CONSTANTINE M. DAFERMOS

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# HYPERBOLIC CONSERVATION LAWS IN CONTINUUM PHYSICS

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Constantine M. Dafermos

# Hyberbolic Conservation Laws in Continuum Physics

Second Edition

With 38 Figures

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For Mihalis and Thalia

# Preface to the Second Edition

It is a testament to the vitality of the field of conservation laws that five years after the original publication of this work it has become necessary to prepare a considerably revised and expanded second edition. A new chapter has been added, recounting the exciting recent developments on the vanishing viscosity method; and numerous new sections have been incorporated in preexisting chapters, to introduce newly derived results or present older material, omitted in the first edition, whose relevance and importance has been underscored by current trends in research. This includes recent work by the author, which has not been published elsewhere. In addition, a substantial portion of the original text has been revamped so as to streamline the exposition, enrich the collection of examples, and improve the notation. The introduction has been revised to reflect these changes. The bibliography has been updated and expanded as well, now comprising over one thousand titles.

Twenty-five years ago, it might have been feasible to compose a treatise surveying the entire area; however, the explosive development of the subject over the past three decades has rendered such a goal unattainable. Thus, even though this work has encyclopedic ambitions, striving to present a panoramic view of the terrain, certain noteworthy features have been sketched very roughly or have been passed over altogether. Fortunately, a number of textbooks and specialized monographs treating some of these subjects in depth are now available. However, additional focused surveys are needed in order to compile a detailed map of the entire field.

# Acknowledgments

My mentors, Jerry Ericksen and Clifford Truesdell, initiated me to continuum physics, as living scientific subject and as formal mathematical structure with fascinating history. I trust that both views are somehow reflected in this work.

I am grateful to many scientists—teachers, colleagues, and students alike—who have helped me, over the past forty years, to learn continuum physics and the theory of hyperbolic conservation laws. Since it would be impossible to list them all here by name, let me single out Stu Antman, John Ball, Alberto Bressan, Gui-Qiang Chen, Bernie Coleman, Ron DiPerna, Jim Glimm, Jim Greenberg, Mort Gurtin, Ling Hsiao, Barbara Keyfitz, Peter Lax, Philippe LeFloch, Tai-Ping Liu, Andy Majda, Pierangelo Marcati, Ingo Müller, Walter Noll, Jim Serrin, Denis Serre, Marshall Slemrod, Joel Smoller, Luc Tartar, Konstantina Trivisa, Thanos Tzavaras, and Zhouping Xin, who have also honored me with their friendship. In particular, Denis Serre's persistent encouragement helped me to carry this arduous project to completion.

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# Introduction

The seeds of continuum physics were planted with the works of the natural philosophers of the eighteenth century, most notably Euler; by the mid-nineteenth century, the trees were fully grown and ready to yield fruit. It was in this environment that the study of gas dynamics gave birth to the theory of quasilinear hyperbolic systems in divergence form, commonly called "hyperbolic conservation laws"; and these two subjects have been traveling hand in hand over the past one hundred and fifty years. This book aims at presenting the theory of hyperbolic conservation laws from the standpoint of its genetic relation to continuum physics. Even though research is still marching at a brisk pace, both fields have attained by now the degree of maturity that would warrant the writing of such an exposition.

In the realm of continuum physics, material bodies are realized as continuous media, and so-called "extensive quantities", such as mass, momentum and energy, are monitored through the fields of their densities, which are related by balance laws and constitutive equations. A self-contained, though skeletal, introduction to this branch of classical physics is presented in Chapter II. The reader may flesh it out with the help of a specialized text on the subject.

In its primal formulation, the typical balance law stipulates that the time rate of change in the amount of an extensive quantity stored inside any subdomain of the body is balanced by the rate of flux of this quantity through the boundary of the subdomain together with the rate of its production inside the subdomain. In the absence of production, a balanced extensive quantity is conserved. The special feature that renders continuum physics amenable to analytical treatment is that, under quite natural assumptions, statements of gross balance, as above, reduce to field equations, i.e., partial differential equations in divergence form.

The collection of balance laws in force demarcates and identifies particular continuum theories, such as mechanics, thermomechanics, electrodynamics, and so on. In the context of a continuum theory, constitutive equations encode the material properties of the medium, for example heat-conducting viscous fluid, elastic solid, elastic dielectric, etc. The coupling of these constitutive relations with the field equations gives birth to closed systems of partial differential equations, dubbed "balance laws" or "conservation laws", from which the equilibrium state or motion of the continuous medium is to be determined. Historically, the vast majority of noteworthy partial differential equations were generated through that process. The central thesis of this book is that the umbilical cord joining continuum physics with the theory of partial differential equations should not be severed, as it is still carrying nourishment in both directions.

Systems of balance laws may be elliptic, typically in statics; hyperbolic, in dynamics, for media with "elastic" response; mixed elliptic-hyperbolic, in statics or dynamics, when the medium undergoes phase transitions; parabolic or mixed parabolic-hyperbolic, in the presence of viscosity, heat conductivity or other diffusive mechanisms. Accordingly, the basic notions shall be introduced, in Chapter I, at a level of generality that would encompass all of the above possibilities. Nevertheless, since the subject of this work is hyperbolic conservation laws, the discussion will eventually focus on such systems, beginning with Chapter III.

Solutions to hyperbolic conservation laws may be visualized as propagating waves. When the system is nonlinear, the profiles of compression waves get progressively steeper and eventually break, generating jump discontinuities which propagate on as shocks. Hence, inevitably, the theory has to deal with weak solutions. This difficulty is compounded further by the fact that, in the context of weak solutions, uniqueness is lost. It thus becomes necessary to devise proper criteria for singling out admissible weak solutions. Continuum physics naturally induces such admissibility criteria through the Second Law of thermodynamics. These may be incorporated in the analytical theory, either directly, by stipulating outright that admissible solutions should satisfy "entropy" inequalities, or indirectly, by equipping the system with a minute amount of diffusion, which has negligible effect on smooth solutions but reacts stiffly in the presence of shocks, weeding out those that are not thermodynamically admissible. The notions of "entropy" and "vanishing diffusion", which will play a central role throughout the book, are first introduced in Chapters III and IV.

From the standpoint of analysis, a very elegant, definitive theory is available for the case of scalar conservation laws, in one or several space dimensions, which is presented in detail in Chapter VI. By contrast, systems of conservation laws in several space dimensions are still terra incognita, as the analysis is currently facing insurmountable obstacles. The limited results derived thus far, pertaining to local existence and stability of smooth or piecewise smooth solutions, underscore the importance of the special structure of the field equations of continuum physics and the stabilizing role of the Second Law of thermodynamics. These issues are discussed in Chapter V.

Beginning with Chapter VII, the focus of the investigation is fixed on systems of conservation laws in one-space dimension. In that setting, the theory has a number of special features that are of great help to the analyst, so major progress has been achieved.

Chapter VIII provides a systematic exposition of the properties of shocks. In particular, various shock admissibility criteria are introduced, compared and contrasted. Admissible shocks are then combined, in Chapter IX, with another class of particular solutions, called centered rarefaction waves, to synthesize wave fans that solve the classical Riemann problem. Solutions of the Riemann problem may in turn be employed as building blocks for constructing solutions to the Cauchy problem, in the class $BV$ of functions of bounded variation. Two construction methods based on this approach will be presented here: The random choice scheme, in Chapter XIII, and a front tracking algorithm, in Chapter XIV. Uniqueness and stability of these solutions will also be established.

Chapter XV outlines an alternative construction of $BV$ solutions to the Cauchy problem, for general strictly hyperbolic systems of conservation laws, by the method of vanishing viscosity.

The above construction methods generally apply when the initial data have sufficiently small total variation. This restriction seems to be generally necessary because, in certain systems, when the initial data are "large" even weak solutions to the Cauchy problem may blow up in finite time. Whether such catastrophes may occur to solutions of the field equations of continuum physics is at present a major open problem. For a limited class of systems, which however contains several important representatives, solutions with large initial data can be constructed by means of the functional analytic method of compensated compactness. This approach, which rests on the notions of measure-valued solution and the Young measure, will be outlined in Chapter XVI.

There are other interesting properties of weak solutions, beyond existence and uniqueness. In Chapter X, the notion of characteristic is extended from classical to weak solutions and is employed for obtaining a very precise description of regularity and long time behavior of solutions to scalar conservation laws, in Chapter XI, as well as to systems of two conservation laws, in Chapter XII.

In order to highlight the fundamental ideas, the discussion proceeds from the general to the particular, notwithstanding the clear pedagogical merits of the reverse course. Even so, under proper guidance, the book may also serve as a text. With that in mind, the pace of the proofs is purposely uneven: slow for the basic, elementary propositions that may provide material for an introductory course; faster for the more advanced technical results that are addressed to the experienced analyst. Even though the various parts of this work fit together to form an integral entity, readers may select a number of independent itineraries through the book. Thus, those principally interested in the conceptual foundations of the theory of hyperbolic conservation laws, in connection to continuum physics, need go through Chapters I-V only. Chapter VI, on the scalar conservation law, may be read virtually independently of the rest. Students intending to study solutions as compositions of interacting elementary waves may begin with Chapters VII-IX and then either continue on to Chapters X-XII or else pass directly to Chapter XIII and/or Chapter XIV. Similarly, Chapter XV relies solely on Chapters VII and VIII. Finally, only Chapter VII is needed as a prerequisite for the functional analytic approach expounded in Chapter XVI.

Certain topics are perhaps discussed in excessive detail, as they are of special interest to the author; and a number of results are published here for the first time. On the other hand, several important aspects of the theory are barely touched upon, or are only sketched very briefly. They include the classical theory of transonic flow in gas dynamics, which is currently undergoing a major revival, the newly developed stability theory of multi-space dimensional shocks and boundary conditions, the derivation of the balance laws of continuum physics from the kinetic theory of gases, and the study of phase transitions. Each one of these areas would warrant the writing of a specialized monograph. The most conspicuous absence is a discussion of numerics, which, beyond its practical applications, also provides valuable insight to the theory. Fortunately, a number of texts on the numerical analysis of hyperbolic conservation laws have recently appeared and may fill this gap.

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# I

# Balance Laws

The ambient space for the balance law will be $\mathbb{R}^k$, with typical point $X$. In the applications to continuum physics, $\mathbb{R}^k$ will stand for physical space, of dimension one, two or three, in the context of statics; and for space-time, of dimension two, three or four, in the context of dynamics.

The generic balance law will be introduced through its primal formulation, as a postulate that the production of an extensive quantity in any domain is balanced by a flux through the boundary; it will then be reduced to a field equation. It is this reduction that renders continuum physics mathematically tractable. It will be shown that the divergence form of the field equation is preserved under change of coordinates, and that the balance law, in its original form, may be retrieved from the field equation.

The field equations for a system of balance laws will be combined with constitutive equations, relating the flux and production density with a state vector, to obtain a closed quasilinear first order system of partial differential equations in divergence form.

It will be shown that symmetrizable systems of balance laws are endowed with companion balance laws which are automatically satisfied by smooth solutions, though not necessarily by weak solutions. The issue of admissibility of weak solutions will be raised.

Solutions will be considered with shock fronts or weak fronts, in which the state vector field or its derivatives experience jump discontinuities across a manifold of codimension one.

The theory of $BV$ functions, which provide the natural setting for solutions with shock fronts, will be surveyed and the geometric structure of $BV$ solutions will be described.

Highly oscillatory weak solutions will be constructed, and a first indication of the stabilizing role of admissibility conditions will be presented.

The setting being Euclidean space, it will be expedient to employ matrix notation at the expense of obscuring the tensorial nature of the fields. The symbol $\mathbb{M}^{r \times s}$ will denote throughout the vector space of $r \times s$ matrices and $\mathbb{R}^r$ shall be identified with $\mathbb{M}^{r \times 1}$. Other standard notation to be used here includes $S^{r-1}$ for the unit sphere in

$\mathbb{R}^r$ and $\mathcal{B}_\rho(X)$ for the ball of radius $\rho$ centered at $X$. In particular, $\mathcal{B}_\rho$ will stand for $\mathcal{B}_\rho(0)$.

## 1.1 Formulation of the Balance Law

Let $\mathcal{X}$ be an open subset of $\mathbb{R}^k$. A *proper domain* in $\mathcal{X}$ is any open bounded subset of $\mathcal{X}$, with Lipschitz boundary. A balance law on $\mathcal{X}$ postulates that the *production* of a (scalar) "extensive" quantity in any proper domain $\mathcal{D}$ is balanced by the *flux* of this quantity through the boundary $\partial\mathcal{D}$ of $\mathcal{D}$.

The salient feature of an extensive quantity is that both its production and its flux are additive over disjoint subsets. Thus, the production in the proper domain $\mathcal{D}$ is given by the value $\mathcal{P}(\mathcal{D})$ of a (signed) Radon measure $\mathcal{P}$ on $\mathcal{X}$. Similarly, with every proper domain $\mathcal{D}$ is associated a countably additive set function $\mathcal{Q}_\mathcal{D}$, defined on Borel subsets of $\partial\mathcal{D}$, such that the flux in or out of $\mathcal{D}$ through any Borel subset $\mathcal{C}$ of $\partial\mathcal{D}$ is given by $\mathcal{Q}_\mathcal{D}(\mathcal{C})$. Hence, the balance law simply states

$$\mathcal{Q}_\mathcal{D}(\partial\mathcal{D}) = \mathcal{P}(\mathcal{D}), \tag{1.1.1}$$

for every proper domain $\mathcal{D}$ in $\mathcal{X}$.

It will be assumed throughout that the set function $\mathcal{Q}_\mathcal{D}$ is absolutely continuous with respect to the $(k-1)$-dimensional Hausdorff measure $\mathcal{H}^{k-1}$, and hence with any proper domain $\mathcal{D}$ in $\mathcal{X}$ is associated a *density flux function* $q_\mathcal{D} \in L^1(\partial\mathcal{D})$ such that

$$\mathcal{Q}_\mathcal{D}(\mathcal{C}) = \int_\mathcal{C} q_\mathcal{D}(X) d\mathcal{H}^{k-1}(X), \tag{1.1.2}$$

for every Borel subset $\mathcal{C}$ of $\partial\mathcal{D}$.

Borel subsets $\mathcal{C}$ of $\partial\mathcal{D}$ are oriented by means of the exterior unit normal $N$ to $\mathcal{D}$, at points of $\mathcal{C}$. The fundamental postulate in the theory of balance laws is that the flux depends solely on the surface and its orientation, i.e., if $\mathcal{C}$ is at the same time a Borel subset of the boundaries of two distinct proper domains $\mathcal{D}_1$ and $\mathcal{D}_2$, sharing the same exterior normal on $\mathcal{C}$, then $\mathcal{Q}_{\mathcal{D}_1}(\mathcal{C}) = \mathcal{Q}_{\mathcal{D}_2}(\mathcal{C})$, and thereby $q_{\mathcal{D}_1}(X) = q_{\mathcal{D}_2}(X)$, for almost all (with respect to $\mathcal{H}^{k-1}$) $X \in \mathcal{C}$.

## 1.2 Reduction to Field Equations

At first glance, the notion of a balance law, as introduced in Section 1.1, appears too general to be of any use. It turns out, however, that the balancing requirement (1.1.1) induces severe restrictions on density flux functions. Namely, the value of $q_\mathcal{D}$ at $X \in \partial\mathcal{D}$ may depend on $\mathcal{D}$ solely through the exterior normal $N$ at $X$, and the dependence is "linear". This renders the balance law quite concrete, reducing it to a field equation.

**1.2.1 Theorem.** *Consider the balance law* (1.1.1) *on $\mathcal{X}$ where $\mathcal{P}$ is a signed Radon measure and the $\mathcal{Q}_{\mathcal{D}}$ are induced, through* (1.1.2), *by density flux functions $q_{\mathcal{D}}$ that are bounded, $|q_{\mathcal{D}}(X)| \le C$, for any $X \in \partial\mathcal{D}$ and all proper domains $\mathcal{D}$. Then,*
*(i) With each $N \in S^{k-1}$ is associated a bounded measurable function $a_N$ on $\mathcal{X}$, with the following property: Let $\mathcal{D}$ be any proper domain in $\mathcal{X}$ and suppose $X$ is some point on $\partial\mathcal{D}$ where the exterior unit normal to $\mathcal{D}$ exists and is $N$. Assume further that $X$ is a Lebesgue point of $q_{\mathcal{D}}$, relative to $\mathcal{H}^{k-1}$, and that the upper derivate of $|\mathcal{P}|$ at $X$, with respect to Lebesgue measure, is finite. Then*

$$q_{\mathcal{D}}(X) = a_N(X). \tag{1.2.1}$$

*(ii) There exists a vector field $A \in L^\infty\big(\mathcal{X};\, \mathbb{M}^{1\times k}\big)$ such that, for any fixed $N \in S^{k-1}$,*

$$a_N(X) = A(X)N, \quad a.e.\ on\ \mathcal{X}. \tag{1.2.2}$$

*(iii) The function $A$ satisfies the field equation*

$$\operatorname{div} A = \mathcal{P}, \tag{1.2.3}$$

*in the sense of distributions on $\mathcal{X}$.*

**Proof.** Fix any $N \in S^{k-1}$ and then take any hyperplane $\mathcal{C}$, of codimension one, with normal $N$ and nonempty intersection with $\mathcal{X}$. Consider any proper domain $\mathcal{D}$ such that $\mathcal{H}^{k-1}(\mathcal{C} \cap \partial\mathcal{D}) > 0$ and define $a_N(X) = q_{\mathcal{D}}(X)$ for $X \in \mathcal{C} \cap \partial\mathcal{D}$. By virtue of our assumptions on flux density functions, the values of $a_N$ do not depend essentially on the particular domain $\mathcal{D}$ used for its construction. We thus end up with a well-defined, bounded, $\mathcal{H}^{k-1}$-measurable function $a_N$ on $\mathcal{C} \cap \mathcal{X}$. For normalization, we require

$$a_N(X) = \lim_{r \downarrow 0} \frac{1}{\mathcal{H}^{k-1}(\mathcal{C} \cap \mathcal{B}_r(X))} \int_{\mathcal{C} \cap \mathcal{B}_r(X)} a_N(Y)\, d\mathcal{H}^{k-1}(Y), \tag{1.2.4}$$

for any $X \in \mathcal{C} \cap \mathcal{X}$ for which the limit on the right-hand side exists. By repeating the above construction for every hyperplane with normal $N$, we define $a_N$ on all of $\mathcal{X}$.

In order to study the properties of $a_N$, we fix $N \in S^{k-1}$, together with a hyperplane $\mathcal{C}$ with normal $N$, and a ball $\mathcal{B}$ in $\mathcal{X}$, centered at some point on $\mathcal{C} \cap \mathcal{X}$. We then apply the balance law to cylindrical domains

$$\mathcal{D} = \bigcup_{-\delta < \tau < \varepsilon} \mathcal{A}_\tau\,, \quad \mathcal{A}_\tau = \{X \,:\, X - \tau N \in \mathcal{C} \cap \mathcal{B}\}\,, \tag{1.2.5}$$

where $\delta$ and $\varepsilon$ are small nonnegative numbers. This yields

$$\int_{\mathcal{A}_\varepsilon} a_N(X)\, d\mathcal{H}^{k-1}(X) + \int_{\mathcal{A}_{-\delta}} a_{-N}(X)\, d\mathcal{H}^{k-1}(X) = \mathcal{P}(\mathcal{D}) + O(\delta) + O(\varepsilon), \tag{1.2.6}$$

where the terms $O(\delta)$ and $O(\varepsilon)$ account for the contribution of the flux through the lateral boundary of the cylindrical domain. Setting $\delta = 0$ and letting $\varepsilon \downarrow 0$, we derive from (1.2.6) an estimate which, applied to all balls $\mathcal{B}$, implies that, as $\tau \downarrow 0$, $a_N(X + \tau N) \to -a_{-N}(X)$, in $L^\infty(\mathcal{C} \cap \mathcal{X})$ weak*. Similarly, setting $\varepsilon = 0$ and letting $\delta \downarrow 0$, we deduce that, as $\tau \uparrow 0$, $a_{-N}(X + \tau N) \to -a_N(X)$, again in $L^\infty(\mathcal{C} \cap \mathcal{X})$ weak*. In particular, this implies that $a_N$ is Lebesgue measurable on $\mathcal{X}$.

Returning to (1.2.6), and now letting both $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we conclude that $a_{-N}(X) = -a_N(X)$, for almost all (with respect to $\mathcal{H}^{k-1}$) $X \in \mathcal{C} \cap \mathcal{X}$, unless $\mathcal{C}$ belongs to the (at most) countable family of *exceptional hyperplanes* with normal $N$ for which $|\mathcal{P}|(\mathcal{C} \cap \mathcal{X}) > 0$.

To show (1.2.1), consider any proper domain $\mathcal{D}$ in $\mathcal{X}$ and fix any $X \in \partial\mathcal{D}$ where the exterior unit normal is $N$ and the tangential hyperplane is $\mathcal{C}$. Assume, further, that $X$ is a Lebesgue point of $q_{\mathcal{D}}$ and that the upper derivate of $|\mathcal{P}|$ at $X$, with respect to Lebesgue measure, is finite. For $r$ positive and small, write the balance law, first for the domain $\mathcal{D} \cap \mathcal{B}_r(X)$, then for the semiball $\{Y \in \mathcal{B}_r(X) : (Y - X) \cdot N < 0\}$; see Fig. 1.2.1.

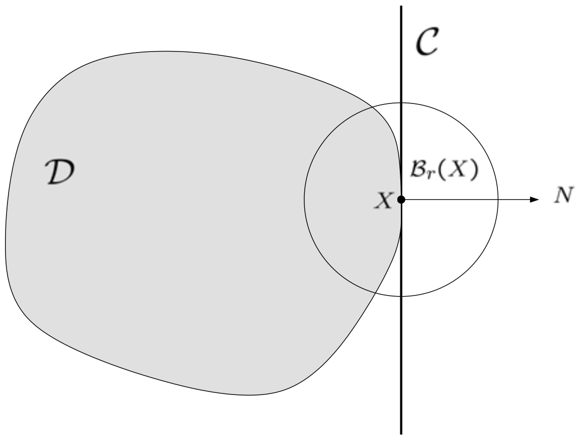


**Fig. 1.2.1**

Subtracting the resulting two equations yields

$$(1.2.7) \qquad \int_{\partial\mathcal{D}\cap\mathcal{B}_r(X)} q_{\mathcal{D}}(Y)d\mathcal{H}^{k-1}(Y) - \int_{\mathcal{C}\cap\mathcal{B}_r(X)} a_N(Y)d\mathcal{H}^{k-1}(Y) = o\big(r^{k-1}\big).$$

Dividing (1.2.7) by $r^{k-1}$, letting $r \downarrow 0$, and recalling (1.2.4), we arrive at (1.2.1), thus establishing assertion (i) of the theorem.

We will verify (1.2.2) by employing the celebrated *Cauchy tetrahedron argument.* We introduce the standard orthonormal basis $\{E_\alpha : \alpha = 1, \cdots, k\}$ in $\mathbb{R}^k$ and assemble the $m$-row vector field $A \in L^\infty\big(\mathcal{X}; \mathbb{M}^{1\times k}\big)$ with components $a_{E_\alpha}$:

$$(1.2.8) \qquad A(X) = \big[a_{E_1}(X), \cdots, a_{E_k}(X)\big].$$

Fix any $N \in S^{k-1}$ with nonzero components $N_\alpha$ (the argument has to be slightly modified when some of the $N_\alpha$ vanish), and take any $X \in \mathcal{X}$ with the following properties: $X$ is a Lebesgue point of the $k+1$ functions $a_{E_1}, \cdots, a_{E_k}$ and $a_N$; the upper derivate of $|\mathcal{P}|$ at $X$, with respect to Lebesgue measure, is finite. For $r$ positive and small, consider the simplex[1]

$$(1.2.9) \qquad \mathcal{D} = \{Y : (Y_\alpha - X_\alpha)\,\mathrm{sgn}\,N_\alpha > -r,\ \alpha = 1, \cdots, k;\ (Y - X) \cdot N < r\}.$$

Notice that $\partial\mathcal{D}$ consists of one face $\mathcal{C}$ with exterior normal $N$ and $k$ faces $\mathcal{C}_\alpha$, for $\alpha = 1, \cdots, k$, with respective exterior normals $(-\mathrm{sgn}\,N_\alpha)E_\alpha$. Furthermore, we have $\mathcal{H}^{k-1}(\mathcal{C}_\alpha) = |N_\alpha|\mathcal{H}^{k-1}(\mathcal{C})$. We select $r$ so that none of the faces of $\mathcal{D}$ lies on an exceptional hyperplane. The balance law for $\mathcal{D}$ then reads

$$(1.2.10) \qquad \int_{\mathcal{C}} a_N d\mathcal{H}^{k-1} - \sum_{\alpha=1}^{k} (\mathrm{sgn}\,N_\alpha) \int_{\mathcal{C}_\alpha} a_{E_\alpha} d\mathcal{H}^{k-1} = \mathcal{P}(\mathcal{D}).$$

Upon dividing (1.2.10) by $\mathcal{H}^{k-1}(\mathcal{C})$ and then letting $r \downarrow 0$ along properly selected sequences arrives at

$$(1.2.11) \qquad a_N(X) = \sum_{\alpha=1}^{k} a_{E_\alpha}(X)N_\alpha = A(X)N,$$

which establishes (1.2.2).

It remains to show (1.2.3). For Lipschitz continuous $A$, (1.2.3) follows directly by applying the divergence theorem to the balance law. In the general case where $A$ is merely in $L^\infty$, we resort to mollification. We fix any test function $\psi \in C_0^\infty(\mathbb{R}^k)$ with total mass one, supported in the unit ball, we rescale it by $\varepsilon$,

$$(1.2.12) \qquad \psi_\varepsilon(X) = \varepsilon^{-k}\psi\big(\varepsilon^{-1}X\big),$$

and employ it to mollify, in the customary fashion, $\mathcal{P}$ and $A$ on the set $\mathcal{X}_\varepsilon \subset \mathcal{X}$ of points whose distance from $\mathcal{X}^c$ exceeds $\varepsilon$:

$$(1.2.13) \qquad p_\varepsilon = \psi_\varepsilon * \mathcal{P}, \qquad A_\varepsilon = \psi_\varepsilon * A.$$

For any hypercube $\mathcal{D} \subset \mathcal{X}_\varepsilon$, we apply the divergence theorem to the smooth field $A_\varepsilon$ and use Fubini's theorem to get

$$(1.2.14) \qquad \begin{aligned} \int_{\mathcal{D}} \mathrm{div} A_\varepsilon(X)dX &= \int_{\partial\mathcal{D}} A_\varepsilon(X)N(X)d\mathcal{H}^{k-1}(X) \\ &= \int_{\partial\mathcal{D}}\int_{\mathbb{R}^k} \psi_\varepsilon(Y)A(X-Y)N(X)dYd\mathcal{H}^{k-1}(X) \\ &= \int_{\mathbb{R}^k} \psi_\varepsilon(Y) \int_{\partial\mathcal{D}_Y} A(Z)N(Z)d\mathcal{H}^{k-1}(Z)dY, \end{aligned}$$

[1] The Cauchy tetrahedron argument derives its name from the special case $k = 3$.

where $\mathcal{D}_Y$ denotes the $Y$-translate of $\mathcal{D}$, that is $\mathcal{D}_Y = \{X : X - Y \in \mathcal{D}\}$. By virtue of the balance law,

$$(1.2.15) \qquad \int_{\partial\mathcal{D}_Y} A(Z)N(Z)d\mathcal{H}^{k-1}(Z) = \int_{\partial\mathcal{D}_Y} a_N(Z)d\mathcal{H}^{k-1}(Z) = \mathcal{P}(\mathcal{D}_Y),$$

for almost all $Y$ in the ball $\{Y : |Y| < \varepsilon\}$. Hence (1.2.14) gives

$$(1.2.16) \qquad \int_{\mathcal{D}} \text{div} A_\varepsilon(X)dX = \int_{I\!R^k} \psi_\varepsilon(Y)\mathcal{P}(\mathcal{D}_Y)dY = \int_{\mathcal{D}} p_\varepsilon(X)dX,$$

whence we infer

$$(1.2.17) \qquad \text{div} A_\varepsilon(X) = p_\varepsilon(X), \quad X \in \mathcal{X}_\varepsilon .$$

Letting $\varepsilon \downarrow 0$ yields (1.2.3), in the sense of distributions on $\mathcal{X}$. This completes the proof.

In the following section we shall see that the course followed in the proof of the above theorem can be reversed: Departing from the field equation (1.2.3), one may retrieve the flux density functions $q_{\mathcal{D}}$ and thereby restore the balance law in its original form (1.1.1).

## 1.3 Change of Coordinates and a Trace Theorem

The divergence form of the field equations of balance laws is preserved under coordinate changes, so long as the fields transform according to appropriate rules. In fact, this even holds when the flux fields are merely locally integrable.

**1.3.1 Theorem.** *Let $\mathcal{X}$ be an open subset of $I\!R^k$ and assume that functions $A \in L^1_{loc}(\mathcal{X}; I\!M^{1\times k})$ and $\mathcal{P} \in \mathcal{M}(\mathcal{X})$ satisfy the field equation*

$$(1.3.1) \qquad \text{div} A = \mathcal{P},$$

*in the sense of distributions on $\mathcal{X}$. Consider any bilipschitz homeomorphism $X^*$ of $\mathcal{X}$ to a subset $\mathcal{X}^*$ of $I\!R^k$, with Jacobian matrix*

$$(1.3.2) \qquad J = \frac{\partial X^*}{\partial X}$$

*such that*

$$(1.3.3) \qquad \det J \geq a > 0, \quad a.e. \text{ on } \mathcal{X}.$$

*Then, $A^* \in L^1_{loc}(\mathcal{X}^*; I\!M^{1\times k})$ and $\mathcal{P}^* \in \mathcal{M}\,(\mathcal{X}^*)$ defined by*

$$A^* \circ X^* = (\det J)^{-1} A J^\top, \tag{1.3.4}$$

$$\langle \mathcal{P}^*, \varphi^* \rangle = \langle \mathcal{P}, \varphi \rangle, \quad \textit{where } \varphi = \varphi^* \circ X^*, \tag{1.3.5}$$

*satisfy the field equation*

$$\operatorname{div} A^* = \mathcal{P}^*, \tag{1.3.6}$$

*in the sense of distributions on* $\mathcal{X}^*$.

**Proof.** It follows from (1.3.1) that

$$\int_{\mathcal{X}} A \operatorname{grad} \varphi \, dX + \langle \mathcal{P}, \varphi \rangle = 0, \tag{1.3.7}$$

for any Lipschitz function $\varphi$ with compact support in $\mathcal{X}$, since one can always construct a sequence $\{\varphi_m\}$ of test functions in $C_0^\infty(\mathcal{X})$, supported in a compact subset of $\mathcal{X}$, such that, as $m \to \infty$, $\varphi_m \to \varphi$, uniformly, and $\operatorname{grad} \varphi_m \to \operatorname{grad} \varphi$, boundedly almost everywhere on $\mathcal{X}$.

Given any test function $\varphi^* \in C_0^\infty(\mathcal{X}^*)$, consider the Lipschitz function $\varphi = \varphi^* \circ X^*$, with compact support in $\mathcal{X}$. Notice that $\operatorname{grad} \varphi = J^\top \operatorname{grad} \varphi^*$. Furthermore, $dX^* = (\det J) dX$. By virtue of these and (1.3.4), (1.3.5), we can write (1.3.7) as

$$\int_{\mathcal{X}^*} A^* \operatorname{grad} \varphi^* dX^* + \langle \mathcal{P}^*, \varphi^* \rangle = 0, \tag{1.3.8}$$

which establishes (1.3.6). The proof is complete.

**1.3.2 Remark.** In the special, yet common, situation where the measure $\mathcal{P}$ is induced by a production density field $p \in L^1_{loc}(\mathcal{X})$, (1.3.5) implies that $\mathcal{P}^*$ is also induced by a production density field $p^* \in L^1_{loc}(\mathcal{X}^*)$, given by

$$p^* \circ X^* = (\det J)^{-1} p. \tag{1.3.9}$$

Even though in general the field $A$ is only defined almost everywhere on an open subset of $\mathbb{R}^k$, it turns out that the field equation induces a modicum of regularity, manifesting itself in trace theorems, which will allow us to identify the flux through surfaces of codimension one, and thus retrieve the balance law in its original form. We begin with planar surfaces.

**1.3.3 Lemma.** *Assume* $A \in L^\infty(\mathcal{K}; \mathbb{M}^{1\times k})$ *and* $\mathcal{P} \in \mathcal{M}(\mathcal{K})$ *satisfy* (1.3.1), *in the sense of distributions, on a cylindrical domain* $\mathcal{K} = \mathcal{B} \times (\alpha, \beta)$, *where* $\mathcal{B}$ *is a ball in* $\mathbb{R}^{k-1}$. *Let* $E_k$ *denote the* $k$*-base vector in* $\mathbb{R}^k$ *and set* $X = (Y, t)$, *with* $Y \in \mathcal{B}$ *and* $t \in (\alpha, \beta)$. *Then, after modifying, if necessary,* $A$ *on a set of measure zero, the* *function* $a(Y,t) = A(Y,t)E_k$ *acquires the following properties: One-sided limits* $a(\cdot,\tau\pm)$ *in* $L^\infty(\mathcal{B})$ *weak* exist, for any* $\tau \in (\alpha,\beta)$*, and can be determined by*

$$(1.3.10)\qquad \begin{cases} a(Y,\tau-) = \operatorname*{ess\,lim}_{t\uparrow\tau} A(Y,t)E_k = \lim_{\varepsilon\downarrow 0} \dfrac{1}{\varepsilon}\displaystyle\int_{\tau-\varepsilon}^{\tau} A(Y,t)E_k dt, \\[2ex] a(Y,\tau+) = \operatorname*{ess\,lim}_{t\downarrow\tau} A(Y,t)E_k = \lim_{\varepsilon\downarrow 0} \dfrac{1}{\varepsilon}\displaystyle\int_{\tau}^{\tau+\varepsilon} A(Y,t)E_k dt, \end{cases}$$

*where the limits are taken in* $L^\infty(\mathcal{B})$ *weak*. Furthermore, for any* $\tau \in (\alpha,\beta)$ *and any Lipschitz continuous function* $\varphi$ *with compact support in* $\mathcal{K}$*,*

$$(1.3.11)\qquad \begin{cases} \displaystyle\int_{\mathcal{B}} a(Y,\tau-)\varphi(Y,\tau)dY = \int_{\mathcal{B}\times(\alpha,\tau)} A(X)\operatorname{grad}\varphi(X)\,dX + \langle\mathcal{P},\varphi\rangle_{\mathcal{B}\times(\alpha,\tau)}\,, \\[2ex] \displaystyle -\int_{\mathcal{B}} a(Y,\tau+)\varphi(Y,\tau)dY = \int_{\mathcal{B}\times(\tau,\beta)} A(X)\operatorname{grad}\varphi(X)\,dX + \langle\mathcal{P},\varphi\rangle_{\mathcal{B}\times(\tau,\beta)}\,. \end{cases}$$

*Thus,* $a(\cdot,\tau-) = a(\cdot,\tau+) = a(\cdot,\tau)$*, unless* $\tau$ *belongs to the (at most) countable set of points with* $|\mathcal{P}|(\mathcal{B}\times\{\tau\}) > 0$*. In particular, when* $\mathcal{P}$ *is absolutely continuous with respect to Lebesgue measure, the function* $\tau \mapsto a(\cdot,\tau)$ *is continuous on* $(\alpha,\beta)$*, in the weak* topology of* $L^\infty(\mathcal{B})$*.*

**Proof.** Fix $\varepsilon$ positive and small. If $r$ is the radius of $\mathcal{B}$, let $\mathcal{B}_\varepsilon$ denote the ball in $\mathbb{R}^{k-1}$ with the same center as $\mathcal{B}$ and radius $r-\varepsilon$. As in the proof of Theorem 1.2.1, we mollify $A$ and $\mathcal{P}$ on $\mathcal{B}_\varepsilon \times (\alpha+\varepsilon, \beta-\varepsilon)$ through (1.2.13). The resulting smooth fields $A_\varepsilon$ and $p_\varepsilon$ satisfy (1.2.17). We also set $a_\varepsilon(Y,t) = A_\varepsilon(Y,t)E_k$ .

We multiply (1.2.17) by any Lipschitz function $\varphi$ on $\mathbb{R}^{k-1}$, with compact support in $\mathcal{B}_\varepsilon$, and integrate the resulting equation over $\mathcal{B}_\varepsilon \times (r,s)$, $\alpha+\varepsilon < r < s < \beta-\varepsilon$. After an integration by parts, this yields

$$(1.3.12)\qquad \int_{\mathcal{B}_\varepsilon} a_\varepsilon(Y,s)\varphi(Y)dY - \int_{\mathcal{B}_\varepsilon} a_\varepsilon(Y,r)\varphi(Y)dY$$

$$= \int_r^s \int_{\mathcal{B}_\varepsilon} \{A_\varepsilon(Y,t)\;\Pi_k \operatorname{grad}\varphi(Y) + p_\varepsilon(Y,t)\varphi(Y)\}\,dY\,dt,$$

where $\Pi_k$ denotes the projection of $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$. It follows that the total variation of the function $t \mapsto \int_{\mathcal{B}_\varepsilon} a_\varepsilon(Y,t)\varphi(Y)dY$, over the interval $(\alpha+\varepsilon, \beta-\varepsilon)$, is bounded, uniformly in $\varepsilon > 0$. Therefore, starting out from some countable family $\{\varphi_\ell\}$ of test functions, with compact support in $\mathcal{B}$, which is dense in $L^1(\mathcal{B})$, we may invoke

Helly's theorem in conjunction with a diagonal argument to extract a sequence $\{\varepsilon_m\}$, with $\varepsilon_m \to 0$ as $m \to \infty$, and identify a countable subset $G$ of $(\alpha, \beta)$, such that, for any $\ell = 1, 2 \cdots$, the sequence $\{\int_{\mathcal{B}} a_{\varepsilon_m}(Y, t)\varphi_\ell(Y)dY\}$ converges, as $m \to \infty$, for all $t \in (\alpha, \beta)\backslash G$, and the limit function has bounded variation over $(\alpha, \beta)$. The resulting limit functions, for all $\ell$, may be collectively represented as $\int_{\mathcal{B}} a(Y, t)\varphi_\ell(Y)dY$, for some function $t \mapsto a(\cdot, t)$ taking values in $L^\infty(\mathcal{B})$. Clearly, $a(Y, t) = A(Y, t)E_k$, a.e. in $\mathcal{K}$. Thus, $a$ does not depend on the particular sequence $\{\varepsilon_m\}$ employed for its construction, and (1.3.10) holds for any $\tau \in (\alpha, \beta)$.

Given any $\tau \in (\alpha, \beta)$ and any Lipschitz function $\varphi$ with compact support in $\mathcal{K}$, we multiply (1.2.17) by $\varphi$ and integrate the resulting equation over $\mathcal{B}_\varepsilon \times (\alpha + \varepsilon, s)$, where $s \in (\alpha + \varepsilon, \tau)\backslash G$. After an integration by parts, this yields

(1.3.13)
$$\int_{\mathcal{B}_\varepsilon} a_\varepsilon(Y, s)\varphi(Y, s)dY$$
$$= \int_{\mathcal{B}_\varepsilon \times (a+\varepsilon, s)} \left[A_\varepsilon(X) \operatorname{grad} \varphi(X) + p_\varepsilon(X)\varphi(X)\right] dX.$$

In (1.3.13) we first let $\varepsilon \downarrow 0$ and then $s \uparrow \tau$ thus arriving at $(1.3.11)_1$. The proof of $(1.3.11)_2$ is similar.

When $\mathcal{P}$ is absolutely continuous with respect to Lebesgue measure, (1.3.12) implies that the family of functions $t \mapsto \int_{\mathcal{B}_\varepsilon} a_\varepsilon(Y, t)\varphi_\ell(Y)dY$, parametrized by $\varepsilon$, is actually equicontinuous, and hence $\int_{\mathcal{B}} a(Y, t)\varphi_\ell(Y)dY$ is continuous on $(\alpha, \beta)$, for $\ell = 1, 2, \cdots$. Thus, $t \mapsto a(\cdot\,; t)$ is continuous on $(\alpha, \beta)$, in $L^\infty(\mathcal{B})$ weak*. This completes the proof.

The $k$-coordinate direction was singled out, in the above proposition, just for convenience. Analogous continuity properties are clearly enjoyed by $AE_\alpha$, in the direction of any base vector $E_\alpha$, and indeed by $AN$, in the direction of any $N \in S^{k-1}$. Thus, departing from the field equation (1.2.3), one may retrieve the flux density functions $a_N$, for planar surfaces, encountered in Theorem 1.2.1. The following proposition demonstrates that even the flux density functions $q_{\mathcal{D}}$, for general proper domains $\mathcal{D}$, may be retrieved by the same procedure.

**1.3.4 Theorem.** *Assume that $A \in L^\infty(\mathcal{X}; \mathbb{M}^{1\times k})$ and $\mathcal{P} \in \mathcal{M}(\mathcal{X})$ satisfy* (1.3.1)*, in the sense of distributions, on an open subset $\mathcal{X}$ of $\mathbb{R}^k$. Then, with any proper domain $\mathcal{D}$ in $\mathcal{X}$ is associated a bounded $\mathcal{H}^{k-1}$-measurable function $q_{\mathcal{D}}$ on $\partial\mathcal{D}$ such that*

$$\int_{\partial\mathcal{D}} q_{\mathcal{D}}(X)\varphi(X)d\mathcal{H}^{k-1}(X) = \int_{\mathcal{D}} A(X)\,\mathrm{grad}\,\varphi(X)\,dX + \langle\mathcal{P},\varphi\rangle_{\mathcal{D}}\,, \tag{1.3.14}$$

*for any Lipschitz continuous function $\varphi$ on $\mathbb{R}^k$, with compact support in $\mathcal{X}$.*

**Proof.** Consider the cylindrical domain $\mathcal{K}^* = \{X^* = (Y,t) : Y \in \mathcal{B},\ t \in (-1,1)\}$, where $\mathcal{B}$ is the unit ball in $\mathbb{R}^{k-1}$. Fix any proper domain $\mathcal{D}$ in $\mathcal{X}$.

Since $\mathcal{D}$ is a Lipschitz domain, with any point $\bar{X} \in \partial\mathcal{D}$ is associated a bilipschitz homeomorphism $X$ from $\mathcal{K}^*$ to some open subset $\mathcal{K}$ of $\mathcal{X}$ such that $X(0) = \bar{X}$, $X(\mathcal{B}\times(-1,0)) = \mathcal{D}\cap\mathcal{K}$ and $X(\mathcal{B}\times\{0\}) = \partial\mathcal{D}\cup\mathcal{K}$.

Consider the inverse map $X^*$ of $X$, with Jacobian matrix $J$, given by (1.3.2) and satisfying (1.3.3). Construct $A^* \in L^\infty(\mathcal{K}^*; \mathbb{M}^{1\times k})$, by (1.3.4), and $\mathcal{P}^* \in \mathcal{M}(\mathcal{K}^*)$, by (1.3.5), which will satisfy (1.3.6) on $\mathcal{K}^*$, in the sense of distributions.

We now apply Lemma 1.3.3 to identify the function $a^*(Y,t)$ on $\mathcal{K}^*$ , which is equal to $A^*(Y,t)E_k$ , a.e on $\mathcal{K}^*$, and by (1.3.10) satisfies

$$a^*(Y,0-) = \operatorname*{ess\,lim}_{t\uparrow 0} A^*(Y,t)E_k = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} A^*(Y,t)E_k dt. \tag{1.3.15}$$

We fix any Lipschitz continuous function $\varphi$ on $\mathbb{R}^k$, with compact support in $\mathcal{K}$, and let $\varphi^* = \varphi\circ X$, for $X^* \in \mathcal{K}^*$. By virtue of (1.3.11),

$$\int_{\mathcal{B}} a^*(Y,0-)\varphi^*(Y,0)dY = \int_{\mathcal{B}\times(-1,0)} A^*(X^*)\,\mathrm{grad}\,\varphi^*(X^*)\,dX^* + \langle\mathcal{P}^*,\varphi^*\rangle_{\mathcal{B}\times(-1,0)}\,. \tag{1.3.16}$$

We employ the homeomorphism $X^*$ in order to transform (1.3.16) into an equation on $\mathcal{X}$. Using that $\mathrm{grad}\,\varphi = J^\top\mathrm{grad}\,\varphi^*$ and recalling (1.3.4) and (1.3.5), we may rewrite (1.3.16) as

$$\int_{\partial\mathcal{D}\cap\mathcal{K}} q_{\mathcal{D}}(X)\varphi(X)d\mathcal{H}^{k-1}(X) = \int_{\mathcal{D}} A(X)\,\mathrm{grad}\,\varphi(X)\,dX + \langle\mathcal{P},\varphi\rangle_{\mathcal{D}}\,, \tag{1.3.17}$$

where we have set

$$q_{\mathcal{D}} = \frac{dY}{d\mathcal{H}^{k-1}}\, a^* \circ X^* = \frac{\det J}{E_k^\top JN}\, a^* \circ X^*, \tag{1.3.18}$$

with $N$ denoting the exterior unit normal to $\mathcal{D}$.

Equation (1.3.17) establishes (1.3.14) albeit only for $\varphi$ with compact support in $\mathcal{K}$. It should be noted, however, that the right-hand side of (1.3.17) does not depend on the homeomorphism $X^*$ and thus the values of $q_{\mathcal{D}}$ on $\partial\mathcal{D} \cap \mathcal{K}$ are intrinsically defined, independently of the particular construction employed above. Hence, one may easily pass from (1.3.17) to (1.3.14), for arbitrary Lipschitz continuous functions $\varphi$ with compact support in $\mathcal{X}$, by a straightforward partition of unity argument. This completes the proof.

The reader can find, in the literature cited in Section 1.10, more refined versions of the above proposition, in which $A$ is assumed to be merely locally integrable or even just a measure, as well as alternative methods of proof. For instance, in a more abstract approach, the existence of $q_{\mathcal{D}}$ follows directly by showing that the right-hand side of (1.3.14) can be realized as a bounded linear functional on $L^1(\partial\mathcal{D})$. The drawback of this, functional analytic, demonstration is that it does not provide any clues on how the $q_{\mathcal{D}}$ may be computed from $A$. In the opposite direction, one may impose slightly more stringent regularity conditions on $\mathcal{D}$ and then derive a representation of $q_{\mathcal{D}}$, in terms of $A$, which is more explicit and does not require passing through (1.3.4), (1.3.5) and (1.3.18), as is done here.

## 1.4 Systems of Balance Laws

We consider the situation where $n$ distinct balance laws, with production measures induced by production density fields, act simultaneously in $\mathcal{X}$, and collect their field equations (1.2.3) into the system

$$\mathrm{div} A(X) = P(X), \tag{1.4.1}$$

where now $A$ is a $n \times k$-matrix field and $P$ is a $n$-column vector field. The divergence operator acts on the row vectors of $A$, yielding as $\mathrm{div} A$ a $n$-column vector field.

We assume that the state of the medium is described by a *state vector* field $U$, taking values in an open subset $\mathcal{O}$ of $\mathbb{R}^n$, which determines the flux density field $A$ and the production density field $P$ at the point $X \in \mathcal{X}$ by *constitutive equations*

$$A(X) = G(U(X), X), \qquad P(X) = \Pi(U(X), X), \tag{1.4.2}$$

where $G$ and $\Pi$ are given smooth functions defined on $\mathcal{O} \times \mathcal{X}$ and taking values in $\mathbb{M}^{n\times k}$ and $\mathbb{R}^n$, respectively.

Combining (1.4.1) with (1.4.2) yields

$$\mathrm{div}\, G(U(X), X) = \Pi(U(X), X), \tag{1.4.3}$$

that is a (formally) closed quasilinear first order system of partial differential equations from which the state vector field is to be determined. Any equation of the form (1.4.3) will henceforth be called a *system of balance laws*, if $n \geq 2$, or a *scalar balance law* when $n = 1$. In the special case where there is no production, $\Pi \equiv 0$, (1.4.3) will be called a *system of conservation laws*, if $n \geq 2$, or a *scalar conservation law* when $n = 1$. This terminology is not quite standard: In lieu of "system of balance laws" certain authors favor the term "system of conservation laws with source". When $G$ and $\Pi$ do not depend explicitly on $X$, the system of balance laws is called *homogeneous*.

Notice that when coordinates are stretched in the vicinity of some fixed point $\bar{X} \in \mathcal{X}$, i.e., $X = \bar{X} + \varepsilon Y$, then, as $\varepsilon \downarrow 0$, the system of balance laws (1.4.3) reduces to a homogeneous system of conservation laws with respect to the $Y$ variable. This is why local properties of solutions of general systems of balance laws may be investigated, without loss of generality, in the simpler setting of homogeneous systems of conservation laws.

A Lipschitz continuous field $U$ that satisfies (1.4.3) almost everwhere on $\mathcal{X}$ will be called a *classical solution*. A measurable field $U$ that satisfies (1.4.3) in the sense of distributions, i.e., $G(U(X), X)$ and $\Pi(U(X), X)$ are locally integrable and

$$\int_{\mathcal{X}} [G(U(X), X) \operatorname{grad} \varphi(X) + \varphi(X)\Pi(U(X), X)]dX = 0, \tag{1.4.4}$$

for any test function $\varphi \in C_0^\infty(\mathcal{X})$, is a *weak solution*. Any weak solution which is Lipschitz continuous is necessarily a classical solution.

**1.4.1 Notation.** For $\alpha = 1, \cdots, k$, $G_\alpha(U, X)$ will denote the $\alpha$-th column vector of the matrix $G(U, X)$.

**1.4.2 Notation.** Henceforth, D will denote the differential with respect to the $U$ variable. When used in conjunction with matrix notations, D shall be regarded as a row operation: $\mathrm{D} = \left[\partial/\partial U^1, \cdots, \partial/\partial U^n\right]$.

## 1.5 Companion Balance Laws

Consider a system (1.4.3) of balance laws on an open subset $\mathcal{X}$ of $\mathbb{R}^k$, resulting from combining the field equation (1.4.1) with constitutive relations (1.4.2). A smooth function $Q$, defined on $\mathcal{O} \times \mathcal{X}$ and taking values in $\mathbb{M}^{1\times k}$, is called a *companion* of $G$ if there is a smooth function $B$, defined on $\mathcal{O} \times \mathcal{X}$ and taking values in $\mathbb{R}^n$, such that, for all $U \in \mathcal{O}$ and $X \in \mathcal{X}$,

$$\mathrm{D}Q_\alpha(U, X) = B(U, X)^\top \mathrm{D}G_\alpha(U, X), \quad \alpha = 1, \cdots, k. \tag{1.5.1}$$

The relevance of (1.5.1) stems from the observation that any classical solution $U$ of the system of balance laws (1.4.3) is automatically also a (classical) solution of the *companion balance law*

$$\text{div}\, Q(U(X), X) = h(U(X), X), \tag{1.5.2}$$

with

$$h(U, X) = B(U, X)^\top \Pi(U, X) + \nabla \cdot Q(U, X) - B(U, X)^\top \nabla \cdot G(U, X). \tag{1.5.3}$$

In (1.5.3) $\nabla\cdot$ denotes divergence with respect to $X$, holding $U$ fixed—as opposed to div, which treats $U$ as a function of $X$.

One determines the companion balance laws (1.5.2) of a given system of balance laws (1.4.3) by identifying the integrating factors $B$ that render the right-hand side of (1.5.1) a gradient of a function of $U$. The relevant integrability condition is

$$\mathrm{D}B(U, X)^\top \mathrm{D}G_\alpha(U, X) = \mathrm{D}G_\alpha(U, X)^\top \mathrm{D}B(U, X), \quad \alpha = 1, \cdots, k, \tag{1.5.4}$$

for all $U \in \mathcal{O}$ and $X \in \mathcal{X}$. Clearly, one can satisfy (1.5.4) by employing any $B$ that does not depend on $U$; in that case, however, the resulting companion balance law (1.5.2) is just a trivial linear combination of the equations of the original system (1.4.3). For nontrivial $B$, which vary with $U$, (1.5.4) imposes $\frac{1}{2}n(n-1)k$ conditions on the $n$ unknown components of $B$. Thus, when $n = 1$ and $k$ is arbitrary one may use any (scalar-valued) function $B$. When $n = 2$ and $k = 2$, (1.5.4) reduces to a system of two equations in two unknowns from which a family of $B$ may presumably be determined. In all other cases, however, (1.5.4) is formally overdetermined and the existence of nontrivial companion balance laws should not be generally expected. Nevertheless, as we shall see in Chapter III, the systems of balance laws of continuum physics are endowed with natural companion balance laws.

The system of balance laws (1.4.3) is called *symmetric* when the $n \times n$ matrices $\mathrm{D}G_\alpha(U, X)$, $\alpha = 1, \cdots, k$, are symmetric, for any $U \in \mathcal{O}$ and $X \in \mathcal{X}$; say $\mathcal{O}$ is simply connected and

$$G(U, X)^\top = \mathrm{D}\Gamma(U, X)^\top, \tag{1.5.5}$$

for some smooth function $\Gamma$, defined on $\mathcal{O} \times \mathcal{X}$ and taking values in $\mathbb{M}^{1\times k}$. In that case one may satisfy (1.5.4) by taking $B(U, X) \equiv U$, which induces the companion

$$Q(U, X) = U^\top G(U, X) - \Gamma(U, X). \tag{1.5.6}$$

Conversely, if (1.5.1) holds for some $B$ with the property that, for every fixed $X \in \mathcal{X}$, $B(\cdot\,, X)$ maps diffeomorphically $\mathcal{O}$ to some open subset $\mathcal{O}^*$ of $\mathbb{R}^n$, then the change $U^* = B(U, X)$ of state vector reduces (1.4.3) to the equivalent system of balance laws

$$\text{div}\, G^*(U^*(X), X) = \Pi^*(U^*(X), X), \tag{1.5.7}$$

with

$$G^*(U^*, X) = G(B^{-1}(U^*, X), X), \quad \Pi^*(U^*, X) = \Pi(B^{-1}(U^*, X), X), \tag{1.5.8}$$

which is symmetric. Indeed, upon setting

(1.5.9) $$Q^*(U^*, X) = Q(B^{-1}(U^*, X), X),$$

(1.5.10) $$\Gamma^*(U^*, X) = U^{*\top} G^*(U^*, X) - Q^*(U^*, X),$$

it follows from (1.5.1) that

(1.5.11) $$G^*(U^*, X)^\top = \mathrm{D}\Gamma^*(U^*, X)^\top.$$

We have thus demonstrated that a system of balance laws is endowed with nontrivial companion balance laws if and only if it is *symmetrizable*.

We shall see that the presence of companion balance laws has major implications on the theory of systems of balance laws arising in physics. Quite often, in order to simplify the analysis, it becomes necessary to make simplifying physical assumptions that truncate the system of balance laws while simultaneously trimming proportionately the size of the state vector. Such truncations cannot be performed arbitrarily without destroying the mathematical structure of the system, which goes hand in hand with its relevance to physics. For a canonical truncation, it is necessary to operate on (or at least think in terms of) the symmetrical form (1.5.7) of the system, and adhere to the rule that dropping the $i$-th balance law should be paired with “freezing” (i.e. assigning fixed values to) the $i$-th component $U^{*i}$ of the special state vector $U^*$. Then, the resulting truncated system will still be symmetric and will inherit the companion

(1.5.12) $$\hat{Q} = Q^* - \sum U^{*i} G^{*i} ,$$

where the summation runs over all $i$ for which the $i$-balance law has been eliminated and $U^{*i}$ has been frozen. $G^{*i}$ denotes the $i$-th row vector of $G$.

Despite (1.5.1), and in contrast to the behavior of classical solutions, weak solutions of (1.4.3) need not satisfy (1.5.2). Nevertheless, one of the tenets of the theory of systems of balance laws is that *admissible weak solutions* should at least satisfy the inequality

(1.5.13) $$\operatorname{div} Q(U(X), X) \leq h(U(X), X),$$

in the sense of distributions, for a designated family of companions. Relating this postulate to the Second Law of thermodynamics and investigating its implications on stability of weak solutions are among the principal objectives of this book.

Notice that an inequality (1.5.13), holding in the sense of distributions, can always be turned into an equality by subtracting from the right-hand side some nonnegative measure $\mathcal{M}$,

(1.5.14) $$\operatorname{div} Q(U(X), X) = h(U(X), X) - \mathcal{M},$$

and may thus be realized, by virtue of Theorem 1.3.4, as the field equation of a balance law.

## 1.6 Weak and Shock Fronts

The regularity of solutions of a system of balance laws will depend on the nature of the constitutive functions. The focus will be on solutions with "fronts", that is singularities assembled on manifolds of codimension one. To get acquainted with this sort of solutions, we consider here two kinds of fronts in a particularly simple setting.

In what follows, $\mathcal{F}$ will be a smooth $(k-1)$-dimensional manifold, embedded in the open subset $\mathcal{X}$ of $\mathbb{R}^k$, with orientation induced by the unit normal field $N$. $U$ will be a (generally weak) solution of the system of balance laws (1.4.3) on $\mathcal{X}$ which is continuously differentiable on $\mathcal{X}\backslash\overline{\mathcal{F}}$, but is allowed to be singular on $\mathcal{F}$. In particular, (1.4.3) holds for any $X \in \mathcal{X}\backslash\overline{\mathcal{F}}$. See Fig. 1.6.1.

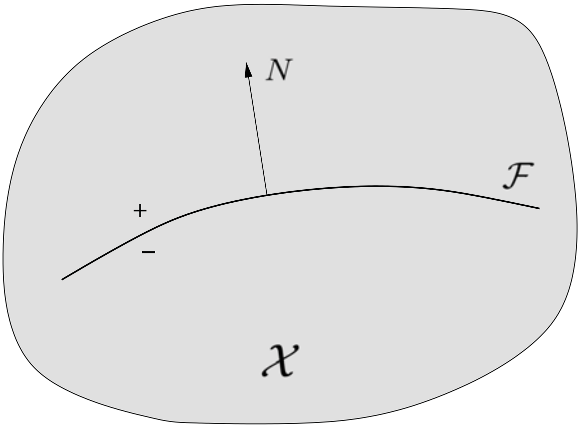


**Fig. 1.6.1**

First we consider the case where $\mathcal{F}$ is a *weak front*, that is, $U$ is Lipschitz continuous on $\mathcal{X}$ and as one approaches $\mathcal{F}$ from either side the gradient of $U$ attains distinct limits $\mathrm{grad}^-U$ , $\mathrm{grad}^+U$. Thus $\mathrm{grad}\,U$ experiences a jump $[\![\mathrm{grad}\,U]\!] = \mathrm{grad}^+U - \mathrm{grad}^-U$ across $\mathcal{F}$. Since $U$ is continuous, tangential derivatives of $U$ cannot jump across $\mathcal{F}$ and hence $[\![\mathrm{grad}\,U]\!] = [\![\partial U/\partial N]\!] \otimes N$, where $[\![\partial U/\partial N]\!]$ denotes the jump of the normal derivative $\partial U/\partial N$ across $\mathcal{F}$. Therefore, taking the jump of (1.4.3) across $\mathcal{F}$ at any point $X \in \mathcal{F}$ yields the following condition on $[\![\partial U/\partial N]\!]$:

$$\mathrm{D}[G(U(X), X)N] \left[\!\!\left[ \frac{\partial U}{\partial N} \right]\!\!\right] = 0. \tag{1.6.1}$$

Next we assume $\mathcal{F}$ is a *shock front*, that is, as one approaches $\mathcal{F}$ from either side, $U$ attains distinct limits $U_-$, $U_+$ and thus experiences a jump $[\![U]\!] = U_+ - U_-$ across $\mathcal{F}$. Both $U_-$ and $U_+$ are continuous functions on $\mathcal{F}$. Since $U$ is a (weak) solution of (1.4.3), we may write (1.4.4) for any $\varphi \in C_0^\infty(\mathcal{X})$. In (1.4.4) integration over $\mathcal{X}$ may be replaced with integration over $\mathcal{X}\backslash\overline{\mathcal{F}}$. Since $U$ is $C^1$ on $\mathcal{X}\backslash\overline{\mathcal{F}}$, we may integrate by parts in (1.4.4). Using that $\varphi$ has compact support in $\mathcal{X}$ and that (1.4.3) holds for any $X \in \mathcal{X}\backslash\overline{\mathcal{F}}$, we get

$$(1.6.2) \qquad \int_{\mathcal{F}} \varphi(X)[G(U_+, X) - G(U_-, X)]N \, d\mathcal{H}^{k-1}(X) = 0,$$

whence we deduce that the following *jump condition* must be satisfied at every point $X$ of the shock front $\mathcal{F}$:

$$(1.6.3) \qquad [G(U_+, X) - G(U_-, X)]N = 0.$$

Notice that (1.6.3) may be rewritten in the form

$$(1.6.4) \qquad \left\{ \int_0^1 \mathrm{D}[G(\tau U_+ + (1-\tau)U_-, X)N]d\tau \right\} [\![U]\!] = 0.$$

Comparing (1.6.4) with (1.6.1) we conclude that weak fronts may be regarded as shock fronts with "infinitesimal" strength: $|[\![U]\!]|$ vanishingly small.

With each $U \in \mathcal{O}$ and $X \in \mathcal{X}$ we associate the variety

$$(1.6.5) \qquad \mathcal{V}(U, X) = \left\{ (N, V) \in \mathcal{S}^{k-1} \times \mathbb{R}^n : \mathrm{D}[G(U, X)N]V = 0 \right\}.$$

The number of weak fronts and shock fronts of small strength that may be sustained by solutions of (1.4.3) will depend on the size of $\mathcal{V}$. In the extreme case where, for all $(U, X)$, the projection of $\mathcal{V}(U, X)$ onto $\mathbb{R}^n$ contains only the vector $V = 0$, (1.4.3) is called *elliptic*. Thus a system of balance laws is elliptic if and only if it cannot sustain any weak fronts or shock fronts of small strength. The opposite extreme to ellipticity, where $\mathcal{V}$ attains the maximal possible size, is hyperbolicity, which will be introduced in Chapter III.

## 1.7 Survey of the Theory of *BV* Functions

In this section we shall get acquainted with $BV$ functions, in which discontinuities assemble on manifolds of codimension one, and thus provide the natural setting for solutions of systems of balance laws with shock fronts. Comprehensive treatment of the theory of $BV$ functions can be found in the references cited in Section 1.10, so only properties relevant to our purposes will be listed here, without proofs.

**1.7.1 Definition.** A scalar function $v$ is of *locally bounded variation* on an open subset $\mathcal{X}$ of $\mathbb{R}^k$ if $v \in L^1_{loc}(\mathcal{X})$ and $\operatorname{grad} v$ is a ($\mathbb{R}^k$-valued) Radon measure $\mathcal{M}$ on $\mathcal{X}$, i.e.,

$$(1.7.1) \qquad -\int_{\mathcal{X}} v \operatorname{div} \Psi(X) dX = \int_{\mathcal{X}} \Psi(X) d\mathcal{M}(X),$$

for any test function $\Psi \in C_0^\infty(\mathcal{X}; I\!M^{1\times k})$. When $v \in L^1(\mathcal{X})$ and $\mathcal{M}$ is finite, $v$ is a function of *bounded variation* on $\mathcal{X}$, with *total variation*

$$TV_{\mathcal{X}} v = |\mathcal{M}|(\mathcal{X}) = \sup_{|\Psi(X)|=1} \int_{\mathcal{X}} v(X)\, \mathrm{div}\, \Psi(X)\, dX. \tag{1.7.2}$$

The set of functions of bounded variation and locally bounded variation on $\mathcal{X}$ will be denoted by $BV(\mathcal{X})$ and $BV_{\mathrm{loc}}(\mathcal{X})$, respectively.

Clearly, the Sobolev space $W^{1,1}(\mathcal{X})$, of $L^1(\mathcal{X})$ functions with derivatives in $L^1(\mathcal{X})$, is contained in $BV(\mathcal{X})$; and $W^{1,1}_{\mathrm{loc}}(\mathcal{X})$ is contained in $BV_{\mathrm{loc}}(\mathcal{X})$.

The following proposition provides a useful criterion for testing whether a given function has bounded variation:

**1.7.2 Theorem.** *Let $\{E_\alpha, \alpha = 1, \cdots, k\}$ denote the standard orthonormal basis of $I\!R^k$. If $v \in BV_{\mathrm{loc}}(\mathcal{X})$, then*

$$\limsup_{h\downarrow 0} \frac{1}{h} \int_{\mathcal{Y}} |v(X + hE_\alpha) - v(X)| dX = |\mathcal{M}_\alpha|(\mathcal{Y})\ ,\ \alpha = 1, \cdots, k, \tag{1.7.3}$$

*for any open bounded set $\mathcal{Y}$ with $\overline{\mathcal{Y}} \subset \mathcal{X}$. Conversely, if $v \in L^1_{\mathrm{loc}}(\mathcal{X})$ and the left-hand side of* (1.7.3) *is finite for every $\mathcal{Y}$ as above, then $v \in BV_{\mathrm{loc}}(\mathcal{X})$.*

As a corollary, the above proposition yields the following result on compactness:

**1.7.3 Theorem.** *Any sequence $\{v_\ell\}$ in $BV_{loc}(\mathcal{X})$, such that $\|v_\ell\|_{L^1(\mathcal{Y})}$ and $TV_{\mathcal{Y}} v_\ell$ are uniformly bounded on every open bounded $\mathcal{Y} \subset \mathcal{X}$, contains a subsequence which converges in $L^1_{loc}(\mathcal{X})$, as well as almost everywhere on $\mathcal{X}$, to some function $v$ in $BV_{loc}(\mathcal{X})$, with $TV_{\mathcal{Y}} v \le \liminf_{\ell\to\infty} TV_{\mathcal{Y}} v_\ell$.*

Functions of bounded variation are endowed with fine geometric structure, as described in

**1.7.4 Theorem.** *The domain $\mathcal{X}$ of any $v \in BV_{\mathrm{loc}}(\mathcal{X})$ is the union of three, pairwise disjoint, subsets $\mathcal{C}$, $\mathcal{J}$, and $\mathcal{I}$ with the following properties*:

(a) *$\mathcal{C}$ is the set of points of approximate continuity of $v$, i.e., with each $\bar{X} \in \mathcal{C}$ is associated $v_0 \in I\!R$ such that*

$$\lim_{r\downarrow 0} \frac{1}{r^k} \int_{\mathcal{B}_r(\bar{X})} |v(X) - v_0| dX = 0. \tag{1.7.4}$$

(b) *$\mathcal{J}$ is the set of points of approximate jump discontinuity of $v$, i.e., with each $\bar{X} \in \mathcal{J}$ are associated $N$ in $S^{k-1}$ and distinct $v_-$, $v_+$ in $I\!R$ such that*

$$\lim_{r\downarrow 0}\frac{1}{r^k}\int_{\mathcal{B}_r^{\pm}(\bar{X})}|v(X)-v_{\pm}|dX=0, \tag{1.7.5}$$

*where* $\mathcal{B}_r^{\pm}(\bar{X})$ *denote the semiballs* $\mathcal{B}_r(\bar{X})\cap\{X : (X-\bar{X})\cdot N \gtrless 0\}$. *Moreover,* $\mathcal{J}$ *is countably rectifiable, i.e. it is essentially covered by the countable union of* $C^1$ $(k-1)$*-dimensional manifolds* $\{\mathcal{F}_i\}$ *embedded in* $I\!R^k$: $\mathcal{H}^{k-1}(\mathcal{J}\backslash\bigcup\mathcal{F}_i)=0$. *Furthermore, when* $\bar{X}\in\mathcal{J}\cap\mathcal{F}_i$ *then* $N$ *is normal on* $\mathcal{F}_i$ *at* $\bar{X}$.
*(c)* $\mathcal{I}$ *is the set of irregular points of* $v$*; its* $(k-1)$*-dimensional Hausdorff measure is zero:* $\mathcal{H}^{k-1}(\mathcal{I})=0$.

Up to this point, the identity of a $BV$ function is unaffected by modifying its values on any set of ($k$-dimensional Lebesgue) measure zero, i.e., $BV_{\rm loc}(\mathcal{X})$ is actually a space of equivalence classes of functions, specified only up to a set of measure zero. However, when dealing with the finer behavior of these functions, it is expedient to designate a canonical representative of each equivalence class, with values specified up to a set of $(k-1)$-dimensional Hausdorff measure zero. This will be effected in the following way.

Suppose $g$ is a continuous function on $I\!R$ and let $v\in BV_{\rm loc}(\mathcal{X})$. With reference to the notation of Theorem 1.7.4, the *normalized composition* $\widetilde{g\circ v}$ of $g$ and $v$ is defined by

$$\widetilde{g\circ v}(X)=\begin{cases} g(v_0), & \text{if } X\in\mathcal{C}\\ \displaystyle\int_0^1 g(\tau v_- +(1-\tau)v_+)d\tau, & \text{if } X\in\mathcal{J}\end{cases} \tag{1.7.6}$$

and arbitrarily on the set $\mathcal{I}$ of irregular points, whose $(k-1)$-dimensional Hausdorff measure is zero. In particular, one may normalize $v$ itself:

$$\tilde{v}(X)=\begin{cases} v_0\,, & \text{if } X\in\mathcal{C}\\ \frac{1}{2}(v_-+v_+), & \text{if } X\in\mathcal{J}.\end{cases} \tag{1.7.7}$$

Thus every point of $\mathcal{C}$ becomes a Lebesgue point.

The appropriateness of the above normalization is indicated by the following generalization of the classical chain rule:

**1.7.5 Theorem.** *Assume* $g$ *is continuously differentiable on* $I\!R$*, with derivative* $\mathrm{D}g$*, and let* $v\in BV_{\rm loc}(\mathcal{X})\cap L^\infty(\mathcal{X})$. *Then* $g\circ v\in BV_{loc}(\mathcal{X})\cap L^\infty(\mathcal{X})$. *The normalized function* $\widetilde{\mathrm{D}g\circ v}$ *is locally integrable with respect to the measure* $\mathcal{M}=\mathrm{grad}\, v$ *and*

$$\mathrm{grad}\,(g\circ v)=(\widetilde{\mathrm{D}g\circ v})\,\mathrm{grad}\, v \tag{1.7.8}$$

*in the sense*

$$-\int_{\mathcal{X}} g(v(X))\,\mathrm{div}\,\Psi(X)\,dX=\int_{\mathcal{X}}(\widetilde{\mathrm{D}g\circ v})(X)\Psi(X)d\mathcal{M}(X), \tag{1.7.9}$$

*for any test function* $\Psi\in C_0^\infty(\mathcal{X};\, I\!M^{1\times k})$.

Next we review certain important geometric properties of a class of sets in $I\!R^k$ that are intimately related to the theory of $BV$ functions.

**1.7.6 Definition.** A subset $\mathcal{D}$ of $I\!R^k$ has (*locally*) *finite perimeter* when its indicator function $\chi_{\mathcal{D}}$ has (locally) bounded variation on $I\!R^k$.

Let us apply Theorem 1.7.4 to the indicator function $\chi_{\mathcal{D}}$ of a set $\mathcal{D}$ with locally finite perimeter. Clearly, the set $\mathcal{C}$ of points of approximate continuity of $\chi_{\mathcal{D}}$ is the union of the sets of density points of $\mathcal{D}$ and $R^k \backslash \mathcal{D}$. The complement of $\mathcal{C}$, i.e., the set of $X$ in $I\!R^k$ that are not points of density of either $\mathcal{D}$ or $R^k \backslash \mathcal{D}$, constitutes the *measure theoretic boundary* $\partial\mathcal{D}$ of $\mathcal{D}$. It can be shown that $\mathcal{D}$ has finite perimeter if and only if $\mathcal{H}^{k-1}(\partial\mathcal{D}) < \infty$, and its perimeter may be measured by $TV_{I\!R^k}\chi_{\mathcal{D}}$ or by $\mathcal{H}^{k-1}(\partial\mathcal{D})$. The set of points of approximate jump discontinuity of $\chi_{\mathcal{D}}$ is called the *reduced boundary* of $\mathcal{D}$ and is denoted by $\partial^*\mathcal{D}$. By Theorem 1.7.4, $\partial^*\mathcal{D} \subset \partial\mathcal{D}$, $\mathcal{H}^{k-1}(\partial\mathcal{D} \backslash \partial^*\mathcal{D}) = 0$, and $\partial^*\mathcal{D}$ is covered by the countable union of $C^1$ $(k-1)$-dimensional manifolds. Moreover, the vector $N \in S^{k-1}$ associated with each point $X$ of $\partial^*\mathcal{D}$ may naturally be interpreted as the *measure theoretic outward normal* to $\mathcal{D}$ at $X$. Sets with Lipschitz boundary have finite perimeter. In fact, the entire theory of balance laws may be reformulated by considering as proper domains sets that are not necessarily Lipschitz, as postulated in Section 1.1, but merely have finite perimeter.

**1.7.7 Definition.** Assume $\mathcal{D}$ has finite perimeter and let $v \in BV_{\mathrm{loc}}(I\!R^k)$. $v$ has *inward* and *outward traces* $v_-$ and $v_+$ at the point $\bar{X}$ of the reduced boundary $\partial^*\mathcal{D}$ of $\mathcal{D}$, where the outward normal is $N$, if

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int_{\mathcal{B}_r^{\pm}} |v(X) - v_\pm| dX = 0. \tag{1.7.10}$$

It can be shown that the traces $v_\pm$ are defined for almost all (with respect to $\mathcal{H}^{k-1}$) points of $\partial^*\mathcal{D}$ and are locally integrable on $\partial^*\mathcal{D}$. Furthermore, the following version of the Gauss-Green theorem holds:

**1.7.8 Theorem.** *Assume* $v \in BV(I\!R^k)$ *so* $\mathcal{M} = \mathrm{grad}\, v$ *is a finite measure. Consider any bounded set* $\mathcal{D}$ *of finite perimeter, with set of density points* $\mathcal{D}^*$ *and reduced boundary* $\partial^*\mathcal{D}$. *Then*

$$\mathcal{M}(\mathcal{D}^*) = \int_{\partial^*\mathcal{D}} v_+ N d\mathcal{H}^{k-1}. \tag{1.7.11}$$

*Furthermore, for any Borel subset* $\mathcal{F}$ *of* $\partial\mathcal{D}$,

$$\mathcal{M}(\mathcal{F}) = \int_{\mathcal{F}} (v_- - v_+) N d\mathcal{H}^{k-1}. \tag{1.7.12}$$

In particular, the set $\mathcal{J}$ of points of approximate jump discontinuity of any $v \in BV_{\mathrm{loc}}(I\!R^k)$ may be covered by the countable union of oriented surfaces and so (1.7.12) will hold for any measurable subset $\mathcal{F}$ of $\mathcal{J}$.

For $v \in BV(\mathcal{X})$, the measure $\mathcal{M} = \operatorname{grad} v$ may be decomposed into the sum of three mutually singular measures: its *continuous part,* which is absolutely continuous with respect to $k$-dimensional Lebesgue measure; its *jump part,* which is concentrated on the set $\mathcal{J}$ of points of approximate jump discontinuity of $v$; and its *Cantor part.* In particular, the Cantor part of the measure of any Borel subset of $\mathcal{X}$ with finite $(k-1)$-dimensional Hausdorff measure vanishes.

**1.7.9 Definition.** $v \in BV(\mathcal{X})$ is a *special function of bounded variation,* namely $v \in SBV(\mathcal{X})$, if the Cantor part of the measure $\operatorname{grad} v$ vanishes.

It turns out that $SBV(\mathcal{X})$ is a proper subspace of $BV(\mathcal{X})$ and it properly contains $W^{1,1}(\mathcal{X})$.

For $k = 1$, the theory of $BV$ functions is intimately related with the classical theory of functions of bounded variation. Assume $v$ is a $BV$ function on a (bounded or unbounded) interval $(a, b) \subset (-\infty, \infty)$. Let $\tilde{v}$ be the normalized form of $v$. Then

$$TV_{(a,b)}v = \sup \sum_{j=1}^{\ell-1} |\tilde{v}(x_{j+1}) - \tilde{v}(x_j)|, \tag{1.7.13}$$

where the supremum is taken over all (finite) meshes $a < x_1 < x_2 < \cdots < x_\ell < b$. Furthermore, (classical) one-sided limits $\tilde{v}(x\pm)$ exist at every $x \in (a, b)$ and are both equal to $\tilde{v}(x)$, except possibly on a countable set of points. When $k = 1$, the compactness Theorem 1.7.3 reduces to the classical *Helly theorem.*

Any $v \in SBV(a, b)$ is the sum of an absolutely continuous function and an (at most) countable sum of step functions. Accordingly, the measure $\operatorname{grad} v$ is the sum of the pointwise derivative $v'$ of $v$, which exists almost everywhere on $(a, b)$, and the (at most) countable sum of weighted Dirac masses, located at the points of jump discontinuity of $v$ and weighted by the jump.

A vector-valued function $U$ is of (locally) bounded variation on $\mathcal{X}$ when each one of its components has (locally) bounded variation on $\mathcal{X}$; and its total variation $TV_{\mathcal{X}}U$ is the sum of the total variations of its components. All of the discussions, above, for scalar-valued functions, and in particular the assertions of Theorems 1.7.2, 1.7.3, 1.7.4, 1.7.5 and 1.7.8, generalize immediately to (and will be used below for) vector-valued functions of bounded variation.

## 1.8 *BV* Solutions of Systems of Balance Laws

We consider here weak solutions $U \in L^\infty(\mathcal{X})$ of the system (1.4.3) of balance laws, which are in $BV_{\mathrm{loc}}(\mathcal{X})$. In that case, by virtue of Theorem 1.7.5, the function $G \circ U$ is also in $BV_{\mathrm{loc}}(\mathcal{X}) \cap L^\infty(\mathcal{X})$ and (1.4.3) is satisfied as an equality of measures. The first task is to examine the local form of (1.4.3), in the light of Theorems 1.7.4, 1.7.5, and 1.7.8.

**1.8.1 Theorem.** *A function* $U \in BV_{\mathrm{loc}}(\mathcal{X}) \cap L^\infty(\mathcal{X})$ *is a weak solution of the system* (1.4.3) *of balance laws if and only if* (a) *the measure equality*

$$[\mathrm{D}G(\tilde{U}(X), X)\,,\ \mathrm{grad}\, U(X)] + \nabla \cdot G(\tilde{U}(X), X) = \Pi(\tilde{U}(X), X) \tag{1.8.1}$$

*holds on the set* $\mathcal{C}$ *of points of approximate continuity of* $U$*; and* (b) *the jump condition*

$$[G(U_+, X) - G(U_-, X)]N = 0 \tag{1.8.2}$$

*is satisfied for almost all (with respect to* $\mathcal{H}^{k-1}$*)* $X$ *on the set* $\mathcal{J}$ *of points of approximate jump discontinuity of* $U$*, with normal vector* $N$ *and one-sided limits* $U_-$, $U_+$.

**Proof.** In (1.8.1) and in (1.8.6), (1.8.7), below, the symbol $\nabla\cdot$ denotes divergence with respect to $X$, holding $U$ fixed—as opposed to div which treats $U$ as a function of $X$. Let $\mathcal{M}$ denote the measure defined by the left-hand side of (1.4.3). On $\mathcal{C}$, $\mathcal{M}$ reduces to the measure on the left-hand side of (1.8.1), by virtue of Theorem 1.7.5. Recalling the Definition 1.7.7 of trace and the characterization of one-sided limits in Theorem 1.7.4, we deduce $(G \circ U)_\pm = G \circ U_\pm$ at every point of $\mathcal{J}$. Thus, if $\mathcal{F}$ is any Borel subset of $\mathcal{J}$, then by account of the remark following the proof of Theorem 1.7.8,

$$\mathcal{M}(\mathcal{F}) = \int_{\mathcal{F}} [G(U_-, X) - G(U_+, X)]N d\mathcal{H}^{k-1}. \tag{1.8.3}$$

Therefore, $\mathcal{M} = \Pi$ in the sense of measures if and only if (1.8.1) and (1.8.2) hold. This completes the proof.

Consequently, the set of points of approximate jump discontinuity of a $BV$ solution is the countable union of *shock fronts*.

As we saw in Section 1.5, when $G$ has a companion $Q$, the companion balance law (1.5.2) is automatically satisfied by any classical solution of (1.4.3). The following proposition describes the situation in the context of $BV$ weak solutions.

**1.8.2 Theorem.** *Assume the system of balance laws* (1.4.3) *is endowed with a companion balance law* (1.5.2). *Let* $U \in BV_{\mathrm{loc}}(\mathcal{X}) \cap L^\infty(\mathcal{X})$ *be a weak solution of* (1.4.3). *Then the measure*

$$\mathcal{N} = \mathrm{div}\, Q(U(X), X) - h(U(X), X) \tag{1.8.4}$$

*is concentrated on the set* $\mathcal{J}$ *of points of approximate jump discontinuity of* $U$ *and the inequality* (1.5.13) *will be satisfied in the sense of measures if and only if*

$$[Q(U_+, X) - Q(U_-, X)]N \geq 0 \tag{1.8.5}$$

*holds for almost all (with respect to $\mathcal{H}^{k-1}$) $X \in \mathcal{J}$.*

**Proof.** By virtue of Theorem 1.7.5, we may write (1.4.3) and (1.8.4) as

$$[\mathrm{D}\widetilde{G \circ}\, U, \operatorname{grad} U] + \nabla \cdot G - \Pi = 0, \tag{1.8.6}$$

$$\mathcal{N} = [\mathrm{D}\widetilde{Q \circ}\, U, \operatorname{grad} U] + \nabla \cdot Q - h. \tag{1.8.7}$$

By account of (1.7.6), if $X$ is in the set $\mathcal{C}$ of points of approximate continuity of $U$,

$$\mathrm{D}\widetilde{G \circ}\, U(X) = \mathrm{D}G(\tilde{U}(X), X), \qquad \mathrm{D}\widetilde{Q \circ}\, U(X) = \mathrm{D}Q(\tilde{U}(X), X). \tag{1.8.8}$$

Combining (1.8.6), (1.8.7), (1.8.8) and using (1.5.1), (1.5.3), we deduce that $\mathcal{N}$ vanishes on $\mathcal{C}$.

From the Definition 1.7.7 of trace and the characterization of one-sided limits in Theorem 1.7.4, we infer $(Q \circ U)_{\pm} = Q \circ U_{\pm}$. If $\mathcal{F}$ is a bounded Borel subset of $\mathcal{J}$, we apply (1.7.12), keeping in mind the remark following the proof of Theorem 1.7.8. This yields

$$\mathcal{N}(\mathcal{F}) = \int_{\mathcal{F}} [Q(U_-, X) - Q(U_+, X)] N d\mathcal{H}^{k-1}. \tag{1.8.9}$$

Therefore, $\mathcal{N} \leq 0$ if and only if (1.8.5) holds. This completes the proof.

## 1.9 Rapid Oscillations and the Stabilizing Effect of Companion Balance Laws

Consider a homogeneous system of conservation laws

$$\operatorname{div} G(U(X)) = 0 \tag{1.9.1}$$

and assume that

$$[G(W) - G(V)]N = 0 \tag{1.9.2}$$

holds for some states $V$, $W$ in $\mathcal{O}$ and $N \in S^{k-1}$. Then one may construct highly oscillatory weak solutions of (1.9.1) on $\mathbb{R}^k$ by the following procedure: Consider any finite family of parallel $(k-1)$-dimensional hyperplanes, all of them orthogonal to $N$, and define a function $U$ on $\mathbb{R}^k$ which is constant on each slab confined between adjacent hyperplanes, taking the values $V$ and $W$ in alternating order. It is clear that $U$ is a weak solution of (1.9.1), by virtue of (1.9.2) and Theorem 1.8.1.

One may thus construct a sequence of solutions that converges in $L^\infty$ weak* to some $U$ of the form $U(X) = \rho(X \cdot N)V + [1 - \rho(X \cdot N)]W$, where $\rho$ is any measurable function from $\mathbb{R}$ to $[0, 1]$. It is clear that, in general, such $U$ will not be a solution of (1.9.1), unless $G(\cdot)N$ happens to be affine along the straight line segment in $\mathbb{R}^n$ that connects $V$ to $W$. This type of instability distinguishes systems that may support shock fronts from elliptic systems that cannot.

Assume now $G$ is equipped with a companion $Q$ and $[Q(W) - Q(V)]N \neq 0$. Notice that imposing the admissibility condition div $Q(U) \leq 0$ would rule out the oscillating solutions constructed above, because, by virtue of Theorem 1.8.2, it would not allow to have jumps both from $V$ to $W$ and from $W$ to $V$, in the direction $N$. Consequently, inequalities (1.5.13) seem to play a stabilizing role. To what extent this stabilizing is effective will be a major issue for discussion in the book.

## 1.10 Notes

The principles of the theory of balance laws were conceived in the process of laying down the foundations of elasticity, in the 1820's. Theorem 1.2.1 has a long and celebrated history. The crucial discovery that the flux density is necessarily a linear function of the exterior normal was made by Cauchy [1,2]. The argument that the flux density through a surface may depend on the surface solely through its exterior normal is attributed to Hamel and to Noll [2]. The proof here borrows ideas from Ziemer [1]. With regard to the issue of retrieving the balance law from its field equation, which is addressed by Theorem 1.3.4, Chen and Frid [1,5,6] have developed a comprehensive theory of divergence measure fields which employs a fairly explicit construction of the trace, under the additional mild technical assumption that the surface may be foliated. For further developments of this approach, see Chen [9,10], Chen and Frid [8,9] and Chen and Torres [1]. An alternative, less explicit, functional analytic approach is found in Anzelotti [1].

The observation that systems of balance laws are endowed with nontrivial companions if and only if they are symmetrizable, is due to Godunov [1,2,3], and to Friedrichs and Lax [1]; see also Boillat [1] and Ruggeri and Strumia [1]. For a discussion of proper truncations of systems of balance laws arising in physics, see Boillat and Ruggeri [1].

In one space dimension, weak fronts are first encountered in the acoustic research of Euler while shock fronts were introduced by Stokes [1]. Fronts in several space dimensions were first studied by Christoffel [1]. The classical reference is Hadamard [1]. For a historical account of the early development of the subject, with emphasis on the contributions of Riemann and Christoffel, see Hölder [1]. The connection between shock fronts and phase transitions will not be pursued here. For references to this active area of research see Section 8.7.

Comprehensive expositions of the theory of $BV$ functions can be found in the treatise of Federer [1], the monographs of Giusti [1], and Ambrosio, Fusco and Pallara [1], and the texts of Evans and Gariepy [1] and Ziemer [2]. Theorems 1.7.5 and 1.7.8 are taken from Volpert [1]. The theory of special functions of bounded variation is elaborated in Ambrosio, Fusco and Pallara [1].

An insightful discussion of the issues raised in Section 1.9 is found in DiPerna [10]. These questions will be elucidated by the presentation of the method of compensated compactness, in Chapter XVI.

# II

# Introduction to Continuum Physics

In continuum physics, material bodies are modeled as continuous media whose motion and equilibrium is governed by balance laws and constitutive relations.

The list of balance laws identifies the theory, for example mechanics, thermomechanics, electrodynamics, etc. The referential (Lagrangian) and the spatial (Eulerian) formulation of the typical balance law will be presented. The balance laws of mass, momentum, energy, and the Clausius-Duhem inequality, which demarcate continuum thermomechanics, will be recorded.

The type of constitutive relation encodes the nature of material response. The constitutive equations of thermoelasticity and thermoviscoelasticity will be introduced. Restrictions imposed by the Second Law of thermodynamics, the principle of material frame indifference, and material symmetry will be discussed.

## 2.1 Bodies and Motions

The ambient space is $\mathbb{R}^m$, of dimension one, two or three. Two copies of $\mathbb{R}^m$ shall be employed, one for the *reference space*, the other for the *physical space.* A *body* is identified by a *reference configuration*, namely an open subset $\mathcal{B}$ of the reference space. Points of $\mathcal{B}$ will be called *particles*. The typical particle will be denoted by $x$ and time will be denoted by $t$.

A *placement* of the body is a bilipschitz homeomorphism of its reference configuration $\mathcal{B}$ to some open subset of the physical space. A *motion* of the body over the time interval $(t_1, t_2)$ is a Lipschitz map $\chi$ of $\mathcal{B} \times (t_1, t_2)$ to $\mathbb{R}^m$ whose restriction to each fixed $t$ in $(t_1, t_2)$ is a placement. Thus, for fixed $x \in \mathcal{B}$ and $t \in (t_1, t_2)$, $\chi(x, t)$ specifies the position in physical space of the particle $x$ at time $t$; for fixed $t \in (t_1, t_2)$, the map $\chi(\cdot, t) : \mathcal{B} \to \mathbb{R}^m$ yields the placement of the body at time $t$; finally, for fixed $x \in \mathcal{B}$, the curve $\chi(x, \cdot) : (t_1, t_2) \to \mathbb{R}^m$ describes the trajectory of the particle $x$ in physical space. See Fig. 2.1.1.

The reference configuration generally renders an abstract representation of the body. In practice, however, one often identifies the reference space with the physical space and employs as reference configuration an actual placement of the body, by

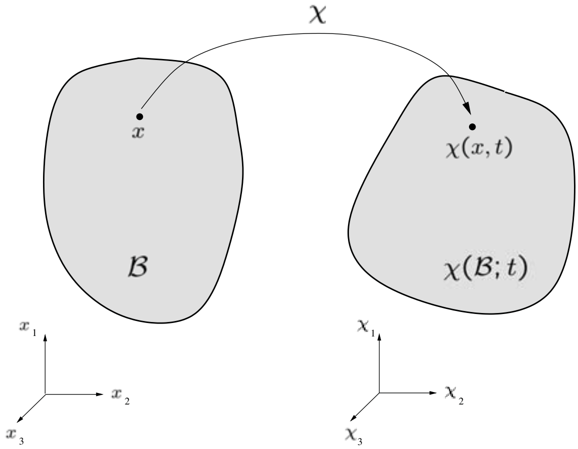


**Fig. 2.1.1**

identifying material particles with the point in physical space that they happen to occupy in that particular placement.

The aim of continuum physics is to monitor the evolution of various fields associated with the body, such as density, stress, temperature, etc. In the *referential* or *Lagrangian* description, one follows the evolution of fields along particle trajectories, while in the *spatial* or *Eulerian* description one monitors the evolution of fields at fixed position in space. The motion allows one to pass from one formulation to the other. For example, considering some illustrative field $w$, we write $w = f(x,t)$ for its referential description and $w = \phi(\chi, t)$ for its spatial description. The motion relates $f$ and $\phi$ by $\phi(\chi(x,t),t) = f(x,t)$, for $x \in \mathcal{B},\ t \in (t_1, t_2)$.

Either formulation has its relative merits, so both will be used here. Thus, in order to keep proper accounting, three symbols should be needed for each field, one to identify it, one for its referential description, and one for its spatial description ($w$, $f$, and $\phi$ in the example, above). However, in order to control the proliferation of symbols and make the physical interpretation of the equations transparent, the standard notational convention is to employ the single identifying symbol of the field for all three purposes. To prevent ambiguity in the notation of derivatives, the following rules will apply: Partial differentiation with respect to $t$ will be denoted by an overdot in the referential description and by a $t$-subscript in the spatial description. Gradient, differential and divergence[1] will be denoted by Grad, $\nabla$ and Div, with respect to the material variable $x$, and by grad, d and div, with respect to the spatial variable $\chi$. Thus, referring again to the typical field $w$ with referential description $w = f(x,t)$ and spatial description $w = \phi(\chi, t)$, $\dot{w}$ will denote $\partial f/\partial t$, $w_t$ will denote $\partial\phi/\partial t$, $\mathrm{Grad}\, w$ will denote $\mathrm{grad}_x f$, and $\mathrm{grad}\, w$ will denote $\mathrm{grad}_\chi \phi$. This

[1] For consistency with matrix notations, gradients will be realized as $m$-column vectors and differentials will be $m$-row vectors, namely the transpose of gradients. As in Chapter I, the divergence operator will be acting on row vectors.

notation may appear confusing at first but the student of the subject soon learns to use it efficiently and correctly.

The motion $\chi$ induces two important kinematical fields, namely the *velocity*

$$v = \dot{\chi}, \tag{2.1.1}$$

in $L^\infty(\mathcal{B} \times (t_1, t_2); \mathbb{R}^m)$, and the *deformation gradient,* which, its name notwithstanding, is the differential of the motion:

$$F = \nabla\chi, \tag{2.1.2}$$

in $L^\infty(\mathcal{B} \times (t_1, t_2); \mathbb{M}^{m\times m})$. In accordance with the definition of placement, we shall be assuming

$$\det F \geq a > 0 \qquad \text{a.e.} \tag{2.1.3}$$

These fields allow one to pass from spatial to material derivatives; for example, assuming $w$ is a Lipschitz field,

$$\dot{w} = w_t + (\mathrm{d}w)v, \tag{2.1.4}$$

$$\mathrm{Grad}\, w = F^\top \mathrm{grad}\, w, \quad \nabla w = (\mathrm{d}w)F. \tag{2.1.5}$$

By virtue of the polar decomposition theorem, the local deformation of the medium, expressed by the deformation gradient $F$, may be realized as the composition of a pure stretching and a rotation:

$$F = RU, \tag{2.1.6}$$

where the symmetric, positive definite matrix

$$U = (F^\top F)^{1/2} \tag{2.1.7}$$

is called the *right stretch tensor* and the proper orthogonal matrix $R$ is called the *rotation tensor.*

Turning to the rate of change of deformation, we introduce the referential and spatial *velocity gradients* (which are actually differentials):

$$\dot{F} = \nabla v, \quad L = \mathrm{d}v. \tag{2.1.8}$$

$L$ is decomposed into the sum of the symmetric *stretching tensor* $D$ and the skew-symmetric *spin tensor* $W$:

$$L = D + W, \quad D = \tfrac{1}{2}(L + L^\top), \quad W = \tfrac{1}{2}(L - L^\top). \tag{2.1.9}$$

The axial vector $\omega = \mathrm{curl}\, v$ of $W$ is the *vorticity.*

The class of Lipschitz continuous motions allows for shocks but is not sufficiently broad to also encompass motions involving cavitation in elasticity, vortices in hydrodynamics, vacuum in gas dynamics, etc. Even so, we shall continue to develop the theory under the assumption that motions are Lipschitz continuous, deferring considerations of generalization until such need arises.

