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Rahul Mukerjee
C.F. Jeff Wu

A Modern Theory of Factorial Designs

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(continued after index)

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Library of Congress Control Number: 2005939038

ISBN-10: 0-387-31991-3
ISBN-13: 978-0387-31991-9

Printed on acid-free paper.

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Printed in the United States of America. (MVY)

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To my father and wife,
and to the memory of my mother, RM

To my wife, CFJW

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(continued from p. ii)

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To my father and wife, and to
the memory of my mother, RM

To my wife, CFJW

Preface

Factorial design has always played a prominent role in the theory and practice of experimental design. It allows efficient and economic experimentation with multiple input variables and has been successfully used in a wide range of applications. Much research has been done and texts have been written on factorial design in the 70 years since its inception. For economic reasons, fractional factorials have been extremely popular, especially when the number of factors is large and the runs are expensive. The first and perhaps the most important issue faced by experimenters is the choice of a fractional factorial design. Given the long history of factorial design, an "optimality" theory for design selection should have emerged long ago. Surprisingly, the first serious attempt in this direction was made only in the early sixties with the notion of resolution. It became apparent later that this notion was not discriminating as a criterion for design selection. Equally surprisingly, it took almost another 20 years to see the birth of the minimum aberration (MA) criterion, which has since become the major criterion for selecting fractional factorial designs. Once the importance of the MA criterion was recognized in the late eighties, research on the theory and algorithms for finding MA and related designs has grown rapidly in the last 15 years. Besides building and improving upon existing techniques in projective geometry and coding theory, such research has led to the development of novel techniques like complementary designs and efficient search algorithms. Factorial designs with increasing complexity in the underlying structure (e.g., mixed-levels, blocking, split-plots, and robust parameter design) have also received considerable attention. A detailed description of this history and an account of the nature and contents of the book appear in Chapter 1.

In 2000, the present authors felt that the time was ripe to start planning and writing a modern book on factorial designs with the MA perspective. Such a book should contain the major theoretical tools and results, and tables of optimal or efficient designs available in the literature. It took several visits by RM to CFJW at the Department of Statistics, University of Michigan, and later at the School of Industrial and Systems Engineering, Georgia Institute of Technology, to complete the project. The hospitality and support of both institutions are gratefully acknowledged.

All the mathematical and theoretical prerequisites are given in Chapter 2 of the book. For mathematically oriented readers, little additional background is required to read and follow the logic and derivations in the book, since each new concept is accompanied with a brief statistical explanation, justification, or reference. For statistically oriented readers, background in basic design and analysis of experiments will help in understanding and appreciating the significance and impact of the concepts and results in the book. The book can be used as a text for a graduate course in design theory in statistics or mathematics programs and also as a reference text for a graduate course in combinatorial mathematics. We hope that it will be a useful reference for design researchers in general. The extensive collection of design tables should endear it to practitioners and others interested in the use of factorial designs.

During the course of writing this book, we have received comments and assistance from colleagues, and former and current students, including Aloke Dey, Tirthankar Dasgupta, Xingwei Deng, Greg Dyson, Ying Hung, Abhyuday Mandal, Zhiguang Qian, Xianggui Qu, and Hongquan Xu. We sincerely thank all of them. The writing of the book was supported in part by NSF grant DMS-0072489. RM also received support for this project from the Centre for Management and Development Studies, Indian Institute of Management Calcutta.

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Contents

1 Introduction and Overview 1
- 1.1 Prologue 1
- 1.2 Why This Book? 3
- 1.3 What Is in the Book? 4
- 1.4 Beyond the Book 6

2 Fundamentals of Factorial Designs 9
- 2.1 Factorial Effects 9
- 2.2 Kronecker Product Formulation for Factorial Effects 12
- 2.3 A Representation for Factorial Effects in Symmetrical Factorials 18
- 2.4 Regular Fractions 24
- 2.5 Optimality Criteria: Resolution and Minimum Aberration 32
- 2.6 Connection with Orthogonal Arrays 35
- 2.7 Connection with Finite Projective Geometry 39
- 2.8 Algebraic Coding Theory 45

3 Two-Level Fractional Factorial Designs 49
- 3.1 Basic Definitions Revisited 49
- 3.2 Minimum Aberration 2^{n-k} Designs with $k \leq 4$ 52
- 3.3 Minimum Aberration Designs via Complementary Designs 57
- 3.4 Clear Effects and the MaxC2 Criterion 63
- 3.5 Description and Use of the Two-Level Design Tables 67

4 Fractional Factorial Designs: General Case 85
- 4.1 Three-Level Designs 85
- 4.2 Minimum Aberration s^{n-k} Designs with Small k 87
- 4.3 A General Result on Complementary Designs 91
- 4.4 Minimum Aberration s^{n-k} Designs via Complementary Designs 98
- 4.5 Description and Use of the Three-Level Design Tables 102

5 Designs with Maximum Estimation Capacity 109
5.1 Preliminaries 109
5.2 Connection with Complementary Sets 113
5.3 Estimation Capacity in 2^n Factorials 116
5.4 Estimation Capacity in s^n Factorials 121

6 Minimum Aberration Designs for Mixed Factorials 125
6.1 Construction of $4^p \times 2^n$ Designs via the Method of Replacement 125
6.2 Minimum Aberration $4^p \times 2^n$ Designs with $p = 1, 2$ 127
6.3 Designs for $(s^r) \times s^n$ Factorials: Preliminaries 133
6.4 Minimum Aberration Designs for $(s^r) \times s^n$ Factorials 139
6.5 Designs for $(s^{r_1}) \times (s^{r_2}) \times s^n$ Factorials 143

7 Block Designs for Symmetrical Factorials 151
7.1 Optimal Block Designs for Full Factorials 151
7.2 Block Designs for Fractional Factorials 154
7.3 A Projective Geometric Formulation 158
7.4 Design Criteria 161
7.5 Description and Use of Block Design Tables 166

8 Fractional Factorial Split-Plot Designs 181
8.1 Description and Salient Features 181
8.2 Design Criteria 184
8.3 A Projective Geometric Formulation 186
8.4 Use of Complementary Sets 189
8.5 Tables of Optimal Designs 191

9 Robust Parameter Design 197
9.1 Control and Noise Factors 197
9.2 Cross Arrays 198
9.3 Modeling Strategies 201
9.4 Single Arrays 202
9.5 Effect Ordering Principle 206
9.6 Design Criteria 207

References 211

Author Index 217

Subject Index 219

1

Introduction and Overview

1.1 Prologue

Design of experiments has played a fundamental role in the statistical curriculum, practice, and research ever since R.A. Fisher founded the modern discipline. It has been successfully applied in many fields of scientific investigation. These include agriculture, medicine, and behavioral research as well as chemical, manufacturing, and high-tech industries. Concepts like randomization, effect confounding, and aliasing, which originated in design and analysis of experiments, have found applications beyond their initial motivation. Work on the mathematical aspects of design theory, such as block designs and orthogonal arrays, has also stimulated new research in some branches of mathematics such as algebra, combinatorics, and coding theory.

Experimental problems can be classified into five broad categories according to their objectives (Wu and Hamada, 2000): (i) treatment comparison, (ii) variable screening, (iii) response surface exploration, (iv) system optimization, and (v) system robustness. Except for treatment comparison with one- or two-way layouts, these problems involve the study of the effects of multiple input variables on the experimental outcome, i.e., the response. These input variables are called *factors* and the experiments are called *factorial experiments.* Each factor must have two or more settings so that the effect of change in factor setting on the response can be explored. These settings are called *levels* of the factor. Any combination of levels of factors is known as a *treatment combination.* A treatment combination is also called a *run* in industrial experimentation. *Factorial design* concerns the *selection and arrangement* of treatment combinations in a factorial experiment. It is the most important class of designs because of its richness in structure, theory, and applications. Applied design texts typically devote a substantial part of their contents to factorial designs. Indeed, any investigation involving multiple factors can benefit from using the concept, theory, and methodology of factorial designs.

A *full factorial design* involves all possible treatment combinations. The number of such combinations grows rapidly as the number of factors increases.

For a factorial experiment with factors at two levels, this number increases from 64 to 512 as the number of factors grows from 6 to 9. When each factor has three levels, this number increases from 27 to 729 as the number of factors grows from 3 to 6. For obvious economic reasons, a full factorial experiment of large size may not be feasible. A practical solution is to choose a fraction of the full factorial for experimentation. Choice of such fractions in an economic and efficient way is the subject of *fractional factorial designs.*

We illustrate fractional factorial designs via a simple example. Consider the 16×15 array, called a design matrix, in Table 1.1. The rows correspond to treatment combinations (or runs) and the columns are used for assigning factors. Each factor has two levels, denoted by 0 and 1 in the table. Columns 1 to 4 consist of all the 16 possible treatment combinations for four factors. Columns 5 to 15 are formed by taking the sum modulo 2 of any two to four columns out of the first four. For example, column 5 is obtained as the sum modulo 2 of columns 1 and 2 and can be interpreted as representing the interaction between the two factors assigned to columns 1 and 2. Suppose the run size is fixed at 16. We can use the first four columns to form a $2 \times 2 \times 2 \times 2$ full factorial (abbreviated as a 2^4 factorial) to study four factors. With five factors, the first four columns can be retained and an additional column out of the remaining 11 (i.e., those numbered 5 to 15 in the table) can be chosen for the fifth factor. Given a design objective, this would be an easy search, since it involves only 11 different choices. The resulting design is a half-fraction of the 2^5 factorial. As the number of factors increases, the size of search becomes more formidable. For nine factors and 16 runs, we will need to find a $1/2^5$ fraction of the 2^9 factorial, i.e., a 2^{9-5} design. In addition to choosing the

Table 1.1 A 16-run design matrix

	Column														
Run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1
3	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
4	0	0	1	1	0	1	1	1	1	0	1	1	0	0	0
5	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
6	0	1	0	1	1	0	1	1	0	1	1	0	1	0	0
7	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0
8	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1
9	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
10	1	0	0	1	1	1	0	0	1	1	1	0	0	1	0
11	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
12	1	0	1	1	1	0	0	1	1	0	0	0	1	0	1
13	1	1	0	0	0	1	1	1	1	0	0	0	1	1	0
14	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1
15	1	1	1	0	0	0	1	0	1	1	1	0	0	0	1
16	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0

first four columns, we then have to choose five more out of the remaining 11 columns, which involves 462 choices. For larger problems, this number can grow astronomically. The same reasoning as before shows that for studying 13 two-level factors with 32 ($= 2^5$) runs, the search involves $\binom{31-5}{13-5} = \binom{26}{8} \approx 1.56 \times 10^6$ choices and the number grows from the millions to 1.65 billion ($\approx \binom{63-6}{14-6} = \binom{57}{8}$) for studying 14 factors with 64 ($= 2^6$) runs. By symmetry arguments and other techniques, the size of search as represented by these numbers can be reduced. But it conveys the message that sheer brute-force search for economic and efficient designs is not feasible and a theory is called for.

1.2 Why This Book?

Before addressing the issue of "optimal" selection of fractions, we must settle on the choice of an optimality criterion. The first major criterion is that of *maximum resolution* proposed by Box and Hunter (1961a,b). A fractional factorial design with a higher resolution is considered to be more desirable (definitions of technical terms are deferred to subsequent chapters). It was later recognized that this criterion is not discriminating in the sense that fractions with the same resolution can have very different properties as judged by further considerations. After a span of nearly 20 years, Fries and Hunter (1980) proposed a more discriminating criterion, called *minimum aberration* (MA), for selecting optimal fractions. Apparently, this seminal work had its roots in a table on two-level fractional factorial designs given by Box, Hunter, and Hunter (1978, p.410). Even though their book did not mention the MA criterion, all the designs in the said table do enjoy the MA property. The classic work at the National Bureau of Standards (1957, 1959) also contained some hints on the MA criterion but did not come close to defining it or capturing its essence.

With the exception of the work by Franklin (1984, 1985), the Fries–Hunter paper and the MA criterion went unnoticed for another decade. In the early nineties, one of the present authors and his collaborators recognized the central role of this criterion in selecting optimal fractional factorial designs and the need to develop a theory and computational algorithms to characterize and search for MA designs. Once the initial papers were published, this new approach to factorial designs received immediate attention in the design community. The last fifteen years have witnessed a copious growth in the literature on MA designs, their extension to more complex situations, and related criteria like estimation capacity. This research has made a significant impact on textbooks and software. The applied design text by Wu and Hamada (2000) was the first one to use the MA criterion extensively in selecting optimal fractions and their tabulation. The second edition of the classic by Box, Hunter, and Hunter (2005) also made reference to the MA criterion and designs. Statistical packages such as SAS/QC, JMP, and Design-Ease now have the option of using this criterion in selecting optimal fractions.

This book aims at providing a comprehensive account of the modern theory of factorial designs with the MA approach at its core. In order to equip the reader with the necessary background, we also develop some foundational concepts and results. In addition, for practical use, extensive tables of MA and related designs are given throughout the book. The combination of foundational work, recent research results with the MA perspective, and design tabulation forms the bulk of the book. Among books on factorial designs and related topics outside the MA paradigm, we refer to Raktoe, Hedayat, and Federer (1981), Dey (1985), John and Williams (1995), Dey and Mukerjee (1999), and Hedayat, Sloane, and Stufken (1999).

1.3 What Is in the Book?

An outline of the book is given in this section. After the introductory chapter, the mathematical foundation of factorial designs is developed at length in Chapter 2. It begins with an introduction to factorial effects following the classic work of Bose (1947) and then moves on to define fractional factorial designs via regular fractions. Optimality criteria based on resolution and aberration are presented and justified by the effect hierarchy principle. Connection of fractional factorial designs with orthogonal arrays, finite projective geometry, and algebraic codes, and the relevant mathematical properties of the latter, are discussed in the last three sections. The contents of this chapter provide the mathematical prerequisite for the modern theory in the subsequent chapters. Moreover, they constitute useful background material for other aspects of experimental design such as incomplete block designs, Latin squares, and response surface designs. Indeed, this chapter alone should be of value to those interested in the general area of combinatorial design.

Chapters 3 to 5 form the next unit. Chapter 3 deals with MA designs in the two-level case and presents the major tools and theoretical results. The related concepts of clear effects and MaxC2 criterion are also discussed. A comprehensive treatment of MA 2^{n-k} designs with $k \leq 4$ is given. For larger k (i.e., highly fractionated experiments), a direct attack on the problem becomes unmanageable. A novel idea then is to relate the properties of a design to those of its complementary design through some combinatorial identities. This approach is particularly useful when the number of factors in the complementary design is smaller than that in the original design. The development and applications of this powerful tool are discussed in detail. It will reappear in the subsequent chapters in increasing complexity. A comprehensive collection of design tables with 16, 32, 64, and 128 runs is given and their practical use is discussed. These tables include not only MA designs but also others that are highly efficient according to the MA or MaxC2 criterion. The design tables can be viewed as the "more applied" part of the book. They also form a data base for possible use by design researchers.

Chapter 4 extends the above work to s-level factorials, s being a prime or prime power. It first discusses the case of three-level designs and points out the extra complication in the mathematical treatment and interpretation of factorial effects while passing from two to three or higher levels. Barring this complication, the contents and flow of this chapter are similar to those of Chapter 3. By employing coding-theoretic tools, a major theorem is developed relating MA designs to their complementary designs. This theorem is one of the deepest and most significant among the recent work in design theory. Three-level design tables with 27 and 81 runs are given and their practical use is discussed. These tables, like those in Chapter 3, include MA designs and other efficient ones.

As the MA criterion became popular, an attempt was made to provide further statistical justification for it via an alternative criterion that relates more directly to the estimability property under various model assumptions. This led to the idea of maximum estimation capacity, which is the focus of Chapter 5. Both two- and general s-level factorials are treated. The main technical tool is again the use of complementary designs, now called complementary sets.

The designs considered in Chapters 3 to 5 concern symmetrical factorials, i.e., all the factors have the *same* number of levels. There are, however, practical situations in which the intrinsic nature of the factors does not allow such symmetry. For example, all factors may have two levels except for a qualitative factor like part supply with four suppliers. The focus of Chapter 6 is on asymmetrical (or mixed-level) designs, i.e., those with factors at *different* numbers of levels. The simplest and most common example is given by the mixed two- and four-level designs, which are constructed by the method of replacement. MA designs with one or two four-level factors and a general number of two-level factors are presented. Designs with s-level and s^r-level factors are then considered. Projective geometry is used for describing such designs and their MA property is investigated using the technique of complementary sets. Tables of MA designs are given.

The next three chapters can be viewed as forming the last unit of the book. They deal with designs that involve two distinct types of factors. In contrast to Chapter 6, where the distinction among factors is in terms of the number of factor levels, the distinction here is of a more subtle nature. Additional care is needed and tools have to be developed to address the new features of asymmetry. Chapter 7 covers full and fractional factorial designs arranged in blocks. These designs incorporate a block structure in addition to the factorial structure for the treatment factors. An extension of the MA criterion is given for full factorials arranged in blocks. The problem of blocking in fractional factorial designs is, however, complicated by the existence of two structures: one defining the fraction for the treatment factors and the other defining the blocking scheme. Projective geometry is used to describe the necessary mathematical formulation, and various optimality criteria, motivated by MA, are discussed. Tables of block designs with desirable properties are given for 16, 32, 64, and 128 runs in the two-level case, and 27 and 81 runs in the three-level case.

Chapter 8 is on fractional factorial split-plot designs, where the whole plot and subplot factors cannot be treated symmetrically because of different error structures. This necessitates supplementing the original MA criterion by another follow-up criterion. As in the previous chapters, projective geometry and complementary sets are used as the main technical tools. This chapter concludes with tables of optimal designs.

Robust parameter design is a statistical/engineering methodology for variation reduction that works by choosing appropriate settings of the control factors so as to make the system insensitive to hard-to-control noise variations. Parameter design experiments are commonly used in quality improvement, and Chapter 9 gives an account of their planning aspects. Two experimental formats, cross arrays and single arrays, are considered. A fundamental result on the estimability property of cross arrays is given. Here the lack of symmetry is due to the different roles played by the control and noise factors in the choice of designs and modeling strategies. A new effect ordering principle is developed to address such asymmetry, taking due cognizance of the experimental priorities. This, in turn, leads to a substantial modification of the MA criterion. The use of the modified criterion in selecting optimal designs is discussed.

Since the focus of the book is on design of experiments, the modeling issues are seldom addressed. For information on modeling, analysis, and applications, we refer to applied design texts like Montgomery (2000), Wu and Hamada (2000), and Box, Hunter, and Hunter (2005).

1.4 Beyond the Book

In this section, we briefly indicate some promising recent topics in factorial designs that are not covered in the book. They are still undergoing rapid development and hence have yet to crystalize.

All the fractional factorial designs considered in the book are called *regular*. A precise definition is given in Section 2.4; see also Section 2.7. To give an idea of what a regular fraction entails, let us refer to the design matrix in Table 1.1. It is easy to see that the sum modulo 2 of any two different columns of this matrix can be found among the remaining columns. In this sense, any fractional factorial design given by a selection of columns from this matrix is regular. On the other hand, designs arising from the 12-run matrix given in Table 1.2 (Plackett and Burman, 1946) are nonregular; observe that the sum modulo 2 of any two of its columns *cannot* be found among the remaining columns. Traditionally, regular fractions have been the primary focus of research in factorial designs. They have a neat mathematical structure that

Table 1.2 A 12-run Plackett–Burman design matrix

	Column										
Run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	0	0	1	0	0	0	1	1	1	0	1
2	1	0	0	1	0	0	0	1	1	1	0
3	0	1	0	0	1	0	0	0	1	1	1
4	1	0	1	0	0	1	0	0	0	1	1
5	1	1	0	1	0	0	1	0	0	0	1
6	1	1	1	0	1	0	0	1	0	0	0
7	0	1	1	1	0	1	0	0	1	0	0
8	0	0	1	1	1	0	1	0	0	1	0
9	0	0	0	1	1	1	0	1	0	0	1
10	1	0	0	0	1	1	1	0	1	0	0
11	0	1	0	0	0	1	1	1	0	1	0
12	1	1	1	1	1	1	1	1	1	1	1

simplifies the derivation and facilitates the understanding of effect aliasing. Moreover, they are the most commonly used designs in practice.

Over the years, nonregular designs had received some attention of researchers mainly from the mathematical point of view. In the recent past, however, there was a realization that they too could be utilized in conducting efficient experiments with flexibility, run size economy, and ability to exploit interactions (Hamada and Wu, 1992; Wu and Hamada, 2000, Chapters 7 and 8). This led to a growth of interest in nonregular designs. The question of their optimal selection was a natural research topic to follow the parallel work in the regular case. Various extensions of the minimum aberration criterion to nonregular designs were proposed. Some major ones are *minimum* G_2-*aberration* (Deng and Tang, 1999; Tang and Deng, 1999), *generalized minimum aberration* (Xu and Wu, 2001), and *minimum moment aberration* (Xu, 2003). Although these criteria are mathematically equivalent to the MA criterion in the regular case, some of them are potentially advantageous even for tackling regular design problems with relatively complex structure such as blocking (Xu, 2006). Through the use of *indicator functions*, Ye (2003) and Cheng and Ye (2004) investigated nonregular designs from a different viewpoint.

Another promising area of research concerns *supersaturated designs*, where the run size is not sufficient even for estimating all the main effects of the factors. This can be a realistic scenario when the runs are expensive and the problem is complex enough to suggest many factors for investigation. None of the designs considered in the book is supersaturated. For example, the design matrix in Table 1.1 can be used to study up to 15 two-level factors with 16 runs. Suppose economic considerations limited the run size to 16 and the investigators were insistent on including, say, as many as 19 factors in the experiment. A supersaturated design in the form of a 16×19 array would then be required. Various construction methods for supersaturated designs have been proposed; see, e.g., Lin (1993, 1995), Wu (1993), Nguyen (1996), Yamada and Lin (1999), and Xu and Wu (2005). On the other hand, much less has been done on their optimal choice from MA or related perspectives. Interestingly, most of the optimality criteria for nonregular designs, as indicated in the last paragraph, are potentially applicable to supersaturated designs.

A further research topic has its motivation in computer experiments. Traditionally, the great majority of experimentation is done by conducting physical experiments, which employ designs with factors typically at two to four levels. As simulation and numerical experiments on a computer have become technically and economically feasible, computer experiments are gradually assuming a significant role in engineering and science. Because the computer models are usually very complex, it is often imperative to consider factors with larger numbers of levels. The size of traditional designs can then become impractically large and hence *space-filling designs* are used instead. Two major classes of space-filling designs are *Latin hypercube designs* and *uniform designs*. We refer to Santner, Williams, and Notz (2003) and Fang, Li, and Sudjianto (2005) for details on design and modeling strategies for computer experiments.

2

Fundamentals of Factorial Designs

The basic definitions and technical tools for factorial design theory are given in this chapter. They include various mathematical definitions of factorial effects, and tools like Galois fields, finite projective geometry, and coding theory. These ideas and tools are used to define and discuss fundamental concepts such as regular fractions, defining pencils, aliasing, resolution, minimum aberration, and orthogonal arrays.

2.1 Factorial Effects

An experiment involving n (≥ 2) factors $F_1, \ldots, F_n$ that appear at $s_1, \ldots, s_n$ (≥ 2) levels is called an $s_1 \times \cdots \times s_n$ factorial experiment (or an $s_1 \times \cdots \times s_n$ factorial for brevity). In particular, if $s_1 = \cdots = s_n = s$, it is called a *symmetrical* s^n factorial; otherwise it is called an *asymmetrical* factorial. For $1 \leq i \leq n$, the s_i levels of the ith factor F_i are denoted by s_i symbols. Suppose these levels are coded as $0, 1, \ldots, s_i - 1$. Then a typical treatment combination, i.e., a combination of the levels of the n factors, will be represented by an ordered n-tuple $j_1 \ldots j_n$, where $j_i \in \{0, 1, \ldots, s_i - 1\}, 1 \leq i \leq n$. Clearly, altogether there are $\Pi_{i=1}^{n} s_i$ treatment combinations.

For example, if there are three factors at two, three, and three levels respectively, then $n = 3,\ s_1 = 2,\ s_2 = 3,\ s_3 = 3$, and there are 18 treatment combinations, namely,

$$\begin{array}{ccccccccc} 000, & 001, & 002, & 010, & 011, & 012, & 020, & 021, & 022, \\ 100, & 101, & 102, & 110, & 111, & 112, & 120, & 121, & 122. \end{array} \tag{2.1.1}$$

Let $\tau(j_1 \ldots j_n)$ denote the treatment effect corresponding to a treatment combination $j_1 \ldots j_n$. These treatment effects are unknown parameters in the context of a factorial experiment. A linear parametric function

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} l(j_1 \ldots j_n)\tau(j_1 \ldots j_n), \tag{2.1.2}$$

where $l(j_1 \ldots j_n)$ are real numbers, not all zero, such that

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} l(j_1 \ldots j_n) = 0, \tag{2.1.3}$$

is called a *treatment contrast*. In factorial experiments, we are concerned with special types of treatment contrasts, namely, those belonging to factorial effects.

To motivate the ideas, consider the simple case of a 2^2 factorial. Then there are two factors F_1 and F_2, each at two levels 0 and 1, and the four treatment combinations are 00, 01, 10, and 11. The effect of changing the factor F_1 from level 0 to level 1, with the factor F_2 held fixed at level 0, is clearly given by

$$L(F_1|F_2 = 0) = \tau(10) - \tau(00). \tag{2.1.4}$$

Similarly, the effect of changing F_1 from level 0 to level 1, with F_2 held fixed at level 1, is given by

$$L(F_1|F_2 = 1) = \tau(11) - \tau(01). \tag{2.1.5}$$

The *main effect* of F_1 is measured by the arithmetic mean of the two quantities in (2.1.4) and (2.1.5), which is given by

$$L(F_1) = \frac{1}{2}[\{\tau(10) - \tau(00)\} + \{\tau(11) - \tau(01)\}]. \tag{2.1.6}$$

Observe that $L(F_1)$ is of the form (2.1.2), with

$$l(00) = l(01) = -\frac{1}{2},\ l(10) = l(11) = \frac{1}{2}. \tag{2.1.7}$$

Clearly $l(00)$, $l(01)$, $l(10)$, and $l(11)$ add up to zero, i.e., satisfy (2.1.3). Thus $L(F_1)$ is a treatment contrast that measures the main effect of F_1. Interchanging the roles of F_1 and F_2, it is obvious that the main effect of the factor F_2 is given by the treatment contrast

$$L(F_2) = \frac{1}{2}[\{\tau(01) - \tau(00)\} + \{\tau(11) - \tau(10)\}]. \tag{2.1.8}$$

Continuing with a 2^2 factorial, we next consider the interaction between F_1 and F_2. This is measured by the influence of the level where F_2 is held fixed on the effect of a level change of F_1. Thus the difference between $L(F_1|F_2 = 1)$ and $L(F_1|F_2 = 0)$ measures this interaction. By (2.1.4) and (2.1.5), the *interaction* F_1F_2 is then measured by

$$\begin{aligned} L(F_1F_2) &= \frac{1}{2}[\{\tau(11) - \tau(01)\} - \{\tau(10) - \tau(00)\}] \\ &= \frac{1}{2}\{\tau(11) - \tau(01) - \tau(10) + \tau(00)\}, \end{aligned} \tag{2.1.9}$$

which is again a treatment contrast. The interaction F_1F_2 could as well be visualized in terms of the influence of the level where F_1 is held fixed on the effect of a level change of F_2. This is reflected in the fact that (2.1.9) remains invariant when the roles of the two factors are interchanged.

The two main effects corresponding to the factors F_1 and F_2 and the interaction F_1F_2 are the factorial effects arising in a 2^2 factorial. Thus in a 2^2 factorial, we will be concerned with the treatment contrasts $L(F_1), L(F_2)$, and $L(F_1F_2)$, which represent these three factorial effects respectively. Incidentally, in most statistical applications, treatment contrasts are scaled appropriately, which is why the multiplier $1/2$ in (2.1.6), (2.1.8), or (2.1.9) will have no special significance in the subsequent development. Indeed, even if this multiplier $1/2$ is replaced by any other nonzero constant, one would still get treatment contrasts proportional to $L(F_1)$, $L(F_2)$, or $L(F_1F_2)$ respectively.

We now turn to the general case of an $s_1 \times \cdots \times s_n$ factorial and introduce the following definition (Bose, 1947) for treatment contrasts belonging to factorial effects.

Definition 2.1.1. *A treatment contrast*

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} l(j_1 \dots j_n)\tau(j_1 \dots j_n)$$

belongs to the factorial effect $F_{i_1} \dots F_{i_g} (1 \le i_1 < \cdots < i_g \le n; 1 \le g \le n)$ *if*

(i) $l(j_1 \dots j_n)$ depends only on $j_{i_1}, \dots, j_{i_g}$,
(ii) writing $l(j_1 \dots j_n) = \bar{l}(j_{i_1} \dots j_{i_g})$ in view of (i) above, the sum of $\bar{l}(j_{i_1} \dots j_{i_g})$ separately over each of the arguments $j_{i_1}, \dots, j_{i_g}$ is zero.

A factorial effect $F_{i_1} \dots F_{i_g}$, as defined above, will be called a main effect if it involves exactly one factor (i.e., $g = 1$) and an interaction if it involves more than one factor (i.e., $g > 1$). Clearly, there are n main effects and $\binom{n}{g}$ g-factor interactions. Thus the total number of factorial effects in an $s_1 \times \cdots \times s_n$ factorial is

$$\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1.$$

This is in agreement with our earlier enumeration of three factorial effects — two main effects and a two-factor interaction (abbreviated as 2fi) — in a 2^2 factorial. The *order* of a factorial effect is the number of factors that it involves. For example, a main effect is of order 1, a 2fi is of order 2, and so on.

We now demonstrate how the ideas developed previously for a 2^2 factorial are encapsulated in Definition 2.1.1. By taking $g = 1$ and $i_1 = 1$ in this definition, a treatment contrast belongs to the main effect of F_1 provided it is of the form

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} \bar{l}(j_1)\tau(j_1 \dots j_n), \qquad (2.1.10)$$

where

$$\sum_{j_1=0}^{s_1-1} \bar{l}(j_1) = 0. \tag{2.1.11}$$

Note that (2.1.10) and (2.1.11) correspond to the requirements (i) and (ii) respectively of Definition 2.1.1. Consider now the contrast $L(F_1)$ given in (2.1.6) and observe that it can be expressed as

$$L(F_1) = -\frac{1}{2}\{\tau(00) + \tau(01)\} + \frac{1}{2}\{\tau(10) + \tau(11)\}.$$

Hence, in compatibility with Definition 2.1.1 (i), the coefficient of $\tau(j_1 j_2)$ in $L(F_1)$ depends only on j_1. In other words, $L(F_1)$ is of the form (2.1.10) with $\bar{l}(0) = -1/2$ and $\bar{l}(1) = 1/2$. Obviously, $\bar{l}(0) + \bar{l}(1)$ equals zero, as it should in view of (2.1.11). Similarly, one can easily check that $L(F_2)$, given in (2.1.8), is also in agreement with Definition 2.1.1.

Turning to the case of the 2fi F_1F_2, we take $g = 2$, $i_1 = 1$, and $i_2 = 2$ in Definition 2.1.1. Then a treatment contrast belongs to F_1F_2 provided it is of the form

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} \bar{l}(j_1 j_2)\tau(j_1 \ldots j_n), \tag{2.1.12}$$

where

$$\sum_{j_1=0}^{s_1-1} \bar{l}(j_1 j_2) = 0 \text{ for each } j_2 (0 \le j_2 \le s_2 - 1), \tag{2.1.13}$$

and

$$\sum_{j_2=0}^{s_2-1} \bar{l}(j_1 j_2) = 0 \text{ for each } j_1 (0 \le j_1 \le s_1 - 1). \tag{2.1.14}$$

As before, (2.1.12) is dictated by the requirement (i) of Definition 2.1.1, whereas (2.1.13) and (2.1.14) are dictated by the requirement (ii). The contrast $L(F_1F_2)$ defined in (2.1.9) is of the form (2.1.12) with $\bar{l}(00) = \bar{l}(11) = 1/2$ and $\bar{l}(01) = \bar{l}(10) = -1/2$. They obviously satisfy (2.1.13) and (2.1.14). Hence $L(F_1F_2)$ is again in agreement with the concept of a treatment contrast belonging to interaction F_1F_2 given in Definition 2.1.1.

2.2 Kronecker Product Formulation for Factorial Effects

Continuing with an $s_1 \times \cdots \times s_n$ factorial, we now discuss some basic properties of treatment contrasts belonging to factorial effects. An alternative formulation for such contrasts, which is equivalent to Definition 2.1.1 but involves Kronecker products of matrices, will be helpful in this context. This formulation was introduced formally by Kurkjian and Zelen (1962, 1963). Some of their ideas were inherent in Zelen (1958) and Shah (1958).

The definition and a few elementary properties of the Kronecker product of matrices are given here; more details are available in Rao (1973, Chapter 1). If $B_1 = \left(\left(b_{ij}^{(1)}\right)\right)$ and B_2 are matrices of orders $p_1 \times q_1$ and $p_2 \times q_2$ respectively, then the *Kronecker product* of B_1 and B_2, denoted by $B_1 \otimes B_2$, is a $(p_1 p_2) \times (q_1 q_2)$ matrix defined as

$$B_1 \otimes B_2 = \left(\left(b_{ij}^{(1)} B_2\right)\right)$$

in the partitioned form. Similarly, the Kronecker product of three matrices B_1, B_2, and B_3 is defined as

$$B_1 \otimes B_2 \otimes B_3 = B_1 \otimes (B_2 \otimes B_3) = (B_1 \otimes B_2) \otimes B_3,$$

and so on. The following properties of Kronecker product will be useful in the sequel:

(i) for any n matrices $B_1, \cdots, B_n$,

$$(B_1 \otimes \cdots \otimes B_n)' = B_1' \otimes \ldots \otimes B_n',$$

where the prime denotes transpose;

(ii) for any n matrices $B_1, \ldots, B_n$,

$$\text{rank}(B_1 \otimes \cdots \otimes B_n) = \Pi_{i=1}^{n} \text{ rank}(B_i);$$

(iii) for any $2n$ matrices $B_{11}, \ldots, B_{1n}, B_{21}, \ldots, B_{2n}$,

$$(B_{11} \otimes \cdots \otimes B_{1n})(B_{21} \otimes \cdots \otimes B_{2n}) = (B_{11}B_{21}) \otimes \cdots \otimes (B_{1n}B_{2n}),$$

provided the ordinary product $B_{1i}B_{2i}$ is well-defined for every i $(1 \le i \le n)$.

We are now in a position to proceed with the Kronecker product formulation for treatment contrasts belonging to factorial effects in an $s_1 \times \cdots \times s_n$ factorial. Some notation and preliminaries will help. First, we write $v = \Pi_{i=1}^{n} s_i$ to denote the total number of treatment combinations. Without loss of generality, assume that the v treatment combinations are arranged lexicographically. For example, if $n = 2$, they are arranged as

$$00, 01, \ldots, 0\bar{s}_2, 10, 11, \ldots, 1\bar{s}_2, \ldots, \bar{s}_1 0, \bar{s}_1 1, \ldots, \bar{s}_1 \bar{s}_2,$$

where $\bar{s}_1 = s_1 - 1$ and $\bar{s}_2 = s_2 - 1$. Another example of a lexicographic arrangement with $n = 3$, $s_1 = 2$, $s_2 = s_3 = 3$ appears in (2.1.1). Let τ be a column vector, of order v, with elements given by the treatment effects $\tau(j_1 \ldots j_n)$ $(0 \le j_i \le s_i - 1, 1 \le i \le n)$, which are lexicographically arranged. Any treatment contrast can then be expressed as $l'\tau$, where l is a nonnull $v \times 1$ vector whose elements add up to zero.

For the Kronecker product formulation, it will be convenient to work with an alternative notational system for factorial effects. Observe that a typical factorial effect $F_{i_1} \dots F_{i_g}$ can be denoted by $F(y)$, where $y = y_1 \dots y_n$ is a binary n-tuple such that

$$y_i = \begin{cases} 1 & \text{if } i \in \{i_1, \dots, i_g\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.1}$$

This establishes a one-to-one correspondence between the set of the $2^n - 1$ factorial effects and the set Ω of the $2^n - 1$ nonnull binary n-tuples. For example, with $n = 3$, the main effect of F_2 can be denoted by $F(010)$, the interaction F_1F_3 by $F(101)$, and so on.

We will need some more notation. For $1 \le i \le n$, let 1_i be the $s_i \times 1$ vector with all elements unity, I_i the identity matrix of order s_i, and M_i an $(s_i - 1) \times s_i$ matrix such that

$$\text{rank}(M_i) = s_i - 1, \quad M_i 1_i = 0. \tag{2.2.2}$$

Of course, these equations do not specify M_i uniquely, but the present discussion does not depend on the specific choice of M_i as long as it satisfies the conditions in (2.2.2). For any $y = y_1 \dots y_n \in \Omega$, the set of nonnull binary n-tuples, define

$$M(y) = M_1^{y_1} \otimes \cdots \otimes M_n^{y_n}, \tag{2.2.3}$$

where, for $1 \le i \le n$,

$$M_i^{y_i} = \begin{cases} 1_i' & \text{if } \; y_i = 0, \\ M_i & \text{if } \; y_i = 1. \end{cases} \tag{2.2.4}$$

It is not hard to see that $M(y)$ involves $m(y)$ rows and v columns, where

$$m(y) = \Pi_{i=1}^{n} (s_i - 1)^{y_i}. \tag{2.2.5}$$

We now present the main result of this section, giving a Kronecker product formulation for treatment contrasts belonging to factorial effects.

Theorem 2.2.1. *For any $y = y_1 \dots y_n \in \Omega$, a treatment contrast $l'\tau$ belongs to the factorial effect $F(y)$ if and only if*

$$l' \in \mathcal{R}[M(y)],$$

where $\mathcal{R}[M(y)]$ stands for the row space of $M(y)$.

The proof of this theorem is somewhat lengthy. Hence we postpone the proof till the end of this section and first discuss the implications of the theorem. Note that by (2.2.2) and (2.2.4), $M_i^{y_i}$ has full row rank for each i ($1 \le i \le n$). Hence by (2.2.3), $M(y)$ has full row rank for every $y \in \Omega$. Since $M(y)$ has $m(y)$ rows, the following result is evident from Theorem 2.2.1.

Theorem 2.2.2. *For any $y = y_1 \dots y_n \in \Omega$, the maximal number of linearly independent treatment contrasts belonging to the factorial effect $F(y)$ is $m(y)$. Furthermore, the $m(y)$ elements of $M(y)\tau$ represent a maximal set of linearly independent treatment contrasts belonging to $F(y)$.*

The concept of orthogonality of treatment contrasts plays a crucial role in factorial experiments. Two treatment contrasts $l^{(1)'}\tau$ and $l^{(2)'}\tau$ are said to be *orthogonal* if

$$l^{(1)'}l^{(2)} = 0. \tag{2.2.6}$$

For example, from (2.1.6), (2.1.8), and (2.1.9), any two of the contrasts $L(F_1)$, $L(F_2)$, and $L(F_1F_2)$ are orthogonal to each other. Since these contrasts belong to different factorial effects, this is actually a consequence of a more general result as presented below.

Theorem 2.2.3. *Any two treatment contrasts belonging to different factorial effects are orthogonal.*

Proof. In view of (2.2.6) and Theorem 2.2.1, it is enough to show that

$$M(y)M(z)' = 0 \tag{2.2.7}$$

whenever $y = y_1 \dots y_n$ and $z = z_1 \dots z_n$ are distinct members of Ω. Now by (2.2.3),

$$M(y)M(z)' = (M_1^{y_1}(M_1^{z_1})') \otimes \cdots \otimes (M_n^{y_n}(M_n^{z_n})'). \tag{2.2.8}$$

If y and z are distinct members of Ω, then $y_i \neq z_i$ for some i. Without loss of generality, let $y_1 \neq z_1$ and suppose $y_1 = 1,\ z_1 = 0$. Then by (2.2.2) and (2.2.4),

$$M_1^{y_1}(M_1^{z_1})' = 0,$$

and (2.2.7) follows from (2.2.8). □

Theorems 2.2.2 and 2.2.3 together have an interesting implication. Since a typical treatment contrast is of the form $l'\tau$, where l is a nonnull $v \times 1$ vector whose elements add up to zero, clearly the maximal number of linearly independent treatment contrasts (belonging to factorial effects or not) is $v-1$. By (2.2.5),

$$v - 1 = \prod_{i=1}^{n} s_i - 1 = \prod_{i=1}^{n}(s_i - 1 + 1) - 1 = \sum_{y \in \Omega} m(y).$$

Hence, in view of Theorems 2.2.2 and 2.2.3, we reach the satisfying conclusion that *treatment contrasts belonging to factorial effects together span all treatment contrasts.*

Theorem 2.2.2, in conjunction with (2.2.3) and (2.2.4), also helps in explicitly describing treatment contrasts belonging to various factorial effects in any given context. Here is an illustrative example.

Example 2.2.1. Consider a $2 \times 3 \times 3$ factorial whose treatment combinations have already been given in (2.1.1). Here $n = 3$ and, following (2.1.1), the vector τ, with lexicographically arranged elements $\tau(j_1j_2j_3)$, is given by $\tau = (\tau(000), \tau(001), \dots, \tau(121), \tau(122))'$. Since $s_1 = 2,\ s_2 = s_3 = 3$, we have $1_1 = (1,1)'$, $1_2 = 1_3 = (1,1,1)'$. Also, following (2.2.2) one can take

$$M_1 = (-1 \;\; 1), M_2 = M_3 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}.$$

By (2.2.3) and (2.2.4),

$$\begin{array}{ll} M(100) = M_1 \otimes 1_2' \otimes 1_3', & M(010) = 1_1' \otimes M_2 \otimes 1_3', \\ M(001) = 1_1' \otimes 1_2' \otimes M_3, & M(110) = M_1 \otimes M_2 \otimes 1_3', \\ M(101) = M_1 \otimes 1_2' \otimes M_3, & M(011) = 1_1' \otimes M_2 \otimes M_3, \\ M(111) = M_1 \otimes M_2 \otimes M_3, & \end{array}$$

where the matrices M_i and the vectors 1_i are stated above.

By Theorem 2.2.2, the elements of $M(100)\tau$, $M(010)\tau$, and $M(001)\tau$ represent maximal sets of linearly independent treatment contrasts belonging to the factorial effects $F(100)$, $F(010)$, and $F(001)$, i.e., the main effects of F_1, F_2, and F_3 respectively. Similarly, the elements of $M(110)\tau$, $M(101)\tau$, $M(011)\tau$, and $M(111)\tau$ represent maximal sets of linearly independent treatment contrasts belonging to interactions F_1F_2, F_1F_3, F_2F_3, and $F_1F_2F_3$ respectively. □

Before concluding this section, we present a proof of Theorem 2.2.1. The following lemma will be useful.

Lemma 2.2.1. *For any g $(1 \le g \le n)$, the row spaces of the matrices $M_1 \otimes \cdots \otimes M_g$ and*

$$H_g = \begin{bmatrix} 1_1' \otimes I_2 \otimes \cdots \otimes I_g \\ \vdots \\ I_1 \otimes \cdots \otimes I_{g-1} \otimes 1_g' \end{bmatrix} \tag{2.2.9}$$

are orthogonal complements of each other.

Proof. To give a flavor of the basic idea of the proof without making the notation too complex, we prove the lemma for $g = 3$. At the expense of heavier notation, the lemma can be proved similarly for any g. By (2.2.9),

$$H_3 = \begin{bmatrix} 1_1' \otimes I_2 \otimes I_3 \\ I_1 \otimes 1_2' \otimes I_3 \\ I_1 \otimes I_2 \otimes 1_3' \end{bmatrix}.$$

For $1 \le i \le 3$, let

$$\overline{M}_i = \begin{bmatrix} 1_i' \\ M_i \end{bmatrix}. \tag{2.2.10}$$

By (2.2.2), $\overline{M}_i$ is nonsingular for every i. Hence premultiplying H_3 by the nonsingular matrix $\mathrm{diag}(\overline{M}_2 \otimes \overline{M}_3, \overline{M}_1 \otimes \overline{M}_3, \overline{M}_1 \otimes \overline{M}_2)$ yields

$$\mathcal{R}(H_3) = \mathcal{R} \begin{bmatrix} 1_1' \otimes \overline{M}_2 \otimes \overline{M}_3 \\ \overline{M}_1 \otimes 1_2' \otimes \overline{M}_3 \\ \overline{M}_1 \otimes \overline{M}_2 \otimes 1_3' \end{bmatrix}, \tag{2.2.11}$$

where, as before, $\mathcal{R}(\cdot)$ stands for the row space of a matrix. But by (2.2.10),

$$1_1' \otimes \overline{M}_2 \otimes \overline{M}_3 = \begin{bmatrix} 1_1' \otimes 1_2' \otimes 1_3' \\ 1_1' \otimes 1_2' \otimes M_3 \\ 1_1' \otimes M_2 \otimes 1_3' \\ 1_1' \otimes M_2 \otimes M_3 \end{bmatrix}.$$

On the basis of similar considerations for $\overline{M}_1 \otimes 1_2' \otimes \overline{M}_3$ and $\overline{M}_1 \otimes \overline{M}_2 \otimes 1_3'$, it follows from (2.2.11) that

$$\mathcal{R}(H_3) = \mathcal{R}(\widetilde{M}), \tag{2.2.12}$$

where

$$\widetilde{M} = \begin{bmatrix} 1_1' \otimes 1_2' \otimes 1_3' \\ 1_1' \otimes 1_2' \otimes M_3 \\ 1_1' \otimes M_2 \otimes 1_3' \\ 1_1' \otimes M_2 \otimes M_3 \\ M_1 \otimes 1_2' \otimes 1_3' \\ M_1 \otimes 1_2' \otimes M_3 \\ M_1 \otimes M_2 \otimes 1_3' \end{bmatrix}. \tag{2.2.13}$$

Now by (2.2.10),

$$\begin{bmatrix} \widetilde{M} \\ M_1 \otimes M_2 \otimes M_3 \end{bmatrix} = \overline{M}_1 \otimes \overline{M}_2 \otimes \overline{M}_3$$

is nonsingular, while by (2.2.2) and (2.2.13), $\widetilde{M}(M_1 \otimes M_2 \otimes M_3)' = 0$. Hence the row spaces of $\widetilde{M}$ and $M_1 \otimes M_2 \otimes M_3$ are orthogonal complements of each other. Therefore, by (2.2.12), so are the row spaces of H_3 and $M_1 \otimes M_2 \otimes M_3$. □

Proof (Proof of Theorem 2.2.1). For notational simplicity, without loss of generality, consider the factorial effect $F(y)$, where $y = y_1 \ldots y_n$ is given by

$$y_1 = \cdots = y_g = 1, y_{g+1} = \cdots = y_n = 0 \tag{2.2.14}$$

for some g. By (2.2.1), this amounts to considering the factorial effect $F_1 \ldots F_g$. Note that by (2.2.3) and (2.2.4),

$$M(y) = M_1 \otimes \cdots \otimes M_g \otimes 1_{g+1}' \otimes \cdots \otimes 1_n' \tag{2.2.15}$$

for y as in (2.2.14). In order to prove the "only if" part, let the treatment contrast $l'\tau$ belong to the factorial effect $F(y)$. Then, interpreting the conditions of Definition 2.1.1 in matrix notation, by condition (i) of the definition and (2.2.14),

$$l = \bar{l} \otimes 1_{g+1} \otimes \cdots \otimes 1_n, \tag{2.2.16}$$

where $\bar{l}$ is a column vector of order $\Pi_{i=1}^{g} s_i$. Furthermore, by condition (ii) of the definition, $\bar{l}$ satisfies $H_g \bar{l} = 0$, where H_g is defined in (2.2.9). Hence by Lemma 2.2.1, $\bar{l}' \in \mathcal{R}(M_1 \otimes \cdots \otimes M_g)$. Therefore, by (2.2.15) and (2.2.16), $l' \in \mathcal{R}[M(y)]$, which proves the "only if" part. The proof of the "if" part follows by reversing the above steps. □

2.3 A Representation for Factorial Effects in Symmetrical Factorials

We now focus on the s^n symmetrical factorial, i.e., $s_1 = \cdots = s_n = s$, where s is a prime or prime power. The theory to be developed here covers, in particular, the 2^n and 3^n factorials, which have been of special interest in the literature from both theoretical and practical considerations. We shall follow Bose (1947) throughout this section. It will be seen that for an s^n factorial there exists a mathematically elegant representation for treatment contrasts belonging to factorial effects. This representation provides, in particular, a significant insight into the issues underlying fractional factorial designs to be introduced later in this chapter.

The developments in this and subsequent sections will heavily involve the use of finite fields. A *field* is a set of elements over which two binary operations, namely addition and multiplication, are defined such that

(i) the elements of the set form a commutative group under addition,
(ii) the nonzero elements of the set form a commutative group under multiplication, and
(iii) the distributive laws hold.

A finite field is nothing but a field having a finite number of elements.

Since s (≥ 2) is a prime or prime power, there exists a finite field with s elements. Such a field is called a *Galois field* of order s and denoted by $GF(s)$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ be the elements of $GF(s)$, where $\alpha_0 (= 0)$ and $\alpha_1 (= 1)$ are the identity elements with respect to the operations of addition and multiplication respectively. A good reference for Galois fields is van der Waerden (1966). In particular, *when s is a prime, addition and multiplication over $GF(s)$ are simply the corresponding operations over the set of integers $\{0, 1, \ldots, s-1\}$ modulo s.*

A typical treatment combination $j_1 \ldots j_n$ $(0 \leq j_i \leq s-1; 1 \leq i \leq n)$ of an s^n factorial is identified with the vector $(\alpha_{j_1}, \ldots, \alpha_{j_n})'$. The s^n treatment combinations are thus represented by the s^n vectors of the form

$$x = (x_1, \ldots, x_n)', \tag{2.3.1}$$

where $x_i \in GF(s)$ for all i. From a geometric viewpoint, the s^n vectors of the form (2.3.1) are points of the n-dimensional finite Euclidean geometry $EG(n, s)$, based on $GF(s)$ (see Raghavarao, 1971, pages 357–359, for details). Thus, from a geometric perspective, the s^n treatment combinations are represented by the s^n points of $EG(n, s)$. The effect of a treatment combination represented by x will be denoted by $\tau(x)$. Some preliminaries are needed at this stage.

Lemma 2.3.1. *Let $b = (b_1, \ldots, b_n)'$ be any fixed nonnull vector over $GF(s)$. Then each of the sets*

$$V_j(b) = \{x = (x_1, \ldots, x_n)' : b'x = \alpha_j\},\ 0 \le j \le s-1, \tag{2.3.2}$$

has cardinality s^{n-1}.

Proof. Without loss of generality, let $b_1 \neq 0$. Then by (2.3.2), $x = (x_1, \ldots, x_n)' \in V_j(b)$ if and only if

$$x_1 = b_1^{-1}\left(\alpha_j - \sum_{i=2}^{n} b_i x_i\right). \tag{2.3.3}$$

By (2.3.3), for any $x = (x_1, \ldots, x_n)' \in V_j(b)$, x_1 is uniquely determined by $x_2, \ldots, x_n$. Since there are s^{n-1} choices of $x_2, \ldots, x_n$, the result follows. □

Clearly, the sets $V_j(b)$, $0 \le j \le s-1$, provide a disjoint partition of the class of all treatment combinations, or equivalently, of the s^n points of the finite Euclidean geometry $EG(n, s)$. These sets are therefore collectively called a *parallel pencil of flats* of $EG(n, s)$. Hence b itself is said to represent a *pencil*.

A treatment contrast L is said to belong to the pencil b if it is of the form

$$L = \sum_{j=0}^{s-1} l_j \left\{ \sum_{x \in V_j(b)} \tau(x) \right\}, \tag{2.3.4}$$

where $l_0, l_1, \ldots, l_{s-1}$ are real numbers, not all zero, satisfying

$$\sum_{j=0}^{s-1} l_j = 0.$$

In other words, a treatment contrast L belongs to b if for all x belonging to the same $V_j(b)$, the coefficient of $\tau(x)$ in L is also the same. By (2.3.4), there are $s-1$ linearly independent treatment contrasts belonging to any pencil b.

Example 2.3.1. Consider a 3^2 factorial, i.e., $s = 3$, $n = 2$. The elements of $GF(3)$ are simply 0, 1, and 2, and as indicated before, addition and multiplication over $GF(3)$ are the corresponding operations over $\{0, 1, 2\}$ modulo 3. (i) First consider the pencil $b = (1, 2)'$. Then by (2.3.2),

$$V_0(b) = \{x = (x_1, x_2)' : x_1 + 2x_2 = 0\} = \{(0,0)', (1,1)', (2,2)'\}. \tag{2.3.5}$$

Similarly,

$$V_1(b) = \{(0,2)', (1,0)', (2,1)'\}, \quad V_2(b) = \{(0,1)', (1,2)', (2,0)'\}. \tag{2.3.6}$$

As such, by (2.3.4), any treatment contrast of the form

$$\begin{aligned} L = \ & l_0\{\tau(0,0) + \tau(1,1) + \tau(2,2)\} + l_1\{\tau(0,2) + \tau(1,0) + \tau(2,1)\} \\ & + l_2\{\tau(0,1) + \tau(1,2) + \tau(2,0)\}, \end{aligned} \tag{2.3.7}$$

where $l_0 + l_1 + l_2 = 0$, belongs to the pencil $b = (1, 2)'$. In particular, the choices $l_0 = -1,\ l_1 = 0,\ l_2 = 1$, and $l_0 = 1,\ l_1 = -2,\ l_2 = 1$, yield two linearly independent (in fact, orthogonal) contrasts belonging to b.

Incidentally, in this section, the treatment combinations are represented by column vectors as in (2.3.1). Hence in (2.3.7), one should have ideally written $\tau((0,0)'), \tau((1,1)')$ etc. in place of $\tau(0,0), \tau(1,1)$ etc., respectively. This minor notational change was made in (2.3.7) for simplicity in presentation. Similar simplified notation will be adopted later on when no ambiguity is caused.

(ii) Consider now the pencil $b = (2, 1)'$. Then as before,

$$\begin{aligned} V_0(b) &= \{(0,0)', (1,1)', (2,2)'\}, \\ V_1(b) &= \{(0,1)', (1,2)', (2,0)'\},\ V_2(b) = \{(0,2)', (1,0)', (2,1)'\}. \end{aligned}$$

These sets are the same as those in (2.3.5) and (2.3.6) with $V_1(b)$ and $V_2(b)$ interchanged. Hence it is easily seen that treatment contrasts belonging to the pencil $(2, 1)'$ also belong to the pencil $(1, 2)'$ and vice versa.

(iii) Consider next the pencil $b = (1, 1)'$. Then

$$\begin{aligned} &V_0(b) = \{(0,0)', (1,2)', (2,1)'\}, \\ &V_1(b) = \{(0,1)', (1,0)', (2,2)'\},\ V_2(b) = \{(0,2)', (1,1)', (2,0)'\}. \end{aligned}$$

By (2.3.4), a typical treatment contrast belonging to the pencil $b = (1, 1)'$ is of the form

$$\begin{aligned} L^* = \;& l_0^*\{\tau(0,0) + \tau(1,2) + \tau(2,1)\} + l_1^*\{\tau(0,1) + \tau(1,0) + \tau(2,2)\} \\ & + l_2^*\{\tau(0,2) + \tau(1,1) + \tau(2,0)\}, \end{aligned} \tag{2.3.8}$$

where $l_0^* + l_1^* + l_2^* = 0$. It is easily seen that the sum of products of the corresponding coefficients in (2.3.7) and (2.3.8) equals

$$(l_0 + l_1 + l_2)(l_0^* + l_1^* + l_2^*) = 0.$$

Thus any treatment contrast belonging to the pencil $(1, 1)'$ is orthogonal to any treatment contrast belonging to the pencil $(1, 2)'$. □

The ideas implicit in the above example will now be formalized. First note that the pencils $(1, 2)'$ and $(2, 1)'$ considered in parts (i) and (ii) of the example are proportional to each other, in the sense that $(1, 2)' = 2(2, 1)'$ over $GF(3)$. In general, consider any two pencils b and b^* such that $b^* = \lambda b$ for some $\lambda (\neq 0) \in GF(s)$. Then by (2.3.2),

$$x \in V_j(b) \Leftrightarrow b'x = \alpha_j \Leftrightarrow b^{*'}x = \lambda\alpha_j. \tag{2.3.9}$$

Since λ is nonzero, $\lambda\alpha_j$ assumes all possible values in $GF(s)$ as j varies over the range $\{0, 1, \ldots, s-1\}$. Therefore, by (2.3.2) and (2.3.9), the pencils b and b^* induce exactly the same partition of the class of all treatment combinations. This is precisely what happened in parts (i) and (ii) of the example. In view of the above, hereafter, *pencils with proportional entries will be considered as identical.* Since the pencils have to be nonnull too, it follows that there are $(s^n - 1)/(s - 1)$ distinct pencils, no two of which are proportional to each other. Thus in a 3^2 factorial, there are $(3^2 - 1)/(3 - 1) = 4$ distinct pencils, namely,

$$(1,0)',\ (0,1)',\ (1,1)',\ (1,2)'. \tag{2.3.10}$$

Hereafter, only distinct pencils will be considered in a given context, even when this is not stated explicitly. The same consideration applies to counting pencils with any specific property. For example, we will simply write "in a 3^2 factorial, there are two pencils with both entries nonzero" to mean that there are two such distinct pencils.

As noted in part (iii) of Example 2.3.1, treatment contrasts belonging to the pencils $(1,1)'$ and $(1,2)'$ are orthogonal. Theorem 2.3.1 below presents a general result in this regard. The following lemma helps in proving the theorem.

Lemma 2.3.2. *If $b^{(1)}$ and $b^{(2)}$ are distinct pencils, then for every j, $j'(0 \leq j, j' \leq s-1)$, the set $V_j(b^{(1)}) \cap V_{j'}(b^{(2)})$ has cardinality s^{n-2}.*

Proof. An argument similar to the proof of Lemma 2.3.1 will be used. Let $b^{(1)} = (b_{11}, b_{12}, \ldots, b_{1n})'$, $b^{(2)} = (b_{21}, b_{22}, \ldots, b_{2n})'$. By (2.3.2),

$$x = (x_1, \ldots, x_n)' \in V_j(b^{(1)}) \cap V_{j'}(b^{(2)}) \tag{2.3.11}$$

if and only if

$$\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \end{bmatrix} x = \begin{pmatrix} \alpha_j \\ \alpha_{j'} \end{pmatrix}. \tag{2.3.12}$$

Since $b^{(1)}$ and $b^{(2)}$ are distinct pencils, they are not proportional to each other. As such, the $2 \times n$ matrix appearing in the left-hand side of (2.3.12) has rank two. Without loss of generality, let its first two columns be linearly independent. Then the 2×2 matrix

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

is nonsingular, and (2.3.12) can be expressed as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_j - \sum_{i=3}^n b_{1i}x_i \\ \alpha_{j'} - \sum_{i=3}^n b_{2i}x_i \end{pmatrix}.$$

Hence for any $x = (x_1, \ldots, x_n)'$ satisfying (2.3.11), x_1 and x_2 are uniquely determined by $x_3, \ldots, x_n$. Since there are s^{n-2} choices of $x_3, \ldots, x_n$, the result follows. □

Theorem 2.3.1. *Treatment contrasts belonging to distinct pencils are orthogonal to each other.*

Proof. Following (2.3.4), consider any two treatment contrasts

$$L_1 = \sum_{j=0}^{s-1} l_{1j} \left\{ \sum_{x \in V_j(b^{(1)})} \tau(x) \right\} \tag{2.3.13}$$

and

$$L_2 = \sum_{j=0}^{s-1} l_{2j} \left\{ \sum_{x \in V_j(b^{(2)})} \tau(x) \right\} \tag{2.3.14}$$

belonging to distinct pencils $b^{(1)}$ and $b^{(2)}$ respectively, where

$$\sum_{j=0}^{s-1} l_{1j} = \sum_{j=0}^{s-1} l_{2j} = 0. \tag{2.3.15}$$

In view of (2.2.6), we consider the sum of products of the corresponding coefficients in (2.3.13) and (2.3.14). Any $x \in V_j(b^{(1)}) \cap V_{j'}(b^{(2)})$ contributes $l_{1j}l_{2j'}$ to this sum. Hence by Lemma 2.3.2, this sum of products equals

$$s^{n-2} \sum_{j=0}^{s-1} \sum_{j'=0}^{s-1} l_{1j} l_{2j'} = s^{n-2} \left(\sum_{j=0}^{s-1} l_{1j} \right) \left(\sum_{j'=0}^{s-1} l_{2j'} \right),$$

and invoking (2.3.15), the result follows. □

The next result links pencils with factorial effects.

Theorem 2.3.2. *Let $b = (b_1, \ldots, b_n)'$ be a pencil such that*

$$b_i \neq 0 \text{ if } i \in \{i_1, \ldots, i_g\}, \text{ and } = 0 \text{ otherwise}, \tag{2.3.16}$$

where $1 \leq i_1 < \cdots < i_g \leq n$ and $1 \leq g \leq n$. Then any treatment contrast belonging to b also belongs to the factorial effect $F_{i_1} \ldots F_{i_g}$.

Proof. Without loss of generality, let $i_1 = 1, \ldots, i_g = g$. Then $b_1, \ldots, b_g$ are nonzero, while $b_{g+1} = \cdots = b_n = 0$, so that by (2.3.2),

$$V_j(b) = \left\{ x = (x_1 \ldots, x_n)' : \sum_{i=1}^{g} b_i x_i = \alpha_j \right\}, \; 0 \leq j \leq s-1. \tag{2.3.17}$$

From (2.3.4), recall that any treatment contrast L belonging to b is of the form

$$L = \sum_{j=0}^{s-1} l_j \left\{ \sum_{x \in V_j(b)} \tau(x) \right\},$$

where $l_0 + \cdots + l_{s-1} = 0$. Clearly, by (2.3.17), for any $x = (x_1, \ldots, x_n)'$, the coefficient of $\tau(x)$ in L depends on x only through $x_1, \ldots, x_g$. In fact, writing $\bar{l}(x_1, \ldots, x_g)$ for the coefficient of $\tau(x)$ in L, by (2.3.17), one gets

$$\bar{l}(x_1, \ldots, x_g) = l_j \text{ if } \sum_{i=1}^{g} b_i x_i = \alpha_j \ (0 \leq j \leq s-1). \tag{2.3.18}$$

Now, since $b_1 \neq 0$, the quantity $\sum_{i=1}^{g} b_i x_i$ equals each of $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ once as x_1 assumes all possible values over $GF(s)$, each exactly once, for any fixed $x_2, \ldots, x_g$. Hence by (2.3.18)

$$\sum_{x_1 \in GF(s)} \bar{l}(x_1, \ldots, x_g) = l_0 + \cdots + l_{s-1} = 0,$$

for any fixed $x_2, \ldots, x_g$. Similarly, for every i $(1 \leq i \leq g)$,

$$\sum_{x_i \in GF(s)} \bar{l}(x_1, \ldots, x_g) = 0,$$

for any fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_g$. Hence by Definition 2.1.1, the treatment contrast L belongs to the factorial effect $F_1 \ldots F_g$. □

In consideration of Theorem 2.3.2, a pencil b as in (2.3.16) is said to belong to the factorial effect $F_{i_1} \ldots F_{i_g}$. Since exactly g of $b_1, \ldots, b_n$ are nonzero in (2.3.16) and pencils with proportional entries are identical, clearly there are $(s-1)^{g-1}$ pencils belonging to $F_{i_1} \ldots F_{i_g}$. Each of these pencils carries $s-1$ linearly independent treatment contrasts. Furthermore, by Theorem 2.3.1, treatment contrasts belonging to distinct pencils are orthogonal to each other. Hence the $(s-1)^{g-1}$ pencils mentioned above provide a representation for the treatment contrasts belonging to $F_{i_1} \ldots F_{i_g}$ in terms of $(s-1)^{g-1}$ mutually orthogonal sets of contrasts with $s-1$ linearly independent contrasts in each set. This accounts for a maximal collection of $(s-1)^g$ linearly independent treatment contrasts belonging to the factorial effect $F_{i_1} \ldots F_{i_g}$ (see Theorem 2.2.2 with $s_1 = \cdots = s_n = s$ in (2.2.5)).

Returning to the 3^2 factorial, the pencils listed in (2.3.10) can now be assigned to factorial effects. Thus $(1,0)'$ and $(0,1)'$ represent the main effects of F_1 and F_2 respectively and the interaction F_1F_2 is represented by $(1,1)'$ and $(1,2)'$.

For the special case of a 2^n factorial, $(s-1)^{g-1} = 1$, so that each factorial effect is represented by a single pencil. Thus, in this case there is practically no distinction between a factorial effect and the associated pencil.

Remark 2.3.1. While the vector notation for pencils as considered above facilitates the mathematical development, another notational system, which is more compact, is useful in other contexts, especially with 2^n and 3^n factorials. A pencil $b = (b_1, \ldots, b_n)'$ can also be denoted by $1^{b_1}2^{b_2} \ldots n^{b_n}$, with the convention that i^{b_i} is dropped for any i with $b_i = 0$. For example, with this notation, the pencils listed in (2.3.10) become $1,\ 2,\ 12$ and 12^2 respectively. This system of notation, popularized by Box and Hunter (1961a), will be referred to as the *compact notation.* □

In the next two sections, we introduce and discuss regular fractions, expanding on Dey and Mukerjee (1999, Chapter 8).

2.4 Regular Fractions

A regular fraction of an s^n symmetrical factorial, where s (≥ 2) is a prime or prime power, is specified by any k ($1 \leq k < n$) linearly independent pencils, say $b^{(1)}, \ldots, b^{(k)}$, and consists of treatment combinations x satisfying $Bx = c$, where B is a $k \times n$ matrix with rows $(b^{(i)})'$, $1 \leq i \leq k$, and c is a fixed $k \times 1$ vector over $GF(s)$. The specific choice of c is inconsequential in what follows. Hence, without loss of generality, it is assumed that $c = 0$, the $k \times 1$ null vector over $GF(s)$. Then a *regular fraction* is given by, say,

$$d(B) = \{x : Bx = 0\}. \tag{2.4.1}$$

Since the rows of the $k \times n$ matrix B are given by linearly independent pencils, the same argument as in proving Lemma 2.3.2 shows that $d(B)$ consists of s^{n-k} treatment combinations. In this sense, $d(B)$ is called a $1/s^k$ *fraction of an* s^n *factorial*, or simply an s^{n-k} *design*. It is easily seen from (2.4.1) that the s^{n-k} treatment combinations in $d(B)$, which can as well be viewed as points of the finite Euclidean geometry $EG(n, s)$, constitute a subgroup of $EG(n, s)$, the group operation being componentwise addition. In the applied literature, these treatment combinations are called *runs* and the number s^{n-k} is called the *run size* of $d(B)$.

Example 2.4.1. The two linearly independent pencils $b^{(1)} = (1, 1, 0, 1, 0)'$ and $b^{(2)} = (1, 0, 1, 0, 1)'$ yield a 2^{5-2} design. Here

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and by (2.4.1), a treatment combination $x = (x_1, \ldots, x_5)'$ is included in $d(B)$ if and only if

$$x_1 + x_2 + x_4 = 0,$$
$$x_1 + x_3 + x_5 = 0,$$

i.e., if and only if

$$\begin{aligned} x_4 &= x_1 + x_2, \\ x_5 &= x_1 + x_3. \end{aligned} \tag{2.4.2}$$

Considering all possibilities for x_1, x_2, and x_3, and then evaluating x_4 and x_5 via (2.4.2), one gets

$$\begin{aligned} d(B) = &\{(0,0,0,0,0)', (0,0,1,0,1)', (0,1,0,1,0)', (0,1,1,1,1)', \\ &(1,0,0,1,1)', (1,0,1,1,0)', (1,1,0,0,1)', (1,1,1,0,0)'\}. \end{aligned}$$

□

Example 2.4.2. The two linearly independent pencils $b^{(1)} = (1, 0, 2, 2)'$ and $b^{(2)} = (0, 1, 1, 2)'$ yield a 3^{4-2} design. Here

$$B = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

and by (2.4.1), a treatment combination $x = (x_1, \ldots, x_4)'$ appears in $d(B)$ if and only if

$$\begin{aligned} x_1 + 2x_3 + 2x_4 &= 0, \\ x_2 + x_3 + 2x_4 &= 0, \end{aligned}$$

i.e., if and only if

$$\begin{aligned} x_1 &= x_3 + x_4, \\ x_2 &= 2x_3 + x_4. \end{aligned} \tag{2.4.3}$$

As before, considering all possibilities for x_3 and x_4, and then evaluating x_1 and x_2 via (2.4.3), one gets

$$\begin{aligned} d(B) =\{&(0,0,0,0)', (1,1,0,1)', (2,2,0,2)', (1,2,1,0)', (2,0,1,1)', (0,1,1,2)', \\ &(2,1,2,0)', (0,2,2,1)', (1,0,2,2)'\}. \end{aligned}$$

□

With reference to $d(B)$, as introduced in (2.4.1), a pencil b is called a *defining pencil* if

$$b' \in \mathcal{R}(B), \tag{2.4.4}$$

where, as before, $\mathcal{R}(\cdot)$ denotes the row space of a matrix. Since the $k \times n$ matrix B over $GF(s)$ has full row rank, the cardinality of $\mathcal{R}(B)$ is s^k. Since pencils are nonnull vectors and pencils with proportional entries are identical, it follows that there are $(s^k - 1)/(s - 1)$ defining pencils. The vectors in $\mathcal{R}(B)$ constitute what is known as the *defining contrast subgroup* of $d(B)$.

In Example 2.4.1, there are three defining pencils, namely,

$$b^{(1)} = (1,1,0,1,0)',\ b^{(2)} = (1,0,1,0,1)',\ b^{(1)} + b^{(2)} = (0,1,1,1,1)'. \tag{2.4.5}$$

Similarly, in Example 2.4.2, there are $(3^2 - 1)/(3 - 1) = 4$ defining pencils, namely,

$$\begin{aligned} b^{(1)} &= (1,0,2,2)',\ b^{(2)} = (0,1,1,2)',\ b^{(1)} + b^{(2)} = (1,1,0,1)', \\ b^{(1)} + 2b^{(2)} &= (1,2,1,0)'. \end{aligned} \tag{2.4.6}$$

It can be readily checked that every other pencil satisfying (2.4.4) is proportional hence identical to one of the above. Using the compact notation introduced in Remark 2.3.1, the defining pencils shown in (2.4.5) can as well be listed as

$$I = 124 = 135 = 2345, \tag{2.4.7}$$

whereas those shown in (2.4.6) can be listed as

$$I = 13^2 4^2 = 234^2 = 124 = 12^2 3. \tag{2.4.8}$$

An equation like (2.4.7) or (2.4.8), listing defining pencils, is called an identity relation or a *defining relation* of an s^{n-k} design. The symbol I here signifies only that defining pencils are being listed and should not be confused with an identity matrix.

By (2.4.1) and (2.4.4), if b is a defining pencil, then $b'x = 0$ for every $x \in d(B)$. In view of (2.3.2), this implies that $d(B) \subset V_0(b)$, i.e., all the treatment combinations appearing in $d(B)$ belong to only one of the sets $V_0(b), V_1(b), \ldots, V_{s-1}(b)$. Hence recalling the definition of treatment contrasts belonging to pencils, the following result is evident.

Theorem 2.4.1. *No treatment contrast belonging to any defining pencil is estimable in* $d(B)$.

As an implication of Theorem 2.4.1, while choosing an s^{n-k} design, it is important that no pencil belonging to a factorial effect of interest be a defining pencil.

The status of $d(B)$, relative to pencils other than the defining ones, will be examined next. The important notion of alias sets needs to be introduced for this purpose. Let $\mathcal{C} = \mathcal{C}(B)$ be the class of pencils that are not defining pencils of the design $d(B)$. Since there are altogether $(s^n - 1)/(s-1)$ pencils of which $(s^k - 1)/(s-1)$ are the defining ones, it follows that there are

$$\frac{s^n - 1}{s-1} - \frac{s^k - 1}{s-1} = \frac{s^k(s^{n-k} - 1)}{s-1}$$

pencils in $\mathcal{C}$. Two members of $\mathcal{C}$, say b and $\widetilde{b}$, are aliases of each other if $(b - \lambda\widetilde{b})' \in \mathcal{R}(\mathcal{B})$ for some $\lambda(\neq 0) \in GF(s)$. However, $\lambda\widetilde{b}$ itself is another representation of the pencil $\widetilde{b}$, since pencils with proportional entries are identical. Hence, equivalently, two pencils in $\mathcal{C}$ are *aliases* of each other if

$$(b - b^*)' \in \mathcal{R}(\mathcal{B}), \tag{2.4.9}$$

for some representations b and b^* of these pencils. Because of its symmetry, hereafter aliasing is defined via (2.4.9) with b and b^* interpreted as appropriate representations of the pencils concerned.

Since $B = \left(b^{(1)'}, \ldots, b^{(k)'}\right)'$ has full row rank, it is not hard to see that any $b \in \mathcal{C}$ has s^k aliases, namely,

$$b(\lambda_1, \ldots, \lambda_k) = b + \sum_{i=1}^{k} \lambda_i b^{(i)}, \ \lambda_i \in GF(s) \quad (1 \leq i \leq k), \tag{2.4.10}$$

including itself.

By (2.4.9), the relationship of being aliased is an equivalence relation (i.e., it is symmetric, reflexive, and transitive) that partitions $\mathcal{C}$ into $(s^{n-k} - 1)/(s-1)$ equivalence classes, each of cardinality s^k. Any such equivalence class is called an *alias set*. Thus the pencils $b(\lambda_1, \ldots, \lambda_k)$, described in (2.4.10), constitute the alias set containing the pencil b. Before presenting the theoretical results on alias sets, we revisit Examples 2.4.1 and 2.4.2 to illustrate how, especially with 2^n and 3^n factorials, the alias sets can be found in a simple manner.

