omega'} \phi \ln u_{\varepsilon}(t) - \int_{\Omega'} \phi \ln u_{0\varepsilon} \right) \right],$$

for all $t > 0$. □

Another useful piece of information is that a condition like (H3) remains satisfied for any $t > 0$.

Lemma 6.2.10 *Let $T > 0$, $M > 0$ and $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ be such that $\|u_{0\varepsilon} - \varepsilon\|_{\Phi, \infty} < \infty$. Then any solution u^ε of (6.22)–(6.24) which satisfies*

$$\int_{\Omega} |\nabla u_\varepsilon(t)|^2 \leq M \quad \text{for any } t \in [0, T],$$

already fulfils

$$\|u_\varepsilon - \varepsilon\|_{\Phi, \infty} \leq \max \{M, \|u_{0\varepsilon} - \varepsilon\|_{\Phi, \infty}\}.$$

Proof Let $C = \max\{M, \|u_{0\varepsilon} - \varepsilon\|_{\Phi, \infty}\}$ and consider $\bar{u} := C\Phi + \varepsilon$ with Φ as in (6.13). Then $\bar{u}_t = 0 \geq (M - C)(C\Phi + \varepsilon) = \bar{u}\Delta\bar{u} + M\bar{u}$, whereas $u_{\varepsilon t} = u_\varepsilon \Delta u_\varepsilon + u_\varepsilon \rho_\varepsilon \left(\int_{\Omega} |\nabla u_\varepsilon|^2\right) \leq u_\varepsilon \Delta u_\varepsilon + Mu_\varepsilon$. Additionally $\bar{u}|_{\partial\Omega} = \varepsilon = u_\varepsilon|_{\partial\Omega}$ and $\bar{u}(x, 0) - \varepsilon = C\Phi(x) \geq \Phi(x) \|u_{0\varepsilon} - \varepsilon\|_{\Phi, \infty} \geq u_{0\varepsilon}(x) - \varepsilon$ and therefore the comparison principle [51] asserts that $u_\varepsilon \leq \bar{u}$ and hence implies the claim. $\square$

With this information at hand, we can proceed to the proof of convergence of the u_ε to a solution of (6.7)–(6.9) that still satisfies an inequality like (6.53).

Lemma 6.2.11 *Suppose that u_0 satisfies (H1)–(H3). Then there exists $T > 0$ depending on bounds on $\|u_0\|_{L^\infty(\Omega)}$ and $\|\nabla u_0\|_{L^2(\Omega)}$ and a locally positive weak solution u of (6.7)–(6.9) in $\Omega \times (0, T)$. This solution can be obtained as the a.e. pointwise limit of a subsequence of the solutions u_ε of (6.22)–(6.24) as $\varepsilon = \varepsilon_j \searrow 0$, and for any smoothly bounded subdomain $\Omega' \subset\subset \Omega$ there is $C_{\Omega'} > 0$ such that*

$$\begin{aligned} & \int_{\Omega} |\nabla u(\cdot, t)|^2 \\ & \leq \int_{\Omega} |\nabla u_0|^2 \exp \left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (0, t)} \int_{\Omega} u(\tau) \right) \left(\int_{\Omega'} \phi \ln u(\cdot, t) - \int_{\Omega'} \phi \ln u_0 + \int_0^t \int_{\Omega'} u \right) \right], \end{aligned} \quad (6.55)$$

as well as

$$\|u(t)\|_{\Phi, \infty} \leq \max \left\{ \|u_0\|_{\Phi, \infty}, \operatorname{ess\,sup}_{\tau \in (0, t)} \int_{\Omega} |\nabla u(\tau)|^2 \right\}, \quad (6.56)$$

for a.e. $t \in (0, T)$.

Proof We set $M_1 := \max\{\|u_0\|_{L^\infty(\Omega)} + 1, \int_{\Omega} |\nabla u_0|^2 + 1\}$ and let $T_1 = T_1(M_1)$ and $c_1 = C_1(M_1)$ be as in Lemma 6.2.7 (i). Then this lemma states that $u_\varepsilon \leq c_1$ in $\Omega \times (0, T_1)$ for all $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$. Accordingly, corresponding to $M_2 = \max\{c_1, \int_{\Omega} |\nabla u_0|^2 + 1\}$, Lemma 6.2.7 (ii) provides $T = T_2(M_2) \in (0, T_1)$ and $c_2 = C_2(M_2) > 0$ such that

$$\int_0^T \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_{\varepsilon}} + \sup_{t \in (0, T)} \int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2 \leq c_2, \quad (6.57)$$

for all $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$, which by $u_{\varepsilon} \leq c_1$ can be turned into a uniform bound on $\|u_{\varepsilon t}\|_{L^2(\Omega \times (0, T))}$, from which it follows by means of the fundamental theorem of calculus that after possibly enlarging c_2 , we also have

$$\|u_{\varepsilon}\|_{C^{\frac{1}{2}}([0, T]; L^2(\Omega))} \leq c_2, \quad (6.58)$$

for such ε .

In order to prove a uniform estimate for u_{ε} from below, locally in space, we follow a standard comparison procedure: Given a compact set $K \subset \Omega$, we pick any smoothly bounded domain $\Omega' \subset \subset \Omega$ such that $K \subset \subset \Omega'$ and let $\phi \in C^2(\bar{\Omega}')$ solve $-\Delta \phi = 1$ in Ω' with $\phi|_{\partial \Omega'} = 0$. Then the lower estimate in (6.19) guarantees that writing $c_3(K) := \frac{1}{2\|\phi\|_{L^{\infty}(\Omega')}} \liminf_{\varepsilon \searrow 0} \inf_K u_{0\varepsilon}$ we can find $\varepsilon_0(K) > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ satisfies $\varepsilon < \varepsilon_0(K)$, we have

$$u_{0\varepsilon}(x) \geq \frac{1}{2} \liminf_{\varepsilon \searrow 0} \inf_K u_{0\varepsilon} \geq c_3(K)\phi(x) \quad \text{for all } x \in \Omega'. \quad (6.59)$$

Letting $z(t) := \frac{c_3(K)}{1+c_3(K)t}$, $t \geq 0$, denote the solution of $z' = -z^2$ with $z(0) = c_3(K)$, we thus find that $\underline{u}(x, t) := z(t)\phi(x)$ satisfies $\underline{u} \leq u_{\varepsilon}$ on the parabolic boundary of $\Omega' \times (0, \infty)$. Since

$$\underline{u}_t - \underline{u} \Delta \underline{u} = z' \phi + z^2 \phi = 0 \quad \text{in } \Omega' \times (0, \infty),$$

and

$$u_{\varepsilon t} - u_{\varepsilon} \Delta u_{\varepsilon} = u_{\varepsilon} \cdot \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \geq 0 \quad \text{in } \Omega \times (0, \infty),$$

we conclude from the comparison principle (see [51] for an adequate version) that $\underline{u} \leq u_{\varepsilon}$ and thus, in particular, that for each $T' > 0$ there exists a suitably small $c_4(K, T') > 0$ such that

$$u_{\varepsilon} \geq c_4(K, T') \quad \text{in } K \times (0, T'), \quad (6.60)$$

holds for all $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ satisfying $\varepsilon < \varepsilon_0(K)$. By positivity of each individual u_{ε} , one can readily verify that upon suitably diminishing $c_4(K, T')$, (6.60) trivially extends so as to actually be valid for all $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$. Now the estimate $u_{\varepsilon} \leq c_1$, (6.57), (6.58) and (6.60) along with standard compactness arguments allow us to extract a subsequence $(\varepsilon_{j_k})_{k \in \mathbb{N}}$ of $(\varepsilon_j)_{j \in \mathbb{N}}$ and a function $u : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that

$$u_\varepsilon \rightarrow u \text{ in } C^0([0, T]; L^2(\Omega)) \text{ and a.e. in } \Omega \times (0, T), \quad (6.61)$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^2_{loc}(\bar{\Omega} \times [0, T]), \quad \text{and} \quad (6.62)$$

$$u_{\varepsilon t} \rightharpoonup u_t \text{ in } L^2(\Omega \times (0, T)), \quad (6.63)$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$. From (6.61), the inequality $u_\varepsilon \leq c_1$ and (6.60), we know that $u \leq c_1$ a.e. in $\Omega \times (0, T)$ and $u \geq c_4(K, T)$ a.e. in $K \times (0, T)$ whenever $K \subset\subset \Omega$. Moreover, since $u_\varepsilon - \varepsilon$ vanishes on $\partial\Omega$, (6.62) implies that $u \in L^2((0, T); W^{1,2}_0(\Omega))$, so that u fulfills all regularity and positivity properties required for a locally positive weak solution in $\Omega \times (0, T)$ in the sense of Definition 6.2.1. In order to verify that u is a weak solution of (6.7)–(6.9) it thus remains to check (6.12). To prepare this, we claim that in addition to (6.62), we also have the strong convergence properties

$$\nabla u_\varepsilon \rightarrow \nabla u \text{ in } L^2_{loc}(\Omega \times [0, T]) \text{ and a.e. in } \Omega \times (0, T), \quad (6.64)$$

as well as

$$\int_\Omega |\nabla u_\varepsilon(x, \cdot)|^2 dx \rightarrow \int_\Omega |\nabla u(x, \cdot)|^2 dx \text{ in } L^1((0, T)), \quad (6.65)$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$. To see (6.64), we let $K \subset\subset \Omega$ be given and fix a nonnegative $\psi \in C^\infty_0(\Omega)$ such that $\psi \equiv 1$ in K . Then

$$\begin{aligned} \int_0^T \int_K |\nabla u_\varepsilon - \nabla u|^2 &\leq \int_0^T \int_\Omega |\nabla u_\varepsilon - \nabla u|^2 \psi \\ &= \int_0^T \int_\Omega \nabla(u_\varepsilon - u) \cdot \nabla u_\varepsilon \cdot \psi - \int_0^T \int_\Omega \nabla u \cdot \nabla(u_\varepsilon - u) \cdot \psi \\ &=: I_1(\varepsilon) - I_2(\varepsilon) \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}, \end{aligned} \quad (6.66)$$

where $I_2(\varepsilon) \rightarrow 0$ as $\varepsilon = \varepsilon_{j_k} \searrow 0$ by (6.62). Using the equation for u_ε , however, after an integration by parts we find that

$$\begin{aligned} I_1(\varepsilon) &= - \int_0^T \int_\Omega (u_\varepsilon - u) \Delta u_\varepsilon \cdot \psi - \int_0^T \int_\Omega (u_\varepsilon - u) \nabla u_\varepsilon \cdot \nabla \psi \\ &= - \int_0^T \int_\Omega (u_\varepsilon - u) \cdot \frac{u_{\varepsilon t}}{u_\varepsilon} \cdot \psi + \int_0^T \int_\Omega (u_\varepsilon - u) \cdot \rho_\varepsilon \left(\int_\Omega |\nabla u_\varepsilon|^2 \right) \cdot \psi \\ &\quad - \int_0^T \int_\Omega (u_\varepsilon - u) \nabla u_\varepsilon \cdot \nabla \psi \\ &=: I_{11}(\varepsilon) + I_{12}(\varepsilon) + I_{13}(\varepsilon) \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}. \end{aligned}$$

Due to (6.61) and (6.62), we have $I_{13}(\varepsilon) \rightarrow 0$, and (6.61) together with (6.57) and Hölder's inequality imply that

$$|I_{12}(\varepsilon)| \leq \left(\int_0^T \int_{\Omega} (u_{\varepsilon} - u)^2 \right)^{\frac{1}{2}} \cdot \left[\int_0^T \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right)^2 \right]^{\frac{1}{2}} \cdot \|\psi\|_{L^2(\Omega)} \rightarrow 0,$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, where we again have used the fact that $\rho_{\varepsilon}(z) \leq z$ for any $z \geq 0$ and all $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$. We now use Hölder's inequality and the local lower estimate (6.60), which in conjunction with (6.57) yields

$$\begin{aligned} |I_{11}(\varepsilon)| &\leq \left(\int_0^T \int_{\Omega} \frac{u_{\varepsilon t}^2}{u_{\varepsilon}} \right)^{\frac{1}{2}} \cdot \left(\int_0^T \int_{\Omega} \frac{(u_{\varepsilon} - u)^2}{u_{\varepsilon}} \cdot \psi^2 \right)^{\frac{1}{2}} \\ &\leq c_2^{\frac{1}{2}} \cdot \frac{\|\psi\|_{L^{\infty}(\Omega)}}{(c_4(\text{supp } \psi, T))^{\frac{1}{2}}} \cdot \left(\int_0^T \int_{\Omega} (u_{\varepsilon} - u)^2 \right)^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, by (6.61). Altogether, we obtain that $I_1(\varepsilon) \rightarrow 0$ and hence, by (6.66), that $\nabla u_{\varepsilon} \rightarrow \nabla u$ in $L^2(K \times (0, T))$ as $\varepsilon = \varepsilon_{j_k} \searrow 0$ for arbitrary $K \subset\subset \Omega$.

Having thus proved (6.64), with the aid of Lemma 6.2.8 we obtain (6.65) as a straightforward consequence:

Given $\delta > 0$, we let $K = K(c_1, c_2, T, \frac{\delta}{4})$ and $\eta > 0$ be the set and the constant provided by Lemma 6.2.8, and employ the convergence asserted by (6.62) to choose $k_0 \in \mathbb{N}$ such that for all $k, l > k_0$ we have $\int_0^T \int_K ||\nabla u_{\varepsilon_k}|^2 - |\nabla u_{\varepsilon_l}|^2| \leq \frac{\delta}{2}$. Then for all $k, l > k_0$,

$$\begin{aligned} \int_0^T \left| \int_{\Omega} |\nabla u_{\varepsilon_k}|^2 - \int_{\Omega} |\nabla u_{\varepsilon_l}|^2 \right| &\leq \int_0^T \int_K ||\nabla u_{\varepsilon_k}|^2 - |\nabla u_{\varepsilon_l}|^2| + \int_0^T \int_{\Omega \setminus K} |\nabla u_{\varepsilon_k}|^2 \\ &\quad + \int_0^T \int_{\Omega \setminus K} |\nabla u_{\varepsilon_l}|^2 \leq \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4}, \end{aligned}$$

and thanks to the completeness of $L^2((0, T))$ we obtain (6.65). We can now proceed to verify that (6.12) holds for all $\varphi \in C_0^{\infty}(\Omega \times (0, T))$. To this end, we multiply (6.22) by $\varphi \in C_0^{\infty}(\Omega \times (0, T))$ and integrate to obtain

$$\int_0^T \int_{\Omega} u_{\varepsilon t} \varphi + \int_0^T \int_{\Omega} |\nabla u_{\varepsilon}|^2 \varphi + \int_0^T \int_{\Omega} u_{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla \varphi = \int_0^T \int_{\Omega} u_{\varepsilon} \cdot \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \cdot \varphi.$$

Here, as $\varepsilon = \varepsilon_{j_k} \searrow 0$ we have

$$\int_0^T \int_{\Omega} u_{\varepsilon t} \varphi \rightarrow \int_0^T \int_{\Omega} u_t \varphi,$$

by (6.63), whereas (6.64) and (6.61) allow us to conclude that

$$\int_0^T \int_{\Omega} |\nabla u_{\varepsilon}|^2 \varphi \rightarrow \int_0^T \int_{\Omega} |\nabla u|^2 \varphi,$$

and

$$\int_0^T \int_{\Omega} u_{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla \varphi \rightarrow \int_0^T \int_{\Omega} u \nabla u \cdot \nabla \varphi,$$

because φ vanishes near $\partial\Omega$ and near $t = T$. Finally,

$$\int_0^T \int_{\Omega} u_{\varepsilon} \cdot \rho_{\varepsilon} \left(\int_{\Omega} |\nabla u_{\varepsilon}|^2 \right) \cdot \varphi \rightarrow \int_0^T \int_{\Omega} u \left(\int_{\Omega} |\nabla u|^2 \right) \cdot \varphi,$$

because of (6.61), (6.65) and the fact that $\rho_{\varepsilon}(z) \rightarrow z$ for all $z \geq 0$ as $\varepsilon \searrow 0$. We thereby see that (6.12) holds and thus infer that u in fact is a weak solution of (6.7)–(6.9) in $\Omega \times (0, T)$. The inequality (6.55) results from Lemma 6.2.9 and the convergence statements. The estimate (6.56) results from Lemma 6.2.10. By (6.57) and (8.92) we have the necessary bounds on gradient and initial value, independent of $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$. Furthermore, for any $t \in [0, T]$ we can find a subsequence $(\varepsilon_{j_k})_{k \in \mathbb{N}}$ of $(\varepsilon_j)_{j \in \mathbb{N}}$ such that

$$\frac{u_{\varepsilon_{j_k}}(t) - \varepsilon_{j_k}}{\Phi} \rightharpoonup^* \frac{u(t)}{\Phi} \quad \text{in } L^{\infty}(\Omega),$$

and finally the same bound as in Lemma 6.2.10 holds for $u(t)$ because

$$\begin{aligned} \|u(t)\|_{\Phi, \infty} &= \left\| \frac{u(t)}{\Phi} \right\|_{\infty} \leq \liminf_{\varepsilon = \varepsilon_{j_k} \searrow 0} \left\| \frac{u_{\varepsilon}(t) - \varepsilon}{\Phi} \right\|_{\infty} \\ &\leq \liminf_{\varepsilon = \varepsilon_{j_k} \searrow 0} \max \left\{ \sup_{0 < \tau < t} \int_{\Omega} |\nabla u_{\varepsilon}(\tau)|^2, \|u_{0\varepsilon} - \varepsilon\|_{\Phi, \infty} \right\} \\ &\leq \liminf_{\varepsilon = \varepsilon_{j_k} \searrow 0} \max \left\{ \sup_{0 < \tau < t} \int_{\Omega} |\nabla u_{\varepsilon}(\tau)|^2, \|u_0\|_{\Phi, \infty} + \varepsilon \right\} \\ &\leq \max \left\{ \operatorname{ess\,sup}_{0 < \tau < t} \int_{\Omega} |\nabla u(\tau)|^2, \|u_0\|_{\Phi, \infty} \right\}, \end{aligned}$$

where for the last inequality we relied on the pointwise a.e. convergence of $\int_{\Omega} |\nabla u_{\varepsilon}|^2$ in $(0, T)$, due to (6.65) valid along a subsequence. $\square$

We are now in the position to prove Theorem 6.2.3, which asserts the existence of a locally positive weak solution and $T_{\max} \in (0, \infty]$ such that the solution blows up at $T_{\max}$ or exists globally.

Proof of Theorem 6.2.3 According to the statement of Lemma 6.2.11 there exists $T > 0$ such that (6.7)–(6.9) possesses a locally positive weak solution u on $\Omega \times (0, T)$ which satisfies (6.15) and (6.16) for a.e. $t \in (0, T)$. Hence, the set

$$S := \left\{ \tilde{T} > 0 \mid \text{there exists a locally positive solution } u \text{ to (6.7)–(6.9) on } \Omega \times (0, \tilde{T}) \right. \\ \left. \text{satisfying (6.15) and (6.16) for a.e. } t \in (0, \tilde{T}) \right\},$$

is not empty and

$$T_{max} = \sup S \in (0, \infty],$$

is well-defined. Assume that $T_{max} < \infty$ and $\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty$.

This implies the existence of a constant $M > 0$ such that $u \leq M$ and hence, due to (6.15), also that there is $C > 0$ with $\int_\Omega |\nabla u|^2 \leq C$ on $[0, T_{max})$. Lemma 6.2.11 provides $T > 0$ such that for any initial data u_0 satisfying $u_0 \leq M$, $\int_\Omega |\nabla u_0|^2 \leq C$, a locally positive weak solution existing on $\Omega \times (0, T)$ can be constructed.

Choose $t_0 \in (T_{max} - \frac{T}{2}, T_{max})$ such that $u(x, t_0) \leq M$ and $\int_\Omega |\nabla u(x, t_0)|^2 \leq C$ and such that u satisfies (6.15) and (6.16) at $t = t_0$.

Let v denote the corresponding solution with initial value $u(\cdot, t_0)$ and define

$$\widehat{u}(x, t) = \begin{cases} u(x, t), & x \in \Omega, t < t_0 \\ v(x, t - t_0), & x \in \Omega, t \in (t_0, t_0 + T) \end{cases}.$$

Then $\widehat{u}$ is a solution of (6.7)–(6.9), and (6.15) and (6.16) obviously hold for a.e. $t \in (0, t_0)$, whereas for $t \in (t_0, t_0 + T)$ we have

$$\begin{aligned} & \int_\Omega |\nabla \widehat{u}(\cdot, t)|^2 \\ & \leq \int_\Omega |\nabla u(t_0)|^2 \times \\ & \quad \times \exp \left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (t_0, t)} \int_\Omega \widehat{u}(\cdot, \tau) \right) \left(\int_{\Omega'} \phi \ln \widehat{u}(\cdot, t) - \int_{\Omega'} \phi \ln u(\cdot, t_0) + \int_{t_0}^t \int_{\Omega'} \widehat{u} \right) \right] \\ & \leq \int_\Omega |\nabla u_0|^2 \exp \left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (0, t_0)} \int_\Omega u(\cdot, \tau) \right) \left(\int_{\Omega'} \phi \ln u(\cdot, t_0) - \int_{\Omega'} \phi \ln u_0 + \int_0^{t_0} \int_{\Omega'} u \right) \right] \times \\ & \quad \times \exp \left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (t_0, t)} \int_\Omega \widehat{u}(\cdot, \tau) \right) \left(\int_{\Omega'} \phi \ln \widehat{u}(\cdot, t) - \int_{\Omega'} \phi \ln u(\cdot, t_0) + \int_{t_0}^t \int_{\Omega'} \widehat{u} \right) \right] \\ & \leq \int_\Omega |\nabla u_0|^2 \exp \left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (0, t)} \int_\Omega \widehat{u}(\cdot, \tau) \right) \right. \\ & \quad \cdot \left. \left(\int_{\Omega'} \phi \ln u(\cdot, t_0) - \int_{\Omega'} \phi \ln u_0 + \int_0^{t_0} \int_{\Omega'} u + \int_{\Omega'} \phi \ln \widehat{u}(\cdot, t) - \int_{\Omega'} \phi \ln u(\cdot, t_0) + \int_{t_0}^t \int_{\Omega'} \widehat{u} \right) \right] \\ & = \int_\Omega |\nabla u_0|^2 \exp \left[\frac{1}{2C_{\Omega'}} \left(\sup_{\tau \in (0, t)} \int_\Omega \widehat{u}(\cdot, \tau) \right) \left(\int_{\Omega'} \phi \ln \widehat{u}(\cdot, t) - \int_{\Omega'} \phi \ln u_0 + \int_0^t \int_{\Omega'} u \right) \right]. \end{aligned}$$

Also, for a.e. $t \in (0, t_0 + T)$,

$$\begin{aligned}
\|\widehat{u}(\cdot, t)\|_{\Phi, \infty} &\leq \max \left\{ \|u(\cdot, t_0)\|_{\Phi, \infty}, \sup_{\tau \in (t_0, t)} \int_{\Omega} |\nabla \widehat{u}(\cdot, \tau)|^2 \right\} \\
&\leq \max \left\{ \max \left\{ \|u_0\|_{\Phi, \infty}, \sup_{\tau \in (0, t_0)} \int_{\Omega} |\nabla u(\cdot, \tau)|^2 \right\}, \sup_{\tau \in (t_0, t)} \int_{\Omega} |\nabla \widehat{u}(\cdot, \tau)|^2 \right\} \\
&\leq \max \left\{ \|u_0\|_{\Phi, \infty}, \sup_{\tau \in (0, t)} \int_{\Omega} |\nabla \widehat{u}(\cdot, \tau)|^2 \right\}.
\end{aligned}$$

Thus $\widehat{u}$ is defined on $(0, T_{\max} + \frac{T}{2})$, contradicting the definition of $T_{\max}$. $\square$

As a direct consequence of (6.16) we obtain that finite-time gradient blow-up cannot occur. More precisely, we have the following.

Corollary 6.2.12 *Let u and $T_{\max}$ be as given by Theorem 6.2.3. If $\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$, then also*

$$\limsup_{t \nearrow T_{\max}} \int_{\Omega} |\nabla u(x, t)|^2 dx = \infty.$$

Combining now Corollary 6.2.12 with the estimate (6.15), we can conclude that if finite-time L^∞ -blow-up occurs, then also L^1 -blow-up takes place at the same finite time.

Corollary 6.2.13 *Let u and $T_{\max}$ be as given by Theorem 6.2.3. If $\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$, then also*

$$\limsup_{t \nearrow T_{\max}} \int_{\Omega} u(x, t) dx = \infty.$$

6.2.2 Global Existence Versus Blow-Up

A natural next topic appears to consist in deriving conditions on the initial data which ensure that the local solutions given by Theorem 6.2.3 either exist for all times, or blow-up in finite time. Here in view of the essentially cubic character of the production term in (6.7) it is not surprising that this may dominate the smoothing effect of the merely quadratic-type diffusion term when the initial data are suitably large in an adequate sense; precedent works indicate that indeed such intuitive considerations are appropriate in related non-degenerate and degenerate parabolic equations with local reaction terms [39, 41, 47, 52]. As a remarkable feature of the precise structure of this interplay in (6.7), we shall see that actually a complete classification of all initial data in this respect is possible, exclusively involving the size of the total initial mass $m := \int_{\Omega} u_0$ as the decisive quantity: In fact, the second of our main results identifies the value $m = 1$ to be critical with regard to global solvability, and moreover gives some basic information on the asymptotic behavior of solutions.

Theorem 6.2.14 *Let u_0 satisfy (H1)–(H3), and let u and $T_{\max}$ denote the corresponding locally positive weak solution of (6.7)–(6.9), as well as its maximal time of existence, provided by Theorem 6.2.3.*

(i) *If $\int_{\Omega} u_0 < 1$, then $T_{\max} = \infty$ and*

$$\int_{\Omega} u(x, t) dx \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

(ii) *Suppose that $\int_{\Omega} u_0 = 1$. Then $T_{\max} = \infty$ and*

$$\int_{\Omega} u(x, t) dx = 1 \quad \text{for all } t > 0.$$

(iii) *In the case $\int_{\Omega} u_0 dx > 1$, we have $T_{\max} < \infty$ and*

$$\limsup_{t \nearrow T_{\max}} \int_{\Omega} u(x, t) dx = \infty.$$

For the new proof of the above theorem we need the following auxiliary result

Lemma 6.2.15 *For any weak solution u of (6.7)–(6.9) on $[0, T]$, then*

$$y(t) = \int_{\Omega} u(x, t) dx, \quad t \in [0, T], \quad (6.67)$$

defines an absolutely continuous function $y: [0, T] \rightarrow \mathbb{R}$ that satisfies

$$y'(t) = (y(t) - 1) \int_{\Omega} |\nabla u(x, t)|^2 dx,$$

for almost every $t \in (0, T)$.

Proof We will show that whenever $0 < s < t < T$,

$$y(t) - y(s) = \int_s^t \left((y(\tau) - 1) \int_{\Omega} |\nabla u(x, \tau)|^2 dx \right) d\tau, \quad (6.68)$$

where absolute continuity follows from the representation as integral and the assertion about the derivative is a direct consequence of division by $t - s$ and passing to the limit $s \rightarrow t$.

Let $0 < s < t < T$ and $0 < \delta < \min\{s, T - t\}$. Define the function $\chi: \mathbb{R} \rightarrow \mathbb{R}$ by setting:

$$\chi(\tau) = \begin{cases} 0, & \tau < s - \delta, \\ 1 + \frac{\tau-s}{\delta}, & s - \delta \leq \tau < s, \\ 1, & s \leq \tau < t, \\ 1 - \frac{\tau-t}{\delta}, & t \leq \tau < t + \delta, \\ 0, & \tau \geq t + \delta. \end{cases}$$

Then, according to standard approximation arguments, $\varphi(x, t) := \chi(t)$ defines an admissible test function for (6.11) and we obtain

$$-\frac{1}{\delta} \int_{s-\delta}^s \int_{\Omega} u + \frac{1}{\delta} \int_t^{t+\delta} \int_{\Omega} u + \int_{s-\delta}^{t+\delta} \int_{\Omega} |\nabla u|^2 \varphi = \int_{s-\delta}^{t+\delta} \left(\int_{\Omega} u \varphi \right) \cdot \left(\int_{\Omega} |\nabla u|^2 \right).$$

Since $u \in C_{loc}([0, T), L^2(\Omega))$, we have

$$\frac{1}{\delta} \int_t^{t+\delta} \int_{\Omega} u \rightarrow y(t) \quad \text{and} \quad \frac{1}{\delta} \int_{s-\delta}^s \int_{\Omega} u \rightarrow y(s),$$

as $\delta \searrow 0$.