## 2.2 Balance Laws in Continuum Physics

Consider a motion $\chi$ of a body with reference configuration $\mathcal{B} \subset \mathbb{R}^m$, over a time interval $(t_1, t_2)$. The typical balance law of continuum physics postulates that the change over any time interval in the amount of a certain extensive quantity stored in any part of the body is balanced by a flux through the boundary and a production in the interior during that time interval. With space and time fused into space-time, the above statement yields a balance law of the type considered in Chapter I, ultimately reducing to a field equation of the form (1.2.3).

To adapt to the present setting the notation of Chapter I, we take space-time $\mathbb{R}^{m+1}$ as the ambient space $\mathbb{R}^k$, and set $\mathcal{X} = \mathcal{B} \times (t_1, t_2)$, $X = (x, t)$. With reference to (1.4.1), we decompose the flux density field $A$ into a $n \times m$ matrix-valued spatial part $\Psi$ and a $\mathbb{R}^n$-valued temporal part $\Theta$, namely $A = [-\Psi, \Theta]$. In the notation introduced in the previous section, (1.4.1) now takes the form

$$\dot{\Theta} = \operatorname{Div} \Psi + P. \tag{2.2.1}$$

This is the referential field equation for the typical balance law of continuum physics. The field $\Theta$ is the density of the balanced quantity; $\Psi$ is the flux density field through material surfaces; and $P$ is the production density.

The corresponding spatial field equation may be derived by appealing to Theorem 1.3.1. The map $X^*$ that carries $(x, t)$ to $(\chi(x, t), t)$ is a bilipschitz homeomorphism of $\mathcal{X}$ to some subset $\mathcal{X}^*$ of $\mathbb{R}^{m+1}$, with Jacobian matrix (cf. (1.3.2), (2.1.1), and (2.1.2)):

$$J = \left[\begin{array}{c|c} F & v \\ \hline 0 & 1 \end{array}\right]. \tag{2.2.2}$$

Notice that (1.3.3) is satisfied by virtue of (2.1.3). Theorem 1.3.1 and Remark 1.3.2 now imply that if $\Theta \in L^1_{loc}(\mathcal{X}; \mathbb{R}^n)$, $\Psi \in L^1_{loc}(\mathcal{X}; \mathbb{M}^{n\times k})$ and $P \in L^1_{loc}(\mathcal{X}; \mathbb{R}^n)$, then (2.2.1) holds in the sense of distributions on $\mathcal{X}$ if and only if

$$\Theta^*_t + \operatorname{div}(\Theta^* v^\top) = \operatorname{div} \Psi^* + P^* \tag{2.2.3}$$

as distributions on $\mathcal{X}^*$, where $\Theta^* \in L^1_{loc}(\mathcal{X}^*; I\!R^n)$, $\Psi^* \in L^1_{loc}(\mathcal{X}^*; I\!M^{n\times m})$ and $P^* \in L^1_{loc}(\mathcal{X}^*, I\!R^n)$ are defined by

$$\Theta^* = (\det F)^{-1}\Theta, \quad \Psi^* = (\det F)^{-1}\Psi F^\top, \quad P^* = (\det F)^{-1}P. \tag{2.2.4}$$

It has thus been established that the referential (Lagrangian) field equations (2.2.1) and the spatial (Eulerian) field equations (2.2.3) of the balance laws of continuum physics are related by (2.2.4) and are equivalent within the function class of fields considered here.

In anticipation of the forthcoming discussion of material symmetry, it is useful to investigate how the fields $\Theta$, $\Psi$ and $\Theta^*$, $\Psi^*$ transform under *isochoric* changes of the reference configuration of the body, induced by a bilipschitz homeomorphism $\bar{x}$ of $\mathcal{B}$ to some subset $\bar{\mathcal{B}}$ of another reference space $I\!R^m$, with Jacobian matrix

$$H = \frac{\partial \bar{x}}{\partial x}, \quad \det H = 1, \tag{2.2.5}$$

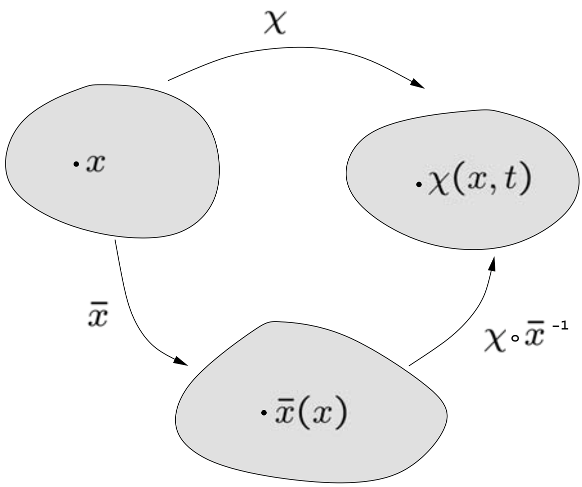


**Fig. 2.2.1**

see Figure 2.2.1. By virtue of Theorem 1.3.1, the Lagrangian field equation (2.2.1) on $\mathcal{B}$ will transform into an equation of exactly the same form on $\bar{\mathcal{B}}$, with fields $\bar{\Theta}$ and $\bar{\Psi}$ related to $\Theta$ and $\Psi$ by

$$\bar{\Theta} = \Theta, \qquad \bar{\Psi} = \Psi H^\top. \tag{2.2.6}$$

In the corresponding Eulerian field equations, the fields $\bar{\Theta}^*$ and $\bar{\Psi}^*$ are obtained through (2.2.4): $\bar{\Theta}^* = (\det \bar{F})^{-1}\bar{\Theta}$ and $\bar{\Psi}^* = (\det \bar{F})^{-1}\bar{\Psi}\,\bar{F}^\top$, where $\bar{F}$ denotes the deformation gradient relative to the new reference configuration $\bar{\mathcal{B}}$. By the chain rule, $\bar{F} = FH^{-1}$ and so

$$\bar{\Theta}^* = \Theta^*, \qquad \bar{\Psi}^* = \Psi^*, \tag{2.2.7}$$

i.e., as was to be expected, the spatial fields are not affected by changing the reference configuration of the body.

In continuum physics, theories are identified by means of the list of balance laws that apply in their context. The illustrative example of thermomechanics will be presented in the next section. It should be noted, however, that in addition to balance laws with physical content there are others that simply express useful, purely kinematic properties. Equation (2.1.8), $\dot{F} = \nabla v$, which expresses the compatibility between the fields $F$ and $v$, provides an example in that direction.

At first reading, one may skip the remainder of this section, which deals with a special topic for future use, and pass directly to the next Section 2.3.

In what follows, we derive, for $m = 3$, a set of kinematic balance laws whose referential form is quite complicated and yet whose spatial form is very simple or even trivial. This will demonstrate the usefulness of switching from the Lagrangian to the Eulerian formulation and vice versa.

A smooth function $\varphi$ on $\mathbb{M}^{3\times 3}$ is called a *null Lagrangian* if the Euler-Lagrange equation

$$\mathrm{Div}\big[\partial_F \varphi(F)\big] = 0, \tag{2.2.8}$$

associated with the functional $\int \varphi(F)dx$, holds for every smooth deformation gradient field $F$. Any null Lagrangian $\varphi$ admits a representation as an affine function

$$\varphi(F) = \mathrm{tr}(AF) + \mathrm{tr}(BF^*) + \alpha \det F + \beta \tag{2.2.9}$$

of $F$ itself, its determinant $\det F$, and its cofactor matrix $F^* = (\det F)F^{-1} = (\partial_F \det F)^\top$.

By combining (2.2.8) with $\dot{F} = \nabla v$, one deduces that if $\varphi$ is any null Langrangian (2.2.9), then the conservation law

$$\dot{\varphi}(F) = \mathrm{Div}\big[v^\top \partial_F \varphi(F)\big] \tag{2.2.10}$$

holds for any smooth motion with deformation gradient $F$ and velocity $v$.

The aim here is to show that, for any null Lagrangian (2.2.9), the "quasi-static" conservation law (2.2.8) as well as the "kinematic" conservation law (2.2.10) actually hold even for motions that are merely Lipschitz continuous, i.e.

$$\mathrm{Div}\big(\partial_F F\big) = 0, \tag{2.2.11}$$

$$\mathrm{Div}\big(\partial_F F^*\big) = 0, \tag{2.2.12}$$

$$\mathrm{Div}\big(\partial_F \det F\big) = 0, \tag{2.2.13}$$

$$\dot{F} = \mathrm{Div}\big(v^\top \partial_F F\big), \tag{2.2.14}$$

$$\dot{F}^* = \mathrm{Div}\big(v^\top \partial_F F^*\big), \tag{2.2.15}$$

$$\overline{\det F}^{\,\cdot} = \mathrm{Div}\big(v^\top \partial_F \det F\big), \tag{2.2.16}$$

for any bounded measurable deformation gradient field $F$ and velocity field $v$.

Clearly, (2.2.11) is obvious and (2.2.14) is just an alternative way of writing the familiar $\dot{F} = \nabla v$. Furthermore, since

$$\frac{\partial F^*_{\alpha i}}{\partial F_{j\beta}} = \sum_{k=1}^{3} \sum_{\gamma=1}^{3} \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} F_{k\gamma}\ , \tag{2.2.17}$$

where $\epsilon_{ijk}$ and $\epsilon_{\alpha\beta\gamma}$ are the standard permutation symbols, (2.2.12) follows from the observation that $\partial F_{k\gamma}/\partial x_\beta = \partial^2\chi_k/\partial x_\beta \partial x_\gamma$ is symmetric in $(\beta, \gamma)$ while $\epsilon_{\alpha\beta\gamma}$ is skew-symmetric in $(\beta, \gamma)$.

To see (2.2.13), consider the trivial balance law (2.2.3), with $\Theta^* = 0$, $\Psi^* = I$, $P^* = 0$, and write its Langrangian form (2.2.1), where by account of (2.2.4), $\Theta = 0$, $\Psi = (\det F)(F^\top)^{-1} = (F^*)^\top = \partial \det F/\partial F$, $P = 0$. Similarly, (2.2.16) is the Lagrangian form (2.2.1) of the trivial balance law (2.2.3), with $\Theta^* = 1$, $\Psi^* = v^\top$, and $P^* = 0$. Indeed, in that case (2.2.4) yields $\Theta = \det F$, $\Psi = (\det F)(F^{-1}v)^\top = (F^* v)^\top = v^\top(\partial \det F/\partial F)$, and $P = 0$.

It remains to verify (2.2.15). We begin with the simple conservation law

$$(F^{-1})_t = (\mathrm{d}x)_t = \mathrm{d}x_t = -\mathrm{d}(F^{-1}v), \tag{2.2.18}$$

in Eulerian coordinates, and derive its Lagrangian form (2.2.1), through (2.2.4). Thus $\Theta = (\det F)F^{-1} = F^*$, while the flux $\Psi$, in components form, reads

$$\Psi_{\alpha i\beta} = \sum_{j=1}^{3} (\det F)\Big[F^{-1}_{\beta j} F^{-1}_{\alpha i} - F^{-1}_{\alpha j} F^{-1}_{\beta i}\Big] v_j\ . \tag{2.2.19}$$

The quantity in brackets vanishes when $\alpha = \beta$ and/or $i = j$; otherwise, it represents a minor of the matrix $F^{-1}$ and thus is equal to $\det F^{-1}$ multiplied by the corresponding entry of the matrix $(F^{-1})^{-1} = F$. Hence, recalling (2.2.17),

$$(\det F)\Big[F^{-1}_{\beta j} F^{-1}_{\alpha i} - F^{-1}_{\alpha j} F^{-1}_{\beta i}\Big] = \sum_{k=1}^{3} \sum_{\gamma=1}^{3} \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} F_{k\gamma} = \frac{\partial F^*_{\alpha i}}{\partial F_{j\beta}}\ , \tag{2.2.20}$$

and this establishes (2.2.15).

## 2.3 The Balance Laws of Continuum Thermomechanics

Continuum thermomechanics, which will serve as a representative model throughout this work, is demarcated by the balance laws of mass, linear momentum, angular momentum, energy, and entropy whose referential and spatial field equations will now be introduced.

In the *balance law of mass,* there is neither flux nor production so the referential and spatial field equations read

$$\dot{\rho}_0 = 0, \tag{2.3.1}$$

$$\rho_t + \operatorname{div}(\rho v^\top) = 0, \tag{2.3.2}$$

where $\rho_0$ is the *reference density* and $\rho$ is the *density* associated with the motion, related through

$$\rho = \rho_0 (\det F)^{-1}. \tag{2.3.3}$$

Note that (2.3.1) implies that the value of the reference density associated with a particle does not vary with time: $\rho_0 = \rho_0(x)$.

In the *balance law of linear momentum*, the production is induced by the *body force* (per unit mass) vector $b$, with values in $\mathbb{R}^m$, while the flux is represented by a *stress tensor* taking values in $\mathbb{M}^{m\times m}$. The referential and spatial field equations read

$$(\rho_0 v)^{\cdot} = \operatorname{Div} S + \rho_0 b, \tag{2.3.4}$$

$$(\rho v)_t + \operatorname{div}(\rho v v^\top) = \operatorname{div} T + \rho b, \tag{2.3.5}$$

where $S$ denotes the *Piola-Kirchhoff stress* and $T$ denotes the *Cauchy stress*, related by

$$T = (\det F)^{-1} S F^\top. \tag{2.3.6}$$

For any unit vector $\nu$, the value of $S\nu$ at $(x,t)$ yields the stress (force per unit area) vector transmitted at the particle $x$ and time $t$ across a material surface with normal $\nu$; while the value of $T\nu$ at $(\chi,t)$ gives the stress vector transmitted at the point $\chi$ in space and time $t$ across a spatial surface with normal $\nu$.

In the *balance law of angular momentum*, production and flux are the moments about the origin of the production and flux involved in the balance of linear momentum. Consequently, the referential field equation is

$$(\chi \wedge \rho_0 v)^{\cdot} = \operatorname{Div}(\chi \wedge S) + \chi \wedge \rho_0 b, \tag{2.3.7}$$

where $\wedge$ denotes cross product. Under the assumption that $\rho_0 v$, $S$ and $\rho_0 b$ are in $L^1_{\text{loc}}$ while the motion $\chi$ is Lipschitz continuous, we may use (2.3.4), (2.1.1) and (2.1.2) to reduce (2.3.7) into

$$S F^\top = F S^\top. \tag{2.3.8}$$

Similarly, the spatial field equation of the balance of angular momentum reduces, by virtue of (2.3.5), to the statement that the Cauchy stress tensor is symmetric:

$$T^\top = T. \tag{2.3.9}$$

There is no need to perform that calculation since (2.3.9) also follows directly from (2.3.6) and (2.3.8).

In the *balance law of energy*, the energy density is the sum of the (specific) *internal energy* (per unit mass) $\varepsilon$ and kinetic energy. The production is the sum of the rate of work of the body force and the *heat supply* (per unit mass) $r$. Finally, the flux is the sum of the rate of work of the stress tensor and the *heat flux*. The referential and spatial field equations thus read

$$(\rho_0\varepsilon + \tfrac{1}{2}\rho_0|v|^2)^{\cdot} = \mathrm{Div}(v^\top S + Q^\top) + \rho_0 v^\top b + \rho_0 r, \tag{2.3.10}$$

$$(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)_t + \mathrm{div}[(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)v^\top] = \mathrm{div}(v^\top T + q^\top) + \rho v^\top b + \rho r, \tag{2.3.11}$$

where the referential and spatial heat flux vectors $Q$ and $q$, with values in $\mathbb{R}^m$, are related by

$$q = (\det F)^{-1} F Q. \tag{2.3.12}$$

Finally, the *balance law of entropy* is expressed by the *Clausius-Duhem inequality*

$$(\rho_0 s)^{\cdot} \geq \mathrm{Div}\left(\frac{1}{\theta} Q^\top\right) + \rho_0 \frac{r}{\theta}\,, \tag{2.3.13}$$

$$(\rho s)_t + \mathrm{div}(\rho s v^\top) \geq \mathrm{div}\left(\frac{1}{\theta} q^\top\right) + \rho \frac{r}{\theta}\,, \tag{2.3.14}$$

in its referential and spatial form, respectively. The symbol $s$ stands for (specific) *entropy* and $\theta$ denotes the (absolute) *temperature*. Thus, the *entropy flux* is just the heat flux divided by temperature. The term $\frac{r}{\theta}$ represents the external entropy supply (per unit mass), induced by the heat supply $r$. However, the fact that (2.3.13) and (2.3.14) are mere inequalities rather than equalities signifies that there may be additional *internal entropy production,* which is not specified a priori in the context of this theory, apart from being constrained to be nonnegative. This last condition is dictated by (and in fact expresses) the *Second Law of thermodynamics*. As a nonnegative distribution, the internal entropy production is necessarily a measure $\mathcal{N}$. Adding $\mathcal{N}$ to the right-hand side, turns the Clausius-Duhem inequality into an equality which, by virtue of Theorem 1.3.3, is the field equation of a balance law.

The motion together with the entropy (or temperature) field constitute a *thermodynamic process.* The fields of internal energy, stress, heat flux, and temperature (or entropy) are determined from the thermodynamic process by means of constitutive relations that characterize the material response of the body. In particular, the constitutive equation for the stress is required to satisfy identically the balance law of angular momentum as expressed by (2.3.8) or (2.3.9). Representative material classes will be introduced in the following Sections, 2.5 and 2.6.

The field equations of the balance laws of mass, linear momentum and energy, coupled with the constitutive relations, render a closed system of evolution equations that should determine the thermodynamic process from assigned body force field $b$, heat supply field $r$, boundary conditions, and initial conditions.

The remaining balance law of entropy plays a markedly different role. The Clausius-Duhem inequality (2.3.13) or (2.3.14) is regarded as a criterion of *thermodynamic admissibility* for thermodynamic processes that already comply with the balance laws of mass, momentum and energy. In this regard, smooth thermodynamic processes are treated differently from thermodynamic processes with discontinuities.

It is a tenet of continuum thermodynamics that the constitutive relations should be constrained by the requirement that any smooth thermodynamic process that balances mass, momentum and energy must be automatically thermodynamically admissible. To implement this requisite, the first step is to derive from the Clausius-Duhem inequality the *dissipation inequality*

$$\rho_0\dot{\varepsilon} - \rho_0\theta\dot{s} - \operatorname{tr}(S\dot{F}^\top) - \frac{1}{\theta}Q\cdot G \le 0, \tag{2.3.15}$$

$$\rho\dot{\varepsilon} - \rho\theta\dot{s} - \operatorname{tr}(TD) - \frac{1}{\theta}q\cdot g \le 0, \tag{2.3.16}$$

in Lagrangian or Eulerian form, respectively, which does not involve the extraneously assigned body force and heat supply. The new symbols $G$ and $g$ appearing in (2.3.15) and (2.3.16) denote the *temperature gradient:*

$$G = \operatorname{Grad}\theta, \quad g = \operatorname{grad}\theta, \quad G = F^\top g. \tag{2.3.17}$$

To establish (2.3.15), one first eliminates the body force $b$ between the field equations (2.3.1), (2.3.4) and (2.3.10) of the balance laws of mass, linear momentum and energy to get

$$\rho_0\dot{\varepsilon} = \operatorname{tr}(S\dot{F}^\top) + \operatorname{Div} Q^\top + \rho_0 r, \tag{2.3.18}$$

and then eliminates the heat supply $r$ between the above equation and the Clausius-Duhem inequality (2.3.13). Similarly, (2.3.16) is obtained by combining (2.3.2), (2.3.5) and (2.3.11) with (2.3.14) in order to eliminate $b$ and $r$. Of course, (2.3.15) and (2.3.16) are equivalent: either one implies the other by virtue of (2.3.3), (2.3.6), (2.3.17), (2.1.9) and (2.3.9). In the above calculations it is crucial that the underlying thermodynamic process is assumed smooth, because this allows us to apply the classical product rule of differentiation on terms like $|v|^2$, $v^\top S$, $\theta^{-1}Q$ etc., which induces substantial cancelation. It should be emphasized that the dissipation inequalities (2.3.15) and (2.3.16) are generally meaningless for thermodynamic processes with discontinuities.

The constitutive equations are required to satisfy identically the dissipation inequality (2.3.15) or (2.3.16), which will guarantee that any smooth thermodynamic process that balances mass, momentum and energy is automatically thermodynamically admissible. The implementation of this requisite for specific material classes will be demonstrated in the following Sections 2.5 and 2.6.

Beyond taking care of smooth thermodynamic processes, as above, the Clausius-Duhem inequality is charged with the additional responsibility of certifying the thermodynamic admissibility of discontinuous processes. This is a central issue, with many facets, which will surface repeatedly in the remainder of the book.

When dealing with continuous media with complex structure, e.g. mixtures of different materials, it becomes necessary to replace the Clausius-Duhem inequality with a more general entropy inequality in which the entropy flux is no longer taken a priori as heat flux divided by temperature but is instead specified by an individual constitutive relation. It turns out, however, that in the context of thermoelastic or thermoviscoelastic media, which are the main concern of this work, the requirement that such an inequality must hold identically for any smooth thermodynamic process that balances mass, momentum and energy implies in particular that entropy flux is necessarily heat flux divided by temperature, so that we fall back to the classical Clausius-Duhem inequality.

To prepare the ground for forthcoming investigation of material symmetry, it is necessary to discuss the law of transformation of the fields involved in the balance laws when the reference configuration undergoes a change induced by an isochoric bilipshitz homeomorphism $\bar{x}$, with unimodular Jacobian matrix $H$ (2.2.5); see Fig. 2.2.1. The deformation gradient $F$ and the stretching tensor $D$ (cf. (2.1.9)) will transform into new fields $\bar{F}$ and $\bar{D}$:

(2.3.19) $$\bar{F} = FH^{-1}, \qquad \bar{D} = D.$$

The reference density $\rho_0$, internal energy $\varepsilon$, Piola-Kirchhoff stress $S$, entropy $s$, temperature $\theta$, referential heat flux vector $Q$, density $\rho$, Cauchy stress $T$, and spatial heat flux vector $q$, involved in the balance laws, will also transform into new fields $\bar{\rho}_0$, $\bar{\varepsilon}$, $\bar{S}$, $\bar{s}$, $\bar{\theta}$, $\bar{Q}$, $\bar{\rho}$, $\bar{T}$, and $\bar{q}$ according to the rule (2.2.6) or (2.2.7), namely,

(2.3.20) $$\bar{\rho}_0 = \rho_0, \quad \bar{\varepsilon} = \varepsilon, \quad \bar{S} = SH^{\top}, \quad \bar{s} = s, \quad \bar{\theta} = \theta, \quad \bar{Q} = HQ,$$

(2.3.21) $$\bar{\rho} = \rho, \quad \bar{T} = T, \quad \bar{q} = q.$$

Also the referential and spatial temperature gradients $G$ and $g$ will transform into $\bar{G}$ and $\bar{g}$ with

(2.3.22) $$\bar{G} = (H^{-1})^{\top}G, \qquad \bar{g} = g.$$

## 2.4 Material Frame Indifference

The body force and heat supply are usually induced by external factors and are assigned in advance, while the fields of internal energy, stress, entropy and heat flux are determined by the thermodynamic process. Motions may influence these fields in as much as they deform the body: Rigid motions, which do not change the distance between particles, should have no effect on internal energy, temperature or referential heat flux and should affect the stress tensor in a manner that the resulting stress vector, observed from a frame attached to the moving body, looks fixed. This requirement is postulated by the fundamental *principle of material frame indifference* which will now be stated with precision.

Consider any two thermodynamic processes $(\chi, s)$ and $(\chi^{\#}, s^{\#})$ of the body such that the entropy fields coincide, $s^{\#} = s$, while the motions differ by a rigid (time dependent) rotation[2]:

$$\chi^{\#}(x,t) = O(t)\chi(x,t), \qquad x \in \mathcal{B}, \ \ t \in (t_1, t_2), \tag{2.4.1}$$

$$O^{\top}(t)O(t) = O(t)O^{\top}(t) = I, \qquad \det O(t) = 1, \quad t \in (t_1, t_2). \tag{2.4.2}$$

Note that the fields of deformation gradient $F$, $F^{\#}$, spatial velocity gradient $L$, $L^{\#}$ and stretching tensor $D$, $D^{\#}$ (cf. (2.1.8), (2.1.9)) of the two processes $(\chi, s)$, $(\chi^{\#}, s^{\#})$ are related by

$$F^{\#} = OF, \quad L^{\#} = OLO^{\top} + \dot{O}O^{\top}, \quad D^{\#} = ODO^{\top}. \tag{2.4.3}$$

Let $(\varepsilon, S, \theta, Q)$ and $(\varepsilon^{\#}, S^{\#}, \theta^{\#}, Q^{\#})$ denote the fields for internal energy, Piola-Kirchhoff stress, temperature and referential heat flux associated with the processes $(\chi, s)$ and $(\chi^{\#}, s^{\#})$. The principle of material frame indifference postulates:

$$\varepsilon^{\#} = \varepsilon, \quad S^{\#} = OS, \quad \theta^{\#} = \theta, \quad Q^{\#} = Q. \tag{2.4.4}$$

From (2.4.4), (2.3.17) and (2.4.3) it follows that the referential and spatial temperature gradients $G$, $G^{\#}$ and $g$, $g^{\#}$ of the two processes are related by

$$G^{\#} = G, \qquad g^{\#} = Og. \tag{2.4.5}$$

Furthermore, from (2.3.6), (2.3.12) and (2.4.3) we deduce the following relations between the Cauchy stress tensors $T$, $T^{\#}$ and the spatial heat flux vectors $q$, $q^{\#}$ of the two processes:

$$T^{\#} = OTO^{\top}, \qquad q^{\#} = Oq. \tag{2.4.6}$$

The principle of material frame indifference should be reflected in the constitutive relations of continuous media, irrespectively of the nature of material response. Illustrative examples will be considered in the following two sections.

## 2.5 Thermoelasticity

In the framework of continuum thermomechanics, a *thermoelastic* medium is identified by the constitutive assumption that, for any fixed particle $x$ and any motion, the

[2] An alternative, albeit equivalent, realization of this setting is to visualize a single thermodynamic process monitored by two observers attached to individual coordinate frames that rotate relative to each other. When adopting that approach, certain authors are allowing for reflections, in addition to proper rotations.

value of the internal energy $\varepsilon$, the Piola-Kirchhoff stress $S$, the temperature $\theta$, and the referential heat flux vector $Q$, at $x$ and time $t$, is determined solely by the value at $(x,t)$ of the deformation gradient $F$, the entropy $s$, and the temperature gradient $G$, through constitutive equations

$$
\begin{cases} \varepsilon = \hat{\varepsilon}(F,s,G), \\ S = \hat{S}(F,s,G), \\ \theta = \hat{\theta}(F,s,G), \\ Q = \hat{Q}(F,s,G), \end{cases} \tag{2.5.1}
$$

where $\hat{\varepsilon}$, $\hat{S}$, $\hat{\theta}$, $\hat{Q}$ are smooth functions defined on the subset of $\mathbb{M}^{m\times m} \times \mathbb{R} \times \mathbb{R}^m$ with $\det F > 0$. Moreover, $\hat{\theta}(F,s,G) > 0$. When the thermoelastic medium is *homogeneous*, the same functions $\hat{\varepsilon}$, $\hat{S}$, $\hat{\theta}$, $\hat{Q}$ and the same value $\rho_0$ of the reference density apply to all particles $x \in \mathcal{B}$.

The Cauchy stress $T$ and the spatial heat flux $q$ are also determined by constitutive equations of the same form, which may be derived from (2.5.1) and (2.3.6), (2.3.12). When employing the spatial description of the motion, it is natural to substitute on the list (2.5.1) the constitutive equations of $T$ and $q$ for the constitutive equations of $S$ and $Q$; also on the list $(F,s,G)$ of the state variables to replace the referential temperature gradient $G$ with the spatial temperature gradient $g$ (cf. (2.3.17)).

The above constitutive equations will have to comply with the conditions stipulated earlier. To begin with, as postulated in Section 2.3, every smooth thermodynamic process that balances mass, momentum and energy must satisfy identically the Clausius-Duhem inequality (2.3.13) or, equivalently, the dissipation inequality (2.3.15). Substituting from (2.5.1) into (2.3.15) yields

$$
\operatorname{tr}[(\rho_0 \partial_F \hat{\varepsilon} - \hat{S})\dot{F}^\top] + \rho_0(\partial_s \hat{\varepsilon} - \hat{\theta})\dot{s} + \rho_0 \partial_G \hat{\varepsilon}\, \dot{G} - \hat{\theta}^{-1}\hat{Q}\cdot G \le 0. \tag{2.5.2}
$$

It is clear that by suitably controlling the body force $b$ and the heat supply $r$ one may construct smooth processes which balance mass, momentum and energy and attain at some point $(x,t)$ arbitrarily prescribed values for $F$, $s$, $G$, $\dot{F}$, $\dot{s}$, $\dot{G}$, subject only to the constraint $\det F > 0$. Hence (2.5.2) cannot hold identically unless the constitutive relations (2.5.1) are of the following special form:

$$
\begin{cases} \varepsilon = \hat{\varepsilon}(F,s), \\ S = \rho_0 \partial_F \hat{\varepsilon}(F,s), \\ \theta = \partial_s \hat{\varepsilon}(F,s), \\ Q = \hat{Q}(F,s,G), \end{cases} \tag{2.5.3}
$$

$$\hat{Q}(F, s, G) \cdot G \geq 0. \tag{2.5.4}$$

Thus the internal energy may depend on the deformation gradient and on the entropy but not on the temperature gradient. The constitutive equations for stress and temperature are induced by the constitutive equation of internal energy, through caloric relations, and are likewise independent of the temperature gradient. Only the heat flux may depend on the temperature gradient, subject to the condition (2.5.4) which implies that heat always flows from the hotter to the colder part of the body.

Another requirement on constitutive relations is that they observe the principle of material frame indifference, formulated in Section 2.4. By combining (2.4.4) and $(2.4.3)_1$ with (2.5.3), we deduce that the functions $\hat{\varepsilon}$ and $\hat{Q}$ must satisfy the conditions

$$\hat{\varepsilon}(OF, s) = \hat{\varepsilon}(F, s), \qquad \hat{Q}(OF, s, G) = \hat{Q}(F, s, G), \tag{2.5.5}$$

for all proper orthogonal matrices $O$. A simple calculation verifies that when (2.5.5) hold, then the remaining conditions in (2.4.4) will be automatically satisfied, by virtue of $(2.5.3)_2$ and $(2.5.3)_3$ .

To see the implications of (2.5.5), we apply it with $O = R^\top$, where $R$ is the rotation tensor in (2.1.6), to deduce

$$\hat{\varepsilon}(F, s) = \hat{\varepsilon}(U, s), \qquad \hat{Q}(F, s, G) = \hat{Q}(U, s, G). \tag{2.5.6}$$

It is clear that, conversely, if (2.5.6) hold then (2.5.5) will be satisfied for any proper orthogonal matrix $O$. Consequently, the principle of material frame indifference is completely encoded in the statement (2.5.6) that the internal energy and the referential heat flux vector may depend on the deformation gradient $F$ solely through the right stretch tensor $U$.

When the spatial description of motion is to be employed, the constitutive equation for the Cauchy stress

$$T = \rho \partial_F \hat{\varepsilon}(F, s) F^\top, \tag{2.5.7}$$

which follows from (2.3.6), (2.3.3) and $(2.5.3)_2$ , will satisfy the principle of material frame indifference $(2.4.6)_1$ so long as (2.5.6) hold. For the constitutive equation of the spatial heat flux vector

$$q = \hat{q}(F, s, g), \tag{2.5.8}$$

the principle of material frame indifference requires (recall $(2.4.6)_2$, $(2.4.3)_1$ and $(2.4.5)_2$):

$$\hat{q}(OF, s, Og) = O\hat{q}(F, s, g), \tag{2.5.9}$$

for all proper orthogonal matrices $O$.

The final general requirement on constitutive relations is that the Piola-Kirchhoff stress satisfy (2.3.8), for the balance of angular momentum. This imposes no additional restrictions, however, because a simple calculation reveals that once $(2.5.5)_1$

holds, $S$ computed through $(2.5.3)_2$ will automatically satisfy (2.3.8). Thus in thermoelasticity, material frame indifference implies balance of angular momentum.

The constitutive equations undergo further reduction when the medium is endowed with *material symmetry*. Recall from Section 2.3 that when the reference configuration of the body is changed by means of an isochoric bilipschitz homeomorphism $\bar{x}$ with unimodular Jacobian matrix $H$ (2.2.5), then the fields transform according to the rules (2.3.19), (2.3.20), (2.3.21) and (2.3.22). It follows, in particular, that any medium that is thermoelastic relative to the original reference configuration will stay so relative to the new one, as well, even though the constitutive functions will generally change. Any isochoric transformation of the reference configuration that leaves invariant all constitutive functions manifests material symmetry of the medium. Consider any such transformation and let $H$ be its Jacobian matrix. By virtue of $(2.3.19)_1$ , $(2.3.20)_2$ and $(2.5.3)_1$ , the constitutive function $\hat{\varepsilon}$ of the internal energy will remain invariant, provided

$$\hat{\varepsilon}(FH^{-1}, s) = \hat{\varepsilon}(F, s). \tag{2.5.10}$$

A simple calculation verifies that when (2.5.10) holds the constitutive functions for $S$ and $\theta$ determined through $(2.5.3)_2$ and $(2.5.3)_3$ satisfy automatically the invariance requirements for that same $H$. The remaining constitutive equation, for the heat flux vector, will be treated for convenience in its spatial description (2.5.8). By account of $(2.3.19)_1$ , $(2.3.22)_2$ and $(2.3.21)_3$ , $\hat{q}$ will remain invariant if

$$\hat{q}(FH^{-1}, s, g) = \hat{q}(F, s, g). \tag{2.5.11}$$

It is clear that the set of matrices $H$ with determinant one for which (2.5.10) and (2.5.11) hold forms a subgroup $\mathcal{G}$ of the special linear group SL($m$), called the *symmetry group* of the medium. In certain media, $\mathcal{G}$ may contain only the identity matrix $I$ in which case material symmetry is minimal. When $\mathcal{G}$ is nontrivial, it dictates through (2.5.10) and (2.5.11) conditions on the constitutive functions of the medium.

Maximal material symmetry is attained when $\mathcal{G} \equiv \mathrm{SL}(m)$. In that case the medium is a *thermoelastic fluid*. Applying (2.5.10) and (2.5.11) with selected matrix $H = (\det F)^{-1/m} F \in \mathrm{SL}(m)$, we deduce that $\hat{\varepsilon}$ and $\hat{q}$ may depend on $F$ solely through its determinant or, equivalently by virtue of (2.3.3), through the density $\rho$:

$$\varepsilon = \tilde{\varepsilon}(\rho, s), \qquad q = \tilde{q}(\rho, s, g). \tag{2.5.12}$$

The Cauchy stress may then be obtained from (2.5.7) and the temperature from $(2.5.3)_3$ . The calculation gives

$$T = -pI, \tag{2.5.13}$$

$$p = \rho^2 \partial_\rho \tilde{\varepsilon}(\rho, s), \qquad \theta = \partial_s \tilde{\varepsilon}(\rho, s). \tag{2.5.14}$$

The constitutive function $\tilde{q}$ in (2.5.12) must also satisfy the requirement (2.5.9) of material frame indifference which now assumes the simple form

$$\tilde{q}(\rho, s, Og) = O\tilde{q}(\rho, s, g), \tag{2.5.15}$$

for all proper orthogonal matrices $O$. The final reduction of $\tilde{q}$ that satisfies (2.5.15) is

$$q = \kappa(\rho, s, |g|)g, \tag{2.5.16}$$

where $\kappa$ is a scalar-valued function. We have thus shown that in a thermoelastic fluid the internal energy depends solely on density and entropy. The Cauchy stress is a *hydrostatic pressure*, likewise depending only on density and entropy. The heat flux obeys Fourier's law with *thermal conductivity* $\kappa$ which may vary with density, entropy and the magnitude of the heat flux.

The simplest classical example of a thermoelastic fluid is the *polytropic gas*, which is determined by Boyle's law

$$p = R\rho\theta, \tag{2.5.17}$$

combined with the constitutive assumption that internal energy is proportional to temperature:

$$\varepsilon = c\theta. \tag{2.5.18}$$

In (2.5.17), $R$ is the *universal gas constant* divided by the molecular weight of the gas, and $c$ in (2.5.18) is the *specific heat*. The constant $\gamma = 1 + R/c$ is the *adiabatic exponent.* The classical kinetic theory predicts $\gamma = 1 + 2/n$, where $n$ is the number of degrees of freedom of the gas molecule.

Combining (2.5.17) and (2.5.18) with (2.5.13) and (2.5.14), one easily deduces that the constitutive relations for the polytropic gas, in normalized units, read

$$\varepsilon = c\rho^{\gamma-1}e^{\frac{s}{c}}, \quad p = R\rho^{\gamma}e^{\frac{s}{c}}, \quad \theta = \rho^{\gamma-1}e^{\frac{s}{c}}. \tag{2.5.19}$$

An *isotropic thermoelastic solid* is a thermoelastic material with symmetry group $\mathcal{G}$ the proper orthogonal group $\mathrm{SO}(m)$. In that case, to obtain the reduced form of the internal energy function $\hat{\varepsilon}$ we combine (2.5.10) with $(2.5.6)_1$. Recalling (2.1.7) we conclude that

$$\hat{\varepsilon}(OUO^{\top}, s) = \hat{\varepsilon}(U, s), \tag{2.5.20}$$

for any proper orthogonal matrix $O$. In particular, we apply (2.5.20) for the proper orthogonal matrices $O$ that diagonalize the symmetric matrix $U : OUO^{\top} = \Lambda$. This establishes that, in consequence of material frame indifference and material symmetry, the internal energy of an isotropic thermoelastic solid may depend on $F$ solely as a symmetric function of the eigenvalues of the right stretch tensor $U$. Equivalently,

$$\varepsilon = \tilde{\varepsilon}(J_1, \cdots, J_m, s), \tag{2.5.21}$$

where $(J_1, \cdots, J_m)$ are invariants of $U$. In particular, when $m = 3$, one may employ $J_1 = |F|^2$, $J_2 = |F^*|^2$ and $J_3 = \det F$, where $F^*$ is the cofactor matrix of $F$. The reduced form of the Cauchy stress for the isotopic thermoelastic solid, computed from (2.5.21) and (2.5.7), is recorded in the references cited in Section 2.9.

In an alternative, albeit equivalent, formulation of thermoelasticity, one regards the temperature $\theta$, rather than the entropy $s$, as a state variable and writes a constitutive equation for $s$ rather than for $\theta$. In that case it is also expedient to monitor the *Helmholtz free energy*

$$\psi = \varepsilon - \theta s \tag{2.5.22}$$

in the place of the internal energy $\varepsilon$. One thus starts out with consititutive equations

$$\begin{cases} \psi = \bar{\psi}(F, \theta, G), \\ S = \bar{S}(F, \theta, G), \\ s = \bar{s}(F, \theta, G), \\ Q = \bar{Q}(F, \theta, G), \end{cases} \tag{2.5.23}$$

in the place of (2.5.1). The requirement that all smooth thermodynamic processes that balance mass, momentum and energy must satisfy identically the dissipation inequality (2.3.15) reduces (2.5.23) to

$$\begin{cases} \psi = \bar{\psi}(F, \theta), \\ S = \rho_0 \partial_F \bar{\psi}(F, \theta), \\ s = -\partial_\theta \bar{\psi}(F, \theta), \\ Q = \bar{Q}(F, \theta, G), \end{cases} \tag{2.5.24}$$

$$\bar{Q}(F, \theta, G) \cdot G \geq 0, \tag{2.5.25}$$

which are the analogs of (2.5.3), (2.5.4). The principle of material frame indifference and the presence of material symmetry further reduce the above constitutive equations. In particular, $\bar{\psi}$ satisfies the same conditions as $\hat{\varepsilon}$, above.

We conclude the discussion of thermoelasticity with remarks on special thermodynamic processes. A process is called *adiabatic* if the heat flux $Q$ vanishes identically; it is called *isothermal* when the temperature field $\theta$ is constant; and it is called *isentropic* if the entropy field $s$ is constant. Note that (2.5.25) implies $\bar{Q}(F, \theta, 0) = 0$ so, in particular, all isothermal processes are adiabatic. Materials that are poor conductors of heat are commonly modeled as *nonconductors of heat*, characterized by the constitutive assumption $\hat{Q} \equiv 0$. Thus every thermodynamic process of a nonconductor is adiabatic.

In an isentropic process, the entropy is set equal to a constant, $s \equiv \bar{s}$; the constitutive relations for the temperature and the heat flux are discarded and those for the internal energy and the stress are restricted to $s = \bar{s}$:

(2.5.26)
$$\begin{cases} \varepsilon = \hat{\varepsilon}(F, \bar{s}), \\ S = \rho_0 \partial_F \hat{\varepsilon}(F, \bar{s}). \end{cases}$$

The motion is determined solely by the balance laws of mass and momentum. In practice this simplifying assumption is made when it is judged that entropy fluctuations have insignificant effect. Later on we shall encounter situations where this is indeed the case. One should keep in mind, however, that, strictly speaking, an isentropic process cannot be sustained unless the heat supply $r$ is regulated in such a manner that the ensuing motion together with the constant entropy field satisfy the balance law of energy.

In particular, in an isentropic process for a polytropic ideal gas, $s = \bar{s}$, (2.5.19) yields

(2.5.27)
$$\varepsilon = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}, \qquad p = \kappa\rho^{\gamma}.$$

Isentropic thermoelasticity rests solely on the balance laws of mass and momentum and this may leave the impression that it is a mechanical, rather than a thermomechanical, theory. In fact the constitutive relations (2.5.19) suggest that isentropic thermoelasticity is isomorphic to a mechanical theory called *hyperelasticity*. It should be noted, however, that isentropic thermoelasticity inherits from thermodynamics the Second Law under the following guise: Assuming that the process is adiabatic as well as isentropic and combining the balance law of energy (2.3.10) with the Clausius-Duhem inequality (2.3.13) yields

(2.5.28)
$$(\rho_0\varepsilon + \tfrac{1}{2}\rho_0|v|^2)^{\cdot} \leq \mathrm{Div}(v^\top S) + \rho_0 v^\top b.$$

The Eulerian form of this inequality is

(2.5.29)
$$(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)_t + \mathrm{div}[(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)v^\top] \leq \mathrm{div}(v^\top T) + \rho v^\top b.$$

The above inequalities play in isentropic thermoelasticity the role played by the Clausius-Duhem inequality (2.3.13), (2.3.14) in general thermoelasticity: For smooth motions, they hold identically, as equalities[3], by virtue of (2.3.4) and (2.5.26). By contrast, in the context of motions that are merely Lipschitz continuous, they are extra conditions serving as the test of *thermodynamic admissibility* of the motion.

In isothermal thermoelasticity, $\theta$ is set equal to a constant $\hat{\theta}$, the heat supply $r$ is regulated to balance the energy equation, and the motion is determined solely by the balance laws of mass and momentum. The only constitutive equations needed are

(2.5.30)
$$\begin{cases} \psi = \bar{\psi}(F, \hat{\theta}), \\ S = \rho_0 \partial_F \bar{\psi}(F, \hat{\theta}), \end{cases}$$

[3] In particular, this implies that smooth isentropic processes may be sustained with $r = 0$, that is without supplying or extracting any amount of heat.

namely the analogs of (2.5.26). The implications of the Second Law of thermodynamics are seen, as before, by combining (2.3.10) with (2.3.13), assuming now $\theta = \hat{\theta}$ =constant. This yields

$$(2.5.31) \qquad (\rho_0\psi + \tfrac{1}{2}\rho_0|v|^2)^{\cdot} \leq \mathrm{Div}(v^\top S) + \rho_0 v^\top b,$$

which should be compared to (2.5.28). We conclude that isothermal and isentropic thermoelasticity are essentially isomorphic, with the Helmholtz free energy at constant temperature, in the former, playing the role of internal energy, at constant entropy, in the latter.

In an isothermal process $\theta = \hat{\theta}$ for a polytropic ideal gas, (2.5.17) implies

$$(2.5.32) \qquad p = \nu\rho\,,$$

where $\nu = R\hat{\theta}$.

## 2.6 Thermoviscoelasticity

We now consider an extension of thermoelasticity, which encompasses materials with *internal dissipation* induced by *viscosity of the rate type*. The internal energy $\varepsilon$, the Piola-Kirchhoff stress $S$, the temperature $\theta$, and the referential heat flux vector $Q$ may now depend not only on the deformation gradient $F$, the entropy $s$ and the temperature gradient $G$, as in (2.5.1), but also on the time rate $\dot{F}$ of the deformation gradient:

$$(2.6.1) \qquad \begin{cases} \varepsilon = \hat{\varepsilon}(F, \dot{F}, s, G), \\ S = \hat{S}(F, \dot{F}, s, G), \\ \theta = \hat{\theta}(F, \dot{F}, s, G), \\ Q = \hat{Q}(F, \dot{F}, s, G). \end{cases}$$

As stipulated in Section 2.3, every smooth thermodynamic process that balances mass, momentum and energy must satisfy identically the dissipation inequality (2.3.15). Substituting from (2.6.1) into (2.3.15) yields

(2.6.2)

$$\mathrm{tr}[(\rho_0\partial_F\hat{\varepsilon} - \hat{S})\dot{F}^\top] + \mathrm{tr}(\rho_0\partial_{\dot{F}}\hat{\varepsilon}\,\ddot{F}^\top) + \rho_0(\partial_s\hat{\varepsilon} - \hat{\theta})\dot{s} + \rho_0\partial_G\hat{\varepsilon}\dot{G} - \hat{\theta}^{-1}\hat{Q}\cdot G \leq 0.$$

By suitably controlling the body force $b$ and heat supply $r$, one may construct smooth processes that balance mass, momentum and energy and attain at some point $(x, t)$ arbitrarily prescribed values for $F, \dot{F}, s, G, \ddot{F}, \dot{s}, \dot{G}$, subject only to the constraint $\det F > 0$. Consequently, the inequality (2.6.2) cannot hold identically unless the constitutive function in (2.6.1) have the following special form:

$$
(2.6.3)\qquad
\begin{cases}
\varepsilon = \hat{\varepsilon}(F, s), \\
S = \rho_0 \partial_F \hat{\varepsilon}(F, s) + Z(F, \dot{F}, s, G), \\
\theta = \partial_s \hat{\varepsilon}(F, s), \\
Q = \hat{Q}(F, \dot{F}, s, G),
\end{cases}
$$

$$
(2.6.4)\qquad \operatorname{tr}[Z(F, \dot{F}, s, G)\dot{F}^\top] + \frac{1}{\hat{\theta}(F, s)} \hat{Q}(F, \dot{F}, s, G) \cdot G \geq 0.
$$

Comparing (2.6.3) with (2.5.3) we observe that, again, the internal energy, which may depend solely on the deformation gradient and the entropy, determines the constitutive equation for the temperature by the same caloric equation of state. On the other hand, the constitutive equation for the stress now includes the additional term $Z$ which contributes the viscous effect and induces internal dissipation manifested in (2.6.4).

The constitutive functions must be reduced further to comply with the principle of material frame indifference, postulated in Section 3.4. In particular, frame indifference imposes to internal energy the same condition $(2.5.5)_1$ as in thermoelasticity, and the resulting reduction is, of course, the same:

$$
(2.6.5)\qquad \hat{\varepsilon}(F, s) = \hat{\varepsilon}(U, s),
$$

where $U$ denotes the right stretch tensor (2.1.7). Furthermore, when (2.6.5) holds the constitutive equation for the temperature, derived through $(2.6.3)_3$, and the term $\rho_0 \partial_F \hat{\varepsilon}(F, s)$, in the constitutive equation for the stress, will be automatically frame indifferent. It remains to investigate the implications of frame indifference on $Z$ and on the heat flux. Since the analysis will focus eventually on thermoviscoelastic fluids, it will be expedient to switch at this point from $S$ and $Q$ to $T$ and $q$; also to replace, on the list $(F, \dot{F}, s, G)$ of state variables, $\dot{F}$ with $L$ (cf. (2.1.8)) and $G$ with $g$ (cf. (2.3.17)). We thus write

$$
(2.6.6)\qquad T = \rho \partial_F \hat{\varepsilon}(F, s) F^\top + \hat{Z}(F, L, s, g),
$$

$$
(2.6.7)\qquad q = \hat{q}(F, L, s, g).
$$

Recalling (2.4.3) and (2.4.5), we deduce that the principle of material frame indifference requires

$$
(2.6.8)\qquad
\begin{cases}
\hat{Z}(OF, OLO^\top + \dot{O}O^\top, s, Og) = O\hat{Z}(F, L, s, g)O^\top \\
\hat{q}(OF, OLO^\top + \dot{O}O^\top, s, Og) = O\hat{q}(F, L, s, g),
\end{cases}
$$

for any proper orthogonal matrix $O$. In particular, for any fixed state $(F, L, s, g)$ with spin $W$ (cf. (2.1.9)), we may pick $O(t) = \exp(-tW)$, in which case $O(0) = I$,

$\dot{O}(0) = -W$. It then follows from (2.6.8) that $\hat{Z}$ and $\hat{q}$ may depend on $L$ solely through its symmetric part $D$ and hence (2.6.6) and (2.6.7) may be written as

$$T = \rho \partial_F \hat{\varepsilon}(F, s) F^\top + \hat{Z}(F, D, s, g), \tag{2.6.9}$$

$$q = \hat{q}(F, D, s, g), \tag{2.6.10}$$

with $\hat{Z}$ and $\hat{q}$ such that

$$\begin{cases} \hat{Z}(OF, ODO^\top, s, Og) = O\hat{Z}(F, D, s, g)O^\top \\ \hat{q}(OF, ODO^\top, s, Og) = O\hat{q}(F, D, s, g), \end{cases} \tag{2.6.11}$$

for all proper orthogonal matrices $O$.

For the balance law of angular momentum (2.3.9) to be satisfied, $\hat{Z}$ must also be symmetric: $\hat{Z}^\top = \hat{Z}$. Notice that in that case the dissipation inequality (2.6.4) may be rewritten in the form

$$\mathrm{tr}\,[\hat{Z}(F, D, s, g)D] + \frac{1}{\hat{\theta}(F, s)} \hat{q}(F, D, s, g) \cdot g \geq 0. \tag{2.6.12}$$

Further reduction of the constitutive functions obtains when the medium is endowed with material symmetry. As in Section 2.5, we introduce here the *symmetry group* $\mathcal{G}$ of the material, namely the subgroup of $\mathrm{SL}(m)$ formed by the Jacobian matrices $H$ of those isochoric transformations $\bar{x}$ of the reference configuration that leave all constitutive functions invariant. The rules of transformation of the fields under change of the reference configuration are recorded in (2.3.19), (2.3.20), (2.3.21) and (2.3.22). Thus, $\mathcal{G}$ is the set of all $H \in \mathrm{SL}(m)$ with the property

$$\begin{cases} \hat{\varepsilon}(FH^{-1}, s) = \hat{\varepsilon}(F, s), \\ \hat{Z}(FH^{-1}, D, s, g) = \hat{Z}(F, D, s, g), \\ \hat{q}(FH^{-1}, D, s, g) = \hat{q}(F, D, s, g). \end{cases} \tag{2.6.13}$$

The material will be called a *thermoviscoelastic fluid* when $\mathcal{G} \equiv \mathrm{SL}(m)$. In that case, applying (2.6.13) with $H = (\det F)^{-1/m} F \in \mathrm{SL}(m)$, we conclude that $\hat{\varepsilon}$, $\hat{Z}$, and $\hat{q}$ may depend on $F$ solely through its determinant or, equivalently, through the density $\rho$. Therefore, the constitutive equations of the thermoviscoelastic fluid reduce to

$$\begin{cases} \varepsilon = \tilde{\varepsilon}(\rho, s), \\ T = -pI + \tilde{Z}(\rho, D, s, g), \\ p = \rho^2 \partial_\rho \tilde{\varepsilon}(\rho, s), \quad \theta = \partial_s \tilde{\varepsilon}(\rho, s), \\ q = \tilde{q}(\rho, D, s, g). \end{cases} \tag{2.6.14}$$

For frame indifference, $\tilde{Z}$ and $\tilde{q}$ should still satisfy, for any proper orthogonal matrix $O$, the conditions

$$\begin{cases} \tilde{Z}(\rho, ODO^\top, s, Og) = O\tilde{Z}(\rho, D, s, g)O^\top, \\ \tilde{q}(\rho, ODO^\top, s, Og) = O\tilde{q}(\rho, D, s, g), \end{cases} \tag{2.6.15}$$

which follow from (2.6.11). It is possible to write down explicitly the form of the most general functions $\tilde{Z}$ and $\tilde{q}$ that conform with (2.6.15). Here, it will suffice to record the most general constitutive relations, for m$=3$, that are compatible with (2.6.15) and are linear in $(D, g)$, namely

$$T = -p(\rho, s)I + \lambda(\rho, s)(\mathrm{tr}D)I + 2\mu(\rho, s)D, \tag{2.6.16}$$

$$q = \kappa(\rho, s)g, \tag{2.6.17}$$

which identify the compressible, heat conducting *Newtonian fluid*.

The *bulk viscosity* $\lambda + \frac{2}{3}\mu$, *shear viscosity* $\mu$ and *thermal conductivity* $\kappa$ of a Newtonian fluid are constrained by the inequality (2.6.12), which here reduces to

$$\lambda(\rho, s)(\mathrm{tr}D)^2 + 2\mu(\rho, s)\mathrm{tr}D^2 + \frac{\kappa(\rho, s)}{\tilde{\theta}(\rho, s)}|g|^2 \geq 0. \tag{2.6.18}$$

This inequality will hold for arbitrary $D$ and $g$ if and only if

$$\mu(\rho, s) \geq 0, \quad 3\lambda(\rho, s) + 2\mu(\rho, s) \geq 0, \quad \kappa(\rho, s) \geq 0. \tag{2.6.19}$$

For actual dissipation, at least one of $\mu$, $3\lambda + 2\mu$ and $\kappa$ should be strictly positive.

## 2.7 Incompressibility

Many fluids, and even certain solids, such as rubber, may be stretched or sheared with relative ease, while exhibiting disproportionately high stiffness when subjected to deformations that would change their volume. Continuum physics treats such materials as incapable of sustaining any volume change, so that the density $\rho$ stays constant along particle trajectories. The incompressibility condition

$$\det F = 1, \quad \mathrm{tr}D = \mathrm{div}\, v^\top = 0, \tag{2.7.1}$$

in Lagrangian or Eulerian coordinates, is then appended to the system of balance laws, as a kinematic constraint. In return, the stress tensor is decomposed into two parts:

$$S = -p(F^{-1})^\top + \hat{S}, \qquad T = -pI + \hat{T}, \tag{2.7.2}$$

where $\hat{S}$ or $\hat{T}$, called the *extra stress*, is determined, as before, by the thermodynamic process, through constitutive equations, while the other term, which represents a *hydrostatic pressure,* is not specified by a constitutive relation but is to be determined, together with the thermodynamic process, by solving the system of balance laws of mass, momentum and energy, subject to the kinematic constraint (2.7.1).

The salient property of the hydrostatic pressure is that it produces no work under isochoric deformations. To motivate (2.7.2) by means of the Second Law of thermodynamics, let us consider an incompressible thermoelastic material with constitutive equations for $\varepsilon, \theta$ and $Q$ as in (2.5.1), but only defined for $F$ with $\det F = 1$, and $S$ unspecified. The dissipation inequality again implies (2.5.2) with $\hat{S}$ replaced by $S$, $\partial_F \hat{\varepsilon}$ replaced by the tangential derivative $\partial_F^{\tau} \hat{\varepsilon}$ on the manifold $\det F = 1$, and $\dot{F}$ constrained to lie on the subspace

$$\operatorname{tr}[(F^{-1})^{\top} \dot{F}^{\top}] = \operatorname{tr}[(F^{*})^{\top} \dot{F}^{\top}] = \operatorname{tr}[(\partial_F \det F) \dot{F}^{\top}] = \dot{\overline{\det F}} = 0. \tag{2.7.3}$$

Therefore, $\operatorname{tr}[(\rho_0 \partial_F^{\tau} \hat{\varepsilon} - S) \dot{F}^{\top}] \leq 0$ for all $\dot{F}$ satisfying (2.7.3) if and only if

$$S = -p(F^{-1})^{\top} + \rho_0 \partial_F^{\tau} \hat{\varepsilon}(F, s), \tag{2.7.4}$$

for some scalar $p$.

In incompressible Newtonian fluids, the stress is still given by (2.6.16), where, however, $\rho$ is constant and $p(\rho, s)$ is replaced by the undetermined hydrostatic pressure $p$. When the incompressible fluid is inviscid, the entire stress tensor is subsumed by the undetermined hydrostatic pressure.

## 2.8 Relaxation

The state variables of continuum physics, introduced in the previous sections, represent statistical averages of certain physical quantities, such as velocity, translational kinetic energy, rotational kinetic energy, chemical energy etc., associated with the molecules of the material. These quantities evolve and eventually settle, or "relax", to states in local equilibrium, characterized by equipartition of energy and other conditions dictated by the laws of statistical physics. The constitutive relations of thermoelasticity and thermoviscoelasticity, considered in earlier sections, are relevant so long as local equilibrium is attained in a time scale much shorter than the time scale of the gross motion of the material body. In the opposite case where the relaxation time is of the same order of magnitude as the time scale of the motion, relaxation mechanisms must be accounted for even within the framework of continuum physics. This is done by introducing additional, *internal state variables,* measuring the deviation from local equilibrium. The states in local equilibrium span a manifold embedded in the extended state space. The internal state variables satisfy special constitutive relations, in the form of balance laws with dissipative source terms that act to drive the state vector towards local equilibrium.

An enormous variety of relaxation theories are discussed in the literature; the reader may catch a glimpse of their common underlying structure through the following example.

We consider a continuous medium that does not conduct heat and whose isentropic response is governed by constitutive relations

$$\varepsilon = \hat{\varepsilon}(F, \Sigma), \tag{2.8.1}$$

$$S = P(F) + \rho_0 \Sigma, \tag{2.8.2}$$

for the internal energy and the Piola-Kirchhoff stress, where $\Sigma$ is an internal variable taking values in $\mathbb{M}^{m\times m}$ and satisfying a balance law of the form

$$\rho_0 \dot{\Sigma} = \frac{1}{\tau}[\Pi(\Sigma) - F]. \tag{2.8.3}$$

Thus, the material exhibits instantaneous elastic response, embodied in the term $P(F)$, combined with viscous response induced by relaxation of $\Sigma$. The positive constant $\tau$ is called the *relaxation time.*

The postulate that any smooth motion of the medium that balances linear momentum (2.3.4) must satisfy identically the entropy inequality (2.5.28) yields

$$S = \rho_0 \partial_F \hat{\varepsilon}(F, \Sigma), \tag{2.8.4}$$

$$\mathrm{tr}\left[\partial_\Sigma \hat{\varepsilon}(F, \Sigma)\dot{\Sigma}^\top\right] \le 0. \tag{2.8.5}$$

Upon combining (2.8.4) and (2.8.5) with (2.8.2) and (2.8.3), we deduce

$$\varepsilon = \sigma(F) + \mathrm{tr}(\Sigma F^\top) + h(\Sigma), \tag{2.8.6}$$

$$P(F) = \rho_0 \partial_F \sigma(F), \qquad \Pi(\Sigma) = -\partial_\Sigma h(\Sigma). \tag{2.8.7}$$

When $h$ is strictly convex, the source term in (2.8.3) is dissipative and acts to drive $\Sigma$ towards local equilibrium $\Sigma = H(F)$, where $H$ is the inverse function of $\Pi$. $\Pi^{-1}$ exists since $-\Pi$ is strictly monotone, namely,

$$\mathrm{tr}\{[\Pi(\Sigma) - \Pi(\bar{\Sigma})][\Sigma - \bar{\Sigma}]^\top\} < 0, \qquad \text{for any } \Sigma \neq \bar{\Sigma}. \tag{2.8.8}$$

In local equilibrium the medium responds like an elastic material with internal energy

$$\varepsilon = \tilde{\varepsilon}(F) = \sigma(F) + \mathrm{tr}[H(F)F^\top] + h(H(F)) \tag{2.8.9}$$

and Piola-Kirchhoff stress

$$S = P(F) + \rho_0 H(F) = \rho_0 \partial_F \tilde{\varepsilon}(F). \tag{2.8.10}$$

## 2.9 Notes

The venerable field of Continuum Physics has been enjoying a resurgence, concomitant with the rise of interest in the behavior of materials with nonlinear response. The encyclopedic works of Truesdell and Toupin [1] and Truesdell and Noll [1] contain reliable historical information as well as massive bibliographies and may serve as excellent guides for following the development of the subject from its inception, in the 18th century, to the mid 1960's. The text by Gurtin [1] provides a clear, elementary introduction to the area. A more advanced treatment, with copious references, is found in the book of Silhavy [1]. The text by Müller [2] is an excellent presentation of thermodynamics from the perspective of modern continuum physics. Other good sources, emphasizing elasticity theory, are the books of Ciarlet [1], Hanyga [1], Marsden and Hughes [1] and Wang and Truesdell [1]. The recent monograph by Antman [3] contains a wealth of material on the theory of elastic strings, rods, shells and three-dimensional bodies, with emphasis on the qualitative analysis of the governing balance laws. On the equivalence of the referential (Lagrangian) and spatial (Eulerian) description of the field equations for the balance laws of Continuum Physics, see Dafermos [17] and Wagner [2,3]. It would be useful to know whether this holds under more general assumptions on the motion than Lipschitz continuity. For instance, when the medium is a thermoelastic gas, it is natural to allow regions of vacuum in the placement of the body. In such a region the density vanishes and the specific volume (determinant of the deformation gradient) becomes infinitely large. For particular results in that direction, see Wagner [2].

The kinematic balance laws (2.2.15) and (2.2.16) were first derived by Qin[1], in the context of smooth motions, by direct calculation. It is interesting that, as we see here, they are valid when the motions are merely Lipschitz continuous and in fact, as shown by Demoulini, Stuart and Tzavaras [2], even under slightly weaker hypotheses. The connection to null Lagrangians was first pointed out in this last reference. For a detailed treatment of null Lagrangians, see Ball, Currie and Olver [1]. See also Wagner [3].

The field equations for the balance laws considered here were originally derived by Euler [1,2], for mass, Cauchy [3,4], for linear and angular momentum, and Kirchhoff [1], for energy. The Clausius-Duhem inequality was postulated by Clausius [1], for the adiabatic case; the entropy flux term was introduced by Duhem [1] and the entropy production term was added by Truesdell and Toupin [1]. More general entropy inequalities were first considered by Müller [1].

The postulate that constitutive equations should be reduced so that the Clausius-Duhem inequality be satisfied automatically by smooth thermodynamic processes that balance mass, momentum and energy was first stated as a general principle by Coleman and Noll [1]. The examples presented here were adapted from Coleman and Noll [1], for thermoelasticity, and Coleman and Mizel [1], for thermoviscoelasticity. Coleman and Gurtin [1] have developed a general theory of thermoviscoelastic materials with internal state variables, of which the example presented in Section 2.8 is a special case. Constitutive relations of this type were first considered by Maxwell [1]. A detailed discussion of relaxation phenomena in gas dynamics is found in the book by Vincenti and Kruger [1].

The use of frame indifference and material symmetry to reduce constitutive equations originated in the works of Cauchy [4] and Poisson [2]. In the ensuing century, this program was implemented (mostly correctly but occasionally incorrectly) by many authors, for a host of special constitutive equations. In particular, the work of the Cosserats [1], Rivlin and Ericksen [1], and others in the 1940's and 1950's contributed to the clarification of the concepts. The principle of material frame indifference and the definition of the symmetry group were ultimately postulated with generality and mathematical precision by Noll [1].

# III

# Hyperbolic Systems of Balance Laws

The ambient space for the system of balance laws, introduced in Chapter I, will be visualized here as space-time, and the central notion of hyperbolicity in the time direction will be motivated and defined. Companions to the flux, considered in Section 1.5, will now be realized as entropy-entropy flux pairs.

Numerous examples will be presented of hyperbolic systems of balance laws arising in continuum physics.

## 3.1 Hyperbolicity

Returning to the setting of Chapter I, we visualize $\mathbb{R}^k$ as $\mathbb{R}^m \times \mathbb{R}$, where $\mathbb{R}^m$, with $m = k - 1$, is "space" with typical point $x$, and $\mathbb{R}$ is "time" with typical value $t$, so $X = (x, t)$. We write $\partial_t$ for $\partial/\partial X_k$ and $\partial_\alpha$ for $\partial/\partial X_\alpha$ , $\alpha = 1, \dots, m$. We retain the symbol div to denote divergence with respect to the $x$-variable in $\mathbb{R}^m$. As in earlier chapters, in matrix operations div will be acting on row vectors. We also recall the Notation 1.4.2, which will remain in force throughout this work: D denotes the differential $[\partial/\partial U^1, \dots, \partial/\partial U^n]$, regarded as a row operation.

We denote $G_k$ by $H$, reassign the symbol $G$ to denote the $n \times m$ matrix with column vectors $(G_1, \dots, G_m)$, and rewrite the system of balance laws (1.4.3) in the form

$$\partial_t H(U(x,t), x, t) + \operatorname{div} G(U(x,t), x, t) = \Pi(U(x,t), x, t). \tag{3.1.1}$$

**3.1.1 Definition.** The system of balance laws (3.1.1) is called *hyperbolic* in the $t$-direction if, for any fixed $U \in \mathscr{O}$, $(x,t) \in \mathscr{X}$ and $\nu \in S^{m-1}$, the $n \times n$ matrix $\mathrm{D}H(U, x, t)$ is nonsingular and the eigenvalue problem

$$\left[ \sum_{\alpha=1}^{m} \nu_\alpha \mathrm{D}G_\alpha(U, x, t) - \lambda DH(U, x, t) \right] R = 0 \tag{3.1.2}$$

has real eigenvalues $\lambda_1(\nu; U, x, t), \cdots, \lambda_n(\nu; U, x, t)$, called *characteristic speeds*, and $n$ linearly independent eigenvectors $R_1(\nu; U, x, t), \cdots, R_n(\nu; U, x, t)$.

A class of great importance are the *symmetric hyperbolic* systems of balance laws (3.1.1), in which, for any $U \in \mathcal{O}$ and $(x, t) \in \mathcal{X}$, the $n \times n$ matrices $\mathrm{D}G_\alpha(U, x, t), \alpha = 1, \cdots, m$, are symmetric and $\mathrm{D}H(U, x, t)$ is symmetric positive definite.

The definition of hyperbolicity may be naturally interpreted in terms of the notion of fronts, introduced in Section 1.6. A front $\mathcal{F}$ of the system of balance laws (3.1.1) may be visualized as a one-parameter family of $m - 1$ dimensional manifolds in $I\!R^m$, parametrized by $t$, i.e., as a surface propagating in space. In that context, if we renormalize the normal $N$ on $\mathcal{F}$ so that $N = (\nu, -s)$ with $\nu \in S^{m-1}$, then the wave will be propagating in the direction $\nu$ with speed $s$. Therefore, comparing (3.1.2) with (1.6.1) we conclude that a system of $n$ balance laws is hyperbolic if and only if $n$ distinct weak waves can propagate in any spatial direction. The eigenvalues of (3.1.2) will determine the speed of propagation of these waves while the corresponding eigenvectors will specify the direction of their amplitude.

When $\mathcal{F}$ is a shock front, (1.6.3) may be written in the current notation as

$$-s[H(U_+, x, t) - H(U_-, x, t)] + [G(U_+, x, t) - G(U_-, x, t)]\nu = 0, \tag{3.1.3}$$

which is called the *Rankine-Hugoniot jump condition*. By virtue of Theorem 1.8.1, this condition should hold at every point of approximate jump discontinuity of any function $U$ of class $BV_{\mathrm{loc}}$ that satisfies the system (3.1.1) in the sense of measures.

It is clear that hyperbolicity is preserved under any change $U^* = U^*(U, x, t)$ of state vector with $U^*(\cdot, x, t)$ a diffeomorphism for every fixed $(x, t) \in \mathcal{X}$. In particular, since $\mathrm{D}H(U, x, t)$ is nonsingular, we may employ, locally at least, $H$ as the new state vector. Thus, without essential loss of generality, one may limit the investigation to hyperbolic systems of balance laws that have the special form

$$\partial_t U(x, t) + \operatorname{div} G(U(x, t), x, t) = \Pi(U(x, t), x, t). \tag{3.1.4}$$

For simplicity and convenience, we shall regard henceforth the special form (3.1.4) as *canonical*. The reader should keep in mind, however, that when dealing with systems of balance laws arising in continuum physics it may be advantageous to keep the state vector naturally provided, even at the expense of having to face the more complicated form (3.1.1) rather than the canonical form (3.1.4).

## 3.2 Entropy-Entropy Flux Pairs

Assume that the system of balance laws (1.4.3), which we now write in the form (3.1.1), is endowed with a companion balance law (1.5.2). We set $Q_k \equiv \eta$, reassign $Q$ to denote the $m$-row vector $(Q_1, \ldots, Q_m)$ and recast (1.5.2) in the new notation:

$$\partial_t \eta(U(x, t), x, t) + \operatorname{div} Q(U(x, t), x, t) = h(U(x, t), x, t). \tag{3.2.1}$$

As we shall see in Section 3.3, in the applications to continuum physics, companion balance laws of the form (3.2.1) are intimately related with the Second Law of thermodynamics. For that reason, $\eta$ is called an *entropy* for the system (3.1.1) of balance laws and $Q$ is called the *entropy flux* associated with $\eta$.

Equation (1.5.1), for $\alpha = k$, should now be written as

$$\text{(3.2.2)} \qquad \mathrm{D}\eta(U,x,t) = B(U,x,t)^\top \mathrm{D}H(U,x,t).$$

Assume the system is in canonical form (3.1.4) so that (3.2.2) reduces to $\mathrm{D}\eta = B^\top$. Then (1.5.1) and the integrability condition (1.5.4) become

$$\text{(3.2.3)} \qquad \mathrm{D}Q_\alpha(U,x,t) = \mathrm{D}\eta(U,x,t)\mathrm{D}G_\alpha(U,x,t), \quad \alpha = 1,\cdots,m,$$

$$\text{(3.2.4)} \quad \mathrm{D}^2\eta(U,x,t)\mathrm{D}G_\alpha(U,x,t) = \mathrm{D}G_\alpha(U,x,t)^\top \mathrm{D}^2\eta(U,x,t), \ \alpha = 1,\cdots,m.$$

Notice that (3.2.4) imposes $\frac{1}{2}n(n-1)m$ conditions on the single unknown function $\eta$. Therefore, as already noted in Section 1.5, the problem of determining a nontrivial entropy-entropy flux pair for (3.1.1) is formally overdetermined, unless either $n = 1$ and $m$ is arbitrary, or $n = 2$ and $m = 1$. However, when the system is symmetric, we may satisfy (3.2.4) with $\eta = \frac{1}{2}|U|^2$. Conversely, if (3.2.4) holds and $\eta(U,x,t)$ is uniformly convex in $U$, then the change $U^* = \mathrm{D}\eta(U,x,t)^\top$ of state vector renders the system symmetric. Thus, systems of balance laws in canonical form (3.1.4) that are endowed with a convex entropy are necessarily hyperbolic.

An interesting, alternative form of the integrability condition obtains by projecting (3.2.4) in the direction of an arbitrary $\nu \in S^{m-1}$ and then multiplying the resulting equation from the left by $R_j(\nu;U,x,t)^\top$ and from the right by $R_k(\nu;U,x,t)$, with $j \neq k$. So long as $\lambda_j(\nu;U,x,t) \neq \lambda_k(\nu;U,x,t)$, this calculation yields

$$\text{(3.2.5)} \qquad R_j(\nu;U,x,t)^\top \mathrm{D}^2\eta(U,x,t)R_k(\nu;U,x,t) = 0, \qquad j \neq k.$$

Moreover, (3.2.5) holds even when $\lambda_j(\nu;U,x,t) = \lambda_k(\nu;U,x,t)$, provided that one selects the eigenvectors $R_j(\nu;U,x,t)$ and $R_k(\nu;U,x,t)$ judiciously in the eigenspace of this multiple eigenvalue.

Notice that (3.2.5) imposes on $\eta$ $\frac{1}{2}n(n-1)$ conditions for each fixed $\nu$, and hence a total of $\frac{1}{2}n(n-1)m$ conditions for $m$ linearly independent—and thereby all—$\nu$ in $S^{m-1}$. A notable exception occurs for systems in which the Jacobian matrices of the components of their fluxes commute:

(3.2.6)
$$\mathrm{D}G_\alpha(U,x,t)\mathrm{D}G_\beta(U,x,t) = \mathrm{D}G_\beta(U,x,t)\mathrm{D}G_\alpha(U,x,t), \ \ \alpha,\beta = 1,\ldots,m.$$

Indeed, in that case the $R_i(\nu;U,x,t)$ do not vary with $\nu$ and hence (3.2.5) represents just $\frac{1}{2}n(n-1)$ conditions on $\eta$. We will revisit this very special class of systems in Section 6.10.

The issue of the overdeterminacy of (3.2.5), in one spatial dimension, will be examined in depth in Section 7.4.

## 3.3 Examples of Hyperbolic Systems of Balance Laws

Out of a host of hyperbolic systems of balance laws in continuum physics, only a small sample will be presented here. They will serve as beacons for guiding the development of the general theory.

**3.3.1 The scalar balance law:**
The single balance law ($n = 1$)

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t),x,t) = \varpi(u(x,t),x,t) \tag{3.3.1}$$

is always hyperbolic. Any function $\eta(u,x,t)$ may serve as entropy, with associated entropy flux and entropy production computed by

$$Q = \int^u \frac{\partial \eta}{\partial u}\frac{\partial G}{\partial u} du, \tag{3.3.2}$$

$$h = \sum_{\alpha=1}^{m} \left[ \frac{\partial \eta}{\partial u}\frac{\partial G_\alpha}{\partial x_\alpha} - \frac{\partial Q_\alpha}{\partial x_\alpha} \right] + \varpi \frac{\partial \eta}{\partial u} + \frac{\partial \eta}{\partial t}. \tag{3.3.3}$$

Equation (3.3.1), the corresponding homogeneous scalar conservation law, and especially their one-space dimensional ($m = 1$) versions will serve extensively as models for developing the theory of general systems.

**3.3.2 Thermoelastic nonconductors of heat:**
The theory of thermoelastic media was discussed in Chapter II. Here we shall employ the referential (Lagrangian) description so the fields will be functions of $(x,t)$. For consistency with the notation of the present chapter, we shall use $\partial_t$ to denote material time derivative (in lieu of the overdot employed in Chapter II) and $\partial_\alpha$ to denote partial derivative with respect to the $\alpha$-component $x_\alpha$ of $x$. For definiteness, we assume the physical space has dimension $m = 3$. We also adopt the standard summation convention: repeated indices are summed over the range 1,2,3.

The constitutive equations are recorded in Section 2.5. Since there is no longer danger of confusion, we may simplify the notation by dropping the "hat" from the symbols of the constitutive functions. Also for simplicity we assume that the medium is homogeneous, with reference density $\rho_0 = 1$.

As explained in Chapter II, a thermodynamic process is determined by a motion $\chi$ and an entropy field $s$. In order to cast the field equations of the balance laws into a first order system of the form (3.1.1), we monitor $\chi$ through its derivatives (2.1.1), (2.1.2) and thus work with the state vector $U = (F, v, s)$, taking values in $\mathbb{R}^{13}$. In that case we must append to the balance laws of linear momentum (2.3.4) and energy (2.3.10) the compatibility condition $(2.1.8)_1$. Consequently, our system of balance laws reads

$$\text{(3.3.4)} \qquad \begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i, \alpha = 1, 2, 3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F, s) = b_i \,, & i = 1, 2, 3 \\ \partial_t\left[\varepsilon(F, s) + \frac{1}{2}|v|^2\right] - \partial_\alpha \left[v_i S_{i\alpha}(F, s)\right] = b_i v_i + r, \end{cases}$$

with (cf. (2.5.3))

$$\text{(3.3.5)} \qquad S_{i\alpha}(F, s) = \frac{\partial \varepsilon(F, s)}{\partial F_{i\alpha}}\,, \qquad \theta(F, s) = \frac{\partial \varepsilon(F, s)}{\partial s}\,.$$

A lengthy calculation verifies that the system (3.3.4) is hyperbolic on a certain region of the state space if for every $(F, s)$ lying in that region

$$\text{(3.3.6)} \qquad \frac{\partial \varepsilon(F, s)}{\partial s} > 0,$$

$$\text{(3.3.7)} \qquad \frac{\partial^2 \varepsilon(F, s)}{\partial F_{i\alpha} \partial F_{j\beta}} \nu_\alpha \nu_\beta \xi_i \xi_j > 0, \quad \text{for all } \nu \text{ and } \xi \text{ in } S^2\,.$$

By account of $(3.3.5)_2$, condition (3.3.6) simply states that the absolute temperature must be positive. (3.3.7), called the *Legendre-Hadamard condition*, means that $\varepsilon$ is *rank-one convex* in $F$, i.e., it is convex along any direction $\xi \otimes \nu$ with rank one. An alternative way of expressing (3.3.7) is to state that for any unit vector $\nu$ the *acoustic tensor* $N(\nu, F, s)$, defined by

$$\text{(3.3.8)} \qquad N_{ij}(\nu, F, s) = \frac{\partial^2 \varepsilon(F, s)}{\partial F_{i\alpha} \partial F_{j\beta}} \nu_\alpha \nu_\beta\,, \quad i, j = 1, 2, 3$$

is positive definite. In fact, for the system (3.3.4), the characteristic speeds are the six square roots of the three eigenvalues of the acoustic tensor, and zero with multiplicity seven.

Recall from Chapter II that, in addition to the system of balance laws (3.3.4), thermodynamically admissible processes should also satisfy the Clausius-Duhem inequality (2.3.13) which here takes the form

$$\text{(3.3.9)} \qquad -\partial_t s \le -\frac{r}{\theta(F, s)}\,.$$

By virtue of (3.3.5), every classical solution of (3.3.4) will satisfy (3.3.9) identically as an equality.[1] Hence, in the terminology of Section 3.2, $-s$ is an entropy for the system (3.3.4) with associated entropy flux zero.[2] Weak solutions of (3.3.4) will not

[1] Thus, for classical solutions it is convenient to substitute the equality (3.3.9) for the third equation in (3.3.4). In particular, if $r \equiv 0$, the entropy $s$ stays constant along particle trajectories and one may determine $F$ and $v$ just by solving the first two equations of (3.3.4).

[2] Identifying $-s$ as the "entropy", rather than $s$ itself which is the physical entropy, may look strange. This convention is adopted because it is more convenient to deal with functionals of the solution that are nonincreasing with time.

necessarily satisfy (3.3.9). Therefore, the role of (3.3.9) is to weed out undesirable weak solutions. The extension of a companion balance law from an identity for classical solutions into an inequality for weak solutions will play a crucial role in the general theory of hyperbolic systems of balance laws.

**3.3.3 Isentropic motion of thermoelastic nonconductors of heat:**
The physical background of isentropic processes was discussed in Section 2.5. The entropy is fixed at a constant value $\bar{s}$ and, for simplicity, is dropped from the notation. The state vector reduces to $U = (F, v)$ with values in $I\!R^{12}$. The system of balance laws results from (3.3.4) by discarding the balance of energy:

$$\text{(3.3.10)} \qquad \begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i, \alpha = 1, 2, 3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F) = b_i \ , & i = 1, 2, 3 \end{cases}$$

and we still have

$$\text{(3.3.11)} \qquad S_{i\alpha}(F) = \frac{\partial \varepsilon(F)}{\partial F_{i\alpha}} \ , \qquad i, \alpha = 1, 2, 3.$$

The system (3.3.10) is hyperbolic if $\varepsilon$ is rank-one convex, i.e., (3.3.7) holds at $s = \bar{s}$.

As explained in Section 2.5, in addition to (3.3.10) thermodynamically admissible isentropic motions must also satisfy the inequality (2.5.28), which in the current notation reads

$$\text{(3.3.12)} \qquad \partial_t[\varepsilon(F) + \tfrac{1}{2}|v|^2] - \partial_\alpha[v_i S_{i\alpha}(F)] \le b_i v_i \ .$$

By virtue of (3.3.11), any classical solution of (3.3.10) satisfies identically (3.3.12) as an equality. Thus, in the terminology of Section 3.2, $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ is an entropy for the system (3.3.10). Note that (3.3.10) is in canonical form (3.1.4) and that $\mathrm{D}\eta = (S(F), v)$. Therefore, as shown in Section 3.2, if the internal energy $\varepsilon(F)$ is uniformly convex, then changing the state vector from $U = (F, v)$ to $U^* = (S, v)$ will render the system (3.3.10) symmetric hyperbolic.

Weak solutions of (3.3.10) will not necessarily satisfy (3.3.12). We thus encounter again the situation in which a companion balance law is extended from an identity for classical solutions into an inequality serving as admissibility condition on weak solutions.

The passage from (3.3.4) to (3.3.10) provides an example of the truncation process that is commonly employed in continuum physics for simplifying systems of balance laws by dropping a number of the equations while simultaneously reducing proportionately the size of the state vector, according to the rules laid down in Section 1.5. In fact, one may derive the companion balance law (3.3.12), for the truncated system (3.3.10), from the companion balance law (3.3.9), of the original system (3.3.4), by using the recipe (1.5.12). Recall that in a canonical truncation, the elimination of any equation should be paired with freezing the corresponding component of the special state vector that symmetrizes the system. Thus, for instance, one may canonically truncate the system (3.3.10) by dropping the $i$-th of the last three equations while freezing the $i$-th component $v_i$ of velocity, or else by dropping the $(i, \alpha)$-th of the first nine equations while freezing the $(i, \alpha)$-th component $S_{i\alpha}(F)$ of the Piola-Kirchhoff stress.

As explained in Section 2.5, the balance laws for isothermal processes of thermoelastic materials are obtained just by replacing in (3.3.10), (3.3.11) and (3.3.12) the internal energy $\varepsilon(F)$, at constant entropy, by the Helmholtz free energy $\psi(F)$, at constant temperature.

**3.3.4 Isentropic motion with relaxation:**
We consider isentropic motions of the material considered in Section 2.8, assuming for simplicity that the reference density $\rho_0 = 1$ and the body force $b = 0$. The state vector is $U = (F, v, \Sigma)$, with values in $I\!R^{21}$. The system of balance laws is composed of the compatibility equation $(2.1.8)_1$, the balance of linear momentum (2.3.4) and the balance law (2.8.3) for the internal variable $\Sigma$:

$$
(3.3.13) \qquad \begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i, \alpha = 1, 2, 3 \\ \partial_t v_i - \partial_\alpha [P_{i\alpha}(F) + \Sigma_{i\alpha}] = 0, & i = 1, 2, 3 \\ \partial_t \Sigma_{i\alpha} = \dfrac{1}{\tau}[\Pi_{i\alpha}(\Sigma) - F_{i\alpha}], & i, \alpha = 1, 2, 3. \end{cases}
$$

Furthermore,

$$
(3.3.14) \qquad P_{i\alpha}(F) = \frac{\partial \sigma(F)}{\partial F_{i\alpha}}, \qquad \Pi_{i\alpha}(\Sigma) = -\frac{\partial h(\Sigma)}{\partial \Sigma_{i\alpha}}.
$$

In addition to (3.3.13), thermodynamically admissible (isentropic) motions should satisfy the entropy inequality (2.5.28), which here takes the form

$$
(3.3.15) \qquad \partial_t \big[\sigma(F) + \mathrm{tr}(\Sigma F^\top) + h(\Sigma) + \tfrac{1}{2}|v|^2\big] - \partial_\alpha [v_i P_{i\alpha}(F) + v_i \Sigma_{i\alpha}] \le 0,
$$

so that, in the terminology of Section 3.2, $\sigma(F) + \mathrm{tr}(\Sigma F^\top) + h(\Sigma) + \frac{1}{2}|v|^2$ is an entropy for (3.3.13).

The system (3.3.13) is hyperbolic when

$$
(3.3.16) \qquad \frac{\partial^2 \sigma(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \nu_\alpha \nu_\beta \xi_i \xi_j + \nu_\alpha \nu_\alpha \xi_i \zeta_i + \frac{\partial^2 h(\Sigma)}{\partial \Sigma_{i\alpha} \partial \Sigma_{j\beta}} \nu_\alpha \nu_\beta \zeta_i \zeta_j > 0,
$$

for all $\nu \in S^2$ and $(\xi, \zeta)^\top \in S^5$.

**3.3.5 Thermoelastic fluid nonconductors of heat:**
The system of balance laws (3.3.4) governs the adiabatic thermodynamic processes of all thermoelastic media, including, in particular, thermoelastic fluids. In the latter case, however, it is advantageous to employ spatial (Eulerian) description. The reason is that, as shown in Section 2.5, the internal energy, the temperature, and the Cauchy stress in a thermoelastic fluid depend on the deformation gradient $F$ solely through

the density $\rho$. We may thus dispense with $F$ and describe the state of the medium through the state vector $U = (\rho, v, s)$ which takes values in the (much smaller) space $I\!R^5$.

The fields will now be functions of $(\chi, t)$. However, for consistency with the notational conventions of this chapter, we will replace the symbol $\chi$ by $x$. Also we will be using $\partial_t$ (rather than a $t$-subscript as in Chapter II) to denote partial derivative with respect to $t$.

The balance laws in force are for mass (2.3.2), linear momentum (2.3.5) and energy (2.3.11). The constitutive relations are (2.5.12), with $\tilde{q} \equiv 0$, (2.5.13) and (2.5.14). To simplify the notation, we drop the "tilde" and write $\varepsilon(\rho, s)$ in place of $\tilde{\varepsilon}(\rho, s)$. Therefore, the system of balance laws takes the form

$$(3.3.17)\qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\[2ex] \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho, s) = \rho b, \\[2ex] \partial_t[\rho\varepsilon(\rho, s) + \frac{1}{2}\rho|v|^2] + \operatorname{div}[(\rho\varepsilon(\rho, s) + \frac{1}{2}\rho|v|^2 + p(\rho, s))v^\top] \\[2ex] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \rho b \cdot v + \rho r, \end{cases}$$

with

$$(3.3.18)\qquad p(\rho, s) = \rho^2 \varepsilon_\rho(\rho, s), \qquad \theta(\rho, s) = \varepsilon_s(\rho, s).$$

The system (3.3.17) will be hyperbolic if

$$(3.3.19)\qquad \varepsilon_s(\rho, s) > 0, \qquad p_\rho(\rho, s) > 0.$$

In addition to (3.3.17), thermodynamically admissible processes must also satisfy the Clausius-Duhem inequality (2.3.14), which here reduces to

$$(3.3.20)\qquad \partial_t(-\rho s) + \operatorname{div}(-\rho s v) \le -\rho \frac{r}{\theta(\rho, s)}\,.$$

When the process is smooth, it follows from (3.3.17) and (3.3.18) that (3.3.20) holds identically, as an equality.[3] Consequently, $\eta = -\rho s$ is an entropy for the system (3.3.17) with associated entropy flux $-\rho s v$. Once again we see that a companion balance law is extended from an identity for classical solutions into an inequality serving as a test for the physical admissibility of weak solutions.

### 3.3.6 Isentropic flow of thermoelastic fluids:

The entropy is fixed at a constant value and is dropped from the notation. The state vector is $U = (\rho, v)$, with values in $I\!R^4$. The system of balance laws results from (3.3.17) by discarding the balance of energy:

[3] Thus for smooth solutions it is often convenient to substitute the simpler equality (3.3.20) for the third equation of (3.3.17).

$$
\left\{
\begin{array}{l}
\partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\
\\
\partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) = \rho b,
\end{array}
\right. \tag{3.3.21}
$$

with

$$
p(\rho) = \rho^2 \varepsilon'(\rho). \tag{3.3.22}
$$

The system (3.3.21) is hyperbolic if

$$
p'(\rho) > 0. \tag{3.3.23}
$$

This condition is indeed satisfied in the case of the polytropic gas (2.5.27).

Thermodynamically admissible isentropic motions must satisfy the inequality (2.5.29), which here reduces to

$$
\partial_t[\rho\varepsilon(\rho) + \tfrac{1}{2}\rho|v|^2] + \operatorname{div}[(\rho\varepsilon(\rho) + \tfrac{1}{2}\rho|v|^2 + p(\rho))v^\top] \le \rho b \cdot v. \tag{3.3.24}
$$

It should be noted that the system (3.3.21) results from the system (3.3.17) by canonical truncation, as described in Section 1.5, and in particular the companion balance law (3.3.24) can be derived from the companion balance law (3.3.20) by means of (1.5.12).

The pattern has by now become familiar: By virtue of (3.3.22), any classical solution of (3.3.21) satisfies identically (3.3.25), as an equality, so that the function $\eta = \rho\varepsilon(\rho) + \frac{1}{2}\rho|v|^2$ is an entropy for the system (3.3.21). At the same time, the inequality (3.3.25) is employed to weed out physically inadmissible weak solutions.

The balance laws for isothermal processes of thermoelastic fluids are obtained by replacing in (3.3.21), (3.3.22) and (3.3.24) the internal energy $\varepsilon(\rho)$, at constant entropy, by the Helmholtz free energy $\psi(\rho)$, at constant termperature.

### 3.3.7 The Boltzmann equation and extended thermodynamics:

In contrast to continuum physics, kinetic theories realize matter as an aggregate of interacting molecules, and characterize the state by means of the molecular density function $f(\xi, x, t)$ of the velocity $\xi \in \mathbb{R}^3$ of molecules occupying the position $x \in \mathbb{R}^3$ at time $t$. In the classical kinetic theory, which applies to monatomic gases, $f(\xi, x, t)$ satisfies the *Boltzmann equation*

$$
\partial_t f + \xi \cdot \operatorname{grad}_x f = \mathcal{Q}[f], \tag{3.3.25}
$$

where $\mathcal{Q}$ stands for a complicated integral operator that accounts for changes in $f$ incurred by collisions between molecules.

A formal connection between the continuum and the kinetic approach can be established by monitoring the family of moments

$$
F_{i_1 \dots i_N} = \int_{\mathbb{R}^3} \xi_{i_1} \cdots \xi_{i_N} f d\xi, \qquad i_1, \cdots, i_N = 1, 2, 3 \tag{3.3.26}
$$

of the density $f$. Indeed, these moments satisfy an infinite system of evolution equations

$$(3.3.27)\qquad \begin{cases} \partial_t F + \partial_j F_j = P \\ \partial_t F_i + \partial_j F_{ij} = P_i\ , & i = 1,2,3 \\ \partial_t F_{ij} + \partial_k F_{ijk} = P_{ij}\ , & i,j = 1,2,3 \\ \partial_t F_{ijk} + \partial_\ell F_{ijk\ell} = P_{ijk}\ , & i,j,k = 1,2,3 \\ \dots\dots\dots\dots\dots \\ \partial_t F_{i_1\dots i_N} + \partial_m F_{i_1\dots i_N m} = P_{i_1\dots i_N}\ , & i_1,\dots,i_N = 1,2,3. \end{cases}$$

In the above equations, and throughout this section, $\partial_i$ denotes $\partial/\partial x_i$ and we employ the summation convention: repeated indices are summed over the range 1,2,3. The term $P_{i_1\dots i_N}$ denotes the integral of $\xi_{i_1}\cdots\xi_{i_N}\mathcal{Q}[f]$ over $\mathbb{R}^3$. Because of the special structure of $\mathcal{Q}$, the terms $P$, $P_i$ and $P_{ii}$ vanish.

We notice that each equation of (3.3.27) may be regarded as a balance law, in the spirit of continuum physics. In that interpretation, the moments of $f$ are playing the role of both density and flux of balanced extensive quantities. In fact, the flux in each equation becomes the density in the following one. Under the identification

$$(3.3.28)\qquad F = \rho\ ,$$

$$(3.3.29)\qquad F_i = \rho v_i\ , \quad i = 1,2,3,$$

$$(3.3.30)\qquad F_{ij} = \rho v_i v_j - T_{ij}\ , \qquad i,j = 1,2,3,$$

$$(3.3.31)\qquad \tfrac{1}{2}F_{ii} = \rho\varepsilon + \tfrac{1}{2}\rho|v|^2,$$

$$(3.3.32)\qquad \tfrac{1}{2}F_{iik} = \big(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2\big)v_k - T_{ki}v_i + q_k\ , \quad k = 1,2,3,$$

in (3.3.27), its first equation renders conservation of mass, its second equation renders conservation of linear momentum, and one half the trace of its third equation renders conservation of energy, for a heat conducting viscous gas with density $\rho$, velocity $v$, internal energy $\varepsilon$, Cauchy stress $T$ and heat flux $q$. We regard $T$ as the composition, $T = -pI + \sigma$, of a pressure $p = -\frac{1}{3}T_{ii}$ and a shearing stress $\sigma$ that is traceless, $\sigma_{ii} = 0$. By virtue of (3.3.30) and (3.3.31),

$$(3.3.33)\qquad \rho\varepsilon = \tfrac{3}{2}p\ ,$$

which is compatible with the constitutive equations (2.5.19) of the polytropic gas, for $\gamma = 5/3$.