Example 2.4.1 (continued). Consider the pencil $b = (1,0,0,0,0)'$, which is not a defining pencil. Since $b^{(1)} = (1,1,0,1,0)'$ and $b^{(2)} = (1,0,1,0,1)'$, by (2.4.10), the alias set containing b consists of the pencils

$$(1,0,0,0,0)' + \lambda_1(1,1,0,1,0)' + \lambda_2(1,0,1,0,1)', \tag{2.4.11}$$

where $\lambda_1, \lambda_2 \in \{0,1\}$. Considering all possible choices of λ_1 and λ_2, this alias set turns out to be

$$\{(1,0,0,0,0)', (0,1,0,1,0)', (0,0,1,0,1)', (1,1,1,1,1)'\}.$$

Using the compact notation, in the spirit of (2.4.7), the above alias set can as well be described as

$$1 = 24 = 35 = 12345. \tag{2.4.12}$$

Since addition over $GF(2)$ uses a binary arithmetic, it is easily seen from (2.4.11) that (2.4.12) can be obtained from the defining relation (2.4.7) by multiplying the latter by 1, which stands for the pencil $(1,0,0,0,0)'$. This multiplication must follow the convention that

(i) any squared symbol is dropped, and
(ii) any string of symbols is invariant under multiplication by I.

Thus $(1)I = 1, (1)(124) = 24$, etc., which yield (2.4.12) from (2.4.7). In a similar manner, the other alias sets in this example turn out to be

$$\begin{array}{lllllll} 2 &=& 14 &=& 1235 &=& 345, \\ 3 &=& 1234 &=& 15 &=& 245, \\ 4 &=& 12 &=& 1345 &=& 235, \\ 5 &=& 1245 &=& 13 &=& 234, \\ 23 &=& 134 &=& 125 &=& 45, \\ 34 &=& 123 &=& 145 &=& 25. \end{array}$$

□

Example 2.4.2 (continued). Consider the pencil $b = (1,0,0,0)'$, which is not a defining pencil. Since $b^{(1)} = (1,0,2,2)'$ and $b^{(2)} = (0,1,1,2)'$, by (2.4.10), the alias set containing b consists of the pencils

$$(1,0,0,0)' + \lambda_1(1,0,2,2)' + \lambda_2(0,1,1,2)', \tag{2.4.13}$$

where $\lambda_1, \lambda_2 \in \{0,1,2\}$. Considering all possible choices of λ_1 and λ_2, this alias set is found to be

$$\{(1,0,0,0)', (2,0,2,2)', (0,0,1,1)', (1,1,1,2)', (1,2,2,1)', (2,1,0,1)', (0,2,0,2)', (2,2,1,0)', (0,1,2,0)'\}.$$

Since pencils with proportional entries are identical, it can be rewritten as

$$\{(1,0,0,0)', (1,0,1,1)', (0,0,1,1)', (1,1,1,2)', (1,2,2,1)', (1,2,0,2)', (0,1,0,1)', (1,1,2,0)', (0,1,2,0)'\},$$

such that the first nonzero entry of each listed pencil is 1. As in (2.4.8), this alias set can be expressed as

$$1 = 134 = 34 = 1234^2 = 12^2 3^2 4 = 12^2 4^2 = 24 = 123^2 = 23^2. \qquad (2.4.14)$$

Since addition over $GF(3)$ is just addition modulo 3, it is not hard to see from (2.4.13) that (2.4.14) can be obtained from the defining relation (2.4.8) by multiplying each term in (2.4.8) and its square by 1, which represents the pencil $(1,0,0,0)'$. The multiplication must follow the convention that

(i) any cubed symbol is dropped, and
(ii) any string of symbols is invariant under multiplication by I or I^2 and is counted only once.

Thus $(1)I = (1)I^2 = 1$ and 1 is listed only once in (2.4.14); similarly, $(1)(13^2 4^2) = 1^2 3^2 4^2 = 134$, $(1)(13^2 4^2)^2 = 34$, $(1)(234^2) = 1234^2$, $(1)(234^2)^2 = 12^2 3^2 4$, etc., which yield (2.4.14) from (2.4.8). The other alias sets in this example can now be obtained as a routine exercise. □

Hereafter, with 2^n or 3^n factorials, often the defining relation and the alias sets will be presented using the compact notation. However, for the mathematical treatment of general s^n factorials, the vector notation for pencils will continue to be useful. The rest of this section presents several results that aim at understanding the consequences of aliasing.

Lemma 2.4.1. *Let the pencils $b, b^* \in \mathcal{C}$ be aliases of each other and let*

$$L = \sum_{j=0}^{s-1} l_j \Big\{ \sum_{x \in V_j(b)} \tau(x) \Big\} \text{ and } L^* = \sum_{j=0}^{s-1} l_j \Big\{ \sum_{x \in V_j(b^*)} \tau(x) \Big\}$$

be the treatment contrasts belonging to b and b^* respectively. Then the parts of L and L^* that involve only the treatment combinations included in $d(B)$ are identical.

Proof. For $0 \le j \le s-1$, let

$$V_j(b, B) = V_j(b) \cap d(B),\ V_j(b^*, B) = V_j(b^*) \cap d(B). \qquad (2.4.15)$$

By (2.3.2) and (2.4.1),

$$\begin{aligned} V_j(b, B) &= \{x : b'x = \alpha_j \text{ and } Bx = 0\}, \\ V_j(b^*, B) &= \{x : {b^*}'x = \alpha_j \text{ and } Bx = 0\}. \end{aligned} \tag{2.4.16}$$

Since b and b^* are aliases of each other, by (2.4.9), $(b - b^*)' \in \mathcal{R}(B)$. Hence

$$V_j(b, B) = V_j(b^*, B). \tag{2.4.17}$$

Now by (2.4.15), the parts of L and L^* that involve only the treatment combinations included in $d(B)$ are given by

$$L(B) = \sum_{j=0}^{s-1} l_j \left\{ \sum_{x \in V_j(b,B)} \tau(x) \right\} \text{ and } L^*(B) = \sum_{j=0}^{s-1} l_j \left\{ \sum_{x \in V_j(b^*,B)} \tau(x) \right\} \tag{2.4.18}$$

respectively. Hence from (2.4.17), the result is evident. □

Remark 2.4.1. Treatment contrasts with matching coefficients, such as L and L^* of Lemma 2.4.1, are called corresponding contrasts. The lemma shows that corresponding contrasts belonging to pencils that are aliased with each other cannot be distinguished on the basis of the design $d(B)$. In this sense, such pencils are said to be *confounded* with each other. Since b, as considered in Lemma 2.4.1, does not belong to $\mathcal{R}(B)$ and B has full row rank, it is clear that the $(k+1) \times n$ matrix $[b, B']'$ also has full row rank. Hence using the argument in the proof of Lemma 2.3.2, the set $V_j(b, B)$ considered in (2.4.16) has cardinality s^{n-k-1} for each j. Therefore, by (2.4.18), $L(B)$ (or, equivalently, $L^*(B)$) itself is a contrast involving the treatment combinations included in $d(B)$. □

Consider now any pencil $b \in \mathcal{C}$ and recall that (2.4.10) describes the alias set containing b. Let $\mathcal{A}$ denote this alias set. For any pencil $a \in \mathcal{A}$, any treatment combination x, and any j $(0 \le j \le s-1)$, let $\phi_j(a, x)$ stand for the indicator that assumes the value 1 if $x \in V_j(a)$ and the value 0 otherwise. Thus by (2.3.2),

$$\phi_j(a, x) = \begin{cases} 1 & \text{if } a'x = \alpha_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.19}$$

Also, let $\sum_a$ denote summation over all $a \in \mathcal{A}$. Then the following two lemmas hold.

Lemma 2.4.2. *For every treatment combination x and every j $(0 \le j \le s-1)$,*

$$\sum_a \phi_j(a, x) = \begin{cases} s^k & \textit{if } x \in V_j(b, B), \\ 0 & \textit{if } x \in d(B) - V_j(b, B), \\ s^{k-1} & \textit{if } x \notin d(B). \end{cases}$$

Proof. Since $\mathcal{A}$ is described by (2.4.10), the pencils in $\mathcal{A}$ are of the form $a = b + B'\lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_k)'$ with $\lambda_i \in GF(s)$ for every i. Hence for every fixed x and j, by (2.4.19),

$$\sum_a \phi_j(a,x) = \#\{\lambda = (\lambda_1, \ldots, \lambda_k)' : b'x + \lambda' Bx = \alpha_j, \lambda_i \in GF(s) \text{ for all } i\}, \tag{2.4.20}$$

where # denotes the cardinality of a set.
(i) If $x \in V_j(b,B)$, then by (2.4.16), $b'x + \lambda'Bx = \alpha_j$ for all $k \times 1$ vectors λ over $GF(s)$. Hence the right-hand side of (2.4.20) equals s^k.
(ii) If $x \in d(B) - V_j(b,B)$, then by (2.4.1) and (2.4.16), $Bx = 0, b'x \neq \alpha_j$. Hence $b'x + \lambda'Bx$ cannot equal α_j for any $k \times 1$ vector λ over $GF(s)$. The right-hand side of (2.4.20), therefore, equals 0.
(iii) Next consider $x \notin d(B)$. Then by (2.4.1), $Bx \neq 0$. Trivially, $b'x + \lambda'Bx = \alpha_j$ if and only if $(Bx)'\lambda = \alpha_j - b'x$. Since $Bx \neq 0$, exactly as in the proof of Lemma 2.3.1, it follows that the right-hand side of (2.4.20) equals s^{k-1}. □

Lemma 2.4.3. *Consider the corresponding treatment contrasts*

$$\sum_{j=0}^{s-1} l_j \left\{ \sum_{x \in V_j(a)} \tau(x) \right\},$$

for $a \in \mathcal{A}$, *where* $\sum_{j=0}^{s-1} l_j = 0$. *Then*

$$\sum_a \left[\sum_{j=0}^{s-1} l_j \left\{ \sum_{x \in V_j(a)} \tau(x) \right\} \right] = s^k \sum_{j=0}^{s-1} l_j \left\{ \sum_{x \in V_j(b,B)} \tau(x) \right\}. \tag{2.4.21}$$

Proof. Let $\mathcal{X}$ denote the set of the s^n treatment combinations. Using Lemma 2.4.2 and the indicator variable $\phi_j(a,x)$ introduced in (2.4.19),

$$\begin{aligned}
\sum_a [\sum_{j=0}^{s-1} l_j \{ \sum_{x \in V_j(a)} \tau(x) \}] &= \sum_a [\sum_{j=0}^{s-1} l_j \{ \sum_{x \in \mathcal{X}} \phi_j(a,x) \tau(x) \}] \\
&= \sum_{j=0}^{s-1} l_j [\sum_{x \in \mathcal{X}} \{ \sum_a \phi_j(a,x) \} \tau(x)] \\
&= \sum_{j=0}^{s-1} l_j \{ s^k \sum_{x \in V_j(b,B)} \tau(x) + s^{k-1} \sum_{x \notin d(B)} \tau(x) \} \\
&= s^k \sum_{j=0}^{s-1} l_j \{ \sum_{x \in V_j(b,B)} \tau(x) \},
\end{aligned}$$

since $\sum_{j=0}^{s-1} l_j = 0$. □

As noted in Remark 2.4.1, the right-hand side of (2.4.21) is a contrast involving only the treatment combinations included in $d(B)$. Hence it is estimable in $d(B)$. Therefore, the same is true for the left-hand side of (2.4.21). Thus while pencils belonging to the same alias set are confounded with one another, the sum of the corresponding contrasts belonging to such pencils is estimable in $d(B)$. Consequently any treatment contrast belonging to a pencil b that is not a defining pencil is estimable in $d(B)$ if and only if the corresponding contrasts belonging to all other pencils that are aliased with b are ignorable.

A pencil is said to be *estimable* in $d(B)$ if every treatment contrast belonging to it is estimable. Similarly, if every treatment contrast belonging to a pencil is ignorable, then the pencil itself is called *ignorable*. Then, summarizing what has been said in the last paragraph, the following result is evident.

Theorem 2.4.2. *A pencil b that is not a defining pencil is estimable in $d(B)$ if and only if all other pencils that are aliased with b are ignorable.*

As noted earlier from Theorem 2.4.1, while choosing an s^{n-k} design, it is important that no pencil belonging to a factorial effect of interest be a defining pencil. Now Theorem 2.4.2 suggests that in addition, no such pencil should be aliased with any other non-ignorable pencil. For example, if interest lies in the main effects and there is reason to assume the absence of all the interactions, then

(i) no main effect pencil should be a defining pencil, and
(ii) no two distinct main effect pencils should be aliased with each other.

These conditions are clearly met in Example 2.4.1; see (2.4.7) and the description of the alias sets. Similarly, by obtaining the alias sets in Example 2.4.2, one can check that the above conditions are met there too.

Interestingly, the above conclusion regarding Examples 2.4.1 and 2.4.2 can be reached directly from the respective defining relations (2.4.7) and (2.4.8) even without explicit determination of the alias sets. This is evident from taking $f = t = 1$ in Theorem 2.4.3 below if one observes from (2.4.7) and (2.4.8) that in both examples, each defining pencil has at least three nonzero entries. In what follows, a factorial effect is called *absent* if all treatment contrasts belonging to it are ignorable.

Theorem 2.4.3. *In an s^{n-k} design, all treatment contrasts belonging to factorial effects involving f or fewer factors are estimable under the absence of all factorial effects involving $t+1$ or more factors $(1 \le f \le t \le n-1)$ if and only if each defining pencil has at least $f+t+1$ nonzero entries.*

Proof. For proving the "if" part, suppose each defining pencil in an s^{n-k} design $d(B)$ has at least $f+t+1$ nonzero entries. Consider any pencil b belonging to a factorial effect involving f or fewer factors. It is enough to show that all treatment contrasts belonging to b are estimable in $d(B)$. Clearly, b is not a defining pencil. Now suppose b is aliased with another pencil b^* belonging to a factorial effect that involves t or fewer factors. Then $b - b^*$ is nonnull and by (2.4.9), $(b - b^*)' \in \mathcal{R}(B)$. Therefore, by (2.4.4), $b - b^*$ is a defining pencil. This is, however, impossible because $b - b^*$ has at most $f + t$ nonzero entries, since b and b^* have at most f and t nonzero entries respectively. Thus b

is neither a defining pencil nor aliased with another pencil belonging to a factorial effect that involves t or fewer factors. Hence by Theorem 2.4.2, all treatment contrasts belonging to b are estimable in $d(B)$. This proves the “if” part.

In order to prove the “only if” part, suppose $d(B)$ allows the estimation of all treatment contrasts belonging to factorial effects involving f or fewer factors under the absence of all factorial effects involving $t+1$ or more factors. Then by Theorems 2.4.1 and 2.4.2,

(i) no pencil having f or fewer nonzero entries is a defining pencil, and
(ii) no pencil having f or fewer nonzero entries is aliased with another pencil having t or fewer nonzero entries.

In view of (i), no defining pencil can have f or fewer nonzero entries. Now suppose there exists a defining pencil, say b_{def}, having exactly p nonzero entries, where $f+1 \leq p \leq f+t$. Without loss of generality, let

$$b_{\text{def}} = (b_1, \ldots, b_p, 0, \ldots, 0)', \tag{2.4.22}$$

where $b_i \neq 0$ $(1 \leq i \leq p)$. By (2.4.4),

$$b'_{\text{def}} \in \mathcal{R}(B). \tag{2.4.23}$$

Consider now the pencils

$$b = (b_1, \ldots, b_f, 0, \ldots, 0)', \tag{2.4.24}$$

$$b^* = (0, \ldots, 0, -b_{f+1}, \ldots, -b_p, 0, \ldots, 0)', \tag{2.4.25}$$

where 0 appears in the first f and the last $n-p$ positions of b^*. By (2.4.22)–(2.4.25), $(b-b^*)' = b'_{\text{def}} \in \mathcal{R}(B)$, so that by definition, the pencils b and b^* are aliased with each other. However, by (2.4.24), b has f nonzero entries, while by (2.4.25), the number of nonzero entries in b^* is $p-f$, which is at most t, since $p \leq f+t$. Consequently, (ii) above is violated. Thus every defining pencil must have at least $f+t+1$ nonzero entries, which proves the “only if” part. □

2.5 Optimality Criteria: Resolution and Minimum Aberration

In view of Theorem 2.4.3, the behavior of an s^{n-k} design depends on the numbers of nonzero entries in the defining pencils and, in particular, on the minimum of these numbers. This minimum number is called the *resolution* of the design (Box and Hunter, 1961a,b). From (2.4.7), each defining pencil in Example 2.4.1 has three or four nonzero entries. Hence the design considered in this example has resolution three. Similarly, by (2.4.8), the design in Example 2.4.2 also has resolution three. (In the literature on applied experimental design, the value of resolution is often indicated by a Roman numeral such as III, IV, or V.) Theorem 2.4.3 implies that a design of resolution one or two fails to ensure the estimability of all treatment contrasts belonging to the main effects even under the absence of all interactions. Since the main effects are almost invariably the objects of interest in a factorial experiment, we will focus primarily on designs of resolution three or higher. The next result then follows as an immediate consequence of Theorem 2.4.3.

Theorem 2.5.1. *An s^{n-k} design of resolution R (≥ 3) keeps all treatment contrasts belonging to factorial effects involving f or fewer factors estimable under the absence of all factorial effects involving $R - f$ or more factors, whenever f satisfies* $1 \leq f \leq \frac{1}{2}(R-1)$.

The above result suggests that given s, n, and k, one should choose an s^{n-k} design with maximum resolution. This is called the *maximum resolution* criterion. In the setup of Examples 2.4.1 and 2.4.2, the maximum possible resolution is three (this will be demonstrated in the next section from a more general result) and the designs considered in these examples achieve this highest resolution.

In many situations, however, there are several designs, each having the maximum possible resolution. Further discrimination among these rival designs is then warranted on the basis of a closer examination of their defining pencils and aliasing patterns. Such discrimination is of utmost practical importance, especially when, as often happens in practice, one is not completely sure about the absence of certain factorial effects. The following example serves to motivate the ideas.

Example 2.5.1. Consider two 3^{5-2} designs, $d(B_1)$ and $d(B_2)$, where

$$B_1 = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

By (2.4.4), the defining pencils in $d(B_1)$ and $d(B_2)$ are given respectively by

$$I = 124^2 = 12^2 35^2 = 13^2 45 = 2345^2 \tag{2.5.1}$$

and

$$I = 124^2 = 135^2 = 12^2 3^2 45 = 23^2 4^2 5. \tag{2.5.2}$$

Both $d(B_1)$ and $d(B_2)$ have resolution three, which as will be seen in the next section, is the highest possible in the present setup.

Now, $d(B_1)$ has only one defining pencil, namely 124^2, with three nonzero entries. Hence the argument in Theorem 2.4.3 shows that the pencils 12, 14^2, and 24^2, each of which belongs to a 2fi, get aliased with the main effect pencils 4, 2, and 1 respectively. No other 2fi pencil is aliased with any main effect pencil in $d(B_1)$. On the other hand, $d(B_2)$ has two defining pencils 124^2 and 135^2 with three nonzero entries. Arguing as before, there are six 2fi pencils that get aliased with the main effect pencils in $d(B_2)$. Thus by Theorem 2.4.2, for the estimation of treatment contrasts belonging to the main effects, one needs to assume the ignorability of three 2fi pencils in $d(B_1)$, and six 2fi pencils in $d(B_2)$. Therefore, if one is not fully confident about the absence of all 2fi's, then $d(B_1)$ is preferable to $d(B_2)$ because the former requires less stringent assumptions. □

More generally, if $A_3\,(\geq 0)$ is the number of defining pencils with three nonzero entries in any design of resolution three or higher, then $3A_3$ 2fi pencils get aliased with main effect pencils in such a design. Hence, given any two resolution three designs, the one with a smaller value of A_3 is preferred. These considerations lead to the criterion of minimum aberration introduced by Fries and Hunter (1980) for s^{n-k} designs. The basic premise underlying this criterion is the following principle.

Effect hierarchy principle:

(i) Lower order factorial effects are more likely to be important than higher order ones.
(ii) Factorial effects of the same order are equally likely to be important.

For $1 \leq i \leq n$, let $A_i(B)$ be the number of (distinct) defining pencils with i nonzero entries in an s^{n-k} design $d(B)$. A defining pencil is also called a *word* (or a *codeword*) in coding theory (see Section 2.8). The number of nonzero entries in a defining pencil is called the *length* of the word. Using this terminology, the sequence

$$W(B) = (A_1(B), A_2(B), A_3(B), \ldots, A_n(B)) \tag{2.5.3}$$

is called the *wordlength pattern* of $d(B)$.

Definition 2.5.1. *Let $d(B_1)$ and $d(B_2)$ be two s^{n-k} designs. Let r be the smallest integer such that $A_r(B_1) \neq A_r(B_2)$. Then $d(B_1)$ is said to have* less aberration *than $d(B_2)$ if $A_r(B_1) < A_r(B_2)$. A design is called a* minimum aberration *(MA) design if no other design has less aberration than it.*

Clearly, the resolution of a design $d(B)$ equals the smallest integer j such that $A_j(B) > 0$. Hence in any given context, *an MA design has the highest possible resolution* as well.

Returning to Example 2.5.1, by (2.5.1) and (2.5.2), the wordlength patterns of $d(B_1)$ and $d(B_2)$ are given by $(0, 0, 1, 3, 0)$ and $(0, 0, 2, 1, 1)$ respectively. Hence $d(B_1)$ has less aberration than $d(B_2)$. MA designs will be discussed extensively in the subsequent chapters. It will be evident that the designs considered in Examples 2.4.1 and 2.4.2 as well as the design $d(B_1)$ in Example 2.5.1 are MA designs.

We conclude this section with a result that will be useful later. Here a factor F_i is said to be involved in a pencil $b = (b_1, \ldots, b_n)'$ if $b_i \neq 0$.

Lemma 2.5.1. *For an s^{n-k} design $d(B)$ to be a minimum aberration design, it is necessary that every factor be involved in some defining pencil of $d(B)$.*

Proof. Suppose some factor, say F_1, is not involved in any defining pencil of $d(B)$. By (2.4.4), then the first column of B is a null vector. Let B^* be a $k \times n$ matrix, over $GF(s)$, with first column given by $(1, 0, \ldots, 0)'$. The other columns of B^* are identical to the corresponding columns of B. Then B^* has full row rank like B, and $d(B^*)$ is also an s^{n-k} design. For any $k \times 1$ vector $\lambda = (\lambda_1, \ldots, \lambda_k)'$ over $GF(s)$, clearly $\lambda' B^*$ has as many nonzero elements as $\lambda' B$ if $\lambda_1 = 0$, and one more nonzero element if $\lambda_1 \neq 0$. From (2.4.4) and Definition 2.5.1, it now follows that $d(B^*)$ has less aberration than $d(B)$, i.e., $d(B)$ is not an MA design. □

2.6 Connection with Orthogonal Arrays

We now introduce the concept of an orthogonal array (Rao, 1947), which facilitates the study of fractional factorials.

Definition 2.6.1. *An orthogonal array $OA(N, n, s, g)$, having N rows, n columns, s symbols, and strength g, is an $N \times n$ array with elements from a set of s symbols in which all possible combinations of symbols appear equally often as rows in every $N \times g$ subarray.*

Since the s symbols can be combined in s^g possible ways among the rows of an $N \times g$ subarray, it is clear that N is a multiple of s^g in an $OA(N, n, s, g)$. The integer N/s^g is called the *index* of the orthogonal array. Without loss of generality, the s symbols may be coded as $0, 1, \ldots, s-1$ or as the elements of $GF(s)$, depending on the context.

Example 2.6.1. In (a), (b), (c) below, we show an $OA(8, 5, 2, 2)$, an $OA(9, 4, 3, 2)$, and an $OA(8, 4, 2, 3)$:

(a) $OA(8, 5, 2, 2)$
$$\begin{bmatrix} 0&0&0&0&0\\ 0&0&1&0&1\\ 0&1&0&1&0\\ 0&1&1&1&1\\ 1&0&0&1&1\\ 1&0&1&1&0\\ 1&1&0&0&1\\ 1&1&1&0&0 \end{bmatrix}$$

(b) $OA(9, 4, 3, 2)$
$$\begin{bmatrix} 0&0&0&0\\ 1&1&0&1\\ 2&2&0&2\\ 1&2&1&0\\ 2&0&1&1\\ 0&1&1&2\\ 2&1&2&0\\ 0&2&2&1\\ 1&0&2&2 \end{bmatrix}$$

(c) $OA(8, 4, 2, 3)$
$$\begin{bmatrix} 0&0&0&0\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0\\ 1&0&0&1\\ 1&0&1&0\\ 1&1&0&0\\ 1&1&1&1 \end{bmatrix}$$

□

The following result (Rao, 1947) presents useful necessary conditions for the existence of orthogonal arrays. The lower bounds for N shown in this result are called *Rao's bounds*. The proof is omitted here, and can be found in Hedayat, Stufken, and Sloane (1999, Chapter 2) or Dey and Mukerjee (1999, Chapter 2).

Theorem 2.6.1. *In an orthogonal array $OA(N, n, s, g)$,*
(a) $N \geq \sum_{i=0}^{p} \binom{n}{i}(s-1)^i$, *if* $g (= 2p, p \geq 1)$ *is even,*
(b) $N \geq \sum_{i=0}^{p} \binom{n}{i}(s-1)^i + \binom{n-1}{p}(s-1)^{p+1}$ *if* $g\,(= 2p+1, p \geq 1)$ *is odd.*

An orthogonal array for which equality holds in either (a) or (b) above is called *saturated* or tight. In particular, taking $g = 2$ or 3 in Theorem 2.6.1, the following corollary is obtained.

Corollary 2.6.1. *(a) In an* $OA(N, n, s, 2), N \geq 1 + n(s-1)$.
(b) In an $OA(N, n, s, 3), N \geq 1 + n(s-1) + (n-1)(s-1)^2$.

The following lemma plays a key role in linking s^{n-k} designs with orthogonal arrays. Later, it will be found to have other important applications as well.

Lemma 2.6.1. *The existence of an s^{n-k} design $d(B)$ is equivalent to the existence of an $(n-k) \times n$ matrix G, defined over $GF(s)$ and having full row rank, such that*

(a) the s^{n-k} treatment combinations included in $d(B)$ are transposes of the s^{n-k} vectors in $\mathcal{R}(G)$,
(b) any pencil b is a defining pencil of $d(B)$ if and only if $Gb = 0$,
(c) any two pencils are aliased with each other in $d(B)$ if and only if $G(b - b^*) = 0$ for some representations b and b^* of these pencils.

Proof. Recall that for any s^{n-k} design $d(B)$, the matrix B is of order $k \times n$ and has full row rank. Hence given $d(B)$, or equivalently B, one can find an $(n-k) \times n$ matrix G, defined over $GF(s)$ and also having full row rank, such that

$$BG' = 0. \tag{2.6.1}$$

The row spaces of B and G are then orthogonal complements of each other. The truth of (a), (b), and (c) now follows from (2.4.1), (2.4.4), and (2.4.9) respectively.

Conversely, given an $(n-k) \times n$ matrix G defined over $GF(s)$ and having full row rank, there exists a $k \times n$ matrix B, again defined over $GF(s)$ and having full row rank, such that (2.6.1) holds. As before, (a), (b), and (c) will remain true for the design $d(B)$ as defined in (2.4.1). □

Theorem 2.6.2. *Let $d(B)$ be an s^{n-k} design of resolution R. If $R \geq g+1$, then the treatment combinations included in $d(B)$, when written as rows, form an orthogonal array $OA(s^{n-k}, n, s, g)$.*

Proof. Since $R \geq g+1$, each defining pencil in $d(B)$ has at least $g+1$ nonzero entries. Hence by Lemma 2.6.1, there exists an $(n-k) \times n$ matrix G, defined over $GF(s)$ and having full row rank, such that

(i) the treatment combinations included in $d(B)$ are transposes of the vectors in $\mathcal{R}(G)$, and
(ii) no g columns of G are linearly dependent.

Note that (ii) follows from Lemma 2.6.1 (b), since if some g columns of G are linearly dependent, then one gets a defining pencil having at most g nonzero entries, in $d(B)$.

Let Q be the array of order $s^{n-k} \times n$ formed by the s^{n-k} vectors in $\mathcal{R}(G)$. In view of (i), it is enough to show that Q is an orthogonal array of strength g. Consider any $s^{n-k} \times g$ subarray, say Q_1, of Q. Let G_1 be the corresponding $(n-k) \times g$ submatrix of G. The rows of Q_1 are then given by the s^{n-k} vectors $\lambda' G_1$, corresponding to the s^{n-k} possible choices of the $(n-k) \times 1$ vector λ over $GF(s)$. Now by (ii), G_1 has full column rank and hence contains a nonsingular $g \times g$ submatrix. Hence, as in the proof of Lemma 2.3.2, there are s^{n-k-g} choices of λ such that $\lambda' G_1$ equals any fixed g-component row vector with elements from $GF(s)$. Consequently, in Q_1 each possible g-component row vector appears with the same frequency s^{n-k-g}. It follows that Q is an orthogonal array of strength g. □

Recall that the designs shown in Examples 2.4.1 and 2.4.2 have resolution three. Hence by the above theorem, the treatment combinations included in these designs, when written as rows, form orthogonal arrays of strength two. These are precisely the arrays $OA(8, 5, 2, 2)$ and $OA(9, 4, 3, 2)$ shown in Example 2.6.1.

Theorem 2.6.2, in conjunction with Theorem 2.6.1 or Corollary 2.6.1, yields necessary conditions for the existence of a design of specified resolution. Two such conditions are presented in the next result.

Theorem 2.6.3. *Let $d(B)$ be an s^{n-k} design of resolution R.*

(a) For $R \geq 3$,

$$n \leq \frac{s^{n-k} - 1}{s - 1}. \tag{2.6.2}$$

(b) For $R \geq 4$,

$$n \leq \frac{s^{n-k-1} - 1}{s - 1} + 1. \tag{2.6.3}$$

Proof. (a) For $R \geq 3$, by Theorem 2.6.2, the treatment combinations in $d(B)$ form an $OA(s^{n-k}, n, s, 2)$. Hence by part (a) of Corollary 2.6.1, the result follows.
(b) For $R \geq 4$, by Theorem 2.6.2 and Corollary 2.6.1 (b),

$$s^{n-k} \geq 1 + n(s-1) + (n-1)(s-1)^2,$$

which, on simplification, yields the desired inequality. □

It will be seen in the next section that the condition (2.6.2) is also sufficient for the existence of a design of resolution three or higher. A similar result on the sufficiency of the condition (2.6.3) for $s = 2$ will also be presented there. For general s (≥ 3), however, (2.6.3) is not sufficient for the existence of a design of resolution four or higher. This will be evident in the next paragraph when Example 2.5.1 is revisited. Incidentally, Theorem 2.6.3(b) shows that no design of resolution four or higher exists in the setup of Examples 2.4.1 or 2.4.2. Thus, as claimed earlier, the designs considered in these examples have the highest possible resolution.

The literature on necessary conditions for the existence of orthogonal arrays is very rich. Extensive reviews of the available results can be found in Hedayat, Sloane, and Stufken (1999, Chapters 2, 4) and Dey and Mukerjee (1999, Chapters 2, 5). In view of Theorem 2.6.2, any such necessary condition is potentially useful for investigating the maximum possible resolution of designs in a given context. For the purpose of illustration, Example 2.5.1 is revisited. Here $s = 3$, $n = 5$, $k = 2$, and (2.6.3) is satisfied. Now if a design of resolution four or higher exists in this setup, then, by Theorem 2.6.2, one gets an $OA(27, 5, 3, 3)$. However, following a result by Bush (1952) as reported in Hedayat, Sloane, and Stufken (1999, p.24) or Dey and Mukerjee (1999, p.38), in an $OA(s^3, n, s, g)$, if $s \leq g$, then $n \leq g+1$. This rules out the existence of an $OA(27, 5, 3, 3)$ and shows that for $s = 3$, $n = 5$, and $k = 2$, the resolution of a design can be at most three. Both $d(B_1)$ and $d(B_2)$, introduced earlier in this example, have resolution three, which is the highest possible. It is also clear now that for $s \geq 3$, the condition (2.6.3) is only necessary but not sufficient for the existence of a design of resolution four or higher.

Theorem 2.6.2 has another important implication from traditional optimality considerations. To motivate the ideas, consider an s^{n-k} design $d(B)$ of resolution three. By Theorem 2.5.1, $d(B)$ allows estimation of all main effect contrasts under the absence of all interactions. One can, however, consider any other selection of s^{n-k} treatment combinations and wonder about the performance of $d(B)$, *vis-à-vis* any such rival fraction, for the estimation of the main effect contrasts. Specifically, one may be interested in comparing the covariance matrix of the estimators arising from $d(B)$ with that for any rival fraction. In particular, if $d(B)$ minimizes the determinant, trace, or maximum eigenvalue of this covariance matrix, then it is called *D-, A-, or E-optimal* respectively, within the class of all fractions of the same size, i.e., having the same number of treatment combinations.

Cheng (1980) addressed the issue of optimality, considering a very general criterion called universal optimality (Kiefer, 1975), which covers the D-, A-, and E-criteria as special cases. He showed that if the treatment combinations in a fractional factorial form an orthogonal array of strength two (when written as rows), then the fraction is universally optimal among all fractions of the same size for estimating the main effect contrasts under the absence of all interactions. Returning to the design $d(B)$ of resolution three, Theorem 2.6.2 shows that its treatment combinations form an orthogonal array of strength two. Hence invoking Cheng's result, one reaches the satisfying conclusion that

$d(B)$ is indeed universally optimal among all fractions of the same size, for the estimation problem considered in the last paragraph.

Cheng's (1980) result was extended by Mukerjee (1982) to orthogonal arrays of general strength. This, in conjunction with Theorem 2.6.2, helps in proving the universal optimality of s^{n-k} designs of general resolution. Specifically, supplementing Theorem 2.5.1, it can be shown that an s^{n-k} design of resolution R (≥ 3) is universally optimal among all fractions of the same size for estimating factorial effects involving f or fewer factors under the absence of all factorial effects involving $R-f$ or more factors, whenever $1 \leq f \leq \frac{1}{2}(R-1)$.

The issue of universal optimality or the specialized D-, A-, or E- optimality will not be considered further in this book. First, optimality results on factorial fractions under these criteria have already been discussed at length in Dey and Mukerjee (1999, Chapters 2, 6, 7). The second and more compelling reason is that while these optimality results involve clear-cut assumptions regarding the absence of certain factorial effects, the present book aims primarily at reviewing and synthesizing good experimental strategies when the validity of such assumptions is not taken for granted. From this perspective, criteria like that of MA will be of greater interest in this book. A hint to this effect has already been given in Example 2.5.1, where the designs $d(B_1)$ and $d(B_2)$ were discriminated on the basis of stringency of assumptions even though both are of resolution three and hence universally optimal for estimating the main effect contrasts under the absence of all interactions.

2.7 Connection with Finite Projective Geometry

Another important tool for the study of s^{n-k} designs is finite projective geometry. The $(r-1)$-dimensional finite projective geometry over $GF(s)$, denoted by $PG(r-1, s)$, consists of points of the form $(x_1, \ldots, x_r)'$, where $x_i \in GF(s)$ $(1 \leq i \leq r)$ and not all of $x_1, \ldots, x_r$ are zero, such that any two points with proportional entries are considered identical. Evidently, the pencils in an s^n factorial are points of $PG(n-1, s)$. As with pencils, there are $(s^r - 1)/(s-1)$ distinct points in $PG(r-1, s)$. *Hereafter, only distinct points of a finite projective geometry are considered in any given context, even when this is not stated explicitly.* For example, when we refer to a collection or set of points, it is implicit that the points are distinct. Along the lines of Remark 2.3.1, in the subsequent chapters it will often be convenient to represent a typical point $(x_1, \ldots, x_r)'$ of $PG(r-1, s)$ using the *compact notation* $1^{x_1} \ldots r^{x_r}$, where i^{x_i} is dropped if $x_i = 0$. More details on finite projective geometry are available in Raghavarao (1971, pages 357–359).

The connection between s^{n-k} designs and finite projective geometry goes much deeper than the mere interpretation of pencils in an s^n factorial as points of $PG(n-1, s)$. Theorem 2.7.1 below, exhibiting a duality between an s^{n-k} design of resolution three or higher and a set of points of a finite projective geometry, underscores this point. This theorem looks quite similar to Lemma

2.6.1 and is actually its consequence. Lemma 2.6.1 and Theorems 2.7.1–2.7.4 are in the spirit of Bose (1947). In what follows, for any nonempty set T of p points of $PG(r-1, s)$, $V(T)$ is an $r \times p$ matrix with columns given by the points of T.

Theorem 2.7.1. *Given any s^{n-k} design $d(B)$ of resolution three or higher, there exists a set T of n points of $PG(n-k-1, s)$ such that $V(T)$ has full row rank, and*

(a) the s^{n-k} treatment combinations included in $d(B)$ are transposes of the s^{n-k} vectors in $\mathcal{R}[V(T)]$,
(b) any pencil b is a defining pencil of $d(B)$ if and only if $V(T)b = 0$,
(c) any two pencils are aliased with each other in $d(B)$ if and only if $V(T)(b-b^*) = 0$ for some representations b and b^* of these pencils.

Conversely, given any set T of n points of $PG(n-k-1, s)$ such that $V(T)$ has full row rank, there exists an s^{n-k} design $d(B)$ of resolution three or higher such that (a)–(c) hold.

Proof. Consider an s^{n-k} design $d(B)$ of resolution three or higher. Then by Lemma 2.6.1, there exists an $(n-k) \times n$ matrix G, defined over $GF(s)$ and having full row rank such that (a)–(c) of this lemma hold. Since $d(B)$ has resolution three or higher, by Lemma 2.6.1(b), no two columns of G are linearly dependent. Thus the columns of the $(n-k) \times n$ matrix G are nonnull and no two of them are proportional to each other. Hence these columns can be interpreted as n points of $PG(n-k-1, s)$. Let T denote the set of these n points ordered in the same manner as the columns of G. Then $G = V(T)$ and $V(T)$ has full row rank as G. The validity of (a)–(c) is now obvious from (a)–(c) of Lemma 2.6.1.

To prove the converse, consider any set T of n points of $PG(n-k-1, s)$ such that the $(n-k) \times n$ matrix $V(T)$ has full row rank. There exists a $k \times n$ matrix B, defined over $GF(s)$ and having full row rank, such that $B[V(T)]' = 0$. As in Lemma 2.6.1, (a)–(c) of this theorem hold for the design $d(B)$. Furthermore, by the definition of T, no two columns of $V(T)$ are linearly dependent. Hence by (b), the design $d(B)$ has resolution three or higher. □

The following corollary is easily obtained from Theorem 2.7.1(b).

Corollary 2.7.1. *Let $g \geq 2$. An s^{n-k} design of resolution $g+1$ or higher exists if and only if there exists a set T of n points of $PG(n-k-1, s)$ such that no g points of T are linearly dependent and $V(T)$ has full row rank.*

Examples 2.4.1 and 2.4.2 are now revisited for illustrating Theorem 2.7.1. In Example 2.4.1, $s = 2$, $n = 5$, $k = 2$ and the matrix B stated there yields a design $d(B)$ of resolution three. The row spaces of B and

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

are orthogonal complements of each other. Interpreting the columns of G as points of $PG(2,2)$, the design $d(B)$ corresponds to a set

$$T = \{(1,0,0)', (0,1,0)', (0,0,1)', (1,1,0)', (1,0,1)'\}$$

of five points of $PG(2,2)$ such that $V(T) = G$ has full row rank. Using the compact notation described in the beginning of the section, the set T can as well be expressed as $T = \{1, 2, 3, 12, 13\}$. Similarly, the design $d(B)$ of resolution three, considered in Example 2.4.2, corresponds to the set

$$T = \{(1,1)', (1,2)', (0,1)', (1,0)'\}$$

of four points of $PG(1,3)$ such that $V(T)$ has full row rank. In either example, it is a simple exercise to verify from first principles that (a)–(c) of Theorem 2.7.1 hold.

The following lemma facilitates the applications of Theorem 2.7.1 to be considered in this section. This lemma is needed to cope with the stipulation regarding full row rank of $V(T)$ in Theorem 2.7.1 or Corollary 2.7.1.

Lemma 2.7.1. *Suppose there are n points of $PG(n-k-1,s)$ such that no g (≥ 2) of these points are linearly dependent. Then there exists a set T of n points of $PG(n-k-1,s)$ such that no g points of T are linearly dependent and $V(T)$ has full row rank.*

Proof. Let $h_1, \ldots, h_n$ be n points of $PG(n-k-1,s)$ such that no g (≥ 2) of these points are linearly dependent. If the $(n-k) \times n$ matrix H, given by the points $h_1, \ldots, h_n$ as columns, has full row rank, then it suffices to take $T = \{h_1, \ldots, h_n\}$. Now suppose $\text{rank}(H) = p < n-k$. Then there exists an $(n-k) \times n$ matrix

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \tag{2.7.1}$$

defined over $GF(s)$ and having full row rank such that Z_1 is $p \times n$, Z_2 is $(n-k-p) \times n$, and

$$\mathcal{R}(Z_1) = \mathcal{R}(H). \tag{2.7.2}$$

By (2.7.1) and (2.7.2), no g columns of Z are linearly dependent, for otherwise the corresponding g columns of H are linearly dependent, which is impossible by the definition of H. Since $g \geq 2$, the n columns of Z represent n points of $PG(n-k-1,s)$. Define T as the set of these n points. Then $V(T) = Z$ has full row rank and no g points of T are linearly dependent. □

Example 2.7.1. In order to illustrate the ideas in the above proof, let $s = 3$, $n = 4$, $k = 1$, and $g = 2$, and consider the points

$$h_1 = (1,0,0)',\ h_2 = (0,1,0)',\ h_3 = (1,1,0)',\ h_4 = (1,2,0)'$$

of $PG(2,3)$. Obviously, no two of these points are linearly dependent. However, the matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

given by these points as columns does not have full row rank. Consider now the matrix

$$Z = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

over $GF(3)$, where

$$Z_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \quad Z_2 = [0\ 0\ 1\ 0].$$

Then Z has full row rank, $\mathcal{R}(Z_1) = \mathcal{R}(H)$, and no two columns of Z, like those of H, are linearly dependent. Interpreting the columns of Z as points, one gets a set

$$T = \{(1,0,0)', (0,1,0)', (1,1,1)', (1,2,0)'\}$$

of four points of $PG(2,3)$ such that no g $(=2)$ points of T are linearly dependent and $V(T) = Z$ has full row rank. □

Theorem 2.7.2 below follows readily from Corollary 2.7.1 and Lemma 2.7.1.

Theorem 2.7.2. *Let $g \geq 2$. An s^{n-k} design of resolution $g+1$ or higher exists if and only if there exist n points of $PG(n-k-1, s)$ such that no g of these points are linearly dependent.*

Theorem 2.7.2 helps in exploring the sufficiency of the conditions (2.6.2) and (2.6.3), which were earlier seen to be necessary for the existence of designs of resolution at least three and four, respectively. By Theorem 2.7.2, an s^{n-k} design of resolution three or higher exists if and only if there exist n points of $PG(n-k-1, s)$, no two of which are linearly dependent. Since there are altogether $(s^{n-k}-1)/(s-1)$ points of $PG(n-k-1, s)$, no two of which are linearly dependent, the sufficiency of (2.6.2) is immediate. Thus one gets the following result.

Theorem 2.7.3. *An s^{n-k} design of resolution three or higher exists if and only if*

$$n \leq \frac{s^{n-k}-1}{s-1}.$$

This theorem is of much importance since, as discussed in Section 2.5, only designs of resolution three or higher are of interest. Turning to (2.6.3), it was noted in the last section that this condition is not in general sufficient for the existence of a design of resolution four or higher. Theorem 2.7.4 below shows that it is, however, sufficient in the special case $s = 2$. Note that for $s = 2$, (2.6.3) reduces to $n \leq 2^{n-k-1}$.

Theorem 2.7.4. *A 2^{n-k} design of resolution four or higher exists if and only if*

$$n \leq 2^{n-k-1}. \tag{2.7.3}$$

Proof. The "only if" part is already proved in Theorem 2.6.3 (b). To prove the "if" part, let (2.7.3) hold. The points of $PG(n-k-1, 2)$ are nonnull binary vectors of order $(n-k) \times 1$. Consider those points with an odd number of 1's. There are

$$\binom{n-k}{1} + \binom{n-k}{3} + \cdots = 2^{n-k-1}$$

such points. Since each of these points has an odd number of 1's, no three of these can add up to the null vector, i.e., no three of these are linearly dependent. Thus there exists a collection of 2^{n-k-1} points of $PG(n-k-1, 2)$ such that no three of these points are linearly dependent. The "if" part now follows from (2.7.3) and Theorem 2.7.2. □

Example 2.7.2. Let $s = 2$, $n = 8$, $k = 4$. Then $n - k - 1 = 3$ and (2.7.3) holds. The points of $PG(3, 2)$ having an odd number of 1's are

$$(1,0,0,0)',\ (0,1,0,0)',\ (0,0,1,0)',\ (0,0,0,1)',$$

$$(1,1,1,0)',\ (1,1,0,1)',\ (1,0,1,1)',\ (0,1,1,1)'.$$

Let T be the set of these eight points, no three of which are linearly dependent. The matrix

$$V(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

has full row rank, and following Theorem 2.7.1, the vectors in $\mathcal{R}[V(T)]$ yield a 2^{8-4} design of resolution at least four. In fact, since the third to sixth columns of $V(T)$ add up to the null vector, this design has resolution exactly four. □

We conclude this section with two results that will be useful in the subsequent chapters.

Lemma 2.7.2. *Let V_r be a matrix with r rows and $(s^r-1)/(s-1)$ columns, such that the columns of V_r are given by the points of $PG(r-1, s)$. Then*

(a) rank$(V_r) = r$,
(b) every nonnull vector in $\mathcal{R}(V_r)$ has exactly s^{r-1} nonzero elements.

Proof. Since the points of $PG(r-1, s)$ are represented by nonnull vectors and two points with proportional entries are considered identical, without loss of generality, the first nonzero element in each column of V_r can be assumed to be 1, the identity element of $GF(s)$ under multiplication. Then $V_1 = (1)$ and for $r = 1, 2, \ldots,$

$$V_{r+1} = \begin{bmatrix} 0' & 1'_{(r)} \\ V_r & M_r \end{bmatrix}, \tag{2.7.4}$$

where $0'$ is the null row vector of order $(s^r - 1)/(s - 1)$, and $1'_{(r)}$ is the row vector of order s^r with each element 1. Also, M_r is an $r \times s^r$ matrix whose columns are given by all possible $r \times 1$ vectors over $GF(s)$. By (2.7.4), V_r has the r unit vectors over $GF(s)$ as columns, and (a) follows.

The proof of (b) will be by induction on r. Since $V_1 = (1)$, obviously (b) holds for $r = 1$. Suppose it holds for $r = t$ and consider any nonnull vector in $\mathcal{R}(V_{t+1})$. By (2.7.4), any such vector must be of the form

$$\xi' = (\lambda' V_t, \lambda_0 1'_{(t)} + \lambda' M_t), \tag{2.7.5}$$

where $\lambda_0 \in GF(s)$ and λ is a $t \times 1$ vector over $GF(s)$ such that (λ_0, λ') is nonnull. By the definition of M_t, the number of zero elements in $\lambda_0 1'_{(t)} + \lambda' M_t$ equals the number of $t \times 1$ vectors x over $GF(s)$ such that $\lambda_0 + \lambda' x = 0$. If $\lambda \neq 0$, then as in the proof of Lemma 2.3.1 this number equals s^{t-1}, so that $\lambda_0 1'_{(t)} + \lambda' M_t$ has $s^t - s^{t-1}$ nonzero elements. Also, by (a), $\lambda' V_t$ is nonnull for $\lambda \neq 0$, and hence has s^{t-1} nonzero elements by the induction hypothesis. Therefore, by (2.7.5), ξ has s^t nonzero elements if $\lambda \neq 0$. On the other hand, if $\lambda = 0$, then $\lambda_0 \neq 0$, and again by (2.7.5), the same conclusion about ξ holds. Thus (b) follows by induction. □

Define the *m-lag* of a row vector W as $lag(W, m) = (0, \ldots, 0, W)$, where W is preceded by m zeros. Also, given s, n, and k, denote the maximum possible resolution of an s^{n-k} design by $R_s(n, k)$. Then the following result, due to Chen and Wu (1991), holds.

Lemma 2.7.3. *(a) Given any s^{n-k} design $d(B)$ with wordlength pattern $W(B)$, there exists an $s^{\left(n+\frac{s^k-1}{s-1}\right)-k}$ design $d(B_k)$, with wordlength pattern $W(B_k) = \left(lag\ \left(W(B), s^{k-1}\right), 0'\right)$, where $0'$ is a null row vector such that $W(B_k)$ has $n + \frac{s^k-1}{s-1}$ elements altogether.*
(b) $R_s\left(n + \frac{s^k-1}{s-1}, k\right) \geq R_s(n, k) + s^{k-1}$.

Proof. Part (b) is a consequence of (a). If $d(B)$ has resolution $R_s(n, k)$, which is indeed possible, then $d(B_k)$ has resolution $R_s(n, k) + s^{k-1}$. Hence the maximum resolution of an $s^{\left(n+\frac{s^k-1}{s-1}\right)-k}$ design is at least $R_s(n, k) + s^{k-1}$.

To prove (a), define the $k \times \left(n + \frac{s^k-1}{s-1}\right)$ matrix $B_k = [B \quad V_k]$, where V_k is as introduced in Lemma 2.7.2. Since both B and B_k have full row rank, the nonnull vectors in $\mathcal{R}(B)$ and $\mathcal{R}(B_k)$ are of the form $\lambda' B$ and $\lambda' B_k = (\lambda' B, \lambda' V_k)$ respectively, where λ is any $k \times 1$ nonnull vector over $GF(s)$. However, by Lemma 2.7.2 (a), V_k has full rank. Therefore, for any such λ, the vector $\lambda' V_k$ is nonnull and hence has s^{k-1} nonzero elements by Lemma 2.7.2 (b). By (2.4.4), it follows that every defining pencil of $d(B)$ corresponds to a defining pencil of $d(B_k)$ such that the latter has s^{k-1} more nonzero entries than the former. Hence (a) is evident from the definition of wordlength pattern. □

2.8 Algebraic Coding Theory

This chapter concludes with algebraic coding theory, another important tool for the study of s^{n-k} designs. Some basic concepts, notation, and results are given in this section. Details and proofs can be found in MacWilliams and Sloane (1977), Pless (1989), and van Lint (1999).

Let B be a $k \times n$ matrix of rank k over $GF(s)$. Then the row space of B,

$$C = \mathcal{R}(B), \tag{2.8.1}$$

is an $[n, k; s]$ *linear code* of length n and dimension k. It is a k-dimensional linear subspace of the finite Euclidean geometry $EG(n, s)$, where the points of $EG(n, s)$ are viewed as row vectors. The matrix B is called a *generator* of C and the elements of C are called *codewords.* Without loss of generality, let $B = [I_k \;\; H]$ and write $G = [-H' \;\; I_{n-k}]$. The row spaces of B and G are then orthogonal complements of each other and clearly the code C is the null space of G. The matrix G is called a *parity check matrix.*

Comparing with Section 2.4, it is easy to see that the $[n, k; s]$ linear code C is equivalent to the defining contrast subgroup of the s^{n-k} design $d(B)$. Specifically, by (2.4.4) and (2.8.1), a nonnull codeword in C is equivalent to a defining pencil of $d(B)$. The mathematical connection between s^{n-k} designs and algebraic codes was established by Bose (1961).

For a codeword (or vector) $u = (u_1, \ldots, u_n)$, the *Hamming weight* $wt(u)$ is the number of its nonzero components. For two codewords $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_n)$, the *Hamming distance*

$$dist(u, w) = wt(u - w)$$

is the number of j's with $w_j \neq u_j$. The *minimum distance* of a code C is the smallest Hamming distance between any two distinct codewords of C. Let $K_i(C)$ be the number of codewords of weight i in C. Then $(K_1(C), K_2(C), \ldots)$ is called the *weight distribution* of C. It is easy to show that the minimum distance of the linear code C is the *minimum weight* of nonzero codewords in C, i.e., the smallest $i > 0$ such that $K_i(C) > 0$. For convenience, a linear code is denoted by $[n, k, d; s]$ if its minimum distance is d.

Continuing the previous interpretation, the minimum distance of a linear code C is mathematically equivalent to the resolution of the corresponding s^{n-k} design. The importance of the concept of resolution was discussed in Section 2.5. The concept of minimum distance plays an equally important role in coding theory, since it determines the capability of error correction of a code.

There is also a mathematical equivalence between the weight distribution of C and the wordlength pattern of the corresponding design as given in (2.5.3). Specifically, the weight distribution of a linear code C defined in (2.8.1) and the wordlength pattern of the corresponding design $d(B)$ are related as

$$K_i(C) = (s-1)A_i(B), \quad 1 \leq i \leq n, \tag{2.8.2}$$

because pencils with proportional entries are identical. Recall that the MA criterion was defined on the basis of the wordlength pattern and that its importance was justified by the effect hierarchy principle. One could similarly define an MA criterion for the weight distribution, but such a definition would lack a meaningful interpretation in coding theory. For this reason, unlike the interplay between minimum distance and resolution, MA designs do not have counterparts in coding theory.

Example 2.4.2 is now revisited for illustrating the connection between s^{n-k} designs and linear codes. Here the row space of B, i.e.,

$$\begin{aligned} C =&\{(0,0,0,0),(1,0,2,2),(0,1,1,2),(1,1,0,1),(1,2,1,0),\\ &(2,0,1,1),(0,2,2,1),(2,2,0,2),(2,1,2,0)\}, \end{aligned}$$

is a $[4,2;3]$ linear code. The weight distribution is

$$K_1(C) = K_2(C) = 0,\ K_3(C) = 8,\ K_4(C) = 0,$$

and the minimum distance is 3. On the other hand, as noted in (2.4.8), the corresponding 3^{4-2} design $d(B)$ has the defining relation $I = 13^2 4^2 = 234^2 = 124 = 12^2 3$. Therefore $d(B)$ has resolution three and wordlength pattern

$$A_1(B) = A_2(B) = 0,\ A_3(B) = 4,\ A_4(B) = 0.$$

Thus, (2.8.2) holds and the equivalence between the linear code C and the design $d(B)$ follows.

An important question in coding theory is the existence of a linear code given n, k, s, d. Let $D_s(n,k)$ be the maximum possible d such that an $[n,k,d;s]$ linear code exists. Brouwer and Verhoeff (1993) gave a comprehensive list of lower and upper bounds for $D_s(n,k)$ of binary (i.e., $s=2$) codes for $1 \leq n \leq 127$.

We now present some concepts and results from coding theory that will be useful later in the book. If C is an $[n,k;s]$ linear code, its *dual code* $C^\perp$ is the set of vectors that are orthogonal to all codewords of C, i.e.,

$$C^\perp = \{u : uw' = 0 \text{ for all } w \in C\}. \tag{2.8.3}$$

If C has the generator matrix B and parity check matrix G, then $C^\perp$ has the generator matrix G and parity check matrix B. Thus $C^\perp$ is an $[n, n-k; s]$ linear code. From the definition, it is easy to see that $C^\perp$ is equivalent to the

s^{n-k} design $d(B)$; the codewords of $C^{\perp}$ are the transposes of the treatment combinations included in $d(B)$.

A fundamental identity relating the weight distributions of a linear code C and its dual code $C^{\perp}$ is given below.

Theorem 2.8.1. *The weight distributions of an $[n, k; s]$ linear code C and its dual code $C^{\perp}$ satisfy the following identities:*

$$K_i(C^{\perp}) = s^{-k} \sum_{j=0}^{n} K_j(C) P_i(j; n, s), \tag{2.8.4}$$

$$K_i(C) = s^{-(n-k)} \sum_{j=0}^{n} K_j(C^{\perp}) P_i(j; n, s), \tag{2.8.5}$$

for $i = 0, \ldots, n$, where

$$P_i(x; n, s) = \sum_{t=0}^{i} (-1)^t (s-1)^{i-t} \binom{x}{t} \binom{n-x}{i-t} \tag{2.8.6}$$

are the Krawtchouk polynomials.

Equations (2.8.4) and (2.8.5) are known as the *MacWilliams identities* (MacWilliams, 1963).

The following identities, known as the *Pless power moment identities* after Pless (1963), relate the moments of the weight distributions of C and $C^{\perp}$.

Theorem 2.8.2. *For an $[n, k; s]$ linear code C and $r = 1, 2, \ldots,$*

$$\sum_{i=0}^{n} i^r K_i(C) = \sum_{i=0}^{n} (-1)^i K_i(C^{\perp}) \left[\sum_{j=0}^{r} j! S(r, j) s^{k-j} (s-1)^{j-i} \binom{n-i}{j-i} \right], \tag{2.8.7}$$

where

$$S(r, j) = (1/j!) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^r$$

for $r \geq j \geq 0$ is a Stirling number of the second kind.

Exercises

2.1 Show that the row space of the matrix $M(y)$ in (2.2.3) does not depend on the specific choice of $M_1, \ldots, M_n$, as long as the latter satisfy (2.2.2).

2.2 Prove Lemma 2.2.1 for $g = 4$.

2.3 Prove the "if" part of Theorem 2.2.1 explicitly.

2.4 List the distinct pencils in a 4^2 factorial. Select any two of them and verify from first principles that treatment contrasts belonging to these pencils are orthogonal to each other.

2.5 Obtain the alias sets in Example 2.4.2.

2.6 Show that the s^k pencils considered in (2.4.10) are distinct.

2.7 Use Theorems 2.6.1 and 2.6.2 to show that no 2^{n-k} design of resolution five or higher exists when $n \geq 2^{n-k-2} + 2$.

2.8 Verify from first principles that (a)–(c) of Theorem 2.7.1 hold in Examples 2.4.1 and 2.4.2.

2.9 Prove Corollary 2.7.1.

2.10 Use Theorem 2.7.1 to obtain the treatment combinations in the 2^{8-4} design considered in Example 2.7.2. Also, find the defining pencils and hence the wordlength pattern of this design.

2.11 For $r = 2$, verify Lemma 2.7.2 from first principles, writing the matrix V_2 explicitly and enumerating all nonnull vectors in the row space of V_2.

2.12 In an s^{n-k} design, let A_{ij} denote the number of defining pencils that involve the ith factor and have j nonzero entries. Following Draper and Mitchell (1970), the *letter pattern matrix* of the design is defined as the $n \times n$ matrix with elements A_{ij}, where each factor is interpreted as a letter. As usual, let $(A_1, \ldots, A_n)$ denote the wordlength pattern of the design. Show that $A_j = j^{-1} \sum_{i=1}^{n} A_{ij}$ for every j.

2.13 Refer to Table 1.1 in Section 1.1 and consider the 4×15 subarray given by the runs numbered $2, 3, 5$, and 9. Verify that the columns of this subarray represent the 15 points of $PG(3, 2)$ and that its four rows span all the 16 rows of Table 1.1. Hence establish a connection between the discussion in the last paragraph of Section 1.1 and Theorem 2.7.1.

2.14 Show that the minimum distance of a linear code C equals the minimum weight of nonzero codewords in C.

3

Two-Level Fractional Factorial Designs

Fractional factorial designs with factors at two levels are the most commonly used in practice. For the same number of factors, they have smaller run size than designs at more than two levels. This run size economy makes them attractive for studying a large number of factors. A fundamental question in this context is the choice of designs. The minimum aberration criterion is commonly used for selecting optimal designs. Theoretical results on minimum aberration designs are given in this chapter. Results on related criteria like maximum resolution and maximum number of clear effects are also considered. A catalogue of two-level fractional factorial designs with 16, 32, 64, and 128 runs is given.

3.1 Basic Definitions Revisited

Two-level factorials have certain simplifying features that facilitate their study even without an extensive use of abstract algebra. The first of these features, evident from (2.2.5) and Theorem 2.2.2, is that each factorial effect is represented by a unique treatment contrast up to proportionality. We begin by showing that any such treatment contrast in a 2^n factorial can have a natural interpretation. This generalizes the ideas in Section 2.1 for the special case $n = 2$.

Consider a *main effect*, say that of the first factor F_1. By (2.1.10) and (2.1.11) with $s = 2$, this main effect is represented by

$$L(F_1) = \sum_{j_1=0}^{1} \cdots \sum_{j_n=0}^{1} \bar{l}(j_1)\tau(j_1 \dots j_n), \tag{3.1.1}$$

where $\bar{l}(0) + \bar{l}(1) = 0$. There are 2^{n-1} treatment combinations corresponding to $j_1 = 0$ or 1. Hence, with $\bar{l}(1) = -\bar{l}(0) = 1/2^{n-1}$, one can interpret (3.1.1) as the difference between the averages of the treatment effects at the levels

1 and 0 of F_1. Note that in applied design texts, level 1 is often assigned to the high level and level 0 to the low level of a factor. While any other choice of $\bar{l}(1)(= -\bar{l}(0))$ does not affect the development of the design theory, the particular choice made above facilitates a natural interpretation.

Consider next the *two-factor interaction* (abbreviated as 2fi) F_1F_2. By (2.1.12)–(2.1.14), this is represented by

$$L(F_1F_2) = \sum_{j_1=0}^{1} \cdots \sum_{j_n=0}^{1} \bar{l}(j_1j_2)\tau(j_1 \ldots j_n), \tag{3.1.2}$$

where

$$\bar{l}(00) = -\bar{l}(01) = -\bar{l}(10) = \bar{l}(11) = l, \tag{3.1.3}$$

and l is any nonzero constant. The choice $l = 1/2^{n-1}$ will render (3.1.2) a natural interpretation similar to the one for the main effects. With l so chosen, (3.1.2) can be expressed as

$$L(F_1F_2) = \frac{1}{2}\{L(F_1|F_2 = 1) - L(F_1|F_2 = 0)\}, \tag{3.1.4}$$

where, for $j_2 = 0, 1$,

$$L(F_1|F_2 = j_2)$$
$$= \frac{1}{2^{n-2}} \sum_{j_3=0}^{1} \cdots \sum_{j_n=0}^{1} \tau(1j_2j_3 \ldots j_n) - \frac{1}{2^{n-2}} \sum_{j_3=0}^{1} \cdots \sum_{j_n=0}^{1} \tau(0j_2j_3 \ldots j_n).$$

In the spirit of the last paragraph, $L(F_1|F_2 = j_2)$ represents the *conditional main effect* of F_1 at level j_2 of F_2. Thus (3.1.4) exhibits $L(F_1F_2)$ in terms of the difference between these conditional main effects. This reinforces the interpretation associated with (2.1.9) that a 2fi measures the influence of the level where one factor is held fixed on the effect of a level change of the other factor. It is not hard to see that (3.1.4) remains invariant, like (2.1.9), when the roles of F_1 and F_2 are interchanged.

Similar considerations apply to any other 2fi or higher order interactions. For example, the treatment contrast representing the three-factor interaction (3fi) $F_iF_jF_r$ for $1 \leq i < j < r \leq n$ can be expressed as

$$L(F_iF_jF_r) = \frac{1}{2}\{L(F_iF_j|F_r = 1) - L(F_iF_j|F_r = 0)\},$$

where $L(F_iF_j|F_r = 1)$ is the conditional 2fi of F_i and F_j at level 1 of F_r, and so on. For a detailed discussion of this approach to defining factorial effects, see Chapter 3 of Wu and Hamada (2000).

We now turn to 2^{n-k} designs. Some additional features of two-level factorials that facilitate the understanding of such designs are as follows:

(a) All pencils are distinct.

(b) Each factorial effect is represented by a single pencil, so that there is practically no distinction between a factorial effect and the associated pencil.

Both (a) and (b) are evident from Section 2.3, and (b) was noted near the end of that section. By (b), a pencil becomes equivalent to a factorial effect. Therefore, the *concept of pencils is not needed explicitly* when one considers 2^{n-k} designs. In particular, the aliasing of pencils amounts to the aliasing of factorial effects.

By (2.4.4), in a 2^{n-k} design, there are $2^k - 1$ defining pencils or words. Using the compact notation introduced in Remark 2.3.1, any such word that corresponds to the factorial effect $F_{i_1} \ldots F_{i_g}$ can be conveniently denoted by $i_1 \ldots i_g$. Thus the defining contrast subgroup, introduced below (2.4.4), consists of the $2^k - 1$ words, together with the identity element I, and is closed under multiplication with the convention of dropping squared symbols. With a view to illustrating the above ideas, Example 2.4.1 is revisited from a different perspective.

Example 3.1.1. Consider the 2^{5-2} design d $(= d(B))$ in Example 2.4.1. Its eight treatment combinations can be represented by the eight rows of the following matrix M:

$$
\begin{array}{c}
 \begin{array}{ccccc} & & & \mathbf{12} & \mathbf{13} \\ & & & \| & \| \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

One can visualize the design as follows. Start from a complete factorial in F_1, F_2, F_3 (based on the first three columns of M). Then add column 4 as the sum of columns 1 and 2 and column 5 as the sum of columns 1 and 3 modulo 2. If column 4 is assigned to F_4, it can be used to estimate the main effect contrast $L(F_4)$ in the spirit of (3.1.1). Because 0 of column 4 corresponds to 00 and 11 of columns 1 and 2, and 1 of column 4 corresponds to 01 and 10 of columns 1 and 2, column 4 can also be used to estimate the interaction contrast $L(F_1F_2)$. Hence, $L(F_4)$ and $L(F_1F_2)$ cannot be disentangled on the basis of d. In other words, the factorial effects F_4 and F_1F_2 are *aliased* in d, and this is denoted by $4 = 12$. Similarly, one gets the aliasing relation $5 = 13$. As in Section 2.4, these two relations can be rewritten as $I = 124 = 135$. By multiplying 124 and 135, one gets a third relation $I = 2345$. Thus the 2^{5-2} design d is completely characterized by the defining relation

$$I = 124 = 135 = 2345, \tag{3.1.5}$$

which describes its defining contrast subgroup and is the same as (2.4.7). The alias sets are now easy to obtain via simple multiplication as discussed and shown below (2.4.12). □

Generally, a 2^{n-k} design is determined by its defining contrast subgroup. The length of each word therein, or *wordlength*, is the number of symbols or letters (i.e., factors) the word contains. The shortest wordlength is the *resolution* of the design. Let A_i be the number of words of length i in the defining contrast subgroup. The wordlength pattern $W = (A_1, A_2, A_3, \ldots, A_n)$ and the *minimum aberration* (MA) criterion are defined as in (2.5.3) and Definition 2.5.1. Thus the 2^{5-2} design in Example 3.1.1 has resolution three and wordlength pattern $(0, 0, 2, 1, 0)$, since its defining contrast subgroup consists of the words $124, 135$, and 2345 (apart from I) with lengths 3, 3, and 4 respectively. As will be shown after Theorem 3.2.1, this design has MA.

Since the properties of a 2^{n-k} design are determined by its defining contrast subgroup, two 2^{n-k} designs are said to be *isomorphic* (or *equivalent*) if the defining contrast subgroup of one of them can be obtained from that of the other by permuting the factor labels. In ranking and selecting designs, isomorphic designs will be treated as the same.

A theoretical question concerning the maximum resolution criterion is to find the highest resolution for a 2^{n-k} design with given n and k (i.e., fixed number of factors and run size). Theorems 2.7.3 and 2.7.4 address this question to some extent. A good summary of existing results on maximum resolution in the statistical literature is given in Draper and Lin (1990). These results will not be reported here for two reasons. First, the maximum resolution criterion is a special case of MA criterion. Results on the latter, which are the main focus of this chapter, imply those on the former. Second, more comprehensive results can be found in the coding-theoretic literature (e.g., Brouwer and Verhoeff, 1993; Brouwer, 1998). Recall from Section 2.8 that the resolution of a 2^{n-k} design is equivalent to the minimum distance of a binary $[n, k; 2]$ linear code and that the concept of minimum distance plays a central role in error-correcting codes. One difference is that for designed experiments, the number of factors n is not usually too large, while n can be quite large for codes. This underscores the need to find good codes (e.g., to maximize the minimum distance) over a wider range of parameters.

3.2 Minimum Aberration 2^{n-k} Designs with $k \leq 4$

When k is small, the defining contrast subgroup of a 2^{n-k} design has fewer elements, which makes it easier to search for MA designs. General explicit results are available for $k \leq 5$. Results for $1 \leq k \leq 4$ are given here. Throughout this section, we consider only those 2^{n-k} designs that have each of the letters $1, \ldots, n$ appearing in some word in the defining contrast subgroup. By Lemma 2.5.1, an MA design must satisfy this requirement.

The case $k = 1$ is straightforward. It is a half-fraction of the 2^n factorial. The MA 2^{n-1} design is also the maximum resolution design with defining relation $I = 12 \ldots n$.

The following lemma, due to Brownlee, Kelly, and Loraine (1948), is useful in exploring MA designs for $k = 2, 3, 4$. It gives some fundamental relationships for the A_i's and the wordlengths. In Lemma 3.2.1, $\sum_i iA_i$ is called the *first moment* of the 2^{n-k} design and is equal to the sum of lengths of the $2^k - 1$ words in the defining contrast subgroup.

Lemma 3.2.1. *For any 2^{n-k} design,*

(a)

$$\sum_{i=1}^{n} A_i = 2^k - 1, \tag{3.2.1}$$

(b)

$$\sum_{i=1}^{n} iA_i = n2^{k-1}, \tag{3.2.2}$$

(c) either all the words in the defining contrast subgroup have even lengths or 2^{k-1} of them have odd lengths.

Proof. (a) This is obvious from the fact that there are $2^k - 1$ words in the defining contrast subgroup, say $\mathcal{G}$, of the design.

(b) Clearly, $\sum_{i=1}^{n} iA_i = \sum_{i=1}^{n} \beta_i$, where β_i is the number of words in $\mathcal{G}$ containing the letter i. It is enough to show that $\beta_i = 2^{k-1}$ for every i. For any fixed i, let $\mathcal{G}_i (\subset \mathcal{G})$ be the set consisting of the β_i words as described above. As indicated in the beginning of this section, $\mathcal{G}_i$ is nonempty. Let B be any fixed word in $\mathcal{G}_i$, and $\bar{\mathcal{G}}_i$ be the complement of $\mathcal{G}_i$ in $\mathcal{G}$. Since $\mathcal{G}$ is a subgroup under multiplication,

$$\mathcal{G}_i = \{BG : G \in \bar{\mathcal{G}}_i\}. \tag{3.2.3}$$

Now, $\mathcal{G}$ has cardinality 2^k inclusive of the identity element. Hence (3.2.3) yields $\beta_i = 2^k - \beta_i$, i.e., $\beta_i = 2^{k-1}$ as desired.

(c) Let the set $\mathcal{G}_0 (\subset \mathcal{G})$ consist of words with odd lengths. If $\mathcal{G}_0$ is empty then (c) holds trivially. Otherwise, the same arguments as in (b) show that $\mathcal{G}_0$ has cardinality 2^{k-1}. □

For $k = 2$, Robillard (1968) gave the following method for constructing MA 2^{n-2} designs. Let $n - 2 = 3m + r$, where $0 \leq r < 3$. Define

$$\begin{aligned}
&(i) && \text{for } r = 0, B_1 = 12\ldots(2m)(n-1), B_2 = (m+1)(m+2)\ldots(3m)n;\\
&(ii) && \text{for } r = 1, B_1 = 12\ldots(2m+1)(n-1), B_2 = (m+1)(m+2)\ldots(3m+1)n;\\
&(iii) && \text{for } r = 2, B_1 = 12\ldots(2m+1)(n-1), B_2 = (m+1)(m+2)\ldots(3m+2)n.
\end{aligned} \tag{3.2.4}$$

Theorem 3.2.1. *The 2^{n-2} design d_0 with the defining relation $I = B_1 = B_2 = B_1B_2$, where B_1 and B_2 are given in (3.2.4), has the maximum resolution $\left[\frac{2n}{3}\right]$ and minimum aberration. Here $[x]$ denotes the integer part of x.*

Proof. The three words B_1, B_2, and B_1B_2 have lengths $\{2m+1, 2m+1, 2m+2\}$ for $r = 0$, $\{2m+2, 2m+2, 2m+2\}$ for $r = 1$, and $\{2m+2, 2m+3, 2m+3\}$ for $r = 2$. Hence d_0 has resolution $\left[\frac{2n}{3}\right]$. From (3.2.2), the sum of the three wordlengths for any 2^{n-2} design equals $2n$. Therefore the shortest wordlength (i.e., the resolution) has $\left[\frac{2n}{3}\right]$ as its upper bound, which proves that d_0 has maximum resolution. The MA property is obvious for $r = 1$, since the three words have the same length. It is evident also for $r = 0$, since by (3.2.2), no design can have only one word of length $2m + 1$ and two words of higher lengths. For $r = 2$, d_0 has only one word of the shortest length $2m + 2$ and two words of the next length $2m + 3$. Again (3.2.2) rules out the existence of designs with less aberration. □

Example 3.2.1. For $n = 5$, the rule in (3.2.4) becomes $B_1 = 124$ and $B_2 = 235$, which leads to the MA 2^{5-2} design with $I = 124 = 235 = 1345$. By mapping $1 \to 2$, $2 \to 1$, $3 \to 3$, $4 \to 4$, $5 \to 5$, it is clear that the defining contrast subgroup of this design reduces to that in (3.1.5), thus showing that the 2^{5-2} design in Example 3.1.1 or 2.4.1 has MA. □

The study of the cases $k = 3$ and 4 is facilitated by periodicity considerations implicit in Lemma 2.7.3. For ease in reference, this lemma is restated below for two-level factorials.

Lemma 3.2.2. *(a) Given any 2^{n-k} design d_1 with wordlength pattern W_1, there exists a $2^{(n+2^k-1)-k}$ design d_2 with wordlength pattern $W_2 = (lag\ (W_1, 2^{k-1}), 0')$, where $0'$ is a null row vector such that W_2 has $n + 2^k - 1$ elements altogether.*
(b) $R_2(n + 2^k - 1, k) \geq R_2(n, k) + 2^{k-1}$.

The next two theorems on MA 2^{n-3} and 2^{n-4} designs are due to Chen and Wu (1991). Considering 2^{n-3} designs first, let $n = 7m + r$, $0 \leq r < 7$. For $i = 1, \ldots, 7$, define

$$B_i = \begin{cases} (im - m + 1)(im - m + 2)\ldots(im)(7m + i), & \text{for } i \leq r, \\ (im - m + 1)(im - m + 2)\ldots(im), & \text{otherwise.} \end{cases} \tag{3.2.5}$$

The B_i's divide the n letters into 7 approximately equal blocks.

Theorem 3.2.2. *The 2^{n-3} design d_0 with the defining relation*

$$\begin{aligned} I &= B_7B_6B_4B_3 = B_7B_5B_4B_2 = B_6B_5B_4B_1 \\ &= B_6B_5B_3B_2 = B_7B_5B_3B_1 = B_7B_6B_2B_1 = B_4B_3B_2B_1, \end{aligned}$$

where the B_i's are given in (3.2.5), has minimum aberration and maximum resolution. Its resolution equals $\left[\frac{4n}{7}\right] - 1$ for $r = 2$ and $\left[\frac{4n}{7}\right]$ for $r \neq 2$.