Also by Lebesgue's dominated convergence theorem,

$$\int_{s-\delta}^{t+\delta} \int_{\Omega} |\nabla u|^2 \varphi \rightarrow \int_s^t \int_{\Omega} |\nabla u|^2,$$

and

$$\int_{s-\delta}^{t+\delta} \left(\int_{\Omega} u \varphi \right) \cdot \left(\int_{\Omega} |\nabla u|^2 \right) \rightarrow \int_s^t \left(\int_{\Omega} u \right) \cdot \left(\int_{\Omega} |\nabla u|^2 \right),$$

as $\delta \searrow 0$. Hence, (6.68) holds. $\square$

Proof of Theorem 6.2.14 (i) In the case of subcritical initial mass Lemma 6.2.15 shows that y as defined in (6.67) is decreasing, which by Corollary 6.2.13 entails global existence, and from the nonnegativity of y we derive that $y(t) \rightarrow c$ as $t \rightarrow \infty$ for some $c \geq 0$. Note that Poincaré's and Hölder's inequalities imply that for some $C_P > 0$ we have

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{C_P} \int_{\Omega} u^2 dx \geq \frac{1}{C_P |\Omega|} \left(\int_{\Omega} u dx \right)^2 = \frac{1}{C_P |\Omega|} y^2 \quad \text{on } (0, \infty),$$

and hence Lemma 6.2.15, due to the negativity of $y(t) - 1$, entails that

$$y'(t) \leq (y(t) - 1) \frac{1}{C_P |\Omega|} y^2(t) \leq -\frac{1 - y(0)}{C_P |\Omega|} y^2(t) \leq -\frac{1 - y(0)}{C_P |\Omega|} c^2,$$

for almost every $t > 0$. This would lead to a contradiction to the nonnegativity of $y(t)$ if c were positive, whence actually $c = 0$.

(ii) If $\int_{\Omega} u_0 = 1$, then Lemma 6.2.15 implies that

$$y(t) - 1 = \int_0^t \left[(y(s) - 1) \int_{\Omega} |\nabla u(x, s)|^2 dx \right] ds,$$

and by virtue of Gronwall's lemma we conclude $y(t) - 1 \equiv 0$ throughout the time interval on which the solution exists, which combined with Corollary 6.2.13 also implies global existence.

(iii) In the case when the total mass is supercritical initially, Lemma 6.2.15 entails that y is nondecreasing, and again Poincaré's and Hölder's inequalities imply that

$$y'(t) \geq \frac{y(0) - 1}{C_P |\Omega|} y^2(t) \quad \text{for a.e. } t \in [0, T_{\max}),$$

with some $C_P > 0$. Now let z denote the solution to

$$z'(t) = \frac{y(0) - 1}{C_P |\Omega|} z(t)^2, \quad z(0) = z_0,$$

for some $1 < z_0 < y(0)$, defined up to its maximal existence time $T_0 > 0$. Then $T := T_{\max} < T_0$, because $y \geq z$, and the assertion follows by Theorem 6.2.3 in combination with Corollary 6.2.13. $\square$

Remark 6.2.3 In case (ii) of Theorem 6.2.14 there actually holds that the global-in-time solution u converges towards the unique steady-state which is a Nash equilibrium for the corresponding evolutionary dynamic game. In particular there holds

$$\lim_{t \rightarrow +\infty} \left\| u(\cdot, t) - \frac{\Phi(\cdot)}{\int_{\Omega} \Phi dx} \right\|_{W_0^{1,2}(\Omega)} = 0,$$

where Φ denotes the solution of the following problem

$$-\Delta \Phi = 1 \quad \text{in } \Omega, \quad \Phi = 0, \quad \text{on } \partial\Omega,$$

see Theorem 1 in [28].

In numerous related equations, involving either linear or degenerate diffusion, blow-up driven by local superlinear production terms is known to occur in thin spatial sets only which in radial settings typically reduce to single points [14, 17, 41]. Only few exceptional situations detected in the literature lead to *regional* or even *global* blow-up, thus referring to cases in which $|\mathcal{B}| > 0$ or even $\mathcal{B} = \overline{\Omega}$ (cf. [15, 16, 26, 47, 53], for instance). In cases of sources which at least partially consist of non-local terms, blow-up in sets of positive measure may occur if the relative size of a possibly contained local contribution at large densities is predominant, as compared to the strength of the respective diffusion term [12, 29, 31, 44, 46, 50]. Our main result in this direction will reveal that any of our non-global solutions in fact blow

up globally in space, thus indicating a certain balance in the competition of diffusion and non-local production in Eq. (6.7):

Theorem 6.2.16 *Suppose that $\int_{\Omega} u_0 dx > 1$, and let u denote the locally positive weak solution of (6.7)–(6.9) from Theorem 6.2.3. Then u blows up globally in the sense that its blow-up set satisfies $\mathcal{B} = \overline{\Omega}$.*

Proof Assume to the contrary that the closed set $\mathcal{B}$ is strictly contained in $\overline{\Omega}$. Then there exists a smoothly bounded subdomain $\Omega' \subset \Omega \setminus \mathcal{B}$ such that u is bounded in $\Omega' \times (0, T_{max})$. Let ϕ be a solution to $-\Delta\phi = 1$ in Ω' , $\phi = 0$ on $\partial\Omega'$.

Consider $T' < T_{max}$. Due to the local positivity of u we have $\frac{\phi}{u} \in L^{\infty}(\Omega \times (0, T'))$ and $\nabla \frac{\phi}{u} = \frac{\nabla\phi}{u} - \frac{\phi}{u^2} \nabla u \in L^2(\Omega' \times (0, T'))$ and hence $\frac{\phi}{u} \in L^2((0, T'), W_0^{1,2}(\Omega')) \cap L^{\infty}(\Omega \times (0, T')) \subset L^2((0, T'), W_0^{1,2}(\Omega)) \cap L^{\infty}(\Omega \times (0, T'))$. Therefore, it can readily be verified by approximation arguments that it is possible to use $\varphi = \frac{\phi}{u}$ as a test function in (6.12), which then leads to

$$\int_0^t \int_{\Omega'} \frac{u_t}{u} \phi dx ds + \int_0^t \int_{\Omega'} \nabla u \cdot \nabla \phi dx ds = \int_0^t \left(\int_{\Omega'} \phi dx \right) \cdot \left(\int_{\Omega} |\nabla u|^2 dx \right) ds,$$

for any $t \in (0, T_{max})$. Hence, with $C_{\Omega'} := \int_{\Omega'} \phi$ and since $\partial_\nu \phi|_{\partial\Omega'} \leq 0$,

$$\int_{\Omega'} \phi \ln u(t) dx - \int_{\Omega'} \phi \ln u_0 dx - \int_0^t \int_{\Omega'} u \cdot \Delta \phi dx ds \geq C_{\Omega'} \int_0^t \int_{\Omega} |\nabla u|^2 dx ds,$$

that is

$$\int_0^t \int_{\Omega'} u dx ds + \int_{\Omega'} \phi \ln u(t) dx - \int_{\Omega'} \phi \ln u_0 dx \geq C_{\Omega'} h(t), \quad (6.69)$$

where $h(t) := \int_0^t \int_{\Omega} |\nabla u(x, s)|^2 dx ds$ and where – due to the choice of Ω' – the left hand side is bounded from above.

On the other hand, from Lemma 6.2.15 we know that

$$\frac{y'(t)}{y(t) - 1} = \int_{\Omega} |\nabla u|^2 dx,$$

for $y(t) = \int_{\Omega} u(x, t) dx$. Therefore

$$h(t) = \int_0^t \int_{\Omega} |\nabla u|^2 dx ds = \int_0^t \frac{y'(\tau)}{y(\tau) - 1} d\tau = \ln(y(t) - 1) - \ln(y(0) - 1) = \ln \frac{\int_{\Omega} u(x, t) dx - 1}{\int_{\Omega} u_0 dx - 1},$$

and, by Theorem 6.2.14 (iii), $\limsup_{t \nearrow T_{max}} h(t) = \infty$, contradicting the boundedness of the left hand side of (6.69). $\square$

References

- Alt, H.W., Luckhaus, S.: Quasilinear elliptic-parabolic differential equations. *Math. Z.* **183**, 311–341 (1983)
- Angenent, S.B., Fila, M.: Interior gradient blow-up in a semilinear parabolic equation. *Differ. Integral Equ.* **9**(5), 865–877 (1996)
- Aronson, D.G.: The porous medium equation. *Nonlinear Diffusion Problems. Lecture Notes in Mathematics*, vol. 1224, pp. 1–46. Springer, Berlin (1986)
- Arrieta, J.M., Rodríguez-Bernal, A., Souplet, P.: Boundedness of global solutions for nonlinear parabolic equations involving gradient blow-up phenomena *Ann. Scu. Norm. Sup. Pisa Cl. Sci.* **3**, 1–15 (2004)
- Bertsch, M., Peletier, L.A.: A positivity property of solutions of nonlinear diffusion equations. *J. Differ. Eq.* **53**, 30–47 (1984)
- Bertsch, M., Ughi, M.: Positivity properties of viscosity solutions of a degenerate parabolic equation. *Nonlinear Anal. TMA* **14**(7), 571–592 (1990)
- Bertsch, M., Dal Passo, R., Ughi, M.: Discontinuous “viscosity” solutions of a degenerate parabolic equation. *Trans. Am. Math. Soc.* **320**(2), 779–798 (1990)
- Bomze, I.: Dynamical aspects of evolutionary stability. *Mon. Math.* **110**, 189–206 (1990)
- Deng, W., Duan, Z., Xie, C.: The blow-up rate for a degenerate parabolic equation with a non-local source. *J. Math. Anal. Appl.* **264**, 577–597 (2001)
- Dlotko, T.: Examples of parabolic problems with blowing-up derivatives. *J. Math. Anal. Appl.* **154**, 226–237 (1991)
- Dugatin, L.A., Reeve, H.K. (eds.): *Game Theory and Animal Behaviour*. Oxford UP, Oxford (1998),
- Du, L., Xiang, Z.: A further blow-up analysis for a localized porous medium equation. *Appl. Math. Comput.* **179**, 200–208 (2006)
- Friedman, A.: *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs (1964)
- Friedman, A., McLeod, B.: Blow-up of positive solutions of semilinear heat equations. *Ind. Univ. Math. J.* **34**, 425–447 (1985)
- Friedman, A., McLeod, B.: Blow-up of solutions of nonlinear degenerate parabolic equations. *Arch. Rational Mech. Anal.* **96**, 55–80 (1987)
- Galaktionov, V.A., Vázquez, J.L.: Regional blow up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation. *SIAM J. Math. Anal.* **24**, 1254–1276 (1993)
- Giga, Y., Kohn, R.V.: Nondegeneracy of blowup for semilinear heat equations. *Commun. Pure Appl. Math.* **42**, 845–884 (1989)
- Haccou, P., Iwasa, Y.: Optical mixed strategies in stochastic environments. *Theor. Popul. Biol.* **47**, 212–243 (1995)
- Haccou, P., Iwasa, Y.: Robustness of optimal mixed strategies. *J. Math. Biol.* **36**, 485–496 (1998)
- Hofbauer, J., Sigmund, K.: *Evolutionary game dynamics*. *Bull. Am. Math. Soc.* **40**, 479–519 (2003)
- Kavallaris, N.I., Lankeit, J., Winkler, M.: On a degenerate non-local parabolic problem describing infinite dimensional replicator dynamics. *SIAM J. Math. Anal.* **49**(2), 954–983 (2017)
- Kravvaritis, C., Papanicolaou, V.G.: Singular equilibrium solutions for a replicator dynamics model. *Electron. J. Differ. Equ.* **87**, 1–8 (2011)
- Kravvaritis, D., Papanicolaou, V.G., Yannacopoulos, A.N.: Similarity solutions for a replicator dynamics equation. *Ind. Univ. Math. J.* **57**, 1929–1946 (2008)
- Kravvaritis, D., Papanicolaou, V.G., Xepapadeas, A., Yannacopoulos, A.N.: On a class of operator equations arising in infinite dimensional replicator dynamics. *Nonlinear Anal. RWA* **11**, 2537–2556 (2010)
- Krugman, P.: *The self-organizing economy. Mitsui Lectures in Economics*, Wiley-Blackwell, Cambridge (1996)

26. Lacey, A.A.: Global blow-up of a nonlinear heat equation. *Proc. Roy. Soc. Edinb. Sect. A* **104**, 161–167 (1986)
27. Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: *Linear and Quasi-linear Equations of Parabolic Type*. AMS, Providence (1968)
28. Lankeit, J.: Equilibration of unit mass solutions to a degenerate parabolic equation with a nonlocal gradient nonlinearity. *Nonlinear Anal.* **135**, 236–248 (2016)
29. Liang, F., Li, Y.: Blow-up for a nonlocal parabolic equation. *Nonlinear Anal.* **71**, 3551–3562 (2009)
30. Li, Y., Souplet, Ph: Single-point gradient blow-up on the boundary for diffusive Hamilton-Jacobi equations in planar domains. *Commun. Math. Phys.* **293**, 499–517 (2010)
31. Liu, Q., Li, Y., Gao, H.: Uniform blow-up rate for a nonlocal degenerate parabolic equations. *Nonlinear Anal.* **66**, 881–889 (2007)
32. Luckhaus, S., Dal Passo, R.: A Degenerate diffusion equation not in divergence form. *J. Differ. Equ.* **69**, 1–14 (1987)
33. Smith, J.M.: *Evolution and The Theory of Games*. Cambridge University Press, Cambridge (1982)
34. Oechssler, J., Riedel, F.: Evolutionary dynamics on infinite strategy spaces. *Econ. Theory* **17**, 141–162 (2001)
35. Oechssler, J., Riedel, F.: On the dynamic foundation of evolutionary stability in continuous models. *J. Econ. Theory* **107**, 223–252 (2002)
36. Papanicolaou, V.G., Smyrlis, G.: Similarity solutions for a multi-dimensional replicator dynamics equation. *Nonlinear Anal. TMA* **71**, 3185–3196 (2009)
37. Papanicolaou, V.G., Vasilakopoulou, K.: Similarity solutions of a replicator dynamics equation associated to a continuum of pure strategies. *Electron. J. Differ. Equ.* **231**, 1–16 (2015)
38. Papanicolaou, V.G., Vasilakopoulou, K.: Similarity solutions of a multidimensional replicator dynamics integrodifferential equation. *J. Dyn. Games* **3**, 51–74 (2016)
39. Quittner, P., Souplet, Ph: *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*. Birkhäuser Advanced Texts, Basel (2007)
40. Schuster, P., Sigmund, K.: Replicator dynamics. *J. Theor. Biol.* **100**, 533–538 (1983)
41. Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., Mihailov, A.P.: *Blow-Up in Quasilinear Parabolic Equations*. De Gruyter Expositions in Mathematics, Berlin (1995)
42. Sigmund, K.: *Games of Life*. Penguin, Harmondsworth (1993)
43. Souplet, Ph: Blow-up in nonlocal reaction-diffusion equations. *SIAM J. Math. Anal.* **29**, 1301–1334 (1998)
44. Souplet, Ph: Uniform blow-up profiles and boundary behaviour for diffusion equations with nonlocal nonlinear source. *J. Differ. Equ.* **153**, 374–406 (1999)
45. Souplet, Ph: Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions. *Differ. Int. Equ.* **15**, 237–256 (2002)
46. Souplet, Ph: Uniform blow-up profile and boundary behaviour for a non-local reaction-diffusion equation with critical damping. *Math. Methods Appl. Sci.* **27**, 1819–1829 (2004)
47. Stinner, C., Winkler, M.: Boundedness vs. blow-up in a degenerate diffusion equation with gradient nonlinearity. *Indiana Univ. Math. J.* **56**(5), 2233–2264 (2007)
48. Stinner, C., Winkler, M.: Finite time vs. infinite time gradient blow-up in a degenerate diffusion equation. *Indiana Univ. Math. J.* **57**(5), 2321–2354 (2008)
49. Taylor, P.D., Jonker, L.: Evolutionary stable strategies and game dynamics. *Math. Biosci.* **40**, 145–156 (1978)
50. Wang, M., Wang, Y.: Properties of positive solutions for non-local reaction-diffusion problems. *Math. Methods Appl. Sci.* **19**(14), 1141–1156 (1996)
51. Wiegner, M.: A degenerate diffusion equation with a nonlinear source term. *Nonlinear Anal. TMA* **28**, 1977–1995 (1997)
52. Winkler, M.: A critical exponent in a degenerate parabolic equation. *Math. Methods Appl. Sci.* **25**(11), 911–925 (2002)
53. Winkler, M.: Blow-up of solutions to a degenerate parabolic equation not in divergence form. *J. Differ. Equ.* **192**(2), 445–474 (2003)

54. Winkler, M.: Boundary behaviour in strongly degenerate parabolic equations. *Acta Math. Univ. Comen.* **72**(1), 129–139 (2003)
55. Winkler, M.: Propagation vs. constancy of support in the degenerate parabolic equation $u_t = f(u)\Delta u$, *Rend. Istit. Mat. Univ. Trieste* **XXXVI**, 1–15 (2004)
56. Winkler, M.: Large time behaviour and stability of equilibria of degenerate parabolic equations. *J. Dyn. Differ. Equ.* **17**(2), 331–351 (2005)
57. Wloka, J.: *Partial Differential Equations*. Cambridge University Press, Cambridge (1987)

Chapter 7

A Non-local Model Arising in Chemotaxis

Abstract The current chapter deals with the biological phenomenon of chemotaxis. In the first place, a parabolic-parabolic Keller–Segel system is considered which describes the movement of some cell population towards a chemo-attractant produced by the population itself. Next, this version of Keller–Segel system is reduced to a non-local parabolic problem for the concentration of chemo-attractant in the case the chemo-attractant diffuses much faster than the cell population. Using the variational structure of the derived non-local parabolic problem we obtain some appropriate a priori estimates permitting us to derive global-in-time solutions when the total cell population is below the threshold 8π . It is also proven that the global-in-time solution converges to the unique steady state solution in the radial symmetric case. When the cell population exceeds the threshold 8π then all the radially symmetric solutions exhibit finite-time blow-up on the origin of the considered sphere, i.e. single-point blow-up occurs.

7.1 Derivation of the Non-local Model

Chemotaxis is the movement of a motile cell or organism, in a direction corresponding to a gradient of increasing or decreasing concentration of a particular substance. Out of the many mathematical models that have been proposed to deal with particular aspects of chemotaxis, the one introduced by Keller and Segel in 1970 (cf. [17]) has received particular attention. The so called Keller–Segel model consists of two equations, describing the evolution of the population density $u(x, t)$ of motile cells (or organisms), and the concentration $v(x, t)$ of a chemical attracting substance, in a bounded domain $\Omega \subset \mathbb{R}^N$, where $N = 2, 3$, and in a time interval $[0, T]$:

$$\varepsilon u_t = \nabla \cdot (D_1 \nabla u - \chi u \nabla v) \quad \text{in } \Omega \times (0, T), \quad (7.1)$$

$$\tau v_t = \Delta v - av + u \quad \text{in } \Omega \times (0, T), \quad (7.2)$$

$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = v = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (7.3)$$

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega, \quad (7.4)$$

where ε, τ are positive constants.

More precisely the first equation describes the random (Brownian) diffusion of the population of cells u , which is biased in the direction of a drift velocity, proportional to the gradient of the concentration of the chemo-attractant ∇v . The diffusion coefficient is denoted by $D_1 > 0$ and the proportionality coefficient of the drift (mobility parameter) is denoted by $\chi > 0$. According to the second equation, the chemo-attractant v , which is directly emitted by the cells, diffuses with a diffusion coefficient $D_2 = 1/\tau > 0$ on the substrate, while is generated proportionally to the density of cells and at the same time is degraded with a rate equal to $a/\tau \geq 0$.

A natural boundary condition, since it guarantees the conservation of total mass, is the no-flux type condition for u , namely the first condition of (7.3) where ν stands for the outer unit normal vector at $\partial\Omega$. As for v , a Dirichlet type boundary condition is assumed. Note that the parabolic system (7.1)–(7.4) preserves the nonnegativity of the initial conditions, i.e. $u, v \geq 0$ for $t > 0$, which is also expected to be true for the physical problem. For simplicity, D_1, χ are considered to be constant and under suitable scaling can be taken $D_1, \chi = 1$.

In view of experimental facts, the degradation of the chemical v is rather small so it can be taken $a = 0$ and therefore we actually focus on the following system

$$\varepsilon u_t = \nabla \cdot (D_1 \nabla u - \chi u \nabla v) \quad \text{in } \Omega \times (0, T), \quad (7.5)$$

$$\tau v_t = \Delta v + u \quad \text{in } \Omega \times (0, T), \quad (7.6)$$

$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = v = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (7.7)$$

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega. \quad (7.8)$$

For $(\varepsilon, \tau) = (1, 0)$ we have an interesting case of the above system, when actually the chemo-attractant diffuses much faster than cell population. In that case system (7.5)–(7.8) has been studied thoroughly in various research papers, see [4, 11, 21] to name a few of them. Additionally, the same limiting case for the following chemotaxis system

$$\varepsilon u_t = \nabla \cdot (\nabla u - u \nabla v) \quad \Omega \times (0, T), \quad (7.9)$$

$$\tau v_t = \Delta v + u - \frac{1}{|\Omega|} \int_{\Omega} u \quad \text{in } \Omega \times (0, T), \quad (7.10)$$

$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (7.11)$$

$$\int_{\Omega} v(x, t) dx = 0 \quad \text{for } t \in (0, T), \quad (7.12)$$

$$u|_{t=0} = u_0(x) \geq 0, \quad v|_{t=0} = v_0(x) \quad \text{in } \Omega, \quad (7.13)$$

has been investigated in [27, 29, 32].

On the other hand, the other limiting case $(\varepsilon, \tau) = (0, 1)$, which was first considered by Wolansky [34] is not so thoroughly studied. The latter limit actually describes the situation when chemo-attractant diffuses much faster compared to the cell population. In that case (7.5) together with non-flux boundary condition (7.7) entail

$$\nabla u - u \nabla v = 0 \quad \text{in } \Omega,$$

which actually gives

$$u = \frac{\lambda e^v}{\int_{\Omega} e^v dx}, \quad (7.14)$$

where λ is the total (conserved) mass of the population u , i.e.

$$\lambda = \int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx.$$

Next by substituting (7.14) into (7.6) we derive the following non-local parabolic equation

$$v_t = \Delta v + \frac{\lambda e^v}{\int_{\Omega} e^v dx} \quad \text{in } \Omega \times (0, T), \quad (7.15)$$

for the chemo-attractant, which is also associated with initial and boundary conditions

$$v = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (7.16)$$

$$v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega. \quad (7.17)$$

In the current chapter we will investigate the long-time behavior of problem (7.15)-(7.17) in the two-dimensional case, i.e. when $\Omega \subset \mathbb{R}^2$ which is a natural setting for species raised in a cell-culture dish. We will mainly focus on its blow-up time behavior which is actually linked with the case where chemo-attractant's concentration becomes quite high.

7.2 Mathematical Analysis

7.2.1 Preliminaries

Since $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, then the functional

$$\mathcal{J}_{\lambda}(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \left(\int_{\Omega} e^v \right) + \lambda (\log \lambda - 1),$$

is C^2 for any $v \in H_0^1(\Omega)$, and we obtain the Trudinger–Moser inequality indicated by

$$\inf_{v \in H_0^1(\Omega)} \mathcal{J}_{8\pi}(v) > -\infty. \quad (7.18)$$

Equation (7.15) is actually the gradient flow of functional $\mathcal{J}$, i.e. there holds

$$v_t = -\delta \mathcal{J}_\lambda(v) \quad \text{in } X = H_0^1(\Omega), \quad (7.19)$$

where $\delta \mathcal{J}_\lambda$ stands for the functional derivative of $\mathcal{J}_\lambda$, hence

$$\frac{d}{dt} \mathcal{J}_\lambda(v(\cdot, t)) = -\|v_t(\cdot, t)\|_2^2. \quad (7.20)$$

Furthermore the mapping

$$v \in X \mapsto \frac{e^v}{\int_\Omega e^v} \in X,$$

is Lipschitz continuous on each bounded set of X , and therefore, (7.15) is well-posed in X . In particular, given initial value $v_0 \in X$, there exists a unique semi-group solution $v = v(\cdot, t) \in X$ locally in time, and henceforth its maximum existence time is denoted by $T = T_{\max} \in (0, +\infty]$, which is estimated from below by $\|\nabla v_0\|_2$. Then, due to (7.18) and via the parabolic regularity we obtain for $\lambda < 8\pi$

$$T = +\infty \quad \text{and} \quad \sup_{t \geq 0} \|v(\cdot, t)\|_\infty < +\infty, \quad (7.21)$$

whereas

$$\lim_{t \uparrow T} \|v(\cdot, t)\|_\infty = +\infty, \quad (7.22)$$

if $T < +\infty$. In the latter case we can define the set

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \rightarrow x_0, \exists t_k \uparrow T, \text{ s.t. } v(x_k, t_k) \rightarrow +\infty\} \neq \emptyset,$$

which is called the blow-up set of v . The blow-up set $\mathcal{S}$ can be also defined in case where (7.22) is fulfilled for $T = \infty$.

Note also that

$$v(\cdot, t) \geq \inf_\Omega v_0,$$

by the comparison principle, and henceforth, $v_0 \geq 0$ is assumed without loss of generality.

The parabolic Brezis–Merle’s inequality [9, 35], on the other hand, is concerned with the linear parabolic equation

$$v_t = \Delta v + f(x, t) \quad \text{in } \Omega \times (0, T), \quad v|_{\partial\Omega} = 0, \quad v|_{t=0} = 0,$$

where Ω is again a two-dimensional bounded domain with smooth boundary $\partial\Omega$. Its global form assures that any $\delta > 0$ admits $p > 1$ and $C > 0$ such that

$$\sup_{t \in (0, T)} \|f(\cdot, t)\|_1 < 4\pi - \delta \Rightarrow \sup_{t \in (0, T)} \|e^{v(\cdot, t)}\|_p \leq C.$$

This induces the local version, and given subdomains ω and $\hat{\omega}$ satisfying $\omega \subset \subset \hat{\omega} \subset \subset \Omega$ and $\delta > 0$, we obtain $p > 1$ and $C > 0$ both determined by $\sup_{t \in (0, T)} \|f(\cdot, t)\|_1$ such that

$$\sup_{t \in (0, T)} \|f(\cdot, t)\|_{L^1(\hat{\omega})} < 4\pi - \delta \Rightarrow \sup_{t \in (0, T)} \|e^{v(\cdot, t)}\|_{L^p(\omega)} \leq C.$$

Remark 7.2.1 Here we should point out that local-existence of problem (7.15)–(7.17) can be also derived by using two iteration schemes with starting point an upper-lower solution pair (z, w) defined as

$$z_t \leq \Delta z + \frac{\lambda e^z}{\int_{\Omega} e^w dx} \quad \text{in } \Omega \times (0, T), \quad (7.23)$$

$$w_t \geq \Delta w + \frac{\lambda e^w}{\int_{\Omega} e^z dx} \quad \text{in } \Omega \times (0, T), \quad (7.24)$$

$$z(x, t) \leq 0 \leq w(x, t), \quad \text{on } \partial\Omega \times (0, T), \quad (7.25)$$

$$z(x, 0) \leq v_0(x) \leq w(x, 0) \quad \text{in } \Omega, \quad (7.26)$$

and following a similar approach to Proposition 1.2.2.

7.2.2 Blow-Up Results

This subsection is devoted to the investigation of the blow-up behavior of problem (7.15)–(7.17). Our first result rules out the possibility of blow-up taking place on the boundary $\partial\Omega$ in case of a convex domain Ω . In particular, the following holds

Lemma 7.1 *If $T < +\infty$ and Ω is convex, then $\mathcal{S} \subset \subset \Omega$.*

Proof Let $v = v(x, t)$ be a solution to (7.15) with $v|_{t=0} = v_0(x)$. Having assumed $v_0 \geq 0$, we obtain $v = v(x, t) \geq 0$, and then

$$\sup_{t \in [0, T)} \|v(\cdot, t)\|_p < +\infty, \quad (7.27)$$

follows for any $p \geq 1$ by virtue of the global parabolic Brezis–Merle inequality.