Motivated by the above observations, one may construct a full hierarchy of continuum theories by truncating the infinite system (3.3.27), retaining only a finite number of equations. The resulting systems, however, will not be closed, because the highest order moments, appearing as flux(es) in the last equation(s), and also the production terms on the right-hand side remain undetermined. In the spirit of continuum physics, *extended thermodynamics* closes these systems by postulating that the highest order moments and the production terms are related to the lower order moments by constitutive equations that are determined by requiring that all smooth solutions of the system satisfy identically a certain inequality, akin to the Clausius-Duhem inequality. This induces a companion balance law which renders the system symmetrizable and thereby hyperbolic. The principle of material frame indifference should also be observed by the constitutive relations.

To see how the program works in practice, let us construct a truncation of (3.3.27) with state vector $U = (\rho, v, p, \sigma, q)$, which has dimension 13, as $\sigma$ is symmetric and traceless. For that purpose, we retain the first three of the equations of (3.3.27), for a total of 10 independent scalar equations, and also extract 3 equations from the fourth equation of (3.3.27) by contracting two of the indices. By virtue of (3.3.28), (3.3.29), (3.3.30), (3.3.31), (3.3.32) and since $P$, $P_i$ and $P_{ii}$ vanish, we end up with the system

$$\left\{\begin{array}{l} \partial_t\rho + \partial_j\big(\rho v_j\big) = 0 \\ \partial_t\big(\rho v_i\big) + \partial_j\big(\rho v_i v_j + p\delta_{ij} - \sigma_{ij}\big) = 0 \\ \partial_t\big(\rho\varepsilon + \frac{1}{2}\rho|v|^2\big) + \partial_k\big\{\big(\rho\varepsilon + \frac{1}{2}\rho|v|^2 + p\big)v_k - \sigma_{kj}v_j + q_k\big\} = 0 \\ \partial_t\big(\rho v_i v_j - \frac{1}{3}\rho|v|^2\delta_{ij} - \sigma_{ij}\big) + \partial_k\big(F_{ijk} - \frac{1}{3}F_{\ell\ell k}\delta_{ij}\big) = P_{ij} \\ \partial_t\big\{\big(\rho\varepsilon + \frac{1}{2}\rho|v|^2 + p\big)v_k - \sigma_{kj}v_j + q_k\big\} + \frac{1}{2}\partial_i F_{jjik} = \frac{1}{2}P_{iik}\,. \end{array}\right. \tag{3.3.34}$$

This system can be closed by postulating that $F_{ijk}$, $F_{jjik}$, $P_{ij}$ and $P_{iik}$ are functions of the state vector $U = (\rho, v, p, \sigma, q)$, which are determined by requiring that all smooth solutions satisfy identically an inequality

$$\partial_t\varphi + \partial_i\psi_i \le 0, \tag{3.3.35}$$

where $\varphi$ and $\psi_i$ are (unspecified) functions of $U$, and $\varphi(U)$ is convex. After a lengthy calculation (see the references cited in Section 3.4), one obtains complicated albeit quite explicit constitutive relations:

$$F_{ijk} = \rho v_i v_j v_k + \big(p v_k + \tfrac{2}{5}q_k\big)\delta_{ij} + \big(p v_i + \tfrac{2}{5}q_i\big)\delta_{jk} + \big(p v_j + \tfrac{2}{5}q_j\big)\delta_{ik}\,, \tag{3.3.36}$$

$$\begin{aligned} F_{jjik} = {}& \big(\rho|v|^2 + 7p\big)v_i v_k + \big(p\delta_{ik} - \sigma_{ik}\big)|v|^2 - \sigma_{ij}v_j v_k \\ & - \sigma_{kj}v_j v_i + \frac{14}{5}\,(q_i v_k + q_k v_i) + \frac{4}{5}q_j v_j\delta_{ik} + \frac{p}{\rho}\,(5p\delta_{ik} - 7\sigma_{ik})\,, \end{aligned} \tag{3.3.37}$$

$$P_{ij} = \tau_0 \sigma_{ij}\ , \qquad P_{iik} = 2\tau_0 \sigma_{ki} v_i - \tau_1 q_k\ . \tag{3.3.38}$$

To complete the picture, $p$, $\tau_0$ and $\tau_1$ must be specified as functions of $(\rho, \theta)$.

The special vector $U^* = B(U)$, in the notation of Section 1.5, that symmetrizes the system has components

$$U^* = \frac{1}{\theta} \begin{pmatrix} \dfrac{5p}{2\rho} - \theta s - \dfrac{1}{2}|v|^2 + \dfrac{1}{2p}\sigma_{ij} v_i v_j - \dfrac{\rho}{5p^2} q_i v_i |v|^2 \\ v_i - \dfrac{1}{p}\sigma_{ij} v_j + \dfrac{\rho}{5p^2}\left(|v|^2 q_i + 2 q_j v_j v_i\right) \\ -1 + \dfrac{2\rho}{3p^2} q_k v_k \\ -\dfrac{1}{2p}\sigma_{ij} - \dfrac{\rho}{5p^2}\left(v_i q_j + v_j q_i - \tfrac{2}{3} v_k q_k \delta_{ij}\right) \\ \dfrac{\rho}{5p^2} q_i \end{pmatrix} \tag{3.3.39}$$

In particular, as explained in Section 1.5, truncating the system (3.3.34) by dropping the last two equations should be paired with "freezing" the last two components of $U^*$, i.e., by setting $q = 0$ and $\sigma = 0$. In that case, the system of the first three equations of (3.3.34) reduces to the system (3.3.17), in the particular situation where $b = 0$, $r = 0$ and $\rho\varepsilon$ and $p$ are related by (3.3.33). If one interprets $(\rho, v, p)$ as the basic state variables and $(\sigma, q)$ as internal state variables, as explained in Section 2.8, then (3.3.17) becomes the relaxed form of the system (3.3.34).

### 3.3.8 Maxwell's equations in nonlinear dielectrics:

Another rich source of interesting systems of hyperbolic balance laws is electromagnetism. The underlying system is Maxwell's equations

$$\begin{cases} \partial_t B = -\operatorname{curl} E \\ \partial_t D = \operatorname{curl} H - J \end{cases} \tag{3.3.40}$$

on $I\!R^3$, relating the *electric field* $E$, the *magnetic field* $H$, the *electric displacement* $D$, the *magnetic induction* $B$ and the *current* $J$, all of them 3-vectors.

Constitutive relations determine $E$, $H$, and $J$ from the state vector $U = (B, D)$. For example, when the medium is a homogeneous electric conductor, with linear dielectric response, at rest relative to the inertial frame, then $D = \varepsilon E$, $B = \mu H$, and $J = \sigma E$, where $\varepsilon$ is the dielectric constant, $\mu$ is the magnetic permeability and $\sigma$ is the electric conductivity. In order to account for (possibly) moving media with nonlinear dielectric response and cross coupling of electromagnetic fields, one postulates general constitutive equations

$$E = E(B, D), \quad H = H(B, D), \quad J = J(B, D), \tag{3.3.41}$$

where the functions $E$ and $H$ satisfy the *lossless condition*

$$\frac{\partial H}{\partial D} = \frac{\partial E}{\partial B}. \tag{3.3.42}$$

Physically admissible fields must also satisfy the dissipation inequality

$$\partial_t \eta(B, D) + \operatorname{div} Q(B, D) \le h(B, D) \tag{3.3.43}$$

where (recall (3.3.42))

$$\eta = \int [H \cdot dB + E \cdot dD], \quad Q = E \wedge H, \quad h = -J \cdot E. \tag{3.3.44}$$

Thus $\eta$ is the *electromagnetic field energy* and $Q$ is the *Poynting vector*. A straightforward calculation shows that smooth solutions of (3.3.40), (3.3.41) satisfy (3.3.43) identically, as an equality. Therefore, $(\eta, Q)$ constitutes an entropy-entropy flux pair for the system of balance laws (3.3.40), (3.3.41). Since $\mathrm{D}\eta = (H, E)$, it follows from the discussion in Section 3.2 that when the electromagnetic field energy function is uniformly convex, then the change of state vector from $U = (B, D)$ to $U^* = (H, E)$ renders the system symmetric hyperbolic.

The dielectric is *isotropic* when the electromagnetic field energy is invariant under rigid rotations of the vectors $B$ and $D$, in which case $\eta$ may depend on $(B, D)$ solely through the three scalar products $B \cdot B$, $D \cdot D$, and $B \cdot D$.

The Born-Infeld medium, with constitutive equations

$$\begin{cases} E = \dfrac{\partial \eta}{\partial D} = \dfrac{1}{\eta}[D + B \wedge Q], \quad H = \dfrac{\partial \eta}{\partial B} = \dfrac{1}{\eta}[B - D \wedge Q] \\ \eta = \sqrt{1 + |B|^2 + |D|^2 + |Q|^2}, \qquad Q = D \wedge B \end{cases} \tag{3.3.45}$$

has remarkably special structure, as we shall see in Section 5.5.

### 3.3.9 Lundquist's equations of magnetohydrodynamics:

Interesting systems of hyperbolic balance laws arise in the context of electromechanical phenomena, where the balance laws of mass, momentum and energy of continuum thermomechanics are coupled with Maxwell's equations. As an illustrative example, we consider here the theory of *magnetohydrodynamics*, which describes the interaction of a magnetic field with an electrically conducting thermoelastic fluid. The equations follow from a number of simplifying assumptions, which will now be outlined.

Beginning with Maxwell's equations, the electric displacement $D$ is considered negligible so (3.3.40) yields $J = \operatorname{curl} H$. The magnetic induction $B$ is related to the magnetic field $H$ by the classical relation $B = \mu H$. The electric field is generated by the motion of the fluid in the magnetic field and so is given by $E = B \wedge v = \mu H \wedge v$.

The fluid is a thermoelastic nonconductor of heat whose thermomechanical properties are still described by the constitutive relations (3.3.18). The balance of mass $(3.3.17)_1$ remains unaffected by the presence of the electromagnetic field. On the other hand, the electromagnetic field exerts a force on the fluid which should be accounted as body force in the balance of momentum $(3.3.17)_2$. The contribution of the electric field $E$ to this force is assumed negligible while the contribution of the magnetic field is $J \wedge B = -\mu H \wedge \operatorname{curl} H$. By account of the identity

$$-H \wedge \operatorname{curl} H = \operatorname{div}[HH^\top - \tfrac{1}{2}(H \cdot H)I], \tag{3.3.46}$$

this body force may be realized as the divergence of the *Maxwell stress tensor*. We assume there is no external body force. The electromagnetic effects on the energy equation $(3.3.17)_3$ are derived by virtue of (3.3.44): The internal energy should be augmented by the electromagnetic field energy $\frac{1}{2}\mu|H|^2$. The Poynting vector $\mu(H \wedge v) \wedge H = \mu|H|^2 v - \mu(H \cdot v)H$ should be added to the flux. Finally, the electromagnetic energy production $-J \cdot E = -\mu(H \wedge v) \cdot \operatorname{curl} H$ and the rate of work $(J \wedge B) \cdot v = -\mu(H \wedge \operatorname{curl} H) \cdot v$ of the electromagnetic body force cancel each other out.

We thus arrive at *Lundquist's equations*:

(3.3.47)

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\ \partial_t(\rho v) + \operatorname{div}[\rho v v^\top - \mu H H^\top] + \operatorname{grad}\big[p(\rho, s) + \tfrac{1}{2}\mu|H|^2\big] = 0 \\ \partial_t\big[\rho\varepsilon(\rho, s) + \tfrac{1}{2}\rho|v|^2 + \tfrac{1}{2}\mu|H|^2\big] \\ \qquad + \operatorname{div}\big[(\rho\varepsilon(\rho, s) + \tfrac{1}{2}\rho|v|^2 + p(\rho, s) + \mu|H|^2)v^\top - \mu(H \cdot v)H^\top\big] = \rho r \\ \partial_t H - \operatorname{curl}(v \wedge H) = 0. \end{cases}$$

The above system of balance laws, with state vector $U = (\rho, v, s, H)$, will be hyperbolic if (3.3.19) hold. Thermodynamically admissible solutions of (3.3.47) should also satisfy the Clausius-Duhem inequality (3.3.20). By virtue of (3.3.18), it is easily seen that any classical solution of (3.3.47) satisfies identically (3.3.20) as an equality. Thus $-\rho s$ is an entropy for the system (3.3.47), with associated entropy flux $-\rho s v$.

## 3.4 Notes

The theory of nonlinear hyperbolic systems of balance laws traces its origins to the mid 19th century and has developed over the years conjointly with gas dynamics. English translations of the seminal papers, with annotations, are collected in the book by Johnson and Chéret [1]. The classic monograph by Courant and Friedrichs [1] amply surveys, in mathematical language, the state of the subject at the end of the

Second World War. It is the distillation of this material that has laid the foundations of the formalized mathematical theory in its present form.

The great number of books on the theoretical and the numerical analysis of hyperbolic systems of conservation laws published in recent years is a testament to the vitality of the field. The fact that these books complement each other, as they differ in scope, style and even content, is indicative of the breadth of the area.

Students who prefer to make their first acquaintance with the subject through a bird's-eye view, may begin with the outlines in the treatise by M.E. Taylor [2], the textbooks by Evans [2] and Hörmander [2], or the lecture notes of Lax [5], Liu [28] and Dafermos [6,10].

On the theoretical side, Jeffrey [2], Rozdestvenski and Janenko [1], and Smoller [3] are early comprehensive texts at an introductory level. The more recent books by Serre [11], Bressan [9], Holden and Risebro [2], and LeFloch [5] combine a general introduction to the subject with advanced, deeper investigations in selected directions. The encyclopedic article by Chen and Wang [1] uses the Euler equations of gas dynamics as springboard for surveying broadly the theory of strictly hyperbolic systems of conservation laws in one-space dimension. Finally, Majda [4], Chang and Hsiao [3], Li, Zhang and Yang [1], Yuxi Zheng [1], Lu [2], and Perthame [2] are specialized monographs, more narrowly focussed. The above books will be cited again, in later chapters, as their content becomes relevant to the discussion, and thus the reader will get some idea of their respective offerings.

Turning to numerical analysis, LeVeque [1] is an introductory text, while the books by Godlewski and Raviart [1,2], and LeVeque [2] provide a more comprehensive and technical coverage together with a voluminous list of references. Other useful sources are the books by Kröner [1], Sod [1], Toro [1], and Holden and Risebro [2], and the lecture notes of Tadmor [2].

Another rich resource is the text by Whitham [2] which presents a panorama of connections of the theory with a host of diverse applications as well as a survey of ideas and techniques devised over the years by applied mathematicians studying wave propagation, of which many are ready for more rigorous analytical development. Zeldovich and Raizer [1,2] are excellent introductions to gas dynamics from the perspective of physicists and may be consulted for building intuition.

The student may get a sense of the evolution of research activity in the field over the past twenty years by consulting the Proceedings of the International Conferences on Hyperbolic Problems which are held biennially. Those that have already appeared at the time of this writing, listed in chronological order and under the names of their editors, are: Carasso, Raviart and Serre [1], Ballmann and Jeltsch [1], Engquist and Gustafsson [1], Donato and Oliveri [1], Glimm, Grove, Graham and Plohr [1], Fey and Jeltsch [1], Freistühler and Warnecke [1], and Hou and Tadmor [1].

An insightful perspective on the state of the subject at the turn of the century is provided by Serre [16].

The term "entropy" in the sense employed here was introduced by Lax [4]. A collection of informative essays on various notions of "entropy" in physics and mathematics is found in the book edited by Greven, Keller and Warnecke [1]. For an interesting discussion of the issue of symmetrizability, see Panov [2].

The systems (3.3.17) and (3.3.21) are commonly called *Euler's equations*. There is voluminous literature on various aspects of their theory, some of which will be cited in subsequent chapters. For a detailed functional analytic study in several space dimensions, together with an extensive bibliography, see the monograph of Lions [2]. A classification of convex entropies is found in Harten [1] and Harten, Lax, Levermore and Morokoff [1].

For a thorough treatment of extended thermodynamics and its relation to the kinetic theory, the reader should consult the monograph by Müller and Ruggeri [1]. The issue of generating simpler systems by truncating more complex ones is addressed in detail by Boillat and Ruggeri [1].

For a systematic development of electrothermomechanics, along the lines of the development of continuum thermomechanics in Chapter II, see Coleman and Dill [1] and Grot [1]. Numerous examples of electrodynamical problems involving hyperbolic systems of balance laws are presented in Bloom [1]. The constitutive equations (3.3.45) were proposed by Born and Infeld [1]. The reader may find some of their remarkable properties, together with relevant references, in Chapter V. For magnetohydrodynamics see for example the texts of Cabannes [1] and Jeffrey [1].

The theory of relativity is a rich source of interesting hyperbolic systems of balance laws, which will not be tapped in this work. When the fluid velocity is comparable to the speed of light, the Euler equations should be modified to account for special relativistic effects; cf. Taub [1], Friedrichs [3] and the book by Christodoulou [1]. The study of these equations from the perspective of the theory of hyperbolic balance laws has already produced a substantial body of literature. Smoller and Temple [3] is a detailed survey article with copious references. See also Ruggeri [1,2], Smoller and Temple [1,2], Pant [1] and Jing Chen [1].

Hyperbolic systems of balance laws, with special structure, govern separation processes, such as chromatography and electrophoresis, employed in chemistry. In that connection the reader may consult the monograph by Rhee, Aris and Amundson [1]. The system of electrophoresis will be recorded later, in Section 7.3, and some of its special properties will be discussed in Chapters VII and VIII.

# IV

# The Cauchy Problem

The theory of the Cauchy problem for hyperbolic conservation laws is confronted with two major challenges. First, classical solutions, starting out from smooth initial values, spontaneously develop discontinuities; hence in general only weak solutions may exist in the large. Next, weak solutions to the Cauchy problem fail to be unique. One does not have to dig too deep in order to encounter these difficulties. As shown in Sections 4.2 and 4.4, they arise even at the level of the simplest nonlinear hyperbolic conservation laws, in one or several space dimensions.

The Cauchy problem for weak solutions will be formulated in Section 4.3. To overcome the obstacle of nonuniqueness, restrictions need to be imposed that will weed out unstable, physically irrelevant, or otherwise undesirable solutions, in hope of singling out a unique admissible solution. Two admissibility criteria will be introduced in this chapter: the requirement that admissible solutions satisfy a designated entropy inequality; and the principle that admissible solutions should be limits of families of solutions to systems containing diffusive terms, as the diffusion asymptotically vanishes. A preliminary comparison of these criteria will be conducted.

The chapter will close with the formulation of the initial-boundary value problem for hyperbolic conservation laws.

## 4.1 The Cauchy Problem: Classical Solutions

To avoid trivial complications, we focus the investigation on homogeneous hyperbolic systems of conservation laws in canonical form,

(4.1.1) $$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) = 0,$$

even though the analysis can be extended in a routine manner to general hyperbolic systems of balance laws (3.1.1). The spatial variable $x$ takes values in $\mathbb{R}^m$ and time $t$ takes values in $[0, T)$, for some $T > 0$ or possibly $T = \infty$. The state vector $U$ takes values in some open subset $\mathscr{O}$ of $\mathbb{R}^n$ and $G = (G_1, \dots, G_m)$ is a given smooth function from $\mathscr{O}$ to $\mathbb{M}^{n\times m}$. The system (4.1.1) is hyperbolic when, for every fixed $U \in \mathscr{O}$ and $\nu \in S^{m-1}$, the $n \times n$ matrix

$$\Lambda(\nu;U) = \sum_{\alpha=1}^{m} \nu_\alpha \mathrm{D}G_\alpha(U) \tag{4.1.2}$$

has real eigenvalues $\lambda_1(\nu;U), \dots, \lambda_n(\nu;U)$ and $n$ linearly independent eigenvectors $R_1(\nu;U), \dots, R_n(\nu;U)$.

To formulate the Cauchy problem, we assign initial conditions

$$U(x,0) = U_0(x), \qquad x \in I\!R^m, \tag{4.1.3}$$

where $U_0$ is a function from $I\!R^m$ to $\mathcal{O}$.

A *classical solution* of (4.1.1) is a locally Lipschitz function $U$, defined on $I\!R^m \times [0,T)$ and taking values in $\mathcal{O}$, which satisfies (4.1.1) almost everywhere. This function solves the Cauchy problem, with initial data $U_0$, if it also satisfies (4.1.3) for all $x \in I\!R^m$.

As we shall see, the theory of the Cauchy problem is greatly enriched when the system is endowed with an entropy $\eta$ with associated entropy flux $Q$, related by

$$\mathrm{D}Q_\alpha(U) = \mathrm{D}\eta(U)\mathrm{D}G_\alpha(U), \qquad \alpha = 1, \cdots, m. \tag{4.1.4}$$

In that case, any classical solution of (4.1.1) will satisfy the additional conservation law

$$\partial_t \eta(U(x,t)) + \mathrm{div}\, Q(U(x,t)) = 0. \tag{4.1.5}$$

As we proceed with the development of the theory, it will become clear that convex entropy functions exert a stabilizing influence on solutions. As a first indication of that effect, the following proposition shows that for systems endowed with a convex entropy, the range of influence of the initial data on solutions of the Cauchy problem is finite.

**4.1.1 Theorem.** *Assume* (4.1.1) *is a hyperbolic system, with characteristic speeds* $\lambda_1(\nu;U) \le \cdots \le \lambda_n(\nu;U)$*, which is endowed with an entropy* $\eta(U)$ *and associated entropy flux* $Q(U)$*. Suppose* $U(x,t)$ *is a classical solution of* (4.1.1) *on* $I\!R^m \times [0,T)$*, with initial data* (4.1.3)*, where* $U_0$ *is constant on a half-space: For some* $\xi \in S^{m-1}$*,* $U_0(x) = \bar{U}$ *=constant whenever* $x \cdot \xi \ge 0$*. Assume, further, that* $\mathrm{D}^2\eta(\bar{U})$ *is positive definite. Then, for any* $t \in [0,T)$*,* $U(x,t) = \bar{U}$ *whenever* $x \cdot \xi \ge \lambda_n(\xi;\bar{U})t$*.*

**Proof.** Without loss of generality, we may assume that $\eta(\bar{U}) = 0$, $\mathrm{D}\eta(\bar{U}) = 0$, $Q_\alpha(\bar{U}) = 0$, $\mathrm{D}Q_\alpha(\bar{U}) = 0$, $\alpha = 1, \dots, m$, since otherwise we just replace the given entropy-entropy flux pair with the new pair

$$\bar{\eta}(U) = \eta(U) - \eta(\bar{U}) - \mathrm{D}\eta(\bar{U})[U - \bar{U}], \tag{4.1.6}$$

$$\bar{Q}(U) = Q(U) - Q(\bar{U}) - \mathrm{D}\eta(\bar{U})\big[G(U) - G(\bar{U})\big]. \tag{4.1.7}$$

For each $s \in I\!R$, $\nu \in S^{m-1}$ and $U \in \mathcal{O}$, we define

$$\Phi(s, \nu; U) = s\eta(U) - Q(U)\nu, \tag{4.1.8}$$

noting that $\Phi(s, \nu; \bar{U}) = 0$ and $\mathrm{D}\Phi(s, \nu; \bar{U}) = 0$. Furthermore, upon using (4.1.4) and (4.1.2),

$$\mathrm{D}^2\Phi(s, \nu; \bar{U}) = \mathrm{D}^2\eta(\bar{U})[sI - \Lambda(\nu; \bar{U})]. \tag{4.1.9}$$

Hence, for $j, k = 1, \dots, n$,

$$\begin{aligned} R_j(\nu; \bar{U})^\top \mathrm{D}^2\Phi(s, \nu; \bar{U}) R_k(\nu; \bar{U}) \\ = [s - \lambda_k(\nu; \bar{U})] R_j(\nu; \bar{U})^\top \mathrm{D}^2\eta(\bar{U}) R_k(\nu; \bar{U}). \end{aligned} \tag{4.1.10}$$

The right-hand side of (4.1.10) vanishes for $j \neq k$, by virtue of (3.2.5), and has the same sign as $s - \lambda_k(\nu; \bar{U})$ for $j = k$, since $\mathrm{D}^2\eta(\bar{U})$ is positive definite.

For $\varepsilon > 0$, we set $\bar{s} = \max\limits_{\nu \in S^{m-1}} \lambda_n(\nu; \bar{U}) + \varepsilon$ and $\hat{s} = \lambda_n(\xi; \bar{U}) + \varepsilon$. Then there exists $\delta = \delta(\varepsilon)$ such that

$$\begin{cases} \Phi(\bar{s}, \nu; U) \geq \frac{1}{4}\varepsilon|U - \bar{U}|^2, \ \ |U - \bar{U}| < \delta(\varepsilon), \ \ \nu \in S^{m-1} \\ \Phi(\hat{s}, \xi; U) \geq \frac{1}{4}\varepsilon|U - \bar{U}|^2, \ \ |U - \bar{U}| < \delta(\varepsilon). \end{cases} \tag{4.1.11}$$

To establish the assertion of the theorem, it suffices to show that for each fixed $\varepsilon > 0$ and $t \in [0, T)$, $U(x, t) = \bar{U}$ whenever $x \cdot \xi \geq \hat{s}t$.

With any point $(y, \tau)$, where $\tau \in (0, T)$ and $y \cdot \xi \geq \hat{s}\tau$, we associate the cone

$$\mathcal{K}_{y,\tau} = \left\{(x, t) : 0 \leq t \leq \tau, \ |x - y| \leq \bar{s}(\tau - t), \ x \cdot \xi \geq y \cdot \xi - \hat{s}(\tau - t)\right\}, \tag{4.1.12}$$

which is contained in the set $\{(x, t) : 0 \leq t < T, \ x \cdot \xi \geq \hat{s}t\}$. Thus, the boundary of the $t$-section of $\mathcal{K}_{y,\tau}$ is the union $\mathcal{P}_t \cup \mathcal{S}_t$ of a subset $\mathcal{P}_t$ of a hyperplane perpendicular to $\xi$, and a subset $\mathcal{S}_t$ of the sphere with center $y$ and radius $\bar{s}(\tau - t)$. The exterior unit normal to the $t$-section at a point $x$ is $-\xi$ if $x \in \mathcal{P}_t$, and $\bar{s}^{-1}(\tau - t)^{-1}(x - y)$ if $x \in \mathcal{S}_t$. Therefore, integrating (4.1.5) over $\mathcal{K}_{y,\tau}$, applying Green's theorem and using the notation (4.1.8) we obtain

$$\int_0^\tau\!\!\int_{\mathcal{P}_t} \Phi(\hat{s}, \xi; U) d\mathcal{H}^{m-1}(x) dt + \int_0^\tau\!\!\int_{\mathcal{S}_t} \Phi\big(\bar{s}, -\bar{s}^{-1}(\tau - t)^{-1}(x - y); U\big) d\mathcal{H}^{m-1}(x) dt = 0. \tag{4.1.13}$$

After this preparation, assume that the assertion of the theorem is false. Since $U(x, t)$ is continuous, and $U(x, 0) = \bar{U}$ for $x \cdot \xi \geq 0$, one can find $(y, \tau)$, with $\tau \in (0, T)$ and $y \cdot \xi \geq \hat{s}\tau$, such that $U(y, \tau) \neq \bar{U}$ and $|U(x, t) - \bar{U}| < \delta(\varepsilon)$ for all $(x, t) \in \mathcal{K}_{y,\tau}$. In that case, (4.1.13) together with (4.1.11) yield a contradiction. The proof is complete.

It is interesting that in the above proof a crude, "energy", estimate provides the sharp value of the rate of growth of the range of influence of the initial data.

As we shall see in Chapter V, the Cauchy problem is well-posed in the class of classical solutions, so long as $U_0$ is sufficiently smooth and $T$ is sufficiently small. In the large, however, the situation is quite different. This will be demonstrated in the following section.

## 4.2 Breakdown of Classical Solutions

Here we shall make the acquaintance of two distinct mechanisms, namely "wave breaking" and "mass explosion", which may induce the breakdown of classical solutions of the Cauchy problem for nonlinear hyperbolic conservation laws.

We shall see first that nonlinearity forces compressive wave profiles to become steeper and eventually break, so that a derivative of the solution blows up. This will be demonstrated in the context of the simplest example of a nonlinear hyperbolic conservation law in one spatial variable, namely the *Burgers equation*

(4.2.1) $$\partial_t u(x,t) + \partial_x\left(\tfrac{1}{2}u^2(x,t)\right) = 0.$$

This deceptively simple-looking equation pervades the theory of hyperbolic conservation laws, as it repeatedly emerges, spontaneously, in the analysis of general systems; see for instance Section 7.6.

Suppose $u(x,t)$ is a smooth solution of the Cauchy problem for (4.2.1), with initial data $u_0(x)$, defined on some time interval $[0,T)$. The characteristics of (4.2.1) associated with this solution are integral curves of the ordinary differential equation $dx/dt = u(x,t)$. Letting an overdot denote differentiation $\cdot = \partial_t + u\partial_x$ in the characteristic direction, we may rewrite (4.2.1) as $\dot{u} = 0$, which shows that $u$ stays constant along characteristics. This implies, in turn, that characteristics are straight lines.

Setting $\partial_x u = v$ and differentiating (4.2.1) with respect to $x$ yields the equation $\partial_t v + u\partial_x v + v^2 = 0$, or $\dot{v} + v^2 = 0$. Therefore, along the characteristic issuing from any point $(\bar{x}, 0)$ where $u_0'(\bar{x}) < 0$, $|\partial_x u|$ will be an increasing function which blows up at $\bar{t} = \left[-u_0'(\bar{x})\right]^{-1}$. It is thus clear that $u(x,t)$ must break down, as classical solution, at or before time $\bar{t}$.

For future reference, we shall compare and contrast the behavior of solutions of (4.2.1) with the behavior of solutions to Burgers's equation with damping:

(4.2.2) $$\partial_t u(x,t) + \partial_x\left(\tfrac{1}{2}u^2(x,t)\right) + u(x,t) = 0.$$

The arguments employed above for (4.2.1), adapted to (4.2.2), yield that the evolution of classical solutions $u$, and their derivatives $v = \partial_x u$, along characteristics $dx/dt = u(x,t)$, is now governed by the equations $\dot{u} + u = 0$, and $\dot{v} + v^2 + v = 0$. The last equation exemplifies the competition between the destabilizing action of nonlinear response and the smoothing effect of damping: When the initial data $u_0$ satisfy $u_0'(x) \geq -1$, for all $x \in (-\infty, \infty)$, then damping prevails, $\partial_x u$ remains bounded, and a global classical solution exists for the Cauchy problem. By contrast, if $u_0'(\bar{x}) < -1$, for some $\bar{x} \in (-\infty, \infty)$, then waves break in finite time, as $v = \partial_x u$ must blow up along the characteristic issuing from the point $(\bar{x}, 0)$.

As we shall see in Section 7.8, the wave-breaking catastrophe occurs generically to solutions of genuinely nonlinear systems of conservation laws in one-space dimension. In fact, waves are more likely to break in a single spatial dimension, where characteristics are confined to a plane and cannot avoid colliding with each other. Nevertheless, in Section 6.1 it is shown that wave breaking commonly occurs even in several spatial dimensions.

Next we consider a different scenario of breakdown of classical solutions, by "mass explosion": Even though the total "mass" is conserved, the restrictions on the rate of growth of the range of influence, imposed by Theorem 4.1.1, prevent the timely dispersion of mass peaks, thus producing sizable fluxes that segregate positive from negative masses, eventually driving them to infinity.

To see whether the above scenario may materialize, consider the Cauchy problem for the Burgers equation (4.2.1), with initial data $u_0(\cdot)$ supported in the interval $[0, 1]$. Suppose a classical solution $u(x, t)$ exists on some time interval $[0, T)$. In that case, by virtue of Theorem 4.1.1, $u(\cdot, t)$ will be supported in $[0, 1]$, for any $t \in [0, T)$. We define the weighted total mass

$$M(t) = \int_0^1 x u(x, t) dx, \tag{4.2.3}$$

and use (4.2.1) and Schwarz's inequality to derive the differential inequality

$$\dot{M}(t) = -\tfrac{1}{2} \int_0^1 x \partial_x \big(u^2(x, t)\big) dx = \tfrac{1}{2} \int_0^1 u^2(x, t) dx \geq \tfrac{3}{2} M^2(t). \tag{4.2.4}$$

Thus, if $M(0) > 0$, $M(t)$ must blow up no later than at time $t^* = \frac{2}{3} M(0)^{-1}$. It is not difficult to see, however, that under the current conditions, waves will start breaking no later than at time $\bar{t} = \frac{1}{6} M(0)^{-1}$, i.e. the wave-breaking catastrophe will occur earlier than the mass explosion catastrophe.

Conditions that may trigger mass explosion are commonly present in hyperbolic systems of conservation laws in several spatial dimensions. We demonstrate this here in the context of the system governing three-dimensional isentropic flow of polytropic gases, namely (3.3.21) with zero body force, $b = 0$, and equation of state $p(\rho) = \kappa \rho^\gamma$, where $\kappa > 0$ and $\gamma > 1$. We write this system in canonical form,

$$\begin{cases} \partial_t \rho + \operatorname{div} m^\top = 0 \\ \partial_t m + \operatorname{div}\big(\rho^{-1} m m^\top\big) + \operatorname{grad}\big(\kappa \rho^\gamma\big) = 0, \end{cases} \tag{4.2.5}$$

using as state variables the mass density $\rho$ and the momentum density $m = \rho v$. The fast characteristic speed is $\lambda(\rho) = \big[p'(\rho)\big]^{1/2} = \big[\kappa \gamma \rho^{\gamma-1}\big]^{1/2}$, in any direction $\nu \in S^2$.

We consider the Cauchy problem for (4.2.5), with initial values $\rho(x,0) = \rho_0$ =constant, for $x \in I\!R^3$, and $m(x,0)$ supported in the unit ball, $m(x,0) = 0$ if $|x| \geq 1$. Suppose there exists a classical solution $(\rho(x,t), m(x,t))$ on some time interval $[0,T)$. By virtue[1] of Theorem 4.1.1, $\rho(x,t) = \rho_0$ and $m(x,t) = 0$, for any $t \in [0,T)$ and $|x| \geq r(t)$, where $r(t) = 1 + \lambda(\rho_0)t$. In particular, from $(4.2.5)_1$ ,

$$\int_{|x|<r(t)} [\rho(x,t) - \rho_0]\, dx = 0, \qquad 0 \leq t < T. \tag{4.2.6}$$

We will monitor the evolution of the weighted radial momentum

$$M(t) = \int_{|x|<r(t)} x \cdot m(x,t) dx. \tag{4.2.7}$$

We differentiate (4.2.7) with respect to $t$, express the time derivative $\partial_t m$ in terms of spatial derivatives, through $(4.2.5)_2$ , and integrate by parts to get

$$\dot{M}(t) = \int_{|x|<r(t)} \left[\rho^{-1}|m|^2 + 3\kappa\left(\rho^\gamma - \rho_0^\gamma\right)\right] dx. \tag{4.2.8}$$

Since $\gamma \geq 1$, (4.2.6) and Jensen's inequality imply

$$\int_{|x|<r(t)} \left(\rho^\gamma - \rho_0^\gamma\right) dx \geq 0. \tag{4.2.9}$$

Furthermore, by (4.2.7), (4.2.6) and Schwarz's inequality,

$$\begin{aligned} M^2(t) &\leq \int_{|x|<r(t)} \rho|x|^2 dx \int_{|x|<r(t)} \rho^{-1}|m|^2 dx \\ &\leq \frac{4\pi}{3}\rho_0 r^5(t) \int_{|x|<r(t)} \rho^{-1}|m|^2 dx. \end{aligned} \tag{4.2.10}$$

Upon combining (4.2.8) with (4.2.9) and (4.2.10), we end up with the differential inequality

$$\dot{M}(t) \geq \frac{3}{4\pi\rho_0}[1 + \lambda(\rho_0)t]^{-5} M^2(t). \tag{4.2.11}$$

After an elementary integration, recalling that $\lambda(\rho_0) = \left[\kappa\gamma\rho_0^{\gamma-1}\right]^{\frac{1}{2}}$, we conclude that if

[1] Recall that (4.2.5) is endowed with the convex entropy $\eta = \frac{\kappa}{\gamma-1}\rho^\gamma + \frac{1}{2\rho}|m|^2$.

$$M(0) > \frac{16\pi}{3}(\kappa\gamma)^{\frac{1}{2}}\rho_0^{\frac{\gamma+1}{2}}, \tag{4.2.12}$$

then $M(t)$ will blow up in finite time. Thus, classical solutions of the Cauchy problem for the system of isentropic gas dynamics, with large initial data, generally break down in finite time.

More refined analysis, reported in the literature cited in Section 4.8, shows that mass explosion may also occur in nonisentropic gas dynamics, and even when the initial data are not necessarily large. Other mechanisms that may induce the breakdown of smooth solutions include wave focusing, formation of vortex sheets, and the appearance of vacuum. It is not easy to determine which kind of catastrophe will occur first in each case.

## 4.3 The Cauchy Problem: Weak Solutions

In view of the examples of breakdown of classical solutions presented in the previous section—and many more that will be encountered in the sequel—it becomes imperative to consider weak solutions to systems of conservation laws (4.1.1). The natural notion for a weak solution should be determined in conjunction with an existence theory. The issue of existence of weak solutions has been settled in a definite manner for scalar conservation laws, in any number of spatial variables (see Chapter VI), and at least partially for systems in one spatial variable (see Chapters XIII–XVI); it remains totally unsettled, however, for systems in several spatial variables. In the absence of a definitive existence theory, and in order to introduce a number of relevant notions, without imposing technical growth conditions on the flux function $G$, we shall define here as *weak solutions* of (4.1.1) locally bounded, measurable functions $U$, defined on $\mathbb{R}^m \times [0, T)$ and taking values in $\mathcal{O}$, which satisfy (4.1.1) in the sense of distributions.

Recalling Lemma 1.3.3, we normalize any weak solution $U$ of (4.1.1) so that the map $t \mapsto U(\cdot, t)$ becomes continuous on $[0, T)$ in $L^\infty(\mathcal{X})$ weak*, for any compact subset $\mathcal{X}$ of $\mathbb{R}^m$. A normalized weak solution of (4.1.1) will then solve the Cauchy problem (4.1.1), (4.1.3) if it also satisfies (4.1.3) almost everywhere on $\mathbb{R}^m$. Lemma 1.3.3 also implies

$$\int_\tau^T \int_{\mathbb{R}^m} \left[\partial_t \Phi\, U + \sum_{\alpha=1}^m \partial_\alpha \Phi\, G_\alpha(U)\right] dx\, dt + \int_{\mathbb{R}^m} \Phi(x, \tau) U(x, \tau) dx = 0, \tag{4.3.1}$$

for every Lipschitz test function $\Phi(x, t)$, with compact support in $\mathbb{R}^m \times [0, T)$ and values in $\mathbb{M}^{1\times n}$, and any $\tau \in [0, T)$. In particular,

$$\int_0^T \int_{\mathbb{R}^m} \left[\partial_t \Phi\, U + \sum_{\alpha=1}^m \partial_\alpha \Phi\, G_\alpha(U)\right] dx\, dt + \int_{\mathbb{R}^m} \Phi(x, 0) U_0(x) dx = 0, \tag{4.3.2}$$

which may serve as an equivalent definition of a weak solution of (4.1.1), (4.1.3). The continuity of $t \mapsto U(\cdot, t)$ also induces the desirable semigroup property: If $U(x,t)$ is a weak solution of (4.1.1) on $[0, T)$, with initial values $U(x, 0)$, then for any $\tau \in [0, T)$ the function $U_\tau(x,t) = U(x, t+\tau)$ is a weak solution of (4.1.1) with initial values $U_\tau(x, 0) = U(x, \tau)$.

As the system is in divergence form, there is a mechanism of regularity transfer from the spatial to the temporal variables:

**4.3.1 Theorem.** *Let $U$ be a bounded weak solution of* (4.1.1) *on* $[0, T)$ *such that, for any fixed $t \in [0, T)$, $U(\cdot, t) \in BV(\mathbb{R}^m)$ and $TV_{\mathbb{R}^m} U(\cdot, t) \le V$, for all $t \in [0, T)$. Then $t \mapsto U(\cdot, t)$ is Lipschitz continuous in $L^1(\mathbb{R}^m)$ on* $[0, T)$,

$$\|U(\cdot, \sigma) - U(\cdot, \tau)\|_{L^1(\mathbb{R}^m)} \le aV|\sigma - \tau|, \quad 0 \le \tau < \sigma < T, \tag{4.3.3}$$

*where $a$ depends solely on the Lipschitz constant of $G$. In particular, $U$ is in $BV_{loc}$ on $\mathbb{R}^m \times [0, T)$.*

**Proof.** Fix $0 \le \tau < \sigma < T$ and any $\Psi \in C_0^\infty\left(\mathbb{R}^m; \mathbb{M}^{1\times n}\right)$, with $|\Psi(x)| \le 1$ for $x \in \mathbb{R}^m$. From (4.3.1) it follows that

$$\int_{\mathbb{R}^m} \Psi(x)[U(x,\sigma) - U(x,\tau)]dx = \int_\tau^\sigma \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \Psi(x) G_\alpha(U(x,t))dx\,dt. \tag{4.3.4}$$

The spatial integral on the right-hand side is majorized by the total variation of $G(U(\cdot, t))$, which in turn is bounded by $aV$. Taking the supremum of (4.3.4) over all $\Psi$ with $|\Psi(x)| \le 1$, we arrive at (4.3.3).

Theorem 1.7.2 together with (4.3.3) imply that $U$ is in $BV_{loc}$ on $\mathbb{R}^m \times [0, T)$. The proof is complete.

For $BV_{loc}$ solutions of the system (4.1.1), which is in canonical form, the Rankine-Hugoniot jump condition (3.1.3) becomes

$$-s\left[U_+ - U_-\right] + \left[G(U_+) - G(U_-)\right]\nu = 0. \tag{4.3.5}$$

## 4.4 Nonuniqueness of Weak Solutions

Extending the notion of solution from classical to weak introduces a new difficulty: nonuniqueness. To see this, consider the Cauchy problem for the Burgers equation (4.2.1), with initial data

$$u(x, 0) = \begin{cases} -1, & x < 0 \\ \phantom{-}1, & x > 0. \end{cases} \tag{4.4.1}$$

This is an example of the celebrated Riemann problem, which will be discussed at length in Chapter IX. The problem (4.2.1), (4.4.1) admits infinitely many weak solutions, including the family

$$(4.4.2) \qquad u_\alpha(x,t) = \begin{cases} -1, & -\infty < x \le -t \\ \dfrac{x}{t}, & -t < x \le -\alpha t \\ -\alpha, & -\alpha t < x \le 0 \\ \alpha, & 0 < x \le \alpha t \\ \dfrac{x}{t}, & \alpha t < x \le t \\ 1, & t < x < \infty, \end{cases}$$

for any $\alpha \in [0,1]$. Indeed, $u_\alpha(x,t)$ satisfies (4.2.1), in the classical sense, provided $x/t \notin \{0, \pm\alpha, \pm 1\}$. The lines $x/t = \pm 1$, for $\alpha \in [0,1]$, and $x/t = \pm\alpha$, for $\alpha$ in $(0,1)$, are just weak fronts, across which $u_\alpha$ is continuous. Finally, for $\alpha \neq 0$, the line $x = 0$ is a stationary shock front across which the Rankine-Hugoniot jump condition (4.3.5) holds.

To resolve the issue of nonuniqueness, additional restrictions, in the form of *admissibility conditions*, shall be imposed on weak solutions. This will require a lengthy discussion, which will begin in the following two sections and culminate in Chapters VIII and IX. In particular, it will turn out that $u_0(x,t)$ is the sole admissible solution of the simple problem (4.2.1), (4.4.1) considered in this section.

## 4.5 Entropy Admissibility Condition

As we saw in Chapter III, every system of balance laws arising in continuum physics is accompanied by an entropy inequality that must be satisfied by any physically meaningful process, as it expresses, explicitly or implicitly, the Second Law of thermodynamics. This motivates the following procedure for characterizing admissible weak solutions of hyperbolic systems of conservation laws.

Assume our system (4.1.1) is endowed with a designated entropy $\eta$, associated with an entropy flux $Q$, so that (4.1.4) holds. A weak solution of (4.1.1), in the sense of Section 4.3, defined on $\mathbb{R}^m \times [0,T)$, will satisfy the *entropy admissibility criterion,* relative to $\eta$, if

$$(4.5.1) \qquad \partial_t \eta(U(x,t)) + \operatorname{div} Q(U(x,t)) \le 0$$

holds, in the sense of distributions, on $\mathbb{R}^m \times [0,T)$.

Clearly, any classical solution of (4.1.1) is admissible, as it satisfies the equality (4.1.5). Another relevant remark is that the entropy admissibility criterion induces a time *irreversibility* condition on solutions: If $U(x,t)$ is an admissible weak solution of (4.1.1) which satisfies (4.5.1) as a strict inequality, then $\bar{U}(x,t) = U(-x,-t)$, which is also a solution, is not admissible.

A natural question is how one may designate an appropriate entropy for the admissibility criterion. For instance, it is clear that a weak solution that is admissible relative to an entropy $\eta$ fails to be admissible relative to the entropy $\bar{\eta} = -\eta$. When the system derives from physics, then it is physics that should designate the natural entropy. In the absence of guidance from physics, one has to use mathematical arguments. It is, of course, desirable that the admissibility criterion induced by the designated entropy should be compatible with admissibility conditions induced by alternative criteria, to be introduced later. Another natural condition is that admissible weak solutions should enjoy reasonable stability properties. As we shall see, all of the above requirements are met when the designated entropy $\eta(U)$ is convex, or at least "convexlike".

The reader should bear in mind that convexity is a relevant property of the entropy only when the system is in canonical form. In the general case, convexity of $\eta$ should be replaced by the condition that the ($n \times n$ matrix-valued) derivative $\mathrm{D}B(U, x, t)$ of the ($n$-vector-valued) function $B(U, x, t)$ in (3.2.2) is positive definite.

A review of the examples considered in Section 3.3 reveals that the entropy, as a function of the state vector that converts the system of balance laws into canonical form, is indeed convex in the case of the thermoelastic fluid (example 3.3.5), the isentropic thermoelastic fluid (example 3.3.6) and magnetohydrodynamics (example 3.3.9). This may raise expectations that in the equations of continuum physics entropy is generally convex. However, as we shall see in Section 5.4, this is not always the case; hence the necessity to consider a broader class of entropy functions.

For any weak solution $U$ satisfying the entropy admissibility criterion, the left-hand side of (4.5.1) is a nonpositive distribution, and thereby a measure, which shall be dubbed the *entropy production measure*. Then Lemma 1.3.3 implies that the map $t \mapsto \eta(U(\cdot, t))$ is continuous on $[0, T)\backslash\mathcal{F}$ in $L^\infty(\mathcal{D})$ weak*, for any compact subset $\mathcal{D}$ of $\mathbb{R}^m$, where $\mathcal{F}$ is at most countable. Furthermore, for every nonnegative Lipschitz test function $\psi(x, t)$, with compact support in $\mathbb{R}^m \times [0, T)$, and any $\tau \in [0, T)\backslash\mathcal{F}$,

(4.5.2)

$$\int_\tau^T \int_{\mathbb{R}^m} \left[\partial_t \psi \, \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(U)\right] dx \, dt + \int_{\mathbb{R}^m} \psi(x, \tau) \, \eta(U(x, \tau)) dx \geq 0.$$

There are good reasons for conjecturing that the set $\mathcal{F}$ is actually empty. Indeed, $\tau \in \mathcal{F}$ if and only if the entropy production measure of the set $\mathbb{R}^m \times \{\tau\}$ is (strictly) negative. However, it is to be expected that sets of Hausdorff dimension $m - 1$ with nonzero entropy production should look like shock fronts, propagating with finite speed. At the time of this writing, there is a rigorous proof that $\mathcal{F} = \emptyset$ only in the scalar case, $n = 1$ (see Section 6.8). As we shall see, it is a great help to the analysis if at least $0 \notin \mathcal{F}$, in which case (4.5.2), with $\tau = 0$, becomes

$$\int_0^T \int_{\mathbb{R}^m} \left[ \partial_t \psi \, \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(U) \right] dx\, dt + \int_{\mathbb{R}^m} \psi(x,0)\, \eta(U_0(x)) dx \ge 0. \tag{4.5.3}$$

Accordingly, it is (4.5.3), rather than the slightly weaker condition (4.5.1), that is often postulated in the literature as the entropy admissibility criterion for the weak solution $U$. It should be noted, however, that admissible weak solutions characterized through (4.5.3) do not necessarily possess the desirable semigroup property, i.e. $U(x,t)$ admissible does not generally imply that $U_\tau(x,t) = U(x, t+\tau)$ is also admissible, for all $\tau \in [0, T)$. Thus, in the author's opinion, admissibility should be defined either through (4.5.1) alone or through (4.5.2), for all $\tau \in [0, T)$. Hopefully, an eventual proof that $\mathcal{F}$ is empty will render the distinction moot.

A first indication of the enhanced regularity enjoyed by admissible weak solutions when the entropy is convex is provided by the following

**4.5.1 Theorem.** *Assume* $U(x,t)$ *is a weak solution of* (4.1.1) *on* $\mathbb{R}^m \times [0, T)$*, which satisfies the entropy admissibility condition* (4.5.1) *relative to a uniformly convex entropy* $\eta(U)$*. Then* $t \mapsto U(\cdot, t)$ *is continuous on* $[0, T) \backslash \mathcal{F}$ *in* $L^p_{loc}(\mathbb{R}^m)$*, for any* $p \in [1, \infty)$*, where* $\mathcal{F}$ *is at most countable. Moreover, if* (4.5.2) *holds for some* $\tau$ *in* $[0, T)$*, then* $t \mapsto U(\cdot, t)$ *is continuous from the right at* $\tau$ *in* $L^p_{loc}(\mathbb{R}^m)$*, for any* $p \in [1, \infty)$.

**Proof.** Since both $t \mapsto U(\cdot, t)$ and $t \mapsto \eta(U(\cdot, t))$ are continuous on $[0, T)\backslash\mathcal{F}$ in $L^\infty(\mathcal{D})$ weak*, for any compact subset $\mathcal{D}$ of $\mathbb{R}^m$, and $\eta(U)$ is uniformly convex, it follows that $t \mapsto U(\cdot, t)$ is strongly continuous on $[0, T)\backslash\mathcal{F}$ in $L^p(\mathcal{D})$, for any $p \in [1, \infty)$.

Assume now (4.5.2) holds, for some $\tau \in [0, T)$. Fix $\varepsilon > 0$ and apply (4.5.2) for $\psi(x,t) = \phi(x) g(t)$, where $\varphi \in C_0^\infty(\mathbb{R}^m)$, with $\varphi(x) \ge 0$ for $x \in \mathbb{R}^m$, and $g(t) = 1 - \varepsilon^{-1}(t - \tau)$, for $0 \le t < \tau + \varepsilon$, and $g(t) = 0$, for $t + \varepsilon \le t < \infty$. This gives

$$\frac{1}{\varepsilon} \int_\tau^{\tau+\varepsilon} \int_{\mathbb{R}^m} \varphi(x) \left[ \eta(U(x,\tau)) - \eta(U(x,t)) \right] dx\, dt \ge O(\varepsilon). \tag{4.5.4}$$

Letting $\varepsilon \downarrow 0$ and recalling Lemma 1.3.3, we deduce that $\operatorname{ess\,lim}_{t \downarrow \tau} \eta(U(\cdot, t)) \le \eta(U(\cdot, \tau))$, where the limit is in $L^\infty(\mathcal{D})$ weak*, for any compact subset $\mathcal{D}$ of $\mathbb{R}^m$. Since $\lim_{t\downarrow\tau} U(\cdot, t) = U(\cdot, \tau)$, again in $L^\infty(\mathcal{D})$ weak*, and $\eta(U)$ is uniformly convex, it follows that, as $t \downarrow \tau$, $U(\cdot, t) \longrightarrow U(\cdot, \tau)$, strongly in $L^p(\mathcal{D})$, for any $p \in [1, \infty)$. The proof is complete.

It should be reiterated here that the set $\mathcal{F}$ in the above theorem is probably empty.

Whenever the admissible solution $U$ is of class $BV_{loc}$, Theorem 1.8.2 implies that the entropy production measure is concentrated on the set of points of approximate jump discontinuity of $U$, i.e. on the shock fronts. In that case, (4.5.1) reduces to the local condition (1.8.5), which in the present notation takes the form

$$(4.5.5)\qquad -s\left[\eta(U_+)-\eta(U_-)\right]+\left[Q(U_+)-Q(U_-)\right]\nu\leq 0.$$

For admissibility of $U$ relative to the entropy $\eta$, (4.5.5) has to be tested at any point of a shock that propagates in the direction $\nu \in S^{m-1}$ with speed $s$.

As an application, let us test the admissibility of the family $u_\alpha(x,t)$ of solutions to (4.2.1), (4.4.1), defined by (4.4.2), relative to the entropy-entropy flux pair $\left(\frac{1}{2}u^2, \frac{1}{3}u^3\right)$. It is clear that, for any $\alpha \in (0,1]$, the stationary shock $x = 0$ violates (4.5.5). Thus, the sole admissible solution in that family is $u_0(x,t)$, which is Lipschitz continuous, away from the origin.

## 4.6 The Vanishing Viscosity Approach

According to the *viscosity criterion,* a solution $U$ of (4.1.1) is admissible provided it is the $\mu \downarrow 0$ limit of solutions $U_\mu$ to a system of conservation laws with diffusive terms:

$$(4.6.1)\qquad \partial_t U(x,t)+\sum_{\alpha=1}^{m}\partial_\alpha G_\alpha(U(x,t))=\mu\sum_{\alpha,\beta=1}^{m}\partial_\alpha[B_{\alpha\beta}(U(x,t))\partial_\beta U(x,t)],$$

where the $B_{\alpha\beta}$ are $n\times n$ matrix-valued functions defined on $\mathcal{O}$.

The motivation for this principle and the term "vanishing viscosity" derive from continuum physics: As we saw in earlier chapters, the balance laws of thermoelastic materials under adiabatic conditions induce first order systems of hyperbolic type. By contrast, the balance laws for thermoviscoelastic, heat-conducting materials, introduced in Section 2.6, generate systems of second order in the spatial variables, containing diffusive terms akin to those appearing on the right-hand side of (4.6.1). In nature, every material has viscous response and conducts heat, to a certain degree. Classifying a particular material as an elastic nonconductor of heat simply means that viscosity and heat conductivity are negligible, albeit not totally absent. Consequently, the theory of adiabatic thermoelasticity may be physically meaningful only as a limiting case of thermoviscoelasticity, with viscosity and heat conductivity tending to zero. It is this premise that underlies the vanishing viscosity approach.

In laying down (4.6.1), the first task is to select the $n\times n$ matrices $B_{\alpha\beta}(U)$, for $\alpha,\beta = 1,\dots,m$. When dealing with systems of physical origin, the natural choice for these matrices is dictated, or at least suggested, by physics. For example, thermoelastic fluid nonconductors of heat should be regarded as Newtonian fluids with constitutive equations (2.6.16), (2.6.17) having vanishingly small viscosity and heat conductivity. Accordingly, when (4.1.1) stands for the system (3.3.17) of balance laws of mass, momentum and energy for thermoelastic fluids that do not conduct heat (with zero body force and heat supply), the appropriate choice for the corresponding system (4.6.1), with diffusive terms, should be[2]

[2] We write this system in components form, let $\partial_i$ denote $\partial/\partial x_i$ and employ the summation convention: Repeated indices are summed over the range 1,2,3.

$$(4.6.2)\qquad \begin{cases} \partial_t \rho + \partial_j(\rho v_j) = 0 \\ \partial_t (\rho v_i) + \partial_j \left(\rho v_i v_j\right) + \partial_i p(\rho, s) = \lambda \partial_i \partial_j v_j + \mu \partial_j \left(\partial_i v_j + \partial_j v_i\right) \\ \partial_t\left[\rho\varepsilon(\rho, s) + \frac{1}{2}\rho|v|^2\right] + \partial_j\left[\left(\rho\varepsilon(\rho, s) + \frac{1}{2}\rho|v|^2 + p(\rho, s)\right)v_j\right] \\ \qquad = \lambda \partial_i \left(v_i \partial_j v_j\right) + \mu \partial_j \left[\left(\partial_i v_j + \partial_j v_i\right)v_i\right] + \kappa \partial_i \partial_i \theta. \end{cases}$$

The reader should take notice that (4.6.2) contains three independent physical parameters, namely the bulk viscosity $\lambda + \frac{2}{3}\mu$, the shear viscosity $\mu$ and the thermal conductivity $\kappa$, which might all be very small albeit of different orders of magnitude. Thus, one should be prepared to consider formulations of the vanishing viscosity principle, more general than (4.6.1), involving several independent small parameters. However, this generalization will not be pursued here.

Physics suggests the natural form for (4.6.1) in every example considered in Section 3.3, including electromagnetism, magnetohydrodynamics, etc. On the other hand, in the absence of guidance from physics, or for mere analytical and computational convenience, one may experiment with *artificial viscosity* added to the right-hand side of (4.1.1). For example, one may add artificial viscosity to (4.2.1) to derive the *Burgers equation with viscosity:*

$$(4.6.3)\qquad \partial_t u(x, t) + \partial_x\left(\tfrac{1}{2}u^2(x, t)\right) = \mu \partial_x^2 u(x, t).$$

It is clear that artificial viscosity should be selected in such a way that the $B_{\alpha\beta}$ induce dissipation and thus render the Cauchy problem for (4.6.1) well-posed. The temptation is to use for $B_{\alpha\beta}$ matrices that would render (4.6.1) parabolic; and in particular the zero matrix if $\alpha \neq \beta$ and the identity matrix if $\alpha = \beta$, which would reduce the right-hand side to $\mu\Delta U$. The physically motivated example (4.6.2) demonstrates, however, that confining attention to the parabolic case would be ill-advised. In general, one has to deal with systems of intermediate parabolic-hyperbolic type, in which case establishing the well-posedness of the Cauchy problem may require considerable effort.

Effective diffusion matrices $B_{\alpha\beta}$ should at least satisfy the *Kawashima condition*

$$(4.6.4)\qquad \sum_{\alpha,\beta=1}^{m} \nu_\alpha \nu_\beta B_{\alpha\beta}(U) R_i(\nu; U) \neq 0, \quad U \in \mathcal{O},\ \nu \in S^{m-1},\ i = 1, \cdots, n,$$

which guarantees that waves of all characteristic families, propagating in any direction, are properly damped. Indeed, if (4.6.4) is violated for some $\bar{U}$, $\nu$ and $i$, then linearizing (4.6.1) about $\bar{U}$ yields a system which admits traveling wave solutions

$$(4.6.5)\qquad U(x, t) = \varphi(\nu \cdot x - \lambda_i(\nu; \bar{U})t) R_i(\nu; \bar{U})$$

that are not attenuated by the diffusion.

Assuming a vanishing viscosity mechanism has been selected, rendering the Cauchy problem (4.6.1), (4.1.3) well-posed, the question arises as to the sense of convergence of the family $\{U_\mu\}$ of solutions, as $\mu \downarrow 0$. This is of course a serious issue: Requiring very strong convergence may raise unreasonable expectations for compactness of the family $\{U_\mu\}$. On the other hand, if the sense of convergence is too weak, it is not clear that $\lim U_\mu$ will be a solution of (4.1.1). Various aspects of this problem will be discussed later, mainly in Chapters VI, XV and XVI.

Another important task is to compare admissibility of solutions in the sense of the vanishing viscosity approach and admissibility in the sense of a designated entropy inequality (4.5.1), as discussed in Section 4.5. In continuum thermodynamics, presented in Chapter II, whenever (4.6.1) results from actual constitutive equations compatible with the Clausius-Duhem inequality, and (4.5.1) is, or derives from, the Clausius-Duhem inequality, solutions of (4.1.1) obtained by the vanishing viscosity approach will automatically satisfy (4.5.1). For example, solutions of (3.3.17) obtained as the $(\lambda, \mu, \kappa) \downarrow 0$ limit of solutions of (4.6.2) will satisfy automatically the inequality (3.3.20).

If $\eta(U)$ is an entropy for (4.1.1), associated with the entropy flux $Q(U)$, then any (classical) solution $U_\mu$ of (4.6.1) satisfies the identity

$$\text{(4.6.6)} \qquad \partial_t \eta(U_\mu) + \sum_{\alpha=1}^{m} \partial_\alpha Q_\alpha(U_\mu) = \mu \sum_{\alpha,\beta=1}^{m} \partial_\alpha \left[ \mathrm{D}\eta(U_\mu) B_{\alpha\beta}(U_\mu) \partial_\beta U_\mu \right]$$
$$- \mu \sum_{\alpha,\beta=1}^{m} \left( \partial_\alpha U_\mu \right)^\top \mathrm{D}^2 \eta(U_\mu) B_{\alpha\beta}(U_\mu) \partial_\beta U_\mu \,.$$

The second term on the right-hand side should be dissipative, so that the quadratic form associated with $\mathrm{D}^2\eta B_{\alpha\beta}$ must be positive semidefinite. Beyond that, however, this term is entrusted with the responsibility of dominating the first term on the right-hand side of (4.6.6) as well as the right-hand side of (4.6.1). A sufficient, though not necessary, condition for this will be

$$\text{(4.6.7)} \qquad \sum_{\alpha,\beta=1}^{m} \xi_\alpha^\top \mathrm{D}^2 \eta(U) B_{\alpha\beta}(U) \xi_\beta \geq a \sum_{\alpha=1}^{m} \left| \sum_{\beta=1}^{m} B_{\alpha\beta}(U) \xi_\beta \right|^2 ,$$

for some positive constant $a$, any $U \in \mathcal{O}$ and all $\xi_\alpha \in I\!R^n$, $\alpha = 1, \cdots, m$. Notice that when $B_{\alpha\beta}$ vanishes for $\alpha \neq \beta$, and is the identity for $\alpha = \beta$, (4.6.7) reduces to the statement that $\eta(U)$ is uniformly convex. Also note the relation between (4.6.7) and the Kawashima condition (4.6.4) $\big($choose $\xi_\alpha = \nu_\alpha R_i\big)$.

Suppose now that the initial data $U_0$ and the solution $U_\mu$ of (4.6.1), (4.1.3) tend sufficiently fast, as $|x| \to \infty$, to a constant state $\bar{U}$. Without loss of generality we may assume that $\eta(\bar{U}) = 0$ and $\mathrm{D}\eta(\bar{U}) = 0$, since otherwise we may replace $\eta(U)$ by $\bar{\eta}(U)$, defined by (4.1.6). We make the further assumption that actually $\bar{U}$ is the minimum of $\eta$ over $\mathcal{O}$. This of course will automatically be the case when $\eta(U)$ is convex. Under these hypotheses, integrating (4.6.6) over $I\!R^m \times [0, T)$ yields the estimate

$$(4.6.8)\quad \mu \int_0^T \int_{\mathbb{R}^m} \sum_{\alpha,\beta=1}^m \big(\partial_\alpha U_\mu\big)^\top \mathrm{D}^2\eta\big(U_\mu\big) B_{\alpha\beta}\big(U_\mu\big)\partial_\beta U_\mu \, dx\, dt \le \int_{\mathbb{R}^m} \eta\,(U_0(x))\, dx.$$

We have now laid the groundwork for showing that the viscosity admissibility criterion implies the entropy admissibility condition.

**4.6.1 Theorem.** *Under the assumptions on $\eta(U)$ and $\{U_\mu\}$ stated above, suppose that a sequence $\{U_{\mu_k}\}$, with $\mu_k \to 0$ as $k \to \infty$, converges boundedly almost everywhere on $\mathbb{R}^m \times [0, T)$ to some function $U$. Then $U$ is a weak solution of* (4.1.1), (4.1.3) *on $\mathbb{R}^m \times [0, T)$, which satisfies the entropy admissibility condition* (4.5.3).

**Proof.** We multiply (4.6.1) by any Lipschitz test function $\Phi(x, t)$, with compact support in $\mathbb{R}^m \times [0, T)$, taking values in $\mathbb{M}^{1\times n}$, and integrate the resulting equation over $\mathbb{R}^m \times [0, T)$. After an integration by parts,

$$(4.6.9)\quad \begin{aligned} &\int_0^T \int_{\mathbb{R}^m} \Big[\partial_t \Phi\, U_\mu + \sum_{\alpha=1}^m \partial_\alpha \Phi\, G_\alpha\big(U_\mu\big)\Big] dx\, dt + \int_{\mathbb{R}^m} \Phi(x, 0)\, U_0(x) dx \\ &\qquad = \mu \int_0^T \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \Phi\, B_{\alpha\beta}\big(U_\mu\big)\partial_\beta U_\mu \, dx\, dt. \end{aligned}$$

By virtue of (4.6.7) and (4.6.8), as $\mu_k \to 0$, the right-hand side tends to zero, and hence the limit function $U$ satisfies the equation (4.3.2).

Next we multiply (4.6.6) by any nonnegative Lipschitz test function $\psi(x, t)$, with compact support in $\mathbb{R}^m \times [0, T)$, and integrate the resulting equation over the strip $\mathbb{R}^m \times [0, T)$. After an integration by parts,

$$(4.6.10)\quad \begin{aligned} &\int_0^T \int_{\mathbb{R}^m} \Big[\partial_t \psi\, \eta\big(U_\mu\big) + \sum_{\alpha=1}^m \partial_\alpha \psi\, Q_\alpha\big(U_\mu\big)\Big] dx\, dt + \int_{\mathbb{R}^m} \psi(x, 0)\, \eta\big(U_0(x)\big) dx \\ &\qquad = \mu \int_0^T \int_{\mathbb{R}^m} \partial_\alpha \psi\, \mathrm{D}\eta\big(U_\mu\big) B_{\alpha\beta}\big(U_\mu\big)\partial_\beta U_\mu \, dx\, dt \\ &\qquad + \mu \int_0^T \int_{\mathbb{R}^m} \psi \sum_{\alpha,\beta=1}^m \big(\partial_\alpha U_\mu\big)^\top \mathrm{D}^2\eta\big(U_\mu\big) B_{\alpha\beta}\big(U_\mu\big)\partial_\beta U_\mu \, dx\, dt. \end{aligned}$$

By account of (4.6.7) and (4.6.8), the first term on the right-hand side tends to zero, as $\mu_k \to 0$, while the second term is nonnegative. Therefore, the limit function $U$ satisfies the inequality (4.5.3). This completes the proof.

More general admissibility conditions, of the same genre as the viscosity criterion, may be formulated by replacing (4.6.1) with a system of the form

(4.6.11)

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = \mu \sum_{\alpha,\beta=1}^{m} \partial_\alpha [B_{\alpha\beta}(U)\partial_\beta U] + \nu \sum_{\alpha,\beta,\gamma=1}^{m} \partial_\alpha [H_{\alpha\beta\gamma}(U)\partial_\beta \partial_\gamma U],$$

involving third, and sometimes even fourth, order differential operators and two "vanishing" parameters $\mu$ and $\nu$. For example, in the place of (4.6.3) one may take

$$\partial_t u(x,t) + \partial_x(\tfrac{1}{2}u^2(x,t)) = \mu \partial_x^2 u(x,t) + \nu \partial_x^3 u(x,t). \tag{4.6.12}$$

The approach to admissibility via (4.6.11) is suggested by physics when the dissipative effect of viscosity coexists with some dispersive mechanism induced, for instance, by capillarity. Accordingly, the admissibility condition associated with (4.6.11) is dubbed the *viscosity-capillarity criterion.* Which solutions of (4.1.1) pass this test of admissibility will generally depend not only on the choice of $B_{\alpha\beta}$ and $H_{\alpha\beta\gamma}$ , but also on the relative speed by which $\mu$ and $\nu$ tend to zero. As a minimum requirement, (4.6.11) must be compatible with the Second Law of thermodynamics, i.e., a proposition analogous to Theorem 4.6.1 must hold for the entropy-entropy flux pair provided by physics.

## 4.7 Initial-Boundary-Value-Problems

Suppose that the hyperbolic system of conservation laws (4.1.1) is posed on a proper, open subset $\mathcal{D}$ of $\mathbb{R}^m$, with Lipschitz boundary $\partial\mathcal{D}$ and outward unit normal field $\nu$. To formulate a well-posed problem for (4.1.1) on the cylinder $\mathcal{X} = \mathcal{D} \times (0,T)$, in addition to assigning initial data $U(x,0) = U_0(x)$ on the base $\mathcal{D} \times \{0\}$, one must also prescribe boundary conditions on the lateral boundary $\mathcal{B} = \partial\mathcal{D} \times (0,T)$. This raises, however, a number of issues.

To begin with, it is not clear a priori on which part of $\mathcal{B}$ one is free to impose boundary conditions, nor is it obvious what the allowable form of such conditions should be. Consider, for example, an initial-boundary-value problem for the linear scalar conservation law $\partial_t u + a\partial_x u = 0$, in a single spatial dimension, on the interval $\mathcal{D} = (-1,1)$. For $a \neq 0$, the values of $u$ along the lines $t = 0$ and $x = -\operatorname{sgn} a$ uniquely determine $u$ on $(-1,1) \times (0,\infty)$, and thus no boundary condition may be imposed along the part $x = \operatorname{sgn} a$ of the boundary. When $a = 0$, the initial data alone determine $u$ on $(-1,1) \times (0,\infty)$, and so no boundary condition may be prescribed on any part of the boundary.

Nonlinearity introduces additional complications, even in the context of classical solutions. Consider, for example, an initial-boundary-value problem on $(-1,1)$ for the Burgers equation (4.2.1). As we saw in Section 4.2, the straight line characteristic issuing from any point $(x_0,0)$ carries along the value $u_0(x_0)$ of the initial data and hence fixes the value of $u$ at the point where it collides with the boundary, which may lie either on $x = -1$ or on $x = 1$, depending on whether $u_0(x_0) < 0$ or $u_0(x_0) > 0$.

The fundamental issue of what constitutes stable boundary conditions for systems of conservation laws, and where should they be imposed, has been the object of intensive recent investigation, but it will be barely touched on in this book. The reader can find pertinent bibliographic information in Sections 4.8 and 5.6.

Another basic question is how boundary conditions should be interpreted in the context of weak solutions. When the solution $U$ is a $BV$ function on $\mathcal{X}$, its inner trace $U_-$ is well-defined on $\mathcal{B}$ (cf. Section 1.7). Consequently, within the $BV$ framework, boundary conditions may be formulated in a virtually classical, pointwise sense. By contrast, when the solution $U$ is merely in $L^\infty$, there is no natural way of defining its trace on a manifold of codimension one, like $\mathcal{B}$. Even so, as shown in Section 1.3, traces on $\mathcal{B}$ may be naturally defined for the normal component of $L^\infty$ vector fields on $\mathcal{X}$ whose space-time divergence is a measure.

Since the space-time divergence of the field $(G_1(U), \dots, G_m(U), U)$ vanishes in $\mathcal{X}$, one may define the trace $G_{\mathcal{B}} \in L^\infty(\mathcal{B}; \mathbb{R}^n)$ of $G(U)\nu$ on $\mathcal{B}$, by means of the equation (1.3.14), which here takes the form

$$\int_0^T \int_{\partial\mathcal{D}} \Phi G_{\mathcal{B}} \, d\mathcal{H}^{m-1}(x)dt - \int_{\mathcal{D}} \Phi(x,0)\, U_0(x)dx \\ = \int_0^T \int_{\mathcal{D}} \Big[ \partial_t \Phi \, U + \sum_{\alpha=1}^m \partial_\alpha \Phi \, G_\alpha(U) \Big] dx\, dt, \tag{4.7.1}$$

for any Lipschitz test function $\Phi$ with compact support in $\mathbb{R}^m \times [0, T)$ and values in $\mathbb{M}^{1\times n}$.

In a similar fashion, when $U$ satisfies an entropy admissibility condition (4.5.1), one may define the trace $Q_{\mathcal{B}} \in L^\infty(\mathcal{B})$ of $Q(U)\nu$ on $\mathcal{B}$ by means of

$$\int_0^T \int_{\partial\mathcal{D}} \psi \, Q_{\mathcal{B}} \, d\mathcal{H}^{m-1}(x)dt = \int_0^T \int_{\mathcal{D}} \Big[ \partial_t \psi \; \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \; Q_\alpha(U) \Big] dx\, dt + \langle \mathcal{P}, \psi \rangle_{\mathcal{X}} \,, \tag{4.7.2}$$

where $\mathcal{P}$ is the nonpositive entropy production measure and $\psi$ is any Lipschitz test function with compact support in $\mathbb{R}^m \times (0, T)$.

A useful device for generating admissible boundary conditions employs the vanishing viscosity approach expounded in Section 4.6. The premise is to prescribe on $\mathcal{B}$ boundary conditions that are suitable for the diffusive system (4.6.1) and let the limiting process $\mu \downarrow 0$ pick natural boundary conditions for the hyperbolic system (4.1.1). As boundary layers typically form near $\mathcal{B}$ when $\mu$ is small, it is not possible to predict what the resulting boundary conditions will be. It is possible, however, to gain some insight when the system is endowed with entropy-entropy flux pairs compatible with (4.6.1). To see this, let us consider the initial-boundary-value problem for the system (4.6.1), with initial conditions $U = U_0$ on $\mathcal{D}$ and boundary conditions $U = \bar{U}$ on $\mathcal{B}$, where $\bar{U}$ is some fixed state. As in Section 4.6, we assume that, for any $\mu > 0$, this problem possesses a classical solution $U_\mu$ on $\mathcal{X}$, and that some sequence $\{U_{\mu_k}\}$, with $\mu_k \to 0$ as $k \to \infty$, converges boundedly almost everywhere on $\mathcal{X}$ to a (weak) solution $U$ of (4.1.1). Suppose $(\eta, Q)$ is an entropy-entropy flux pair that satisfies (4.6.7). We write (4.6.6) for the normalized entropy-entropy flux pair $(\bar{\eta}, \bar{Q})$, defined by (4.1.6), (4.1.7), multiply the resulting equation by any nonnegative Lipschitz test function $\psi$ with compact support in $\mathbb{R}^m \times [0, T)$, integrate over $\mathcal{D} \times (0, T)$, integrate by parts, and use the initial and boundary conditions to get the analog of (4.6.10), namely

$$\int_0^T \int_{\mathcal{D}} \left[ \partial_t \psi \, \bar{\eta}(U_\mu) + \sum_{\alpha=1}^m \partial_\alpha \psi \, \bar{Q}_\alpha(U_\mu) \right] dx\, dt + \int_{\mathcal{D}} \psi(x, 0)\, \bar{\eta}(U_0(x)) dx \tag{4.7.3}$$

$$= \mu \int_0^T \int_{\mathcal{D}} \partial_\alpha \psi \, \mathrm{D}\bar{\eta}(U_\mu) B_{\alpha\beta}(U_\mu) \partial_\beta U_\mu dx\, dt$$

$$+ \mu \int_0^T \int_{\mathcal{D}} \psi \sum_{\alpha,\beta=1}^m \left(\partial_\alpha U_\mu\right)^\top \mathrm{D}^2 \bar{\eta}(U_\mu) B_{\alpha\beta}(U_\mu) \partial_\beta U_\mu dx\, dt.$$

The argument employed in Section 4.6 shows that, as $\mu \to 0$, the first term on the right-hand side of (4.7.3) tends to zero while the second term stays nonnegative. Therefore,

$$\int_0^T \int_{\mathcal{D}} \left[ \partial_t \psi \, \bar{\eta}(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, \bar{Q}_\alpha(U) \right] dx\, dt + \int_{\mathcal{D}} \psi(x, 0)\, \bar{\eta}(U_0(x)) dx \geq 0. \tag{4.7.4}$$

To return to the original entropy-entropy flux pair $(\eta, Q)$, we write (4.7.1) for $\Phi = \psi \mathrm{D}\eta(\bar{U})$ and combine it with (4.7.4) to get

$$\int_0^T \int_{\mathcal{D}} \left[ \partial_t \psi \, \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(U) \right] dx\, dt + \int_{\mathcal{D}} \psi(x, 0)\, \eta(U_0(x)) dx \tag{4.7.5}$$

$$\geq \int_0^T \int_{\partial\mathcal{D}} \psi \left\{ \bar{Q}_{\mathcal{B}} - \mathrm{D}\eta(\bar{U})[\bar{G}_{\mathcal{B}} - G_{\mathcal{B}}] \right\} d\mathcal{H}^{m-1}(x) dt,$$

where we have set

$$\bar{G}_{\mathcal{B}} = G(\bar{U})\nu, \qquad \bar{Q}_{\mathcal{B}} = Q(\bar{U})\nu. \tag{4.7.6}$$

Finally, we combine (4.7.5) with (4.7.2):

$$(4.7.7)\qquad \int_0^T \int_{\partial\mathcal{D}} \psi \left\{Q_{\mathcal{B}} - \bar{Q}_{\mathcal{B}} - \mathrm{D}\eta(\bar{U})\left[G_{\mathcal{B}} - \bar{G}_{\mathcal{B}}\right]\right\} d\mathcal{H}^{m-1}(x)dt \geq \langle \mathcal{P}, \psi \rangle_{\mathcal{X}} \,,$$

assuming $\psi(x, 0) = 0$, $x \in I\!R^m$ . By letting the support of $\psi$ shrink about points of $\mathcal{B}$, we deduce the pointwise condition

$$(4.7.8)\qquad Q_{\mathcal{B}} - \bar{Q}_{\mathcal{B}} - \mathrm{D}\eta(\bar{U})[G_{\mathcal{B}} - \bar{G}_{\mathcal{B}}] \geq 0.$$

The quantity on the left-hand side of (4.7.8) may be interpreted as the density of a surface measure that represents the entropy loss in the boundary layer.

The inequality (4.7.8) furnishes some information on the boundary conditions induced by the vanishing viscosity approach. Naturally, this information becomes more precise when the system (4.1.1) is endowed with multiple independent entropies compatible with (4.6.1). In particular, as we shall see in Section 6.9, for the scalar conservation law a sufficiently large collection of inequalities (4.7.5) characterizes completely the solution to the initial-boundary-value problem constructed by the vanishing viscosity approach.

## 4.8 Notes

Apparently, it was Bateman [1] who first suggested, in a little noticed paper, that (4.2.1) and (4.6.3) should be employed as models for the system of conservation laws of inviscid and viscous gases. The commonly used name of Burgers [1] was attached to these equations by Hopf [1].

The breaking of waves was first noticed by Challis [1], in the context of a particular solution of the system of isothermal gas dynamics. For a systematic development and references, see Sections 7.7, 7.8 and 7.10. The breakdown of classical solutions of the system of nonisentropic gas dynamics, and other systems of conservation laws, through mass explosion was demonstrated by Sideris [1]. See also John [2], Beale, Kato and Majda [1], and Liu and Yang [1]. Development of singularities in the complex Burgers equation was shown by Noelle [1]. For a discussion of various mechanisms of blowing up, see Alihnac [1].

The issue of admissibility of weak solutions to hyperbolic systems of conservation laws stirred up a debate quite early in the development of the subject. Responding to the introduction of shock fronts in gas dynamics by Stokes [1], Kelvin (in private correspondence with Stokes) and Rayleigh [1] raised the objection that, in the presence of shocks, (isentropic) flows that conserve mass and momentum fail to conserve (mechanical) energy; in other words, weak solutions of the system (3.3.21) do not generally satisfy (3.3.24) as an equality. Intimidated by this criticism, Stokes [2] revised his paper, renouncing the idea of a shock. By the turn of the century, following the development of thermodynamics, weak solutions had been reinstated in physics, albeit under conditions of admissibility, in the form of inequalities derived from the Second Law, as we saw in Section 3.3 (cf. Burton [1], Weber [1], Rayleigh [2]). The jump conditions associated with entropy inequalities were first written down by Jouguet [1], for the equations of gas dynamics. In the framework of the general theory of hyperbolic systems of conservation laws, the use of entropy inequalities to characterize admissible solutions was proposed by Kruzkov [1] and elaborated by Lax [4]. We shall return to this topic on several occasions.

The idea of regarding inviscid gases as viscous gases with vanishingly small viscosity is quite old; there are hints even in the aforementioned seminal paper by Stokes [1]. The important contributions of Rankine [1], Hugoniot [1,2], and Rayleigh [3] helped to clarify the issue. The Kawashima condition was first formulated in Shizuta and Kawashima [1]. In later chapters, we shall have frequent encounters with the vanishing viscosity approach, as a method for constructing solutions or as a means of identifying admissible shocks. References to relevant papers will be provided in the proper context.

An exposition of the theory of systems of intermediate parabolic-hyperbolic type is given in the monographs by Songmu Zheng [1] and Hsiao [3], and in the recent survey article by Hsiao and Jiang [1], where the reader will find an extensive list of references.

Inequalities (4.7.8) were first derived by Bardos, Leroux and Nédélec [1], for scalar conservation laws, and were then extended to systems, in one spatial dimension, by DuBois and LeFloch [1]. As we shall see in Section 6.9, they completely characterize admissible boundary conditions in the scalar case. For the case of systems, see Section 5.6.

# V

# Entropy and the Stability of Classical Solutions

It is a tenet of continuum physics that the Second Law of thermodynamics is essentially a statement of stability. In the examples discussed in the previous chapters, the Second Law manifests itself in the presence of companion balance laws, to be satisfied identically, as equalities, by classical solutions, and to be imposed as inequality thermodynamic admissibility constraints on weak solutions of the systems of balance laws. A recurring theme in the exposition of the theory of hyperbolic systems of balance laws in this book will be that companion balance laws induce stability under various guises. Here the reader will get a glimpse of the implications of entropy inequalities on the stability of classical solutions.

It will be shown that when the system of balance laws is endowed with a companion balance law induced by a convex entropy, the initial-value problem is locally well-posed in the context of classical solutions: Sufficiently smooth initial data generate a classical solution defined on a maximal time interval, typically of finite duration. However, in the presence of frictional damping, and when the initial data are sufficiently small, the classical solution exists globally in time. Classical solutions are unique and depend continuously on their initial values, not only within the class of classical solutions but even within the broader class of weak solutions that satisfy the companion balance law as an inequality admissibility constraint.

Similar existence and stability results will be established, even when the entropy fails to be convex, in the following two situations: (a) the entropy is convex only in the direction of a certain cone in state space but the system is equipped with special companion balance laws, called involutions, whose presence compensates for the lack of convexity in complementary directions; or (b) the system is endowed with complementary entropies and the principal entropy is polyconvex. This structure arises in elastodynamics and electromagnetism.

From the standpoint of analytical technique, this chapter presents the aspects of the theory of quasilinear hyperbolic systems of balance laws that can be tackled by the methodology of the linear theory, namely energy estimates and Fourier analysis.

## 5.1 Convex Entropy and the Existence of Classical Solutions

As in Chapter IV, we consider here the Cauchy problem

$$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) = 0, \qquad x \in \mathbb{R}^m,\ t > 0, \tag{5.1.1}$$

$$U(x,0) = U_0(x), \qquad x \in \mathbb{R}^m, \tag{5.1.2}$$

for a homogeneous hyperbolic system of conservation laws in canonical form. The results may be extended to general hyperbolic systems of balance laws (3.1.1) at the expense of trivial technical complications.

It will be assumed that (5.1.1) is endowed with an entropy $\eta(U)$, so that (3.2.4) holds,

$$\mathrm{D}^2\eta(U)\mathrm{D}G_\alpha(U) = \mathrm{D}G_\alpha(U)^\top \mathrm{D}^2\eta(U), \qquad \alpha = 1,\dots,m, \tag{5.1.3}$$

and it will be shown that if $\eta(U)$ is convex, then a classical solution of the initial-value problem exists on a maximal time interval, provided the initial data are sufficiently smooth.

In what follows, a *multi-index* $r$ will stand for a $m$-tuple of nonnegative integers: $r = (r_1, \cdots, r_m)$. We put $|r| = r_1 + \cdots + r_m$ and $\partial^r = \partial_1^{r_1} \cdots \partial_m^{r_m}$. Thus $\partial^r$ is a differential operator of order $|r|$. We also employ the notation $\nabla = (\partial_1, \dots, \partial_m)$.

For $\ell = 0, 1, 2, \cdots$, $H^\ell$ will be the Sobolev space $W^{\ell,2}(\mathbb{R}^m; \mathbb{M}^{n\times m})$ of $n \times m$ matrix-valued functions. The norm of $H^\ell$ will be denoted by $\|\cdot\|_\ell$. By the Sobolev embedding theorem, for $\ell > m/2$, $H^\ell$ is continuously embedded in the space of continuous $n \times m$ matrix-valued functions on $\mathbb{R}^m$.

**5.1.1 Theorem.** *Assume the system of conservation laws* (5.1.1) *is endowed with a $C^3$ entropy $\eta$ with $\mathrm{D}^2\eta(U)$ positive definite on $\mathcal{O}$. Suppose the initial data $U_0$ are continuously differentiable on $\mathbb{R}^m$, take values in some compact subset of $\mathcal{O}$ and $\nabla U_0 \in H^\ell$ for some $\ell > m/2$. Moreover, let $G \in C^{\ell+2}$. Then there exists $T_\infty \leq \infty$, and a unique continuously differentiable function $U$ on $\mathbb{R}^m \times [0, T_\infty)$, taking values in $\mathcal{O}$, which is a classical solution of the Cauchy problem* (5.1.1), (5.1.2) *on* $[0, T_\infty)$. *Furthermore,*

$$\nabla U(\cdot, t) \in C^0([0, T_\infty); H^\ell). \tag{5.1.4}$$

*The interval* $[0, T_\infty)$ *is maximal, in the sense that whenever* $T_\infty < \infty$

$$\int_0^{T_\infty} \|\nabla U(\cdot,t)\|_{L^\infty} dt = \infty \tag{5.1.5}$$

*and/or the range of $U(\cdot, t)$ escapes from every compact subset of $\mathcal{O}$ as $t \uparrow T_\infty$.*

**Proof.** It is lengthy and tedious. Just an outline will be presented here, so as to illustrate the role of the convex entropy. For the details the reader may consult the references cited in Section 5.6.

Fix any open subset $\mathcal{B}$ of $\mathbb{R}^m$ which contains the closure of the range of $U_0$ and whose closure $\overline{\mathcal{B}}$ is in turn contained in $\mathcal{O}$. With positive constants $\omega$ and $T$, to be fixed later, we associate the class $\mathcal{F}$ of Lipschitz continuous functions $V$, defined on $\mathbb{R}^m \times [0, T]$, taking values in $\mathcal{B}$, satisfying the initial condition (5.1.2) and

$$\nabla V(\cdot, t) \in L^\infty([0, T]; H^\ell), \qquad \partial_t V(\cdot, t) \in L^\infty([0, T]; L^2 \cap L^\infty) \tag{5.1.6}$$

with

$$\sup_{[0,T]} \|\nabla V(\cdot, t)\|_\ell \le \omega, \tag{5.1.7}$$

$$\sup_{[0,T]} \|\partial_t V(\cdot, t)\|_{L^\infty} \le b\omega, \qquad \sup_{[0,T]} \|\partial_t V(\cdot, t)\|_{L^2} \le b\omega, \tag{5.1.8}$$

where

$$b^2 = \max_{V \in \bar{\mathcal{B}}} \sum_{\alpha=1}^m |\mathrm{D}G_\alpha(V)|^2. \tag{5.1.9}$$

For $\omega$ sufficiently large, $\mathcal{F}$ is nonempty; for instance, $V(x, t) \equiv U_0(x)$ is a member of it.

By standard weak lower semicontinuity of norms, $\mathcal{F}$ is a complete metric space under the metric

$$\rho(V, \bar{V}) = \sup_{[0,T]} \|V(\cdot, t) - \bar{V}(\cdot, t)\|_{L^2}\,. \tag{5.1.10}$$

Notice that, even though $V(\cdot, t)$ and $\bar{V}(\cdot, t)$ are not necessarily in $L^2$, we still have $\rho(V, \bar{V}) \le 2b\omega T < \infty$, by virtue of $V(\cdot, 0) - \bar{V}(\cdot, 0) = 0$ and (5.1.8).

We now linearize (5.1.1) about any fixed $V \in \mathcal{F}$:

$$\partial_t U(x, t) + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(V(x, t))\partial_\alpha U(x, t) = 0. \tag{5.1.11}$$

The existence of a solution to (5.1.1), (5.1.2) on $[0, T]$ will be established by showing that

(a) When $\omega$ is sufficiently large and $T$ is sufficiently small, the initial-value problem (5.1.11), (5.1.2) admits a solution $U \in \mathcal{F}$ on $[0, T]$.
(b) The aforementioned solution $U$ is endowed with regularity (5.1.4), slightly better than $(5.1.6)_1$ which mere membership in $\mathcal{F}$ would guarantee.
(c) For $T$ sufficiently small, the map that carries $V \in \mathcal{F}$ to the solution $U \in \mathcal{F}$ of (5.1.11), (5.1.2) is a contraction in the metric (5.1.10) and thus possesses a unique fixed point in $\mathcal{F}$, which is the desired solution of (5.1.1), (5.1.2).

In the following sketch of proof of assertion (a), above, we shall take for granted that the solution $U$ of (5.1.11), (5.1.2), with the requisite regularity, exists, and will proceed to establish that it belongs to $\mathcal{F}$. In a complete proof, one should first mollify $V$ and the initial data, then employ the classical theory of symmetrizable linear hyperbolic systems, and finally pass to the limit. The Hessian $\mathrm{D}^2\eta(V)$ will serve as symmetrizer, since it is symmetric, positive definite, and symmetrizes $\mathrm{D}G_\alpha(V)$ by virtue of (5.1.3).

We fix any multi-index $r$ of order $1 \leq |r| \leq \ell + 1$, set $\partial^r U = U_r$, and apply $\partial^r$ to equation (5.1.11) to get

$$\partial_t U_r + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(V)\partial_\alpha U_r = \sum_{\alpha=1}^{m} \{\mathrm{D}G_\alpha(V)\partial^r\partial_\alpha U - \partial^r[DG_\alpha(V)\partial_\alpha U]\}. \tag{5.1.12}$$

The $L^2$ norm of the right-hand side of (5.1.12) may be majorized with the help of Moser-type inequalities combined with (5.1.7):

$$\begin{aligned} &\|\sum_{\alpha=1}^{m}\{\mathrm{D}G_\alpha(V)\partial^r\partial_\alpha U - \partial^r[\mathrm{D}G_\alpha(V)\partial_\alpha U]\}\|_{L^2} \\ &\leq c\|\nabla V\|_{L^\infty}\|\nabla U\|_\ell + c\|\nabla U\|_{L^\infty}\|\nabla V\|_\ell \leq 2ac\omega\|\nabla U\|_\ell\,. \end{aligned} \tag{5.1.13}$$

Here and below $c$ will stand for a generic positive constant, which may depend on $\mathcal{B}$, but is independent of $\omega$ and $T$.[1]

Let us now multiply (5.1.12), from the left, by $2U_r^\top \mathrm{D}^2\eta(V)$, sum over all multi-indices $r$ with $1 \leq |r| \leq \ell + 1$ and integrate the resulting equation over $I\!R^m \times [0, t]$. Note that

$$2U_r^\top \mathrm{D}^2\eta(V)\partial_t U_r = \partial_t[U_r^\top \mathrm{D}^2\eta(V)U_r] - U_r^\top\partial_t \mathrm{D}^2\eta(V)U_r\,. \tag{5.1.14}$$

Moreover, by virtue of (5.1.3),

$$\begin{aligned} 2U_r^\top \mathrm{D}^2\eta(V)\mathrm{D}G_\alpha(V)\partial_\alpha U_r = \partial_\alpha[U_r^\top \mathrm{D}^2\eta(V)\mathrm{D}G_\alpha(V)U_r] \\ -U_r^\top\partial_\alpha[\mathrm{D}^2\eta(V)\mathrm{D}G_\alpha(V)]U_r\,. \end{aligned} \tag{5.1.15}$$

Recall that $\mathrm{D}^2\eta(V)$ is positive definite, uniformly on compact sets, so that

$$U_r^\top \mathrm{D}^2\eta(V)U_r \geq \delta|U_r|^2, \qquad V \in \overline{\mathcal{B}}, \tag{5.1.16}$$

for some $\delta > 0$. Therefore, combining the above we end up with an estimate

$$\|\nabla U(\cdot,t)\|_\ell^2 \leq c\|\nabla U_0(\cdot)\|_\ell^2 + c\omega\int_0^t \|\nabla U(\cdot,\tau)\|_\ell^2 d\tau \tag{5.1.17}$$

whence, by Gronwall's inequality,

---

[1] To be precise, $c$ solely depends on the maximum on $\bar{\mathcal{B}}$ of $|U|$ and all derivatives $|\mathrm{D}^k G(U)|$, up to order $k = \ell + 2$.

$$\sup_{[0,T]} \|\nabla U(\cdot,t)\|_\ell^2 \le ce^{c\omega T}\|\nabla U_0(\cdot)\|_\ell^2 \,. \tag{5.1.18}$$

It follows from (5.1.18) that if $\omega$ is sufficiently large and $T$ is sufficiently small, $\sup_{[0,T]}\|\nabla U(\cdot,t)\|_\ell \le \omega$. Then (5.1.11) implies $\sup_{[0,T]}\|\partial_t U(\cdot,t)\|_{L^\infty} \le b\omega$, $\sup_{[0,T]}\|\partial_t U(\cdot,T)\|_{L^2} \le b\omega$, with $b$ given by (5.1.9). Finally, for $T$ sufficiently small, $U$ will take values in $\overline{\mathcal{B}}$ on $I\!R^m \times [0,T]$. Thus $U \in \mathcal{F}$.