Proof. For $n = 4$, d_0 has the wordlength pattern $W = (0, 6, 0, 1)$. Hence we consider only those 2^{4-3} designs that have $A_1 = 0$. By (3.2.1) and (3.2.2), for any such design,

$$\begin{aligned} A_2 + A_3 + A_4 &= 7, \\ 2A_2 + 3A_3 + 4A_4 &= 16, \end{aligned} \tag{3.2.6}$$

which together with Lemma 3.2.1(c) yield the unique solution $A_2 = 6$, $A_3 = 0$, $A_4 = 1$, thus proving the MA property of d_0.

For $n = 5$, d_0 has the wordlength pattern $W = (0, 2, 4, 1, 0)$. Hence we consider only those 2^{5-3} designs that have $A_1 = 0$ and $A_2 \leq 2$. As before,

$$\begin{aligned} A_2 + A_3 + A_4 + A_5 &= 7, \\ 2A_2 + 3A_3 + 4A_4 + 5A_5 &= 20, \end{aligned} \tag{3.2.7}$$

and $A_3 + A_5 = 0$ or 4 by Lemma 3.2.1(c). Thus the unique solution of (3.2.7) with $A_2 \leq 2$ is $A_2 = 2, A_3 = 4, A_4 = 1, A_5 = 0$, and the MA property of d_0 follows.

For $6 \leq n \leq 10$, the proofs are similar. For $n \geq 11$, the proofs are essentially the same as those for $4 \leq n \leq 10$ because from Lemma 3.2.2(a), they involve the same type of equations with a period of length 4 $(= 2^{3-1})$. For example, if $n = 4 + 7m$, then one word in the defining contrast subgroup of d_0 has length $4 + 4m$ and the rest have length $2 + 4m$ each. Hence considering only those designs satisfying $A_i = 0$ for $i \leq 1 + 4m$, (3.2.1) and (3.2.2) become

$$\begin{aligned} A_{2+4m} + A_{3+4m} + A_{4+4m} + \cdots &= 7, \\ (2+4m)A_{2+4m} + (3+4m)A_{3+4m} + (4+4m)A_{4+4m} + \cdots &= 4(4+7m). \end{aligned} \tag{3.2.8}$$

Subtracting $(2 + 4m)$ times the first equation from the second equation yields

$$A_{3+4m} + 2A_{4+4m} + 3A_{5+4m} + \cdots = 2,$$

which forces $A_{5+4m} = A_{6+4m} = \cdots = 0$ and reduces the equations in (3.2.8) to those in (3.2.6). Therefore the proof for $n = 4 + 7m$ reduces to that for $n = 4$.

The MA property of d_0 implies that it has maximum resolution. One can easily check that the resolution of d_0 is as claimed. □

To construct 2^{n-4} MA designs, we use the same idea as with $k = 3$. Let $n = 15m + r$, $0 \le r < 15$. Divide the n letters into 15 approximately equal blocks given by $B_1, \dots, B_{15}$, where

$$B_i = \begin{cases} (im - m + 1)(im - m + 2)\dots(im)(15m + i) & \text{for } i \le r, \\ (im - m + 1)(im - m + 2)\dots(im) & \text{otherwise.} \end{cases}$$

When $r \ne 5$, let

$$\begin{aligned} \mathcal{B} \;=\; \{&B_{15}B_{14}B_{12}B_9B_8B_7B_6B_1,\;\; B_{15}B_{13}B_{11}B_9B_8B_7B_5B_2, \\ &B_{15}B_{14}B_{11}B_{10}B_8B_6B_5B_3,\;\; B_{15}B_{13}B_{12}B_{10}B_7B_6B_5B_4\}. \end{aligned} \tag{3.2.9}$$

When $r = 5$, switch B_{15} and B_5 in (3.2.9).

Theorem 3.2.3. *The 2^{n-4} design d_0, whose defining contrast subgroup is generated by the four words in (3.2.9), has minimum aberration and maximum resolution. Its resolution equals $\left[\frac{8n}{15}\right]$ for $r \ne 2, 3, 4, 6, 10$ and $\left[\frac{8n}{15}\right] - 1$ otherwise.*

Proof. For $n = 5$, d_0 has wordlength pattern $(0, 10, 0, 5, 0)$. By Lemma 3.2.1, there are only two possibilities, $(0, 6, 8, 1, 0)$ and $(0, 7, 7, 0, 1)$, for the wordlength pattern of any design with less aberration. Consider the first one. Let l be a letter shared by a shortest word and the longest word. Following the proof of Lemma 3.2.1(b), there are $8\,(= 2^{4-1})$ words containing l. If these words are deleted, then by (3.2.1) and Lemma 3.2.1(c), the wordlength pattern of the resulting $2^{n'-3}$ design ($n' \le 4$) must be $(0, 3, 4, 0)$, which violates (3.2.2). The second possibility is also eliminated in the same manner. Thus the MA property of d_0 follows.

Similarly, for $n = 6$, the only possibilities for the wordlength pattern of any design having less aberration than d_0 are

(a) $W = (0, 2, 8, 5, 0, 0)$,
(b) $W = (0, 3, 7, 4, 1, 0)$.

In case (a), suppose letter l occurs in a shortest word. Delete all the words containing l. The remaining words will define a $2^{n'-3}$ design with $n' \le 5$. According to (3.2.2), its first moment is at most $5 \times 4 = 20$. On the other hand, it has $A_2 \le 1$, $A_4 \le 5$, and $A_i = 0$ for $i \ne 2, 3, 4$. Hence by (3.2.1) and Lemma 3.2.1(c), $A_3 = 4$. Using (3.2.1) again, $(A_2, A_4) = (1, 2)$ or $(0, 3)$. Therefore $2A_2 + 3A_3 + 4A_4 = 22$ or 24, each of which is greater than 20 and thus violates (3.2.2). In case (b), notice that there is a letter that occurs in at least two of the three shortest words. This is because $A_6 = 0$ in (b) forces the product of these three words to have at most length 5. Deleting the words containing this letter would lead to the same violation as in case (a). Thus, no other design can have less aberration than d_0.

By Lemma 3.2.1, for $n = 7$, the only possibilities for the wordlength pattern of any design having less aberration than d_0 are

$$W = (0, 0, 6, 7, 2, 0, 0), \quad W = (0, 0, 7, 6, 1, 1, 0).$$

To prove that there is no design with either of these wordlength patterns, we assume the contrary. Suppose d_1 is a 2^{7-4} design with $W = (0, 0, 6, 7, 2, 0, 0)$. Its defining contrast subgroup contains words of odd lengths and hence cannot be generated exclusively by even-length words. By adding a new letter to all odd-length generators, we have a 2^{8-4} design d_2. Every word in the defining contrast subgroup of d_2 has even length since all its generators have even lengths. Because the corresponding words in d_2 are at least as long as the words in d_1, from (3.2.1) and (3.2.2), the wordlength pattern of d_2 has to be

(c) $W = (0, 0, 0, 13, 0, 2, 0, 0)$.

Now we prove the impossibility of (c). Clearly, there exists a letter l appearing in one of the two longest words but not in the other. By deleting all the words containing l, the remaining words define a $2^{n'-3}$ design with $n' \leq 7$. From (3.2.1), the wordlength pattern of this design must be $(0, 0, 0, 6, 0, 1, 0)$ with its first moment being 30, which violates (3.2.2). Therefore such a design does not exist. The proof of the impossibility of $W = (0, 0, 7, 6, 1, 1, 0)$ is similar. This proves the MA property of d_0 for $n = 7$.

The proofs are similar for $8 \leq n \leq 19$. Using the same periodicity argument as with $k = 3$ (see (3.2.8)), one can show that the proofs for $n \geq 20$ are the same as those for $5 \leq n \leq 19$.

The MA property of d_0 implies that it has maximum resolution. It is also easy to check that the resolution of d_0 is as claimed. □

Chen and Wu (1991) tabulated the wordlength patterns of the MA design d_0 in Theorem 3.2.3 for $5 \leq n \leq 19$.

The combinatorial arguments used for $k = 3$ and 4 are not completely adequate in the exploration of MA 2^{n-5} designs. They are effective in substantially reducing the number of candidate designs, but the remaining work has to be done by computer search. Chen (1992) used a combination of these two techniques to find MA 2^{n-5} designs in closed form.

Since a word of length two in the defining contrast subgroup would entail the aliasing of main effects, we will not consider designs of resolution two. *In the rest of this chapter, only designs of resolution three or higher are considered.*

3.3 Minimum Aberration Designs via Complementary Designs

The approach taken in the previous section is useful for finding MA 2^{n-k} designs with $k \leq 4$. As k becomes larger, the number of candidate designs grows exponentially and therefore a different approach is required. To motivate the new approach, consider the search for MA 2^{n-k} designs with 16 runs.

Because $n-k=4$, we can rewrite them as $2^{n-(n-4)}$ designs. Based on the results for 2^{n-k} designs with $k \le 4$, the cases $5 \le n \le 8$ are solved. For $n \ge 9$, the search for MA designs is greatly simplified if one recalls from Theorem 2.7.1 that each $2^{n-(n-4)}$ design is equivalent to a selection of n points out of the 15 points of the finite projective geometry $PG(3,2)$. Since the number of remaining points, namely $15-n$, is much smaller than n for $n \ge 9$, it is easier to conduct design search over them. Clearly, these remaining points form the complement of the set of n points for the 2^{n-k} design. This *complementary set* is often referred to as the *complementary design* of the original one. The terms complementary set and complementary design are used interchangeably in what follows.

Let $m = n-k$. Using the compact notation introduced in Section 2.7, $PG(m-1,2)$ can be represented by

$$H_m = \{u_1 \dots u_r : 1 \le u_1 < \cdots < u_r \le m, 1 \le r \le m\}, \qquad (3.3.1)$$

where the element $u_1 \dots u_r$ corresponds to the point of $PG(m-1,2)$ that has 1 in the u_1th,...,u_rth positions, and 0 elsewhere. For example,

$$H_3 = \{1, 2, 12, 3, 13, 23, 123\}.$$

(In coding theory H_m can be viewed as a Hamming code.) Addition of any two points of $PG(m-1,2)$ is equivalent to multiplication of the corresponding elements of H_m with the convention of dropping squared symbols. Indeed, $\{I\} \cup H_m$ forms a group under such multiplication where I is an identity element. Any g elements of H_m are *independent* if none of them equals the product of some or all of the rest. Thus the elements 13, 23, and 123 of H_3 are independent, while 12, 13, and 23 are not so because $23 = (12)(13)$.

Since only designs of resolution three or higher are being considered, the following result is evident from Theorem 2.7.1.

Theorem 3.3.1. *(a) Let $m = n-k$. Then a 2^{n-k} design is equivalent to a set $T(\subset H_m)$ of cardinality n such that there are m independent elements in T.*

(b) Furthermore, with $T = \{c_1, \dots, c_n\}$, the defining contrast subgroup of the design contains the word $i_1 \dots i_g$ if and only if $c_{i_1} \dots c_{i_g} = I$.

Throughout this section, a 2^{n-k} design is denoted simply by the corresponding set T as envisaged above. Its wordlength pattern is also denoted by $W(T) = (A_1(T), \dots, A_n(T))$. The complementary set $\overline{T} = H_m - T$, of cardinality $f = 2^m - 1 - n$, represents the complementary design of T. If $f = 0$, then T is called a *saturated* design. Obviously, in this case there is only one choice of T, namely, $T = H_m$. Hence to avoid trivialities, let $f \ge 1$. Then $\overline{T}$ is nonempty and Theorem 3.3.1(b), with T there replaced by $\overline{T}$, dictates the defining contrast subgroup and hence the wordlength pattern of $\overline{T}$. The latter is denoted by $W(\overline{T}) = (A_1(\overline{T}), \dots, A_f(\overline{T}))$. Clearly $A_i(T) = A_i(\overline{T}) = 0$ for $i = 1, 2$.

Note that $\overline{T}$ may not always represent a design in a strict sense. For instance, it may so happen that $f \leq m$ or that $\overline{T}$ does not contain m independent elements (cf.Theorem 3.3.1(a)). Notwithstanding these possibilities, the terminology "complementary design" for $\overline{T}$ is a common usage. At any rate, the wordlength pattern of $\overline{T}$ is always well-defined via Theorem 3.3.1(b). For example, if $f \leq m$ and the elements of $\overline{T}$ are independent, then $A_i(\overline{T}) = 0$ for every i.

Example 3.3.1. Let $n = 4$, $k = 1$, $m = n - k = 3$. Consider two 2^{4-1} designs represented by the sets $T_1 = \{1, 2, 3, 12\}$ and $T_2 = \{1, 2, 3, 123\}$ of H_3. The only product involving the elements of T_1 that equals I is (1)(2)(12). Hence by Theorem 3.3.1(b), the only word in the defining contrast subgroup of T_1 has length 3, giving $W(T_1) = (0, 0, 1, 0)$. Similarly, $W(T_2) = (0, 0, 0, 1)$, so that T_2 has higher resolution and hence less aberration than T_1. Consider now the complements $\overline{T}_1 = H_3 - T_1 = \{13, 23, 123\}$ and $\overline{T}_2 = H_3 - T_2 = \{12, 13, 23\}$. The three elements of $\overline{T}_1$ are independent, while those of $\overline{T}_2$ satisfy the relation $(12)(13)(23) = I$. Hence, using Theorem 3.3.1(b) again, $W(\overline{T}_1) = (0, 0, 0)$ and $W(\overline{T}_2) = (0, 0, 1)$. Intuitively, this example suggests that when the elements of the complementary set $\overline{T}$ are more "dependent," those of T should be less "dependent" and thus T may have less aberration. A rigorous version of this intuitive observation will be developed later in the section. □

Following Tang and Wu (1996), we now describe how the use of complementary designs can simplify the study of MA designs, especially when the complementary sets are not too large. The method first employs isomorphism to reduce the design search. An *isomorphism* ϕ is a one-to-one mapping from H_m to H_m such that $\phi(xy) = \phi(x)\phi(y)$ for every $x \neq y$. Two sets T_1 and T_2 of H_m are said to be *isomorphic* if there is an isomorphism ϕ that maps T_1 onto T_2. Two 2^{n-k} designs are called *isomorphic* if the corresponding sets are isomorphic. Note that this definition of design isomorphism is equivalent to the one given in Section 3.1. Isomorphic designs are treated as the same design. For example, they are equivalent according to the MA criterion. The following crucial result is evident from the definition of isomorphism. It shows that while looking for an MA design, one can reduce the class of complementary designs by isomorphism and then restrict the search to the reduced class.

Lemma 3.3.1. *Let $\overline{T}_i$ be the complement of T_i in H_m $(i = 1, 2)$. If $\overline{T}_1$ and $\overline{T}_2$ are isomorphic, then T_1 and T_2 are also isomorphic.*

Theorem 3.3.2. *Any two 2^{n-k} designs with $n = 2^{n-k} - 2$ are isomorphic. The same holds for $n = 2^{n-k} - 3$.*

Proof. Any 2^{n-k} design with $n = 2^m - 2$, where $m = n - k$, is given by a set T of $2^m - 2$ elements of H_m. Its complement $\overline{T}$ consists of only one element of H_m. Since any two singleton sets are isomorphic, sets like T must be isomorphic to one another according to Lemma 3.3.1. Similarly, for $n = 2^m - 3$, the complementary set $\overline{T}$ consists of two elements. Since all such sets of two elements are isomorphic, all designs with $n = 2^m - 3$ are also isomorphic. □

The next simplest case is $n = 2^{n-k} - 4$, i.e., the complementary set $\overline{T}$ has three elements. There are two nonisomorphic choices of $\overline{T}$: (i) $\{a, b, c\}$, where a, b, c are three independent elements, and (ii) $\{a, b, ab\}$, where ab is the product of a and b. From Example 3.3.1, one would expect the design whose complementary design has the form (ii) to be superior. The following identities enable us to verify whether an intuitive guess like this is correct. They relate the values of A_3, A_4, A_5 for a 2^{n-k} design T to those of its complementary design $\overline{T}$:

$$\begin{aligned} A_3(T) &= \text{constant} - A_3(\overline{T}), \\ A_4(T) &= \text{constant} + A_3(\overline{T}) + A_4(\overline{T}), \\ A_5(T) &= \text{constant} - (2^{n-k-1} - n)A_3(\overline{T}) - A_4(\overline{T}) - A_5(\overline{T}). \end{aligned} \tag{3.3.2}$$

The constants in (3.3.2) may depend on n and k but not on the particular choice of T. These identities are special cases of a more general result to be given in Chapter 4 (see Corollary 4.3.2). Hence their proofs are omitted.

The following rules for identifying MA designs emerge from (3.3.2). Recall that $f = 2^{n-k} - 1 - n$ is the cardinality of $\overline{T}$.

Rule 1. A design T^* has minimum aberration if

(i) $A_3(\overline{T}^*) = \max A_3(\overline{T})$ over all $\overline{T}$ of cardinality f, and
(ii) $\overline{T}^*$ is the unique set (up to isomorphism) satisfying (i).

Rule 2. A design T^* has minimum aberration if

(i) $A_3(\overline{T}^*) = \max A_3(\overline{T})$ over all $\overline{T}$ of cardinality f,
(ii) $A_4(\overline{T}^*) = \min\{A_4(\overline{T}) : A_3(\overline{T}) = A_3(\overline{T}^*)\}$, and
(iii) $\overline{T}^*$ is the unique set (up to isomorphism) satisfying (ii).

Rule 3. A design T^* has minimum aberration if

(i) $A_3(\overline{T}^*) = \max A_3(\overline{T})$ over all $\overline{T}$ of cardinality f,
(ii) $A_4(\overline{T}^*) = \min\{A_4(\overline{T}) : A_3(\overline{T}) = A_3(\overline{T}^*)\}$,
(iii) $A_5(\overline{T}^*) = \max\{A_5(\overline{T}) : A_3(\overline{T}) = A_3(\overline{T}^*)$ and $A_4(\overline{T}) = A_4(\overline{T}^*)\}$, and
(iv) $\overline{T}^*$ is the unique set (up to isomorphism) satisfying (iii).

The following example serves to illustrate the use of Rule 1.

Example 3.3.2. Let $f = 2^w - 1$. Clearly, $A_3(\overline{T})$ is maximized if and only if $\{I\} \cup \overline{T}$ is a subgroup of $\{I\} \cup H_m$. Since this subgroup is unique (up to isomorphism), one obtains a sequence of MA 2^{n-k} designs with $n = 2^m - 2^w$, where $w = 1, \ldots, m-1$ and $m = n - k$. In particular, if $w = 2$, then $\overline{T}$ of the form $\{a, b, ab\}$ yields the MA design. Since $w = 2$ amounts to $f = 3$, i.e., $n = 2^{n-k} - 4$, this shows the validity of the intuitive guess made above (3.3.2) for such n. □

Use of Rule 1 and design isomorphism yields MA 2^{n-k} designs for general n and k with $f = 2^{n-k} - 1 - n = 1, 2, \ldots, 9$. The corresponding sets $\overline{T}$ are summarized in Table 3.1. The results for $f = 10$ and 11 can be obtained via Rule 2 and design isomorphism. The explanatory notes (i)–(ix) below indicate the proofs for most of the cases. Details for all cases can be found in Tang and Wu (1996). In Table 3.1 as well as the explanatory notes, a, b, c, d are independent elements of H_m.

Table 3.1 The sets $\overline{T}$ for MA designs with $f = 1$ to 9

f	1	2	3	4	5	6	7	8	9
$\overline{T}$	a	b	ab	c	ac	bc	abc	d	ad

(Note: For each f, the optimum $\overline{T}$ consists of the first f elements of the second row.)

(i) $f = 1, 2$ are covered by Theorem 3.3.2; $f = 3$ is covered by Example 3.3.2.

(ii) $f = 4$. There are three nonisomorphic choices of $\overline{T}$:

$$\overline{T}_1 = \{a, b, c, ab\}, \quad \overline{T}_2 = \{a, b, c, abc\}, \quad \overline{T}_3 = \{a, b, c, d\}.$$

Since $\overline{T}_1$ is the only one with positive A_3, it yields the MA design.

(iii) $f = 5$. First consider $\overline{T}_1 = \{a, b, c, ab, ac\}$. Because $\overline{T}_1$ can be viewed as a 2^{5-2} design and from Theorem 3.3.2 all 2^{5-2} designs are isomorphic, other choices of $\overline{T}$ must have four or five independent elements, say, a, b, c, d, e. Among them, the following four are nonisomorphic:

$$\overline{T}_2 = \{a, b, c, d, ab\}, \qquad \overline{T}_3 = \{a, b, c, d, abc\},$$
$$\overline{T}_4 = \{a, b, c, d, abcd\}, \qquad \overline{T}_5 = \{a, b, c, d, e\}.$$

Since $A_3(\overline{T}_1) = 2 > A_3(\overline{T}_2) = 1 > A_3(\overline{T}_i) = 0$ for $i = 3, 4, 5$, $\overline{T}_1$ gives the MA design.

(iv) The proof for $f = 6$ is similar to that for f =5.

(v) $f = 7$. The most "dependent" $\overline{T}$ is of the form

$$\overline{T}_1 = \{a, b, c, ab, ac, bc, abc\}.$$

From Example 3.3.2, $\overline{T}_1$ gives the MA design with $A_3(\overline{T}_1) = 7$. For use in (vi) below, we give the A_3 values associated with the other nonisomorphic $\overline{T}$. They must have at least four independent elements and are given by

$$\overline{T}_2 = \{a, b, c, d, x_1, x_2, x_3\}, \qquad \overline{T}_3 = \{a, b, c, d, e, x_1, x_2\},$$
$$\overline{T}_4 = \{a, b, c, d, e, g, x\}, \qquad \overline{T}_5 = \{a, b, c, d, e, g, h\},$$

where a, b, c, d, e, g, h denote independent elements and x_1, x_2, x_3, x denote products of the independent elements in the corresponding sets. It is easy to show that the maximum A_3 for $\overline{T}_2$ is 4, which is attained uniquely (up to isomorphism) by choosing $x_1 = ab$, $x_2 = ac$, $x_3 = bc$. It is also readily seen that $\max A_3(\overline{T}_3) = 2$, $\max A_3(\overline{T}_4) = 1$, and $A_3(\overline{T}_5) = 0$.

(vi) $f = 8$. It will be shown that the set

$$\overline{T}^* = \{a, b, c, d, ab, ac, bc, abc\}$$

uniquely attains the maximum A_3 value 7, and thus gives the MA design. To this end, we write any set of eight elements as $\overline{T} = Q \cup \{x_8\}$, where $Q = \{x_1, \ldots, x_7\}$. As noted in (v) above, $A_3(Q) \le 7$, with equality if and only if Q has the form $\{a, b, c, ab, ac, bc, abc\}$, in which case $\overline{T}$ becomes the same as $\overline{T}^*$.
From (v) again, the next largest value of $A_3(Q)$ is 4 and is uniquely attained by the set $Q_0 = \{a, b, c, d, ab, ac, bc\}$. Additional relations involving three elements of $\overline{T}$ and hence contributing to $A_3(\overline{T})$ must involve x_8 and take the form

$$x_i x_j = x_8, \quad 1 \le i \ne j \le 7. \tag{3.3.3}$$

With $Q = Q_0$ (and $x_8 \ne abc$ so as to avoid reversal to $\overline{T}^*$), there is at most one pair of x_i and x_j to satisfy (3.3.3). Therefore $\max A_3(Q_0 \cup \{x_8\}) = 5$. For any other choice of Q, $A_3(Q) \le 3$ as seen in (v). Noting that (3.3.3) has at most three solutions, it follows that the maximum A_3 for $Q \cup \{x_8\}$ with $A_3(Q) \le 3$ is 6, thus completing the proof.

(vii) The proof for $f = 9$ is similar to but more elaborate than that for $f = 8$.

(viii) $f = 11$. It will be shown that Rule 2 yields the MA design. Note that $m = n - k \ge 5$, for otherwise $n = 2^{n-k} - 1 - f \le 4$, which is impossible since $k \ge 1$. Hence H_4 is embedded in H_m. By complete enumeration it can be shown that $\overline{T}$ must be a subset of H_4, up to isomorphism; otherwise, $A_3(\overline{T})$ is not maximized. Since the size of $\overline{T}$ is larger than that of $\overline{\overline{T}} = H_4 - \overline{T}$, it is easier to work with the smaller set $\overline{\overline{T}}$. Since $\overline{\overline{T}}$ has only four elements, according to the case of $f = 4$, there are only three nonisomorphic choices for it, namely,

$$\overline{\overline{T}}_1 = \{x, y, z, u\}, \;\; \overline{\overline{T}}_2 = \{x, y, z, xyz\}, \;\; \overline{\overline{T}}_3 = \{x, y, z, xy\},$$

where x, y, z, u are independent elements. Let $\overline{T}_i = H_4 - \overline{\overline{T}}_i$ $(i = 1, 2, 3)$. Since $A_3(\overline{\overline{T}}_1) = A_4(\overline{\overline{T}}_1) = 0$, $A_3(\overline{\overline{T}}_2) = 0$, $A_4(\overline{\overline{T}}_2) = 1$, $A_3(\overline{\overline{T}}_3) =$

1, $A_4(\overline{\overline{T}}_3) = 0$, applying (3.3.2) to the pairs $\left(\overline{T}_i, \overline{\overline{T}}_i\right)$ that are complementary with reference to H_4, one gets

$$A_3(\overline{T}_1) = A_3(\overline{T}_2) = A_3(\overline{T}_3) + 1, \quad A_4(\overline{T}_1) = A_4(\overline{T}_2) - 1.$$

Hence, by Rule 2, $\overline{T}_1$ yields the MA design. Note that $\overline{T}_1$ is isomorphic to

$$\{a, b, ab, c, ac, bc, abc, d, ad, bd, cd\}. \tag{3.3.4}$$

(ix) f =10. Arguments similar to those for f =11 show that

$$\overline{T}^* = \{a, b, ab, c, ac, bc, d, ad, bd, cd\} \tag{3.3.5}$$

satisfies Rule 2 and yields the MA design.

For each of the cases $1 \leq f \leq 11$ considered above, it is satisfying to note that the set T, representing the MA design, contains m independent elements, as it should in view of Theorem 3.3.1(a). For illustration, consider $f = 10$ and suppose $a, b, c, d, e_5, \ldots, e_m$ are m independent elements of H_m. Then by (3.3.5), the set representing the MA design includes the m elements $abc, abd, acd, bcd, e_5, \ldots, e_m$, which are independent.

The above results can be applied to complete the search for MA 16-run $2^{n-(n-4)}$ designs discussed in the beginning of the section. Among the remaining cases, i.e., $9 \leq n \leq 15$, the case n =15 is trivial since there is only one design, namely, $T = H_4$. The cases $n = 14$ and 13 correspond to $f = 1$ and 2 respectively. By Theorem 3.3.2 or (i) above, all designs are isomorphic in these two cases. Finally, the cases $n = 12, 11, 10$, and 9 correspond to $f = 3, 4, 5$, and 6 and are settled respectively by (i)–(iv) above. For instance, with $n = 10$, one may take $a = 123$, $b = 24$, $c = 34$ in (iii) to get $\overline{T}_1 = \{123, 24, 34, 134, 124\}$, which shows that the 2^{10-6} design $T_1 = \{1, 2, 12, 3, 13, 23, 4, 14, 234, 1234\}$ has MA. Since the result for each f, $1 \leq f \leq 11$, does not depend on the value of n, the same rules can be used to find MA 32-run $2^{n-(n-5)}$ designs with $20 \leq n \leq 30$ and MA 64-run $2^{n-(n-6)}$ designs with $52 \leq n \leq 62$, etc.

Chen and Hedayat (1996) defined a weak MA design as one that minimizes $A_3(T)$, or equivalently, maximizes $A_3(\overline{T})$ in view of (3.3.2). Obviously, an MA design has weak MA as well. These authors gave a necessary and sufficient condition for the maximization of $A_3(\overline{T})$. This result will be presented in Chapter 5 (See Lemma 5.3.1) using a projective geometric language. The findings concerning the maximization of $A_3(\overline{T})$ in the case-by-case discussion above are in agreement with this result and hence demonstrate, from first principles, its validity for smaller f.

3.4 Clear Effects and the MaxC2 Criterion

From the applications point of view, the properties of a design with regard to the estimability of factorial effects, especially the lower-order ones, are of direct interest. As discussed in Section 2.4, these properties are influenced by the aliasing pattern. The resolution of a design indeed sheds some light on the aliasing pattern. Thus in a resolution four design, no main effect is aliased with another main effect or any 2fi, but some 2fi's are aliased with other 2fi's. Similarly, in a resolution five design, no main effect is aliased with another main effect or any 2fi or any 3fi, and no 2fi is aliased with any main effect or any other 2fi. However, note that the resolution of a design gives only partial information about the nature of aliasing. For example, the mere fact that a design has resolution four does not determine the exact number of 2fi's that are aliased with other 2fi's.

These considerations led Wu and Chen (1992) to propose the following classification of effects. A main effect or 2fi is called *clear* if it is not aliased with any other main effect or 2fi. By Theorem 2.4.2, a clear main effect or 2fi is estimable under the assumption that the interactions involving three or more factors are absent. A main effect or 2fi is called *strongly clear* if it is not aliased with any other main effect or 2fi or any 3fi. As before, a strongly clear main effect or 2fi is estimable under the assumption that the interactions involving four or more factors are absent. Observe that the latter assumption is less stringent than the former.

For any 2^{n-k} design, let $C1$ be the *number of clear main effects* and $C2$ be the *number of clear 2fi's*. The following important and useful rules are now evident from the facts noted in the beginning of the section.

Rules for 2^{n-k} designs of resolution four or five:
(i) In any resolution four design, the main effects are clear but the 2fi's are not all clear.
(ii) In any resolution five design, the main effects are strongly clear and the 2fi's are clear.
(iii) Among the resolution four designs with given n and k, those with the largest $C2$ are the best.

Rule (iii) was proposed in Wu and Hamada (2000, Section 4.2), who justified it as follows. In a resolution four design, all main effects are clear but some 2fi's are aliased with other 2fi's. One can therefore use $C2$, the number of clear 2fi's, to compare and rank-order resolution four designs. It is called the *MaxC2 criterion*. Resolution four designs that achieve the maximum $C2$ value are called *MaxC2 designs*. A natural question is whether the MaxC2 criterion is consistent with the MA criterion. In many cases they are, but the following example shows that the two criteria can be in conflict.

Example 3.4.1. Following (3.2.9) and Theorem 3.2.3, consider the MA 2^{9-4} design d_0 whose defining contrast subgroup is generated by the four independent words 16789, 25789, 3568, and 4567. The design has resolution four with $A_4 = 6$. The six words of length four in its defining contrast subgroup are 1238, 1247, 1256, 3478, 3568, and 4567. All pairs formed out of the letters $1, \ldots, 8$ appear in one or more of these six words, while the letter 9 does not appear in any one of them. Hence the only clear 2fi's are the ones that involve the letter 9, so that $C2 = 8$ for d_0.

Consider now the rival 2^{9-4} design d_1, whose defining contrast subgroup is generated by the four independent words 1236, 1247, 1345, and 23489. This design also has resolution four but its defining contrast subgroup contains seven words of length four, namely, 1236, 1247, 1345, 1567, 2357, 2456, and 3467. Each pair formed out of the letters $1, \ldots, 7$ appears in one or more of these seven words, while the letters 8 and 9 do not appear in any of them. Hence $A_4 = 7$ and $C2 = 15$ for d_1. Consequently, d_0 has less aberration than d_1 but d_1 has a much larger $C2$ value than d_0. Thus the MA design d_0 is not a MaxC2 design. Using an exhaustive computer search or a proof in Wu and Wu (2002), it can be shown that d_1 is a MaxC2 design and that it is the second best in terms of the MA criterion. □

From the catalogue of designs at the end of the chapter, there are many designs with higher $C2$ values than the MA designs. Before we proceed to this discussion, we need to state some fundamental properties about clear effects. For 2^{n-k} designs with a fixed run size, write $m = n-k$ and denote the run size by 2^m. According to Theorem 2.7.4, a 2^{n-k} design can have resolution four or higher if and only if $n \leq 2^{m-1}$, i.e., the number of factors does not exceed half the run size. Therefore *any* 2^{n-k} *design with* $n > 2^{m-1}$ *has resolution three.* In addition, such designs do not have any clear 2fi, as the following theorem reveals.

Theorem 3.4.1. *No* 2^{n-k} *design with* $n > 2^{m-1}$*, where* $m = n - k$*, has any clear 2fi.*

Proof. Consider a 2^{n-k} design, which, by Theorem 3.3.1(a), is equivalent to a set $T = \{c_1, \ldots, c_n\}$ of H_m. Suppose there is a clear 2fi that involves, say, the first two factors. Then no word of length three or four contains both the letters 1 and 2. Hence by Theorem 3.3.1(b), $c_1c_2 \in \overline{T}$ and $c_1c_2c_i \in \overline{T}$ $(3 \leq i \leq n)$, where $\overline{T} = H_m - T$. Since the elements c_1c_2 and $c_1c_2c_i$ $(3 \leq i \leq n)$ are distinct, the cardinality of $\overline{T}$ is at least $n-1$. On the other hand, by definition, $\overline{T}$ has cardinality $2^m - 1 - n$. Hence $2^m - 1 - n \geq n - 1$, i.e., $n \leq 2^{m-1}$, a contradiction. □

By Theorem 2.7.4, there exist designs of resolution four or higher for $n \leq 2^{m-1}$. The next result says that when the number of factors n is between half run size and quarter run size plus two, no resolution four design has any clear 2fi.

Theorem 3.4.2. *No* 2^{n-k} *design of resolution four has any clear 2fi if* $2^{m-2} + 2 \leq n \leq 2^{m-1}$*, where* $m = n - k$.

Proof. Using the same notation as in the proof of Theorem 3.4.1, consider the design represented by T. Let T have resolution four and suppose there is a clear 2fi involving, say, the first two factors. As before, then $c_1c_2c_i \in \overline{T}$

$(3 \leq i \leq n)$. Also $c_1c_i \in \overline{T}$ $(2 \leq i \leq n)$ and $c_2c_i \in \overline{T}$ $(3 \leq i \leq n)$ since resolution four rules out words of length three. It is easy to see that the $3n-5$ elements just mentioned are distinct. Hence the cardinality of $\overline{T}$ is at least $3n-5$, i.e., $2^m - 1 - n \geq 3n - 5$, which is equivalent to $n \leq 2^{m-2} + 1$, a contradiction. □

The last two theorems are due to Chen and Hedayat (1998).

Since all 2fi's in resolution five designs are clear, we focus on resolution four designs. For $n \leq 2^{m-2} + 1$, are there resolution four designs with clear 2fi's? For a fixed run size 2^m, $m = n-k$, let $n_{\max}(m)$ denote the maximum possible n for which there is a 2^{n-k} design of resolution five or higher. For $n_{\max}(m) < n \leq 2^{m-2} + 1$, there are resolution four designs. Chen and Hedayat (1998) showed by a simple construction that for each n in this range, there exists a resolution four design with some clear 2fi's. Therefore Rule (iii) above is particularly relevant in this case. The result just stated and those in Theorems 2.7.4, 3.4.1, and 3.4.2 can be verified with the designs given in the appendix of this chapter. Take, for example, the case of 32-run designs listed in Table 3A.3. For $n > 16$, all designs have resolution three and none has any clear 2fi; for $10 \leq n \leq 16$, no resolution four design has any clear 2fi (but resolution three designs can have clear 2fi's). For $6 < n \leq 9$, there are resolution four designs with clear 2fi's. The 2^{9-4} designs in Example 3.4.1 fall in this range. Here $n_{\max}(5) = 6$ and there is a 2^{6-1} design of resolution six.

For $m = 5$ and 6, i.e., with 32 and 64 runs, MA designs have been obtained in the literature over the range $n_{\max}(m) < n \leq 2^{m-2} + 1$. MA designs with 128 runs are also known for $12 \leq n \leq 14$ (Chen, 1992, 1998). See the catalogue at the end of the chapter or the tables in Wu and Hamada (2000, Chapter 4). Wu and Wu (2002) showed that *these MA designs, which all have resolution four, are also MaxC2 designs, except for* the 2^{9-4}, 2^{13-7}, 2^{14-8}, 2^{15-9}, 2^{16-10} and 2^{17-11} designs. They also showed that the designs 9-4.2, 13-7.2, 16-10.6, 17-11.6 and 15-8.3 in Tables 3A.3 – 3A.5 are MaxC2 designs though not MA designs. The proofs in Wu and Wu (2002) will not be given here. They are quite complicated and vary from case to case. While the proofs for MA designs generally use a few techniques, there is no unified approach for obtaining MaxC2 designs. This difference can be explained by the fact that $C2$ is a complicated function of the defining contrast subgroup of the design while the MA criterion is based on the lengths of words in the subgroup.

A resolution three design with some clear 2fi's may be preferred to a resolution four design with no clear 2fi. Many such examples can be found in the catalogue of designs at the end of the chapter. This is also anticipated from Theorem 3.4.2. For illustration, the MA 2^{6-2} design d_0 with $I = 1235 = 2346 = 1456$ has resolution four. All its six main effects are clear but no 2fi is clear. Consider now the resolution three 2^{6-2} design d_1 with $I = 125 = 1346 = 23456$. It has three clear main effects 3, 4, 6 and six clear 2fi's 23, 24, 26, 35, 45, 56. Which one is better? If only main effects are of interest, d_0 is preferred. On the other hand, d_1 has altogether nine clear effects, while d_0 has only six. If only three factors (out of six) and some 2fi's involving them are believed to be important *a priori*, d_1 may be preferred. This and other examples demonstrate that use of the resolution criterion *alone* may give a very rough measure of the estimability properties of designs. A more quantitative measure of such properties is provided by the values of $C1$ and $C2$.

Since the MA criterion and the MaxC2 criterion can be in conflict, we suggest that MA be used as the primary criterion, supplemented by the use of $C1$ and $C2$ values. If a MaxC2 design is not an MA design, it may be preferred when the difference in the $C2$ values is large. Further discussion on these criteria will be given at the end of Section 5.3, where another criterion called estimation capacity is considered.

3.5 Description and Use of the Two-Level Design Tables

A catalogue of 2^{n-k} designs with 16, 32, 64, and 128 runs is given in the appendix of this chapter. The designs with 16, 32, and 64 runs are taken from Chen, Sun, and Wu (1993) based on an algorithm and a search program developed by the same authors. The 8-run designs are not included, since the case is quite straightforward. The listing of 16-run designs in Table 3A.2 is complete, i.e., it contains all the nonisomorphic designs. For 32 or 64 runs, a complete listing is too long to be included. To save space, at most ten designs are given in Tables 3A.3 and 3A.4 for each combination of $n - k$ and n. Selection of designs for inclusion in these two tables is based primarily on the MA criterion and supplemented by the MaxC2 criterion. The 128-run designs for $12 \leq n \leq 40$ in Table 3A.5 are adapted mainly from Block and Mee (2005), who also gave designs for $n > 40$. A few entries in the table are taken from other sources. Because of the large number of nonisomorphic designs in this case and the difficulties in verifying the MA property of a design, following these authors, only one or a few designs are listed in Table 3A.5 for each n. These are resolution four designs and, except for the one numbered 15-8.3, have been claimed by Block and Mee (2005) to have the smallest A_4 value. These designs tend to perform well also under the MA, MaxC2, or other criteria not discussed in the book.

This catalogue of designs can be useful for design search based on criteria other than the MA. For example, in studying more complex situations such as fractional factorial designs with blocking or split-plot structure or with the distinction of control and noise factors (to be considered in Chapters 7, 8, and 9 respectively), the optimality criteria are more elaborate than the MA criterion but can include it as a major component. The catalogue can serve as the basis for searching optimal designs in such situations.

Recall from Theorem 3.3.1 that a 2^{n-k} design is equivalent to a set of n elements of H_m, where $m = n - k$ and H_m is defined in (3.3.1), such that the set contains m independent elements. Up to isomorphism, the independent elements can always be taken as $1, 2, \ldots, m$. Thus a 2^{n-k} design can be represented by the elements $1, 2, \ldots, m$, together with k additional elements of H_m. This representation is followed in the catalogue for tabulating designs. Furthermore, instead of using the notation $1, 2, 12, 3, 13, 23, \ldots$ for the elements of H_m, to save space, we denote them by the corresponding serial numbers $1, 2, 3, 4, 5, 6, \ldots$. This numbering scheme is shown in Table 3A.1. For instance, the independent elements $1, 2, 3, 4$ of H_4 are numbered $1, 2, 4, 8$ according to this scheme. Consequently, in listing any 16-run design (i.e., $m = n - k = 4$) in the catalogue, it is implied that the elements numbered $1, 2, 4, 8$ are included but only the serial numbers of the additional k elements are listed under "Additional Elements". Similar considerations apply to 32-, 64-, and 128-run designs. To save space, a notation like $19 - 22$ is used to denote elements numbered 19 to 22.

For clarity, the ith 2^{n-k} design in the catalogue is denoted by $n - k.i$. The wordlength pattern W and $C2$, the number of clear 2fi's, appear in the last two columns of the design tables. Again to save space, for 32-, 64-, and 128-run designs, at most five components of W are shown. For any given $n-k$ and n, the first design $n-k.1$ in Tables 3A.2 through 3A.4 is the MA design, and this is in agreement with the tabulated wordlength patterns. The designs 12-5.1, 13-6.1, and 14-7.1 in Table 3A.5 are also known to have MA (Chen, 1992, 1998). Ordering of the remaining designs in Tables 3A.3 and 3A.4 is not strictly according to the MA criterion. Designs with more aberration but much higher $C2$ values may be placed ahead of others with less aberration. For example, designs 14-8.4 and 14-8.5 have more aberration than designs 14-8.6 to 14-8.10. Use of the design tables is illustrated in the following example.

Example 3.5.1. Consider the 32-run MA design 9-4.1 in Table 3A.3. It is given by the elements of H_5 that are numbered 1, 2, 4, 8, 16, 7, 11, 19, and 29. Table 3A.1 identifies these elements and shows that the design is represented by the set $\{1, 2, 3, 4, 5, 123, 124, 125, 1345\}$ of H_5. The nine factors can be associated with the elements of this set in the order stated. Then the following aliasing relations are immediate: $6 = 123$, $7 = 124$, $8 = 125$, $9 = 1345$. It can be verified that this design is isomorphic to the MA design d_0 in Example 3.4.1. As demonstrated there, the latter design has $A_4 = 6$ and $C2 = 8$, which are in agreement with the entries under W and $C2$ for design 9-4.1 in the table. □

Exercises

3.1 Derive the wordlength patterns for the MA designs in Theorem 3.2.2 for $6 \leq n \leq 10$.

3.2 Prove Theorem 3.2.2 for $6 \leq n \leq 10$ by following the proofs for $n = 4$ and 5.

3.3 Derive the wordlength patterns for the MA designs in Theorem 3.2.3 for $6 \leq n \leq 19$.

3.4 (a) Prove that the definition of design isomorphism in Section 3.1 is equivalent to the definition of design isomorphism in Section 3.3.
(b) Show that if two 2^{n-k}designs are isomorphic, then their letter pattern matrices, as defined in Exercise 2.12, must be identical up to permutation of rows.

3.5 Prove Lemma 3.3.1.

3.6 (a) Prove the first identity in (3.3.2) $A_3(T) = \text{constant} - A_3(\overline{T})$ using definitions and a simple combinatorial argument.
(b) Prove the second identity in (3.3.2) $A_4(T) = \text{constant} + A_3(\overline{T}) + A_4(\overline{T})$ using a more elaborate combinatorial argument.

3.7 Give an example to demonstrate that part (ii) of Rule 1 in Section 3.3 is indispensable.

3.8 Fill in the derivation for the MA design with $f = 6$ by following the lines of proof for $f = 5$.

3.9 Fill in the derivation for the MA design with $f = 9$ by following the lines of proof for $f = 8$.

3.10 Fill in the derivation for the MA design with $f = 10$ by following the lines of proof for $f = 11$.

3.11 Consider two 2^{8-3} designs d_1 and d_2 whose defining contrast subgroups are generated by the independent words 126, 137, 23458 and 126, 347, 1358, respectively.
(a) Show that both designs have the same wordlength pattern $W = (0, 0, 2, 1, 2, 2, 0, 0)$.
(b) Find the letter pattern matrices of d_1 and d_2. Verify that they are not identical up to permutation of rows. Conclude that d_1 and d_2 are not isomorphic even though they have the same wordlength pattern. Hence infer that the letter pattern matrix provides a more explicit description of a design than the wordlength pattern.

Appendix 3A. Catalogue of 2^{n-k} Designs with 16, 32, 64, and 128 Runs

Table 3A.1 Numbering of the elements of H_m for 16-, 32-, 64-, and 128-run designs

(The table gives the serial numbers of the elements of H_7; the first 63 entries describe the serial numbers of the elements of H_6, the first 31 entries describe the serial numbers of the elements of H_5, and the first 15 entries describe the serial numbers of the elements of H_4. Independent elements are numbered **1, 2, 4, 8, 16, 32, 64** in boldface.)

Number	**1**	**2**	3	**4**	5	6	7	**8**	9
Element	1	2	12	3	13	23	123	4	14
Number	10	11	12	13	14	15	**16**	17	18
Element	24	124	34	134	234	1234	5	15	25
Number	19	20	21	22	23	24	25	26	27
Element	125	35	135	235	1235	45	145	245	1245
Number	28	29	30	31	**32**	33	34	35	36
Element	345	1345	2345	12345	6	16	26	126	36
Number	37	38	39	40	41	42	43	44	45
Element	136	236	1236	46	146	246	1246	346	1346
Number	46	47	48	49	50	51	52	53	54
Element	2346	12346	56	156	256	1256	356	1356	2356
Number	55	56	57	58	59	60	61	62	63
Element	12356	456	1456	2456	12456	3456	13456	23456	123456
Number	**64**	65	66	67	68	69	70	71	72
Element	7	17	27	127	37	137	237	1237	47
Number	73	74	75	76	77	78	79	80	81
Element	147	247	1247	347	1347	2347	12347	57	157
Number	82	83	84	85	86	87	88	89	90
Element	257	1257	357	1357	2357	12357	457	1457	2457
Number	91	92	93	94	95	96	97	98	99
Element	12457	3457	13457	23457	123457	67	167	267	1267
Number	100	101	102	103	104	105	106	107	108
Element	367	1367	2367	12367	467	1467	2467	12467	3467
Number	109	110	111	112	113	114	115	116	117
Element	13467	23467	123467	567	1567	2567	12567	3567	13567
Number	118	119	120	121	122	123	124	125	126
Element	23567	123567	4567	14567	24567	124567	34567	134567	234567
Number	127								
Element	1234567								

Table 3A.2 Complete catalogue of 16-run designs

(Each design is represented by 1, 2, 4, 8 and the numbers specified under "Additional Elements". $W = (A_3, A_4, \ldots)$ is the wordlength pattern of the design. $C2$ is the number of clear 2fi's. Designs for $n = 13, 14, 15$ are unique up to isomorphism and hence omitted.)

Design	Additional Elements	W	$C2$
5-1.1	15	0 0 1	10
5-1.2	7	0 1 0	4
5-1.3	3	1 0 0	7
6-2.1	7 11	0 3 0 0	0
6-2.2	3 13	1 1 1 0	6
6-2.3	3 12	2 0 0 1	9
6-2.4	3 5	2 1 0 0	5
7-3.1	7 11 13	0 7 0 0 0	0
7-3.2	3 5 14	2 3 2 0 0	2
7-3.3	3 5 10	3 2 1 1 0	4
7-3.4	3 5 9	3 3 0 0 1	0
7-3.5	3 5 6	4 3 0 0 0	6
8-4.1	7 11 13 14	0 14 0 0 0 1	0
8-4.2	3 5 9 14	3 7 4 0 1 0	1
8-4.3	3 5 10 12	4 5 4 2 0 0	0
8-4.4	3 5 6 15	4 6 4 0 0 1	0
8-4.5	3 5 6 9	5 5 2 2 1 0	2
8-4.6	3 5 6 7	7 7 0 0 1 0	7
9-5.1	3 5 9 14 15	4 14 8 0 4 1 0	0
9-5.2	3 5 10 12 15	6 9 9 6 0 0 1	0
9-5.3	3 5 6 9 14	6 10 8 4 2 1 0	0
9-5.4	3 5 6 9 10	7 9 6 6 3 0 0	0
9-5.5	3 5 6 7 9	8 10 4 4 4 1 0	0
10-6.1	3 5 6 9 14 15	8 18 16 8 8 5 0 0	0
10-6.2	3 5 6 9 10 13	9 16 15 12 7 3 1 0	0
10-6.3	3 5 6 9 10 12	10 15 12 15 10 0 0 1	0
10-6.4	3 5 6 7 9 10	10 16 12 12 10 3 0 0	0
11-7.1	3 5 6 9 10 13 14	12 26 28 24 20 13 4 0 0	0
11-7.2	3 5 6 7 9 10 12	13 25 25 27 23 10 3 1 0	0
11-7.3	3 5 6 7 9 10 11	13 26 24 24 26 13 0 0 1	0
12-8.1	3 5 6 9 10 13 14 15	16 39 48 48 48 39 16 0 0 1	0
12-8.2	3 5 6 7 9 10 11 12	17 38 44 52 54 33 12 4 1 0	0

Table 3A.3 Selected 32-run designs for $n = 6$ to 28

(Each design is represented by 1, 2, 4, 8, 16 and the numbers specified under "Additional Elements". $W = (A_3, \ldots, A_7)$ when $n < 17$ and $W = (A_3, \ldots, A_6)$ when $n \geq 17$. $C2$ is the number of clear 2fi's. Designs for $n = 29$, 30, and 31 are unique up to isomorphism and hence omitted.)

Design	Additional Elements	W	$C2$
6-1.1	31	0 0 0 1 0	15
7-2.1	7 27	0 1 2 0 0	15
7-2.2	7 25	0 2 0 1 0	9
7-2.3	7 11	0 3 0 0 0	6
7-2.4	3 29	1 0 1 1 0	18
7-2.5	3 28	1 1 0 0 1	12
7-2.6	3 13	1 1 1 0 0	12
7-2.7	3 12	2 0 0 1 0	15
7-2.8	3 5	2 1 0 0 0	11
8-3.1	7 11 29	0 3 4 0 0	13
8-3.2	7 11 21	0 5 0 2 0	4
8-3.3	7 11 19	0 6 0 0 0	0
8-3.4	7 11 13	0 7 0 0 0	7
8-3.5	3 13 22	1 2 3 1 0	13
8-3.6	3 5 30	2 1 2 2 0	18
8-3.7	3 13 21	1 3 2 0 1	10
8-3.8	3 12 21	2 1 2 2 0	16
8-3.9	3 5 26	2 2 1 1 1	12
8-3.10	3 5 25	2 2 2 0 0	12
9-4.1	7 11 19 29	0 6 8 0 0	8
9-4.2	7 11 13 30	0 7 7 0 0	15
9-4.3	7 11 21 25	0 9 0 6 0	0
9-4.4	7 11 13 19	0 10 0 4 0	2
9-4.5	7 11 13 14	0 14 0 0 0	8
9-4.6	3 13 21 26	1 5 6 2 1	9
9-4.7	3 13 21 25	1 7 4 0 3	12
9-4.8	3 12 21 26	2 3 6 4 0	12
9-4.9	3 5 9 30	3 3 4 4 1	15
9-4.10	3 5 10 28	3 3 4 4 1	13

Table 3A.3 (continued)

Design	Additional Elements	W	$C2$
10-5.1	7 11 19 29 30	0 10 16 0 0	0
10-5.2	7 11 21 25 31	0 15 0 15 0	0
10-5.3	7 11 13 19 21	0 16 0 12 0	0
10-5.4	7 11 13 14 19	0 18 0 8 0	0
10-5.5	3 13 21 25 28	1 14 7 0 7	14
10-5.6	3 13 21 25 30	1 10 11 4 3	8
10-5.7	3 12 21 26 31	2 7 12 7 2	6
10-5.8	3 5 14 22 25	2 8 12 4 2	4
10-5.9	3 5 14 23 26	2 9 9 6 4	5
10-5.10	3 5 9 14 31	3 8 11 4 1	12
11-6.1	7 11 13 19 21 25	0 25 0 27 0	0
11-6.2	7 11 13 14 19 21	0 26 0 24 0	0
11-6.3	3 5 14 22 25 31	2 14 22 8 6	0
11-6.4	3 5 14 22 26 29	2 16 16 12 10	6
11-6.5	3 5 14 22 26 28	2 18 14 8 14	6
11-6.6	3 5 10 23 27 28	3 13 19 11 9	3
11-6.7	3 5 9 22 26 29	3 15 13 15 13	4
11-6.8	3 5 9 22 26 28	3 16 12 12 16	4
11-6.9	3 5 9 14 22 26	3 16 13 12 13	4
11-6.10	3 5 9 14 18 29	4 12 18 12 8	5
12-7.1	7 11 13 14 19 21 25	0 38 0 52 0	0
12-7.2	7 11 13 14 19 21 22	0 39 0 48 0	0
12-7.3	3 5 9 14 22 26 29	3 25 23 27 25	5
12-7.4	3 5 9 14 22 26 28	3 26 22 24 28	5
12-7.5	3 5 10 12 22 27 29	4 20 32 22 20	0
12-7.6	3 5 10 12 22 25 31	4 22 28 20 28	0
12-7.7	3 5 6 15 23 25 30	4 23 28 16 28	0
12-7.8	3 5 9 14 17 22 26	4 25 19 27 31	3
12-7.9	3 5 9 14 15 22 26	4 26 20 24 28	3
12-7.10	3 5 9 14 18 20 31	5 19 29 25 23	2

Table 3A.3 (continued)

Design	Additional Elements	W	$C2$
13-8.1	7 11 13 14 19 21 22 25	0 55 0 96 0	0
13-8.2	3 5 9 14 17 22 26 28	4 38 32 52 56	4
13-8.3	3 5 9 14 15 22 26 29	4 38 33 52 52	4
13-8.4	3 5 9 14 15 22 26 28	4 39 32 48 56	4
13-8.5	3 5 9 14 15 17 22 26	5 38 28 52 62	2
13-8.6	3 5 10 12 15 22 27 29	6 28 51 42 42	0
13-8.7	3 5 9 14 18 20 24 31	6 29 46 46 50	0
13-8.8	3 5 9 15 18 20 24 30	6 30 44 44 56	0
13-8.9	3 5 9 15 18 20 24 31	7 28 42 50 56	2
13-8.10	3 5 6 9 14 17 26 29	7 29 42 46 56	2
14-9.1	7 11 13 14 19 21 22 25 26	0 77 0 168 0	0
14-9.2	3 5 9 14 15 17 22 26 28	5 55 45 96 106	3
14-9.3	3 5 9 14 15 17 22 23 26	6 55 40 96 116	1
14-9.4	3 5 9 15 18 20 24 30 31	8 42 64 85 112	0
14-9.5	3 5 9 14 15 18 20 24 31	8 42 65 84 108	0
14-9.6	3 5 6 9 14 17 22 26 29	8 43 64 80 112	0
14-9.7	3 5 9 14 15 18 20 24 30	8 43 64 80 112	0
14-9.8	3 5 6 9 14 15 23 26 29	8 45 64 72 112	0
14-9.9	3 5 6 9 14 17 22 26 27	9 42 60 84 118	2
14-9.10	3 5 6 9 14 15 17 26 29	9 43 61 80 114	2
15-10.1	7 11 13 14 19 21 22 25 26 28	0 105 0 280 0	0
15-10.2	3 5 9 14 15 17 22 23 26 28	6 77 62 168 188	2
15-10.3	3 5 9 14 15 17 22 23 26 27	7 77 56 168 203	0
15-10.4	3 5 6 9 14 17 22 26 27 28	10 60 90 141 212	0
15-10.5	3 5 6 9 14 15 17 22 26 29	10 61 90 136 212	0
15-10.6	3 5 6 9 14 15 17 22 26 27	11 60 85 141 222	2
15-10.7	3 5 9 14 18 20 23 24 27 29	12 49 108 144 176	0
15-10.8	3 5 6 9 14 18 23 24 29 31	12 51 102 144 192	0
15-10.9	3 5 9 14 15 18 20 23 24 30	12 51 102 144 192	0
15-10.10	3 5 6 9 14 15 17 22 23 26	12 61 80 136 232	2

Table 3A.3 (continued)

Design	Additional Elements	W	$C2$
16-11.1	7 11 13 14 19 21 22 25 26 28 31	0 140 0 448 0	0
16-11.2	3 5 9 14 15 17 22 23 26 27 28	7 105 84 280 315	1
16-11.3	3 5 6 9 14 15 17 22 26 27 28	12 83 124 230 376	0
16-11.4	3 5 6 9 14 15 17 22 23 26 29	12 84 124 224 376	0
16-11.5	3 5 6 9 14 15 17 22 23 26 27	13 83 118 230 391	2
16-11.6	3 5 9 14 18 20 23 24 27 29 31	15 65 156 232 315	0
16-11.7	3 5 6 9 10 14 17 22 27 28 29	15 70 141 231 358	0
16-11.8	3 5 6 9 10 14 17 22 23 26 29	15 71 140 226 363	0
16-11.9	3 5 6 9 10 14 15 17 22 26 29	15 73 140 216 363	0
16-11.10	3 5 6 9 10 14 17 22 26 29 31	16 65 148 236 336	0
17-12.1	3 5 9 14 15 17 22 23 26 27 28 29	8 140 112 448	0
17-12.2	3 5 6 9 14 15 17 22 23 26 27 28	14 112 168 364	0
17-12.3	3 5 6 9 10 14 17 22 23 26 27 28	18 95 192 354	0
17-12.4	3 5 6 9 10 14 15 17 22 27 28 29	18 95 193 354	0
17-12.5	3 5 6 9 10 14 15 17 22 23 26 29	18 96 192 348	0
18-13.1	3 5 6 9 14 15 17 22 23 26 27 28 29	16 148 224 560	0
18-13.2	3 5 6 9 10 14 15 17 22 23 26 27 28	21 126 259 532	0
18-13.3	3 5 6 7 9 10 11 17 18 19 28 29 30	22 126 252 532	0
18-13.4	3 5 6 9 14 15 18 21 23 24 27 28 31	24 108 288 552	0
18-13.5	3 5 6 9 10 14 17 22 23 24 27 28 29	24 113 272 547	0
19-14.1	3 5 6 9 10 14 15 17 22 23 26 27 28 29	24 164 344 784	0
19-14.2	3 5 6 7 9 10 11 17 18 19 28 29 30 31	25 164 336 784	0
19-14.3	3 5 6 9 10 14 15 17 18 22 23 26 27 28	28 147 364 791	0
19-14.4	3 5 6 9 10 13 14 15 17 22 23 26 27 28	28 148 364 784	0
19-14.5	3 5 6 9 10 13 14 17 22 23 24 26 29 31	30 136 378 816	0
20-15.1	3 5 6 9 10 14 15 17 18 22 23 26 – 29	32 188 480 1128	0
20-15.2	3 5 6 9 10 13 14 15 17 22 23 26 – 29	32 189 480 1120	0
20-15.3	3 5 6 7 9 – 12 17 18 19 28 – 31	33 188 472 1128	0
20-15.4	3 5 6 9 10 14 15 17 18 22 23 26 27 28 31	35 175 491 1155	0
20-15.5	3 5 6 9 10 13 14 15 17 18 22 23 26 27 28	35 176 490 1148	0
21-16.1	3 5 6 9 10 14 15 17 18 22 23 26 – 29 31	40 220 641 1608	0
21-16.2	3 5 6 9 10 13 14 15 17 18 22 23 26 – 29	40 221 640 1600	0
21-16.3	3 5 6 7 9 – 12 17 – 20 28 – 31	41 220 632 1608	0
21-16.4	3 5 6 9 10 13 14 17 19 22 23 24 26 28 29 31	42 210 651 1638	0
21-16.5	3 5 6 9 10 13 14 15 17 18 21 – 25 26 29	42 213 644 1624	0

Table 3A.3 (continued)

Design	Additional Elements	W	$C2$
22-17.1	3 5 6 9 10 13 – 15 17 18 21 – 23 25 26 29 30	48 263 832 2224	0
22-17.2	3 5 6 9 10 13 – 15 17 18 21 – 23 25 – 28	49 259 833 2240	0
22-17.3	3 5 6 7 9 – 12 17 – 20 25 28 – 31	49 261 825 2240	0
22-17.4	3 5 6 7 9 – 12 17 – 20 24 28 29 30 31	50 260 816 2249	0
22-17.5	3 5 6 7 9 – 13 17 – 20 28 – 31	50 261 816 2240	0
23-18.1	3 5 6 9 10 13 14 15 17 18 21 22 23 25 – 29	56 315 1064 3024	0
23-18.2	3 5 6 7 9 – 13 17 – 20 26 28 – 31	58 311 1050 3056	0
23-18.3	3 5 6 7 9 – 13 17 18 19 20 21 26 27 28 30	59 308 1047 3073	0
23-18.4	3 5 6 7 9 – 13 17 – 20 22 28 – 31	59 310 1041 3065	0
23-18.5	3 5 6 7 9 – 13 17 – 21 26 – 29	59 311 1040 3056	0
24-19.1	3 5 6 9 10 13 – 15 17 18 21 – 23 25 – 30	64 378 1344 4032	0
24-19.2	3 5 6 7 9 – 13 17 – 21 26 – 30	67 371 1324 4088	0
24-19.3	3 5 6 7 9 – 13 17 18 20 21 22 24 26 27 30 31	68 369 1316 4106	0
24-19.4	3 5 6 7 9 – 14 17 – 20 27 – 31	68 370 1316 4096	0
24-19.5	3 5 6 7 9 – 13 17 – 20 22 24 27 – 30	69 366 1311 4129	0
25-20.1	3 5 6 7 9 – 13 17 – 21 26 – 31	76 442 1656 5376	0
25-20.2	3 5 6 7 9 – 13 17 – 20 22 24 27 – 31	78 437 1641 5422	0
25-20.3	3 5 6 7 9 – 14 17 – 21 26 – 30	78 438 1640 5412	0
25-20.4	3 5 6 7 9 – 14 17 – 22 25 – 28	79 436 1632 5430	0
25-20.5	3 5 6 7 9 – 14 17 – 22 25 26 28 31	79 437 1630 5422	0
26-21.1	3 5 6 7 9 – 14 17 – 21 26 – 31	88 518 2032 7032	0
26-21.2	3 5 6 7 9 – 14 17 – 22 25 – 29	89 516 2023 7052	0
26-21.3	3 5 6 7 9 – 14 17 – 22 24 – 26 28 31	90 515 2012 7063	0
26-21.4	3 5 6 7 9 – 15 17 – 22 24 – 26 28	90 515 2013 7062	0
26-21.5	3 5 6 7 9 – 15 17 – 26	90 516 2012 7052	0
27-22.1	3 5 6 7 9 – 14 17 – 22 25 – 30	100 606 2484 9064	0
27-22.2	3 5 6 7 9 – 15 17 – 26 28	101 605 2473 9075	0
27-22.3	3 5 6 7 9 – 15 17 – 27	101 606 2472 9064	0
28-23.1	3 5 6 7 9 – 14 17 – 22 25 – 31	112 707 3024 11536	0
28-23.2	3 5 6 7 9 – 15 17 – 28	113 706 3012 11548	0

Table 3A.4 Selected 64-run designs for $n = 7$ to 32

(Each design is represented by 1, 2, 4, 8, 16, 32 and the numbers specified under "Additional Elements". Each design has $A_3 = 0$. $W = (A_4, \ldots, A_7)$ when $n < 18$ and $W = (A_4, A_5, A_6)$ when $n \geq 18$. $C2$ is the number of clear 2fi's.)

Design	Additional Elements	W	$C2$
7-1.1	63	0 0 0 1	21
8-2.1	15 51	0 2 1 0	28
9-3.1	7 27 45	1 4 2 0	30
9-3.2	7 25 43	2 3 1 1	24
9-3.3	7 27 43	2 4 0 0	24
9-3.4	7 11 61	3 0 4 0	21
9-3.5	7 25 42	3 0 4 0	18
9-3.6	7 11 53	3 2 0 2	21
9-3.7	7 11 51	3 3 0 0	21
9-3.8	7 11 29	3 4 0 0	21
9-3.9	7 11 49	4 0 2 0	15
9-3.10	7 11 21	5 0 2 0	12
10-4.1	7 27 43 53	2 8 4 0	33
10-4.2	7 25 42 53	3 6 4 2	27
10-4.3	7 11 29 51	3 7 4 0	30
10-4.4	7 11 29 46	3 8 3 0	30
10-4.5	7 11 29 49	4 6 2 2	24
10-4.6	7 11 29 45	4 8 0 0	24
10-4.7	7 25 42 52	5 0 10 0	15
10-4.8	7 11 21 57	5 4 2 4	21
10-4.9	7 11 21 45	5 5 2 2	21
10-4.10	7 11 13 62	7 0 7 0	24

Table 3A.4 (continued)

Design	Additional Elements	W	$C2$
11-5.1	7 11 29 45 51	4 14 8 0	34
11-5.2	7 25 42 52 63	5 10 10 5	25
11-5.3	7 11 29 46 49	5 12 7 4	28
11-5.4	7 11 21 46 56	6 10 8 4	25
11-5.5	7 11 29 45 49	6 12 4 4	25
11-5.6	7 11 19 29 62	6 12 8 0	27
11-5.7	7 11 21 38 57	7 8 7 8	22
11-5.8	7 11 21 41 51	7 9 6 6	22
11-5.9	7 11 13 30 49	8 10 4 4	28
11-5.10	7 11 13 30 46	8 14 0 0	28
12-6.1	7 11 29 45 51 62	6 24 16 0	36
12-6.2	7 11 21 46 54 56	8 20 14 8	27
12-6.3	7 11 21 41 51 63	9 18 13 12	24
12-6.4	7 11 21 41 54 56	10 15 16 11	21
12-6.5	7 11 13 30 46 49	10 20 8 8	30
12-6.6	7 11 19 37 57 63	10 16 12 16	20
12-6.7	7 11 19 29 37 59	10 16 16 8	20
12-6.8	7 11 19 29 37 57	10 18 10 12	20
12-6.9	7 11 21 25 38 58	11 14 15 12	21
12-6.10	7 11 13 19 46 49	12 14 12 12	23

Table 3A.4 (continued)

Design	Additional Elements	W	$C2$
13-7.1	7 11 21 25 38 58 60	14 28 24 24	20
13-7.2	7 11 13 30 46 49 63	14 33 16 16	36
13-7.3	7 11 19 29 37 59 62	15 24 32 16	12
13-7.4	7 11 19 29 37 41 60	15 27 21 27	16
13-7.5	7 11 13 19 46 49 63	15 28 20 24	22
13-7.6	7 11 19 30 37 41 52	16 22 30 22	17
13-7.7	7 11 13 19 37 57 63	16 24 22 32	18
13-7.8	7 11 19 37 41 60 63	16 26 18 30	12
13-7.9	7 11 19 29 37 41 47	18 20 28 24	20
13-7.10	7 11 13 19 35 49 63	18 21 24 24	21
14-8.1	7 11 19 30 37 41 49 60	22 40 36 56	8
14-8.2	7 11 19 29 30 37 41 47	22 40 41 48	16
14-8.3	7 11 13 19 21 25 35 60	29 26 46 50	19
14-8.4	7 11 13 14 19 21 25 54	38 17 52 44	25
14-8.5	7 11 13 14 19 21 22 57	39 16 48 48	25
14-8.6	7 11 19 29 30 37 41 49	22 41 36 52	8
14-8.7	7 11 19 30 37 41 52 56	23 32 56 40	13
14-8.8	7 11 13 19 21 41 54 63	23 38 38 54	16
14-8.9	7 11 13 19 21 46 54 56	23 40 36 48	16
14-8.10	7 11 19 29 37 41 47 49	24 31 54 42	16

Table 3A.4 (continued)

Design	Additional Elements	W	$C2$
15-9.1	7 11 19 30 37 41 49 60 63	30 60 60 105	0
15-9.2	7 11 19 29 30 37 41 49 60	30 61 60 100	0
15-9.3	7 11 19 29 37 41 47 49 55	33 44 96 72	14
15-9.4	7 11 13 14 19 21 35 41 63	39 38 80 88	19
15-9.5	7 11 13 14 19 21 22 25 58	55 22 96 72	27
15-9.6	7 11 13 19 21 35 37 57 58	33 54 60 108	6
15-9.7	7 11 13 19 21 25 35 60 63	34 52 65 100	12
15-9.8	7 11 13 19 21 35 41 49 63	35 42 88 80	14
15-9.9	7 11 13 19 21 25 35 37 63	37 40 84 84	17
15-9.10	7 11 13 14 19 21 25 35 60	43 34 80 88	18
16-10.1	7 11 13 19 21 35 37 57 58 60	43 81 96 189	0
16-10.2	7 11 19 29 37 41 47 49 55 59	45 60 160 120	15
16-10.3	7 11 13 19 21 25 35 37 41 63	49 56 144 136	15
16-10.4	7 11 13 14 19 21 25 35 37 63	53 52 136 144	18
16-10.5	7 11 13 14 19 21 22 25 35 60	61 44 136 144	17
16-10.6	7 11 13 14 19 21 22 25 26 60	77 28 168 112	29
16-10.7	7 11 13 14 19 21 35 37 57 58	47 72 98 192	4
16-10.8	7 11 13 14 19 21 25 35 60 63	49 68 108 176	8
16-10.9	7 11 13 14 19 21 22 35 57 60	51 64 102 192	4
16-10.10	7 11 13 14 19 21 22 35 37 57	57 48 120 160	15
17-11.1	7 11 13 14 19 21 35 37 57 58 60	59 108 150 324	0
17-11.2	7 11 19 29 37 41 47 49 55 59 62	60 80 256 192	16
17-11.3	7 11 13 19 21 25 35 37 41 49 63	65 75 232 216	16
17-11.4	7 11 13 14 19 21 25 35 37 41 63	68 72 224 224	16
17-11.5	7 11 13 14 19 21 22 25 35 37 63	73 67 216 232	19
17-11.6	7 11 13 14 19 21 22 25 26 28 63	105 35 280 168	31
17-11.7	7 11 13 14 19 21 22 35 37 38 57	76 64 192 256	16
17-11.8	7 11 13 19 21 25 35 37 42 61 62	79 0 394 0	0
17-11.9	7 11 13 14 19 21 35 41 49 50 61	80 0 388 0	0
17-11.10	7 11 13 14 19 21 22 25 26 35 60	84 56 224 224	16

Table 3A.4 (continued)

Design	Additional Elements	W	$C2$
18-12.1	7 11 13 14 19 21 22 35 37 57 58 60	78 144 228	0
18-12.2	7 11 13 14 19 21 22 35 37 38 57 58	84 128 240	0
18-12.3	7 11 13 14 19 21 22 25 26 35 60 63	92 112 280	0
18-12.4	7 11 13 19 21 25 35 37 42 49 61 62	102 0 588	0
18-12.5	7 11 13 14 19 21 25 35 44 49 52 62	103 0 582	0
19-13.1	7 11 13 14 19 21 22 35 37 38 57 58 60	100 192 336	0
19-13.2	7 11 13 14 19 21 22 35 41 44 49 55 56	131 0 847	0
19-13.3	7 11 13 14 19 21 25 35 37 42 49 50 61	131 0 847	0
19-13.4	7 11 13 14 19 21 22 35 41 42 49 52 56	132 0 840	0
19-13.5	7 11 13 14 19 21 25 35 37 41 49 50 61	132 0 840	0
20-14.1	7 11 13 14 19 21 22 35 37 38 57 58 60 63	125 256 480	0
20-14.2	7 11 13 14 19 21 22 35 41 42 49 52 56 62	164 0 1208	0
20-14.3	7 11 13 14 19 21 22 35 41 42 49 52 56 61	165 0 1200	0
20-14.4	7 11 13 14 19 21 22 35 41 42 49 50 61 62	165 0 1200	0
20-14.5	7 11 13 14 19 21 25 35 37 42 49 52 59 61	165 0 1200	0
21-15.1	7 11 13 14 19 21 22 25 35 41 42 49 52 56 62	204 0 1680	0
21-15.2	7 11 13 14 19 21 22 25 35 37 41 42 49 50 61	205 0 1672	0
21-15.3	7 11 13 14 19 21 22 25 35 37 41 42 49 52 56	205 0 1672	0
21-15.4	7 11 13 14 19 21 22 25 35 41 42 49 50 61 62	205 0 1672	0
21-15.5	7 11 13 14 19 21 22 25 26 37 41 44 49 52 59	206 0 1666	0
22-16.1	7 11 13 14 19 21 22 25 35 37 41 42 49 52 56 62	250 0 2304	0
22-16.2	7 11 13 14 19 21 22 25 26 35 37 41 44 49 52 59	251 0 2296	0
22-16.3	7 11 13 14 19 21 22 25 26 37 41 44 49 52 59 62	251 0 2296	0
22-16.4	7 11 13 14 19 21 22 25 26 35 37 38 41 44 49 56	252 0 2288	0
22-16.5	7 11 13 14 19 21 22 25 26 35 37 38 41 44 49 55	252 0 2289	0
23-17.1	7 11 13 14 19 21 22 25 26 35 37 41 44 49 52 56 62	304 0 3105	0
23-17.2	7 11 13 14 19 21 22 25 26 35 37 38 41 44 49 55 56	304 0 3105	0
23-17.3	7 11 13 14 19 21 22 25 26 35 37 38 41 42 49 52 56	305 0 3096	0
23-17.4	7 11 13 14 19 21 22 25 26 28 35 37 38 41 42 49 52	306 0 3089	0
23-17.5	7 11 13 14 19 21 22 25 26 28 35 37 38 41 42 49 50	307 0 3080	0

Table 3A.4 (continued)

Design	Additional Elements	W	$C2$
24-18.1	7 11 13 14 19 21 22 25 26 35 37 38 41 42 49 52 56 62	365 0 4138	0
24-18.2	7 11 13 14 19 21 22 25 26 35 37 38 41 42 49 52 56 61	366 0 4128	0
24-18.3	7 11 13 14 19 21 22 25 26 28 35 37 38 41 42 49 52 56	366 0 4129	0
24-18.4	7 11 13 14 19 21 22 25 26 28 35 37 38 41 42 44 49 50	367 0 4120	0
24-18.5	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 49 52	369 0 4106	0
25-19.1	7 11 13 14 19 21 22 25 26 28 35 37 38 41 42 49 52 56 62	435 0 5440	0
25-19.2	7 11 13 14 19 21 22 25 26 28 35 37 38 41 42 44 49 50 52	436 0 5430	0
25-19.3	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 49 52 56	437 0 5422	0
25-19.4	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 49 50	438 0 5412	0
25-19.5	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49	442 0 5376	0
26-20.1	7 11 13 14 19 21 22 25 26 28 35 37 38 41 42 44 49 50 52 56	515 0 7062	0
26-20.2	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 49 52 56 62	515 0 7063	0
26-20.3	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 49 50 52	516 0 7052	0
26-20.4	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50	518 0 7032	0
27-21.1	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 49 50 52 56	605 0 9075	0
27-21.2	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50 52	606 0 9064	0
28-22.1	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50 52 56	706 0 11548	0
28-22.2	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50 52 55	707 0 11536	0
29-23.1	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50 52 55 56	819 0 14560	0
30-24.1	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50 52 55 56 59	945 0 18200	0
31-25.1	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50 52 55 56 59 61	1085 0 22568	0
32-26.1	7 11 13 14 19 21 22 25 26 28 31 35 37 38 41 42 44 47 49 50 52 55 56 59 61 62	1240 0 27776	0

Table 3A.5 Selected 128-run designs for $n = 12$ to 40

(Each design is represented by 1, 2, 4, 8, 16, 32, 64 and the numbers specified under "Additional Elements". Each design has $A_3 = 0$. $W = (A_4, A_5, A_6)$ and $C2$ is the number of clear 2fi's.)