Notably v is a solution to

$$v_t = \Delta v + \sigma(t)e^v \quad \text{in } \Omega \times (0, T), \quad v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x)$$

for $\sigma(t) = \frac{\lambda}{\int_{\Omega} e^{v(\cdot, t)}}$ and the method of the moving plane, [8], is applicable. Using (7.27), now we can apply the argument of [5], and consequently, there is an open set $\hat{\omega}$ containing $\partial\Omega$ and a constant $C > 0$ such that

$$\sup_{t \in [0, T)} \|v(\cdot, t)\|_{L^\infty(\Omega \cap \hat{\omega})} \leq C,$$

and therefore, $\mathcal{S} \subset \Omega$. □

Remark 7.2.2 Due to a uniform L^1 -estimate of the solution of (7.15) the result of the above Lemma can be extended to higher dimensions $N \geq 2$. In fact, multiplying the equation of (7.15) by the eigenfunction $\phi_1(x) > 0$ corresponding to the principal eigenvalue λ_1 of $-\Delta_D$ and integrating all over Ω we obtain

$$\frac{d}{dt} \int_{\Omega} v(x, t) \phi_1(x) dx \leq -\lambda_1 \int_{\Omega} v(x, t) \phi_1(x) dx + \lambda M, \quad 0 < t < T,$$

for $M = \max_{\bar{\Omega}} \phi_1(x) > 0$, which implies

$$\int_{\Omega} v(x, t) \phi_1(x) dx \leq C := C(v_0, \lambda_1, \lambda, M) \quad \text{for any } 0 < t < T. \quad (7.28)$$

Since Ω is convex, using the method of moving planes, [8], we can find $\bar{\Omega}_0 \subset \Omega$ such that

$$\int_{\Omega} v(x, t) dx \leq \frac{k+1}{m} \int_{\Omega_0} v(x, t) \phi_1(x) dx \leq \frac{k+1}{m} \int_{\Omega} v(x, t) \phi_1(x) dx < C_1, \quad (7.29)$$

by (7.28) for any $0 < t < T$ and $m = \min_{\bar{\Omega}_0} \phi_1(x) > 0$. Using now estimate (7.29), which for $N = 2$ stems from the parabolic version of the Brezis–Merle’s inequality, along with the arguments introduced in [5] we derive the desired result. If $v = v(x, t)$ is radially symmetric and decreasing in $r = |x|$, and if (7.22) holds, then $\mathcal{S} = \{0\}$ by the same reasoning. We also note that in the case of $T_{\max} = +\infty$, any radially symmetric solution becomes decreasing in $r = |x|$ eventually, see [25].

Note that stationary problem to (7.15)–(7.17) is described by

$$-\Delta v_* = \frac{\lambda e^{v_*}}{\int_{\Omega} e^{v_*}} \quad \text{in } \Omega, \quad v_* = 0 \quad \text{on } \partial\Omega. \quad (7.30)$$

Let now

$$E = \{v_* \mid v_* \text{ is a classical solution to (7.30)}\}$$

then we have the following result due to Wolansky, see Theorem 8 in [34]. Since the local well-posedness in time of (7.15)–(7.17) in $X = H_0^1(\Omega)$ together with the Trudinger–Moser inequality are used then its proof is valid only to $N = 1, 2$. However, here a simpler proof is provided.

Lemma 7.2 *If $E = \emptyset$, then it holds that*

- (i)

$$\lim_{t \uparrow T} \int_{\Omega} e^{v(\cdot, t)} = +\infty, \quad (7.31)$$

- (ii) $\mathcal{S} \neq \emptyset$,

where the case $T = +\infty$ is also permitted.

Proof We first prove statement (ii). If we assume that $\mathcal{S} = \emptyset$, then

$$\sup_{t \in [0, T)} \|v(\cdot, t)\|_{\infty} < +\infty.$$

The latter implies $T = +\infty$ and $E \neq \emptyset$ by a standard argument using Lyapunov function [10], which leads to a contradiction.

Next we proceed with the proof of (i). Therefore if we assume that

$$\liminf_{t \uparrow T} \int_{\Omega} e^{v(\cdot, t)} < +\infty,$$

then we obtain

$$\lim_{t \uparrow T} \mathcal{J}_{\lambda}(v(\cdot, t)) \geq \limsup_{t \uparrow T} \left\{ -\lambda \log \left(\int_{\Omega} e^{v(\cdot, t)} \right) \right\} + \lambda(\log \lambda - 1) > -\infty, \quad (7.32)$$

and finally

$$\begin{aligned} & \frac{1}{2} \liminf_{t \uparrow T} \|\nabla v(\cdot, t)\|_2^2 \\ & \leq \mathcal{J}_{\lambda}(v_0) + \lambda \liminf_{t \uparrow T} \log \left(\int_{\Omega} e^{v(\cdot, t)} \right) - \lambda(\log \lambda + 1) < +\infty. \end{aligned}$$

The latter estimation guarantees $T = +\infty$ owing to the well-posedness of (7.15)–(7.17) in $X = H_0^1(\Omega)$. Furthermore, there are $\delta \in (0, 1)$, $t_k \uparrow +\infty$, and $C > 0$ such that $t_{k+1} > t_k + 1$ and

$$\sup_{t \in (t_k, t_k + \delta)} \|\nabla v(\cdot, t)\|_2 \leq C.$$

The latter implies

$$\sum_{k=1}^{\infty} \int_{t_k}^{t_k + \delta} \|v_t(\cdot, t)\|_2^2 dt \leq \int_0^{\infty} \|v_t(\cdot, t)\|_2^2 dt < +\infty,$$

by (7.32) and (7.20), and therefore,

$$\lim_{k \rightarrow \infty} \int_{t_k}^{t_k + \delta} \|v_t(\cdot, t)\|_2^2 dt = 0.$$

Then, we can find $t'_k \in (t_k, t_k + \delta)$ satisfying $\|v_t(\cdot, t'_k)\|_2 \rightarrow 0$. Since $\|\nabla v(\cdot, t'_k)\|_2 \leq C$, we obtain a subsequence, denoted by the same symbol, such that $v(\cdot, t'_k) \rightharpoonup^3 w$ weakly in X . This implies $e^{v(\cdot, t_k)} \rightarrow e^w$ strongly in $L^p(\Omega)$, $p \geq 1$, by the Trudinger–Moser inequality, and then $w \in E$ follows, which is a contradiction. $\square$

The following result which is due to Wolansky, [34], actually says that above the threshold 8π a mass concentration occurs at the origin $x = 0$ in the case of the unit disc $\Omega = B_1(0) = \{x \in \mathbb{R}^2 \mid 0 \leq |x| < 1\}$.

Theorem 7.1 *If $\Omega = B_1(0)$, $v_0 = v_0(|x|)$ is smooth and $\lambda \geq 8\pi$, then*

$$\frac{e^v}{\int_{\Omega} e^v} \rightharpoonup \delta_0(dx) \quad \text{as } t \uparrow T, \quad (7.33)$$

for the solution v of (7.15)–(7.17), where $T \leq +\infty$.

Proof Let $\Omega = B_1(0)$, $v_0 = v_0(|x|) \geq 0$, and $\lambda \geq 8\pi$. We obtain $E = \emptyset$ in this case, and therefore, $\mathcal{S} \neq \emptyset$ and also (7.31) holds true. Since $v = v(|x|, t)$, then any $x_0 \in \Omega \setminus \{0\}$ admits $0 < r \ll 1$ such that

$$\sup_{t \in (0, T)} \left\| \frac{\lambda e^{v(\cdot, t)}}{\int_{\Omega} e^{v(\cdot, t)}} \right\|_{L^1(B_{2r}(x_0))} < 4\pi.$$

Accordingly, the local parabolic Brezis–Merle’s inequality applied to problem (7.15)–(7.17) guarantees

$$\sup_{t \in (0, T)} \|e^{v(\cdot, t)}\|_{L^p(B_r(x_0))} < +\infty,$$

with $p > 1$, and therefore, there holds that

$$\sup_{t \in (0, T)} \left\| \frac{\lambda e^{v(\cdot, t)}}{\int_{\Omega} e^{v(\cdot, t)}} \right\|_{L^p(B_r(x_0))} < +\infty,$$

since $v \geq 0$. The latter implies

$$\sup_{t \in (0, T)} \|v(\cdot, t)\|_{W^{2,p}(B_{r/2}(x_0))} < +\infty,$$

via the local parabolic regularity, and hence $x_0 \notin \mathcal{S}$ by Sobolev’s imbedding theorem. Consequently, we have $\mathcal{S} = \{0\}$ and so (7.33) is valid. $\square$

Although the case $T = +\infty$ is also admitted in the above theorem, in [16] is proven the following more delicate result.

Theorem 7.2 *If $\Omega = B_1(0)$ and $v_0 = v_0(|x|)$ is smooth, then finite-time blow-up occurs for the solution of (7.15)–(7.17), i.e. $T < +\infty$, provided that $\lambda > 8\pi$.*

Proof First note, that v is radial symmetric, i.e. $v(x, t) = v(r, t)$ where $r = |x|$, since $\Omega = B(0, 1)$. Furthermore, for u defined by (7.14) we have $\nabla u = u \nabla v$ and therefore via integration by parts,

$$-\int_{\Omega} u \nabla \cdot \psi \, dx = \int_{\Omega} u \psi \cdot \nabla v \, dx, \quad (7.34)$$

for any $\psi \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$. In view of the following problem

$$v_t = \Delta v + u \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \quad (7.35)$$

we can express v as follows

$$v = (-\Delta_D)^{-1}u - (-\Delta_D)^{-1}v_t,$$

where $w = (-\Delta_D)^{-1}u$ denotes the solution of the following Dirichlet problem

$$-\Delta w = u \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

Next we define the function

$$\rho(x, x') = \psi(x) \cdot \nabla_x G(x, x') + \psi(x') \cdot \nabla_{x'} G(x, x'),$$

where $G = G(x, x')$ is the Green's function of $-\Delta_D$ satisfying $G(x', x) = G(x, x')$ and

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|} + K(x, x'), \quad (7.36)$$

with $K \in C_{loc}^{2+\theta}(\overline{\Omega} \times \Omega \cup \Omega \times \overline{\Omega})$ for $0 < \theta < 1$.

Thus, the right-hand side of (7.34), see [32], is equal to

$$\begin{aligned} \int_{\Omega} u \psi \cdot \nabla v \, dx &= \int_{\Omega} u \psi \cdot \nabla \{(-\Delta_D)^{-1}u - (-\Delta_D)^{-1}v_t\} \, dx \\ &= \frac{1}{2} \int \int_{\Omega \times \Omega} \rho(x, x') u(x, t) u(x', t) \, dx \, dx' - \int_{\Omega} u \psi \cdot \nabla (-\Delta_D)^{-1}v_t \, dx. \end{aligned}$$

Considering now a test function of the form $\psi(x) = x\varphi(|x|)$ for $\varphi = \varphi(|x|) \in C_0^\infty(\Omega)$ satisfying $0 \leq \varphi \leq 1$ and $\varphi = 1$ near $x = 0$ then, $\nabla \cdot \psi|_{x=0} = 2$ and therefore,

$$\int_{\Omega} u \nabla \cdot \psi \, dx = 2\lambda + o(1), \quad (7.37)$$

holds as $t \uparrow T$ by (7.33).

The relation (7.36), on the other hand, guarantees

$$\rho(x, x') = -\frac{1}{2\pi} + L(x, x'),$$

with $L = L(x, x') \in C(\overline{\Omega} \times \overline{\Omega})$ satisfying $L(0, 0) = 0$, and therefore,

$$\frac{1}{2} \int \int_{\Omega \times \Omega} \rho(x, x') u(x, t) u(x', t) dx dx' = -\frac{\lambda^2}{4\pi} + o(1),$$

as $t \uparrow T$. Thus, by virtue of (7.34) and (7.37) we obtain

$$-\int_{\Omega} u(\psi \cdot \nabla)(-\Delta_D)^{-1} v_t dx = \frac{\lambda^2}{4\pi} - 2\lambda + o(1), \quad (7.38)$$

as $t \uparrow T$.

In addition, the relation $V = (-\Delta_D)^{-1} v_t$ implies $(r V_r)_r = -r v_t$, and hence

$$r V_r(r, t) = -\int_0^r s v_t(s, t) ds.$$

The latter implies

$$\begin{aligned} (\psi \cdot \nabla)(-\Delta_D)^{-1} v_t &= \varphi(r) r \partial_r (-\Delta_D)^{-1} v_t = -\varphi(r) \int_0^r s v_t(s, t) ds \\ &= -\frac{\varphi(r)}{2\pi} \int_{B(0, r)} v_t(s, t) dx, \end{aligned}$$

and thus we derive

$$-\int_{\Omega} u(\psi \cdot \nabla)(-\Delta_D)^{-1} v_t dx \leq \frac{\lambda}{2\pi} \sup_r \varphi(r) \int_{B(0, r)} v_t dx \leq \frac{\lambda}{2\pi} \|v_t\|_1,$$

since $u \geq 0$ and

$$\|u\|_1 = \lambda. \quad (7.39)$$

Finally by virtue of (7.38) and taking also into account that $\lambda > 8\pi$ we end up with

$$\liminf_{t \uparrow T} \|v_t(\cdot, t)\|_1 \geq \frac{\lambda}{2} - 4\pi > 0, \quad (7.40)$$

which entails

$$\|v_t(\cdot, t)\|_2 \geq \delta, \quad (7.41)$$

with $\delta > 0$ independent of $t \geq 1$, in case of a global-in-time solution, i.e. for $T = +\infty$.

Let now $0 < \mu_1 < \mu_2 \leq \dots$ be the eigenvalues of $-\Delta_D$ with corresponding L^2 -normalized eigenfunctions $\varphi_j = \varphi_j(x)$, $j = 1, 2, \dots$, i.e.

$$-\Delta\varphi_j = \mu_j\varphi_j \quad \text{in } \Omega, \quad \varphi_j = 0 \quad \text{on } \partial\Omega, \quad \text{with } \|\varphi_j\|_2 = 1.$$

Then, we have the following asymptotic behavior, [30],

$$\mu_j \sim j \quad \text{and} \quad \|\varphi_j\|_\infty \leq Cj^{1/4} \quad \text{as } j \rightarrow \infty. \quad (7.42)$$

If $T = +\infty$, then by virtue of (7.27) for $p = 2$, we can find $t_k \rightarrow +\infty$ and a $v_* \in L^2(\Omega)$ such that

$$v(\cdot, t_k) \rightharpoonup v_*(\cdot) \quad \text{weakly in } L^2(\Omega) \quad \text{as } k \rightarrow \infty. \quad (7.43)$$

Set $g_j(t) = \langle v(\cdot, t), \varphi_j \rangle$ then there holds

$$\dot{g}_j = -\mu_j g_j + \langle u, \varphi_j \rangle,$$

and thus

$$g_j(t + t_k) = e^{-\mu_j t} g_j(t_k) + \int_0^t e^{-(t-s)\mu_j} \langle u(\cdot, s + t_k), \varphi_j \rangle ds.$$

The above relation leads, taking also into account (7.39), to the estimate

$$|\langle v(\cdot, t + t_k), \varphi_j \rangle| \leq e^{-\mu_j t} \|v(\cdot, t_k)\|_2 + \lambda \|\varphi_j\|_\infty \mu_j^{-1},$$

which by virtue of (7.27) and (7.42) implies

$$|\langle v(\cdot, t + t_k), \varphi_j \rangle| \leq A_j,$$

for $t \geq 1$ and $k = 1, 2, \dots$ where $A_j = C(e^{-\alpha j} + \lambda j^{-\frac{3}{4}}) > 0$ for $j \gg 1$, $\alpha > 0$, and $C > 0$, satisfying $\sum_{j=1}^\infty A_j^2 < +\infty$.

Now using (7.33) and (7.43) we derive

$$\begin{aligned} g_j(t + t_k) &\rightarrow e^{-\mu_j t} \langle v_*, \varphi_j \rangle + \int_0^t e^{-(t-s)\mu_j} \lambda \varphi_j(0) ds \\ &= e^{-\mu_j t} \langle v_*, \varphi \rangle + \frac{\lambda \varphi_j(0)}{\mu_j} (1 - e^{-\mu_j t}), \end{aligned}$$

as $k \rightarrow \infty$ uniformly in $t \geq 1$ for each j , and therefore,

$$v(\cdot, t + t_k) \rightarrow V(\cdot, t) \quad \text{as } k \rightarrow \infty,$$

in $L^2(\Omega)$ uniformly in $t \geq 1$, where

$$V(\cdot, t) = \sum_{j=1}^{\infty} \left\{ e^{-\mu_j t} \langle v_*, \varphi_j \rangle + \frac{\lambda \varphi_j(0)}{\mu_j} (1 - e^{-\mu_j t}) \right\} \varphi_j.$$

Similarly,

$$v_t(\cdot, t + t_k) \rightarrow W(\cdot, t) \quad \text{as } k \rightarrow \infty,$$

in $L^2(\Omega)$ uniformly in $t \geq 1$, where

$$W(\cdot, t) = \sum_{j=1}^{\infty} e^{-\mu_j t} \{ -\mu_j \langle v_*, \varphi_j \rangle + \lambda \varphi_j(0) \} \varphi_j = V_t, \quad (7.44)$$

and thus

$$\|W(\cdot, t)\|_2^2 \geq \delta^2, \quad (7.45)$$

by virtue of (7.41).

On the other hand, (7.44) implies

$$\|W(\cdot, t)\|_2^2 = \sum_{j=1}^{\infty} e^{-2\mu_j t} | -\mu_j \langle v_*, \varphi_j \rangle + \lambda \varphi_j(0) |^2 \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

which actually contradicts (7.45). This completes the proof of the theorem.

The above theorem entails that the formation of collapse with dis-quantized mass occurs in finite time for the limiting case $\varepsilon = 0$, in contrast with what happens in other limiting case $\tau = 0$, regarding the problem (7.5)–(7.8).

7.3 An Associated Competition-Diffusion System

In the current subsection a brief investigation of a non-local reaction-diffusion system stems from (7.15)–(7.17) is delivered. Specifically, we consider the following

$$z_t = \Delta z + f(z, w) \quad \text{in } \Omega \times (0, T), \quad (7.46)$$

$$w_t = \Delta w + g(z, w) \quad \text{in } \Omega \times (0, T), \quad (7.47)$$

$$z(x, t) = w(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T), \quad (7.48)$$

$$z(x, 0) = z_0(x), \quad w(x, 0) = w_0(x), \quad \text{in } \Omega, \quad (7.49)$$

where

$$f(z, w) = \frac{\lambda e^z}{\int_{\Omega} e^w dx} \quad \text{and} \quad g(z, w) = \frac{\lambda e^w}{\int_{\Omega} e^z dx}.$$

Notably, in case where $z(x, t) = v(x, t)$ then system (7.23)–(7.26) is reduced to problem (7.15)–(7.17).

Next, we observe that

$$\frac{\partial f(z, w)}{\partial w} \leq 0 \quad \text{and} \quad \frac{\partial g(z, w)}{\partial z} \leq 0, \quad (7.50)$$

and thus (7.23)–(7.26) is a competition-diffusion system with non-local competition terms according to [19].

As it is pointed out in [19] one of the prominent characteristics of competition-diffusion systems for two species is the so-called comparison principle, which stems from the maximum principle. Owing to this property, the general theory of strongly order-preserving local semiflows, [12] implements to system (7.23)–(7.26), thereby providing a number of results on the dynamical structure of this system. To this end, we first introduce the comparison principle for system (7.23)–(7.26) as follows.

Comparison Principle: Let $(z, w), (\bar{z}, \bar{w})$ be solutions to (7.23)–(7.26) with initial data $(z_0, w_0), (\bar{z}_0, \bar{w}_0)$ respectively. Suppose $\bar{z}_0(x) \geq z_0(x)$, $\bar{w}_0(x) \leq w_0(x)$ for all $x \in \bar{\Omega}$ and $\bar{z}(x, t) \geq z(x, t)$, $\bar{w}(x, t) \leq w(x, t)$ for all $(x, t) \in \partial\Omega \times (0, T)$. Then $\bar{w}(x, t) \geq w(x, t)$, and $\bar{z}(x, t) \leq z(x, t)$ for all $(x, t) \in \Omega \times (0, T)$.

Motivated by the above comparison principle we determine, see also [19], the following order relation in the space $C(\bar{\Omega}) \times C(\bar{\Omega})$

$$\begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} \geq \begin{pmatrix} z \\ w \end{pmatrix} \Leftrightarrow \bar{z}(x) \geq z(x) \quad \text{and} \quad \bar{w}(x) \leq w(x) \quad \text{for any } x \in \bar{\Omega}. \quad (7.51)$$

The strict form of relation (7.51) is defined as follows:

$$\begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} \gg \begin{pmatrix} z \\ w \end{pmatrix} \Leftrightarrow \bar{z}(x) \geq z(x) \quad \text{and} \quad \bar{w}(x) \leq w(x) \quad \text{with} \quad \bar{z}(x) \not\equiv z(x) \quad \text{and} \quad \bar{w}(x) \not\equiv w(x),$$

for any $x \in \bar{\Omega}$.

Owing to (7.50) the local semiflow designated by system (7.23)–(7.26) preserves the order relation defined above. Moreover, since we actually have

$$\frac{\partial f(z, w)}{\partial w} < 0 \quad \text{and} \quad \frac{\partial g(z, w)}{\partial z} < 0,$$

then there finally holds that the local semiflow is strongly order-preserving with respect to the order relation provided above, [19].

Property (7.50) suggests that some interesting properties of the determined local semiflow hold as described by Theorem 7.3.2, see also [12, 19]. Before stating the result of this theorem we need to introduce the following definition

Definition 7.3.1 We say that

$$\phi(x) = \begin{pmatrix} z(x) \\ w(x) \end{pmatrix},$$

is a (time-independent) super-solution to (7.23)–(7.26) if the following inequalities are satisfied

$$\begin{aligned} -\Delta z &\geq \frac{\lambda e^z}{\int_{\Omega} e^w dx}, & -\Delta w &\leq \frac{\lambda e^w}{\int_{\Omega} e^z dx} & \text{in } \Omega, \\ z &\geq 0 \geq w, & & & \text{on } \partial\Omega. \end{aligned}$$

In case the reverse inequalities are fulfilled then ϕ is called a (time-independent) sub-solution. Once, the above inequalities are strict then we end up with strict super and sub-solutions.

Theorem 7.3.2 *The following statements hold true:*

- (i) *The ω -limit sets of almost bounded orbits of system (7.23)–(7.26) are contained in the set of the stationary solutions designated as:*

$$-\Delta v_1 = \frac{\lambda e^{v_1}}{\int_{\Omega} e^{v_2} dx}, \quad -\Delta v_2 = \frac{\lambda e^{v_2}}{\int_{\Omega} e^{v_1} dx} \quad \text{in } \Omega, \quad (7.52)$$

$$v_1 = v_2 = 0, \quad \text{on } \partial\Omega. \quad (7.53)$$

- (ii) *Any periodic orbit of system (7.23)–(7.26) is unstable.*
 (iii) *Any unstable stationary solution*

$$v^*(x) = \begin{pmatrix} v_1^*(x) \\ v_2^*(x) \end{pmatrix},$$

(which solves (7.52)–(7.53)) has a non-trivial unstable set, i.e. there exists

$$\phi(x, t) = \begin{pmatrix} z(x, t) \\ w(x, t) \end{pmatrix} \neq v^*(x) = \begin{pmatrix} v_1^*(x) \\ v_2^*(x) \end{pmatrix},$$

such that $\phi(t) \rightarrow v^*$ as $t \rightarrow -\infty$ in $C(\bar{\Omega})$.

- (iv) *Assume that $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in C(\bar{\Omega}) \times C(\bar{\Omega})$ be a strict super-solution and $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in C(\bar{\Omega}) \times C(\bar{\Omega})$ be a strict sub-solution of (7.23)–(7.26) such that*

$\xi(x) \gg \zeta(x)$ in $\overline{\Omega}$, then there exists a stable stationary solution $v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix}$ such that $\xi(x) \leq v^*(x) \leq \zeta(x)$ in $\overline{\Omega}$, i.e.

$$\xi_1(x) \leq v_1^*(x) \leq \zeta_1(x) \quad \text{and} \quad \zeta_2(x) \leq v_2^*(x) \leq \xi_2(x) \quad \text{for any } x \text{ in } \overline{\Omega}.$$

An immediate consequence of Theorem 7.3.2 is the following

Corollary 7.3.3 *If for system (7.23)–(7.26) we consider initial data*

$$\begin{pmatrix} z_0 \\ w_0 \end{pmatrix} = v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix},$$

where v^* is the stable stationary point provided by Theorem 7.3.2 (iv) then system (7.23)–(7.26) has a global-in-time solution.

Proof Indeed a straightforward application of the comparison principles entails that

$$\xi(x) = \begin{pmatrix} \xi_1(x) \\ \xi_2(x) \end{pmatrix} \leq \begin{pmatrix} z(x, t) \\ v(x, t) \end{pmatrix} \leq \begin{pmatrix} \zeta_1(x) \\ \zeta_2(x) \end{pmatrix} = \zeta(x) \quad \text{for any } x \in \Omega \quad \text{and for any } t > 0.$$

□

which evidently entails the existence of a global-in-time solution for problem (7.23)–(7.26).

Remark 7.3.4 In case where

$$\xi(x) = \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} \quad \text{and} \quad \zeta(x) = \begin{pmatrix} 0 \\ \psi(x) \end{pmatrix},$$

for ψ being a steady-state solution of (7.15)–(7.17) then Corollary 7.3.3 guarantees the existence of a global-in-time solutions of this problem.

We close this section with a brief investigation of the steady-state problem (7.52)–(7.53) in the radial symmetric case, i.e. when $\Omega = B_1(0) = \{x \in \mathbb{R}^N \mid |x| < 1\}$, for $N \geq 3$. We first observe that in this case the solutions of (7.52)–(7.53) are radially symmetric and thus problem (7.52)–(7.53) reduces to

$$-\Delta_r v_1 = \frac{\lambda e^{v_1}}{N\omega_N \int_0^1 r^{N-1} e^{v_2(r)} dr}, \quad -\Delta_r v_2 = \frac{\lambda e^{v_2}}{N\omega_N \int_0^1 r^{N-1} e^{v_1(r)} dr} \quad \text{in } (0, 1), \quad (7.54)$$

$$\frac{\partial v_1}{\partial r} \Big|_{r=0} = \frac{\partial v_2}{\partial r} \Big|_{r=0} = 0, \quad v_1(1) = v_2(1) = 0, \quad (7.55)$$

where $\Delta_r := r^{N-1} \frac{\partial^2}{\partial r^2} + (N-1)r^{N-2} \frac{\partial^2}{\partial r^2}$ and $r = |x|$, recalling that ω_N denotes the volume of the N -dimensional unit sphere. Let us set

$$\sigma_1 = \frac{\lambda}{N\omega_N \int_0^1 r^{N-1} e^{v_2(r)} dr} \quad \text{and} \quad \sigma_2 = \frac{\lambda}{N\omega_N \int_0^1 r^{N-1} e^{v_1(r)} dr}, \quad (7.56)$$

then the non-local system (7.52)–(7.53) is reduced to the following local one

$$-\Delta_r v_1 = \sigma_1 e^{v_1}, \quad -\Delta_r v_2 = \sigma_2 e^{v_2} \quad \text{in } (0, 1), \quad (7.57)$$

$$\left. \frac{\partial v_1}{\partial r} \right|_{r=0} = \left. \frac{\partial v_2}{\partial r} \right|_{r=0} = 0, \quad v_1(1) = v_2(1) = 0, \quad v_1(1) = v_2(1) = 0, \quad (7.58)$$

and those two problems are equivalent through (7.56). Moreover, (7.56) also yields

$$\sigma_1 N\omega_N \int_0^1 r^{N-1} e^{v_1(r)} dr = \frac{\sigma_1}{\sigma_2} \lambda \quad \text{and} \quad \sigma_2 N\omega_N \int_0^1 r^{N-1} e^{v_2(r)} dr = \frac{\sigma_2}{\sigma_1} \lambda. \quad (7.59)$$

Remarkably, the solution set

$$\mathcal{C}_i^r = \{(\sigma_i, v_i) | v_i(x) \text{ is a classical solution to (7.57)–(7.58) for } \sigma_i > 0\}, \quad i = 1, 2,$$

is a one-dimensional open manifold with end points $(0, 0)$ and $(2(N-2), \log \frac{1}{r})$, where the latter one is a weak solution of (7.57)–(7.58), see also [15, 20]. Therefore, the solution set

$$\mathcal{S}^r = \left\{ (\lambda, v_1, v_2) \left| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ is a classical solution to (7.54)–(7.55) for } \lambda > 0 \right. \right\},$$

is a two-dimensional manifold with end points $(0, 0, 0)$ and $(2\alpha_N, \log \frac{1}{r}, \log \frac{1}{r})$, where $\alpha_N = N\omega_N$ denotes the $(N-1)$ -dimensional volume of the surface of the unit ball in $\mathbb{R}^N$. Again the latter end point is a weak solution of (7.54)–(7.55).