As a by-product of the derivation of (5.1.7), one obtains that the function $t \mapsto \int U_r^\top \mathrm{D}^2\eta(V)U_r dx$ is continuous on $[0,T]$. Moreover, $t \mapsto U_r(\cdot,t)$ is at least weakly continuous in $L^2$. Consequently, for any fixed $t \in [0,T]$, the integral over $I\!R^m$ of the right-hand side of the identity

$$\begin{aligned}
&[U_r(\cdot,t) - U_r(\cdot,\tau)]^\top \mathrm{D}^2\eta(V(\cdot,t))[U_r(\cdot,t) - U_r(\cdot,\tau)] \\
&\quad = U_r^\top(\cdot,t)\mathrm{D}^2\eta(V(\cdot,t))U_r(\cdot,t) + U_r^\top(\cdot,\tau)\mathrm{D}^2\eta(V(\cdot,\tau))U_r(\cdot,\tau) \\
&\quad - 2U_r^\top(\cdot,t)\mathrm{D}^2\eta(V(\cdot,t))U_r(\cdot,\tau) \\
&\quad + U_r^\top(\cdot,\tau)[\mathrm{D}^2\eta(V(\cdot,t)) - \mathrm{D}^2\eta(V(\cdot,\tau))]U_r(\cdot,\tau)
\end{aligned} \tag{5.1.19}$$

tends to zero, as $\tau \to t$. Thus, $t \mapsto U_r(\cdot,t)$ is actually strongly continuous in $L^2$. This in turn implies assertion (b), namely $\nabla U(\cdot,t)$ is in fact continuous, and not merely bounded, in $H^\ell$ on $[0,T]$.

Turning now to assertion (c), let us fix $V$ and $\bar V$ in $\mathcal{F}$ which induce solutions $U$ and $\bar U$ of (5.1.11), (5.1.2), also in $\mathcal{F}$. Thus

$$\partial_t[U - \bar U] + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(V)\partial_\alpha[U - \bar U] = -\sum_{\alpha=1}^m [\mathrm{D}G_\alpha(V) - \mathrm{D}G_\alpha(\bar V)]\partial_\alpha \bar U. \tag{5.1.20}$$

Multiply (5.1.20), from the left, by $2(U-\bar U)^\top \mathrm{D}^2\eta(V)$ and integrate the resulting equation over $I\!R^m \times [0,t]$, $0 \le t \le T$. Notice that

$$\begin{aligned}
2(U-\bar U)^\top \mathrm{D}^2\eta(V)\partial_t(U-\bar U) &= \partial_t[(U-\bar U)^\top \mathrm{D}^2\eta(V)(U-\bar U)] \\
&\quad -(U-\bar U)^\top \partial_t \mathrm{D}^2\eta(V)(U-\bar U),
\end{aligned} \tag{5.1.21}$$

and also, by virtue of (5.1.3),

$$\begin{aligned}
2(U-\bar U)^\top \mathrm{D}^2\eta(V)\mathrm{D}G_\alpha(V)\partial_\alpha(U-\bar U) &= \partial_\alpha[(U-\bar U)^\top \mathrm{D}^2\eta(V)\mathrm{D}G_\alpha(V)(U-\bar U)] \\
&\quad -(U-\bar U)^\top \partial_\alpha[\mathrm{D}^2\eta(V)\mathrm{D}G_\alpha(V)](U-\bar U).
\end{aligned} \tag{5.1.22}$$

Since $\mathrm{D}^2\eta(V)$ is positive definite,

$$(U-\bar U)^\top \mathrm{D}^2\eta(V)(U-\bar U) \ge \delta|U-\bar U|^2. \tag{5.1.23}$$

Therefore, combining the above with (5.1.7), (5.1.8) and the Sobolev embedding theorem, we arrive at the estimate

$$\|(U - \bar{U})(\cdot, t)\|_{L^2}^2 \le c\omega \int_0^t \|(U - \bar{U})(\cdot, \tau)\|_{L^2}^2 d\tau + c\omega \int_0^t \|(V-\bar{V})(\cdot, \tau)\|_{L^2} \|(U-\bar{U})(\cdot, \tau)\|_{L^2} d\tau. \tag{5.1.24}$$

Using (5.1.10) and Gronwall's inequality, we infer from (5.1.24) that

$$\rho(U, \bar{U}) \le c\omega T e^{c\omega T} \rho(V, \bar{V}). \tag{5.1.25}$$

Consequently, for $T$ sufficiently small, the map that carries $V$ in $\mathcal{F}$ to the solution $U$ of (5.1.11), (5.1.2) is a contraction on $\mathcal{F}$ and thus possesses a unique fixed point $U$ which is the unique solution of (5.1.1), (5.1.2) on $[0, T]$, in the function class $\mathcal{F}$.

Since the restriction $U(\cdot, T)$ of the constructed solution to $t = T$ belongs to the same function class as $U_0(\cdot)$, we may repeat the above construction and extend $U$ to a larger time interval $[0, T']$. Continuing the process, we end up with a solution $U$ defined on a maximal interval $[0, T_\infty)$ with $T_\infty \le \infty$. Furthermore, if $T_\infty < \infty$, then the range of $U(\cdot, t)$ must escape from every compact subset of $\mathcal{O}$, as $t \uparrow T_\infty$, and/or

$$\|\nabla U(\cdot, t)\|_\ell \to \infty, \quad \text{as } t \uparrow T_\infty . \tag{5.1.26}$$

In order to see the implications of (5.1.26), we retrace the steps that led to (5.1.17). We use again (5.1.13), (5.1.14), (5.1.15), and (5.1.16), setting $V \equiv U$, but we no longer majorize $\|\nabla U\|_{L^\infty}$ by $a\omega$. Thus, in the place of (5.1.17) we now get

$$\|\nabla U(\cdot, t)\|_\ell^2 \le c\|\nabla U_0(\cdot)\|_\ell^2 + c \int_0^t \|\nabla U(\cdot, \tau)\|_{L^\infty} \|\nabla U(\cdot, \tau)\|_\ell^2 d\tau. \tag{5.1.27}$$

Gronwall's inequality then implies that (5.1.26) cannot occur unless (5.1.5) does. This completes the proof.

We already saw, in Chapter IV, that finite life span for classical solutions is the rule rather than the exception.

## 5.2 The Role of Damping and Relaxation

In this section we consider the Cauchy problem

$$\partial_t U(x, t) + \operatorname{div} G(U(x, t)) + P(U(x, t)) = 0, \qquad x \in I\!R^m, \quad t > 0, \tag{5.2.1}$$

$$U(x, 0) = U_0(x), \qquad x \in I\!R^m, \tag{5.2.2}$$

for a homogeneous hyperbolic system of balance laws in canonical form, where $G(U)$ and $P(U)$ are smooth functions defined on $\mathcal{O}$. We assume that $P(\bar{U}) = 0$, for some $\bar{U} \in \mathcal{O}$, so that $U \equiv \bar{U}$ is a constant equilibrium solution of (5.2.1).

Suppose (5.2.1) is endowed with a $C^3$ entropy-entropy flux pair $(\eta, Q)$, where $\eta(U)$ is locally uniformly convex, so that any classical solution satisfies the additional balance law

$$\partial_t \eta(U(x,t)) + \operatorname{div} Q(U(x,t)) + \mathrm{D}\eta(U(x,t))P(U(x,t)) = 0. \tag{5.2.3}$$

Without loss of generality, we may assume $\eta(\bar{U}) = 0$, $\mathrm{D}\eta(\bar{U}) = 0$, $Q(\bar{U}) = 0$, $\mathrm{D}Q_\alpha(\bar{U}) = 0$, $\alpha = 1, \dots, m$, since otherwise we simply replace $(\eta, Q)$ with the pair $(\bar{\eta}, \bar{Q})$ defined by (4.1.6), (4.1.7).

For initial data $U_0$ with $\nabla U_0 \in H^\ell$, $\ell > m/2$, a straightforward extension of Theorem 5.1.1 yields the existence of a classical solution to (5.2.1), (5.2.2) on a maximal time interval $[0, T_\infty)$. The aim is to investigate whether the mechanism that causes the breaking of waves may be offset by a dissipative source term that keeps $\|\nabla U(\cdot,t)\|_{L^\infty}$ bounded for all $t > 0$. Our experience with Equation (4.2.2), in Section 4.2, indicates that dissipation is likely to prevail near equilibrium.

Damping manifests itself in that the entropy production is nonnegative on some open neighborhood $\mathcal{B} \subset \mathcal{O}$ of $\bar{U}$:

$$\mathrm{D}\eta(U)P(U) \geq 0, \qquad U \in \mathcal{B}. \tag{5.2.4}$$

Under this assumption, for as long as $U$ takes values in $\mathcal{B}$,

$$\|U(\cdot,t) - \bar{U}\|_{L^2} \leq a\|U_0(\cdot) - \bar{U}\|_{L^2}, \tag{5.2.5}$$

which is obtained by integrating (5.2.3) over $\mathbb{R}^m \times (0,t)$. This, combined with the "interpolation" estimate

$$\|U(\cdot,t) - \bar{U}\|_{L^\infty} \leq b\|\nabla U(\cdot,t)\|^{\rho}_{L^\infty}\|U(\cdot,t) - \bar{U}\|^{1-\rho}_{L^2}, \tag{5.2.6}$$

where $\rho = \frac{1}{2}m(\ell+1)$, in turn implies that $U(\cdot,t)$ will lie in $\mathcal{B}$ for as long as $\|\nabla U(\cdot,t)\|_{L^\infty}$ stays sufficiently small.

As in Section 5.1, we fix any multi-index $r$ of order $1 \leq |r| \leq \ell + 1$, then set $\partial^r U = U_r$, and apply $\partial^r$ to the equation (5.2.1) to get

$$\begin{aligned} \partial_t U_r + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(U)\partial_\alpha U_r + \mathrm{D}P(U)U_r \\ = \sum_{\alpha=1}^m \{\mathrm{D}G_\alpha(U)\partial^r\partial_\alpha U - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\} \\ + \{\mathrm{D}P(U)\partial^s\partial_\beta U - \partial^s[\mathrm{D}P(U)\partial_\beta U]\}, \end{aligned} \tag{5.2.7}$$

where $\beta$ is any fixed index in $\{1, \dots, m\}$ with $r_\beta \geq 1$, and $s$ is the multi-index with $s_\gamma = r_\gamma$, for $\gamma \neq \beta$, and $s_\beta = r_\beta - 1$. We recall (5.1.13) and note its analog

$$\|\mathrm{D}P(U)\partial^s\partial_\beta U - \partial^s[\mathrm{D}P(U)\partial_\beta U]\|_{L^2} \leq c\|\nabla U\|_{L^\infty}\|\nabla U\|_\ell\,. \tag{5.2.8}$$

Here and below $c$ stands for a generic constant depending solely on the maximum on $\overline{\mathcal{B}}$ of $U$, all derivatives $|\mathrm{D}^k G(U)|$ up to order $k = \ell + 2$, and all derivaties $|\mathrm{D}^k P(U)|$ up to order $k = \ell + 1$.

When (5.2.4) holds, the matrix $\mathrm{D}^2\eta(\bar{U})\mathrm{D}P(\bar{U})$ is at least positive semi-definite. In particular, $P$ is strongly dissipative at $\bar{U}$ if

$$W^\top \mathrm{D}^2\eta(\bar{U})\mathrm{D}P(\bar{U})W \geq \mu > 0, \qquad W \in S^{m-1}. \tag{5.2.9}$$

In that case, multiplying (5.2.7), from the left, by $2U_r^\top \mathrm{D}^2\eta(U)$, summing over all multi-indices $r$ with $1 \leq |r| \leq \ell + 1$, and integrating the resulting equation over $\mathbb{R}^m \times (0, t)$, we arrive at the following analog of (5.1.27):

$$\begin{aligned} &\|\nabla U(\cdot, t)\|_\ell^2 + 2\mu \int_0^t \|\nabla U(\cdot, \tau)\|_\ell^2 d\tau \\ &\qquad \leq c\|\nabla U_0(\cdot)\|_\ell^2 + c\int_0^t \{\|\nabla U(\cdot, \tau)\|_{L^\infty} + \|U(\cdot, \tau) - \bar{U}\|_{L^\infty}\}\|\nabla U(\cdot, \tau)\|_\ell^2 d\tau. \end{aligned} \tag{5.2.10}$$

So long as $\|\nabla U(\cdot, \tau)\|$ stays small, the integral on the left-hand side of (5.2.10) dominates the integral on the right-hand side and induces $\|\nabla U\|_\ell^2 \leq c\|\nabla U_0\|_\ell^2$. Since $\|\nabla U\|_{L^\infty} \leq \kappa\|\nabla U\|_\ell$, we conclude that if $\|\nabla U_0\|_\ell$ is sufficiently small, then $\|\nabla U\|_\ell$, and thereby $\|\nabla U\|_{L^\infty}$, stay small throughout the life span of the solution and thus the life span cannot be finite.

Unfortunately, assumption (5.2.9) is too stringent, as it generally rules out the type of source term associated with the dissipative mechanisms encountered in continuum physics. A typical example is the system that governs isentropic gas flow through a porous medium, namely (3.3.21) with body force $-v$:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v^\top) = 0 \\ \partial_t(\rho v) + \mathrm{div}(\rho v v^\top) + \mathrm{grad}\, p(\rho) + \rho v = 0. \end{cases} \tag{5.2.11}$$

This difficulty is also encountered in systems with source terms induced by relaxation effects, for instance (3.3.13). Typically, in the applications, $\mathrm{D}^2\eta(\bar{U})\mathrm{D}P(\bar{U})$ is merely positive semidefinite. In this situation, the damping fails to be effective, unless the source term satisfies an additional condition which ensures that waves of any characteristic family, propagating in any direction $\nu \in S^{m-1}$, are properly damped. The appropriate assumption, similar to the Kawashima condition (4.6.4), reads

$$\mathrm{D}P(\bar{U})R_i(\nu; \bar{U}) \neq 0, \qquad \nu \in S^{m-1}, \;\; i = 1, \dots, n, \tag{5.2.12}$$

where $R_i(\nu; U)$ is an eigenvector of the matrix $\Lambda(\nu; U)$, in (4.1.2), associated with the eigenvalue $\lambda_i(\nu; U)$. To see the implications of (5.2.12), linearize (5.2.1) about $\bar{U}$:

(5.2.13) $$\partial_t V(x,t) + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(\bar{U})\partial_\alpha V(x,t) + \mathrm{D}P(\bar{U})V(x,t) = F(x,t).$$

Notice that when (5.2.12) is violated for some $i$ and $\nu$, then (5.2.13), with $F \equiv 0$, admits traveling wave solutions (4.6.5), which are not attenuated by the damping. On the other hand, it can be shown that when (5.2.12) holds, then, for any $\nu \in S^{m-1}$, there exists a skew symmetric $n \times n$ matrix $K(\nu)$ such that the matrix

(5.2.14) $$K(\nu)\Lambda(\nu;\bar{U}) + \mathrm{D}^2\eta(\bar{U})\mathrm{D}P(\bar{U})$$

is positive definite. This in turn implies that solutions of (5.2.13) satisfy an estimate

(5.2.15)

$$\int_0^t \int_{I\!R^m} |V|^2(x,\tau)dx\,d\tau \le \kappa \int_0^t \int_{I\!R^m} V^\top(x,\tau)\mathrm{D}^2\eta(\bar{U})\mathrm{D}P(\bar{U})V(x,\tau)dx\,d\tau$$

$$+\kappa \int_{I\!R^m} [|V|^2(x,t) + |V|^2(x,0)]dx + \kappa \int_0^t \int_{I\!R^m} |F|^2(x,\tau)dx\,d\tau.$$

As before, we multiply (5.2.7), from the left, by $2U_r^\top \mathrm{D}^2\eta(U)$, we sum over all multi-indices $r$ with $1 \le |r| \le \ell+1$, and integrate over $I\!R^m \times (0,t)$. Upon combining the resulting equation with the estimate (5.2.15), one reestablishes (5.2.10), for some $\mu > 0$, thus proving the following

**5.2.1 Theorem.** *Consider the hyperbolic system of balance laws* (5.2.1)*, with $G$ in $C^{\ell+2}$ and $P$ in $C^{\ell+1}$, for some $\ell > m/2$. Assume $P(\bar{U}) = 0$ and $\mathrm{D}P(\bar{U})$ satisfies* (5.2.12)*. Furthermore, let $\eta$ be a $C^3$ entropy for* (5.2.1) *such that $\mathrm{D}^2\eta(\bar{U})$ is positive definite and* (5.2.4) *holds on some neighborhood $\mathcal{B}$ of $\bar{U}$. When $U_0 - \bar{U} \in L^2(I\!R^m)$, $\nabla U_0 \in H^\ell$ and $\|\nabla U_0\|_\ell$ is sufficiently small, then the Cauchy problem* (5.2.1), (5.2.2) *admits a unique classical solution $U$ on the upper half-space, such that*

(5.2.16) $$\nabla U(\cdot,t) \in C^0([0,\infty); H^\ell) \cap L^2([0,\infty); H^\ell).$$

When $\bar{U}$ is a strict minimum of $\mathrm{D}\eta(U)P(U)$, it is expected that dissipation will drive the solution obtained in the above theorem to this isolated equilibrium point, as $t \to \infty$. Of far greater interest is the long time behavior of solutions of (5.2.1), (5.2.2) when the source vanishes on a manifold in state space. This is typically the case with systems governing relaxation phenomena.

Upon rescaling the coordinates by $(x,t) \mapsto (\mu x, \mu t)$, where $\mu > 0$ is the so-called *relaxation parameter*, we recast (5.2.1) in the form

(5.2.17) $$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) + \frac{1}{\mu} P(U(x,t)) = 0, \qquad x \in I\!R^m,\ t > 0.$$

Thus, the asymptotic behavior of solutions of (5.2.1), as $t \uparrow \infty$, will be derived from the asymptotic behavior of solutions of (5.2.17), as $\mu \downarrow 0$.

The following assumptions on $P$ embody the structure typically encountered in systems governing relaxation phenomena:

(a) For some $k < n$, there is a constant $k \times n$ matrix $K$ such that $KP(U) = 0$, for all $U \in \mathcal{O}$.
(b) There is a $k$-dimensional *local equilibrium manifold,* embedded in $\mathcal{O}$, which is defined by a smooth function $U = E(V)$, $V \in \mathcal{V} \subset I\!R^k$, such that $P(E(V)) = 0$ and $KE(V) = V$, for all $V \in \mathcal{V}$.

As a representative example, consider the system

$$\begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \partial_t v(x,t) + \partial_x p(u(x,t)) + \frac{1}{\mu}[v(x,t) - f(u(x,t))] = 0 \end{cases} \tag{5.2.18}$$

of two balance laws in one spatial variable, where $p'(u) = a^2(u)$, $a(u) > 0$. Here $K = (1, 0)$, $V = u$, and $E(u) = (u, f(u))^\top$.

The expectation is that, as $\mu \downarrow 0$, the stiff source will induce $U$ to relax on its local equilibrium manifold $U = E(V)$, with $V$ satisfying the *relaxed system* of conservation laws

$$\partial_t V(x,t) + \operatorname{div} \hat{G}(V(x,t)) = 0, \qquad x \in I\!R^m, \quad t > 0, \tag{5.2.19}$$

where

$$\hat{G}(V) = KG(E(V)), \qquad V \in \mathcal{V}. \tag{5.2.20}$$

For the system (5.2.18), (5.2.19) reduces to the scalar conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0. \tag{5.2.21}$$

We now explore the implications of the dissipativeness of the source $P$ as encoded in the existence of an entropy-entropy flux pair $(\eta, Q)$ for (5.2.17) which satisfies (5.2.4), for all $U \in \mathcal{O}$. In particular, $\mathrm{D}\eta(U)P(U)$ is minimized on the local equilibrium manifold, and so

$$\mathrm{D}\eta(E(V))\mathrm{D}P(E(V)) = 0, \qquad V \in \mathcal{V}. \tag{5.2.22}$$

We also have $K\mathrm{D}P(U) = 0$, $U \in \mathcal{O}$. Hence, assuming that the rank of $\mathrm{D}P(E(V))$ is $n - k$, for any $V \in \mathcal{V}$, we conclude

$$\mathrm{D}\eta(E(V)) = M(V)K, \qquad V \in \mathcal{O}, \tag{5.2.23}$$

for some $k$-row vector-valued function $M$ on $\mathcal{O}$.

We now set

$$\hat{\eta}(V) = \eta(E(V)), \qquad \hat{Q}(V) = Q(E(V)), \tag{5.2.24}$$

and show that $(\hat{\eta}, \hat{Q})$ is an entropy-entropy flux pair for the relaxed system (5.2.19). Indeed, recalling (4.1.4), (5.2.20), (5.2.23) and noting that $KE(V) = V$ implies $K\mathrm{D}_V E = I$, we deduce, by the chain rule,

$$\text{(5.2.25)} \quad \mathrm{D}_V \hat{\eta}\, \mathrm{D}_V \hat{G}_\alpha = \mathrm{D}_U \eta\, \mathrm{D}_V E\, K\, \mathrm{D}_U G_\alpha\, \mathrm{D}_V E = MK\, \mathrm{D}_V E\, K\, \mathrm{D}_U G_\alpha\, \mathrm{D}_V E$$
$$= M\, K\, \mathrm{D}_U G_\alpha\, \mathrm{D}_V E = \mathrm{D}_U \eta\, \mathrm{D}_U G_\alpha\, \mathrm{D}_V E = \mathrm{D}_U Q_\alpha\, \mathrm{D}_V E = \mathrm{D}_V \hat{Q}_\alpha\, .$$

It has also been shown (references in Section 5.6) that if $\mathrm{D}(U)P(U)$ is strictly positive away from the local equilibrium manifold and $\mathrm{D}^2\eta(U)$ is positive definite on $\mathcal{O}$, then $\mathrm{D}^2\hat{\eta}(V)$ is positive definite on $\mathcal{V}$, in which case the relaxed system (5.2.19) is hyperbolic. Moreover, all characteristic speeds of (5.2.19), in any direction $\nu \in S^{m-1}$ and state $V \in \mathcal{V}$, are confined between the minimum and the maximum characteristic speed of (5.2.17), in the direction $\nu$ and state $U = E(V)$. This last property expresses the *subcharacteristic condition* which has important implications for stability.

As noted above, the objective is to demonstrate that, as $\mu \downarrow 0$, the solution $U_\mu$ of (5.2.17), (5.2.2) converges to $E(V)$, where $V$ is the solution of the relaxed system (5.2.19) with initial value $V_0 = KU_0$. When the initial data $U_0$ do not lie on the local equilibrium manifold, i.e. $U_0 \neq E(V_0)$, then as $\mu \downarrow 0$, $U_\mu$ will develop a boundary layer across $t = 0$, connecting $U_0$ to $E(V_0)$.

The asymptotic behavior of $U_\mu$, as $\mu \downarrow 0$, has been analyzed within the context of classical solutions, for quite general systems. The reader should consult the relevant references cited in Section 5.6. Additional information can be found in Sections 6.6 and 16.5.

An intimate relation exists between dissipation induced by relaxation and dissipation induced by viscosity. The reader may catch a first glimpse through the following formal calculation for the simple system (5.2.18).

We set

$$\text{(5.2.26)} \qquad v = f(u) + \mu w$$

and substitute into (5.2.18). Dropping, formally, all terms of order $\mu$ and then eliminating $\partial_t u$ between the two equations of the system yields

$$\text{(5.2.27)} \qquad w = [f'(u)^2 - a(u)^2]\partial_x u.$$

Upon combining $(5.2.18)_1$ with (5.2.26) and (5.2.27), we deduce that, formally, to leading order, $u$ satisfies the equation

$$\text{(5.2.28)} \qquad \partial_t u + \partial_x f(u) = \mu \partial_x\{[a^2(u) - f'(u)^2]\partial_x u\}.$$

For well-posedness we need

$$\text{(5.2.29)} \qquad -a(u) < f'(u) < a(u).$$

Since $\pm a(u)$ are the characteristic speeds of (5.2.18) and $f'(u)$ is the characteristic speed of (5.2.21), (5.2.29) expresses the subcharacteristic condition encountered above.

An analogous calculation, with analogous conclusions, applies to the general system (5.2.17) as well. In fact the Kawashima-type conditions (4.6.4) and (5.2.12) are intimately related. The reader can find details in the literature cited in Section 5.6.

## 5.3 Convex Entropy and the Stability of Classical Solutions

The aim here is to show that the presence of a convex entropy guarantees that classical solutions of the initial-value problem depend continuously on the initial data, even within the broader class of admissible bounded weak solutions.

**5.3.1 Theorem.** *Assume the system of conservation laws* (5.1.1) *is endowed with an entropy-entropy flux pair* $(\eta, Q)$, *where* $\mathrm{D}^2\eta(U)$ *is positive definite on* $\mathcal{O}$. *Suppose* $\bar{U}$ *is a classical solution of* (5.1.1) *on* $[0, T)$, *taking values in a convex compact subset* $\mathcal{D}$ *of* $\mathcal{O}$, *with initial data* $\bar{U}_0$. *Let* $U$ *be any admissible weak solution of* (5.1.1) *on* $[0, T)$, *taking values in* $\mathcal{D}$, *with initial data* $U_0$. *Then*

$$\int_{|x|<r} |U(x,t) - \bar{U}(x,t)|^2 dx \le ae^{bt} \int_{|x|<r+st} |U_0(x) - \bar{U}_0(x)|^2 dx \tag{5.3.1}$$

*holds for any* $r > 0$ *and* $t \in [0, T)$, *with positive constants* $s, a$, *depending solely on* $\mathcal{D}$, *and* $b$ *that also depends on the Lipschitz constant of* $\bar{U}$. *In particular,* $\bar{U}$ *is the unique admissible weak solution of* (5.1.1) *with initial data* $\bar{U}_0$ *and values in* $\mathcal{D}$.

**Proof.** On $\mathcal{D} \times \mathcal{D}$ we define the functions

$$h(U, \bar{U}) = \eta(U) - \eta(\bar{U}) - \mathrm{D}\eta(\bar{U})[U - \bar{U}], \tag{5.3.2}$$

$$Y_\alpha(U, \bar{U}) = Q_\alpha(U) - Q_\alpha(\bar{U}) - \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})], \tag{5.3.3}$$

$$Z_\alpha(U, \bar{U}) = \mathrm{D}^2\eta(\bar{U})\big\{G_\alpha(U) - G_\alpha(\bar{U}) - \mathrm{D}G_\alpha(\bar{U})[U - \bar{U}]\big\}, \tag{5.3.4}$$

all of quadratic order in $U - \bar{U}$ (recall (4.1.4)). Consequently, since $\mathrm{D}^2\eta(U)$ is positive definite, uniformly on $\mathcal{D}$, there is a positive constant $s$ such that

$$|Y(U, \bar{U})| \le sh(U, \bar{U}). \tag{5.3.5}$$

Let us fix any nonnegative, Lipschitz continuous test function $\psi$, with compact support, on $I\!R^m \times [0, T)$ and evaluate $h, Y$ and $Z$ along the two solutions $U(x,t), \bar{U}(x,t)$. Recalling that $U$, as an admissible weak solution, must satisfy inequality (4.5.3), while $\bar{U}$, being a classical solution, will identically satisfy (4.5.3) as an equality, we deduce

(5.3.6)

$$\int_0^T \int_{I\!R^m} [\partial_t \psi \, h(U, \bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Y_\alpha(U, \bar{U})] dx dt + \int_{I\!R^m} \psi(x, 0) \, h(U_0(x), \bar{U}_0(x)) \, dx$$

$$\geq - \int_0^T \int_{I\!R^m} \{\partial_t \psi \, \mathrm{D}\eta(\bar{U})[U - \bar{U}] + \sum_{\alpha=1}^m \partial_\alpha \psi \, \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})]\} dx dt$$

$$- \int_{I\!R^m} \psi(x, 0) \, \mathrm{D}\eta(\bar{U}_0(x))[U_0(x) - \bar{U}_0(x)] dx.$$

Next we write (4.3.2) for both solutions $U$ and $\bar{U}$, using the Lipschitz continuous vector field $\psi \mathrm{D}\eta(\bar{U})$ as test function $\Phi$, to get

$$\text{(5.3.7)} \quad \int_0^T \int_{I\!R^m} \{\partial_t [\psi \mathrm{D}\eta(\bar{U})][U - \bar{U}] + \sum_{\alpha=1}^m \partial_\alpha [\psi \mathrm{D}\eta(\bar{U})][G_\alpha(U) - G_\alpha(\bar{U})]\} dx dt$$

$$+ \int_{I\!R^m} \psi(x, 0) \mathrm{D}\eta(\bar{U}_0(x))[U_0(x) - \bar{U}_0(x)] dx = 0.$$

Since $\bar{U}$ is a classical solution of (5.1.1), and by virtue of (5.1.3),

$$\text{(5.3.8)} \qquad \partial_t \mathrm{D}\eta(\bar{U}) = \partial_t \bar{U}^\top \mathrm{D}^2 \eta(\bar{U}) = - \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top \mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}^2 \eta(\bar{U})$$

$$= - \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top \mathrm{D}^2 \eta(\bar{U}) \mathrm{D}G_\alpha(\bar{U})$$

so that, recalling (5.3.4),

(5.3.9)

$$\partial_t \mathrm{D}\eta(\bar{U})[U - \bar{U}] + \sum_{\alpha=1}^m \partial_\alpha \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})] = \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U, \bar{U}).$$

Combining (5.3.6), (5.3.7) and (5.3.9) yields

(5.3.10)

$$\int_0^T \int_{I\!R^m} [\partial_t \psi \, h(U, \bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Y_\alpha(U, \bar{U})] dx dt + \int_{I\!R^m} \psi(x, 0) \, h(U_0(x), \bar{U}_0(x)) \, dx$$

$$\geq \int_0^T \int_{I\!R^m} \psi \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U, \bar{U}) dx dt.$$

We now fix $r > 0$ and any point $t \in (0, T)$ of $L^\infty$ weak* continuity of $\eta(U(\cdot, \tau))$; see Section 4.5. For $\varepsilon$ positive small, write (5.3.10) for the test function $\psi(x, \tau) = \chi(x, \tau)\omega(\tau)$, with

$$\omega(\tau) = \begin{cases} 1 & 0 \le \tau < t \\ \varepsilon^{-1}(t - \tau) + 1 & t \le \tau < t + \varepsilon \\ 0 & t + \varepsilon \le \tau < \infty \end{cases} \tag{5.3.11}$$

$$\chi(x, \tau) = \begin{cases} 1 & |x| - r - s(t - \tau) < 0 \\ \varepsilon^{-1}[r + s(t - \tau) - |x|] + 1 & 0 \le |x| - r - s(t - \tau) < \varepsilon \\ 0 & |x| - r - s(t - \tau) \ge \varepsilon \end{cases} \tag{5.3.12}$$

where $s$ is the constant appearing in (5.3.5). The calculation gives

(5.3.13)

$$\begin{aligned} \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<r} h(U(x, \tau), \bar{U}(x, \tau))dxd\tau \le \int_{|x|<r+st} h(U_0(x), \bar{U}_0(x))dx \\ - \frac{1}{\varepsilon}\int_0^t \int_{r+s(t-\tau)<|x|<r+s(t-\tau)+\varepsilon} \left[ sh(U, \bar{U}) + \frac{Y(U, \bar{U})x}{|x|} \right] dxd\tau \\ - \int_0^t \int_{|x|<r+s(t-\tau)} \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U, \bar{U})dxd\tau + O(\varepsilon). \end{aligned}$$

We let $\varepsilon \downarrow 0$. The second integral on the right-hand side of (5.3.13) is nonnegative by account of (5.3.5). Using that $h(U(\cdot, \tau), \bar{U}(\cdot, \tau))$ is weak* continuous in $L^\infty$ at $\tau = t$, we deduce

$$\begin{aligned} \int_{|x|<r} h(U(x, t), \bar{U}(x, t))dx \le \int_{|x|<r+st} h(U_0(x), \bar{U}_0(x))dx \\ - \int_0^t \int_{|x|<r+s(t-\tau)} \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U, \bar{U})dxd\tau. \end{aligned} \tag{5.3.14}$$

As noted above, $h(U, \bar{U})$ and the $Z_\alpha(U, \bar{U})$ are of quadratic order in $U - \bar{U}$ and, in addition, $h(U, \bar{U})$ is positive definite, due to the convexity of $\eta$. Therefore, (5.3.14) in conjunction with Gronwall's inequality imply (5.3.1). Notice that $a$ and $s$ depend solely on $\mathcal{D}$ while $b$ depends also on the Lipschitz constant of $\bar{U}$. This completes the proof.

It is remarkable that a single entropy inequality, with convex entropy, manages to weed out all but one solution of the initial-value problem, so long as a classical solution exists. As we shall see, however, when no classical solution exists, just one entropy inequality is no longer generally sufficient to single out any particular weak solution. The issue of uniqueness of weak solutions is knotty and will be a major issue for discussion in subsequent chapters.

The functions $h(U, \bar{U})$ and $Y(U, \bar{U})$, defined by (5.3.2) and (5.3.3), are commonly called the *relative entropy* and associated *relative entropy flux*, with respect to the state $\bar{U}$.

Notice that in the proof of Theorem 5.3.1 one only needs that $h(U, \bar{U})$ be positive definite for all $\bar{U}$ in the range of the classical solution. This may well hold, even for $\eta$ that fail to be convex, when the classical solution is special, e.g. it is a constant state $\bar{U}$ which is a strong minimum of $\eta$.

## 5.4 Involutions

The previous three sections have illustrated the beneficent role of convex entropies. Nevertheless, the entropy associated with systems of balance laws in continuum physics is not always convex. An illustrative case is example 3.3.3 of Section 3.3, namely isentropic adiabatic thermoelasticity, with system of balance laws (3.3.10) and entropy function $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$, which would be convex if $\varepsilon$ were convex. Even though $\varepsilon$ may indeed be convex on certain regions of state space, global convexity is incompatible with experience and in particular would violate the principle of material frame indifference, which requires $\varepsilon(OF) = \varepsilon(F)$ for all proper orthogonal matrices $O$ (cf. (2.5.5)).

It will be shown here that the failure of $\varepsilon$ to be convex in certain directions is compensated by the property that solutions of the system (3.3.10) satisfy identically the additional conservation law

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0, \qquad i = 1, \cdots, m; \;\; \alpha, \beta = 1, \cdots, m, \tag{5.4.1}$$

whenever the initial data do so. By virtue of (2.1.2), these are the only solutions that are physically relevant.

Systems exhibiting such behavior arise quite commonly in continuum physics. For example, solutions of Maxwell's equations (3.3.40), with current $J \equiv 0$, satisfy identically the additional conservation laws

$$\mathrm{div} B = 0, \qquad \mathrm{div} D = 0, \tag{5.4.2}$$

as long as the initial data do so, and again these are the solutions with physical relevance. It is thus warranted to investigate systems of balance laws with this special structure, in a general framework:

**5.4.1 Definition.** The first order system

$$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U = 0 \tag{5.4.3}$$

of differential equations, with $M_\alpha$ $k \times n$ matrices, $\alpha = 1, \dots, m$, is called an *involution* of the system (5.1.1) of conservation laws if any (generally weak) solution of the Cauchy problem (5.1.1),(5.1.2) satisfies (5.4.3) identically, whenever the initial data do so.[2]

Thus (5.4.1) is an involution of (3.3.10) and (5.4.2) is an involution of (3.3.40).

A sufficient condition for (5.4.3) to be an involution of (5.1.1) is

$$M_\alpha G_\beta(U) + M_\beta G_\alpha(U) = 0, \qquad \alpha, \beta = 1, \cdots, m, \tag{5.4.4}$$

for any $U \in \mathcal{O}$. We shall focus our investigation to this special case which covers, in particular, the prototypical examples (5.4.1) and (5.4.2). The aim is to demonstrate that, in the presence of involutions, one may establish existence and stability of classical solutions under the weaker hypothesis that the entropy is convex just in the direction of a certain cone in state space, which is constructed by the following procedure:

With any $\nu \in S^{m-1}$, we associate the $k \times n$ matrix

$$N(\nu) = \sum_{\alpha=1}^{m} \nu_\alpha M_\alpha \ . \tag{5.4.5}$$

Recalling the notation (4.1.2), (5.4.4) implies

$$N(\nu)\Lambda(\nu; U) = 0. \tag{5.4.6}$$

We impose the condition, valid in the prototypical examples, that for any $\nu \in S^{m-1}$ the rank of $N(\nu)$ equals the dimension of the kernel of $\Lambda(\nu; U)$, i.e., the row vectors of $N(\nu)$ span the left eigenspace of the matrix $\Lambda(\nu; U)$ associated with the eigenvalue $\lambda(\nu; U) = 0$.

**5.4.2 Definition.** The *involution cone* in $\mathbb{R}^n$ of the involution (5.4.3) is

$$\mathcal{C} = \bigcup_{\nu \in S^{m-1}} \ker N(\nu), \tag{5.4.7}$$

with $N(\nu)$ given by (5.4.5).

[2] This should be contrasted with the vanishing vorticity condition curl $v = 0$, characterizing irrotational or potential flow, which is sustained by classical solutions of the Euler equations (3.3.21), when the body force derives from a potential, but generally breaks down after discontinuities develop.

**5.4.3 Lemma.** *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.3) *with involution cone* $\mathcal{C}$. *Suppose* $P$ *is a symmetric* $n \times n$ *matrix-valued* $L^\infty$ *function on* $\mathbb{R}^m$ *which is uniformly positive definite in the direction of* $\mathcal{C}$, *i.e.,*

$$Z^\top P(x) Z \geq \mu |Z|^2, \qquad Z \in \mathcal{C}, \;\; x \in \mathbb{R}^m, \tag{5.4.8}$$

*for some* $\mu > 0$, *and its local oscillation is less than* $\mu$, *i.e.,*

$$\limsup_{\varepsilon \downarrow 0} \sup_{|y-x|<\varepsilon} |P(y) - P(x)| < \mu - 2\delta, \tag{5.4.9}$$

*for some* $\delta > 0$. *If* $W$ *is any* $L^2$ *function from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$ *which is compactly supported in the hypercube* $\mathcal{K} = \{x \in \mathbb{R}^m : |x_\alpha| < p, \; \alpha = 1, \dots, m\}$ *and satisfies the involution*

$$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha W = 0 \tag{5.4.10}$$

*in the sense of distributions, then*

$$\int_{\mathbb{R}^m} W(x)^\top P(x) W(x) dx \geq \delta \|W\|^2_{L^2(\mathcal{K})} - b\|W\|^2_{W^{-1,2}(\mathcal{K})}, \tag{5.4.11}$$

*where* $b$ *does not depend on* $W$.

**Proof.** Expand $W$ in Fourier series over $\mathcal{K}$:

$$W(x) = \sum_{\xi \in \mathbb{Z}^m} \exp\left\{ \frac{i\pi}{p} (\xi \cdot x) \right\} X(\xi). \tag{5.4.12}$$

Fix $\hat{U} \in \mathcal{O}$ and consider the differential operator

$$\mathcal{L} = \sum_{\beta=1}^{m} \mathrm{D}G_\beta(\hat{U}) \partial_\beta \,. \tag{5.4.13}$$

Construct a $2p$-periodic $W^{1,2}_{loc}$ function $V$, mapping $\mathbb{R}^m$ to $\mathbb{R}^n$, such that

$$\mathcal{L} V = W - X(0). \tag{5.4.14}$$

Such a $V$ is derived via its Fourier series expansion

$$V(x) = \sum_{\xi \in \mathbb{Z}^m \setminus \{0\}} \exp\left\{ \frac{i\pi}{p} (\xi \cdot x) \right\} Y(\xi), \tag{5.4.15}$$

with coefficients determined by solving the linear systems

$$\Lambda\left(|\xi|^{-1}\xi;\hat{U}\right)Y(\xi)=-\frac{ip}{\pi}\frac{1}{|\xi|}X(\xi). \tag{5.4.16}$$

Solutions to (5.4.16) exist because by (5.4.10)

$$|\xi|N(|\xi|^{-1}\xi)X(\xi)=\left[\sum_{\alpha=1}^{m}\xi_\alpha M_\alpha\right]X(\xi)=0, \tag{5.4.17}$$

so that $X(\xi)$ lies in the kernel of $N(|\xi|^{-1}\xi)$; and the rank of $N(|\xi|^{-1}\xi)$ is assumed equal to the dimension of the kernel of $\Lambda(|\xi|^{-1}\xi;\hat{U})$. Furthermore, since

$$\|W\|^2_{W^{-1,2}(\mathcal{K})}=(2p)^m\sum_{\xi\in\mathbb{Z}^m}(1+|\xi|^2)^{-1}|X(\xi)|^2\,, \tag{5.4.18}$$

$$\|V\|^2_{L^2(\mathcal{K})}=(2p)^m\sum_{\xi\in\mathbb{Z}^m}|Y(\xi)|^2\,, \tag{5.4.19}$$

we conclude that

$$\|V\|_{L^2(\mathcal{K})}\le a\,p\|W\|_{W^{-1,2}(\mathcal{K})}\,, \tag{5.4.20}$$

where $a$ is independent of $p$ and $W$.

Next we cover $\overline{\mathcal{K}}$ by the union of a finite collection $\mathcal{K}_1,\cdots,\mathcal{K}_J$ of open hypercubes, centered at points $y^1,\cdots,y^J$, such that

$$\sup_{x\in\mathcal{K}_I}|P(x)-P(y^I)|\le\mu-2\delta,\qquad I=1,\cdots,J. \tag{5.4.21}$$

With the above covering we associate a partition of unity induced by $C^\infty$ functions $\theta_1,\dots,\theta_J$ on $\mathbb{R}^m$ such that $\operatorname{spt}\theta_I\subset\mathcal{K}_I\cap\mathcal{K}$, $I=1,\dots,J$, and

$$\sum_{I=1}^{J}\theta_I^2(x)=1,\qquad x\in\operatorname{spt}W. \tag{5.4.22}$$

Then

$$\int_{\mathbb{R}^m}W(x)^\top P(x)W(x)dx=\sum_{I=1}^{J}\int_{\mathcal{K}_I}\theta_I^2(x)W(x)^\top P(x)W(x)dx \tag{5.4.23}$$

$$=\sum_{I=1}^{J}\int_{\mathcal{K}_I}\theta_I^2(x)W(x)^\top P(y^I)W(x)dx+\sum_{I=1}^{J}\int_{\mathcal{K}_I}\theta_I^2(x)W(x)^\top[P(x)-P(y^I)]W(x)dx.$$

By virtue of (5.4.21) and (5.4.22),

$$\sum_{I=1}^{J}\int_{\mathcal{K}_I}\theta_I^2(x)W(x)^\top[P(x)-P(y^I)]W(x)dx\ge-(\mu-2\delta)\|W\|^2_{L^2}\,. \tag{5.4.24}$$

For each $I = 1, \dots, J$, we split $\theta_I W$ into

$$\theta_I W = S_I + T_I \,, \tag{5.4.25}$$

where

$$S_I = \mathcal{L}(\theta_I V), \tag{5.4.26}$$

$$T_I(x) = \theta_I(x) X(0) - \left[ \sum_{\beta=1}^{m} \partial_\beta \theta_I(x) \mathrm{D}\, G_\beta(\hat{U}) \right] V(x). \tag{5.4.27}$$

Clearly, $S_I$ is square integrable, has compact support in $\mathcal{K}_I$ and zero mean over $\mathcal{K}_I$ . Moreover, by account of (5.4.13) and (5.4.4),

$$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha S_I = 0. \tag{5.4.28}$$

Consequently, $S_I$ may be expanded in Fourier series over $\mathcal{K}_I$ ,

$$S_I(x) = \sum_{\xi \in \mathbb{Z}^m \setminus \{0\}} \exp\left\{ \frac{i\pi}{p^I} [\xi \cdot (x - y^I)] \right\} Z(\xi), \tag{5.4.29}$$

with

$$|\xi| N(|\xi|^{-1}\xi) Z(\xi) = \left[ \sum_{\alpha=1}^{m} \xi_\alpha M_\alpha \right] Z(\xi) = 0. \tag{5.4.30}$$

Thus $X(\xi)$ lies in the complexification of the involution cone $\mathcal{C}$ and so, by Parseval's relation and (5.4.8),

$$\begin{aligned} \int_{\mathcal{K}_I} S_I(x)^\top P(y^I) S_I(x) dx &= (2p^I)^m \sum_{\xi \in \mathbb{Z}^m} Z(\xi)^* P(y^I) Z(\xi) \\ &\geq \mu (2p^I)^m \sum_{\xi \in \mathbb{Z}^m} |Z(\xi)|^2 = \mu \int_{\mathcal{K}_I} |S_I(x)|^2 dx. \end{aligned} \tag{5.4.31}$$

Moreover, from (5.4.27), (5.4.18) and (5.4.20) we infer

$$\int_{\mathcal{K}_I} |T_I(x)|^2 dx \leq c \|W\|^2_{W^{-1,2}(\mathcal{K})} \,. \tag{5.4.32}$$

We now return to (5.4.23). From (5.4.25), (5.4.31) and (5.4.32) it follows that

$$\int_{\mathcal{K}_I} \theta_I^2(x) W(x)^\top P(y^I) W(x) dx \tag{5.4.33}$$

$$\geq (1 - \frac{\delta}{2\mu}) \int_{\mathcal{K}_I} S_I(x)^\top P(y^I) S_I(x) dx - \frac{2\mu}{\delta} \int_{\mathcal{K}_I} T_I(x)^\top P(y^I) T_I(x) dx$$

$$\geq (\mu - \frac{\delta}{2}) \int_{\mathcal{K}_I} |S_I(x)|^2 dx - c\|W\|^2_{W^{-1,2}} .$$

Again by (5.4.25) and (5.4.32),

$$\int_{\mathcal{K}_I} |S_I(x)|^2 dx \geq (1 - \frac{\delta}{2\mu}) \int_{\mathcal{K}_I} \theta_I^2(x) |W(x)|^2 dx - \frac{2\mu}{\delta} \int_{\mathcal{K}_I} |T_I(x)|^2 dx \tag{5.4.34}$$

$$\geq (1 - \frac{\delta}{2\mu}) \int_{\mathcal{K}_I} \theta_I^2(x) |W(x)|^2 dx - c\|W\|^2_{W^{-1,2}} .$$

Combining (5.4.23), (5.4.24), (5.4.33), (5.5.34) and (5.4.22), we arrive at (5.4.11). This completes the proof.

In the presence of involutions, local existence of classical solutions to the Cauchy problem is obtained even when the entropy is merely convex in the direction of the involution cone:

**5.4.4 Theorem.** *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.3) *and is equipped with a $C^3$ entropy $\eta$, with $\mathrm{D}^2\eta(U)$ positive definite in the direction of the involution cone $\mathcal{C}$. Suppose the initial data $U_0$ are continuously differentiable on $\mathbb{R}^m$, take values in a compact subset of $\mathcal{O}$, are constant, say $\tilde{U}$, outside a bounded subset of $\mathbb{R}^m$, satisfy the involution on $\mathbb{R}^m$, and $\nabla U_0 \in H^\ell$ for some $\ell > m/2$. Moreover, let $G \in C^{\ell+2}$. Then there exists $T_\infty$, $0 < T_\infty \leq \infty$, and a unique continuously differentiable function $U$ on $\mathbb{R}^m \times [0, T_\infty)$, taking values in $\mathcal{O}$, which is a classical solution of the Cauchy problem* (5.1.1), (5.1.2) *on* $[0, T_\infty)$. *Furthermore,*

$$\nabla U(\cdot, t) \in C^0([0, T_\infty); H^\ell). \tag{5.4.35}$$

*The interval* $[0, T_\infty)$ *is maximal, in the sense that whenever $T_\infty < \infty$ then*

$$\int_0^{T_\infty} \|\nabla U(\cdot, t)\|_{L^\infty} dt = \infty \tag{5.4.36}$$

*and/or the range of $U(\cdot, t)$ escapes from every compact subset of $\mathcal{O}$ as $t \uparrow T_\infty$.*

**Sketch of Proof.** It is a slight variant of the proof of Theorem 5.1.1. In the place of (5.1.11) one should now employ some alternative linearization of (5.1.1), such as

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha [G_\alpha(V) + \mathrm{D}G_\alpha(V)(U - V)] = 0, \tag{5.4.37}$$

whose solutions satisfy the involution (5.4.3) when their initial values do so. Nevertheless, in order to avoid the tedium of recasting all the estimates established earlier to this new setting, we shall carry out the proof under the extraneous assumption that the matrices $M_\alpha \mathrm{D}G_\beta(V)$ are constant, which holds, in particular, for the system (3.3.10) of elastodynamics. In that case, the involution (5.4.3) will be satisfied even by solutions of (5.1.11), and we may thus continue using that linearization of (5.1.1).

We retrace the steps in the proof of Theorem 5.1.1. In the definition of the metric space $\mathcal{F}$ the stipulation should be added that its members are constant, $\tilde{U}$, outside some ball in $\mathbb{R}^m$.

The first snag we hit is that (5.1.16) no longer applies, as $\mathrm{D}^2\eta(V)$ is now positive only in the direction of the involution cone $\mathcal{C}$. In its place we use

$$\int_{\mathbb{R}^m} U_r^\top \mathrm{D}^2\eta(V) U_r dx \geq \delta \|U_r\|_{L^2}^2 - c\|U_r\|_{W^{-1,2}}^2 , \tag{5.4.38}$$

which follows from Lemma 5.4.3. To estimate $\|U_r\|_{W^{-1,2}}$ , we integrate (5.1.12) with respect to $t$. This yields

$$\|U_r(\cdot, t)\|_{W^{-1,2}} \leq c\|\nabla U_0(\cdot)\|_\ell + c\omega \int_0^t \|\nabla U(\cdot, \tau)\|_\ell d\tau. \tag{5.4.39}$$

By employing (5.4.38), (5.4.39) as a substitute for (5.1.16), we establish, in the place of (5.1.17), the new estimate

$$\|\nabla U(\cdot, t)\|_\ell^2 \leq c\|\nabla U_0(\cdot)\|_\ell^2 + c\omega(1 + \omega T) \int_0^t \|\nabla U(\cdot, \tau)\|_\ell^2 d\tau, \tag{5.4.40}$$

whence we deduce that when $\omega$ is sufficiently large and $T$ is sufficiently small, $\sup_{[0,T]} \|\nabla U(\cdot, t)\|_\ell < \omega$, as required for the proof.

A similar argument is used to compensate for the failure of (5.1.23). In its place, by using Lemma 5.4.3, we have

$$\int_{\mathbb{R}^m} (U - \bar{U})^\top \mathrm{D}^2\eta(V)(U - \bar{U}) dx \geq \delta \|U - \bar{U}\|_{L^2}^2 - c\|U - \bar{U}\|_{W^{-1,2}}^2 . \tag{5.4.41}$$

Integrating (5.1.20) with respect to $t$ yields the estimate

$$\|(U - \bar{U})(\cdot, t)\|_{W^{-1,2}} \leq c\omega \int_0^t \{\|(U - \bar{U})(\cdot, \tau)\|_{L^2} + \|(V - \bar{V})(\cdot, \tau)\|_{L^2}\} d\tau. \tag{5.4.42}$$

By virtue of (5.4.41) and (5.4.42), we obtain, in the place of (5.1.24),

(5.4.43)
$$\|(U-\bar{U})(\cdot,t)\|_{L^2}^2 \le c\omega(1+\omega T)\int_0^t \{\|(U-\bar{U})(\cdot,\tau)\|_{L^2}^2 + \|(V-\bar{V})(\cdot,\tau)\|_{L^2}^2\}d\tau.$$

From (5.1.10), (5.4.43) and Gronwall's inequality, we deduce

$$\rho(U,\bar{U}) \le [c\omega T(1+\omega T)]^{1/2}\exp[c\omega T(1+\omega T)]\rho(V,\bar{V}). \tag{5.4.44}$$

Thus, for $T$ small, the map that carries $V \in \mathcal{F}$ to the solution $U \in \mathcal{F}$ of (5.1.11), (5.1.2) is a contraction.

This completes the proof.

In the presence of involutions, classical solutions to the Cauchy problem are stable within the class of admissible weak solutions, even when the entropy is convex only in the direction of the involution cone:

**5.4.5 Theorem.** *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.3)*, and is equipped with an entropy-entropy flux pair* $(\eta, Q)$*, where* $\mathrm{D}^2\eta(U)$ *is positive definite in the direction of the involution cone* $\mathcal{C}$*. Suppose* $\bar{U}$ *is a classical solution of* (5.1.1) *on a bounded time interval* $[0,T)$*, which takes values in a convex, compact subset* $\mathcal{D}$ *of* $\mathcal{O}$*, and has initial data* $\bar{U}_0$ *satisfying the involution. Let* $U$ *be any weak solution of* (5.1.1)*, which also takes values in* $\mathcal{D}$*, coincides with* $\bar{U}$ *outside some ball of* $\mathbb{R}^m$*, has sufficiently small local oscillation*

$$\limsup_{\varepsilon\downarrow 0}\ \sup_{|y-x|<\varepsilon} |U(y,t)-U(x,t)| < \kappa, \qquad 0 \le t < T, \tag{5.4.45}$$

*meets the entropy admissibility criterion* (4.5.3)*, and has initial values* $U_0$ *satisfying the involution. Then*

$$\int_{\mathbb{R}^m} |U(x,t)-\bar{U}(x,t)|^2 dx \le a \int_{\mathbb{R}^m} |U_0(x)-\bar{U}_0(x)|^2 dx \tag{5.4.46}$$

*holds for* $t \in [0,T)$*, where* $a$ *depends on* $\mathcal{D}$*,* $T$ *and the Lipschitz constant of* $\bar{U}$*.*

**Proof.** Retracing the steps in the proof of Theorem 5.3.1, we derive (5.3.14), with $r=\infty$:

$$\int_{\mathbb{R}^m} h(U(x,t),\bar{U}(x,t))dx \le \int_{\mathbb{R}^m} h(U_0(x),\bar{U}_0(x))dx - \int_0^t\int_{\mathbb{R}^m}\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U})dxd\tau. \tag{5.4.47}$$

From (5.3.2),

$$h(U,\bar{U}) = (U-\bar{U})^\top P(U,\bar{U})(U-\bar{U}), \tag{5.4.48}$$

where

$$P(U,\bar{U}) = \int_0^1 \int_0^w \mathrm{D}^2\eta(\bar{U}+z(U-\bar{U}))dzdw. \tag{5.4.49}$$

In particular,

$$Z^\top P(U,\bar{U})Z \geq \mu |Z|^2, \qquad Z \in \mathcal{C}, \tag{5.4.50}$$

for some $\mu > 0$. Therefore, when $\kappa$ in (5.4.45) is so small that the local oscillation of $P(U(x,t),\bar{U}(x,t))$ is less than $\mu$, we may apply Lemma 5.4.3 with $W = U - \bar{U}$ to get

$$\int_{\mathbb{R}^m} h(U(x,t),\bar{U}(x,t))dx \geq \delta \|U(\cdot,t)-\bar{U}(\cdot,t)\|^2_{L^2} - c\|U(\cdot,t)-\bar{U}(\cdot,t)\|^2_{W^{-1,2}} \tag{5.4.51}$$

for some $\delta > 0$. We estimate the second term on the right-hand side of (5.4.51) as follows:

$$\|U(\cdot,t)-\bar{U}(\cdot,t)\|_{W^{-1,2}} \leq \|U_0(\cdot)-\bar{U}_0(\cdot)\|_{W^{-1,2}} + \int_0^t \|\partial_t\{U(\cdot,\tau)-\bar{U}(\cdot,\tau)\}\|_{W^{-1,2}} d\tau, \tag{5.4.52}$$

$$\|\partial_t\{U(\cdot,\tau)-\bar{U}(\cdot,\tau)\}\|_{W^{-1,2}} = \|\sum_{\alpha=1}^m \partial_\alpha\{G_\alpha(U(\cdot,\tau))-G_\alpha(\bar{U}(\cdot,\tau))\}\|_{W^{-1,2}} \tag{5.4.53}$$

$$\leq \sum_{\alpha=1}^m \|G_\alpha(U(\cdot,\tau))-G_\alpha(\bar{U}(\cdot,\tau))\|_{L^2} \leq c\|U(\cdot,\tau)-\bar{U}(\cdot,\tau)\|_{L^2}\,.$$

Combining (5.4.51), (5.4.52) and (5.4.53), we deduce from (5.4.47),

$$\|U(\cdot,t)-\bar{U}(\cdot,t)\|^2_{L^2} \leq c\|U_0(\cdot)-\bar{U}_0(\cdot)\|^2_{L^2} + c\int_0^t \|U(\cdot,\tau)-\bar{U}(\cdot,\tau)\|^2_{L^2}d\tau + c\{\int_0^t \|U(\cdot,\tau)-\bar{U}(\cdot,\tau)\|_{L^2}d\tau\}^2\,, \tag{5.4.54}$$

whence (5.4.46) follows. The proof is complete.

For the system of elastodynamics (3.3.10), with involution (5.4.1), the involution cone $\mathcal{C}$ consists of states $(F, v)$ in $I\!R^{12}$ with $F = \xi \otimes \nu$, for $\xi$, $\nu$ and $v$ in $I\!R^3$. Thus, Theorems 5.4.4 and 5.4.5 establish the existence and stability of classical solutions to the Cauchy problem, under the assumption that the internal energy $\varepsilon(F)$ is rank-one convex, i.e. it is convex in the direction of matrices $F = \xi \otimes \nu$ of rank one. As noted in Section 3.3.3, $\varepsilon(F)$ is rank-one convex if and only if the system (3.3.10) is hyperbolic.

The following discussion will shed some light on the relationship between convexity of the entropy in the direction of the involution cone and the important notion of quasiconvexity.

**5.4.6 Definition.** An entropy $\eta$ for the system of conservation laws (5.1.1), endowed with an involution (5.4.3), is called *quasiconvex* if for any $U \in L^\infty(I\!R^m; \mathcal{O})$, which is $2p$-periodic, satisfies (5.4.3) and has mean

$$\hat{U} = (2p)^{-m} \int_{\mathcal{K}} U(y) dy \tag{5.4.55}$$

over the standard hypercube $\mathcal{K}$ in $I\!R^m$ with edge length $2p$, it is

$$\eta(\hat{U}) \le (2p)^{-m} \int_{\mathcal{K}} \eta(U(y)) dy. \tag{5.4.56}$$

Roughly, quasiconvexity stipulates that the uniform state minimizes the total entropy, among all states that are compatible with the involution and have the same "mass". This is in the spirit of the fundamental law of classical thermostatics, which affirms that the physical entropy is maximized at the equilibrium state.

The relevance of quasiconvexity is demonstrated by the following propostion, whose proof may be found in the references cited in Section 5.6:

**5.4.7 Theorem.** *Assume the system of conservation laws* (5.1.1) *is endowed with an entropy $\eta$ and an involution* (5.4.3)*, such that the rank of $N(\nu)$ is constant, for any $\nu \in S^{m-1}$, and equal to the dimension of the kernel of $\Lambda(\nu; U)$. Then $\int_{|x|<r} \eta(U) dx$ is weak* lower semicontinuous on the space of $L^\infty$ vector fields $U$ that satisfy* (5.4.3)*, if and only if $\eta$ is quasiconvex. Furthermore, any quasiconvex $\eta$ is necessarily convex in the direction of the involution cone $\mathcal{C}$.*

Because of the above proposition, the notion of quasiconvexity plays a fundamental role in the calculus of variations. Unfortunately, Definition 5.4.6 does not provide any clue as to how to test whether a given entropy is quasiconvex. The conjecture that convexity in the direction of the involution cone is also sufficient for quasiconvexity is valid when the entropy is quadratic: $\eta = U^\top A U$. In general, however, quasiconvexity is a more stringent condition than mere convexity in the direction of the involution cone.

The above may be illustrated in the context of our prototypical example, namely the system (3.3.10) of isentropic elastodynamics, with involution (5.4.1) and entropy $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$. In that case, $\eta$ is quasiconvex when $\varepsilon(F)$ is quasiconvex in the sense of Morrey: For any constant deformation gradient $\hat{F}$ and any Lipschitz function $\chi$ from $\mathcal{K}$ to $I\!R^3$, with compact support in $\mathcal{K}$,

$$\varepsilon(\hat{F}) \leq (2p)^{-3} \int_{\mathcal{K}} \varepsilon(\hat{F} + \nabla\chi)dy. \tag{5.4.57}$$

In other words, a homogeneous deformation of $\mathcal{K}$ minimizes the total internal energy among all placements of $\mathcal{K}$ with the same boundary values. Any quasiconvex internal energy is rank-one convex (3.3.7). On the other hand, there exist rank-one convex functions $\varepsilon(F)$ that fail to be quasiconvex.

A placement of an elastic body is in (isentropic) equilibrium when its total internal energy $\int \varepsilon(F)dx$ is minimum over all placements with the same boundary conditions. Thus, quasiconvexity is necessary and sufficient for attaining equilibria by minimizing sequences of placements that are merely bounded in $W^{1,\infty}$.

It is easily seen, from (2.2.17), that the involution (5.4.1) is equivalent to (2.2.12). Accordingly, one may regard (2.2.13) as a nonlinear involution. Nonlinear involutions are often encountered in mathematical physics, most notably in the theory of relativity.

## 5.5 Contingent Entropies and Polyconvexity

As pointed out in the previous section, the natural entropy $\varepsilon(F)+\frac{1}{2}|v|^2$ for the system of conservation laws (3.3.10) of elastodynamics fails to be convex on the entire state space. The same difficulty is encountered in Maxwell's equations (3.3.40) for the Born-Infeld medium, with constitutive relations (3.3.45), as the entropy defined by $[1 + |B|^2 + |D|^2 + |D \wedge B|^2]^{\frac{1}{2}}$ is not a convex function of $(B, D)$, far from the origin. We have seen that when the internal energy $\varepsilon(F)$ is rank-one convex, the presence of involutions (5.4.1) provides some relief, but only partial: Theorem 5.4.5 is weaker than Theorem 5.3.1, as it requires the small local oscillation assumption (5.4.45) and asserts global, but not necessarily local, stability. Moreover, when it comes to Maxwell's equations, Theorems 5.4.4 and 5.4.5 are of no help, because the involution cone associated with the involutions (5.4.2) spans the entire state space $I\!R^6$. Fortunately, in the above systems the failure of the entropy to be convex is compensated by the presence of supplementary entropy-entropy flux pairs. These are induced by the kinematic conservation laws (2.2.15), (2.2.16) for the system of elastodynamics; and they derive from the extra conservation law

$$\partial_t Q = \mathrm{div}\big[\eta^{-1}(I + BB^\top + DD^\top - QQ^\top)\big], \tag{5.5.1}$$

for the electrodynamics equations of the Born-Infeld medium. It should be noted, however, that (2.2.15), (2.2.16) do not hold for arbitrary solutions of (3.3.10), but only for those that satisfy the involution (5.4.1). Similarly, (5.5.1) holds identically only for classical solutions of (3.3.40) (with $J = 0$) and (3.3.45) that satisfy the involution (5.4.2). We thus need an extended notion of entropy that is contingent on involutions:

**5.5.1 Definition.** For a system of conservation laws (5.1.1) endowed with an involution (5.4.3), a scalar-valued function $\eta(U)$, together with a $m$-row vector-valued function $Q(U)$, constitute a *contingent entropy-entropy flux pair* if

$$\mathrm{D}Q_\alpha(U) = \mathrm{D}\eta(U)\mathrm{D}G_\alpha(U) + \Xi(U)^\top M_\alpha\,, \quad \alpha = 1, \dots, m, \tag{5.5.2}$$

for some $k$-column vector Lagrange multiplier $\Xi(U)$.

Notice that (5.5.2) still implies

$$\partial_t \eta(U(x,t)) + \operatorname{div} Q(U(x,t)) = 0, \tag{5.5.3}$$

for any classical solution $U$ of (5.1.1) that satisfies the involution (5.4.3). On the other hand, (5.1.3) is here replaced by the symmetry condition

$$\mathrm{D}^2\eta(U)\mathrm{D}G_\alpha(U) + \mathrm{D}\Xi(U)^\top M_\alpha = \mathrm{D}G_\alpha(U)^\top \mathrm{D}^2\eta(U) + M_\alpha^\top \mathrm{D}\Xi(U). \tag{5.5.4}$$

Thus, $F^*$ and $\det F$ are contingent entropies of the system (3.3.10) of elastodynamics, with involution (5.4.1); and $Q = \mathrm{D} \wedge B$ is a contingent entropy of the system of electrodynamics (3.3.40) for the Born-Infeld medium (3.3.45), with involution (5.4.2).

We now lay down a general framework that will eventually encompass the above applications. We consider a system of conservation laws (5.1.1) which is endowed with an involution (5.4.3) and is equipped with a principal contingent entropy-entropy flux pair $(\eta(U), Q(U))$ as well as $N$ additional contingent entropy-entropy flux pairs $(\Phi_I(U), \Psi_I(U))$, $I = 1, \dots, N$, which we group together into a single pair $(\Phi(U), \Psi(U))$ of a $N$-column vector and a $N \times m$ matrix. In particular, for any $i = 1, \dots, n$, the $i$-th component $U^i$ of $U$ may be viewed as a contingent entropy for (5.1.1), with associated flux the $i$-th row vector $G^i$ of $G$, and we include all these pairs in $(\Phi, \Psi)$. Thus, for the equations (3.3.10) of elastodynamics one employs as $\Phi$ $(F, v, F^*, \det F)$ arranged as a 22-column vector. Similarly, in the case of Maxwell's equations (3.3.40) for the Born-Infeld medium (3.3.45), $\Phi$ shall be $(B, D, D \wedge B)$ arranged as a 9-column vector. As usual, the $\alpha$-th column of $\Psi$ will be denoted by $\Psi_\alpha$. The Lagrange multiplier vectors $\Omega_I(U)$ associated with $(\Phi_I(U), \Psi_I(U))$ will be assembled into a $k \times N$ matrix-valued function $\Omega(U)$. We thus have

$$\mathrm{D}\Psi_\alpha(U) = \mathrm{D}\Phi(U)\mathrm{D}G_\alpha(U) + \Omega(U)^\top M_\alpha\,, \qquad \alpha = 1, \dots, m, \tag{5.5.5}$$

$$\mathrm{D}^2\Phi_I(U)\mathrm{D}G_\alpha(U) + \mathrm{D}\Omega_I(U)^\top M_\alpha = \mathrm{D}G_\alpha(U)^\top \mathrm{D}^2\Phi_I(U) + M_\alpha^\top \mathrm{D}\Omega_I(U). \tag{5.5.6}$$

Furthermore,

$$\partial_t \Phi(U(x,t)) + \operatorname{div} \Psi(U(x,t)) = 0 \tag{5.5.7}$$

will hold for any classical solution $U$ of (5.1.1) which satisfies the involution (5.4.3). Since $(U, G)$ is part of $(\Phi, \Psi)$, $\Phi(U)$ will be *complete* in the sense

$$\operatorname{rank} \mathrm{D}\Phi(U) = n, \qquad U \in \mathcal{O}. \tag{5.5.8}$$

The special structure that induces existence and stability of classical solutions is introduced in the following

**5.5.2 Definition.** In the above setting, the principal contingent entropy $\eta(U)$ is called *polyconvex*, relative to the contingent entropies $\Phi(U)$, if it admits a representation

$$\eta(U) = \theta(\Phi(U)), \qquad U \in \mathcal{O}, \tag{5.5.9}$$

where $\theta$ is uniformly convex on $\mathbb{R}^N$.

In the conservation laws for the Born-Infeld medium, with $\Phi = (B, D, Q)$, the principal entropy $\eta = [1 + |B|^2 + |D|^2 + |Q|^2]^{\frac{1}{2}}$ is indeed polyconvex.

For the system of elastodynamics, with $\Phi = (F, v, F^*, \det F)$, the principal entropy $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ will be polyconvex if

$$\varepsilon(F) = \sigma(F, F^*, \det F), \tag{5.5.10}$$

where $\sigma$ is uniformly convex on $\mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{R}^+$. Functions of this form provide a realistic representation of the internal energy functions of rubberlike materials at constant temperature or at constant (physical) entropy.

It can be shown that any null Lagrangian (2.2.9) is continuous in $L^\infty$ weak*. Consequently, any polyconvex internal energy function (5.5.10) is necessarily lower semicontinuous in $L^\infty$ weak*, and thereby quasiconvex and rank-one convex, by virtue of Theorem 5.4.7. However, the converse is not generally true: Quasiconvexity does not necessarily imply polyconvexity of $\varepsilon$.

In what follows, $\theta_{\Phi_I}(\Phi)$ will denote the partial derivative $\partial\theta(\Phi)/\partial\Phi_I$ ; $\theta_\Phi$ will stand for the $N$-row vector $(\theta_{\Phi_1}, \dots, \theta_{\Phi_N})$; and $\theta_{\Phi\Phi}$ will denote the $N \times N$ Hessian matrix of $\theta(\Phi)$.

For any $U \in \mathcal{O}$, we define the symmetric $n \times n$ matrix

$$A(U) = \mathrm{D}^2\eta(U) - \sum_{I=1}^{N} \theta_{\Phi_I}(\Phi(U))\mathrm{D}^2\Phi_I(U). \tag{5.5.11}$$

Using (5.5.9),

$$A(U) = \mathrm{D}\Phi(U)^\top \theta_{\Phi\Phi}(\Phi(U))\mathrm{D}\Phi(U), \tag{5.5.12}$$

so $A(U)$ is positive definite when $\eta(U)$ is polyconvex and $\Phi(U)$ is complete. Furthermore, by virtue of (5.5.4) and (5.5.6),

$$A(U)\mathrm{D}G_\alpha(U) + \Gamma(U)^\top M_\alpha = \mathrm{D}G_\alpha(U)^\top A(U) + M_\alpha^\top \Gamma(U), \tag{5.5.13}$$

where

$$\Gamma(U) = \mathrm{D}\Xi(U) - \sum_{I=1}^{N} \theta_{\Phi_I}(\Phi(U))\mathrm{D}\Omega_I(U). \tag{5.5.14}$$

We now show that polyconvexity induces local existence of classical solutions to the Cauchy problem:

**5.5.3 Theorem.** *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.3) *and is equipped with a principal contingent entropy $\eta(U)$, which is polyconvex* (5.5.9) *relative to a complete family $\Phi(U)$ of contingent entropies. Suppose the initial data $U_0$ are continuously differentiable on $\mathbb{R}^m$, take values in a compact subset of $\mathcal{O}$, satisfy the involution on $\mathbb{R}^m$, and $\nabla U_0 \in H^\ell$ for some $\ell > m/2$. Moreover, let $G \in C^{\ell+2}$. Then there exists $T_\infty$, $0 < T_\infty \le \infty$, and a unique continuously differentiable function $U$ on $\mathbb{R}^m \times [0, T_\infty)$, taking values in $\mathcal{O}$, which is a classical solution of the Cauchy problem* (5.1.1), (5.1.2) *on* $[0, T_\infty)$. *Furthermore,*

$$\nabla U(\cdot, t) \in C^0([0, T_\infty); H^\ell). \tag{5.5.15}$$

*The interval* $[0, T_\infty)$ *is maximal, in the sense that whenever $T_\infty < \infty$ then*

$$\int_0^{T_\infty} \|\nabla U(\cdot, t)\|_{L^\infty} dt = \infty \tag{5.5.16}$$

*and/or the range of $U(\cdot, t)$ escapes from every compact subset of $\mathcal{O}$ as $t \uparrow T_\infty$.*

**Sketch of Proof.** It is a slight variant of the proof of Theorem 5.1.1. As in the proof of Theorem 5.4.4, in order to avoid the replication of cumbersome routine estimations, we shall proceed under the extraneous assumption that the matrices $M_\alpha \mathrm{D}G_\beta(V)$ are constant, in which case solutions of (5.1.11), (5.1.2) satisfy the involution (5.4.3), so long as $U_0$ does. This will allow us to use (5.1.11), rather than (5.4.37), as the designated linearization of (5.1.1).

In carrying out the proof, one may no longer use $\mathrm{D}^2\eta(V)$ as symmetrizer. In its place, we employ the symmetric positive definite matrix $A(V)$, defined by (5.5.12). Indeed, the key role of the equations (5.1.15) and (5.1.22) which embody the symmetrizability of (5.1.11) when $\eta$ is a convex entropy, is here undertaken by

$$\sum_{\alpha=1}^{m} 2U_r^\top A(V)\mathrm{D}G_\alpha(V)\partial_\alpha U_r$$

$$(5.5.17)\qquad = \sum_{\alpha=1}^{m} \partial_\alpha \big\{ U_r^\top [A(V)\mathrm{D}G_\alpha(V) + \Gamma(V)^\top M_\alpha] U_r \big\}$$

$$- \sum_{\alpha=1}^{m} U_r^\top \partial_\alpha [A(V)\mathrm{D}G_\alpha(V) + \Gamma(V)^\top M_\alpha] U_r \,,$$

$$\sum_{\alpha=1}^{m} 2(U-\bar{U})^\top A(V)\mathrm{D}G_\alpha(V)\partial_\alpha (U-\bar{U})$$

$$(5.5.18)\qquad = \sum_{\alpha=1}^{m} \partial_\alpha \big\{ (U-\bar{U})^\top [A(V)\mathrm{D}G_\alpha(V) + \Gamma(V)^\top M_\alpha] (U-\bar{U}) \big\}$$

$$- \sum_{\alpha=1}^{m} (U-\bar{U})^\top \partial_\alpha [A(V)\mathrm{D}G_\alpha(V) + \Gamma(V)^\top M_\alpha] (U-\bar{U}),$$

which are verified by using (5.5.13) in conjunction with $\sum M_\alpha \partial_\alpha U_r = 0$ and $\sum M_\alpha \partial_\alpha (U-\bar{U}) = 0$.

The remaining equations in the proof of Theorem 5.1.1 carry over to the present setting by simply substituting $A(V)$ for $\mathrm{D}^2\eta(V)$. This completes the proof.

In certain systems, (5.5.7) may hold, as an equality, even for weak solutions. This is certainly the case in elastodynamics, as the kinematic conservation laws (2.2.15) and (2.2.16) hold for any $L^\infty$ solution satisfying the involution (5.4.1). This is also true in electrodynamics for the Born-Infeld medium, at least in the realm of $BV$ solutions, the reason being that all shocks turn out to be linearly degenerate (see Section 7.5) and thus do not incur any entropy production (Theorem 8.5.2). Under these circumstances, polyconvexity of the entropy implies uniqueness and stability of classical solutions, even within the class of admissible weak solutions:

**5.5.4 Theorem.** *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.3) *and is equipped with a principal contingent entropy* $\eta(U)$*, which is polyconvex* (5.5.9)*, relative to a complete family* $\Phi(U)$ *of contingent entropies. Suppose* $\bar{U}$ *is a classical solution of* (5.1.1) *on* $[0, T)$*, taking values in a convex compact subset* $\mathcal{D}$ *of* $\mathcal{O}$*, with initial data* $\bar{U}_0$ *that satisfy the involution. Let* $U$ *be any weak solution of* (5.1.1) *on* $[0, T)$*, also taking values in* $\mathcal{D}$*, which satisfies* (5.5.7)*, meets the entropy admissibliity criterion* (4.5.3)*, and has initial data* $U_0$ *satisfying the involution. Then*

$$(5.5.19)\qquad \int_{|x|<r} |U(x,t) - \bar{U}(x,t)|^2 dx \le a e^{bt} \int_{|x|<r+st} |U_0(x) - \bar{U}_0(x)|^2 dx$$

*holds for any $r > 0$ and $t \in [0, T)$, with positive constants $s, a$, depending solely on $\mathcal{D}$, and $b$ that also depends on the Lipschitz constant of $\bar{U}$. In particular, $\bar{U}$ is the unique admissible weak solution of* (5.1.1) *with initial data $\bar{U}_0$ and values in $\mathcal{D}$.*

**Proof.** It is a variant of the proof of Theorem 5.3.1. In the place of (5.3.2), (5.3.3) and (5.3.4), we now set

$$h(U,\bar{U}) = \eta(U) - \eta(\bar{U}) - \theta_\Phi(\Phi(\bar{U}))[\Phi(U) - \Phi(\bar{U})], \tag{5.5.20}$$

$$\begin{aligned} Y_\alpha(U,\bar{U}) = Q_\alpha(U) - Q_\alpha(\bar{U}) - \theta_\Phi(\Phi(\bar{U}))[\Psi_\alpha(U) - \Psi_\alpha(\bar{U})] \\ + [\theta_\Phi(\Phi(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top]M_\alpha[U-\bar{U}], \end{aligned} \tag{5.5.21}$$

$$\begin{aligned} Z_\alpha(U,\bar{U}) = &- \mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}\Phi(\bar{U})^\top \theta_{\Phi\Phi}(\Phi(\bar{U}))[\Phi(U) - \Phi(\bar{U})] \\ &+ \mathrm{D}\Phi(\bar{U})^\top \theta_{\Phi\Phi}(\Phi(\bar{U}))[\Psi_\alpha(U) - \Psi_\alpha(\bar{U})] \\ &- \mathrm{D}\Phi(\bar{U})^\top \theta_{\Phi\Phi}(\Phi(\bar{U}))\Omega(\bar{U})^\top M_\alpha[U-\bar{U}] \\ &+ \Gamma(\bar{U})^\top M_\alpha[U-\bar{U}], \end{aligned} \tag{5.5.22}$$

where $\Gamma(U)$ is defined by (5.5.14).

Clearly, $h(U,\bar{U})$ is of quadratic order in $U-\bar{U}$, and positive definite. Upon using (5.5.2), (5.5.5) and (5.5.9), we deduce

$$\begin{aligned} \mathrm{D}Y_\alpha(U,\bar{U}) = [\theta_\Phi(\Phi(U)) - \theta_\Phi(\Phi(\bar{U}))]\mathrm{D}\Phi(U)\mathrm{D}G_\alpha(U) \\ +[\Xi(U)-\Xi(\bar{U})]^\top M_\alpha - \theta_\Phi(\Phi(\bar{U}))[\Omega(U)-\Omega(\bar{U})]^\top M_\alpha\,, \end{aligned} \tag{5.5.23}$$

which vanishes at $U = \bar{U}$, so that $Y(U,\bar{U})$ is also of quadratic order in $U - \bar{U}$. In particular, we still have (5.3.5).

Turning to $Z(U,\bar{U})$, and by virtue of (5.5.5),

$$\begin{aligned} \mathrm{D}Z_\alpha(U,\bar{U}) = &- \mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}\Phi(\bar{U})^\top \theta_{\Phi\Phi}(\Phi(\bar{U}))\mathrm{D}\Phi(U) \\ &+ \mathrm{D}\Phi(\bar{U})^\top \theta_{\Phi\Phi}(\Phi(\bar{U}))\mathrm{D}\Phi(U)\mathrm{D}G_\alpha(U) \\ &+ \mathrm{D}\Phi(\bar{U})^\top \theta_{\Phi\Phi}(\Phi(\bar{U}))[\Omega(U) - \Omega(\bar{U})]^\top M_\alpha \\ &+ \Gamma(\bar{U})^\top M_\alpha\,. \end{aligned} \tag{5.5.24}$$

Recalling (5.5.12) and (5.5.13), we conclude that

$$\mathrm{D}Z_\alpha(\bar{U},\bar{U}) = -\mathrm{D}G_\alpha(\bar{U})^\top A(\bar{U}) + A(\bar{U})\mathrm{D}G_\alpha(\bar{U}) + \Gamma(\bar{U})^\top M_\alpha = M_\alpha^\top \Gamma(\bar{U}). \tag{5.5.25}$$

Retracing the steps in the proof of Theorem 5.3.1, we fix a nonnegative, Lipschitz continuous test function $\psi$ on $\mathbb{R}^m \times [0, T)$, with compact support, and evaluate $h$, $Y$ and $Z$ along the two solutions $U(x,t)$, $\bar{U}(x,t)$. As an admissible weak solution, $U$ satisfies the inequality (4.5.3), while $\bar{U}$, being a classical solution, will identically satisfy (4.5.3) as an equality. Thus, in the place of (5.3.6) we now have

(5.5.26)

$$\int_0^T\int_{IR^m}[\partial_t\psi\, h(U,\bar U)+\sum_{\alpha=1}^m\partial_\alpha\psi\, Y_\alpha(U,\bar U)]dxdt+\int_{IR^m}\psi(x,0)\,h(U_0(x),\bar U_0(x))\,dx$$
$$\geq -\int_0^T\int_{IR^m}\Big\{\partial_t\psi\,\theta_\Phi(\Phi(\bar U))[\Phi(U)-\Phi(\bar U)]+\sum_{\alpha=1}^m\partial_\alpha\psi\{\theta_\Phi(\Phi(\bar U))$$
$$\times[\Psi_\alpha(U)-\Psi_\alpha(\bar U)]-[\theta_\Phi(\Phi(\bar U))S(\bar U)^\top-\Xi(\bar U)^\top]M_\alpha[U-\bar U]\}\Big\}dxdt$$
$$-\int_{IR^m}\psi(x,0)\theta_\Phi(\Phi(\bar U_0(x)))[\Phi(U_0(x))-\Phi(\bar U_0(x))]dx.$$

Next we recall that both $U$ and $\bar U$ satisfy (5.5.7), in the sense of distributions. Hence, using $\psi\theta_\Phi(\Phi(\bar U))$ as test function,

(5.5.27)

$$\int_0^T\int_{IR^m}\Big\{\partial_t[\psi\theta_\Phi(\Phi(\bar U))][\Phi(U)-\Phi(\bar U)]+\sum_{\alpha=1}^m\partial_\alpha[\psi\theta_\Phi(\Phi(\bar U))][\Psi_\alpha(U)-\Psi_\alpha(\bar U)]\Big\}dxdt$$
$$+\int_{IR^m}\psi(x,0)\theta_\Phi(\Phi(\bar U_0(x)))[\Phi(U_0(x))-\Phi(\bar U_0(x))]dx=0.$$

Furthermore, since (5.4.3) holds for both $U$ and $\bar U$,

$$\int_0^T\int_{IR^m}\sum_{\alpha=1}^m\partial_\alpha\big\{\psi[\theta_\Phi(\Phi(\bar U))\Omega(\bar U)^\top-\Xi(\bar U)^\top]M_\alpha[U-\bar U]\big\}dxdt=0. \tag{5.5.28}$$

By virtue of (5.5.5) and $\sum M_\alpha\partial_\alpha\bar U=0$,

(5.5.29)

$$\begin{aligned}\partial_t\theta_\Phi(\Phi(\bar U))&=\partial_t\Phi(\bar U)^\top\theta_{\Phi\Phi}(\Phi(\bar U))\\&=-\sum_{\alpha=1}^m\partial_\alpha\Psi_\alpha(\bar U)^\top\theta_{\Phi\Phi}(\Phi(\bar U))\\&=-\sum_{\alpha=1}^m\partial_\alpha\bar U^\top\mathrm{D}\Psi_\alpha(\bar U)^\top\theta_{\Phi\Phi}(\Phi(\bar U))\\&=-\sum_{\alpha=1}^m\partial_\alpha\bar U^\top[\mathrm{D}\Phi(\bar U)\mathrm{D}G_\alpha(\bar U)+\Omega(\bar U)^\top M_\alpha]^\top\theta_{\Phi\Phi}(\Phi(\bar U))\\&=-\sum_{\alpha=1}^m\partial_\alpha\bar U^\top\mathrm{D}G_\alpha(\bar U)^\top\mathrm{D}\Phi(\bar U)^\top\theta_{\Phi\Phi}(\Phi(\bar U)).\end{aligned}$$

Similarly,

$$\partial_\alpha \theta_\Phi(\Phi(\bar U)) = \partial_\alpha \bar U^\top \mathrm{D}\Phi(\bar U)^\top \theta_{\Phi\Phi}(\Phi(\bar U)), \tag{5.5.30}$$

$$\begin{aligned}\partial_\alpha[\theta_\Phi(\Phi(\bar U))\Omega(\bar U)^\top - \Xi(\bar U)^\top] \\ = \partial_\alpha \bar U^\top \mathrm{D}\Phi(\bar U)^\top \theta_{\Phi\Phi}(\Phi(\bar U))\Omega(\bar U)^\top - \partial_\alpha \bar U^\top \Gamma(\bar U)^\top .\end{aligned} \tag{5.5.31}$$

Therefore, recalling (5.5.22),

$$\begin{aligned}\partial_t \theta_\Phi(\Phi(\bar U))[\Phi(U) - \Phi(\bar U)] + \sum_{\alpha=1}^m \partial_\alpha \theta_\Phi(\Phi(\bar U))[\Psi_\alpha(U) - \Psi_\alpha(\bar U)] \\ - \sum_{\alpha=1}^m \partial_\alpha[\theta_\Phi(\Phi(\bar U))\Omega(\bar U)^\top - \Xi(\bar U)^\top] M_\alpha[U - \bar U] \\ = \sum_{\alpha=1}^m \partial_\alpha \bar U^\top Z_\alpha(U, \bar U).\end{aligned} \tag{5.5.32}$$

By account of (5.2.25),

$$\sum_{\alpha=1}^m \partial_\alpha \bar U^\top \mathrm{D}Z_\alpha(\bar U, \bar U) = \left[\sum_{\alpha=1}^m M_\alpha \partial_\alpha \bar U\right]^\top \Gamma(\bar U) = 0. \tag{5.5.33}$$

Consequently, the right-hand side of (5.5.32) is of quadratic order in $U - \bar U$. Upon combining (5.5.26), (5.5.27), (5.5.28) and (5.5.32), we reestablish (5.3.10) and thereby (5.3.14). Then, an application of Gronwall's inequality yields (5.5.19). The proof is complete.

In particular, Theorems 5.5.3 and 5.5.4 apply to the class of systems of conservation laws that are endowed with an involution and are equipped with a convex contingent entropy $\eta(U)$ (just take $\Phi(U) \equiv U$). One may attempt to reduce the more general class of systems endowed with an involution and equipped with a polyconvex contingent entropy to the above special class by means of the following procedure. Assume that the system (5.1.1) is endowed with the involution (5.4.3) and is equipped with a principal contingent entropy-entropy flux pair $(\eta(U), Q(U))$ which is polyconvex (5.5.9), relative to a complete family $(\Phi(U), \Psi(U))$ of contingent entropy-entropy flux pairs. We seek functions $S(X)$ and $\Pi(X)$, defined on $\mathbb{R}^N$ and taking values in $\mathbb{M}^{N\times m}$ and $\mathbb{M}^{1\times m}$, respectively, such that

$$S(\Phi(U)) = \Psi(U), \qquad \Pi(\Phi(U)) = Q(U) \tag{5.5.34}$$

and, in addition, $(\theta(X),\ \Pi(X))$ is a (generally contingent) entropy-entropy flux pair for the *extended system*

$$\partial_t X(x,t) + \operatorname{div} S(X(x,t)) = 0. \tag{5.5.35}$$

When functions satisfying the above specifications can be found, one may construct solutions to the Cauchy problem (5.1.1), (5.1.2) by first solving (5.5.35) with initial conditions

$$X(x,0) = \Phi(U_0(x)), \tag{5.5.36}$$

and then getting $U$ from the equation $\Phi(U) = X$. The merit of this approach lies in that (5.5.35) is now equipped with a convex (possibly contingent) entropy $\theta$.

The above program has been implemented successfully for the systems of elastodynamics and electrodynamics.

In elastodynamics, $U = (F, v)^\top$, $X = (F, v, \Theta, \omega)^\top$, $\sigma = \sigma(F, \Theta, \omega)$, the extended system reads

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & \alpha = 1,2,3; \quad i = 1,2,3 \\ \partial_t v_i - \partial_\alpha \left( \dfrac{\partial \sigma}{\partial F_{i\alpha}} + \dfrac{\partial \sigma}{\partial \Theta_{\beta j}} \dfrac{\partial F^*_{\beta j}}{\partial F_{1\alpha}} + \dfrac{\partial \sigma}{\partial \omega} \dfrac{\partial \det F}{\partial F_{i\alpha}} \right) = 0, & i = 1,2,3 \\ \partial_t \Theta_{\beta i} - \partial_\alpha \left( \dfrac{\partial F^*_{\beta i}}{\partial F_{j\alpha}} v_j \right) = 0, & \beta = 1,2,3; \quad i = 1,2,3 \\ \partial_t \omega - \partial_\alpha \left( \dfrac{\partial \det F}{\partial F_{j\alpha}} v_j \right) = 0, & \end{cases} \tag{5.5.37}$$

and the entropy-entropy flux pair is

$$\theta = \tfrac{1}{2}|v|^2 + \sigma(F, \Theta, \omega), \tag{5.5.38}$$

$$\Pi_\alpha = -\left( \frac{\partial \sigma}{\partial F_{i\alpha}} + \frac{\partial \sigma}{\partial \Theta_{\beta j}} \frac{\partial F^*_{\beta j}}{\partial F_{i\alpha}} + \frac{\partial \sigma}{\partial \omega} \frac{\partial \det F}{\partial F_{i\alpha}} \right) v_i \,. \tag{5.5.39}$$

On the "manifold" $X = \Phi(U) = (F, v, F^*, \det F)^\top$, (5.5.37) reduces to the system (3.3.10) (with $b = 0$) together with the kinematic conservation laws (2.2.15), (2.2.16), while $(\theta, \Pi)$ reduces to the classical entropy-entropy flux pair recorded in Section 3.3.3.

In electrodynamics, for the Born-Infeld medium, where $U = (B, D)^\top$, $X = (B, D, P)^\top$, the extended system reads

$$\begin{cases} \partial_t B + \operatorname{curl}\left[\theta^{-1}(D + B \wedge P)\right] = 0 \\ \partial_t D - \operatorname{curl}\left[\theta^{-1}(B - D \wedge P)\right] = 0 \\ \partial_t P - \operatorname{div}\left[\theta^{-1}(I + BB^\top + DD^\top - PP^\top)\right] = 0, \end{cases} \tag{5.5.40}$$

and the entropy-entropy flux pair is

$$\theta = (1 + |B|^2 + |D|^2 + |P|^2)^{\frac{1}{2}}, \tag{5.5.41}$$

$$\Pi = P - \theta^{-2}[P - D \wedge B - (D \cdot P)D - (B \cdot P)B]. \tag{5.5.42}$$

Again, on the "manifold" $X = \Phi(U) = (B, D, D \wedge B)^{\top}$ (5.5.40) reduces to Maxwell's equations (3.3.40) (with $J = 0$), (3.3.45), together with the supplementary conservation law (5.5.1), while $(\theta, \Pi)$ reduces to the entropy-entropy flux pair $(\eta, Q)$ recorded in (3.3.44).

## 5.6 Notes

The proof of Theorem 5.1.1 has been adapted from Majda [3]. This approach is in the spirit of the theory of symmetric hyperbolic systems developed by Friedrichs [2]. For an alternative, functional analytic approach, see Kato [1]. By employing more sophisticated symmetrizers, one may establish local existence for the Cauchy problem in a more general class of hyperbolic systems, including those with the sole property that the multiplicity of each characteristic speed $\lambda_i(\nu; U)$ does not vary with the direction $\nu$; cf. Lax [1], M.E. Taylor [1,2], and Métivier [1]. Quite often, systems arising in continuum physics exhibit particular features that require special treatment; cf. Godunov [3], Makino, Ukai and Kawashima [1], and Chemin [1].

Remarks on the initial-boundary-value problem for systems are here in order. An informative discussion of proper boundary conditions is found in the book [11] and the survey article [23] by Serre. In the case of linear systems, the initial-boundary-value problem is well posed when the boundary conditions satisfy the so-called *Lopatinski condition,* which rules out the existence of sinusoidal waves with amplitude that grows exponentially in time (see Kreiss [1] and the survey article by Higdon [1]). For quasilinear systems, a uniform Lopatinski condition suffices for local existence of solutions to the initial-boundary-value problem, constructed by linearization, in the spirit of Theorem 5.1.1. In one spatial dimension, the uniform Lopatinski condition is essentially equivalent to the statement that the number of independent, scalar boundary conditions equals the number of characteristics impinging on the boundary. For that case, solutions to the initial-boundary-value problem are constructed by the method of characteristics in the monograph by Li and Yu [1]. Unfortunately, the uniform Lopatinski condition requires that the amplitude of sinusoidal waves decay exponentially in time, thus ruling out the system of elastodynamics and other systems of physical interest, which admit Rayleigh wave solutions with persistent, nondecaying amplitudes. To treat such systems, one has to exploit their special structure; see Schochet [1] and references therein for the Euler equations of gas dynamics. The construction of solutions to systems by the vanishing viscosity method, and the study of the resulting boundary layer, has been mainly considered in one spatial dimension and before shocks impinge upon the boundary; see Serre [11,17], Benabdallah and Serre [1], Gisclon [1], Gisclon and Serre [1], Grenier [1], Joseph and LeFloch [1,2,3], Serre and Zumbrun [1], Rousset [1,2,3] and Xin [4]. The case where a shock is impinging on the boundary is discussed in Serre [14]. References to weak solutions of initial-boundary-value problems are found in Sections 6.11, 13.10, 14.13, 15.9 and 16.9.