Design	Additional Elements	W	$C2$
12-5.1	25 39 95 106 116	1 8 12	60
12-5.2	7 27 45 78 121	1 10 10	60
12-5.3	7 27 45 86 120	1 10 11	60
13-6.1	31 39 78 112 123 125	2 16 18	66
13-6.2	7 27 43 53 78 120	2 16 20	66
14-7.1	7 27 45 56 94 107 117	3 24 36	73
15-8.1	7 25 42 53 78 83 111 120	7 32 52	63
15-8.2	7 25 42 53 75 87 116 120	7 34 46	63
15-8.3	7 11 13 14 51 85 105 127	14 28 28	77
16-9.1	7 25 42 53 75 87 108 118 120	10 48 72	60
17-10.1	7 25 42 53 62 78 83 92 99 120	15 60 130	46
17-10.2	7 11 29 45 51 78 81 100 118 120	15 66 110	52
17-10.3	Same as design 17-10.2 except replace 118 with 62	15 68 106	52
17-10.4	7 11 25 45 51 62 78 84 90 120	15 72 102	58
18-11.1	Same as design 17-10.1, plus 111	20 80 200	33
18-11.2	Same as design 17-10.4, plus 101	20 92 160	45
19-12.1	7 11 21 41 54 58 79 86 92 99 101 120	27 120 235	36
20-13.1	Same as design 19-12.1, plus 123	36 152 340	24
21-14.1	13 14 23 43 51 53 63 75 81 86 93 103 123 124	51 200 414	26
21-14.2	15 27 39 50 53 71 81 84 93 105 106 108 112 126	51 202 400	28
22-15.1	7 11 19 29 37 41 55 59 74 82 84 102 108 120 126	65 248 572	25
22-15.2	7 11 19 30 38 41 52 61 74 87 93 101 111 114 120	65 256 552	12
23-16.1	7 11 19 25 26 31 35 45 46 77 81 92 100 106 118 120	83 316 744	12
23-16.2	7 11 19 30 38 57 60 70 73 76 84 93 99 110 118 120	83 318 734	14

Table 3A.5 (Continued)

Design	Additional Elements	W	$C2$
24-17.1	7 11 19 29 35 46 53 57 73 76 82 87 100 109 118 120 123	102 384 992	0
24-17.2	Same as design 23-16.2, plus 81	102 394 985	7
25-18.1	7 11 19 29 37 41 47 49 55 59 62 77 78 82 84 91 102 120	124 482 1312	0
26-19.1	7 11 19 29 35 46 53 57 70 73 76 82 87 94 100 109 118 120 123	152 568 1704	0
27-20.1	Same as design 26-19.1, plus 97	180 690 2200	0
28-21.1	Same as design 27-20.1, plus 60	210 840 2800	0
29-22.1	Same as design 28-21.1, plus 69	266 945 3472	0
30-23.1	23 25 26 39 43 45 46 51 53 56 63 71 73 74 76 81 84 88 99 101 102 104 112	335 972 4662	0
31-24.1	7 11 19 21 22 25 26 35 45 46 49 60 67 77 78 81 95 101 105 108 116 120 123 126	391 1134 5826	0
31-24.2	Same as for design 30-23.1, plus 28	391 1134 5827	0
32-25.1	Same as design 31-24.2, plus 82	452 1322 7219	0
32-25.2	7 11 19 29 30 35 41 42 47 53 54 56 59 77 82 84 88 102 104 107 112 121 122 124 127	452 1323 7218	0
32-25.3	Same as design 32-25.2 except replace 29 with 101	452 1324 7219	0
33-26.1	Same as design 32-25.1, plus 54	518 1543 8863	0
33-26.2	Same as design 32-25.2, plus 44	518 1544 8863	0
34-27.1	Same as design 33-26.1, plus 95	589 1800 10788	0
34-27.2	7 11 19 29 30 35 45 46 53 54 57 58 60 63 67 77 86 89 97 98 100 103 104 107 112 115 125	589 1801 10788	0
35-28.1	Same as design 34-27.1, plus 111	665 2100 13020	0
35-28.2	15 23 25 26 28 39 43 45 46 51 53 54 56 71 73 74 76 81 82 84 88 101 104 111 112 119 123 126	665 2101 13020	0
36-29.1	Same as design 35-28.1, plus 15	756 2401 15736	0
37-30.1	Same as design 36-29.1, plus 119	854 2744 18886	0
38-31.1	Same as design 37-30.1, plus 123	959 3136 22512	0
39-32.1	Same as design 38-31.1, plus 125	1071 3584 26656	0
40-33.1	Same as design 39-32.1, plus 126	1190 4096 31360	0

4

Fractional Factorial Designs: General Case

Fractional factorial designs with factors at s levels, $s > 2$, are used in practice, especially when the investigator anticipates a curvature effect of a quantitative factor or when a qualitative factor has several levels. Extension of the work in Chapter 3 to such designs with s being a prime or prime power is considered here. A general discussion on minimum aberration designs and the method of complementary designs are presented. A catalogue of three-level designs with 27 and 81 runs is given.

4.1 Three-Level Designs

In this chapter, we consider s^{n-k} designs, where s (≥ 2) is a prime or prime power. In order to highlight the difference between two-level and s-level ($s > 2$) designs, we focus in this section on the simplest case of three-level designs. Consider the 3^2 factorial with factors F_1 and F_2. As noted near the end of Section 2.3, the pencils $(1, 0)'$ and $(0, 1)'$ represent the main effects of F_1 and F_2 respectively, and $(1, 1)'$ and $(1, 2)'$ together represent the interaction F_1F_2. In applied design texts, this is described in another way without use of projective geometry. Consider the following 9×4 array with four columns represented by 1, 2, 12, and 12^2. Columns 1 and 2 correspond to factors F_1 and F_2 and generate all nine treatment combinations of the 3^2 factorial. Column 12 corresponds to the pencil $(1, 1)'$ and column 12^2 to the pencil $(1, 2)'$. Together they represent the interaction effect F_1F_2. Column 12 is obtained as the sum of columns 1 and 2 over $GF(3)$ (or equivalently, the sum of columns 1 and 2 modulo 3); similarly column 12^2 is the sum of column 1 and two times column 2 modulo 3. We can construct a 3^{3-1} design by assigning factor F_3 to column 12 and a 3^{4-2} design by further assigning factor F_4 to column 12^2. This representation has the advantage that it aids the planning of experiments by spelling out all the treatment combinations in the 3^{3-1} and 3^{4-2} designs mentioned above.

1	**2**	**12**	$\mathbf{12^2}$
0	0	0	0
0	1	1	2
0	2	2	1
1	0	1	1
1	1	2	0
1	2	0	2
2	0	2	2
2	1	0	1
2	2	1	0

The distinction between two-level and three-level designs lies in the correspondence between factorial effects and pencils. In the former, there is a one-to-one correspondence and use of pencils to represent factorial effects is unnecessary. In the latter, a two-factor interaction (2fi) corresponds to two pencils. (For s levels, a 2fi corresponds to $s-1$ pencils.) Therefore use of pencils is indispensable in the theoretical development. However, in describing 3^{n-k} designs, use of the vector notation for pencils can often be avoided. A more compact notational system was suggested in Remark 2.3.1. This system is more user-friendly, as can been seen by considering the 3^3 factorial. Along the lines of Example 2.3.1, it has 13 pencils. Instead of using vectors, these can be represented by $1, 2, 3, 12, 12^2, 13, 13^2, 23, 23^2, 123, 123^2, 12^23, 12^23^2$. In this chapter both notational systems will be used depending on the context.

In the 3^2 factorial, the decomposition of the 2fi into two components corresponding to the pencils 12 and 12^2 follows naturally from the algebraic structure. It also gives justification for the analysis of variance (ANOVA) method. The total sum of squares (SS) in ANOVA can be decomposed into four terms, each with two degrees of freedom: SS for factor 1, SS for factor 2, SS for the pencil 12, and SS for the pencil 12^2. The contrasts belonging to the pencils 12 and 12^2 can be explained as follows. Denote the levels of the two factors by x_1 and x_2. As seen in Example 2.3.1, the pencil 12 is represented by the contrasts among the three sets of treatment combinations satisfying $x_1 + x_2$ = 0, 1, 2 modulo 3. These sets are denoted by the Latin letters A, B, C in the following table. Similarly the pencil 12^2 is represented by the contrasts among the three sets of treatment combinations satisfying $x_1 + 2x_2 = 0, 1, 2$ modulo 3. These sets are denoted by the Greek letters α, β, γ in the same table. Observe that the table so obtained is a Graeco-Latin square. This is well anticipated since treatment contrasts belonging to distinct pencils are orthogonal to each other.

	x_2		
x_1	0	1	2
0	$A\alpha$	$B\gamma$	$C\beta$
1	$B\beta$	$C\alpha$	$A\gamma$
2	$C\gamma$	$A\beta$	$B\alpha$

From the data analysis point of view, there is, however, some difficulty in interpreting the pencils 12 and 12^2. Suppose the sum of squares for 12 in ANOVA is significant while that for 12^2 is not. How can this be properly interpreted? As noted above, the pencil 12 is represented by the contrasts among the three sets of treatment combinations denoted by A, B, C. From the positions of A, B, C in the table, there is no obvious physical interpretation of the contrasts among these three sets. These difficulties and some remedial measures are discussed in Chapter 5 of Wu and Hamada (2000).

4.2 Minimum Aberration s^{n-k} Designs with Small k

For ease in reference, recall from (2.4.1) that an s^{n-k} design is given by $d(B) = \{x : Bx = 0\}$, where x is a typical treatment combination and B is a $k \times n$ matrix of full row rank over $GF(s)$. From (2.4.4), a defining pencil b of $d(B)$ is one that satisfies $b' \in \mathcal{R}(B)$, where $\mathcal{R}(\cdot)$ is the row space of a matrix, and there are $(s^k - 1)/(s - 1)$ defining pencils.

We begin by exploring minimum aberration (MA) s^{n-k} designs for relatively small k. The objective is to generalize some of the results in Section 3.2. Throughout this section, attention is restricted to designs where each factor is involved in some defining pencil. By Lemma 2.5.1, an MA design must satisfy this requirement. Let $(A_1, \ldots, A_n)$ denote the wordlength pattern of any s^{n-k} design, where A_i is the number of defining pencils (or words) with i nonzero entries. As a partial generalization of Lemma 3.2.1, the following result holds.

Lemma 4.2.1. *For any s^{n-k} design,*

$$(a) \sum_{i=1}^{n} A_i = (s^k - 1)/(s - 1), \tag{4.2.1}$$

$$(b) \sum_{i=1}^{n} iA_i = ns^{k-1}. \tag{4.2.2}$$

Proof. Since there are altogether $(s^k-1)/(s-1)$ defining pencils, (a) is evident. To prove (b), consider any s^{n-k} design $d(B)$. With reference to this design, let β_i be the number of defining pencils that involve the ith factor, i.e., the ith entry is nonzero. As in Lemma 3.2.1, $\sum_{i=1}^{n} iA_i = \sum_{i=1}^{n} \beta_i$. Hence it suffices to show that $\beta_i = s^{k-1}$ for each i. Consider any fixed i, and let c_i denote the ith column of B. If $c_i = 0$, then by (2.4.4), every defining pencil has 0 in the ith position, i.e., no such pencil involves the ith factor, which is impossible. Hence $c_i \neq 0$, and arguing as in Lemma 2.3.1, there are s^{k-1} choices of the vector λ over $GF(s)$ such that $\lambda' c_i = 0$, i.e., $\lambda' B$ has 0 in the ith position. Since pencils are nonnull vectors and pencils with proportional entries are identical, from (2.4.4) it now follows that there are $(s^{k-1} - 1)/(s - 1)$ defining pencils with 0 in the ith position. Since there are $(s^k - 1)/(s - 1)$ defining pencils altogether, one gets

$$\beta_i = \frac{s^k - 1}{s - 1} - \frac{s^{k-1} - 1}{s - 1} = s^{k-1},$$

which proves (b). □

The next lemma plays a key role in obtaining the results of this section.

Lemma 4.2.2. *Suppose there exists an s^{n-k} design d_0 such that*

(i) each factor is involved in some defining pencil of d_0, and
(ii) the numbers of nonzero entries in the defining pencils of d_0 differ by at most one.

Then d_0 has minimum aberration and maximum resolution. Its resolution is given by

$$R_0 = \left[\frac{ns^{k-1}(s-1)}{s^k - 1}\right], \tag{4.2.3}$$

where $[z]$ denotes the integer part of z.

Proof. Let $(A_1^0, \ldots, A_n^0)$ denote the wordlength pattern of d_0. By (ii), there exists a positive integer p $(< n)$ such that

$$A_i^0 = 0 \ (i \neq p, p+1). \tag{4.2.4}$$

Hence by (i) and Lemma 4.2.1,

$$A_p^0 + A_{p+1}^0 = (s^k - 1)/(s-1), \quad pA_p^0 + (p+1)A_{p+1}^0 = ns^{k-1},$$

yielding the unique solution

$$A_p^0 = \frac{(p+1)(s^k - 1)}{s - 1} - ns^{k-1}, \quad A_{p+1}^0 = ns^{k-1} - \frac{p(s^k - 1)}{s - 1}. \tag{4.2.5}$$

Since both A_p^0 and A_{p+1}^0 are nonnegative, from (4.2.5) one gets

$$p \leq \frac{ns^{k-1}(s-1)}{s^k - 1} \leq p + 1. \tag{4.2.6}$$

If the second inequality in (4.2.6) is strict, then by (4.2.5), $A_p^0 > 0$, and hence by (4.2.4), the resolution of d_0 equals p. In this case, by (4.2.6), the right-hand side of (4.2.3) also reduces to p, and the truth of (4.2.3) follows. On the other hand, if equality is attained in the second inequality in (4.2.6), then, by (4.2.5), $A_p^0 = 0$, $A_{p+1}^0 > 0$, and (4.2.3) follows again in a similar manner.

It will now be shown that d_0 has MA and maximum resolution. By (4.2.4), d_0 has less aberration than every design for which $A_i > 0$ for some $i \leq p - 1$. Therefore, consider s^{n-k} designs satisfying

$$A_i = 0 \text{ for } i \leq p - 1. \tag{4.2.7}$$

For any such design, by (4.2.1), (4.2.2), and the first identity in (4.2.5),

$$\sum_{i=p}^{n}(p+2-i)A_i = (p+2)\sum_{i=p}^{n}A_i - \sum_{i=p}^{n} iA_i$$
$$= \frac{(p+2)(s^k-1)}{s-1} - ns^{k-1} = \frac{s^k-1}{s-1} + A_p^0. \qquad (4.2.8)$$

But

$$\sum_{i=p}^{n}(p+2-i)A_i \le \sum_{i=p}^{p+1}(p+2-i)A_i = 2A_p + A_{p+1},$$

because $p+2-i \le 0$ unless $i \le p+1$. Hence (4.2.8) yields

$$2A_p + A_{p+1} \ge \frac{s^k-1}{s-1} + A_p^0. \qquad (4.2.9)$$

On the other hand, by (4.2.1),

$$A_p + A_{p+1} \le \frac{s^k-1}{s-1}. \qquad (4.2.10)$$

For any design satisfying (4.2.7), from (4.2.9) and (4.2.10), one gets $A_p \ge A_p^0$. Furthermore, by (4.2.5), (4.2.9), and (4.2.10), if $A_p = A_p^0$, then $A_{p+1} = A_{p+1}^0$. In view of (4.2.4), it follows that d_0 has MA. Hence it has maximum resolution as well. □

As in Section 3.2, the case $k = 1$ is straightforward. There is only one defining pencil; cf. (4.2.1). Any design for which this defining pencil has all entries nonzero has MA and maximum resolution.

Next, consider $k = 2$. The following construction yields a design as envisaged in Lemma 4.2.2. Let

$$p_1 = [n/(s+1)],\ p_2 = n - p_1(s+1). \qquad (4.2.11)$$

Then $0 \le p_2 \le s$. Define the integers $u_0, u_1, \ldots, u_s$ as

$$u_i = \begin{cases} p_1 & \text{if } i \le s - p_2, \\ p_1 + 1 & \text{otherwise.} \end{cases} \qquad (4.2.12)$$

These integers are nonnegative and the second identity in (4.2.11) yields

$$u_0 + u_1 + \cdots + u_s = n. \qquad (4.2.13)$$

From (4.2.11) and (4.2.12), it is also seen easily that

$$0 < u_s < n. \qquad (4.2.14)$$

For example, if $u_s = n$, then either $p_1 = n$, $p_2 = 0$ or $p_1 = n-1$, $p_2 > 0$, but neither is possible. Let $\alpha_0(=0)$, $\alpha_1(=1)$, $\alpha_2, \ldots, \alpha_{s-1}$ be the elements of $GF(s)$. Define the matrix

$$B_0 = [B_{00}, B_{01}, \ldots, B_{0,s-1}, B_{0s}], \tag{4.2.15}$$

where

$$B_{00} = \begin{bmatrix} \alpha_1 \ldots \alpha_1 \\ 0 \ldots 0 \end{bmatrix}, \quad B_{0i} = \begin{bmatrix} \alpha_1 \ldots \alpha_1 \\ \alpha_i \ldots \alpha_i \end{bmatrix}, \; 1 \le i \le s-1, \quad B_{0s} = \begin{bmatrix} 0 \ldots 0 \\ \alpha_1 \ldots \alpha_1 \end{bmatrix},$$

and B_{0i} has u_i columns for $i = 0, \ldots, s$. If $u_i = 0$ for any i, then the corresponding set of columns does not appear in B_0. By (4.2.13), B_0 is of order $2 \times n$, while by (4.2.14), it has full row rank. Following (2.4.1), one can therefore consider the s^{n-2} design $d_0 = d(B_0)$.

Theorem 4.2.1. *The s^{n-2} design $d_0 = d(B_0)$, where B_0 is given by (4.2.15), has minimum aberration and maximum resolution. Its resolution equals $[ns/(s+1)]$.*

Proof. Let $b^{(1)'}$ and $b^{(2)'}$ denote the two rows of B_0. Then by (2.4.4), the defining pencils of d_0 are

$$b^{(2)},\ b^{(2)} - \alpha_1 b^{(1)},\ \ldots,\ b^{(2)} - \alpha_{s-1} b^{(1)},\ b^{(1)}.$$

Since $\alpha_1 (= 1)$ is the identity element of $GF(s)$ under multiplication, from (4.2.15) it is seen that these pencils have $n - u_0, n - u_1, \ldots, n - u_{s-1}$, and $n - u_s$ nonzero entries respectively. Hence by (4.2.12), d_0 satisfies condition (ii) of Lemma 4.2.2. From (4.2.15), it also satisfies condition (i) of Lemma 4.2.2. Therefore, d_0 has MA and maximum resolution. Taking $k = 2$ in (4.2.3), the resolution of d_0 is as stated. □

The above result extends Theorem 3.2.1 to the case of general s. For $s = 2$, it is not hard to verify that the designs considered in Theorems 3.2.1 and 4.2.1 are isomorphic.

Example 4.2.1. Let $s = 3$, $n = 6$, $k = 2$. Then by (4.2.11), (4.2.12), and (4.2.15), $p_1 = 1$, $p_2 = 2$, $u_0 = u_1 = 1$, $u_2 = u_3 = 2$, and

$$B_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 1 \end{bmatrix}.$$

Theorem 4.2.1 shows that the 3^{6-2} design $d(B_0)$, with B_0 as above, has MA and resolution four. A look at the rows of the matrix B_0 shows that the defining relation of $d(B_0)$ is

$$I = 1234 = 23^2 4^2 56 = 12^2 56 = 13^2 4^2 5^2 6^2. \qquad \square$$

Now suppose $k \ge 3$. Then it is difficult to obtain a general result like Theorem 4.2.1. However, Theorem 4.2.2 below gives MA designs in some cases. As in Lemma 2.7.2, let V_k be a matrix with k rows and $(s^k - 1)/(s-1)$ columns, such that the columns of V_k are given by the points of the finite projective geometry $PG(k-1, s)$. Define $B^* = V_k$ and $B_+ = [V_k \; c]$, where c is any nonnull $k \times 1$ vector over $GF(s)$. Also, let B_- be obtained from V_k by deleting any one of its columns. By Lemma 2.7.2 (a), B^* and B_+ have full row rank. It is easy to see that the same holds for B_-. Therefore, following (2.4.1), $d(B^*)$, $d(B_+)$, and $d(B_-)$ represent s^{n-k} designs, with $n = (s^k-1)/(s-1)$, $n = (s^k-1)/(s-1)+1$, and $n = (s^k-1)/(s-1)-1$, respectively.

Theorem 4.2.2. *Let $k \geq 3$. The designs $d(B^*)$, $d(B_+)$, and $d(B_-)$ have minimum aberration and maximum resolution. The resolutions of these designs equal s^{k-1}, s^{k-1} and $s^{k-1}-1$ respectively.*

Proof. From Lemma 2.7.2 (b), every nonnull vector in $\mathcal{R}(V_k)$ has s^{k-1} nonzero elements. Hence by the definition of B^*, B_+, and B_-, every nonnull vector in $\mathcal{R}(B^*)$ has s^{k-1} nonzero elements, every nonnull vector in $\mathcal{R}(B_+)$ has s^{k-1} or $s^{k-1}+1$ nonzero elements, and every nonnull vector in $\mathcal{R}(B_-)$ has s^{k-1} or $s^{k-1}-1$ nonzero elements. Consequently, by (2.4.4), the designs $d(B^*)$, $d(B_+)$, and $d(B_-)$ meet condition (ii) of Lemma 4.2.2. It is easily seen that they also meet condition (i) of this lemma. Hence the result follows. □

4.3 A General Result on Complementary Designs

The results in the last section give MA s^{n-k} designs for small k. For a fixed run size, these results are useful only when n is also relatively small. For example, consider 3^{n-k} designs with 27 runs. Then $n-k=3$, and the cases $k=1$ and 2 discussed in Section 4.2 yield MA designs for $n=4$ and 5 respectively. For $n \geq 6$, however, none of the results in Section 4.2 are applicable. The approach based on complementary designs, introduced in Section 3.3 for $s=2$, can be of help in these situations. This technique is now explored for general s, and the necessary derivation, which was omitted in Chapter 3, is given. The finite projective geometric formulation as well as the results from coding theory presented in Chapter 2 constitute the basic tools for this purpose. As in Section 2.7, for any nonempty set Q of points of $PG(n-k-1, s)$, let $V(Q)$ denote a matrix with columns given by the points of Q. The developments in this and the next section follow Suen, Chen, and Wu (1997).

It was seen in Section 2.5 that a design of resolution one or two fails to ensure the estimability of the main effects even under the absence of all interactions. Hence, throughout the rest of this chapter, only s^{n-k} designs of resolution three or higher are considered. By Theorem 2.7.1, any such design is equivalent to a set T of n points of $PG(n-k-1, s)$, with $V(T)$ having full row rank and the conditions (a)–(c) of the theorem being satisfied. In view of this, the design itself is denoted by the corresponding set T, and its wordlength pattern is denoted by $(A_1(T), \ldots, A_n(T))$. Since pencils with proportional entries are identical, it follows from Theorem 2.7.1(b) that

$$A_i(T) = (s-1)^{-1}\#\{\lambda : \lambda \in \Omega_{in}, V(T)\lambda = 0\}, \qquad 1 \le i \le n, \qquad (4.3.1)$$

where Ω_{in} is the set of $n \times 1$ vectors over $GF(s)$ with i nonzero elements and $\#$ denotes the cardinality of a set.

Let $\overline{T}$ denote the complementary set of T in $PG(n-k-1, s)$. The cardinality of $\overline{T}$ equals

$$f = (s^{n-k} - 1)/(s-1) - n. \qquad (4.3.2)$$

The design T is called *saturated* if $f = 0$ and *nearly saturated* if f is positive but small. Indeed, in the saturated case, n attains the upper bound in Theorem 2.7.3. Then $\overline{T}$ is empty and T consists of all the points of $PG(n-k-1, s)$, i.e., there is only one choice of T. To avoid trivialities, hereafter it is assumed that $f \ge 1$. The matrix $V(\overline{T})$, of order $(n-k) \times f$, is then well defined. If this matrix has full row rank and $f > n-k$, then following Theorem 2.7.1, $\overline{T}$ represents an s^{f-k^*} design where $k^* = f - (n-k)$. This is called the *complementary design* of T and its wordlength pattern is denoted by $(A_1(\overline{T}), \ldots, A_f(\overline{T}))$. Analogously to (4.3.1),

$$A_i(\overline{T}) = (s-1)^{-1}\#\{\lambda : \lambda \in \Omega_{if}, V(\overline{T})\lambda = 0\}, \qquad 1 \le i \le f. \qquad (4.3.3)$$

If $V(\overline{T})$ does not have full row rank or $f \le n-k$, then $\overline{T}$ does not represent a design in a strict sense. However, it is still called the complementary design according to common usage. At any rate, the quantities $A_i(\overline{T})$ remain well-defined via (4.3.3). Since no two points of $PG(n-k-1, s)$ are proportional to each other, it is clear from (4.3.1) and (4.3.3) that $A_i(T) = A_i(\overline{T}) = 0$, for $i = 1, 2$. We also write

$$A_0(\overline{T}) = (s-1)^{-1}. \qquad (4.3.4)$$

Example 4.3.1. Consider the 3^{10-7} design represented by the set $T = \{(1,2,0)', (0,0,1)', (1,0,1)', (1,0,2)', (0,1,1)', (0,1,2)', (1,1,1)', (1,1,2)', (1,2,1)', (1,2,2)'\}$ of ten points of $PG(2,3)$. By (4.3.2), $f = 3$. Also $\overline{T} = \{(1,0,0)', (0,1,0)', (1,1,0)'\}$, and

$$V(\overline{T}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $V(\overline{T})$ does not have full row rank. Nevertheless, the quantities $A_i(\overline{T})$ remain well defined via (4.3.3) and are given by $A_1(\overline{T}) = A_2(\overline{T}) = 0$, $A_3(\overline{T}) = 1$. □

Since $A_1(T) = A_2(T) = 0$, the complementary design theory to be developed now aims at expressing the $A_i(T)$, $3 \le i \le n$, in terms of the $A_i(\overline{T})$, $0 \le i \le f$. This is done in Theorem 4.3.1 below. When f is small compared to n, it is easier to work with $\overline{T}$ rather than T, and hence this result and its corollaries facilitate the study of MA designs. Some preliminaries help in establishing Theorem 4.3.1.

From Section 2.8, recall that the defining contrast subgroup of an s^{n-k} design is equivalent to an $[n,k;s]$ linear code, say C. Let $C^{\perp}$ be the dual code of C. Then by (2.8.2) and the MacWilliam's identity (2.8.5) (note that $A_i(B)$ in (2.8.2) is now denoted by $A_i(T)$),

$$A_i(T) = (s-1)^{-1}s^{-(n-k)}\sum_{j=0}^{n} K_j(C^{\perp})P_i(j;n,s), \qquad 3 \le i \le n, \quad (4.3.5)$$

where $K_j(C^{\perp})$ is the number of codewords of weight j in $C^{\perp}$, and following (2.8.6),

$$P_i(j;n,s) = \sum_{t=0}^{i}(-1)^t(s-1)^{i-t}\binom{j}{t}\binom{n-j}{i-t}. \qquad (4.3.6)$$

As noted below (2.8.3), the codewords in $C^{\perp}$ are equivalent to the treatment combinations in the design T, i.e., by Theorem 2.7.1(a),

$$C^{\perp} = \mathcal{R}[V(T)]. \qquad (4.3.7)$$

Let $r = \text{rank}[V(\overline{T})]$. Since $V(\overline{T})$ has $n-k$ rows and f nonnull columns, $1 \le r \le n-k$ and the s^r vectors in $\mathcal{R}[V(\overline{T})]$ form an $[f,r;s]$ linear code, say M. Following (2.8.3), its dual code is given by

$$M^{\perp} = \{\lambda' : \lambda \text{ is an } f \times 1 \text{ vector over } GF(s) \text{ satisfying } V(\overline{T})\lambda = 0\}. \quad (4.3.8)$$

Let $K_j(M)$ and $K_j(M^{\perp})$ be the numbers of codewords of weight j in M and $M^{\perp}$ respectively. By (4.3.3), (4.3.4), and (4.3.8), $K_j(M^{\perp}) = (s-1)A_j(\overline{T})$, $0 \le j \le f$. Hence (2.8.5) yields

$$K_j(M) = (s-1)s^{-(f-r)}\sum_{u=0}^{f} A_u(\overline{T})P_j(u;f,s), \qquad 0 \le j \le f, \quad (4.3.9)$$

where

$$P_j(u;f,s) = \sum_{q=0}^{j}(-1)^q(s-1)^{j-q}\binom{u}{q}\binom{f-u}{j-q}, \qquad (4.3.10)$$

analogously to (4.3.6). For any integers t_1 and t_2, throughout this section, $\binom{t_1}{t_2}$ is interpreted as

$$\binom{t_1}{t_2} = \begin{cases} \frac{t_1(t_1-1)\cdots(t_1-t_2+1)}{t_2(t_2-1)\cdots 1}, & \text{for } t_2 > 0, \\ 1, & \text{for } t_2 = 0, \\ 0, & \text{for } t_2 < 0. \end{cases}$$

In view of (4.3.5) and (4.3.9), if one can connect the $K_j(C^{\perp})$ with the $K_j(M)$, then the objective of expressing the $A_i(T)$ in terms of the $A_i(\overline{T})$ is achieved. The next lemma is an important step in this direction.

Lemma 4.3.1. *Let* $\theta = s^{n-k-1}$ *and* $\mu = s^{n-k-r}$. *Then*

(a) $K_0(C^{\perp}) = 1 + \mu K_{\theta}(M)$,
(b) $K_j(C^{\perp}) = \mu K_{\theta - j}(M)$, *for* $1 \le j \le \theta - 1$,
(c) $K_{\theta}(C^{\perp}) = \mu K_0(M) - 1$,
(d) $K_j(C^{\perp}) = K_j(M) = 0$, *for* $j > \theta$.

Proof. Consider the matrix V_{n-k}, with columns given by the points of $PG(n-k-1, s)$. Clearly,

$$V_{n-k} = [V(T) \quad V(\overline{T})]. \tag{4.3.11}$$

By Lemma 2.7.2(a), V_{n-k} has full row rank. Let H be a matrix, with s^{n-k} rows and $(s^{n-k} - 1)/(s - 1)$ columns, such that the rows of H are given by the s^{n-k} vectors in $\mathcal{R}(V_{n-k})$. Note that

(i) H has one null row, and
(ii) by Lemma 2.7.2(b), each other row of H has θ nonzero elements.

Partition H as

$$H = [H(T) \quad H(\overline{T})], \tag{4.3.12}$$

where the columns of $H(T)$ and $H(\overline{T})$ correspond to those of $V(T)$ and $V(\overline{T})$ in (4.3.11).

Let ξ_j be the number of rows of $H(T)$ with j nonzero elements. Similarly, define $\overline{\xi}_j$ with respect to $H(\overline{T})$. From (4.3.12) and the facts (i) and (ii), the following are evident:

$$\xi_0 = 1 + \overline{\xi}_{\theta}, \tag{4.3.13}$$

$$\xi_j = \overline{\xi}_{\theta - j}, \qquad \text{for } 1 \le j \le \theta - 1, \tag{4.3.14}$$

$$\xi_{\theta} = \overline{\xi}_0 - 1, \tag{4.3.15}$$

$$\xi_j = \overline{\xi}_j = 0, \qquad \text{for } j > \theta. \tag{4.3.16}$$

For instance, (4.3.15) follows by noting that each null row of $H(\overline{T})$, excluding the one embedded in the null row of H, corresponds to a row of $H(T)$ with θ nonzero elements.

Since $V(T)$ has full row rank, the rows of $H(T)$ are given by the vectors in $\mathcal{R}[V(T)]$. Hence by (4.3.7),

$$K_j(C^{\perp}) = \xi_j, \tag{4.3.17}$$

for each j. Similarly, since $\text{rank}[V(\overline{T})] = r$, each vector in $\mathcal{R}[V(\overline{T})]$, i.e., each codeword of M, appears μ times among the rows of $H(\overline{T})$. Hence

$$\overline{\xi}_j = \mu K_j(M), \tag{4.3.18}$$

for each j. The result is immediate from (4.3.13)–(4.3.18). □

Theorem 4.3.1. *For* $3 \le i \le n$,

$$A_i(T) = \rho_i + \sum_{j=0}^{f} \rho_{ij} A_j(\overline{T}), \tag{4.3.19}$$

where

$$\rho_i = (s-1)^{-1} s^{-(n-k)} \{P_i(0;n,s) - P_i(\theta;n,s)\}, \tag{4.3.20}$$
$$\rho_{ij} = 0 \text{ for } j > i, \tag{4.3.21}$$

and

$$\rho_{ij} = \sum\nolimits^{*} \binom{\theta-f}{t_1}\binom{n-\theta}{t_2}\binom{f-j}{t_3}(-1)^{t_1+j}(s-1)^{t_2}(s-2)^{t_3}, \text{ for } j \le i, \tag{4.3.22}$$

with $\sum^*$ *denoting the sum over nonnegative integers* t_1, t_2, t_3 *that satisfy*

$$t_1 + t_2 + t_3 = i - j. \tag{4.3.23}$$

Proof. By definition, $K_j(C^{\perp}) = 0$ for $j > n$. Together with Lemma 4.3.1(d), this implies that

$$K_j(C^{\perp}) = 0, \qquad \text{for } j > \min(n, \theta). \tag{4.3.24}$$

Similarly,

$$K_j(M) = 0, \qquad \text{for } j > \min(f, \theta). \tag{4.3.25}$$

By (4.3.5) and (4.3.24),

$$A_i(T) = (s-1)^{-1} s^{-(n-k)} \sum_{j=0}^{\theta} K_j(C^{\perp}) P_i(j;n,s), \qquad 3 \le i \le n.$$

Since $\mu = s^{n-k-r}$, using Lemma 4.3.1(a)–(c) and then (4.3.25),

$$\begin{aligned} A_i(T) &= (s-1)^{-1} s^{-(n-k)} \Big\{ P_i(0;n,s) - P_i(\theta;n,s) + \mu \sum_{j=0}^{\theta} K_{\theta-j}(M) P_i(j;n,s) \Big\} \\ &= \rho_i + \{(s-1)s^r\}^{-1} \sum_{j=0}^{\theta} K_j(M) P_i(\theta-j;n,s) \\ &= \rho_i + \{(s-1)s^r\}^{-1} \sum_{j=0}^{f} K_j(M) P_i(\theta-j;n,s), \end{aligned} \tag{4.3.26}$$

where ρ_i is given by (4.3.20). Recalling (4.3.9), this yields

$$\begin{aligned} A_i(T) &= \rho_i + s^{-f} \sum_{j=0}^{f} \sum_{u=0}^{f} A_u(\overline{T}) P_j(u; f, s) P_i(\theta - j; n, s) \\ &= \rho_i + s^{-f} \sum_{j=0}^{f} \sum_{u=0}^{f} A_j(\overline{T}) P_u(j; f, s) P_i(\theta - u; n, s) \\ &= \rho_i + \sum_{j=0}^{f} \rho_{ij} A_j(\overline{T}), \qquad 3 \le i \le n, \end{aligned}$$

where

$$\rho_{ij} = s^{-f} \sum_{u=0}^{f} P_u(j; f, s) P_i(\theta - u; n, s), \qquad 0 \le j \le f. \tag{4.3.27}$$

Thus $A_i(T)$, $3 \le i \le n$, are of the form (4.3.19) and it remains to show that (4.3.27) is equivalent to (4.3.21) or (4.3.22) depending on whether $j > i$ or $j \le i$. To that effect, note that

(i) by (4.3.10), for $0 \le j, u \le f$,

$$P_u(j; f, s) = \binom{f}{u} (s-1)^{u-j} P_j(u; f, s) \Big/ \binom{f}{j},$$

and $P_j(u; f, s)$ is the coefficient of y^j in the expansion of $(1-y)^u\{1 + (s-1)y\}^{f-u}$;

(ii) by (4.3.6), $P_i(\theta - u; n, s)$ is the coefficient of z^i in the expansion of

$$(1-z)^{\theta-u}\{1 + (s-1)z\}^{n-\theta+u};$$

(iii) by (i) and (ii), the right-hand side of (4.3.27) equals $\left\{s^f (s-1)^j \binom{f}{j}\right\}^{-1}$ times the coefficient of $y^j z^i$ in

$$\Delta(y, z) = \sum_{u=0}^{f} \binom{f}{u} (s-1)^u (1-y)^u \{1+(s-1)y\}^{f-u} (1-z)^{\theta-u} \{1+(s-1)z\}^{n-\theta+u}.$$

Now,

$$\begin{aligned} \Delta(y, z) &= \{1 + (s-1)y\}^f (1-z)^\theta \{1 + (s-1)z\}^{n-\theta} \\ &\quad \times \sum_{u=0}^{f} \binom{f}{u} \left[\frac{(s-1)(1-y)\{1 + (s-1)z\}}{\{1 + (s-1)y\}(1-z)} \right]^u \\ &= \{1 + (s-1)y\}^f (1-z)^\theta \{1 + (s-1)z\}^{n-\theta} \\ &\quad \times \left[1 + \frac{(s-1)(1-y)\{1 + (s-1)z\}}{\{1 + (s-1)y\}(1-z)} \right]^f \end{aligned}$$

$$= s^f(1-z)^{\theta-f}\{1+(s-1)z\}^{n-\theta}\{1+(s-2)z-(s-1)yz\}^f$$
$$= s^f\left[\sum_{t_1\geq 0}\binom{\theta-f}{t_1}(-z)^{t_1}\right]\left[\sum_{t_2\geq 0}\binom{n-\theta}{t_2}\{(s-1)z\}^{t_2}\right]$$
$$\times\left[\sum_{t_3,t_4\geq 0}\binom{f}{t_4}\binom{f-t_4}{t_3}\{(s-2)z\}^{t_3}\{-(s-1)yz\}^{t_4}\right]$$
$$= s^f\sum_{t_1,t_2,t_3,t_4\geq 0}\binom{\theta-f}{t_1}\binom{n-\theta}{t_2}\binom{f}{t_4}\binom{f-t_4}{t_3}$$
$$\times(-1)^{t_1+t_4}(s-1)^{t_2+t_4}(s-2)^{t_3}y^{t_4}z^{t_1+t_2+t_3+t_4}.$$

The coefficient of y^jz^i in the above equals 0 when $j>i$. On the other hand, if $j\leq i$, then this coefficient equals

$$s^f(s-1)^j\binom{f}{j}\sum{}^{*}\binom{\theta-f}{t_1}\binom{n-\theta}{t_2}\binom{f-j}{t_3}(-1)^{t_1+j}(s-1)^{t_2}(s-2)^{t_3},$$

where the sum $\sum^*$ is as defined in (4.3.23). From (iii) above, it is now clear that (4.3.27) is equivalent to (4.3.21) or (4.3.22) for $j>i$ or $j\leq i$, respectively. □

Corollary 4.3.1. *In the setup of Theorem 4.3.1,*

$$\rho_{ij}=\begin{cases}(-1)^i, & \text{if } j=i,\\ (-1)^i\{s(i-1)-2i+3\}, & \text{if } j=i-1,\\ \frac{1}{2}(-1)^i\{s^{n-k}-2(s-1)n+(s-2)^2(i-1)(i-2) & \\ +(s-2)(2i-3)\}, & \text{if } j=i-2.\end{cases}$$

Proof. If $j=i$, then by (4.3.23), the sum $\sum^*$ extends only over $t_1=t_2=t_3=0$. Hence by (4.3.22), ρ_{ij} is as claimed.

If $j=i-1$, then by (4.3.23), the sum $\sum^*$ extends over $(t_1,t_2,t_3)=(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Hence by (4.3.22), ρ_{ij} equals

$$(\theta-f)(-1)^i+(n-\theta)(-1)^{i-1}(s-1)+(f-i+1)(-1)^{i-1}(s-2).$$

Since $\theta=s^{n-k-1}$, the claim about ρ_{ij} follows after a little algebra using (4.3.2).

The proof is similar for $j=i-2$. □

Corollary 4.3.2.

(*a*) $A_3(T)=\text{constant}-A_3(\overline{T})$,
(*b*) $A_4(T)=\text{constant}+(3s-5)A_3(\overline{T})+A_4(\overline{T})$,
(*c*) $A_5(T)=\text{constant}-\frac{1}{2}\{s^{n-k}-2(s-1)n+(s-2)(12s-17)\}A_3(\overline{T})$
$-(4s-7)A_4(\overline{T})-A_5(\overline{T})$,

where the constants may depend on s, n, and k but not on the particular choice of T.

Proof. Consider (a). First suppose $f \geq 3$. Recall that $A_1(\overline{T}) = A_2(\overline{T}) = 0$ and that by (4.3.4), $A_0(\overline{T}) = (s-1)^{-1}$. Hence from (4.3.19), (4.3.21), and Corollary 4.3.1,

$$A_3(T) = \rho_3 + \rho_{30}(s-1)^{-1} + \rho_{33}A_3(\overline{T}) = \rho_3 + \rho_{30}(s-1)^{-1} - A_3(\overline{T}).$$

By (4.3.20) and (4.3.22), $\rho_3 + \rho_{30}(s-1)^{-1}$ is a constant that may depend on s, n, and k but not on the choice of T. Hence (a) follows for $f \geq 3$. On the other hand, for $f \leq 2$, a similar argument yields $A_3(T) = \rho_3 + \rho_{30}(s-1)^{-1}$ and trivially, $A_3(\overline{T}) = 0$; cf. (4.3.3). Thus (a) follows again.

The identities in (b) and (c) can be proved similarly using Corollary 4.3.1 with $i = 4$ and 5 respectively. □

In particular, for $s = 2$, Corollary 4.3.2 yields the identities in (3.3.2) that played a key role in Chapter 3.

Observe that the rank of $V(\overline{T})$ does not influence the conclusions of Theorem 4.3.1 and the above corollaries. Similarly, these results remain valid even when $V(T)$ does not have full row rank and hence T lacks interpretation as an s^{n-k} design. This fact, useful later in Chapter 6, can be established by a minor modification of the present derivation with $V(T)$ handled in the same way as $V(\overline{T})$.

4.4 Minimum Aberration s^{n-k} Designs via Complementary Designs

The results of the last section, especially Corollary 4.3.2, yield simple rules for finding MA s^{n-k} designs. This development is parallel to that in Section 3.3 and the concept of isomorphism again helps in reducing the design search.

Consider two sets of points of a finite projective geometry, with the same cardinality. The sets are called *isomorphic* if there exists a nonsingular transformation that maps each point of one set to some point of the other set up to proportionality. Two s^{n-k} designs T_1 and T_2 are isomorphic if the corresponding sets T_1 and T_2 are isomorphic. Writing $T_i = \left\{h_1^{(i)}, \ldots, h_n^{(i)}\right\}$, $i = 1, 2$, then there exists a nonsingular matrix Λ, of order $n-k$ and defined over $GF(s)$, such that for each j, $\Lambda h_j^{(1)}$ is proportional to some $h_t^{(2)}$. Since points with proportional entries are considered identical in a finite projective geometry, this means that there is an appropriate representation of the points of T_2 such that

$$\Lambda V(T_1) = V(T_2)R, \tag{4.4.1}$$

where R is a permutation matrix of order n over $GF(s)$, and, as usual, $V(T_i) = \left(h_1^{(i)}, \ldots, h_n^{(i)}\right)$, $i = 1, 2$. By (4.4.1), for any $n \times 1$ vector b,

$$V(T_1)b = 0 \Longleftrightarrow V(T_2)Rb = 0, \tag{4.4.2}$$

i.e., by Theorem 2.7.1(b), b is a defining pencil of the design T_1 if and only if Rb is a defining pencil of the design T_2. Obviously, the roles of T_1 and T_2 are interchangeable in the above. Hence by (4.4.2), if T_1 and T_2 are isomorphic, then they have the same defining pencils, or equivalently the same defining contrast subgroup, up to a permutation of factor labels. For $s = 2$, this is in agreement with the definition of design isomorphism given in Section 3.1. Obviously, isomorphic designs have the same wordlength pattern and are considered equivalent in the sequel. The following lemmas are easily obtained from the definition of isomorphism.

Lemma 4.4.1. *Let T_1 and T_2 be sets of points of $PG(n-k-1, s)$ with the same cardinality, and $\overline{T}_1$ and $\overline{T}_2$ their complementary sets respectively. If $\overline{T}_1$ and $\overline{T}_2$ are isomorphic, then $\overline{T}_1$ and $\overline{T}_2$ are also isomorphic.*

Lemma 4.4.2. *(a) All singleton sets of $PG(n-k-1, s)$ are isomorphic.*
(b) All sets of $PG(n-k-1, s)$ with cardinality two are isomorphic.

The above lemmas imply that all s^{n-k} designs are isomorphic when the cardinality f of the complementary sets equals 1 or 2, i.e., by (4.3.2), all such designs are isomorphic when $n = (s^{n-k}-1)/(s-1)-1$ or $n = (s^{n-k}-1)/(s-1)-2$. This extends Theorem 3.3.2 to the case of general s. Turning to the situation $f \geq 3$, Corollary 4.3.2 and Lemma 4.4.1 yield the same rules as in Section 3.3 for the identification of MA designs. For ease in reference, Rules 1 and 2 are reproduced below.

Rule 1. An s^{n-k} design T^* has minimum aberration if

(i) $A_3(\overline{T}^*) = \max A_3(\overline{T})$ over all $\overline{T}$ of cardinality f, and
(ii) $\overline{T}^*$ is the unique set (up to isomorphism) satisfying (i).

Rule 2. An s^{n-k} design T^* has minimum aberration if

(i) $A_3(\overline{T}^*) = \max A_3(\overline{T})$ over all $\overline{T}$ of cardinality f,
(ii) $A_4(\overline{T}^*) = \min\{A_4(\overline{T}) : A_3(\overline{T}) = A_3(\overline{T}^*)\}$, and
(iii) $\overline{T}^*$ is the unique set (up to isomorphism) satisfying (ii).

In general, by Theorem 4.3.1 and Corollary 4.3.1,

$$A_i(T) = \rho_i + \sum_{j=0}^{i-1} \rho_{ij} A_j(\overline{T}) + (-1)^i A_i(\overline{T}), \quad 3 \leq i \leq n, \tag{4.4.3}$$

where ρ_i and ρ_{ij} are constants that do not depend on T, $A_0(T) = (s-1)^{-1}$, and $A_j(\overline{T}) = 0$ for $j = 1, 2$ or $j > f$. Hence further rules, in the spirit of the ones stated above, are easy to develop. By (4.4.3), these call for maximization of $A_3(\overline{T})$, then minimization of $A_4(\overline{T})$, followed by maximization of $A_5(\overline{T})$, and so on. However, one rarely has to go beyond Rules 1 and 2 for relatively small f.

The concept of a flat of a finite projective geometry as well as a few lemmas are needed for obtaining further results. Consider any w $(1 \leq w \leq n-k)$ linearly independent points of $PG(n-k-1, s)$. These w points generate, as their linear combinations, $(s^w-1)/(s-1)$ points, including themselves, of the finite projective geometry. Such a collection of $(s^w-1)/(s-1)$ points is called a $(w-1)$-*flat* of $PG(n-k-1, s)$. Evidently, a flat is closed, up to proportionality, under the formation of nonnull linear combinations of the points therein. A 0-flat trivially consists of a single point, whereas a 1-flat, consisting of $s+1$ points, is also called a *line*. On the other extreme, an $(n-k-1)$-flat is identical to the entire $PG(n-k-1, s)$. It is easily seen that for a fixed w, all $(w-1)$-flats of $PG(n-k-1, s)$ are isomorphic.

Lemma 4.4.3. *Let* $f \geq 3$. *Then*

(*a*) $A_3(\overline{T}) \leq \frac{1}{6} f(f-1)\min\{f-2, s-1\}$. (4.4.4)

(*b*) *For* $3 \leq f \leq s+1$, *equality holds in (4.4.4) if and only if* rank$[V(\overline{T})] = 2$.

(*c*) *For* $f > s+1$, *equality holds in (4.4.4) if and only if* $f = (s^w-1)/(s-1)$ *and* $\overline{T}$ *is a* $(w-1)$*-flat with* $w \geq 3$.

Lemma 4.4.4. *All* s^{n-k} *designs* T *with rank*$[V(\overline{T})] = 2$ *have the same wordlength pattern.*

Lemma 4.4.3 follows from a result to be presented in Chapter 5 (see Lemma 5.4.1). Lemma 4.4.4 is a consequence of Theorem 4.3.1 and the fact that $A_j(\overline{T})$, $0 \leq j \leq f$, do not depend on $\overline{T}$ as long as rank$[V(\overline{T})] = 2$. In view of Lemma 4.4.4, one may wonder whether all s^{n-k} designs T with rank$[V(\overline{T})] = 2$ are isomorphic. Interestingly, this is not the case in general. For example, let $s = 7$, $n = 4$, $k = 2$, and consider the designs $T_1 = \{(1,3)', (1,4)', (1,5)', (1,6)'\}$, $T_2 = \{(1,2)', (1,4)', (1,5)', (1,6)'\}$. Then both $V(\overline{T}_1)$ and $V(\overline{T}_2)$ have rank two, and T_1 and T_2 have the same wordlength pattern, but a complete enumeration of all nonsingular transformations reveals that they are not isomorphic.

In particular, if $n-k = 2$ and $f \geq 3$, then the $2 \times f$ matrix $V(\overline{T})$ has rank two for every design T. Consequently, by Lemma 4.4.4, all designs have the same wordlength pattern and hence are equivalent under the MA criterion. The next two results hold for $n-k \geq 3$. The first follows readily from Corollary 4.3.2(a), Lemma 4.4.3(b), and Lemma 4.4.4. The second one is immediate from Rule 1 and Lemma 4.4.3(c).

Theorem 4.4.1. *Let* $n-k \geq 3$ *and* $3 \leq f \leq s+1$. *Then an* s^{n-k} *design* T *has minimum aberration if and only if rank*$[V(\overline{T})] = 2$.

Theorem 4.4.2. *Let* $f = (s^w-1)/(s-1)$, *where* $3 \leq w < n-k$. *Then an* s^{n-k} *design* T *has minimum aberration if and only if* $\overline{T}$ *is* $(w-1)$*-flat*

The condition $w \geq 3$ in Theorem 4.4.2 is not restrictive. If $w = 1$, then $f = 1$, which is trivial. If $w = 2$, then $f = s + 1$, which is covered by Theorem 4.4.1. In the setup of either theorem, the matrix $V(T)$ corresponding to an MA design T has full row rank, as it should in view of Theorem 2.7.1 . To see this for Theorem 4.4.2, let $h^{(1)}, \ldots, h^{(n-k)}$ be linearly independent points of $PG(n-k-1, s)$, and suppose the first w of these generate the $(w-1)$-flat $\overline{T}$. Then T includes in particular the $n-k$ linearly independent points $h^{(1)} + h^{(w+1)}, \ldots, h^{(w)} + h^{(w+1)}, h^{(w+1)}, \ldots, h^{(n-k)}$, and hence $V(T)$ must have full row rank. A similar argument works for Theorem 4.4.1.

Theorem 4.4.1 shows that the 3^{10-7} design in Example 4.3.1 has MA. Two more examples, illustrating the use of these theorems, are considered below.

Example 4.4.1. Let $s = 4$, $n = 17$, $k = 14$. By (4.3.2), $f = 4$. Let $\overline{T} = \{(1,0,0)', (0,1,0)', (1,1,0)', (1,\alpha,0)'\}$, where α is a primitive element of $GF(4)$. Then rank$[V(\overline{T})] = 2$. Here $n - k = 3$ and $3 < f < s+1$. Hence by Theorem 4.4.1, the 4^{17-14} design T, with $\overline{T}$ as shown above, has MA. □

Example 4.4.2. Let $s = 3$, $n = 27$, $k = 23$. By (4.3.2), $f = 13[= (3^3 - 1)/(3-1)]$. Let $\overline{T}$ be any 2-flat. Then by Theorem 4.4.2, the 3^{27-23} design T has MA. □

For $s = 3$ and $3 \leq f \leq 13$, Suen, Chen, and Wu (1997) obtained MA designs using Rule 1. Table 4.1 shows the set $\overline{T}$ for these MA designs. In this table, the compact notation is used. For example, 13^2 denotes the point $(1,0,2,0,\ldots,0)'$, and so on. For each f, one can check that the matrix $V(T)$ corresponding to the MA design given by Table 4.1 has full row rank.

Using Table 4.1, one can obtain an MA 3^{n-k} design for every n when the run size is 27, i.e., $n - k = 3$. Clearly $n \leq 13$. The cases $n = 13$, 12, and 11 are trivial, since they correspond to $f = 0$, 1, and 2, respectively. The cases $n = 4, 5, \ldots, 10$ correspond to $f = 9, 8, \ldots, 3$ respectively, and are covered by Table 4.1. Similarly, Table 4.1 yields 81-run $3^{n-(n-4)}$ designs for $27 \leq n \leq 37$. The cases $n = 38$, 39, and 40 are trivial, since they correspond to $f = 2$, 1, and 0, respectively. For example, using Table 4.1, an MA 3^{28-24} design can be readily obtained. Noting that $n = 28$ and $f = 12 (= 40 - 28)$, an MA 3^{28-24} design T can be constructed by taking its complementary set $\overline{T}$ to be the set of points given in Table 4.1 under $f = 12$.

Table 4.1 The sets $\overline{T}$ for MA 3^{n-k} designs

f	$\overline{T}$
3	$\{1, 2, 12\}$
4	$\{1, 2, 12, 12^2\}$
5	$\{1, 2, 12, 12^2, 3\}$
6	$\{1, 2, 12, 12^2, 3, 13\}$
7	$\{1, 2, 12, 12^2, 3, 12^2 3, 12^2 3^2\}$
8	$\{1, 2, 12, 12^2, 3, 23^2, 12^2 3, 12^2 3^2\}$
9	$\{1, 2, 12^2, 3, 13^2, 23^2, 123^2, 12^2 3, 12^2 3^2\}$
10	$\{1, 2, 12, 12^2, 3, 13, 13^2, 23, 23^2, 123\}$
11	$\{1, 2, 12, 12^2, 3, 13, 13^2, 23, 23^2, 123, 123^2\}$
12	$\{1, 2, 12, 12^2, 3, 13, 13^2, 23, 23^2, 123, 123^2, 12^2 3^2\}$
13	$\{1, 2, 12, 12^2, 3, 13, 13^2, 23, 23^2, 123, 123^2, 12^2 3, 12^2 3^2\}$

4.5 Description and Use of the Three-Level Design Tables

A catalogue of 3^{n-k} designs with 27 and 81 runs is given in the appendix of the chapter. The listing of the 27-run designs in Table 4A.2 is complete, i.e., it contains all the nonisomorphic designs. It is taken from Chen, Sun, and Wu (1993) with a few corrections. For 81 runs, a complete listing of designs is too long to be included. Therefore, only a selection of such designs, including all designs of resolution four or higher, is presented in Table 4A.3 for $5 \leq n \leq 20$. This table is adapted from Xu (2005), who also gave 243- and 729-run designs. Selection of designs for inclusion in Table 4A.3 is based on the MA, MaxC2, and other criteria not discussed in the book. As in Chapter 3, the MaxC2 criterion aims at maximizing $C2$, the number of clear 2fi's. A 2fi, say, between the first two factors, is now called *clear* if neither of the interaction pencils 12 and 12^2 is aliased with any main effect pencil or any other 2fi pencil.

From Section 4.3, recall that a 3^{n-k} design is equivalent to a set of n points of $PG(n-k-1, 3)$. The set must contain $n-k$ $(= m,$ say) independent points, which can be taken as $1, 2, \ldots, m$, using the compact notation. Thus a 3^{n-k} design can be represented by the independent points $1, 2, \ldots, m$, together with k additional points. This representation is used in the catalogue for tabulating designs. Furthermore, instead of using the explicit notation $1, 2, 12, 12^2, 3, 13, \ldots$ for the points of $PG(n-k-1, 3)$, to save space, we denote them by the corresponding serial numbers $1, 2, 3, 4, 5, 6, \ldots,$ the numbering scheme being shown in Table 4A.1. For instance, the independent points 1, 2, 3 of $PG(2, 3)$ are numbered 1, 2, 5, according to this scheme. Consequently, in listing any 27-run design (i.e., $m = n-k = 3$) in Table 4A.2, it is implied that the points numbered 1, 2, 5 are included but we list only the serial numbers of the additional k points under "Additional Points". Similar considerations apply to 81-run designs.

For clarity, the ith 3^{n-k} design in the catalogue is denoted by $n-k.i$. The wordlength pattern W and $C2$ appear in the last two columns of the design tables. To save space, for 81-run designs, at most four components of W are shown. For any given $n-k$ and n, the first design $n-k.1$ is the MA design. Use of the design tables is illustrated in the following example.

Example 4.5.1. Consider the 27-run MA design 6-3.1 in Table 4A.2. It is given by the points of $PG(2,3)$ with serial numbers 1, 2, 5, 3, 9, 13. Table 4A.1 identifies these points and shows that the design is given by the set $\{1, 2, 3, 12, 12^2 3, 12^2 3^2\}$. The six factors can be associated with the points of the set in the order stated. Then the following aliasing relations are immediate: $4 = 12$, $5 = 12^2 3$, $6 = 12^2 3^2$. In other words, 124^2, $12^2 35^2$, and $12^2 3^2 6^2$ are three independent defining pencils of this design. From this, one can easily obtain the other defining pencils and check that $A_3 = 2$, $A_4 = 9$, $A_5 = 0$, and $A_6 = 2$, which agree with the wordlength pattern W as given in Table 4A.2. □

Exercises

4.1 (a) Use Theorem 4.2.1 to find an MA 5^{4-2} design.
(b) Rewrite the design in (a) as a 5×5 Graeco-Latin square.

4.2 (a) Use Theorem 4.2.1 to find an MA 3^{5-2} design.
(b) Show that this design has the same wordlength pattern as the design $d(B_1)$ in Example 2.5.1. Hence conclude that the latter design also has MA.
(c) Verify that the two designs in (b) are isomorphic.

4.3 (a) Find an MA 3^{9-6} design using Table 4.1. Denote it by T_1.
(b) Find two other nonisomorphic 3^{9-6} designs and verify that they are inferior to T_1 according to the MA criterion. Denote them by T_2 and T_3.
(c) The theory of complementary designs can be used to confirm the findings in (b). In particular, compute the values of $A_3(\overline{T}_i)$ for $i = 1$, 2, 3. By applying Rule 1 in Section 4.4 to these values, show that T_1 has MA.

4.4 For $f = 5$, verify that the set $\overline{T}$ in Table 4.1 corresponds to an MA design.

4.5 Prove Corollary 4.3.2(b),(c).

4.6 Prove Lemma 4.4.1.

4.7 Prove Lemma 4.4.2.

4.8 Show that the $A_j(\overline{T})$, $0 \le j \le f$, do not depend on $\overline{T}$ if $\text{rank}[V(\overline{T})] = 2$.

Appendix 4A. Catalogue of 3^{n-k} Designs with 27 and 81 Runs

Table 4A.1 Numbering of points for 27- and 81-run designs

(The table gives the serial numbers of the points of $PG(3,3)$; the first 13 entries describe the serial numbers of the points of $PG(2,3)$. Independent points are numbered **1**, **2**, **5**, **14** in boldface.)

Number	**1**	**2**	3	4	**5**	6	7	8	9	10
Point	1	2	12	12^2	3	13	23	123	12^23	13^2
Number	11	12	13	**14**	15	16	17	18	19	20
Point	23^2	123^2	12^23^2	4	14	24	124	12^24	34	134
Number	21	22	23	24	25	26	27	28	29	30
Point	234	1234	12^234	13^24	23^24	123^24	12^23^24	14^2	24^2	124^2
Number	31	32	33	34	35	36	37	38	39	40
Point	12^24^2	34^2	134^2	234^2	1234^2	12^234^2	13^24^2	23^24^2	123^24^2	$12^23^24^2$

Table 4A.2 Complete catalogue of 27-run designs

(Each design is represented by 1, 2, 5 and the numbers specified under "Additional Points". $W = (A_3, A_4, \ldots)$ is the wordlength pattern of the design. $C2$ is the number of clear 2fi's. Designs for $n = 11, 12, 13$ are unique up to isomorphism and hence omitted.)

Design	Additional Points	W	$C2$
4-1.1	8	0 1	0
4-1.2	3	1 0	3
5-2.1	3 9	1 3 0	0
5-2.2	3 6	2 1 1	0
5-2.3	3 4	4 0 0	4
6-3.1	3 9 13	2 9 0 2	0
6-3.2	3 6 7	3 6 3 1	0
6-3.3	3 6 11	4 3 6 0	0
6-3.4	3 4 6	5 3 3 2	0
7-4.1	3 10 11 13	5 15 9 8 3	0
7-4.2	4 8 9 11	6 11 15 4 4	0
7-4.3	4 8 10 11	7 10 12 9 2	0
7-4.4	3 4 9 13	8 9 9 14 0	0
8-5.1	3 8 9 10 11	8 30 24 32 24 3	0
8-5.2	4 8 9 10 11	10 23 32 30 22 4	0
8-5.3	3 4 9 11 13	11 21 30 38 15 6	0
9-6.1	3 8 9 10 11 13	12 54 54 96 108 27 13	0
9-6.2	3 4 8 9 10 11	15 42 69 96 93 39 10	0
9-6.3	4 9 10 11 12 13	16 39 69 106 78 48 8	0
10-7.1	3 6 7 8 10 11 12	21 72 135 240 315 189 103 18	0
10-7.2	3 4 6 7 8 10 11	22 68 138 250 290 213 92 20	0

Table 4A.3 Selected 81-run designs for $n = 5$ to 20

(Each design is represented by 1, 2, 5, 14 and the numbers specified under "Additional Points". $W = (A_3, A_4, A_5)$ when $n = 5$ and $W = (A_3, \ldots, A_6)$ when $n > 5$. $C2$ is the number of clear 2fi's.)

Design	Additional Points	W	$C2$
5-1.1	22	0 0 1	10
5-1.2	8	0 1 0	4
5-1.3	3	1 0 0	7
6-2.1	9 22	0 2 2 0	4
6-2.2	8 17	0 3 0 1	0
6-2.3	4 22	1 0 3 0	12
7-3.1	9 22 24	0 5 6 1	0
7-3.2	9 18 22	0 6 3 4	0
7-3.3	9 15 22	1 3 6 3	3
7-3.4	9 10 22	1 4 6 0	6
7-3.5	4 22 26	2 0 9 2	15
7-3.6	3 4 22	4 1 3 3	9
8-4.1	9 22 24 31	0 10 16 4	0
8-4.2	9 22 24 25	0 11 12 10	0
8-4.3	9 18 22 38	0 12 8 16	0
8-4.4	9 10 22 35	2 6 18 2	9
8-4.5	9 15 22 28	4 4 12 12	4
8-4.6	3 4 22 26	5 3 9 17	12
8-4.7	3 4 19 32	8 0 0 32	16
9-5.1	9 22 24 31 34	0 18 36 12	0
9-5.2	3 9 22 24 31	1 18 27 28	0
9-5.3	7 9 22 24 25	1 20 20 36	0
9-5.4	6 9 22 24 25	2 17 23 34	0
9-5.5	9 16 22 24 29	5 11 26 31	1
9-5.6	8 9 10 22 23	5 12 27 26	9
9-5.7	3 9 10 13 22	5 18 24 23	2
9-5.8	3 9 10 12 22	6 15 27 21	4
9-5.9	4 6 8 11 12	8 30 24 32	8

Table 4A.3 (continued)

Design	Additional Points	W	$C2$
10-6.1	9 22 24 31 34 39	0 30 72 30	0
10-6.2	3 9 22 24 31 34	2 28 57 65	0
10-6.3	3 9 22 24 25 31	2 30 48 80	0
10-6.4	3 6 9 22 24 31	3 30 42 84	0
10-6.5	4 7 9 12 22 24	5 28 48 68	0
10-6.6	3 9 10 11 13 22	8 34 48 62	0
10-6.7	3 6 9 10 13 22	10 28 51 67	2
10-6.8	4 6 7 8 12 17	10 29 48 67	4
10-6.9	4 6 8 11 12 13	12 54 54 96	9
11-7.1	3 9 22 24 31 34 39	3 42 111 132	0
11-7.2	3 9 13 22 24 25 31	3 48 84 177	0
11-7.3	7 9 12 18 22 24 25	3 54 63 195	0
11-7.4	3 6 9 13 22 24 31	5 47 77 182	0
11-7.5	3 4 7 9 12 22 24	10 40 91 154	0
11-7.6	3 4 9 10 11 13 22	15 48 99 162	0
11-7.7	3 4 6 7 9 13 22	15 49 95 165	2
11-7.8	3 4 6 8 11 12 13	21 72 135 240	10
12-8.1	3 9 13 22 24 25 31 37	4 72 144 354	0
12-8.2	7 9 12 18 22 24 25 38	4 81 108 390	0
12-8.3	3 9 13 22 24 25 31 38	5 69 141 375	0
12-8.4	3 6 7 9 13 22 24 31	8 73 124 364	0
12-8.5	3 4 6 7 9 12 13 22	21 81 171 357	2
12-8.6	3 4 6 9 10 11 13 22	22 76 178 364	0
12-8.7	3 4 6 7 8 11 12 13	30 108 252 546	11
13-9.1	3 6 9 13 22 24 25 31 37	7 102 219 690	0
13-9.2	3 7 9 12 18 22 24 25 38	7 105 207 696	0
13-9.3	3 9 13 15 22 24 25 31 37	8 92 249 654	0
13-9.4	3 6 7 9 12 13 22 24 31	12 109 198 672	0
13-9.5	3 4 6 7 9 10 12 13 22	30 118 306 726	0
13-9.6	3 4 6 7 8 9 11 12 13	40 162 432 1092	12
14-10.1	3 6 9 13 18 22 24 25 31 37	10 140 334 1236	0
14-10.2	3 7 9 12 18 22 24 25 31 38	10 141 330 1236	0
14-10.3	3 6 7 9 13 22 24 25 31 37	10 144 330 1209	0
14-10.4	3 6 7 9 12 13 22 24 25 31	13 147 315 1200	0
14-10.5	3 4 6 7 8 9 10 11 12 13	52 234 702 2028	13

Table 4A.3 (continued)

Design	Additional Points	W	$C2$
15-11.1	3 6 7 9 13 18 22 24 25 31 37	13 192 495 2055	0
15-11.2	3 6 7 9 12 13 22 24 25 31 37	14 198 486 2009	0
15-11.3	3 6 9 13 22 23 24 25 30 31 37	15 171 564 1963	0
16-12.1	3 6 7 9 13 18 22 24 25 31 35 37	16 256 720 3288	0
16-12.2	3 6 7 9 12 13 18 22 24 25 31 37	17 258 711 3275	0
16-12.3	3 6 7 9 13 18 21 22 24 25 31 37	19 232 789 3201	0
17-13.1	3 6 7 9 12 13 18 22 24 25 31 35 37	20 336 1014 5072	0
17-13.2	3 6 7 9 13 16 18 22 24 25 31 35 37	23 306 1107 4952	0
17-13.3	3 6 7 9 13 15 18 22 24 25 31 35 37	24 304 1096 4984	0
18-14.1	3 6 7 9 12 13 18 22 24 25 31 35 37 38	24 432 1404 7608	0
18-14.2	3 6 7 9 12 13 15 18 22 24 25 31 35 37	28 396 1518 7438	0
18-14.3	3 6 9 13 15 16 22 23 24 25 31 34 37 38	30 369 1602 7443	0
19-15.1	3 6 7 9 12 13 15 18 22 24 25 31 35 37 38	33 504 2052 10884	0
19-15.2	3 6 7 9 12 13 15 16 18 22 24 25 31 35 37	36 480 2112 10875	0
19-15.3	3 6 7 9 12 13 15 18 22 24 25 31 35 36 37	37 464 2202 10600	0
20-16.1	3 6 7 9 12 13 15 16 18 22 24 25 31 35 37 38	42 603 2808 15537	0
20-16.2	3 6 7 9 13 15 16 18 22 23 24 25 31 34 37 38	44 584 2852 15608	0
20-16.3	3 6 7 9 12 13 15 17 18 22 24 25 31 35 37 38	44 584 2900 15212	0

5

Designs with Maximum Estimation Capacity

This chapter continues with regular fractions of symmetrical factorials. A criterion of model robustness, called estimation capacity, is introduced and explored. Results on s^{n-k} designs with maximum estimation capacity are given. In many situations, these results are consistent with those under the minimum aberration criterion.

5.1 Preliminaries

The notion of estimation capacity is due to Sun (1993). It was studied in fuller detail by Cheng, Steinberg, and Sun (1999) and Cheng and Mukerjee (1998). The development in this chapter follows the last two papers. The criterion of estimation capacity aims at selecting a design that retains full information on the main effects, and as much information as possible on the two-factor interactions (2fi's) in the sense of entertaining the *maximum possible model diversity*, under the assumption of absence of interactions involving three or more factors. This approach will be seen to provide further statistical justification for the more common criterion of minimum aberration (MA).

As before, we consider s^{n-k} designs of resolution three or higher. By Theorem 2.7.1, any such design is equivalent to a set T of n points of $PG(n-k-1, s)$, so that the matrix $V(T)$ with columns given by these points has full row rank. Hence as in Chapter 4, an s^{n-k} design is denoted by the corresponding set of points T.

From Section 2.4, recall that any s^{n-k} design T involves $(s^{n-k}-1)/(s-1)$ ($= q$, say) alias sets, each containing s^k pencils. Since T has resolution three or higher, no two distinct main effect pencils are aliased with each other. Thus there are n alias sets of T, each of which contains one main effect pencil. Let $f = q - n$, and for $1 \leq i \leq f$, let $m_i(T)$ be the number of 2fi pencils in the ith of the remaining f alias sets of T. To avoid trivialities, throughout this chapter it is assumed that $f \geq 1$. If $f = 0$, then there is only one choice of T, namely, the entire $PG(n-k-1, s)$.

To introduce the criterion of estimation capacity, recall that it evaluates a design on the basis of its capability to handle model diversity. Note that in an s^n factorial, there are $\nu = \binom{n}{2}(s-1)$ 2fi pencils altogether. Thus, for $1 \le r \le \nu$, there are $\binom{\nu}{r}$ possible models that include all the main effects and r 2fi pencils; of course, any of these models assumes the ignorability of the remaining $\nu - r$ 2fi pencils and the absence of all interactions involving three or more factors. For any fixed r, let $E_r(T)$ be the number of models of this kind that can be estimated by a design T. In particular, for $s = 2$, there is practically no distinction between a factorial effect and the associated pencil. Hence for two-level factorials, $E_r(T)$ simply represents the number of models containing all the main effects and r 2fi's, which a design T can estimate when the remaining 2fi's and higher order interactions are absent.

An expression for $E_r(T)$ follows readily from Theorem 2.4.2. By this theorem, any model involving all the main effects and r 2fi pencils is estimable in T if and only if the r 2fi pencils occur in the f alias sets that do not contain any main effect pencil and no two of them occur in the same alias set. Thus, $E_r(T)$ equals the number of ways of choosing r 2fi pencils from these f alias sets such that no two of them are chosen from the same alias set. Therefore,

$$E_r(T) = \begin{cases} \sum \cdots \sum_{1 \le i_1 < \cdots < i_r \le f} \prod_{j=1}^{r} m_{i_j}(T) & \text{if } r \le f, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1.1}$$

The criterion of estimation capacity aims at maximizing $E_r(T)$ for every r ($1 \le r \le \nu$). An s^{n-k} design that achieves this is said to have *maximum estimation capacity.* Furthermore, given any two s^{n-k} designs T_1 and T_2, T_1 dominates T_2 with respect to estimation capacity if $E_r(T_1) \ge E_r(T_2)$ for every r, with strict inequality for some r.

Some ideas from the theory of majorization greatly facilitate the study of estimation capacity; see Marshall and Olkin (1979) for more details. Consider vectors $u = (u_1, \ldots, u_f)'$ and $w = (w_1, \ldots, w_f)'$ with real-valued elements. Then u is said to be *majorized* by w if

$$\sum_{i=1}^{f} u_i = \sum_{i=1}^{f} w_i \text{ and } \sum_{i=1}^{j} u_{[i]} \ge \sum_{i=1}^{j} w_{[i]} \ (1 \le j \le f-1),$$

where $u_{[1]} \le \cdots \le u_{[f]}$ and $w_{[1]} \le \cdots \le w_{[f]}$ are the ordered elements of u and w respectively. In this definition, if the condition $\sum_{i=1}^{f} u_i = \sum_{i=1}^{f} w_i$ is replaced by $\sum_{i=1}^{f} u_i \ge \sum_{i=1}^{f} w_i$, then u is said to be *upper weakly majorized* by w. A real-valued function g is called *Schur concave* if $g(u) \ge g(w)$ whenever u is majorized by w.