Using Emden's transformation

$$v_i(r) = w_i(\tau) - 2\tau + A, \quad r = Be^\tau, \quad B = \left(\frac{2(N-2)}{\sigma_i e^A} \right)^{1/2}, \quad i = 1, 2, \quad (7.60)$$

then system (7.57)–(7.58) is transformed to

$$\ddot{w}_i + (N-2)\dot{w}_i + 2(N-2)(e^{w_i} - 1) = 0, \quad \lim_{\tau \rightarrow -\infty} w_i(\tau) = -\infty, \quad \lim_{\tau \rightarrow -\infty} \dot{w}_i(\tau) = 2, \quad i = 1, 2, \quad (7.61)$$

where $\dot{w}_i = \frac{dw_i}{d\tau}$, $\ddot{w}_i = \frac{d^2 w_i}{d\tau^2}$, and by this transformation any $(\sigma_i, v_i) \in \mathcal{C}_i^r$, $i = 1, 2$, corresponds to a point $P(w_i, \dot{w}_i) \in \mathcal{O} = \{(w(t), \dot{w}(t)), t \in \mathbb{R}\}$.

Additionally, by virtue of (7.56), (7.61) we also have

$$\lambda = \sigma_i \alpha_N \int_0^1 r^{N-1} e^{v_i(r)} dr = -\alpha_N \frac{\partial v_i}{\partial r} \Big|_{r=1} = \alpha_N (2 - \dot{w}_i), \quad i = 1, 2, \quad (7.62)$$

which in conjunction with (7.59) entails

$$2 - \dot{w}_1 = \frac{\lambda}{\alpha_N} \frac{\sigma_1}{\sigma_2} = \frac{\lambda}{\alpha_N} e^{w_1 - w_2} \quad \text{and} \quad 2 - \dot{w}_2 = \frac{\lambda}{\alpha_N} \frac{\sigma_2}{\sigma_1} = \frac{\lambda}{\alpha_N} e^{w_2 - w_1}. \quad (7.63)$$

Now for any given $0 < \lambda < \lambda^*$ where

$\lambda^* := \sup\{\lambda > 0 : \text{problem (7.54) – (7.55) has a solution corresponding to } \lambda\}$,

we consider a point $P(w_0, \dot{w}_0) \in \mathcal{O}$ determined by $2 - \dot{w}_0 = \frac{\lambda}{\alpha_N}$. Assuming that $w_1 > w_2$, then there exists $\gamma > 1$ such that $w_2 = w_1 - \log \gamma$, and thus

$$2 - \dot{w}_1 = \gamma \frac{\lambda}{\alpha_N} \quad \text{and} \quad 2 - \dot{w}_2 = \frac{1}{\gamma} \frac{\lambda}{\alpha_N}, \quad (7.64)$$

by virtue of (7.62) and (7.63).

Writing $w_2 = w_2(\gamma)$, we then have

$$2 - \dot{w}_2(\gamma) = \frac{1}{\gamma} \frac{\lambda}{\alpha_N},$$

by (7.64), and if $N \geq 10$ the mapping $\gamma \in (1, 2) \mapsto \dot{w}_2(\gamma)$ is monotone increasing and there holds

$$\lim_{\gamma \downarrow 1} \dot{w}_2(\gamma) = \frac{\lambda}{\alpha_N} \quad \text{and} \quad \lim_{\gamma \uparrow 2} \dot{w}_2(\gamma) = 2.$$

Consequently, the following statement holds true

Theorem 7.3.5 *For any $0 < \lambda \leq \alpha_N$ and $N \geq 10$ there exists a solution $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to (7.54)–(7.55) such that*

$$v_2 \leq v_0 \leq v_1, \quad (7.65)$$

where v_0 is a solution to the following

$$-\Delta_r v = \frac{\lambda e^v}{N \omega_N \int_0^1 r^{N-1} e^{v(r)} dr}, \quad \text{in } (0, 1),$$

$$\frac{\partial v}{\partial r} \Big|_{r=0} = 0, \quad v(1) = 0.$$

In case $\lambda = \alpha_N$, then $v_1 = v_2 = v_0$, otherwise the inequalities in (7.65) are strict.

7.4 Miscellanea

The current section is devoted to several remarks. First of all we note that the Green's function of the operator $-\Delta_D$ for $\Omega = B_1(0)$ is given explicitly by

$$G(x, x') = -\frac{1}{4\pi} \log \left| \frac{z - z'}{1 - \bar{z}z'} \right|^2,$$

where z and z' are complex numbers corresponding to x and x' , respectively. The latter implies

$$x \cdot \nabla_x G(x, x') + x' \cdot \nabla_{x'} G(x, x') = -\frac{1}{2\pi} \cdot \frac{1 - |\zeta|^2}{|1 - \zeta|^2},$$

for $\zeta = \bar{z}z'$. We can then see that this function does not belong to $L^\infty(\Omega \times \Omega)$ in contrast with the corresponding function derived from the Green's function to $-\Delta_{JL}$, see also [32]. There is a similar difficulty in (7.5)–(7.8), and the control of the boundary blow-up points has not been completed even in the case of $(\varepsilon, \tau) = (1, 0)$.

Next, if $0 < \lambda < 8\pi$, the stationary problem (8.103) admits a unique solution [31]. Since (7.21) holds in this range of λ , then solution $v = v(x, t)$ to (7.15)–(7.17) converges uniformly to this steady solution. Now, what is anticipated for the case $\lambda = 8\pi$, $\Omega = B_1(0)$ is that an infinite-time blow-up occurs for the solution v , i.e., $T = +\infty$ and $v(x, t) \rightarrow 4 \log \frac{1}{|x|}$ locally uniformly in $x \in \bar{\Omega} \setminus \{0\}$ as $t \uparrow +\infty$.

In the limiting case $\varepsilon = 0$ system (7.9)–(7.13) is described by

$$v_t = \Delta v + \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \quad \text{in } \Omega \times (0, T), \quad (7.66)$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (7.67)$$

with the smooth initial value $v_0 = v_0(x)$ satisfying $\int_\Omega v_0 dx = 0$.

In the case where either $0 < \lambda < 4\pi$ or $0 < \lambda < 8\pi$, $v_0 = v_0(|x|)$, and $\Omega = B_1(0)$, we obtain (7.21) similarly to the full system, see [2, 7, 23]. In contrast to the case of problem (7.30) the stationary problem of (7.66)–(7.67) has the trivial solution $v = 0$ for any λ . Furthermore, multiple existence of the stationary (non-radially symmetric) solution arises even for $0 < \lambda < 4\pi$ and $\Omega = B_1(0)$ [26, 32]. On the other hand, either $T = +\infty$ or (7.33) with $T < +\infty$ holds for problem (7.66)–(7.67), similarly to Theorem 7.2. There is, furthermore, a bifurcation of non-constant radially symmetric stationary solutions at $\lambda = \lambda_* > 8\pi$. Then, we conjecture that any $\lambda > 8\pi$ admits a radially symmetric stationary solution (possibly the trivial one $v = 0$), stable in the space of radially symmetric functions. The possibility of the occurrence of a mass concentration at the origin $x = 0$ as in (7.33) holding with $T < +\infty$ is left open for problem (7.66)–(7.67) even for $\lambda > 8\pi$ and $\Omega = B_1(0)$ in contrast to Theorem 7.2.

A simple blow-up criterion is obtained for the semilinear parabolic equation

$$v_t = \Delta v + |v|^{q-1} v \quad \text{in } \Omega \times (0, T), \quad v = 0 \quad \text{on } \partial\Omega \times (0, T),$$

with $1 < q < \infty$. In fact, this equation admits the properties

$$\begin{aligned} \frac{d}{dt} J(v(\cdot, t)) &\leq 0, \\ \frac{1}{4} \frac{d}{dt} \|v(\cdot, t)\|_2^2 &= -J(v(\cdot, t)) + \left(\frac{1}{2} - \frac{1}{q+1} \right) \|v(\cdot, t)\|_{q+1}^{q+1}, \end{aligned}$$

for

$$J(v) = \frac{1}{2} \|\nabla v\|_2^2 - \frac{1}{q+1} \|v\|_{q+1}^{q+1}.$$

Using the above, we can infer $T < +\infty$ by $J(v_0) \leq 0$. The same argument is valid to the following non-local problem

$$\begin{aligned} v_t &= \Delta v + \frac{\lambda e^v}{\left(\int_{\Omega} e^v\right)^p} \quad \text{in } \Omega \times (0, T), \\ v(x, t) &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ v(x, 0) &= v_0(x) \quad \text{in } \Omega, \end{aligned}$$

with $0 < p < 1$, see [1], but this is not the case for problem (7.15)–(7.17).

In fact, problem (7.15)–(7.17) can be written in the form (7.35) with $u = \frac{\lambda e^v}{\int_{\Omega} e^v}$ satisfying $\|u\|_1 = \lambda$. Set $\mathcal{J}_{\lambda}(v) = L(u, v)$, where

$$\begin{aligned} \mathcal{J}_{\lambda}(v) &= \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \left(\int_{\Omega} e^v \right) + \lambda(\log \lambda - 1), \\ L(u, v) &= \int_{\Omega} u(\log u - 1) + \frac{1}{2} \|\nabla v\|_2^2 - \langle v, u \rangle. \end{aligned}$$

Then, it holds that

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} \|v\|_2^2 &= -L(u, v) + \int_{\Omega} u(\log u - 1) - \frac{1}{2} uv \, dx \\ &= -\mathcal{J}_{\lambda}(v) + K(u, v), \end{aligned}$$

with

$$\begin{aligned} K(u, v) &= \int_{\Omega} u(\log u - 1) - \frac{1}{2} uv \, dx \\ &\geq K|_{u=\frac{\lambda e^{v/2}}{\int_{\Omega} e^{v/2}}} = -\lambda \log \int_{\Omega} e^{v/2} + \lambda(\log \lambda - 1), \end{aligned}$$

i.e.,

$$\frac{1}{4} \frac{d}{dt} \|v(\cdot, t)\|_2^2 \geq -\mathcal{J}_\lambda(v(\cdot, t)) - \lambda \log \left(\int_{\Omega} e^{v(\cdot, t)/2} \right) + \lambda(\log \lambda - 1).$$

In spite of

$$\frac{d}{dt} \mathcal{J}_\lambda(v(\cdot, t)) \leq 0,$$

the above inequality is not sufficient to guarantee $T < +\infty$ because of the negativity of the second term on its right-hand side.

References

1. Bebernes, J.W., Lacey, A.A.: Global existence and finite-time blow-up for a class of nonlocal parabolic problems. *Adv. Differ. Equ.* **6**, 927–953 (1997)
2. Biler, P.: Local and global solvability of some parabolic systems modelling chemotaxis. *Adv. Math. Sci. Appl.* **8**, 715–743 (1998)
3. Brezis, H., Merle, F.: Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. *Commun. Partial. Differ. Equ.* **16**, 1223–1253 (1991)
4. Childress, S., Percus, J.K.: Nonlinear aspects of chemotaxis. *Math. Biosci.* **56**, 217–237 (1981)
5. DeFigueiredo, D.G., Lions, P.-L., Nussbaum, R.D.: A priori estimates and existence of positive solutions of semilinear elliptic equations. *J. Math. Pures Appl.* **61**, 41–63 (1982)
6. Doi, M., Edwards, S.F.: *The Theory of Polymer Dynamics*. Carendon Press, Oxford, (1986)
7. Gajewski, H., Zacharias, K.: Global behaviour of a reaction-diffusion system modelling chemotaxis. *Math. Nachr.* **195**, 77–114 (1998)
8. Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry and related properties via the maximum principle. *Commun. Math. Phys.* **68**, 209–243 (1979)
9. Harada, G., Nagai, T., Senba, T., Suzuki, T.: Concentration lemma, Brezis-Merle type inequality, and a parabolic system of chemotaxis. *Adv. Differ. Equ.* **6**, 1255–1280 (2001)
10. Henry, D.: *Geometric Theory of Semilinear Parabolic Equations*. Springer, Berlin (1981)
11. Herrero, M.A., Velázquez, J.J.L.: A blowup mechanism for a chemotaxis model. *Ann. Scuola Norm. Sup. Pisa IV* **24**, 633–683 (1997)
12. Hirsch, M.: Differential equations and convergence almost everywhere in strongly monotone flows. *Contemp. Math.* **17**, 267–285 (1983)
13. Horstmann, D.: From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I. *Jahresbericht der DMV* **105**, 103–165 (2003)
14. Horstmann, D.: From 1970 until present: The Keller-Segel model in chemotaxis and its consequences II. *Jahresbericht der DMV* **106**, 51–69 (2004)
15. Joseph, D.D., Lundgren, T.S.: Quasilinear Dirichlet problems driven by positive sources. *Arch. Ration. Mech. Anal.* **49**, 241–269 (1973)
16. Kavallaris, N.I., Suzuki, T.: On the finite-time blow-up of a non-local parabolic equation describing chemotaxis. *Differ. Int. Equ.* **20**(3), 293–308 (2007)
17. Keller, E.F., Segel, L.A.: Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.* **26**, 399–415 (1970)
18. Li, Y.Y., Shafrir, I.: Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. *Indiana Univ. Math. J.* **43**, 1255–1270 (1994)
19. Matano, H., Mimura, M.: Pattern formation in competition-diffusion systems in nonconvex domains. *Publ RIMS, Kyoto University* **19**, 1049–1079 (1983)

20. Miyasita, T., Suzuki, T.: Non-local Gelfand problem in higher dimensions, Nonlocal elliptic and parabolic problems, Banach Center Publications, Warsaw. Polish Acad. Sci. Inst. Math. **66**, 221–235 (2004)
21. Nagai, T.: Blow-up of radially symmetric solutions to a chemotaxis systems. Adv. Math. Sci. Appl. **5**, 581–601 (1995)
22. Nagai, T.: Blowup of nonradial solutions to parabolic-elliptic systems modelling chemotaxis in two-dimensional domains. J. Inequal Appl. **6**, 37–55 (2001)
23. Nagai, T., Senba, T., Yoshida, K.: Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. Funkcial. Ekvac. **40**, 411–433 (1997)
24. Nagasaki, K., Suzuki, T.: Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities. Asymptot. Anal. **3**, 173–188 (1990)
25. Ni, W.-M., Sacks, P.: The number of peaks of positive solutions of semilinear parabolic equations. SIAM J. Math. Anal. **16**, 460–471 (1985)
26. Senba, T., Suzuki, T.: Some structures of the solution set for a stationary system of chemotaxis. Adv. Math. Sci. Appl. **10**, 191–224 (2000)
27. Senba, T., Suzuki, T.: Chemotactic collapse in a parabolic-elliptic system of mathematical biology. Adv. Differ. Equ. **6**, 21–50 (2001)
28. Senba, T., Suzuki, T.: Parabolic system of chemotaxis: blowup in a finite and the infinite time. Method Appl. Anal. **8**, 349–368 (2001)
29. Senba, T., Suzuki, T.: Time global solutions to a parabolic-elliptic system modelling chemotaxis. Asymptot. Anal. **32**, 63–89 (2002)
30. Sogge, C.D.: Fourier Integrals in Classical Analysis. Cambridge University Press, Cambridge (1993)
31. Suzuki, T.: Global analysis for a two-dimensional elliptic eigenvalue problem the exponential nonlinearity. Ann. Inst. Henri Poincaré, Analyse non linéaire **9**, 367–398 (1992)
32. Suzuki, T.: Free Energy and Self-Interacting Particles. Birkhäuser, Boston (2005)
33. Suzuki, T.: Mean Field Theories and Dual Variation - Mathematical Structures of the Mesoscopic Model. Atlantis Press, Atlantis Studies in Mathematics for Engineering and Science (2015)
34. Wolansky, G.: A critical parabolic estimate and application to nonlocal equations arising in chemotaxis. Appl. Anal. **66**, 291–321 (1997)
35. Wolansky, G.: A concentration theorem for the heat equation. Monatsch. Math. **132**, 255–261 (2001)
36. Zheng, G-F.: On finite-time blow-up for a nonlocal parabolic problem arising from shear bands in metals. Proc. Amer. Math. Soc. **135**(5): 1487–1494 (2007)

Chapter 8

A Non-local Reaction-Diffusion System

Illustrating Cell Dynamics

Abstract Initially a reaction-diffusion system with non-local reaction terms is build up as a mathematical model to illustrate the evolution of protein dimers within human cells. The derived system inspects the situation when chemical reactions occur when the two chemicals within a cell are in distance R , where such a distance is the reaction radius. Next, the long-time behavior of the solutions of the preceding non-local system is investigated as well as the phase separation phenomenon occurring when the reaction takes place very fast is also examined. It is actually shown that a two-phase Stefan problem is derived in the limit of infinite chemical reaction rate. Next the convergence of the global-in-time solution to the preceding system towards the unique stationary solution is derived. The chapter closes with some results on the determination of the decay rate of the above convergence towards the unique stationary solution.

8.1 Derivation of the Non-local Reaction-Diffusion System

We first consider the system describing the dynamics of the spatially distributed fundamental chemical reaction between the reactants A and B



with a reaction rate k . Taking also into account the diffusion, this process is usually modeled by the following reaction-diffusion system

$$\frac{\partial q_A}{\partial t} = D_A \Delta q_A - k_A q_A q_B, \quad \frac{\partial q_B}{\partial t} = D_B \Delta q_B - k_B q_A q_B, \quad (8.2)$$

using the diffusion constants D_A , D_B , the renormalized reaction rates $k_A = k[B]_*$, $k_B = k[A]_*$, and the relative concentrations $0 \leq q_A = [A]/[A]_* \leq 1$, $0 \leq q_B = [B]/[B]_* \leq 1$, where $[A]$ and $[B]$ are the concentrations of the reactants A and B , respectively, and $[A]_*$ and $[B]_*$ are those in saturated states. The law of mass action which actually interprets the fact that the chemical reaction occurs with the probability

proportional to the collision of the molecules of the reactants A , B and that this probability is proportional to the product of $[A]$ and $[B]$ entails

$$\frac{d[A]}{dt} = -k[A][B], \quad \frac{d[B]}{dt} = -k[A][B],$$

which implies

$$\frac{dq_A}{dt} = -k_A q_A q_B, \quad \frac{dq_B}{dt} = -k_B q_A q_B. \quad (8.3)$$

On the other hand, in a microscopic level, and differently from [16], we consider $0 \leq q_A = q_A(x, t) \leq 1$ and $0 \leq q_B = q_B(x, t) \leq 1$ as the existence probabilities of A and B molecular, respectively and thus the diffusion process can be described as a random walk of many interacting particles. We assume that this walk is provided with the constant jump length Δx and let $0 \leq \sigma = \sigma(x, t; \omega) \leq 1$ be its transient probability in the direction $\omega \in \mathcal{S}^{N-1} = \{\omega \in \mathbb{R}^N \mid |\omega| = 1\}$ during the calculation time Δt . This setting establishes the master equation

$$\begin{aligned} q_i(x, t + \Delta t) - q_i(x, t) &= \int_{\mathcal{S}^{N-1}} \sigma_i(x + \omega \Delta x, t; -\omega) q_i(x + \omega \Delta x, t) d\omega \\ &\quad - \int_{\mathcal{S}^{N-1}} \sigma_i(x, t; \omega) d\omega \cdot q_i(x, t), \end{aligned} \quad (8.4)$$

for $i = A, B$ and then the mean waiting time τ is defined by

$$\frac{1}{\Delta t} \int_{\mathcal{S}^{N-1}} \sigma_i(x, t; \omega) d\omega = \tau^{-1}. \quad (8.5)$$

Here and henceforth, $d\omega$ stands for a probability measure on $\mathcal{S}^{N-1}$, and thus

$$\int_{\mathcal{S}^{N-1}} d\omega = 1.$$

Einstein's formula asserts that τ is independent of (x, t) and is given by the relation

$$\tau = \frac{(\Delta x)^2}{2ND},$$

with the diffusion coefficient D , see [3]. In the simplest case that $\sigma_i(x, t; \omega)$ is a constant denoted by σ_i , it follows that

$$\frac{\sigma_i}{\Delta t} = \frac{2ND_i}{(\Delta x)^2},$$

and hence the master equation (8.4) is reduced to

$$\begin{aligned} \frac{1}{\Delta t} \{q_i(x, t + \Delta t) - q_i(x, t)\} &= \frac{\sigma_i}{\Delta t} \int_{\mathcal{S}^{N-1}} [q_i(x + \omega \Delta x, t) - q_i(x, t)] d\omega \\ &= \frac{2ND_i}{(\Delta x)^2} \int_{\mathcal{S}^{N-1}} [q_i(x + \omega \Delta x, t) - q_i(x, t)] d\omega. \end{aligned} \quad (8.6)$$

Then

$$\frac{\partial q_i}{\partial t} = D_i \Delta q_i, \quad i = A, B, \quad (8.7)$$

arises in the mean field limit $\Delta t \downarrow 0$, $\Delta x \downarrow 0$, assuming that $q_i = q_i(x, t)$ is smooth in (x, t) and using the following elementary result (see [11])

Lemma 8.1 *If $d\omega$ is isotropic, that is*

$$\int_{\mathcal{S}^{N-1}} \omega_i \omega_j d\omega = \frac{\delta_{ij}}{N}, \quad i, j = 1, \dots, N,$$

then there holds

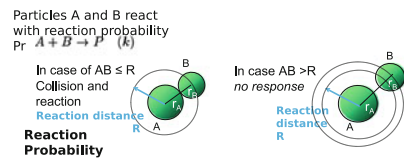
$$\int_{\mathcal{S}^{N-1}} [f(x + \omega \Delta x) - f(x)] d\omega = \frac{1}{2N} (\Delta x)^2 \{\Delta f(x) + o(1)\}, \quad \Delta x \downarrow 0 \quad (8.8)$$

for each $f = f(x) \in C^2(\mathcal{S}^{N-1})$ where $\omega_j = x_j/|x|$.

Using accordingly Lemma 8.1 then system (8.2) arises as a combination of (8.3) and (8.7).

Next we adopt the ansatz that the chemical reaction takes place with definite rate and it occurs if and only if a pair of A - B molecules are in a distance less than R , see Fig. 8.1 Under this law of chemical reaction the master equation for A molecule is now formulated by

Fig. 8.1 Chemical reaction by distance R



$$\begin{aligned} \frac{\partial q_A}{\partial t} &= D_A \Delta q_A - \frac{k^{B \rightarrow A}}{v} \int_{|y|, |y-A| \leq R} q_B \, dy \cdot q_A \\ \frac{\partial q_B}{\partial t} &= D_B \Delta q_B - \frac{k^{A \rightarrow B}}{v} \int_{|y|, |y-B| \leq R} q_A \, dy \cdot q_B \end{aligned}$$

Reaction diffusion system

$$\begin{aligned}
& q_A(x, t + \Delta t) - q_A(x, t) \\
&= \int_{\mathcal{S}^{N-1}} \left[\sigma_A(x + \omega \Delta x, t; -\omega) q_A(x + \omega \Delta x, t) - \sigma_A(x, t; \omega) q_A(x, t) \right] d\omega \\
&\quad - \frac{Q_r^{A \rightarrow B} \Delta t}{v} \int_{B(x, R) \cap \Omega} q_B(y, t) dy \cdot q_A(x, t),
\end{aligned}$$

where $0 \leq \sigma_A = \sigma_A(x, t; \omega) \leq 1$ denotes the transient probability of A molecule, and, more importantly, $Q_r^{A \rightarrow B}$ stands for the rate by which A molecule hits B molecule to cause chemical reaction per unit time. The normalizing factor v has thus the dimension of volume. It must hold that

$$0 \leq \frac{1}{v} \int_{B(x, R) \cap \Omega} q_B dy \leq 1$$

because this $\frac{1}{v} \int_{B(x, R) \cap \Omega} q_B dy$ stands for the existence probability of B molecule inside $B(x, R)$ with q_B regarded to be 0 outside Ω and consequently it follows that $v = |B(x, R)| = \omega_N R^N$. Similarly we derive

$$\begin{aligned}
q_B(x, t + \Delta t) - q_B(x, t) &= \int_{\mathcal{S}^{N-1}} \sigma_B(x + \omega \Delta x, t; -\omega) q_B(x + \omega \Delta x, t) \\
&\quad - \sigma_B(x, t; \omega) q_B(x, t) d\omega \\
&\quad - \frac{Q_r^{B \rightarrow A} \Delta t}{|B(x, R)|} \int_{B(x, R) \cap \Omega} q_A(y, t) dy \cdot q_B(x, t),
\end{aligned}$$

where $Q_r^{B \rightarrow A}$ denotes the rate by which B molecule hits A molecule to cause chemical reaction per unit time. In the case of constant transient probabilities, then the limit system

$$\frac{\partial q_A}{\partial t} = D_A \Delta q_A - \frac{Q_r^{A \rightarrow B}}{|B(\cdot, R)|} \int_{B(\cdot, R) \cap \Omega} q_B dy \cdot q_A, \quad (8.9)$$

$$\frac{\partial q_B}{\partial t} = D_B \Delta q_B - \frac{Q_r^{B \rightarrow A}}{|B(\cdot, R)|} \int_{B(\cdot, R) \cap \Omega} q_A dy \cdot q_B, \quad (8.10)$$

arises in the mean field limit $\Delta t \downarrow 0$, $\Delta x \downarrow 0$ using again Lemma 8.1. Then (8.9)–(8.10) is transformed to (8.2) if and only if

$$Q_r^{A \rightarrow B} = k_B = k[B]_*, \quad \text{and} \quad Q_r^{B \rightarrow A} = k_A = k[A]_*. \quad (8.11)$$

More precisely, when the solution (q_A, q_B) to (8.9)–(8.10) is independent of x then it satisfies (8.3) in $\Omega_R \times (0, T)$, $\Omega_R = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > R\}$, provided that (8.11) is satisfied.

Next using the notation

$$\begin{aligned} u &= [A] = q_A[A]_*, & d_1 &= D_A, \\ v &= [B] = q_B[B]_*, & d_2 &= D_B, \end{aligned}$$

then problem (8.9)–(8.10) with (8.11) reads

$$u_t = d_1 \Delta u - ku \cdot \int_{B(\cdot, R) \cap \Omega} v \quad \text{in } \Omega \times (0, T), \quad (8.12)$$

$$v_t = d_2 \Delta v - kv \cdot \int_{B(\cdot, R) \cap \Omega} u \quad \text{in } \Omega \times (0, T), \quad (8.13)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (8.14)$$

$$u|_{t=0} = u_0(x) \geq 0, \quad v|_{t=0} = v_0(x) \geq 0 \quad \text{in } \Omega, \quad (8.15)$$

where

$$\int = \frac{1}{\omega_N R^N} \int.$$

8.2 Mathematical Analysis

8.2.1 Preliminary Results

We first note that by a standard argument we obtain a unique local-in-time classical solution

$$(u, v) \in C^{2,1}(\overline{\Omega} \times [0, T])^2,$$

of problem (8.12)–(8.15), for $0 < T < 1$, under the assumption

$$0 \leq u_0, v_0 \in C^2(\overline{\Omega}), \quad \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \quad \text{on } \partial \Omega. \quad (8.16)$$

Then it follows that

$$0 \leq u \leq \|u_0\|_\infty, \quad 0 \leq v \leq \|v_0\|_\infty \quad \text{in } Q_T = \Omega \times (0, T), \quad (8.17)$$

by virtue of the comparison principle. The later a priori estimate and the standard parabolic regularity entail that the solution of problem (8.12)–(8.15) is extended globally in time, [14].