The discussion of the effects of damping, culminating in Theorem 5.2.1, is here adapted from Hanouzet and Natalini [1] and Yong [6]. See also Sideris, Thomases and Wang [1]. For the effect of damping on the long time behavior of solutions, see Ruggeri and Serre [1]. The setting of the general relaxation framework has been taken from Chen, Levermore and Liu [1]. For applications to continuum physics, see Boillat and Ruggeri [1]. There is voluminous literature investigating relaxation of smooth solutions to local equilibrium, e.g. Yong [2,3,5], Lattanzio and Tzavaras [1]. Surveys providing examples and extensive bibliography are found in Natalini [3] and Yong [4]. For additional discussion and references on relaxation, see Chapters VI and XVI. The connection between relaxation and diffusion was first recognized in the kinetic theory of gases, where it is effected by means of the Chapman-Enskog expansion (e.g. Cercignani [1]). Chapman-Enskog type expansions have also been employed in order to relate classes of hyperbolic balance laws (5.2.1) with parabolic systems of the form (4.6.1); see Kawashima and Yong [1,2]. The intimate relation between relaxation and diffusion also emerges in the large time behavior of solutions to hyperbolic systems with frictional damping; see Liu [25], Hsiao and Liu [1], Hsiao and Li [1,2], Hsiao, Li and Pan [1], Hsiao and Pan [1,2,3], Pan [1], Marcati and Pan [1], Serre and Xiao [1], Liu and Natalini [1], He and Li [1], Huang and Pan [1,2,3], and Lattanzio and Rubino [1].

Dispersion is an alternative decay mechanism that may delay or even prevent the breaking of waves, thus rendering smooth solutions in the large to the Cauchy problem. This is particularly effective when the dimension of the space is large, the initial data are "small" and the system satisfies the so-called *null condition;* cf. Christodoulou [1], Klainerman [1], Sideris [2,3], and Chae and Huh [1]. See also the monograph by Ta-tsien Li [1]. An interesting family of global classical solutions to the Euler equations is presented in Serre [5] and Grassin and Serre [1].

The proof of Theorem 5.3.1 combines ideas of DiPerna [7] and Dafermos [9]. This approach even applies in certain cases where $\bar{U}$ is a special weak solution to special systems; see Chen [7], Chen and Frid [7] and Chen and Yachun Li [1,2]. A connection between relative entropy and relaxation is established by Tzavaras [6].

Hyperbolic systems of conservation laws with involutions were considered by Boillat [4] and by Dafermos [12]. In particular, Boillat [4] exhibits sufficient conditions that are more general that (5.4.4) and presents examples arising in the theory of general relativity. The analysis in Section 5.4 is intimately related to the theory of compensated compactness, as formalized by Murat and Tartar; cf. Tartar [1,2]. The "involution cone" corresponds to the "characteristic cone", in the terminology of that theory. Theorem 5.4.4 originally appeared in the first edition of this book; however, typical examples, such as the system (3.3.10) of balance laws of (isentropic) elastodynamics, had been studied earlier, for example by Hughes, Kato and Marsden [1] and by Dafermos and Hrusa [1]. Theorem 5.4.5 is taken from Dafermos [21].

The notion of quasiconvexity, introduced by Definition 5.4.6, is a generalization of quasiconvexity in the sense of Morrey [1] due to Dacorogna [1]; for a detailed study and proof of Theorem 5.4.7, see Müller and Fonseca [1].

The notion of polyconvexity in elasticity was introduced by Ball [1], as a condition rendering the internal energy function weakly lower semicontinuous. The author originally heard from P.G. LeFloch the idea of extending the system of conservation laws in elastodynamics by appending conservation laws for the invariants of the stretch tensor. Explicit extensions were first published by Qin [1] and by Demoulini, Stuart and Tzavaras [2]. Brennier [2] presents two distinct extensions of the equations of electrodynamics, for the Born-Infeld medium, including the one recorded here. See also Neves and Serre [1]. Serre [22] discovered that the Poynting vector $Q$ satisfies a conservation law, akin to (5.5.1), for general media (3.3.41), under the lossless condition (3.3.42), and devised the proper extension of Maxwell's equations for that general class of media. See also Boillat [3,5]. Existence and stability of classical solutions to the Cauchy problems for systems of conservation laws endowed with an involution and equipped with a convex contingent entropy was established by Serre [22]. The general polyconvex case, Theorems 5.5.3 and 5.5.4, is treated here for the first time (for elastodynamics, see Lattanzio and Tzavaras [1]). The author thanks Denis Serre for his helpful remarks.

In the context of elasticity, the proof that polyconvexity implies quasiconvexity and, in turn, quasiconvexity implies rank-one convexity, was established in the aforementioned, pioneering paper by Ball [1]. The question of whether, conversely, rank-one convexity generally implies quasiconvexity was settled, in the negative, by Šverak [1].

# VI

# The $L^1$ Theory for Scalar Conservation Laws

The theory of the scalar balance law, in several spatial dimensions, has reached a state of virtual completeness. In the framework of classical solutions, the elementary, yet effective, method of characteristics yields a sharper version of Theorem 5.1.1, determining explicitly the life span of solutions with Lipschitz continuous initial data and thereby demonstrating that in general this life span is finite. Thus one must deal with weak solutions, even when the initial data are very smooth.

In regard to weak solutions, the special feature that sets the scalar balance law apart from systems of more than one equation is the size of its family of entropies. It will be shown that the abundance of entropies induces an effective characterization of admissible weak solutions as well as very strong $L^1$-stability and $L^\infty$-monotonicity properties. Armed with such powerful a priori estimates, one can construct admissible weak solutions in a number of ways. As a sample, construction by the method of vanishing viscosity, the theory of $L^1$-contraction semigroups, the layering method, a relaxation method and an approach motivated by the kinetic theory will be presented here. The method of vanishing viscosity will also be employed for solving the initial-boundary-value problem. When the initial data are functions of locally bounded variation then so are the solutions. Remarkably, however, even solutions that are merely in $L^\infty$ exhibit the same geometric structure as $BV$ functions, with jump discontinuities assembling on "manifolds" of codimension one.

The chapter will close with a description of the insurmountable obstacles encountered in the study of weak solutions for hyperbolic systems of conservation laws in several spatial dimensions, and an account of current efforts to bypass these obstructions.

In order to expose the elegance of the theory, the discussion will be restricted to the homogeneous scalar conservation law, even though the general, inhomogeneous balance law (3.3.1) may be treated by the same methodology, at the expense of rather minor technical complications.

The issue of stability of weak solutions with respect to the weak* topology of $L^\infty$ will be addressed in Chapter XVI. The special case of a single space variable, $m = 1$, has a very rich theory of its own, certain aspects of which will be presented in later chapters and especially in Chapter XI.

## 6.1 The Cauchy Problem: Perseverance and Demise of Classical Solutions

We consider the Cauchy problem for a homogeneous scalar conservation law:

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t)) = 0, \qquad x \in I\!R^m, \quad t > 0, \tag{6.1.1}$$

$$u(x,0) = u_0(x), \qquad x \in I\!R^m \,. \tag{6.1.2}$$

The flux $G(u) = (G_1(u), \dots, G_m(u))$ is a given smooth function on $I\!R$, taking values in $I\!M^{1\times m}$.

A *characteristic* of (6.1.1), associated with a continuously differentiable solution $u$, is an orbit $\xi : [0,T) \to I\!R^m$ of the system of ordinary differential equations

$$\frac{dx}{dt} = G'(u(x,t))^\top \,. \tag{6.1.3}$$

With every characteristic $\xi$ we associate the differential operator

$$\frac{d}{dt} = \partial_t + G'(u(\xi(t),t)) \operatorname{grad} , \tag{6.1.4}$$

which determines the directional derivative along $\xi$. In particular, since $u$ satisfies (6.1.1), $du/dt = 0$, i.e., $u$ is constant along any characteristic. By virtue of (6.1.3), this implies that the slope of the characteristic is constant. Thus all characteristics are straight lines along which the solution is constant. With the help of this property, one may study classical solutions of (6.1.1), (6.1.2) in minute detail. In particular, for scalar conservation laws Theorem 5.1.1 admits the following refinement:

**6.1.1 Theorem.** *Assume that $u_0$, defined on $I\!R^m$, is bounded and Lipschitz continuous. Let*

$$\kappa = \operatorname*{ess\,inf}_{y \in I\!R^m} \operatorname{div} G'(u_0(y)). \tag{6.1.5}$$

*Then there exists a classical solution $u$ of* (6.1.1), (6.1.2) *on the maximal interval* $[0, T_\infty)$*, where $T_\infty = \infty$ when $\kappa \geq 0$ and $T_\infty = -\kappa^{-1}$ when $\kappa < 0$. Furthermore, if $u_0$ is $C^k$ so is $u$.*

**Proof.** Suppose first $\nabla u_0 \in H^\ell$, for $\ell$ very large, so that, by Theorem 5.1.1, the solution $u$ of (6.1.1), (6.1.2) exists on some maximal interval $[0, T_\infty)$ and is a smooth function. Since $u$ is constant along characteristics, its value at any point $(x,t)$, with $x \in I\!R^m$, $t \in [0, T_\infty)$, satisfies the implicit relation

$$u(x,t) = u_0(x - tG'(u(x,t))^\top). \tag{6.1.6}$$

In particular, the range of $u$ coincides with the range of $u_0$.

Differentiating (6.1.6) yields

$$\partial_\alpha u(x,t) = \frac{\partial_\alpha u_0(y)}{1 + t \operatorname{div} G'(u_0(y))}, \qquad \alpha = 1, \cdots, m, \tag{6.1.7}$$

where $y = x - tG'(u(x,t))^\top$. Thus, by virtue of Theorem 5.1.1, $T_\infty = \infty$ when $\kappa \geq 0$ and $T_\infty \geq -\kappa^{-1}$ when $\kappa < 0$. For $\kappa < 0$, derivatives of the solution along characteristics emanating from points $(y, 0)$ with $\operatorname{div} G'(u_0(y)) < \kappa + \varepsilon < 0$ will have to blow up no later than $t = -(\kappa + \varepsilon)^{-1}$. Hence $T_\infty = -\kappa^{-1}$.

When $u_0$ is merely Lipschitz continuous, we approximate it in $L^\infty(I\!R^m)$, via mollification, by a sequence $\{u_n\}$ of smooth functions with $\nabla u_n \in H^\ell$ and

$$\operatorname{ess\,inf} \operatorname{div} G'(u_n(y)) \geq \kappa - \frac{1}{n}. \tag{6.1.8}$$

Classical solutions of (6.1.1) with initial data $u_n$ are defined on $I\!R^m \times [0, T_n)$, where $T_n \geq n$ when $\kappa \geq 0$ and $T_n \geq -(\kappa - 1/n)^{-1}$ when $\kappa < 0$, and are Lipschitz equicontinuous on any compact subset of $I\!R^m \times [0, T_\infty)$. Therefore, we may extract a subsequence that converges, uniformly on compact sets, to some function $u$ on $I\!R^m \times [0, T_\infty)$. Clearly, $u$ is at least a weak solution of (6.1.1), (6.1.2) and, being locally Lipschitz continuous, it is actually a classical solution on $[0, T_\infty)$. The limiting process also implies that $u$ still satisfies (6.1.6) for $x \in I\!R^m$ and $t \in [0, T_\infty)$. In particular, if $u_0$ is differentiable at a point $y \in I\!R^m$ then $u$ is differentiable along the straight line $x = y + tG'(u_0(y))^\top$, the derivatives being given by (6.1.7). Consequently, $[0, T_\infty)$ is the life span of the classical solution.

When $u_0$ is $C^k$, (6.1.6) together with (6.1.7) and the implicit function theorem imply that $u$ is also $C^k$ on $I\!R^m \times [0, T_\infty)$. This completes the proof.

An instructive way of viewing classical solutions $u$ to (6.1.1) is by realizing them as "level surfaces" of functions $f(v; x, t)$, defined on $I\!R \times I\!R^m \times I\!R$; that is

$$f(u(x,t); x, t) = 0 \tag{6.1.9}$$

whenever $u$ satisfies (6.1.1). It is easy to see that for that purpose $f$ must satisfy the transport equation

$$\partial_t f(v; x, t) + \sum_{\alpha=1}^{m} \mathrm{DG}_\alpha(v) \partial_\alpha f(v; x, t) = 0. \tag{6.1.10}$$

In particular, to solve the initial-value problem (6.1.1), (6.1.2), one should solve a Cauchy problem for (6.1.10) with initial condition $f(v; x, 0) = v - u_0(x)$. Since (6.1.10) is linear, a solution of this Cauchy problem will exist on $I\!R \times I\!R^m \times I\!R$. The resulting $f$ will in turn induce, through (6.1.9), the classical solution $u$ to (6.1.1), (6.1.2), which will be valid up until $f_v$ vanishes for the first time. We shall return to the transport equation (6.1.10), in the context of weak solutions, in Section 6.7.

From the above considerations it becomes clear that the life span of classical solutions is generally finite. It is thus imperative to deal with weak solutions.

## 6.2 Admissible Weak Solutions and their Stability Properties

In Section 4.2, we saw that the initial-value problem for a scalar conservation law may admit more than one weak solution, thus raising the need to impose admissibility conditions. In Section 4.5, we discussed how entropy inequalities may serve that purpose. Recall from Section 3.3.1 that for the scalar conservation law (6.1.1) any smooth function $\eta$ may serve as an entropy, with associated entropy flux

$$Q(u) = \int^u \eta'(\omega) G'(\omega) d\omega, \tag{6.2.1}$$

and entropy production zero. It will be convenient to relax slightly the regularity condition and allow entropies (and thereby entropy fluxes) that are merely locally Lipschitz continuous. Similarly, $G$ need only be locally Lipschitz continuous. It turns out that in order to characterize properly admissible weak solutions, one has to impose the entropy inequality

$$\partial_t \eta(u(x,t)) + \operatorname{div} Q(u(x,t)) \leq 0 \tag{6.2.2}$$

for every convex entropy-entropy flux pair:

**6.2.1 Definition.** A bounded measurable function $u$ on $\mathbb{R}^m \times [0, \infty)$ is an *admissible weak solution* of (6.1.1), (6.1.2), with $u_0$ in $L^\infty(\mathbb{R}^m)$, if the inequality

(6.2.3)

$$\int_0^\infty \int_{\mathbb{R}^m} [\partial_t \psi \, \eta(u) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(u)] dx dt + \int_{\mathbb{R}^m} \psi(x,0) \, \eta(u_0(x)) \, dx \geq 0$$

holds for every convex function $\eta$, with $Q$ determined through (6.2.1), and all nonnegative Lipschitz continuous test functions $\psi$ on $\mathbb{R}^m \times [0, \infty)$, with compact support.

Applying (6.2.3) with $\eta(u) = \pm u, \ Q(u) = \pm G(u)$ shows that (6.2.3) implies (4.3.2), i.e., any admissible weak solution in the sense of Definition 6.2.1 is in particular a weak solution as defined in Section 4.3. Also note that if $u$ is a classical solution of (6.1.1), (6.1.2), then (6.2.3) holds automatically, as an equality, i.e., all classical solutions are admissible. Several motivations for (6.2.3) will be presented in subsequent sections.

To verify (6.2.3) for all convex $\eta$, it would suffice to test it just for some family of convex $\eta$ with the property that the set of linear combinations of its members, with nonnegative coefficients, spans the entire set of convex functions. To formulate examples, consider the following standard notation: For $w \in \mathbb{R}$, $w^+$ denotes $\max\{w, 0\}$ and $\operatorname{sgn} w$ stands for $-1$, 1 or 0, as $w$ is negative, positive or zero. Notice that any Lipschitz continuous function is the limit of a sequence of piecewise linear convex functions

$$c_0 u + \sum_{i=1}^k c_i (u - u_i)^+ \tag{6.2.4}$$

with $c_i > 0, i = 1, \cdots, k$. Consequently, it would suffice to verify (6.2.3) for the entropies $\pm u$, with entropy flux $\pm G$, together with the family of entropy-entropy flux pairs

$$\eta(u; \bar{u}) = (u - \bar{u})^+, \qquad Q(u; \bar{u}) = \operatorname{sgn}(u - \bar{u})^+[G(u) - G(\bar{u})], \tag{6.2.5}$$

where $\bar{u}$ is a parameter taking values in $I\!R$. Equally well, one may use the celebrated family of entropy-entropy flux pairs of Kruzkov:

$$\eta(u; \bar{u}) = |u - \bar{u}|, \qquad Q(u; \bar{u}) = \operatorname{sgn}(u - \bar{u})[G(u) - G(\bar{u})]. \tag{6.2.6}$$

The fundamental existence and uniqueness theorem, which will be demonstrated by several methods in subsequent sections, is

**6.2.2 Theorem.** *For each $u_0 \in L^\infty(I\!R^m)$, there exists a unique admissible weak solution $u$ of* (6.1.1), (6.1.2) *and*

$$u(\cdot, t) \in C^0([0, \infty); L^1_{\mathrm{loc}}(I\!R^m)). \tag{6.2.7}$$

The following proposition establishes the most important properties of admissible weak solutions of the scalar conservation law, namely, stability in $L^1$ and monotonicity in $L^\infty$:

**6.2.3 Theorem.** *Let $u$ and $\bar{u}$ be admissible weak solutions of* (6.1.1) *with respective initial data $u_0$ and $\bar{u}_0$ taking values in a compact interval* $[a,b]$. *There is $s > 0$, depending solely on* $[a,b]$, *such that, for any $t > 0$ and $r > 0$*

$$\int_{|x|<r} [u(x,t) - \bar{u}(x,t)]^+ dx \le \int_{|x|<r+st} [u_0(x) - \bar{u}_0(x)]^+ dx, \tag{6.2.8}$$

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathcal{B}_r)} \le \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathcal{B}_{r+st})}\,. \tag{6.2.9}$$

*Furthermore, if*

$$u_0(x) \le \bar{u}_0(x), \qquad a.e.\ on\ I\!R^m\,, \tag{6.2.10}$$

*then*

$$u(x,t) \le \bar{u}(x,t), \qquad a.e.\ on\ I\!R^m \times [0,\infty). \tag{6.2.11}$$

*In particular, the (essential) range of both $u$ and $\bar{u}$ is contained in* $[a,b]$.

**Proof.** The salient feature of the scalar conservation law that induces (6.2.8) is that the functions $\eta(u; \bar{u})$, $Q(u; \bar{u})$, defined through (6.2.5), constitute entropy-entropy flux pairs not only in the variable $u$, for fixed $\bar{u}$, but also in the variable $\bar{u}$, for fixed $u$.

Consider any nonnegative Lipschitz continuous function $\phi(x,t,\bar{x},\bar{t})$, defined on $I\!R^m \times [0,\infty) \times I\!R^m \times [0,\infty)$ and having compact support. Fix $(\bar{x},\bar{t})$ in $I\!R^m \times [0,\infty)$ and write (6.2.3) for the entropy-entropy flux pair $\eta(u;\bar{u}(\bar{x},\bar{t}))$, $Q(u;\bar{u}(\bar{x},\bar{t}))$, and the test function $\psi(x,t) = \phi(x,t,\bar{x},\bar{t})$:

$$
\begin{aligned}
(6.2.12) \qquad \int_0^\infty \int_{I\!R^m} &\{\partial_t \phi(x,t,\bar{x},\bar{t})\, \eta(u(x,t);\bar{u}(\bar{x},\bar{t})) \\
&+ \sum_{\alpha=1}^m \partial_{x_\alpha} \phi(x,t,\bar{x},\bar{t})\; Q_\alpha(u(x,t);\bar{u}(\bar{x},\bar{t}))\} dx dt \\
&+ \int_{I\!R^m} \phi(x,0,\bar{x},\bar{t})\, \eta(u_0(x);\bar{u}(\bar{x},\bar{t})) dx \ge 0.
\end{aligned}
$$

Interchanging the roles of $u$ and $\bar{u}$, we similarly obtain, for any fixed point $(x,t)$ in $I\!R^m \times [0,\infty)$:

$$
\begin{aligned}
(6.2.13) \qquad \int_0^\infty \int_{I\!R^m} &\{\partial_{\bar{t}} \phi(x,t,\bar{x},\bar{t})\, \eta(u(x,t);\bar{u}(\bar{x},\bar{t})) \\
&+ \sum_{\alpha=1}^m \partial_{\bar{x}_\alpha} \phi(x,t,\bar{x},\bar{t})\; Q_\alpha(u(x,t);\bar{u}(\bar{x},\bar{t}))\} d\bar{x} d\bar{t} \\
&+ \int_{I\!R^m} \phi(x,t,\bar{x},0)\, \eta(u(x,t);\bar{u}_0(\bar{x}))\, d\bar{x} \ge 0.
\end{aligned}
$$

Integrating over $I\!R^m \times [0,\infty)$ (6.2.12), with respect to $(\bar{x},\bar{t})$, and (6.2.13), with respect to $(x,t)$, and then adding the resulting inequalities yields

$$
\begin{aligned}
(6.2.14) \qquad \int_0^\infty \int_{I\!R^m} \int_0^\infty \int_{I\!R^m} &\{(\partial_t + \partial_{\bar{t}}) \phi(x,t,\bar{x},\bar{t})\, \eta(u(x,t);\bar{u}(\bar{x},\bar{t})) \\
&+ \sum_{\alpha=1}^m (\partial_{x_\alpha} + \partial_{\bar{x}_\alpha}) \phi(x,t,\bar{x},\bar{t})\; Q_\alpha(u(x,t);\bar{u}(\bar{x},\bar{t}))\} dx dt d\bar{x} d\bar{t} \\
&+ \int_0^\infty \int_{I\!R^m} \int_{I\!R^m} \phi(x,0,\bar{x},\bar{t})\, \eta(u_0(x);\bar{u}(\bar{x},\bar{t}))\, dx d\bar{x} d\bar{t} \\
&+ \int_0^\infty \int_{I\!R^m} \int_{I\!R^m} \phi(x,t,\bar{x},0)\, \eta(u(x,t);\bar{u}_0(\bar{x}))\, dx d\bar{x} dt \ge 0.
\end{aligned}
$$

We fix a smooth nonnegative function $\rho$ on $I\!R$ with compact support and total mass one:

$$
(6.2.15) \qquad \int_{-\infty}^\infty \rho(\xi) d\xi = 1.
$$

Consider any nonnegative Lipschitz test function $\psi$ on $I\!R^m \times [0,\infty)$, with compact support. For positive small $\varepsilon$, write (6.2.14) with

$$\phi(x,t,\bar{x},\bar{t}) = \varepsilon^{-(m+1)}\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\rho(\frac{t-\bar{t}}{2\varepsilon})\prod_{\beta=1}^{m}\rho(\frac{x_\beta-\bar{x}_\beta}{2\varepsilon}) \tag{6.2.16}$$

and then let $\varepsilon \downarrow 0$. Noting that

(6.2.17)

$$(\partial_t + \partial_{\bar{t}})\phi(x,t,\bar{x},\bar{t}) = \varepsilon^{-(m+1)}\partial_t\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\rho(\frac{t-\bar{t}}{2\varepsilon})\prod_{\beta=1}^{m}\rho(\frac{x_\beta-\bar{x}_\beta}{2\varepsilon}),$$

(6.2.18)

$$\left(\partial_{x_\alpha} + \partial_{\bar{x}_\alpha}\right)\phi(x,t,\bar{x},\bar{t}) = \varepsilon^{-(m+1)}\partial_\alpha\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\rho(\frac{t-\bar{t}}{2\varepsilon})\prod_{\beta=1}^{m}\rho(\frac{x_\beta-\bar{x}_\beta}{2\varepsilon}),$$

$$|\eta(u(x,t);\bar{u}_0(\bar{x})) - \eta(u_0(x);\bar{u}_0(\bar{x}))| \leq |u(x,t)-u_0(x)|, \tag{6.2.19}$$

$$|\eta(u_0(x);\bar{u}(\bar{x},\bar{t})) - \eta(u_0(x);\bar{u}_0(\bar{x}))| \leq |\bar{u}(\bar{x},\bar{t})-\bar{u}_0(\bar{x})|, \tag{6.2.20}$$

recalling Theorem 4.5.1, and using standard convergence theorems, we conclude that

(6.2.21)

$$\int_0^\infty\int_{I\!R^m}\{\partial_t\psi(x,t)\,\eta(u(x,t);\bar{u}(x,t)) + \sum_{\alpha=1}^{m}\partial_\alpha\psi(x,t)\;Q_\alpha(u(x,t);\bar{u}(x,t))\}dxdt$$

$$+\int_{I\!R^m}\psi(x,0)\,\eta(u_0(x);\bar{u}_0(x))\,dx \geq 0.$$

From (6.2.5) it is clear that there is $s > 0$ such that

$$|Q(u;\bar{u})| \leq s\eta(u;\bar{u}), \tag{6.2.22}$$

for all $u$ and $\bar{u}$ in the range of the solutions.

Fix $r > 0$, $t \geq 0$, and $\varepsilon > 0$ small; write (6.2.21) for the test function $\psi(x,\tau) = \chi(x,\tau)\omega(\tau)$, with $\chi$ and $\omega$ defined by (5.3.12) and (5.3.11) to get

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<r}[u(x,\tau)-\bar{u}(x,\tau)]^+dxd\tau \leq \int_{|x|<r+st}[u_0(x)-\bar{u}_0(x)]^+dx \tag{6.2.23}$$

$$-\frac{1}{\varepsilon}\int_0^t\int_{r+s(t-\tau)<x<r+s(t-\tau)+\varepsilon}\left[s\eta(u;\bar{u}) + \frac{Q(u;\bar{u})x}{|x|}\right]dxd\tau + O(\varepsilon).$$

On account of (6.2.22), the second integral on the right-hand side of (6.2.23) is nonnegative. Thus, letting $\varepsilon \downarrow 0$, recalling Theorem 4.5.1, and using that $(u-\bar{u})^+$ is a convex function of $u-\bar{u}$, we arrive at (6.2.8).

Interchanging the roles of $u$ and $\bar{u}$ in (6.2.8) we deduce a similar inequality which added to (6.2.8) yields (6.2.9).

Clearly, (6.2.10) implies (6.2.11), by virtue of (6.2.8). In particular, applying this monotonicity property, first for $\bar{u}_0(x) \equiv b$ and then for $u_0(x) \equiv a$, we deduce

$u(x,t) \le b$ and $\bar{u}(x,t) \ge a$ a.e. Interchanging the roles of $u$ and $\bar{u}$, we conclude that the essential range of both solutions is contained in $[a,b]$. Thus $s$ in (6.2.22) depends solely on $[a,b]$. This completes the proof.

From (6.2.9) we immediately draw the following conclusion on uniqueness and finite dependence:

**6.2.4 Corollary.** *There is at most one admissible weak solution of* (6.1.1), (6.1.2).

**6.2.5 Corollary.** *The value of the admissible weak solution at any point $(\bar{x},\bar{t})$ depends solely on the restriction of the initial data to the ball $\{\mathcal{B}_{s\bar{t}}(\bar{x})\}$.*

Another important consequence of (6.2.9) is that any admissible weak solution of (6.1.1) with initial data of locally bounded variation is itself a function of locally bounded variation:

**6.2.6 Theorem.** *Let $u$ be an admissible weak solution of* (6.1.1) *with initial data $u_0 \in BV_{\mathrm{loc}}(I\!R^m)$ taking values in an interval $[a,b]$. Then $u \in BV_{\mathrm{loc}}(I\!R^m \times (0,\infty))$. For any fixed $t > 0$, $u(\cdot,t)$ is in $BV_{\mathrm{loc}}(I\!R^m)$ and*

$$TV_{\mathcal{B}_r} u(\cdot,t) \le TV_{\mathcal{B}_{r+st}} u_0(\cdot), \tag{6.2.24}$$

*for every $r > 0$, where $s$ depends solely on $[a,b]$.*

**Proof.** Let $\{E_\alpha,\ \alpha = 1,\cdots,m\}$ denote the standard orthonormal basis of $I\!R^m$. Note that, for $\alpha = 1,\cdots,m$, the function $\bar{u}$, defined by $\bar{u}(x,t) = u(x+hE_\alpha,t)$, $h > 0$, is an admissible weak solution of (6.1.1) with initial data $\bar{u}_0$, $\bar{u}_0(x) = u_0(x+hE_\alpha)$. Therefore, by virtue of (6.2.9), for any $t \in (0,T)$,

$$\int_{|x|<r} |u(x+hE_\alpha,t) - u(x,t)|dx \le \int_{|x|<r+st} |u_0(x+hE_\alpha) - u_0(x)|dx. \tag{6.2.25}$$

Since $u_0 \in BV_{\mathrm{loc}}(I\!R^m)$, Theorem 1.7.2 and (1.7.3) yield that $u(\cdot,t) \in BV_{\mathrm{loc}}(I\!R^m)$ and (6.2.24) holds.

Thus $\partial_\alpha u(\cdot,t)$ is a Radon measure which is bounded on any ball of radius $r$ in $I\!R^m$, uniformly on compact time intervals. Since $u$ is bounded, it follows from Theorem 1.7.5 that $\operatorname{div} G(u(\cdot,t))$ has the same property. In particular, the distributions $\partial_\alpha u$ and $\operatorname{div} G(u)$ are locally finite measures on $I\!R^m \times (0,\infty)$. Because (6.1.1) is satisfied in the sense of distributions, $\partial_t u$ will also be a measure on $I\!R^m \times (0,\infty)$. Consequently, $u \in BV_{\mathrm{loc}}(I\!R^m \times (0,\infty))$. This completes the proof.

The trivial, constant, solutions of (6.1.1) are stable, not only in $L^1$ but also in any $L^p$. Since $u$ may be renormalized, it suffices to establish $L^p$-stability for the zero solution.

**6.2.7 Theorem.** *Let $u$ be an admissible weak solution of* (6.1.1), (6.1.2), *with initial data taking values in a compact interval $[a,b]$. There is $s > 0$, depending solely on $[a,b]$, such that, for any $1 \le p \le \infty$, $t \ge 0$, and $r > 0$,*

$$\|u(\cdot,t)\|_{L^p(\mathcal{B}_r)} \le \|u_0(\cdot)\|_{L^p(\mathcal{B}_{r+st})} \,. \tag{6.2.26}$$

**Proof.** For $1 \le p < \infty$, consider the convex entropy $\eta(u) = |u|^p$, with entropy flux $Q$ determined through (6.2.1). Note that there is $s > 0$, independent of $p$, such that

$$|Q(u)| \le s\eta(u), \qquad u \in [a,b]. \tag{6.2.27}$$

Fix $r > 0$, $t \ge 0$, and $\varepsilon > 0$ small; write (6.2.3) for the above entropy-entropy flux pair and the test function $\psi(x,\tau) = \chi(x,\tau)\omega(\tau)$, with $\chi$ and $\omega$ defined by (5.3.12) and (5.3.11). This yields

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<r} |u(x,\tau)|^p dx d\tau \le \int_{|x|<r+st} |u_0(x)|^p dx \tag{6.2.28}$$

$$-\frac{1}{\varepsilon}\int_0^t \int_{r+s(t-\tau)<|x|<r+s(t-\tau)+\varepsilon}\left[ s\eta(u) + \frac{Q(u)x}{|x|}\right] dx d\tau + O(\varepsilon).$$

We know that the range of $u$ is contained in $[a,b]$ and so, by (6.2.27), the second integral on the right-hand side of (6.2.28) is nonnegative. Thus, letting $\varepsilon \downarrow 0$ and using that $|u|^p$ is convex, we arrive at (6.2.26). This completes the proof.

The following sections will present various methods of constructing admissible weak solutions of (6.1.1), (6.1.2), inducing alternative proofs of Theorem 6.1.1.

## 6.3 The Method of Vanishing Viscosity

The aim here is to construct admissible weak solutions of the scalar hyperbolic conservation law (6.1.1) as the $\mu \downarrow 0$ limit of solutions of the family of parabolic equations

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t)) = \mu \Delta u(x,t), \qquad x \in I\!R^m\,,\ t \in [0,\infty), \tag{6.3.1}$$

where $\Delta$ stands for Laplace's operator with respect to the spatial variables, namely $\Delta = \sum_{\alpha=1}^m \partial_\alpha^2$ , and $\mu$ is a positive parameter.

The motivation for this approach has already been presented in Section 4.6. Note that (6.3.1) is not necessarily related to any specific physical model and so the term $\mu\Delta u$ should be regarded as “artificial viscosity”.

Because (6.3.1) is parabolic, the initial value problem (6.3.1), (6.1.2) always has a unique solution, which is smooth for $t > 0$ (assuming $G$ is regular) even when the initial data $u_0$ are merely in $L^\infty$. For example, if the derivative $G'$ is Hölder continuous, then the solution $u$ of (6.3.1), (6.1.2) is continuously differentiable with respect to $t$ and twice continuously differentiable with respect to the spatial variables, on $I\!R^m \times (0,\infty)$.

Espousing the premise that “relevant” solutions of (6.1.1), (6.1.2) are $\mu \downarrow 0$ limits of solutions of (6.3.1), (6.1.2) provides the first justification of the notion of admissible weak solution postulated by Definition 6.2.1:

**6.3.1 Theorem.** *Let $u_\mu$ denote the solution of* (6.3.1), (6.1.2). *Assume that for some sequence $\{\mu_k\}$, with $\mu_k \downarrow 0$ as $k \to \infty$, $\{u_{\mu_k}\}$ converges to some function $u$, boundedly almost everywhere on $I\!R^m \times [0, \infty)$. Then $u$ is an admissible weak solution of* (6.1.1), (6.1.2) *on $I\!R^m \times [0, \infty)$.*

**Proof.** Consider any smooth convex entropy function $\eta$, with associated entropy flux $Q$ determined through (6.2.1). Multiply (6.3.1) by $\eta'(u_\mu(x,t))$ and use (6.2.1) to get

$$\partial_t \eta(u_\mu) + \operatorname{div} Q(u_\mu) = \mu \Delta \eta(u_\mu) - \mu \eta''(u_\mu) |\nabla u_\mu|^2. \tag{6.3.2}$$

Multiply (6.3.2) by any smooth nonnegative test function $\psi$, with compact support in $I\!R^m \times [0, \infty)$, integrate over $I\!R^m \times [0, \infty)$, and integrate by parts. Taking into account that the last term in (6.3.2) is nonnegative yields the inequality

$$\int_0^\infty \int_{I\!R^m} [\partial_t \psi\, \eta(u_\mu) + \sum_{\alpha=1}^m \partial_\alpha \psi\, Q_\alpha(u_\mu)] dx dt + \int_{I\!R^m} \psi(x,0)\, \eta(u_0(x))\, dx \geq -\mu \int_0^\infty \int_{I\!R^m} \Delta \psi\, \eta(u_\mu)\, dx dt. \tag{6.3.3}$$

Setting $\mu = \mu_k$ in (6.3.3) and letting $k \to \infty$, we conclude that the limit $u$ of $\{u_{\mu_k}\}$ satisfies (6.2.3) for all smooth convex entropy functions $\eta$ and all smooth nonnegative test functions $\psi$. By completion we infer that (6.2.3) holds even when $\eta$ and $\psi$ are merely Lipschitz continuous. This completes the proof.

That (6.1.1) and (6.3.1) are perfectly matched becomes clear by comparing Theorem 6.2.3 with

**6.3.2 Theorem.** *Let $u_\mu$ and $\bar{u}_\mu$ be solutions of* (6.3.1) *with respective initial data $u_0$ and $\bar{u}_0$ that are in $L^1(I\!R^m)$ and take values in a compact interval $[a,b]$. Then, for any $t > 0$,*

$$\int_{I\!R^m} [u_\mu(x,t) - \bar{u}_\mu(x,t)]^+ dx \leq \int_{I\!R^m} [u_0(x) - \bar{u}_0(x)]^+ dx, \tag{6.3.4}$$

$$\|u_\mu(\cdot,t) - \bar{u}_\mu(\cdot,t)\|_{L^1(I\!R^m)} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(I\!R^m)}\,. \tag{6.3.5}$$

*Furthermore, if*

$$u_0(x) \leq \bar{u}_0(x), \qquad a.e. \text{ on } I\!R^m, \tag{6.3.6}$$

*then*

$$u_\mu(x,t) \leq \bar{u}_\mu(x,t), \qquad \text{on } I\!R^m \times (0, \infty). \tag{6.3.7}$$

*In particular, the range of both $u_\mu$ and $\bar{u}_\mu$ is contained in $[a,b]$.*

**Proof.** To simplify the notation, we drop the subscript $\mu$ and denote $u_\mu$, $\bar{u}_\mu$ by $u$, $\bar{u}$. From standard theory of parabolic equations it follows that when $u_0(\cdot)$, $\bar{u}_0(\cdot)$ are in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, then $u(\cdot,t)$, $\bar{u}(\cdot,t)$ and their spatial derivatives of any order are also in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, with norms uniformly bounded with respect to $t$ on compact subsets of $(0,\infty)$.

For $\varepsilon > 0$, we define the function $\eta_\varepsilon$ on $\mathbb{R}$ by

$$\eta_\varepsilon(w) = \begin{cases} 0 & \text{for } -\infty < w \le 0 \\ \dfrac{w^2}{4\varepsilon} & \text{for } 0 < w \le 2\varepsilon \\ w - \varepsilon & \text{for } 2\varepsilon < w < \infty. \end{cases} \tag{6.3.8}$$

Using that both $u$ and $\bar{u}$ satisfy (6.3.1), one easily verifies the equation

$$\begin{aligned} &\partial_t \eta_\varepsilon(u-\bar{u}) + \sum_{\alpha=1}^m \partial_\alpha\{\eta'_\varepsilon(u-\bar{u})[G_\alpha(u) - G_\alpha(\bar{u})]\} \\ &- \sum_{\alpha=1}^m \eta''_\varepsilon(u-\bar{u})[G_\alpha(u)-G_\alpha(\bar{u})]\partial_\alpha(u-\bar{u}) \\ &= \mu\Delta\eta_\varepsilon(u-\bar{u}) - \mu\eta''_\varepsilon(u-\bar{u})|\nabla(u-\bar{u})|^2. \end{aligned} \tag{6.3.9}$$

Fix $0 < s < t < \infty$ and integrate (6.3.9) over $\mathbb{R}^m \times (s,t)$. Considering that the last term on the right-hand side of (6.3.9) is nonnegative, we thus obtain the inequality

$$\begin{aligned} &\int_{\mathbb{R}^m} \eta_\varepsilon(u(x,t)-\bar{u}(x,t))dx - \int_{\mathbb{R}^m} \eta_\varepsilon(u(x,s)-\bar{u}(x,s))dx \\ &\le \sum_{\alpha=1}^m \int_s^t \int_{\mathbb{R}^m} \eta''_\varepsilon(u-\bar{u})\,[G_\alpha(u)-G_\alpha(\bar{u})]\,\partial_\alpha(u-\bar{u})\,dxd\tau. \end{aligned} \tag{6.3.10}$$

Notice that $\eta''_\varepsilon(u-\bar{u})[G_\alpha(u)-G_\alpha(\bar{u})]$ is bounded, uniformly for $\varepsilon > 0$. Also, it is clear that as $\varepsilon \downarrow 0$, $\eta_\varepsilon(u(x,t)-\bar{u}(x,t))$ converges pointwise to $[u(x,t)-\bar{u}(x,t)]^+$ while $\eta''_\varepsilon(u(x,t)-\bar{u}(x,t))[G_\alpha(u(x,t))-G_\alpha(\bar{u}(x,t))]$ converges pointwise to zero. Therefore, (6.3.10) and the Lebesgue dominated convergence theorem imply

$$\int_{\mathbb{R}^m} [u(x,t)-\bar{u}(x,t)]^+ dx - \int_{\mathbb{R}^m} [u(x,s)-\bar{u}(x,s)]^+ dx \le 0, \tag{6.3.11}$$

whence we deduce (6.3.4), by letting $s \downarrow 0$.

Interchanging the roles of $u$ and $\bar{u}$ in (6.3.4) we derive a similar inequality which added to (6.3.4) yields (6.3.5).

Clearly, (6.3.6) implies (6.3.7), by virtue of (6.3.4). In particular, applying this monotonicity property, first for $\bar{u}_0(x) \equiv b$ and then for $u_0(x) \equiv a$, we deduce

$u(x,t) \le b$ and $\bar{u}(x,t) \ge a$. Interchanging the roles of $u$ and $\bar{u}$, we conclude that the range of both solutions is contained in $[a,b]$. This completes the proof.

Estimate (6.3.5) may be employed to estimate the modulus of continuity in the mean of solutions of (6.3.1) with initial data in $L^\infty(I\!R^m) \cap L^1(I\!R^m)$.

**6.3.3 Lemma.** *Let $u_\mu$ be the solution of* (6.3.1), (6.1.2), *where $u_0$ is in $L^1(I\!R^m)$ and takes values in a compact interval* $[a,b]$. *In particular,*

$$\int_{I\!R^m} |u_0(x+y) - u_0(x)|dx \le \omega(|y|), \qquad y \in I\!R^m\,, \tag{6.3.12}$$

*for some nondecreasing function $\omega$ on* $[0,\infty)$, *with $\omega(r) \downarrow 0$ as $r \downarrow 0$. There is a constant c, depending solely on* $[a,b]$, *such that, for any $t > 0$,*

$$\int_{I\!R^m} |u_\mu(x+y,t) - u_\mu(x,t)|dx \le \omega(|y|), \qquad y \in I\!R^m\,, \tag{6.3.13}$$

(6.3.14)

$$\int_{I\!R^m} |u_\mu(x,t+h) - u_\mu(x,t)|dx \le c(h^{2/3} + \mu h^{1/3})\|u_0\|_{L^1(I\!R^m)} + 2\omega(h^{1/3}),\ \ h > 0.$$

**Proof.** Fix $t > 0$. For any $y \in I\!R^m$, the function $\bar{u}_\mu(x,t) = u_\mu(x+y,t)$ is the solution of (6.3.1) with initial data $\bar{u}_0(x) = u_0(x+y)$. Applying (6.3.5) yields

$$\int_{I\!R^m} |u_\mu(x+y,t) - u_\mu(x,t)|dx \le \int_{I\!R^m} |u_0(x+y) - u_0(x)|dx \tag{6.3.15}$$

whence (6.3.13) follows.

We now fix $h > 0$. We normalize $G$ by subtracting $G(0)$ so henceforth we may assume, without loss of generality, that $G(0) = 0$. We multiply (6.3.1) by a bounded smooth function $\phi$, defined on $I\!R^m$, and integrate the resulting equation over the strip $I\!R^m \times (t, t+h)$. Integration by parts yields

$$\int_{I\!R^m} \phi(x)[u_\mu(x,t+h) - u_\mu(x,t)]dx \tag{6.3.16}$$

$$= \int_t^{t+h} \int_{I\!R^m} \{\sum_{\alpha=1}^m \partial_\alpha \phi(x)\, G_\alpha(u_\mu(x,\tau)) + \mu \Delta\phi(x)\, u_\mu(x,\tau)\} dx d\tau.$$

Let us set

$$v(x) = u_\mu(x,t+h) - u_\mu(x,t). \tag{6.3.17}$$

One may establish (6.3.14) formally by inserting $\phi(x) = \operatorname{sgn} v(x)$ in (6.3.16). However, since the function sgn is discontinuous, we have to mollify it first, with the help of a smooth, nonnegative function $\rho$ on $I\!R$, with support contained in $[-m^{-1/2}, m^{-1/2}]$ and total mass one, (6.2.15):

(6.3.18)
$$\phi(x) = \int_{\mathbb{R}^m} h^{-m/3} \prod_{\beta=1}^{m} \rho(\frac{x_\beta - z_\beta}{h^{1/3}}) \operatorname{sgn} v(z)\, dz.$$

Notice that $|\partial_\alpha \phi| \leq c_1 h^{-1/3}$ and $|\Delta\phi| \leq c_2 h^{-2/3}$. Moreover, by virtue of (6.3.5), with $\bar{u} \equiv 0$, $\|u(\cdot,\tau)\|_{L^1(\mathbb{R}^m)} \leq \|u_0(\cdot)\|_{L^1(\mathbb{R}^m)}$. Therefore, (6.3.16) implies

(6.3.19)
$$\int_{\mathbb{R}^m} \phi(x)v(x)dx \leq c(h^{2/3} + \mu h^{1/3})\|u_0\|_{L^1(\mathbb{R}^m)},$$

where $c$ depends solely on $[a, b]$. On the other hand, observing that

(6.3.20)
$$|v(x)| - v(x) \operatorname{sgn} v(z) = |v(x)| - |v(z)| + [v(z) - v(x)] \operatorname{sgn} v(z) \leq 2|v(x) - v(z)|,$$

we obtain from (6.3.18):

(6.3.21)
$$|v(x)| - \phi(x)v(x) = \int_{\mathbb{R}^m} h^{-m/3} \prod_{\beta=1}^{m} \rho(\frac{x_\beta - z_\beta}{h^{1/3}})[|v(x)| - v(x) \operatorname{sgn} v(z)]dz$$
$$\leq 2\int_{|\xi|<1} \prod_{\beta=1}^{m} \rho(\xi_\beta)|v(x) - v(x - h^{1/3}\xi)|d\xi.$$

Combining (6.3.17), (6.3.21), (6.3.19), and (6.3.13), we arrive at (6.3.14). This completes the proof.

We have now laid the groundwork for presenting a

**Proof of Theorem 6.2.2.** Assume first that $u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. Let $u_\mu$ denote the solution of (6.3.1), (6.1.2). By Theorem 6.3.2 and Lemma 6.3.3, the family $\{u_\mu\}$ is uniformly bounded and equicontinuous in the mean on any compact subset of $\mathbb{R}^m \times (0, \infty)$. Consequently, any sequence $\{\mu_k\}$, with $\mu_k \downarrow 0$ as $k \to \infty$, will contain a subsequence, denoted again by $\{\mu_k\}$, such that $\{u_{\mu_k}\}$ converges in $L^1_{\text{loc}}$, as well as boundedly almost everywhere on $\mathbb{R}^m \times [0, \infty)$, to some function $u$. On account of Theorem 6.3.1, $u$ is an admissible weak solution of (6.1.1), (6.1.2). Since there may exist at most one such solution (cf. Corollary 6.2.4), we conclude that the whole family $\{u_\mu\}$ converges to $u$, as $\mu \downarrow 0$. Furthermore, by virtue of Lemma 6.3.3, for $h > 0$,

(6.3.22)
$$\int_{\mathbb{R}^m} |u(x, t + h) - u(x, t)|dx \leq ch^{2/3}\|u_0\|_{L^1(\mathbb{R}^m)} + 2\omega(h^{1/3}),$$

so $u(\cdot, t) \in C^0([0, \infty); L^1(\mathbb{R}^m))$.

Suppose now $u_0 \in L^\infty(\mathbb{R}^m)$. For $r > 0$, let $\chi_r$ denote the characteristic function of the ball $\mathcal{B}_r(0)$, and $u^r$ denote the admissible weak solution of (6.1.1), with initial data $\chi_r u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. As $r \to \infty$, $\chi_r u_0 \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Therefore, on account of (6.2.9), the family $\{u^r\}$ will converge in $L^1_{\mathrm{loc}}$ to some function $u$. Clearly, $u$ is an admissible weak solution of (6.1.1), (6.1.2). By Corollary 6.2.4, this solution is unique. Now, by Corollary 6.2.5, $u \equiv u^r$ on any compact subset of $\mathbb{R}^m \times [0, \infty)$, if $r$ is sufficiently large. Since $u^r(\cdot, t) \in C^0([0, \infty); L^1(\mathbb{R}^m))$, it follows that $u(\cdot, t) \in C^0([0, \infty); L^1_{\mathrm{loc}}(\mathbb{R}^m))$. This completes the proof.

## 6.4 Solutions as Trajectories of a Contraction Semigroup

For $t \in [0, \infty)$, consider the map $S(t)$ that carries $u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ to the admissible weak solution $u$ of (6.1.1), (6.1.2) restricted to $t$, i.e., $S(t)u_0(\cdot) = u(\cdot, t)$. By virtue of the properties of admissible weak solutions demonstrated in the previous two sections, $S(t)$ is well-defined as a map from $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ to $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ and

$$S(0) = I \quad \text{(the identity)}, \tag{6.4.1}$$

$$S(t + \tau) = S(t)S(\tau), \qquad \text{for any } t \text{ and } \tau \text{ in } [0, \infty), \tag{6.4.2}$$

$$S(\cdot)u_0 \in C^0([0, \infty); L^1(\mathbb{R}^m)), \tag{6.4.3}$$

$$\|S(t)u_0 - S(t)\bar{u}_0\|_{L^1(\mathbb{R}^m)} \le \|u_0 - \bar{u}_0\|_{L^1(\mathbb{R}^m)}, \quad \text{for any } t \text{ in } [0, \infty). \tag{6.4.4}$$

Consequently, $S(\cdot)$ is a $L^1$-contraction semigroup on $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$.

Naturally, the question arises whether one may construct $S(\cdot)$ *ab initio*, through the theory of nonlinear contraction semigroups in Banach space. This would provide a direct, independent proof of existence of admissible weak solutions of (6.1.1), (6.1.2) as well as an alternative derivation of their properties.

To construct the semigroup, we must realize (6.1.1) as an abstract differential equation

$$\frac{du}{dt} + A(u) \ni 0, \tag{6.4.5}$$

for a suitably defined nonlinear transformation $A$, with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in $L^1(\mathbb{R}^m)$. This operator may, in general, be multivalued, i.e., for each $u \in \mathcal{D}(A)$, $A(u)$ will be a nonempty subset of $L^1(\mathbb{R}^m)$ which may contain more than one point.

For $u$ smooth, one should expect $A(u) = \operatorname{div} G(u)$. However, the task of extending $\mathcal{D}(A)$ to $u$ that are not smooth is by no means straightforward, because the construction should somehow reflect the admissibility condition encoded in Definition 6.2.1. First we perform a preliminary extension. For convenience, we normalize $G$ so that $G(0) = 0$.

**6.4.1 Definition.** The (possibly multivalued) transformation $\hat{A}$, with domain $\mathcal{D}(\hat{A})$ in $L^1(\mathbb{R}^m)$, is defined by $u \in \mathcal{D}(\hat{A})$ and $w \in \hat{A}(u)$ if $u$, $w$ and $G(u)$ are all in $L^1(\mathbb{R}^m)$ and the inequality

$$\int_{\mathbb{R}^m} \{\sum_{\alpha=1}^{m} \partial_\alpha \psi(x)\, Q_\alpha(u(x)) + \psi(x)\, \eta'(u(x)) w(x)\} dx \geq 0 \tag{6.4.6}$$

holds for any convex entropy function $\eta$, such that $\eta'$ is bounded on $\mathbb{R}$, with associated entropy flux $Q$ determined through (6.2.1), and for all nonnegative Lipschitz continuous test functions $\psi$ on $\mathbb{R}^m$, with compact support.

Applying (6.4.6) for the entropy-entropy flux pairs $\pm u$, $\pm G(u)$, verifies that

$$\hat{A}(u) = \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(u) \tag{6.4.7}$$

holds, in the sense of distributions, for any $u \in \mathcal{D}(\hat{A})$. In particular, $\hat{A}$ is single-valued. Furthermore, the identity

$$\int_{\mathbb{R}^m} \{\sum_{\alpha=1}^{m} \partial_\alpha \psi\, Q_\alpha(u) + \psi\, \eta'(u) \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(u)\} dx = 0, \tag{6.4.8}$$

which is valid for any $u \in C_0^1(\mathbb{R}^m)$ and every entropy-entropy flux pair, implies that $C_0^1(\mathbb{R}^m) \subset \mathcal{D}(\hat{A})$. In particular, $\mathcal{D}(\hat{A})$ is dense in $L^1(\mathbb{R}^m)$. For $u \in C_0^1(\mathbb{R}^m)$, $\hat{A}(u)$ is given by (6.4.7). Thus $\hat{A}$ is indeed an extension of (6.4.7).

The reader may have already noticed the similarity between (6.4.6) and (6.2.3). Similar to (6.2.3), to verify (6.4.6) it would suffice to test it just for the entropies $\pm u$ and the family (6.2.5) or (6.2.6) of entropy-entropy flux pairs.

**6.4.2 Definition.** The (possibly multivalued) transformation $A$, with domain $\mathcal{D}(A)$ in $L^1(\mathbb{R}^m)$, is the graph closure of $\hat{A}$, i.e., $u \in \mathcal{D}(A)$ and $w \in A(u)$ if $(u, w)$ is the limit in $L^1(\mathbb{R}^m) \times L^1(\mathbb{R}^m)$ of a sequence $\{(u_k, w_k)\}$ such that $u_k \in \mathcal{D}(\hat{A})$ and $w_k \in \hat{A}(u_k)$.

The following propositions establish properties of $A$, implying that it is the generator of a contraction semigroup on $L^1(\mathbb{R}^m)$.

**6.4.3 Theorem.** *The transformation $A$ is accretive, that is if $u$ and $\bar{u}$ are in $\mathcal{D}(A)$, then*

$$\|(u + \lambda w) - (\bar{u} + \lambda \bar{w})\|_{L^1(\mathbb{R}^m)} \geq \|u - \bar{u}\|_{L^1(\mathbb{R}^m)}, \lambda > 0, w \in A(u), \bar{w} \in A(\bar{u}). \tag{6.4.9}$$

**Proof.** It is the property of accretiveness that renders the semigroup generated by $A$ contractive. Consequently, the proof of Theorem 6.4.3 bears close resemblance to the demonstration of the $L^1$-contraction estimate (6.2.9) in Theorem 6.2.3.

In view of Definition 6.4.2, it would suffice to show that the "smaller" transformation $\hat{A}$ is accretive. Accordingly, fix $u, \bar{u}$ in $\mathcal{D}(\hat{A})$ and let $w = \hat{A}(u)$, $\bar{w} = \hat{A}(\bar{u})$. Consider any nonnegative Lipschitz continuous function $\phi$ on $I\!R^m \times I\!R^m$, with compact support. Fix $\bar{x}$ in $I\!R^m$ and write (6.4.6) for the entropy-entropy flux pair $\eta(u; \bar{u}(\bar{x}))$, $Q(u; \bar{u}(\bar{x}))$ of the Kruzkov family (6.2.6) and the test function $\psi(x) = \phi(x, \bar{x})$ to obtain

$$\int_{I\!R^m} \text{sgn}[u(x) - \bar{u}(\bar{x})]\{\sum_{\alpha=1}^{m} \partial_{x_\alpha}\phi(x,\bar{x})\,[G_\alpha(u(x)) - G_\alpha(\bar{u}(\bar{x}))] + \phi(x,\bar{x})\,w(x)\}dx \geq 0. \tag{6.4.10}$$

We may interchange the roles of $u$ and $\bar{u}$ and derive the analog of (6.4.10), for any fixed $x$ in $I\!R^m$:

$$\int_{I\!R^m} \text{sgn}[\bar{u}(\bar{x}) - u(x)]\{\sum_{\alpha=1}^{m} \partial_{\bar{x}_\alpha}\phi(x,\bar{x})\,[G_\alpha(\bar{u}(\bar{x})) - G_\alpha(u(x))] + \phi(x,\bar{x})\,\bar{w}(\bar{x})\}d\bar{x} \geq 0. \tag{6.4.11}$$

Integrating over $I\!R^m$ (6.4.10), with respect to $\bar{x}$, and (6.4.11), with respect to $x$, and then adding the resulting inequalities yields

$$\int_{I\!R^m}\int_{I\!R^m} \text{sgn}[u(x)-\bar{u}(\bar{x})]\{\sum_{\alpha=1}^{m}(\partial_{x_\alpha}+\partial_{\bar{x}_\alpha})\phi(x,\bar{x})\,[G_\alpha(u(x))-G_\alpha(\bar{u}(\bar{x}))] + \phi(x,\bar{x})[w(x) - \bar{w}(\bar{x})]\}dxd\bar{x} \geq 0. \tag{6.4.12}$$

Fix a smooth nonnegative function $\rho$ on $I\!R$ with compact support and total mass one, (6.2.15). Take any nonnegative Lipschitz continuous test function $\psi$ on $I\!R^m$, with compact support. For positive small $\varepsilon$, write (6.4.12) with

$$\phi(x,\bar{x}) = \varepsilon^{-m}\psi(\frac{x+\bar{x}}{2})\prod_{\beta=1}^{m}\rho(\frac{x_\beta - \bar{x}_\beta}{2\varepsilon}), \tag{6.4.13}$$

and let $\varepsilon \downarrow 0$. Noting that

$$(\partial_{x_\alpha} + \partial_{\bar{x}_\alpha})\phi(x,\bar{x}) = \varepsilon^{-m}\partial_\alpha\psi(\frac{x+\bar{x}}{2})\prod_{\beta=1}^{m}\rho(\frac{x_\beta - \bar{x}_\beta}{2\varepsilon}), \tag{6.4.14}$$

$$\int_{I\!R^m} \sigma(x)\{\sum_{\alpha=1}^{m}\partial_\alpha\psi(x)\,[G_\alpha(u(x)) - G_\alpha(\bar{u}(x))] + \psi(x)[w(x) - \bar{w}(x)]\}dx \geq 0, \tag{6.4.15}$$

where $\sigma$ is some function such that

$$\sigma(x)\begin{cases} = 1 & \text{if } u(x) > \bar{u}(x) \\ \in [-1, 1] & \text{if } u(x) = \bar{u}(x) \\ = -1 & \text{if } u(x) < \bar{u}(x). \end{cases} \tag{6.4.16}$$

In particular, choosing $\psi$ with $\psi(x) = 1$ for $|x| < r$, $\psi(x) = 1 + r - |x|$ for $r \le |x| < r + 1$ and $\psi(x) = 0$ for $r + 1 \le |x| < \infty$, and letting $r \to \infty$, we obtain

$$\int_{I\!R^m} \sigma(x)[w(x) - \bar{w}(x)]dx \ge 0, \tag{6.4.17}$$

for some function $\sigma$ as in (6.4.16).

Take now any $\lambda > 0$ and use (6.4.17), (6.4.16) to conclude

(6.4.18)
$$\|(u + \lambda w) - (\bar{u} + \lambda \bar{w})\|_{L^1(I\!R^m)} \ge \int_{I\!R^m} \sigma(x)\{u(x) - \bar{u}(x) + \lambda[w(x) - \bar{w}(x)]\}dx$$
$$\ge \int_{I\!R^m} \sigma(x)[u(x) - \bar{u}(x)]dx = \|u - \bar{u}\|_{L^1(I\!R^m)}\,.$$

This completes the proof.

An immediate consequence (actually an alternative, equivalent restatement) of the assertion of Theorem 6.4.3 is

**6.4.4 Corollary.** *For any $\lambda > 0$, $(I + \lambda A)^{-1}$ is a well-defined, single-valued, $L^1$-contractive transformation, defined on the range $\mathcal{R}(I + \lambda A)$ of $I + \lambda A$.*

**6.4.5 Theorem.** *The transformation $A$ is maximal, that is*

$$\mathcal{R}(I + \lambda A) = L^1(I\!R^m), \qquad for\ any\ \lambda > 0. \tag{6.4.19}$$

**Proof.** By virtue of Definition 6.4.2 and Corollary 6.4.4, it will suffice to show that $\mathcal{R}(I + \lambda \hat{A})$ is dense in $L^1(I\!R^m)$; for instance that it contains $L^1(I\!R^m) \cap L^\infty(I\!R^m)$. We thus fix $f \in L^1(I\!R^m) \cap L^\infty(I\!R^m)$ and seek solutions $u \in \mathcal{D}(\hat{A})$ of the equation

$$u + \lambda \hat{A}(u) = f. \tag{6.4.20}$$

Recall that $\hat{A}(u)$ admits the representation (6.4.7), in the sense of distributions. Thus, solving (6.4.20) amounts to determining an admissible weak solution of a first order quasilinear partial differential equation, namely the stationary analog of (6.1.1).

Motivated by the method of vanishing viscosity, discussed in Section 6.3, we shall construct solutions to (6.4.20) as the $\mu \downarrow 0$ limit of solutions of the family of elliptic equations

$$u(x) + \lambda \operatorname{div} G(u(x)) - \mu \Delta u(x) = f(x), \qquad x \in I\!R^m. \tag{6.4.21}$$

For any fixed $\mu > 0$, (6.4.21) admits a solution in $H^2(I\!R^m)$. We have to show that, as $\mu \downarrow 0$, the family of solutions of (6.4.21) converges, boundedly almost everywhere, to some function $u$ which is the solution of (6.4.20). The proof will be partitioned into the following steps.

**6.4.6 Lemma.** *Let $u_\mu$ and $\bar{u}_\mu$ be solutions of* (6.4.21) *with respective right-hand sides $f$ and $\bar{f}$ that are in $L^1(I\!R^m)$ and take values in a compact interval* $[a, b]$. *Then*

$$\int_{I\!R^m} [u_\mu(x) - \bar{u}_\mu(x)]^+ dx \le \int_{I\!R^m} [f(x) - \bar{f}(x)]^+ dx, \tag{6.4.22}$$

$$\|u_\mu - \bar{u}_\mu\|_{L^1(I\!R^m)} \le \|f - \bar{f}\|_{L^1(I\!R^m)} . \tag{6.4.23}$$

*Furthermore, if*

$$f(x) \le \bar{f}(x), \qquad a.e. \text{ on } I\!R^m , \tag{6.4.24}$$

*then*

$$u_\mu(x) \le \bar{u}_\mu(x), \qquad on\ I\!R^m . \tag{6.4.25}$$

*In particular, the range of both $u$ and $\bar{u}$ is contained in* $[a, b]$.

**Proof.** It is very similar to the proof of Theorem 6.3.2 and so it will be left to the reader.

**6.4.7 Lemma.** *Let $u_\mu$ denote the solution of* (6.4.21), *with right-hand side $f$ in $L^\infty(I\!R^m) \cap L^1(I\!R^m)$. Then, as $\mu \downarrow 0$, $\{u_\mu\}$ converges boundedly a.e. to the solution $u$ of* (6.4.20).

**Proof.** For any $y \in I\!R^m$, the function $\bar{u}_\mu$, defined by $\bar{u}_\mu(x) = u_\mu(x + y)$, is a solution of (6.4.21) with right-hand side $\bar{f}$, $\bar{f}(x) = f(x + y)$. Hence, by (6.4.23),

$$\int_{I\!R^m} |u_\mu(x + y) - u_\mu(x)| dx \le \int_{I\!R^m} |f(x + y) - f(x)| dx. \tag{6.4.26}$$

Thus the family $\{u_\mu\}$ is uniformly bounded and uniformly equicontinuous in $L^1$. It follows that every sequence $\{\mu_k\}$, with $\mu_k \to 0$ as $k \to \infty$, will contain a subsequence, labeled again as $\{\mu_k\}$, such that

$$u_{\mu_k} \to u, \qquad \text{boundedly a.e. on } I\!R^m, \tag{6.4.27}$$

where $u$ is in $L^\infty(I\!R^m) \cap L^1(I\!R^m)$.

Consider now any smooth convex entropy function $\eta$, with associated entropy flux $Q$, determined by (6.2.1). Then $u_\mu$ will satisfy the identity

$$\eta'(u_\mu)u_\mu + \lambda \text{div}\, Q(u_\mu) - \mu\Delta\eta(u_\mu) + \mu\eta''(u_\mu)|\nabla u_\mu|^2 = \eta'(u_\mu)f. \tag{6.4.28}$$

Multiplying (6.4.28) by any nonnegative smooth test function $\psi$ on $\mathbb{R}^m$, with compact support, and integrating over $\mathbb{R}^m$ yields

$$\int_{\mathbb{R}^m} \{\lambda \sum_{\alpha=1}^{m} \partial_\alpha \psi \; Q_\alpha(u_\mu) + \psi \, \eta'(u_\mu)(f - u_\mu)\} dx \geq -\mu \int_{\mathbb{R}^m} \Delta\psi \, \eta \, dx. \tag{6.4.29}$$

From (6.4.27) and (6.4.29),

$$\int_{\mathbb{R}^m} \{\sum_{\alpha=1}^{m} \partial_\alpha \psi \; Q_\alpha(u) + \psi \, \eta'(u) \lambda^{-1}(f - u)\} dx \geq 0, \tag{6.4.30}$$

which shows that $u$ is indeed a solution of (6.4.20).

By virtue of Corollary 6.4.4, the solution of (6.4.20) is unique and so the entire family $\{u_\mu\}$ converges to $u$, as $\mu \downarrow 0$. This completes the proof.

Once accretiveness and maximality have been established, the Crandall-Liggett theory of semigroups in nonreflexive Banach space ensures that $A$ generates a contraction semigroup $S(\cdot)$ on $\overline{\mathcal{D}(A)} = L^1(\mathbb{R}^m)$. $S(\cdot)u_0$ can be constructed by solving the differential equation (6.4.5) through the implicit difference scheme

$$\begin{cases} \dfrac{1}{\varepsilon}[u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)] + A(u_\varepsilon(t)) \ni 0, & t > 0 \\ u_\varepsilon(t) = u_0\,, & t < 0. \end{cases} \tag{6.4.31}$$

For any $\varepsilon > 0$, a unique solution $u_\varepsilon$ of (6.4.31) exists on $[0, \infty)$, by virtue of Theorem 6.4.5 and Corollary 6.4.4. It can be shown, further, that Corollary 6.4.4 provides the necessary stability to ensure that, as $\varepsilon \downarrow 0$, $u_\varepsilon(\cdot)$ converges, uniformly on compact subsets of $[0, \infty)$, to some function that we denote by $S(\cdot)u_0$.

The general properties of $S(\cdot)$ follow from the Crandall-Liggett theory: When $u_0 \in \mathcal{D}(A)$, $S(t)u_0$ stays in $\mathcal{D}(A)$ for all $t \in [0, \infty)$. In general, $S(t)u_0$ may fail to be differentiable with respect to $t$, even when $u_0 \in \mathcal{D}(A)$. Thus $S(\cdot)u_0$ should be interpreted as a weak solution of the differential equation (6.4.5).

The special properties of $S(\cdot)$ are consequences of the special properties of $A$ induced by the propositions recorded above (e.g. Lemma 6.4.6). The following theorem, whose proof can be found in the references cited in Section 6.11, summarizes the properties of $S(\cdot)$ and in particular provides an alternative proof for the existence of a unique admissible weak solution to (6.1.1), (6.1.2) (Theorem 6.2.2) and its basic properties (Theorems 6.2.3 and 6.2.7).

**6.4.8 Theorem.** *The transformation $A$ generates a contraction semigroup $S(\cdot)$ in $L^1(\mathbb{R}^m)$, namely a family of maps $S(t) : L^1(\mathbb{R}^m) \to L^1(\mathbb{R}^m), t \in [0, \infty)$, which satisfy the semigroup property* (6.4.1), (6.4.2); *the continuity property* (6.4.3), *for any $u_0 \in L^1(\mathbb{R}^m)$; and the contraction property* (6.4.4), *for any $u_0, \bar{u}_0$ in $L^1(\mathbb{R}^m)$. If*

$$u_0 \leq \bar{u}_0\,, \qquad a.e. \text{ on } \mathbb{R}^m, \tag{6.4.32}$$

*then*

$$S(t)u_0 \leq S(t)\bar{u}_0\,, \qquad a.e.\ on\ \mathbb{R}^m. \tag{6.4.33}$$

*For* $1 \leq p \leq \infty$, *the sets* $L^p(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ *are positively invariant under* $S(t)$ *and, for any* $t \in [0, \infty)$,

$$\|S(t)u_0\|_{L^p(\mathbb{R}^m)} \leq \|u_0\|_{L^p(\mathbb{R}^m)}\,, \qquad for\ all\ u_0 \in L^p(\mathbb{R}^m) \cap L^1(\mathbb{R}^m). \tag{6.4.34}$$

*If* $u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$, *then* $S(\cdot)u_0$ *is the admissible weak solution of* (6.1.1), (6.1.2), *in the sense of Definition* 6.2.1.

The reader should note that the approach via semigroups suggests a notion of admissible weak solution to (6.1.1), (6.1.2) for any, even unbounded, $u_0$ in $L^1(\mathbb{R}^m)$. These are not necessarily distributional solutions of (6.1.1), unless the flux $G$ exhibits linear growth at infinity.

## 6.5 The Layering Method

The admissible weak solution of (6.1.1), (6.1.2) will here be determined as the $h \downarrow 0$ limit of a family $\{u_h\}$ of functions constructed by patching together classical solutions of (6.1.1) in a stratified pattern. In addition to providing another method for constructing solutions and thereby an alternative proof of the existence Theorem 6.2.2, this approach also offers a different justification of the admissibility condition, Definition 6.2.1.

The initial data $u_0$ are in $L^\infty(\mathbb{R}^m)$, taking values in a compact interval $[a, b]$. The construction of approximate solutions will involve mollification of functions on $\mathbb{R}^m$ by forming their convolution with a kernel $\lambda_h$ constructed as follows. We start out with a nonnegative, smooth function $\rho$ on $\mathbb{R}$, supported in $[-1, 1]$, which is even, $\rho(-\xi) = \rho(\xi)$ for $\xi \in \mathbb{R}$, and has total mass one, (6.2.15). For $h > 0$, we set

$$\lambda_h(x) = (ph)^{-m} \prod_{\beta=1}^{m} \rho(\frac{x_\beta}{ph}), \tag{6.5.1}$$

with

$$p = \sqrt{m}\, q\gamma \|u_0\|_{L^\infty(\mathbb{R}^m)}\,, \tag{6.5.2}$$

where $q$ denotes the total variation of the function $\rho$ and $\gamma$ is the maximum of $|G''(u)|$ over the interval $[a, b]$. We employ $\lambda_h$ to mollify functions $f \in L^\infty(\mathbb{R}^m)$:

$$(\lambda_h * f)(x) = \int_{\mathbb{R}^m} \lambda_h(x - y) f(y) dy, \qquad x \in \mathbb{R}^m. \tag{6.5.3}$$

From (6.5.3) and (6.5.1) it follows easily

$$\inf(\lambda_h * f) \geq \operatorname{ess\,inf} f, \qquad \sup(\lambda_h * f) \leq \operatorname{ess\,sup} f, \tag{6.5.4}$$

$$\|\lambda_h * f\|_{L^1(\mathcal{B}_r)} \leq \|f\|_{L^1(\mathcal{B}_{r+\sqrt{m}ph})}, \qquad \text{for any } r > 0, \tag{6.5.5}$$

$$\|\partial_\alpha(\lambda_h * f)\|_{L^\infty(I\!R^m)} \leq \frac{q}{ph}\|f\|_{L^\infty(I\!R^m)}, \qquad \alpha = 1, \cdots, m. \tag{6.5.6}$$

A somewhat subtler estimate, which depends crucially on $\lambda_h$ being an even function, and whose proof can be found in the references cited in Section 6.11, is

$$\left|\int_{I\!R^m} \chi(x)[(\lambda_h * f)(x) - f(x)]dx\right| \leq ch^2\|\chi\|_{C^2(I\!R^m)}\|f\|_{L^\infty(I\!R^m)}, \tag{6.5.7}$$

for all $\chi \in C_0^\infty(I\!R^m)$.

The construction of the approximate solutions proceeds as follows. After the parameter $h > 0$ has been fixed, $I\!R^m \times [0, \infty)$ is partitioned into layers:

$$I\!R^m \times [0, \infty) = \bigcup_{\ell=0}^{\infty} I\!R^m \times [\ell h, \ell h + h). \tag{6.5.8}$$

The initial value $u_h(\cdot, 0)$ is determined by

$$u_h(\cdot, 0) = \lambda_h * u_0(\cdot). \tag{6.5.9}$$

By virtue of (6.5.6) and (6.5.2), $u_h(\cdot, 0)$ is Lipschitz continuous, with Lipschitz constant $\omega = 1/p\gamma$. Hence, by Theorem 6.1.1, (6.1.1) with initial data $u_h(\cdot, 0)$ admits a classical solution $u_h$ on the layer $I\!R^m \times [0, h)$.

Next we determine $u_h(\cdot, h)$ by mollifying the limit $u_h(\cdot, h-)$ of $u_h(\cdot, t)$ as $t \uparrow h$:

$$u_h(\cdot, h) = \lambda_h * u_h(\cdot, h-). \tag{6.5.10}$$

We extend $u_h$ to the layer $I\!R^m \times [h, 2h)$ by solving (6.1.1) with data $u_h(\cdot, h)$ at $t = h$.

Continuing this process, we determine $u_h$ on the general layer $[\ell h, \ell h + h)$ by solving (6.1.1) with data

$$u_h(\cdot, \ell h) = \lambda_h * u_h(\cdot, \ell h-) \tag{6.5.11}$$

at $t = \ell h$. We thus end up with a measurable function $u_h$ on $I\!R^m \times [0, \infty)$ which takes values in the interval $[a, b]$. Inside each layer $I\!R^m \times [\ell h, \ell h + h)$, $u_h$ is a classical solution of (6.1.1). However, as one crosses the border $t = \ell h$ between adjacent layers, $u_h$ experiences jump discontinuities, from $u_h(\cdot, \ell h-)$ to $u_h(\cdot, \ell h)$.

**6.5.1 Theorem.** *As $h \downarrow 0$, the family $\{u_h\}$ constructed above converges boundedly almost everywhere on $I\!R^m \times [0, \infty)$ to the admissible solution $u$ of* (6.1.1), (6.1.2).

The proof is an immediate consequence of the following two propositions together with uniqueness of the admissible solution, Corollary 6.2.4. The fact that the limit of classical solutions yields the admissible weak solution provides another justification of Definition 6.2.1.

**6.5.2 Lemma.** (*Consistency*). *Assume that for some sequence $\{h_k\}$, with $h_k \to 0$ as $k \to \infty$,*

$$u_{h_k}(x,t) \to u(x,t), \qquad a.e. \text{ on } I\!R^m \times [0,\infty). \tag{6.5.12}$$

*Then $u$ is an admissible weak solution of* (6.1.1), (6.1.2).

**Proof.** Consider any convex entropy function $\eta$ with associated entropy flux $Q$ determined through (6.2.1). In the interior of each layer, $u_h$ is a classical solution of (6.1.1) and so it satisfies the identity

$$\partial_t \eta(u_h(x,t)) + \operatorname{div} Q(u_h(x,t)) = 0. \tag{6.5.13}$$

Fix any nonnegative smooth test function $\psi$ on $I\!R^m \times [0, T)$, with compact support. Multiply (6.5.13) by $\psi$, integrate over each layer, integrate by parts, and then sum the resulting equations over all layers to get

(6.5.14)

$$\int_0^\infty \int_{I\!R^m} [\partial_t \psi\, \eta(u_h) + \sum_{\alpha=1}^m \partial_\alpha \psi\, Q_\alpha(u_h)] dx dt + \int_{I\!R^m} \psi(x,0)\, \eta(u_h(x,0)) dx$$

$$= -\sum_{\ell=1}^\infty \int_{I\!R^m} \psi(x,\ell h)[\eta(u_h(x,\ell h)) - \eta(u_h(x,\ell h-))] dx.$$

Combining (6.5.11) with Jensen's inequality and using (6.5.7) yields

$$\int_{I\!R^m} \psi(x,\ell h)[\eta(u_h(x,\ell h)) - \eta(u_h(x,\ell h-))] dx \tag{6.5.15}$$

$$\leq \int_{I\!R^m} \psi(x,\ell h)[\lambda_h * \eta(u_h)(x,\ell h-)) - \eta(u_h(x,\ell h-))] dx \leq Ch^2.$$

The summation on the right-hand side of (6.5.14) contains $O(1/h)$ many nonzero terms. Therefore, passing to the $k \to \infty$ limit along the sequence $\{h_k\}$ in (6.5.14) and using (6.5.12), (6.5.9), and (6.5.15), we conclude that $u$ satisfies (6.2.3). This completes the proof.

**6.5.3 Lemma.** (*Compactness*). *There is a sequence* $\{h_k\}$, *with* $h_k \to 0$ *as* $k \to \infty$, *and a* $L^\infty$ *function* $u$ *on* $I\!R^m \times [0, \infty)$ *such that* (6.5.12) *holds.*

**Proof.** The first step is to establish the weaker assertion that for some sequence $\{h_k\}$, with $h_k \to 0$ as $k \to \infty$, and a function $u$,

$$u_{h_k}(\cdot, t) \to u(\cdot, t), \quad \text{as } k \to \infty, \quad \text{in } L^\infty(I\!R^m) \text{ weak}^*, \tag{6.5.16}$$

for almost all $t$ in $[0, \infty)$. To this end, fix any smooth test function $\chi$ on $I\!R^m$, with compact support, and consider the function

$$v_h(t) = \int_{I\!R^m} \chi(x) u_h(x, t) dx, \quad t \in [0, \infty). \tag{6.5.17}$$

Notice that $v_h$ is smooth on $[\ell h, \ell h + h)$ and satisfies

$$\begin{aligned} \int_{\ell h}^{\ell h+h} \left| \frac{d}{dt} v_h(t) \right| dt &= \int_{\ell h}^{\ell h+h} \left| - \int_{I\!R^m} \chi(x) \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(u(x, t)) dx \right| dt \\ &= \int_{\ell h}^{\ell h+h} \left| \int_{I\!R^m} \sum_{\alpha=1}^{m} \partial_\alpha \chi(x)\, G_\alpha(u(x, t)) dx \right| dt \le Ch. \end{aligned} \tag{6.5.18}$$

On the other hand, $v_h$ experiences jump discontinuities across the points $t = \ell h$ which can be estimated with the help of (6.5.11) and (6.5.7):

$$|v_h(\ell h) - v_h(\ell h-)| = \left| \int_{I\!R^m} \chi(x)[u_h(x, \ell h) - u_h(x, \ell h-)] dx \right| \le Ch^2. \tag{6.5.19}$$

From (6.5.18) and (6.5.19) it follows that the total variation of $v_h$ over any compact subinterval of $[0, \infty)$ is bounded, uniformly in $h$. Therefore, by Helly's theorem (cf. Section 1.7), there is a sequence $\{h_k\}$, $h_k \to 0$ as $k \to \infty$, such that $v_{h_k}(t)$ converges for almost all $t$ in $[0, \infty)$.

By Cantor's diagonal process, we may construct a subsequence of $\{h_k\}$, which will be denoted again by $\{h_k\}$, such that the sequence

$$\left\{ \int_{I\!R^m} \chi(x) u_{h_k}(x, t) dx \right\} \tag{6.5.20}$$

converges for almost all $t$, for every member $\chi$ of any given countable family of test functions. Consequently, the sequence (6.5.20) converges for any $\chi$ in $L^1(I\!R^m)$. Thus, for almost any $t$ in $[0, \infty)$ there is a bounded measurable function on $I\!R^m$, denoted by $u(\cdot, t)$, such that (6.5.16) holds.

We now strengthen the mode of convergence in (6.5.16). For any $y \in I\!R^m$, the functions $u_h$ and $\bar{u}_h$, $\bar{u}_h(x, t) = u_h(x + y, t)$ are both solutions of (6.1.1) in every layer. Let us fix $t > 0$ and $r > 0$. Suppose $t \in [\ell h, \ell h + h)$. Applying repeatedly (6.2.9) and (6.5.5) (recalling (6.5.11)), we conclude

(6.5.21)

$$\begin{aligned}\int_{|x|<r} |u_h(x+y,t)-u_h(x,t)|dx &\le \int_{|x|<r+s(t-\ell h)} |u_h(x+y,\ell h)-u_h(x,\ell h)|dx\\ &\le \int_{|x|<r+s(t-\ell h)+\sqrt{m}ph} |u_h(x+y,\ell h-)-u_h(x,\ell h-)|dx\\ &\le \dots \le \int_{|x|<r+st+\sqrt{m}p(t+h)} |u_0(x+y)-u_0(x)|dx.\end{aligned}$$

It follows that the family $\{u_h(\cdot,t)\}$ is equicontinuous in the mean on every compact subset of $I\!R^m$. Therefore, the convergence in (6.5.16) is upgraded to strongly in $L^1_{\mathrm{loc}}(I\!R^m)$. Thus, passing to a final subsequence we arrive at (6.5.12). This completes the proof.

## 6.6 Relaxation

Another interesting method for constructing admissible weak solutions of (6.1.1) is through *relaxation*. The point of departure is a semilinear system of $m+1$ equations,

(6.6.1)

$$\begin{cases} \partial_t v(x,t) + \sum_{\alpha=1}^m c_\alpha \partial_\alpha v(x,t) = \dfrac{1}{\mu}\sum_{\alpha=1}^m [F_\alpha(v(x,t)) - Z_\alpha(x,t)] \\ \partial_t Z_\alpha(x,t) - c_\alpha \partial_\alpha Z_\alpha(x,t) = \dfrac{1}{\mu}[F_\alpha(v(x,t)) - Z_\alpha(x,t)], \qquad \alpha = 1,\cdots,m, \end{cases}$$

in the $m+1$ unknowns $(v, Z_1,\cdots,Z_m)$, where $\mu$ is a small positive parameter while, for $\alpha = 1,\cdots,m$, the $c_\alpha$ are given constants and the $F_\alpha$ are specified smooth functions such that

(6.6.2) $$F'_\alpha(v) < 0, \qquad -\infty < v < \infty, \qquad \alpha = 1,\cdots,m,$$

(6.6.3) $$F_\alpha(0) = 0, \qquad F_\alpha(v) \to \pm\infty \text{ as } v \longrightarrow \mp\infty, \qquad \alpha = 1,\cdots,m.$$

Notice that solutions of (6.6.1) satisfy the conservation law

(6.6.4) $$\partial_t[v(x,t) - \sum_{\alpha=1}^m Z_\alpha(x,t)] + \sum_{\alpha=1}^m c_\alpha \partial_\alpha [v(x,t) + Z_\alpha(x,t)] = 0.$$

Because of the form of the right-hand side of (6.6.1), one should expect that, as $\mu \downarrow 0$, the variables $Z_\alpha$ "relax" to their equilibrium states $F_\alpha(v)$, in which case (6.6.4) reduces to a scalar conservation law (6.1.1) with[1]

[1] By virtue of (6.6.2), the transformation $(6.6.5)_1$ may be inverted to express $v$ as a smooth, increasing function of $u$, and it is in that sense that $G_\alpha$, defined by $(6.6.5)_2$, should be realized as a function of $u$.

$$u = v - \sum_{\alpha=1}^{m} F_\alpha(v), \qquad G_\alpha(u) = c_\alpha[v + F_\alpha(v)], \quad \alpha = 1, \cdots, m. \tag{6.6.5}$$

The above considerations suggest a program for constructing solutions of (6.1.1) as asymptotic limits of solutions of (6.6.1).

The first step is to examine the Cauchy problem for (6.6.1), under assigned initial conditions

$$v(x, 0) = v_0(x), \qquad Z_\alpha(x, 0) = Z_{\alpha 0}(x), \quad \alpha = 1, \cdots, m, \quad x \in I\!R^m. \tag{6.6.6}$$

Since (6.6.1) is semilinear hyperbolic, when the initial data $(v_0, Z_{10}, \cdots, Z_{m0})$ are in $C_0^1(I\!R^m)$ there exists a unique classical solution $(v, Z_1, \cdots, Z_m)$ defined on a maximal time interval $[0, T)$, for some $0 < T \leq \infty$. For any $t \in [0, T)$, the functions $(v(\cdot, t), Z_1(\cdot, t), \cdots, Z_m(\cdot, t))$ are in $C_0^1(I\!R^m)$. Furthermore, if $T < \infty$,

$$\|v(\cdot, t)\|_{L^\infty(I\!R^m)} + \sum_{\alpha=1}^{m} \|Z_\alpha(\cdot, t)\|_{L^\infty(I\!R^m)} \to \infty, \qquad \text{as } t \uparrow T. \tag{6.6.7}$$

Here we need (possibly weak) solutions, under a broader class of initial data, which exist globally in time. Such solutions do indeed exist because, under our assumptions (6.6.2), (6.6.3), the effect of the right-hand side in (6.6.1) is dissipative. This is manifested in the following

**6.6.1 Theorem.** *For any initial data* $(v_0, Z_{10}, \cdots, Z_{m0})$ *in* $L^1(I\!R^m) \cap L^\infty(I\!R^m)$, *there exists a unique weak solution* $(v, Z_1, \cdots, Z_m)$ *of* (6.6.1), (6.6.6) *on* $I\!R^m \times [0, \infty)$ *such that* $(v(\cdot, t), Z_1(\cdot, t), \cdots, Z_m(\cdot, t))$ *are in* $C^0([0, \infty); L^1(I\!R^m))$. *If*

$$a \leq v_0(x) \leq b, \quad F_\alpha(b) \leq Z_{\alpha 0}(x) \leq F_\alpha(a), \quad \alpha = 1, \cdots, m, \quad x \in I\!R^m, \tag{6.6.8}$$

*then*

(6.6.9)
$$a \leq v(x, t) \leq b, \; F_\alpha(b) \leq Z_\alpha(x, t) \leq F_\alpha(a), \; \alpha = 1, \cdots, m, \; (x, t) \in I\!R^m \times [0, \infty).$$

*Furthermore, if* $(\bar{v}, \bar{Z}_1, \cdots, \bar{Z}_m)$ *is another solution of (6.6.1), with initial data* $(\bar{v}_0, \bar{Z}_{10}, \cdots, \bar{Z}_{m0})$ *in* $L^1(I\!R^m) \cap L^\infty(I\!R^m)$, *then, for any* $t \in [0, \infty)$,

$$\int_{I\!R^m} \{[v(x, t) - \bar{v}(x, t)]^+ + \sum_{\alpha=1}^{m} [\bar{Z}_\alpha(x, t) - Z_\alpha(x, t)]^+\} dx \tag{6.6.10}$$

$$\leq \int_{I\!R^m} \{[v_0(x) - \bar{v}_0(x)]^+ + \sum_{\alpha=1}^{m} [\bar{Z}_{\alpha 0}(x) - Z_{\alpha 0}(x)]^+\} dx,$$

$$\|v(\cdot,t)-\bar{v}(\cdot,t)\|_{L^1(I\!R^m)}+\sum_{\alpha=1}^{m}\|Z_\alpha(\cdot,t)-\bar{Z}_\alpha(\cdot,t)\|_{L^1(I\!R^m)} \tag{6.6.11}$$
$$\leq \|v_0(\cdot)-\bar{v}_0(\cdot)\|_{L^1(I\!R^m)}+\sum_{\alpha=1}^{m}\|Z_{\alpha 0}(\cdot)-\bar{Z}_{\alpha 0}(\cdot)\|_{L^1(I\!R^m)}\,.$$

*In particular, if*

$$v_0(x)\leq \bar{v}_0(x),\quad Z_{\alpha 0}(x)\geq \bar{Z}_{\alpha 0}(x),\quad \alpha=1,\cdots,m,\quad x\in I\!R^m, \tag{6.6.12}$$

*then*

$$v(x,t)\leq \bar{v}(x,t),\quad Z_\alpha(x,t)\geq \bar{Z}_\alpha(x,t),\quad \alpha=1,\cdots,m,\quad (x,t)\in I\!R^m\times[0,\infty). \tag{6.6.13}$$

**Proof.** The first objective is to establish (6.6.10) under the assumption that both solutions $(v,Z_1,\cdots,Z_m)$ and $(\bar{v},\bar{Z}_1,\cdots,\bar{Z}_m)$ are classical, with initial data $(v_0,Z_{10},\cdots,Z_{m0})$ and $(\bar{v}_0,\bar{Z}_{10},\cdots,\bar{Z}_{m0})$ in $C_0^1(I\!R^m)$. For $\varepsilon>0$, we recall the function $\eta_\varepsilon$ defined through (6.3.8) and note that

$$\partial_t[\eta_\varepsilon(v-\bar{v})+\sum_{\alpha=1}^{m}\eta_\varepsilon(\bar{Z}_\alpha-Z_\alpha)]+\sum_{\alpha=1}^{m}c_\alpha\partial_\alpha[\eta_\varepsilon(v-\bar{v})-\eta_\varepsilon(\bar{Z}_\alpha-Z_\alpha)] \tag{6.6.14}$$
$$=\frac{1}{\mu}\sum_{\alpha=1}^{m}[\eta_\varepsilon'(v-\bar{v})-\eta_\varepsilon'(\bar{Z}_\alpha-Z_\alpha)][F_\alpha(v)-F_\alpha(\bar{v})+\bar{Z}_\alpha-Z_\alpha]$$

follows readily from (6.6.1). For fixed values of $v$, $\bar{v}$, $Z_\alpha$, $\bar{Z}_\alpha$, of any sign, the right-hand side of (6.6.14) has a nonpositive limit as $\varepsilon\downarrow 0$. Therefore, integrating (6.6.14) over $I\!R^m\times(0,t)$ and letting $\varepsilon\downarrow 0$ we arrive at (6.6.10).

When (6.6.12) holds, (6.6.10) immediately implies (6.6.13). Notice that, for any constants $a$ and $b$, $(a,F_1(a),\cdots,F_m(a))$ and $(b,F_1(b),\cdots,F_m(b))$ are particular solutions of (6.6.1) and hence (6.6.8) implies (6.6.9). In particular, blow-up (6.6.7) cannot occur for any $T$ and thus the solutions exist on $I\!R^m\times[0,\infty)$.

To get (6.6.11), it suffices to write (6.6.10) with the roles of $(v,Z_1,\cdots,Z_m)$ and $(\bar{v},\bar{Z}_1,\cdots,\bar{Z}_m)$ reversed and then add the resulting inequality to the original (6.6.10).

We have now verified all the assertions of the theorem, albeit within the context of classical solutions, with initial data in $C_0^1(I\!R^m)$. Nevertheless, by virtue of the $L^1$-contraction estimate (6.6.11), weak solutions of (6.6.1), with any initial data in $L^1(I\!R^m)\cap L^\infty(I\!R^m)$, satisfying the asserted properties, may readily be constructed as $L^1$ limits of sequences of classical solutions. This completes the proof.

Our next task is to investigate the limiting behavior of solutions of (6.6.1) as $\mu\downarrow 0$. The mechanism that induces the $Z_\alpha$ to relax to their equilibrium values $F_\alpha(v)$ will be captured through an entropy-like inequality. We define the family

$$\Phi_\alpha(Z_\alpha) = -\int_0^{Z_\alpha} F_\alpha^{-1}(w)dw, \qquad \alpha = 1, \cdots, m \tag{6.6.15}$$

of nonnegative, convex functions on $(-\infty, \infty)$. Assuming $(v, Z_1, \cdots, Z_m)$ is a classical solution of (6.6.1), with initial data $(v_0, Z_{10}, \cdots, Z_{m0})$ in $C_0^1(I\!R^m)$, we readily verify that

$$\partial_t\Big[\tfrac{1}{2}v^2 + \sum_{\alpha=1}^m \Phi_\alpha(Z_\alpha)\Big] + \sum_{\alpha=1}^m c_\alpha \partial_\alpha \Big[\tfrac{1}{2}v^2 - \Phi_\alpha(Z_\alpha)\Big] = \frac{1}{\mu}\sum_{\alpha=1}^m [v - F_\alpha^{-1}(Z_\alpha)][F_\alpha(v) - Z_\alpha]. \tag{6.6.16}$$

Since $v - F_\alpha^{-1}(Z_\alpha) = F_\alpha^{-1}(F_\alpha(v)) - F_\alpha^{-1}(Z_\alpha)$, the mean-value theorem implies

$$-[v - F_\alpha^{-1}(Z_\alpha)][F_\alpha(v) - Z_\alpha] \geq \frac{1}{k}[F_\alpha(v) - Z_\alpha]^2\,, \tag{6.6.17}$$

where $k$ is any upper bound of $-F'_\alpha$ over the range of $v$. Therefore, upon integrating (6.6.16) over $I\!R^m \times [0, \infty)$ we deduce the inequality

$$\int_0^\infty \int_{I\!R^m} \sum_{\alpha=1}^m [F_\alpha(v) - Z_\alpha]^2 dx dt \leq k\mu \int_{I\!R^m} \Big[\tfrac{1}{2}v_0^2 + \sum_{\alpha=1}^m \Phi_\alpha(Z_{\alpha_0})\Big]dx. \tag{6.6.18}$$

As explained in the proof of Theorem 6.6.1, weak solutions of (6.6.1) are constructed as $L^1$ limits of sequences of classical solutions, and hence the inequality (6.6.18) will hold even for weak solutions with initial data in $L^1(I\!R^m) \cap L^\infty(I\!R^m)$.