Lemma 5.1.1. *If u is upper weakly majorized by w, then $g(u) \ge g(w)$ for all functions g that are Schur concave and nondecreasing in each element.*

Proof. Since u is upper weakly majorized by w, one gets $\sum_{i=1}^{f} u_i \ge \sum_{i=1}^{f} w_i$. If equality holds here, then u is majorized by w and the result follows by the definition of Schur concavity. Otherwise, let w^* be a vector obtained from w, replacing the element of w corresponding to $w_{[f]}$ by $(w_{[f]}+\sum_{i=1}^{f} u_i-\sum_{i=1}^{f} w_i)$, and keeping the other elements unchanged. Then u is majorized by w^* and $g(u) \geq g(w^*)$, again by the definition of Schur concavity. The result follows by noting that $g(w^*) \geq g(w)$ for all functions g that are nondecreasing in each element. □

For any s^{n-k} design T, let $m(T) = (m_1(T), \ldots, m_f(T))'$. Then the following result, which plays a fundamental role in the study of estimation capacity, holds.

Theorem 5.1.1. *Given two s^{n-k} designs T_1 and T_2, if $m(T_1)$ is upper weakly majorized by $m(T_2)$ and not obtainable from $m(T_2)$ by permuting its elements, then T_1 dominates T_2 with respect to estimation capacity.*

Proof. By (5.1.1), using a result of Marshall and Olkin (1979, p.78, Proposition F.1), it can be shown that $E_r(T)$ is a Schur concave function of $m(T)$ for each r. It is also easily seen that $E_r(T)$ is nondecreasing in each element of $m(T)$. Hence if $m(T_1)$ is upper weakly majorized by $m(T_2)$, then by Lemma 5.1.1,

$$E_r(T_1) \geq E_r(T_2) \tag{5.1.2}$$

for every r. If equality holds in (5.1.2) for every r, then by (5.1.1), the polynomials $\Pi_{i=1}^{f}\{y-m_i(T_1)\}$ and $\Pi_{i=1}^{f}\{y-m_i(T_2)\}$ have identical expanded forms and hence the same set of zeros; i.e., $m(T_1)$ can be obtained from $m(T_2)$ by permuting its elements. Since this is not the case here, the inequality must be strict for some r in (5.1.2). This completes the proof. □

Example 5.1.1. Consider the 3^{5-2} designs $d(B_1)$ and $d(B_2)$ introduced in Example 2.5.1. In accordance with the notational system of this chapter, these are denoted by T_1 and T_2, where T_1 and T_2 are the corresponding sets of points of $PG(2,3)$. The defining relations of T_1 and T_2 are given by (2.5.1) and (2.5.2) respectively. Here $n = 5$, $k = 2$, $q = 13$, and $f = q - n = 8$. From (2.5.1), it can be seen that the eight alias sets of T_1 that contain no main effect pencil are given by

$$\begin{array}{rl} 12^2 = 14 = 24 = 35^2 = \cdots, & 13 = 25 = \cdots, \\ 13^2 = 45 = \cdots, & 23 = 45^2 = \cdots, \\ 23^2 = 15^2 = \cdots, & 34 = 25^2 = \cdots, \\ 34^2 = 15 = \cdots, & 35 = \cdots. \end{array}$$

Only the 2fi pencils in each of these alias sets are shown here and the dots represent pencils belonging to interactions that involve three or more factors. Thus $m(T_1) = (4,2,2,2,2,2,2,1)'$. Similarly, by (2.5.2), $m(T_2) = (3,3,2,2,1,1,1,1)'$. Clearly, $m(T_1)$ and $m(T_2)$ satisfy the conditions of Theorem 5.1.1. Hence T_1 dominates T_2 with respect to estimation capacity, a fact that can be verified directly from (5.1.1) as well. It will be seen in Section 5.4 that T_1, in fact, has maximum estimation capacity. In addition, T_1 has MA because it is isomorphic to the design 5-2.1 of Table 4A.2. □

In general, do MA designs have an edge over others under the criterion of estimation capacity as well? Theorem 5.1.1 suggests that a design T should behave well under the latter criterion if $\sum_{i=1}^{f} m_i(T)$ is large and $m_1(T), \ldots, m_f(T)$ are close to one another. The next result shows why an MA design can be expected to meet these requirements. In what follows, $m_{f+1}(T), \ldots, m_q(T)$ denote the numbers of 2fi pencils in the n $(= q - f)$ alias sets of T, each of which contains a main effect pencil. As in Chapter 4, the wordlength pattern of a design T is represented by $(A_1(T), \ldots, A_n(T))$, and $A_1(T) = A_2(T) = 0$.

Theorem 5.1.2. *For any s^{n-k} design T,*

$$(a)\quad \sum_{i=1}^{f} m_i(T) = \binom{n}{2}(s-1) - 3A_3(T),$$

$$(b)\quad \sum_{i=1}^{q} \{m_i(T)\}^2 = \binom{n}{2}(s-1) + 6(s-2)A_3(T) + 6A_4(T).$$

Proof. Part (a) follows from noting that there are $\nu = \binom{n}{2}(s-1)$ 2fi pencils, that none of them appear in the defining relation of T with resolution three or higher, and that $3A_3(T)$ of them are aliased with the main effect pencils.

To prove (b), note that

$$\frac{1}{2}\sum_{i=1}^{q}\{m_i(T)\}^2 = \frac{1}{2}\sum_{i=1}^{q} m_i(T) + \frac{1}{2}\sum_{i=1}^{q} m_i(T)\{m_i(T) - 1\}. \tag{5.1.3}$$

As in the proof of (a),

$$\sum_{i=1}^{q} m_i(T) = \binom{n}{2}(s-1). \tag{5.1.4}$$

Also, the second term in the right-hand side of (5.1.3) equals the number of unordered pairs that can be formed out of the distinct 2fi pencils appearing in the same alias set of T. In accordance with (2.4.4) and (2.4.9), any defining pencil b_{def} of T is said to account for such an unordered pair if b_{def} equals $b - b^*$ or $b^* - b$ for some representations b and b^* of the 2fi pencils in the pair. Clearly, then b_{def} has three or four nonzero entries since b and b^* have two nonzero entries each and T has resolution three or higher. Now observe the following:

(i) Each defining pencil of T with three or four nonzero entries accounts respectively for $3(s-2)$ or 3 unordered pairs of 2fi pencils appearing in the same alias set of T. For example, a defining pencil $(b_1, b_2, b_3, 0, \ldots, 0)'$, where $b_i \neq 0$ $(i = 1, 2, 3)$, accounts for $3(s-2)$ such unordered pairs as shown below:

$$\{((b_{11}, b_2, 0, \ldots, 0)', (b_{11} - b_1, 0, -b_3, 0, \ldots, 0)') : \quad b_{11}(\neq 0, b_1) \in GF(s)\},$$
$$\{((b_1, b_{21}, 0, \ldots, 0)', (0, b_{21} - b_2, -b_3, 0, \ldots, 0)') : \quad b_{21}(\neq 0, b_2) \in GF(s)\},$$
$$\{((b_1, 0, b_{31}, 0, \ldots, 0)', (0, -b_2, b_{31} - b_3, 0, \ldots, 0)') : \quad b_{31}(\neq 0, b_3) \in GF(s)\}.$$

(ii) No two distinct defining pencils can account for the same unordered pair as considered here.

The above argument shows that the second term in the right-hand side of (5.1.3) equals $3(s-2)A_3(T) + 3A_4(T)$. Hence (b) follows from (5.1.3) and (5.1.4). □

An MA design minimizes $A_3(T)$ and, subject to that condition, minimizes $A_4(T)$ as well. Hence by Theorem 5.1.2, it maximizes $\sum_{i=1}^{f}\{m_i(T)\}$ and, subject to that condition, minimizes $\sum_{i=1}^{q}\{m_i(T)\}^2$. Therefore, one would also expect $\sum_{i=1}^{f}\{m_i(T)\}^2$ to be small in an MA design. Thus, in addition to maximizing $\sum_{i=1}^{f} m_i(T)$, an MA design is expected to keep $m_1(T), \ldots, m_f(T)$ close to one another. As noted earlier, in view of Theorem 5.1.1, these are precisely the requirements for a design to behave well under the criterion of estimation capacity. MA designs are therefore expected to perform well under the latter criterion. These issues will be explored in more detail in the subsequent sections and the sufficient condition in Theorem 5.1.1 will be seen to be particularly helpful in the derivation of the results.

5.2 Connection with Complementary Sets

As in the last two chapters, complementary sets play a key role in the study of estimation capacity. It is seen below that they arise naturally in this context.

Theorem 2.7.1 suggests a one-to-one correspondence between the q $[= (s^{n-k} - 1)/(s-1)]$ points of $PG(n-k-1, s)$ and the q alias sets of an s^{n-k} design T. Any alias set of T corresponds to the point $V(T)b$ of $PG(n-k-1, s)$, where b is any pencil in the alias set. In particular, if an alias set contains a main effect pencil, say b, then $V(T)b$ reduces to a column of $V(T)$, i.e., a point of T. Thus the n alias sets of T containing the main effect pencils correspond to the n points of T. The remaining $f\ (= q - n)$ alias sets of T, therefore, correspond to the f points of $\overline{T}$, the *complementary set* of T in $PG(n-k-1, s)$. Let $\overline{T} = \{h_1, \ldots, h_f\}$ and $T = \{h_{f+1}, \ldots, h_q\}$, where $h_1, \ldots, h_q$ are the points of $PG(n-k-1, s)$ and, for $1 \leq i \leq f$, h_i corresponds to the ith alias set of T that contains no main effect pencil. The following lemma is then obvious.

Lemma 5.2.1. *For $1 \leq i \leq f$, $m_i(T)$ equals the number of 2fi pencils b such that $V(T)b$ is proportional to, and hence representative of, h_i.*

Some more notation will be needed for presenting the subsequent results. For $\lambda(\neq 0) \in GF(s)$ and distinct i, j, r $(1 \leq i, j, r \leq q)$, define the indicators

$$\zeta_{ijr}(\lambda) = \begin{cases} 1, & \text{if } h_j + \lambda h_r \text{ is proportional to } h_i, \\ 0, & \text{otherwise,} \end{cases} \tag{5.2.1}$$

$$\theta_{ijr} = \begin{cases} 1, & \text{if } h_i, h_j \text{ and } h_r \text{ are linearly dependent,} \\ 0, & \text{otherwise.} \end{cases} \tag{5.2.2}$$

It is easily seen that for any fixed distinct i, j, r, there exists at most one $\lambda\,(\neq 0) \in\ GF(s)$ such that $\zeta_{ijr}(\lambda) = 1$, and that such a λ exists if and only if h_i, h_j, and h_r are linearly dependent. Hence

$$\sum_{\lambda(\neq 0) \in GF(s)} \zeta_{ijr}(\lambda) = \theta_{ijr}. \tag{5.2.3}$$

Furthermore, linear combinations of any two points of $PG(n{-}k{-}1, s)$ generate additional $s - 1$ points, so that for any fixed i, j $(1 \leq i \neq j \leq q)$,

$$\sum_{\substack{r=1 \\ r \neq i,j}}^{q} \theta_{ijr} = s - 1. \tag{5.2.4}$$

For $1 \leq i \leq f$, define

$$\begin{aligned} \phi_i = {} & \text{number of linearly dependent triplets } \{h_i, h_j, h_r\} \text{ such that } i, j, r \text{ are} \\ & \text{distinct members of } \{1, \ldots, f\} \text{ and } j < r. \end{aligned} \tag{5.2.5}$$

The following identity shows that the computation of $m_i(T)$ is equivalent to the computation of ϕ_i.

Lemma 5.2.2. *For* $1 \leq i \leq f,\ m_i(T) = \frac{1}{2}(s-1)(q - 2f + 1) + \phi_i$.

Proof. For any fixed i $(1 \leq i \leq f)$, let $\Delta_{1i}, \ldots, \Delta_{6i}$ denote sums over θ_{ijr} with respect to j and r, the ranges of summation being

$$\begin{aligned} &(i) && f + 1 \leq j < r \leq q, \\ &(ii) && f + 1 \leq j \neq r \leq q, \\ &(iii) && 1 \leq j \neq r \leq q,\ \ j \neq i,\ \ r \neq i, \\ &(iv) && 1 \leq j \neq r \leq f,\ \ j \neq i,\ \ r \neq i, \\ &(v) && 1 \leq j \leq f,\ \ f + 1 \leq r \leq q,\ \ j \neq i, \\ &(vi) && f + 1 \leq j \leq q,\ \ 1 \leq r \leq f,\ \ r \neq i, \end{aligned}$$

respectively. By (5.2.2) and the definition of ϕ_i,

$$\Delta_{1i} = \frac{1}{2}\Delta_{2i},\ \Delta_{5i} = \Delta_{6i},\ \phi_i = \frac{1}{2}\Delta_{4i}. \tag{5.2.6}$$

Now, $T = \{h_{f+1}, \ldots, h_q\}$ and hence $V(T) = (h_{f+1}, \ldots, h_q)$. Since any 2fi pencil b has two nonzero entries the first of which can be taken as 1, it is clear that $V(T)b$ must be of the form $h_j + \lambda h_r$ for some j, r $(f+1 \le j < r \le q)$, where λ is the second nonzero entry of b. Hence by Lemma 5.2.1, for $1 \le i \le f$, $m_i(T)$ equals the number of choices of j, r $(f+1 \le j < r \le q)$ and $\lambda (\neq 0) \in GF(s)$ such that $h_j + \lambda h_r$ is proportional to h_i, i.e., by (5.2.1),

$$m_i(T) = \sum_{f+1 \le j} \sum_{< r \le q} \sum_{\lambda(\neq 0) \in GF(s)} \zeta_{ijr}(\lambda),$$

so that using (5.2.3), (5.2.6), and (i)–(vi) above,

$$\begin{aligned} m_i(T) &= \sum_{f+1 \le j} \sum_{< r \le q} \theta_{ijr} = \Delta_{1i} = \frac{1}{2}\Delta_{2i} \\ &= \frac{1}{2}(\Delta_{3i} - \Delta_{4i} - \Delta_{5i} - \Delta_{6i}) = \frac{1}{2}(\Delta_{3i} - \Delta_{4i} - 2\Delta_{5i}). \end{aligned} \tag{5.2.7}$$

For fixed i, j $(1 \le i \neq j \le f)$, by (5.2.4),

$$\sum_{r=f+1}^{q} \theta_{ijr} = \sum_{\substack{r=1 \\ r \neq i,j}}^{q} \theta_{ijr} - \sum_{\substack{r=1 \\ r \neq i,j}}^{f} \theta_{ijr} = s - 1 - \sum_{\substack{r=1 \\ r \neq i,j}}^{f} \theta_{ijr}. \tag{5.2.8}$$

Summing (5.2.8) over j $(1 \le j \le f, j \neq i)$ and recalling (iv) and (v) above,

$$\Delta_{5i} = (s-1)(f-1) - \Delta_{4i}. \tag{5.2.9}$$

Similarly, for fixed i $(1 \le i \le f)$, summing (5.2.4) over j $(1 \le j \le q, j \neq i)$,

$$\Delta_{3i} = (s-1)(q-1). \tag{5.2.10}$$

If one substitutes (5.2.9) and (5.2.10) in (5.2.7) and then employs the last relation in (5.2.6), the result follows. □

Example 5.1.1 (continued). To illustrate the above ideas, consider again the design T_1 in Example 5.1.1. Recall from Example 2.5.1 that the treatment combinations x in T_1 satisfy $B_1 x = 0$, where

$$B_1 = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 & 2 \end{bmatrix}.$$

The row spaces of B_1 and

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

are orthogonal complements of each other. As in the proof of Theorem 2.7.1, interpreting the columns of G_1 as points of $PG(2,3)$, the set T_1 can be explicitly described as

$$T_1 = \{(1,0,0)', (0,1,0)', (0,0,1)', (1,1,0)', (1,2,1)'\}.$$

Thus $V(T_1) = G_1$, and

$$\overline{T}_1 = \{(1,2,0)', (1,0,1)', (1,0,2)', (0,1,1)', (0,1,2)', (1,1,1)', (1,1,2)', (1,2,2)'\}.$$

The eight points of $\overline{T}_1$ correspond to the eight alias sets of T_1 listed in the previous section. Consider the first set, which is given by $12^2 = 14 = 24 = 35^2 = \cdots$. Writing any pencil b in this alias set in vector notation, $V(T_1)b$ $(= G_1 b)$ represents the point $(1,2,0)'$ of $\overline{T}_1$. Thus this alias set corresponds to the first point $(1,2,0)'$ of $\overline{T}_1$. Denote the eight points of $\overline{T}_1$ by $h_1, \ldots, h_8$ in the order in which they are listed above, and observe that there are six linearly dependent triplets containing $h_1 = (1,2,0)'$ and two other points of $\overline{T}_1$. These triplets are $\{h_1,h_2,h_4\}, \{h_1,h_2,h_7\}, \{h_1,h_4,h_7\}, \{h_1,h_3,h_5\}, \{h_1,h_3,h_6\}, \{h_1,h_5,h_6\}$. Thus $\phi_1 = 6$. Since $s = 3$, $q = 13$, and $f = 8$ in this example, for $i = 1$, the right-hand side of the identity in Lemma 5.2.2 now equals $\phi_1 - 2 = 4$, which is the same as the left-hand side, namely, $m_1(T_1)$. Similarly, one can verify from first principles that the conclusion of Lemma 5.2.2 holds for each of the remaining seven alias sets listed earlier. □

Let $\phi(\overline{T}) = (\phi_1, \ldots, \phi_f)'$, where ϕ_i is defined in (5.2.5). Theorem 5.1.1 and Lemma 5.2.2 yield the following important result.

Theorem 5.2.1. *Given two s^{n-k} designs T_1 and T_2, if $\phi(\overline{T}_1)$ is upper weakly majorized by $\phi(\overline{T}_2)$ and not obtainable from $\phi(\overline{T}_2)$ by permuting its elements, then T_1 dominates T_2 with respect to estimation capacity.*

One needs to consider only the complementary set $\overline{T}$, of cardinality f, to get $\phi(\overline{T})$. Thus Lemma 5.2.2 and Theorem 5.2.1 can substantially simplify the study of estimation capacity, particularly when f is small, a situation that corresponds to the nearly saturated cases. While this is in the spirit of the last two chapters, a new feature is that the aliasing pattern has to be explicitly taken care of.

In particular, for $f = 1$ or 2, all designs are isomorphic and hence, as with the MA criterion, they are equivalent with respect to estimation capacity. This is clear also from (5.1.1) and Lemma 5.2.2 if one notes that $\phi_i = 0$ for each i when $f = 1$ or 2. In the rest of this chapter, therefore, attention will be focused on the situation $f \geq 3$.

5.3 Estimation Capacity in 2^n Factorials

Throughout this section, the case $s = 2$ is considered. Three points of $PG(n-k-1, 2)$ are linearly dependent if and only if they add up to the null vector. Clearly, three such points form a 1–flat or a line; cf. Section 4.4. Thus for $1 \leq i \leq f$, one can geometrically interpret ϕ_i as the number of lines that pass through the ith point of $\overline{T}$ and two other points of $\overline{T}$. Moreover, since each line contains three points, $\sum_{i=1}^{f} \phi_i$ equals thrice the total number of lines contained in $\overline{T}$. Theorem 5.2.1 therefore suggests that a 2^{n-k} design should perform well with regard to estimation capacity if it keeps the number of lines contained in $\overline{T}$ large and distributes these lines over the points of $\overline{T}$ as uniformly as possible. Taking $s = 2$ in (4.3.3) or directly from Section 3.3, it is evident that the number of lines contained in $\overline{T}$ is the same as $A_3(\overline{T})$.

Lemma 5.3.1 (Chen and Hedayat, 1996), on subsets of $PG(n-k-1, 2)$ that contain the maximum number of lines, or equivalently maximize $A_3(\overline{T})$, will be very useful in this section. Its proof is ingenious but somewhat long and hence omitted here. In what follows, an r-set or an r-subset means a set or a subset of cardinality r.

Lemma 5.3.1. *Let $2^{w-1} \leq f < 2^w (2 \leq w \leq n-k)$. Then an f-subset $\overline{T}$ of $PG(n-k-1, 2)$ contains the maximum number of lines if and only if $\overline{T} = \mathcal{F} - \mathcal{G}$, i.e., the complement of $\mathcal{G}$ in $\mathcal{F}$, where $\mathcal{F}$ is any $(w-1)$-flat of $PG(n-k-1, 2)$ and $\mathcal{G}$ is a $(2^w - 1 - f)$-subset of $\mathcal{F}$ such that $\mathcal{G}$ contains no line at all.*

It is not hard to see that one can always find a subset $\mathcal{G}$ as envisaged in Lemma 5.3.1. Let $h^{(1)}, \ldots, h^{(w)}$ be linearly independent points in a $(w-1)$-flat $\mathcal{F}$, and define $\mathcal{G}_0 = \mathcal{F} - \mathcal{F}_0$, where $\mathcal{F}_0$ is the $(w-2)$-flat generated by $h^{(1)}, \ldots, h^{(w-1)}$. Then $\mathcal{G}_0$ contains no line at all since its points are of the form

$$h^{(w)} + \text{a linear combination of } h^{(1)}, \ldots, h^{(w-1)}.$$

Now $f \geq 2^{w-1}$, i.e., $2^w - 1 - f \leq 2^{w-1} - 1$, and $\mathcal{G}_0$ has cardinality 2^{w-1}. So it is enough to choose $\mathcal{G}$ as any $(2^w - 1 - f)$-subset of $\mathcal{G}_0$.

Theorem 5.3.1. *Suppose $2^{w-1} \leq f < 2^w (2 \leq w \leq n-k)$ and $\mathcal{F}$ is a $(w-1)$-flat of $PG(n-k-1, 2)$. Furthermore, suppose there exists a $(2^w - 1 - f)$-subset $\mathcal{G}^*$ of $\mathcal{F}$ such that no four points of $\mathcal{G}^*$ are linearly dependent. Then the 2^{n-k} design T^*, where $\overline{T}^* = \mathcal{F} - \mathcal{G}^*$, has maximum estimation capacity (MEC). In this case, if a design T has MEC, then $\overline{T}$ must have the structure described above.*

Proof. Consider any $(2^w - 1 - f)$-subset of $\mathcal{F}$, say $\mathcal{G}$, that contains no line, and let $\overline{T} = \mathcal{F} - \mathcal{G}$. Since $\mathcal{F}$ is a $(w-1)$-flat, each of its points belongs to $2^{w-1} - 1$ lines that are contained in $\mathcal{F}$. Hence for $1 \leq i \leq f$, there are $2^{w-1} - 1$ lines, contained in $\mathcal{F}$, passing through the ith point of $\overline{T} (\subset \mathcal{F})$. Three mutually exclusive and exhaustive possibilities arise regarding the other two points in any such line:

(i) both belong to $\overline{T}$,
(ii) one belongs to $\overline{T}$ and the other to $\mathcal{G}$,
(iii) both belong to $\mathcal{G}$.

By definition, the possibility (i) accounts for ϕ_i lines. Also, let there be r_i lines of type (iii), i.e., passing through the ith point of $\overline{T}$ and two points of $\mathcal{G}$. Since two lines can have at most one point in common, these r_i lines together cover $2r_i$ points of $\mathcal{G}$. Each of the remaining $(2^w-1-f-2r_i)$ points of $\mathcal{G}$ accounts for a line of type (ii). This enumeration yields $2^{w-1}-1=\phi_i+(2^w-1-f-2r_i)+r_i$, that is,

$$\phi_i = f - 2^{w-1} + r_i,\ 1 \le i \le f. \tag{5.3.1}$$

Now, no four points of $\mathcal{G}^*$ are linearly dependent. Hence $\mathcal{G}^*$ contains no line, and for any four points $h_{(1)},\ldots,h_{(4)} \in \mathcal{G}^*$, the lines determined by $(h_{(1)},h_{(2)})$ and $(h_{(3)},h_{(4)})$ do not intersect. Consequently, for $\overline{T}^* = \mathcal{F}-\mathcal{G}^*$, each r_i is either 1 or 0, i.e., by (5.3.1), $\phi_1,\ldots,\phi_f$ differ form one another by at most unity. Also by Lemma 5.3.1, $\overline{T}^*$ contains the maximum possible number of lines, i.e., maximizes $\sum_{i=1}^{f}\phi_i$. It is therefore clear that $\phi(\overline{T}^*)$ is upper weakly majorized by $\phi(\overline{T})$ for all f-subsets $\overline{T}$ of $PG(n-k-1,2)$. Hence by Theorem 5.2.1, the design T^*, where T^* is the complement of $\overline{T}^*$ in $PG(n-k-1,2)$, has MEC.

If any other T also represents a design with MEC, then $\phi(\overline{T})$ can be obtained from $\phi(\overline{T}^*)$ by permuting its elements. Then $\overline{T}$, like $\overline{T}^*$, contains the maximum possible number of lines, so that by Lemma 5.3.1, $\overline{T}=\mathcal{F}-\mathcal{G}$, where $\mathcal{F}$ is some $(w-1)$-flat and $\mathcal{G}(\subset \mathcal{F})$ contains no line at all. Also, since $\phi(\overline{T})$ is a permutation of $\phi(\overline{T}^*)$, by (5.3.1) each r_i is either 1 or 0 with reference to such $\mathcal{G}$. Thus given any point of $\overline{T}$, there exists at most one line passing through that point and two points of $\mathcal{G}$. Since $\mathcal{G}$ contains no line, it follows that for any four points $h_{(1)},\ldots,h_{(4)} \in \mathcal{G}$, the lines determined by the pairs $(h_{(1)},h_{(2)})$ and $(h_{(3)},h_{(4)})$ do not intersect. Consequently, no four points of $\mathcal{G}$ are linearly dependent. □

In the setup of Theorem 5.3.1, $V(T^*)$ satisfies the requirement of having full row rank whenever $w<n-k$. This follows using the same argument as with Theorem 4.4.2.

Some applications of Theorem 5.3.1 are now considered. For $f=2^w-1$ $(2\le w<n-k)$, the design given by T^* has MEC if and only if $\overline{T}^*$ is a $(w-1)$-flat. For $f=2^w-2$, 2^w-3, or 2^w-4 $(3\le w<n-k)$, T^* represents a design with MEC if and only if $\overline{T}^*$ is obtained by deleting (i) any one point, (ii) any two points, or (iii) any three noncollinear points from a $(w-1)$-flat. In each of these cases, by Lemma 5.3.1, the structure of $\overline{T}^*$ is the only one that can maximize the number of lines therein. Hence recalling the complementary design theory developed in Chapter 3, T^* has MA as well. The cases considered above cover, in particular, the nearly saturated situations given by $3\le f\le 7$, since $3=2^2-1, 4=2^3-4$, and so on.

For $f=2^w-5$ $(4\le w<n-k)$, T^* has MEC if and only if $\overline{T}^*$ is obtained by deleting any four linearly independent points from a $(w-1)$-flat. The same arguments as with the case $f=11$ in Section 3.3 show that such a design has MA.

For $f=2^w-6$ $(4\le w<n-k)$, T^* has MEC if $\overline{T}^*$ is obtained by deleting from a $(w-1)$-flat any five points of the form $h^{(1)}$, $h^{(2)}$, $h^{(3)}$, $h^{(4)}$,

and $h^{(1)} + h^{(2)} + h^{(3)} + h^{(4)}$, where $h^{(1)}, \ldots, h^{(4)}$ are linearly independent. However, unless $w = 4$, this is not the only structure of $\overline{T}$ to ensure MEC. For example, when $w \geq 5$, let $\overline{T}_1^*$ be obtained by deleting five linearly independent points from a $(w-1)$-flat. Then T_1^* also has MEC. It can be seen as before that T_1^* has MA as well. On the other hand, T^* is not an MA design unless $w = 4$. For $f = 2^w - 6$, if $w\,(\geq 4)$ equals $n - k$, will $V(T^*)$ or $V(T_1^*)$ still have full row rank? To answer this question, note that if $w = n - k$, then $n = (2^{n-k} - 1) - f = 5$, and since $n - k = w \geq 4$, one must have $k = 1$. Thus $w = n - k = 4$, so that T_1^* does not arise and T^* consists of the five points $h^{(1)}, \ldots, h^{(4)}$ and $h^{(1)} + \cdots + h^{(4)}$. Since $n - k = 4$ and $h^{(1)}, \ldots, h^{(4)}$ are linearly independent, $V(T^*)$ indeed has full row rank.

Example 5.3.1. Let $n = 21$, $k = 16$. Then $f = (2^{n-k} - 1) - n = 10 = 2^4 - 6$. Using the compact notation, let $\mathcal{F}$ be the 3-flat generated by the points 1, 2, 3, and 4. Then by the discussion in the last paragraph, the 2^{21-16} design T^*, where

$$\overline{T}^* = \{1, 2, 3, 4, 12, 13, 14, 23, 24, 34\}$$

is obtained by deleting the four linearly independent points 123, 124, 134, 234 and their sum 1234 from $\mathcal{F}$, has MEC. Here $f = 10$ and from Section 3.3 it is clear that T^* has MA as well. □

Further examples showing the agreement between the two criteria of MA and MEC abound. For illustration, we revisit the 16-run 2^{n-k} designs. For $5 \leq n \leq 12$, all nonisomorphic 16-run designs were listed and ranked with respect to aberration in Table 3A.2. Using the majorization argument of Theorem 5.2.1, from that table it can be seen that the MA design uniquely has MEC over this range of n except for $n = 6$ and 7. The case $n = 9$ is illustrated in Example 5.3.2 below. One exceptional case $n = 6$ is treated in Example 5.3.3. The other exceptional case $n = 7$ is given in an exercise.

Example 5.3.2. Let $n = 9$, $k = 5$. Consider the five 2^{9-5} designs 9-5.1, ..., 9-5.5 listed in Table 3A.2. These are ranked according to aberration, i.e., 9-5.1 has MA whereas 9-5.5 performs the worst. Let $\overline{T}_1, \ldots, \overline{T}_5$ represent the set $\overline{T}$ for the five designs. Using the same notation as in Example 5.3.1, then

$$\overline{T}_1 = \{23, 123, 24, 124, 34, 134\}, \qquad \overline{T}_2 = \{23, 123, 14, 124, 134, 234\},$$
$$\overline{T}_3 = \{123, 24, 124, 34, 134, 1234\}, \quad \overline{T}_4 = \{123, 124, 34, 134, 234, 1234\},$$
$$\overline{T}_5 = \{24, 124, 34, 134, 234, 1234\}.$$

Observe that $\overline{T}_1$ contains four lines, namely, $\{23, 24, 34\}$, $\{23, 124, 134\}$, $\{123, 24, 134\}$, and $\{123, 124, 34\}$. Out of these four lines, two pass through any point of $\overline{T}_1$. Hence $\phi(\overline{T}_1) = (2, 2, 2, 2, 2, 2)'$. Similarly,

$$\phi(\overline{T}_2) = (1, 1, 1, 1, 1, 1)', \quad \phi(\overline{T}_3) = (2, 1, 1, 1, 1, 0)',$$
$$\phi(\overline{T}_4) = (1, 1, 1, 0, 0, 0)', \quad \phi(\overline{T}_5) = (0, 0, 0, 0, 0, 0)'.$$

Thus, for $1 \le j \le 4$, $\phi(\overline{T}_j)$ is upper weakly majorized by $\phi(\overline{T}_{j+1})$ and not obtainable from $\phi(\overline{T}_{j+1})$ by permuting its elements. Hence by Theorem 5.2.1, the MA design 9-5.1 has MEC as well. Moreover, in this example the two criteria of aberration and estimation capacity yield identical ranking of the designs. □

Example 5.3.3. Let $n = 6$, $k = 2$. Consider the four 2^{6-2} designs 6-2.1, ..., 6-2.4 listed in Table 3A.2. Let $\overline{T}_1, \ldots, \overline{T}_4$ represent the set $\overline{T}$ for these four designs respectively. As in the last example, it can be checked that

$$\phi(\overline{T}_1) = (1,1,3,3,3,3,3,3,4)', \quad \phi(\overline{T}_2) = (2,2,2,2,2,2,3,3,3)',$$
$$\phi(\overline{T}_3) = (2,2,2,2,2,2,2,2,2)', \quad \phi(\overline{T}_4) = (1,1,2,2,2,2,2,3,3)',$$

up to a permutation of their elements. The majorization argument of Theorem 5.2.1 shows that both 6-2.3 and 6-2.4 are dominated by 6-2.2 with respect to estimation capacity. Therefore, one needs only to compare 6-2.1 and 6-2.2. Since neither of $\phi(\overline{T}_1)$ and $\phi(\overline{T}_2)$ is upper weakly majorized by the other, one has to compute $E_r(T_1)$ and $E_r(T_2)$ explicitly for various r, using (5.1.1) and Lemma 5.2.2. Here $s = 2$, $q = 15$, $f = 9$, and Lemma 5.2.2 yields $m_i(T) = \phi_i - 1$ for each i. Hence

$$m(T_1) = (0,0,2,2,2,2,2,2,3)', \quad m(T_2) = (1,1,1,1,1,1,2,2,2)'$$

and by (5.1.1),

$$(E_i(T_1))_{i=1}^{9} = (15, 96, 340, 720, 912, 640, 192, 0, 0)$$

and

$$(E_i(T_2))_{i=1}^{9} = (12, 63, 190, 363, 456, 377, 198, 60, 8).$$

Since $f = 9$, clearly $E_r(T_1) = E_r(T_2) = 0$ for $r > 9$. Hence the MA design 6-2.1 maximizes $E_r(T)$ for $1 \le r \le 6$, whereas the next best design 6-2.2 maximizes $E_r(T)$ for $r = 7, 8, 9$. □

Cheng, Steinberg, and Sun (1999) and Cheng and Mukerjee (1998) reported similar studies concerning 32-run 2^{n-k} designs. Again the MA designs have MEC whenever $n \le 8$ or $n \ge 16$.

The results in this section show that the criteria of MA and MEC are in general agreement. On the other hand, the MaxC2 criterion as introduced in Section 3.4 can be in conflict with the MA criterion as demonstrated in Example 3.4.1 and supported by the design tables at the end of Chapter 3. More specifically, in Example 3.4.1, the 2^{9-4} MA design d_0 has eight clear 2fi's, while the second best 2^{9-4} design d_1, according to the MA criterion, has 15 clear 2fi's. It can be verified that the $E_r(T)$ values of d_0 are larger than those of d_1 and thus d_0 dominates d_1 with respect to estimation capacity (details left in an exercise). Therefore the two criteria of MEC and MaxC2 are not in general agreement and give different goodness measures of a design.

The aim of MaxC2 is to find a *single* model consisting of all the main effects and as many 2fi's as possible that can be estimated without being aliased. The definition of clear effects ensures that the estimability of effects here does not require the 2fi's not in the model to be absent. Through the maximization of the $E_r(T)$ values, the aim of MEC is to get as *many models* as possible in which all the main effects and r 2fi's are estimable. For MEC the estimability of effects requires all the 2fi's not in the model to be absent. Because of this major difference, the behavior of the two criteria can be very different. To further understand why these criteria perform differently, more research is needed on their implications in data analysis.

5.4 Estimation Capacity in s^n Factorials

We now turn to the case of general prime or prime power s. Consider an s^{n-k} design T and as before, write $\overline{T} = \{h_1, \ldots, h_f\}$, $T = \{h_{f+1}, \ldots, h_q\}$, and $V(\overline{T}) = (h_1, \ldots, h_f)$. To avoid trivialities, let $f \geq 3$. Then the following lemma holds.

Lemma 5.4.1. *(a) For* $1 \leq i \leq f$*,*

$$\phi_i \leq \frac{1}{2}(f-1)\min\{f-2, s-1\}. \tag{5.4.1}$$

(b) For $3 \leq f \leq s+1$*, equality holds in (5.4.1) for every* i *if and only if* $\text{rank}[V(\overline{T})] = 2$.
(c) For $f > s+1$*, equality holds in (5.4.1) for every* i *if and only if* $f = (s^w - 1)/(s-1)$ *and* $\overline{T}$ *is a* $(w-1)$*-flat with* $w \geq 3$.

Proof. (a) By the last identity in (5.2.6), for $1 \leq i \leq f$,

$$\phi_i = \frac{1}{2} \sum_{\substack{1 \leq j \neq \\ j,r}} \sum_{\substack{r \leq f \\ \neq\ i}} \theta_{ijr}, \tag{5.4.2}$$

where the indicators θ_{ijr} are defined in (5.2.2). Since $\theta_{ijr} \leq 1$ for each i, j, r, (5.4.2) yields

$$\phi_i \leq \frac{1}{2}(f-1)(f-2). \tag{5.4.3}$$

Again, by (5.2.4) and (5.4.2),

$$\phi_i = \frac{1}{2}\sum_{\substack{j=1 \\ j \neq i}}^{f} \left(\sum_{\substack{r=1 \\ r \neq i,j}}^{f} \theta_{ijr} \right) \leq \frac{1}{2}\sum_{\substack{j=1 \\ j \neq i}}^{f} \left(\sum_{\substack{r=1 \\ r \neq i,j}}^{q} \theta_{ijr} \right) = \frac{1}{2}(f-1)(s-1). \tag{5.4.4}$$

Combining (5.4.3) and (5.4.4), the inequality (5.4.1) follows.

(b) For $3 \leq f \leq s+1$, (5.4.1) reduces to (5.4.3), where equality holds for every i if and only if $\theta_{ijr} = 1$ for every choice of distinct i, j, r from $\{1, \ldots, f\}$, i.e., if and only if every three points of $\overline{T}$ are linearly dependent. Obviously this happens if and only if rank$[V(\overline{T})] = 2$.

(c) For $f > s+1$, (5.4.1) reduces to (5.4.4), where equality holds if and only if $\sum_{r=f+1}^{q} \theta_{ijr} = 0$ for every $i, j \in \{1, \ldots, f\}$, $i \neq j$. This happens if and only if no two points of $\overline{T}$ generate any point outside $\overline{T}$ as a linear combination; that is, $\overline{T}$ is a $(w-1)$-flat and $f = (s^w - 1)/(s-1)$ where $w \geq 3$ as $f > s+1$. □

A proof of Lemma 4.4.3 follows from Lemma 5.4.1. From (4.3.3), observe that $A_3(\overline{T})$ equals the number of linearly dependent triplets that can be formed out of the points of $\overline{T}$. On the other hand, by the definition of ϕ_i, any such dependent triplet is counted thrice in $\sum_{i=1}^{f} \phi_i$. Therefore, $A_3(\overline{T}) = \frac{1}{3} \sum_{i=1}^{f} \phi_i$, and Lemma 4.4.3 is immediate from Lemma 5.4.1.

In particular, if $n-k=2$, then $q = s+1$. Hence $f < s+1$ and trivially for every choice of T, the $2 \times f$ matrix $V(\overline{T})$ has rank two. Consequently, by Lemma 5.4.1 (a), (b), together with (5.1.1) and Lemma 5.2.2, all designs are equivalent with respect to estimation capacity. As noted in Section 4.4, they are all equivalent under the MA criterion as well. For $n-k \geq 3$, the following results hold as immediate consequences of Theorem 5.2.1 and Lemma 5.4.1.

Theorem 5.4.1. *Let $n-k \geq 3$ and $3 \leq f \leq s+1$. Then an s^{n-k} design T has maximum estimation capacity if and only if rank $[V(\overline{T})] = 2$.*

Theorem 5.4.2. *Let $f = (s^w - 1)/(s-1)$, where $3 \leq w < n-k$. Then an s^{n-k} design T has maximum estimation capacity if and only if $\overline{T}$ is a $(w-1)$-flat.*

A comparison with Theorems 4.4.1 and 4.4.2 reveals that under the setup of the last two theorems, the two criteria of MEC and MA are in perfect agreement. In particular, the 4^{17-14} and 3^{27-23} MA designs considered in Examples 4.4.1 and 4.4.2 have MEC as well.

Further evidence in support of the agreement between the two criteria of MA and MEC is obtained if one revisits the 27-run 3^{n-k} designs. Then $4 \leq n \leq 13$, $n-k=3$, and all such designs are isomorphic for $n = 11$, 12, and 13 (i.e., $f = 2$, 1, and 0). For $4 \leq n \leq 10$, all nonisomorphic 27-run 3^{n-k} designs were listed and ranked with respect to aberration in Table 4A.2. One can check that for each n in this range not only the MA design has MEC, but also the two criteria lead to identical ranking of the designs. For this purpose, the majorization argument of Theorem 5.2.1 works throughout, except for two pairs of designs, where (5.1.1) has to be invoked explicitly. Returning to the design T_1 in Example 5.1.1, note that T_1 is isomorphic to the design 5-2.1 of Table 4A.2. Hence as discussed above, T_1 has MEC.

We conclude this section with the following example.

Example 5.4.1. Let $n = 8$, $k = 5$. Consider the three 3^{8-5} designs 8-5.1, 8.5-2, and 8-5.3 listed in Table 4A.2. These are ranked according to aberration, i.e., 8-5.1 has MA, 8-5.2 is the next best, and so on. Let $\overline{T}_1$, $\overline{T}_2$, and $\overline{T}_3$ represent the set $\overline{T}$ for these three designs respectively. Using the compact notation, we have

$$\overline{T}_1 = \{12^2, 13, 23, 123^2, 12^23^2\}, \qquad \overline{T}_2 = \{12, 13, 23, 123^2, 12^23^2\},$$
$$\overline{T}_3 = \{13, 23, 123, 13^2, 123^2\}.$$

Now $\overline{T}_1$ contains four linearly dependent triplets, namely,

$$\{12^2, 13, 23\}, \{12^2, 13, 123^2\}, \{12^2, 23, 123^2\}, \{13, 23, 123^2\}.$$

Hence $\phi(\overline{T}_1) = (3, 3, 3, 3, 0)'$. Similarly, $\phi(\overline{T}_2) = (1, 2, 1, 1, 1)'$, $\phi(\overline{T}_3) = (1, 1, 0, 0, 1)'$. Thus both $\phi(\overline{T}_1)$ and $\phi(\overline{T}_2)$ are upper weakly majorized by $\phi(\overline{T}_3)$. By Theorem 5.2.1, both 8-5.1 and 8-5.2 dominate 8-5.3 with respect to estimation capacity. On the other hand, neither of $\phi(\overline{T}_1)$ and $\phi(\overline{T}_2)$ is upper weakly majorized by the other. Therefore, in order to compare 8-5.1 and 8-5.2, one has to obtain $E_r(T_1)$ and $E_r(T_2)$ explicitly for various r (details left as exercise). Based on these values, 8-5.1 dominates 8-5.2. Therefore, the two criteria yield identical ranking of the designs considered. □

Exercises

5.1 Show that every s^{n-k} design of resolution five or higher has MEC.

5.2 For $f = 2$, verify from first principles that $E_2(T)$ is a Schur concave function of $m(T)$.

5.3 Obtain the alias sets of the 2^{6-2} design with the defining relation $I = 1234 = 1256 = 3456$. For each alias set, identify the corresponding point of $PG(3, 2)$.

5.4 Verify Lemma 5.2.2 for the alias set $13 = 25 = \cdots$ in Example 5.1.1.

5.5 For $f = 2^w - 2$ $(2 \leq w \leq n-k)$, show that a set $\overline{T}$ as envisaged in Lemma 5.3.1 contains $\frac{2}{3}(2^{w-1} - 1)(2^{w-1} - 2)$ lines.

5.6 Consider the five 2^{7-3} designs 7-3.1, ..., 7-3.5 listed in Table 3A.2. Let $\overline{T}_1, \ldots, \overline{T}_5$ be the set $\overline{T}$ for these five designs respectively.
(a) Compute $\phi(\overline{T}_i)$ for $1 \leq i \leq 5$.
(b) Based on the results in (a) and Theorem 5.2.1, show that each of 7-3.3, 7-3.4, and 7-3.5 is dominated by 7-3.2 with respect to estimation capacity.
(c) Use (5.1.1) and Lemma 5.2.2 to compute $E_r(T_1)$ and $E_r(T_2)$ for $r \leq 8$. Show that the MA design 7-3.1 maximizes $E_r(T)$ for $1 \leq r \leq 7$, whereas the next best design 7-3.2 maximizes $E_8(T)$.

5.7 For the designs 8-5.1 and 8-5.2 in Example 5.4.1, obtain $m(T_1)$, $m(T_2)$, and hence $E_r(T_1)$ and $E_r(T_2)$ for various r by using (5.1.1) and Lemma 5.2.2. Based on these, show that 8-5.1 dominates 8-5.2 and has MEC.

5.8 Represent the 2^{9-4} designs d_0 and d_1 in Example 3.4.1 by sets T_0 and T_1 of $PG(4, 2)$. Compute $\phi(\overline{T}_0)$ and $\phi(\overline{T}_1)$, and hence $E_r(T_0)$ and $E_r(T_1)$ for various r by using (5.1.1) and Lemma 5.2.2. Based on these, show that d_0 dominates d_1 according to the criterion of estimation capacity.

6

Minimum Aberration Designs for Mixed Factorials

Extension of the ideas in Chapters 3 and 4 to designs with factors at different numbers of levels is the focus of this chapter. The important special case of mixed two- and four-level designs is first discussed. An extension of the minimum aberration criterion is considered. More generally, designs with one factor at s^r levels and n factors at s levels, or one factor at s^{r_1} levels, a second factor at s^{r_2} levels, and n factors at s levels, where s is a prime or prime power, are considered. These designs can be conveniently described and their properties obtained using finite projective geometry. The method of complementary sets is again seen to provide a general approach for finding minimum aberration designs in such settings.

6.1 Construction of $4^p \times 2^n$ Designs via the Method of Replacement

Among fractional factorial designs with factors at different numbers of levels, those with factors at two and four levels have the simplest mathematical structure. We refer to these designs as mixed two- and four-level designs, or simply as $4^p \times 2^n$ designs, where p denotes the number of four-level factors and n the number of two-level factors. To facilitate the discussion on $4^p \times 2^n$ designs, we need to extend the definition of orthogonal arrays as given in Section 2.6.

Definition 6.1.1. *An orthogonal array* $OA(N, n, s_1^{n_1} \dots s_u^{n_u}, g)$ *of strength* g *is an* $N \times n$ *array,* $n = n_1 + \cdots + n_u$*, in which* n_i *columns have* s_i *symbols each* $(1 \le i \le u)$*, and all possible combinations of symbols appear equally often as rows in every* $N \times g$ *subarray.*

For $u > 1$, the array is called an *asymmetrical* or *mixed-level orthogonal array.* For $u = 1$, all columns have the same number of symbols and the array reduces to the one given in Definition 2.6.1. In the latter case, it is called a *symmetrical orthogonal array.*

In particular, an $OA(N, n, s_1^{n_1} \dots s_u^{n_u}, 2)$ of strength two is denoted simply by $OA(N, s_1^{n_1} \dots s_u^{n_u})$.

The simplest way to construct a fractional factorial $4^p \times 2^n$ design is to start with a two-level design given by a symmetrical orthogonal array with two symbols (cf. Theorem 2.6.2) and replace three of its columns by a four-symbol column. To illustrate this method, consider the 2^{7-4} design given by the array on the right-hand side of Table 6.1. Since column 3 of the array is equal to the sum of columns 1 and 2 modulo 2, we can replace these three columns by a four-symbol column with the following *replacement rule* for each row corresponding to these columns:

$$(0\ 0\ 0) \to 0,\ (0\ 1\ 1) \to 1,\ (1\ 0\ 1) \to 2,\ (1\ 1\ 0) \to 3. \qquad (6.1.1)$$

The resulting array, given on the left-hand side of Table 6.1, has one four-symbol column denoted by T_0 and four two-symbol columns denoted by 4, 5, 6, 7. Following Definition 6.1.1, this array is an $OA(8, 4^1 2^4)$ of strength two. Interpreting its rows as treatment combinations, one gets a 4×2^4 design with 8 runs. Note that the 2^{7-4} design can also be denoted by $OA(8, 2^7)$, and the latter notation is used in Table 6.1 for consistency.

Table 6.1 Construction of $OA(8, 4^1 2^4)$ from $OA(8, 2^7)$

$\mathbf{T_0}$	**4**	**5**	**6**	**7**		**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	0	0	0	0		0	0	0	0	0	0	0
0	1	1	1	1		0	0	0	1	1	1	1
1	0	0	1	1		0	1	1	0	0	1	1
1	1	1	0	0	$\longleftarrow$	0	1	1	1	1	0	0
2	0	1	0	1		1	0	1	0	1	0	1
2	1	0	1	0		1	0	1	1	0	1	0
3	0	1	1	0		1	1	0	0	1	1	0
3	1	0	0	1		1	1	0	1	0	0	1

The replacement rule in (6.1.1) can be repeatedly applied to generate additional four-symbol columns. First, note that the three two-symbol columns in the above construction correspond to three dependent elements of the set H_3 introduced in Section 3.3, such that the product of any two of these elements equals the third. To describe the general procedure, we use the $2^m - 1$ elements of H_m in (3.3.1) to represent the $2^m - 1$ factors of the saturated $2^{\nu-k}$ design with $\nu = 2^m - 1$ and $k = \nu - m$. By Theorem 2.6.2, this saturated design is represented by a two-symbol symmetrical orthogonal array OA(2^m, 2^ν) of strength two, where each column of the array corresponds to a factor. Suppose that among the $2^m - 1$ elements of H_m, there are p mutually exclusive sets of elements of the form $\{a_i,\ b_i,\ a_i b_i\}$. We can apply the rule in (6.1.1) to each of the corresponding p sets of columns of the $OA(2^m, 2^\nu)$ to generate p four-symbol columns. By retaining the other $\nu - 3p$ two-symbol columns of the original array, we obtain an $OA(2^m, 4^p 2^n)$, with $n = \nu - 3p$. As before, this mixed-level array gives a $4^p \times 2^n$ design that is *saturated* in the sense that $3p + n = \nu$. Unsaturated designs can be obtained by dropping some of the factors (columns) in the saturated $4^p \times 2^n$ design. This technique of construction is called the *method of replacement*. The maximum number of four-symbol columns p attainable by this method is $(2^m - 1)/3$ for even m and $(2^m - 5)/3$ for odd m. For a proof of this result and the explicit construction of the p mutually exclusive sets, see Wu (1989).

Example 6.1.1. The method is illustrated by the construction of a $4^3 \times 2^6$ design with 16 runs. Start with the 2^{15-11} design whose 15 factors (columns) correspond to the elements of $H_4 = \{1, 2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234\}$. By replacing the three sets of columns represented by $\{1, 2, 12\}$, $\{3, 4, 34\}$, and $\{123, 134, 24\}$ by three four-symbol columns and retaining the other six two-symbol columns corresponding to $13, 23, 14, 124, 234, 1234$, we obtain the desired $4^3 \times 2^6$ design. □

Among the $4^p \times 2^n$ designs so constructed, which one is "optimal" and what optimality criterion should be used? An obvious approach is to define an extension of the minimum aberration (MA) criterion and to find MA designs accordingly. This will be addressed in the next section.

6.2 Minimum Aberration $4^p \times 2^n$ Designs with $p = 1, 2$

The MA criterion for 2^{n-k} designs requires suitable modification for $4^p \times 2^n$ designs because the words in the defining relation that involve the four-level factors need to be treated differently from those involving only the two-level factors. To illustrate this difference, we first consider the problem of selecting a 4×2^4 design with 16 runs. Using the same notation as in Example 6.1.1, the four-level factor can be represented by the set

$$T_0 = \{1, 2, 12\}$$

of H_4. The choice of a 4×2^4 design now amounts to choosing four elements from the remaining 12 elements of H_4 for the four two-level factors. First, consider the design

$$d_1 = d\,(T_0, 3, 4, 23, 134), \tag{6.2.1}$$

which consists of the four-level factor given by T_0 and four two-level factors represented by 3, 4, 23, 134. It is easy to see that the four elements 3, 4, 23, 134 are independent. In order to obtain the defining relation of the design d_1, we note that the three degrees of freedom associated with the main effect of the four-level factor correspond to the elements 1, 2, and 12 of T_0. This is intuitively clear from the replacement rule in (6.1.1) and will also be evident from the more general discussion in the next section. For notational convenience, write

$$\gamma_1 = 1,\ \gamma_2 = 2,\ \gamma_3 = 12.$$

Obviously, $\gamma_3 = \gamma_1\gamma_2$. We also denote the elements representing the four two-level factors by c_1, c_2, c_3, c_4, i.e., $c_1 = 3$, $c_2 = 4$, $c_3 = 23$, and $c_4 = 134$. Then the aliasing relation $\gamma_2 = c_1c_3$ follows from $2 = (3)(23)$. Thus the word $\gamma_2c_1c_3$ appears in the defining relation of d_1. In this manner, it is easy to see that d_1 has the following defining relation:

$$I = \gamma_1c_1c_2c_4 = \gamma_2c_1c_3 = \gamma_3c_2c_3c_4. \tag{6.2.2}$$

For comparison, consider an alternative design

$$d_2 = d\,(T_0, 3, 4, 34, 124).$$

A similar argument shows that d_2 has the following defining relation:

$$I = c_1c_2c_3 = \gamma_3c_2c_4 = \gamma_3c_1c_3c_4. \tag{6.2.3}$$

If the MA criterion for two-level designs is adopted, d_1 would have less aberration than d_2 because the A_3 value for d_1 is one, while the A_3 value for d_2 is two. This conclusion is based on the assumption that all words of the same length are of equal importance. Wu and Zhang (1993) proposed a refinement of the standard MA criterion by classifying the words into different types. For 4×2^n designs, there are two types of words: those involving only two-level factors are called *type* 0, and those involving the four-level factor (as represented by one of the γ_i's) and some two-level factors are called *type* 1. (Note that any two γ_i's that appear in a word can be replaced by the third γ_i because of the relation $I = \gamma_1\gamma_2\gamma_3$. This justifies the consideration of only one γ_i in the definition of type-1 words.) It can be argued that a type-1 word is usually less serious than a type-0 word of the same length. Because the four-level factor has three degrees of freedom as represented by γ_1, γ_2, and γ_3, it rarely happens that all three γ_i's are important. Therefore, *a priori* knowledge may allow the experimenter to choose the least important γ_i to be included in a type-1 word in the defining relation. Such an assignment would make the effect aliasings implied by a type-1 word less severe than those implied by a type-0 word of the same length. This consideration leads to the following extension of the MA criterion for 4×2^n designs.

For a 4×2^n design d, let $A_{i0}(d)$ and $A_{i1}(d)$ be the numbers of type-0 and type-1 words of length i in its defining relation. The vector

$$W(d) = \{A_i(d)\}_{i \geq 1}, \tag{6.2.4}$$

where $A_i(d) = (A_{i0}(d), A_{i1}(d))$, is the *wordlength pattern* of d. The *resolution* of d is defined to be the smallest i such that $A_{ij}(d)$ is positive for at least one j. In view of the foregoing discussion on the two types of words, it is more important to have a smaller A_{i0} than a smaller A_{i1} for the same i. This leads to the following criterion (Wu and Zhang, 1993).

Definition 6.2.1. *Let d_1 and d_2 be two 4×2^n designs with the same run size and u be the smallest integer such that $A_u(d_1) \neq A_u(d_2)$. If $A_{u0}(d_1) < A_{u0}(d_2)$ or $A_{u0}(d_1) = A_{u0}(d_2)$ but $A_{u1}(d_1) < A_{u1}(d_2)$, then d_1 is said to have less aberration of type 0 than d_2. A design d has minimum aberration of type 0 if no other design has less aberration of type 0 than d.*

It is easy to see that for the design d_1 in (6.2.2), $A_{30} = 0$, $A_{31} = 1$, $A_{40} = 0$, $A_{41} = 2$, while for d_2 in (6.2.3), $A_{30} = A_{31} = 1$, $A_{40} = 0, A_{41} = 1$. Thus d_1 has less aberration of type 0 than d_2.

To extend Definition 6.2.1 to $4^2 \times 2^n$ designs, we first note that the two four-level factors can be represented by

$$T_{01} = \{1, 2, 12\} \text{ and } T_{02} = \{3, 4, 34\}.$$

Now, there are three types of words. *Type* 0 is defined as before. *Type* 1 involves one four-level factor, as represented by one element of T_{01} or T_{02}, and some two-level factors. Finally, *type* 2 involves both four-level factors, as represented by one element of T_{01} and one element of T_{02}, and some two level factors. For a $4^2 \times 2^n$ design d, let $A_{ij}(d)$ be the number of type-j words of length i in its defining relation, and $W(d) = \{A_i(d)\}_{i \geq 1}$, where $A_i(d) = (A_{i0}(d), A_{i1}(d), A_{i2}(d))$. The *resolution* of d is defined to be the smallest i such that $A_{ij}(d)$ is positive for at least one j. As argued previously, for the same length, a word of type 0 is most serious while a word of type 2 is least serious. This leads to the following criterion.

Definition 6.2.2. *Let d_1 and d_2 be two $4^2 \times 2^n$ designs with the same run size and u be the smallest integer such that $A_u(d_1) \neq A_u(d_2)$. Suppose that one of the following three conditions holds: (i) $A_{u0}(d_1) < A_{u0}(d_2)$; (ii) $A_{u0}(d_1) = A_{u0}(d_2)$, $A_{u1}(d_1) < A_{u1}(d_2)$; (iii) $A_{u0}(d_1) = A_{u0}(d_2)$, $A_{u1}(d_1) = A_{u1}(d_2)$, $A_{u2}(d_1) < A_{u2}(d_2)$. Then d_1 is said to have less aberration of type 0 than d_2. A design d has minimum aberration of type 0 if no other design has less aberration of type 0 than d.*

Example 6.2.1. Consider two $4^2 \times 2^3$ designs with 16 runs, $d_1 = d(T_{01}, T_{02}, 14, 23, 234)$ and $d_2 = d(T_{01}, T_{02}, 14, 23, 1234)$. For either design, denote the three elements representing the two-level factors by c_1, c_2, c_3; e.g., $c_1 = 14$, $c_2 = 23$, $c_3 = 234$ for d_1. As before, the three elements of T_{01} can be represented by $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 12$. Similarly, the three elements of T_{02} can be represented by $\beta_1 = 3$, $\beta_2 = 4$, and $\beta_3 = 34$ with the relation $\beta_3 = \beta_1\beta_2$. It can be easily verified that d_1 has the following defining relation:

$$I = \gamma_1\beta_2c_1 = \gamma_2\beta_1c_2 = \gamma_2\beta_3c_3 = \gamma_3\beta_3c_1c_2 = \gamma_3\beta_1c_1c_3 = \beta_2c_2c_3 = \gamma_1c_1c_2c_3.$$

Therefore, for d_1, $A_{30} = 0$, $A_{31} = 1$, $A_{32} = 3$, $A_{40} = 0$, $A_{41} = 1$, and $A_{42} = 2$. Similar calculations show that for d_2, $A_{30} = 1$, $A_{31} = 0$, $A_{32} = 3$, $A_{40} = A_{41} = 0$, and $A_{42} = 3$. Thus d_1 has a smaller value of A_{30}, and hence has less aberration of type 0 than d_2. □

Tables of 4×2^n and $4^2 \times 2^n$ designs having MA of type 0 and with 16, 32, and 64 runs were reported in Wu and Hamada (2000). For completeness, these optimal designs are adapted from the original source and presented in Tables 6.2–6.7. In these tables, the column "Design Generators" lists the elements that generate (i.e., define) the design. For example, the design d_1 in (6.2.1) has T_0, 3, 4, 23, 134 as its generators. It can be found in Table 6.2 with $n = 4$. It is thus an MA design of type 0. Table 6.5 with $n = 3$ shows that the design d_1 in Example 6.2.1 also enjoys the same property.

Most of the theoretical results on MA designs for mixed factorials are obtained by employing the technique of complementary sets, which will be discussed in the remaining sections under a general framework. Following Wu and Zhang (1993), here we will present one result on 4×2^n designs that does not rely on the use of this technique and helps in reducing the design search in the context of Tables 6.2–6.4, which appear at the end of this section.

Consider a 4×2^n design d^* with 2^m runs, where $m < n+2$. Suppose d^* is represented by $T_0 = \{\gamma_1, \gamma_2, \gamma_3\}$, where $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 12$, and n other elements $c_1, \ldots, c_n$ of H_m. As before, T_0 corresponds to the four-level factor and $c_1, \ldots, c_n$ correspond to the n two-level factors in d^*. Let $k = n + 2 - m$, so that d^* is a $1/2^k$ fraction of a 4×2^n factorial. Write $l = n + 2$, and let d be a 2^{l-k} design represented by the l elements $\gamma_1, \gamma_2, c_1, \ldots, c_n$ of H_m; cf. Theorem 3.3.1(a). Of course, the set $\{\gamma_1, \gamma_2, c_1, \ldots, c_n\}$ is supposed to contain m independent elements.

Theorem 6.2.1 below shows how the MA property of the 4×2^n design d^* is influenced by the characteristics of the two-level design d. Denote the defining relations of d and d^* by $DR(d)$ and $DR(d^*)$ respectively. Let M_0, M_1, M_2, and M_{12} denote respectively the sets of words in $DR(d)$ that involve neither γ_1 nor γ_2, only γ_1 but not γ_2, only γ_2 but not γ_1, and both γ_1 and γ_2. The following facts are now evident from the correspondence between d and d^*:

(i) any word in $M_0 \cup M_1 \cup M_2$ appears as it is in $DR(d^*)$;
(ii) any word in M_{12} appears in $DR(d^*)$ with $\gamma_1\gamma_2$ replaced by γ_3;
(iii) any word in M_0 becomes a word of type 0 in $DR(d^*)$;
(iv) any word in $M_1 \cup M_2 \cup M_{12}$ becomes a word of type 1 in $DR(d^*)$.

As an illustration, if d^* is taken as the design d_1 in (6.2.1), then d is represented by the elements $\gamma_1 = 1$, $\gamma_2 = 2$, $c_1 = 3$, $c_2 = 4$, $c_3 = 23$, and $c_4 = 134$ of H_4. Thus $DR(d)$ is given by

$$I = \gamma_1 c_1 c_2 c_4 = \gamma_2 c_1 c_3 = \gamma_1 \gamma_2 c_2 c_3 c_4, \tag{6.2.5}$$

and each of M_1, M_2, and M_{12} is a singleton set consisting of the words $\gamma_1 c_1 c_2 c_4$, $\gamma_2 c_1 c_3$, and $\gamma_1 \gamma_2 c_2 c_3 c_4$ respectively. A comparison between (6.2.2) and (6.2.5) illustrates the facts (i)–(iv) above.

Theorem 6.2.1. *(a) Let $k = 1$. Then a 4×2^n design d^* has minimum aberration of type 0 if and only if the only word in $DR(d)$ is either $\gamma_1 c_1 \ldots c_n$ or $\gamma_2 c_1 \ldots c_n$ or $\gamma_1 \gamma_2 c_1 \ldots c_n$.*

(b) Let $k \geq 2$. Then in order that d^* have minimum aberration of type 0, it is necessary that each of the sets M_1, M_2, and M_{12} be nonempty.

Proof. (a) In this case, there is only one word in $DR(d^*)$. By (i), (ii), and (iv) above, this word has maximum length, $n+1$, and is of type 1 if and only if $DR(d)$ is as envisaged.

(b) For notational simplicity, we prove the result for $k = 2$. The proof for general k (≥ 2) involves the same ideas and is left as an exercise. With $k = 2$, there are two independent words in $DR(d)$, say ω_1 and ω_2, i.e., $DR(d)$ is given by

$$I = \omega_1 = \omega_2 = \omega_1\omega_2. \tag{6.2.6}$$

Suppose at least one of M_1, M_2, and M_{12} is empty. By (6.2.6), then either each of these three sets is empty or exactly one of them is nonempty.

First suppose each of M_1, M_2, and M_{12} is empty. Then $\omega_1, \omega_2 \in M_0$. Let $\widetilde{d}$ be a 2^{l-k} design, $k = 2$, given by the defining relation

$$I = \gamma_1\omega_1 = \omega_2 = \gamma_1\omega_1\omega_2. \tag{6.2.7}$$

Define $\widetilde{d}^*$ as the 4×2^n design corresponding to $\widetilde{d}$. By the fact (i) above, $DR(d^*)$ and $DR(\widetilde{d}^*)$ are again given by (6.2.6) and (6.2.7) respectively. Since $\omega_1, \omega_2 \in M_0$, by (iii) and (iv) above, the words ω_1 and $\omega_1\omega_2$ in $DR(d^*)$ are of type 0, whereas the words $\gamma_1\omega_1$ and $\gamma_1\omega_1\omega_2$ in $DR(\widetilde{d}^*)$ are of type 1; moreover, the lengths of the latter two words are one more than those of the former two words respectively. Hence $\widetilde{d}^*$ has less aberration of type 0 than d^*.

Consider next the situation in which exactly one of M_1, M_2, and M_{12} is nonempty. Suppose only M_{12} is nonempty (the other cases can be treated similarly). Then both ω_1 and ω_2 belong to $M_0 \cup M_{12}$ and at least one of them belongs to M_{12}. Let $\omega_2 \in M_{12}$, i.e.,

$$\omega_2 = \gamma_1\gamma_2\overline{\omega}_2, \tag{6.2.8}$$

where $\overline{\omega}_2$ involves neither γ_1 nor γ_2. Without loss of generality, it may be assumed that $\omega_1 \in M_0$, for otherwise, $\omega_1 \in M_{12}$, $\omega_1\omega_2 \in M_0$, and one can take $\{\omega_1\omega_2, \omega_2\}$ as a spanning set of $DR(d)$. Define a two-level design $\widetilde{d}$ and the corresponding 4×2^n design $\widetilde{d}^*$ exactly as in the last paragraph. Using (6.2.8), the defining relations (6.2.6) and (6.2.7) of d and $\widetilde{d}$ can be expressed as

$$I = \omega_1 = \gamma_1\gamma_2\overline{\omega}_2 = \gamma_1\gamma_2\omega_1\overline{\omega}_2$$

and

$$I = \gamma_1\omega_1 = \gamma_1\gamma_2\overline{\omega}_2 = \gamma_2\omega_1\overline{\omega}_2,$$

respectively. Hence by (i) and (ii) above, the defining relations of the corresponding 4×2^n designs d^* and $\widetilde{d}^*$ are given by

$$I = \omega_1 = \gamma_3\overline{\omega}_2 = \gamma_3\omega_1\overline{\omega}_2$$

and

$$I = \gamma_1\omega_1 = \gamma_3\overline{\omega}_2 = \gamma_2\omega_1\overline{\omega}_2,$$

respectively. As in the last paragraph, now it is easily seen that $\widetilde{d}^*$ has less aberration of type 0 than d^*. □

Table 6.2 MA(type 0) 4×2^n designs with 16 runs, $3 \le n \le 11$

n	Resolution	Design Generators
3	4	T_0, 3, 4, 134
4	3	T_0, 3, 4, 23, 134
5	3	T_0, 3, 4, 23, 24, 134
6	3	T_0, 3, 4, 23, 24, 134, 1234
7	3	T_0, 3, 4, 13, 14, 23, 24, 124
8	3	T_0, 3, 4, 13, 14, 23, 24, 123, 124
9	3	T_0, 3, 4, 13, 23, 34, 123, 134, 234, 1234
10	3	T_0, 3, 4, 13, 14, 23, 34, 123, 134, 234, 1234
11	3	T_0, 3, 4, 13, 14, 23, 24, 34, 123, 134, 234, 1234

Note: $T_0 = \{1, 2, 12\}$ for Tables 6.2–6.4.

Table 6.3 MA(type 0) 4×2^n designs with 32 runs, $4 \le n \le 9$

n	Resolution	Design Generators
4	5	T_0, 3, 4, 5, 1345
5	4	T_0, 3, 4, 5, 245, 1345
6	4	T_0, 3, 4, 5, 235, 245, 1345
7	4	T_0, 3, 4, 5, 234, 235, 245, 1345
8	3	T_0, 3, 4, 5, 13, 145, 234, 235, 12345
9	3	T_0, 3, 4, 5, 13, 14, 234, 235, 245, 1345

Table 6.4 MA(type 0) 4×2^n designs with 64 runs, $5 \le n \le 9$

n	Resolution	Design Generators
5	6	T_0, 3, 4, 5, 6, 123456
6	5	T_0, 3, 4, 5, 6, 1345, 2456
7	4	T_0, 3, 4, 5, 6, 1345, 2346, 12356
8	4	T_0, 3, 4, 5, 6, 356, 1345, 2456, 12346
9	4	T_0, 3, 4, 5, 6, 356, 1235, 1345, 2456, 12346

Table 6.5 MA(type 0) $4^2 \times 2^n$ designs with 16 runs, $1 \le n \le 8$

n	Design Generators
1	T_{01}, T_{02}, 14
2	T_{01}, T_{02}, 14, 23
3	T_{01}, T_{02}, 14, 23, 234
4	T_{01}, T_{02}, 14, 23, 124, 234
5	T_{01}, T_{02}, 14, 23, 24, 124, 234
6	T_{01}, T_{02}, 13, 14, 23, 24, 134, 234
7	T_{01}, T_{02}, 13, 14, 23, 24, 124, 134, 234
8	T_{01}, T_{02}, 13, 14, 23, 24, 124, 134, 234, 1234

Note: All designs in this table have resolution three. $T_{01} = \{1, 2, 12\}$ and $T_{02} = \{3, 4, 34\}$ for Tables 6.5–6.7.

Table 6.6 MA(type 0) $4^2 \times 2^n$ designs with 32 runs, $2 \le n \le 10$

n	Resolution	Design Generators
2	4	T_{01}, T_{02}, 5, 235
3	4	T_{01}, T_{02}, 5, 235, 1245
4	4	T_{01}, T_{02}, 5, 235, 1245, 1345
5	3	T_{01}, T_{02}, 5, 14, 235, 1245, 1345
6	3	T_{01}, T_{02}, 5, 14, 234, 235, 1245, 1345
7	3	T_{01}, T_{02}, 5, 13, 14, 234, 235, 1245, 1345
8	3	T_{01}, T_{02}, 5, 13, 14, 234, 235, 1234, 1245, 1345
9	3	T_{01}, T_{02}, 5, 13, 14, 15, 234, 235, 1234, 1245, 1345
10	3	T_{01}, T_{02}, 5, 13, 14, 15, 234, 235, 345, 1234, 1245, 1345

Table 6.7 MA(type 0) $4^2 \times 2^n$ designs with 64 runs, $3 \le n \le 7$

n	Resolution	Design Generators
3	5	T_{01}, T_{02}, 5, 6, 123456,
4	4	T_{01}, T_{02}, 5, 6, 1356, 2456
5	4	T_{01}, T_{02}, 5, 6, 1356, 2456, 2346
6	4	T_{01}, T_{02}, 5, 6, 1356, 2456, 2346, 1235
7	4	T_{01}, T_{02}, 5, 6, 1356, 2456, 2346, 1235, 1246

6.3 Designs for $(s^r) \times s^n$ Factorials: Preliminaries

Consider an $(s^r) \times s^n$ factorial with one factor, say F_0, at s^r levels and n factors, say $F_1, \ldots, F_n$, at s levels each. Here $r\,(\ge 2)$ is an integer and s is a prime or prime power. The special case $s = r = 2$ covers 4×2^n factorials discussed in Section 6.2. Typically s is small, say 2 or 3, and n is large. This is in keeping with some practical applications of mixed factorials that involve a large number of factors, each with a small number of levels and very few factors, like F_0, with more levels. The developments in this and the next two sections follow Mukerjee and Wu (2001).

As in the previous chapters, a finite projective geometric formulation plays a key role in the study of regular fractions of an $(s^r) \times s^n$ factorial. This formulation calls for an appropriate representation for the s^{r+n} treatment combinations in such a factorial. Let $t = (s^r - 1)/(s-1)$ and let $\mathcal{R}\ (\cdot)$ denote the row space of a matrix. Then one has the following lemma, which helps in handling the s^r-level factor F_0.

Lemma 6.3.1. *Let V_r be an $r \times t$ matrix with columns given by the points of $PG(r-1, s)$. Then*

(a) there are s^r vectors in $\mathcal{R}(V_r)$,
(b) for any fixed $\alpha \in GF(s)$ and any j $(1 \le j \le t)$, there are s^{r-1} vectors in $\mathcal{R}(V_r)$ with jth element equal to α.

Proof. By Lemma 2.7.2 (a), V_r has full row rank and hence (a) follows. Also, by the definition of V_r, no two of its columns are linearly dependent. Hence as in the proof of Theorem 2.6.2, the s^r vectors in $\mathcal{R}(V_r)$ form a symmetrical orthogonal array $OA(s^r, t, s, 2)$. Since each element of $GF(s)$ occurs s^{r-1} times in every column of this array, (b) follows. □

In view of Lemma 6.3.1(a), the s^r levels of F_0 can be identified with the s^r vectors in $\mathcal{R}(V_r)$; for a 4×2^n factorial, this is in agreement with (6.1.1) since then $s = r = 2$ and

$$\mathcal{R}(V_2) = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}.$$

As usual, the s levels of each other factor can be represented by the elements of $GF(s)$. Thus

$$\mathcal{X} = \{(x_1, \ldots, x_t, x_{t+1}, \ldots, x_{t+n})' : (x_1, \ldots, x_t) \in \mathcal{R}(V_r), \\ x_{t+1}, \ldots, x_{t+n} \in GF(s)\} \qquad (6.3.1)$$

represents the collection of the s^{r+n} treatment combinations in an $(s^r) \times s^n$ factorial. Clearly, $(x_1, \ldots, x_t)$ refers to a level of F_0 and x_{t+i} refers to a level of F_i $(1 \le i \le n)$.

Example 6.3.1. For a 9×3^3 factorial, $s = 3$, $r = 2$, $n = 3$, $t = 4$, and consideration of the points of $PG(1, 3)$ yields

$$V_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Hence $\mathcal{R}(V_2)$ consists of nine vectors, namely,

$$(0,0,0,0),\quad (0,1,1,2),\quad (0,2,2,1),\quad (1,0,1,1),\quad (1,1,2,0),$$
$$(1,2,0,2),\quad (2,0,2,2),\quad (2,1,0,1),\quad (2,2,1,0).$$

Consequently, in accordance with (6.3.1), the treatment combinations in a 9×3^3 factorial can be represented by $(x_1,\ldots,x_7)'$, where $(x_1,\ldots,x_4) \in \mathcal{R}(V_2)$, and x_5, x_6, $x_7 \in \{0,1,2\}$. □

We are now in a position to introduce a regular fraction of an $(s^r) \times s^n$ factorial. In the spirit of Theorem 2.7.1, this is done via a geometric approach. Suppose it is desired to have a fraction consisting of s^m treatment combinations, where $r < m < r+n$. Let P denote the set of the $(s^m-1)/(s-1)$ points of $PG(m-1,s)$. As in the preceding chapters, for any nonempty subset Q of P, let $V(Q)$ be a matrix with columns given by the points of Q. Define T_0 as the $(r-1)$-flat of P that is generated by $e_1,\ldots,e_r$, where $e_1,\ldots,e_m$ are the $m\times 1$ unit vectors over $GF(s)$. Since $r < m$, the flat T_0 is well defined. Furthermore, it is easily seen that

$$V(T_0) = \begin{bmatrix} V_r \\ 0 \end{bmatrix}, \tag{6.3.2}$$

where V_r is defined in Lemma 6.3.1 and 0 is the null matrix of order $(m-r)\times t$. Let T be an n-subset of P such that T_0 and T are disjoint and the matrix

$$V(T_0 \cup T) = [V(T_0) \quad V(T)] \tag{6.3.3}$$

has full row rank. Then there are s^m vectors in $\mathcal{R}[V(T_0\cup T)]$. By (6.3.1)–(6.3.3), the transpose of each of these vectors belongs to $\mathcal{X}$ and hence represents a treatment combination of an $(s^r)\times s^n$ factorial. The collection of the s^m treatment combinations (or runs) so obtained gives a *regular fraction*, to be denoted by $d = d(T_0,T)$, of such a factorial. For the special case of a 4×2^n factorial considered in Section 6.2, the elements of H_m represent the points of $PG(m-1,2)$, $T_0 = \{1,2,12\}$ represents the four-level factor, and T, consisting of n elements from the rest of H_m, represents the n two-level factors. In the remainder of the chapter, a regular fraction as introduced here is simply called a design. The number s^m is called its run size.