Furthermore, by Schauder-type arguments we obtain

$$\|u\|_{C^{1+\theta, 1/2+\theta/2}(Q_T)} + \|v\|_{C^{1+\theta, 1/2+\theta/2}(Q_T)} \leq C,$$

for $0 < \theta < 1$ with a constant $C = C(k, T)$ independent of $R > 0$, which guarantees the convergence as $R \downarrow 0$, i.e.

$$u_R \rightarrow u, \quad v_R \rightarrow v \quad \text{in } C^{1,0}(\overline{Q_T}), \quad (8.18)$$

passing to a subsequence. This $(u, v) \in C^{1,0}(\overline{Q_T})$ is a weak solution to the following (local) reaction-diffusion system

$$u_t = d_1 \Delta u - kuv \quad \text{in } \Omega \times (0, T), \quad (8.19)$$

$$v_t = d_2 \Delta v - kvu \quad \text{in } \Omega \times (0, T), \quad (8.20)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (8.21)$$

$$u|_{t=0} = u_0(x) \geq 0, \quad v|_{t=0} = v_0(x) \geq 0 \quad \text{in } \Omega, \quad (8.22)$$

and, hence it is derived, by means of parabolic regularity, that it is the classical solution. Due to the uniqueness of the solution of problem (8.19)–(8.22), we finally derive that the limit in (8.18) is the (unique) solution of (8.19)–(8.22).

Several properties are known regarding the dynamics of the solutions of problem (8.19)–(8.22). First if $\|u_0\|_1 = \|v_0\|_1$, [7, 8, 10], then the solution decays with rate which resembles that of the ODE part

$$\begin{aligned} \frac{dU}{dt} &= -kUV, \quad U(0) = \bar{u}_0 \equiv \int_{\Omega} u_0, \\ \frac{dV}{dt} &= -kUV, \quad V(0) = \bar{v}_0 \equiv \int_{\Omega} v_0. \end{aligned}$$

More precisely, as $t \rightarrow \infty$ we have

$$\begin{aligned} \|(u, v)(\cdot, t)\|_{\infty} &= O(t^{-1}), \\ \|(u, v)(\cdot, t) - (U, V)(t)\|_{\infty} &= O(t^{-2}), \\ \|(u, v)(\cdot, t) - (\bar{u}, \bar{v})(t)\|_{\infty} &= O(e^{-d\mu_2 t}), \end{aligned}$$

for

$$\bar{u}(t) = \int_{\Omega} u(\cdot, t), \quad \bar{v}(t) = \int_{\Omega} v(\cdot, t),$$

where

$$d = \min\{d_1, d_2\}, \quad (8.23)$$

and $\mu_2 > 0$ denotes the second eigenvalue of $-\Delta$ provided with the Neumann boundary condition. In the complementary case, when $\|u_0\|_1 > \|v_0\|_1$, only u survives as $t \uparrow +\infty$ and there holds

$$\begin{aligned} \|u(\cdot, t) - u_\infty\|_\infty &= \begin{cases} O(e^{-\beta t}), & d_1\mu_2 \neq u_\infty \\ O(te^{-\beta t}), & d_1\mu_2 = u_\infty \end{cases}, \\ \|v(\cdot, t)\|_\infty &= O(e^{-u_\infty t}), \end{aligned}$$

where $\beta = \min\{d_1\mu_2, u_\infty\}$ and

$$u_\infty = \int_{\Omega} (u_0 - v_0) > 0.$$

The second property characterizing problem (8.19)–(8.22) is the phase separation occurring as $k \uparrow +\infty$, see [4]. In particular, if we put as $k \uparrow +\infty$ $z = -u$ and assume $z_0 \cdot v_0 = 0$, $z_0 = -u_0$ then (z^k, v^k) converges strongly in $L^1(Q_T)$ with the limit denoted by (z, v) .

Set $w = z + v$ then w solves the following two-phases problem

$$w_t = \nabla \cdot d(w) \nabla w \quad \text{in } \Omega \times (0, T), \quad (8.24)$$

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (8.25)$$

$$w|_{t=0} = w_0(x) \quad \text{in } \Omega, \quad (8.26)$$

where $w_0 = z_0 + v_0$ and

$$d(w) = \begin{cases} d_1, & w > 0 \\ \frac{d_1+d_2}{2}, & w = 0 \\ d_2, & w < 0. \end{cases}$$

There also holds that

$$w^+ = v, \quad w^- = -z, \quad w^+ \cdot w^- = 0, \quad (8.27)$$

where $w^\pm = \max\{\pm w, 0\}$.

The above observation actually means that the interaction of A, B molecules is thus reduced to the non-uniform diffusion as $k \uparrow +\infty$ in the case where $z_0 \cdot v_0 = 0$.

Indeed $w = w(x, t) \in L^1(Q_T)$ is the weak solution of (8.24)–(8.26) with $\nabla w \in L^2(Q_T)$ satisfying

$$\iint_{Q_T} w \xi_t - d(w) \nabla w \cdot \nabla \xi \, dx dt + \int_{\Omega} w_0(x) \xi(x, 0) dx = 0, \quad (8.28)$$

for any

$$\xi = \xi(x, t) \in C^1(\overline{\Omega} \times [0, T)), \quad \xi = 0, \quad 0 < T - t \ll 1. \quad (8.29)$$

The value $d(w) = \frac{d_1+d_2}{2}$ at $w = 0$ does not contribute in (8.24) because $\nabla w = 0$ a.e. on $\{w = 0\}$. Furthermore w is also the solution to the free boundary problem, [2],

$$\begin{aligned}
z_t &= d_1 \Delta z && \text{in } Q_+, \\
v_t &= d_2 \Delta v && \text{in } Q_-, \\
d_1 \frac{\partial z}{\partial \nu} + d_2 \frac{\partial v}{\partial \nu} &= 0 && \text{on } \Gamma, \\
\frac{\partial z}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 && \text{on } \partial \Omega \times (0, T), \\
z|_{t=0} &= z_0(x), \quad v|_{t=0} = v_0(x) && \text{in } \Omega,
\end{aligned}$$

where

$$\begin{aligned}
Q_- &= \{(x, t) \in Q_T \mid z(x, t) < 0 = v(x, t)\}, \\
Q_+ &= \{(x, t) \in Q_T \mid z(x, t) = 0 < v(x, t)\}, \\
\Gamma &= \{(x, t) \in Q_T \mid z(x, t) = v(x, t) = 0\}.
\end{aligned}$$

Regarding to the regularity of w there actually holds $w = w(x, t) \in C^{\theta, \theta/2}(\overline{Q_T})$, $0 < \theta < 1$, whilst an improved partial regularity result is valid. In particular there is an open set $\mathcal{O}$ in Q_T such that $w = w(x, t) \in C^{2+\theta, 1+\theta/2}(\mathcal{O})$ with $C^{2+\theta, 1+\theta/2}$ -portion of the free boundary $\{(x, t) \in \mathcal{O} \mid w(x, t) = 0\} \subset \Gamma$, see [22]. The residual set $W = Q_T \setminus \mathcal{O}$, furthermore, is composed of $W_i, i = 1, 2$ satisfying $\mathcal{P}^N(W_1) = 0$ and

$$\lim_{r \downarrow 0} \frac{1}{r^{n+2}} \int_{P_r(x, t)} |\nabla w|^2 dx dt = 0, \quad (x, t) \in W_2,$$

where

$$P_r(x, t) = \{(y, s) \in Q_T \mid |y - x| < r, |s - t| < r^2\},$$

and

$$\mathcal{P}^N(W) = \liminf_{\delta \downarrow 0} \left\{ \sum_j r_j^N \mid W \subset \bigcup_j P_{r_j}(x_j, t_j), \ 2r_j < \delta \text{ for all } j \right\}.$$

Finally we note that the phase field model associated to (8.24)–(8.26), see [18], is

$$w_t = \nabla \cdot d(\varphi) \nabla w \quad \text{in } \Omega \times (0, T), \tag{8.30}$$

$$\tau \varphi_t = -\delta \mathcal{F}_w(\varphi) \quad \text{in } \Omega \times (0, T), \tag{8.31}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, T), \tag{8.32}$$

$$w(x, 0) = w_0(x), \quad \varphi(x, 0) = \varphi_0(x) \quad \text{in } \Omega, \tag{8.33}$$

where

$$d(\varphi) = \frac{1}{2}(d_1 - d_2)\varphi + \frac{1}{2}(d_1 + d_2),$$

$$\mathcal{F}_w(\varphi) = \int_{\Omega} \frac{\xi^2}{2} |\nabla \varphi|^2 + W(\varphi) - 2w\varphi \, dx,$$

$$W(\varphi) = \frac{1}{4}(1 - \varphi^2)^2,$$

for ξ being a constant related to the molecular distance.

This phase field model actually realizes the continuously varying diffusion coefficient which takes values d_1 , $\frac{d_1+d_2}{2}$, and d_2 according as $\varphi = 1$, $\varphi = 0$, and $\varphi = -1$, respectively. Conversely, $\varphi = 1$ and $\varphi = -1$ are stable according to $w > 0$ and $w < 0$, respectively.

8.2.2 Phase Separation

In the current subsection we are investigating whether the phenomenon of phase separation can be predicted by the non-local reaction-diffusion system (8.12)–(8.14) when the reaction rate k is extremely big, i.e. when the indicated reactions happen very rapidly, see Fig. 8.2. The approach of the current section goes along the lines of [4, 10]. However, here, we have to take into consideration the non-local interaction kernel, which stems from the action-reaction law.

We first put $z = -u$ into (8.12)–(8.14) to obtain

$$z_t = d_1 \Delta z - \frac{1}{\varepsilon} z \cdot \int_{B(\cdot, R) \cap \Omega} v \quad \text{in } \Omega \times (0, T), \quad (8.34)$$

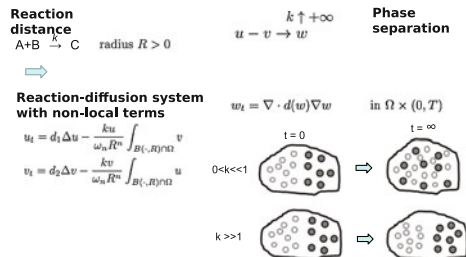
$$v_t = d_2 \Delta v + \frac{1}{\varepsilon} v \cdot \int_{B(\cdot, R) \cap \Omega} z \quad \text{in } \Omega \times (0, T), \quad (8.35)$$

$$\frac{\partial z}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (8.36)$$

$$z|_{t=0} = z_0(x) = -u_0(x) \leq 0, \quad v|_{t=0} = v_0(x) \geq 0 \quad \text{in } \Omega, \quad (8.37)$$

where $\varepsilon = k^{-1}$. So in the following we prove that if the initial states of A , B molecules are separated as

Fig. 8.2 Phase separation process for fast reaction



$$v_0(x) \cdot z_0(y) = 0, \quad |x - y| < R, \quad x, y \in \Omega, \quad (8.38)$$

then there is the limit (z, v) of $(z^\varepsilon, v^\varepsilon)$ as $\varepsilon \downarrow 0$ which stabilizes at the same state, that is

$$v(x, t) \cdot z(y, t) = 0, \quad |x - y| < R, \quad x, y \in \Omega, \quad t \geq 0.$$

The above result can be stated into the following

Theorem 8.1 *If (z_0, v_0) satisfies (8.16) and (8.38) then any $\varepsilon_j \downarrow 0$ admits a subsequence denoted by the same symbol such that*

$$\begin{aligned} v^{\varepsilon_j} &\rightarrow w^+, \quad z^{\varepsilon_j} \rightarrow -w^-, \quad v^{\varepsilon_j} + z^{\varepsilon_j} \rightarrow w && \text{in } L^1(Q_T), \\ \nabla v^{\varepsilon_j} &\rightarrow \nabla w^+, \quad \nabla z^{\varepsilon_j} \rightarrow -\nabla w^- && \text{in } L^2(Q_T). \end{aligned}$$

Furthermore, there holds

$$w^+(x, t) \cdot w^-(y, t) = 0, \quad \text{for almost everywhere (a.e.) } x, y \in \Omega, \quad |x - y| < R, \quad t \geq 0,$$

and $w^+ = v$, $w^- = -z$.

Before we proceed to the proof of Theorem 8.1 we first need to prove some auxiliary lemmas

Lemma 8.2 *There is $C_1 = C_1(d_1, d_2, \|z_0\|_\infty, \|v_0\|_\infty) > 0$ independent of ε such that*

$$\begin{aligned} &\iint_{Q_T} |\nabla z|^2 + |\nabla v|^2 \, dx dt + \frac{1}{d\varepsilon} \int_0^T dt \cdot \\ &\int_\Omega \left[z^2(x, t) \int_{B(x, R) \cap \Omega} v(\cdot, t) - v^2(x, t) \int_{B(x, R) \cap \Omega} z(\cdot, t) \right] dx \leq C_1, \end{aligned} \quad (8.39)$$

where $d > 0$ is defined by (8.23) and $(z, v) = (z^\varepsilon(x, t), v^\varepsilon(x, t))$ is the solution to (8.34)–(8.37).

Proof We first note that

$$\|z(\cdot, t)\|_\infty, \|v(\cdot, t)\|_\infty \leq C_2, \quad 0 \leq t < T, \quad (8.40)$$

by (8.17). Then we observe

$$\begin{aligned} &\int_\Omega [v^2(x, T) + z^2(x, T) - v_0^2(x) - z_0^2(x)] \, dx = \iint_{Q_T} \frac{\partial}{\partial t} (v^2 + z^2) \, dx dt \\ &= 2 \iint_{Q_T} v \left(d_2 \Delta v + \frac{1}{\varepsilon} v \int_{B(\cdot, R) \cap \Omega} z \right) + z \left(d_1 \Delta z - \frac{1}{\varepsilon} z \int_{B(\cdot, R) \cap \Omega} v \right) \, dx dt, \end{aligned}$$

where the boundary condition is applicable. There also holds that

$$\begin{aligned}
& 2d_2 \iint_{Q_T} |\nabla v|^2 dx dt + 2d_1 \iint_{Q_T} |\nabla z|^2 dx dt \\
& + \frac{2}{\varepsilon} \int_0^T dt \int_{\Omega} \left[z^2(x, t) \fint_{B(x, R) \cap \Omega} v(\cdot, t) - v^2(x, t) \fint_{B(x, R) \cap \Omega} z(\cdot, t) \right] dx \\
& = \int_{\Omega} [v_0^2(x) + z_0^2(x) - v^2(x, T) - z^2(x, T)] dx \\
& \leq 2|\Omega|C_2^2,
\end{aligned}$$

and hence

$$\begin{aligned}
& \iint_{Q_T} (|\nabla v|^2 + |\nabla z|^2) dx dt \\
& + \frac{1}{d\varepsilon} \int_0^T dt \int_{\Omega} \left[z^2(x, t) \fint_{B(x, R) \cap \Omega} v(\cdot, t) - v^2(x, t) \fint_{B(x, R) \cap \Omega} z(\cdot, t) \right] dx \\
& \leq \frac{2|\Omega|}{d} C_2^2 = C_1.
\end{aligned}$$

□

Lemma 8.3 *Whenever (8.38) holds then*

$$\|v_t(\cdot, t)\|_1 + \|z_t(\cdot, t)\|_1 \leq C_3, \quad 0 \leq t < T, \quad (8.41)$$

where $C_3 = d_2 \|\Delta v_0\|_1 + d_1 \|\Delta z_0\|_1$ and $(z, v) = (z^\varepsilon(x, t), v^\varepsilon(x, t))$ is the solution to (8.34)–(8.37).

Proof We take a convex function $\Phi(x)$ such that $\Phi(0) = \Phi'(0) = 0$ to obtain

$$\begin{aligned}
& \int_0^T \frac{d}{dt} \int_{\Omega} [\Phi(v_t) + \Phi(z_t)] dx dt \\
& = \iint_{Q_T} \Phi'(v_t) \left[d_2 \Delta v_t + \frac{1}{\varepsilon} \left(v \fint_{B(x, R) \cap \Omega} z \right)_t \right] dx dt \\
& + \iint_{Q_T} \Phi'(z_t) \left[d_1 \Delta z_t - \frac{1}{\varepsilon} \left(z \fint_{B(x, R) \cap \Omega} v \right)_t \right] dx dt \\
& = - \iint_{Q_T} d_2 \Phi''(v_t) |\nabla v_t|^2 + d_1 \Phi''(z_t) |\nabla z_t|^2 dx dt \\
& + \frac{1}{\varepsilon} \iint_{Q_T} \Phi'(v_t) \left(v \fint_{B(x, R) \cap \Omega} z \right)_t - \Phi'(z_t) \left(z \fint_{B(x, R) \cap \Omega} v \right)_t dx dt.
\end{aligned}$$

Since

$$\begin{aligned} \int_{\Omega} \Phi'(v_t) \left(v \int_{B(x, R) \cap \Omega} z \right)_t dx &= \iint_{\mathbf{R}^N \times \mathbf{R}^N} \Phi'(v_t(x, t)) \\ &\quad \times \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) \{v(x, t) z(y, t)\}_t dx dy, \end{aligned}$$

and

$$\begin{aligned} &\int_{\Omega} \Phi'(z_t) \left(z \int_{B(x, R) \cap \Omega} v \right)_t dx \\ &= \iint_{\mathbf{R}^N \times \mathbf{R}^N} \Phi'(z_t(x, t)) \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) \{v(y, t) z(x, t)\}_t dx dy \\ &= \iint_{\mathbf{R}^N \times \mathbf{R}^N} \Phi'(z_t(y, t)) \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) \{v(x, t) z(y, t)\}_t dx dy, \end{aligned}$$

we derive that

$$\begin{aligned} &\int_{\Omega} [\Phi(v_t(x, T)) + \Phi(z_t(x, T))] dx \leq \int_{\Omega} [\Phi(v_t(x, 0)) + \Phi(z_t(x, 0))] dx \\ &+ \frac{1}{\varepsilon} \int_0^T dt \cdot \iint_{\mathbf{R}^N \times \mathbf{R}^N} \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) \\ &\quad \cdot [\Phi'(v_t(x, t)) - \Phi'(z_t(y, t))] [v_t(x, t) z(y, t) + v(x, t) z_t(y, t)] dx dy, \end{aligned} \quad (8.42)$$

taking also into consideration that $\Phi'' \geq 0$.

Next we place Φ into (8.42) by a sequence of convex functions $\Phi_n = \Phi_n(s)$ such that $|\Phi'_n(s)| \leq 1$ to derive

$$\begin{aligned} \Phi_n(s) &\rightarrow |s| && \text{locally uniformly in } s \in \mathbf{R}, \\ \Phi'_n(s) &\rightarrow \text{sgn}(s) && \text{pointwise in } s \in \mathbf{R} \setminus \{0\}, \end{aligned}$$

and then we take the limit $n \rightarrow \infty$. Since (z, v) is the classical solution we have

$$\begin{aligned} &\int_0^T dt \cdot \iint_{\mathbf{R}^N \times \mathbf{R}^N} \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) \\ &\quad \cdot \Phi'_n(v_t(x, t)) v_t(x, t) z(y, t) dx dy \rightarrow \\ &\int_0^T dt \cdot \iint_{\mathbf{R}^N \times \mathbf{R}^N} \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) |v_t(x, t)| z(y, t) dx dy, \end{aligned}$$

and hence

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \int_0^T dt \cdot \iint_{\mathbf{R}^N \times \mathbf{R}^N} \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) \\ &\quad \cdot [\Phi'_n(v_t(x, t)) - \Phi'_n(z_t(y, t))] v_t(x, t) z(y, t) dx dy \leq 0, \end{aligned}$$

by $z \leq 0$ and

$$\left[\Phi'_n(v_t(x, t)) - \Phi'_n(z_t(y, t)) \right] v_t(x, t) \geq \Phi'_n(v_t(x, t)) v_t(x, t) - |v_t(x, t)|.$$

Similarly we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_0^T dt \cdot \iint_{\mathbf{R}^N \times \mathbf{R}^N} \frac{\chi_{|x-y| < R}(x, y)}{\omega_N R^N} \chi_\Omega(x) \chi_\Omega(y) \\ \cdot \left[\Phi'_n(v_t(x, t)) - \Phi'_n(z_t(y, t)) \right] v(x, t) z_t(y, t) dx dy \leq 0, \end{aligned}$$

and thus

$$\int_\Omega |v_t(x, T)| + |z_t(x, T)| dx \leq \int_\Omega |v_{0t}(x)| + |z_{0t}(x)| dx.$$

Consequently, we deduce

$$\int_\Omega |z_{0t}(x)| dx \leq d_1 \|\Delta z_0\|_1 + \frac{1}{\varepsilon} \int_\Omega \left| z_0(x) \int_{B(x, R) \cap \Omega} v_0(y) dy \right| dx = d_1 \|\Delta z_0\|_1,$$

by $z_0 \leq 0 \leq v_0$ and (8.38), and, similarly,

$$\int_\Omega |v_{0t}(x)| dx \leq d_2 \|\Delta v_0\|_1,$$

and the proof is complete. $\square$

Proof of Theorem 8.1 By virtue of Lemmas 8.2 and 8.3, we obtain $z^\varepsilon \leq 0 \leq v^\varepsilon$, and owing to the compact imbedding $W^{1,1}(Q_T) \hookrightarrow L^1(Q_T)$ we can find a subsequence $\varepsilon_j \rightarrow 0$, henceforth denoted with $\varepsilon \rightarrow 0$ for simplicity, such that for the solution $(v^\varepsilon, z^\varepsilon)$ to (8.34)–(8.37) there holds

$$\begin{aligned} v^\varepsilon &\rightharpoonup v, \quad z^\varepsilon \rightarrow z && \text{a.e. and strongly in } L^1(Q_T), \\ \nabla v^\varepsilon &\rightharpoonup \nabla v, \quad \nabla z^\varepsilon \rightharpoonup \nabla z && \text{weakly in } L^2(Q_T). \end{aligned} \quad (8.43)$$

(Note that owing to the uniqueness of the weak solution w of problem (8.24)–(8.26) it will be eventually verified in the proof of Theorem 8.3 that $v = w^+$ and $z = -w^-$.)

Since $z^\varepsilon \leq 0 \leq v^\varepsilon$ we finally get

$$z \leq 0 \leq v, \quad \text{a.e. in } Q_T. \quad (8.44)$$

We have, on the other hand,

$$\int_0^T dt \int_\Omega \left[z^\varepsilon(x, t)^2 \int_{B(x, R) \cap \Omega} v^\varepsilon(\cdot, t) - v^\varepsilon(x, t)^2 \int_{B(x, R) \cap \Omega} z^\varepsilon(\cdot, t) \right] dx \leq C_1 d \varepsilon,$$

by Lemma 8.2 which implies

$$\int_0^T dt \int_{\Omega} \left[z^2(x, t) \int_{B(x, R) \cap \Omega} v(\cdot, t) - v^2(x, t) \int_{B(x, R) \cap \Omega} z(\cdot, t) \right] dx \leq 0.$$

Thus we obtain

$$z^2 \int_{B(\cdot, R) \cap \Omega} v = v^2 \int_{B(\cdot, R) \cap \Omega} z = 0 \quad \text{a.e. in } Q_T,$$

and hence

$$z(x, t) \cdot v(y, t) = 0 \quad \text{a.e. } x, y \in \Omega, \quad |x - y| < R, \quad t \geq 0.$$

□

In the following we provide a similar to Theorem 8.1 result but for the regularized reaction radius problem

$$z_t = d_1 \Delta z - \frac{1}{\varepsilon} z \cdot \bar{v} \quad \text{in} \quad \Omega \times (0, T), \quad (8.45)$$

$$v_t = d_2 \Delta v + \frac{1}{\varepsilon} v \cdot \bar{z} \quad \text{in} \quad \Omega \times (0, T), \quad (8.46)$$

$$\frac{\partial z}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0, T), \quad (8.47)$$

$$z(x, 0) = z_0(x) = -u_0(x) \leq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \quad (8.48)$$

where

$$\begin{aligned} \bar{v}(x, t) &= \int_{\Omega} \chi(x, y) v(y, t) dy, \\ \bar{z}(x, t) &= \int_{\Omega} \chi(x, y) z(y, t) dy, \end{aligned}$$

using a smooth function $\chi = \chi(x, y)$ such that

$$\chi(x, x) > 0, \quad 0 \leq \chi(x, y) = \chi(y, x) \leq 1. \quad (8.49)$$

In particular there holds

Theorem 8.2 *If (z_0, v_0) satisfies (8.16) and*

$$z_0(x) \chi(x, y) v_0(y) = 0, \quad x, y \in \Omega, \quad (8.50)$$

then any $\varepsilon_j \downarrow 0$ admits a subsequence denoted by the same symbol such that

$$\begin{aligned} v^{\varepsilon_j} &\rightarrow w^+, \quad z^{\varepsilon_j} \rightarrow -w^-, \quad v^{\varepsilon_j} + z^{\varepsilon_j} \rightarrow w && \text{in } L^1(Q_T), \\ \nabla v^{\varepsilon_j} &\rightarrow \nabla w^+, \quad \nabla z^{\varepsilon_j} \rightarrow -\nabla w^- && \text{in } L^2(Q_T), \end{aligned}$$

where $(z^\varepsilon, v^\varepsilon)$ denotes the solution to (8.45)–(8.48). There also holds that

$$z(x, t)\chi(x, y)v(y, t) = 0, \quad \text{a.e. } x, y \in \Omega, t \geq 0, \quad (8.51)$$

where $z = -w^-$ and $v = w^+$, and, furthermore,

$$\bar{w}_t = d_1 \Delta \bar{z} + d_2 \Delta \bar{v} \quad \text{in } Q_T, \quad (8.52)$$

where $\bar{w} = \bar{z} + \bar{v}$.

Proof We can actually follow a similar approach with the one employed for the system (8.45)–(8.48) associated with regularized reaction radius. We then deduce

$$\begin{aligned} &\iint_{Q_T} |\nabla z|^2 + |\nabla v|^2 \, dxdt + \frac{1}{d\varepsilon} \int_0^T dt \cdot \\ &\iint_{\Omega \times \Omega} \left[z^2(x, t)\chi(x, y)v(y, t) - v^2(x, t)\chi(x, y)z(y, t) \right] dx dy \leq C_1, \end{aligned}$$

in the same spirit with estimation (8.39). Additionally a similar relation to (8.41) holds under the assumption (8.50) where the symmetry $\chi(x, y) = \chi(y, x)$ is used essentially. Then the limit (8.43) is valid together with (8.44) and (8.51) due to $\chi(x, x) > 0$, $x \in \bar{\Omega}$.

Let

$$\bar{w}^\varepsilon = \bar{z}^\varepsilon + \bar{v}^\varepsilon = \int_{\Omega} \chi(\cdot, y)[z^\varepsilon(y, t) + v^\varepsilon(y, t)]dy.$$

Since the equality $\chi(x, y) = \chi(y, x)$ implies $\overline{z \cdot \bar{v}} = \overline{v \cdot \bar{z}}$ then we have

$$\begin{aligned} \bar{w}_t^\varepsilon(x, t) &= - \int_{\Omega} (d_1 \nabla z^\varepsilon(y, t) + d_2 \nabla v^\varepsilon(y, t)) \cdot \nabla_y \chi(x, y) dy \\ &= - \int_{\Omega} (d_1 \nabla z^\varepsilon(y, t) + d_2 \nabla v^\varepsilon(y, t)) \cdot \nabla_x \chi(x, y) dy. \end{aligned}$$

Testing by $\xi = \xi(x, t) \in C_0^1(\Omega \times [0, T))$ and using Green's identity implies

$$\begin{aligned} &\iint_{Q_T} \bar{w}^\varepsilon \xi_t dxdt + \int_{\Omega} \bar{w}_0^\varepsilon(x) \xi(x, 0) dx \\ &= \int_0^T dt \iint_{\Omega \times \Omega} (d_1 \nabla z^\varepsilon(y, t) + d_2 \nabla v^\varepsilon(y, t)) \cdot \nabla_x \chi(x, y) \xi(x, t) dx dy \end{aligned}$$

$$\begin{aligned}
&= - \int_0^T dt \iint_{\Omega \times \Omega} (d_1 \nabla z^\varepsilon(y, t) + d_2 \nabla v^\varepsilon(y, t)) \chi(x, y) \cdot \nabla_x \xi(x, t) dx dy \\
&= - \int_{Q_T} (d_1 \nabla \bar{z}^\varepsilon(y, t) + d_2 \nabla \bar{v}^\varepsilon(y, t)) \cdot \nabla_x \xi(x, t) dx, \tag{8.53}
\end{aligned}$$

and thus (8.52) is derived in a distributional form by taking the limit $\varepsilon \downarrow 0$. $\square$

The limit equation (8.52) arises by means of $(z^\varepsilon, v^\varepsilon)$ under an additional assumption for the weighted function $\chi(x, y)$ which is not the case of (8.34)–(8.37) for $\chi(x, y) = \chi_{B(x, R) \cap \Omega}(y) / \omega_N R^N$; here $\chi_{B(x, R) \cap \Omega}$ denotes the characteristic (indicator) function of $B(x, R) \cap \Omega$. More precisely there holds.