**6.6.2 Theorem.** *Let* $(v^\mu, Z_1^\mu, \cdots, Z_m^\mu)$ *denote the family of solutions of* (6.6.1), (6.6.6), *with parameter* $\mu > 0$, *and initial data* $(v_0, F_1(v_0), \cdots, F_m(v_0))$, *where* $v_0$ *is in* $L^1(I\!R^m) \cap L^\infty(I\!R^m)$. *Then there is a bounded measurable function* $v$ *on* $I\!R^m \times [0, \infty)$ *such that, as* $\mu \downarrow 0$,

$$v^\mu(x,t) \longrightarrow v(x,t), \qquad Z_\alpha^\mu(x,t) \longrightarrow F_\alpha(v(x,t)), \qquad \alpha = 1, \cdots, m, \tag{6.6.19}$$

*almost everywhere on* $I\!R^m \times [0, \infty)$. *The function*

$$u(x,t) = v(x,t) - \sum_{\alpha=1}^m F_\alpha(v(x,t)) \tag{6.6.20}$$

*is the admissible weak solution of the conservation law* (6.1.1), *with flux functions* $G_\alpha$ *defined through* (6.6.5), *and initial data*

$$u_0(x) = v_0(x) - \sum_{\alpha=1}^m F_\alpha(v_0(x)), \qquad x \in I\!R^m. \tag{6.6.21}$$

**Proof.** Let us set, for $(x, t) \in I\!R^m \times [0, \infty)$,

$$u^\mu(x,t) = v^\mu(x,t) - \sum_{\alpha=1}^{m} Z_\alpha^\mu(x,t), \tag{6.6.22}$$

$$G_\alpha^\mu(x,t) = c_\alpha[v^\mu(x,t) + Z_\alpha^\mu(x,t)]. \tag{6.6.23}$$

By virtue of (6.6.4),

$$\partial_t u^\mu(x,t) + \operatorname{div} G^\mu(x,t) = 0. \tag{6.6.24}$$

First we show that there is a bounded measurable function $u$ on $I\!R^m \times [0, \infty)$ and some sequence $\{\mu_n\}$, with $\mu_n \downarrow 0$ as $n \to \infty$, such that

$$u^{\mu_n}(\cdot, t) \longrightarrow u(\cdot, t), \qquad n \to \infty, \tag{6.6.25}$$

in $L^\infty(I\!R^m)$ weak*, for all $t \in [0, \infty)$. To that end, let us fix any test function $\chi \in C_0^\infty(I\!R^m)$ and define the family of functions

$$w^\mu(t) = \int_{I\!R^m} \chi(x) u^\mu(x,t) dx, \qquad t \in [0, \infty), \tag{6.6.26}$$

which, by account of (6.6.24), are continuously differentiable with derivative

$$\frac{d}{dt} w^\mu(t) = \sum_{\alpha=1}^{m} \int_{I\!R^m} \partial_\alpha \chi(x)\, G_\alpha^\mu(x,t) \tag{6.6.27}$$

bounded, uniformly in $\mu > 0$. It then follows from Arzela's theorem that there is a sequence $\{\mu_n\}$, with $\mu_n \downarrow 0$ as $n \to \infty$, such that $\{w^{\mu_n}\}$ converges for all $t \in [0, \infty)$. By Cantor's diagonal process we may construct a subsequence of $\{\mu_n\}$, denoted again by $\{\mu_n\}$, such that the sequence

$$\left\{ \int_{I\!R^m} \chi(x) u^{\mu_n}(x,t) dx \right\} \tag{6.6.28}$$

is convergent for all $t \in [0, \infty)$ and every member $\chi$ of any given countable family of test functions. Consequently, (6.6.28) is convergent for any $\chi \in L^1(I\!R^m)$. Thus, for each $t \in [0, \infty)$ there is a bounded measurable function on $I\!R^m$, denoted by $u(\cdot, t)$, such that (6.6.25) holds in $L^\infty(I\!R^m)$ weak*. Next we note that, by the $L^1$ contraction estimate (6.6.11), for any fixed $t$ in $[0, \infty)$ the family of functions $(v^\mu(\cdot, t), Z_1^\mu(\cdot, t), \cdots, Z_m^\mu(\cdot, t))$ is equicontinuous in the mean. Hence, the convergence in (6.6.25) is upgraded to strongly in $L^1(I\!R^m)$. In particular,

$$u^{\mu_n}(x,t) \longrightarrow u(x,t), \qquad n \to \infty, \tag{6.6.29}$$

almost everywhere on $I\!R^m \times [0, \infty)$.

We now apply (6.6.18) for our solutions $(v^{\mu_n}, Z_1^{\mu_n}, \cdots, Z_m^{\mu_n})$ and, passing if necessary to a subsequence, denoted again by $\{\mu_n\}$, we obtain

$$(6.6.30)\qquad F_\alpha(v^{\mu_n}(x,t)) - Z_\alpha^{\mu_n}(x,t) \to 0, \quad n\to\infty, \quad \alpha=1,\cdots,m,$$

almost everywhere on $I\!R^m \times [0,\infty)$.

Combining (6.6.22), (6.6.29) and (6.6.30), we deduce

$$(6.6.31)\qquad v^{\mu_n}(x,t) - \sum_{\alpha=1}^m F_\alpha(v^{\mu_n}(x,t)) \to u(x,t), \quad n\to\infty,$$

almost everywhere on $I\!R^m\times[0,\infty)$. Because of the monotonicity assumption (6.6.2), (6.6.31) implies that the sequence $\{v^{\mu_n}\}$ itself must be convergent, say

$$(6.6.32)\qquad v^{\mu_n}(x,t) \longrightarrow v(x,t), \quad n\to\infty,$$

almost everywhere on $I\!R^m \times [0,\infty)$, where $v$ is a function related to $u$ through (6.6.20). Furthermore, (6.6.30) and (6.6.32) together imply

$$(6.6.33)\qquad Z_\alpha^{\mu_n}(x,t) \to F_\alpha(v(x,t)), \quad n\to\infty, \quad \alpha=1,\cdots,m,$$

almost everywhere on $I\!R^m \times [0,\infty)$.

By virtue of (6.6.22), (6.6.23), (6.6.24), (6.6.32) and (6.6.33), $u$ is a weak solution of (6.1.1), with fluxes $G_\alpha$ defined through (6.6.5). We proceed to show that this solution is admissible. We fix any constant $\bar{v}$ and write (6.6.14) for the two solutions $(v^{\mu_n}, Z_1^{\mu_n},\cdots,Z_m^{\mu_n})$ and $(\bar{v}, F_1(\bar{v}),\cdots,F_m(\bar{v}))$. We apply this (distributional) equation to any nonnegative Lipschitz continuous test function $\psi$, with compact support on $I\!R^m \times [0,\infty)$ and let $\varepsilon\downarrow 0$. Since the $\varepsilon\downarrow 0$ limit of the right-hand side of (6.6.14) is nonpositive, this calculation gives

$$(6.6.34)\qquad \begin{aligned} &\int_0^\infty\int_{I\!R^m} \partial_t\psi\,[(v^{\mu_n}-\bar{v})^+ + \sum_{\alpha=1}^m (F_\alpha(\bar{v}) - Z_\alpha^{\mu_n})^+]\,dxdt \\ &+\int_0^\infty\int_{I\!R^m}\sum_{\alpha=1}^m c_\alpha\partial_\alpha\psi\,[(v^{\mu_n}-\bar{v})^+ - (F_\alpha(\bar{v}) - Z_\alpha^{\mu_n})^+]\,dxdt \\ &+\int_{I\!R^m}\psi(x,0)[(v_0-\bar{v})^+ + \sum_{\alpha=1}^m (F_\alpha(\bar{v}) - F_\alpha(v_0))^+]dx \geq 0. \end{aligned}$$

Letting $n\to\infty$ and using (6.6.32) and (6.6.33), (6.6.34) yields

$$(6.6.35)\qquad \begin{aligned} &\int_0^\infty\int_{I\!R^m} \partial_t\psi\,[(v-\bar{v})^+ + \sum_{\alpha=1}^m (F_\alpha(\bar{v}) - F_\alpha(v))^+]\,dxdt \\ &+\int_0^\infty\int_{I\!R^m}\sum_{\alpha=1}^m c_\alpha\partial_\alpha\psi\,[(v-\bar{v})^+ - (F_\alpha(\bar{v}) - F_\alpha(v))^+]\,dxdt \\ &+\int_{I\!R^m}\psi(x,0)[(v_0-\bar{v})^+ + \sum_{\alpha=1}^m (F_\alpha(\bar{v}) - F_\alpha(v_0))^+]dx \geq 0. \end{aligned}$$

On account of (6.6.2), $v - \bar{v}$ and $F_\alpha(\bar{v}) - F_\alpha(v)$ have the same sign. Furthermore, if we set $\bar{u} = \bar{v} - \sum F_\alpha(\bar{v})$, then $v - \bar{v}$ and $u - \bar{u}$ also have the same sign. Therefore, upon using (6.6.20), (6.6.21), and (6.6.5), we may rewrite (6.6.35) as

$$\int_0^\infty \int_{I\!R^m} [\partial_t \psi\, \eta(u;\bar{u}) + \sum_{\alpha=1}^m \partial_\alpha \psi\, Q_\alpha(u;\bar{u})] dxdt \tag{6.6.36}$$

$$+ \int_{I\!R^m} \psi(x,0)\eta(u_0;\bar{u})dx \geq 0,$$

where $(\eta(u;\bar{u}), Q(u;\bar{u}))$ is the entropy-entropy flux pair defined by (6.2.5). As noted in Section 6.2, the set of entropy-entropy flux pairs (6.2.5), with $\bar{u}$ arbitrary, is "complete" and hence (6.6.36) implies that (6.2.3) will hold for any entropy-entropy flux pair $(\eta, Q)$ with $\eta$ convex. This verifies that $u$ is the admissible weak solution of (6.1.1), with initial data $u_0$ given by (6.6.21). Since $u$ is unique, the convergence in (6.6.29), (6.6.32) and (6.6.33) applies not only along the particular sequence $\{\mu_n\}$ but also along the whole family $\{\mu\}$, as $\mu \downarrow 0$. This completes the proof.

Theorem 6.6.2 demonstrates how, starting out from a given system (6.6.1), one may construct, by relaxation, admissible solutions of a particular scalar conservation law induced by (6.6.1). Of course, we are interested in the reverse process, namely to determine the appropriate system (6.6.1) whose relaxed form is a given scalar conservation law (6.1.1). This may be accomplished when, given the fluxes $G_\alpha(u)$, it is possible to select constants $c_\alpha$ in such a way that the transformations (6.6.5) determine implicitly functions $F_\alpha(v)$ that satisfy the assumptions (6.6.2) and (6.6.3). Let us normalize the given fluxes by $G_\alpha(0) = 0, \alpha = 1, \cdots, m$. Since our solutions will be a priori bounded, let us assume, without loss of generality, that the $G'_\alpha(u)$ are uniformly bounded on $(-\infty, \infty)$. From (6.6.5),

$$(m+1)v = u + \sum_{\alpha=1}^m \frac{1}{c_\alpha} G_\alpha(u). \tag{6.6.37}$$

Therefore, the first constraint is to fix the $|c_\alpha|$ so large that

$$(m+1)\frac{dv}{du} = 1 + \sum_{\alpha=1}^m \frac{1}{c_\alpha} G'_\alpha(u) \geq \frac{1}{2}\,, \tag{6.6.38}$$

in order to secure that the map $v \mapsto u$ will possess a smooth inverse. Next we note

$$F'_\alpha(v) = -1 + \frac{1}{c_\alpha} G'_\alpha(u)\frac{du}{dv} = -1 + \frac{m+1}{c_\alpha}[1 + \sum_{\beta=1}^m \frac{1}{c_\beta} G'_\beta(u)]^{-1} G'_\alpha(u), \tag{6.6.39}$$

so that, by selecting the $|c_\alpha|$ sufficiently large, we can satisfy both assumptions (6.6.2) and (6.6.3). Restrictions on $c_\alpha$ that maintain that the convective characteristic speeds $c_\alpha$ should be high relative to the characteristic speeds $G'_\alpha$ of the relaxed conservation law are called *subcharacteristic conditions*.

## 6.7 A Kinetic Formulation

This section discusses an alternative, albeit equivalent, characterization of admissible weak solutions to (6.1.1), (6.1.2), which, as we shall see below, is motivated by the kinetic theory.

It has already been noted that the entropy production for any solution of (6.1.1) satisfying (6.2.2) is a nonpositive measure. In particular, if $u$ is an admissible solution of (6.1.1), (6.1.2) in the sense of Definition 6.2.1, then for any $v \in (-\infty, \infty)$,

$$\partial_t\{|u-v|-|v|\}+\mathrm{div}\{\mathrm{sgn}\,(u-v)\,[G(u)-G(v)]-\mathrm{sgn}\,v\;G(v)\} = -2\nu_v\,, \tag{6.7.1}$$

where $\nu_v$ is a nonnegative measure on $I\!R^m \times I\!R^+$. Notice that for $|v| > \sup|u_0| = \sup|u|$ it is $-2\nu_v = \partial_t u + \mathrm{div}\, G(u) = 0$.

We realize $\{\nu_v\}$ as a nonnegative measure $\nu$ on $I\!R \times I\!R^m \times I\!R^+$ and differentiate (6.7.1), in the sense of distributions, with respect to $v$ to deduce

$$\partial_t \chi(v; u) + \sum_{\alpha=1}^{m} G'_\alpha(v)\partial_\alpha \chi(v; u) = \partial_v \nu, \tag{6.7.2}$$

where $\chi$ denotes the function

$$\chi(v; u) = \begin{cases} 1 & \text{if } 0 < v < u \\ -1 & \text{if } u < v < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6.7.3}$$

The entropy production by any entropy-entropy flux pair $(\eta, Q)$ is easily expressed in terms of $\nu$. Indeed, let us multiply (6.7.2) by $\eta'(v)$ and integrate with respect to $v$ over $(-\infty, \infty)$. Recalling (6.2.1) and after an integration by parts, we obtain

$$\begin{aligned} \partial_t \int_{-\infty}^{\infty} \eta'(v)\chi(v; u)dv + \mathrm{div} \int_{-\infty}^{\infty} Q'(v)\chi(v; u)dv \\ = -\int_{-\infty}^{\infty} \eta''(v)d\nu(v; \cdot, \cdot). \end{aligned} \tag{6.7.4}$$

One easily verifies that if $p(v)$ is any $C^1$ function, then

$$\int_{-\infty}^{\infty} p'(v)\chi(v; u)dv = p(u) - p(0), \tag{6.7.5}$$

and so (6.7.4) yields

$$\partial_t \eta(u) + \mathrm{div}\, Q(u) = -\int_{-\infty}^{\infty} \eta''(v)d\nu(v; \cdot, \cdot). \tag{6.7.6}$$

In particular, when $\eta(u)$ is convex the right-hand side of (6.7.6) is nonpositive. Furthermore, applying (6.7.6) for $\eta(u) = \frac{1}{2}u^2$ and integrating with respect to $(x, t)$ over $I\!R^m \times [0, \infty)$ we deduce

$$(6.7.7)\qquad \int_0^\infty \int_{I\!R^m} \int_{-\infty}^\infty d\nu(v; x, t) \leq \frac{1}{2} \int_{I\!R^m} u_0^2(x)dx.$$

It is remarkable that (6.7.2) fully characterizes admissible weak solutions of (6.1.1), as shown in the following

**6.7.1 Theorem.** *A bounded measurable function $u(x, t)$ on $I\!R^m \times [0, \infty)$ which satisfies* (6.7.2), *for some bounded, nonnegative measure $\nu$, together with the initial condition*

$$(6.7.8)\qquad \chi(v; u(x, 0)) = \chi(v; u_0(x)),$$

*is the admissible solution of* (6.1.1), (6.1.2).

**Proof.** Equation (6.7.2) admits solutions $\chi(\cdot\,; u(\cdot, t)) \in C^0([0, \infty); I\!R^{m+1})$ and thus the initial condition (6.1.2) is attained strongly in $L^1(I\!R^m)$. Hence it remains to show that (6.2.2) holds for every entropy-entropy flux pair $(\eta, Q)$ with $\eta$ convex. Since $u$ is bounded, it will suffice to establish (6.2.2) for entropies with linear growth, i.e., with $|\eta'(u)|$ bounded on $(-\infty, \infty)$.

Starting out from (6.7.2), one can show, as above, that (6.7.6) holds, albeit only for functions $\eta(v)$ whose derivative $\eta'(v)$ vanishes for $|v|$ large (in order to perform the integration by parts, as it is no longer known that $\nu$ vanishes for $|v| > \sup |u_0|$).

Fix any convex function $\eta$, with linear growth. For $k = 1, 2, \cdots$, set $\eta_k(v) = \eta(v)\phi(v/k)$, where $\phi$ is a smooth even function on $(-\infty, \infty)$, with $\phi(v) = 1$ for $|v| \leq 1$, $\phi(v) = 0$ for $|v| \geq 2$, and $\phi'(v) < 0$ for $v \in (1, 2)$. We thus have

$$(6.7.9)\qquad \partial_t \eta_k(u) + \operatorname{div} Q_k(u) = -\int_{-\infty}^\infty \eta_k''(v) d\nu(v; \cdot, \cdot)$$

$$= -\int_{-\infty}^\infty \left[\eta''(v)\phi\Big(\frac{v}{k}\Big) + \frac{2}{k}\eta'(v)\phi'\Big(\frac{v}{k}\Big) + \frac{1}{k^2}\eta(v)\phi''\Big(\frac{v}{k}\Big)\right] d\nu(v; \cdot, \cdot).$$

For $k$ large, $\eta_k(u) = \eta(u)$ and $Q_k(u) = Q(u)$, on the range of the solution. Furthermore, $\eta''(v)\phi(v/k) \to \eta''(v)$ monotonically, as $k \to \infty$. Finally, it is clear that $\eta'(v)\phi'(v/k) = O(1)$ and $\eta(v)\phi''(v/k) = O(k)$, as $k \to \infty$. Thus, letting $k \to \infty$ in (6.7.9), we arrive at (6.7.6), and thereby at (6.2.2). This completes the proof.

The kinetic formulation (6.7.2), which may serve as an alternative, albeit equivalent, definition of admissible weak solutions of (6.1.1), provides a powerful instrument for discovering properties of these solutions. In particular, one obtains an alternative, direct proof of the $L^1$ contraction property (6.2.9), even under the more general assumption that the initial data are merely in $L^1(I\!R^m)$ and not necessarily in $L^\infty(I\!R^m)$; see references in Section 6.11.

Up to this point, we have been facing nonlinearity as an agent that provokes the development of discontinuities in solutions with smooth initial values. It turns out, however, that nonlinearity may also play the opposite role, of smoothing out solutions with rough initial data. In the course of the book, we shall encounter various manifestations of such behavior. The kinetic formulation provides valuable insight into the compactifying and smoothing effects of nonlinearity in scalar conservation laws. This will become evident in the next Section 6.8, but it is also seen in the following regularity theorem whose (hard and technical) proof is found in the references cited in Section 6.11.

**6.7.2 Theorem.** *Assume there are* $r \in (0, 1]$ *and* $C \geq 0$ *such that*

$$\text{meas}\{v : |v| \leq \|u_0\|_{L^\infty}, \ \ |p + G'(v)P| \leq \delta\} \leq C\delta^r, \tag{6.7.10}$$

*for all* $\delta \in (0, 1)$, $p \in I\!R$, $P \in I\!R^m$ *with* $p^2 + |P|^2 = 1$. *Then the admissible weak solution* $u$ *of* (6.1.1), (6.1.2) *satisfies*

$$u(\cdot, t) \in C^0((0, \infty); W^{s,1}_{\text{loc}}(I\!R^m)), \tag{6.7.11}$$

*for any* $s \in \left(0, \frac{r}{r+2}\right)$.

It is condition (6.7.10) that encodes the aspect of nonlinearity of $G$ responsible for the regularizing effect. For example, (6.7.10) fails, for any $r$, when $G$ is linear, but it is satisfied, with $r = 1$, if the $G_\alpha$ are uniformly convex functions, $G''_\alpha(u) > 0$, $\alpha = 1, \cdots, m$.

The section closes with a discussion on how the kinetic formulation (6.7.2) of the scalar conservation law may be motivated by the kinetic theory of matter. As we saw in Chapter III, Example 3.3.7, in the classical kinetic theory of gases the state of the gas at the point $x$ and time $t$ is described by the molecular density function $f(\xi, x, t)$ of the molecular velocity $\xi$. The evolution of $f$ is governed by the Boltzmann equation (3.3.25), which monitors the changes in the distribution of molecular velocities due to transport and collisions. The connection between the kinetic and the continuum approach is established by identifying intensive quantities, such as density, velocity, pressure, temperature, etc., with appropriate moments of the molecular density function $f$, and then showing that these fields satisfy the balance laws of continuum physics. Thus, in principle one may construct solutions to systems of balance laws by treating the fields as moments of a molecular density in an underlying kinetic model with density function whose zero moment satisfies the scalar conservation law (6.1.1).

In the model, the "velocity" $v$ is scalar-valued and the "molecular density" $f(v; x, t)$, at the point $x$ and time $t$, is allowed to take positive and negative values. Then $u$ is obtained from $f$ by

$$u(x, t) = \int_{-\infty}^{\infty} f(v; x, t)dv. \tag{6.7.12}$$

In turn, $f$ satisfies the transport equation

$$\text{(6.7.13)}\quad \partial_t f(v;x,t) + \sum_{\alpha=1}^{m} G'_\alpha(v)\partial_\alpha f(v;x,t) = \frac{1}{\mu}\left[\chi(v;u(x,t)) - f(v;x,t)\right],$$

where $\mu$ is a small positive parameter and $\chi(v;u)$ is the function defined by (6.7.3). Readers familiar with the kinetic theory will recognize in (6.7.13) a model of the BGK approximation to the classical Boltzmann equation. Hopefully, as $\mu \downarrow 0$, the stiff term on the right-hand side will force $f(v;x,t)$ to "relax" to $\chi(v;u(x,t))$ which satisfies (6.7.2). Before verifying that this expectation will be fulfilled, let us discuss properties of solutions of (6.7.13), (6.7.12).

**6.7.3 Theorem.** *Let $u_0 \in L^\infty(I\!R^m)\bigcap L^1(I\!R^m)$. For any $\mu > 0$, there exist bounded measurable functions*

$$\text{(6.7.14)}\quad f_\mu(\cdot;\cdot,t) \in C^0([0,\infty);L^1(I\!R\times I\!R^m)),\ u_\mu(\cdot,t)\in C^0([0,\infty);L^1(I\!R^m))$$

*which provide the unique solution of* (6.7.13), (6.7.12) *with initial data induced by*

$$\text{(6.7.15)}\qquad f_\mu(v;x,0) = \chi(v;u_0(x)),\quad v\in(-\infty,\infty),\ x\in I\!R^m.$$

*For any $(x,t)\in I\!R^m\times[0,\infty)$,*

$$\text{(6.7.16)}\qquad f(v;x,t)\in\begin{cases}[0,1] & \text{if } v>0\\ [-1,0] & \text{if } v<0.\end{cases}$$

*If $(\bar f_\mu,\bar u_\mu)$ is another solution of* (6.7.13), (6.7.12), *with initial data induced by $\bar u_0$* in $L^\infty(I\!R^m)\bigcap L^1(I\!R^m)$, *then, for any $t>0$,*

$$\text{(6.7.17)}\quad \|f_\mu(\cdot;\cdot,t)-\bar f_\mu(\cdot;\cdot,t)\|_{L^1(I\!R\times I\!R^m)} \le \|f_\mu(\cdot;\cdot,0)-\bar f_\mu(\cdot;\cdot,0)\|_{L^1(I\!R\times I\!R^m)}$$

$$\text{(6.7.18)}\qquad \|u_\mu(\cdot,t)-\bar u_\mu(\cdot,t)\|_{L^1(I\!R^m)} \le \|u_0(\cdot)-\bar u_0(\cdot)\|_{L^1(I\!R^m)}.$$

*Furthermore, if*

$$\text{(6.7.19)}\qquad u_0(x)\le\bar u_0(x),\qquad x\in I\!R^m,$$

*then*

$$\text{(6.7.20)}\qquad f_\mu(v;x,t)\le\bar f_\mu(v;x,t),\quad v\in(-\infty,\infty),\ x\in I\!R^m,\ t\in[0,\infty),$$

$$\text{(6.7.21)}\qquad u_\mu(x,t)\le\bar u_\mu(x,t),\qquad x\in I\!R^m,\ t\in[0,\infty).$$

**Proof.** Taking, for the time being, the existence of $(f_\mu, u_\mu)$ and $(\bar{f}_\mu, \bar{u}_\mu)$ for granted, we integrate (6.7.13) along characteristics $dx/dt = G'(v)^\top$ , $dv/dt = 0$ to deduce

$$f_\mu(v; x, t) = e^{-\frac{t}{\mu}} f_\mu(v; x - tG'(v)^\top, 0) + \frac{1}{\mu}\int_0^t e^{-\frac{t-\tau}{\mu}} \chi(v; u_\mu(x - (t-\tau)G'(v)^\top, \tau) d\tau. \tag{6.7.22}$$

Thus (6.7.16) readily follows from (6.7.22), (6.7.15) and the properties of the function $\chi$.

We write the analog of (6.7.22) for the other solution $(\bar{f}_\mu, \bar{u}_\mu)$ and subtract the resulting equation from (6.7.22) to get

(6.7.23)

$$f_\mu(v; x, t) - \bar{f}_\mu(v; x, t) = e^{-\frac{t}{\mu}} \Big[ f_\mu(v; x - tG'(v)^\top, 0) - \bar{f}_\mu(v; x - tG'(v)^\top, 0) \Big] + \frac{1}{\mu}\int_0^t e^{-\frac{t-\tau}{\mu}} [\chi(v; u_\mu(x - (t-\tau)G'(v)^\top, \tau) - \chi(v; \bar{u}_\mu(x - (t-\tau)G'(v)^\top, \tau)] d\tau$$

whence

(6.7.24)

$$\begin{aligned} &\| f_\mu(\cdot\,;\cdot, t) - \bar{f}_\mu(\cdot\,;\cdot, t)\|_{L^1(I\!R \times I\!R^m)} \le e^{-\frac{t}{\mu}} \| f_\mu(\cdot\,;\cdot, 0) - \bar{f}_\mu(\cdot\,;\cdot, 0)\|_{L^1(I\!R \times I\!R^m)} \\ &+ \frac{1}{\mu}\int_0^t e^{-\frac{t-\tau}{\mu}} \|\chi(v; u_\mu(x - (t-\tau)G'(v)^\top, \tau) - \chi(v; \bar{u}_\mu(x - (t-\tau)G'(v)^\top, \tau)\|_{L^1(I\!R \times I\!R^m)} d\tau \\ &\le e^{-\frac{t}{\mu}} \| f_\mu(\cdot\,;\cdot, 0) - \bar{f}_\mu(\cdot\,;\cdot, 0)\|_{L^1(I\!R \times I\!R^m)} \\ &\quad + (1 - e^{-\frac{t}{\mu}}) \max_{0 \le \tau \le t} \| f_\mu(\cdot\,;\cdot, \tau) - \bar{f}_\mu(\cdot\,;\cdot, \tau)\|_{L^1(I\!R \times I\!R^m)}. \end{aligned}$$

Clearly, (6.7.24) implies (6.7.17) and this in turn yields (6.7.18). In particular, there is at most one solution to (6.7.13), (6.7.12), (6.7.15). Furthermore, this solution can be constructed from the integral equation (6.7.22) by Picard iteration.

Since $\chi(v; u)$ is increasing in $u$, (6.7.23) and (6.7.12) guarantee that (6.7.19) implies (6.7.20) and (6.7.21). This completes the proof.

We now turn to the limiting behavior of solutions as $\mu \downarrow 0$.

**6.7.4 Theorem.** *For* $\mu > 0$, *let* $(f_\mu, u_\mu)$ *denote the solution of* (6.7.13), (6.7.12), (6.7.15) *with* $u_0 \in L^\infty(I\!R^m) \bigcap L^1(I\!R^m)$. *Then, as* $\mu \downarrow 0$,

$$u_\mu(x, t) \to u(x, t), \tag{6.7.25}$$

(6.7.26) $$f_\mu(v; x, t) \to \chi(v; u(x, t)),$$

*in* $L^1_{\mathrm{loc}}$ , *where* $\chi(v; u)$ *satisfies* (6.7.2) *for some bounded, nonnegative measure* $\nu$, *and hence* $u$ *is the admissible weak solution of* (6.1.1.), (6.1.2).

**Proof.** The first step is to demonstrate that the family $\{(f_\mu, u_\mu)\colon \mu > 0\}$ is equicontinuous in the mean. This is clearly the case in the $v$ and $x$ directions by virtue of the contraction property (6.7.17), (6.7.18). For any $w \in I\!R$ and $y \in I\!R^m$, the functions $(\bar{f}_\mu, \bar{u}_\mu)$ defined by $\bar{f}_\mu(v; x, t) = f_\mu(v + w; x + y, t)$, $\bar{u}_\mu(x, t) = u_\mu(x + y, t)$ are solutions of (6.7.13), (6.7.12) with initial data $\bar{f}_\mu(v; x, 0) = \chi(v + w; u_0(x + y))$, and so

(6.7.27) $$\int_0^\infty \int_{I\!R^m} |f_\mu(v + w; x + y, t) - f_\mu(v; x, t)| dx dv$$
$$\leq \int_0^\infty \int_{I\!R^m} |\chi(v + w; u_0(x + y)) - \chi(v; u_0(x))| dx dv,$$

(6.7.28) $$\int_{I\!R^m} |u_\mu(x + y, t) - u_\mu(x, t)| dx \leq \int_{I\!R^m} |u_0(x + y) - u_0(x)| dx.$$

Equicontinuity in the $t$-direction easily follows from the above, in conjunction with the transport equation (6.7.13) itself; the details are omitted.

Next we consider the function

(6.7.29) $$\omega_\mu(v; x, t) = \int_{-\infty}^{v} \big[\chi(w; u_\mu(x, t)) - f_\mu(w; x, t)\big] dw.$$

Let us fix $(x, t)$, assuming for definiteness $u_\mu(x, t) > 0$ (the other cases being similarly treated). Clearly, $\omega_\mu(-\infty; x, t) = 0$. By virtue of (6.7.3) and (6.7.16), $\omega_\mu(\cdot\,; x, t)$ is nondecreasing on the interval $(-\infty, u_\mu(x, t))$ and nonincreasing on the interval $(u_\mu(x, t), \infty)$. Finally, by account of (6.7.12), $\omega_\mu(\infty; x, t) = 0$. Consequently, we may write

(6.7.30) $$\frac{1}{\mu}\big[\chi(v; u_\mu(x, t)) - f_\mu(v; x, t)\big] = \partial_v \nu_\mu\ ,$$

where $\nu_\mu$ is a nonnegative measure which is bounded, uniformly in $\mu > 0$.

It follows that from any sequence $\{\mu_k\}$, $\mu_k \to 0$ as $k \to \infty$, we may extract a subsequence, denoted again by $\{\mu_k\}$, so that $\{(f_{\mu_k}\ ,\ u_{\mu_k})\}$ converges in $L^1_{\mathrm{loc}}$ to functions $(f, u)$, and $\{\nu_{\mu_k}\}$ converges weakly in the space of measures to a bounded non-negative measure $\nu$. Clearly, $f(v; x, t) = \chi(v; u(x, t))$ and (6.7.2) holds. By uniqueness, the whole family $\{(f_\mu, u_\mu)\}$ converges to $(\chi(\cdot\,; u), u)$, as $\mu \downarrow 0$. This completes the proof.

## 6.8 Fine Structure of $L^\infty$ Solutions

According to Theorem 6.2.6, admissible solutions $u$ to the scalar conservation law (6.1.1), with initial values $u_0$ of locally bounded variation on $\mathbb{R}^m$, have locally bounded variation on the upper half-space, and thereby inherit the fine structure of $BV$ functions described in Sections 1.7 and 1.8. In particular, the points of approximate jump discontinuity of $u$ assemble on the (at most) countable union of $C^1$ manifolds of codimension one. Furthermore, $u$ has (generally distinct) traces on both sides of any oriented manifold of codimension one. However, when $u_0$ is merely in $L^\infty$ the above structure is generally lost, as may be seen by considering the case where (6.1.1) is linear. On the other hand, we saw in Section 6.7 (Theorem 6.7.2) that nonlinearity in the flux function may exert a smoothing influence on $L^\infty$ solutions. As another manifestation of this phenomenon, we shall see here that when the conservation law is linearly nondegenerate, in a sense to be made precise below, admissible solutions that are merely in $L^\infty$ are nevertheless endowed with fine structure that closely resembles the structure of $BV$ functions.

For the present purposes, the distinction between spatial and temporal variables is irrelevant, so it will be convenient to revert to the formulation and notations of Chapter I, by fusing the $m$-dimensional space and 1-dimensional time into $k$-dimensional space-time, $k = m + 1$, and representing $(x, t)$ by the vector $X$, with $X_\alpha = x_\alpha$, $\alpha = 1, \cdots, m$ and $X_k = t$. In what follows, div will denote the divergence operator in $\mathbb{R}^k$, acting on $k$-row vectors.

On some open subset $\mathscr{X}$ of $\mathbb{R}^k$, we consider scalar balance laws in the form

$$\operatorname{div} G(u(X)) = \nu_G \,, \tag{6.8.1}$$

where $\nu_G$ is a locally bounded Radon measure. A function $u \in L^\infty(\mathscr{X})$ will be called an *admissible solution* of (6.8.1) if, for any companion $Q$ of $G$,

$$\operatorname{div} Q(u(X)) = \nu_Q \,, \tag{6.8.2}$$

where $\nu_Q$ is a locally bounded Radon measure on $\mathscr{X}$.

We recall, from Section 1.5, that companions $Q$ are related to $G$ by

$$Q'(u) = \eta'(u)G'(u), \tag{6.8.3}$$

where $\eta$ is some scalar-valued function.

In the setting of Section 6.2, $G_k(u) = u$, $Q_k(u) = \eta(u)$ and $\nu_G = 0$. For companions $Q$ induced by convex entropies $\eta$, admissible solutions of (6.1.1), in the sense of Definition 6.2.1, satisfy (6.8.2) with $\nu_Q$ a nonpositive, locally bounded measure. Thus, admissible solutions in the sense of Definition 6.2.1 are in particular admissible solutions in the above sense.

In order to expunge linear systems, we introduce the following notion (compare with (6.7.10)):

**6.8.1 Definition.** The balance law (6.8.1) is called *linearly nondegenerate* if for each $N \in S^{k-1}$

$$G'(u)N \neq 0, \quad \text{for almost all } u \in (-\infty, \infty). \tag{6.8.4}$$

The fine structure of admissible solutions of linearly nondegenerate scalar balance laws is described by the following

**6.8.2 Theorem.** *Assume* (6.8.1) *is linearly nondegenerate and let $u$ be an admissible solution on $\mathcal{X}$. Then $\mathcal{X}$ is the union of three pairwise disjoint subsets $\mathcal{C}$, $\mathcal{J}$ and $\mathcal{I}$ with the following properties:*

(a) *$\mathcal{C}$ is the set of points of vanishing mean oscillation of $u$, i.e., for $\bar{X} \in \mathcal{C}$*

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int_{\mathcal{B}_r(\bar{X})} |u(X) - \bar{u}_r(\bar{X})| dX = 0, \tag{6.8.5}$$

*where $\bar{u}_r(\bar{X})$ denotes the average of $u$ on the ball $\mathcal{B}_r(\bar{X})$.*

(b) *$\mathcal{J}$ is rectifiable, namely it is essentially covered by the countable union of $C^1$ $(k-1)$-dimensional manifolds $\{\mathcal{F}_i\}$ embedded in $I\!R^k$: $\mathcal{H}^{k-1}(\mathcal{J} \backslash \bigcup \mathcal{F}_i) = 0$. When $\bar{X} \in \mathcal{J} \bigcap \mathcal{F}_i$ , then the normal on $\mathcal{F}_i$ at $\bar{X}$ is interpreted as the normal on $\mathcal{J}$ at $\bar{X}$. The function $u$ has distinct inward and outward traces $u_-$ and $u_+$ , in the sense of Definition* 1.7.7, *at any point $\bar{X} \in \mathcal{J}$.*

(c) *The $(k-1)$-dimensional Hausdorff measure of $\mathcal{I}$ is zero: $\mathcal{H}^{k-1}(\mathcal{I}) = 0$.*

A comparison between Theorems 1.7.4 and 6.8.2 reveals the striking similarity in the fine structure of admissible $L^\infty$ solutions and $BV$ functions. The reader should note, however, that there are some differences as well: Points in the set $\mathcal{C}$ have merely vanishing mean oscillation in admissible $L^\infty$ solutions, whereas they are Lebesgue points in the $BV$ case. Furthermore, if $u$ is a $BV$ solution of (6.8.1), with $\nu_G = 0$, then, on account of Theorem 1.8.2, for any companion $Q$, $\nu_Q$ is concentrated on the set $\mathcal{J}$ of points of jump discontinuity. However, it is not known at the present time whether this important property carries over to $L^\infty$ admissible solutions.

The reader should consult the references in Section 6.11 for the proof of Theorem 6.8.2, which is lengthy and technical. Even so, a brief outline of some of the key ingredients is here in order.

Admissible $L^\infty$ solutions $u$ to (6.8.1) on $\mathcal{X}$ may be characterized by the kinetic formulation, discussed in Section 6.7. In the present setting, (6.7.2) takes the form

$$\sum_{\alpha=1}^{k} G'_\alpha(v) \partial_\alpha \chi(v; u) = \partial_v \nu, \tag{6.8.6}$$

where $\chi$ is the function defined by (6.7.3) and $\nu$ is a bounded measure on $I\!R \times \mathcal{X}$. Notice that here, in contrast to Section 6.7, the measure $\nu$ need not be nonnegative, as the notion of admissible solution adopted in this section is broader.

In analogy to (6.7.6), the measure $\nu_Q$ associated with any companion $Q$ induced by some $\eta$ through (6.8.3) is related to the measure $\nu$ by

$$\nu_Q = -\int_{-\infty}^{\infty} \eta''(v) d\nu(v; \cdot). \tag{6.8.7}$$

The measure $\nu$ also determines the "jump set" $\mathcal{J}$, in Theorem 6.8.2, by

$$\mathcal{J} = \{X \in \mathcal{X} : \limsup_{r\downarrow 0} \frac{|\nu|(I\!R \times \mathcal{B}_r(X))}{r^{k-1}} > 0\}, \tag{6.8.8}$$

where $|\nu|$ denotes the total variation measure of $\nu$.

The resolution of the fine structure of $u$ is achieved by "blowing up" the neighborhood of any point $X \in \mathcal{X}$, that is by rescaling $u$ and $\nu$ in the vicinity of $X$ in a manner that leaves (6.8.6) invariant. The linear nondegeneracy condition (6.8.4), in conjunction with velocity averaging estimates for the transport equation (6.8.6), induces the requisite compactness, so that the limits $u_\infty$ and $\nu_\infty$ of $u$ and $\nu$ under rescaling exist and satisfy (6.8.6). When $X \notin \mathcal{J}$, the measure $\nu_\infty$ vanishes. On the other hand, when $X \in \mathcal{J}$, $\nu_\infty$ is the tensor product of a measure on $I\!R$ and a measure on $\mathcal{X}$. It is by studying solutions of (6.8.6) with $\nu$ having this special tensor product structure that the assertion of Theorem 6.8.2 is established.

By the same techniques one verifies that admissible solutions of linearly nondegenerate scalar balance laws share another important property with $BV$ functions, namely they have one-sided traces on manifolds of codimension one:

**6.8.3 Theorem.** *Let $u$ be an admissible solution of the linearly nondegenerate balance law* (6.8.1) *on a Lipschitz subset $\mathcal{X}$ of $I\!R^k$ with boundary $\mathcal{B}$. Assume that for any companion $Q$ the measure $\nu_Q$ in* (6.8.2) *is finite on $\mathcal{X}$. Then $u$ has a strong trace $u_{\mathcal{B}} \in L^\infty(\mathcal{B})$ on $\mathcal{B}$.*

The strong trace is realized in $L^1_{\text{loc}}$, roughly as follows: Suppose that $\mathcal{B}$ contains a compact subset $\mathcal{P}$ of a $(k-1)$-dimensional hyperplane with outward unit normal $N$. Then the restriction of $u_{\mathcal{B}}$ to $\mathcal{P}$ is characterized by

$$\operatorname*{ess\,lim}_{\tau\downarrow 0} \int_{\mathcal{P}} |u(X - \tau N) - u_{\mathcal{B}}(X)| d\mathcal{H}^{k-1}(X) = 0. \tag{6.8.9}$$

In the general case, one employs Lipschitz transformations on $I\!R^k$ to map "pieces" of $\mathcal{B}$ into "pieces" $\mathcal{P}$ of a hyperplane, and then uses the above characterization.

Theorem 6.8.3 plays an important role in the theory of boundary-value problems for scalar conservation laws, as we shall see in Section 6.9. Another important implication of Theorem 6.8.3 is the following

**6.8.4 Corollary.** *Assume that the scalar conservation law* (6.1.1) *is linearly nondegenerate, and let $u$ be an $L^\infty$ weak solution of the Cauchy problem* (6.1.1), (6.1.2), *on the upper half-space, which satisfies the inequalities* (6.2.2), *in the sense of distributions, for every convex entropy $\eta$. Then the map $t \mapsto u(\cdot, t)$ is strongly continuous in $L^1_{\text{loc}}(I\!R^m)$, for any $t \in [0, \infty)$.*

In particular, for linearly nondegenerate scalar conservation laws, admissible solutions to the Cauchy problem may be characterized merely by the set of inequalities (6.2.2), rather than by the stronger condition (6.2.3). Thus, referring back to the discussion on entropy admissibility, in Section 4.5, we conclude that for scalar, linearly nondegenerate conservation laws, the set $\mathcal{F}$ is empty.

## 6.9 Initial-Boundary-Value Problems

Let $\mathcal{D}$ be an open bounded subset of $\mathbb{R}^m$, with smooth boundary $\partial\mathcal{D}$ and outward unit normal field $\nu$. Here we consider the initial-boundary-value problem

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t)) = 0, \qquad (x,t) \in \mathcal{X}, \tag{6.9.1}$$

$$u(x,t) = 0, \qquad (x,t) \in \mathcal{B}, \tag{6.9.2}$$

$$u(x,0) = u_0(x), \qquad x \in \mathcal{D}, \tag{6.9.3}$$

in the domain $\mathcal{X} = \mathcal{D} \times (0,\infty)$, with lateral boundary $\mathcal{B} = \partial\mathcal{D} \times (0,\infty)$.

The boundary condition (6.9.2) shall be interpreted in the context of the vanishing viscosity approach, as explained in Section 4.7. The inequality (4.7.5) motivates the following notion of admissible weak solution:

**6.9.1 Definition.** A bounded measurable function $u$ on $\mathcal{X}$ is an *admissible weak solution* of (6.9.1), (6.9.2), (6.9.3), with initial data $u_0 \in L^\infty(\mathcal{D})$, if the inequality

$$\int_0^\infty \int_{\mathcal{D}} [\partial_t \psi\, \eta(u) + \sum_{\alpha=1}^m \partial_\alpha \psi\, Q_\alpha(u)] dx dt + \int_{\mathcal{D}} \psi(x,0)\eta(u_0(x))dx$$
$$\geq \int_0^\infty \int_{\partial\mathcal{D}} \psi \left\{ Q^0_{\mathcal{B}} - \eta'(0)[G^0_{\mathcal{B}} - G_{\mathcal{B}}] \right\} d\mathcal{H}^{m-1}(x)dt \tag{6.9.4}$$

holds for every convex entropy $\eta$, with associated entropy flux $Q$ determined by (6.2.1), and all nonnegative Lipschitz continuous test functions $\psi$ with compact support in $\mathbb{R}^m \times [0,\infty)$. $G_{\mathcal{B}}$ denotes the trace of the normal component of $G$ on $\mathcal{B}$, while $G^0_{\mathcal{B}}$ and $Q^0_{\mathcal{B}}$ stand for $G(0)\nu$ and $Q(0)\nu$, respectively.

Notice that (6.9.4) implies $\partial_t \eta + \operatorname{div} Q \leq 0$, and in particular $\partial_t u + \operatorname{div} G = 0$, so that the traces $Q_{\mathcal{B}}$ and $G_{\mathcal{B}}$ of the normal components of $Q$ and $G$ on $\mathcal{B}$ are well defined. Furthermore, (4.7.8) holds on $\mathcal{B}$, in the form

$$Q_{\mathcal{B}} - Q^0_{\mathcal{B}} - \eta'(0)[G_{\mathcal{B}} - G^0_{\mathcal{B}}] \geq 0. \tag{6.9.5}$$

At the price of technical complications, but without any essential difficulty, the special boundary condition $u = 0$ may be replaced with $u = \hat{u}(x,t)$, for any sufficiently smooth function $\hat{u}$.

The justification of Definition 6.9.1 is provided by

**6.9.2 Theorem.** *For each $u_0 \in L^\infty(\mathcal{D})$, there exists a unique admissible weak solution $u$ of* (6.9.1), (6.9.2), (6.9.3), *and*

$$u(\cdot,t) \in C^0([0,\infty); L^1(\mathcal{D})). \tag{6.9.6}$$

*Furthermore, if $u_0 \in BV(\mathcal{D})$, then $u \in BV_{\mathrm{loc}}(\mathcal{X})$.*

Before establishing the existence of solutions by proving the above theorem, we demonstrate uniqueness and stability by means of the following analog of Theorem 6.2.3:

**6.9.3 Theorem.** *Let $u$ and $\bar{u}$ be admissible weak solutions of* (6.9.1), (6.9.2) *with respective initial values $u_0$ and $\bar{u}_0$. Then, for any $t > 0$,*

(6.9.7)
$$\int_{\mathcal{D}} [u(x,t) - \bar{u}(x,t)]^+ \, dx \leq \int_{\mathcal{D}} [u_0(x) - \bar{u}_0(x)]^+ \, dx,$$

(6.9.8)
$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathcal{D})} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathcal{D})} .$$

*Furthermore, if*

(6.9.9)
$$u_0(x) \leq \bar{u}_0(x), \qquad a.e. \text{ on } \mathcal{D},$$

*then*

(6.9.10)
$$u(x,t) \leq \bar{u}(x,t), \qquad a.e. \text{ on } \mathcal{X}.$$

**Proof.** We sketch the proof under the simplifying assumption that both $u$ and $\bar{u}$ attain strong traces $u_{\mathcal{B}}$ and $\bar{u}_{\mathcal{B}}$ on $\mathcal{B}$, in which case the traces of the normal components of $G$ and $Q$ on $\mathcal{B}$ are obtained via ordinary composition:

(6.9.11)
$$G_{\mathcal{B}} = G(u_{\mathcal{B}})\nu, \quad Q_{\mathcal{B}} = Q(u_{\mathcal{B}})\nu, \quad \bar{G}_{\mathcal{B}} = G(\bar{u}_{\mathcal{B}})\nu, \quad \bar{Q}_{\mathcal{B}} = Q(\bar{u}_{\mathcal{B}})\nu.$$

The above assumption will hold when $u$ and $\bar{u}$ are $BV$ functions or when $u$ and $\bar{u}$ are merely in $L^\infty$ and $G$ is linearly nondegenerate; see Theorem 6.8.3.

We retrace the steps in the proof of Theorem 6.2.3, employing the same entropy-entropy flux pair $\eta(u;\bar{u})$, $Q(u;\bar{u})$, defined by (6.2.5), and the same test function $\phi(x,t,\bar{x},\bar{t})$, given by (6.2.16). However, we now integrate over $\mathcal{D} \times [0,\infty)$, instead of $I\!R^m \times [0,\infty)$, and substitute (6.9.4) for (6.2.3). We thus obtain, in the place of (6.2.21),

(6.9.12)
$$\int_0^\infty \int_{\mathcal{D}} \{\partial_t \psi \; \eta(u;\bar{u}) + \sum_{\alpha=1}^m \partial_\alpha \psi \; Q_\alpha(u;\bar{u})\} dx dt + \int_{\mathcal{D}} \psi(x,0) \eta(u_0(x);\bar{u}_0(x)) dx$$
$$\geq \int_0^\infty \int_{\partial\mathcal{D}} \psi \; \mathrm{sgn}[u_{\mathcal{B}} - \bar{u}_{\mathcal{B}}]^+ [G_{\mathcal{B}} - \bar{G}_{\mathcal{B}}] d\mathcal{H}^{m-1}(x) dt.$$

We verify that, as a consequence of the boundary condition (6.9.5), the integral on the right-hand side of (6.9.12) is nonnegative. Indeed, the integrand vanishes where $u_{\mathcal{B}} \leq \bar{u}_{\mathcal{B}}$, and has the sign of $G_{\mathcal{B}} - \bar{G}_{\mathcal{B}}$ where $u_{\mathcal{B}} > \bar{u}_{\mathcal{B}}$. In the latter case, we examine, separately, the following three subcases:

(a) $u_{\mathcal{B}} > \bar{u}_{\mathcal{B}} \geq 0$: (6.9.5), written for the solution $u$ and the entropy-entropy flux pair $\eta(u; \bar{u}_{\mathcal{B}})$, $Q(u; \bar{u}_{\mathcal{B}})$, yields $G_{\mathcal{B}} \geq \bar{G}_{\mathcal{B}}$.
(b) $0 \geq u_{\mathcal{B}} > \bar{u}_{\mathcal{B}}$ : (6.9.5), written for the solution $\bar{u}$ and the entropy-entropy flux pair $\eta(u_{\mathcal{B}}; \bar{u})$, $Q(u_{\mathcal{B}}; \bar{u})$, again yields $G_{\mathcal{B}} \geq \bar{G}_{\mathcal{B}}$.
(c) $u_{\mathcal{B}} > 0 > \bar{u}_{\mathcal{B}}$ : (6.9.5), written for the solution $u$ and the entropy-entropy flux pair $\eta(u; 0)$, $Q(u; 0)$, yields $G_{\mathcal{B}} \geq G^0_{\mathcal{B}}$. Similarly, (6.9.5), written for the solution $\bar{u}$ and the entropy-entropy flux pair $\eta(0; \bar{u})$, $Q(0; \bar{u})$, yields $\bar{G}_{\mathcal{B}} \leq G^0_{\mathcal{B}}$. In particular, $G_{\mathcal{B}} \geq \bar{G}_{\mathcal{B}}$.

We apply (6.9.12) for the test function $\psi(x, \tau) = \chi(x)\omega(\tau)$, where $\chi(x) = 1$ for $x \in \mathcal{D}$, and $\omega$ is defined by (5.3.11). Since the right-hand side of (6.9.12) is nonnegative, we deduce

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathcal{D}} [u(x, \tau) - \bar{u}(x, \tau)]^+ \, dx d\tau \leq \int_{\mathcal{D}} [u_0(x) - \bar{u}_0(x)]^+ \, dx. \tag{6.9.13}$$

Letting $\varepsilon \downarrow 0$, we arrive at (6.9.7). In turn, (6.9.7) readily implies the remaining assertions of the theorem. The proof is complete.

The next task is to construct the solution to (6.9.1), (6.9.2), (6.9.3) by the vanishing viscosity method. We thus consider the family of parabolic equations

$$\partial_t u(x, t) + \operatorname{div} G(u(x, t)) = \mu \Delta u(x, t), \qquad (x, t) \in \mathcal{X}, \tag{6.9.14}$$

with boundary condition (6.9.2) and initial condition (6.9.3). For any $\mu > 0$, (6.9.14), (6.9.2), (6.9.3) admits a unique solution $u_\mu$ which is smooth on $\bar{\mathcal{D}} \times (0, \infty)$. By the maximum principle,

$$\left|u_\mu(x, t)\right| \leq \sup |u_0(\cdot)|, \qquad x \in \mathcal{D}, \quad t \in (0, \infty). \tag{6.9.15}$$

Upon retracing the steps in the proof of Theorem 6.3.2, except that now (6.3.10) should be integrated over $\mathcal{D} \times (s, t)$ instead of $\mathbb{R}^m \times (s, t)$, one readily obtains

**6.9.4 Theorem.** *Let $u_\mu$ and $\bar{u}_\mu$ be solutions of* (6.9.14), (6.9.2) *with respective initial data $u_0$ and $\bar{u}_0$. Then, for any $t > 0$,*

$$\int_{\mathcal{D}} \left[u_\mu(x, t) - \bar{u}_\mu(x, t)\right]^+ dx \leq \int_{\mathcal{D}} [u_0(x) - \bar{u}_0(x)]^+ \, dx, \tag{6.9.16}$$

$$\|u_\mu(\cdot, t) - \bar{u}_\mu(\cdot, t)\|_{L^1(\mathcal{D})} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathcal{D})}. \tag{6.9.17}$$

*Furthermore, if*

$$u_0(x) \leq \bar{u}_0(x), \qquad a.e. \ on \ \mathcal{D}, \tag{6.9.18}$$

*then*

$$u_\mu(x, t) \leq \bar{u}_\mu(x, t), \qquad (x, t) \in \mathcal{D} \times (0, \infty). \tag{6.9.19}$$

We proceed to show that the family $\{u_\mu : \mu > 0\}$ of solutions to (6.9.14), (6.9.2), (6.9.3) is relatively compact in $L^1$.

**6.9.5 Lemma.** *Let $u_\mu$ be the solution of* (6.9.14), (6.9.2), (6.9.3) *with initial data $u_0 \in L^\infty(\mathcal{D}) \bigcap W^{2,1}(\mathcal{D})$. Then, for any $t > 0$,*

$$\left\| \partial_t u_\mu(\cdot, t) \right\|_{L^1(\mathcal{D})} \le c_0 \left\| u_0(\cdot) \right\|_{W^{1,1}(\mathcal{D})} + \mu \left\| u_0(\cdot) \right\|_{W^{2,1}(\mathcal{D})} , \tag{6.9.20}$$

$$\sum_{\beta=1}^{m} \left\| \partial_\beta u_\mu(\cdot, t) \right\|_{L^1(\mathcal{D})} \le a(t) \| u_0(\cdot) \|_{W^{1,1}(\mathcal{D})} + \mu b(t) \| u_0(\cdot) \|_{W^{2,1}(\mathcal{D})} , \tag{6.9.21}$$

*where $c_0$ and the continuous functions $a(t)$, $b(t)$ do not depend on $\mu$.*

**Proof.** For $h > 0$, we apply (6.9.17) for the two solutions $u_\mu(x, t)$, with initial value $u_0(x)$, and $\bar{u}_\mu(x, t) = u_\mu(x, t + h)$, with initial value $\bar{u}_0(x) = u_\mu(x, h)$. Upon dividing the resulting inequality by $h$, and then letting $h \downarrow 0$, we deduce $\left\| \partial_t u_\mu(\cdot, t) \right\|_{L^1(\mathcal{D})} \le \left\| \partial_t u_\mu(\cdot, 0) \right\|_{L^1(\mathcal{D})}$ , whence (6.9.20) follows with the help of (6.9.14).

One cannot use the same procedure for estimating spatial derivatives, because shifting in the spatial direction no longer carries solutions into solutions. We thus have to employ a different argument.

For $\varepsilon > 0$, we define the function

$$\nu_\varepsilon(w) = \begin{cases} -w - \varepsilon & \text{for } \; -\infty < w \le -2\varepsilon \\ \dfrac{w^2}{4\varepsilon} & \text{for } \; -2\varepsilon < w \le 2\varepsilon \\ w - \varepsilon & \text{for } \quad 2\varepsilon < w < \infty. \end{cases} \tag{6.9.22}$$

We set $w = \partial_\beta u_\mu$ , differentiate (6.9.14) with respect to $x_\beta$ , multiply the resulting equation by $\eta'_\varepsilon(w)$ and integrate over $\mathcal{D}$. After an integration by parts, this yields

$$\frac{d}{dt} \int_{\mathcal{D}} \eta_\varepsilon(w) dx = \int_{\mathcal{D}} \left[ \eta_\varepsilon(w) - \eta'_\varepsilon(w) w \right] \operatorname{div} G'(u_\mu) \, dx \tag{6.9.23}$$

$$-\mu \int_{\mathcal{D}} \eta''_\varepsilon(w) \, |\nabla w|^2 \, dx + \int_{\partial\mathcal{D}} [\mu \eta'_\varepsilon(w) \frac{\partial w}{\partial \nu} - \eta_\varepsilon(w) G'(0) \nu] d\mathcal{H}^{m-1}(x).$$

As $\varepsilon \downarrow 0$, the integrand on the left-hand side of (6.9.23) tends to $|w|$. On the right-hand side, the first integral is $O(\varepsilon)$ and the second integral is nonnegative. To estimate the integral over $\partial\mathcal{D}$, we note that since $u_\mu$ vanishes on the boundary, $\partial_\alpha u_\mu = \dfrac{\partial u_\mu}{\partial \nu} \nu_\alpha$ , $\alpha = 1, \cdots, m$. In particular, $w = \dfrac{\partial u_\mu}{\partial \nu} \nu_\beta$ . Then (6.9.14) implies

$\frac{\partial u_\mu}{\partial \nu} G'(0)\nu = \mu \Delta u_\mu$ . Finally, it is clear that $\frac{\partial w}{\partial \nu} = \frac{\partial^2 u_\mu}{\partial \nu^2} \nu_\beta + O(1) \frac{\partial u_\mu}{\partial \nu}$ and $\Delta u_\mu = \frac{\partial^2 u_\mu}{\partial \nu^2} + O(1) \frac{\partial u_\mu}{\partial \nu}$ . We thus have

$$\mu \eta'_\varepsilon(w) \frac{\partial w}{\partial \nu} - \eta_\varepsilon(w) G'(0)\nu = \mu \Big[ \eta'_\varepsilon(w) - \frac{\eta_\varepsilon(w)}{w} \Big] \frac{\partial^2 u_\mu}{\partial \nu^2} \nu_\beta + O(1) \mu \frac{\partial u_\mu}{\partial \nu} , \tag{6.9.24}$$

which tends to $O(1)\mu \frac{\partial u_\mu}{\partial \nu}$ as $\varepsilon \downarrow 0$. Therefore, in the limit, as $\varepsilon \downarrow 0$, (6.9.23) yields

$$\frac{d}{dt} \int_{\mathcal{D}} \left| \partial_\beta u_\mu \right| dx \le c \int_{\partial \mathcal{D}} \mu \left| \frac{\partial u_\mu}{\partial \nu} \right| d\mathcal{H}^{m-1}(x) \le c' \int_{\mathcal{D}} \mu \left| \Delta u_\mu \right| dx. \tag{6.9.25}$$

We sum (6.9.25) over $\beta = 1, \cdots, m$, and also substitute $\mu \Delta u_\mu$ by $\partial_t u_\mu + \operatorname{div} G(u_\mu)$. Using (6.9.15), (6.9.20) and applying Gronwall's inequality, we arrive at (6.9.21). The proof is complete.

**Proof of Theorem 6.9.2.** Assume first $u_0 \in L^\infty(\mathcal{D}) \bigcap W^{2,1}(\mathcal{D})$. By virtue of Lemma 6.9.5, the family $\{u_\mu : \mu > 0\}$ of solutions to (6.9.14), (6.9.2), (6.9.3) is relatively compact in $L^1(\mathcal{D} \times (0, T))$, for any $T > 0$. Therefore, recalling (6.9.15), we may extract a sequence $\{u_{\mu_k}\}$, with $\mu_k \downarrow 0$ as $k \to \infty$, which converges boundedly almost everywhere on $\mathcal{D} \times (0, \infty)$ to some function $u$. As shown in Section 4.7, $u$ satisfies (6.9.4) and hence is the unique solution of (6.9.1), (6.9.2), (6.9.3). In particular, the entire family $\{u_\mu \colon \mu > 0\}$ converges to $u$, as $\mu \downarrow 0$. Moreover, it follows from (6.9.20), (6.9.21) that $u$ is in $BV_{\mathrm{loc}}(\mathcal{D} \times (0, \infty))$ and, for any $T > 0$,

$$TV_{\mathcal{D} \times (0,T)} u \le c(T) \|u_0\|_{W^{1,1}(\mathcal{D})} . \tag{6.9.26}$$

In addition, $u$ inherits from (6.9.15) the maximum principle: $|u(x, t)| \le \sup |u_0(\cdot)|$.

Assume now $u_0 \in L^\infty(\mathcal{D})$. We construct a sequence of functions $\{u_{0n}\}$ in $L^\infty(\mathcal{D}) \bigcap W^{2,1}(\mathcal{D})$ with $\|u_{0n}\|_{L^\infty(\mathcal{D})} \le \|u_0\|_{L^\infty(\mathcal{D})}$ and $u_{0n} \to u_0$ in $L^1(\mathcal{D})$. By virtue of (6.9.8), the sequence $\{u_n\}$ of admissible solutions to (6.9.1), (6.9.2), with initial data $u_{0n}$ , converges in $L^1$ to a function $u$ which satisfies (6.9.4) and hence is the admissible solution of (6.9.1), (6.9.2), (6.9.3). Moreover, when $u_0$ is in $BV(\mathcal{D})$, the sequence $\{u_{0n}\}$ may be constructed with the additional requirement that $\|u_{0n}\|_{W^{1,1}(\mathcal{D})} \le C \left[ TV_{\mathcal{D}} u_0 + \|u_0\|_{L^\infty(\mathcal{D})} \right]$, in which case (6.9.26) implies that $u$ is in $BV(\mathcal{D} \times (0, T))$, for any $T > 0$. This completes the proof.

## 6.10 The $L^1$ Theory for Systems of Conservation Laws

The successful treatment of the scalar conservation law, based on $L^1$ and $L^\infty$ estimates, which we witnessed in the previous sections, naturally raises the expectation that a similar approach may also be effective for systems of conservation laws. Unfortunately, this does not seem to be the case. In order to gain some insight into the difficulty, let us consider the Cauchy problem for a symmetrizable system of conservation laws:

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0, \qquad x \in I\!R^m, \quad t > 0, \tag{6.10.1}$$

$$U(x,0) = U_0(x), \qquad x \in I\!R^m. \tag{6.10.2}$$

In analogy to Definition 6.2.1, for the scalar case, we shall require that admissible solutions of (6.10.1), (6.10.2) satisfy (4.5.3), for any entropy-entropy flux pair $(\eta, Q)$ with $\eta$ convex. The first test of this should be whether the trivial, constant solutions $\bar{U}$ of (6.10.1) are $L^p$-stable in the class of admissible solutions:

$$\|U(\cdot,t) - \bar{U}\|_{L^p(\mathcal{B}_r)} \le c_p \|U_0(\cdot) - \bar{U}\|_{L^p(\mathcal{B}_{r+st})}. \tag{6.10.3}$$

Since the system is symmetrizable, and thereby endowed with a convex entropy of quadratic growth, (6.10.3) will be satisfied at least for $p = 2$, by virtue of Theorem 5.3.1. The question then arises whether such an estimate may also hold for $p \neq 2$, with the cases $p = 1$ and $p = \infty$ being of particular interest.

For the linear system

$$\partial_t V + \sum_{\alpha=1}^{m} DG_\alpha(\bar{U}) \partial_\alpha V = 0, \tag{6.10.4}$$

resulting from linearizing (6.10.1) about a constant state $\bar{U}$, it is known (references in Section 6.11) that the following three statements are equivalent: (a) the zero solution is $L^p$-stable for some $p \neq 2$; (b) the zero solution is $L^p$-stable for all $1 \le p \le \infty$; (c) the Jacobian matrices $DG_\alpha(\bar{U})$ commute:

$$DG_\alpha(\bar{U}) DG_\beta(\bar{U}) = DG_\beta(\bar{U}) DG_\alpha(\bar{U}), \quad \alpha, \beta = 1, \cdots, m. \tag{6.10.5}$$

The nonlinear system (6.10.1) inherits (6.10.5) as a necessary condition for $L^p$-stability:

**6.10.1 Theorem.** *Assume that the constant state $\bar{U}$ is $L^p$-stable,* (6.10.3) *for some $p \neq 2$, within the class of classical solutions. Then* (6.10.5) *holds.*

**Sketch of Proof.** For $\varepsilon$ small, let $U_\varepsilon(x,t)$ denote the solution of (6.10.1) with initial values $U_\varepsilon(x,0) = \bar{U} + \varepsilon V_0(x)$, where $\nabla V_0 \in H^\ell$ for $\ell > \frac{m}{2}$. By Theorem 5.1.1, $U_\varepsilon$ exists, as a classical solution, on a time interval with length $O(\varepsilon^{-1})$. Furthermore,

$$U_\varepsilon(x,t) = \bar{U} + \varepsilon V(x,t) + O(\varepsilon^2), \tag{6.10.6}$$

where $V(x,t)$ is the solution of (6.10.4) with initial value $V_0(x)$. Now if (6.10.3) is satisfied by the solutions $U_\varepsilon$, for any $\varepsilon > 0$, it follows that the zero solution of (6.10.4) is $L^p$-stable and hence (6.10.5) must hold. This completes the proof.

A similar argument shows that (6.10.3) is also necessary for stability of solutions of (6.10.1), (6.10.2) in the space $BV$:

$$TV_{\mathcal{B}_r} U(\cdot, t) \le c\, TV_{\mathcal{B}_{r+st}} U_0(\cdot). \tag{6.10.7}$$

The above results douse any hope that the elegant $L^1$ and $BV$ theory of the scalar conservation law may be readily extended to general systems of conservation laws for which (6.10.5) is violated. A question of some relevance is whether (6.10.3) may at least hold in the special class of systems that satisfy (6.10.5). This is indeed the case, at least for systems of just two conservation laws:

**6.10.2 Theorem.** *Let* (6.10.1) *be a symmetrizable system of two conservation laws* ($n = 2$) *with the property that* (6.10.5) *holds for all* $\bar{U}$. *Then, for any* $1 \le p \le 2$, *there are* $\delta > 0$ *and* $c_p > 0$ *such that* (6.10.3) *holds for any admissible solution* $U$ *of* (6.10.1), (6.10.2), *taking values in the ball* $\mathcal{B}_\delta(\bar{U})$.

The proof, which is found in the references cited in Section 6.11, employs a convex entropy $\eta$ for (6.10.1) such that

$$c|U - \bar{U}|^p \le \eta(U) \le C|U - \bar{U}|^p, \qquad U \in \mathcal{B}_\delta(\bar{U}). \tag{6.10.8}$$

Recall that in order to construct an entropy for a system of $n$ conservation laws in $m$ spatial variables, one has to solve the generally overdetermined system (3.2.4) of $\frac{1}{2}n(n-1)m$ equations for the single scalar $\eta$. However, as noted in Section 3.2, when (6.10.5) holds, the number of independent equations is reduced to $\frac{1}{2}n(n-1)$, and in the special case $n = 2$ to just one. It thus becomes possible to construct a convex entropy with the requisite property (6.10.8), by solving a Goursat problem on $\mathcal{B}_\delta(\bar{U})$. In fact, under additional assumptions on the system, it is even possible to construct convex entropies that satisfy (6.10.8) for any $1 \le p \le \infty$, and for such systems constant solutions are $L^p$-stable over the full range $1 \le p \le \infty$.

The class of systems that satisfy (6.10.5) includes, in particular, the scalar conservation laws ($n = 1$), in any spatial dimension $m$, as well as the systems of arbitrary size $n$, in a single spatial dimension ($m = 1$); but beyond that it contains very few representatives of (even modest) physical interest. An example is the system

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha [F_\alpha(|U|)U] = 0, \tag{6.10.9}$$

which governs the flow of a fluid in an anisotropic porous medium. The special features of this system make it analytically tractable, so that it may serve as a vehicle for exhibiting some of the issues facing the study of hyperbolic systems of conservation laws in several space dimensions.

If $U$ is a classical solution of (6.10.9), it is easy to see that its "density" $\rho = |U|$ satisfies the scalar conservation law

$$\partial_t \rho + \sum_{\alpha=1}^{m} \partial_\alpha [\rho F_\alpha(\rho)] = 0, \tag{6.10.10}$$

while its directional unit vector field $\Theta = \rho^{-1}U$ satisfies the transport equation

$$\partial_t \Theta + \sum_{\alpha=1}^{m} F_\alpha(\rho)\partial_\alpha \Theta = 0. \tag{6.10.11}$$

Thus, classical solutions to the Cauchy problem (6.10.9), (6.10.2) can be constructed by first solving (6.10.10), with initial data $\rho(\cdot, 0) = |U_0(\cdot)|$, say by the method of characteristics expounded in Section 6.1, and then determining $\Theta$ by its property of staying constant along the trajectories of the ordinary differential equation

$$\frac{dx}{dt} = F(\rho(x,t)). \tag{6.10.12}$$

It is not obvious how to adapt the above procedure to weak solutions. It is of course still possible to determine $\rho$ as the admissible weak solution of (6.10.10) with initial data $|U_0|$ merely in $L^\infty$, but it is by no means clear how one should interpret (6.10.12) when $F(\rho(x,t))$ is just an $L^\infty$ function. A relevant, powerful theory of ordinary differential equations $\dot{X} = P(X)$ exists, but it requires that $P$ be a divergence-free vector field in $BV$. In order to use that theory, we restrict the initial data so that $|U_0|$ is a positive function of locally bounded variation on $I\!R^m$. This will guarantee, by virtue of Theorems 6.2.3 and 6.2.6, that $\rho$ is a positive function of locally bounded variation on the upper half-space. Next, we rescale the time variable and rewrite (6.10.12) in the implicit form

$$\begin{cases} \dfrac{dt}{d\tau} = \rho(x,t) \\[2ex] \dfrac{dx}{d\tau} = \rho(x,t)F(\rho(x,t)), \end{cases} \tag{6.10.13}$$

which has the desired feature that the vector field $(\rho, \rho F(\rho))$ is divergence-free on the upper half-space, by virtue of (6.10.10).

By eliminating $\tau$ in the family of solutions $(t(\tau), x(\tau))$ of (6.10.13), one obtains the family of curves $x = x(t)$, namely the formal trajectories of (6.10.12), along which $\Theta$ stays constant. Thus $\Theta$ can be determined from its initial data, which may merely be in $L^\infty$. Finally, it can be shown (references in Section 6.11) that $U = \rho\Theta$ is an admissible weak solution of (6.10.9), (6.10.2):

**6.10.3 Theorem.** *Let* $U_0 \in L^\infty(I\!R^m; I\!R^n)$, $|U_0| \in BV_{\mathrm{loc}}(I\!R^m)$, *and* $|U_0| \geq a > 0$ *a.e. on* $I\!R^m$. *Then there exists an admissible weak solution* $U$ *of* (6.10.9), (6.10.2) *on* $[0, \infty)$. *Furthermore,* $\rho = |U|$ *is the admissible weak solution of* (6.10.10) *with initial data* $\rho(\cdot, 0) = |U_0(\cdot)|$.

An example has been concocted (references in Section 6.11) demonstrating that when $|U_0|$ is merely in $L^\infty$ solutions to the Cauchy problem (6.10.9), (6.10.2) may fail to exist.

The reader should bear in mind that (6.10.9) is so special that the above should not necessarily be interpreted as representative of the behavior of generic systems.

The theory of hyperbolic systems of conservation laws in several spatial variables is still in its infancy.

## 6.11 Notes

More extensive discussion on the breakdown of classical solutions of scalar conservation laws can be found in Majda [4]. Theorem 6.1.1 is due to Conway [1]. For a systematic study of the geometric features of shock formation and propagation, see Izumiya and Kossioris [1]. The reduction of (6.1.1) to the linear transport equation (6.1.10) is classical; see Courant-Hilbert [1,§I.5].

There is voluminous literature on weak solutions of the scalar conservation law. The investigation was initiated in the 1950's, in the framework of the single space dimension, stimulated by the seminal paper of Hopf [1], already cited in Section 4.8. References to this early work will be provided, as they become relevant, in Section 11.12.

The first existence proof in several space dimensions is due to Conway and Smoller [1], who recognized the relevance of the space $BV$ and constructed solutions with bounded variation through the Lax-Friedrichs difference scheme. The definitive treatment in the space $BV$ was later given by Volpert [1], who was apparently the first to realize the $L^1$ contraction property in several space dimensions. Building on Volpert's work, Kruzkov [1] proposed the characterization of admissible weak solutions recorded in Section 6.2, derived the $L^1$ contraction estimate, and established the convergence of the method of vanishing viscosity along the lines of our discussion in Section 6.3. More delicate treatment is needed when the flux is merely continuous in $u$; see Bénilan and Kruzkov [1]. On the other hand, the analysis extends routinely to inhomogeneous scalar balance laws (3.3.1), though solutions may blow up in finite time when the production grows superlinearly with $u$; see Natalini, Sinestrari and Tesei [1]. In particular, the inhomogeneous conservation law of "transport type," with flux $G(u,x) = f(u)V(x)$, has interesting structure, especially when $\operatorname{div} V = 0$; see Caginalp [1] and Otto [2].

The theory of nonlinear contraction semigroups in general, not necessarily reflexive, Banach space is due to Crandall and Liggett [1]. The application to the scalar conservation law presented in Section 6.4 is taken from Crandall [1]. For an alternative functional analytic characterization of admissible solutions, see Portilheiro [1].

The construction of solutions by the layering method, discussed in Section 6.5, was suggested by Roždestvenskii [1] and was carried out by Kuznetsov [1] and Douglis [1].

The program of realizing hyperbolic conservation laws as the "relaxed" form of larger, but simpler, systems that govern, or model, relaxation phenomena in physics (see Section 5.2) is currently undergoing active development. Further discussion and references are found in Chapter XVI. The presentation in Section 6.6 follows Katsoulakis and Tzavaras [1]. Though artificially constructed for the purposes of the analysis, (6.6.1) may be interpreted a posteriori as a system governing the evolution of an ensemble of interacting particles, at the mesoscopic scale. An alternative construction of solutions to multidimensional scalar conservation laws by a relaxation scheme is discussed in Natalini [2].

The kinetic formulation described in Section 6.7 is due to Perthame and Tadmor [1] and Lions, Perthame and Tadmor [2]. A detailed discussion, with extensions, applications and an extensive bibliography, is found in the recent monograph and survey article by Perthame [2,3]. For related results, see Giga and Miyakawa [1], Bäcker and Dressler [1], Brenier [1], James, Peng and Perthame [1], Natalini [2], Perthame [1], and Perthame and Pulvirenti [1]. The mechanism that induces the regularizing effect stated in Theorem 6.7.2 plays a prominent role in the theory of nonlinear transport equations in general, including the classical Boltzmann equation (cf. DiPerna and Lions [1]).

There are several other methods for constructing solutions, most notably by fractional stepping, spectral viscosity approximation, or through various difference schemes that may also be employed for efficient computation. See, for example, Bouchut and Perthame [1], Chen, Du and Tadmor [1], Cockburn, Coquel and LeFloch [1], and Crandall and Majda [1]. For references on the numerics the reader should consult LeVeque [1], Godlewski and Raviart [1,2], and Kröner [1].

In addition to $L^1$ and $BV$, other function spaces are relevant to the theory. DeVore and Lucier [1] show that solutions of (6.1.1) reside in Besov spaces. Perthame and Westdickenberg [1] establish a total oscillation diminishing property for solutions.

The fine structure of $L^\infty$ solutions, and in particular Theorem 6.8.2, is discussed in De Lellis, Otto and Westdickenberg [1]. See also De Lellis and Rivière [1] and De Lellis and Golse [1]. Theorem 6.8.3 is due to Vasseur [2]. See also Chen and Rascle [1], and Panov [3].

The construction of $BV$ solutions to the initial-boundary-value problem by the method of vanishing viscosity, expounded in Section 6.9, is taken from Bardos, Leroux and Nédélec [1]. For a proof of Theorem 6.9.3 when $u$ and $\bar{u}$ are merely in $L^\infty$, see Otto [1] and Málek, Nečas, Rokyta and Růžička [1]. For an alternative approach, see Szepessy [1]. Solutions in $L^\infty$ have been constructed via the kinetic formulation by Nouri, Omrane and Villa [1] and Tidriri [1]. For measure-valued solutions, see Kondo and LeFloch [1].

The large time behavior of solutions of (6.1.1), (6.1.2) is discussed in Conway [1], Engquist and E [1], Bauman and Phillips [1], and Feireisl and Petzeltová [1]. Chen and Frid [1,3,4,6] set a framework for investigating, in general systems of conservation laws, decay of solutions induced by scale invariance and compactness. In particular, this theory establishes the long time behavior of solutions of (6.1.1), (6.1.2) when $u_0$ is either periodic or of the form $u_0(x) = v(|x|^{-1}x) + w(x)$, with $w \in L^1(I\!R^m)$.

The proof that (6.10.5) is necessary and sufficient for $L^p$-stability in symmetrizable linear systems, is due to Brenner [1]. Rauch [1] demonstrated Theorem 6.10.1, and Dafermos [19] proved Theorem 6.10.2. Theorem 6.10.3 is due to Ambrosio and De Lellis [1]. See also Ambrosio, Bouchut and De Lellis [1]. Finally, Bressan [11] and De Lellis [1] explain why the Cauchy problem for the system (6.10.9) is not generally well-posed in $L^\infty$ or in $BV$, when $m > 1$. By contrast, when $m = 1$

the Cauchy problem for this system is well-posed and has an interesting theory; see Temple [2], Isaacson and Temple [1], Liu and Wang [1], Tveito and Winther [1], and Freistühler [7].

# VII

# Hyperbolic Systems of Balance Laws in One-Space Dimension

The remainder of the book will be devoted to the study of systems of balance laws in one-space dimension. This narrowing of focus is principally dictated by necessity: At the present time the theory of multidimensional systems is terra incognita. Eventually, research should turn to that vastly unexplored area, which is replete with fascinating problems. In any event, the reader should bear in mind that certain multidimensional phenomena, with special symmetry, such as wave focussing, may be studied in the context of the one-space dimensional theory.

This chapter introduces many of the concepts that serve as foundation of the theory of hyperbolic systems of balance laws in one space dimension: strict hyperbolicity; Riemann invariants and their relation to entropy; simple waves; genuine nonlinearity and its role in the breakdown of classical solutions.

In order to set the stage, the chapter opens with the presentation of a number of illustrative examples of hyperbolic systems of balance laws in one-space dimension, arising in physics or other branches of science and technology.

## 7.1 Balance Laws in One-Space Dimension

When $m = 1$, the general system of balance laws (3.1.1) reduces to

$$\partial_t H(U(x,t),x,t) + \partial_x F(U(x,t),x,t) = \Pi(U(x,t),x,t). \tag{7.1.1}$$

Systems (7.1.1) naturally arise in the study of gas flow in ducts, vibration of elastic bars or strings, etc., in which the medium itself is modeled as one-dimensional. The simplest examples are homogeneous systems of conservation laws, beginning with the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{7.1.2}$$

Despite its apparent simplicity, the scalar conservation law provides valuable insight into complex processes, in physics and elsewhere. The simple hydrodynamic theory of traffic flow in a stretch of highway is a case in point.

The state of the traffic at the location $x$ and time $t$ is described by the traffic density $\rho(x,t)$ (measured, say, in vehicles per mile) and the traffic speed $v(x,t)$ (in miles per hour). The fields $\rho$ and $v$ are related by the law of conservation of vehicles, which is identical to mass conservation (2.3.2), in one-space dimension:

$$\partial_t \rho + \partial_x(\rho v) = 0. \tag{7.1.3}$$

This equation is then closed by the behavioral assumption that drivers set their vehicles' speed according to the local density, $v = g(\rho)$. In order to account for the congestion effect, $g$ must be decreasing with $\rho$, for instance $g(\rho) = v_0(1 - \rho/\rho_0)$, where $v_0$ is the speed limit and $\rho_0$ is the saturation density beyond which traffic crawls to a standstill. For that $g(\rho)$, (7.1.3) becomes

$$\partial_t \rho + \partial_x\left[v_0\rho\left(1 - \frac{\rho}{\rho_0}\right)\right] = 0. \tag{7.1.4}$$

This simplistic model manages, nevertheless, to capture some of the qualitative features of traffic flow in congested highways, and serves as the springboard for more sophisticated models, developed in the references cited in Section 7.10.

Thermoelasticity is a rich source of interesting examples of systems. A classical one is the one-dimensional version of (3.3.4), in Lagrangian coordinates,

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u,s) = 0 \\ \partial_t\left[\varepsilon(u,s) + \frac{1}{2}v^2\right] - \partial_x[v\sigma(u,s)] = 0, \end{cases} \tag{7.1.5}$$

with

$$\sigma(u,s) = \varepsilon_u(u,s), \qquad \theta(u,s) = \varepsilon_s(u,s), \tag{7.1.6}$$

which governs the adiabatic flow of a thermoelastic gas in a duct, or the longitudinal oscillation of a thermoelastic solid bar, or even the shearing motion of a thermoelastic layer. In the context of gas flow, $u$ is *specific volume* (notice that by virtue of (2.3.3) $u = 1/\rho$), thus constrained by $u > 0$. In the context of the thermoelastic bar, $u$ is the *strain*, likewise constrained by $u > 0$. Finally, in the context of shearing motion, $u$ is *shearing*, which may take both positive and negative values. In the gas case, it is traditional to use the *pressure* $p = -\sigma$, instead of $\sigma$.

The system (7.1.5) is hyperbolic if

$$\varepsilon_s(u,s) > 0, \qquad \varepsilon_{uu}(u,s) > 0, \tag{7.1.7}$$

that is, the absolute temperature $\theta$ is positive and the internal energy $\varepsilon$ is convex in $u$. Equivalently, $\sigma$ is increasing in $u$, $\sigma_u(u,s) > 0$, or $p$ is decreasing in $u$, $p_u(u,s) < 0$.

In the isentropic case, (7.1.5) reduces to

(7.1.8)
$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u) = 0, \end{cases}$$

which is hyperbolic when $\sigma'(u) > 0$. Again, in the context of gas dynamics one uses $p = -\sigma$, instead of $\sigma$, in which case (7.1.8) becomes the so-called "$p$-system". As with (7.1.5), when (7.1.8) is interpreted as governing the longitudinal oscillation of elastic bars, the natural range of $u$ is $(0, \infty)$, with $\sigma$ becoming unbounded as $u \downarrow 0$. However, when (7.1.8) governs the shearing motion of an elastic layer, the shearing $u$ is no longer constrained by $u > 0$ but may take any value in $(-\infty, \infty)$. Accordingly, in our use of (7.1.8) as a mathematical model we shall be assuming that $\sigma$ is defined as a smooth monotone increasing function on $(-\infty, \infty)$.

The spatial form of (7.1.8), namely the one-space dimensional version of (3.3.21), which governs in Eulerian coordinates the isentropic flow of a thermoelastic fluid in a duct, reads

(7.1.9)
$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x[\rho v^2 + p(\rho)] = 0. \end{cases}$$

This system is hyperbolic when $p'(\rho) > 0$. In particular, when the fluid is a polytropic gas (2.5.27), (7.1.9) becomes

(7.1.10)
$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x[\rho v^2 + \kappa\rho^\gamma] = 0. \end{cases}$$

For $\gamma > 1$, hyperbolicity breaks down at the vacuum state $\rho = 0$.

The so called system of *pressureless gas dynamics*

(7.1.11)
$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x\left(\rho v^2\right) = 0, \end{cases}$$

which is not hyperbolic, governs the flow of an aggregate of "sticky" particles: Colliding particles fuse into a single particle that combines their masses and moves with velocity that conserves the total linear momentum. The propensity of solutions of (7.1.11) to develop mass concentrations may serve as an explanation for the formation of large scale structures in the universe.

Next we derive the system that governs isentropic, planar oscillations of a three-dimensional, homogeneous thermoelastic medium, with reference density $\rho_0 = 1$. In the terminology and notation of Chapter II, we consider motions of the particular form $\chi = x + \phi(x \cdot \nu, t)$, where $\nu$ is the (constant) unit vector pointing in the direction of the oscillation. For consistency with the notation of this chapter, we shall denote the scalar variable $x \cdot \nu$ by $x$, so that $\partial_x = \sum_{\alpha=1}^{3} \nu_\alpha \partial_\alpha$. The velocity in the $\nu$-direction is $v(x, t) = \partial_t \phi(x, t)$. We also set $u(x, t) = \partial_x \phi(x, t)$, in which case the deformation gradient is $F = I + u \otimes \nu$. The stress vector, per unit area, on planes perpendicular to $\nu$ is $\sigma(u) = S(I + u \otimes \nu)\nu$, where $S(F)$ is the Piola-Kirchhoff stress. We thus end up with a system of six conservation laws

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u) = 0, \end{cases} \tag{7.1.12}$$

which looks identical to (7.1.8), except that here $u$, $v$ and $\sigma$ are no longer scalars but 3-vectors.

The internal energy $\varepsilon(F)$ also becomes a function of $u$: $\varepsilon(I+u\otimes\nu) = e(u)$. Then, (2.5.26) yields $\sigma(u) = \partial e(u)/\partial u$. Thus the Jacobian matrix of $\sigma(u)$ is the Hessian matrix of $e(u)$, which in turn is the acoustic tensor (3.3.8) evaluated at $F = I+u\otimes\nu$. The system (7.1.12) is hyperbolic when the function $e(u)$ is convex.

As explained in Section 2.5 (recall (2.5.21)), when the medium is an isotropic solid, the internal energy depends on $F$ solely through the invariants $|F|$, $|F^*|$ and $\det F$. Here $F = I+u\otimes\nu$ and so $|F|^2 = 3+2u\cdot\nu+|u|^2$, $|F^*|^2 = (u\cdot\nu)^2+4u\cdot\nu+|u|^2$ and $\det F = 1 + u \cdot \nu$. Thus, the internal energy depends on just two variables, $|u|$ and $u \cdot \nu$. If, in addition, the material is incompressible, the kinematic constraint (2.7.1) becomes $u \cdot \nu = 0$, in which case the internal energy depends solely on $|u|$, $e(u) = h(|u|)$. The stress tensor is now given by (2.7.2), where $p$ is the hydrostatic pressure. After a short calculation, recalling that $\sigma = S\nu$, we deduce that (7.1.12) takes the form

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x p\nu - \partial_x \left( \dfrac{h'(|u|)}{|u|} u \right) = 0. \end{cases} \tag{7.1.13}$$

However, the incompressibility condition $u \cdot \nu = 0$ implies $\partial_x v \cdot \nu = 0$; let us take $v \cdot \nu = 0$ to eliminate a trivial rigid motion in the direction $\nu$. Then $(7.1.13)_2$ yields $\partial_x p = 0$, and thus (7.1.13) reduces to

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \left( \dfrac{h'(|u|)}{|u|} u \right) = 0. \end{cases} \tag{7.1.14}$$

The special symmetry encoded in the flux function of (7.1.14) induces rich geometric structure which is a gift to the geometer that must be paid by the analyst, who has to deal with particular analytical difficulties. A taste of these is coming later. The next example indicates that the same symmetry structure arises in other contexts as well.

We now derive the system that governs the oscillation of a flexible, extensible elastic string. The reference configuration of the string lies along the $x$-axis, and is assumed to be a natural state of (linear) density one. The motion $\chi = \chi(x,t)$ is monitored through the *velocity* $v = \partial_t \chi$ and the *stretching* $u = \partial_x \chi$ which take values in $I\!R^3$ or in $I\!R^2$, depending on whether the string is free to move in 3-dimensional space or is constrained to undergo planar oscillations. The tension $\tau$ of the string is assumed to depend solely on $|u|$, $\tau = \tau(|u|)$, which measures the stretch of the string. Since the string cannot sustain any compression, the natural range of $|u|$ is $[1,\infty)$, and $\tau$ is assumed to satisfy $\tau(r) > 0$, $[\tau(r)/r]' > 0$, for $r > 1$. The compatibility relation between $u$ and $v$ together with balance of momentum, in Lagrangian coordinates, yield the hyperbolic system

$$
(7.1.15) \qquad \begin{cases} \partial_t u - \partial_x v = 0 \\[2ex] \partial_t v - \partial_x \left( \dfrac{\tau(|u|)}{|u|} u \right) = 0, \end{cases}
$$

which is identical to (7.1.14).

Our next example is the classical system of conservation laws that governs the propagation of long gravity waves in shallow water. It may be derived either by asymptotic analysis of the Euler equations or *ab initio,* by appealing to gross balance of mass and momentum. We follow here the latter approach.

An incompressible, inviscid fluid of density one flows isentropically in an open channel with horizontally level bottom and unit width. The atmospheric pressure on the free surface is taken to be zero. The flow is driven by the hydrostatic pressure gradient induced by variations in the height of the free surface. Assume the channel lies along the $x$-axis, the $y$-axis is vertical, pointing upwards, and the bottom rests on the $x$-$z$ plane. It is assumed that the height of the free surface is constant in the $z$-direction and thus is described by a function $h$ of $(x,t)$ alone. Moreover, the velocity vector points in the $x$-direction and is constant on any cross section of the channel, so its length is likewise described by a function $v$ of $(x,t)$.

As explained in Section 2.7, the stress tensor for an incompressible, inviscid fluid is just a hydrostatic pressure $-pI$. The balance of linear momentum in the $y$ and the $z$-direction yields $\partial_y p = -g$ and $\partial_z p = 0$, respectively, where $g$ is the acceleration of gravity. Thus, $p = g[h(x,t) - y]$, for $0 \le y \le h(x,t)$. Integrating with respect to $y$ and $z$, we find that the total pressure force exerted on the $x$-cross section at time $t$ is $P(x,t) = \frac{1}{2} g h^2(x,t)$.

We treat the flow in the channel as a motion of a one-dimensional continuum governed by conservation of mass and linear momentum, exactly as in (7.1.9), where now the role of density is naturally played by the cross sectional area $h$ and the role of pressure is played by the pressure force $P$. We thus arrive at the system of shallow water waves:

$$
(7.1.16) \qquad \begin{cases} \partial_t h + \partial_x (hv) = 0 \\[2ex] \partial_t (hv) + \partial_x \left( hv^2 + \frac{1}{2} g h^2 \right) = 0. \end{cases}
$$

Notice that (7.1.16) is identical to (7.1.10), with $\gamma = 2$.

Systems with interesting features govern the propagation of planar electromagnetic waves through special isotropic dielectrics in which the electromagnetic energy depends on the magnetic induction $B$ and the electric displacement $D$ solely through the scalar $r = (B\cdot B + D\cdot D)^{\frac{1}{2}}$; i.e., in the notation of Section 3.3.8, $\eta(B, D) = \psi(r)$, with $\psi'(0) = 0$, $\psi''(0) > 0$, and $\psi'(r) > 0$, $\psi''(r) > 0$ for $r > 0$. Waves propagating in the direction of the 3-axis are represented by solutions of Maxwell's equations (3.3.40), with $J = 0$, in which the fields $B$, $D$, $E$ and $H$ depend solely on the single spatial variable $x = x_3$ and on time $t$. In particular, (3.3.40) imply $B_3 = 0$ and $D_3 = 0$ so that $B$ and $D$ should be regarded as vectors in $\mathbb{R}^2$ satisfying the hyperbolic system

$$\text{(7.1.17)} \qquad \begin{cases} \partial_t B - \partial_x \left[ \dfrac{\psi'(r)}{r} A D \right] = 0 \\[2ex] \partial_t D + \partial_x \left[ \dfrac{\psi'(r)}{r} A B \right] = 0, \end{cases}$$

where $A$ is the alternating $2 \times 2$ matrix, with $A_{11} = A_{22} = 0$, $A_{12} = -A_{21} = 1$.

Returning to the general balance law (7.1.1), we note that $H$ and/or $F$ may depend explicitly on $x$, to account for inhomogeneity of the medium. For example, isentropic gas flow through a duct of (slowly) varying cross section $a(x)$ is governed by the system

$$\text{(7.1.18)} \qquad \begin{cases} \partial_t[a(x)\rho] + \partial_x[a(x)\rho v] = 0 \\[2ex] \partial_t[a(x)\rho v] + \partial_x[a(x)\rho v^2 + a(x)p(\rho)] = a'(x)p(\rho), \end{cases}$$

which reduces to (7.1.9) in the homogeneous case $a$ =constant. On the other hand, explicit dependence of $H$ or $F$ on $t$, indicating "ageing" of the medium, is fairly rare. By contrast, dependence of $\Pi$ on $t$ is not uncommon, because external forcing is generally time-dependent.

The source $\Pi$ may depend on the state vector $U$, to account for relaxation or reaction effects. A simple example of the latter case is provided by the system

$$\text{(7.1.19)} \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\[2ex] \partial_t(\rho v) + \partial_x[\rho v^2 + (\gamma - 1)c\rho\theta] = 0 \\[2ex] \partial_t\left[c\rho\theta + \beta\rho z + \frac{1}{2}\rho v^2\right] + \partial_x\left[\left(\gamma c\rho\theta + \beta\rho z + \frac{1}{2}\rho v^2\right)v\right] = 0 \\[2ex] \partial_t(\rho z) + \partial_x(\rho z v) = -\delta h(\theta - \theta_i)\rho z, \end{cases}$$

which governs the flow of a combustible polytropic gas in a duct. In addition to density $\rho$, velocity $v$ and temperature $\theta$, the state vector here comprises the *mass fraction* $z$ of the unburnt gas, which takes values in [0, 1]. The first three equations in (7.1.19) express the balance of mass, momentum and energy. As in (2.5.17), the equation of state for the pressure is $p = R\rho\theta = (\gamma - 1)c\rho\theta$, where $\gamma$ is the adiabatic exponent and $c$ is the specific heat. On the other hand, unlike (2.5.18), the internal energy here depends also on $z$, $\varepsilon = c\theta + \beta z$, where $\beta > 0$ is the *heat of reaction* (assumed exothermic). In the fourth equation of (7.1.19), which governs the reaction, $h$ is the standard Heaviside function (i.e. $h(\zeta) = 0$ for $\zeta < 0$ and $h(\zeta) = 1$ for $\zeta \geq 0$), $\theta_i$ is the *ignition temperature* and $\delta > 0$ is the *reciprocal activation energy*.

A simple model system that captures the principal features of (7.1.19) is

$$\begin{cases} \partial_t(u + \beta z) + \partial_x f(u) = 0 \\ \partial_t z = -\delta h(u) z, \end{cases} \tag{7.1.20}$$

where both $u$ and $z$ are scalar variables, and $f(u)$ is a strictly increasing convex function.

As an example of a source that manifests relaxation, consider the isothermal flow of a binary mixture of polytropic gases in a duct. Both constituents of the mixture satisfy partial balance laws of mass and momentum: For $\alpha = 1, 2$,

$$\begin{cases} \partial_t \rho_\alpha + \partial_x(\rho_\alpha v_\alpha) = 0 \\ \partial_t(\rho_\alpha v_\alpha) + \partial_x\left[\rho_\alpha v_\alpha^2 + \nu_\alpha \rho_\alpha\right] = \chi_\alpha . \end{cases} \tag*{(7.1.21)$_\alpha$}$$

The coupling is induced by the source term $\chi_\alpha$ , which accounts for the momentum transfer to the $\alpha$-constituent by the other constituent, as a result of the disparity between $v_1$ and $v_2$ . In particular, $\chi_1 + \chi_2 = 0$. In nonisothermal flow, the coupling is enhanced by the balance law of energy. In more sophisticating modeling of mixtures, the density gradient appears, along with the density, as a state variable (Fick's law), in which case second order spatial derivatives of the concentrations emerge in the field equations. Such terms induce diffusion, similar to the effect of heat conduction or viscosity. Here, however, we shall deal with the simple system $(7.1.21)_1$–$(7.1.21)_2$ , which is hyperbolic.