Considering the cardinalities of T_0, T, and P, the above construction is possible if and only if

$$\frac{s^r-1}{s-1} + n \le \frac{s^m-1}{s-1}, \quad \text{i.e.,} \quad s^r + n(s-1) \le s^m.$$

This condition is supposed to hold throughout this and the next section. This construction is motivated by the approach of Wu, Zhang, and Wang (1992) for the construction of asymmetrical orthogonal arrays. Indeed, if the s^m runs in $d(T_0,T)$ are written as rows and in each of them the subvector $(x_1,\ldots,x_t)$ is replaced by a single symbol representing the corresponding level of F_0, then one gets an asymmetrical orthogonal array $OA(s^m,(s^r)^1 s^n)$ of strength two.

Example 6.3.1 (continued). Continuing with the 9×3^3 factorial, suppose it is desired to have a design with 27 runs. Then $m = 3$ and $T_0 = \{(1,0,0)', (0,1,0)', (1,1,0)', (1,2,0)'\}$, which is compatible with (6.3.2). Take $T = \{(1,1,2)', (1,2,1)', (1,2,2)'\}$. Then T_0 and T are disjoint and the matrix

$$V(T_0 \cup T) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 & 2 \end{bmatrix} \qquad (6.3.4)$$

has full row rank. Hence consideration of the vectors in $\mathcal{R}[V(T_0 \cup T)]$ serves the purpose. □

In order to study the properties of a regular fraction or design as introduced above, one needs to extend the concept of pencils to the present setup. With reference to an $(s^r) \times s^n$ factorial, a *pencil* is a nonnull vector of the form $b = (b_1, \ldots, b_t, b_{t+1}, \ldots, b_{t+n})'$, where $b_i \in GF(s)$ for every i and among $b_1, \ldots, b_t$, at most one is nonzero.

Lemma 6.3.2. *For any pencil $b = (b_1, \ldots, b_t, b_{t+1}, \ldots, b_{t+n})'$ and any $\alpha \in GF(s)$, the set $\{x : x \in \mathcal{X}, b'x = \alpha\}$ has cardinality s^{r+n-1}.*

Proof. First suppose $b_1 = \cdots = b_t = 0$. Then among $b_{t+1}, \ldots, b_{t+n}$ at least one, say b_{t+1}, is nonzero. Therefore, as in the proof of Lemma 2.3.1, for any $x = (x_1, \ldots, x_t, x_{t+1}, \ldots, x_{t+n})'$ belonging to the set under consideration, x_{t+1} is uniquely determined by $x_{t+2}, \ldots, x_{t+n}$. Since there are s^{n-1} choices of $x_{t+2}, \ldots, x_{t+n}$ and s^r choices of $(x_1, \ldots, x_t)$, the result follows.

Next suppose $b_1, \ldots, b_t$ are not all zeros. Without loss of generality, let $b_1 \neq 0$. Then $b_2 = \cdots = b_t = 0$. Again as in the proof of Lemma 2.3.1, for any x belonging to the set under consideration, x_1 is uniquely determined by $x_{t+1}, \ldots, x_{t+n}$. There are now s^n choices of $x_{t+1}, \ldots, x_{t+n}$. Furthermore, by (6.3.1) and Lemma 6.3.1(b), corresponding to the unique x_1 associated with any such choice, there are s^{r-1} possibilities for $(x_1, \ldots, x_t)$. Hence the result follows. □

In view of Lemma 6.3.2, *treatment contrasts belonging to a pencil* b can be defined as in (2.3.4). Clearly, there are $s - 1$ linearly independent treatment contrasts belonging to any pencil. Also, as in Section 2.3, pencils with proportional entries induce the same partition of $\mathcal{X}$ and are hence considered identical. *Hereafter only distinct pencils are considered in any given context even when this is not stated explicitly.*

In particular, if $b_{t+1} = \cdots = b_{t+n} = 0$ in a pencil b, then one of $b_1, \ldots, b_t$ is nonzero. In this case, proceeding along the lines of Theorem 2.3.2 and using Lemma 6.3.1(b), it can be seen that any treatment contrast belonging to b also belongs to the main effect of F_0. Hence the pencil b itself is said to belong to the main effect of F_0. Since there are $t = (s^r - 1)/(s - 1)$ pencils of this kind, each of which carries $s-1$ linearly independent treatment contrasts, this accounts for the $s^r - 1$ linearly independent treatment contrasts belonging to the main effect of F_0. This agrees with Section 6.2, where for a 4×2^n factorial (i.e., $s = r = 2$), it was noted that the main effect of the four-level factor has three $\bigl(=(2^2-1)/(2-1)\bigr)$ components represented by 1, 2, and 12. Similarly, a pencil b with $b_{t+i} \neq 0$ for some i $(1 \leq i \leq n)$ and $b_j = 0$ for every $j \neq t+i$ represents the main effect of the s-level factor F_i.

Any pencil with i (≥ 2) nonzero entries belongs to an i-factor interaction in the same sense as above. Such a pencil b can involve only some of the s-level factors $F_1, \ldots, F_n$ or the s^r-level factor F_0 together with some of $F_1, \ldots, F_n$. In the former case, $b_1 = \cdots = b_t = 0$ and in the latter case one of $b_1, \ldots, b_t$ is nonzero. Following the same terminology as in Section 6.2, pencils of these two types are called *type* 0 and *type* 1 respectively.

As in Theorem 2.7.1(b), a pencil b is a *defining pencil* of the design $d = d(T_0, T)$ if

$$V(T_0 \cup T)b = 0. \tag{6.3.5}$$

Since T_0 and T are disjoint sets of points of P, the columns of $V(T_0 \cup T)$ are nonnull and no two of them are proportional to each other. Hence every defining pencil of d has at least three nonzero entries, i.e., belongs to an interaction involving at least three factors. Thus d has resolution three or higher. For $i \geq 3$, in the spirit of Section 6.2, let $A_{i0}(d)$ and $A_{i1}(d)$ denote the numbers of (distinct) defining pencils of d that have i nonzero entries and are of types 0 and 1 respectively.

Example 6.3.1 (continued). By (6.3.4) and (6.3.5), the defining pencils of the design considered in the example are

$$(1,0,0,0,1,1,0)',\ (0,1,0,0,1,0,2)',\ (0,0,0,1,0,1,1)',\ (0,0,1,0,2,1,2)'.$$

Each of these is of type 1. Hence counting the numbers of nonzero entries in these pencils, one gets $A_{30}(d) = 0$, $A_{31}(d) = 3$, $A_{40}(d) = 0$, $A_{41}(d) = 1$. □

Along the lines of Section 4.4, two designs $d(T_0, T_1)$ and $d(T_0, T_2)$ in the present setup are *isomorphic* if there exists a nonsingular transformation that maps each point of T_0 to some point of T_0 up to proportionality, and each point of T_1 to some point of T_2 up to proportionality. From (6.3.5), it can be seen that isomorphic designs have the same A_{i0} and A_{i1} for every i.

We now present some notation and lemmas that will be needed in the next section. Consider any nonempty subset Q of P. Let q be the cardinality of Q, and for $i \geq 1$, let Ω_{iq} be the set of $q \times 1$ vectors over $GF(s)$ having i nonzero elements. For $i \geq 1$, define

$$G_i(Q) = (s-1)^{-1}\#\{\lambda : \lambda \in \Omega_{iq}, V(Q)\lambda = 0\}, \tag{6.3.6}$$

where $\#$ denotes the cardinality of a set. Furthermore, when Q and T_0 are disjoint, define for $i \geq 1$,

$$H_i(T_0, Q) = (s-1)^{-1}\#\{\lambda : \lambda \in \Omega_{iq}, V(Q)\lambda \text{ is nonnull and proportional to some point of } T_0\}. \tag{6.3.7}$$

Clearly $G_1(Q) = G_2(Q) = 0$. Similarly, when Q and T_0 are disjoint, $H_1(T_0, Q) = 0$. Also,

$$G_i(Q) = H_i(T_0, Q) = 0 \quad \text{for} \quad i > q. \tag{6.3.8}$$

Since pencils with proportional entries are identical, it is not hard to see that (6.3.5)–(6.3.7) lead to the important relationships for any design $d = d(T_0, T)$,

$$A_{i0}(d) = G_i(T), \qquad A_{i1}(d) = H_{i-1}(T_0, T), \text{ for } i \geq 3. \tag{6.3.9}$$

Lemma 6.3.3. *If T_0 and Q are disjoint, then*
(a) $G_3(T_0 \cup Q) = \text{constant} + G_3(Q) + H_2(T_0, Q)$,
(b) $G_4(T_0 \cup Q) = \text{constant} + G_4(Q) + H_3(T_0, Q) + \frac{1}{2}(s^r - s)H_2(T_0, Q)$.

Lemma 6.3.4. *Let $\overline{Q} = P - Q$ be nonempty. Then*
(a) $G_3(Q) = \text{constant} - G_3(\overline{Q})$,
(b) $G_4(Q) = \text{constant} + (3s - 5)G_3(\overline{Q}) + G_4(\overline{Q})$.

The constants in these lemmas may depend on s, r, q, and m, but not on the particular choice of Q. Lemma 6.3.3 is a special case of a more general result reported by Mukerjee and Wu (1999) in a different context; the interested reader may see the original source for details. Comparing (6.3.6) with (4.3.1), Lemma 6.3.4 is immediate from Corollary 4.3.2 and the fact noted in the concluding paragraph of Section 4.3.

For a design $d = d(T_0, T)$, let $\widetilde{T} = P - (T_0 \cup T)$. The cardinality of $\widetilde{T}$ is

$$f = (s^m - s^r)/(s - 1) - n.$$

For $f = 0$, there is only one design; for $f = 1$, all designs are isomorphic. Hence only $f \geq 2$ is considered hereafter. Also, to avoid trivialities, let $n \geq 3$. Then, as a consequence of the last two lemmas, the following result holds.

Lemma 6.3.5. *With reference to an $(s^r) \times s^n$ factorial, let $d = d(T_0, T)$ be a design having run size s^m and $\widetilde{T} = P - (T_0 \cup T)$. Then*

(a) $A_{30}(d) = \text{constant} - G_3(T_0 \cup \widetilde{T})$,
(b) $A_{31}(d) = \text{constant} + G_3(T_0 \cup \widetilde{T}) - G_3(\widetilde{T})$,
(c) $A_{40}(d) = \text{constant} + (3s - 5)G_3(T_0 \cup \widetilde{T}) + G_4(T_0 \cup \widetilde{T})$,
(d) $A_{41}(d) = \text{constant} - \frac{1}{2}(s^r + 5s - 10)\{G_3(T_0 \cup \widetilde{T}) - G_3(\widetilde{T})\}$
$\qquad -G_4(T_0 \cup \widetilde{T}) + G_4(\widetilde{T})$.

Proof. Since $T_0 \cup \widetilde{T} = P - T$, (a) and (c) are immediate from (6.3.9) and Lemma 6.3.4. Next by (6.3.9) and Lemma 6.3.3,

$$\begin{aligned} A_{31}(d) &= H_2(T_0, T) = \text{constant} + G_3(T_0 \cup T) - G_3(T), \\ A_{41}(d) &= H_3(T_0, T) = \text{constant} + G_4(T_0 \cup T) - G_4(T) - \frac{1}{2}(s^r - s)H_2(T_0, T) \\ &= \text{constant} + G_4(T_0 \cup T) - G_4(T) - \frac{1}{2}(s^r - s)\{G_3(T_0 \cup T) - G_3(T)\}. \end{aligned}$$

Hence recalling the definition of $\widetilde{T}$, (b) and (d) follow from Lemma 6.3.4. □

From the proof of Lemma 6.3.5, it is clear that the constants involved therein may depend on s, r, n, and m but not on the particular choice of T. In the practically important nearly saturated cases that correspond to relatively small f, it is much easier to handle $\widetilde{T}$ than T. Thus the above lemma greatly facilitates the study of MA designs. This will be considered in the next section.

6.4 Minimum Aberration Designs for $(s^r) \times s^n$ Factorials

We begin by considering the situation in which pencils of type 0 are considered more important than those of type 1. As indicated in Section 6.2, this can happen commonly in practice. The case in which pencils of the two types are equally important will also be discussed briefly later in this section.

With reference to an $(s^r) \times s^n$ factorial, consider designs d_1 and d_2 both having run size s^m. Let u be the smallest integer such that $(A_{u0}(d_1), A_{u1}(d_1)) \neq (A_{u0}(d_2), A_{u1}(d_2))$. As in Definition 6.2.1, if $A_{u0}(d_1) < A_{u0}(d_2)$ or $A_{u0}(d_1) = A_{u0}(d_2)$ but $A_{u1}(d_1) < A_{u1}(d_2)$, then d_1 is said to have less aberration of type 0 than d_2. An *MA design of type* 0 is a design such that no other design has less aberration of type 0 than it.

Define the following classes of designs:

$$\begin{aligned}
D_1 &= \{d = d(T_0, T) : d \text{ maximizes } G_3(T_0 \cup \widetilde{T})\},\\
D_2 &= \{d : d \in D_1,\ d \text{ maximizes } G_3(\widetilde{T}) \text{ over } D_1\},\\
D_3 &= \{d : d \in D_2,\ d \text{ minimizes } G_4(T_0 \cup \widetilde{T}) \text{ over } D_2\},\\
D_4 &= \{d : d \in D_3,\ d \text{ minimizes } G_4(\widetilde{T}) \text{ over } D_3\}.
\end{aligned}$$

Then the following useful result is evident from Lemma 6.3.5.

Theorem 6.4.1. *For any i $(1 \leq i \leq 4)$, suppose d belongs to D_i and, up to isomorphism, is the unique member of D_i. Then d is a minimum aberration design of type 0.*

Corollary 6.4.1. *Let $f = 2$. Then $d = d(T_0, T)$ is a minimum aberration design of type 0 provided $\widetilde{T} = P - (T_0 \cup T)$ is of the form*

$$\widetilde{T} = \{h_1, h_1 + \alpha h_0\}, \tag{6.4.1}$$

for some $h_1 \notin T_0$, $h_0 \in T_0$, and $\alpha (\neq 0) \in GF(s)$.

Proof. Since $f = 2$, by (6.3.8) and Lemma 6.3.3(a),

$$G_3(T_0 \cup \widetilde{T}) = \text{constant} + H_2(T_0, \widetilde{T}). \tag{6.4.2}$$

Since T_0 is a flat and T_0 and $\widetilde{T}$ are disjoint, it follows from (6.3.7) that $H_2(T_0, \widetilde{T})$ equals unity if $\widetilde{T}$ is as in (6.4.1) and zero otherwise. The result now follows from Theorem 6.4.1 (with $i = 1$) and (6.4.2) noting that all designs with $\widetilde{T}$ as in (6.4.1) are isomorphic. □

For $f = 2$ and $m - r \geq 2$, not all designs have $\widetilde{T}$ as in (6.4.1). Another choice of $\widetilde{T}$ is $\{h_1, h_2\}$, where $h_1 \notin T_0$, $h_2 \notin T_0$, and $V(T_0 \cup \{h_1, h_2\})$ has rank $r + 2$. Hence even for $f = 2$, one can discriminate among rival designs with respect to the MA criterion of type 0. This may be contrasted with the case of symmetrical factorials, where as noted in Section 4.4, all designs are equivalent under the MA criterion when $f = 2$. With $i = 1$, Theorem 6.4.1 yields another corollary as follows.

Corollary 6.4.2. *Let $f = (s^w - s^r)/(s - 1)$, where $w > r$. Then $d = d(T_0, T)$ is a minimum aberration design of type 0 provided $T_0 \cup \widetilde{T}$ is a $(w-1)$-flat of P, where $\widetilde{T} = P - (T_0 \cup T)$.*

Proof. For f as stated, the cardinality of $T_0 \cup \widetilde{T}$ equals $(s^w - 1)/(s - 1)$, which is the same as the cardinality of a $(w - 1)$-flat of P. Hence by (6.3.6), $G_3(T_0 \cup \widetilde{T})$ is maximum if and only if $T_0 \cup \widetilde{T}$ is a $(w-1)$-flat of P; cf. Lemma 4.4.3. Since all such choices of $\widetilde{T}$ yield isomorphic designs, the result follows from Theorem 6.4.1 (with $i = 1$). □

As a consequence of Theorem 6.4.1, Mukerjee and Wu (2001) also reported the following result. Its proof is somewhat long and hence omitted. Recall that T_0 is the $(r - 1)$-flat spanned by $e_1, \ldots, e_r$, where $e_1, \ldots, e_m$ are the $m \times 1$ unit vectors over $GF(s)$.

Theorem 6.4.2. *Let $s = 2$ and $f = 2^w - 2^r - u$, where $w > r$ and $1 \leq u \leq 3$. Let $h_{r+1}, \ldots, h_w$ be any $w - r$ points of P such that the w points $e_1, \ldots, e_r, h_{r+1}, \ldots, h_w$ are linearly independent, and let T_1 be a $(w - 1)$-flat of P spanned by these w points. Let $\widetilde{T} = T_1 - (T_0 \cup Q)$, where*

(a) $Q = \{h_{r+1}\}$ if $u = 1$,
(b) $Q = \{h_{r+1}, e_1 + h_{r+1}\}$ if $u = 2$ and $w = r + 1$,
(c) $Q = \{h_{r+1}, h_{r+2}\}$ if $u = 2$ and $w > r + 1$,
(d) $Q = \{h_{r+1}, e_1 + h_{r+1}, e_2 + h_{r+1}\}$ if $u = 3$ and $w = r + 1$,
(e) $Q = \{h_{r+1}, h_{r+2}, e_1 + h_{r+1} + h_{r+2}\}$ if $u = 3$ and $w = r + 2$,
(f) $Q = \{h_{r+1}, h_{r+2}, h_{r+3}\}$ if $u = 3$ and $w > r + 2$.

Then $d = d(T_0, T)$ is a minimum aberration design of type 0, where $T = P - (T_0 \cup \widetilde{T})$.

Example 6.4.1. With reference to a 4×2^{25} factorial, suppose it is desired to have a design with run size 32. Then $s = r = 2$, $n = 25$, $m = 5$, and $f = 32 - 4 - 25 = 3$. Since $f = 3 = 2^3 - 2^2 - 1$, Theorem 6.4.2 is applicable with $w = 3$, $u = 1$. Take $h_3 = e_1 + e_2 + e_3$. Then e_1, e_2, and h_3 are linearly independent, and

$$T_1 = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3\}, \quad T_0 = \{e_1, e_2, e_1 + e_2\}.$$

By Theorem 6.4.2(a), $Q = \{h_3\} = \{e_1 + e_2 + e_3\}$, so that $\widetilde{T} = T_1 - (T_0 \cup Q) = \{e_3, e_1 + e_3, e_2 + e_3\}$. Here $P \equiv PG(4, 2)$. Therefore $d(T_0, T)$, where

$T = PG(4,2) - (T_0 \cup \widetilde{T})$ with T_0 and $\widetilde{T}$ as shown above is an MA design of type 0. □

Some useful special cases are now discussed. Considering 4×2^n factorials, one has $s = r = 2$ and for $f = 3, 4, 9, 10, 11, 12$, MA designs of type 0 are given by Corollary 6.4.2 or Theorem 6.4.2. This is evident because $4 = 2^3 - 2^2$, $9 = 2^4 - 2^2 - 3$, and so on. For f = 5, 6, 7, and 8, MA designs of type 0 can be obtained directly from Theorem 6.4.1. Table 6.8 lists the set $\widetilde{T} = P - (T_0 \cup T)$ for these MA designs of type 0 and indicates how they are obtained. Using the compact notation, $T_0 = \{1, 2, 12\}$ in Table 6.8 as $r = s = 2$. Hence if m is given, then for any f covered by this table, the set T corresponding to an MA design of type 0 can be easily obtained as $T = P - (T_0 \cup \widetilde{T})$.

Table 6.8 The sets $\widetilde{T}$ for MA designs of type 0 for 4×2^n factorials

f	$\widetilde{T}$	Source
3	{3, 13, 23}	Theorem 6.4.2(a)
4	{3, 13, 23, 123}	Corollary 6.4.2
5	{3, 13, 23, 123, 4}	Theorem 6.4.1 ($i = 1$)
6	{3, 13, 4, 14, 34, 134}	Theorem 6.4.1 ($i = 2$)
7	{3, 13, 4, 14, 24, 34, 134}	Theorem 6.4.1 ($i = 2$)
8	{3, 13, 23, 4, 14, 24, 34, 134}	Theorem 6.4.1 ($i = 2$)
9	{3, 23, 123, 4, 14, 24, 34, 134, 234}	Theorem 6.4.2 (e)
10	{3, 13, 23, 123, 4, 14, 24, 34, 134, 234}	Theorem 6.4.2 (c)
11	{3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234}	Theorem 6.4.2 (a)
12	{3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234}	Corollary 6.4.2

Note: $T_0 = \{1, 2, 12\}$.

In particular, if $m = 5$ in the setup of Table 6.8, i.e., the run size is 32, then $f = 2^5 - 2^2 - n = 28 - n$. Thus Table 6.8 gives MA 4×2^n designs of type 0 over the range $16 \leq n \leq 25$ and hence supplements Tables 6.3, which covers $4 \leq n \leq 9$.

For 8×2^n factorials, if $5 \leq f \leq 8$, then MA designs of type 0 are given by Corollary 6.4.2 or Theorem 6.4.2. On the other hand, for $f = 3$ or 4, such designs are given by $T_0 = \{1, 2, 12, 3, 13, 23, 123\}$ and $\widetilde{T} = \{4, 14, 24\}$ or $\widetilde{T} = \{4, 14, 24, 34\}$ respectively; this follows from Theorem 6.4.1 with $i = 1$ or 3 respectively.

Turning to 9×3^n factorials, Corollary 6.4.2 yields an MA design of type 0 for $f = 9$ $[= (3^3 - 3^2)/(3 - 1)]$. On the other hand, Table 6.9 gives the set $\widetilde{T} = P - (T_0 \cup T)$ for such designs over the range $3 \leq f \leq 8$. This table is obtained using Theorem 6.4.1 with $i = 1$. In the setup of Table 6.9, $T_0 = \{1, 2, 12, 12^2\}$. Hence if m is given, then for any f covered by this table, the set T corresponding to an MA design of type 0 can be easily obtained as $T = P - (T_0 \cup \widetilde{T})$.

Table 6.9 The sets $\widetilde{T}$ for MA designs of type 0 for 9×3^n factorials

f	$\widetilde{T}$
3	$\{3, 12^23, 12^23^2\}$
4	$\{3, 12^23, 12^23^2, 23^2\}$
5	$\{3, 13^2, 23, 12^23, 12^23^2\}$
6	$\{3, 13, 23, 123, 13^2, 23^2\}$
7	$\{3, 13, 23, 123, 13^2, 23^2, 123^2\}$
8	$\{3, 13, 23, 123, 13^2, 23^2, 123^2, 12^23^2\}$

Note: $T_0 = \{1, 2, 12, 12^2\}$.

As an illustration, we revisit Example 6.3.1. Then $s = 3$, $r = 2$, $n = 3$, $m = 3$, and $f = (3^3 - 3^2)/(3-1) - 3 = 6$. For $f = 6$, Table 6.9 shows that $\widetilde{T} = \{3, 13, 23, 123, 13^2, 23^2\}$. Here $P = PG(2, 3)$. Hence the set T corresponding to an MA design of type 0 is given by $P - (T_0 \cup \widetilde{T}) = \{123^2, 12^23, 12^23^2\}$. It is now immediate from (6.3.4) that the design considered earlier in this example is an MA design of type 0.

Before concluding this section, we briefly discuss the situation in which pencils of types 0 and 1 are equally important. Then it is appropriate to consider minimum overall aberration designs, which are defined as follows. With reference to an $(s^r) \times s^n$ factorial, consider designs d_1 and d_2 both having run size s^m. Let u be the smallest integer such that $A_{u0}(d_1) + A_{u1}(d_1) \neq A_{u0}(d_2) + A_{u1}(d_2)$. If $A_{u0}(d_1) + A_{u1}(d_1) < A_{u0}(d_2) + A_{u1}(d_2)$, then d_1 is said to have less overall aberration than d_2. A *minimum overall aberration* (MOA) design is a design such that no other design has less overall aberration than it.

Lemma 6.4.1. *With reference to an* $(s^r) \times s^n$ *factorial, let* $d = d(T_0, T)$ *be a design having run size* s^m *and* $\widetilde{T} = P - (T_0 \cup T)$. *Then*

(a) $A_{30}(d) + A_{31}(d) = \text{constant} - G_3(\widetilde{T})$,

(b) $A_{40}(d) + A_{41}(d) = \text{constant} + (3s-5)G_3(\widetilde{T}) + G_4(\widetilde{T}) - \frac{1}{2}(s^r - s)H_2(T_0, \widetilde{T})$.

Proof. Part (a) is immediate from Lemma 6.3.5 (a) and (b). By Lemma 6.3.5 (c) and (d),

$$A_{40}(d) + A_{41}(d) = \text{constant} - \frac{1}{2}(s^r - s)G_3(T_0 \cup \widetilde{T}) + \frac{1}{2}(s^r + 5s - 10)G_3(\widetilde{T}) + G_4(\widetilde{T}).$$

Applying Lemma 6.3.3(a) to the second term in the right-hand side, (b) follows. □

One can obtain an analogue of Theorem 6.4.1 from the above lemma and use it to derive further results. For example, with $f = 2$, it can be seen that $d(T_0, T)$ is an MOA design if and only if $\widetilde{T} = P - (T_0 \cup T)$ is of the form (6.4.1). For 4×2^n factorials, Table 6.10 gives the set $\widetilde{T}$ for MOA designs over the range $3 \leq f \leq 12$. It also indicates the parts of Lemma 6.4.1 that are needed for the derivation of such designs. In Table 6.10, $T_0 = \{1, 2, 12\}$. If m is given, then for any f covered by this table, the set T corresponding to an MOA design can be easily obtained as before. A comparison of Tables 6.10 and 6.8 reveals that in most cases the criteria of MOA and MA of type 0 yield different results.

Table 6.10 The sets $\widetilde{T}$ for MOA designs for 4×2^n factorials

f	$\widetilde{T}$	Needed Part(s) of Lemma 6.4.1
3	$\{3, 4, 34\}$	(a)
4	$\{3, 4, 34, 13\}$	(a),(b)
5	$\{3, 4, 34, 14, 134\}$	(a),(b)
6	$\{3, 4, 34, 13, 14, 134\}$	(a),(b)
7 $(m = 4)$	$\{3, 4, 34, 13, 14, 134, 24\}$	(a)
7 $(m \geq 5)$	$\{3, 4, 34, 5, 35, 45, 345\}$	(a)
8 $(m = 4)$	$\{3, 4, 34, 13, 14, 134, 23, 24\}$	(a)
8 $(m \geq 5)$	$\{3, 4, 34, 5, 35, 45, 345, 13\}$	(a),(b)
9 $(m = 4)$	$\{3, 4, 34, 13, 14, 134, 23, 24, 234\}$	(a)
9 $(m \geq 5)$	$\{3, 4, 34, 5, 35, 45, 345, 14, 134\}$	(a),(b)
10	$\{3, 4, 34, 5, 35, 45, 134, 135, 145, 1345\}$	(a),(b)
11	$\{3, 4, 34, 5, 35, 45, 345, 134, 135, 145, 1345\}$	(a),(b)
12	$\{3, 4, 34, 5, 35, 45, 345, 13, 14, 134, 15, 1345\}$	(a),(b)

Note: $T_0 = \{1, 2, 12\}$.

We further remark that the matrix $V(T_0 \cup T)$ has full row rank for any MA design of type 0 as envisaged in Corollary 6.4.1, Corollary 6.4.2, Theorem 6.4.2, and Tables 6.8, 6.9. The same holds also for any MOA design given in Table 6.10.

6.5 Designs for $(s^{r_1}) \times (s^{r_2}) \times s^n$ Factorials

As noted in Section 6.3, mixed factorials typically involve several factors, each with a small number of levels and rather few factors with more levels. From this perspective, regular fractions of an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial are considered in this section. Such a factorial has one factor, say F_{01}, at s^{r_1} levels, another factor, say F_{02}, at s^{r_2} levels, and additional n factors, say $F_1, \ldots, F_n$, at s levels each. Here r_1 (≥ 2), r_2 (≥ 2) are integers and s (≥ 2) is a prime or prime power. The special case $s = r_1 = r_2 = 2$ covers $4^2 \times 2^n$ factorials discussed in Section 6.2.

The developments in this section closely follow the last two sections. For $i = 1, 2$, let $t_i = (s^{r_i} - 1)/(s - 1)$ and V_{r_i} be the $r_i \times t_i$ matrix with columns given by points of $PG(r_i - 1, s)$. Then, in view of Lemma 6.3.1 and analogously to (6.3.1),

$$\begin{aligned} \mathcal{X} = \{ & (x_1, \ldots, x_{t_1}, x_{t_1+1}, \ldots, x_{t_1+t_2}, x_{t_1+t_2+1}, \ldots, x_{t_1+t_2+n})' : \\ & (x_1, \ldots, x_{t_1}) \in \mathcal{R}(V_{r_1}), (x_{t_1+1}, \ldots, x_{t_1+t_2}) \in \mathcal{R}(V_{r_2}), \\ & x_{t_1+t_2+1}, \ldots, x_{t_1+t_2+n} \in GF(s)\} \end{aligned} \tag{6.5.1}$$

represents the collection of the $s^{r_1+r_2+n}$ treatment combinations in an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial. Here $(x_1, \ldots, x_{t_1})$ refers to a level of F_{01}, $(x_{t_1+1}, \ldots, x_{t_1+t_2})$ refers to a level of F_{02}, and $x_{t_1+t_2+i}$ refers to a level of F_i $(1 \le i \le n)$.

Example 6.5.1. For a $4^2 \times 2^3$ factorial, $s = 2$, $r_1 = r_2 = 2$, $n = 3$, $t_1 = t_2 = 3$, and as in Section 6.3, $\mathcal{R}(V_2) = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$. Thus by (6.5.1), the treatment combinations in a $4^2 \times 2^3$ factorial can be represented by $(x_1, \ldots, x_9)'$, where $(x_1, x_2, x_3) \in \mathcal{R}(V_2), (x_4, x_5, x_6) \in \mathcal{R}(V_2)$ and $x_7, x_8, x_9 \in \{0, 1\}$. □

With reference to an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, a pencil is a nonnull vector of the form

$$b = (b_1, \ldots, b_{t_1}, b_{t_1+1}, \ldots, b_{t_1+t_2}, b_{t_1+t_2+1}, \ldots, b_{t_1+t_2+n})',$$

where $b_i \in GF(s)$ for all i, among $b_1, \ldots, b_{t_1}$ at most one is nonzero, and among $b_{t_1+1}, \ldots, b_{t_1+t_2}$ at most one is nonzero. Pencils with proportional entries are considered identical. As in Section 6.3, a pencil with i nonzero entries belongs to a main effect if $i = 1$ and to an i-factor interaction if $i \ge 2$. For any pencil b belonging to an interaction, one of the following three cases arises:

(0) it involves only some of $F_1, \ldots, F_n$, i.e., $b_1 = \cdots = b_{t_1+t_2} = 0$;
(1) it involves one of F_{01} and F_{02} and some of $F_1, \ldots, F_n$, i.e., either one of $b_1, \ldots, b_{t_1}$ is nonzero and $b_{t_1+1} = \cdots = b_{t_1+t_2} = 0$, or one of $b_{t_1+1}, \ldots, b_{t_1+t_2}$ is nonzero and $b_1 = \cdots = b_{t_1} = 0$;
(2) it involves both F_{01} and F_{02} and possibly some of $F_1, \ldots, F_n$, i.e., one of $b_1, \ldots, b_{t_1}$ is nonzero and one of $b_{t_1+1}, \ldots, b_{t_1+t_2}$ is also nonzero.

Following the terminology of Section 6.2, pencils of these three types are called type 0, type 1, and type 2 respectively.

The concept of a regular fraction will now be extended to an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial. Suppose it is desired to have a fraction consisting of s^m treatment combinations, where $r_1 + r_2 \le m < r_1 + r_2 + n$. Let P denote the set of the $(s^m - 1)/(s - 1)$ points of $PG(m - 1, s)$, and for any nonempty subset Q of P, let the matrix $V(Q)$ be defined as in Section 6.3. Let T_{01} and T_{02} be the $(r_1 - 1)$- and $(r_2 - 1)$-flats of P spanned by $e_1, \ldots, e_{r_1}$ and $e_{r_1+1}, \ldots, e_{r_1+r_2}$ respectively, where $e_1, \ldots, e_m$ are the $m \times 1$ unit vectors over $GF(s)$. Since $r_1 + r_2 \le m$, both T_{01} and T_{02} are well defined. Furthermore, analogously to (6.3.2),

$$V(T_{01}) = \begin{bmatrix} V_{r_1} \\ 0 \end{bmatrix}, \tag{6.5.2}$$

where 0 is the null matrix of order $(m - r_1) \times t_1$, and

$$V(T_{02}) = \begin{bmatrix} 0^{(1)} \\ V_{r_2} \\ 0^{(2)} \end{bmatrix}, \tag{6.5.3}$$

where $0^{(1)}$ and $0^{(2)}$ are null matrices of orders $r_1 \times t_2$ and $(m - r_1 - r_2) \times t_2$ respectively. Let T be an n-subset of P such that T_{01}, T_{02}, and T are disjoint and the matrix

$$V(T_{01} \cup T_{02} \cup T) = [V(T_{01}) \;\; V(T_{02}) \;\; V(T)] \tag{6.5.4}$$

has full row rank. Then there are s^m vectors in $\mathcal{R}[V(T_{01} \cup T_{02} \cup T)]$. By (6.5.1)–(6.5.4), the transpose of each of these vectors belongs to $\mathcal{X}$ and hence represents a treatment combination of an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial. The collection of the s^m treatment combinations (or runs) so obtained gives a *regular fraction*, to be denoted by $d = d(T_{01}, T_{02}, T)$, of such a factorial. For the special case of a $4^2 \times 2^n$ factorial (i.e., $s = r_1 = r_2 = 2$), the elements of H_m represent the points of $PG(m-1, 2)$; $T_{01} = \{1, 2, 12\}$, and $T_{02} = \{3, 4, 34\}$ represent the two four-level factors; and T, consisting of n elements from the rest of H_m, represents the n two-level factors. Again, hereafter in this section, a regular fraction is simply called a design.

Considering the cardinalities of T_{01}, T_{02}, T, and P, the above construction is possible if and only if $r_1 + r_2 \le m$, and

$$\frac{s^{r_1} - 1}{s - 1} + \frac{s^{r_2} - 1}{s - 1} + n \le \frac{s^m - 1}{s - 1}, \quad \text{i.e.,} \quad s^{r_1} + s^{r_2} + n(s-1) - 1 \le s^m.$$

These conditions are supposed to hold throughout this section. This construction is motivated by the approach of Wu, Zhang, and Wang (1992).

A pencil b is a *defining pencil* of $d = d(T_{01}, T_{02}, T)$ if

$$V(T_{01} \cup T_{02} \cup T)b = 0. \tag{6.5.5}$$

As in Section 6.3, any pencil satisfying (6.5.5) has at least three nonzero entries, i.e., d has resolution three or higher. For $i \ge 3$, let $A_{i0}(d)$, $A_{i1}(d)$, and $A_{i2}(d)$ denote the numbers of (distinct) defining pencils of d that have i nonzero entries and are of types 0, 1, and 2 respectively. In the present setup, two designs $d(T_{01}, T_{02}, T_1)$ and $d(T_{01}, T_{02}, T_2)$ are *isomorphic* if there exists a nonsingular transformation that maps each point of T_{0j} to some point of $T_{0j} (j = 1, 2)$ up to proportionality, and each point of T_1 to some point of T_2 up to proportionality. Furthermore, if $r_1 = r_2$, then two such designs are isomorphic also when there exists a nonsingular transformation that maps each point of T_{01} to some point of T_{02}, each point of T_{02} to some point of T_{01}, and each point of T_1 to some point of T_2, up to proportionality. From (6.5.5), it can be seen that isomorphic designs have the same A_{i0}, A_{i1}, and A_{i2} for every i.

Example 6.5.2. Using the projective geometric formulation, we revisit the $4^2 \times 2^3$ design d_1 with 16 runs, as considered in Example 6.2.1. Then $s = r_1 = r_2 = 2$, $n = 3$, $m = 4$, and $d_1 = d(T_{01}, T_{02}, T)$, where $T_{01} = \{(1,0,0,0)', (0,1,0,0)', (1,1,0,0)'\}$, $T_{02} = \{(0,0,1,0)', (0,0,0,1)', (0,0,1,1)'\}$, and $T = \{(1,0,0,1)', (0,1,1,0)', (0,1,1,1)'\}$. Clearly, T_{01}, T_{02}, and T are disjoint and the matrix

$$V(T_{01} \cup T_{02} \cup T) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \tag{6.5.6}$$

has full row rank. Hence consideration of $\mathcal{R}[V(T_{01} \cup T_{02} \cup T)]$ yields the design d_1. By (6.5.5) and (6.5.6) the defining pencils of d_1 are

$$\begin{aligned} &(1,0,0,0,1,0,1,0,0)', \quad (0,1,0,1,0,0,0,1,0)', \quad (0,1,0,0,0,1,0,0,1)', \\ &(0,0,1,0,0,1,1,1,0)', \quad (0,0,1,1,0,0,1,0,1)', \quad (0,0,0,0,1,0,0,1,1)', \\ &(1,0,0,0,0,0,1,1,1)', \end{aligned}$$

which agree with the defining relation of d_1 shown in Example 6.2.1. Among the pencils listed above, the last two are of type 1 and the rest are of type 2. Hence counting the numbers of nonzero entries in these pencils, one gets $A_{30}(d_1) = 0$, $A_{31}(d_1) = 1$, $A_{32}(d_1) = 3$, $A_{40}(d_1) = 0$, $A_{41}(d_1) = 1$, $A_{42}(d_1) = 2$, $A_{50}(d_1) = A_{51}(d_1) = A_{52}(d_1) = 0$. These again agree with Example 6.2.1. □

As argued in Section 6.2, we suppose that pencils of type 0 are most important and those of type 2 are least important. Then it is appropriate to consider MA designs of type 0, which are defined as follows. With reference to an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, consider designs d_1 and d_2 both having run size s^m. Let u be the smallest integer such that $(A_{u0}(d_1), A_{u1}(d_1), A_{u2}(d_1)) \neq (A_{u0}(d_2), A_{u1}(d_2), A_{u2}(d_2))$. As in Definition 6.2.2, if either (i) $A_{u0}(d_1) < A_{u0}(d_2)$ or (ii) $A_{u0}(d_1) = A_{u0}(d_2)$ but $A_{u1}(d_1) < A_{u1}(d_2)$ or (iii) $A_{u0}(d_1) = A_{u0}(d_2)$, $A_{u1}(d_1) = A_{u1}(d_2)$ but $A_{u2}(d_1) < A_{u2}(d_2)$, then d_1 is said to have less aberration of type 0 than d_2. An *MA design of type 0* is a design such that no other design has less aberration of type 0 than it.

Consideration of complementary sets again facilitates the study of MA designs of type 0. To that effect, some notation is needed. For any nonempty $Q(\subset P)$ of cardinality q, define $G_i(Q)$ via (6.3.6). If, in addition, Q and T_{01} or Q and T_{02} are disjoint, define $H_i(T_{01}, Q)$ or $H_i(T_{02}, Q)$ as in (6.3.7), replacing T_0 there by T_{01} or T_{02} respectively. Furthermore, if Q, T_{01}, and T_{02} are all disjoint, define for $i \geq 1$,

$$K_i(T_{01}, T_{02}, Q) = (s-1)^{-1} \#\{\lambda : \lambda \in \Omega_{iq}, \text{ there exist nonzero } \alpha_j \in GF(s) \text{ and } h_j \in T_{0j} \ (j = 1, 2) \text{ such that } V(Q)\lambda = \alpha_1 h_1 + \alpha_2 h_2\}.$$

Note that $K_i(T_{01}, T_{02}, Q) = 0$ if $i > q$.

Considering a design $d = d(T_{01}, T_{02}, T)$, by (6.5.5) and analogously to (6.3.9),

$$A_{i0}(d) = G_i(T), \ A_{i1}(d) = H_{i-1}(T_{01}, T) + H_{i-1}(T_{02}, T),$$
$$A_{i2}(d) = K_{i-2}(T_{01}, T_{02}, T), \text{ for } i \geq 3. \tag{6.5.7}$$

Let $\widetilde{T} = P - (T_{01} \cup T_{02} \cup T)$. The cardinality of $\widetilde{T}$ equals

$$f = (s^m - s^{r_1} - s^{r_2} + 1)/(s-1) - n.$$

Let

$$\Psi_1(T_{01}, T_{02}, \widetilde{T}) = K_1(T_{01}, T_{02}, \widetilde{T}) - G_3(\widetilde{T}),$$
$$\Psi_2(T_{01}, T_{02}, \widetilde{T}) = 2G_4(\widetilde{T}) + H_3(T_{01}, \widetilde{T}) + H_3(T_{02}, \widetilde{T}).$$

Since there is only one design when $f = 0$, hereafter suppose $f \geq 1$. Also to avoid trivialities, let $n \geq 2$.

Lemma 6.5.1. *With reference to an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, let $d = d(T_{01}, T_{02}, T)$ be a design having run size s^m. Then*

$$\begin{aligned}
(a) \quad A_{30}(d) &= \text{constant} - G_3(T_{01} \cup T_{02} \cup \widetilde{T}),\\
(b) \quad A_{31}(d) &= \text{constant} + G_3(T_{01} \cup T_{02} \cup \widetilde{T}) + \Psi_1(T_{01}, T_{02}, \widetilde{T}),\\
(c) \quad A_{32}(d) &= \text{constant} - K_1(T_{01}, T_{02}, \widetilde{T}),\\
(d) \quad A_{40}(d) &= \text{constant} + (3s-5)G_3(T_{01} \cup T_{02} \cup \widetilde{T}) + G_4(T_{01} \cup T_{02} \cup \widetilde{T}),\\
(e) \quad A_{41}(d) &= \text{constant} - (3s-5)\{G_3(T_{01} \cup T_{02} \cup \widetilde{T}) + \Psi_1(T_{01}, T_{02}, \widetilde{T})\}\\
&\quad -\frac{1}{2}(s^{r_1} + s^{r_2} - 2s)K_1(T_{01}, T_{02}, \widetilde{T}) - 2G_4(T_{01} \cup T_{02} \cup \widetilde{T})\\
&\quad +\Psi_2(T_{01}, T_{02}, \widetilde{T}),\\
(f) \quad A_{42}(d) &= \text{constant} + (s^{r_1} + s^{r_2} + s - 5)K_1(T_{01}, T_{02}, \widetilde{T})\\
&\quad +K_2(T_{01}, T_{02}, \widetilde{T}).
\end{aligned}$$

The constants in Lemma 6.5.1 may depend on s, r_1, r_2, n, and m, but not on the particular choice of T. The proof of this lemma utilizes (6.5.7). It is available in Mukerjee and Wu (2001) and omitted here. Define the following classes of designs:

$$\Delta_1 = \{d = d(T_{01}, T_{02}, T) : d \text{ maximizes } G_3(T_{01} \cup T_{02} \cup \widetilde{T})\},$$
$$\Delta_2 = \{d : d \in \Delta_1,\ d \text{ minimizes } \Psi_1(T_{01}, T_{02}, \widetilde{T}) \text{ over } \Delta_1\},$$
$$\Delta_3 = \{d : d \in \Delta_2,\ d \text{ maximizes } K_1(T_{01}, T_{02}, \widetilde{T}) \text{ over } \Delta_2\},$$
$$\Delta_4 = \{d : d \in \Delta_3,\ d \text{ minimizes } G_4(T_{01} \cup T_{02} \cup \widetilde{T}) \text{ over } \Delta_3\},$$
$$\Delta_5 = \{d : d \in \Delta_4,\ d \text{ minimizes } \Psi_2(T_{01}, T_{02}, \widetilde{T}) \text{ over } \Delta_4\},$$
$$\Delta_6 = \{d : d \in \Delta_5,\ d \text{ minimizes } K_2(T_{01}, T_{02}, \widetilde{T}) \text{ over } \Delta_5\}.$$

Recalling the definition of MA designs of type 0, Lemma 6.5.1 yields the following result, which serves as a useful tool for finding such designs.

Theorem 6.5.1. *For any i $(1 \le i \le 6)$, suppose d belongs to Δ_i and, up to isomorphism, is the unique member of Δ_i. Then d is a minimum aberration design of type 0.*

The above result is analogous to Theorem 6.4.1 and, like the latter, leads to the following corollaries when applied with $i = 1$.

Corollary 6.5.1. *Let $f = 1$. Then $d = d(T_{01}, T_{02}, T)$ is a minimum aberration design of type 0 provided $\widetilde{T} = P - (T_{01} \cup T_{02} \cup T)$ is of the form $\widetilde{T} = \{h_1 + \alpha h_2\}$ for some $h_j \in T_{0j} (j = 1, 2)$ and $\alpha (\neq 0) \in GF(s)$.*

Corollary 6.5.2. *Let $f = (s^w - s^{r_1} - s^{r_2} + 1)/(s - 1)$, where $w \ge r_1 + r_2$. Then $d = d(T_{01}, T_{02}, T)$ is a minimum aberration design of type 0 provided $T_{01} \cup T_{02} \cup \widetilde{T}$ is a $(w - 1)$- flat of P, where $\widetilde{T} = P - (T_{01} \cup T_{02} \cup T)$.*

If $m > r_1 + r_2$ then for $f = 1$, not all designs have $\widetilde{T}$ as in Corollary 6.5.1, so that even for $f = 1$ discrimination among designs is possible on the basis of the MA criterion of type 0. In general, Theorem 6.5.1 considerably simplifies the identification of MA designs of type 0 when f is small. As a specific application, consider $4^2 \times 2^n$ factorials. Then $s = r_1 = r_2 = 2$ and Corollaries 6.5.1 and 6.5.2 settle the cases $f = 1$ and $f = 9$ respectively. For $2 \le f \le 8$, Table 6.11 gives the set $\widetilde{T} = P - (T_{01} \cup T_{02} \cup T)$ for MA designs of type 0 and indicates how they are obtained via Theorem 6.5.1. This table uses the same notation as in Tables 6.8 and 6.10. Thus, in this table, $T_{01} = \{1, 2, 12\}$, $T_{02} = \{3, 4, 34\}$, so that if m is given, then for any f covered by the table, the set T corresponding to an MA design of type 0 can be easily obtained as $T = P - (T_{01} \cup T_{02} \cup \widetilde{T})$. For any MA design, as given by Corollary 6.5.1, Corollary 6.5.2, and Table 6.11, it can be seen that the matrix $V(T_{01} \cup T_{02} \cup T)$ has full row rank.

Table 6.11 The sets $\widetilde{T}$ for MA designs of type 0 for $4^2 \times 2^n$ factorials

f	$\widetilde{T}$	Use Theorem 6.5.1 with $i =$
2	$\{13, 23\}$	1
3	$\{13, 23, 123\}$	1
4	$\{13, 23, 14, 24\}$	4
5	$\{13, 23, 14, 24, 1234\}$	2
6	$\{13, 23, 123, 14, 24, 1234\}$	2
7	$\{13, 23, 123, 14, 24, 134, 1234\}$	2
8	$\{13, 23, 123, 14, 24, 124, 134, 1234\}$	1

Note: $T_{01} = \{1, 2, 12\}$ and $T_{02} = \{3, 4, 34\}$.

As an illustration, we consider the problem of finding an MA design of type 0, with 16 runs, for a $4^2 \times 2^6$ factorial. Then $s = 2$, $r_1 = r_2 = 2$, $n = 6$, $m = 4$, and $f = (2^4 - 2^2 - 2^2 + 1) - 6 = 3$. For $f = 3$, Table 6.11 shows that $\widetilde{T} = \{13, 23, 123\}$. Here $P \equiv PG(3, 2)$. Hence the set T corresponding to an MA design of type 0 is given by $T = P - (T_{01} \cup T_{02} \cup \widetilde{T}) = \{14, 24, 124, 134, 234, 1234\}$.

In particular, if $m = 5$ in the setup of Table 6.11, i.e., the run size is 32, then $f = (2^5 - 2^2 - 2^2 + 1) - n = 25 - n$. Table 6.11 then gives MA designs of type 0 for $4^2 \times 2^n$ factorials over the range $17 \leq n \leq 23$ and hence supplements Table 6.6, which covers $2 \leq n \leq 10$.

To conclude the chapter, we mention some other related work. Zhang and Shao (2001) reported further results on MA designs for mixed factorials. Mukerjee, Chan, and Fang (2000) extended the criterion of maximum estimation capacity to mixed factorials taking cognizance of the distinction among pencils of type 0, 1, etc., and, in the spirit of Chapter 5, observed that the results tend to agree with those under the MA criterion.

Exercises

6.1 Show that the four-symbol column obtained by applying the method of replacement to any two-symbol orthogonal array of strength two is orthogonal to the remaining two-symbol columns of the array.

6.2 Derive the defining relation for the $4^2 \times 2^3$ design d_2 in Example 6.2.1 and use it to confirm the A_{ij} values in Example 6.2.1.

6.3 Prove from first principles that the $4^2 \times 2^3$ design d_1 in Example 6.2.1 has MA of type 0.

6.4 Prove Theorem 6.2.1(b) for $k = 3$. Hence indicate the proof for general $k \geq 2$.

6.5 Using Theorem 6.2.1 (b), show that the design d_2 whose defining relation is shown in (6.2.3) does not have MA of type 0.

6.6 Consider a 4×2^n design d^* with run size 2^m. Suppose d^* is represented by $T_0 = \{\gamma_1, \gamma_2, \gamma_3\}$, where $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 12$, and n other elements $c_1, \ldots, c_n$ of H_m. Assume that

(i) each of $\gamma_3, c_1, \ldots, c_n$ appears in some word in the defining relation $DR(d^*)$ of d^*,

(ii) the set $\{\gamma_1, \gamma_2, c_1, \ldots, c_n\}$ contains m independent elements.

(a) Show that $\sum_{i \geq 3} iA_i(d^*) = (n+2)2^{n+1-m} - C$, where $A_i(d^*) = A_{i0}(d^*) + A_{i1}(d^*)$, and C is the number of words in $DR(d^*)$ containing γ_3. [Hint: use (3.2.2) and the discussion before (6.2.5).]

(b) Show that $C = 2^{n+1-m}$ if γ_1 and γ_2 do not appear in any word in $DR(d^*)$, and that $C = 2^{n-m}$ otherwise.

6.7 Under appropriate assumptions, extend the results in the previous exercise to $4^2 \times 2^n$ designs with run size 2^m.

6.8 The MA criterion given in Definition 6.2.1 is called type 0 because it first compares the values of A_{u0}. For 4×2^n designs, if the roles of A_{u0} and A_{u1} are reversed in the definition, the resulting criterion is called MA of *type* 1. A third alternative is to use the minimum overall aberration criterion introduced in Section 6.4.

(a) Describe situations in which either of these two alternative criteria make more sense.

(b) Considering 4×2^7 designs with 16 runs, show that the design $d(T_0, 3, 4, 14, 23, 34, 123, 134)$ has less aberration of type 1 than the design $d(T_0, 3, 4, 13, 14, 23, 24, 124)$. Which design has less overall aberration?

6.9 For $4^p \times 2^n$ designs with general p, extend the definition of type-j words to $0 \leq j \leq p$. Based on the extended definitions, define the MA criterion of type 0 for $4^p \times 2^n$ designs.

6.10 Prove Lemma 6.3.3(a) by using definitions and combinatorial arguments.

6.11 Considering 4×2^n factorials with $17 \leq n \leq 19$, use Theorem 6.4.2 to obtain MA designs of type 0 when the run size is 32.

6.12 Prove Lemma 6.5.1(c) by using definitions and combinatorial arguments.

6.13 Prove Corollary 6.5.1.

7

Block Designs for Symmetrical Factorials

Block designs for factorial experiments are studied in this chapter. They are useful when the experimental units are not homogeneous. The simpler case of two-level full factorials is considered first. Then the problem of blocking for fractions of general symmetrical factorials is taken up and a finite projective geometric formulation is obtained. Various optimality criteria are discussed and the method of complementary sets is developed for finding optimal designs. A collection of design tables is given.

7.1 Optimal Block Designs for Full Factorials

The designs studied in the previous chapters involve a *completely random allocation* of the selected treatment combinations to the experimental units. This kind of allocation is appropriate only if the experimental units are homogeneous. However, such homogeneity may not always be guaranteed especially when the size of the experiment is relatively large. A practical design strategy is then to partition the experimental units into homogeneous groups, known as *blocks*, and restrict randomization separately to each block. Consequently, a design should dictate not only the treatment combinations to be included in the experiment but also their allocation to the blocks. Blocks are often formed as natural groupings of experimental units such as batches of materials, plots of land, time periods, etc. Block designs for symmetrical factorials are considered in this chapter. *Throughout, the additivity of the block and treatment effects is assumed.*

While blocking is expected to partition the experimental units into homogeneous groups, there can be significant heterogeneity from one block to another. Therefore, the block effects are potentially as substantial as even the factorial main effects. On the other hand, although the block effects can be quite nontrivial, they are, as such, of no interest to the experimenter. Interest lies only in the factorial effects that are rank-ordered according to the effect hierarchy principle. A major concern now is the extent to which the presence of blocks can affect the estimation of the factorial effects. This distinction between the block and factorial effects is fundamental to the development of the design criteria to be discussed in this chapter.

Following Sun, Wu, and Chen (1997), we consider in this section the simpler problem of finding optimal block designs for two-level full factorials. As in Section 3.1, any factorial effect can be conveniently denoted by $i_1 \dots i_g$.

Example 7.1.1. To motivate the ideas, consider the problem of designing a full 2^5 factorial in $8(=2^3)$ blocks. Start with three independent factorial effects, say,

$$b_1 = 135, \quad b_2 = 235, \quad b_3 = 1234. \tag{7.1.1}$$

For any treatment combination, there are $2^3(=8)$ possibilities, depending on whether it appears with a plus or minus sign in b_1, b_2, and b_3. This entails a partitioning of the 2^5 treatment combinations into eight classes, each of size four. One can assign each class so formed to one block to get the eight blocks. Clearly, the treatment combinations in the same block appear with the same sign in any $b_i (i = 1, 2, 3)$. Hence the treatment contrasts representing b_1, b_2, b_3 are also contrasts among blocks. As a consequence, these factorial effects are entangled or *confounded* with blocks. They cannot be estimated in the presence of block effects. In fact, it is easily seen that this happens for not only b_1, b_2, b_3 but also the factorial effects that they generate. In other words, the factorial effects

$$\begin{array}{ll} b_1 b_2 = (135)(235) = 12, & b_1 b_3 = (135)(1234) = 245, \\ b_2 b_3 = (235)(1234) = 145, & b_1 b_2 b_3 = (135)(235)(1234) = 34, \end{array} \tag{7.1.2}$$

are also confounded with blocks.

What happens for the remaining factorial effects? One can check that half the treatment combinations in any block appear with a plus sign and the remaining half with a minus sign in any of these factorial effects. Hence they remain estimable even in the presence of block effects, i.e., they are *unconfounded* with blocks. □

A more rigorous explanation for the phenomenon of confounding appears in the next section with reference to a general setting. In Example 7.1.1, there are altogether seven confounded factorial effects as shown in (7.1.1) and (7.1.2). Together with the identity element I, they form a group called the *block defining contrast subgroup*. It is the blocking counterpart of the (treatment) defining contrast subgroup of a 2^{n-k} design. Generally, as with 2^{n-k} designs, we can apply the minimum aberration (MA) criterion to the block defining contrast subgroup on the basis of the effect hierarchy principle. Formally, for a block design d, let $\beta_i(d)$ be the number of words of length i in the block defining contrast subgroup, i.e., $\beta_i(d)$ is the number of factorial effects of order i that are confounded with blocks. We may use the *wordlength pattern* $(\beta_1(d), \beta_2(d), \beta_3(d), \dots)$ to characterize the properties of d. Obviously, it is required that $\beta_1(d) = 0$. The MA criterion, based on sequential minimization of $\beta_2(d), \beta_3(d), \ldots$, can be applied to rank-order block designs. A best design according to this criterion is called a *minimum aberration block design.* Thus all the tools and results for MA designs in the previous chapters can be used here. Since the block defining contrast subgroup of a 2^5 design in 2^3 blocks is equivalent to the (treatment) defining contrast subgroup of a 2^{5-3} design, from (7.1.1), (7.1.2), and the proof of Theorem 3.2.2 (for $n = 5$), we conclude that the block design in Example 7.1.1 has MA because it has wordlength pattern $(0, 2, 4, 1, 0)$.

Because of the mathematical equivalence between 2^n designs in 2^k blocks and 2^{n-k} designs, there is no need to separately list tables of optimal block designs for full factorials. These tables can be found in Sun, Wu, and Chen (1997). But there are some differences that are worth noting. First, the MA criterion has a stronger justification for the blocking problem because $\beta_i(d)$ has a straightforward interpretation as the number of factorial effects of order i that are "sacrificed" for blocking. The counterpart of $\beta_i(d)$ in a 2^{n-k} design d is $A_i(d)$; see (2.5.3). The implication of $A_i(d)$ is more complicated and based on the stringency of model assumptions under the effect hierarchy principle; cf.Section 2.5. Second, while $\beta_2(d) > 0$ is allowable, $A_2(d) > 0$ is not. The latter would imply the aliasing of two main effects, whereas the former merely means that some two-factor interactions are confounded with blocks.

The previous discussion suggests that a direct definition of estimability can be given for block designs for full factorials. Any such design is said to have *estimability of order* E if E is the largest integer such that all factorial effects of order E or lower are estimable (i.e., not confounded with blocks). For example, the design in Example 7.1.1 has $E = 1$ since two words of length two appear in (7.1.2). Clearly, $E + 1$ equals the smallest integer R such that $\beta_R(d) > 0$. For a 2^{n-k} design, this R would be the resolution of the design. So, why do we not use the notion of resolution in this context? Recall the implications of resolution as summarized in Theorem 2.5.1. For resolution three designs, the main effects are estimable under the assumption that all interactions are absent, while for resolution four designs, the *same* estimability holds under the *weaker* assumption that all interactions involving three or more factors are absent. Therefore, from the point of view of estimability, resolutions three and four differ only in the assumptions on the absence of interactions. Both correspond to $E = 1$ in the context of blocking. The main difference in the ideas underlying E and R is that the former does not require any assumption regarding the absence of factorial effects, while the latter does. If a connection has to be made between them, we can say that estimability of order E for 2^n designs in 2^k blocks corresponds to resolution $2E + 1$ or $2E + 2$ for 2^{n-k} designs.

For simplicity in presentation, we have so far restricted our discussion to the two-level case. Because the extension of the block defining contrast subgroup from $s = 2$ to general prime power s is straightforward, it is clear that the concepts of MA and estimability order are applicable more generally to s^n block designs. To avoid repetition, this extension is not discussed here.

The rest of this chapter is devoted to block designs for fractional factorials with general s. Their study is far more complicated and difficult due to the presence of two wordlength patterns, one arising from the choice of the fraction and the other arising from blocking. An effective treatment of this problem calls for the employment of formal mathematical tools such as those in Chapter 2.

7.2 Block Designs for Fractional Factorials

Consider a symmetrical s^n factorial, where s is a prime or prime power. As in Chapter 2, a typical treatment combination is denoted by an $n \times 1$ vector x over $GF(s)$. Also, an s^{n-k} design is given by $d(B) = \{x : Bx = 0\}$, where B is a $k \times n$ matrix, of full row rank, over $GF(s)$. Suppose it is desired to conduct the experiment in s^r blocks, each of size s^{n-k-r}, where $k + r < n$. This can be achieved as follows. Let B_0 be another matrix, of order $r \times n$ and defined over $GF(s)$, such that

$$\operatorname{rank}\begin{pmatrix} B \\ B_0 \end{pmatrix} = k + r, \tag{7.2.1}$$

and let Λ_r be the set of $r \times 1$ vectors over $GF(s)$. Index the s^r blocks by the s^r vectors in Λ_r. For any $\lambda \in \Lambda_r$, place the treatment combinations x satisfying

$$Bx = 0, \quad B_0 x = \lambda, \tag{7.2.2}$$

in the block indexed by λ. The first equation in (7.2.2) dictates the treatment combinations included in the experiment, while the second equation dictates their allocation to the blocks. For $s = 2$, any pattern of plus and minus signs, as considered in Example 7.1.1, corresponds to some λ. By (7.2.1), the $(k+r) \times n$ matrix $(B' \; B_0')'$ has full row rank. Hence from (7.2.2), it is easy to see that each block has the desired size s^{n-k-r}; cf. Lemma 2.3.1. Thus (7.2.2) defines an s^{n-k} design in s^r blocks of equal size, or briefly, an (s^{n-k}, s^r) *block design*, which will be denoted by $d(B, B_0)$.

Example 7.2.1. Let $s = 2$, $n = 6$, $k = 2$, $r = 2$. With

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{7.2.3}$$

the rank condition (7.2.1) holds. Consider, for instance, the block indexed by $(1,1)'$. By (7.2.2) and (7.2.3), this block consists of treatment combinations $x = (x_1, \ldots, x_6)'$ that satisfy

$$x_1 + x_2 + x_3 + x_5 = 0, \quad x_2 + x_3 + x_4 + x_6 = 0,$$

and

$$x_1 + x_3 + x_4 = 1, \quad x_1 + x_2 + x_4 = 1.$$

It is easily seen that these treatment combinations are $(0,1,1,0,0,0)'$, $(0,0,0,1,0,1)'$, $(1,0,0,0,1,0)'$, and $(1,1,1,1,1,1)'$. Similarly, the blocks indexed by $(0,0)'$, $(0,1)'$, and $(1,0)'$ are given by

$$\{(0,0,0,0,0,0)', (0,1,1,1,0,1)', (1,1,1,0,1,0)', (1,0,0,1,1,1)'\},$$

$$\{(1,1,0,1,0,0)', (1,0,1,0,0,1)', (0,0,1,1,1,0)', (0,1,0,0,1,1)'\},$$

and

$$\{(1,0,1,1,0,0)', (1,1,0,0,0,1)', (0,1,0,1,1,0)', (0,0,1,0,1,1)'\},$$

respectively. □

Example 7.2.2. Let $s = 3$, $n = 4$, $k = 1$, $r = 2$. With

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \tag{7.2.4}$$

the rank condition (7.2.1) holds. As before, the block indexed by $(2,2)'$ consists of treatment combinations $x = (x_1, \dots, x_4)'$ that satisfy

$$x_1 + x_2 + x_3 + x_4 = 0, \quad x_1 + x_2 = 2, \quad x_1 + x_3 = 2.$$

Thus this block is given by $\{(0,2,2,2)', (1,1,1,0)', (2,0,0,1)'\}$. Similarly, the other blocks can be obtained. □

The concepts of defining pencil and aliasing in the (s^{n-k}, s^r) block design $d(B, B_0)$ are the same as those in the s^{n-k} design $d(B)$. In other words, a defining pencil of $d(B)$ is also a defining pencil of $d(B, B_0)$, whereas pencils that are aliases of each other in $d(B)$ remain so in $d(B, B_0)$. As in Section 2.4, no treatment contrast belonging to any defining pencil is estimable in $d(B, B_0)$. We now examine the status of $d(B, B_0)$ relative to the estimation of the other pencils, taking cognizance of the block effects. Recall that these pencils are grouped into alias sets.

Consider any alias set $\mathcal{A}$ of $d(B)$, or equivalently, of $d(B, B_0)$. For any pencil $b \in \mathcal{A}$, following (2.4.16), define the sets

$$V_j(b, B) = \{x : b'x = \alpha_j \text{ and } Bx = 0\}, \ 0 \le j \le s-1, \tag{7.2.5}$$

where $\alpha_0, \alpha_1, \dots, \alpha_{s-1}$ are the elements of $GF(s)$. As noted in Remark 2.4.1, $V_j(b, B)$ has cardinality s^{n-k-1} for every j. Let $\mathcal{R}(\cdot)$ denote the row space of a matrix. Then one gets the following result, which is useful in the sequel.

Lemma 7.2.1. *(a) Either*

$$b' \in \mathcal{R}\begin{pmatrix} B \\ B_0 \end{pmatrix} - \mathcal{R}(B) \text{ for every } b \in \mathcal{A} \tag{7.2.6}$$

or

$$b' \notin \mathcal{R}\begin{pmatrix} B \\ B_0 \end{pmatrix} \text{ for every } b \in \mathcal{A}. \tag{7.2.7}$$

(b) If (7.2.6) holds, then for every $b \in \mathcal{A}$, each block of $d(B, B_0)$ is contained in one of the sets $V_j(b, B)$ $(0 \le j \le s-1)$.
(c) If (7.2.7) holds, then for every $b \in \mathcal{A}$ and every j $(0 \le j \le s-1)$, each block of $d(B, B_0)$ intersects $V_j(b, B)$ in $s^{n-k-r-1}$ treatment combinations.

Proof.

(a) Consider any two pencils in $\mathcal{A}$. Then

$$(b - b^*)' \in \mathcal{R}(B), \tag{7.2.8}$$

for some representations b and b^* of these pencils; cf.(2.4.9). Since b is not a defining pencil, it does not belong to $\mathcal{R}(B)$. Therefore, b must satisfy (7.2.6) or (7.2.7). Similarly, b^* also satisfies (7.2.6) or (7.2.7). Again, by (7.2.8), b satisfies (7.2.6) if and only if so does b^*. This proves (a).
(b) If (7.2.6) holds, then for every $b \in \mathcal{A}$, it is evident from (7.2.2) that $b'x$ remains the same for all treatment combinations x in the same block. Since these treatment combinations also satisfy $Bx = 0$, (b) follows from (7.2.5).
(c) If (7.2.7) holds, then by (7.2.1), the $(k + r + 1) \times n$ matrix $(B' \ B_0' \ b)'$ has full row rank for every $b \in \mathcal{A}$. Hence (c) is immediate from (7.2.2) and (7.2.5). □

Consider now any treatment contrast L belonging to a pencil b in the alias set $\mathcal{A}$. Following (2.4.18), the part of L that involves only the treatment combinations in $d(B, B_0)$ is of the form

$$L(B) = \sum_{j=0}^{s-1} l_j \Big\{ \sum_{x \in V_j(b,B)} \tau(x) \Big\}, \tag{7.2.9}$$

where $\tau(x)$ is the effect of the treatment combination x and $l_0, l_1, \ldots, l_{s-1}$ are real numbers, not all zero, satisfying

$$l_0 + l_1 + \cdots + l_{s-1} = 0. \tag{7.2.10}$$

If (7.2.6) holds, then by Lemma 7.2.1(b), every block is contained in one of the sets $V_j(b, B)$. Consequently, by (7.2.9), the expectation of the observational contrast corresponding to $L(B)$ involves the block effects, and this vitiates the estimation of $L(B)$. On the other hand, if (7.2.7) holds, then by Lemma 7.2.1(c), every block has $s^{n-k-r-1}$ treatment combinations in common with $V_j(b, B)$, $0 \le j \le s-1$. Therefore, by (7.2.9) and (7.2.10), the block effects cancel out in the expectation of the observational contrast corresponding to $L(B)$, i.e., this contrast estimates $L(B)$ even in the presence of the block effects. From the points just noted, the following facts are evident:

(I) If (7.2.6) holds for any alias set $\mathcal{A}$, then no treatment contrast belonging to any pencil in $\mathcal{A}$ remains estimable in $d(B, B_0)$ under the presence of block effects. In this sense, the alias set $\mathcal{A}$, as well as all pencils therein, are said to be *confounded* with blocks.

(II)If (7.2.7) holds for any alias set $\mathcal{A}$, then Theorem 2.4.2 is applicable to $\mathcal{A}$ even in the presence of block effects, i.e., any pencil $b \in \mathcal{A}$ is estimable in $d(B, B_0)$ if and only if all other pencils in $\mathcal{A}$ are ignorable. In this sense, the alias set $\mathcal{A}$, as well as all pencils therein, are said to remain *unconfounded* with blocks.

It is easy to see that there are $(s^r - 1)/(s - 1)$ alias sets that satisfy (7.2.6) and hence are confounded with blocks. Since there are $(s^{n-k} - 1)/(s - 1)$ alias sets altogether, this leaves $(s^{n-k} - s^r)/(s - 1)$ alias sets that satisfy (7.2.7) and hence remain unconfounded.

Example 7.2.2 (continued). Here $s = 3$, $r = 2$. Thus there are four alias sets that are confounded with blocks. By (7.2.4) and (7.2.6), these are

$$\{(1,1,0,0)', (0,0,1,1)', (1,1,2,2)'\}, \quad \{(1,0,1,0)', (0,1,0,1)', (1,2,1,2)'\},$$

$$\{(0,1,2,0)', (1,2,0,1)', (1,0,2,1)'\}, \quad \{(1,0,0,2)', (0,1,1,2)', (1,2,2,0)'\}.$$

None of the above alias sets contains a main effect pencil, i.e., all alias sets containing main effect pencils satisfy (7.2.7). Also, from (7.2.4) it is easily seen that no main effect pencil is aliased with another pencil involving fewer than three factors. Therefore, by (II) above, every main effect pencil is estimable in this block design when interactions involving three or more factors are absent. □

We now indicate an extension of the concept of resolution to block designs. With reference to a block design $d(B, B_0)$, let R be the minimum number of nonzero entries in any defining pencil. Also, let $\theta + 1$ be the minimum number of nonzero entries in any pencil that is confounded with blocks, i.e., satisfies (7.2.6). If there are no blocks, then the resolution of a design is given by R. In the same spirit, Mukerjee and Wu (1999) defined the *resolution* of a block design $d(B, B_0)$ as R^*, where

$$R^* = \begin{cases} \min(R, 2\theta + 1), & \text{if } R \text{ is odd,} \\ \min(R, 2\theta + 2), & \text{if } R \text{ is even.} \end{cases} \tag{7.2.11}$$

If $R^* \leq 2$ then either $R \leq 2$ or $\theta = 0$, i.e., either the (unblocked) design $d(B)$ has resolution at most two or a main effect pencil is confounded with blocks in $d(B, B_0)$. Thus a block design of resolution one or two fails to ensure the estimability of all treatment contrasts belonging to the main effects even under the absence of all interactions. Hence we focus on designs of resolution three or higher. Note that the concept of estimability order in Section 7.1 cannot be extended here because a pencil is no longer estimable merely if it is unconfounded with blocks; as seen in (II) above, the status of its aliases also plays a role in this regard.

The following result, analogous to Theorem 2.5.1, is not hard to deduce from the fact (II) above using the same arguments as in Theorem 2.4.3.

Theorem 7.2.1. *An* (s^{n-k}, s^r) *block design of resolution* $R^*(\geq 3)$ *keeps all treatment contrasts belonging to factorial effects involving* f *or fewer factors estimable under the absence of all factorial effects involving* $R^* - f$ *or more factors, whenever* f *satisfies* $1 \leq f \leq \frac{1}{2}(R^* - 1)$.

Theorem 7.2.1 provides further justification for (7.2.11). For the block design in Example 7.2.2, $R = 4$, and one can check that $\theta = 1$. Thus $R^* = 4$, and from Theorem 7.2.1 it is clear why every main effect pencil is estimable in this design under the absence of interactions involving three or more factors.

7.3 A Projective Geometric Formulation

As with s^{n-k} designs, a projective geometric formulation facilitates the study and tabulation of block designs. Following Mukerjee and Wu (1999) and Chen and Cheng (1999), the formulation is given in Theorem 7.3.1 below. This extends Theorem 2.7.1 to the present setup. Given any nonempty set of points Q of a finite projective geometry, we continue to write $V(Q)$ for the matrix with columns given by the points of Q.

Definition 7.3.1. *An ordered pair of subsets* (T_0, T) *of* $PG(n - k - 1, s)$ *is called an* $(r-1, n)$ blocking pair *if (a)* T_0 is *an* $(r-1)$*-flat, (b)* T *has cardinality* n*, (c)* T_0 *and* T *are disjoint, and (d)* $V(T)$ *has full row rank.*

Theorem 7.3.1. *Given any* (s^{n-k}, s^r) *block design* $d(B, B_0)$ *of resolution three or higher, there exists an* $(r - 1, n)$ *blocking pair of subsets* (T_0, T) *of* $PG(n - k - 1, s)$ *such that*

(a) for every $\lambda \in \Lambda_r$*, a treatment combination* x *appears in the block, indexed by* λ*, of* $d(B, B_0)$ *if and only if*

$$x' = \xi' V(T) \quad \text{and} \quad \lambda' = \xi' V(T_0^*), \tag{7.3.1}$$

for some $(n - k) \times 1$ *vector* ξ *over* $GF(s)$*, where* T_0^* *is a set of* r *linearly independent points of* T_0*,*

(b) any pencil b *is a defining pencil of* $d(B, B_0)$ *if and only if* $V(T)b = 0$*,*

(c) any two pencils are aliased with each other in $d(B, B_0)$ *if and only if* $V(T)(b - b^*) = 0$ *for some representations* b *and* b^* *of these pencils,*

(d) *any pencil b that is not a defining pencil is confounded with blocks in $d(B, B_0)$ if and only if $V(T)b$ is nonnull and proportional to some point of T_0.*

Conversely, given any $(r-1, n)$ blocking pair of subsets (T_0, T) of $PG(n-k-1, s)$, there exists an (s^{n-k}, s^r) block design $d(B, B_0)$ of resolution three or higher such that (a)–(d) hold.