Theorem 8.3 *Assume that the assumptions of Theorem 8.2 are still valid, then if*

$$\frac{\partial \chi}{\partial v_x}(x, y) = 0, \quad (x, y) \in \partial \Omega \times \Omega, \tag{8.54}$$

there holds

$$\begin{aligned}
\bar{v}^\varepsilon &\rightarrow \bar{w}^+, \quad \bar{z}^\varepsilon \rightarrow -\bar{w}^-, \quad \bar{v}^\varepsilon + \bar{z}^\varepsilon \rightarrow \bar{w} && \text{in } L^1(Q_T), \\
\nabla \bar{v}^\varepsilon &\rightarrow \nabla \bar{w}^+, \quad \nabla \bar{z}^\varepsilon \rightarrow -\nabla \bar{w}^- && \text{in } L^2(Q_T),
\end{aligned} \tag{8.55}$$

as $\varepsilon \downarrow 0$ where $\bar{w} = \bar{z} + \bar{v}$ is the weak solution to (8.26).

Proof Due to (8.54) we obtain $\frac{\partial \bar{z}^\varepsilon}{\partial v} = \frac{\partial \bar{v}^\varepsilon}{\partial v} = 0$ on $\partial \Omega$. Then $\xi = \xi(x, t)$ in (8.53) is extended as in (8.29), and, therefore, the above limit $w = w(x, t)$ is a weak solution to (8.24)–(8.26). Since this weak solution is unique the convergence (8.55) is valid. $\square$

8.2.3 Long-Time Behavior

Next we focus on the investigation of the long-time behavior of the solution of our non-local reaction-diffusion system. For simplicity, we go back again to the initial problem (8.12)–(8.15) for which the following convergence result is valid

Theorem 8.4 *If $\|u_0\|_{L^1(\Omega)} \geq \|v_0\|_1$ and (8.16) still holds then the global-in-time solution (u, v) to (8.12)–(8.15) satisfies*

$$u(\cdot, t) \rightarrow u_\infty, \quad v(\cdot, t) \rightarrow 0 \quad \text{in } C^m(\bar{\Omega}), \quad t \uparrow +\infty, \tag{8.56}$$

for every $m \in [0, 2)$, where

$$u_\infty = \frac{\|u_0\|_{L^1(\Omega)} - \|v_0\|_{L^1(\Omega)}}{|\Omega|} \geq 0.$$

Proof We first note that problem (8.34)–(8.37) determines a conservation law, that is

$$\frac{d}{dt} \int_{\Omega} (u - v) dx = k \int_{\Omega} \left[u \oint_{B(\cdot, R) \cap \Omega} v - v \oint_{B(\cdot, R) \cap \Omega} u \right] dx,$$

with the right-hand side equal to

$$k \iint_{\mathbf{R}^N \times \mathbf{R}^N} \frac{1}{\omega_N R^N} \chi_{\Omega}(x) \chi_{\Omega}(y) \chi_{|x-y| < R}(x, y) \cdot [u(x, t)v(y, t) - u(y, t)v(x, t)] dx dy = 0,$$

by Fubini's theorem and the symmetricity of the characteristic function

$$\chi_{|x-y| < R}(x, y).$$

Hence,

$$\int_{\Omega} [u(x, t) - v(x, t)] dx = \int_{\Omega} [u_0(x) - v_0(x)] dx,$$

which yields

$$\oint_{\Omega} [u(x, t) - v(x, t)] dx = \bar{u}_0 - \bar{v}_0 = u_{\infty}. \quad (8.57)$$

Next we define the operator

$$\mathcal{B}_p(w) = (-d_2 \Delta + \alpha)w, \quad \text{with domain } D(\mathcal{B}_p) = \left\{ w \in W^{2,p}(\Omega) \mid \frac{\partial w}{\partial \nu} \Big|_{\partial \Omega} = 0 \right\},$$

where $\alpha > 0$ and $1 < p < \infty$. The spectrum of operator $\mathcal{B}_p$ settles in the positive axis owing to the resolvent estimate and hence the fractional powers $\mathcal{B}_p^{\gamma}$, $0 \leq \gamma < 1$ are defined. It also generates the analytic semigroup $\{e^{-t\mathcal{B}_p}\}_{t \geq 0}$ and there holds

$$\|\mathcal{B}_p^{\gamma} e^{-t\mathcal{B}_p} w\|_p \leq C_2(\gamma) q^{-\gamma}(t) e^{-\alpha t} \|w\|_p, \quad (8.58)$$

for $t > 0$, see [6, 17] where $0 \leq \gamma < 1$ and $0 < q(t) = \min\{t, 1\} \leq 1$. Then it is readily seen

$$\int_0^t q^{-\gamma}(\sigma) e^{\delta \sigma} d\sigma \leq \begin{cases} C_3(\gamma, \delta) e^{\delta t}, & \delta > 0 \\ C_3(\gamma, \delta) (t + 1), & \delta = 0 \\ C_3(\gamma, \delta), & \delta < 0. \end{cases} \quad (8.59)$$

Finally, given $m \in [0, 2)$ and $p \in (1, +\infty)$ such that

$$m < 2 - N/p, \quad (\text{resp. } m < 2\beta - N/p, \quad \beta \in (0, 1)),$$

we obtain the imbedding, see [6],

$$D(\mathcal{B}_p) \subset C^m(\overline{\Omega}) \quad (\text{resp. } D(\mathcal{B}_p^\beta) \subset C^m(\overline{\Omega}), \quad \beta \in (0, 1)). \quad (8.60)$$

Using the variation of constants formula, then equation (8.35) can be written in the integral form

$$v(t) = e^{-t\mathcal{B}_p} v_0 + \alpha \int_0^t e^{-(t-s)\mathcal{B}_p} v(s) ds - k \int_0^t e^{-(t-s)\mathcal{B}_p} [v(s) \cdot \int_{B(\cdot, R) \cap \Omega} u(s)] ds. \quad (8.61)$$

Operating with $\mathcal{B}_p^\gamma$, we derive

$$\begin{aligned} \mathcal{B}_p^\gamma v(t) &= \mathcal{B}_p^\gamma e^{-t\mathcal{B}_p} v_0 + \alpha \int_0^t \mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} v(s) ds \\ &\quad - k \int_0^t \mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} [v(s) \cdot \int_{B(\cdot, R) \cap \Omega} u(s)] ds. \end{aligned}$$

Let $t \geq \delta > 0$ then due to (8.58) and (8.17) we deduce

$$\|\mathcal{B}_p^\gamma e^{-t\mathcal{B}_p} v_0\|_p \leq C(\gamma) q^{-\gamma}(t) e^{-\alpha t} \|v_0\|_p \leq C(\gamma, p, \delta), \quad (8.62)$$

while

$$\int_0^t \|\mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} v(s)\|_p ds \leq C(\gamma, p, \Omega) \int_0^t q^{-\gamma}(t-s) ds \leq C(\gamma, \delta, p, \Omega), \quad (8.63)$$

is derived by virtue of (8.17) and (8.58). Finally, we have

$$\int_0^t \left\| \mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} v(s) \int_{B(\cdot, R) \cap \Omega} u(s) \right\|_p ds \leq C(\gamma, \delta, p, \Omega), \quad (8.64)$$

since (8.17) implies

$$\|v(\cdot, t) \cdot \int_{B(\cdot, R) \cap \Omega} u(\cdot, t)\|_\infty \leq \|u_0\|_\infty \|v_0\|_\infty.$$

Combining (8.62), (8.63), and (8.64), we obtain

$$\left\| \mathcal{B}_p^\gamma v(t) \right\|_p \leq C(\gamma, \delta, p, \Omega), \quad t \geq \delta, \quad (8.65)$$

and hence the orbit $\{v(t)\}_{t \geq \delta}$ is compact in $C(\overline{\Omega})$ by Morrey's imbedding theorem. Therefore there exist $t_j \uparrow +\infty$ and $v^* \in C(\overline{\Omega})$ such that

$$v(\cdot, t_j) \longrightarrow v^* \text{ in } C(\overline{\Omega}), \quad j \rightarrow \infty. \quad (8.66)$$

Following the arguments of Lemma 3.1 in [9] it is also proved that the function

$$t \longmapsto \mathcal{B}_p^\gamma v(t) \text{ is Hölder continuous in } [\delta, +\infty). \quad (8.67)$$

Moreover, by (8.67) also follows that the function

$$t \longmapsto \|\nabla v(t)\|_2^2 \text{ is uniformly continuous in } [\delta, +\infty), \quad (8.68)$$

see also Lemma 8 in [15].

Now, multiplying the second equation of (8.35) by v and integrating over space and time, we obtain

$$\begin{aligned} \|v(t)\|_2^2 + 2d_2 \int_0^t \|\nabla v(s)\|_2^2 ds \\ + 2k \int_0^t \int_\Omega v^2(x, t) \left[\oint_{B(x, R) \cap \Omega} u(y, t) dy \right] dx ds = \|v_0\|_2^2, \end{aligned} \quad (8.69)$$

which yields

$$\int_0^\infty \|\nabla v(s)\|_2^2 ds < +\infty. \quad (8.70)$$

An immediate consequence of (8.68) and (8.70) is

$$\|\nabla v(t)\|_2^2 \longrightarrow 0, \quad t \uparrow +\infty. \quad (8.71)$$

Taking into account Poincaré's inequality

$$\mu_2 \|v(t) - \bar{v}(t)\|_2 \leq \|\nabla v(t)\|_2,$$

as well as (8.66) and (8.71), we derive

$$v(\cdot, t_j) \longrightarrow v^* = \text{constant in } C(\overline{\Omega}), \quad j \rightarrow \infty. \quad (8.72)$$

Analogous properties to (8.65) and (8.66) arise also for u , and, therefore, there is a subsequence of $\{t_j\}$ denoted by the same symbol and $u^* \in C(\overline{\Omega})$ such that

$$u(\cdot, t_j) \longrightarrow u^* = \text{constant in } C(\overline{\Omega}), \quad j \rightarrow \infty. \quad (8.73)$$

By (8.67) and Morrey's imbedding theorem, we finally obtain

$$t \longmapsto \|v(t)\|_\infty^2 \quad \text{is uniformly continuous in } [\delta, +\infty),$$

and, similarly,

$$t \longmapsto \|u(t)\|_\infty^2 \quad \text{is uniformly continuous in } [\delta, +\infty).$$

Thus there holds that

$$t \longmapsto \left\| v(\cdot, t)^2 \cdot \int_{B(\cdot, R) \cap \Omega} u(t) \right\|_1 \quad \text{is uniformly continuous in } [\delta, +\infty).$$

We also have

$$v(\cdot, t)^2 \cdot \int_{B(\cdot, R) \cap \Omega} u(t) \in L^1(0, +\infty; L^1(\Omega)),$$

by (8.69), and, hence

$$\left\| v(\cdot, t)^2 \cdot \int_{B(\cdot, R) \cap \Omega} u(t) \right\|_1 \rightarrow 0, \quad t \uparrow +\infty. \quad (8.74)$$

Therefore we derive

$$v(\cdot, t_j)^2 \cdot \int_{B(\cdot, R) \cap \Omega} u(t_j) \rightarrow F(\cdot, R)(v^*)^2 u^* \quad \text{in } L^1(\Omega), \quad j \rightarrow \infty$$

due to (8.72) and (8.73) where $F(x, R) = |B(x, R) \cap \Omega|/|B(x, R)|$. Then (8.74) entails

$$(v^*)^2 u^* = 0, \quad (8.75)$$

since $F(x, R) \neq 0$. On the other hand, we obtain

$$u^* - v^* = u_\infty, \quad (8.76)$$

due to (8.72), (8.73), and (8.57). In the case where $u_\infty > 0$ then (8.75) and (8.76) imply $u^* = u_\infty$, $v^* = 0$, while in the complimentary case $u_\infty = 0$ it follows that $u^* = v^* = 0$. Then the uniqueness of the above limits yields

$$u(\cdot, t) \longrightarrow u_\infty, \quad v(\cdot, t) \rightarrow 0 \text{ in } C(\overline{\Omega}), \quad t \uparrow +\infty. \quad (8.77)$$

In order to prove the last part of the theorem we use the following interpolation inequality [6],

$$\|\mathcal{B}_p^\theta v(t)\|_p \leq C \|\mathcal{B}_p^\gamma v(t)\|_p^{\theta/\gamma} \|v(t)\|_p^{1-\theta/\gamma}, \quad (8.78)$$

where $0 < \theta < \gamma < 1$. Since $\|\mathcal{B}_p^\gamma v(t)\|_p$ is bounded by (8.65), inequality (8.78) yields

$$\|\mathcal{B}_p^\theta v(t)\|_p \rightarrow 0, \quad t \uparrow +\infty, \quad (8.79)$$

for every $\theta \in (0, 1)$. Then (8.89) implies

$$\|v(t)\|_{C^m(\overline{\Omega})} \rightarrow 0, \quad t \uparrow +\infty,$$

for every $m \in [0, 2)$. Similarly, we obtain

$$\|u(t) - u_\infty\|_{C^m(\overline{\Omega})} \rightarrow 0, \quad t \uparrow +\infty,$$

which completes the proof. $\square$

In the context of biology, Theorem 8.4 actually means that if the initial concentrations $[A]$ and $[B]$ of the A and B molecules are equal then they will eventually die out. Otherwise, only the reactant with the highest concentration will survive.

Our motivation is mainly in cell biology associated with protein dimers, which is the reason why we restrict our study on the fundamental process (8.1). Our results, however, extend to the process $nA + mB \rightarrow C$ with minor modifications.

8.2.4 Decay Rate Towards the Steady States

The topic of the current subsection is the investigation of the rate of convergence of the solution of (8.12)–(8.15) towards the unique stationary solution.

Before we proceed with the derivation of the decay rate towards the steady states for the non-local reaction-diffusion system (8.12)–(8.15) we review the main existing results regarding the decay rates of the (local) reaction diffusion system (8.19)–(8.22). The reason for that is that we will eventually compare the decay rates for both systems (8.12)–(8.15) and (8.19)–(8.22) since the former system converges to the later as $R \downarrow 0$, see the above section as well as [12].

Theorem 8.5 ([7, 8, 10]) *The solution $(u, v) = (u(\cdot, t), v(\cdot, t))$ of (8.19)–(8.22) exists globally in time and decays towards the unique steady with the*

following rate. Below we assume $\|u_0\|_1 \geq \|v_0\|_1$ without loss of generality and put $u_\infty = (\|u_0\|_1 - \|v_0\|_1)/|\Omega| \geq 0$:

1. If $u_\infty > 0$ it holds that

$$\begin{aligned} \|u(\cdot, t) - u_\infty\|_\infty &= \begin{cases} O(e^{-\beta t}), & d_1\mu_2 \neq ku_\infty \\ O(te^{-\beta t}), & d_1\mu_2 = ku_\infty \end{cases} \\ \|v(\cdot, t)\|_\infty &= O(e^{-k u_\infty t}) \end{aligned}$$

as $t \uparrow +\infty$, where $\beta = \min\{d_1\mu_2, k u_\infty\}$.

2. If $u_\infty = 0$ it holds that

$$\begin{aligned} \|(u, v)(\cdot, t)\|_\infty &= O(t^{-1}), \\ \|(u, v)(\cdot, t) - (U, V)(t)\|_\infty &= O(t^{-2}), \\ \|(u, v)(\cdot, t) - (\bar{u}, \bar{v})(t)\|_\infty &= O(e^{-d\mu_2 t}), \end{aligned}$$

as $t \uparrow +\infty$, where (U, V) is the solution to the ODE system

$$\begin{aligned} \frac{dU}{dt} &= -kUV, \quad U(0) = \bar{u}_0 \equiv \int_\Omega u_0, \\ \frac{dV}{dt} &= -kUV, \quad V(0) = \bar{v}_0 \equiv \int_\Omega v_0, \end{aligned}$$

and

$$\bar{u}(t) = \int_\Omega u(\cdot, t), \quad \bar{v}(t) = \int_\Omega v(\cdot, t).$$

Moreover, $\mu_2 > 0$ is the second eigenvalue of $-\Delta$ associated with Neumann boundary condition, and

$$d = \min\{d_1, d_2\}.$$

Our result concerning the non-local reaction-diffusion system (8.12)–(8.15) is provided in accordance with the principal eigenvalue $\mu_1(R)$ of

$$-d_2\Delta w + u_\infty k F(R, x)w = \mu w \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad (8.80)$$

where

$$F(R, x) = \frac{|B(x, R) \cap \Omega|}{|B(x, R)|}.$$

Since Ω is a bounded domain with smooth boundary $\partial\Omega$, there holds

$$\bar{F}_R = \min_{x \in \bar{\Omega}} F(R, x) > 0.$$

We also have $F(R, x) \leq 1$, and thus

$$0 \leq u_\infty k \overline{F}_R \leq \mu_1(R) \leq u_\infty k, \quad (8.81)$$

follows from the Rayleigh principle

$$\mu_1(R) = \inf \left\{ \int_{\Omega} |\nabla w|^2 + u_\infty k F(R, x) w^2 dx \mid w \in H^1(\Omega), \|w\|_2 = 1 \right\}.$$

In particular, $\mu_1(R) = 0$ if and only if $u_\infty = 0$ by (8.81).

It also follows that

$$\lim_{R \downarrow 0} F(R, x) = \begin{cases} 1, & x \in \Omega \\ 1/2, & x \in \partial\Omega \end{cases},$$

and hence

$$\lim_{R \downarrow 0} \mu_1(R) = k u_\infty, \quad (8.82)$$

by Rellich's compactness theorem, [5], applied to

$$\begin{aligned} -d_2 \Delta w_R + u_\infty k F(R, x) w_R &= \mu_1(R) w_R, \quad w_R > 0 \text{ in } \Omega, \\ \frac{\partial w_R}{\partial \nu} \Big|_{\partial\Omega} &= 0, \quad \int_{\Omega} w_R^2 dx = 1. \end{aligned}$$

Owing to (8.82), the next result is regarded as a natural extension of some parts of Theorem 8.5.

Theorem 8.6 *The global-in-time solution $(u, v) = (u(\cdot, t), v(\cdot, t))$ of (8.12)–(8.15) satisfies the following decay rates towards the stationary state as $t \uparrow +\infty$ provided that $\|u_0\|_1 \geq \|v_0\|_1$:*

1. *If $u_\infty > 0$ there holds*

$$\begin{aligned} \|u(\cdot, t) - u_\infty\|_\infty &= \begin{cases} O(e^{-\beta_0 t}), & d_1 \mu_2 \neq \mu_1(R) \\ O(t e^{-\beta_0 t}), & d_1 \mu_2 = \mu_1(R) \end{cases}, \\ \|v(\cdot, t)\|_\infty &= O(e^{-\mu_1(R)t}), \end{aligned}$$

where $\beta_0 = \min\{d_1 \mu_2, \mu_1(R)\}$.

2. *If $u_\infty = 0$ then*

$$\|(u, v)(\cdot, t)\|_\infty = O(t^{-1}). \quad (8.83)$$

Comparing Theorems 8.5 and 8.6, we observe that the asymptotic equivalence of the solution to the non-local system (8.12)–(8.15) with the ODE solution and the spatial mean is absent. Actually, the difference between the preceding reaction

diffusion systems is that we have no spatially homogeneous solution for the non-local one (8.12)–(8.15).

We first provide some preliminaries and then by employing ideas initially developed in [7, 10] we prove the first and the second cases of Theorem 8.6. Henceforth, C_i , $i = 1, 2, \dots$, are used to denote various positive constants independent of elements involved in the corresponding inequality. If they depend on the parameters, say, $\alpha, \beta, \dots$, we shall write them as $C_i(\alpha, \beta, \dots)$.

In the following we also use the notation

$$\|(w_1, w_2)\|_p = \left(\|w_1\|_p^2 + \|w_2\|_p^2 \right)^{1/2},$$

for $1 \leq p \leq \infty$, where

$$\|w\|_p = \begin{cases} \left(\int_{\Omega} |w(x)|^p dx \right)^{1/p} & 1 \leq p < \infty, \\ \text{ess. sup}_{x \in \Omega} |w(x)| & p = \infty. \end{cases}$$

Moreover in the following, we use the operators

$$\begin{aligned} \mathcal{A}_p(w) &= -d_1 \Delta w, \\ \mathcal{B}_p(w) &= (-d_2 \Delta + \alpha)w, \end{aligned}$$

in $L^p(\Omega)$, $1 < p < \infty$, provided with the domains

$$D(\mathcal{A}_p) = D(\mathcal{B}_p) = \left\{ w \in W^{2,p}(\Omega) \mid \frac{\partial w}{\partial \nu} \Big|_{\partial \Omega} = 0 \right\},$$

where $\alpha > 0$. The spectrum of the operator $\mathcal{B}_p$ settles in the positive axis, while the operators $-\mathcal{A}_p$ and $-\mathcal{B}_p$ are sectorial which generate analytic semigroups denoted by $\{e^{-t\mathcal{A}_p}\}_{t \geq 0}$ and $\{e^{-t\mathcal{B}_p}\}_{t \geq 0}$, respectively. Next, we consider the projection

$$P_0 : L^p(\Omega) \rightarrow L^p(\Omega), \quad P_0(w) = \frac{1}{|\Omega|} \int_{\Omega} w(x) dx = \bar{w},$$

and put

$$P_+ = I - P_0.$$

Then the spectrum of the operator $\mathcal{A}_{p+}$, defined by

$$\mathcal{A}_{p+} = \mathcal{A}_p|_{P_+L^p(\Omega)} : P_+L^p(\Omega) \rightarrow P_+L^p(\Omega),$$

settles on the positive axis, and $-\mathcal{A}_{p+}$ generates an analytic semigroup $\{e^{-t\mathcal{A}_{p+}}\}_{t \geq 0}$ on $P_+L^p(\Omega)$. In accordance with the fractional powers of $\mathcal{B}_p$ and $\mathcal{A}_{p+}$, we have

$$\|\mathcal{A}_{p+}^\gamma e^{-t\mathcal{A}_{p+}} P_+ w\|_p \leq C_1(\gamma) q(t)^{-\gamma} e^{-d_1 \mu_2 t} \|P_+ w\|_p, \quad (8.84)$$

$$\|\mathcal{B}_p^\gamma e^{-t\mathcal{B}_p} w\|_p \leq C_2(\gamma) q(t)^{-\gamma} e^{-\alpha t} \|w\|_p, \quad (8.85)$$

$$\|e^{-tA} P_+ w\|_p \leq C_3 q(t)^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p})} e^{-\mu_2 t} \|P_+ w\|_q, \quad (8.86)$$

for $t > 0$, $1 \leq q \leq p \leq \infty$, and $0 \leq \gamma < 1$, where

$$0 < q(t) = \min\{t, 1\} \leq 1,$$

see [6, 17]. Here, the operator A used in (8.86) denotes $-\Delta$ associated with Neumann boundary condition, and therefore, μ_2 is nothing but the second eigenvalue of A .

Notably a direct consequence of inequalities (8.59) is

$$\int_0^t q(t-s)^{-\gamma} e^{\delta s} ds \leq C_4(\gamma, \delta), \quad \delta < 0. \quad (8.87)$$

In fact, the first inequality of (8.59) with δ replaced by $-\delta > 0$ implies

$$\int_0^t q(t-s)^{-\gamma} e^{\delta s} ds = e^{\delta t} \int_0^t q(s)^{-\gamma} e^{-\delta s} ds \leq C_4(\gamma, \delta).$$

Next, given $\mu \in [0, 2)$, we take $p \in (1, +\infty)$ and $\gamma \in (0, 1)$ satisfying

$$\mu < 2\gamma - N/p. \quad (8.88)$$

Then there holds, see [6],

$$D(\mathcal{A}_{p+}^\gamma) \subset D(\mathcal{B}_p^\gamma) \subset C^\mu(\overline{\Omega}). \quad (8.89)$$

8.2.4.1 Proof of Theorem 8.6 for $u_\infty > 0$

We split the proof of Theorem 8.6 in two parts. We first prove the case $u_\infty > 0$ and later on we provide the proof of the second case $u_\infty = 0$.

We first recall that $\mu_1(R) > 0$ is equivalent to $u_\infty > 0$ by (8.81). To prove Theorem 8.6 in this case we begin with the following lemma.

Lemma 8.4 *Given $1 \leq p < \infty$, any $0 < \varepsilon \ll 1$ admits $T \gg 1$ such that*

$$\|v(\cdot, t)\|_p \leq C_5(p) e^{-(\mu_1(R) - \varepsilon)(t-T)} \|v(\cdot, T)\|_p, \quad t \geq T. \quad (8.90)$$

Proof By (8.56), the behavior of the solution $(u, v) = (u(\cdot, t), v(\cdot, t))$ to (8.12)–(8.15) is controlled by that of the linear part around the stationary solution $(u_\infty, 0)$ as $t \uparrow +\infty$. More precisely, the perturbation $(u(\cdot, t) - u_\infty, v(\cdot, t))$ will be approximated for $t \gg 1$ by $(\tilde{u}, \tilde{v}) = (\tilde{u}(\cdot, t), \tilde{v}(\cdot, t))$ solving the following linearized system

$$\begin{aligned}
\tilde{u}_t &= d_1 \Delta \tilde{u} - k u_\infty \cdot \int_{B(\cdot, R) \cap \Omega} \tilde{v} \quad \text{in} \quad \Omega \times (T, +\infty), \\
\tilde{v}_t &= d_2 \Delta \tilde{v} - k F(R, x) u_\infty \tilde{v} \quad \text{in} \quad \Omega \times (T, +\infty), \\
\frac{\partial \tilde{u}}{\partial \nu} &= \frac{\partial \tilde{v}}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (T, +\infty), \\
\tilde{u}(x, T) &= u(x, T) - u_\infty, \quad \tilde{v}(x, T) = v(x, T) \geq 0 \quad \text{in} \quad \Omega,
\end{aligned}$$

where $T \gg 1$.