So as to realize the mixture as a single continuous medium, it is expedient to replace the original state vector $(\rho_1, \rho_2, v_1, v_2)$ with new state variables $(\rho, c, v, m)$, where $\rho$ and $v$ are the density and mean velocity of the mixture, that is, $\rho = \rho_1 + \rho_2$ , $\rho v = \rho_1 v_1 + \rho_2 v_2$ , $c$ is the concentration of the first constituent, i.e. $c = \rho_1/\rho$, and $m = (-1)^\alpha \rho_\alpha(v - v_\alpha)$. It is assumed that $\chi_\alpha = \beta\rho_\alpha(v_\alpha - v) = (-1)^\alpha \beta m$, where $\beta$ is a positive constant. One may then rewrite the system (7.1.21) in the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho c) + \partial_x(\rho c v + m) = 0 \\ \partial_t(\rho v) + \partial_x\left[\rho v^2 + (\nu_2 + (\nu_1 - \nu_2)c)\rho + \dfrac{m^2}{\rho c(1-c)}\right] = 0 \\ \partial_t(\rho c v + m) + \partial_x\left[\rho c v^2 + 2mv + \dfrac{m^2}{\rho c} + \nu_1 c \rho\right] = -\beta m. \end{cases} \tag{7.1.22}$$

Indeed, the second and fourth equations in the above system are just $(7.1.21)_1$, rewritten in terms of the new state variables, while the first and the third equations are obtained by adding the corresponding equations of $(7.1.21)_1$ and $(7.1.21)_2$ .

Single-space-dimensional systems (7.1.1) also derive from multispace-dimensional systems (3.1.1), in the presence of symmetry (planar, cylindrical, radial, etc.) that reduces spatial dependence to a single parameter. In that process, parent multidimensional homogeneous systems of conservation laws may yield one-dimensional inhomogeneous systems of balance laws, as a reflection of multidimensional geometric effects. For example, the single-space-dimensional system governing radial, isentropic gas flow, which results from the homogeneous Euler equations (3.3.21) is inhomogeneous:

$$\begin{cases} \partial_t \rho + \partial_r(\rho v) + \dfrac{2\rho v}{r} = 0 \\ \\ \partial_t(\rho v) + \partial_r[\rho v^2 + p(\rho)] + \dfrac{2\rho v^2}{r} = 0. \end{cases} \tag{7.1.23}$$

In particular, certain multidimensional phenomena, such as wave focusing, may be investigated in the framework of one-space dimension.

## 7.2 Hyperbolicity and Strict Hyperbolicity

As in earlier chapters, to avoid inessential technical complications, the theory will be developed in the context of homogeneous systems of conservation laws in canonical form:

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0. \tag{7.2.1}$$

$F$ is a $C^3$ map from an open convex subset $\mathcal{O}$ of $\mathbb{R}^n$ to $\mathbb{R}^n$.

Often in the applications, systems (7.2.1) govern planar front solutions, namely, $U = U(\nu \cdot x, t)$, in the spatial direction $\nu \in S^{m-1}$, of multispace-dimensional systems of conservation laws (4.1.1). In that connection,

$$F(U) = \sum_{\alpha=1}^{m} \nu_\alpha G_\alpha(U), \qquad U \in \mathcal{O}. \tag{7.2.2}$$

Referring to the examples introduced in Section 7.1, in order to cast the system (7.1.5) of thermoelasticity to canonical form, we have to switch from $(u, v, s)$ to new state variables $(u, v, E)$, where $E = \varepsilon + \frac{1}{2}v^2$ is the total energy. Similarly, the system (7.1.9) of isentropic gas flow is written in canonical form in terms of the state variables $(\rho, m)$, where $m = \rho v$ is the momentum.

By Definition 3.1.1, the system (7.2.1) is hyperbolic if for every $U \in \mathcal{O}$ the $n \times n$ Jacobian matrix $\mathrm{D}F(U)$ has real eigenvalues $\lambda_1(U) \le \cdots \le \lambda_n(U)$ and $n$ lineary independent eigenvectors $R_1(U), \cdots, R_n(U)$. For future use, we also introduce left (row) eigenvectors $L_1(U), \cdots, L_n(U)$ of $\mathrm{D}F(U)$, normalized by

$$L_i(U)R_j(U) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j. \end{cases} \tag{7.2.3}$$

Henceforth, the symbols $\lambda_i$ , $R_i$ and $L_i$ will be reserved to denote these quantities.

Clearly, the multispace-dimensional system (4.1.1) is hyperbolic if and only if all one-space-dimensional systems (7.2.1) resulting from it through (7.2.2), for arbitrary $\nu \in S^{m-1}$, are hyperbolic. Thus hyperbolicity is essentially a one-space-dimensional notion.

For the system (7.1.8) of one-dimensional isentropic elasticity, in Lagrangian coordinates, which will serve throughout as a vehicle for illustrating the general concepts, we have

$$\lambda_1 = -\sigma'(u)^{1/2}, \qquad \lambda_2 = \sigma'(u)^{1/2}, \tag{7.2.4}$$

$$R_1 = \tfrac{1}{2}\begin{pmatrix} -\sigma'(u)^{-1/2} \\ -1 \end{pmatrix}, \qquad R_2 = \tfrac{1}{2}\begin{pmatrix} -\sigma'(u)^{-1/2} \\ 1 \end{pmatrix}, \tag{7.2.5}$$

$$L_1 = (-\sigma'(u)^{1/2}\ ,\ -1), \qquad L_2 = (-\sigma'(u)^{1/2}\ ,\ 1). \tag{7.2.6}$$

The eigenvalue $\lambda_i$ of $\mathrm{D}F$, $i = 1, \cdots, n$, is called the *$i$-characteristic speed* of the system (7.2.1). The term derives from the following

**7.2.1 Definition.** An *$i$-characteristic*, $i = 1, \cdots, n$, of the system (7.2.1), associated with a classical solution $U$, is a $C^1$ function $x = x(t)$, with graph contained in the domain of $U$, which is an integral curve of the ordinary differential equation

$$\frac{dx}{dt} = \lambda_i(U(x,t)). \tag{7.2.7}$$

The standard existence-uniqueness theory for ordinary differential equations (7.2.7) implies that through any point $(\bar{x}, \bar{t})$ in the domain of a classical solution of (7.2.1) passes precisely one characteristic of each characteristic family.

Characteristics are carriers of waves of various types. For example, Eq. (1.6.1), for the general system (1.4.3) of balance laws, specialized to (7.2.1), implies that weak fronts propagate along characteristics. As a result, the presence of multiple eigenvalues of $\mathrm{D}F$ may induce severe complexity in the behavior of solutions, because of resonance. It is thus natural to single out systems that are free from such complication:

**7.2.2 Definition.** The system (7.2.1) is *strictly hyperbolic* if for any $U \in \mathcal{O}$ the Jacobian $\mathrm{D}F(U)$ has real, distinct eigenvalues

$$\lambda_1(U) < \cdots < \lambda_n(U). \tag{7.2.8}$$

By virtue of (7.2.4), the system (7.1.8) of isentropic elasticity in Lagrangian coordinates is strictly hyperbolic. The same is true for the system (7.1.5) of adiabatic thermoelasticity, for which the characteristic speeds are

$$\lambda_1 = -\sigma_u(u,s)^{1/2}, \qquad \lambda_2 = 0, \qquad \lambda_3 = \sigma_u(u,s)^{1/2}. \tag{7.2.9}$$

The system (7.1.10) for the polytropic gas has characteristic speeds

$$\lambda_1 = v - (\kappa\gamma)^{1/2}\rho^{\frac{\gamma-1}{2}}, \qquad \lambda_2 = v + (\kappa\gamma)^{1/2}\rho^{\frac{\gamma-1}{2}}, \tag{7.2.10}$$

and so it is strictly hyperbolic on the part of the state space with $\rho > 0$.

Furthermore, any one-dimensional system resulting, through (7.2.2), from the Euler equations for two-dimensional isentropic flow is strictly hyperbolic.

In view of the above examples, the reader may form the impression that strict hyperbolicity is the norm in systems arising in continuum physics. However, this is not the case. For example, the system (7.1.12) of planar elastic oscillations fails to be strictly hyperbolic in those directions $\nu$ for which the acoustic tensor (3.3.8) has multiple eigenvalues. Indeed, it has been shown that in one-space dimensional systems (7.2.1), of size $n = \pm 2, \pm 3, \pm 4 \pmod 8$, which result from parent three-space-dimensional systems (4.1.1) through (7.2.2), strict hyperbolicity necessarily fails, at least in some spatial direction $\nu \in S^2$. In particular, one-dimensional systems resulting from the Euler equations for two-dimensional non-isentropic flow ($n = 4$), or for three-dimensional isentropic or non-isentropic flow ($n = 4$ or $n = 5$) are not strictly hyperbolic. Actually, failure of strict hyperbolicity is often a byproduct of symmetry. For instance, the systems (7.1.14) and (7.1.15) are not strictly hyperbolic.

In systems of size $n = 2$, strict hyperbolicity typically fails at isolated *umbilic points*, at which $\mathrm{D}F$ reduces to a multiple of the identity matrix. Even the presence of a single umbilic point is sufficient to create havoc in the behavior of solutions. This will be demonstrated in following chapters by means of the simple system

$$\begin{cases} \partial_t u + \partial_x[(u^2+v^2)u] = 0 \\ \partial_t v + \partial_x[(u^2+v^2)v] = 0, \end{cases} \tag{7.2.11}$$

which is a caricature of (7.1.14) and (7.1.15). The characteristic speeds of (7.2.11) are

$$\lambda_1 = u^2 + v^2, \qquad \lambda_2 = 3(u^2+v^2), \tag{7.2.12}$$

with corresponding eigenvectors

$$R_1 = \begin{pmatrix} v \\ -u \end{pmatrix}, \qquad R_2 = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{7.2.13}$$

so this system is strictly hyperbolic, except at the origin $(0,0)$ which is an umbilic point.

We close this section with the derivation of a useful identity. We apply D to both sides of the equation $\mathrm{D}FR_j = \lambda_j R_j$ and then multiply, from the left, by $R_k^\top$; we also apply D to $\mathrm{D}FR_k = \lambda_k R_k$ and then multiply, from the left, by $R_j^\top$. Upon subtracting the resulting two equations, we deduce

$$(7.2.14)\qquad (\mathrm{D}\lambda_j R_k)R_j - (\mathrm{D}\lambda_k R_j)R_k$$
$$= \mathrm{D}F[R_j, R_k] - \lambda_j \mathrm{D}R_j R_k + \lambda_k \mathrm{D}R_k R_j\,, \qquad j,k = 1,\cdots,n,$$

where $[R_j, R_k]$ denotes the Lie bracket:

$$(7.2.15)\qquad [R_j, R_k] = \mathrm{D}R_j R_k - \mathrm{D}R_k R_j\,.$$

In particular, at a point $U \in \mathscr{O}$ where strict hyperbolicity fails, say $\lambda_j(U) = \lambda_k(U)$, (7.2.14) yields

$$(7.2.16)\qquad (\mathrm{D}\lambda_j R_k)R_j - (\mathrm{D}\lambda_k R_j)R_k = (\mathrm{D}F - \lambda_j I)[R_j, R_k].$$

Upon multiplying (7.2.16), from the left, by $L_j(U)$ and by $L_k(U)$, we conclude from (7.2.3):

$$(7.2.17)\qquad \mathrm{D}\lambda_j(U)R_k(U) = \mathrm{D}\lambda_k(U)R_j(U) = 0.$$

## 7.3 Riemann Invariants

Consider a hyperbolic system (7.2.1) of conservation laws on $\mathscr{O} \subset \mathbb{R}^n$. A very important concept is introduced by the following

**7.3.1 Definition.** An *$i$-Riemann invariant* of (7.2.1) is a smooth scalar-valued function $w$ on $\mathscr{O}$ such that

$$(7.3.1)\qquad \mathrm{D}w(U)R_i(U) = 0, \qquad U \in \mathscr{O}.$$

For example, recalling (7.2.5), one readily verifies that the functions

$$(7.3.2)\qquad w = -\int^u \sigma'(\omega)^{\frac{1}{2}} d\omega + v, \qquad z = -\int^u \sigma'(\omega)^{\frac{1}{2}} d\omega - v$$

are, respectively, 1- and 2-Riemann invariants of the system (7.1.8). Similarly, it can be shown that

$$(7.3.3)\qquad w = v + \frac{2(\kappa\gamma)^{1/2}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}, \qquad z = v - \frac{2(\kappa\gamma)^{1/2}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}$$

are 1- and 2-Riemann invariants of the system (7.1.10) of isentropic flow of a polytropic gas.[1]

[1] In the isothermal case, $\gamma = 1$, $w = v + \kappa^{1/2}\log\rho$, $z = v - \kappa^{1/2}\log\rho$.

By solving the first order linear differential equation (7.3.1) for $w$, one may construct in the vicinity of any point $U \in \mathscr{O}$ $n-1$ $i$-Riemann invariants whose gradients are linearly independent and span the orthogonal complement of $R_i$. For example, the reader may verify as an exercise that the three pairs of functions

$$\begin{cases} s \ , \ -\int^{u} \sigma_{\omega}(\omega, s)^{\frac{1}{2}} d\omega + v \\ v \ , \ \sigma(u, s) \\ s \ , \ -\int^{u} \sigma_{\omega}(\omega, s)^{\frac{1}{2}} d\omega - v \end{cases} \tag{7.3.4}$$

are, respectively, 1-, 2-, and 3-Riemann invariants of the system (7.1.5) of adiabatic thermoelasticity.

Riemann invariants are particularly useful in systems with the following special structure:

**7.3.2 Definition.** The system (7.2.1) is endowed with a *coordinate system of Riemann invariants* if there exist $n$ scalar-valued functions $(w_1, \cdots, w_n)$ on $\mathscr{O}$ such that, for any $i, j = 1, \cdots, n$, with $i \neq j$, $w_j$ is an $i$-Riemann invariant of (7.2.1).

An immediate consequence of Definitions 7.3.1 and 7.3.2 is

**7.3.3 Theorem.** *The functions* $(w_1, \cdots, w_n)$ *form a coordinate system of Riemann invariants for* (7.2.1) *if and only if*

$$\mathrm{D}w_i(U) R_j(U) \begin{cases} = 0 \text{ if } i \neq j \\ \neq 0 \text{ if } i = j \end{cases} \tag{7.3.5}$$

*i.e., if and only if, for* $i = 1, \cdots, n$, $\mathrm{D}w_i(U)$ *is a left eigenvector of the matrix* $\mathrm{D}F(U)$, *associated with the characteristic speed* $\lambda_i(U)$. *Equivalently, the tangent hyperplane to the level surface of* $w_i$ *at any point* $U$, *is spanned by the vectors* $R_1(U), \ldots, R_{i-1}(U), R_{i+1}(U), \ldots, R_n(U)$.

Assuming (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants and multiplying from the left by $\mathrm{D}w_i$, $i = 1, \cdots, n$, we reduce this system to diagonal form:

$$\partial_t w_i + \lambda_i \partial_x w_i = 0, \qquad i = 1, \cdots, n, \tag{7.3.6}$$

which is equivalent to the original form (7.2.1), albeit only in the context of classical solutions. The left-hand side of (7.3.6) is just the derivative of $w_i$ in the $i$-characteristic direction. Therefore,

**7.3.4 Theorem.** *Assume* $(w_1, \cdots, w_n)$ *form a coordinate system of Riemann invariants for* (7.2.1). *For* $i = 1, \cdots, n$, $w_i$ *stays constant along every* $i$*-characteristic associated with any classical solution* $U$ *of* (7.2.1).

Clearly, any hyperbolic system of two conservation laws is endowed with a coordinate system of Riemann invariants. By contrast, in systems of size $n \geq 3$, coordinate systems of Riemann invariants will exist only in the exceptional case where the formally overdetermined system (7.3.5), with $n(n-1)$ equations for the $n$ unknown $(w_1, \cdots, w_n)$, has a solution. By the Frobenius theorem, the hyperplane to the level surface of $w_i$ will be spanned by $R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n$ if and only if, for $i \neq j \neq k \neq i$, the Lie bracket $[R_j, R_k]$ (cf. (7.2.15)) lies in the span of $\{R_1, \cdots, R_{i-1}, R_{i+1}, \cdots, R_n\}$. Consequently, the system (7.2.1) is endowed with a coordinate system of Riemann invariants if and only if

$$[R_j, R_k] = \alpha_j^k R_j - \alpha_k^j R_k \,, \qquad j, k = 1, \cdots, n, \tag{7.3.7}$$

where the $\alpha_i^\ell$ are scalar fields.

When a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants exists for (7.2.1), it is convenient to normalize the eigenvectors $R_1, \cdots, R_n$ so that

$$\mathrm{D}w_i(U) R_j(U) = \begin{cases} 0 \text{ if } i \neq j \\ \\ 1 \text{ if } i = j. \end{cases} \tag{7.3.8}$$

In that case we note the identity

$$\begin{aligned} \mathrm{D}w_i \mathrm{D}R_j R_k &= \mathrm{D}(\mathrm{D}w_i R_j) R_k - R_j^\top \mathrm{D}^2 w_i R_k \\ &= -R_j^\top \mathrm{D}^2 w_i R_k \,, \qquad i, j, k = 1, \cdots, n, \end{aligned} \tag{7.3.9}$$

which implies, in particular, $\mathrm{D}w_i [R_j, R_k] = 0$, $i = 1, \cdots, n$, i.e.,

$$[R_j, R_k] = 0, \qquad j, k = 1, \cdots, n. \tag{7.3.10}$$

Recalling the identity (7.2.14) and using (7.2.15), (7.3.10), we deduce that whenever $\lambda_j(U) \neq \lambda_k(U)$, $\mathrm{D}R_j(U) R_k(U)$ lies in the span of $\{R_j(U), R_k(U)\}$. This, together with (7.3.8) and (7.3.9), yields

$$R_j^\top \mathrm{D}^2 w_i R_k = -\mathrm{D}w_i \mathrm{D}R_j R_k = 0, \qquad i \neq j \neq k \neq i. \tag{7.3.11}$$

When (7.2.1) possesses a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants, the map that carries $U$ to $W = (w_1, \cdots, w_n)^\top$ is locally a diffeomorphism. It is often convenient to regard $W$ rather than $U$ as the state vector. To avoid proliferation of symbols, when there is no danger of confusion we shall be using the same symbol to denote fields as functions of either $U$ or $W$. By virtue of (7.3.8), $\partial U / \partial w_i = R_i$ and so the chain rule yields, for the typical function $\phi$,

$$\frac{\partial \phi}{\partial w_i} = \mathrm{D}\phi R_i \,, \qquad i = 1, \cdots, n. \tag{7.3.12}$$

For example, (7.3.10) reduces to $\partial R_j / \partial w_k = \partial R_k / \partial w_j = \partial^2 U / \partial w_j \partial w_k$ .

We proceed to derive certain identities that will help us later to establish other remarkable properties of systems endowed with a coordinate system of Riemann invariants. Upon combining (7.2.14), (7.2.15), (7.3.10) and (7.3.12), we deduce

$$-\frac{\partial R_j}{\partial w_k} = g_{jk} R_j + g_{kj} R_k \,, \qquad j,k = 1,\cdots,n; \quad j \neq k, \tag{7.3.13}$$

where we have set

$$g_{jk} = \frac{1}{\lambda_j - \lambda_k} \frac{\partial \lambda_j}{\partial w_k} \,, \qquad j,k = 1,\cdots,n \;\; ; \quad j \neq k. \tag{7.3.14}$$

Notice that $g_{jk}$ may be defined even when $\lambda_j = \lambda_k$, because at such points $\partial\lambda_j/\partial w_k = 0$, by virtue of (7.2.17) and (7.3.12). From (7.3.13),

$$\begin{aligned} -\frac{\partial^2 R_j}{\partial w_i \partial w_k} = {} & \frac{\partial g_{jk}}{\partial w_i} R_j - g_{jk}(g_{ji} R_j + g_{ij} R_i) \\ & + \frac{\partial g_{kj}}{\partial w_i} R_k - g_{kj}(g_{ki} R_k + g_{ik} R_i). \end{aligned} \tag{7.3.15}$$

Since $R_i$, $R_j$, $R_k$ are linearly independent for $i \neq j \neq k \neq i$, and the right-hand side of (7.3.15) has to be symmetric in $(i, k)$, we deduce

$$\frac{\partial g_{jk}}{\partial w_i} = \frac{\partial g_{ji}}{\partial w_k} \,, \quad i \neq j \neq k \neq i, \tag{7.3.16}$$

$$\frac{\partial g_{ij}}{\partial w_k} + g_{ij} g_{jk} - g_{ij} g_{ik} + g_{ik} g_{kj} = 0, \quad i \neq j \neq k \neq i. \tag{7.3.17}$$

Of the hyperbolic systems of conservation laws of size $n \geq 3$ that arise in the applications, few possess coordinate systems of Riemann invariants. A noteworthy example is the system of *electrophoresis*:

$$\partial_t U^i + \partial_x \left[ \frac{c_i U^i}{\sum_{j=1}^{n} U^j} \right] = 0, \qquad i = 1,\cdots,n, \tag{7.3.18}$$

where $c_1 < c_2 < \cdots < c_n$ are positive constants. This system governs the process used to separate $n$ ionized chemical compounds in solution by applying an electric field. In that context, $U^i$ denotes the concentration and $c_i$ measures the electrophoretic mobility of the $i$-th species. In particular, $U^i \geq 0$. As an exercise, the reader may verify that the characteristic speeds of (7.3.18) are given by

$$\lambda_i = \mu_i \sum_{j=1}^{n} U^j \,, \qquad i = 1,\cdots,n, \tag{7.3.19}$$

where for $i = 1,\cdots,n-1$ the value of $\mu_i$ at $U$ is the solution of the equation

(7.3.20)
$$\sum_{j=1}^{n} \frac{c_j U^j}{c_j - \mu} = \sum_{j=1}^{n} U^j$$

lying in the interval $(c_i, c_{i+1})$; and $\mu_n = 0$. Moreover, (7.3.18) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants, where, for $i = 1, \cdots, n-1$, the value of $w_i$ at $U$ is the solution of the equation

(7.3.21)
$$\sum_{j=1}^{n} \frac{U^j}{c_j - w} = 0$$

that lies in the interval $(c_i, c_{i+1})$; and

(7.3.22)
$$w_n = \sum_{j=1}^{n} \frac{1}{c_j} U^j \,.$$

Later we shall see that the system (7.3.18) has very special structure and a host of interesting properties.

Another interesting system endowed with coordinate systems of Riemann invariants is (7.1.17), which, as we recall, governs the propagation of planar electromagnetic waves through special isotropic dielectrics. This is seen by passing from $(B_1, B_2, D_1, D_2)$ to the new state vector $(p, q, a, b)$ defined through

(7.3.23)
$$\begin{cases} \sqrt{2}p \exp(ia) = B_2 + D_1 - i(B_1 - D_2) \\ \\ \sqrt{2}q \exp(ib) = -B_2 + D_1 + i(B_1 + D_2). \end{cases}$$

In particular, $p^2 + q^2 = r^2$. A simple calculation shows that, at least in the context of classical solutions, (7.1.17) reduces to

(7.3.24)
$$\begin{cases} \partial_t p + \partial_x \left[ \dfrac{\psi'(r)}{r} p \right] = 0 \\ \\ \partial_t q - \partial_x \left[ \dfrac{\psi'(r)}{r} q \right] = 0, \end{cases}$$

(7.3.25)
$$\begin{cases} \partial_t a + \dfrac{\psi'(r)}{r} \partial_x a = 0 \\ \\ \partial_t b - \dfrac{\psi'(r)}{r} \partial_x b = 0. \end{cases}$$

Notice that (7.3.24) constitutes a closed system of two conservation laws, from which $p, q$, and thereby $r$, may be determined. Subsequently (7.3.25) may be solved, as two independent nonhomogeneous scalar conservation laws, to determine $a$ and $b$. In particular, $a$ and $b$ together with any pair of Riemann invariants of (7.3.24) will constitute a coordinate system of Riemann invariants for (7.1.17).

## 7.4 Entropy-Entropy Flux Pairs

Entropies play a central role in the theory of hyperbolic systems of conservation laws in one-space dimension. Adapting the discussion of Section 3.2 to the present setting, we infer that functions $\eta$ and $q$ on $\mathcal{O}$ constitute an entropy-entropy flux pair for the system (7.2.1) if

$$\mathrm{D}q(U) = \mathrm{D}\eta(U)\mathrm{D}F(U), \qquad U \in \mathcal{O}. \tag{7.4.1}$$

Furthermore, the integrability condition (3.2.4) here reduces to

$$\mathrm{D}^2\eta(U)\mathrm{D}F(U) = \mathrm{D}F(U)^\top \mathrm{D}^2\eta(U), \qquad U \in \mathcal{O}. \tag{7.4.2}$$

Upon multiplying (7.4.2) from the left by $R_j(U)^\top$ and from the right by $R_k(U)$, $j \neq k$, we deduce that (7.4.2) is equivalent to

$$R_j(U)^\top \mathrm{D}^2\eta(U) R_k(U) = 0, \qquad j,k = 1,\cdots,n; \quad j \neq k, \tag{7.4.3}$$

with the understanding that (7.4.3) holds automatically when $\lambda_j(U) \neq \lambda_k(U)$ but may require renormalization of eigenvectors $R_i$ associated with multiple characteristic speeds. (Compare with (3.2.5).) Note that the requirement that some entropy $\eta$ is convex may now be conveniently expressed as

$$R_j(U)^\top \mathrm{D}^2\eta(U) R_j(U) > 0, \qquad j = 1,\cdots,n. \tag{7.4.4}$$

When the system (7.2.1) is symmetric,

$$\mathrm{D}F(U)^\top = \mathrm{D}F(U), \qquad U \in \mathcal{O}, \tag{7.4.5}$$

it admits two interesting entropy-entropy flux pairs:

$$\eta = \tfrac{1}{2}|U|^2, \qquad q = U \cdot F(U) - h(U), \tag{7.4.6}$$

$$\eta = h(U), \qquad q = \tfrac{1}{2}|F(U)|^2, \tag{7.4.7}$$

where $h$ is defined by the condition

$$\mathrm{D}h(U) = F(U)^\top. \tag{7.4.8}$$

As explained in Chapter III, the systems (7.1.5), (7.1.8), (7.1.10) are endowed with entropy-entropy flux pairs, respectively,

$$\eta = -s, \qquad q = 0, \tag{7.4.9}$$

$$\eta = \tfrac{1}{2}v^2 + e(u), \qquad q = -v\sigma(u), \qquad e(u) = \int^u \sigma(\omega)d\omega, \tag{7.4.10}$$

(7.4.11) $$\eta = \frac{1}{2}\rho v^2 + \frac{\kappa}{\gamma - 1}\rho^\gamma, \qquad q = \frac{1}{2}\rho v^3 + \frac{\kappa\gamma}{\gamma - 1}\rho^\gamma v,$$

induced by the Second Law of thermodynamics.[2] In fact, (7.4.10), with $v\sigma$ and $\sigma d\omega$ interpreted as $v \cdot \sigma$ and $\sigma \cdot d\omega$, constitutes an entropy-entropy flux pair even for the system (7.1.12). When expressed as functions of the canonical state variables, that is $(u, v, E)$ for (7.4.9), $(u, v)$ for (7.4.10), and $(\rho, m)$ for (7.4.11), the above entropies are convex.

In developing the theory of systems (7.2.1), it will be useful to construct entropies with given specifications. These must be solutions of (7.4.2), which is a linear, second order system of $\frac{1}{2}n(n-1)$ partial differential equations in a single unknown $\eta$. Thus, when $n = 2$, (7.4.2) reduces to a single linear hyperbolic equation which may be solved to produce an abundance of entropies. By contrast, for $n \geq 3$, (7.4.2) is formally overdetermined. Notwithstanding the presence of special solutions such as (7.4.6) and (7.4.7), one should not expect an abundance of entropies, unless (7.2.1) is special. It is remarkable that the overdeterminacy of (7.4.2) vanishes when (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants. In that case it is convenient to seek $\eta$ and $q$ as functions of the state vector $W = (w_1, \cdots, w_n)^\top$. Upon multiplying (7.4.1), from the right, by $R_j(U)$ and by using (7.3.12), we deduce that (7.4.1) is now equivalent to

(7.4.12) $$\frac{\partial q}{\partial w_j} = \lambda_j \frac{\partial \eta}{\partial w_j}, \qquad j = 1, \cdots, n.$$

The integrability condition associated with (7.4.12) takes the form

(7.4.13) $$\frac{\partial^2 \eta}{\partial w_j \partial w_k} + g_{jk}\frac{\partial \eta}{\partial w_j} + g_{kj}\frac{\partial \eta}{\partial w_k} = 0, \qquad j, k = 1, \cdots, n; \quad j \neq k,$$

where $g_{jk}, g_{kj}$ are the functions defined through (7.3.14). An alternative, useful expression for $g_{jk}$ arises if one derives (7.4.13) directly from (7.4.3). Indeed, for $j, k = 1, \cdots, n$,

(7.4.14) $$\begin{aligned} R_j^\top \mathrm{D}^2\eta R_k &= \mathrm{D}(\mathrm{D}\eta R_j)R_k - \mathrm{D}\eta \mathrm{D}R_j R_k \\ &= \mathrm{D}(\mathrm{D}\eta R_j)R_k - \sum_{i=1}^{n} \frac{\partial \eta}{\partial w_i}\mathrm{D}w_i \mathrm{D}R_j R_k. \end{aligned}$$

Combining (7.4.3), (7.3.12), (7.3.10) and (7.3.9), we arrive at an equation of the form (7.4.13) with

(7.4.15) $$g_{jk} = R_j^\top \mathrm{D}^2 w_j R_k\,, \qquad j, k = 1, \cdots, n; \quad j \neq k.$$

The reader may verify directly, as an exercise, with the help of (7.2.14), (7.3.8), (7.3.11), (7.3.10), (7.3.9) and (7.3.12) that (7.3.14) and (7.4.15) are equivalent.

[2] In the isothermal case, $\gamma = 1$, the entropy-entropy flux pair of (7.1.10) takes the following form: $\eta = \frac{1}{2}\rho v^2 + \kappa\rho\log\rho,\ q = \frac{1}{2}\rho v^3 + \kappa\rho v\log\rho + \kappa\rho v$.

Applying (7.4.14) with $k = j$, using (7.3.12), (7.3.9) and recalling (7.4.4), we deduce that, in terms of Riemann invariants, the convexity condition on $\eta$ is expressed by the set of inequalities

$$\frac{\partial^2 \eta}{\partial w_j^2} + \sum_{i=1}^{n} a_{ij} \frac{\partial \eta}{\partial w_i} \geq 0, \qquad j = 1, \cdots, n, \tag{7.4.16}$$

where

$$a_{ij} = R_j^\top \mathrm{D}^2 w_i R_j \,, \qquad i, j = 1, \cdots, n. \tag{7.4.17}$$

The system (7.4.13) contains $\frac{1}{2}n(n-1)$ equations in the single unknown $\eta$ and thus looks overdetermined when $n \geq 3$. It turns out, however, that this set of equations is internally consistent. To see this, differentiate (7.4.13) with respect to $w_i$ , $i \neq j \neq k \neq i$, to get

$$\begin{aligned} \frac{\partial^3 \eta}{\partial w_i \partial w_j \partial w_k} = \frac{\partial g_{jk}}{\partial w_i} \frac{\partial \eta}{\partial w_j} - g_{jk} \left( g_{ji} \frac{\partial \eta}{\partial w_j} + g_{ij} \frac{\partial \eta}{\partial w_i} \right) \\ + \frac{\partial g_{kj}}{\partial w_i} \frac{\partial \eta}{\partial w_k} - g_{kj} \left( g_{ki} \frac{\partial \eta}{\partial w_i} + g_{ik} \frac{\partial \eta}{\partial w_i} \right). \end{aligned} \tag{7.4.18}$$

The system (7.4.13) will be integrable if and only if, for $i \neq j \neq k \neq i$, the right-hand side of (7.4.18) is symmetric in $(i, j, k)$. But this is always the case, on account of the identities (7.3.16) and (7.3.17). Consequently, in a neighborhood of any given state $\bar{W} = (\bar{w}_1, \cdots, \bar{w}_n)^\top$, there exists a unique entropy $\eta$ with arbitrarily prescribed values $\{\eta(w_1, \bar{w}_2, \cdots, \bar{w}_n), \eta(\bar{w}_1, w_2, \cdots, \bar{w}_n)\ , \cdots, \eta(\bar{w}_1, \cdots, \bar{w}_{n-1}, w_n)\}$ along straight lines parallel to the coordinate axes. When $n = 2$, this amounts to solving a classical Goursat problem.

We have thus shown that systems endowed with coordinate systems of Riemann invariants are also endowed with an abundance of entropies. For this reason, such systems are called *rich*. In particular, the system (7.3.18) of electrophoresis and the system (7.1.17) of electromagnetic waves are rich. The reader will find how to construct the family of entropies of these systems in the references cited in Section 7.10.

## 7.5 Genuine Nonlinearity and Linear Degeneracy

The feature distinguishing the behavior of linear and nonlinear hyperbolic systems of conservation laws is that in the former, characteristic speeds being constant, all waves of the same family propagate with fixed speed; while in the latter, wave speeds vary with wave-amplitude. As we proceed with our study, we will encounter various manifestations of nonlinearity, and in every case we shall notice that its effects will be particularly pronounced when the characteristic speeds $\lambda_i$ vary in the direction of the corresponding eigenvectors $R_i$ . This motivates the following

**7.5.1 Definition.** For the hyperbolic system (7.2.1) of conservation laws on $\mathcal{O}$, $U$ in $\mathcal{O}$ is called a *state of genuine nonlinearity of the i-characteristic family* if

$$\mathrm{D}\lambda_i(U)R_i(U) \neq 0, \tag{7.5.1}$$

or a *state of linear degeneracy of the i-characteristic family* if

$$\mathrm{D}\lambda_i(U)R_i(U) = 0. \tag{7.5.2}$$

When (7.5.1) holds for all $U \in \mathcal{O}$, $i$ is a *genuinely nonlinear characteristic family* while if (7.5.2) is satisfied for all $U \in \mathcal{O}$, then $i$ is a *linearly degenerate characteristic family*. When every characteristic family is genuinely nonlinear, (7.2.1) is a *genuinely nonlinear system.*

It is clear that the $i$-characteristic family is linearly degenerate if and only if the $i$-characteristic speed $\lambda_i$ is constant along the integral curves of the vector field $R_i$.

The scalar conservation law (7.1.2), with characteristic speed $\lambda = f'(u)$, is genuinely nonlinear when $f$ has no inflection points: $f''(u) \neq 0$. In particular, the Burgers equation (4.2.1) is genuinely nonlinear.

Using (7.2.4) and (7.2.5), one readily checks that the system (7.1.8) is genuinely nonlinear when $\sigma''(u) \neq 0$. As an exercise, the reader may verify that the system (7.1.9) is genuinely nonlinear if $2p'(\rho) + \rho p''(\rho) > 0$ so, in particular, the system (7.1.10) for the polytropic gas is genuinely nonlinear. The system (7.1.16) of waves in shallow water is likewise genuinely nonlinear.

By account of (7.2.9), the 2-characteristic family of the system (7.1.5) of thermoelasticity is linearly degenerate. It turns out that the other two characteristic families are genuinely nonlinear, provided $\sigma_{uu}(u, s) \neq 0$.

Consider next the system (7.1.12) of planar elastic oscillations in the direction $\nu$, recalling that $\sigma(u) = \partial e(u)/\partial u$, with $e(u)$ convex. The six characteristic speeds are the square roots $\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \pm\sqrt{\mu_3}$ of the eigenvalues $\mu_1(u), \mu_2(u), \mu_3(u)$ of the Hessian matrix of $e(u)$, namely the eigenvalues of the acoustic tensor (3.3.8) evaluated at $F = I + u \otimes \nu$. A simple calculation shows that the characteristic families associated with the characteristic speeds $\pm\sqrt{\mu_\ell}$ are genuinely nonlinear at $u = F\nu$ if

$$\sum_{i,j,k=1}^{3} \frac{\partial^3 e(u)}{\partial u_i \partial u_j \partial u_k} \xi_i \xi_j \xi_k = \sum_{i,j,k=1}^{3} \sum_{\alpha,\beta,\gamma=1}^{3} \frac{\partial^3 \varepsilon(F)}{\partial F_{i\alpha} \partial F_{j\beta} \partial F_{k\gamma}} \xi_i \xi_j \xi_k \nu_\alpha \nu_\beta \nu_\gamma \neq 0, \tag{7.5.3}$$

where $\xi$ is the eigenvector of the acoustic tensor associated with the eigenvalue $\mu_\ell$.

Applying the above to the special system (7.1.14), one finds that $\mu_1 = h''(|u|)$ is a simple eigenvalue, with eigenvector $u$, and $\mu_2 = \mu_3 = h'(|u|)/|u|$ is a double eigenvalue, with eigenspace the orthogonal complement of $u$. Thus, the characteristic speeds $\pm\left[h''(|u|)\right]^{1/2}$ are associated with longitudinal oscillations, while $\pm\left[h'(|u|)/|u|\right]^{1/2}$ are associated with transverse oscillations. However, only transverse oscillations that are also orthogonal to $\nu$ are compatible with incompressibility.

The characteristic families associated with $\pm\left[h''(|u|)\right]^{1/2}$ are genuinely nonlinear at $u$ if $h'''(|u|) \neq 0$, while the characteristic families associated with $\pm\left[h'(|u|)/|u|\right]^{1/2}$ are linearly degenerate. Clearly, the same conclusions apply to the system of elastic string oscillations (7.1.15), with $\tau(|u|)$ replacing $h'(|u|)$. For this system, all transverse oscillations are physically meaningful, as the incompressibility constraint is no longer relevant. The model system (7.2.11) exhibits similar behavior, as its 1-characteristic family is linearly degenerate, while its 2-characteristic family is genuinely nonlinear, except at the origin.

Finally, in the system (7.3.18) of electrophoresis the $n$-characteristic family is linearly degenerate while the rest are genuinely nonlinear.

The system of Maxwell's equations (3.3.40) for the Born-Infeld medium (3.3.45) has the remarkable property that planar oscillations in any spatial direction $\nu \in S^2$ are governed by a system whose characteristic families are all linearly degenerate.

Quite often, linear degeneracy results from the loss of strict hyperbolicity. Indeed, an immediate consequence of (7.2.17) is

**7.5.2 Theorem.** *In the hyperbolic system* (7.2.1) *of conservation laws, assume that the $j$- and $k$-characteristic speeds coincide*: $\lambda_j(U) = \lambda_k(U)\ ,\quad U \in \mathcal{O}$. *Then both the $j$- and the $k$-characteristic families are linearly degenerate.*

When the system (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants and one uses $W = (w_1, \cdots, w_n)^\top$ as state vector, the conditions of genuine nonlinearity and linear degeneracy assume an elegant and suggestive form. Indeed, upon using (7.3.12), we deduce that (7.5.1) and (7.5.2) are respectively equivalent to

(7.5.4) $$\frac{\partial \lambda_i}{\partial w_i} \neq 0$$

and

(7.5.5) $$\frac{\partial \lambda_i}{\partial w_i} = 0.$$

## 7.6 Simple Waves

In the context of classical solutions, the scalar conservation law (7.1.2), with characteristic speed $\lambda = f'(u)$, takes the form

(7.6.1) $$\partial_t u(x,t) + \lambda(u(x,t))\partial_x u(x,t) = 0.$$

As noted already in Section 6.1, by virtue of (7.6.1) $u$ stays constant along characteristics and this, in turn, implies that each characteristic propagates with constant speed, i.e., it is a straight line. It turns out that general hyperbolic systems (7.2.1) of conservation laws admit special solutions with the same features:

**7.6.1 Definition.** A classical, $C^1$ solution $U$ of the hyperbolic system (7.2.1) of conservation laws is called an *$i$-simple wave* if $U$ stays constant along any $i$-characteristic associated with it.

Thus a $C^1$ function $U$, defined on an open subset of $\mathbb{R}^2$ and taking values in $\mathcal{O}$, is an $i$-simple wave if it satisfies (7.2.1) together with

$$\partial_t U(x,t) + \lambda_i(U(x,t))\partial_x U(x,t) = 0. \tag{7.6.2}$$

In particular, in an $i$-simple wave each $i$-characteristic propagates with constant speed and so it is a straight line.

If $U$ is an $i$-simple wave, combining (7.2.1) with (7.6.2) we deduce

$$\begin{cases} \partial_x U(x,t) = a(x,t)R_i(U(x,t)) \\ \partial_t U(x,t) = -a(x,t)\lambda_i(U(x,t))R_i(U(x,t)), \end{cases} \tag{7.6.3}$$

where $a$ is a scalar field. Conversely, any $C^1$ function $U$ that satisfies (7.6.3) is necessarily an $i$-simple wave.

It is possible to give still another characterization of simple waves, in terms of Riemann invariants:

**7.6.2 Theorem.** *A classical, $C^1$ solution $U$ of* (7.2.1) *is an $i$-simple wave if and only if every $i$-Riemann invariant is constant on each connected component of the domain of $U$.*

**Proof.** For any $i$-Riemann invariant $w$, $\partial_x w = \mathrm{D}w\partial_x U$ and $\partial_t w = \mathrm{D}w\partial_t U$. If $U$ is an $i$-simple wave, $\partial_x w$ and $\partial_t w$ vanish identically, by virtue of (7.6.3) and (7.3.1), so that $w$ is constant on any connected component of the domain of $U$.

Conversely, recalling that the gradients of $i$-Riemann invariants span the orthogonal complement of $R_i$, we infer that when $\partial_x w = \mathrm{D}w\partial_x U$ vanishes identically for all $i$-Riemann invariants $w$, $\partial_x U$ must satisfy $(7.6.3)_1$. Substituting $(7.6.3)_1$ into (7.2.1) we conclude that $(7.6.3)_2$ holds as well, i.e. $U$ is an $i$-simple wave. This completes the proof.

Any constant function $U = \bar{U}$ qualifies, according to Definition 7.6.1, to be viewed as an $i$-simple wave, for every $i = 1, \cdots, n$. It is expedient, however, to refer to such trivial solutions as *constant states* and reserve the term simple wave for solutions that are not constant on any open subset of their domain. The following proposition, which demonstrates that simple waves are the natural neighbors of constant states, is stated informally, in physical rather than mathematical terminology. The precise meaning of assumptions and conclusions may be extracted from the proof.

**7.6.3 Theorem.** *Any weak front moving into a constant state propagates with constant characteristic speed of some family $i$. Furthermore, the wake of this front is necessarily an $i$-simple wave.*

**Proof.** The setting is as follows: The system (7.2.1) is assumed strictly hyperbolic. $U$ is a classical, Lipschitz solution which is $C^1$ on its domain, except along the graph of a $C^1$ curve $x = \chi(t)$. $U$ is constant, $\bar{U}$, at any point of its domain lying on one side, say to the right, of the graph of $\chi$. By contrast, $\partial_x U$ and $\partial_t U$ attain nonzero limits from the left along the graph of $\chi$. Thus, according to the terminology of Section 1.6, $\chi$ is a weak front propagating with speed $\dot{\chi} = d\chi/dt$. In particular, (1.6.1) here reduces to

$$[\mathrm{D}F(\bar{U}) - \dot{\chi} I][\![\partial U/\partial N]\!] = 0, \tag{7.6.4}$$

which shows that $\dot{\chi}$ is constant and equal to $\lambda_i(\bar{U})$ for some $i$.

Next we show that to the left of, and sufficently close to, the graph of $\chi$ the solution $U$ is an $i$-simple wave. By virtue of Theorem 7.6.2, it suffices to prove that $n-1$ independent $i$-Riemann invariants, which will be denoted by $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$, are constant.

For $U$ near $\bar{U}$, the $n-1$ vectors $\{\mathrm{D}w_1(U), \cdots, \mathrm{D}w_{i-1}(U), \mathrm{D}w_{i+1}(U), \cdots, \mathrm{D}w_n(U)\}$ span the orthogonal complement of $R_i(U)$ and so do the vectors $\{L_1(U), \cdots, L_{i-1}(U), L_{i+1}(U), \cdots, L_n(U)\}$. Consequently, there is a nonsingular $(n-1) \times (n-1)$ matrix $B(U)$ such that

$$L_j(U) = \sum_{k \neq i} B_{jk}(U)\mathrm{D}w_k(U), \qquad j = 1, \cdots, i-1, i+1, \cdots, n. \tag{7.6.5}$$

Multiplying (7.2.1), from the left, by $L_j(U)$ yields

$$L_j(U)\partial_t U + \lambda_j(U) L_j(U)\partial_x U = 0, \qquad j = 1, \cdots, n. \tag{7.6.6}$$

Combining (7.6.5) with (7.6.6), we conclude

$$\sum_{k \neq i} B_{jk}\partial_t w_k + \sum_{k \neq i} \lambda_j B_{jk}\partial_x w_k = 0, \qquad j = 1, \cdots, i-1, i+1, \cdots, n. \tag{7.6.7}$$

We regard (7.6.7) as a first order linear inhomogeneous system of $n-1$ equations in the $n-1$ unknowns $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$. In that sense, (7.6.7) is strictly hyperbolic, with characteristic speeds $\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n$. Along the graph of $\chi$, the $n-1$ Riemann invariants are constant, namely, equal to their values at $\bar{U}$ : $w_1(\bar{U}), \cdots, w_{i-1}(\bar{U}), w_{i+1}(\bar{U}), \cdots, w_n(\bar{U})$. Also the graph of $\chi$ is non-characteristic for the system (7.6.7). Consequently, the standard uniqueness theorem for the Cauchy problem for linear hyperbolic systems implies that (7.6.7) may admit only one solution compatible with the Cauchy data, namely the trivial one: $w_1 = w_1(\bar{U}), \cdots, w_{i-1} = w_{i-1}(\bar{U}), w_{i+1} = w_{i+1}(\bar{U}), \cdots, w_n = w_n(\bar{U})$. This completes the proof.

At any point $(x,t)$ in the domain of an $i$-simple wave $U$ of (7.2.1), we let $\xi(x,t)$ denote the slope at $(x,t)$ of the $i$-characteristic associated with $U$, i.e.,

$$\xi(x,t) = \lambda_i(U(x,t)). \tag{7.6.8}$$

The derivative of $\xi$ in the direction of the line with slope $\xi$ is zero, that is

$$\partial_t \xi + \xi \partial_x \xi = 0. \tag{7.6.9}$$

Thus $\xi$ satisfies the Burgers equation (4.2.1).

In the vicinity of any point $(\bar{x}, \bar{t})$ in the domain of $U$, we shall say that the $i$-simple wave is an *$i$-rarefaction wave* if $\partial_x \xi(\bar{x}, \bar{t}) > 0$, i.e., if the $i$-characteristics diverge, or an *$i$-compression wave* if $\partial_x \xi(\bar{x}, \bar{t}) < 0$, i.e., if the $i$-characteristics converge. This terminology originated in the context of gas dynamics.

Since in an $i$-simple wave $U$ stays constant along $i$-characteristics, on a small neighborhood $\mathcal{X}$ of any point $(\bar{x}, \bar{t})$ where $\partial_x \xi(\bar{x}, \bar{t}) \neq 0$ we may use the single variable $\xi$ to label $U$, i.e., there is a function $V_i$, defined on an interval $(\bar{\xi} - \varepsilon, \bar{\xi} + \varepsilon)$, with $\bar{\xi} = \lambda_i(U(\bar{x}, \bar{t}))$, taking values in $\mathcal{O}$ and such that

$$U(x,t) = V_i(\xi(x,t)), \qquad (x,t) \in \mathcal{X}. \tag{7.6.10}$$

Furthermore, by virtue of (7.6.3) and (7.6.8), $V_i$ satisfies

$$\dot{V}_i(\xi) = b(\xi) R_i(V(\xi)), \qquad \xi \in (\bar{\xi} - \varepsilon,\ \bar{\xi} + \varepsilon), \tag{7.6.11}$$

$$\lambda_i(V_i(\xi)) = \xi, \qquad \xi \in (\bar{\xi} - \varepsilon,\ \bar{\xi} + \varepsilon), \tag{7.6.12}$$

where $b$ is a scalar function and an overdot denotes derivative with respect to $\xi$.

Conversely, if $V_i$ satisfies (7.6.11), (7.6.12) and $\xi$ is any $C^1$ solution of (7.6.9) taking values in the interval $(\bar{\xi} - \varepsilon, \bar{\xi} + \varepsilon)$, then $U = V_i(\xi(x,t))$ is an $i$-simple wave. The above considerations motivate the following

**7.6.4 Definition.** An *$i$-rarefaction wave curve* in the state space $\mathbb{R}^n$, for the hyperbolic system (7.2.1), is a curve $U = V_i(\cdot)$, where the function $V_i$ satisfies (7.6.11) and (7.6.12).

Rarefaction wave curves will provide one of the principal tools for solving the Riemann problem in Chapter IX. The construction of these curves is particularly simple in the neighborhood of states of genuine nonlinearity:

**7.6.5 Theorem.** *Assume $\bar{U} \in \mathcal{O}$ is a state of genuine nonlinearity of the $i$-characteristic family of the hyperbolic system* (7.2.1) *of conservation laws. Then there exists a unique $i$-rarefaction wave curve $V_i$ through $\bar{U}$. If $R_i$ is normalized on a neighborhood of $\bar{U}$ through*

$$\mathrm{D}\lambda_i(U) R_i(U) = 1, \tag{7.6.13}$$

*and $V_i$ is reparametrized by $\tau = \xi - \bar{\xi}$, where $\bar{\xi} = \lambda_i(\bar{U})$, then $V_i$ is the solution of the ordinary differential equation*

$$\dot{V}_i = R_i(V_i) \tag{7.6.14}$$

*with initial condition $V_i(0) = \bar{U}$. In particular, $V_i$ is $C^3$. The more explicit notation $V_i(\tau; \bar{U})$ shall be employed when one needs to display the point of origin of this rarefaction wave curve.*

**Proof.** Any solution $V_i$ of (7.6.14) clearly satisfies (7.6.11) with $b = 1$. At $\xi = \bar{\xi}$, i.e. $\tau = 0$, $\lambda_i(V_i) = \lambda_i(\bar{U}) = \bar{\xi}$. Furthermore, $\dot{\lambda}_i(V_i) = \mathrm{D}\lambda_i(V_i)\dot{V}_i = 1$, by virtue of (7.6.14) and (7.6.13). This establishes (7.6.12) and completes the proof.

By contrast, when the $i$-characteristic family is linearly degenerate, differentiating (7.6.12) with respect to $\xi$ and combining the resulting equation with (7.6.11), yields a contradiction: $0 = 1$. In that case, $i$-characteristics in any $i$-simple wave are necessarily parallel straight lines. It is still true, however, that any $i$-simple wave takes values along some integral curve of the differential equation (7.6.14).

Motivated by Theorem 7.6.2, we may characterize rarefaction wave curves in terms of Riemann invariants:

**7.6.6 Theorem.** *Every $i$-Riemann invariant is constant along any $i$-rarefaction wave curve of the system* (7.2.1). *Conversely, if $\bar{U}$ is any state of genuine nonlinearity of the $i$-characteristic family of* (7.2.1) *and $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$ are independent $i$-Riemann invariants on some neighborhood of $\bar{U}$, then the $i$-rarefaction curve through $\bar{U}$ is determined implicitly by the system of equations $w_j(U) = w_j(\bar{U})$, for $j = 1, \cdots, i-1, i+1, \cdots, n$.*

**Proof.** Any $i$-rarefaction curve $V_i$ satisfies (7.6.11). If $w$ is an $i$-Riemann invariant of (7.2.1), multiplying (7.6.11), from the left, by $\mathrm{D}w(V_i(\xi))$ and using (7.3.1) yields $\dot{w}(V_i(\xi)) = 0$, i.e., $w$ stays constant along $V_i$.

Assume now $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$ are $i$-Riemann invariants such that $\mathrm{D}w_1, \cdots, \mathrm{D}w_{i-1}, \mathrm{D}w_{i+1}, \cdots, \mathrm{D}w_n$ are linearly independent. Then the $n-1$ surfaces $w_j(U) = w_j(\bar{U})$, $j = 1, \cdots, i-1, i+1, \cdots, n$, intersect transversely to form a $C^1$ curve $V_i$ through $\bar{U}$, parametrized by arclength $s$, whose tangent $V_i'$ must satisfy, on account of Definition 7.3.1, $V_i'(s) = c(s)R_i(V(s))$, for some nonzero scalar function $c$. For as long as $V_i$ is a state of genuine nonlinearity of the $i$-characteristic field, $\lambda_i'(V_i) = \mathrm{D}\lambda_i V_i' = c\mathrm{D}\lambda_i R_i \neq 0$. We may thus find the proper parametrization $s = s(\xi)$ so that $V_i$ satisfies both (7.6.11) and (7.6.12). This completes the proof.

As an application of Theorem 7.6.6, we infer that the 1- and 2-rarefaction wave curves of the system (7.1.8) through a point $(\bar{u}, \bar{v})$, with $\sigma''(\bar{u}) \neq 0$, are determined, in terms of the Riemann invariants (7.3.2), by the equations

$$v = \bar{v} + \int_{\bar{u}}^{u} \sqrt{\sigma'(\omega)}d\omega, \qquad v = \bar{v} - \int_{\bar{u}}^{u} \sqrt{\sigma'(\omega)}d\omega. \tag{7.6.15}$$

When the system (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants and we use $W = (w_1, \cdots, w_n)^\top$, instead of $U$, as our state variable, the rarefaction wave curves assume a very simple form. Indeed, by virtue of Theorem 7.6.4, the $i$-rarefaction wave curve through the point $\bar{W} = (\bar{w}_1, \cdots, \bar{w}_n)^\top$ is the straight line $w_j = \bar{w}_j$ , $j \neq i$, parallel to the $i$-axis.

## 7.7 Explosion of Weak Fronts

The aim here is to expose the decisive role played by genuine nonlinearity in the amplification and eventual explosion of weak fronts.

We consider a Lipschitz continuous solution $U$ of the strictly hyperbolic system (7.2.1), defined on a strip $(-\infty, \infty) \times [0, T)$ and having the following structure: A $C^1$ curve $x = \chi(t)$ issues from the origin, and $U(x, t) = \bar{U}$ = constant on the set $\{(x, t) : 0 \leq t < T,\ x > \chi(t)\}$, while on the set $\{(x, t) : 0 \leq t < T,\ x < \chi(t)\}$ $U$ is $C^2$ and its first and second partial derivatives attain non-zero limits, as $x \uparrow \chi(t)$. Thus, $\chi(\cdot)$ is a weak front moving into a constant state.

On the set $\{(x, t) : 0 \leq t < T,\ x < \chi(t)\}$,

$$\partial_t U(x, t) + \mathrm{D}F(U(x, t))\partial_x U(x, t) = 0. \tag{7.7.1}$$

Since $U(\chi(t)-, t) = \bar{U}$,

$$\partial_t U(\chi(t)-, t) + \dot{\chi}(t)\partial_x U(\chi(t)-, t) = 0. \tag{7.7.2}$$

By combining (7.7.1) with (7.7.2),

$$\big[\mathrm{D}F(\bar{U}) - \dot{\chi}(t)I\big]\partial_x U(\chi(t)-, t) = 0. \tag{7.7.3}$$

Therefore, $\dot{\chi}(t)$ is constant, equal to $\lambda_i(\bar{U})$, for some characteristic family $i$, and

$$\partial_x U(\chi(t)-, t) = a(t)R_i(\bar{U}). \tag{7.7.4}$$

The function $a(t)$ measures the strength of the weak front.

We multiply (7.7.4), from the left, by $L_i(\bar{U})$, use (7.2.3) and differentiate with respect to $t$ to get

$$\frac{da(t)}{dt} = L_i(\bar{U})\big[\partial_x\partial_t U(\chi(t))-, t) + \lambda_i(\bar{U})\partial_x\partial_x U(\chi(t)-, t)\big]. \tag{7.7.5}$$

Next, we multiply (7.7.1), from the left, by $L_i(U(x, t))$,

$$L_i(U(x, t))\,[\partial_t U(x, t) + \lambda_i(U(x, t))\partial_x U(x, t)] = 0, \tag{7.7.6}$$

then differentiate with respect to $x$ and let $x \uparrow \chi(t)$. Upon combining $\dot{\chi}(t) = \lambda_i(\bar{U})$, (7.7.2), (7.7.5), (7.7.4) and (7.2.3), we conclude that $a(t)$ satisfies an ordinary differential equation of Bernoulli type:

$$\frac{da}{dt} + \mathrm{D}\lambda_i(\bar{U})R_i(\bar{U})a^2 = 0. \tag{7.7.7}$$

Thus, if $\bar{U}$ is a state of genuine nonlinearity for the $i$-characteristic family and $\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U})a(0) < 0$, then the strength of the weak wave increases with time and eventually explodes as $t \uparrow \big[-\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U})a(0)\big]^{-1}$. The issue of breakdown of classical solutions will be discussed from a broader perspective in the following section.

## 7.8 Breakdown of Classical Solutions

When the system (7.2.1) is equipped with a convex entropy, Theorem 5.1.1 guarantees the existence of a unique, locally defined, classical solution, with initial data $U_0$ in the Sobolev space $H^2$. In one-space dimension, however, there is a sharper existence theory which applies to quasilinear hyperbolic systems in general, not necessarily conservation laws, and does not rely on the existence of entropies:

**7.8.1 Theorem.** *Let $U_0$ be a $C^1$ function, defined on $(-\infty, \infty)$ and taking values in a ball of $\mathbb{R}^n$ with closure contained in $\mathcal{O}$. Assume $dU_0/dx$ is bounded on $(-\infty, \infty)$. Then there exists a unique $C^1$ function $U$ defined on $(-\infty, \infty) \times [0, T_\infty)$, for some $T_\infty$, $0 < T_\infty \leq \infty$, and taking values in $\mathcal{O}$, which satisfies (7.2.1) on the strip $(-\infty, \infty) \times (0, T_\infty)$ together with the initial condition $U(x, 0) = U_0(x)$ on $(-\infty, \infty)$. Furthermore, the life span interval $[0, T_\infty)$ is maximal in the sense that if $T_\infty < \infty$, then, as $t \uparrow T_\infty$, $\|\partial_x U(\cdot, t)\|_{L^\infty} \to \infty$ and/or the range of $U(\cdot, t)$ escapes from every compact subset of $\mathcal{O}$.*

The proof of the above theorem, which may be found in the references cited in Section 7.10, relies on pointwise bounds for $U$ and $\partial_x U$ obtained by monitoring the evolution of $U$ and its derivatives along characteristics. Estimates of this nature will be established below but they will be employed not for establishing the existence of classical solutions but for demonstrating that classical solutions break down in finite time.

We have already encountered a number of examples of breakdown of classical solutions, notably for scalar conservation laws, in Section 6.1, and for weak fronts, in Section 7.7. Breakdown also occurs in the presence of compressive simple waves. Indeed, as shown in Section 7.6, an $i$-simple wave solution $U$ is obtained by taking the composition (7.6.10) of a (smooth) solution $V_i$ to the ordinary differential equation (7.6.11) with a classical solution $\xi$ to the Burgers equation (7.6.9). When that solution of (7.6.9) breaks down, so does the $i$-simple wave. The above examples involve a single characteristic family. The aim here is to demonstrate that, in the presence of genuine nonlinearity, the interaction of waves from different characteristic families cannot prevent the breakdown of smooth solutions.

Any classical, $C^2$ solution $U$ of (7.2.1) on $(-\infty, \infty) \times [0, T)$ may be written as

(7.8.1)
$$\begin{cases} \partial_x U = \sum_{j=1}^n a_j R_j(U) \\ \partial_t U = -\sum_{j=1}^n a_j \lambda_j(U) R_j(U) \end{cases}$$

with

(7.8.2)
$$a_j = L_j(U)\partial_x U, \qquad j = 1, \cdots, n.$$

In view of (7.6.3), one may interpret (7.8.1) as a decomposition of $U$ into simple waves, one for each characteristic family, with respective strengths $a_1, \cdots, a_n$. Our aim is to study the evolution of $a_i$ along the $i$-characteristics associated with $U$. We let

$$\frac{d}{dt} = \partial_t + \lambda_i \partial_x \tag{7.8.3}$$

denote differentiation in the $i$-characteristic direction. Combining (7.8.2) with (7.8.1) yields

$$\begin{aligned} \partial_t a_i &= L_i \partial_t \partial_x U + \partial_x U^\top \mathrm{D} L_i^\top \partial_t U \\ &= \partial_x (L_i \partial_t U) - \partial_t U^\top \mathrm{D} L_i^\top \partial_x U + \partial_x U^\top \mathrm{D} L_i^\top \partial_t U \\ &= \partial_x (L_i \partial_t U) + \sum_{j,k=1}^{n} (\lambda_j - \lambda_k) R_j^\top \mathrm{D} L_i^\top R_k a_j a_k \ , \end{aligned} \tag{7.8.4}$$

$$\begin{aligned} \lambda_i \partial_x a_i &= \partial_x (\lambda_i L_i \partial_x U) - (\mathrm{D}\lambda_i \partial_x U)(L_i \partial_x U) \\ &= \partial_x (\lambda_i L_i \partial_x U) - \sum_{j,k=1}^{n} (\mathrm{D}\lambda_i R_j) \delta_{ik} a_j a_k \ , \end{aligned} \tag{7.8.5}$$

where $\delta_{ik}$ is the Kronecker delta. From (7.2.1), $L_i \partial_t U + \lambda_i L_i \partial_x U = 0$. Also, by virtue of (7.2.3), $R_j^\top \mathrm{D} L_i^\top R_k = -L_i \mathrm{D} R_j R_k$. Therefore, combining (7.8.3), (7.8.4), (7.8.5) and symmetrizing we conclude

$$\frac{da_i}{dt} = \sum_{j,k=1}^{n} \gamma_{ijk} a_j a_k \tag{7.8.6}$$

with

$$\gamma_{ijk} = -\tfrac{1}{2}(\lambda_j - \lambda_k) L_i [R_j, R_k] - (\mathrm{D}\lambda_i R_j) \delta_{ik} \ , \tag{7.8.7}$$

where $[R_j, R_k]$ denotes the Lie bracket (7.2.15). Note, in particular, that

$$\gamma_{iii} = -\mathrm{D}\lambda_i R_i \ , \tag{7.8.8}$$

$$\gamma_{ijj} = 0, \quad j \neq i. \tag{7.8.9}$$

It is clear that in any argument showing blow-up of $a_i$ through (7.8.6), the coefficient $\gamma_{iii}$ will play a pivotal role. By virtue of (7.8.8), $\gamma_{iii}$ never vanishes when the $i$-characteristic family is genuinely nonlinear, and vanishes identically when the $i$-characteristic family is linearly degenerate.

To gain some insight, let us consider first the case where $U$ is just an $i$-simple wave, i.e., $a_i \neq 0$ and $a_j = 0$ for $j \neq i$. In that case, (7.8.6) reduces to

$$\frac{da_i}{dt} = \gamma_{iii} a_i^2 \,. \tag{7.8.10}$$

Furthermore, since $U$ is constant along characteristics, $\gamma_{iii}$ in (7.8.10) is a constant. When $\gamma_{iii} \neq 0$ and $a_i$ has the same sign as $\gamma_{iii}$, (7.8.10) induces blow-up of $a_i$ in a finite time.

Another noteworthy special case is when the system (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants. In that case $L_j = \mathrm{D}w_j$ and so, by (7.8.2),

$$a_j = \partial_x w_j \,. \tag{7.8.11}$$

Moreover, in virtue of (7.8.7), (7.3.10) and (7.3.12), (7.8.6) reduces to

$$\frac{da_i}{dt} = -\sum_{j=1}^{n} \frac{\partial \lambda_i}{\partial w_j} a_i a_j \,. \tag{7.8.12}$$

We seek an integrating factor for (7.8.12). If $\phi$ is any smooth scalar function of $U$, we get from (7.8.1):

$$\begin{aligned} \frac{d\phi}{dt} &= \mathrm{D}\phi(\partial_t U + \lambda_i \partial_x U) = \sum_{j\neq i} (\lambda_i - \lambda_j)(\mathrm{D}\phi R_j) a_j \\ &= \sum_{j \neq i} (\lambda_i - \lambda_j) \frac{\partial \phi}{\partial w_j} a_j \,. \end{aligned} \tag{7.8.13}$$

Combining (7.8.12) with (7.8.13) yields

$$\frac{d}{dt}(e^{\phi} a_i) = -e^{\phi} \frac{\partial \lambda_i}{\partial w_i} a_i^2 - \sum_{j \neq i} e^{\phi} [\frac{\partial \lambda_i}{\partial w_j} - (\lambda_i - \lambda_j) \frac{\partial \phi}{\partial w_j}] a_i a_j \,. \tag{7.8.14}$$

From (7.3.14) and (7.3.16), it follows that there exists $\phi$ that satisfies

$$\frac{\partial \phi}{\partial w_j} = \frac{1}{\lambda_i - \lambda_j} \frac{\partial \lambda_i}{\partial w_j} \,, \qquad j = 1, \cdots, i-1, i+1, \cdots, n. \tag{7.8.15}$$

For that $\phi$, (7.8.14) reduces to

$$\frac{d}{dt}(e^{\phi} a_i) = -e^{-\phi} \frac{\partial \lambda_i}{\partial w_i} (e^{\phi} a_i)^2. \tag{7.8.16}$$

When the $i$-characteristic family is genuinely nonlinear, $\partial \lambda_i / \partial w_i \neq 0$. Whenever $e^{-\phi} \partial \lambda_i / \partial w_i$ is bounded away from zero, uniformly on the range of the solution, (7.8.16) will induce blowup of $a_i$, in finite time, along any characteristic emanating from a point $\bar{x}$ of the $x$-axis where $a_i$ has the opposite sign of $\partial\lambda_i/\partial w_i$. Uniform boundedness of $e^{-\phi}\partial\lambda_i/\partial w_i$ is maintained, because, by Theorem 7.3.4, the range of any classical solution in the state space of Riemann invariants coincides with the range of its initial values. We have thus established

**7.8.2 Theorem.** *Assume* (7.2.1) *is endowed with a coordinate system of Riemann invariants* $(w_1, \cdots, w_n)$. *Suppose the* $i$*-characteristic family is genuinely nonlinear. Then any classical solution* $U$ *with bounded initial values* $U_0$, *such that* $dw_i(U_0)/dx$ *has the opposite sign from* $\partial\lambda_i/\partial w_i$ *at some point* $\bar{x} \in (-\infty, \infty)$, *breaks down in finite time.*

We now return to the general situation. When the $i$-characteristic field is genuinely nonlinear, and thus, by (7.8.8), $\gamma_{iii} \neq 0$, the term $\gamma_{iii}a_i^2$ in (7.8.6) will have a destabilizing effect. Any expectation that this may be offset by the remaining terms in (7.8.6), which account for the interaction effects with the other characteristic fields, is not likely to be fulfilled, at least when the initial data have compact support, for the following reason. Equation (7.8.9) rules out the possibility of self-interactions of the remaining characteristic fields: All interactions, other than $\gamma_{iii}a_i^2$, involve two distinct characteristic families. Now, when the initial data have compact support, mutual interactions eventually become insignificant, because waves of distinct characteristic families propagate with different speeds and thus eventually separate. Consequently, in the long run the term $\gamma_{iii}a_i^2$ becomes the dominant factor and drives $a_i$ to infinity in finite time. The above heuristic arguments can be formalized and lead to the following

**7.8.3 Theorem.** *Assume* (7.2.1) *is a genuinely nonlinear strictly hyperbolic system of conservation laws. When the initial data* $U_0$ *are* $C^2$, *have compact support, and* $\max|dU_0/dx|$ *is sufficiently small, the classical solution of the initial-value problem breaks down in finite time.*

The long and technical proof of Theorem 7.8.3, together with various extensions addressing the situation where some (or all) of the characteristic fields are linearly degenerate or weakly linearly degenerate, may be found in the references cited in Section 7.10.

## 7.9 Weak Solutions

In view of the breakdown of classical solutions, demonstrated in the previous section, in order to solve the initial-value problem in the large, for nonlinear hyperbolic systems of conservation laws, one has to resort to weak solutions. As explained in Chapter IV, the issue of the admissibility of weak solutions will have to be addressed.

In earlier chapters, we mainly considered weak solutions that are merely bounded measurable functions. Existence in that function class will indeed be established, for certain systems, in Chapter XVI, through the functional analytic method of compensated compactness. On the other hand, there are systems of three conservation laws for which the Cauchy problem is not well-posed in $L^1$. Apparently, the function class of choice for hyperbolic systems of conservation laws is $BV$, which provides the natural framework for envisioning the most important features of weak solutions, namely shocks and their interactions.

The finite domain of dependence property for solutions of hyperbolic systems, combined with the fact that our system (7.2.1) is invariant under uniform stretching of coordinates: $x = \bar{x} + ay \ , \ t = \bar{t} + a\tau \ , \ a > 0$, suggests that the admissibility of $BV$ weak solutions may be decided locally, through examination of shocks and wave fans. These issues will be discussed thoroughly in the following two chapters.

## 7.10 Notes

The general mathematical framework of the theory of hyperbolic systems of conservation laws in one-space dimension was set in the seminal paper of Lax [2], which distills the material collected over the years in the context of special systems. The notions of Riemann invariants, genuine nonlinearity, simple waves and simple wave curves, at the level of generality considered here, were introduced in that paper. The books by Smoller [3] and Serre [11] contain expositions of these topics, illustrated by interesting examples.

The simple hydrodynamic model of traffic flow was introduced by Lighthill and Whitham [1]. For elaborations and extensions, see the book by Whitham [1] and the papers by Tong Li [1,2], Aw and Rascle [1], Colombo [1], Coclite, Garavello and Piccoli [1], Benzoni-Gavage and Colombo [1], and Greenberg, Klar and Rascle [1].

The connection of the system (7.1.11) of pressureless gas dynamics with astrophysics is discussed in Shandarin and Zeldovich [1].

A systematic, rigorous exposition of the theory of one-dimensional elastic continua (strings, rods, etc.) is found in the book by Antman [1]. See also Antman [2]. The system (7.1.14) was studied by Freistühler.

The shallow water wave system (7.1.16), originally derived (in a somewhat different form) by Lagrange [1], has been used extensively in hydraulic theory to model flood and tidal waves and bores. A few relevant references, out of an immense bibliography, are Airy [1], Saint Venant [1], Stoker [1], Whitham [1], Gerbeau and Perthame [1], and Holden and Risebro [2].

The system (7.1.17) for planar electromagnetic waves was studied thoroughly by Serre [4].

Combustion theory, in connection to system (7.1.19), is expounded in the book by Williams [1]. The model system (7.1.20) was proposed by Majda [1].

For a general thermodynamic theory of mixtures, see Müller [2] and Müller and Ruggeri [1]. A thorough treatment of the mathematical properties of the nonisothermal version of the system (7.1.22) is given in Ruggeri and Simić [1].

There are many other interesting examples of hyperbolic systems of conservation laws, for example the equations governing sedimentation and suspension flows

(Bürger and Wendland [1]), the system of chemical chromatography (Rhee, Aris and Amundson [1]), the system of flood waves (Whitham [2]), the equations of multi-phase flow in porous media and the system of polymer flooding (Holden, Risebro and Tveito [1]).

The failure of strict hyperbolicity in one-space dimensional systems deriving from three-space dimensional parent systems is discussed by Lax [6]. The system (7.2.11) has been used extensively as a vehicle for demonstrating the features of non-strictly hyperbolic systems of conservation laws, beginning with the work of Keyfitz and Kranzer [2].

Riemann invariants were first considered by Earnshaw [1] and by Riemann [1], in the context of the system (7.1.9) of isentropic gas dynamics. Conditions for existence of coordinate systems of Riemann invariants and its implications on the existence of entropies were investigated by Conlon and Liu [1] and by Sévennec [1]. The calculation of the characteristic speeds and Riemann invariants of the system (7.3.18) of electrophoresis is due to Alekseyevskaya [1] and Fife and Geng [1]. A detailed exposition of the noteworthy properties of this system is contained in Serre [11]. Serre [4] shows that the system (7.1.17) is equivalent to (7.3.24), (7.3.25) even within the realm of weak solutions.

As already mentioned in Section 1.10, the special entropy-entropy flux pair (7.4.6), for symmetric systems, was noted by Godunov [1,2,3] and by Friedrichs and Lax [1]. Over the years, a great number of entropy-entropy flux pairs with special properties have been constructed, mainly for systems of two conservation laws, beginning with the pioneering paper of Lax [4]. We shall see some of that work in later chapters. The characterization of systems of size $n \geq 3$ endowed with an abundance of entropies is due to Tsarev [1], who calls them *semi-Hamiltonian*, and Serre [6], who named them *rich*. A comprehensive exposition of their theory is contained in Serre [11].

Theorem 7.5.2 is due to Boillat [2].

The earliest example of a simple wave, in the context of the system of isothermal gas dynamics, appears in a memoir by Poisson [1]. See also Earnshaw [1]. Theorem 7.6.3 is taken from Lax [2], who attributes the proof to Friedrichs.

A thorough discussion on the explosion of weak waves in continuum physics, together with extensive bibliography, are found in the encyclopedic article by Peter Chen [1].

Local existence of $C^1$ solutions to the initial-value problem in one-space dimension was first established by Schauder [1] and Friedrichs [1]. For a comprehensive treatment of the initial as well as the initial-boundary value problem see the monograph by Li Ta-tsien and Yu Wen-ci [1].

The breakdown of classical solutions was first noticed by Challis [1], in the context of the compressible simple wave solution of the system of isothermal gas dynamics derived by Poisson [1]. It is this paper that provided the stimulus for the introduction of weak solutions with shocks, by Stokes [1] (see Sections 1.10 and 4.8). The earliest result on generic breakdown of solutions is due to Lax [3], who proved directly the case $n = 2$ of Theorem 7.8.2. This work was extended in several directions: Klainerman and Majda [1] established breakdown in the case $n = 2$ so long as none of the two characteristic families is linearly degenerate. John [1] derived[3] (7.8.6) and used it to prove Theorem 7.8.3. A detailed discussion is found in Hörmander [1,2]. Liu [13] gives an extension of Theorem 7.8.3 covering the case where some of the characteristic families are linearly degenerate. Li Ta-tsien, Zhou Yi and Kong De-xing [1] consider the case of weakly linearly degenerate characteristic families. See also Li Ta-tsien and Kong De-xing [1]. A direct proof of Theorem 7.8.2, for any $n$, is found in Serre [11]. Additional results are presented in Chemin [2] and in the monograph by Alinhac [1].

Examples of systems for which the Cauchy problem is not well-posed in $L^1$ are found in Bressan and Shen [1].

[3] John's formula for $\gamma_{ijk}$ is different from (7.8.7) but, of course, the two expressions are equivalent.

# VIII

# Admissible Shocks

Shock fronts were introduced in Section 1.6, for general systems of balance laws, and were placed in the context of $BV$ solutions in Section 1.8. They were encountered again, briefly, in Section 3.1, where the governing Rankine-Hugoniot condition was recorded.

Since shock fronts have codimension one, important aspects of their local behavior may be investigated, without loss of generality, within the framework of systems in one-space dimension. This will be the object of the present chapter. The discussion will begin with an exploration of the geometric features of the Rankine-Hugoniot condition, leading to the introduction of the Hugoniot locus.

The necessity of imposing admissibility conditions on weak solutions was pointed out in Chapter IV. These in turn induce, or at least motivate, admissibility conditions on shocks. Indeed, the prevailing view is that the issue of admissibility of general $BV$ weak solutions should be resolved through a test applied to every point of the shock set. In particular, the shock admissibility conditions associated with the entropy condition of Section 4.5 and the vanishing viscosity approach of Section 4.6 will be introduced, and they will be compared with each other as well as with other important shock admissibility conditions proposed by Lax and by Liu.

## 8.1 Strong Shocks, Weak Shocks, and Shocks of Moderate Strength

For the hyperbolic system

$$\partial_t U + \partial_x F(U) = 0, \tag{8.1.1}$$

in one-space dimension, the Rankine-Hugoniot jump condition (3.1.3) reduces to

$$F(U_+) - F(U_-) = s(U_+ - U_-). \tag{8.1.2}$$

Actually, (8.1.2) is as general as the multi-space-dimensional version (3.1.3), once the direction $\nu$ of propagation of the shock has been fixed and $F$ has been defined through (7.2.2).

When (8.1.2) holds, we say that *the state $U_-$, on the left, is joined to the state $U_+$, on the right, by a shock of speed $s$*. Note that "left" and "right" may be interchanged in (8.1.2), in consequence of the invariance of (8.1.1) under the transformation $(x,t) \mapsto (-x,-t)$. Nevertheless, later on we shall introduce admissibility conditions inducing irreversibility, as a result of which the roles of $U_-$ and $U_+$ cannot be interchanged.

The jump $U_+ - U_-$ is the *amplitude* and its size $|U_+ - U_-|$ is the *strength* of the shock. Properties established without restriction on the strength are said to hold even for *strong shocks*. Quite often, however, we shall have to impose limitations on the strength of shocks: $|U_+ - U_-| < \delta$, with $\delta$ depending on $\mathrm{D}F$ through parameters such as the size of the gaps between characteristic speeds of distinct families, which induce the separation of waves of different families, and the size of derivatives of the functions $\lambda_i$ and $R_i$, which manifest the nonlinearity of the system. In particular, when $\delta$ depends solely on the size of the first derivatives of the $\lambda_i$ and $R_i$, the shock is of *moderate strength*; while if $\delta$ also depends on the size of second derivatives, the shock is dubbed *weak*. Of course, the size of these parameters may be changed by rescaling the variables $x$, $t$ and $U$, so the relevant factor is the relative rather than the absolute size of $\delta$.

Notice that (8.1.2) may be written as

$$[A(U_-, U_+) - sI](U_+ - U_-) = 0, \tag{8.1.3}$$

where we are using the notation

$$A(V, U) = \int_0^1 \mathrm{D}F(\tau U + (1-\tau)V)d\tau. \tag{8.1.4}$$

For $i = 1, \cdots, n$, let $\mu_i(V,U)$ denote the eigenvalues and $S_i(V,U)$ the corresponding eigenvectors of $A(V,U)$. In particular, $A(U,U) = \mathrm{D}F(U)$ and so $\mu_i(U,U) = \lambda_i(U)$, $S_i(U,U) = R_i(U)$. Notice that $A(V,U)$, and thereby also $\mu_i(V,U)$ and $S_i(V,U)$ are symmetric in $(V,U)$. Therefore, (finite) Taylor expanding of these functions about the midpoint $\frac{1}{2}(V+U)$ yields

$$\mu_i(V,U) = \lambda_i(\tfrac{1}{2}(V+U)) + O(|V-U|^2), \tag{8.1.5}$$

$$S_i(V,U) = R_i(\tfrac{1}{2}(V+U)) + O(|V-U|^2). \tag{8.1.6}$$

Clearly, (8.1.3) will hold if and only if

$$s = \mu_i(U_-, U_+), \tag{8.1.7}$$

$$U_+ - U_- = \zeta S_i(U_-, U_+), \tag{8.1.8}$$

for some $i = 1, \cdots, n$ and some nonzero $\zeta \in I\!R$. In particular, the speed $s$ of any shock of moderate strength must be close to some characteristic speed $\lambda_i$. Such a shock is then called an *$i$-shock*.

An interesting implication of (8.1.5), (8.1.7) is the useful identity

$$s = \tfrac{1}{2}[\lambda_i(U_-) + \lambda_i(U_+)] + O(|U_- - U_+|^2). \tag{8.1.9}$$

In special systems it is possible to associate even strong shocks with a particular characteristic family. For example, the Rankine-Hugoniot condition

$$\begin{cases} v_+ - v_- + s(u_+ - u_-) = 0 \\ \sigma(u_+) - \sigma(u_-) + s(v_+ - v_-) = 0 \end{cases} \tag{8.1.10}$$

for the system (7.1.8) of isentropic elasticity implies

$$s = \pm\sqrt{\frac{\sigma(u_+) - \sigma(u_-)}{u_+ - u_-}} \;. \tag{8.1.11}$$

Recalling the characteristic speeds (7.2.4) of this system, it is natural to call shocks propagating to the left ($s < 0$) 1-shocks and shocks propagating to the right ($s > 0$) 2-shocks.

## 8.2 The Hugoniot Locus

The set of points $U$ in state space that may be joined to a fixed point $\bar{U}$ by a shock is called the *Hugoniot locus* of $\bar{U}$. It has a simple geometric structure in the vicinity of any point $U$ of strict hyperbolicity of the system.

**8.2.1 Theorem.** *For a given state $\bar{U} \in \mathcal{O}$, assume that the characteristic speed $\lambda_i(\bar{U})$ is a simple eigenvalue of $\mathrm{D}F(\bar{U})$. Then there is a $C^3$ curve $U = W_i(\tau)$ in state space, called the $i$-shock curve through $\bar{U}$, and a $C^2$ function $s = s_i(\tau)$, both defined for $\tau$ in some neighborhood of $0$, with the following property: A state $U$ can be joined to $\bar{U}$ by an $i$-shock of moderate strength and speed $s$ if and only if $U = W_i(\tau)$, $s = s_i(\tau)$, for some $\tau$. Furthermore, $W_i(0) = \bar{U}$ and*

$$s_i(0) = \lambda_i(\bar{U}), \tag{8.2.1}$$

$$\dot{s}_i(0) = \tfrac{1}{2}\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U}), \tag{8.2.2}$$

$$\dot{W}_i(0) = R_i(\bar{U}), \tag{8.2.3}$$

$$\ddot{W}_i(0) = \mathrm{D}R_i(\bar{U})R_i(\bar{U}). \tag{8.2.4}$$

*The more explicit notation $W_i(\tau;\bar{U})$, $s_i(\tau;\bar{U})$ shall be employed when one needs to identify the point of origin of this shock curve.*

**Proof.** Recall the notation developed in Section 8.1 and, in particular, Equations (8.1.7), (8.1.8). A state $U$ may be joined to $\bar{U}$ by an $i$-shock of speed $s$ if and only if

$$U = \bar{U} + \tau S_i(\bar{U}, U), \tag{8.2.5}$$

$$s = \mu_i(\bar{U}, U). \tag{8.2.6}$$

Accordingly, we consider the function

$$H(U, \tau) = U - \bar{U} - \tau S_i(\bar{U}, U), \tag{8.2.7}$$

defined on $\mathcal{O} \times I\!R$, and note that $H(\bar{U}, 0) = 0$ , $\mathrm{D}H(\bar{U}, 0) = I$. Consequently, by the implicit function theorem, there is a curve $U = W_i(\tau)$ in state space, with $W_i(0) = \bar{U}$, such that $H(U, \tau) = 0$ for $\tau$ near 0 if and only if $U = W_i(\tau)$. We then define

$$s_i(\tau) = \mu_i(\bar{U}, W_i(\tau)). \tag{8.2.8}$$

In particular, $s_i(0) = \mu_i(\bar{U}, \bar{U}) = \lambda_i(\bar{U})$. Furthermore, differentiating (8.2.5) with respect to $\tau$ and setting $\tau = 0$, we deduce $\dot{W}_i(0) = S_i(\bar{U}, \bar{U}) = R_i(\bar{U})$. To establish the remaining equations (8.2.2) and (8.2.4), we appeal to (8.1.5) and (8.1.6) to get

$$\begin{aligned} s_i(\tau) &= \lambda_i(\tfrac{1}{2}(\bar{U} + W_i(\tau))) + O(\tau^2) \\ &= \lambda_i(\bar{U}) + \tfrac{1}{2}\tau \mathrm{D}\lambda_i(\bar{U}) R_i(\bar{U}) + O(\tau^2), \end{aligned} \tag{8.2.9}$$

$$\begin{aligned} W_i(\tau) &= \bar{U} + \tau R_i(\tfrac{1}{2}(\bar{U} + W_i(\tau))) + O(\tau^3) \\ &= \bar{U} + \tau R_i(\bar{U}) + \tfrac{1}{2}\tau^2 \mathrm{D}R_i(\bar{U}) R_i(\bar{U}) + O(\tau^3). \end{aligned} \tag{8.2.10}$$

This completes the proof.

In particular, if $\bar{U}$ is a point of strict hyperbolicity of the system (8.1.1), Theorem 8.2.1 implies that the Hugoniot locus of $\bar{U}$ is the union of $n$ shock curves, one for each characteristic family.

The shock curve constructed above is generally confined in the regime of shocks of moderate strength, because of the use of the implicit function theorem, which applies only when the strength of the shock, measured by $|\tau|$, is sufficiently small: $|\tau| < \delta$ with $\delta$ depending on the $C^1$ norm of $S_i$ , which in turn can be estimated in terms of the $C^1$ norm of $\mathrm{D}F$ and the inverse of the gap between $\lambda_i$ and the other characteristic speeds. Nevertheless, in special systems one may often use more delicate analytical or topological arguments or explicit calculation to extend shock curves to the range of strong shocks. For example, in the case of the system (7.1.8), combining (8.1.10) with (8.1.11) we deduce that the Hugoniot locus of any point $(\bar{u}, \bar{v})$ in state space consists of two curves

$$(8.2.11) \qquad v = \bar{v} \pm \sqrt{[\sigma(u) - \sigma(\bar{u})](u - \bar{u})}\,,$$

defined on the whole range of $u$.

The $i$-shock curves introduced here have common features with the $i$-rarefaction wave curves defined in Section 7.6. Indeed, recalling Theorems 7.6.5 and 8.2.1, and, in particular, comparing (7.6.14) with (8.2.3), (8.2.4), we deduce

**8.2.2 Theorem.** *Assume $\bar{U} \in \mathcal{O}$ is a point of genuine nonlinearity of the $i$-characteristic family of the hyperbolic system* (8.1.1) *of conservation laws, and $\lambda_i(\bar{U})$ is a simple eigenvalue of* $\mathrm{D}F(\bar{U})$. *Normalize $R_i$ so that* (7.6.13) *holds on some neighborhood of $\bar{U}$. Then the $i$-rarefaction wave curve $V_i$, defined through Theorem* 7.6.5, *and the $i$-shock curve $W_i$, defined through Theorem* 8.2.1, *have a second order contact at $\bar{U}$.*

Recall that, by Theorem 7.6.6, $i$-Riemann invariants are constant along $i$-rarefaction wave curves. At the same time, as shown above, $i$-shock curves are very close to $i$-rarefaction wave curves. It is then to be expected that $i$-Riemann invariants vary very slowly along $i$-shock curves. Indeed,

**8.2.3 Theorem.** *The jump of any $i$-Riemann invariant across a weak $i$-shock is of third order in the strength of the shock.*

**Proof.** Assume $\lambda_i(\bar{U})$ is a simple eigenvalue of $\mathrm{D}F(\bar{U})$ and consider the $i$-shock curve $W_i$ through $\bar{U}$. For any $i$-Riemann invariant $w$, differentiating along the curve $W_i(\cdot)$,

$$(8.2.12) \qquad \dot{w} = \mathrm{D}w\dot{W}_i\,,$$

$$(8.2.13) \qquad \ddot{w} = \dot{W}_i^{\top}\mathrm{D}^2 w\dot{W}_i + \mathrm{D}w\ddot{W}_i\,.$$

By virtue of (8.2.3) and (7.3.1), $\dot{w} = 0$ at $\tau = 0$.

We now apply D to (7.3.1) and then multiply the resulting equation from the right by $R_i$ to deduce the identity

$$(8.2.14) \qquad R_i^{\top}\mathrm{D}^2 wR_i + \mathrm{D}w\mathrm{D}R_iR_i = 0.$$

Combining (8.2.13), (8.2.3), (8.2.4) and (8.2.14), we conclude that $\ddot{w} = 0$ at $\tau = 0$. This completes the proof.

In the special case where the system (8.1.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants, we may calculate the leading term in the jump of $w_j$ across a weak $i$-shock, $i \neq j$, as follows. The Rankine-Hugoniot condition reads

$$(8.2.15) \qquad F(W_i(\tau)) - F(\bar{U}) = s_i(\tau)[W_i(\tau) - \bar{U}].$$

Differentiating with respect to $\tau$ yields

$$[\mathrm{D}F(W_i(\tau)) - s_i(\tau)I]\dot{W}_i(\tau) = \dot{s}_i(\tau)[W_i(\tau) - \bar{U}]. \tag{8.2.16}$$

Multiplying (8.2.16), from the left, by $\mathrm{D}w_j(W_i)$ gives

$$(\lambda_j - s_i)\dot{w}_j = \dot{s}_i \mathrm{D}w_j[W_i - \bar{U}]. \tag{8.2.17}$$

Next we differentiate (8.2.17), with respect to $\tau$, thus obtaining

$$(\lambda_j - s_i)\ddot{w}_j + (\dot{\lambda}_j - 2\dot{s}_i)\dot{w}_j = \ddot{s}_i \mathrm{D}w_j[W_i - \bar{U}] + \dot{s}_i \dot{W}_i^\top \mathrm{D}^2 w_j[W_i - \bar{U}]. \tag{8.2.18}$$

We differentiate (8.2.18), with respect to $\tau$, and then set $\tau = 0$. We use (8.2.1), (8.2.2), (8.2.3), (7.3.12) and that both $\dot{w}_j$ and $\ddot{w}_j$ vanish at 0, by virtue of Theorem 8.2.3, to conclude

$$\dddot{w}_j = \frac{1}{2}\frac{1}{\lambda_j - \lambda_i}\frac{\partial \lambda_i}{\partial w_i} R_i^\top \mathrm{D}^2 w_j R_i\,, \tag{8.2.19}$$

where $\dddot{w}_j$ is evaluated at 0 and the right-hand side is evaluated at $\bar{U}$.

Returning to the general case, we next investigate how the shock speed function $s_i(\tau)$ evolves along the $i$-shock curve. We multiply (8.2.16), from the left, by $L_i(W_i(\tau))$ to get

$$[\lambda_i(W_i(\tau)) - s_i(\tau)]L_i(W_i(\tau))\dot{W}_i(\tau) = \dot{s}_i(\tau)L_i(W_i(\tau))[W_i(\tau) - \bar{U}]. \tag{8.2.20}$$

For $\tau$ sufficiently close to 0, but $\tau \neq 0$,

$$L_i(W_i(\tau))\dot{W}_i(\tau) > 0, \qquad \tau L_i(W_i(\tau))[W_i(\tau) - \bar{U}] > 0, \tag{8.2.21}$$

by virtue of (8.2.3). In the applications it turns out that (8.2.21) continue to hold for a broad range of $\tau$, often extending to the regime of strong shocks. In that case, (8.2.20) and (8.2.16) immediately yield the following

**8.2.4 Lemma.** *Assume* (8.2.21) *hold. Then*

$$\dot{s}_i(\tau) > 0 \;\; if \;\; and \;\; only \;\; if \;\; \tau[\lambda_i(W_i(\tau)) - s_i(\tau)] > 0, \tag{8.2.22}$$

$$\dot{s}_i(\tau) = 0 \;\; if \;\; and \;\; only \;\; if \;\; \lambda_i(W_i(\tau)) = s_i(\tau). \tag{8.2.23}$$

*Moreover,* $\dot{s}_i(\tau) = 0$ *implies that* $\dot{W}_i(\tau)$ *is collinear to* $R_i(W_i(\tau))$.

In order to see how $s_i$ varies across points where $\dot{s}_i$ vanishes, we differentiate (8.2.20) with respect to $\tau$ and then evaluate the resulting expression at any $\tau$ where $\dot{s}_i(\tau) = 0$. Since $s_i(\tau) = \lambda_i\,(W_i(\tau))$ and $\dot{W}_i(\tau) = aR_i\,(W_i(\tau))$, upon recalling (7.2.3) we deduce

$$\ddot{s}_i(\tau)L_i\big(W_i(\tau)\big)\big[W_i(\tau) - \bar{U}\big] = a^2 \mathrm{D}\lambda_i\big(W_i(\tau)\big)R_i\big(W_i(\tau)\big), \tag{8.2.24}$$

whence it follows that at points where $\dot{s}_i = 0$, $\ddot{s}_i$ has the same sign as $\tau \mathrm{D}\lambda_i R_i$ .

By Lemma 8.2.4, $s_i$ constant implies that the $i$-shock curve is an integral curve of the vector field $R_i$, along which $\lambda_i$ is constant. Consequently, all points along such a shock curve are states of linear degeneracy of the $i$-characteristic family. The converse of this statement is also valid:

**8.2.5 Theorem.** *Assume the $i$-characteristic family of the hyperbolic system* (8.1.1) *of conservation laws is linearly degenerate and $\lambda_i(\bar{U})$ is a simple eigenvalue of* $\mathrm{D}F(\bar{U})$. *Then the $i$-shock curve $W_i$ through $\bar{U}$ is the integral curve of $R_i$ through $\bar{U}$. In fact, under the proper parametrization, $W_i$ is the solution of the differential equation*

$$\dot{W}_i = R_i(W_i) \tag{8.2.25}$$

*with initial condition $W_i(0) = \bar{U}$. Along $W_i$, the characteristic speed $\lambda_i$ and all $i$-Riemann invariants are constant. The shock speed function $s_i$ is also constant:*

$$s_i(\tau) = \lambda_i(W_i(\tau)) = \lambda_i(\bar{U}). \tag{8.2.26}$$

**Proof.** Let $W_i$ denote the solution of (8.2.25) with initial condition $W_i(0) = \bar{U}$. Then

$$[\mathrm{D}F(W_i(\tau)) - \lambda_i(W_i(\tau))I]\dot{W}_i(\tau) = 0. \tag{8.2.27}$$

Since $\mathrm{D}\lambda_i(U)R_i(U) = 0$, $\dot{\lambda}_i = 0$ and so $\lambda_i(W_i(\tau)) = \lambda_i(\bar{U})$. Integrating (8.2.27) from 0 to $\tau$ yields

$$F(W_i(\tau)) - F(\bar{U}) = \lambda_i(\bar{U})[W_i(\tau) - \bar{U}], \tag{8.2.28}$$

which establishes that $W_i$ is the $i$-shock curve through $\bar{U}$, with corresponding shock speed function $s_i$ given by (8.2.26). This completes the proof.