Proof. Consider an (s^{n-k}, s^r) block design $d(B, B_0)$ of resolution three or higher. Since the $k \times n$ matrix B has full row rank, there exists an $(n-k) \times n$ matrix G, defined over $GF(s)$, such that

$$\text{rank}(G) = n - k \quad \text{and} \quad BG' = 0, \tag{7.3.2}$$

i.e., the row spaces of B and G are orthogonal complements of each other. Define the $(n-k) \times r$ matrix

$$G_0 = GB_0'. \tag{7.3.3}$$

The following facts now emerge:

(i) no two columns of G are linearly dependent,
(ii) the r columns of G_0 are linearly independent,
(iii) no column of G is spanned by the columns of G_0.

As in Theorem 2.7.1, fact (i) is obvious from (7.3.2) because $d(B, B_0)$ has resolution three or higher. In Exercises 7.6 and 7.7, the reader is asked to verify (ii) and (iii).

By (i), the n columns of G represent points of $PG(n-k-1, s)$. Let T be the set of these n points. Similarly, by (ii), the r columns of G_0 represent linearly independent points of $PG(n-k-1, s)$. Let T_0^* be the set of these r points and T_0 be the $(r-1)$-flat generated by the points of T_0^*. Then by (iii), T_0 and T are disjoint, and by (7.3.2), $V(T) = G$ has full row rank. Hence (T_0, T) is an $(r-1, n)$ blocking pair of subsets of $PG(n-k-1, s)$.

With T_0, T_0^*, and T as above, it is not hard to verify (a)–(d). While (b) and (c) are evident from (7.3.2), we ask the reader to verify (a) and (d) in Exercises 7.8 and 7.9.

The converse can be proved by reversing the above steps. □

To illustrate Theorem 7.3.1, we revisit the block design in Example 7.2.1. It can be seen that the design has resolution four. The matrices B and B_0 for this design are shown in (7.2.3). Hence

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

satisfies (7.3.2), and by (7.3.3),

$$G_0 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}'.$$

The columns of G and G_0 represent the points of T and T_0^* respectively. Therefore, $T_0 = \{(1,0,1,1)', (1,1,0,1)', (0,1,1,0)'\}$, and (T_0, T) is an $(r-1, n)$ blocking pair of subsets, with $r = 2$, $n = 6$. We now verify Theorem 7.3.1(a) for the block indexed by $(1,1)'$. Since $V(T_0^*) = G_0$, with $\lambda = (1,1)'$, the solutions of the second equation in (7.3.1) for ξ are $(0,1,1,0)'$, $(0,0,0,1)'$, $(1,0,0,0)'$, and $(1,1,1,1)'$. Since $V(T) = G$, the first equation in (7.3.1) now shows that this block consists of the treatment combinations $(0,1,1,0,0,0)'$, $(0,0,0,1,0,1)'$, $(1,0,0,0,1,0)'$, and $(1,1,1,1,1,1)'$. This agrees with what was noted earlier in Example 7.2.1 directly from (7.2.2). Similarly, with T_0, T_0^*, and T as above, one can verify (a) for the other blocks, as well as (b)–(d).

Considering the cardinalities of T_0 and T, it follows from Theorem 7.3.1 that given s, n, k, and r, an (s^{n-k}, s^r) block design of resolution three or higher exists if and only if

$$\frac{s^r - 1}{s - 1} + n \leq \frac{s^{n-k} - 1}{s - 1}, \quad \text{i.e.,} \quad s^r + n(s-1) \leq s^{n-k}. \tag{7.3.4}$$

Observe that with $m = n - k$, (7.3.4) is the same as the corresponding condition in Section 6.3 in the context of an s^m-run design for an $(s^r) \times s^n$ factorial. Indeed, (7.3.1) suggests that in an (s^{n-k}, s^r) block design, one can identify the blocks with the levels of an s^r-level factor and thus the block design formally corresponds to an s^{n-k}-run design for an $(s^r) \times s^n$ factorial. However, there is one major difference between the two settings. Whereas the s^r-level factor is itself of interest in Section 6.3, no such interest lies here in the block effects. Consequently, as will be seen in the next section, the design criteria for block designs are different from those in the last chapter.

Given an s^{n-k} design $d(B)$ of resolution at least three, Theorem 7.3.1 can be employed to find the *maximum* possible blocking of $d(B)$ so that the resulting block design also has resolution three or higher. This can be done in two steps:

(i) Following Theorem 2.7.1, represent $d(B)$ by a set T of n points of $PG(n - k - 1, s)$ such that $V(T)$ has full row rank.

(ii) Given T, find the largest r such that the complement of T in $PG(n - k - 1, s)$ contains an $(r-1)$-flat, say T_0. Then (T_0, T) is an $(r-1, n)$ blocking pair of subsets. The block design corresponding to (T_0, T), as envisaged in Theorem 7.3.1, has resolution three or higher and gives a maximum possible blocking of $d(B)$.

The following example is from Mukerjee and Wu (1999), who addressed the issue of maximum blocking in some detail.

Example 7.3.1. Let $s = 2$, $k = 1$, and to avoid trivialities, suppose $n \geq 4$. Then (7.3.4) yields

$$r \leq n - 2. \tag{7.3.5}$$

Let T_0 be the $(n-3)$-flat of $PG(n-2,2)$ consisting of the points whose coordinates add up to zero, e.g., $T_0 = \{(1,1,0)', (1,0,1)', (0,1,1)'\}$ for $n = 4$. The 2^{n-1} design $d^{(1)}$ with the highest resolution n is represented by the set of points $T^{(1)} = \{e_1, \ldots, e_{n-1}, y\}$ of $PG(n-2,2)$, where $e_1, \ldots, e_{n-1}$ are the unit vectors, of order $n-1$, over $GF(2)$, and $y = e_1 + \cdots + e_{n-1}$. If n is even, then T_0 and $T^{(1)}$ are disjoint. Since T_0 is an $(n-3)$-flat, it follows that the bound (7.3.5) is then attainable for $d^{(1)}$, i.e., *for even n, $d^{(1)}$ can be partitioned into 2^{n-2} blocks so that the resulting block design continues to have resolution three or higher.*

For odd n, however, $y \in T_0$, i.e., T_0 and $T^{(1)}$ are no longer disjoint. Nevertheless, suppose the complement of $T^{(1)}$ in $PG(n-2,2)$ contains an $(n-3)$-flat, say $\widehat{T}$. Clearly, then the points of $\widehat{T}$, in conjunction with $y(\notin \widehat{T})$, span $PG(n-2,2)$. This implies that $e_i = y + q_i$, $1 \leq i \leq n-1$, for some $q_1, \ldots, q_{n-1} \in \widehat{T}$. Since n is odd, the $n-1$ points $q_i(= e_i + y)$, $1 \leq i \leq n-1$, are linearly independent. But this is impossible because they belong to an $(n-3)$-flat. *Hence for odd n, $d^{(1)}$ cannot be partitioned into 2^{n-2} blocks so that the resulting block design still has resolution three or higher.* In this case, however, it is possible to partition $d^{(1)}$ into 2^{n-3} blocks so that the resulting block design has the desired resolution. To see this, consider the $(n-4)$-flat T_0' consisting of the points of $PG(n-2,2)$ whose last coordinate is zero and all other coordinates add up to zero, and observe that T_0' and $T^{(1)}$ are disjoint.

Interestingly, for odd n, the bound (7.3.5) can be attained for the 2^{n-1} design $d^{(2)}$ with the second highest resolution $n-1$. This happens because $d^{(2)}$ is represented by the set $T^{(2)} = \{e_1, \ldots, e_{n-1}, e_1 + \cdots + e_{n-2}\}$, and T_0 and $T^{(2)}$ are disjoint for odd n. □

7.4 Design Criteria

From now on, we consider only block designs of resolution three or higher, i.e., the condition (7.3.4) is supposed to hold. By Theorem 7.3.1, any such design is equivalent to a blocking pair of subsets (T_0, T) such that (a)–(d) of the theorem are met. Thus the design itself may be denoted by $d(T_0, T)$. Given a block design $d = d(T_0, T)$, for $1 \leq i \leq n$, let $A_i(d)$ denote the number of (distinct) defining pencils with i nonzero entries; also, let $A_i^*(d)$ denote the number of (distinct) pencils, with i nonzero entries, that are not defining pencils but confounded with blocks. Thus one gets *two* wordlength patterns

$$W(d) = (A_1(d), A_2(d), A_3(d), \ldots, A_n(d))$$

and

$$W^*(d) = (A_1^*(d), A_2^*(d), A_3^*(d), \ldots, A_n^*(d)),$$

where $A_1(d) = A_2(d) = A_1^*(d) = 0$, since the design has resolution three or higher. Note that $A_i^*(d)$ is the natural counterpart of $\beta_i(d)$ of Section 7.1 in the present context.

Under the effect hierarchy principle, a good design aims at keeping $A_i(d)$ and $A_i^*(d)$ small for smaller values of i. To that effect, as in Section 2.5, one can define an MA criterion *separately* with respect to either $W(d)$ or $W^*(d)$. The resulting designs are called MA(W) or MA(W^*) designs respectively. Quite often, however, there is no single design that is both MA(W) and MA(W^*). Example 7.4.1 below illustrates this point. Along the lines of the previous chapters, in what follows, two (s^{n-k}, s^r) block designs $d(T_{01}, T_1)$ and $d(T_{02}, T_2)$ are called *isomorphic* if there exists a nonsingular transformation that maps each point of T_{01} to some point of T_{02} up to proportionality, and each point of T_1 to some point of T_2 up to proportionality. From Theorem 7.3.1, it is evident that isomorphic designs have the same $W(d)$ as well as the same $W^*(d)$.

Example 7.4.1. Let $s = 3$, $n = 4$, $k = r = 1$. Up to isomorphism, there are five block designs. These are $d_i = d(T_{0i}, T_i),\ 1 \le i \le 5$, where

$$T_1 = T_2 = \{1, 2, 3, 123\}, \quad T_3 = T_4 = T_5 = \{1, 2, 3, 12\},$$

$$T_{01} = \{12^2\},\ T_{02} = \{12\},\ T_{03} = \{12^23\},\ T_{04} = \{13\},\ T_{05} = \{12^2\},$$

using the compact notation for the points of a finite projective geometry. From Theorem 7.3.1(b) and (d), one can check that

$$\begin{aligned}
W(d_1) &= (0,0,0,1), & W^*(d_1) &= (0,1,2,0),\\
W(d_2) &= (0,0,0,1), & W^*(d_2) &= (0,2,0,1),\\
W(d_3) &= (0,0,1,0), & W^*(d_3) &= (0,0,3,0),\\
W(d_4) &= (0,0,1,0), & W^*(d_4) &= (0,1,1,1),\\
W(d_5) &= (0,0,1,0), & W^*(d_5) &= (0,3,0,0).
\end{aligned}$$

Thus d_1 and d_2 are the MA(W) designs, while d_3 is the MA(W^*) design. No design has MA with respect to both $W(d)$ and $W^*(d)$. □

Examples similar to the above abound. Hence it is unrealistic to look for a design that is both MA(W) and MA(W^*). As a way out, Sun, Wu, and Chen (1997) and Mukerjee and Wu, (1999) considered the concept of admissibility.

Definition 7.4.1. *Given s, n, k, and r, a block design d is said to be admissible if there exists no other block design d' that is at least as good as d with respect to one of the two wordlength patterns and better than d with respect to the other, i.e., with reference to (i)–(iv) below, neither (i) and (iv) hold, nor (ii) and (iii) hold, nor (iii) and (iv) hold.*

(i) $A_i(d') = A_i(d),\ 1 \le i \le n$;

(ii) $A_i^*(d') = A_i^*(d)$, $1 \le i \le n$;
(iii) *there exists a positive integer* u *such that* $A_i(d') = A_i(d)$ *for* $i < u$, *and* $A_u(d') < A_u(d)$;
(iv) *there exists a positive integer* u^* *such that* $A_i^*(d') = A_i^*(d)$ *for* $i < u^*$, *and* $A_{u^*}^*(d') < A_{u^*}^*(d)$.

The only admissible designs in Example 7.4.1 are d_1 and d_3. A comparison with d_1 shows that d_2 is inadmissible. Similarly, the inadmissibility of d_4 and d_5 follows from comparison with d_3.

Theorem 7.4.1 below gives a simple sufficient condition for admissibility as defined above. Consider an (s^{n-k}, s^r) block design $d(T_0, T)$, where (T_0, T) is an $(r-1, n)$ blocking pair of subsets of $PG(n-k-1, s)$. Let

$$N = \frac{s^r - 1}{s - 1} + n, \quad K = \frac{s^r - 1}{s - 1} + k.$$

Then $N - K = n - k$, and $T_0 \cup T$ is a set of N points of $PG(N-K-1, s)$. Also, $V(T_0 \cup T)$, like $V(T)$, has full row rank. Therefore, by Theorem 2.7.1, $T_0 \cup T$ represents an s^{N-K} design (with no blocks) of resolution three or higher.

Theorem 7.4.1. *An* (s^{n-k}, s^r) *block design* $d(T_0, T)$ *is admissible in the sense of Definition 7.4.1 if* $T_0 \cup T$ *represents an* s^{N-K} *design with minimum aberration.*

The proof of the above theorem is available in Mukerjee and Wu (1999) and omitted here. The reader may verify that in Example 7.4.1, the designs d_1 and d_3 satisfy the condition of Theorem 7.4.1, while the other designs do not. It is particularly easy to apply this theorem in the nearly saturated cases, where

$$f = \frac{s^{N-K} - 1}{s - 1} - N = \frac{s^{n-k} - s^r}{s - 1} - n \tag{7.4.1}$$

is small, and consideration of complementary sets facilitates the understanding of the s^{N-K} design represented by $T_0 \cup T$.

Example 7.4.2. (a) Let $s = 2$, $n = 8$, $k = 4$, $r = 2$. Consider an $(r-1, n)$ blocking pair of subsets (T_0, T), where

$$T_0 = \{1, 2, 12\}, \qquad T = \{23, 123, 14, 24, 124, 134, 234, 1234\}.$$

Here $N = 11$, $K = 7$, $f = 4$, and the complement of $T_0 \cup T$ in $PG(3, 2)$ is $\widetilde{T} = \{3, 4, 34, 13\}$. From Table 3.1 with $f = 4$, it is now clear that the 2^{11-7} design represented by $T_0 \cup T$ has MA. Hence by Theorem 7.4.1, the $(2^{8-4}, 2^2)$ block design $d(T_0, T)$ is admissible.
(b) Let $s = 3$, $n = 6$, $k = 3$, $r = 2$. Consider an $(r-1, n)$ blocking pair of subsets (T_0, T), where

$$T_0 = \{1, 2, 12, 12^2\}, \qquad T = \{23, 23^2, 123, 123^2, 12^23, 12^23^2\}.$$

Here $N = 10$, $K = 7$, $f = 3$, and the complement of $T_0 \cup T$ in $PG(2,3)$ is $\widetilde{T} = \{3, 13, 13^2\}$. Hence Theorem 4.4.1 shows that the 3^{10-7} design represented by $T_0 \cup T$ has MA. Therefore, by Theorem 7.4.1, the $(3^{6-3}, 3^2)$ block design $d(T_0, T)$ is admissible. □

In the spirit of the previous chapters, the wordlength patterns of a block design can be expressed in terms of the complementary set $\widetilde{T}$ as considered in the last example. From Theorem 7.3.1(b) and (d), observe that for an (s^{n-k}, s^r) block design $d = d(T_0, T)$,

$$A_i(d) = G_i(T), \qquad A_i^*(d) = H_i(T_0, T), \quad 1 \le i \le n, \tag{7.4.2}$$

with $G_i(T)$ and $H_i(T_0, T)$ as in (6.3.6) and (6.3.7). Let $\widetilde{T}$ be the complement of $T_0 \cup T$ in $PG(n-k-1, s)$. Then by Lemma 6.3.3(a) and Lemma 6.3.4(a),

$$A_3(d) = G_3(T) = \text{constant} - G_3(T_0 \cup \widetilde{T}) = \text{constant} - G_3(\widetilde{T}) - H_2(T_0, \widetilde{T}), \tag{7.4.3}$$

$$\begin{aligned} A_2^*(d) &= H_2(T_0, T) = \text{constant} + G_3(T_0 \cup T) - G_3(T) \\ &= \text{constant} - G_3(\widetilde{T}) + G_3(T_0 \cup \widetilde{T}) = \text{constant} + H_2(T_0, \widetilde{T}). \end{aligned} \tag{7.4.4}$$

Incidentally, Lemma 6.3.3 was stated in the last chapter with reference to a particular $(r-1)$-flat and a set Q, but it holds generally for any $(r-1)$-flat T_0 as long as Q and T_0 are disjoint. If $V(\widetilde{T})$ has full row rank, then $(T_0, \widetilde{T})$ is an $(r-1, f)$ blocking pair and $\widetilde{d} = d(T_0, \widetilde{T})$ is an (s^{f-k^*}, s^r) block design, where $k^* = f - (n-k)$ and f is as in (7.4.1). In this case, using (7.4.2), one can express (7.4.3) and (7.4.4) as

$$A_3(d) = \text{constant} - A_3(\widetilde{d}) - A_2^*(\widetilde{d}), \tag{7.4.5}$$

and

$$A_2^*(d) = \text{constant} + A_2^*(\widetilde{d}), \tag{7.4.6}$$

respectively. Note that $\widetilde{d}$ may be regarded as the *complementary design* of d. The constants in (7.4.3)–(7.4.6) may depend on s, n, k, and r, but not on the particular choice of (T_0, T). Counterparts of (7.4.5) and (7.4.6) for $A_4(d)$ and $A_3^*(d)$ can be obtained with a little more algebra using (7.4.2) and part (b) of Lemmas 6.3.3 and 6.3.4, in conjunction with part (a) of these lemmas. The corresponding expressions for general $A_i(d)$ and $A_i^*(d)$ are, of course, more involved. For two-level factorials, these are available in Chen and Cheng (1999), who employed a coding-theoretic approach akin to that in Chapter 4.

For two-level factorials, Chen and Cheng (1999) proposed a design criterion that *combines* the two wordlength patterns $W(d)$ and $W^*(d)$, and then applies the MA criterion to the combined wordlength pattern. Since $A_1(d) = A_2(d) = A_1^*(d) = 0$, in essence, their criterion calls for sequential minimization of

$$A_3^{\text{comb}}(d) = 3A_3(d) + A_2^*(d), A_4^{\text{comb}}(d) = A_4(d), A_5^{\text{comb}}(d) = 10A_5(d) + A_3^*(d), \tag{7.4.7}$$

and so on. This was motivated by consideration of estimation capacity as applied to block designs. For instance, analogously to Theorem 5.1.2, Chen and Cheng (1999) noted that there are $\binom{n}{2} - A_3^{\text{comb}}(d)$ two-factor interactions (2fi's) that are neither aliased with main effects nor confounded with blocks, and that these 2fi's tend to be uniformly distributed over the relevant alias sets when $A_4^{\text{comb}}(d)$ is small. If $A_3^{\text{comb}}(d) = A_4^{\text{comb}}(d) = 0$, then no 2fi is confounded with blocks or aliased with either any main effect or another 2fi. They further observed that in this situation $A_5^{\text{comb}}(d)$ equals the number of three-factor interactions that are aliased with lower-order factorial effects or confounded with blocks.

The next example illustrates the use of complementary sets in exploring the MA criterion based on (7.4.7). We refer to Cheng and Mukerjee (2001) for results on the estimation capacity of block designs, and to Cheng and Tang (2005) for discussion on some variations of (7.4.7) from other perspectives.

Example 7.4.3. Let $s = 2$, $n = 9$, $k = 5$, $r = 2$. Up to isomorphism, there are four block designs, which correspond to

$$\begin{array}{ll} \text{(a)} & T_0 = \{1, 2, 12\}, \quad \widetilde{T} = \{3, 13, 23\}, \\ \text{(b)} & T_0 = \{1, 2, 12\}, \quad \widetilde{T} = \{3, 4, 34\}, \\ \text{(c)} & T_0 = \{1, 2, 12\}, \quad \widetilde{T} = \{3, 4, 13\}, \\ \text{(d)} & T_0 = \{1, 2, 12\}, \quad \widetilde{T} = \{3, 4, 134\}, \end{array}$$

where $\widetilde{T}$ is the complement of $T_0 \cup T$ in $PG(3, 2)$. By (7.4.3), (7.4.4), and (7.4.7), minimization of $A_3^{\text{comb}}(d)$ is equivalent to maximization of $3G_3(\widetilde{T}) + 2H_2(T_0, \widetilde{T})$. Now, for (a)–(d) above, the pair $(G_3(\widetilde{T}), H_2(T_0, \widetilde{T}))$ equals (0, 3), (1, 0), (0, 1), and (0, 0) respectively. Thus, up to isomorphism, the design in (a) uniquely minimizes $A_3^{\text{comb}}(d)$. Hence this is the MA design with respect to the combined wordlength pattern considered in (7.4.7). □

While Chen and Cheng (1999) suggested combining $W(d)$ and $W^*(d)$, criteria based on *interpenetration* of the two wordlength patterns have also been proposed in the literature. Sitter, Chen, and Feder (1997) considered the interpenetration

$$W_{\text{int}}^{(1)}(d) = (A_3(d), A_2^*(d), A_4(d), A_3^*(d), A_5(d), A_4^*(d), \ldots). \tag{7.4.8}$$

On the other hand, Cheng and Wu (2002) considered

$$W_{\text{int}}^{(2)}(d) = (A_3(d), A_4(d), A_2^*(d), A_5(d), A_6(d), A_3^*(d), A_7(d), \ldots) \tag{7.4.9}$$

and

$$W_{\text{int}}^{(3)}(d) = (A_3(d), A_2^*(d), A_4(d), A_5(d), A_3^*(d), A_6(d), A_7(d), \ldots). \tag{7.4.10}$$

Chen and Cheng (1999) also mentioned (7.4.10). The MA criterion can be applied as usual to any of (7.4.8)–(7.4.10). In each of the three criteria, the ordering of $A_i(d)$ for $i \geq 3$ and of $A_i^*(d)$ for $i \geq 2$ follows from the usual effect hierarchy principle. Where to insert an $A_i^*(d)$ within the sequence $(A_3(d), A_4(d), \ldots)$

will depend on a construal of the effect hierarchy principle for block designs. Consider, for example, the justification for $W^{(2)}_{\text{int}}(d)$. Note that any pencil contributing to $A_4(d)$ entails *three pairs* of aliased 2fi pencils, while any pencil contributing to $A^*_2(d)$ entails the confounding of *one* 2fi pencil with blocks. From this perspective, $A_4(d)$ is more serious than $A^*_2(d)$. A similar argument justifies viewing $A_6(d)$ more seriously than $A^*_3(d)$. Again, $A^*_2(d)$ is more serious than $A_5(d)$ because the latter does not entail the aliasing of any 2fi pencil with main effect pencils or other 2fi pencils. A similar argument justifies viewing $A^*_3(d)$ more seriously than $A_7(d)$. On the other hand, in $W^{(1)}_{\text{int}}(d)$ and $W^{(3)}_{\text{int}}(d)$, $A^*_2(d)$ is considered to be more serious than $A_4(d)$ from a different viewpoint. Details can be found in the three papers cited above.

It is not hard to verify that an MA design arising from any of (7.4.7)–(7.4.10) is admissible in the sense of Definition 7.4.1. Do these approaches for combining or interpenetrating $W(d)$ and $W^*(d)$ lead to the same MA design? This happens in Example 7.4.3, where the design in (a), which was earlier seen to enjoy the MA property with respect to (7.4.7), uniquely maximizes $G_3(\widetilde{T}) + H_2(T_0, \widetilde{T})$. By (7.4.3), therefore, it uniquely minimizes $A_3(d)$ and hence is the MA design with respect to (7.4.8)–(7.4.10) as well. However, there are plenty of examples showing that this is not the case in general. One such example is indicated in Exercise 7.12.

7.5 Description and Use of Block Design Tables

Tables of admissible block designs for $s = 2$ and 3 are presented in the appendix of this chapter. These tables are adapted from Sun, Wu, and Chen (1997) and Cheng and Wu (2002). Following these authors, a more elaborate definition of admissibility is adopted in the tables by bringing in the $C1$ and $C2$ criteria in addition to $W(d)$ and $W^*(d)$, where $C1$ and $C2$ are the numbers of clear main effect and 2fi pencils respectively. In the spirit of Section 3.4, a main effect or 2fi pencil is called *clear* if it is neither confounded with blocks nor aliased with any other main effect or 2fi pencil. Notice that unlike in Section 4.5, $C2$ here refers to 2fi pencils rather than the 2fi's themselves. This change is imperative for borrowing the three-level block design tables from Cheng and Wu (2002).

A good design should aim at keeping $C1$ and $C2$ large in addition to achieving less aberration with respect to $W(d)$ and $W^*(d)$. From this viewpoint, a block design d_1 is now called *admissible* if there exists no other block design d_2 that is at least as good as d_1 with respect to each of $W(d)$, $W^*(d)$, $C1$, and $C2$, and better than d_1 with respect to one or more of these four criteria.

For illustration, we revisit Example 7.4.1. For the designs $d_1, \ldots, d_5$ in this example, the pair $(C1, C2)$ equals $(4, 5)$, $(4, 6)$, $(1, 6)$, $(1, 5)$, and $(1, 6)$ respectively. From this, together with the wordlength patterns for $d_1, \ldots, d_5$ in Example 7.4.1, it is evident that d_1, d_2, and d_3 are admissible according to the above definition. Recall that d_1 and d_3 were earlier seen to be admissible according to Definition 7.4.1. Consideration of $C1$ and $C2$ in addition to $W(d)$ and $W^*(d)$ leads to the admissibility of d_2 as well, because d_2 has higher $C2$ than d_1 and higher $C1$ than d_3, d_4, d_5.

Six tables of admissible block designs are presented in the appendix of this chapter. Tables 7A.1–7A.4 give 16-, 32-, 64-, and 128-run designs for $s = 2$. Tables 7A.5 and 7A.6 give 27- and 81-run designs for $s = 3$. Table 7A.5 is complete, i.e., it lists all nonisomorphic admissible block designs with 27 runs. The cases $n = 11, 12$ are omitted in Table 7A.5 since the designs are unique up to isomorphism. The other tables display selected admissible designs.

Designs of resolution three or higher are considered in Tables 7A.1–7A.6. Accordingly, any design is described via an $(r-1, n)$ blocking pair of subsets (T_0, T) of $PG(n-k-1, s)$. Given n, k, and r, the ith admissible design in any table is denoted by $n-k/\mathrm{B}r.i$, where B signifies that it is a block design. As with the design tables in Chapters 3 and 4, T is supposed to contain the independent points $1, 2, \ldots, m$ of $PG(n-k-1, s)$, where $m = n-k$, and only the additional k points of T are shown in the tables under the column heading T_{add}. Also, only r independent points of T_0 are displayed under the column heading T_0^*. For ease in presentation, the points of the finite projective geometry are indicated in the tables via their serial numbers, the numbering schemes for the two- and three-level designs being as in Tables 3A.1 and 4A.1 respectively. Against each tabulated design, the wordlength patterns $W(d)$ and $W^*(d)$ as well as $C1$ and $C2$ are displayed. Since $A_1(d) = A_2(d) = A_1^*(d) = 0$ for each design, we show $A_3(d), \ldots, A_n(d)$ under $W(d)$ and $A_2^*(d), \ldots, A_n^*(d)$ under $W^*(d)$. Use of the design tables is illustrated in the following example.

Example 7.5.1. Let $s = 3$, $n = 4$, $k = r = 1$. Table 7A.5 lists three nonisomorphic admissible block designs, namely 4-1/B1.1, 4-1/B1.2, and 4-1/B1.3. Consider 4-1/B1.1. For this design, the columns T_{add} and T_0^* show the points with serial numbers 8 and 4 respectively. The numbering scheme displayed in Table 4A.1 identifies these points as 123 and 12^2. Since T also contains the independent points $1, 2, 3$ of $PG(2, 3)$, it follows that $T = \{1, 2, 3, 123\}$ and $T_0 = \{12^2\}$ for this design. Thus 4-1/B1.1 is the same as d_1 considered in Example 7.4.1. Similarly, 4-1/B1.2 and 4-1/B1.3 are the same as d_2 and d_3 respectively in that example. This agrees with what was noted earlier in this section about d_1, d_2, and d_3. □

Exercises

7.1 Obtain all the blocks in Example 7.2.2.

7.2 In Example 7.2.1, obtain the alias sets that are confounded with blocks.

7.3 Show that there are $(s^r - 1)/(s - 1)$ alias sets that satisfy (7.2.6).

7.4 Show that the block design in Example 7.2.1 has resolution four.

7.5 Prove Theorem 7.2.1.

7.6 Refer to the proof of Theorem 7.3.1. Suppose a column of G is spanned by the columns of G_0.
(a) From (7.3.3), observe that $G(e - B_0'\delta_0) = 0$, where δ_0 is some $r \times 1$ vector and e is an $n \times 1$ unit vector, both over $GF(s)$.
(b) Use (7.3.2) to conclude that

$$e' \in \mathcal{R}\begin{pmatrix} B \\ B_0 \end{pmatrix}.$$

Argue that this is impossible in view of (7.2.6) since e represents a main effect pencil and $d(B, B_0)$ has resolution three or higher. Hence infer the truth of fact (iii) in the proof.

7.7 Refer again to the proof of Theorem 7.3.1. Use (7.2.1) and arguments similar to those in Exercise 7.6 to establish fact (ii) in the proof.

7.8 In order to verify Theorem 7.3.1(a), it suffices to show that (7.2.2) and (7.3.1) are equivalent. Establish this using (7.3.2), (7.3.3), and the facts that $V(T) = G$ and $V(T_0^*) = G_0$.

7.9 Observe that any pencil b satisfies (7.2.6) if and only if $b = B'\delta + B_0'\delta_0$, for some vectors δ and δ_0 such that $\delta_0 \neq 0$. Hence use (7.3.2) and (7.3.3) to verify Theorem 7.3.1(d).

7.10 For the block design in Example 7.2.2, obtain the sets T_0 and T as envisaged in Theorem 7.3.1 and verify, from first principles, (a)–(d) of the theorem.

7.11 For $s \geq 3$ and $n \geq 3$, consider any s^{n-1} design of resolution three or higher.
(a) Show that the design is represented, up to isomorphism, by a set of points $T = \{e_1, \ldots, e_{n-1}, y\}$ of $PG(n-2, s)$, where $e_1, \ldots, e_{n-1}$ are the unit vectors of order $n-1$ over $GF(s)$, and $y = (y_1, \ldots, y_{n-1})'$ is some other point.
(b) Without loss of generality, suppose $y_1 \neq 0$. Because $s \geq 3$, there exists $\alpha(\neq 0) \in GF(s)$ such that $\alpha \neq y_1^{-1}(y_2 + \cdots + y_{n-1})$. Show that the $n-2$ points $e_1 + \alpha e_i$, $2 \leq i \leq n-1$, are linearly independent.
(c) Show that the $(n-3)$-flat spanned by the $n-2$ points in (b) does not intersect T. Hence conclude that the s^{n-1} design can be partitioned into s^{n-2} blocks such that the resulting block design also has resolution three or higher.

7.12 Enumerate, up to isomorphism, all $(2^{5-1}, 2^1)$ block designs of resolution three or higher. Verify that none of these designs has MA with respect to both $W(d)$ and $W^*(d)$. Identify the admissible designs in the sense of Definition 7.4.1. Show that the same MA design does not arise from each of (7.4.7)–(7.4.10).

7.13 Verify that the designs d_1 and d_3 in Example 7.4.1 satisfy the condition of Theorem 7.4.1.

7.14 Derive the counterparts of (7.4.5) and (7.4.6) for $A_4(d)$ and $A_3^*(d)$.

Appendix 7A Tables of Two- and Three-Level Admissible Block Designs

Table 7A.1 Selected two-level admissible block designs with 16 runs

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
5-1/B1.1	15	0 0 1	3	1 1 0 0	5	9
5-1/B1.2	7	0 1 0	11	0 2 0 0	5	4
5-1/B1.3	3	1 0 0	13	0 1 1 0	2	7
5-1/B2.1	15	0 0 1	3 5	3 3 0 0	5	7
5-1/B2.2	7	0 1 0	13 14	2 4 0 0	5	4
5-1/B2.3	3	1 0 0	5 15	2 3 1 0	2	5
5-1/B3.1	7	0 1 0	9 10 12	10 0 4 0	5	0
6-2/B1.1	7 11	0 3 0 0	13	0 4 0 0 0	6	0
6-2/B1.2	3 13	1 1 1 0	6	1 2 1 0 0	3	5
6-2/B1.3	3 13	1 1 1 0	5	2 1 0 1 0	3	6
6-2/B1.4	3 5	2 1 0 0	14	0 2 2 0 0	1	5
6-2/B2.1	7 11	0 3 0 0	13 14	3 8 0 0 1	6	0
6-2/B2.2	3 13	1 1 1 0	10 15	4 5 2 1 0	3	4
6-2/B2.3	3 13	1 1 1 0	5 9	6 3 0 3 0	3	6
6-2/B3.1	7 11	0 3 0 0	5 6 9	15 0 12 0 1	6	0
7-3/B1.1	7 11 13	0 7 0 0 0	14	0 7 0 0 0 1	7	0
7-3/B1.2	3 5 14	2 3 2 0 0	9	1 4 2 0 1 0	2	1
7-3/B1.3	3 5 14	2 3 2 0 0	10	2 2 2 2 0 0	2	2
7-3/B1.4	3 5 10	3 2 1 1 0	12	1 3 3 1 0 0	0	3
7-3/B1.5	3 5 10	3 2 1 1 0	6	2 3 1 1 1 0	0	4
7-3/B1.6	3 5 9	3 3 0 0 1	14	0 4 4 0 0 0	0	0
7-3/B1.7	3 5 6	4 3 0 0 0	15	0 3 4 0 0 1	1	6

Table 7A.1 (continued)

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
7-3/B2.1	7 11 13	0 7 0 0 0	3 5	9 0 12 0 3 0	7	0
7-3/B2.2	3 5 14	2 3 2 0 0	9 15	5 10 4 2 3 0	2	0
7-3/B2.3	3 5 14	2 3 2 0 0	10 13	6 7 6 4 0 1	2	2
7-3/B2.4	3 5 6	4 3 0 0 0	9 14	5 8 4 4 3 0	1	4
7-3/B3.1	7 11 13	0 7 0 0 0	3 5 9	21 0 28 0 7 0	7	0
8-4/B1.1	7 11 13 14	0 14 0 0 0 1	3	4 0 8 0 4 0 0	8	0
8-4/B1.2	3 5 9 14	3 7 4 0 1 0	15	1 7 4 0 3 1 0	1	0
8-4/B1.3	3 5 9 14	3 7 4 0 1 0	6	3 3 4 4 1 1 0	1	1
8-4/B1.4	3 5 6 15	4 6 4 0 0 1	9	2 4 4 4 2 0 0	2	0
8-4/B1.5	3 5 6 7	7 7 0 0 1 0	9	1 3 4 4 3 1 0	1	6
8-4/B2.1	7 11 13 14	0 14 0 0 0 1	3 5	12 0 24 0 12 0 0	8	0
8-4/B2.2	3 5 9 14	3 7 4 0 1 0	6 10	9 9 12 12 3 3 0	1	1
8-4/B2.3	3 5 10 12	4 5 4 2 0 0	11 13	7 14 10 8 7 2 0	0	0
8-4/B2.4	3 5 6 15	4 6 4 0 0 1	9 14	8 12 8 12 8 0 0	2	0
8-4/B2.5	3 5 6 9	5 5 2 2 1 0	10 13	7 13 10 10 7 1 0	0	2
8-4/B3.1	7 11 13 14	0 14 0 0 0 1	3 5 9	28 0 56 0 28 0 0	8	0
9-5/B1.1	3 5 9 14 15	4 14 8 0 4 1 0	6	4 4 8 8 4 4 0 0	0	0
9-5/B1.2	3 5 10 12 15	6 9 9 6 0 0 1	6	3 7 6 6 7 3 0 0	0	0
9-5/B1.3	3 5 6 9 14	6 10 8 4 2 1 0	15	2 8 8 4 6 4 0 0	0	0
9-5/B1.4	3 5 6 9 10	7 9 6 6 3 0 0	13	2 7 9 6 4 3 1 0	0	0
9-5/B2.1	3 5 9 14 15	4 14 8 0 4 1 0	6 10	12 12 24 24 12 12 0 0	0	0
9-5/B2.2	3 5 10 12 15	6 9 9 6 0 0 1	6 11	9 21 18 18 21 9 0 0	0	0

Table 7A.2 Selected two-level admissible block designs with 32 runs

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
6-1/B1.1	31	0 0 0 1	7	0 2 0 0 0	6	15
6-1/B1.2	15	0 0 1 0	19	0 1 1 0 0	6	15
6-1/B1.3	7	0 1 0 0	27	0 0 2 0 0	6	9
6-1/B2.1	31	0 0 0 1	13 14	1 4 1 0 0	6	14
6-1/B2.2	15	0 0 1 0	21 22	1 3 2 0 0	6	14
6-1/B2.3	7	0 1 0 0	11 30	0 4 2 0 0	6	9
6-1/B3.1	31	0 0 0 1	21 22 25	3 8 3 0 0	6	12
6-1/B3.2	7	0 1 0 0	13 14 27	3 8 2 0 1	6	8
6-1/B3.3	3	1 0 0 0	15 22 28	3 7 3 0 1	3	9
6-1/B4.1	31	0 0 0 1	3 5 9 17	15 0 15 0 0	6	0
6-1/B4.2	7	0 1 0 0	3 5 9 17	15 0 14 0 1	6	0
7-2/B1.1	7 27	0 1 2 0 0	13	0 2 2 0 0 0	7	15
7-2/B1.2	7 11	0 3 0 0 0	29	0 0 4 0 0 0	7	6
7-2/B1.3	3 29	1 0 1 1 0	14	0 2 2 0 0 0	4	18
7-2/B1.4	3 13	1 1 1 0 0	22	0 1 2 1 0 0	4	12
7-2/B1.5	3 5	2 1 0 0 0	30	0 0 2 2 0 0	2	11
7-2/B2.1	7 27	0 1 2 0 0	21 30	1 6 4 0 1 0	7	14
7-2/B2.2	7 27	0 1 2 0 0	13 14	2 5 4 0 0 1	7	15
7-2/B2.3	7 11	0 3 0 0 0	19 30	0 7 4 0 0 1	7	6
7-2/B2.4	3 29	1 0 1 1 0	22 26	1 5 5 1 0 0	4	17
7-2/B2.5	3 28	1 1 0 0 1	13 22	0 6 6 0 0 0	4	12

Table 7A.2 (continued)

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
7-2/B3.1	7 27	0 1 2 0 0	14 22 29	5 12 6 2 3 0	7	12
7-2/B3.2	7 25	0 2 0 1 0	21 22 28	5 12 5 4 2 0	7	8
7-2/B3.3	3 29	1 0 1 1 0	18 23 27	5 11 7 3 2 0	4	13
7-2/B4.1	7 25	0 2 0 1 0	3 5 9 17	21 0 33 0 6 0	7	0
7-2/B4.2	7 11	0 3 0 0 0	3 5 9 17	21 0 32 0 7 0	7	0
8-3/B1.1	7 11 29	0 3 4 0 0 0	19	0 3 4 0 0 1 0	8	13
8-3/B1.2	7 11 19	0 6 0 0 0 1	29	0 0 8 0 0 0 0	8	0
8-3/B1.3	7 11 13	0 7 0 0 0 0	30	0 0 7 0 0 0 1	8	7
8-3/B1.4	3 13 22	1 2 3 1 0 0	25	0 3 3 1 1 0 0	5	13
8-3/B1.5	3 5 30	2 1 2 2 0 0	15	0 3 4 0 0 1 0	3	18
8-3/B1.6	3 12 21	2 1 2 2 0 0	26	0 2 4 2 0 0 0	2	16
8-3/B2.1	7 11 29	0 3 4 0 0 0	19 30	1 10 8 0 3 2 0	8	12
8-3/B2.2	7 11 29	0 3 4 0 0 0	5 9	6 3 6 6 0 3 0	8	13
8-3/B2.3	3 13 22	1 2 3 1 0 0	25 28	2 8 7 3 3 1 0	5	13
8-3/B2.4	3 5 30	2 1 2 2 0 0	23 25	1 8 10 2 1 2 0	3	17
8-3/B3.1	7 11 29	0 3 4 0 0 0	14 17 18	8 16 11 12 8 0 1	8	8
8-3/B3.2	7 11 29	0 3 4 0 0 0	18 23 30	9 12 16 12 3 4 0	8	10
8-3/B3.3	7 11 29	0 3 4 0 0 0	5 6 9	15 6 12 16 1 6 0	8	13
8-3/B3.4	3 13 22	1 2 3 1 0 0	17 26 31	7 17 13 9 8 2 0	5	10
8-3/B3.5	3 5 30	2 1 2 2 0 0	9 15 18	7 16 14 10 7 2 0	3	13
8-3/B3.6	3 12 21	2 1 2 2 0 0	10 27 30	8 14 13 14 6 0 1	2	14
8-3/B3.7	3 5 24	3 1 0 2 1 0	15 18 20	7 15 14 12 7 1 0	0	10
8-3/B4.1	7 11 21	0 5 0 2 0 0	3 5 9 17	28 0 65 0 26 0 1	8	0
8-3/B4.2	7 11 19	0 6 0 0 0 1	3 5 9 17	28 0 64 0 28 0 0	8	0

Table 7A.2 (continued)

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
9-4/B1.1	7 11 19 29	0 6 8 0 0 1 0	30	0 4 8 0 0 4 0 0	9	8
9-4/B1.2	7 11 13 30	0 7 7 0 0 0 1	17	1 3 4 4 3 1 0 0	9	14
9-4/B1.3	7 11 13 30	0 7 7 0 0 0 1	3	3 1 4 4 1 3 0 0	9	15
9-4/B1.4	7 11 13 14	0 14 0 0 0 1 0	19	0 4 0 8 0 4 0 0	9	8
9-4/B1.5	3 13 21 26	1 5 6 2 1 0 0	28	0 5 5 2 2 1 1 0	6	9
9-4/B1.6	3 13 21 25	1 7 4 0 3 0 0	30	0 3 7 4 0 1 1 0	6	12
9-4/B1.7	3 5 9 30	3 3 4 4 1 0 0	15	0 5 7 0 0 3 1 0	2	15
9-4/B1.8	3 5 6 31	4 3 3 4 0 0 1	9	1 3 4 4 3 1 0 0	3	20
9-4/B1.9	3 5 6 31	4 3 3 4 0 0 1	7	3 5 0 0 5 3 0 0	3	21
9-4/B1.10	3 5 6 24	5 3 0 4 3 0 0	15	0 3 7 4 0 1 1 0	0	18
9-4/B2.1	7 11 19 29	0 6 8 0 0 1 0	5 30	4 8 16 8 4 8 0 0	9	8
9-4/B2.2	7 11 13 30	0 7 7 0 0 0 1	17 31	3 13 8 8 13 3 0 0	9	12
9-4/B2.3	7 11 13 30	0 7 7 0 0 0 1	17 18	5 7 12 12 7 5 0 0	9	13
9-4/B2.4	7 11 13 30	0 7 7 0 0 0 1	3 5	9 3 12 12 3 9 0 0	9	15
9-4/B2.5	3 5 6 31	4 3 3 4 0 0 1	9 17	3 9 12 12 9 3 0 0	3	18
9-4/B2.6	3 5 6 31	4 3 3 4 0 0 1	9 14	5 11 8 8 11 5 0 0	3	19
9-4/B3.1	7 11 19 29	0 6 8 0 0 1 0	5 9 30	12 16 32 24 12 16 0 0	9	8
9-4/B3.2	7 11 13 30	0 7 7 0 0 0 1	17 18 20	13 15 28 28 15 13 0 0	9	11
9-4/B3.3	7 11 13 30	0 7 7 0 0 0 1	3 5 9	21 7 28 28 7 21 0 0	9	15
9-4/B3.4	7 11 21 25	0 9 0 6 0 0 0	13 14 31	9 27 18 27 21 9 0 1	9	0
9-4/B3.5	7 11 13 19	0 10 0 4 0 1 0	25 26 28	10 24 18 32 18 8 2 0	9	2
9-4/B3.6	3 5 6 31	4 3 3 4 0 0 1	9 14 18	9 23 24 24 23 9 0 0	3	15
9-4/B4.1	7 11 21 25	0 9 0 6 0 0 0	3 5 9 17	36 0 117 0 78 0 9 0	9	0
9-4/B4.2	7 11 13 19	0 10 0 4 0 1 0	3 5 9 17	36 0 116 0 80 0 8 0	9	0

Table 7A.3 Selected two-level admissible block designs with 64 runs

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
7-1/B1.1	63	0 0 0 0 1	7	0 1 1 0 0 0	7	21
7-1/B1.2	31	0 0 0 1 0	39	0 0 2 0 0 0	7	21
7-1/B1.3	15	0 0 1 0 0	51	0 0 1 1 0 0	7	21
7-1/B1.4	7	0 1 0 0 0	59	0 0 0 2 0 0	7	15
7-1/B2.1	63	0 0 0 0 1	7 25	0 3 3 0 0 0	7	21
7-1/B2.2	31	0 0 0 1 0	41 46	0 3 2 1 0 0	7	21
7-1/B2.3	7	0 1 0 0 0	27 45	0 0 6 0 0 0	7	15
7-1/B3.1	63	0 0 0 0 1	7 25 42	0 7 7 0 0 0	7	21
7-1/B3.2	7	0 1 0 0 0	30 45 56	0 7 6 0 0 1	7	15
7-1/B4.1	63	0 0 0 0 1	3 5 24 40	6 9 9 6 0 0	7	15
7-1/B4.2	31	0 0 0 1 0	3 12 21 33	5 12 7 4 2 0	7	16
7-1/B4.3	15	0 0 1 0 0	3 5 24 41	5 12 7 3 3 0	7	16
7-1/B5.1	31	0 0 0 1 0	3 5 9 17 33	21 0 35 0 6 0	7	0
7-1/B5.2	7	0 1 0 0 0	3 5 9 17 33	21 0 34 0 7 0	7	0
8-2/B1.1	15 51	0 0 2 1 0 0	21	0 1 2 1 0 0 0	8	28
8-2/B1.2	7 59	0 1 0 2 0 0	29	0 0 4 0 0 0 0	8	22
8-2/B1.3	7 27	0 1 2 0 0 0	45	0 0 2 2 0 0 0	8	22
8-2/B1.4	7 11	0 3 0 0 0 0	61	0 0 0 4 0 0 0	8	13
8-2/B2.1	15 51	0 0 2 1 0 0	21 42	0 4 5 2 1 0 0	8	28
8-2/B2.2	7 59	0 1 0 2 0 0	25 53	0 4 4 4 0 0 0	8	22
8-2/B2.3	7 57	0 1 1 0 1 0	26 44	0 3 6 3 0 0 0	8	22
8-2/B2.4	7 56	0 2 0 0 0 1	27 45	0 0 12 0 0 0 0	8	16
8-2/B3.1	15 51	0 0 2 1 0 0	41 42 61	2 8 10 6 1 0 1	8	26
8-2/B3.2	7 59	0 1 0 2 0 0	37 46 54	1 10 10 4 1 2 0	8	21
8-2/B4.1	15 51	0 0 2 1 0 0	5 9 18 34	7 18 15 10 8 2 0	8	21
8-2/B4.2	7 59	0 1 0 2 0 0	5 9 18 35	7 18 14 12 7 2 0	8	17
8-2/B4.3	7 57	0 1 1 0 1 0	3 12 21 37	7 18 14 11 9 1 0	8	17
8-2/B5.1	7 59	0 1 0 2 0 0	3 5 9 17 33	28 0 69 0 26 0 1	8	0
8-2/B5.2	7 56	0 2 0 0 0 1	3 5 9 17 33	28 0 68 0 28 0 0	8	0

Table 7A.3 (continued)

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
9-3/B1.1	7 27 45	0 1 4 2 0 0 0	51	0 1 4 2 0 1 0 0	9	30
9-3/B1.2	7 25 43	0 2 3 1 1 0 0	52	0 1 3 3 1 0 0 0	9	24
9-3/B1.3	7 27 43	0 2 4 0 0 1 0	53	0 0 4 4 0 0 0 0	9	24
9-3/B1.4	7 11 13	0 7 0 0 0 0 0	62	0 0 0 7 0 0 0 1	9	15
9-3/B2.1	7 27 45	0 1 4 2 0 0 0	49 63	0 6 8 5 4 0 0 1	9	30
9-3/B2.2	7 25 43	0 2 3 1 1 0 0	49 60	0 6 8 5 3 1 1 0	9	24
9-3/B2.3	7 27 43	0 2 4 0 0 1 0	13 62	0 4 12 4 0 4 0 0	9	24
9-3/B2.4	7 11 53	0 3 2 0 2 0 0	45 59	0 4 11 6 0 2 1 0	9	21
9-3/B3.1	7 27 45	0 1 4 2 0 0 0	35 53 62	2 14 17 8 8 6 1 0	9	28
9-3/B3.2	7 27 45	0 1 4 2 0 0 0	49 50 60	3 13 14 11 11 3 0 1	9	29
9-3/B3.3	7 27 45	0 1 4 2 0 0 0	49 50 52	6 10 9 16 12 2 1 0	9	30
9-3/B3.4	7 25 43	0 2 3 1 1 0 0	37 46 49	2 14 16 9 9 5 1 0	9	22
9-3/B3.5	7 27 43	0 2 4 0 0 1 0	49 50 60	2 14 16 8 10 6 0 0	9	24
9-3/B4.1	7 27 45	0 1 4 2 0 0 0	3 13 17 37	9 27 26 23 25 9 0 1	9	23
9-3/B4.2	7 27 45	0 1 4 2 0 0 0	3 5 9 48	12 20 25 36 18 4 5 0	9	24
9-3/B5.1	7 11 61	0 3 0 4 0 0 0	3 5 9 17 33	36 0 123 0 80 0 9 0	9	0
9-3/B5.2	7 11 49	0 4 0 2 0 1 0	3 5 9 17 33	36 0 122 0 82 0 8 0	9	0

Table 7A.4 Selected two-level admissible block designs with 128 runs

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
8-1/B1.1	127	0 0 0 0 0 1	15	0 0 2 0 0 0 0	8	28
8-1/B1.2	63	0 0 0 0 1 0	71	0 0 1 1 0 0 0	8	28
8-1/B1.3	31	0 0 0 1 0 0	103	0 0 0 2 0 0 0	8	28
8-1/B1.4	7	0 1 0 0 0 0	123	0 0 0 0 2 0 0	8	22
8-1/B2.1	127	0 0 0 0 0 1	15 51	0 0 6 0 0 0 0	8	28
8-1/B2.2	31	0 0 0 1 0 0	39 108	0 0 5 0 1 0 0	8	28
8-1/B2.3	15	0 0 1 0 0 0	51 85	0 0 3 3 0 0 0	8	28
8-1/B3.1	127	0 0 0 0 0 1	15 51 85	0 0 14 0 0 0 0	8	28
8-1/B3.2	7	0 1 0 0 0 0	27 45 120	0 0 13 0 0 0 1	8	22
8-1/B4.1	127	0 0 0 0 0 1	3 12 49 84	2 8 10 8 2 0 0	8	26
8-1/B4.2	63	0 0 0 0 1 0	7 25 42 65	1 10 11 4 3 1 0	8	27
8-1/B4.3	31	0 0 0 1 0 0	3 13 52 85	1 10 11 4 2 2 0	8	27
8-1/B4.4	15	0 0 1 0 0 0	3 21 41 77	1 10 11 3 3 2 0	8	27
8-1/B5.1	127	0 0 0 0 0 1	3 5 9 48 81	8 16 14 16 8 0 0	8	20
8-1/B5.2	63	0 0 0 0 1 0	3 5 24 40 73	7 18 15 12 9 1 0	8	21
8-1/B5.3	31	0 0 0 1 0 0	3 12 21 33 68	7 18 15 12 8 2 0	8	21
8-1/B5.4	15	0 0 1 0 0 0	3 5 24 40 73	7 18 15 11 9 2 0	8	21
8-1/B6.1	127	0 0 0 0 0 1	3 5 9 17 33 65	28 0 70 0 28 0 0	8	0
8-1/B6.2	31	0 0 0 1 0 0	3 5 9 17 33 65	28 0 70 0 27 0 1	8	0
9-2/B1.1	31 103	0 0 0 3 0 0 0	43	0 0 3 0 1 0 0 0	9	36
9-2/B1.2	15 115	0 0 1 1 1 0 0	53	0 0 2 2 0 0 0 0	9	36
9-2/B1.3	15 51	0 0 2 1 0 0 0	85	0 0 1 2 1 0 0 0	9	36
9-2/B1.4	7 27	0 1 2 0 0 0 0	109	0 0 0 2 2 0 0 0	9	30
9-2/B1.5	7 11	0 3 0 0 0 0 0	125	0 0 0 0 4 0 0 0	9	21
9-2/B2.1	31 103	0 0 0 3 0 0 0	43 85	0 0 9 0 3 0 0 0	9	36
9-2/B2.2	15 113	0 0 2 0 0 1 0	54 90	0 0 6 6 0 0 0 0	9	36
9-2/B2.3	7 27	0 1 2 0 0 0 0	45 120	0 0 5 6 0 0 1 0	9	30

Table 7A.4 (continued)

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
9-2/B3.1	31 103	0 0 0 3 0 0 0	41 46 85	0 6 9 9 3 0 0 1	9	36
9-2/B3.2	15 113	0 0 2 0 0 1 0	19 54 90	0 4 14 6 0 4 0 0	9	36
9-2/B4.1	31 103	0 0 0 3 0 0 0	3 13 37 84	2 14 18 12 7 6 1 0	9	34
9-2/B4.2	15 115	0 0 1 1 1 0 0	5 18 35 73	2 14 18 11 9 5 1 0	9	34
9-2/B4.3	15 113	0 0 2 0 0 1 0	6 19 35 74	2 14 18 10 10 6 0 0	9	34
9-2/B4.4	15 51	0 0 2 1 0 0 0	3 20 41 69	2 14 18 10 9 6 1 0	9	34
9-2/B5.1	31 103	0 0 0 3 0 0 0	7 9 18 33 66	9 27 27 27 24 9 0 1	9	27
9-2/B5.2	15 115	0 0 1 1 1 0 0	3 5 24 40 73	9 27 27 26 26 8 0 1	9	27
9-2/B5.3	15 51	0 0 2 1 0 0 0	5 9 18 34 67	9 27 27 25 26 9 0 1	9	27
9-2/B6.1	31 103	0 0 0 3 0 0 0	3 5 9 17 33 65	36 0 126 0 81 0 9 0	9	0
9-2/B6.2	7 121	0 1 0 1 0 1 0	3 5 9 17 33 65	36 0 125 0 83 0 8 0	9	0

Table 7A.5 Complete catalogue of three-level admissible block designs with 27 runs

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
4-1/B1.1	8	0 1	4	1 2 0	4	5
4-1/B1.2	8	0 1	3	2 0 1	4	6
4-1/B1.3	3	1 0	9	0 3 0	1	6
4-1/B2.1	8	0 1	4 7	6 4 2	4	4
4-1/B2.2	3	1 0	6 7	6 3 3	1	3
5-2/B1.1	3 9	1 3 0	13	1 6 0 2	2	0
5-2/B1.2	3 9	1 3 0	6	2 3 3 1	2	1
5-2/B1.3	3 6	2 1 1	8	2 2 5 0	0	4
5-2/B1.4	3 6	2 1 1	7	1 5 2 1	0	3
5-2/B1.5	3 4	4 0 0	6	1 3 3 2	1	7
5-2/B2.1	3 9	1 3 0	6 7	10 9 12 5	2	1
5-2/B2.2	3 6	2 1 1	4 8	10 8 14 4	0	4
6-3/B1.1	3 9 13	2 9 0 2	6	3 6 9 6 3	0	0
6-3/B1.2	3 6 11	4 3 6 0	8	2 8 9 4 4	0	0
6-3/B1.3	3 6 7	3 6 3 1	9	2 9 6 7 3	0	0
6-3/B1.4	3 4 6	5 3 3 2	7	2 7 9 7 2	0	0
6-3/B2.1	3 9 13	2 9 0 2	6 7	15 18 36 30 9	0	0
6-3/B2.2	3 6 7	3 6 3 1	4 8	15 17 39 27 10	0	0
7-4/B1.1	3 10 11 13	5 15 9 8 3	8	3 15 15 24 21 3	0	0
7-4/B1.2	4 8 10 11	7 10 12 9 2	9	3 13 20 21 20 4	0	0
7-4/B1.3	3 4 9 13	8 9 9 14 0	6	3 12 21 24 15 6	0	0
7-4/B2.1	3 10 11 13	5 15 9 8 3	4 7	21 30 90 96 69 18	0	0
8-5/B1.1	3 8 9 10 11	8 30 24 32 24 3	13	4 24 30 64 84 24 13	0	0
8-5/B1.2	3 4 9 11 13	11 21 30 38 15 6	8	4 21 39 58 78 33 10	0	0
8-5/B2.1	3 8 9 10 11	8 30 24 32 24 3	4 7	28 48 180 256 276 144 40	0	0
9-6/B1.1	3 8 9 10 11 13	12 54 54 96 108 27 13	4	9 18 81 144 207 162 90 18	0	0
9-6/B1.2	3 4 8 9 10 11	15 42 69 96 93 39 10	13	6 30 66 144 222 150 93 18	0	0
9-6/B1.3	4 9 10 11 12 13	16 39 69 106 78 48 8	3	6 29 69 144 212 165 84 20	0	0
9-6/B2.1	3 8 9 10 11 13	12 54 54 96 108 27 13	4 7	36 72 324 576 828 648 360 72	0	0
10-7/B1.1	3 6 7 8 10 11 12	21 72 135 240 315 189 103 18	4	9 36 117 306 495 576 414 198 36	0	0
10-7/B1.2	3 4 6 7 8 10 11	22 68 138 250 290 213 92 20	9	8 40 114 296 520 552 425 196 36	0	0

Table 7A.6 Selected three-level admissible block designs with 81 runs

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
5-1/B1.1	22	0 0 1	9	0 2 1 0	5	20
5-1/B1.2	8	0 1 0	18	0 1 2 0	5	14
5-1/B2.1	22	0 0 1	3 24	1 7 3 1	5	19
5-1/B2.2	8	0 1 0	4 20	1 7 2 2	5	13
5-1/B3.1	22	0 0 1	3 6 15	10 10 15 4	5	10
5-1/B3.2	8	0 1 0	3 6 15	10 10 14 5	5	8
6-2/B1.1	8 18	0 2 2 0	21	0 3 4 1 1	6	18
6-2/B1.2	8 17	0 3 0 1	27	0 2 6 0 1	6	15
6-2/B1.3	3 23	1 0 3 0	24	0 3 3 3 0	3	24
6-2/B2.1	8 18	0 2 2 0	3 20	2 12 10 8 4	6	18
6-2/B2.2	8 17	0 3 0 1	4 19	2 12 9 10 3	6	13
6-2/B2.3	3 23	1 0 3 0	4 24	3 10 9 12 2	3	24
6-2/B2.4	3 23	1 0 3 0	6 17	2 11 12 7 4	3	22
6-2/B3.1	8 18	0 2 2 0	3 6 15	15 20 43 28 11	6	11
6-2/B3.2	8 17	0 3 0 1	4 6 15	15 20 42 30 10	6	12
6-2/B3.3	3 23	1 0 3 0	4 6 15	15 19 45 27 11	3	12
7-3/B1.1	8 17 27	0 5 6 1 1	36	0 5 10 3 7 2	7	15
7-3/B1.2	8 17 20	0 6 3 4 0	21	0 5 9 6 4 3	7	18
7-3/B2.1	8 17 27	0 5 6 1 1	6 15	6 11 31 33 19 8	7	15
7-3/B2.2	8 17 27	0 5 6 1 1	4 22	3 20 25 27 28 5	7	14
7-3/B2.3	8 17 20	0 6 3 4 0	7 16	6 11 30 36 16 9	7	18
7-3/B3.1	8 17 27	0 5 6 1 1	3 6 15	21 35 100 99 76 20	7	11
7-3/B3.2	8 17 20	0 6 3 4 0	3 6 15	21 35 99 102 73 21	7	12
8-4/B1.1	8 17 27 36	0 10 16 4 8 2	37	0 8 20 8 28 16 1	8	8
8-4/B1.2	8 17 20 21	0 11 12 10 4 3	27	2 4 18 16 26 12 3	8	16
8-4/B1.3	8 17 20 21	0 11 12 10 4 3	9	1 7 15 18 23 15 2	8	15
8-4/B1.4	8 17 20 40	0 12 8 16 0 4	7	2 4 17 20 20 16 2	8	16

Table 7A.6 (continued)

Design	T_{add}	$W(d)$	T_0^*	$W^*(d)$	$C1$	$C2$
8-4/B2.1	8 17 27 36	0 10 16 4 8 2	4 22	4 32 50 72 112 40 14	8	8
8-4/B2.2	8 17 20 21	0 11 12 10 4 3	4 27	8 20 58 80 100 44 14	8	16
8-4/B2.3	8 17 20 21	0 11 12 10 4 3	3 24	7 20 64 76 91 56 10	8	14
8-4/B2.4	8 17 20 21	0 11 12 10 4 3	4 24	5 29 49 86 91 53 11	8	13
8-4/B2.5	8 17 20 40	0 12 8 16 0 4	4 27	7 20 63 80 85 60 9	8	15
8-4/B2.6	8 17 20 40	0 12 8 16 0 4	4 24	4 32 48 80 100 48 12	8	12
8-4/B3.1	8 17 27 36	0 10 16 4 8 2	3 6 15	28 56 200 264 304 160 41	8	8
8-4/B3.2	8 17 20 21	0 11 12 10 4 3	3 6 16	28 56 199 268 298 164 40	8	12
8-4/B3.3	8 17 20 40	0 12 8 16 0 4	3 6 15	28 56 198 272 292 168 39	8	12
9-5/B1.1	8 17 27 36 37	0 18 36 12 36 18 1	38	0 12 36 18 84 72 9 12	9	0
9-5/B1.2	3 9 18 20 25	1 18 27 28 27 18 2	19	2 8 27 47 68 54 32 5	6	7
9-5/B1.3	3 9 18 20 25	1 18 27 28 27 18 2	22	1 10 30 37 73 60 25 7	6	6
9-5/B1.4	3 9 18 20 21	1 20 20 36 25 16 3	24	2 8 25 54 60 56 34 4	6	9
9-5/B1.5	3 9 18 20 21	1 20 20 36 25 16 3	25	1 10 28 44 65 62 27 6	6	8
9-5/B2.1	8 17 27 36 37	0 18 36 12 36 18 1	3 20	9 30 117 162 291 234 99 30	9	0
9-5/B2.2	3 9 18 20 25	1 18 27 28 27 18 2	13 17	8 39 87 206 270 219 119 24	6	7
9-5/B2.3	3 9 18 20 25	1 18 27 28 27 18 2	10 17	7 40 93 196 265 240 103 28	6	6
9-5/B2.4	3 9 18 20 25	1 18 27 28 27 18 2	13 22	6 44 90 186 290 216 114 26	6	5
9-5/B2.5	3 9 18 20 21	1 20 20 36 25 16 3	6 15	10 28 109 188 272 224 117 24	6	9
9-5/B2.6	3 9 18 20 21	1 20 20 36 25 16 3	8 17	7 40 91 203 257 242 105 27	6	8
9-5/B2.7	3 9 18 20 21	1 20 20 36 25 16 3	10 22	6 44 88 193 282 218 116 25	6	7
9-5/B3.1	8 17 27 36 37	0 18 36 12 36 18 1	3 7 16	36 84 360 594 912 720 369 84	9	0
9-5/B3.2	3 9 18 20 25	1 18 27 28 27 18 2	4 6 15	36 83 360 603 896 729 369 83	6	6
9-5/B3.3	3 9 18 20 21	1 20 20 36 25 16 3	4 6 15	36 83 358 610 888 731 371 82	6	7
10-6/B1.1	8 17 27 36 37 38	0 30 72 30 120 90 10 12	3	3 12 39 102 165 192 138 66 12	10	0
10-6/B2.1	8 17 27 36 37 38	0 30 72 30 120 90 10 12	3 6	12 48 156 408 660 768 552 264 48	10	0
10-6/B3.1	3 6 18 21 24 35	2 28 57 65 100 78 27 7	4 8 15	45 118 602 1203 2245 2420 1857 823 164	5	0
10-6/B3.2	3 9 18 19 24 25	2 30 48 80 90 78 30 6	4 6 15	45 118 600 1212 2230 2430 1857 820 165	4	2
10-6/B3.3	3 9 13 18 20 34	2 31 48 74 94 87 18 10	4 6 15	45 118 599 1212 2236 2426 1848 832 161	4	4
10-6/B3.4	3 9 13 18 20 21	2 34 36 89 94 72 30 7	4 6 15	45 118 596 1224 2221 2426 1863 820 164	4	4
10-6/B3.5	3 6 7 18 22 38	5 28 48 68 100 87 21 7	4 8 15	45 115 602 1212 2242 2420 1848 829 164	4	5
10-6/B3.6	3 6 7 9 12 27	8 34 48 62 88 87 24 13	4 8 15	45 112 596 1212 2248 2432 1848 826 158	2	16

8

Fractional Factorial Split-Plot Designs

Fractional factorial split-plot designs are considered in this chapter. They are used when the levels of some factors are difficult to change, and as a result, a completely random allocation of the treatment combinations to the experimental units is not feasible. The special features of these designs are discussed and a finite projective geometric formulation is given. An extension of the minimum aberration criterion is considered for selecting optimal designs. Tables of optimal designs are also provided.

8.1 Description and Salient Features

Regular s^{n-k} designs, introduced in Chapter 2 and studied at length in Chapters 3–5, have the following important characteristics:

(a) All factors have the *same* status.
(b) The experiment based on any such design involves a *completely random allocation* of the s^{n-k} treatment combinations to the experimental units.

Of course, (b) is appropriate only when the experimental units are homogeneous; otherwise, one has to adopt a restricted randomization via blocking as discussed in the previous chapter.

We now focus on situations in which neither (a) holds nor (b) is appropriate even with homogeneous experimental units. Consider an s^n factorial involving factors $F_1, \ldots, F_n$, each at s levels, where s is a prime or prime power. Suppose that among the n factors, there are n_1 $(1 \leq n_1 < n)$ whose levels are very difficult or expensive to change. Without loss of generality, let these be $F_1, \ldots, F_{n_1}$. The levels of the remaining n_2 $(= n - n_1)$ factors are easy to change. For reasons to be explained later, the hard-to-change factors $F_1, \ldots, F_{n_1}$ are called *whole plot (WP) factors* and the rest are called *subplot (SP) factors*. Obviously, the factors no longer have the same status and (a) is violated. Furthermore, a completely random allocation as in (b) is inadvisable since that might entail too many level changes of the WP factors, thus making the experiment unduly expensive or infeasible. *Fractional factorial split-plot (FFSP) designs* represent a practical option in such situations. These designs take due cognizance of the distinction between the WP and SP factors and have two salient features, both targeted to reducing the experimental cost:

(i) They involve not only a fixed number of treatment combinations but also a fixed number of WP factor settings.
(ii) They allow a *two-phase randomization*, as opposed to a complete randomization.

In (i), a "WP factor setting" means a combination of levels of the WP factors. In a similar sense, the term "SP factor setting" will be used later. It will be seen in the next section that additional distinguishing features of FFSP designs, concerning isomorphism and estimation efficiency, emerge from (i) and (ii) above. An illuminating discussion on the use of split-plot designs in industrial experiments is available in Box and Jones (1992). Further practical applications have been indicated by Huang, Chen, and Voelkel (1998) and Bingham and Sitter (1999a, 2001), among others.

Continuing with the above setup, a fuller description of FFSP designs is now in order. In the spirit of Chapter 2, only regular FFSP designs are considered. As before, a typical treatment combination x is an $n \times 1$ vector over $GF(s)$. The experimental units are assumed to be homogeneous and blocking is not considered.

Suppose the available resources allow experimentation with a total of $s^{n_1+n_2-k_1-k_2}$ treatment combinations. It is also stipulated that these treatment combinations should involve a fixed number, $s^{n_1-k_1}$, of WP factor settings. Here $0 \le k_1 < n_1$, $0 \le k_2 < n_2$, and $k_1 + k_2 \ge 1$. An FFSP design meeting the above specifications is given by

$$d(B) = \{x : Bx = 0\}, \tag{8.1.1}$$

where

$$B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \tag{8.1.2}$$

is a matrix over $GF(s)$, with B_{11}, B_{21}, and B_{22} of orders $k_1 \times n_1$, $k_2 \times n_1$, and $k_2 \times n_2$ respectively such that

$$\text{rank}(B_{11}) = k_1, \qquad \text{rank}(B_{22}) = k_2. \tag{8.1.3}$$

If $k_1 = 0$ or $k_2 = 0$, then the corresponding block of rows in (8.1.2), as well as the corresponding rank condition in (8.1.3), do not arise.

Indeed, (8.1.1) is formally similar to equation (2.4.1) describing an s^{n-k} design, but the important new feature is that B must have the structure (8.1.2) and satisfy (8.1.3). A design, as specified by (8.1.1)–(8.1.3), is called an $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP design.

Theorem 8.1.1. *An $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP design has the following properties:*

(a) The design involves a total of $s^{(n_1+n_2)-(k_1+k_2)}$ treatment combinations.
(b) These treatment combinations involve $s^{n_1-k_1}$ WP factor settings.
(c) Each such WP factor setting appears in conjunction with $s^{n_2-k_2}$ SP factor settings.

Proof. By (8.1.2) and (8.1.3), the k_1+k_2 rows of B are linearly independent. Hence, (a) follows from (8.1.1) using the same argument as in Lemma 2.3.2.

To prove (b) and (c), partition any treatment combination x as $x = (x^{(1)'}, x^{(2)'})'$, where $x^{(i)}$ has n_i elements $(i = 1, 2)$. Clearly, the elements of $x^{(1)}$ represent the levels of the WP factors, while those of $x^{(2)}$ represent the levels of the SP factors. Suppose k_1 and k_2 are both positive. Then by (8.1.2), (8.1.1) holds if and only if

$$B_{11}x^{(1)} = 0 \tag{8.1.4}$$

and

$$B_{22}x^{(2)} = -B_{21}x^{(1)}. \tag{8.1.5}$$

Since B_{11} has order $k_1 \times n_1$, the first rank condition in (8.1.3) implies that there are $s^{n_1-k_1}$ solutions of (8.1.4) for $x^{(1)}$, proving (b). Given any such solution for $x^{(1)}$, the second rank condition in (8.1.3) similarly shows that there are $s^{n_2-k_2}$ solutions of (8.1.5) for $x^{(2)}$. This proves (c). It is easily seen that similar arguments work when either $k_1 = 0$ or $k_2 = 0$. □

Equation (8.1.5) suggests that in an FFSP design, different WP factor settings can appear in conjunction with different SP factor settings. This is also evident from the following example.

Example 8.1.1. Let $s = 2$, $n_1 = 2$, $n_2 = 4$, $k_1 = 0$, $k_2 = 2$, and consider the $2^{(2+4)-(0+2)}$ FFSP design $d_0 = d(B_0)$, where

$$B_0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}. \tag{8.1.6}$$

Here $k_1 = 0$ and hence the first k_1 rows in (8.1.2) as well as the first rank condition in (8.1.3) do not arise. Therefore, (8.1.6) is compatible with (8.1.2), with the matrices B_{21} and B_{22} given respectively by the first $n_1 (= 2)$ and the last $n_2 (= 4)$ columns of (8.1.6). Clearly, then the second rank condition in (8.1.3) is satisfied. By (8.1.1) and (8.1.6), d_0 consists of the 16 treatment combinations $x = (x_1, \ldots, x_6)'$ satisfying

$$x_5 = x_1 + x_2 + x_3, \qquad x_6 = x_1 + x_3 + x_4. \tag{8.1.7}$$

Hence it is readily seen that in keeping with Theorem 8.1.1, d_0 involves four settings of the WP factors F_1 and F_2, and that each such setting appears in conjunction with four settings of the SP factors $F_3, \ldots, F_6$. For example, the WP factor setting $(0,0)$ appears in conjunction with the SP factor settings $(0,0,0,0)$, $(0,1,0,1)$, $(1,0,1,1)$, and $(1,1,1,0)$. Similarly, the WP factor setting $(0,1)$ appears in conjunction with the SP factor settings $(0,0,1,0)$, $(0,1,1,1)$, $(1,0,0,1)$, $(1,1,0,0)$, and so on. □

We are now in a position to describe the two-phase randomization in an $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP design. This is done as follows:

(i) Randomly choose any of the $s^{n_1-k_1}$ WP factor settings.
(ii) Run the experiment with the associated $s^{n_2-k_2}$ SP factor settings. In the process, keep the WP factors fixed at the setting chosen in (i) and randomize only SP factor settings.
(iii) Repeat steps (i) and (ii) till all the $s^{n_1-k_1}$ WP factor settings are covered.

The two-phase randomization is cost-effective since it is parsimonious in changing the WP factor settings. Under this kind of randomization, each of the $s^{n_1-k_1}$ WP factor settings defines a *whole plot* (WP). Any WP incorporates $s^{n_2-k_2}$ individual treatment combinations, each representing a *subplot* (SP), obtained through change in the SP factor settings. This explains the rationale behind the terminology "WP factors" and "SP factors".

8.2 Design Criteria

In view of (8.1.1), the concepts of defining pencil, defining contrast subgroup, and aliasing for an FFSP design $d(B)$ remain the same as in Section 2.4. The concepts of resolution, wordlength pattern, and minimum aberration (MA) are also the same as in Section 2.5.

Consider, for instance, the design d_0 in Example 8.1.1. By (8.1.6), d_0 has the defining relation

$$I = 1235 = 1346 = 2456. \tag{8.2.1}$$

Thus it has resolution four and wordlength pattern $(0,0,0,3,0,0)$.

The concept of isomorphism for FFSP designs is similar to that for s^{n-k} designs, with the additional requirement that the distinction between the WP and SP factors be considered. Two $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs are *isomorphic* if the defining contrast subgroup of one design can be obtained from that of the other by permuting the WP factor labels and/or the SP factor labels. While this is in the spirit of the corresponding definition in Section 3.1, permutation of factor labels has now to be considered separately for the WP and SP factors. A more general definition of isomorphism for s-level FFSP designs will be given in the next section.