Next we consider the operator

$$\mathcal{C}_p(w) = (-d_2 \Delta + k F(R, x) u_\infty) w,$$

in $L^p(\Omega)$, $1 < p < \infty$, associated with the domain

$$D(\mathcal{C}_p) = \left\{ w \in W^{2,p}(\Omega) \mid \frac{\partial w}{\partial \nu} \Big|_{\partial \Omega} = 0 \right\}.$$

This operator is associated with the $\tilde{v}$ -component. Namely, using the analytic semigroup $\{e^{-\mathcal{C}_p t}\}_{t \geq 0}$ generated by $-\mathcal{C}_p$, we obtain

$$\tilde{v}(\cdot, t) = e^{-\mathcal{C}_p(t-T)} v(\cdot, T), \quad t \geq T.$$

Then the semigroup estimate

$$\|e^{-\mathcal{C}_p t}\|_{L^p \rightarrow L^p} \leq C_6(p) e^{-\mu_1(R)t}, \quad t \geq 0, \quad (8.91)$$

see [1] page 250, establishes

$$\|\tilde{v}(\cdot, t)\|_p \leq C_6(p) e^{-\mu_1(R)(t-T)} \|v(\cdot, T)\|_p, \quad t \geq T. \quad (8.92)$$

Next observe that $z = v - \tilde{v}$ satisfies

$$\begin{aligned}
z_t &= -\mathcal{C}_p z + k[F(R, \cdot) u_\infty - \int_{B(\cdot, R) \cap \Omega} u(y, s) dy] v, \quad x \in \Omega, \quad t \geq T, \\
z(x, T) &= 0, \quad x \in \Omega,
\end{aligned}$$

which implies

$$z(\cdot, t) = k \int_T^t e^{-\mathcal{C}_p(t-s)} [F(R, \cdot) u_\infty - \int_{B(\cdot, R) \cap \Omega} u(y, s) dy] v(\cdot, s) ds.$$

Hence there holds

$$\begin{aligned}
\|z(\cdot, t)\|_p &\leq k \int_T^t \|e^{-\mathcal{C}_p(t-s)}[F(R, \cdot)u_\infty - \int_{B(\cdot, R) \cap \Omega} u(y, s)dy]v(\cdot, s)\|_p ds \\
&\leq C_7(p) \int_T^t e^{-\mu_1(R)(t-s)} \|F(R, \cdot)u_\infty - \int_{B(\cdot, R) \cap \Omega} u(y, s)dy\|_\infty \cdot \|v(\cdot, s)\|_p ds.
\end{aligned} \tag{8.93}$$

We also have

$$F(R, \cdot)u_\infty - \int_{B(\cdot, R) \cap \Omega} u(y, s)dy = \int_{B(\cdot, R) \cap \Omega} (u_\infty - u(y, s))dy,$$

and hence

$$\begin{aligned}
\left\| F(R, \cdot)u_\infty - \int_{B(\cdot, R) \cap \Omega} u(y, s)dy \right\|_\infty &= \left\| \int_{B(\cdot, R) \cap \Omega} (u_\infty - u(y, s))dy \right\|_\infty \\
&\leq \|u_\infty - u(\cdot, s)\|_\infty.
\end{aligned} \tag{8.94}$$

It thus follows

$$\|z(\cdot, t)\|_p \leq C_7(p) \cdot \max_{s \geq T} \|u_\infty - u(\cdot, s)\|_\infty \cdot \int_T^t e^{-\mu_1(R)(t-s)} \|v(\cdot, s)\|_p ds.$$

Now, given $\varepsilon > 0$, we choose $T \gg 1$ such that

$$C_7(p) \cdot \sup_{s \geq T} \|u_\infty - u(\cdot, s)\|_p < \varepsilon,$$

recalling (8.56). Then we obtain

$$\begin{aligned}
\|v(\cdot, t)\|_p &\leq \|\tilde{v}(\cdot, t)\|_p + \|z(\cdot, t)\|_p \\
&\leq C_6(p)e^{-\mu_1(R)(t-T)} \|v(\cdot, T)\|_p + \varepsilon \int_T^t e^{-\mu_1(R)(t-s)} \|v(\cdot, s)\|_p ds,
\end{aligned}$$

and hence

$$X(t) \leq C_6(p)e^{\mu_1(R)T} \|v(\cdot, T)\|_p + \varepsilon \int_T^t X(s)ds, \tag{8.95}$$

where

$$X(t) = e^{\mu_1(R)t} \|v(\cdot, t)\|_p.$$

The latter implies

$$Y' - \varepsilon Y \leq C_6(p)e^{\mu_1(R)T} \|v(\cdot, T)\|_p, \quad t \geq T,$$

for

$$Y(t) = \int_T^t X(s)ds,$$

which yields

$$e^{-\varepsilon t} Y \leq C_7(p, \varepsilon) e^{\mu_1(R)T} \|v(\cdot, T)\|_p, \quad t \geq T,$$

since $Y(T) = 0$.

Therefore inequality (8.90) thus holds since (8.95) yields

$$X(t) \leq [C_6(p) + C_7(p, \varepsilon) e^{\varepsilon t}] e^{\mu_1(R)T} \|v(\cdot, T)\|_p, \quad t \geq T.$$

The proof is complete. $\square$

In the following lemma, the rate $u(\cdot, t) \rightarrow u_\infty, t \uparrow +\infty$, is estimated in C^m -norm. We emphasize again that $\mu_1(R) > 0$ follows from the assumption $u_\infty > 0$.

Lemma 8.5 *For every $m \in [0, 2)$ there holds*

$$\|u(\cdot, t) - u_\infty\|_{C^m(\overline{\Omega})} \leq C_8(m) e^{-\beta_1(t-T)} \|v(\cdot, T)\|_\infty, \quad t \geq T, \quad (8.96)$$

where

$$\beta_1 = \begin{cases} d_1 \mu_2, & d_1 \mu_2 < \mu_1(R) \\ \mu_1(R) - \varepsilon, & d_1 \mu_2 \geq \mu_1(R) \end{cases},$$

with $0 < \varepsilon \ll 1$.

Proof By virtue of relations (8.12) and (8.14) we derive

$$u(\cdot, t) = e^{-(t-T)\mathcal{A}_p} u(\cdot, T) - k \int_T^t e^{-(t-s)\mathcal{A}_p} \left[u(\cdot, s) \int_{B(\cdot, R) \cap \Omega} v(y, s) dy \right] ds, \quad (8.97)$$

for $t \geq T$ and $1 \leq p < \infty$.

By using the decomposition

$$u(\cdot, t) = P_0 u(\cdot, t) + P_+ u(\cdot, t),$$

and by virtue of (8.57), we first derive

$$P_0 u(\cdot, t) = \frac{1}{|\Omega|} \int_\Omega u(y, t) dy = \frac{1}{|\Omega|} \int_\Omega v(y, t) dy + u_\infty,$$

and hence

$$|P_0 u(\cdot, t) - u_\infty| \leq \frac{1}{|\Omega|} \|v(\cdot, t)\|_1 \leq C_9 e^{-(\mu_1(R) - \varepsilon)(t-T)} \|v(\cdot, T)\|_1, \quad t \geq T, \quad (8.98)$$

by $0 < F(R, x) \leq 1$. Combining (8.100) and (8.101), we arrive at

$$\begin{aligned}
& \int_T^t \|\mathcal{A}_{p+}^\gamma e^{-(t-s)\mathcal{A}_{p+}} P_+[u(\cdot, s)] \frown_{B(\cdot, R) \cap \Omega} v(y, s) dy\|_p ds \\
& \leq C_1(\gamma) \|P_+\|_{L^p \rightarrow L^p} \|u_0\|_\infty \int_T^t q(t-s)^{-\gamma} e^{-d_1 \mu_2(t-s)} \|v(\cdot, s)\|_p ds \\
& \leq C_{10}(\gamma) \|P_+\|_{L^p \rightarrow L^p} \|u_0\|_\infty \|v(\cdot, T)\|_p \\
& \quad \cdot \int_0^{t-T} q(t-T-s)^{-\gamma} e^{-d_1 \mu_2(t-T-s)} e^{-(\mu_1(R)-\varepsilon)s} ds, \tag{8.102}
\end{aligned}$$

for $t \geq T$, using again Lemma 8.4. Hence it suffices to estimate the term

$$I_\varepsilon(t) = \int_0^{t-T} q(t-T-s)^{-\gamma} e^{-d_1 \mu_2(t-T-s)} e^{-(\mu_1(R)-\varepsilon)s} ds.$$

If $\mu_1(R) - d_1 \mu_2 > 0$ is the case we then take $0 < \varepsilon \ll 1$ in $0 < \varepsilon < \mu_1(R) - d_1 \mu_2$ to use the first inequality of (8.59). Then it follows that

$$\begin{aligned}
I_\varepsilon(t) &= e^{-(\mu_1(R)-\varepsilon)(t-T)} \int_0^{t-T} q(t-T-s)^{-\gamma} e^{(\mu_1(R)-d_1 \mu_2-\varepsilon)(t-T-s)} ds \\
&\leq C_{11} e^{-(\mu_1(R)-\varepsilon)(t-T)} \cdot e^{(\mu_1(R)-\varepsilon-d_1 \mu_2)(t-T)} \\
&= C_{11} e^{-d_1 \mu_2(t-T)}, \quad t \geq T.
\end{aligned}$$

In the other case when $d_1 \mu_2 \geq \mu_1(R)$, we just use the third inequality of (8.59) to get

$$I_\varepsilon(t) \leq C_{12} e^{-(\mu_1(R)-\varepsilon)(t-T)}.$$

Using $\|v(\cdot, T)\|_p \leq |\Omega|^{1/p} \|v(\cdot, T)\|_\infty$, we complete the proof. $\square$

Notably Lemma 8.5 improves Lemma 8.4 as follows.

Lemma 8.6 *For every $1 \leq p < \infty$ there holds*

$$\|v(\cdot, t)\|_p \leq C_{13}(p) e^{-\mu_1(R)(t-T)} \|v(\cdot, T)\|_p, \quad t \geq T. \tag{8.103}$$

Proof By (8.93) and (8.94) we have

$$\|z(\cdot, t)\|_p \leq C_7(p) \int_T^t e^{-\mu_1(R)(t-s)} \|u_\infty - u(\cdot, s)\|_\infty \cdot \|v(\cdot, s)\|_p ds.$$

Here, any given $\varepsilon > 0$ admits $T \gg 1$ such that

$$C_7(p) \|u_\infty - u(\cdot, s)\|_\infty \leq \varepsilon e^{-\beta_1(t-T)}, \quad t \geq T,$$

by (8.96) and (8.56). Therefore we derive

$$\|z(\cdot, t)\|_p \leq \varepsilon \int_T^t e^{-\mu_1(R)(t-s)} e^{-\beta_1(s-T)} \|v(\cdot, s)\|_p ds, \quad t \geq T,$$

which, in conjunction with (8.92), implies

$$\begin{aligned} \|v(\cdot, t)\|_p &\leq \|\tilde{v}(\cdot, t)\|_p + \|z(\cdot, t)\|_p \leq C_6(p) e^{-\mu_1(R)(t-T)} \|v(\cdot, T)\|_p \\ &\quad + \varepsilon \int_T^t e^{-\mu_1(R)(t-s)} e^{-\beta_1(s-T)} \|v(\cdot, s)\|_p ds. \end{aligned}$$

The latter inequality entails

$$e^{\mu_1(R)t} \|v(\cdot, t)\|_p \leq C_6(p) e^{\mu_1(R)T} \|v(\cdot, T)\|_p + \varepsilon \int_T^t e^{-\beta_1(s-T)} \cdot e^{\mu_1(R)s} \|v(\cdot, s)\|_p ds,$$

and hence

$$X(t) \leq C_6(p) e^{\mu_1(R)T} \|v(\cdot, T)\|_p + \varepsilon \int_T^t e^{-\beta_1(s-T)} X(s) ds,$$

for

$$X(t) = e^{\mu_1(R)t} \|v(\cdot, t)\|_p.$$

Then, Gronwall's lemma implies

$$X(t) \leq C_6(p) e^{\mu_1(R)T} \|v(\cdot, T)\|_p \cdot \exp\left(\varepsilon \int_T^t e^{-\beta_1(s-T)} ds\right) = C_{14}(p, \varepsilon) e^{\mu_1(R)T} \|v(\cdot, T)\|_p,$$

and the proof is complete. $\square$

Lemma 8.6 now provides the following improvement of Lemma 8.5.

Lemma 8.7 *It holds that*

$$\|u(\cdot, t) - u_\infty\|_{C^\mu(\overline{\Omega})} \leq C_{15}(\mu) \cdot \begin{cases} e^{-\beta_0(t-T)} \|v(\cdot, T)\|_\infty, & d_1 \mu_2 \neq \mu_1(R) \\ (t - T + 1) e^{-\beta_0(t-T)} \|v(\cdot, T)\|_\infty, & d_1 \mu_2 = \mu_1(R) \end{cases}, \quad (8.104)$$

for $t \geq T$ and $\mu \in [0, 2)$, where $\beta_0 = \min\{d_1 \mu_2, \mu_1(R)\}$ is the exponent defined in Theorem 8.6.

Proof We prove the current lemma by repeating the proof of Lemma 8.5. In fact, once Lemma 8.6 is established, it suffices to estimate

$$I(t) = \int_0^{t-T} q(t - T - s)^{-\gamma} e^{-d_1 \mu_2(t-T-s)} e^{-\mu_1(R)s} ds,$$

where inequality (8.59) is available.

The argument is essential the same for $d_1\mu_2 \neq \mu_1(R)$. If $d_1\mu_2 = \mu_1(R)$ then

$$I(t) = \int_0^{t-T} q(t-T-s)^{-\gamma} e^{-d_1\mu_2(t-T)} ds \leq C_{20}(t-T+1)e^{-d_1\mu_2(t-T)},$$

by the second inequality of (8.59). Thus the proof is complete. $\square$

We conclude the proof of Theorem 8.6 for the case of $u_\infty > 0$ with the following lemma.

Lemma 8.8 *For any $m \in [0, 2)$ there holds*

$$\|v(\cdot, t)\|_{C^m(\overline{\Omega})} = O(e^{-\mu_1(R)t}), \quad t \uparrow +\infty. \quad (8.105)$$

Proof We take $\mathcal{B}_p$ for $\alpha > \mu_1(R)$ to derive

$$\begin{aligned} v(\cdot, t) &= e^{-(t-T)\mathcal{B}_p} v(\cdot, T) + \alpha \int_T^t e^{-(t-s)\mathcal{B}_p} v(\cdot, s) ds \\ &\quad - k \int_T^t e^{-(t-s)\mathcal{B}_p} [v(\cdot, s) \cdot \oint_{B(\cdot, R) \cap \Omega} u(y, s) dy] ds, \quad t \geq T, \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{B}_p^\gamma v(\cdot, t) &= \mathcal{B}_p^\gamma e^{-(t-T)\mathcal{B}_p} v(\cdot, T) + \alpha \int_T^t \mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} v(\cdot, s) ds \\ &\quad - k \int_T^t \mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} [v(\cdot, s) \cdot \oint_{B(\cdot, R) \cap \Omega} u(y, s) dy] ds, \end{aligned}$$

for $0 < \gamma < 1$. It is evident that

$$\begin{aligned} \|\mathcal{B}_p^\gamma e^{-(t-T)\mathcal{B}_p} v(\cdot, T)\|_p &\leq C_1(\gamma) q(t-T)^{-\gamma} e^{-\alpha(t-T)} \|v(\cdot, T)\|_p \\ &\leq C_{16} e^{-\alpha(t-T)} \leq C_{17} e^{-\mu_1(R)(t-T)}, \quad t \geq T+1 \end{aligned}$$

by $\alpha > \mu_1(R)$. Next, by virtue of (8.103) we obtain

$$\begin{aligned} &\int_T^t \|\mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} v(\cdot, s)\|_p ds \\ &\leq C_{17} \int_T^t q(t-s)^{-\gamma} e^{-\alpha(t-s)} e^{-\mu_1(R)(s-T)} ds \\ &= C_{17} e^{-\mu_1(R)(t-T)} \int_T^t q(t-s)^{-\gamma} e^{-(\alpha-\mu_1(R))(t-s)} ds \\ &\leq C_{18} e^{-\mu_1(R)(t-T)}. \end{aligned} \quad (8.106)$$

Finally, using

$$\left\| \oint_{B(\cdot, R) \cap \Omega} u(y, s) dy \right\|_{\infty} \leq \|u(\cdot, s)\|_{\infty} \leq \|u_0\|_{\infty},$$

we obtain

$$\begin{aligned} & \int_T^t \|\mathcal{B}_p^\gamma e^{-(t-s)\mathcal{B}_p} [v(\cdot, s) \cdot \oint_{B(\cdot, R) \cap \Omega} u(y, s) dy]\|_p ds \\ & \leq C_2(\gamma) \int_T^t q(t-s)^{-\gamma} e^{-\alpha(t-s)} \|v(\cdot, s)\|_p \|u_0\|_{\infty} ds \\ & \leq C_{19} \|u_0\|_{\infty} \int_T^t q(t-s)^{-\gamma} e^{-\alpha(t-s)} \cdot e^{-\mu_1(R)(s-T)} ds \\ & \leq C_{20} e^{-\mu_1(R)(t-T)} \int_T^t q(t-s)^{-\gamma} e^{-(\alpha-\mu_1(R))(t-s)} ds \\ & \leq C_{21} e^{-\mu_1(R)(t-T)}, \quad t \geq T, \end{aligned} \tag{8.107}$$

by Lemma 8.7 and (8.59). Taking the exponents $1 < p < \infty$ and $0 < \gamma < 1$ as in (8.88), we obtain (8.105). $\square$

8.2.4.2 Proof of Theorem 8.6 for $u_{\infty} = 0$

We recall that $u_{\infty} = 0$ is equivalent to $\mu_1(R) = 0$. In this case, there also holds that

$$\bar{u}(t) = \bar{v}(t). \tag{8.108}$$

We begin with the following lemma.

Lemma 8.9 *There is $0 < \delta_1 \ll 1$ such that*

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_2^2 + d_1 \|\nabla u(\cdot, t)\|_2^2 + \delta_1 \|u(\cdot, t)\|_3^3 \leq C_{22} M(t) \|(\nabla u(\cdot, t), \nabla v(\cdot, t))\|_2^2, \tag{8.109}$$

for $t > 0$ and

$$\frac{1}{2} \frac{d}{dt} \|v(\cdot, t)\|_2^2 + d_2 \|\nabla v(\cdot, t)\|_2^2 + \delta_1 \|v(\cdot, t)\|_3^3 \leq C_{22} M(t) \|(\nabla u(\cdot, t), \nabla v(\cdot, t))\|_2^2, \tag{8.110}$$

for $t > 0$ where

$$M(t) = \max\{\|u(\cdot, t) - \bar{u}(t)\|_{\infty}, \|v(\cdot, t) - \bar{v}(t)\|_{\infty}\}.$$

Proof First, multiplying (8.12) by u and integrating by parts over Ω , we have

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_2^2 + d_1 \|\nabla u(\cdot, t)\|_2^2 + k \int_{\Omega} u^2(x, t) \left(\oint_{B(x, R) \cap \Omega} v(y, t) dy \right) dx = 0.$$

Here, the term

$$I(t) = \int_{\Omega} u^2(x, t) \left(\oint_{B(x, R) \cap \Omega} v(y, t) dy \right) dx,$$

is treated by Fubini's theorem, hence

$$I(t) = \frac{1}{\omega_N R^N} \iint_{\mathbf{R}^N \times \mathbf{R}^N} u^2(x, t) v(y, t) \chi_{\Omega}(x) \chi_{\Omega}(y) \chi_{|x-y| < R}(x, y) dx dy.$$

Then we use

$$\min_{z \geq 0} (z + |1 - z|) = 1,$$

to derive

$$u^2(x, t) v(y, t) \geq u^3(x, t) - u^2(x, t) |u(x, t) - v(y, t)|.$$

Hence it follows that

$$\begin{aligned} I(t) &\geq \frac{1}{\omega_N R^N} \iint_{\mathbf{R}^N \times \mathbf{R}^N} u^3(x, t) \chi_{\Omega}(x) \chi_{\Omega}(y) \chi_{|x-y| < R}(x, y) dx dy \\ &\quad \chi_{\Omega}(x) \chi_{\Omega}(y) \chi_{|x-y| < R}(x, y) dx dy \\ &= \int_{\Omega} F(R, x) u^3(x, t) dx - \frac{1}{\omega_N R^N} \iint_{\Omega \times \Omega} u^2(x, t) |u(x, t) \\ &\quad - v(y, t)| \chi_{|x-y| < R}(x, y) dx dy \\ &\geq \bar{F}_R \|u(\cdot, t)\|_3^3 - I_1(t), \end{aligned} \tag{8.111}$$

with

$$I_1(t) = \frac{1}{\omega_N R^N} \iint_{\Omega \times \Omega} u^2(x, t) |u(x, t) - v(y, t)| \chi_{|x-y| < R}(x, y) dx dy.$$

Now we estimate $I_1(t)$, using (8.108). Namely, by Hölder's and Young's inequalities, we derive

$$\begin{aligned} I_1(t) &\leq \frac{|\Omega|^{2/3} \|u(\cdot, t)\|_3^2}{\omega_N R^N} \left(\iint_{\Omega \times \Omega} |u(x, t) - v(y, t)|^3 \chi_{|x-y| < R}(x, y) dx dy \right)^{1/3} \\ &\leq \frac{|\Omega|^{2/3} \|u(\cdot, t)\|_3^2}{\omega_N R^N} \left[\left(\iint_{\Omega \times \Omega} |u(x, t) - \bar{u}(t)|^3 \chi_{|x-y| < R}(x, y) dx dy \right)^{1/3} \right. \\ &\quad \left. + \left(\iint_{\Omega \times \Omega} |v(y, t) - \bar{v}(t)|^3 \chi_{|x-y| < R}(x, y) dx dy \right)^{1/3} \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|\Omega|^{2/3} \|u(\cdot, t)\|_3^2}{(\omega_N R^N)^{2/3}} \left(\|u(\cdot, t) - \bar{u}(t)\|_3 + \|v(\cdot, t) - \bar{v}(t)\|_3 \right) \\
&\leq \frac{|\Omega| \|u(\cdot, t)\|_3^2}{(\omega_N R^N)^{2/3}} \|u(\cdot, t) - \bar{u}(t)\|_\infty^{1/3} \|u(\cdot, t) - \bar{u}(t)\|_2^{2/3} \\
&\quad + \frac{|\Omega| \|v(\cdot, t)\|_3^2}{(\omega_N R^N)^{2/3}} \|v(\cdot, t) - \bar{v}(t)\|_\infty^{1/3} \|v(\cdot, t) - \bar{v}(t)\|_2^{2/3} \\
&\leq \frac{4\delta |\Omega|^{3/2}}{3\omega_N R^N} \|u(\cdot, t)\|_3^3 + \frac{1}{3\delta^2} \left(\|u(\cdot, t) - \bar{u}(t)\|_\infty \|u(\cdot, t) - \bar{u}(t)\|_2^2 \right. \\
&\quad \left. + \|v(\cdot, t) - \bar{v}(t)\|_\infty \|v(\cdot, t) - \bar{v}(t)\|_2^2 \right),
\end{aligned}$$

where $0 < \delta \ll 1$. Then, by Poincaré-Wirtinger's inequality; we obtain

$$\begin{aligned}
I_1(t) &\leq \frac{4\delta |\Omega|^{3/2}}{3\omega_N R^N} \|u(\cdot, t)\|_3^3 + \frac{1}{3\mu_2 \delta^2} \left(\|u(\cdot, t) - \bar{u}(t)\|_\infty \|\nabla u(\cdot, t)\|_2^2 \right. \\
&\quad \left. + \|v(\cdot, t) - \bar{v}(t)\|_\infty \|\nabla v(\cdot, t)\|_2^2 \right) \\
&\leq \frac{4\delta |\Omega|^{2/3}}{3\omega_N R^N} \|u(\cdot, t)\|_3^3 + C_{23}(\delta) M(t) \|(\nabla u(\cdot, t), \nabla v(\cdot, t))\|_2^2,
\end{aligned}$$

recalling that $\mu_2 > 0$ denotes the second eigenvalue of A . Taking $0 < \delta \ll 1$ such that

$$\bar{F}_R - \frac{4\delta |\Omega|^{2/3}}{3\omega_N R^N} = \delta_1 > 0,$$

we obtain (8.109). The proof of (8.110) is similar. $\square$

Next, we show the following lemma.

Lemma 8.10 *There exists $T \gg 1$ such that*

$$\|(u(\cdot, t), v(\cdot, t))\|_p \leq C_{24}(t - T + 1)^{-1}, \quad \text{for } t \geq T, \quad (8.112)$$

for any $1 \leq p \leq 2$.

Proof Adding (8.109) and (8.110), we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|(u(\cdot, t), v(\cdot, t))\|_2^2 + d_0 \|(\nabla u(\cdot, t), \nabla v(\cdot, t))\|_2^2 + \delta_1 \|(u(\cdot, t), v(\cdot, t))\|_3^3 \\
&\leq C_{22} M(t) \|(\nabla u(\cdot, t), \nabla v(\cdot, t))\|_2^2, \quad t > 0,
\end{aligned}$$

where $d_0 = \min\{d_1, d_2\}$. Regarding (8.56) with $u_\infty = 0$, we take $T \gg 1$ satisfying

$$M(t) \leq \frac{d_0}{C_{22}}, \quad t \geq T,$$

which implies

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(u(\cdot, t), v(\cdot, t))\|_2^2 + |\Omega|^{-1/2} \delta_1 \|(u(\cdot, t), v(\cdot, t))\|_2^3 \\ & \leq \frac{1}{2} \frac{d}{dt} \|(u(\cdot, t), v(\cdot, t))\|_2^2 + \delta_1 \|(u(\cdot, t), v(\cdot, t))\|_3^3 \leq 0, \quad t \geq T. \end{aligned}$$

Thus

$$\frac{dX}{dt} + \delta_2 X^2 \leq 0, \quad t \geq T,$$

for $\delta_2 = |\Omega|^{-1/2} \delta_1$ and $X = \|(u, v)\|_2$. This inequality, together with

$$\|(u(\cdot, T), v(\cdot, T))\|_2 \leq |\Omega|^{1/2} \|(u(\cdot, T), v(\cdot, T))\|_\infty \leq |\Omega|^{1/2} \|(u_0, v_0)\|_\infty,$$

implies (8.112) for $p = 2$. Accordingly (8.112) for any $1 \leq p \leq 2$ by Hölder's inequality. $\square$

Turning to the proof of the second case of Theorem 8.6, we use, again the decomposition

$$(u, v) = P_0(u, v) + P_+(u, v).$$

First, by Lemma 8.10 we obtain

$$|P_0(u(\cdot, t), v(\cdot, t))| \leq \frac{1}{|\Omega|} \|(u(\cdot, t), v(\cdot, t))\|_1 \leq C_{25}(t - T - 1)^{-1}, \quad t \geq T. \quad (8.113)$$

To find a similar estimate for the $P_+(u, v)$, we apply P_+ to the integral equation equivalent to (8.12), that is,

$$P_+u(\cdot, t) = e^{-(t-T)d_1A} P_+u(\cdot, T) - k \int_T^t e^{-(t-s)d_1A} P_+[u(\cdot, s) \fint_{B(\cdot, R) \cap \Omega} v(y, s) dy] ds, \quad t \geq T.$$

A similar integral equation holds for the second component v where now d_1 is replaced by d_2 . Adding these two integral equations and applying the $L^p - L^q$ estimate (8.86), for $1 \leq q \leq p \leq \infty$, we derive

$$\begin{aligned} & \|P_+(u(\cdot, t), v(\cdot, t))\|_p \leq C_{26} e^{-d_0 \mu_2(t-T)} \|P_+(u(\cdot, T), v(\cdot, T))\|_p \\ & + C_{26} \int_T^t q(t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\mu_2 d_0(t-s)} \|P_+[u(\cdot, s) \fint_{B(\cdot, R) \cap \Omega} v(y, s) dy \\ & + v(\cdot, s) \fint_{B(\cdot, R) \cap \Omega} u(y, s) dy]\|_q ds, \end{aligned} \quad (8.114)$$

recalling $d_0 = \min\{d_1, d_2\}$.

Since

$$\|P_+(u(\cdot, T), v(\cdot, T))\|_p \leq |\Omega|^{1/p} \|P_+(u(\cdot, T), v(\cdot, T))\|_\infty \leq C_{27} \|(u_0, v_0)\|_\infty,$$

it suffices to estimate the second term in the right-hand side of (8.114). The following inequality, comparable to (3.20) of [7], will be appropriate for this purpose.