Example 8.2.1. In the setup of Example 8.1.1, consider the $2^{(2+4)-(0+2)}$ FFSP design $d_1 = d(B_1)$, where

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

This design has the defining relation

$$I = 1235 = 1246 = 3456. \tag{8.2.2}$$

Observe that (8.2.1) and (8.2.2) can be obtained from each other by interchanging the letters 2 and 3. Thus d_0 and d_1 would have been isomorphic to each other had these been ordinary 2^{n-k} designs. However, these are FFSP designs and the letters 2 and 3, which correspond to a WP and an SP factor respectively, are not interchangeable. Consequently, d_0 and d_1 are not isomorphic. A comparison with Theorem 3.2.1 shows that both designs have MA. □

Because of the lack of interchangeability of the WP and SP factors, the phenomenon observed in the last example is not uncommon in FFSP designs. Hence an additional criterion is required for discrimination among rival nonisomorphic MA designs. The two-phase randomization provides a clue in this regard. For ease in further elucidation, a few simple concepts are introduced first. A pencil is said to be of the WP type if it involves only the WP factors (i.e., has the last n_2 entries zero), and of the SP type otherwise. Note that a WP type pencil cannot involve any SP factor, whereas an SP type pencil can involve both types of factors. An alias set is said to be a *WP alias set* if it contains at least one WP type pencil, and an *SP alias set* if it contains only SP type pencils. For example, it is readily seen from (8.2.1) that all the three defining pencils of the design d_0 are of SP type. Furthermore,

$$\begin{aligned} 1 &= 235 = 346 = 12456, \\ 2 &= 135 = 12346 = 456, \\ 12 &= 35 = 2346 = 1456 \end{aligned}$$

are the WP alias sets of d_0 (since they contain the WP type pencils 1, 2, and 12 respectively), while

$$3 = 125 = 146 = 23456$$

is an SP alias set of d_0.

It is well known (Kempthorne, 1952) that the two-phase randomization results in two sources of error in analysis of variance, one at the WP level and the other at the SP level, the former being larger than the latter. If we were considering a full factorial (i.e., $k_1 = k_2 = 0$), this would entail lower estimation efficiency for WP type pencils compared to SP type pencils. In the same manner, in an FFSP design, estimation from a WP alias set has a lower efficiency than that from an SP alias set (Bingham and Sitter, 1999a, 2001). As such, a good FFSP design should avoid assignment of SP type pencils, especially those representing lower-order factorial effects, to WP alias sets. In particular, no pencil representing the main effect of an SP factor should appear in a WP alias set.

The aforesaid considerations led Bingham and Sitter (2001) to suggest a follow-up criterion for FFSP designs as a supplement to the MA criterion.

With reference to an FFSP design, for $i = 1, 2, \ldots$, let N_i be the number of pencils of the SP type that involve i factors and appear in WP alias sets. The requirement mentioned in the last sentence of the previous paragraph is equivalent to $N_1 = 0$. In case there are nonisomorphic MA designs, Bingham and Sitter (2001) proposed *selecting one with the smallest* N_2.

The points noted above can be summarized as follows:

(i) an FFSP design must have $N_1 = 0$;
(ii) in case there are nonisomorphic MA designs, one with the smallest N_2 should be selected.

As an illustration, the $2^{(2+4)-(0+2)}$ MA designs d_0 and d_1 are considered again. From (8.2.1) and (8.2.2), it is readily seen that both designs satisfy $N_1 = 0$. A look at the WP alias sets of d_0, as shown before, reveals that the only SP type pencil that involves two factors and appears in a WP alias set is 35. Hence $N_2 = 1$ for d_0. Similarly, the SP type pencils 35 and 46 appear in a WP alias set of d_1, so that $N_2 = 2$ for d_1. Hence although both d_0 and d_1 have MA, d_0 is preferred to d_1 in the sense of having a smaller N_2. The fact that d_0 and d_1 have different N_2 values also highlights that these designs are not isomorphic when one distinguishes between the WP and SP factors.

In the remainder of this chapter, FFSP designs are studied under the MA criterion. The follow-up criterion of minimizing N_2 is also used whenever necessary. Throughout, attention is restricted to designs that have resolution three or higher and satisfy $N_1 = 0$.

8.3 A Projective Geometric Formulation

A projective geometric formulation that facilitates the study and tabulation of optimal FFSP designs is given in this section. Let

$$t_1 = n_1 - k_1, \qquad t_2 = n_2 - k_2, \qquad t = t_1 + t_2, \tag{8.3.1}$$

and let P denote the set of the $(s^t - 1)/(s - 1)$ points of the finite projective geometry $PG(t-1, s)$. Let $e_1, \ldots, e_t$ represent the $t \times 1$ unit vectors over $GF(s)$. Define P_1 as the $(t_1 - 1)$-flat of P that is generated by e_i $(1 \le i \le t_1)$, and P_2 as the complement of P_1 in P. As usual, for any nonempty subset Q of P, let $V(Q)$ be a matrix with columns given by the points of Q. Also, let $\mathcal{R}(\cdot)$ denote the row space of a matrix.

Definition 8.3.1. *An ordered pair of subsets* (T_1, T_2) *of* P *is called an eligible* (n_1, n_2)*-pair if (a)* T_i *has cardinality* n_i $(i = 1, 2)$, *(b)* $T_i \subset P_i$ $(i = 1, 2)$, *(c) rank*$[V(T_1)] = t_1$, *and (d) rank*$[V(T)] = t$, *where* $T = T_1 \cup T_2$.

As a counterpart of Theorem 2.7.1 for s^{n-k} designs, the following result, due to Mukerjee and Fang (2002), holds.

Theorem 8.3.1. *Given any $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP design d, having resolution three or higher and $N_1 = 0$, there exists an eligible (n_1, n_2)-pair of subsets (T_1, T_2) of P such that with $T = T_1 \cup T_2$ and*

$$V(T) = [V(T_1) \quad V(T_2)], \tag{8.3.2}$$

the following hold:

(a) the treatment combinations included in d are transposes of the vectors in $\mathcal{R}[V(T)]$,
(b) the WP factor settings in d are given by the vectors in $\mathcal{R}[V(T_1)]$,
(c) any pencil b is a defining pencil of d if and only if $V(T)b = 0$,
(d) any two pencils are aliased with each other in d if and only if $V(T)(b - b^*) = 0$ for some representations b and b^* of these pencils.

Conversely, given any eligible (n_1, n_2)-pair of subsets (T_1, T_2) of P, there exists an $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP design d, having resolution three or higher and $N_1 = 0$, such that (a)–(d) hold, with $V(T)$ defined as in (8.3.2).

Proof. Consider an $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP design $d = d(B)$ as specified by (8.1.1)–(8.1.3). Then it is not hard to see that there exists a matrix

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix} \tag{8.3.3}$$

over $GF(s)$, with G_{11}, G_{12}, and G_{22} of orders $t_1 \times n_1, t_1 \times n_2$, and $t_2 \times n_2$ respectively, such that

$$\text{rank}(G_{11}) = t_1, \qquad \text{rank}(G_{22}) = t_2, \tag{8.3.4}$$

and

$$BG' = 0. \tag{8.3.5}$$

By (8.1.2) and (8.1.3), B has full row rank. Similarly, by (8.3.3) and (8.3.4), G has full row rank. Also, $[B' \;\; G']'$ is a square matrix in view of (8.3.1). Hence by (8.3.5), the row spaces of B and G are orthogonal complements of each other and the following hold:

(i) the treatment combinations included in d are transposes of the vectors in $\mathcal{R}(G)$,
(ii) the WP factor settings in d are given by the vectors in $\mathcal{R}(G_{11})$,
(iii) any pencil b is a defining pencil of d if and only if $Gb = 0$,
(iv) any two pencils are aliased with each other in d if and only if $G(b-b^*) = 0$ for some representations b and b^* of these pencils,
(v) no column of G_{22} is null.

Of these, (i), (iii), and (iv) are precisely as in Lemma 2.6.1, while (ii) is immediate from (i) and (8.3.3). In Exercise 8.3, the reader is asked to establish (v).

Since d has resolution three or higher, (iii) implies that the columns of G are nonnull and that no two of them are proportional to each other. Hence these columns can be interpreted as points of $P(\equiv PG(t-1,s))$. Let T_1 and T_2 be the sets of points given by the first n_1 and last n_2 columns, respectively, of G. Then, with $T = T_1 \cup T_2$,

$$V(T_1) = \begin{bmatrix} G_{11} \\ 0 \end{bmatrix}, \quad V(T_2) = \begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix}, \quad V(T) = G. \tag{8.3.6}$$

The first two equations in (8.3.6), together with fact (v), imply that $T_i \subset P_i$ $(i = 1, 2)$. From (8.3.4) and (8.3.6), it is now clear that (T_1, T_2) is an eligible (n_1, n_2)-pair of subsets of P. Since $V(T) = G$ and $\mathcal{R}[V(T_1)] = \mathcal{R}(G_{11})$, the validity of (a)–(d) of the theorem is immediate from (i)–(iv) above.

The converse can be proved by reversing the above steps. □

In order to illustrate Theorem 8.3.1, we consider the design $d_0 = d(B_0)$ in Example 8.1.1. This design has resolution four and satisfies $N_1 = 0$. By (8.3.1), $t_1 = n_1 = 2$, $t_2 = 2$, $n_2 = 4$, and $t = 4$. The matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

has the form (8.3.3), meets the rank conditions in (8.3.4), and satisfies $B_0 G' = 0$. Hence interpreting the columns of G as points of $PG(3, 2)$, the design d_0 is equivalent to the eligible $(2, 4)$-pair of subsets (T_1, T_2), where

$$T_1 = \{(1, 0, 0, 0)', (0, 1, 0, 0)'\},$$
$$T_2 = \{(0, 0, 1, 0)', (0, 0, 0, 1)', (1, 1, 1, 0)', (1, 0, 1, 1)'\}.$$

It is easy to verify from first principles that (a)–(d) of Theorem 8.3.1 hold in this example.

As indicated at the end of the last section, the only FFSP designs of interest are those having resolution three or higher and $N_1 = 0$. Theorem 8.3.1 shows that in studying such designs, it is enough to consider eligible (n_1, n_2)-pairs of subsets of P. The design corresponding to an eligible pair of subsets (T_1, T_2) will be denoted by $d(T_1, T_2)$. Evidently, by (8.3.2) and Theorem 8.3.1(a) and (b), the WP and SP factors correspond to the points of T_1 and T_2 respectively. Considering the cardinalities of T_1, T_2, P_1, and P_2, given n_1, n_2, k_1, and k_2, such a design exists if and only if

$$n_1 \le \frac{s^{t_1} - 1}{s - 1}, \qquad n_2 \le \frac{s^t - s^{t_1}}{s - 1}, \tag{8.3.7}$$

where $t_1 = n_1 - k_1$ and $t = n_1 + n_2 - k_1 - k_2$; cf. (8.3.1). Hereafter, the above conditions are supposed to hold.

Theorem 8.3.1 also paves the way for defining isomorphism of FFSP designs for general s. Two $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs $d(T_{11}, T_{12})$ and $d(T_{21}, T_{22})$ are *isomorphic* if there exists a nonsingular transformation that maps each point of T_{11} to some point of T_{21} up to proportionality, and each point of T_{12} to some point of T_{22} up to proportionality. While this is along the lines of the definition given in Section 4.4 for s^{n-k} designs, a new feature is the distinction between the WP and SP factors. From Theorem 8.3.1, it follows that isomorphic FFSP designs have not only the same wordlength pattern but also the same N_i for every i. For $s = 2$, one can verify that the above definition of isomorphism is equivalent to the one given in the last section.

8.4 Use of Complementary Sets

As in the previous chapters, use of complementary sets often facilitates the exploration of optimal FFSP designs. For an $s^{(n_1+n_2)-(k_1+k_2)}$ FFSP design $d = d(T_1, T_2)$, let $\overline{T}_1$ and $\overline{T}_2$ be the complements of T_1 and T_2 in P_1 and P_2 respectively, and write $\overline{T} = \overline{T}_1 \cup \overline{T}_2$. The cardinalities of $\overline{T}_1$, $\overline{T}_2$, and $\overline{T}$ are given respectively by

$$f_1 = \frac{s^{t_1} - 1}{s - 1} - n_1, \quad f_2 = \frac{s^t - s^{t_1}}{s - 1} - n_2, \quad f = f_1 + f_2. \tag{8.4.1}$$

If $f = 0$, then there is only one design, i.e., the one corresponding to $T_1 = P_1$ and $T_2 = P_2$. Also, it is not hard to see that all designs are isomorphic when $f = 1$. Hence only $f \geq 2$ is considered in what follows.

Let $(A_1(d), \ldots, A_n(d))$ denote the wordlength pattern of $d = d(T_1, T_2)$, where $A_1(d) = A_2(d) = 0$. The following identities, involving the complementary set $\overline{T}$, are immediate from Corollary 4.3.2:

$$(a) \quad A_3(d) = \text{constant} - A_3(\overline{T}), \tag{8.4.2}$$

$$(b) \quad A_4(d) = \text{constant} + (3s - 5)A_3(\overline{T}) + A_4(\overline{T}), \tag{8.4.3}$$

$$\begin{aligned}(c) \quad A_5(d) = \text{constant} - \frac{1}{2}\{s^{n-k} - 2(s-1)n + (s-2)(12s-17)\}A_3(\overline{T}) \\ -(4s-7)A_4(\overline{T}) - A_5(\overline{T}),\end{aligned} \tag{8.4.4}$$

where the $A_i(\overline{T})$ are as in (4.3.3). Furthermore, denoting the N_2 value of d by $N_2(d)$, it was shown in Mukerjee and Fang (2002) that

$$N_2(d) = \text{constant} + H_2(P_1, \overline{T}_2), \tag{8.4.5}$$

where as in (6.3.7),

$$\begin{aligned}H_2(P_1, \overline{T}_2) = (s-1)^{-1}\#\{\lambda : \lambda \text{ is } f_2 \times 1 \text{ over } GF(s) \text{ with two nonzero} \\ \text{elements such that } V(\overline{T}_2)\lambda \text{ is nonnull and proportional to some} \\ \text{point of } P_1\},\end{aligned} \tag{8.4.6}$$

and # denotes the cardinality of a set. The constants in (8.4.2)–(8.4.5) may depend on s, n_1, n_2, k_1, and k_2 but not on the particular choice of d.

The identities (8.4.2)–(8.4.5) are particularly useful when f_1 and f_2 are relatively small compared to n_1 and n_2, and hence the complementary sets are easier to handle. As in Chapter 4, (8.4.2)–(8.4.4) help in finding MA designs, and then (8.4.5) may be needed for further discrimination if nonisomorphic MA designs emerge. The possibility of employing (8.4.5) is a new feature of FFSP designs. Another feature, compared to s^{n-k} designs, is that an arbitrary set of cardinality f cannot be a candidate for $\overline{T}$ when (8.4.2)–(8.4.4) are used. It is required that $\overline{T}$ be decomposable as

$$\overline{T} = \overline{T}_1 \cup \overline{T}_2, \tag{8.4.7}$$

where

$$\#\overline{T}_i = f_i \text{ and } \overline{T}_i \subset P_i, \quad i = 1, 2. \tag{8.4.8}$$

The next two examples illustrate these ideas. In the rest of this chapter including the tables, the uniqueness of a design is up to isomorphism.

Example 8.4.1. Let $s = 2$, $f_1 = 4$, $f_2 = 1$. Since $f_1 = 4$, from (8.4.1) one gets $t_1 \geq 3$. There is a unique design associated with the maximum possible $A_3(\overline{T})$. This corresponds to

$$\overline{T}_1 = \{e_1, e_2, e_3, e_1 + e_2\}, \quad \overline{T}_2 = \{e_{t_1+1}\},$$

where, as before, $e_1, e_2, \ldots$ are the $t \times 1$ unit vectors. By (8.4.2), this is the unique MA design. Here $f = 5$, and interpreting Table 3.1 in the present notation, a 2^{n-k} MA design is given by

$$\overline{T} = \{h_1, h_2, h_3, h_1 + h_2, h_1 + h_3\}, \tag{8.4.9}$$

where h_1, h_2, h_3 are linearly independent points of the finite projective geometry P. With $f_1 = 4$ and $f_2 = 1$, however, the set $\overline{T}$ in (8.4.9) is not decomposable as in (8.4.7) and (8.4.8) and hence it cannot arise in the context of FFSP designs. □

Example 8.4.2. Let $s = 3$, $f_1 = 1$, $f_2 = 4$. Since $f_1 = 1$, from (8.4.1) one gets $t_1 \geq 2$.

(a) If $t_2 = 1$, then as in the last example, the unique MA design is given by

$$\overline{T}_1 = \{e_1\}, \quad \overline{T}_2 = \{e_{t_1+1}, e_1 + e_{t_1+1}, e_1 + 2e_{t_1+1}, e_2 + e_{t_1+1}\}. \tag{8.4.10}$$

(b) If $t_2 \geq 2$, then up to isomorphism, there are three different designs associated with the maximum possible $A_3(\overline{T})$. One of these, say d_1, is given by (8.4.10). The other two, say, d_2 and d_3, correspond to

$$\overline{T}_1 = \{e_1\}, \quad \overline{T}_2 = \{e_{t_1+1}, e_1 + e_{t_1+1}, e_1 + 2e_{t_1+1}, e_{t_1+2}\}, \tag{8.4.11}$$

and

$$\overline{T}_1 = \{e_1\}, \quad \overline{T}_2 = \{e_{t_1+1}, e_{t_1+2}, e_{t_1+1} + e_{t_1+2}, e_{t_1+1} + 2e_{t_1+2}\}, \quad (8.4.12)$$

respectively. Here $f = 5$, and all these designs have the same $A_4(\overline{T})$ and $A_5(\overline{T})$. Hence by Theorem 4.3.1, they all have the same wordlength pattern, and consequently, they are all MA designs. The follow-up criterion of minimizing N_2 is therefore needed at this stage. By (8.4.6) and (8.4.10)–(8.4.12), the quantity $H_2(P_1, \overline{T}_2)$ equals 6, 3, and 0 for d_1, d_2, and d_3 respectively. Hence (8.4.5) implies that d_3 uniquely minimizes the value of N_2 among MA designs. □

For the optimal designs in the last two examples, it is not hard to see that the pair (T_1, T_2) satisfies the rank conditions of Definition 8.3.1. In particular, taking $t_1 = 3$ and $t_2 = 2$, or $t_1 = 4$ and $t_2 = 1$ in Example 8.4.1, one gets the $2^{(3+23)-(0+21)}$ and $2^{(11+15)-(7+14)}$ FFSP designs with MA. Similarly, with $t_1 = t_2 = 2$, Example 8.4.2 yields the $3^{(3+32)-(1+30)}$ FFSP design with MA and minimum N_2.

Further theoretical results on optimal FFSP designs are available in Bingham and Sitter (1999b) and Mukerjee and Fang (2002). The latter also considered the criterion of maximum estimation capacity for FFSP designs.

8.5 Tables of Optimal Designs

Tables 8.1 and 8.2, adapted from Bingham and Sitter (1999a) and Mukerjee and Fang (2002), show the sets T_1 and T_2 for optimal FFSP designs with (i) $s = 2$ and 16 runs, and (ii) $s = 3$ and 27 runs, respectively. All possibilities for n_1, n_2, k_1, k_2 satisfying (8.3.7) are considered, except for the cases $n_1 + n_2 = 14, 15$ in Table 8.1, and $n_1 + n_2 = 12, 13$ in Table 8.2. These cases correspond to $f = 1$ or 0 and are hence trivial. Interestingly, (8.3.7) rules out the case $(n_1, n_2) = (4, 9)$ in Table 8.1. Each tabulated design has MA; it is either the unique MA design or minimizes N_2 among all MA designs. The footnotes of the tables explain the sense in which each design is optimal. As in the previous chapters, the tables use the compact notation for the points of P.

For smaller n_1 and n_2, the optimal designs are obtained by direct search. For relatively large n_1 and n_2, use of complementary sets helps. The following example illustrates the latter situation.

Example 8.5.1. Let $s = 3$. Consider the case $n_1 = 3$, $n_2 = 5$, $k_1 = 1$, $k_2 = 4$ in Table 8.2. By (8.3.1) and (8.4.1), $t_1 = 2$, $t_2 = 1$, $t = 3$, $f_1 = 1$, $f_2 = 4$. Therefore, (8.4.10) shows that the unique MA design is given by $\overline{T}_1 = \{1\}$, $\overline{T}_2 = \{3, 13, 13^2, 23\}$, or equivalently, by $T_1 = \{2, 12, 12^2\}$, $T_2 = \{23^2, 123, 123^2, 12^2 3, 12^2 3^2\}$, as reported in Table 8.2. □

Table 8.1 The sets T_1 and T_2 for optimal two-level FFSP designs with 16 runs

n_1	n_2	k_1	k_2	t_1	t_2	T_1	T_2	Optimality property
1	4	0	1	1	3	{1}	{2,3,4,1234}	1
2	3	0	1	2	2	{1,2}	{3,4,1234}	1
3	2	1	0	2	2	{1,2,12}	{3,4}	1
3	2	0	1	3	1	{1,2,3}	{4,1234}	1
4	1	1	0	3	1	{1,2,3,123}	{4}	1
1	5	0	2	1	3	{1}	{2,3,123,4,124}	1
2	4	0	2	2	2	{1,2}	{3,123,4,134}	2
3	3	1	1	2	2	{1,2,12}	{3,4,134}	1
3	3	0	2	3	1	{1,2,3}	{4,124,134}	1
4	2	1	1	3	1	{1,2,3,123}	{4,124}	1
5	1	2	0	3	1	{1,2,12,3,13}	{4}	1
1	6	0	3	1	3	{1}	{2,3,123,4,124,134}	1
2	5	0	3	2	2	{1,2}	{3,123,4,124,134}	1
3	4	1	2	2	2	{1,2,12}	{3,13,4,234}	1
3	4	0	3	3	1	{1,2,3}	{4,124,134,234}	1
4	3	1	2	3	1	{1,2,3,123}	{4,124,134}	1
5	2	2	1	3	1	{1,2,12,3,13}	{4,234}	1
6	1	3	0	3	1	{1,2,12,3,13,23}	{4}	1
1	7	0	4	1	3	{1}	{2,3,123,4,124,134,234}	1
2	6	0	4	2	2	{1,2}	{3,123,4,124,134,234}	1
3	5	1	3	2	2	{1,2,12}	{3,13,4,14,234}	1
3	5	0	4	3	1	{1,2,3}	{4,14,24,34,1234}	1
4	4	1	3	3	1	{1,2,3,123}	{4,124,134,234}	1
5	3	2	2	3	1	{1,2,12,3,13}	{4,14,234}	1
6	2	3	1	3	1	{1,2,12,3,13,23}	{4,1234}	1
7	1	4	0	3	1	{1,2,12,3,13,23,123}	{4}	1
1	8	0	5	1	3	{1}	{2,12,3,23,4,24,134,1234}	2
2	7	0	5	2	2	{1,2}	{3,13,23,4,124,34,1234}	1
3	6	1	4	2	2	{1,2,12}	{3,13,4,14,234,1234}	1
3	6	0	5	3	1	{1,2,3}	{4,14,24,134,234,1234}	1
4	5	1	4	3	1	{1,2,3,123}	{4,14,24,34,1234}	1
5	4	2	3	3	1	{1,2,12,3,13}	{4,14,234,1234}	1
6	3	3	2	3	1	{1,2,12,3,13,23}	{4,14,234}	1
7	2	4	1	3	1	{1,2,12,3,13,23,123}	{4,14}	1

Table 8.1(continued) The sets T_1 and T_2 for optimal two-level FFSP designs with 16 runs

n_1	n_2	k_1	k_2	t_1	t_2	T_1	T_2	Optimality property
1	9	0	6	1	3	{1}	{2,12,3,13,23,4,24,134,1234}	2
2	8	0	6	2	2	{1,2}	{3,13,23,4,14,24,134,234}	2
3	7	1	5	2	2	{1,2,12}	{3,13,23,4,14,234,1234}	1
3	7	0	6	3	1	{1,2,3}	{4,14,24,124,34,134,234}	1
4	6	1	5	3	1	{1,2,3,123}	{4,14,24,124,34,1234}	1
5	5	2	4	3	1	{1,2,12,3,13}	{4,14,24,234,1234}	1
6	4	3	3	3	1	{1,2,12,3,13,23}	{4,14,234,1234}	1
7	3	4	2	3	1	{1,2,12,3,13,23,123}	{4,14,24}	1
1	10	0	7	1	3	{1}	{2,12,3,13,23,4,14,24,134,234}	2
2	9	0	7	2	2	{1,2}	{3,13,23,4,14,24,134,234,1234}	2
3	8	1	6	2	2	{1,2,12}	{3,13,23,4,14,24,134,234}	1
3	8	0	7	3	1	{1,2,3}	{4,14,24,124,34,134,234,1234}	1
4	7	1	6	3	1	{1,2,3,123}	{4,14,24,124,34,134,234}	1
5	6	2	5	3	1	{1,2,12,3,13}	{4,14,24,34,234,1234}	1
6	5	3	4	3	1	{1,2,12,3,13,23}	{4,14,24,134,234}	1
7	4	4	3	3	1	{1,2,12,3,13,23,123}	{4,14,24,34}	1
1	11	0	8	1	3	{1}	{2,12,3,13,23,4,14,24,134,234,1234}	1
2	10	0	8	2	2	{1,2}	{3,13,23,123,4,14,24,124,34,1234}	1
3	9	1	7	2	2	{1,2,12}	{3,13,23,4,14,24,134,234,1234}	1
4	8	1	7	3	1	{1,2,3,123}	{4,14,24,124,34,134,234,1234}	1
5	7	2	6	3	1	{1,2,12,3,13}	{4,14,24,124,34,134,234}	1
6	6	3	5	3	1	{1,2,12,3,13,23}	{4,14,24,134,234,1234}	1
7	5	4	4	3	1	{1,2,12,3,13,23,123}	{4,14,24,124,34}	1
1	12	0	9	1	3	{1}	{2,12,3,13,23,123,4,14,24,124,34,234}	2
2	11	0	9	2	2	{1,2}	{3,13,23,123,4,14,24,124,34,134,1234}	1
3	10	1	8	2	2	{1,2,12}	{3,13,23,123,4,14,24,34,134,234}	2
5	8	2	7	3	1	{1,2,12,3,13}	{4,14,24,124,34,134,234,1234}	1
6	7	3	6	3	1	{1,2,12,3,13,23}	{4,14,24,124,34,134,234}	1
7	6	4	5	3	1	{1,2,12,3,13,23,123}	{4,14,24,124,34,134}	1

1. Unique MA design.
2. MA design; uniquely minimizes N_2 among all MA designs.

Table 8.2 The sets T_1 and T_2 for optimal three-level FFSP designs with 27 runs

n_1	n_2	k_1	k_2	t_1	t_2	T_1	T_2	Optimality property
1	3	0	1	1	2	{1}	{2,3,123}	1
2	2	0	1	2	1	{1,2}	{3,123}	1
3	1	1	0	2	1	{1,2,12}	{3}	1
1	4	0	2	1	2	{1}	$\{2,3,23,12^23\}$	2
2	3	0	2	2	1	{1,2}	$\{3,13,123^2\}$	3
3	2	1	1	2	1	{1,2,12}	$\{3,12^23\}$	1
4	1	2	0	2	1	$\{1,2,12,12^2\}$	{3}	1
1	5	0	3	1	2	{1}	$\{2,12,3,12^23,12^23^2\}$	1
2	4	0	3	2	1	{1,2}	$\{3,13,123^2,12^23^2\}$	1
3	3	1	2	2	1	{1,2,12}	$\{3,12^23,12^23^2\}$	1
4	2	2	1	2	1	$\{1,2,12,12^2\}$	{3,13}	1
1	6	0	4	1	2	{1}	$\{2,12,3,13^2,23^2,12^23^2\}$	2
2	5	0	4	2	1	{1,2}	$\{3,13^2,23,123,123^2\}$	1
3	4	1	3	2	1	{1,2,12}	$\{3,13^2,23^2,12^23^2\}$	3
4	3	2	2	2	1	$\{1,2,12,12^2\}$	{3,13,23}	1
1	7	0	5	1	2	{1}	$\{12^2,13,13^2,123,123^2,12^23,12^23^2\}$	1
2	6	0	5	2	1	$\{12,12^2\}$	$\{23, 23^2,123,123^2,12^23,12^23^2\}$	1
3	5	1	4	2	1	$\{2,12,12^2\}$	$\{23^2,123,123^2,12^23,12^23^2\}$	1
4	4	2	3	2	1	$\{1,2,12,12^2\}$	$\{123,123^2,12^23,12^23^2\}$	1
1	8	0	6	1	2	{1}	$\{12,12^2,13,13^2,123,123^2,12^23,12^23^2\}$	1
2	7	0	6	2	1	$\{12,12^2\}$	$\{13^2, 23,23^2,123,123^2,12^23,12^23^2\}$	1
3	6	1	5	2	1	$\{2,12,12^2\}$	$\{23,23^2,123,123^2,12^23,12^23^2\}$	1
4	5	2	4	2	1	$\{1,2,12,12^2\}$	$\{23^2,123,123^2,12^23,12^23^2\}$	1
1	9	0	7	1	2	{1}	$\{12,12^2,13,13^2,23^2,123,123^2,12^23,12^23^2\}$	2
2	8	0	7	2	1	$\{12,12^2\}$	$\{13,13^2, 23,23^2,123,123^2,12^23,12^23^2\}$	1
3	7	1	6	2	1	$\{2,12,12^2\}$	$\{13^2,23,23^2,123,123^2,12^23,12^23^2\}$	1
4	6	2	5	2	1	$\{1,2,12,12^2\}$	$\{23,23^2,123,123^2,12^23,12^23^2\}$	1
1	10	0	8	1	2	{1}	$\{12,12^2,13,13^2,23, 23^2,123,123^2,12^23,12^23^2\}$	2
2	9	0	8	2	1	$\{12,12^2\}$	$\{3,13,13^2, 23, 23^2,123,123^2,12^23,12^23^2\}$	1
3	8	1	7	2	1	$\{2,12,12^2\}$	$\{13,13^2,23,23^2,123,123^2,12^23,12^23^2\}$	1
4	7	2	6	2	1	$\{1,2,12,12^2\}$	$\{13^2,23,23^2,123,123^2,12^23,12^23^2\}$	1

1. Unique MA design.
2. MA design; uniquely minimizes N_2 among all MA designs.
3. MA design; minimizes N_2 among all MA designs. But there is another nonisomorphic MA design with the same N_i as the tabulated one for every i.

Exercises

8.1 For the design in Example 8.2.1, obtain the sets T_1 and T_2 as envisaged in Theorem 8.3.1. Hence, from first principles, verify the truth of (a)–(d) of the theorem for this design.

8.2 Verify the existence of a matrix G as envisaged in the proof of Theorem 8.3.1.

8.3 Refer again to the proof of Theorem 8.3.1. Suppose a column of G_{22}, say the first column, is null.
(a) Use fact (iii) in the proof to show that the first column of G_{12}, say ξ, is nonnull. Hence use (8.3.4) to conclude that there exists a nonnull vector δ over $GF(s)$ such that $G_{11}\delta = \xi$.
(b) Let e be the $n_2 \times 1$ unit vector over $GF(s)$ with 1 in the first position. Define $b^{(1)} = (0', e')'$ and $b^{(2)} = (\delta', 0')'$, where the null subvectors in $b^{(1)}$ and $b^{(2)}$ are of orders n_1 and n_2 respectively. With G as in (8.3.3), show that $G(b^{(1)} - b^{(2)}) = 0$.
(c) Hence use fact (iv) in the proof to conclude that a pencil representing the main effect of an SP factor appears in a WP alias set of the design.
Observe the impossibility of the conclusion in (c) and hence infer the truth of fact (v) in the proof.

8.4 Prove (8.4.5).

8.5 Using the method of complementary sets, show that there are two non-isomorphic $2^{(3+26)-(1+23)}$ FFSP designs with MA. Which of these has a smaller N_2?

8.6 For $n_1 = 1, n_2 = 10, k_1 = 0, k_2 = 7$, use the method of complementary sets to show that the FFSP design in Table 8.1 has MA and that it minimizes N_2 among all MA designs.

8.7 Do the same for the FFSP design shown in Table 8.2 with $n_1 = 1, n_2 = 10$, $k_1 = 0$, $k_2 = 8$.

8.8 For $s = 3$, $n_1 = 2$, $n_2 = 3$, $k_1 = 0$, $k_2 = 2$, obtain an FFSP design that is not isomorphic to the one shown in Table 8.2 but has the same wordlength pattern as well as the same N_i for every i.

9

Robust Parameter Design

Robust parameter design is an effective tool for variation reduction. The planning aspect of this methodology is studied in this chapter. The distinction between the control factors and the noise factors in parameter design experiments is discussed. This leads to new priorities and criteria in the choice of designs. Two experimental strategies, cross and single arrays, which play major roles in this context, are explored in the light of the new priorities.

9.1 Control and Noise Factors

Robust parameter design (or parameter design) is a statistical/engineering methodology (Taguchi, 1987) that aims at reducing the performance variation of a product or process by appropriately choosing the setting of its control factors so as to make it less sensitive to noise variation. Its effectiveness lies in the exploitation of some significant control-by-noise interactions. Because it is usually much easier and less costly to change the control factor setting than to tighten the noise variation directly, the methodology has been widely adopted in engineering practice and is now a commonly used tool in quality engineering. Its success in manufacturing and high-tech industries has been documented in many case studies compiled by companies and professional societies.

The factors in parameter design experiments are divided into two types: *control factors* and *noise factors*. Control factors are variables whose values (i.e., levels) remain fixed once they are chosen. They include the design parameters in product and process design. By contrast, noise factors are variables whose values (i.e., levels) are hard to control during the normal process or use conditions. They include variation in product and process parameters, environmental variation, load factors, user conditions, and degradation. Consider the problem of improving the yield of a chemical process. Its control factors can include reaction temperature and time, and type and concentration of catalyst; and its noise factors can include purity of reagent and purity of solvent stream. The latter two are treated as noise factors because purity varies from batch to batch and is thus hard to control.

From the design-theoretic point of view, the most interesting aspect of robust parameter design is the difference between the roles played by the control and noise factors. As hinted above, the control-by-noise interactions are crucial in achieving robustness. Thus this type of two-factor interactions (2fi's) must be placed in the same category of importance as the main effects. This obviously violates the effect hierarchy principle. The importance of the control main effects and the control-by-noise interactions is underscored by the roles they play in the two-step procedure for parameter design optimization given in (9.3.1) below. In the first step, the setting of some control factors is chosen to reduce the sensitivity (e.g., variance) of the response to the noise variation; in the second step, the setting of other control factors is chosen to adjust the mean response value on target. The success of the first step implicitly hinges on the existence of some significant control-by-noise interactions, whereas to carry out the second step, the control main effects should be estimable.

Two experimental strategies, cross arrays and single arrays, are discussed in this chapter. The distinction between the control and noise factors results in new design priorities with an appropriate modification of the effect hierarchy principle. The focus continues to be on the planning aspects. For details on robust parameter design, see Chapters 10 and 11 of Wu and Hamada (2000).

9.2 Cross Arrays

A simple example is first presented to motivate the ideas. Suppose there are four control factors $F_1, \ldots, F_4$ and three noise factors F_5, F_6, F_7, each of the seven factors being at two levels. Let d_C be a 2^{4-1} design involving the control factors alone and specified by the defining relation $I = 1234$. Similarly, let d_N be a 2^{3-1} design involving only the noise factors and specified by the defining relation $I = 567$. Writing the treatment combinations in d_C and d_N as rows, these designs can be expressed as

$$d_C = \begin{bmatrix} 0000 \\ 1100 \\ 1010 \\ 1001 \\ 0110 \\ 0101 \\ 0011 \\ 1111 \end{bmatrix}, \quad d_N = \begin{bmatrix} 000 \\ 110 \\ 101 \\ 011 \end{bmatrix}.$$

An array such as d_C that involves exclusively the control factors is called a *control array*. Similarly, an array such as d_N that involves exclusively the noise factors is called a *noise array.* A cross array, involving all the seven factors, is now given by the direct product of d_C and d_N. In other words, it is obtained by combining every row of d_C with every row of d_N and thus has run size $8 \times 4 = 32$. One can readily check that the cross array is a 2^{7-2} design with the defining relation

$$I = 1234 = 567 = 1234567. \tag{9.2.1}$$

In general, a *cross array* d is the direct product of a control array d_C and a noise array d_N. Typically, as in the above example, d_C and d_N are chosen as orthogonal arrays. This ensures reasonable uniformity over the levels of the noise factors so that running an experiment based on a cross array amounts to performing a systematic Monte Carlo over the noise variation. For practical reasons like small sample size, irregular design region, and large number of factor levels, nonorthogonal arrays like Latin hypercubes or other space-filling designs may sometimes be chosen for d_C and d_N; see Santner, Williams, and Notz (2003) for details.

Suppose there are altogether $n(= n_1 + n_2)$ factors $F_1, \ldots, F_n$, each at s levels, s being a prime or prime power. The first n_1 factors are control factors and the last n_2 are noise factors. Consider an $s^{n_1-k_1}$ design d_C for the control factors and an $s^{n_2-k_2}$ design d_N for the noise factors as given by

$$d_C = \{x^{(1)} : B_1 x^{(1)} = 0\}, \quad d_N = \{x^{(2)} : B_2 x^{(2)} = 0\},$$

where B_1 and B_2 are $k_1 \times n_1$ and $k_2 \times n_2$ matrices, of full row rank, over $GF(s)$. The cross array d, arising from d_C and d_N, consists of the s^{n-k} treatment combinations $x = (x^{(1)'}, x^{(2)'})'$, where $x^{(1)}$ is any solution of $B_1 x^{(1)} = 0$, $x^{(2)}$ is any solution of $B_2 x^{(2)} = 0$, and $k = k_1 + k_2$. Thus d itself is an s^{n-k} design and one can write

$$d = \{x : Bx = 0\},$$

with

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}. \tag{9.2.2}$$

In the above development, d_C or d_N can as well be a full factorial, in which case the corresponding block of rows in (9.2.2) does not arise.

For any $n \times 1$ vector b over $GF(s)$, let $b^{(1)}$ and $b^{(2)}$ be subvectors that correspond to the control and noise factors. Then $b^{(1)}$ and $b^{(2)}$ consist respectively of the first n_1 and the last n_2 entries of b. Denoting the row space of a matrix by $\mathcal{R}(.)$, it is clear from (9.2.2) that $b' \in \mathcal{R}(B)$ if and only if $b^{(i)'} \in \mathcal{R}(B_i)$ for $i = 1, 2$. This is equivalent to the following result.

Theorem 9.2.1. *The vector b' belongs to the defining contrast subgroup of d if and only if $b^{(1)'}$ and $b^{(2)'}$ belong to the defining contrast subgroups of d_C and d_N respectively.*

The above theorem is in agreement with (9.2.1) and, although straightforward, has interesting implications. For instance, it shows that the defining pencils of d are of the form $(b^{(1)'}, 0')'$ or $(0', b^{(2)'})'$ or $(b^{(1)'}, b^{(2)'})'$, where $b^{(1)}$ and $b^{(2)}$ are any defining pencils of d_C and d_N. Thus the resolution of d equals the minimum of the resolutions of d_C and d_N. We have noted earlier that the main effects and the 2fi's involving a control and a noise factor are the factorial effects of greatest interest in parameter design experiments. Hence, as in the previous chapters, d is hereafter stipulated to have resolution three or higher, and to ensure this, it is assumed that the same holds for both d_C and d_N. The next result shows that d enjoys an attractive property.

Theorem 9.2.2. *No 2fi pencil involving a control and a noise factor is aliased in d with any main effect pencil or any other 2fi pencil.*

Proof. Consider any 2fi pencil b involving a control and a noise factor. Let $\widetilde{b}$ be any other pencil with which b is aliased in d. As before, define the subvectors $b^{(1)}$ and $b^{(2)}$ relative to b, and the subvectors $\widetilde{b}^{(1)}$ and $\widetilde{b}^{(2)}$ relative to $\widetilde{b}$. Since b and $\widetilde{b}$ are aliased in d, by (2.4.9), $(b - \widetilde{b})' \in \mathcal{R}(B)$, so that by Theorem 9.2.1,

$$(b^{(1)} - \widetilde{b}^{(1)})' \in \mathcal{R}(B_1), \quad (b^{(2)} - \widetilde{b}^{(2)})' \in \mathcal{R}(B_2). \tag{9.2.3}$$

Because d_C and d_N have resolution three or higher, for $i = 1, 2$, it follows from (9.2.3) that $b^{(i)} - \widetilde{b}^{(i)}$ is either null or has at least three nonzero entries. On the other hand, by the definition of b, there is exactly one nonzero entry in $b^{(i)}$. Thus for $i = 1, 2$, either $\widetilde{b}^{(i)}$ equals $b^{(i)}$ and hence has exactly one nonzero entry or $\widetilde{b}^{(i)}$ has at least two nonzero entries. The first possibility, however, cannot arise for both $i = 1$ and $i = 2$, because then $\widetilde{b} = b$, which is impossible. Therefore, $\widetilde{b}$ must have at least three nonzero entries and the theorem follows. □

It is possible to work out further ramifications and generalizations of Theorem 9.2.2. In fact, this result has counterparts when d_C and d_N are mixed-level designs, nonregular designs, or even nonorthogonal arrays. The details under appropriate model assumptions are available in Shoemaker, Tsui, and Wu (1991).

Theorem 9.2.2 can be restated as *"in a cross array, all 2fi's involving a control and a noise factor are clear."* In view of the importance of such interactions in achieving robustness, this is a very desirable property of cross arrays. There is, however, no guarantee that the equally important control main effects are clear. To ensure further estimability properties for the control main effects, the control array needs to be chosen according to criteria like resolution or minimum aberration (MA). This may lead to a large size of the control array and thus of the cross array. (An economic alternative is to use single arrays to be discussed in later sections.) A similar remark holds on the choice of the noise array, but the required increase in run size may be less prohibitive. This is especially so if the estimation of the noise main effects is not required as in the location–dispersion modeling approach to be discussed in Section 9.3.

Another appealing feature of the cross array format is that it has a natural layout for experimentation. Each setting of its control factors (i.e., level combination in the control array) is *crossed* with all settings of its noise factors (i.e., level combinations in the noise array). This makes it convenient to carry out the experiment, especially when the levels of the noise factors are easier to change than those of the control factors, e.g., in an experiment on plasma etching, where the control factors, plasma temperature and etching time, are applied to all wafers in a batch, while the location of the chips within each wafer represents a noise factor.

9.3 Modeling Strategies

Two approaches for modeling data from a cross array experiment are now discussed. These are called location and dispersion modeling and response modeling. Details can be found in Wu and Hamada (2000, Chapter 10).

A typical observation arising from a cross array d may be denoted by Y_{ij}, which corresponds to the treatment combination given by the ith row of d_C and the jth row of d_N. For every fixed i, let $\overline{Y}_i$ be the mean and S_i the standard deviation of the Y_{ij} over j. Clearly, the $\overline{Y}_i$ and S_i refer to the control factor settings in d_C. The *location and dispersion modeling* approach builds models *separately* for the $\overline{Y}_i$ and the log S_i in terms of the control factor main effects and interactions. Observe that only the control factors can appear in these models. With the objective of attaining a target value for the response Y while minimizing performance variation due to noise factors, these models can be used as follows.

Two-step procedure:

(i) Select the levels of the control factors appearing in the dispersion model to minimize dispersion.
(ii) Select the levels of control factors appearing in the location model but not in the dispersion model to bring the location on target.

(9.3.1)

Note that factors appearing in both models are not considered in step (ii) because a change in the setting of any such factor will affect location as well as dispersion.

Since the location and dispersion models involve the control factors alone, the effect hierarchy principle of Section 2.5, as applied to the control factors, is followed in building these models. Consequently, given d_N, the choice of d_C is guided by the same ideas as for ordinary s^{n-k} designs. In other words, the same criteria and techniques as discussed in Chapters 2–5 can be used in choosing d_C, and therefore this point is not further elaborated.

A disadvantage of the location and dispersion modeling approach is that modeling in terms of the control factors alone may mask important interactions between the control and noise factors. Moreover, log S_i may have a nonlinear relationship with the control factors even if the original response bears a linear relationship with the control and noise factors. From these considerations, the *response modeling* approach (Welch, Yu, Kang, and Sacks, 1990; Shoemaker, Tsui and Wu, 1991) provides a viable alternative. In this approach, the response is modeled, directly on the basis of the Y_{ij}, as a function of both the control and noise factors. One can study the control-by-noise interactions appearing in such a model to identify control factor settings at which the fitted response has a relatively flat relationship with the noise factors. These are called robust settings. Then a robust setting at which the fitted response tends to be close to the target (irrespective of the variation in the noise factors) may be chosen for the control factors. Alternatively, from the response model, one can obtain the variance of the fitted response under suitable assumptions on the variation in the noise factors. This leads to a *transmitted variance model*, which involves only the control factors and hence can be used to find control factor settings with small transmitted variance. From among the settings so identified, the one that brings the expectation of the fitted response on target may be chosen for the control factors.

A response model can potentially involve any factor and not just the control factors. While this is as in the analysis of an ordinary s^{n-k} design, a key difference lies in the relative importance of the factorial effects for inclusion in the model because the control and noise factors do not play the same role. This necessitates a new effect ordering principle that takes care of this distinction and yields new design criteria for cross arrays under the response modeling approach. Since the choice of single arrays introduced in the next section is also guided by the same principle, it makes sense to discuss this in a unified framework. Therefore, a discussion of this principle and its application to the study of optimal designs is postponed till Sections 9.5 and 9.6.

9.4 Single Arrays

The direct product structure of a cross array may sometimes result in an unduly large run size. Suppose, with three control factors and two noise factors, each at two levels, it is desired to keep all main effects clear. It is easily seen that then both d_C and d_N have to be full factorials. Consequently, the resulting cross array is a full factorial with run size $8 \times 4 = 32$. On the other hand, the 2^{5-1} design $I = 12345$, with run size merely 16, has resolution five and hence keeps all main effects as well as 2fi's clear. The latter design, which incorporates the control and noise factors without using a direct product structure, is known as a single array and illustrates how single arrays can significantly reduce the run size without sacrificing factorial effects of interest.

As in the last section, now suppose there are $n(= n_1 + n_2)$ s-level factors. The first n_1 of these are control factors and the rest are noise factors. A *single array* is simply an s^{n-k} design

$$d = \{x : Bx = 0\}, \tag{9.4.1}$$

where B is a $k \times n$ matrix, of full row rank, over $GF(s)$. The distinction between the control and noise factors is, however, a new feature, which must be carefully accounted for in defining design isomorphism or developing design criteria for single arrays. For the same reasons as before, single arrays of resolution three or higher are considered in the sequel.

In particular, if B in (9.4.1) has the block diagonal structure (9.2.2), then a single array reduces to a cross array. Of course, B need not have this form in general, and consequently single arrays are more flexible than cross arrays. The location and dispersion modeling approach of the last section requires that the same noise factor settings appear in conjunction with every setting of the control factors. Hence this approach cannot be employed to a single array unless it is also a cross array. The response modeling approach is, therefore, recommended for experiments based on single arrays. In the same manner as indicated in Section 9.3, it is applicable to any single array irrespective of whether B has the form (9.2.2).

Because single arrays cover cross arrays as a special case and the response modeling approach is applicable to both, the optimal design problem under this approach may conveniently be studied in the unified framework of single arrays with the understanding that an optimal design may well turn out to be a cross array in a specific situation. Before this is taken up in some detail in the next two sections, it will be helpful to examine the notion of isomorphism for single arrays.

With all factors at two levels, two single arrays are *isomorphic* if the defining contrast subgroup of one can be obtained from that of the other by permuting the control factor labels and/or the noise factor labels. Observe that permutation of factor labels is restricted separately to the control and noise factors. This is the same as in the last chapter, where the whole plot and subplot factors were considered separately in defining isomorphism. In a similar manner, for s-level factors, isomorphism of single arrays can be defined following Section 8.3. Clearly, if two ordinary s^{n-k} designs are nonisomorphic when all factors have the same status, then single arrays given by these designs are again nonisomorphic. However, it is noteworthy that the same s^{n-k} design may also entail nonisomorphic single arrays depending on which factors are taken as control factors and which are taken as noise factors. The next example (Wu and Zhu, 2003) illustrates these points.

Example 9.4.1. Suppose there are three control factors F_1, F_2, F_3 and three noise factors F_4, F_5, F_6, each at two levels. All nonisomorphic single arrays are enumerated below for $k = 2$. Here $n = 6$ and Table 3A.2 lists all nonisomorphic

2^{6-2} designs. It is easily seen that the first of these, namely the design 6-2.1, has the defining relation

$$I = 1235 = 1246 = 3456. \tag{9.4.2}$$

From (9.4.2), the mappings

$$1 \to F_1,\ 2 \to F_2,\ 3 \to F_3,\ 4 \to F_4,\ 5 \to F_5,\ 6 \to F_6$$

and

$$1 \to F_1,\ 2 \to F_4,\ 3 \to F_3,\ 4 \to F_2,\ 5 \to F_5,\ 6 \to F_6$$

yield single arrays d_1 and d_2 with the defining relations

$$d_1 : I = 1235 = 1246 = 3456 \tag{9.4.3}$$

and

$$d_2 : I = 1345 = 1246 = 2356. \tag{9.4.4}$$

In (9.4.3) and (9.4.4), the letters $1, \ldots, 6$ correspond to the factors $F_1, \ldots, F_6$ respectively. Although (9.4.3) and (9.4.4) can be obtained from each other by interchanging the letters 2 and 4, the single arrays d_1 and d_2 are not isomorphic because 2, representing the control factor F_2, and 4, representing the noise factor F_4, are not interchangeable. One can check that these are the only nonisomorphic single arrays that arise from (9.4.2).

Similarly, the design 6-2.2 in Table 3A.2 yields the six nonisomorphic single arrays

$$\begin{aligned}
d_3 &: \quad I = 123 = 1456 = 23456,\\
d_4 &: \quad I = 124 = 1356 = 23456,\\
d_5 &: \quad I = 145 = 1236 = 23456,\\
d_6 &: \quad I = 456 = 1234 = 12356,\\
d_7 &: \quad I = 145 = 2346 = 12356,\\
d_8 &: \quad I = 124 = 3456 = 12356,
\end{aligned}$$

while the design 6-2.3 leads to the two nonisomorphic single arrays

$$\begin{aligned}
d_9 &: \quad I = 123 = 456 = 123456,\\
d_{10} &: \quad I = 124 = 356 = 123456.
\end{aligned}$$

Finally, the design 6-2.4 entails six nonisomorphic single arrays:

$$\begin{aligned}
d_{11} &: \quad I = 123 = 145 = 2345,\\
d_{12} &: \quad I = 124 = 135 = 2345,\\
d_{13} &: \quad I = 124 = 156 = 2456,\\
d_{14} &: \quad I = 124 = 456 = 1256,\\
d_{15} &: \quad I = 145 = 246 = 1256,\\
d_{16} &: \quad I = 124 = 345 = 1235.
\end{aligned}$$

Thus there are altogether 16 nonisomorphic single arrays in this example. □

Before concluding this section, we briefly discuss a version of a single array, known as a *compound array*. Following Rosenbaum (1994, 1996) and Hedayat and Stufken (1999), a compound array can be defined generally as a kind of orthogonal array that may not even be a regular fraction. In the present context, this definition boils down to that of a single array d for which the matrix B in (9.4.1) has the form

$$B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}, \tag{9.4.5}$$

where $B_{11},\ B_{21}$, and B_{22} are of orders $k_1 \times n_1,\ k_2 \times n_1$, and $k_2 \times n_2$ respectively, with $k_1 + k_2 = k$, and B_{11} and B_{22} have full row rank. The structure (9.4.5) is formally the same as (8.1.2) for fractional factorial split-plot (FFSP) deigns, and hence by Theorem 8.1.1, the following hold:

(i) the treatment combinations in d involve $s^{n_1-k_1}$ control factor settings;
(ii) each such control factor setting appears in conjunction with $s^{n_2-k_2}$ noise factor settings in d.

Despite the formal similarity between (9.4.5) and (8.1.2), the control and noise factors do not quite play the roles of the whole plot and subplot factors in FFSP designs. For example, there is no two-phase randomization here.

In particular, if $B_{21} = 0$ then a compound array reduces to a cross array. In this case, every control factor setting appears in conjunction with the same set of noise factor settings in d. However, this need not happen in general, since B_{21} may be nonnull. It is easily seen that among the single arrays listed in Example 9.4.1, d_3, d_9, and d_{11} are compound arrays and d_9 is a cross array.

With reference to (9.4.5), suppose the $s^{n_1-k_1}$, $s^{n_2-k_2}$, and s^{n-k} designs given by

$$\{x^{(1)} : B_{11}x^{(1)} = 0\},\ \{x^{(2)} : B_{22}x^{(2)} = 0\},\ \text{and } \{x : Bx = 0\}$$

have resolutions R_1, R_2, and R respectively. Let $g_1 = R_1 - 1$, $g_2 = R_2 - 1$, and $g = R - 1$. Then by Theorem 2.6.2, the $s^{n_1-k_1}$ control factor settings in (i) above, when written as rows, form an orthogonal array of strength g_1. Similarly, each set of $s^{n_2-k_2}$ noise factor settings, which appear with the same control factor setting as indicated in (ii) above, represents an orthogonal array of strength g_2. Furthermore, the compound array itself forms an orthogonal array of strength g. Although the strengths g_1, g_2, and g are important combinatorial features of a compound array, they do not completely determine the design characteristics that are relevant to the present context. For example, both the compound arrays d_3 and d_9 arising in Example 9.4.1 have $(g_1, g_2, g) = (2, 2, 2)$, but as seen in Section 9.6 (see (9.6.3)), they do not behave identically with regard to clear effects of interest. For this reason, from now on, single arrays are considered in general without any specific attention to compound arrays.

9.5 Effect Ordering Principle

The twin objectives of bringing the mean response on target and reducing variation dictate the effect ordering principle in robust parameter design. A generic notation is followed to denote any control factor main effect by C, any noise factor main effect by N, any 2fi involving a control and a noise factor by CN, and so on.

Among the factorial effects, C is the most crucial one for mean response adjustment, whereas CN is expected to have the maximum impact on variation reduction. If one works with a response model, then N can also play a role as important as CN in variation reduction. Besides, being a main effect, N is potentially substantial and its aliasing with other effects of interest can severely impair the utility of a design. From these considerations, C, N, and CN are viewed as the most important factorial effects. In contrast with the effect hierarchy principle, any 2fi CN now enjoys the same status as the main effects. In order of importance, next come CC and CCN, and then come $CCNN$, CNN, and NN. The impact of CC on mean response adjustment is supposed to be on a par with that of CCN in variation reduction, and hence they are grouped together. On the other hand, CCN involves only one noise factor, and therefore it is perceived as more important than $CCNN$, CNN, and NN with regard to variation reduction.

In general, following Wu and Zhu (2003), it is possible to develop a numerical rule for ranking the factorial effects in order of importance. The weight of any effect involving i control factors and j noise factors is defined as

$$W(i,j) = \begin{cases} 1 & \text{if } \max(i,j) = 1, \\ i & \text{if } i > j \text{ and } i \geq 2, \\ j + \frac{1}{2} & \text{if } i \leq j \text{ and } j \geq 2. \end{cases} \tag{9.5.1}$$

For $w = 1, 2, 2.5, \ldots$, let K_w be the set of factorial effects with weight w. Then

$$K_1 = \{C, N, CN\},\ K_2 = \{CC, CCN\},\ K_{2.5} = \{CCNN, CNN, NN\}, \tag{9.5.2}$$

and so on. The above discussion may now be summarized by the following *effect ordering principle*:

(i) Factorial effects with smaller weights are considered more important than those with larger weights.
(ii) Factorial effects with the same weight are considered equally important.

Although the effect ordering principle is based on special consideration for parameter design, it should be used with some discretion. For example, there may be practical situations in which adequate prior knowledge is available about the relative importance of the factorial effects so as to warrant a modification of this principle. Indeed, Bingham and Sitter (2003) proposed an alternative effect ordering scheme via a different argument and weight-assignment. At any rate, even with such other effect ordering, the development of design criteria should essentially follow the lines of the next section.

9.6 Design Criteria

The effect ordering principle leads to design criteria appropriate for single arrays, and hence for cross arrays, under the response modeling approach. In order to give a flavor of the main ideas while avoiding heavy notation and algebra, we consider two-level factorials and assume the absence of interactions involving three or more factors. Then by (9.5.1) and (9.5.2),

$$K_1 = \{C, N, CN\}, \quad K_2 = \{CC\}, \quad K_{2.5} = \{NN\}, \tag{9.6.1}$$

and K_w is empty for $w > 2.5$. The following six types of aliasing patterns arise from (9.6.1):

$$\begin{array}{llll} \text{(I)}\ \ 1 \sim 1, & \text{(II)}\ \ 1 \sim 2, & \text{(III)}\ \ 1 \sim 2.5, \\ \text{(IV)}\ \ 2 \sim 2, & \text{(V)}\ \ 2 \sim 2.5, & \text{(VI)}\ \ 2.5 \sim 2.5. \end{array}$$

Here aliasing of two different effects from K_1 is denoted by $1 \sim 1$, aliasing of an effect from K_1 with an effect from K_2 is denoted by $1 \sim 2$, and so on. In keeping with the effect ordering principle, (I) is considered most severe, (II) less severe than (I), and so forth, with (VI) being the least severe. For any single array d, defining $J_1(d), \ldots, J_6(d)$ as the numbers of aliased pairs of types (I)–(VI) respectively, the following design criterion, due to Wu and Zhu (2003), now emerges naturally.

Definition 9.6.1. *Let d_1 and d_2 be two single arrays for the same n_1, n_2, and k.*

(a) If $J_i(d_1) = J_i(d_2)$, $1 \le i \le 6$, then d_1 and d_2 are said to be J-equivalent.
(b) Otherwise, let r be the smallest integer such that $J_r(d_1) \neq J_r(d_2)$. If $J_r(d_1) < J_r(d_2)$, then d_1 is said to have less J-aberration than d_2.
(c) If no other single array has less J-aberration than d_1, then d_1 is said to have minimum J-aberration.

Application of the above criterion calls for expressing the $J_i = J_i(d)$ $(1 \le i \le 6)$ in terms of the defining relation of d. For $i \ge 0,\ j \ge 0$, and $(i, j) \neq (0, 0)$, let $A_{ij} = A_{ij}(d)$ be the number of words in the defining relation of d that involve i control factors and j noise factors. Note that $A_{ij} = 0$ for $i + j \le 2$, because single arrays of resolution three or higher are being considered. Then

$$\begin{array}{ll} J_1 = 2A_{21} + 2A_{12} + 2A_{22}, & J_2 = A_{21} + 3A_{30} + 3A_{31}, \\ J_3 = A_{12} + 3A_{03} + 3A_{13}, & J_4 = 3A_{40},\ J_5 = A_{22},\ J_6 = 3A_{04}. \end{array} \tag{9.6.2}$$

Since no two distinct main effects are aliased in d, by (9.6.1), any aliased pair of type (I) must be either (C, CN) or (N, CN) or (CN, CN). The equation for J_1 in (9.6.2) now follows by noting that a word in the defining relation of d accounts for

(i) two aliased pairs (C, CN) if it is of the form CCN,

(ii) two aliased pairs (N, CN) if it is of the form CNN,
(iii) two aliased pairs (CN, CN) if it is of the form $CCNN$.

Other equations in (9.6.2) follow similarly and are left as an exercise. Example 9.6.1 below illustrates the use of these equations in applying the minimum J-aberration criterion.

Practical considerations in specific situations may warrant minor rearrangement of the aliasing types (I)–(VI) with regard to their severity. For instance, one may wish to interchange the ordering of (III) and (IV). This kind of rearrangement will, in turn, necessitate a corresponding change in the ordering of $J_1, \ldots, J_6$ in Definition 9.6.1. Nevertheless, (9.6.2) will remain valid, and even with any such modified criterion, optimal single arrays may be explored along the lines of the following example.

Example 9.6.1. Consider the setup of Example 9.4.1 with three control factors and three noise factors, each at two levels, and $k = 2$. Among the 16 nonisomorphic single arrays listed there, only d_3, d_6, and d_9 have $A_{21} = A_{12} = A_{22} = 0$. Hence by (9.6.2), J_1 equals zero for these arrays and is positive for the rest. All the three arrays d_3, d_6, and d_9 have $A_{40} = A_{04} = 0$, and their A_{30}, A_{03}, A_{31}, and A_{13} values are as follows:

$$\begin{aligned} d_3:&\quad A_{30}=1,\quad A_{03}=0,\quad A_{31}=0,\quad A_{13}=1,\\ d_6:&\quad A_{30}=0,\quad A_{03}=1,\quad A_{31}=1,\quad A_{13}=0,\\ d_9:&\quad A_{30}=1,\quad A_{03}=1,\quad A_{31}=0,\quad A_{13}=0. \end{aligned}$$

Therefore, by (9.6.2), the vector $(J_1, \ldots, J_6)$ equals $(0, 3, 3, 0, 0, 0)$ for each of them. Thus d_3, d_6, and d_9 are J-equivalent and they all have minimum J-aberration. Recall that d_6 is actually a cross array. Interestingly, the minimum J-aberration criterion eliminates the single arrays d_1 and d_2, which when viewed as ordinary 2^{6-2} designs, have MA. □

The single arrays d_3, d_6, and d_9, which were seen to have minimum J-aberration in the last example, enjoy another attractive property. It can be checked that among all nonisomorphic single arrays, they alone maximize $cl(C) + cl(N) + cl(CN)$, where for any such array, $cl(C)$, $cl(N)$, and $cl(CN)$ are the numbers of clear effects of types C, N, and CN respectively. In fact, the values of $cl(C)$, $cl(N)$, and $cl(CN)$ for d_3, d_6, and d_9 are as follows:

$$\begin{aligned} d_3:&\quad cl(C)=0,\quad cl(N)=3,\quad cl(CN)=6,\\ d_6:&\quad cl(C)=3,\quad cl(N)=0,\quad cl(CN)=6,\\ d_9:&\quad cl(C)=0,\quad cl(N)=0,\quad cl(CN)=9. \end{aligned} \tag{9.6.3}$$

While they all have the same $cl(C) + cl(N) + cl(CN)$, the above details may be useful in further discrimination among them when additional knowledge about the system is available. For instance, in many practical situations, it is unlikely that all 2fi's of type CN are important. If the six clear 2fi's of this type in d_6 can be chosen to represent the important ones, then d_6 has an edge over d_9. If, in addition, one can reasonably assume the absence of 2fi's of type NN, then the noise main effects also become clear in d_6 and its advantage over d_9 and d_3 becomes more pronounced.

Tables of single arrays that perform well with respect to the J-criterion and clear effects are available in Wu and Zhu (2003) and Wu and Hamada (2000, Chapter 10). These tables are obtained via enumeration of nonisomorphic single arrays as demonstrated in Examples 9.4.1 and 9.6.1. It is not hard to extend the J-criterion and the equations in (9.6.2) to the situation in which only interactions involving four or more factors are assumed to be absent. Although this can be done from first principles, certain general formulas given by Wu and Zhu (2003) may help.

As in the previous chapters, it is possible to develop a theory, based on complementary sets, for single arrays. For this purpose, one needs to consider, along the lines of Theorem 2.7.1, a finite projective geometric formulation in which the control and noise factors are identified with two disjoint sets of points of the geometry. Let T_C and T_N be these two sets and $\overline{T}$ be the complement of their union. Formulas for the A_{ij} in terms of either T_C and $\overline{T}$ or T_N and $\overline{T}$ can be derived. These results are useful when $\overline{T}$ has a relatively small size. The details in this regard are available in Zhu (2003) and Zhu and Wu (2006).

Exercises

9.1 Develop a stronger version of Theorem 9.2.2 when both d_C and d_N are known to have resolution four or higher.

9.2 Suppose d_C and d_N are $2^{n_1-k_1}$ and $2^{n_2-k_2}$ designs respectively in the control and noise factors. Let H_C be a set of factorial effects that are estimable in d_C under the absence of all other factorial effects. Similarly, define H_N with respect to d_N. Write H_{CN} for the collection of factorial effects of the form $E_C E_N$, where $E_C \in H_C$ and $E_N \in H_N$. For example, if $n_1 = n_2 = 3,\ H_C = \{F_1, F_2, F_3\}$, and $H_N = \{F_4, F_4F_6\}$, then

$$H_{CN} = \{F_1F_4, F_2F_4, F_3F_4, F_1F_4F_6, F_2F_4F_6, F_3F_4F_6\}.$$

Let d be the cross array obtained from d_C and d_N. Show that all factorial effects in $H_C \cup H_N \cup H_{CN}$ are estimable in d under the absence of all other factorial effects.

9.3 Verify (9.6.3).

9.4 In Example 9.6.1, show that among all nonisomorphic single arrays, d_3, d_6, and d_9 alone maximize $cl(C) + cl(N) + cl(CN)$.

9.5 (a) With four control factors and two noise factors, each at two levels, and $k = 2$, use Table 3A.2 to enumerate all nonisomorphic single arrays.
(b) From among the single arrays obtained in (a), find the one or ones with minimum J-aberration.

9.6 Verify the last five equations in (9.6.2).

9.7 Describe the sets $K_w (w = 1, 2, 2.5, \ldots)$ assuming only the absence of interactions involving four or more factors. Under the same assumption, obtain counterparts of the equations in (9.6.2) for two-level factorials.

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Author Index

Bingham, D.R. 182, 185, 186, 191, 206
Block, R.M. 67
Bose, R.C. 4, 11, 18, 40, 45
Box, G.E.P. 3, 6, 23, 32, 182
Brouwer, A.E. 46, 52
Brownlee, K.A. 53
Burman, J.P. 6
Bush, K.A. 38

Chan, L.Y. 149
Chen, D. 182
Chen, H. 63, 66, 91, 101, 117, 158, 164, 165
Chen, J. 44, 54, 57, 66, 67, 68, 102, 165
Chen, Y. 64, 152, 153, 162, 166
Cheng, C.S. 38, 39, 109, 120, 158, 164, 165
Cheng, S.W. 7, 165, 166

Deng, L.Y. 7
Dey, A. 4, 23, 36, 38, 39
Draper, N.R. 48, 52

Fang, K.T. 8, 149, 186, 189, 191
Feder M. 165
Federer, W.T. 4
Franklin, M.F. 3
Fries, A. 3, 34

Hamada, M. 1, 3, 6, 7, 50, 64, 66, 87, 130, 198, 201, 209
Hedayat, A.S. 4, 36, 38, 63, 66, 117, 205
Huang, P. 182
Hunter, J.S. 3, 6, 23, 32
Hunter, W.G. 3, 6, 34

John, J.A. 4
Jones, S. 182

Kang, S.M. 202
Kelly, B.K. 53
Kempthorne, O. 185
Kiefer, J.C. 38
Kurkjian, B. 12

Li, R. 8
Lin, D.K.J. 7, 8, 52
Loraine, P.K. 53

MacWilliams, F.J. 45, 47
Marshall, A. 110, 111
Mee, R.W. 67
Mitchell, T. 48
Montgomery, D. 6
Mukerjee, R. 4, 23, 36, 38, 39, 109, 120, 134, 138, 140, 147, 149, 157, 158, 160, 162, 163, 165, 186, 189, 191

National Bureau of Standards 3
Nguyen, N.K. 7
Notz, W. 8, 199

Olkin, I. 110, 111

Plackett, R.L. 6
Pless, V. 45, 47

Raghavarao, D. 18, 39

Raktoe, B.L. 4
Rao, C.R. 13, 35
Robillard, P. 53
Rosenbaum, P.R. 205

Sacks, J. 202
Santner, T. 8, 199
Shah, B.V. 12
Shao Q. 149
Shoemaker, A.C. 200, 202
Sitter, R.R. 165, 182, 185, 186, 191, 206
Sloane, N.J.A. 4, 36, 38, 45
Steinberg, D.M. 109, 120
Stufken, J. 4, 36, 38, 205
Sudjianto, A. 8
Suen, C. 91, 101
Sun, D.X. 67, 102, 109, 120, 152, 153, 162, 166

Taguchi, G. 197
Tang, B. 7, 59, 61, 165
Tsui, K.L. 200, 202

van der Waerden, B.L. 18
van Lint, J.H. 45

Verhoeff, T. 46, 52
Voelkel, J.O. 182

Wang, R. 135, 145
Welch, W.J. 202
Williams, B. 8, 199
Williams, E.R. 4
Wu, C.F.J. 1, 3, 6, 7, 8, 44, 50, 54, 57, 59, 61, 64, 65, 66, 67, 87, 91, 101, 102, 127, 128, 130, 134, 135, 138, 140, 145, 147, 152, 153, 157, 158, 160, 162, 163, 165, 166, 198, 200, 201, 202, 203, 206, 207, 209
Wu, H. 65, 66

Xu, H. 7, 8, 102

Yamada, S. 7
Ye, K.Q. 7
Yu, T.K. 202

Zelen, M. 12
Zhang, R. 128, 130, 135, 145, 149
Zhu, Y. 203, 206, 207, 209

Subject Index

$4^p \times 2^n$ designs, 125
Alias set, 27
Aliases, 26
Aliasing, 155
Analysis of variance, 86
A-optimality, 38
Asymmetrical factorial, 9

Block defining contrast subgroup, 152
Block designs, 154
  admissible, 162, 166
  fractional factorials, 154
  full factorials, 151
  order of estimability, 153

Clear effects, 64, 200
  robust designs, 200
  three-level, 102
  two-level, 64
Code
  dual, 46
  linear, 45
  MacWilliams identities, 47
  minimum distance, 45
  Pless power moment identities, 47
  weight distribution, 45
Codeword, 34, 45
Compact notation, 23, 39
Complementary design, 58, 92, 164
Complementary set, 58, 92, 113, 189
Compound array, 205
Conditional main effect, 50
Confounded, 29, 152, 157
Control-by-noise interaction, 198, 202
Control factors, 197
Cross arrays, 199

Designs
  $4^p \times 2^n$, 125
  three-level, 85
Defining contrast subgroup, 25
Defining pencil, 25, 137, 145, 155, 158
Defining relation, 26
D-optimality, 38
Dual code, 46

Effect hierarchy principle, 34, 162, 198
  block designs, 166
Effect ordering principle, 206
Effects
  clear, 64
  strongly clear, 64
E-optimality, 38
Estimable, 31
Estimation capacity, 109

Factorials
  $(s^r) \times s^n$, 133
  $(s^{r_1}) \times (s^{r_2}) \times s^n$, 143
Factorial designs, 1
Factorial effects
  representation, 18
Fractional factorial designs, 2
  split-plot, 181
  three-level, 85
  two-level, 49
Fractional factorial split-plot designs, 182, 205

optimal, 189
Finite Projective geometry, 39
flat, 100
Full factorial designs, 1

Galois field, 18
Generalized minimum aberration criterion, 7
Graeco-Latin square, 86

Hamming distance, 45

Indicator functions, 7
Interaction, 10, 11
control-by-noise, 198, 202
Isomorphic designs
2^{n-k}, 52, 59
blocking, 162
single arrays, 203
s-level fractional factorial split-plot, 189
s^{n-k}, 98
$(s^r) \times s^n$, 137
$(s^{r_1}) \times (s^{r_2}) \times s^n$, 145
two-level fractional factorial split-plot, 184

J-equivalent single arrays, 207

Krawtchouk polynomials, 47
Kronecker product, 13

Latin hypercube designs, 8
Linear code, 45
Location and dispersion modeling, 201

MacWilliams identities, 47, 93
MaxC2 criterion, 64, 102
MaxC2 designs, 64
Main effect, 10, 11, 49
Maximum blocking, 160
Maximum estimation capacity, 110
2^{n-k} designs, 117
s^{n-k} designs, 122
Maximum resolution criterion, 33, 52
Method of replacement, 127
Minimum aberration block designs
full factorials, 153
fractional factorials, 164, 165
Minimum aberration criterion, 3, 184, 200
supplement to, 185
type 0, 129
type 1, 150
Minimum aberration designs, 34
2^{n-k}, 52
$4^p \times 2^n$, 127
s^{n-k}, 34, 87, 98
$(s^r) \times s^n$, 139
$(s^{r_1}) \times (s^{r_2}) \times s^n$, 146
type 0, 129, 139, 146
weak, 63
Minimum G_2-aberration criterion, 7
Minimum J-aberration criterion, 207
Minimum moment aberration criterion, 7
Minimum overall aberration criterion, 142
Minimum overall aberration designs, 142

Nearly saturated designs, 92
Noise array, 199
Noise factors, 197
Nonregular designs, 6

Optimal single arrays, 208
Order of estimability, 153
Orthogonal arrays, 35, 125
asymmetrical, 125
index, 35
mixed-level, 125
Rao's bound, 36
saturated, 36
symmetrical, 125

Parallel pencil of flats, 19
Parameter design, 197
Pencils, 19, 136, 144
confounded, 29
defining, 25
distinct, 21
estimable, 31
subplot type, 185
type 0, 137
type 1, 137
whole plot type, 185
Pless power moment identities, 47

Rao's bound, 36
Regular designs, 6
Regular fractions, 24
  $(s^r) \times s^n$ factorial, 135
  $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, 144
Resolution, 32, 37, 40, 44, 64, 88
  $4^p \times 2^n$ designs, 128, 129
  block designs, 157
  split-plot designs, 184
  two-level designs, 52
Response modeling, 202
Restricted randomization, 181
Robust parameter design, 197
Robust settings, 202
Run, 24
Run size, 24

Saturated designs, 92
Schur concave function, 110
Single arrays, 203
  J-equivalent, 207
  optimal, 208
Space-filling designs, 8
Split-plot designs, 181
Strongly clear effects, 64
Subplot factors, 181, 184
Supersaturated designs, 7
Symmetrical factorial, 9
s^{n-k} designs, 24

Three-level designs, 85
Transmitted variance model, 202
Treatment contrasts, 10
  belong to factorial effects, 11, 14
  orthogonality, 15
Treatment effect, 9
Two-factor interaction (2fi), 50
Two-phase randomization, 182
Two-step procedure, 201

Unconfounded, 152, 157
Uniform designs, 8
Universal optimality, 38

Variation reduction, 197

Weight distribution, 45
Whole plot factors, 181, 184
Word, 34
  length of, 34
  type 0, 128
  type 1, 128
  type 2, 129
Wordlength, 52
Wordlength pattern, 34, 46, 87, 152