Lemma 8.11 *There holds*

$$\left\| P_+ \left[u \cdot \int_{B(\cdot, R) \cap \Omega} v dy + v \int_{B(\cdot, R) \cap \Omega} u dy \right] \right\|_q \leq C_{28} \left(\| (u, v) \|_\infty \| P_+(u, v) \|_q + |P_0 u| \cdot |P_0 v| \right). \quad (8.115)$$

Proof First, we have

$$\begin{aligned} P_+ \left[u \cdot \int_{B(\cdot, R) \cap \Omega} v dy \right] &= P_+ \left[(P_0 u + P_+ u) \cdot \int_{B(\cdot, R) \cap \Omega} (P_0 v + P_+ v) dy \right] \\ &= P_+ \left[P_0 u \cdot \int_{B(\cdot, R) \cap \Omega} P_+ v dy \right. \\ &\quad \left. + P_+ u \cdot \int_{B(\cdot, R) \cap \Omega} v dy \right] + P_0 u \cdot P_0 v \cdot P_+ F(R, \cdot). \end{aligned}$$

Then we derive

$$\begin{aligned} \left\| P_+ \left[u \cdot \int_{B(\cdot, R) \cap \Omega} v dy \right] \right\|_q &\leq \| P_+ \|_{L^q \rightarrow L^q} \left(|P_0 u| \cdot \left\| \int_{B(\cdot, R) \cap \Omega} P_+ v dy \right\|_q \right. \\ &\quad \left. + \| P_+ u \|_q \cdot \left\| \int_{B(\cdot, R) \cap \Omega} v dy \right\|_\infty \right) + |P_0 u| \cdot |P_0 v| \cdot \| F_R(\cdot) \|_q \\ &\leq C_{29} \left(\| u \|_\infty \| P_+ v \|_q + \| v \|_\infty \| P_+ u \|_q + |P_0 u| \cdot |P_0 v| \right), \end{aligned}$$

by (8.101). The proof of

$$\left\| P_+ \left[v \cdot \int_{B(\cdot, R) \cap \Omega} u dy \right] \right\|_q \leq C_{30} \left(\| u \|_\infty \| P_+ v \|_q + \| v \|_\infty \| P_+ u \|_q + |P_0 u| \cdot |P_0 v| \right),$$

is similar. □

Plugging (8.115) into (8.114), we have

$$\begin{aligned} \|P_+(u(\cdot, t), v(\cdot, t))\|_p &\leq C_{31} \left(e^{-d_0 \mu_2(t-T)} \| (u(\cdot, T), v(\cdot, T)) \|_\infty \right. \\ &\quad \left. + \int_T^t e^{-d_0 \mu_2(t-s)} q(t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} I_q(s) ds \right), \quad t \geq T, \end{aligned} \quad (8.116)$$

for $1 \leq q \leq p \leq \infty$ where

$$I_q(s) = \|P_+(u(\cdot, s), v(\cdot, s))\|_q + |P_0 u(s)| \cdot |P_0 v(s)|.$$

To ensure integrability near $s = t$ of the integrand terms appearing in (8.116) we impose

$$\frac{N}{2} \left(\frac{1}{q} - \frac{1}{p} \right) < 1. \quad (8.117)$$

Here, (8.117) is satisfied for $p = \infty$ and $q = 2$ whenever $N \leq 3$. For this choice (8.116) yields

$$\begin{aligned} \|P_+(u(\cdot, t), v(\cdot, t))\|_\infty &\leq C_{31} \left(e^{-d_0 \mu_2(t-T)} \| (u(\cdot, T), v(\cdot, T)) \|_\infty \right. \\ &\quad \left. + \int_T^t e^{-d_0 \mu_2(t-s)} q(t-s)^{-N/4} I_2(s) ds \right), \end{aligned}$$

which via (8.112) first gives

$$\begin{aligned} &\|P_+(u(\cdot, t), v(\cdot, t))\|_\infty \\ &\leq C_{32} \left(e^{-d_0 \mu_2(t-T)} + \int_T^t e^{-d_0 \mu_2(t-s)} q(t-s)^{-N/4} (s-T+1)^{-1} ds \right) \\ &= C_{32} \left(e^{-d_0 \mu_2(t-T)} + \int_0^{t-T} e^{-d_0 \mu_2 s} q(s)^{-N/4} (t-T-s+1)^{-1} ds \right), \end{aligned}$$

since

$$I_2(t) \leq C_{33} \{ (t-T+1)^{-1} + (t-T+1)^{-2} \} \leq C_{34} (t-T+1)^{-1}.$$

Next we use the elementary inequality

$$(T+1) \int_0^T e^{-\delta s} q(s)^{-\gamma} (T-s+1)^{-1} ds \leq C_{35}(\gamma, \delta), \quad T \geq 0, \quad (8.118)$$

valid for $0 < \gamma < 1$ and $\delta > 0$. Since $\frac{N}{4} < 1$ it follows that

$$\|P_+(u(\cdot, t), v(\cdot, t))\|_\infty \leq C_{36} (t-T+1)^{-1}, \quad t \geq T,$$

and the proof is complete for $N \leq 3$.

To confirm (8.118) we use

$$\int_0^{T/2} e^{-\delta s} q(s)^{-\gamma} \frac{T+1}{T-s+1} ds \leq 2 \int_0^{T/2} e^{-\delta s} q(s)^{-\gamma} ds \leq 2C_4(\gamma, -\delta),$$

and

$$\begin{aligned} \int_{T/2}^T e^{-\delta s} q(s)^{-\gamma} \frac{T+1}{T-s+1} ds &\leq (T+1)e^{-\delta T/2} \int_{T/2}^T e^{-\delta s/2} q(s)^{-\gamma} ds \\ &\leq (T+1)e^{-\delta T/2} C_4(\gamma, -\delta/2) \leq C_{37}(\gamma, \delta). \end{aligned}$$

Then (8.118) is obtained.

In the complementary case $N \geq 4$ we define a sequence of exponents $\{p_j\}$ by

$$p_0 = 1, \quad \frac{1}{p_j} - \frac{1}{p_{j+1}} = \frac{1}{N}, \quad j = 0, 1, \dots, N-1.$$

It is easily seen that $\{p_j\}$ is an increasing sequence with $p_N = \infty$. Setting $p = p_{j+1}$ and $q = p_j$ in (8.116), we obtain

$$\|P_+(u(\cdot, t), v(\cdot, t))\|_{p_{j+1}} \leq C_{31} \left(e^{-d_0 \mu_2(t-T)} + \int_T^t q(t-s)^{-1/2} I_{p_j}(s) ds \right), \quad (8.119)$$

for $j = 0, 1, \dots, N-1$. Based on (8.112) for $p = 1$ we can use (8.119) recursively for j to derive

$$\|P_+(u(\cdot, t), v(\cdot, t))\|_{p_j} \leq C_{37}(t-T+1)^{-1}, \quad t \geq T, \quad j = 1, 2, \dots, N.$$

Then (8.83), which is actually (8.112) with $p = \infty$, follows for $N \geq 4$ since $p_N = \infty$.

References

1. Amann, H.: Dual semigroups and second order linear elliptic boundary value problems. *Israel J. Math.* **45**, 225–254 (1983)
2. Cannon, J.R., Hill, C.D.: On the movement of a chemical reaction interface. *Indiana Univ. Math. J.* **20**, 429–454 (1970)
3. Egelstaff, P.A.: *An Introduction to the Liquid State*. Academic Press, London (1967)
4. Evans, L.C.: A convergence theorem for a chemical diffusion-reaction system. *Houston J. Math.* **6**, 259–267 (1980)
5. Evans, L.C.: *Partial Differential Equations*, 2nd edn. American Mathematical Society, Providence, RI (2010)
6. Henry, D.: *Geometric Theory of Semilinear Parabolic Equations*. Springer, Berlin (1981)
7. Hoshino, H., Kawashima, K.: Asymptotic equivalence of a reaction-diffusion system to the corresponding system of ordinary differential equations. *Math. Model. Meth. Appl. Sci.* **5**, 813–834 (1995)
8. Hoshino, H., Kawashima, S.: Exponential decaying component of global solution to a reaction-diffusion system. *Math. Model. Meth. Appl. Sci.* **8**, 897–904 (1998)
9. Hoshino, H., Yamada, Y.: Solvability and smoothing effect for semilinear parabolic equations. *Funkcialaj Ekvacioj* **34**, 639–650 (1991)

10. Hoshino, H., Yamada, Y.: Asymptotic behavior of global solutions for some reaction-diffusion systems. *Nonl. Anal.* **23**, 639–650 (1994)
11. Ichikawa, K., Rouzaimaiti, M., Suzuki, T.: Reaction diffusion equation with non-local term arises as a mean field limit of the master equation. *Disc. Cont. Dyn. Syst. S* **5**(1), Special Issue 115–126 (2012)
12. Kavallaris, N.I., Suzuki, T.: Nonlocal reaction-diffusion system involved by reaction radius I. *IMA J. Appl. Math.* **78**(3), 614–632 (2013)
13. Kavallaris, N.I., Suzuki, T.: Nonlocal reaction-diffusion system involved by reaction radius II: rate of convergence. *IMA J. Appl. Math.* **79**(1), 1–21 (2014)
14. Ladyženskaja, O.A., Solonikov, V.A., Ural'ceva, N.N.: *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society Translations. vol. 23, American Mathematical Society, Providence, RI (1968)
15. Masuda, K.: On the global existence and asymptotic behavior of solutions of reaction-diffusion equations. *Hokkaido Math. J.* **12**, 360–370 (1983)
16. Othmer, H.G., Stevens, A.: Aggregation, blowup, and collapse: the ABC's of taxis in reinforced random walks. *SIAM J. Appl. Math.* **57**, 1044–1081 (1997)
17. Rothe, F.: *Global Solutions of Reaction-diffusion Systems*. Lecture Notes in Mathematics, vol. 1072. Springer, Berlin (1984)
18. Suzuki, T.: *Mean Field Theories and Dual Variation-Mathematical Structures of the Mesoscopic Model*, 2nd edn. Atlantis Press, Paris (2015)
19. Suzuki, T.: Mathematical models of tumor growth systems. *J. Mathematica Bohemica*, devoted to EQUADIFF12 (Bruno 2009), **137**(2), 201–218 (2012)
20. Suzuki, T.: *Mathematical Methods for Cancer Evolution*. Springer, Berlin (2017)
21. Suzuki, T., Senba, T.: *Applied Analysis - Mathematical Methods in Natural Science*, 2nd edn. Imperial College Press, London (2011)
22. Tonegawa, Y.: On the regularity of chemical reaction interface. *Commun. Partial Differ. Equ.* **23**, 1181–1207 (1998)

Appendix

Here we review several other systems provided with non-local terms modeling elastic string, point vortices, and geometric deformation.

A.1 Kirchhoff Equation

We first consider Kirchhoff's equation which in one dimension takes the form

$$u_{tt} - \left(\varepsilon^2 + \frac{1}{2\ell} \int_0^1 u_x^2 dx \right) u_{xx} = 0 \quad \text{in} \quad (0, 1) \times (0, T), \quad T > 0, \quad (\text{A.1})$$

where $u = u(x, t)$ denotes the deformation of an elastic string under the oscillation of small amplitude and ℓ is a constant reflects the elastic features of the string. Equation (A.1) was introduced by G.R. Kirchhoff in 19th century but global-in-time wellposedness in Sobolev spaces is still an open problem in spite of several results on real analytic initial values [2, 6, 20]. The problem in higher dimensions can be written as

$$\begin{aligned} u_{tt} - \Phi(\|\nabla u\|_2^2) \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T), \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \mathbb{R}^N, \end{aligned}$$

with $\Phi = \Phi(s) \in C[0, \infty)$ satisfying

$$\inf_{s>0} \Phi(s) > 0, \quad \text{and} \quad \int_0^\infty \Phi(s) ds = \infty.$$

The total energy is provided by

$$E(t) = \frac{1}{2} \left(F(\|\nabla u\|_2^2) + \|u_t\|_2^2 \right),$$

where $F(s) = \int_0^s \Phi(s')ds'$ and it is preserved, i.e. $E(t) = E(0)$ for any $t > 0$. The energy conservation implies uniform bounds of the norms $\|\nabla u(\cdot, t)\|_2$ and $\|u_t(\cdot, t)\|_2$. Higher order norms, on the other hand, must be controlled for the global-in-time wellposedness, because those norms are necessary for the short time solvability. For this purpose we consider

$$E_m(t) = \frac{1}{2} \left(\Phi(\|\nabla u\|_2^2) \|\nabla u\|_{H^m}^2 + \|u_t\|_{H^m}^2 \right),$$

for $m \geq \frac{1}{2}$ and we deduce

$$\frac{dE_m}{dt} = \Phi'(\|\nabla u\|_2^2) \left(|\nabla|^{\frac{1}{2}} u_t, |\nabla|^{\frac{3}{2}} u \right) \|\nabla u\|_{H^m}^2 \leq C_0 E_{\frac{1}{2}} E_m,$$

owing to the energy conservation. Therefore, the control of $E_{\frac{1}{2}}$ is the key estimate to guarantee the global-in-time existence of the solution. Closely related linear theory is concerned on

$$\begin{aligned} (\partial_t^2 - a^2(t)\Delta)u &= 0 \quad \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } \mathbb{R}^N, \end{aligned}$$

with $a = a(t) > 0$ being singular at $t = T < \infty$, say $a \in C^1[0, T) \cap L^1(0, T)$ and $|a'(t)| = O((T-t)^{-1})$ as $t \uparrow T$. Then a class of non-analytic function spaces provided with global-in-time wellposedness of (A.1) is introduced [10, 11, 16].

A.2 Equilibrium and Relaxation States of Point Vortices

The motion of the perfect fluid occupied in a bounded, simply connected domain $\Omega \subset \mathbb{R}^2$ and with high Reynolds number is described by the Euler–Poisson equation

$$\omega_t + \nabla \cdot u\omega = 0, \quad \Delta\psi = -\omega, \quad u = \nabla^\perp \psi, \quad \psi|_{\partial\Omega} = 0 \quad (\text{A.2})$$

where $\nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}$ for $x = (x_1, x_2)$. Assuming the point vortex model

$$\omega(x, t) = \sum_{i=1}^N \alpha_i \delta_{x_i(t)}(dx),$$

we can reduce system (A.2) to

$$\alpha_i \frac{dx_i}{dt} = \nabla_{x_i}^\perp H_N, \quad 1 \leq i \leq N, \quad (\text{A.3})$$

associated with the Hamiltonian

$$H_N(x_1, \dots, x_N) = \frac{1}{2} \sum_{i=1}^N \alpha_i^2 R(x_i) + \sum_{1 \leq i < j \leq N} \alpha_i \alpha_j G(x_i, x_j),$$

where $G = G(x, x')$ is the Green's function of $-\Delta$ associated with Dirichlet boundary condition and

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x},$$

stands for the Robin function.

Onsager [19] proposed the use of statistical mechanics to handle (A.3). Indeed, considering the limit $N \rightarrow \infty$ with $\alpha N = 1$, local mean of vortex distribution is given by

$$\tilde{\omega}(x) = \int_I \tilde{\alpha} \rho^{\tilde{\alpha}}(x) P(d\tilde{\alpha}), \quad x \in \Omega,$$

where $\alpha_i = \tilde{\alpha}^i \alpha$, $\tilde{\alpha}^i \in I = [-1, 1]$ is the intensity of the i -th vortex, $\rho^{\tilde{\alpha}}(x)$ is the existence probability of the vortex at x with relative intensity $\tilde{\alpha}$ which satisfies

$$\int_{\Omega} \rho^{\tilde{\alpha}}(x) dx = 1, \quad \tilde{\alpha} \in I,$$

and $P(d\tilde{\alpha})$ is the numerical density of the vortices with the relative intensity $\tilde{\alpha}$. Assuming $H_N = E = \text{constant}$, $\alpha^2 N \beta_N = \beta = \text{constant}$, and taking the limit as $N \rightarrow \infty$, then by a series of arguments the following mean field equation is derived

$$-\Delta \psi = \int_I \alpha \frac{e^{-\beta \alpha \psi}}{\int_{\Omega} e^{-\beta \alpha \psi}} P(d\alpha), \quad \psi|_{\partial\Omega} = 0, \quad (\text{A.4})$$

with

$$\omega = -\Delta \psi, \quad \rho^{\alpha} = \frac{e^{-\beta \alpha \psi}}{\int_{\Omega} e^{-\beta \alpha \psi}},$$

see [25]. The non-local term in (A.4) is associated with the canonical measure transformed from the micro-canonical measure using thermodynamical relations. The simplest case $\tilde{\alpha}^1 = 1$ is particularly studied in details for this model. There is a recursive hierarchy with mass quantization, which means that the singular limit of the family of solutions arises only when $\beta \rightarrow -8\pi N$ with exactly N blow-up points denoted by $x_1^*, \dots, x_N^*$, whose location coincides with that of the critical point of $H_N = H_N(x_1, \dots, x_N)$: $\nabla_{x_i} H(x_1^*, \dots, x_N^*) = 0$, $1 \leq i \leq N$, see [17].

There is a quasi-equilibrium in several isolated systems with many components. The observation is that there is a relatively stationary state, different from the equilibrium, which eventually approaches the latter. Relaxation indicates this time

interval, from quasi-equilibrium to equilibrium. There is a kinetic mean field model assuming the Brownian point vortices, see [5],

$$\frac{\partial \rho^\alpha}{\partial t} + \nabla \cdot \rho^\alpha u = \nabla \cdot D(\nabla \rho^\alpha + \beta \alpha \rho^\alpha \nabla \psi), \quad \frac{\partial \rho^\alpha}{\partial \nu} + \beta \alpha \rho^\alpha \frac{\partial \psi}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad (\text{A.5})$$

$$\omega = \int_I \alpha \rho^\alpha P(d\alpha) = -\Delta \psi, \quad \psi|_{\partial \Omega} = 0, \quad u = \nabla^\perp \psi, \quad (\text{A.6})$$

$$\beta = -\frac{\int_\Omega D \nabla \omega \cdot \nabla \psi}{\int_\Omega \int_I \alpha^2 \rho^\alpha P(d\alpha) |\nabla \psi|^2}, \quad (\text{A.7})$$

where D is the diffusion coefficient. Inverse temperature β is associated with the non-local term, which realizes the thermodynamical law valid to the isolated system, that is, conservations of total mass and energy as well as increase of entropy.

In fact, averaging (A.5)–(A.7) entails

$$\begin{aligned} \frac{\partial \omega}{\partial t} + \nabla \cdot \omega u &= \nabla \cdot D(\nabla \omega + \beta \omega_2 \nabla \psi), \quad \frac{\partial \omega}{\partial \nu} + \beta \omega_2 \frac{\partial \psi}{\partial \nu} \Big|_{\partial \Omega} = 0, \\ \omega &= -\Delta \psi, \quad \psi|_{\partial \Omega} = 0, \quad u = \nabla^\perp \psi, \quad \beta = -\frac{\int_\Omega D \nabla \omega \cdot \nabla \psi}{\int_\Omega D \omega_2 |\nabla \psi|^2}, \end{aligned}$$

with

$$\omega = \int_I \omega \rho^\alpha P(d\alpha), \quad \omega_2 = \int_I \alpha^2 \rho^\alpha P(d\alpha).$$

Then we obtain

$$\frac{d}{dt} \int_\Omega \omega = 0, \quad (\omega_t, \psi) = \frac{1}{2} \frac{d}{dt} (\omega, (-\Delta)^{-1} \omega) = 0.$$

We also write

$$\frac{\partial \rho^\alpha}{\partial t} = \nabla \cdot \rho^\alpha u = \nabla \cdot D \rho^\alpha \nabla (\log \rho^\alpha + \beta \alpha \psi),$$

for (A.5) and thus it follows that

$$\frac{d}{dt} \int_\Omega \Phi(\rho^\alpha) + \beta \alpha (\rho_t^\alpha, \psi) = - \int_\Omega D \rho^\alpha |\nabla (\log \rho^\alpha + \beta \alpha \psi)|^2,$$

where $\Psi(s) = s(\log s - 1) + 1 \geq 0$, $s > 0$. Consequently, we deduce

$$\frac{d}{dt} \int_\Omega \left(\int_I \Phi(\rho^\alpha) P(d\alpha) \right) = - \int_\Omega \left(\int_I D \rho^\alpha |\nabla (\log \rho^\alpha + \beta \alpha \psi)|^2 P(d\alpha) \right) \leq 0.$$

The patch model

$$\omega(x, t) = \sum_{i=1}^N \sigma_i I_{\Omega_i(t)}(x),$$

is used to describe detailed vortex distribution, where N , σ_i , and $\Omega_i(t)$ denote the number of patches, the vorticity of the i -th patch, and the domain of the i -th patch, respectively. Mean field equations for equilibrium and relaxation time are derived by the principles of maximum entropy and maximum entropy production, respectively, see [21–24]. At the scaling limit those patch models converge to the point vortex models, and in this way the Brownian point vortex model for relaxation dynamics has a physical reality [26].

In the canonical setting the inverse temperature β is a given constant in (A.5). Study of the asymptotic behavior of the solution is done in [29]. Without the vortex term $\nabla \cdot \rho^\alpha u$, and particularly, for the one intensity case that $P(d\alpha) = \delta_1(d\alpha)$, this is the Smoluchowski–Poisson equation for self-gravitating Brownian particles, [7], as well as it serves as form of chemotaxis system in mathematical biology [12], see also Chap. 7. Then there is a quantized blow-up mechanism for both blow-up in finite and in infinite time described as in Theorems 1.2.2 and 1.9.1 of [28], see also [27].

A.3 Normalized Ricci Flow on Surfaces

The normalized Ricci flow describes an evolution in time of the metric $g = g(t)$ on a compact Riemannian manifold. If Ω is a compact Riemannian surface without boundary, this flow is described by

$$\frac{\partial g}{\partial t} = (r - R)g, \quad (\text{A.8})$$

where $R = R(\cdot, t)$ stands for the scalar curvature of $(\Omega, g(t))$ and $r = r(t)$ represents the average scalar curvature:

$$r = \frac{\int_{\Omega} R(\cdot, t) d\mu_t}{\int_{\Omega} d\mu_t}.$$

R. Hamilton, [8], was first introduced this flow to approach the Poincaré conjecture. The case that Ω is a compact Riemannian surface is described in [9]. It is shown that the solution to (A.8) is global-in-time, and converges in C^∞ -topology to a metric as $t \uparrow \infty$, and the scalar curvature of this limit metric is constant.

An analytic approach was undertaken by [1], assuming $R(\cdot, 0) > 0$ everywhere. In fact, then R continues to be positive by the maximum principle applied to

$$\frac{\partial R}{\partial t} = \Delta_t R + R(R - r),$$

where Δ_t is the Laplace–Beltrami operator associated with $g(t)$. Then, from Gauss–Bonnet’s theorem follows

$$\int_{\Omega} R(\cdot, t) d\mu_t = 4\pi \chi(\Omega) > 0,$$

where $\chi(\Omega) = 2 - 2k(\Omega)$ stands for the Euler characteristic of Ω . Hence $k(\Omega)$ is the genus of Ω and it holds that $k(\Omega) = 0$. Then the uniformization theorem reduces the problem to the case $\Omega = S^2$, $g(t) = e^{w(\cdot, t)} g_0$, where S^2 the two dimensional sphere, g_0 is its standard metric, and $w = w(\cdot, t)$ is a smooth function. In this case (A.8) is reduced to

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda \left(\frac{e^w}{\int_{\Omega} e^w} - \frac{1}{|\Omega|} \right), \quad (\text{A.9})$$

with $\lambda = 8\pi$. This equation is studied in [13, 14] for general Ω and λ . The fundamental property is

$$\frac{d}{dt} \int_{\Omega} e^w = 0 \quad \text{and} \quad \frac{d}{dt} J_{\lambda}(w) = - \int_{\Omega} e^w w_t^2,$$

which implies that $r = \frac{\lambda}{\int_{\Omega} e^w}$ is a constant.

In fact, under the change of variables $u = r e^w$ and $t = r^{-1} \tau$, and writing t for τ , we obtain

$$u_t = \Delta \log u + u - \bar{u}, \quad \bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u.$$

Then, rewriting $w = \log u$, we reach to

$$\frac{\partial e^w}{\partial t} = \Delta w + e^w - \frac{\lambda}{|\Omega|}, \quad (\text{A.10})$$

which implies

$$e^w w_t^2 + e^w w_{tt} = \Delta w_t + e^w w_t.$$

Hence

$$p_t = e^{-w} \Delta p + p - p^2, \quad p = w_t,$$

and then follows that

$$p = w_t \leq \frac{e^t}{e^t - 1}, \quad (\text{A.11})$$

from the comparison theorem. By virtue of (A.10) we also derive

$$\frac{d}{dt} J_{\lambda}(w) = - \int_{\Omega} e^w w_t^2 \leq 0, \quad (\text{A.12})$$

for

$$J_\lambda(w) = \frac{1}{2} \|\nabla w\|_2^2 - \lambda \left(\log \int_\Omega e^w - \bar{w} \right), \quad \bar{w} = \frac{1}{|\Omega|} \int_\Omega w.$$

Using (A.11), (A.12), and the concentration compactness lemma [18] (see also [3]), we obtain a uniformly bounded global-in-time solution, provided that $\lambda \leq 8\pi$. Therefore, (A.12) guarantees the inclusion of the ω -limit set in the set of stationary solutions. Since the stationary solution of the original problem (A.9) for $\Omega = S^2$ and $\lambda = 8\pi$ is unique (see [4, 15] for the analytic proof), we now recover the result [9] for $g(\Omega) = 0$.

References

1. Bartz, J., Struwe, M., Ye, R.: A new approach to the Ricci flow on S^2 . *Ann. Scuola Norm. Sup. Pisa Cl. Sci. IV* **21**, 475–482 (1994)
2. Bernstein, S.: Sur une classe d'équations fonctionnelles aux dérivées partielles. *Izv. Akad. Nauk SSSR Sér. Mat.* **4**, 17–26 (1940)
3. Caglioti, E., Lions, P.L., Marchioro, C., Pulvirenti, M.: A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, part II. *Comm. Math. Phys.* **174**, 229–260 (1995)
4. Chanillo, S., Kiessling, M.: Rotational symmetry of solutions for some nonlinear problems in statistical mechanics and in geometry. *Comm. Math. Phys.* **160**, 217–238 (1994)
5. Chavanis, P.H.: Two-dimensional Brownian vortices. *Phys. A* **387**, 6917–6942 (2008)
6. Chavanis, P.H., Sire, C.: Anomalous diffusion and collapse of self-gravitating Langevin particles in D dimensions. *Phys. Rev. E* **69**, 016116 (2004)
7. D'Ancona, P., Spagnolo, S.: Global solvability for the degenerate Kirchhoff equation with real analytic data. *Invent. Math.* **108**, 247–262 (1992)
8. Hamilton, R.S.: Three manifolds with positive Ricci curvature. *J. Differ. Geom.* **17**, 255–306 (1982)
9. Hamilton, R.S.: The Ricci flow on surfaces. *Contem. Math.* **71**, 237–262 (1988)
10. Hirose, F.: Global solvability for Kirchhoff equation in special classes of non-analytic functions. *J. Differ. Equ.* **230**, 49–70 (2006)
11. Hirose, F.: A class of non-analytic functions for the global solvability of Kirchhoff equation. *Nonlinear Anal.* **116**, 37–63 (2015)
12. Jäger, W., Luckhaus, S.: On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.* **329**, 819–824 (1992)
13. Kavallaris, N.I., Suzuki, T.: An analytic approach to the normalized Ricci flow-like equation. *Nonlinear Anal.* **72**, 2300–2317 (2010)
14. Kavallaris, N.I., Suzuki, T.: An analytic approach to the normalized Ricci flow-like equation. *Appl. Math. Lett.* **44**, 30–33 (2015)
15. Lin, C.S.: Uniqueness of solutions to the mean field equations for the spherical Onsager vortex. *Arch. Ration. Mech. Anal.* **153**, 153–176 (2000)
16. Manfrin, R.: On the global solvability of Kirchhoff equation for non-analytic initial data. *J. Differ. Equ.* **211**, 38–60 (2005)
17. Nagasaki, K., Suzuki, T.: Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities. *Asymptot. Anal.* **3**, 173–188 (1990)
18. Ohtsuka, H., Senba, T., Suzuki, T.: Blowup in infinite time in the simplified system of chemotaxis. *Adv. Math. Sci. Appl.* **17**, 445–472 (2007)
19. Onsager, L.: Statistical hydrodynamics. *Suppl. Nuovo Cimento* **6**, 279–287 (1949)

20. Pohozaev, S.I.: A certain class of quasilinear hyperbolic equations. *Math. Sb. (N.S.)* **96**, 152–166 (1975)
21. Robert, R.: A maximum-entropy principle for two-dimensional perfect fluid dynamics. *J. Stat. Phys.* **65**, 531–553 (1991)
22. Robert, R., Sommeria, J.: Statistical equilibrium states for two-dimensional flows. *J. Fluid Mech.* **229**, 291–310 (1991)
23. Robert, R., Sommeria, J.: Relaxation towards a statistical equilibrium state in two-dimensional perfect fluid dynamics. *Phys. Rev. Lett.* **69**, 2776–2779 (1992)
24. Robert, R., Rosier, C.: The modeling of small scales in two-dimensional turbulent flows. *J. Stat. Phys.* **86**, 481–515 (1997)
25. Sawada, K., Suzuki, T.: Derivation of the equilibrium mean field equations of point vortex system and vortex filament system. *Theor. Appl. Mech. Jpn.* **56**(2008), 285–290 (2008)
26. Sawada, K., Suzuki, T.: Chapter 11 Relaxation theory for point vortices. In: Perez-de-Tejada, H. (ed.) *Vortex Structures in Fluid Dynamics Problems*, pp. 205–224. INTECH (2017)
27. Suzuki, T.: *Free Energy and Self-Interacting Particles*. Birkhäuser, Boston (2005)
28. Suzuki, T.: Brownian point vortices and DD-model. *Disc. Cont. Dyn. Syst. S* **7**(1), 161–176 (2014)
29. Suzuki, T.: *Mean Field Theories and Dual Variation-Mathematical Structures of the Mesoscopic Model*, 2nd edn. Atlantis Press, Paris (2015)

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