The following important implication of Theorem 8.2.5 provides an alternative characterization of linear degeneracy:

**8.2.6 Corollary.** *When the $i$-characteristic family of the hyperbolic system* (8.1.1) *is linearly degenerate, there exist traveling wave solutions*

$$U(x,t) = V(x - \sigma t), \tag{8.2.29}$$

*for any $\sigma$ in the range of the $i$-characteristic speed $\lambda_i$.*

**Proof.** Let $\sigma = \lambda_i(\bar{U})$, for some state $\bar{U}$. Consider the $i$-shock curve $W_i$ through $\bar{U}$, which satisfies (8.2.25). Take any $C^1$ function $\tau = \tau(\xi)$ and define $U$ by (8.2.29), with $V(\xi) = W_i(\tau(\xi))$. By account of (8.2.25) and (8.2.26),

$$\partial_t U + \partial_x F(U) = \frac{d\tau}{d\xi}\left[\mathrm{D}F\big(W_i(\tau)\big) - \lambda_i\big(W_i(\tau)\big)I\right]R_i\big(W_i(\tau)\big) = 0. \tag{8.2.30}$$

The proof is complete.

It is natural to inquire whether an $i$-shock curve may be an integral curve of the vector field $R_i$ in the absence of linear degeneracy. It turns out that this may only occur under very special circumstances:

**8.2.7 Theorem.** *For the hyperbolic system* (8.1.1), *assume* $\bar{U}$ *is a state of genuine nonlinearity for the i-characteristic family and* $\lambda_i(\bar{U})$ *is a simple eigenvalue of* $\mathrm{D}F(\bar{U})$. *The i-shock curve through* $\bar{U}$ *coincides with the integral curve of the field* $R_i$ *(i.e. the i-rarefaction wave curve) through* $\bar{U}$ *if and only if the latter is a straight line in state space.*

**Proof.** If the $i$-shock curve $W_i$ through $\bar{U}$ coincides with the integral curve of $R_i$ through $\bar{U}$, then $\dot{W}_i(\tau)$ must be collinear to $R_i(W_i(\tau))$. In that case, (8.2.16) imples

$$[\lambda_i(W_i(\tau)) - s_i(\tau)]\dot{W}_i(\tau) = \dot{s}_i(\tau)[W_i(\tau) - \bar{U}]. \tag{8.2.31}$$

For $\tau$ near 0, but $\tau \neq 0$, it is $\lambda_i(W_i(\tau)) \neq s_i(\tau)$, by genuine nonlinearity. Therefore, (8.2.31) implies that the graph of $W_i$ is a straight line through $\bar{U}$.

Conversely, assume the integral curve of $R_i$ through $\bar{U}$ is a straight line, which may be parametrized as $U = W_i(\tau)$, where $W_i$ is some smooth function satisfying $W_i(0) = \bar{U}$, as well as (8.2.3) and (8.2.4) (note that $\mathrm{D}R_i(\bar{U})R_i(\bar{U})$ is necessarily collinear to $R_i(\bar{U})$). We may then determine a scalar-valued function $s_i(\tau)$ such that

$$\begin{aligned} F(W_i(\tau)) - F(\bar{U}) &= \int_0^\tau \mathrm{D}F(W_i(\zeta))\dot{W}_i(\zeta)d\zeta \\ &= \int_0^\tau \lambda_i(W_i(\zeta))\dot{W}_i(\zeta)d\zeta = s_i(\tau)[W_i(\tau) - \bar{U}]. \end{aligned} \tag{8.2.32}$$

Thus $W_i$ is the $i$-shock curve through $\bar{U}$. This completes the proof.

Special as it may be, the class of hyperbolic systems of conservation laws with coinciding shock and rarefaction wave curves of each characteristic family includes some noteworthy examples. Consider, for instance, the system (7.3.18) of electrophoresis. Notice that, for $i = 1, \cdots, n$, the level surfaces of the $i$-Riemann invariant $W_i$, determined through (7.3.21) or (7.3.22), are hyperplanes. In particular, for $i = 1, \cdots, n$, the integral curves of the vector field $R_i$ are the straight lines produced by the intersection of the level hyperplanes of the $n - 1$ Riemann invariants $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$. Consequently, the conditions of Theorem 8.2.7 apply to the system (7.3.18).

In the presence of multiple characteristic speeds, the Hugoniot locus may contain multi-dimensional varieties, in the place of shock curves. In that connection it is instructive to consider the model system (7.2.11), for which the origin is an umbilic point. When a state $(\bar{u}, \bar{v})$ is joined to a state $(u, v)$ by a shock of speed $s$, the Rankine-Hugoniot condition reads

$$\left\{ \begin{aligned} &(u^2 + v^2)u - (\bar{u}^2 + \bar{v}^2)\bar{u} = s(u - \bar{u}) \\ &(u^2 + v^2)v - (\bar{u}^2 + \bar{v}^2)\bar{v} = s(v - \bar{v}). \end{aligned} \right. \tag{8.2.33}$$

Notice that when $(\bar{u}, \bar{v}) \neq (0, 0)$, the Hugoniot locus of $(\bar{u}, \bar{v})$ consists of the circle $u^2 + v^2 = \bar{u}^2 + \bar{v}^2$, along which the shock speed is constant, $s = \bar{u}^2 + \bar{v}^2$, and the straight line $\bar{v}u = \bar{u}v$, which connects $(\bar{u}, \bar{v})$ to the origin. Thus, the 1-characteristic family provides an example of the application of Theorem 8.2.5 while the 2-characteristic family satisfies the assumptions of Theorem 8.2.7. On the other hand, the Hugoniot locus of the umbilic point $(0, 0)$ is the entire plane, because any point $(u, v)$ can be joined to $(0, 0)$ by a shock of speed $s = u^2 + v^2$.

Not all systems in which strict hyperbolicity fails exhibit the same behavior. For instance, for the system

$$\begin{cases} \partial_t u + \partial_x[2(u^2 + v^2)u] = 0 \\ \partial_t v + \partial_x[(u^2 + v^2)v] = 0, \end{cases} \tag{8.2.34}$$

in which strict hyperbolicity also fails at the origin, the Hugoniot locus of $(0, 0)$ consists of two lines, namely the $u$-axis and the $v$-axis.

## 8.3 The Lax Shock Admissibility Criterion; Compressive, Overcompressive and Undercompressive Shocks

An $i$-shock of speed $s$ which joins the state $U_-$, on the left, to the state $U_+$, on the right, is said to satisfy the *Lax E-condition* if

$$\lambda_i(U_-) \geq s \geq \lambda_i(U_+). \tag{8.3.1}$$

In particular, when the left or the right part of (8.3.1) is satisfied as an equality, the shock is called a *left* or a *right i-contact discontinuity*; and when both parts of (8.3.1) hold as equalities, the shock is called an *i-contact discontinuity*. For example, by account of Theorem 8.2.5, any weak shock associated with a linearly degenerate characteristic family is necessarily a contact discontinuity. Notice that, with the exception of contact discontinuities, (8.3.1) induces an *irreversibility* condition that fixes the roles of $U_-$ and $U_+$ as left and right states of the shock.

When the above shock is embedded in an otherwise smooth solution, the meaning of (8.3.1) is that $i$-characteristics from the left catch up with $i$-characteristics from the right and they collide at the shock. Thus "information" from the past propagating along $i$-characteristics is absorbed and lost into admissible shocks. In contrast, shocks that violate (8.3.1) become sources of new "information" which is then carried along $i$-characteristics into the future. Postulating the Lax $E$-condition may appear *ad hoc* at this point, but justification is provided by its implications for stability of weak solutions as well as through its connection with other, physically motivated, shock admissibility criteria. These issues will be discussed at length in following sections.

Let us begin the investigation with the scalar conservation law (7.1.2). The characteristic speed is $\lambda(u) = f'(u)$ and so (8.3.1) takes the form

$$f'(u_-) \ge s \ge f'(u_+), \tag{8.3.2}$$

where $s$ is the shock speed computed through the Rankine-Hugoniot jump condition:

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \,. \tag{8.3.3}$$

The reader will immediately realize the geometric interpretation of (8.3.2) upon noticing that $f'(u_-)$ and $f'(u_+)$ are the slopes of the graph of $f$ at the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$ while $s$ is the slope of the chord that connects $(u_-, f(u_-))$ with $(u_+, f(u_+))$. In particular, when (7.1.2) is genuinely nonlinear, i.e., $f''(u) \neq 0$ for all $u$, then (8.3.2) reduces to $u_- < u_+$ if $f''(u) < 0$, and $u_- > u_+$ if $f''(u) > 0$.

Next we consider the system (7.1.8) of isentropic elasticity. The characteristic speeds are recorded in (7.2.4) and the shock speeds in (8.1.11), so that (8.3.1) assumes the form

(8.3.4)
$$\sigma'(u_-) \le \frac{\sigma(u_+) - \sigma(u_-)}{u_+ - u_-} \le \sigma'(u_+) \;\text{ or }\; \sigma'(u_-) \ge \frac{\sigma(u_+) - \sigma(u_-)}{u_+ - u_-} \ge \sigma'(u_+),$$

for 1-shocks or 2-shocks, respectively. The geometric interpretation of (8.3.4) is again clear. When (7.1.8) is genuinely nonlinear, i.e., $\sigma''(u) \neq 0$ for all $u$, (8.3.4) reduces to $u_- < u_+$ or $u_- > u_+$ if $\sigma''(u) > 0$, and to $u_- > u_+$ or $u_- < u_+$ if $\sigma''(u) < 0$. Equivalently, in terms of velocity, by virtue of (8.1.10): $v_- < v_+$ if $\sigma''(u) > 0$ and $v_- > v_+$ if $\sigma''(u) < 0$, for both shock families.

A similar analysis applies to the system (7.1.10) of isentropic flow of a polytropic gas, with characteristic speeds given by (7.2.10), and yields that a 1-shock (or 2-shock) that joins the state $(\rho_-, v_-)$, on the left, to the state $(\rho_+, v_+)$, on the right, satisfies the Lax $E$-condition if and only if $\rho_- < \rho_+$ (or $\rho_- > \rho_+$). In other words, the passing of an admissible shock front compresses the gas.

When $\lambda_i$ is a simple eigenvalue of D$F$ and we are dealing with $i$-shocks of (at most) moderate strength, the remaining characteristic speeds are well-separated and do not interfere, i.e., (8.3.1) may be extended into

$$\lambda_j(U_\pm) > \lambda_i(U_-) \ge s \ge \lambda_i(U_+) > \lambda_k(U_\pm), \qquad j > i > k. \tag{8.3.5}$$

Because classical gas dynamics has served as the prototype for the development of the general theory, shocks that satisfy (8.3.5) as strict inequalities are called *compressive*. Thus, $n+1$ characteristics are impinging on compressive shocks. In many special systems, such as those considered above, (8.3.5) may hold even in the realm of strong shocks. On the other hand, in the presence of umbilic points and/or strong shocks, one may encounter the situation in which a shock satisfies the Lax $E$-condition simultaneously for two distinct characteristic families $i$ and $j$, say

$$\lambda_j(U_-) > \lambda_i(U_-) > s > \lambda_j(U_+) > \lambda_i(U_+), \tag{8.3.6}$$

in which case more than $n + 1$ characteristics are impinging on the shock. Such shocks are called *overcompressive*. An example is provided by system (7.2.11). Recalling the form of the Hugoniot locus, described in Section 8.2, we consider a shock of speed $s$, joining, on the left, a state $(u_-, v_-)$, lying on the unit circle, to a state $(u_+, v_+) = a(u_-, v_-)$, on the right, where $a$ is some constant. From (7.2.12), $\lambda_1(u_-, v_-) = 1, \lambda_2(u_-, v_-) = 3, \lambda_1(u_+, v_+) = a^2, \lambda_2(u_+, v_+) = 3a^2$. Furthermore, the Rankine-Hugoniot condition (8.2.33) yields $s = a^2 + a + 1$. Therefore, if $a \in (-\frac{1}{2}, 0)$,

$$\lambda_2(u_-, v_-) > \lambda_1(u_-, v_-) > s > \lambda_2(u_+, v_+) > \lambda_1(u_+, v_+), \tag{8.3.7}$$

i.e., the shock is overcompressive.

The opposite case of *undercompressive* shocks, in which fewer than $n+1$ characteristics impinge on the shock, may arise as well. In that situation all $j$-characteristics are crossing the shock, from right to left for $j = 1, \dots, i$ and from left to right for $j = i+1, \dots, n$. Therefore, the Rankine-Hugoniot conditions must be supplemented with an additional jump condition, which is dubbed "kinetic relation".

The occurrence of overcompressive or undercompressive shocks raises serious difficulties in the theory, which, at the time of this writing, have only been partially resolved. To avoid such complications, we shall limit our investigation to the range of shock strength in which the assumptions of Theorem 8.2.2 are satisfied. In particular, this will encompass the case of weak shocks. Thus, with reference to the system (8.1.1), let us consider a state $U_-$, on the left, which is joined to a state $U_+$, on the right, by an $i$-shock of speed $s$. Assuming $\lambda_i(U_-)$ is a simple eigenvalue of $\mathrm{D}F(U_-)$, let $W_i$ denote the $i$-shock curve through $U_-$ (cf. Theorem 8.2.1), so that $U_- = W_i(0)$ and $U_+ = W_i(\tau)$. Furthermore, $\lambda_i(U_-) = s_i(0)$ and $s = s_i(\tau)$. We show that if $\tau < 0$ and $\dot{s}_i(\cdot) \geq 0$ on $(\tau, 0)$, then the shock satisfies the Lax $E$-condition. Indeed, $\dot{s}_i(\cdot) \geq 0$ implies $s = s_i(\tau) \leq s_i(0) = \lambda_i(U_-)$, which is the left half of (8.3.1). At the same time, so long as (8.2.22) and (8.2.23) hold at $\tau$, $\dot{s}_i(\cdot) \geq 0$ implies, by virtue of Theorem 8.2.4, that $s = s_i(\tau) \geq \lambda_i(W_i(\tau)) = \lambda_i(U_+)$, namely, the right half of (8.3.1). A similar argument demonstrates that the Lax $E$-condition also holds when $\tau > 0$ and $\dot{s}_i(\cdot) \leq 0$ on $(0, \tau)$, but it is violated if either $\tau < 0$ and $\dot{s}_i(\cdot) < 0$ on $(\tau, 0)$ or $\tau > 0$ and $\dot{s}_i(\cdot) > 0$ on $(0, \tau)$. The implications of the above statements to the genuine nonlinear case, in which, by virtue of (8.2.2), $\dot{s}_i(\cdot)$ does not change sign across 0, are recorded in the following

**8.3.1 Theorem.** *Assume $U_-$ is a point of genuine nonlinearity of the $i$-characteristic family of the system* (8.1.1), *with* $\mathrm{D}\lambda_i(U_-)R_i(U_-) > 0$ (or $< 0$). *Suppose $\lambda_i(U_-)$ is a simple eigenvalue of* $\mathrm{D}F(U_-)$ *and let $W_i$ denote the $i$-shock curve through $U_-$, with $U_- = W_i(0)$. Then a weak $i$-shock that joins $U_-$ to a state $U_+ = W_i(\tau)$ satisfies the Lax $E$-condition if and only if $\tau < 0$ (or $\tau > 0$).*

Thus, in the genuinely nonlinear case, one half of the shock curve is compatible with the Lax $E$-condition (8.3.1), as strict inequalities, and the other half is incompatible with it. When $U_-$ is a point of linear degeneracy of the $i$-characteristic field, so that $\dot{s}_i(0) = 0$, the situation is more delicate: If $\ddot{s}_i(0) < 0$, $\dot{s}_i(\tau)$ is positive for

$\tau < 0$ and negative for $\tau > 0$, so that weak $i$-shocks that join $U_-$ to $U_+ = W_i(\tau)$ are admissible, regardless of the sign of $\tau$. On the other hand, if $\ddot{s}_i(0) > 0$, $\dot{s}_i(\tau)$ is negative for $\tau < 0$ and positive for $\tau > 0$, in which case all (sufficiently) weak $i$-shocks violate the Lax $E$-condition. As noted above, when the $i$-characteristic family itself is linearly degenerate, $i$-shocks are $i$-contact discontinuities satisfying (8.3.1) as equalities.

Experience indicates that the primary role of the Lax $E$-condition is to secure the stability of the interaction of the shock, as an entity, with its adjacent "smoother" parts of the solution. This view is corroborated by the following

**8.3.2 Theorem.** *Assume the system* (8.1.1) *is strictly hyperbolic. Consider initial data $U_0$ such that $U_0(x) = U_L(x)$ for $x$ in* $(-\infty, 0)$ *and $U_0(x) = U_R(x)$ for $x$ in* $(0, \infty)$, *where $U_L$ and $U_R$ are smooth functions that are bounded, together with their first derivatives, on* $(-\infty, \infty)$. *Assume, further, that the state $U_- = U_L(0)$, on the left, is joined to the state $U_+ = U_R(0)$, on the right, by an $i$-shock of moderate strength and speed $s$, which satisfies the strict Lax $E$-condition*

$$\lambda_i(U_-) > s > \lambda_i(U_+). \tag{8.3.8}$$

*Then there exist: $T > 0$; a smooth function $x = \chi(t)$ on $[0, T)$, with $\chi(0) = 0$; and a function $U$ on $(-\infty, \infty) \times [0, T)$ with initial values $U_0$ and the following properties. $U$ is smooth and satisfies* (8.1.1), *in the classical sense, for any $(x, t)$, with $t \in [0, T)$ and $x \neq \chi(t)$. Furthermore, for $t \in [0, T)$ one-sided limits $U(\chi(t)-, t)$ and $U(\chi(t)+, t)$ exist and are joined by a weak $i$-shock of speed $\dot{\chi}(t)$, which satisfies the Lax $E$-condition.*

The proof, which is found in the references cited in Section 8.8, employs pointwise bounds on $U$ and its derivatives, obtained by monitoring the evolution of these functions along characteristics; i.e., it is of the same genre as the proof of Theorem 7.8.1. One may gain some insight from the very simple special case $n = 1$.

We thus consider the scalar conservation law (7.1.2) and assign initial data $u_0$ such that $u_0(x) = u_L(x)$ for $x \in (-\infty, 0)$ and $u_0(x) = u_R(x)$ for $x \in (0, \infty)$, where $u_L$ and $u_R$ are bounded and uniformly Lipschitz continuous functions on $(-\infty, \infty)$. Furthermore, $u_- = u_L(0)$ and $u_+ = u_R(0)$ satisfy

$$f'(u_-) > \frac{f(u_+) - f(u_-)}{u_+ - u_-} > f'(u_+). \tag{8.3.9}$$

Let $u_-(x, t)$ and $u_+(x, t)$ be the classical solutions of (7.1.2) with initial data $u_L(x)$ and $u_R(x)$, respectively, which exist on $(-\infty, \infty) \times [0, T)$, for some $T > 0$, by virtue of Theorem 6.1.1. On $[0, T)$ we define the function $\chi$ as solution of the ordinary differential equation

$$\frac{dx}{dt} = \frac{f(u_+(x,t)) - f(u_-(x,t))}{u_+(x,t) - u_-(x,t)} \tag{8.3.10}$$

with $\chi(0) = 0$. Finally, we define the function $u$ on $(-\infty, \infty) \times [0, T)$ by

$$u(x,t) = \begin{cases} u_-(x,t),\ t \in [0,T),\ x < \chi(t) \\ \\ u_+(x,t),\ t \in [0,T),\ x > \chi(t). \end{cases} \tag{8.3.11}$$

Clearly, $u$ satisfies (7.1.2), in the classical sense, for any $(x,t)$ with $t \in [0,T)$ and $x \neq \chi(t)$. Furthermore, $u(\chi(t)-,t)$ and $u(\chi(t)+,t)$ are joined by a shock of speed $\dot\chi(t)$. Finally, for $T$ sufficiently small, the Lax $E$-condition

$$f'(u(\chi(t)-,t)) > \dot\chi(t) > f'(u(\chi(t)+,t)), \qquad t \in [0,T), \tag{8.3.12}$$

holds by continuity, since it is satisfied at $t = 0$. Notice that it is because of the Lax $E$-condition that the solution $u$ solely depends on the initial data, i.e., it is independent of the "extraneous" information carried by $u_L(x)$ for $x > 0$ and $u_R(x)$ for $x < 0$.

It turns out that the assertion of Theorem 8.3.2 remains valid even in the more general situation where the states $U_-$ and $U_+$ are joined by a strong $i$-shock, provided that, in addition to the strong Lax $E$-condition (8.3.8), the following shock stability conditions hold:

$$\lambda_{i+1}(U_+) > s > \lambda_{i-1}(U_-), \tag{8.3.13}$$

$$\det[R_1(U_-),\dots,R_{i-1}(U_-),U_+ - U_-,R_{i+1}(U_+),\dots,R_n(U_+)] \neq 0. \tag{8.3.14}$$

For shocks of moderate strength, (8.3.13) follows from (8.3.8), by strict hyperbolicity, and (8.3.14) holds automatically, as $U_+ - U_-$ is nearly collinear to $R_i(U_\pm)$. Assumption (8.3.14) is a version of the *Lopatinski condition,* which plays a pivotal role in the stability theory of shock fronts and boundary-value problems.

As noted in Section 7.1, one-dimensional systems of conservation laws arise either in connection to media that are inherently one-dimensional or in the context of multispace-dimensional media wherein the fields stay constant in all but one spatial dimension. In the latter situation, Theorem 8.3.2 establishes the stability of planar shock fronts, albeit only for perturbations that likewise vary solely in the normal spatial direction. Naturally, it is important to investigate the stability of multispace-dimensional planar shocks under a broader class of perturbations and, more generally, the stability of non-planar shock fronts in $\mathbb{R}^m$.

The type of problem addressed by Theorem 8.3.2 may be formulated for hyperbolic systems (4.1.1) of conservation laws in $\mathbb{R}^m$ as follows. Let $U_L$, $U_R$ be smooth functions on $\mathbb{R}^m$, and $\mathcal{F}$ a smooth $(m-1)$-dimensional hypersurface embedded in $\mathbb{R}^m$ and oriented by means of its unit normal vector field $\nu$. Assume that the traces $U_-$ and $U_+$ of $U_L$ and $U_R$ on $\mathcal{F}$ satisfy the Rankine-Hugoniot jump condition (4.3.5). Denote by $U_0$ the function on $\mathbb{R}^m$ which coincides with $U_L$ on the negative side of $\mathcal{F}$ and with $U_R$ on the positive side of $\mathcal{F}$. It is required to construct an $m$-dimensional hypersurface $\mathcal{S}$ embedded in $\mathbb{R}^m \times [0,T)$, with trace $\mathcal{F}$ at $t = 0$, together with a piecewise smooth solution $U$ of (4.1.1) on $\mathbb{R}^m \times [0,T)$, with initial values $U_0$, such that $U$ is smooth for $(x,t) \notin \mathcal{S}$. Thus $\mathcal{S}$ will be a shock evolving out of $\mathcal{F}$.

The above problem has been solved under the following assumptions. The system (4.1.1) is endowed with a uniformly convex entropy and satisfies a certain structural condition, valid in particular for the Euler equations of isentropic or nonisentropic gas dynamics. At each point of $\mathcal{F}$, where the unit normal is $\nu$, the Lax $E$-condition (8.3.8) together with the stability conditions (8.3.13) and (8.3.14) hold, with $\lambda_j(\nu; U)$ and $R_j(\nu; U)$ in the place of $\lambda_j(U)$ and $R_j(U)$. Finally, a complicated set of compatibility conditions, involving the normal derivatives $\partial^p U_L/\partial\nu^p$ and $\partial^p U_R/\partial\nu^p$ of $U_L$ and $U_R$, up to a certain order depending on $m$, is satisfied on $\mathcal{F}$. These are needed in order to avert the emission of spurious waves from $\mathcal{F}$.

The construction of $\mathcal{S}$ and $U$ is performed within the framework of Sobolev spaces and involves quite sophisticated tools (pseudodifferential operators, paradifferential calculus, etc.). The relevant references are listed in Section 8.8.

Another serious issue of concern is the internal stability of shocks. It turns out that the Lax $E$-condition is effective in that direction as well, so long as the system is genuinely nonlinear and the shocks are weak; however, it is insufficient in more general situations. For that purpose, we have to consider additional, more discriminating shock admissibility criteria, which will be introduced in the following sections.

## 8.4 The Liu Shock Admissibility Criterion

The Liu shock admissibility test is more discriminating than the Lax $E$-condition and strives to capture the internal stability of shocks. By its very design, it makes sense only in the context of shocks joining states that may be connected by shock curves. Thus, for general systems, its applicability is limited to shocks of moderate strength. Nevertheless, in special systems it also applies to strong shocks.

For a given state $U_-$, assume $\lambda_i(U_-)$ is a simple eigenvalue of $\mathrm{D}F(U_-)$ so that the $i$-shock curve $W_i(\tau; U_-)$ through $U_-$ is well defined, by Theorem 8.2.1. An $i$-shock that joins $U_-$, on the left, to a state $U_+ = W_i(\tau_+; U_-)$, on the right, of speed $s$, satisfies the *Liu $E$-condition* if

$$s = s_i(\tau_+; U_-) \leq s_i(\tau; U_-), \quad \text{for all } \tau \text{ between } 0 \text{ and } \tau_+ . \tag{8.4.1}$$

Similar to the Lax $E$-condition, the justification of the above admissibility criterion will be established a posteriori, through its connection to other, physically motivated, shock admissibility criteria, as well as by its role in the construction of stable solutions to the Riemann problem, in Chapter IX.

As $U_-$ and $U_+$ are joined by an $i$-shock, $U_-$ must also lie on the $i$-shock curve emanating from $U_+$, say $U_- = W_i(\tau_-; U_+)$. So long as (8.2.21) holds along the above shock curves, (8.4.1) is equivalent to the dual statement

$$s = s_i\big(\tau_-; U_+\big) \geq s_i\big(\tau; U_+\big), \quad \text{for all } \tau \text{ between } 0 \text{ and } \tau_- . \tag{8.4.2}$$

We proceed to verify that (8.4.1) implies (8.4.2) under the hypothesis that all minima of the function $s_i(\tau; U_-)$ are nondegenerate. (The general case may be reduced to the above by a perturbation argument, and the proof of the converse statement is similar.) For definiteness, we assume that $\tau_+ > 0$, in which case $\tau_- < 0$. From (8.4.1) it follows that either $\dot{s}_i(\tau_+; U_-) < 0$ or $\dot{s}_i(\tau_+; U_-) = 0$ and $\ddot{s}_i(\tau_+; U_-) > 0$. Thus, by virtue of (8.2.22), (8.2.23) and (8.2.24), either $\lambda_i(U_+) < s$ or $\lambda_i(U_+) = 0$ and $\mathrm{D}\lambda_i(U_+)R_i(U_+) > 0$. In either case, recalling (8.2.1) and (8.2.2), we deduce that $s_i(\tau; U_+) < s$ for $\tau < 0$, near zero. If (8.4.2) is violated for some $\tau \in (\tau_-, 0)$, $s_i(\tau; U_+) - s$ must be changing sign across some $\tau_0 \in (\tau_-, 0)$. Let $U_0 = s_i(\tau_0; U_+)$. Since both $U_-$ and $U_0$ are joined to $U_+$ by shocks of speed $s$, $U_-$ and $U_0$ can also be joined to each other by a shock of speed $s$, i.e., $U_0 = W_i(\tau_1; U_-)$, for some $\tau_1 \in (0, \tau_+)$, and $s_i(\tau_1; U_-) = s$. Thus, (8.4.1) implies $\dot{s}_i(\tau_1; U_-) = 0$ and $\ddot{s}_i(\tau_1; U_-) > 0$, and so, by account of (8.2.23) and (8.2.24), $\lambda_i(U_0) = s$ and $\mathrm{D}\lambda_i(U_0)R_i(U_0) > 0$. But then (8.2.23) and (8.2.24) again yield $\dot{s}_i(\tau_0; U_+) = 0$ and $\ddot{s}_i(\tau_0; U_+) < 0$ so that, contrary to our hypothesis, $s_i(\tau; U_+) - s$ cannot change sign across $\tau_0$.

In particular, applying (8.4.1) and (8.4.2) for $\tau = 0$ and recalling (8.2.1), we arrive at (8.3.1). We have thus established

**8.4.1 Theorem.** *Within the range where* (8.2.21) *holds, any shock satisfying the Liu $E$-condtion also satisfies the Lax $E$-condition.*

When the system is genuinely nonlinear, these two criteria coincide, at least in the realm of weak shocks:

**8.4.2 Theorem.** *Assume the $i$-characteristic family is genuinely nonlinear and $\lambda_i$ is a simple characteristic speed. Then weak $i$-shocks satisfy the Liu $E$-condition if and only if they satisfy the Lax $E$-condition.*

**Proof.** The Liu $E$-condition implies the Lax $E$-condition by Theorem 8.4.1. To show the converse, assume the state $U_-$, on the left, is joined to the state $U_+ = W_i(\tau_+; U_-)$, on the right, by a weak $i$-shock of speed $s$, which satisfies the Lax $E$-condition (8.3.1). Suppose, for definiteness, $\mathrm{D}\lambda_i(U_-)R_i(U_-) > 0$ (the case of the opposite sign is similar). By virtue of Theorem 8.3.1, $\tau_+ < 0$. Since the shock is weak, by Theorem 8.2.1, $\dot{s}_i(\tau; U_-) > 0$ on the interval $(\tau_+, 0)$. Then $s = s_i(\tau_+; U_-) < s_i(\tau; U_-)$ for $\tau \in (\tau_+, 0)$, i.e., the Liu $E$-condition holds. This completes the proof.

When the system is not genuinely nonlinear and/or the shocks are not weak, the Liu $E$-condition is stricter than the Lax $E$-condition. This will be demonstrated by means of the following examples.

Let us first consider the scalar conservation law (7.1.2). The shock curve is the $u$-axis and we may use $u$ as the parameter $\xi$. The shock speed is given by (8.3.3). It is then clear that a shock joining the states $u_-$ and $u_+$ will satisfy the Liu $E$-condition (8.4.1), (8.4.2) if and only if

$$\text{(8.4.3)} \qquad \frac{f(u_0) - f(u_-)}{u_0 - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \geq \frac{f(u_+) - f(u_0)}{u_+ - u_0}$$

holds for every $u_0$ between $u_-$ and $u_+$. This is the celebrated *Oleinik E-condition*. It is easily memorized as a geometric statement: When $u_- < u_+$ (or $u_- > u_+$) the shock that joins $u_-$, on the left, to $u_+$, on the right, is admissible if the arc of the graph of $f$ with endpoints $(u_-, f(u_-))$ and $(u_+, f(u_+))$ lies above (or below) the chord that connects the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$. Letting $u_0$ converge to $u_-$ and to $u_+$, we deduce that (8.4.3) implies (8.3.2). The converse, of course, is generally false, unless $f$ is convex or concave. We have thus demonstrated that in the scalar conservation law the Liu $E$-condition is stricter than the Lax $E$-condition when $f$ contains inflection points. In the genuinely nonlinear case, the Liu and Lax $E$-conditions are equivalent.

We now turn to the system (7.1.8) of isentropic elasticity. The shock curves are determined by (8.2.11) so we may use $u$ as parameter instead of $\xi$. The shock speed is given by (8.1.11). Therefore, a shock joining the states $(u_-, v_-)$ and $(u_+, v_+)$ will satisfy the Liu $E$-condition (8.4.1), (8.4.2) if and only if

(8.4.4) $$\frac{\sigma(u_0)-\sigma(u_-)}{u_0-u_-} \lesseqgtr \frac{\sigma(u_+)-\sigma(u_-)}{u_+-u_-} \lesseqgtr \frac{\sigma(u_+)-\sigma(u_0)}{u_+-u_0}$$

holds for all $u_0$ between $u_-$ and $u_+$, where "$\leq$" applies for 1-shocks and "$\geq$" applies for 2-shocks. This is called the *Wendroff E-condition*. In geometric terms, it may be stated as follows: When $s(u_+ - u_-) < 0$ (or $> 0$) the shock that joins $(u_-, v_-)$, on the left, to $(u_+, v_+)$, on the right, is admissible if the arc of the graph of $\sigma$ with endpoints $(u_-, \sigma(u_-))$ and $(u_+, \sigma(u_+))$ lies below (or above) the chord that connects the points $(u_-, \sigma(u_-))$ and $(u_+, \sigma(u_+))$. Clearly, there is close analogy with the Oleinik $E$-condition. Letting $u_0$ in (8.4.4) converge to $u_-$ and to $u_+$, we deduce that the Wendroff $E$-condition implies the Lax $E$-condition (8.3.4). The converse is true when $\sigma$ is convex or concave, but false otherwise. Thus, for the system (7.1.8) the Liu $E$-condition is stricter than the Lax $E$-condition when $\sigma$ contains inflection points. In the genuinely nonlinear case, the Liu and Lax $E$-conditions are equivalent.

As we shall see, the Oleinik $E$-condition and the Wendroff $E$-condition follow naturally from other admissibility criteria. To a great extent these special $E$-conditions provided the motivation for postulating the general Liu $E$-condition.

## 8.5 The Entropy Shock Admissibility Criterion

The idea of employing entropy inequalities to weed out spurious weak solutions of general hyperbolic systems of conservation laws was introduced in Section 4.5 and was used repeatedly in Chapters IV, V, and VI. It was observed that in the context of $BV$ weak solutions the entropy condition reduces to the set of inequalities (4.5.5), to be tested at every point of the shock set. For the system (8.1.1), in one-space dimension, (4.5.5) assumes the form

(8.5.1) $$-s[\eta(U_+) - \eta(U_-)] + q(U_+) - q(U_-) \leq 0,$$

where $(\eta, q)$ is an entropy-entropy flux pair satisfying (7.4.1), $\mathrm{D}q = \mathrm{D}\eta\mathrm{D}F$. The quantity on the left-hand side of (8.5.1) will be called henceforth the *entropy production across the shock*.

The fact that the entropy condition reduces to a pointwise test on shocks has played a dominant role in shaping the prevailing view that admissibility need be tested only at the level of shocks, i.e., that a general $BV$ weak solution will be admissible if and only if each one of its shocks is admissible.

In setting up an entropy admissibility condition (8.5.1), the first task is to designate the appropriate entropy-entropy flux pair $(\eta, q)$. Whenever (8.1.1) arises in connection to physics, the physically appropriate entropy should always be designated. In particular, the pairs (7.4.9), (7.4.10) and (7.4.11) must be designated for the systems (7.1.5), (7.1.8) and (7.1.10), respectively[1].

In the absence of guidelines from physics, or when the entropy-entropy flux pair supplied by physics is inadequate to rule out all spurious shocks, additional entropy-entropy flux pairs must be designated (whenever available), motivated by other admissibility criteria, such as viscosity. In that connection, we should bear in mind that, as demonstrated in earlier chapters, convexity of the entropy function is a desirable feature.

Let us begin the investigation with the scalar conservation law (7.1.2). The shock speed $s$ is given by (8.3.3). In accordance with the discussion in Chapter VI, admissible shocks must satisfy (8.5.1) for all convex functions $\eta$. However, as explained in Section 6.2, (8.5.1) need only be tested for the family (6.2.5) of entropy-entropy flux pairs, namely

$$\eta(u;\bar{u}) = (u-\bar{u})^+ \,, \qquad q(u;\bar{u}) = \operatorname{sgn}(u-\bar{u})^+[f(u)-f(\bar{u})]. \tag{8.5.2}$$

It is immediately seen that (8.5.1) will be satisfied for every $(\eta, q)$ in the family (8.5.2) if and only if (8.4.3) holds for all $u_0$ between $u_-$ and $u_+$. We have thus rederived the Oleinik $E$-condition encountered in Section 8.4. This implies that, for the scalar conservation law, the entropy admissibility condition, applied for all convex entropies, is equivalent to the Liu $E$-condition.

It is generally impossible to recover the Oleinik $E$-condition from the entropy condition (8.5.1) for a single entropy-entropy flux pair. Take for example

$$\eta(u) = \tfrac{1}{2}u^2 \,, \qquad q(u) = \int_0^u \omega f'(\omega) d\omega. \tag{8.5.3}$$

By virtue of (8.3.3) and after a short calculation, (8.5.1) takes the form

$$\tfrac{1}{2}[f(u_+) + f(u_-)](u_+ - u_-) - \int_{u_-}^{u_+} f(\omega) d\omega \le 0. \tag{8.5.4}$$

Notice that the entropy production across the shock is here measured by the signed area of the domain bordered by the arc of the graph of $f$ with endpoints $(u_-, f(u_-))$,

[1] In applying (8.5.1) to the system (7.1.5), with entropy-entropy flux pair (7.4.9), one should not confuse $s$ in (7.4.9), namely the physical entropy, with $s$ in (8.5.1), the shock speed. Since $q = 0$, (8.5.1) here states that "after a shock passes, the physical entropy must increase." The reader is warned that this statement is occasionally misinterpreted as a general physical principle and is applied even when it is no longer appropriate.

$(u_+, f(u_+))$, and the chord that connects $(u_-, f(u_-))$ with $(u_+, f(u_+))$. Clearly, the Oleinik $E$-condition (8.4.3) implies (8.5.4) but the converse is generally false. Moreover, neither (8.5.4) generally implies the Lax $E$-condition (8.3.2) nor the other way around. However, when $f$ is convex or concave, (8.5.4), (8.4.3) and (8.3.2) are all equivalent.

Next we turn to the system (7.1.8) of isentropic elasticity. We employ the entropy-entropy flux pair $(\eta, q)$ given by (7.4.10). An interesting, rather lengthy, calculation, which involves the Rankine-Hugoniot condition (8.1.10), shows that (8.5.1) here reduces to

$$s\{\tfrac{1}{2}[\sigma(u_+) + \sigma(u_-)](u_+ - u_-) - \int_{u_-}^{u_+} \sigma(\omega)d\omega\} \leq 0. \tag{8.5.5}$$

The quantity in braces on the left-hand side of (8.5.5) measures the signed area of the set bordered by the arc of the graph of $\sigma$ with endpoints $(u_-, \sigma(u_-))$ , $(u_+, \sigma(u_+))$ and the chord that connects $(u_-, \sigma(u_-))$ with $(u_+, \sigma(u_+))$. Hence, the Wendroff $E$-condition (8.4.4) implies (8.5.5) but the converse is generally false. Neither (8.5.5) necessarily implies the Lax $E$-condition (8.3.4) nor the other way around. However, when $\sigma$ is convex or concave, (8.5.5), (8.4.4) and (8.3.4) are all equivalent. Of course, the system (7.1.8) is endowed with a rich collection of entropies, so one may employ additional entropy-entropy flux pairs to recover the Wendroff $E$-condition from the entropy condition, but this shall not be attempted here.

We now consider the entropy shock admissibility condition (8.5.1) for a general system (8.1.1), under the assumption that $U_-$ and $U_+$ are connected by a shock curve. In particular, this will encompass the case of shocks of moderate strength. We thus assume $\lambda_i(U_-)$ is a simple characteristic speed, consider the $i$-shock curve $W_i(\tau; U_-)$ through $U_-$, and let $U_+ = W_i(\tau_+; U_-)$, $s = s_i(\tau_+; U_-)$. The entropy production along the $i$-shock curve is given by

$$E(\tau) = -s_i(\tau)[\eta(W_i(\tau)) - \eta(U_-)] + q(W_i(\tau)) - q(U_-)\,. \tag{8.5.6}$$

Differentiating (8.5.6) and using (7.4.1) yields

$$\dot{E} = -\dot{s}_i[\eta(W_i) - \eta(U_-)] - s_i \mathrm{D}\eta(W_i)\dot{W}_i + \mathrm{D}\eta(W_i)\mathrm{D}F(W_i)\dot{W}_i\ . \tag{8.5.7}$$

Combining (8.5.7) with (8.2.16) (for $\bar{U} = U_-$), we deduce

$$\dot{E} = -\dot{s}_i\,\{\eta(W_i) - \eta(U_-) - \mathrm{D}\eta(W_i)[W_i - U_-]\}\,. \tag{8.5.8}$$

Notice that the right-hand side of (8.5.8) is of quadratic order in the strength of the shock. Therefore, the entropy production $E(\tau_+)$ across the shock, namely the integral of $\dot{E}(\tau)$ from 0 to $\tau_+$, is of cubic order in $\tau_+$. We have thus established the following

**8.5.1 Theorem.** *The entropy production across a weak shock is of third order in the strength of the shock.*

When $U_-$ is a point of linear degeneracy of the $i$-characteristic family, $\dot{s}_i(0; U_-)$ vanishes and so the entropy production across the shock will be of (at most) fourth order in the strength of the shock. In particular, when the $i$-characteristic family is linearly degenerate, $\dot{s}_i$ vanishes identically, by Theorem 8.2.5, and so

**8.5.2 Theorem.** *When the $i$-characteristic family is linearly degenerate, the entropy production across any $i$-shock ($i$-contact discontinuity) is zero.*

Turning now to the issue of admissibility of the shock, we observe that when $\eta$ is a convex function the expression in braces on the right-hand side of (8.5.8) is nonpositive. Thus $\dot{E}$ and $\dot{s}_i$ have the same sign. Consequently, the entropy admissibility condition $E(\tau_+) \leq 0$ will hold if $\tau_+ < 0$ and $\dot{s}_i \geq 0$ on $(\tau_+, 0)$, or if $\tau_+ > 0$ and $\dot{s}_i \leq 0$ on $(0, \tau_+)$; while it will be violated when either $\tau_+ < 0$ and $\dot{s}_i < 0$ on $(\tau_+, 0)$ or $\tau_+ > 0$ and $\dot{s}_i > 0$ on $(0, \tau_+)$. Recalling our discussion in Section 8.3, we conclude that the entropy admissibility condition and the Lax $E$-condition are equivalent in the range of $\tau$, on either side of 0, where $\dot{s}_i(\tau)$ does not change sign. In particular, this will be the case when the characteristic family is genuinely nonlinear and the shocks are weak:

**8.5.3 Theorem.** *When the $i$-characteristic family is genuinely nonlinear and $\lambda_i$ is a simple characteristic speed, the entropy admissibility condition and the Lax $E$-condition for weak $i$-shocks are equivalent.*

In order to escape from the realm of genuine nonlinearity and weak shocks, let us consider the condition

$$\tau \dot{W}_i^\top(\tau; U_-)\mathrm{D}^2\eta(W_i(\tau; U_-))[W_i(\tau; U_-) - U_-] \geq 0. \tag{8.5.9}$$

Recalling (7.4.3), (7.4.4) and Theorem 8.2.1, we conclude that when the entropy $\eta$ is convex (8.5.9) will always hold for weak $i$-shocks; it will also be satisfied for shocks of moderate strength when $i$-shock curves extend into that regime; and may even hold for strong shocks, so long as $\dot{W}_i$ and $W_i - U_-$ keep pointing nearly in the direction of $R_i$.

**8.5.4 Theorem.** *Assume that the $i$-shock curve $W_i(\tau; U_-)$ through $U_-$, and corresponding shock speed function $s_i(\tau; U_-)$, are defined on an interval $(\alpha, \beta)$ containing 0, and satisfy (8.5.9) for $\tau \in (\alpha, \beta)$, where $\eta$ is a convex entropy of the system. Then any $i$-shock joining $U_-$, on the left, to $U_+ = W_i(\tau_+; U_-)$, on the right, with speed $s$, which satisfies the Liu $E$-condition (8.4.1) also satisfies the entropy admissibility condition (8.5.1).*

**Proof.** We set

$$Q(\tau) = \eta(W_i(\tau; U_-)) - \eta(U_-) - \mathrm{D}\eta(W_i(\tau; U_-))[W_i(\tau; U_-) - U_-]. \tag{8.5.10}$$

By virtue of (8.5.9),

$$\tau \dot{Q}(\tau) \le 0. \tag{8.5.11}$$

Integrating (8.5.8) from 0 to $\tau_+$, integrating by parts and using (8.5.10), (8.5.11) and (8.4.1) we obtain

$$\begin{aligned} E(\tau_+) &= -\int_0^{\tau_+} \dot{s}_i(\tau; U_-) Q(\tau) d\tau = -s Q(\tau_+) + \int_0^{\tau_+} s_i(\tau; U_-) \dot{Q}(\tau) d\tau \\ &\le -s Q(\tau_+) + s \int_0^{\tau_+} \dot{Q}(\tau) d\tau = 0, \end{aligned} \tag{8.5.12}$$

which shows that the shock satisfies (8.5.1). This completes the proof.

## 8.6 Viscous Shock Profiles

The idea of using the vanishing viscosity approach for identifying admissible weak solutions of hyperbolic systems of conservation laws was introduced in Section 4.6. In the present setting of one-space dimension, for the system (8.1.1), Equation (4.6.1) reduces to

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = \mu \partial_x [B(U(x,t)) \partial_x U(x,t)]. \tag{8.6.1}$$

As already explained in Section 4.6, the selection of the $n \times n$ matrix-valued function $B$ may be suggested by the physical context of the system or it may just be an artifact of the analysis. Consider for example the dissipative systems

$$\partial_t u + \partial_x f(u) = \mu \partial_x^2 u, \tag{8.6.2}$$

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u) = \mu \partial_x (u^{-1} \partial_x v), \end{cases} \tag{8.6.3}$$

$$\begin{cases} \partial_t u + \partial_x [(u^2 + v^2) u] = \mu \partial_x^2 u \\ \partial_t v + \partial_x [(u^2 + v^2) v] = \mu \partial_x^2 v, \end{cases} \tag{8.6.4}$$

associated with the hyperbolic systems (7.1.2), (7.1.8), and (7.2.11). In so far as (7.1.8) is interpreted as the system of isentropic gas dynamics, the selection of viscosity in (8.6.3) is dictated by physics[2]. On the other hand, in (8.6.2) and (8.6.4) the viscosity is artificial.

In contrast to the entropy criterion, it is not at all clear that admissibility of weak solutions by means of the vanishing viscosity criterion is decided solely at the level

[2] Compare with (4.6.2). The variable viscosity coefficient $\mu u^{-1}$ is adopted so that in the spatial setting, where measurements are usually performed, viscosity will be constant $\mu$. Of course this will make sense only when $u > 0$.

of the shock set. However, taking that premise for granted, it will suffice to test admissibility in the context of solutions in the simple form

$$U(x,t) = \begin{cases} U_-, & x < st \\ U_+, & x > st, \end{cases} \tag{8.6.5}$$

namely a shock of constant speed $s$ joining the constant state $U_-$, on the left, to the constant state $U_+$, on the right. Presumably, functions (8.6.5) may be approximated, as $\mu \downarrow 0$, by a family of solutions $U_\mu$ of (8.6.1) in the form of *traveling waves*, namely functions of the single variable $x - st$. Taking advantage of the scaling in (8.6.1), we seek a family of solutions in the form

$$U_\mu(x,t) = V\left(\frac{x - st}{\mu}\right). \tag{8.6.6}$$

Substituting in (8.6.1), we deduce that $V$ should satisfy the ordinary differential equation

$$[B(V(\tau))\dot{V}(\tau)]^{\cdot} = \dot{F}(V(\tau)) - s\dot{V}(\tau), \tag{8.6.7}$$

where the overdot denotes differentiation with respect to $\tau = (x - ct)/\mu$. We are interested in solutions in which $\dot{V}$ vanishes at $V = U_-$ and so, upon integrating (8.6.7) once with respect to $\tau$,

$$B(V)\dot{V} = F(V) - F(U_-) - s[V - U_-]. \tag{8.6.8}$$

Notice that the right-hand side of (8.6.8) vanishes on the set of $V$ that may be joined to $U_-$ by a shock of speed $s$. This set includes, in particular, the state $U_+$.

We say that $U_-$, on the left, is connected to $U_+$, on the right, by a *viscous shock profile* if there is a smooth arc joining $U_-$ to $U_+$ which is an invariant set for the differential equation (8.6.8) and, in addition, at any point where there is motion, the flow is directed from $U_-$ to $U_+$.

The shock that joins $U_-$, on the left, to $U_+$, on the right, is said to satisfy the *viscous shock admissibility criterion* if $U_-$ can be connected to $U_+$ by a viscous shock profile.

Determining viscous shock profiles is important not only because they shed light on the issue of admissibility but also because they provide information (at least when the matrix $B$ is physically motivated) on the nature of the sharp transition modelled by the shock, the so-called *structure of the shock*. Indeed, the stretching of coordinates involved in (8.6.6), as $\mu \downarrow 0$, allows one, as it were, to observe the shock under the microscope.

Any contact discontinuity associated with a linearly degenerate characteristic family satisfies the viscous shock admissibility criterion. Indeed, in that case, by virtue of Theorem 8.2.5, the shock curve itself serves as the viscous shock profile and all of its points are equilibria of the differential equation (8.6.8). The opposite extreme arises when $U_-$ and $U_+$ are the only equilibrium points on the viscous shock profile, in which case $U_-$ is the $\alpha$-limit set and $U_+$ is the $\omega$-limit set of an orbit of the differential equation (8.6.8). In general, the viscous shock profile may contain a (finite or infinite) number of equilibrium points with any two consecutive ones connected by orbits of (8.6.8).

Let us illustrate the above by means of the scalar conservation law (7.1.2) and the corresponding dissipative equation (8.6.2). System (8.6.8) now reduces to the scalar equation

$$\dot{u} = f(u) - f(u_-) - s(u - u_-). \tag{8.6.9}$$

It is clear that $u_-$ will be connected to $u_+$ by a viscous shock profile if and only if the right-hand side of (8.6.9) does not change sign between $u_-$ and $u_+$, and indeed it is nonnegative when $u_- < u_+$ and nonpositive when $u_- > u_+$. Recalling (8.3.3), we conclude that in the scalar conservation law (7.1.2) a shock satisfies the viscous shock admissibility criterion if and only if the Oleinik $E$-condition (8.4.3) holds. When (8.4.3) holds as a strict inequality for any $u_0$ (strictly) between $u_-$ and $u_+$, then $u_-$ is connected to $u_+$ with a single orbit. By contrast, when (8.4.3) becomes equality for a set of intermediate $u_0$, we need more than one orbit and perhaps even a number of contact discontinuities in order to build the viscous shock profile. In that case one may prefer to visualize the shock as a composite of several shocks and/or contact discontinuities, all travelling with the same speed.

Next we turn to the system (7.1.8) and the corresponding dissipative system (8.6.3). In that case (8.6.8) reads

$$\begin{cases} 0 = -v + v_- - s(u - u_-) \\ \\ u^{-1}\dot{v} = -\sigma(u) + \sigma(u_-) - s(v - v_-). \end{cases} \tag{8.6.10}$$

The reason we end up here with a combination of algebraic and differential equations, rather than just differential equations, is that $B$ is a singular matrix. In any event, upon eliminating $v$ between the two equations in (8.6.10), we deduce

$$su^{-1}\dot{u} = \sigma(u) - \sigma(u_-) - s^2(u - u_-). \tag{8.6.11}$$

Since $u > 0$, $(u_-, v_-)$ will be connected to $(u_+, v_+)$ by a viscous shock profile if and only if the right-hand side of (8.6.11) does not change sign between $u_-$ and $u_+$ and is in fact nonnegative when $s(u_+ - u_-) > 0$ and nonpositive when $s(u_+ - u_-) < 0$. In view of (8.1.11), we conclude that in the system (7.1.8) of isentropic elasticity a shock satisfies the viscous shock admissibility criterion if and only if the Wendroff $E$-condition (8.4.4) holds.

It was the Oleinik $E$-condition and the Wendroff $E$-condition, originally derived through the above argument, that motivated the general Liu $E$-condition. We now proceed to show that the viscous shock admissibility criterion is generally equivalent to the Liu $E$-condition, at least in the range of shocks of moderate strength. For simplicity, only the special case $B = I$ will be discussed here; the case of more general $B$ is treated in the references cited in Section 8.8.

**8.6.1 Theorem.** *Assume $\lambda_i$ is a simple eigenvalue of* $\mathrm{D}F$*. Then an $i$-shock of moderate strength satisfies the viscous shock admissibility criterion, with $B = I$, if and only if it satisfies the Liu $E$-condition.*

**Proof.** Assume the state $U_-$, on the left, is joined to the state $U_+$, on the right, by an $i$-shock of moderate strength and speed $s$. In order to apply the viscous shock admissibility test, the first task is to construct a curve in state space that connects $U_+$ with $U_-$ and is invariant under the flow generated by (8.6.8), for $B = I$. To that end, we embed (8.6.8) into a larger, autonomous, system by introducing a new (scalar) variable $r$:

$$\begin{cases} \dot{V} = F(V) - F(U_-) - r[V - U_-] \\ \\ \dot{r} = 0. \end{cases} \tag{8.6.12}$$

Notice that the Jacobian matrix of the right-hand side of (8.6.12), evaluated at the equilibrium point $V = U_-$, $r = \lambda_i(U_-)$, is

$$J = \left( \begin{array}{c|c} \mathrm{D}F(U_-) - \lambda_i(U_-)I & 0 \\ \hline 0 & 0 \end{array} \right), \tag{8.6.13}$$

with eigenvalues $\lambda_j(U_-) - \lambda_i(U_-)$, $j = 1, \cdots, n$, and 0; the corresponding eigenvectors being

$$\begin{pmatrix} R_j(U_-) \\ 0 \end{pmatrix}, \quad j = 1, \cdots, n, \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8.6.14}$$

We see that $J$ has two zero eigenvalues, associated with a two-dimensional eigenspace, while the remaining eigenvalues are nonzero real numbers. The center manifold theorem then implies that any trajectory of (8.6.12) that is confined in a small neighborhood of the point $(U_-, \lambda_i(U_-))$ must lie on a two-dimensional manifold $\mathcal{M}$, which is invariant under the flow generated by (8.6.12), and may be parametrized by

$$V = \Phi(\zeta, r) = U_- + \zeta R_i(U_-) + S(\zeta, r), \qquad r = r, \tag{8.6.15}$$

with

$$S(0, \lambda_i(U_-)) = 0, \quad S_\zeta(0, \lambda_i(U_-)) = 0, \quad S_r(0, \lambda_i(U_-)) = 0. \tag{8.6.16}$$

In particular, the equilibrium point $(U_+, s)$ of (8.6.12) must lie on $\mathcal{M}$, in which case $U_+ = \Phi(\rho, s)$, for some $\rho$ near zero. Thus $U_-$ and $U_+$ are connected by the curve $V = \Phi(\zeta, s)$, for $\zeta$ between 0 and $\rho$, and this curve is invariant under the flow generated by (8.6.8), for $B = I$.

The flow induced by $(8.6.12)_1$ along the invariant curve $V = \Phi(\cdot, r)$ is represented by a function $\zeta = \zeta(\cdot)$ which satisfies the scalar ordinary differential equation

$$\dot{\zeta} = g(\zeta, r), \tag{8.6.17}$$

with $g$ defined through

$$g(\zeta, r)\Phi_\zeta(\zeta, r) = F(\Phi(\zeta, r)) - F(U_-) - r[\Phi(\zeta, r) - U_-]. \tag{8.6.18}$$

In particular, recalling (8.6.15) and (8.6.16),

$$g(0, r) = 0, \qquad g_\zeta(0, r) = \lambda_i(U_-) - r. \tag{8.6.19}$$

Clearly, the viscous shock admissibility criterion will be satisfied if and only if $\rho g(\zeta, s) \geq 0$ for all $\zeta$ between 0 and $\rho$.

Suppose now the shock satisfies the Liu $E$-condition. Thus, if $W_i$ denotes the $i$-shock curve through $U_-$ and $s_i$ is the corresponding shock speed function, so that $U_- = W_i(0)$, $U_+ = W_i(\tau_+)$, $s = s_i(\tau_+)$, we must have $s_i(\tau) \geq s$ for $\tau$ between 0 and $\rho$. For definiteness, let us assume $U_+ - U_-$ points nearly in the direction of $R_i(U_-)$, in which case both $\rho$ and $\tau_+$ are positive.

We fix $r < s$, with $s - r$ very small, consider the curve $\Phi(\cdot, r)$ and identify $\kappa > 0$ such that $[\Phi(\kappa, r) - U_+]^\top R_i(U_-) = 0$. We show that $g(\zeta, r) > 0$, $0 < \zeta < \kappa$. Indeed, if $g(\zeta, r) = 0$ for some $\zeta$, $0 < \zeta < \kappa$, then, by virtue of (8.6.18), the state $\Phi(\zeta, r)$ may be joined to the state $U_-$ by a shock of speed $r$. Thus, $\Phi(\zeta, r)$ lies on the shock curve $W_i$, say $\Phi(\zeta, r) = W_i(\tau)$, for some $\tau$. By the construction of $\kappa$, since $0 < \zeta < \kappa$, it is necessarily $0 < \tau < \tau_+$. However, in that case it is $r = s_i(\tau) \geq s$, namely, a contradiction to our assumption $r < s$. This establishes that $g(\zeta, r)$ does not change sign on $(0, \kappa)$. At the same time, on account of (8.6.19), $g_\zeta(0, r) = s_i(0) - r \geq s - r > 0$, which shows that $g(\zeta, r) > 0$, $0 < \zeta < \kappa$. Finally, we let $r \uparrow s$, in which case $\kappa \to \rho$. Hence $g(\zeta, s) \geq 0$ for $\zeta \in (0, \rho)$.

By a similar argument one shows the converse, namely that $\rho g(\zeta, s) \geq 0$, for $\zeta$ between 0 and $\rho$, implies $s_i(\tau) \geq s$, for $\tau$ between 0 and $\tau_+$. This completes the proof.

Combining Theorems 8.4.1, 8.4.2 and 8.6.1, we conclude that the viscous shock admissibility criterion generally implies the Lax $E$-condition but the converse is generally false, unless the system is genuinely nonlinear and the shocks are weak.

Our next task is to compare the viscous shock admissibility criterion with the entropy shock admissibility criterion. We thus assume that the system (8.1.1) is equipped with an entropy-entropy flux pair $(\eta, q)$, satisfying (7.4.1), $\mathrm{D}q = \mathrm{D}\eta\mathrm{D}F$. The natural compatibility condition between the entropy and the viscosity matrix $B$ was already discussed in Section 4.6. We write (a weaker form of) the condition (4.6.7) in the present, one-dimensional setting:

$$H^\top \mathrm{D}^2\eta(U)B(U)H \geq 0, \quad H \in I\!R^n, \quad U \in \mathcal{O}. \tag{8.6.20}$$

As already noted in Section 4.6, when $B = I$, (8.6.20) will hold if and only if $\eta$ is convex.

**8.6.2 Theorem.** *When* (8.6.20) *holds, any shock that satisfies the viscous shock admissibility criterion also satisfies the entropy shock admissibility criterion.*

**Proof.** Consider a shock of speed $s$ which joins the state $U_-$, on the left, with the state $U_+$, on the right, and satisfies the viscous shock admissibility condition.

Assume first $U_-$ is connected to $U_+$ with a single orbit of (8.6.8), i.e., there is a function $V$ which satisfies (8.6.8), and thereby also (8.6.7), on $(-\infty, \infty)$, together with the conditions $V(\tau) \to U_\pm$, as $\tau \to \pm\infty$. We multiply (8.6.7), from the left, by $\mathrm{D}\eta(V(\tau))$ and use (7.4.1) to get

$$[\mathrm{D}\eta(V)B(V)\dot{V}]^{\cdot} - \dot{V}^{\top}\mathrm{D}^2\eta(V)B(V)\dot{V} = \dot{q}(V) - s\dot{\eta}(V). \tag{8.6.21}$$

Integrating (8.6.21) over $(-\infty, \infty)$ and using (8.6.20), we arrive at (8.5.1). We have thus proved that the shock satisfies the entropy condition.

In the general case where the viscous shock profile contains intermediate equilibrium points, we realize the shock as a composite of a (finite or infinite) number of simple shocks of the above type and/or contact discontinuities, all propagating with the same speed $s$. As shown above, the entropy production across each simple shock is nonpositive. On the other hand, by Theorem 8.5.2, the entropy production across any contact discontinuity will be zero. Therefore, combining the partial entropy productions we conclude that the total entropy production (8.5.1) is nonpositive. This completes the proof.

The converse of Theorem 8.6.2 is generally false. Consider for example the system (7.1.8) of isentropic elasticity, with corresponding dissipative system (8.6.3) and entropy-entropy flux pair (7.4.10), which satisfy the compatibility condition (8.6.20). As shown in Section 8.5, the entropy shock admissibility criterion is tested through the inequality (8.5.5), which follows from, but does not generally imply, the Wendroff $E$-condition (8.4.4).

One may plausibly argue that mere existence of a viscous shock profile should not constitute grounds for admissibility of the shock unless the profile itself is stable under perturbations of the states $U_\pm$ and perhaps even under perturbations of the flux function $F$. For simplicity, let us focus attention to the case $B = I$ and let us consider weak shocks, of speed $s$, joining $U_-$, on the left, to $U_+$, on the right, with shock profile consisting of a single connecting orbit of (8.6.8). Clearly, the shock profile must lie on the intersection of the unstable manifold $\mathcal{U}$ of (8.6.8) at $U_-$ and the stable manifold $\mathcal{S}$ of (8.6.8) at $U_+$. The Jacobian of the right-hand side of (8.6.8) is the matrix $DF(V) - sI$, with eigenvalues $\lambda_1(V) - s, \cdots, \lambda_n(V) - s$, and corresponding eigenvectors $R_1(V), \cdots, R_n(V)$. Therefore, $\mathcal{U}$ is equidimensional, and tangential at $U_-$, to the subspace spanned by $R_j(U_-)$ for all $j = 1, \cdots, n$ with $\lambda_j(U_-) > s$; and $\mathcal{S}$ is equidimensional, and tangential at $U_+$, to the subspace spanned by $R_k(U_+)$ for all $k = 1, \cdots, n$ with $\lambda_k(U_+) < s$. In a strictly hyperbolic system with weak shocks, we have $\lambda_1(U_\pm) < \lambda_2(U_\pm) < \cdots < \lambda_n(U_\pm)$ and so the Lax $E$-condition

$$\lambda_n(U_-) > \cdots > \lambda_i(U_-) > s > \lambda_i(U_+) > \cdots > \lambda_1(U_+) \tag{8.6.22}$$

implies $\dim \mathcal{U} + \dim \mathcal{S} = n + 1$, in which case $\mathcal{U}$ and $\mathcal{S}$ intersect transversely to produce a unique shock profile joining $U_-$ with $U_+$, which is stable under perturbations of $U_\pm$ and $F$. By contrast, when the shock is overcompressive or undercompressive,

$\dim \mathcal{U} + \dim \mathcal{S}$ is larger or smaller than $n+1$, and thus the existence, uniqueness and stability of viscous shock profiles is no longer guaranteed. In particular, for systems of two conservation laws, a compressive shock profile connects a node with a saddle, an overcompressive shock profile connects two nodes, and an undercompressive shock profile connects two saddles. Later in this section, it will be shown by means of an example that admissibility of undercompressive shocks depends sensitively on the particular selection of the viscosity matrix $B$.

One may argue, further, that viscous shock profiles employed to test the admissibility of shocks must derive from traveling wave solutions of the system (8.6.1) that are asymptotically stable. This issue has been investigated thoroughly in recent years and a complete theory has emerged, warranting the writing of a monograph on the subject. A detailed presentation would lie beyond the scope of the present book, so only the highlights shall be reported here. For details and proofs the reader may consult the references cited in Section 8.8.

For simplicity, we limit our discussion to viscosity matrix $B = I$ and normalize (8.6.1) by setting $\mu = 1$. We consider a weak $i$-shock, joining the states $U_-$, on the left, and $U_+$, on the right, which admits a viscous shock profile $V$. A change of variable $x \mapsto x + st$ renders the shock stationary. The viscous shock profile $V$ is called *asymptotically stable* if the solution $U(x,t)$ of (8.6.1) with initial values $U(x,0) = V(x) + U_0(x)$, where $U_0$ is a “small” perturbation decaying at $\pm\infty$, satisfies

$$U(x,t) \to V(x+h), \quad \text{as } t \to \infty, \tag{8.6.23}$$

for some appropriate phase shift $h \in \mathbb{R}$.

Motivated by the observation that the total mass of solutions of (8.6.1) is conserved, it seems natural to require that the convergence in (8.6.23) be in $L^1(-\infty,\infty)$. In particular, this would imply that $V(x+h)$ carries the excess mass introduced by the perturbation:

$$\int_{-\infty}^{\infty} U_0(x)dx = \int_{-\infty}^{\infty} [V(x+h) - V(x)]dx = h[U_+ - U_-]. \tag{8.6.24}$$

In the scalar case, $n = 1$, any viscous shock profile is asymptotically stable in $L^1(-\infty,\infty)$, under arbitrary perturbations $U_0 \in L^1(-\infty,\infty)$, with $h$ determined through (8.6.24).

For systems, $n \geq 2$, the single scalar parameter $h$ is generally inadequate to balance the vectorial equation (8.6.24), in which case (8.6.23) cannot hold in $L^1(-\infty,\infty)$, as no $h$-translate of $V$ alone may carry the excess mass. Insightful analysis of the asymptotics of (8.6.1) suggests that, for large $t$, the solution $U$ should develop a viscous shock profile accompanied by a family of so-called diffusion waves, which share the burden of carrying the mass:

(8.6.25)

$$U(x,t) \sim V(x+h) + W(x,t) + \sum_{j<i} \theta_j(x,t) R_j(U_-) + \sum_{j>i} \theta_j(x,t) R_j(U_+) \,.$$

The $j$-term in the summation on the right-hand side of (8.6.25) represents a *decoupled $j$-diffusion wave*. The scalar function $\theta_j$ is a self-similar solution,

$$\theta_j(x,t) = \frac{1}{\sqrt{t}}\phi_j\left(\frac{x-\lambda_j t}{\sqrt{t}}\right), \tag{8.6.26}$$

of the nonlinear diffusion equation

$$\partial_t\theta_j + \partial_x\left[\lambda_j\theta_j + \tfrac{1}{2}(\mathrm{D}\lambda_j R_j)\theta_j^2\right] = \partial_x^2\theta_j\,. \tag{8.6.27}$$

In (8.6.26) and (8.6.27), $\lambda_j$, $\mathrm{D}\lambda_j$ and $R_j$ are evaluated at $U_-$, for $j = 1, \cdots, i-1$, or at $U_+$, for $j = i+1, \cdots, n$. Thus the $j$-diffusion wave has a bell-shaped profile which propagates at characteristic speed $\lambda_j$; its peak decays like $O(t^{-\frac{1}{2}})$, while its mass stays constant, say $m_j R_j$. The remaining term $W$ on the right-hand side of (8.6.25) represents the *coupled diffusion wave*, which satisfies a complicated linear diffusion equation, decays at the same rate as the uncoupled diffusion waves, but carries no mass. Therefore, mass conservation as $t \to \infty$ yields, in lieu of (8.6.24), the equation

$$\int_{-\infty}^{\infty} U_0(x)dx = \sum_{j<i} m_j R_j(U_-) + h[U_+ - U_-] + \sum_{j>i} m_j R_j(U_+), \tag{8.6.28}$$

which dictates how the excess mass is distributed among the viscous shock profile and the decoupled diffusion waves. Since $U_+ - U_-$ and $R_i(U_\pm)$ are nearly collinear, (8.6.28) determines explicitly and uniquely the phase shift $h$ of the viscous shock profile as well as the masses $m_j$ of the $j$-diffusion waves.

It has been established that the viscous shock profile $V$ is asymptotically stable (8.6.23) in $L^\infty(-\infty,\infty)$, for the $h$ determined through (8.6.28), under any perturbation $U_0 \in H^1(-\infty,\infty)$ of $V$ with

$$\int_{-\infty}^{\infty} |U_0(x)|dx + \int_{-\infty}^{\infty} (1+x^2)|U_0(x)|^2 dx << 1, \tag{8.6.29}$$

provided only that the eigenvalue $\lambda_i$ is simple and the shock satisfies the strict form of the Lax $E$-condition. It should be noted that this assertion holds even when the $i$-characteristic family fails to be genuinely nonlinear.

The orderly structure depicted above disintegrates when dealing with overcompressive or undercompressive shocks, and occasionally even with strong compressive shocks. In order to catch a glimpse of the geometric complexity that may arise in such cases, let us discuss the construction of viscous shock profiles for 2-shocks of the simple system (7.2.11), with dissipative form (8.6.4). The properties of shocks were already discussed in Section 8.3. Taking advantage of symmetry under rotations and scaling properties of the system, we may fix, without loss of generality, the left state $(u_-, v_-)$ at the point $(1,0)$. The right state $(u_+, v_+)$ will be located at a point $(a, 0)$, with $a \in (-\frac{1}{2}, 0)$. In that case, as shown in Section 8.3, the shock speed is $s = a^2 + a + 1$ and the shock is overcompressive (8.3.7). Notice that the state

$(b, 0)$, where $b = -1 - a$, is also joined to $(1, 0)$ by a 2-shock of the same speed $s$, which satisfies the Lax $E$-condition, is not overcompressive, but does not satisfy the Liu $E$-condition.

The system (8.6.8) associated with (8.6.4) reads:

$$\begin{cases} \dot{u} = -s(u-1) + u(u^2+v^2) - 1 \\ \dot{v} = -sv + v(u^2+v^2); \end{cases} \tag{8.6.30}$$

or, equivalently, in polar coordinates $(\rho, \theta)$, $u = \rho\cos\theta$, $v = \rho\sin\theta$:

$$\begin{cases} \dot{\rho} = \rho(\rho^2 - s) + (s-1)\cos\theta \\ \rho\dot{\theta} = -(s-1)\sin\theta. \end{cases} \tag{8.6.31}$$

Notice that (8.6.30) possesses three equilibrium points: (a) $(1, 0)$ which is an unstable node; (b) $(a, 0)$ which is a stable node; and (c) $(b, 0)$ which is a saddle. The phase portrait, which may be easily determined through elementary analysis of (8.6.30) and (8.6.31), is depicted in Fig. 8.6.1.

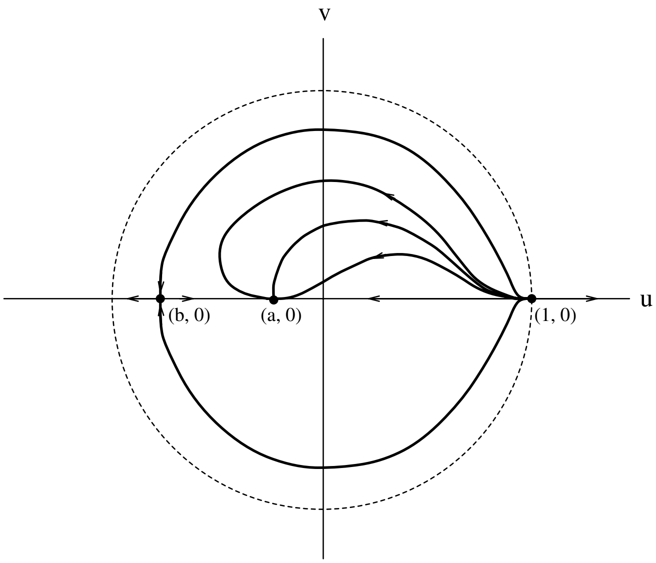


**Fig. 8.6.1**

Even though the shock joining $(1, 0)$ to $(b, 0)$ violates the Liu $E$-condition, these states are connected by two viscous shock profiles, symmetric with respect to the $u$-axis. By contrast, the states $(1, 0)$ and $(a, 0)$ are connected by infinitely many viscous shock profiles. To test the asymptotic stability of any one of these viscous shock profiles, say $(\bar{u}(\tau), \bar{v}(\tau))$, in the light of our discussion above, we introduce a small perturbation $(u_0(x), v_0(x))$ and inquire whether the solution $(u,v)(x,t)$ of (8.6.4) with initial values

(8.6.32)
$$(u,v)(x,0) = \left(\bar{u}(\frac{x}{\mu}) + u_0(x),\ \bar{v}(\frac{x}{\mu}) + v_0(x)\right)$$

satisfies

(8.6.33)
$$(u,v)(x,t) \to \left(\hat{u}\left(\frac{x-st}{\mu}\right),\ \hat{v}\left(\frac{x-st}{\mu}\right)\right), \qquad \text{as } t \to \infty,$$

where $(\hat{u}(\tau), \hat{v}(\tau))$ is a (generally different) viscous shock profile. Because no diffusion waves are possible here, the convergence in (8.6.33) must be in $L^1(-\infty,\infty)$. In particular, the $v$-component of the excess mass conservation yields

(8.6.34)
$$\int_{-\infty}^{\infty} v_0(x)dx = \int_{-\infty}^{\infty} \left[\hat{v}\left(\frac{x-st}{\mu}\right) - \bar{v}\left(\frac{x-st}{\mu}\right)\right]dx = \mu \int_{-\infty}^{\infty} \left[\hat{v}(\tau) - \bar{v}(\tau)\right]d\tau.$$

It may be shown that the integral on the right-hand side of (8.6.34) is uniformly bounded, independently of the choice of $\bar{v}$ and $\hat{v}$. Consequently, when $v_0$ is fixed so that $\int v_0 dx \neq 0$, (8.6.34) cannot hold when $\mu$ is sufficiently small. Thus, insofar as shock admissibility hinges on stability of the connecting shock profiles, the overcompressive shocks of the system (7.2.11) should be termed inadmissible.

The following simple example demonstrates how sensitively the admissibility of shocks may depend on the particular choice of viscosity matrix when strict hyperbolicity fails. Consider the simple system

(8.6.35)
$$\partial_t(u,v,w)^\top + \partial_x(u^2,v^2,w^2)^\top = 0,$$

which consists of three uncoupled copies of the Burgers equation. The undercompressive shock joining the state $(-3,7,-1)^\top$, on the left, with the state $(5,-5,3)^\top$, on the right, and propagating with speed $s = 2$, violates the Lax $E$-condition and also the viscous shock admissibility criterion when the viscosity matrix is $B = I$. However, this shock does satisfy the viscous shock admissibility condition for the symmetric and positive definite viscosity matrix

(8.6.36)
$$B = \begin{pmatrix} 9 & 8 & 2 \\ 8 & 9 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

Indeed, the corresponding viscous shock profile is given by

(8.6.37)
$$V(\tau) = (1,1,1)^\top + \tanh(2\tau)(4,-6,2)^\top, \qquad -\infty < \tau < \infty.$$

A very technical theory of linear stability for multispace-dimensional viscous shock profiles has emerged in recent years, paralleling the corresponding theory for multispace-dimensional shocks, briefly outlined at the end of Section 8.3. Expositions are found in the references cited in Section 8.8.

As pointed out in Section 4.6, in certain cases the physically relevant admissibility condition is provided not by the viscosity criterion but by the viscosity-capillarity criterion, in which (8.6.1) is replaced by

(8.6.38)

$$\partial_t U(x,t)+\partial_x F(U(x,t)) = \mu\partial_x[B(U(x,t))\partial_x U(x,t)]+\nu\partial_x[H(U(x,t))\partial_x^2 U(x,t)].$$

In general, diffusion is dominant when $\nu = o(\mu^2)$, while dispersion prevails if $\mu = o(\sqrt{\nu})$. The two effects are balanced when $\nu = \mu^2$. In that case, shock profiles are governed by the ordinary differential equation

(8.6.39) $$H(V)\ddot{V} + B(V)\dot{V} = F(V) - F(U_-) - s[V - U_-],$$

replacing (8.6.8). A theory of these profiles is gradually emerging in the literature.

## 8.7 Nonconservative Shocks

In continuum physics one occasionally encounters quasilinear hyperbolic systems

(8.7.1) $$\partial_t U(x,t) + A(U(x,t))\partial_x U(x,t) = 0$$

that are not in conservative form. In that case, it is not possible to characterize weak solutions within the setting of the theory of distributions. It is still possible, however, to introduce a notion of weak solution within the class $BV$ of functions of bounded variation by postulating jump conditions that play the role of the Rankine-Hugoniot jump condition (8.1.2) at the points of approximate jump discontinuity.

Appropriate jump conditions can be motivated by prior information on shock profiles, deriving from the vanishing viscosity approach, the vanishing viscosity-capillarity argument, or from (so-called) *kinetic relations* in the theory of phase transitions. The formulation of this theory proceeds along the following lines.

To (8.7.1) one links a function $V(\tau; U_-, U_+)$, defined on $(-\infty,\infty) \times I\!R^n \times I\!R^n$ and taking values in $I\!R^n$, which has the following properties:

(8.7.2) $$V(-\infty; U_-, U_+) = U_-\,, \qquad V(\infty; U_-, U_+) = U_+\,,$$

(8.7.3) $$V(\tau; U, U) = U,$$

(8.7.4) $$|\partial_\tau V(\tau; U_-, U_+) - \partial_\tau V(\tau; \bar{U}_-, \bar{U}_+)| \le a|(U_+ - \bar{U}_+) - (U_- - \bar{U}_-)|,$$

for all $U, U_-, U_+, \bar{U}_-, \bar{U}_+$ in $\mathbb{R}^n$, any $\tau \in (-\infty, \infty)$, and some $a > 0$. One then requires that a shock joining the state $U_-$, on the left, with the state $U_+$, on the right, and propagating with speed $s$ must satisfy the jump condition

$$(8.7.5) \qquad \int_{-\infty}^{\infty} A(V(\tau; U_-, U_+))\partial_\tau V(\tau; U_-, U_+)d\tau = s[U_+ - U_-].$$

In the above setting, $V(\cdot\,; U_-, U_+)$ represents the shock profile. Notice that in the conservative case, $A(U) = \mathrm{D}F(U)$, (8.7.5) reduces to the Rankine-Hugoniot jump condition (8.1.2), regardless of the particular choice of $V$.

The literature cited in Section 8.8 explains how the above device naturally leads to a notion of weak solution to (8.7.1), within the framework of $BV$ functions.

## 8.8 Notes

The study of shock waves originated in the context of gas dynamics. The book by Courant and Friedrichs [1], already cited in Chapter III, presented a coherent, mathematical exposition of material from the physical and engineering literature, accumulated over the past 150 years, paving the way for the development of a general theory by Lax [2].

For as long as gas dynamics remained the prototypical example, the focus of the research effort was set on strictly hyperbolic, genuinely nonlinear systems. The intricacy of shock patterns in nonstrictly hyperbolic systems was not recognized until recently, and this subject is currently undergoing active development.

Expositions of many of the topics covered in this chapter are also contained in the books of Smoller [3] and Serre [11].

The notion of Hugoniot locus, in gas dynamics, may be traced back to the work of Riemann [1] and Hugoniot [2]; but the definition of shock curves in the general setting is due to Lax [2], who first established the properties stated in Theorems 8.2.1, 8.2.2 and 8.2.3. The elegant proof of Theorem 8.2.1 is here taken from Serre [11]. The significance of systems with coinciding shock and rarefaction wave curves was first recognized by Temple [3], who conducted a thorough study of their noteworthy properties. A detailed discussion is also contained in Serre [11].

For gas dynamics, the statement that admissible shocks should be subsonic relative to their left state and supersonic relative to their right state is found in the pioneering paper of Riemann [1]. This principle was postulated as a general shock admissibility criterion, namely the Lax $E$-condition, by Lax [2], who also proved Theorem 8.3.1. A proof of Theorem 8.3.2 is given in Li and Yu [1]. See also Hsiao and Chang [1]. A different connection between the Lax $E$-condition and stability is established in Smoller, Temple and Xin [1].

The construction of multispace-dimensional shocks and their stability theory was pioneered by Majda [2,3,4]. This seminal work has been further developed and extended in various directions and now encompasses compressive, undercompressive and overcompressive shocks for hyperbolic systems, as well as phase boundaries for systems that change type. By a suitable change of variables, the construction of the shock is reduced to solving an initial-boundary-value problem. Accordingly, Lopatinski type conditions, touched upon in Section 5.6, play a prominent role in the theory. A small sample of relevant references out of a voluminous literature includes Benzoni-Gavage [2,3], Corli and Sablé-Tougeron [1], Franchéteau and Métivier [1], Freistühler [8], Godin [1], Métivier [2], Serre [18], and Coulombel [1]. Detailed expositions are found in the English edition of the book by Serre [11] and in the survey article by Métivier [3], while Benzoni-Gavage, Rousset, Serre and Zumbrun [1] provides informative historical perspective.

Shock admissibility in the absense of genuine nonlinearity was first discussed by Bethe [1] and Weyl [1], for the system of gas dynamics. The Liu $E$-condition and related Theorems 8.4.1, 8.4.2 and 8.5.4 are due to Liu [2]. The motivation was provided by the Oleinik $E$-condition, derived in Oleinik [4], and the Wendroff $E$-condition, established in Wendroff [1]. This admissibility criterion seems to have been anticipated in the 1960's by Chang and Hsiao [1,2] (see also Hsiao and Zhang [1]) but their work was not published until much later.

The entropy shock admissibility condition has been part of the basic theory of continuum thermomechanics for over a century. The form (8.5.1), for general systems (8.1.1), was postulated by Lax [4], who established Theorems 8.5.1, 8.5.3, and 8.6.2. The proofs of Theorems 8.5.1, 8.5.2, 8.5.3 and 8.5.4 here, based on Equation (8.5.8), are taken from Dafermos [10]. Stricter versions of the entropy admissibility criterion, that are equivalent to the Liu $E$-condition, have been proposed by Dafermos (see Section 9.7) and by Liu and Ruggeri [1].

The notion of viscous shock profile was introduced to gas dynamics by Rankine [1], Rayleigh [3] and G.I. Taylor [1]. For the physical background, see e.g. Zeldovich and Raizer [1]. A seminal reference is Gilbarg [1]. The general form (8.6.8), for systems (8.1.1), was first written down by Gelfand [1]. Theorem 8.6.1 is due to Majda and Pego [1]. See also Conlon [2]. An earlier paper by Foy [1] had established the result in the special case where the system is genuinely nonlinear and the shocks are weak. Also Mock [1] has proved a similar result under the assumption that the system is genuinely nonlinear and it is endowed with a uniformly convex entropy. The issue of characterizing appropriate viscosity matrices $B$ has been discussed by several authors, including Conley and Smoller [1,3], Majda and Pego [1], Pego [1], and Serre [11]. See also Conley and Smoller [2] for an early study of profiles induced by viscosity combined with capillarity. For a detailed study of viscous shock profiles in isentropic (or isothermal) elastodynamics, under physically appropriate assumptions, see Antman and Malek-Madani [1]. The case of general, nonisentropic gas dynamics, with nonconvex equation of state, was investigated by Pego [2], who established that strong shocks satisfying the Liu E-condition do not necessarily admit viscous shock profiles when heat conductivity dominates viscosity.

The literature on asymptotic stability of viscous shock profiles is so vast that it would be impossible to provide here a comprehensive list of references. An informative presentation is contained in the book by Serre [11]. Of the seminal papers in that area, it will suffice to cite Ilin and Oleinik [1], on the scalar case; Goodman

[1], on systems for perturbations with zero excess mass; Liu [19], which introduces the decoupled diffusion waves; and Szepessy and Xin [1], which adds the coupled diffusion waves. For further developments, dealing with the case of contact discontinuities, rarefaction waves, undercompressive shocks, boundary effects on stationary shocks, nonstrictly hyperbolic systems, and various types of physical viscosity and relaxation, see Liu [20,26], Liu and Nishihara [1], Liu and Xin [2,3], Liu and Yu [1], Liu and Zeng [1,2], Liu and Zumbrun [1], Chern and Liu [1], Goodman, Szepessy and Zumbrun [1], Kawashima and Matsumura [1], Xin [1,3], Zeng [1,2,3], and Luo and Serre [1]. A definitive treatment of the scalar case is found in the survey paper by Serre [11], which presents, among other topics, the original contributions of Freistühler and Serre [1,2] on the subject. The most general result on the stability of weak shocks that merely satisfy the Lax $E$-condition, without any assumption of genuine nonlinearity of the system, is due to Fries [1,2].

A parallel theory is currently emerging on the stability of shock profiles in the context of solutions to the Boltzmann equation. See Caflisch and Nicolaenko [1] and Liu and Yu [4].

The stability theory of multispace-dimensional viscous shock profiles is currently undergoing rapid development and extensive bibliography is already available. A detailed exposition of the stability of planar viscous shocks, together with an exhaustive list of references, is found in the survey article by Zumbrun [3]. For an investigation of the stability of curved viscous shocks, which is very technical, see Gues and Williams [1] and Gues, Métivier, Williams and Zumbrun [1,2]. Other interesting contributions include Gardner and Zumbrun [1], Zumbrun and Howard [1], Freistühler and Szmolyan [2], and Zumbrun and Serre [1]. A parallel stability theory of capillary or viscocapillary shock profiles is also under development; see Benzoni-Gavage [4,5]. Closely related to the above is the stability theory of shock profiles associated with finite difference systems resulting from discretizing hyperbolic systems of conservation laws; see Liu and Yu [2] and Benzoni-Gavage [6].

The class of hyperbolic systems of conservation laws with rotational invariance has interesting mathematical structure as well as applications to elasticity and magnetohydrodynamics. Various aspects of the existence and stability of shock waves in that class are discussed in Brio and Hunter [1], Freistühler [1,2,3,4,5,6], Freistühler and Liu [1], and Freistühler and Szmolyan [1].

The admissibility of overcompressive and undercompressive shocks has been investigated in great detail, especially in the context of systems of two conservation laws, with quadratic or cubic flux functions, which are not strictly hyperbolic. Viscosity or viscosity-capillarity conditions, as well as kinetic relations have been employed as admissibility criteria. The standard test for success is whether the selection renders the Riemann problem well-posed. This will be discussed in Chapter IX. Out of an extensive body of literature, a sample is Shearer [2], Schaeffer and Shearer [1], Schecter and Shearer [1], Jacobs, MacKinney and Shearer [1], Schulze and Shearer [1], and Čanić and Plohr [1]. For additional, relevant references, see Section 9.12. In particular, the discussion here of the stability of overcompressive shocks for the system (7.2.11) was borrowed from Liu [24]; (see also Liu [27]). Furthermore, the system (8.6.35) with viscosity matrix (8.6.36) is treated in detail by Mailybaev and Marchesin [1].

The question of admissibility of jump discontinuities that represent phase boundaries or transonic shocks arises in systems of conservation laws of mixed, elliptic-hyperbolic type which govern phase transitions or transonic gas flow. A prototypical example is the system (7.1.8) with nonmonotone $\sigma(u)$; in particular the classical van der Waals fluid. Entropy, viscosity and viscosity-capillarity admissibility criteria have been tried in that context, in combination with a new criterion based on "kinetic relations" motivated by considerations at the microscale. Shocks induced by such kinetic relations are typically undercompressive. See Abeyaratne and Knowles [1,2], Asakura [2], Bedjaoui and LeFloch [1], Benzoni-Gavage [2,3,4,5], Benzoni-Gavage and Freistühler [1], Fan [3,4], Freistühler [8], Hagan and Slemrod [1], Hatton and Mischaikow [1], Hayes and LeFloch [2,3], Hayes and Shearer [3], Hoff and Khodja [1], R.D. James [1], Keyfitz [2], Keyfitz and Warnecke [1], Pego [3], Pence [1], Rosakis [1], Shearer [2], Slemrod [1,2,3], and Truskinovsky [1,2]. An informative discussion and a comprehensive list of references are found in the survey article and monograph by LeFloch [4,5].

The notion of weak solution for quasilinear hyperbolic systems that are not in conservative form, outlined in Section 8.7, was introduced by LeFloch [2] and Dal Maso, LeFloch and Murat [1]. For developments and applications of these ideas, see Amadori, Baiti, LeFloch and Piccoli [1], Hayes and LeFloch [2,3], and LeFloch and Tzavaras [1,2]. For a survey, see LeFloch [4]. See also Xiao-Biao Lin [1].

# IX

# Admissible Wave Fans and the Riemann Problem

The property of systems of conservation laws to be invariant under uniform stretching of the space-time coordinates induces the existence of self-similar solutions, which stay constant along straight-line rays emanating from some focal point in space-time. Such solutions depict a collection of waves converging to the focal point and interacting there to produce a jump discontinuity which is in turn resolved into an outgoing wave fan.

This chapter investigates the celebrated Riemann problem, whose object is the resolution of jump discontinuities into wave fans. A solution will be constructed in three different ways, namely: (a) by the classical method of piecing together elementary centered solutions encountered in earlier chapters, i.e., constant states, shocks joining constant states, and centered rarefaction waves bordered by constant states or contact discontinuities; (b) by minimizing the total entropy production of the outgoing wave fan; and (c) by a vanishing viscosity approach which employs time-dependent viscosity so that the resulting dissipative system is invariant under stretching of coordinates, just like the original hyperbolic system. A new type of discontinuity, called a delta shock, will emerge in the process.

The issue of admissibility of wave fans will be raised. In particular, it will be examined whether shocks contained in solutions constructed by any one of the above methods are necessarily admissible.

Next, the wave fan that best approximates the complex wave pattern generated by the interaction of two wave fans will be determined.

A system will be exhibited in which bounded initial data generate a resonating wave pattern that drives the solution amplitude to infinity, in finite time.

The chapter will close with a brief introduction to the theory of self-similar solutions to hyperbolic systems of conservation laws in two space dimensions.

## 9.1 Self-similar Solutions and the Riemann Problem

The hyperbolic system of conservation laws

$$\partial_t U + \partial_x F(U) = 0 \tag{9.1.1}$$

is invariant under uniform stretching of coordinates: $(x, t) \mapsto (\alpha x, \alpha t)$; hence it admits *self-similar solutions*, defined on the space-time plane and constant along straight-line rays emanating from the origin. Since (9.1.1) is also invariant under translations of coordinates: $(x, t) \mapsto (x + \bar{x}, t + \bar{t})$, the focal point of self-similar solutions may be translated from the origin to any fixed point $(\bar{x}, \bar{t})$ in space-time.

If $U$ is a (generally weak) self-similar solution of (9.1.1), focused at the origin, its restriction to $t > 0$ admits the representation

$$U(x,t) = V\left(\frac{x}{t}\right), \quad -\infty < x < \infty, \quad 0 < t < \infty, \tag{9.1.2}$$

where $V$ is a bounded measurable function on $(-\infty, \infty)$, which satisfies the ordinary differential equation

$$[F(V(\xi)) - \xi V(\xi)]^{\cdot} + V(\xi) = 0, \tag{9.1.3}$$

in the sense of distributions. Indeed, if $U$ is given by (9.1.2) and $\phi$ is any $C^\infty$ test function with compact support on $(-\infty, \infty) \times (0, \infty)$, then, after a short calculation,

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty [\partial_t \phi(x,t) U(x,t) + \partial_x \phi(x,t) F(U(x,t))] dx dt \\ &= \int_{-\infty}^\infty \{\dot{\psi}(\xi)[F(V(\xi)) - \xi V(\xi)] - \psi(\xi) V(\xi)\} d\xi, \end{aligned} \tag{9.1.4}$$

where

$$\psi(\xi) = \int_0^\infty \phi(\xi t, t) dt, \qquad -\infty < \xi < \infty. \tag{9.1.5}$$

The restriction of $U$ to $t < 0$ similarly admits a representation like (9.1.2), for a (generally different) function $V$, which also satisfies (9.1.3).

From (9.1.3) we infer that $F(V) - \xi V$ is Lipschitz continuous on $(-\infty, \infty)$ and (9.1.3) holds, in the classical sense, at any Lebesgue point $\xi$ of $V$.

Henceforth, we shall consider self-similar solutions $U$ of class $BV_{\text{loc}}$. In that case, the function $V$, above, has bounded variation on $(-\infty, \infty)$. We assume $V$ is normalized, as explained in Section 1.7, so that one-sided limits $V(\xi\pm)$ exist for every $\xi \in (-\infty, \infty)$ and $V(\xi) = V(\xi-) = V(\xi+)$ except possibly on a countable set of $\xi$.

By account of Theorem 1.7.5, (9.1.3) may be written as

$$[\widetilde{\mathrm{D}F}(V) - \xi I]\dot{V} = 0, \tag{9.1.6}$$

in the sense of measures, with

$$\widetilde{\mathrm{D}F}(V)(\xi) = \int_0^1 \mathrm{D}F(\tau V(\xi-) + (1-\tau)V(\xi+)) d\tau. \tag{9.1.7}$$

Furthermore, as a function of bounded variation $V$ is differentiable almost everywhere on $(-\infty, \infty)$ and (9.1.6) will be satisfied at any point $\xi$ of continuity of $V$ where $\dot{V}(\xi)$ exists.

In view of the above, $(-\infty, \infty)$ is decomposed into the union of three pairwise disjoint sets $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{W}$ as follows:

$\mathcal{C}$ is the maximal open subset of $(-\infty, \infty)$ on which the measure $\dot{V}$ vanishes, i.e., it is the complement of the support of $\dot{V}$. Thus $\mathcal{C}$ is the (at most) countable union of disjoint open intervals, on each of which $V$ is constant.

$\mathcal{S}$ is the (at most) countable set of points of jump discontinuity of $V$. The Rankine-Hugoniot jump condition

$$F(V(\xi+)) - F(V(\xi-)) = \xi[V(\xi+) - V(\xi-)] \tag{9.1.8}$$

holds at any $\xi \in \mathcal{S}$. This may be inferred from the continuity of $F(V) - \xi V$, noted above, or it may be deduced by comparing (9.1.6), (9.1.7) with (8.1.3), (8.1.4).

$\mathcal{W}$ is the (possibly empty) set of points of continuity of $V$ that lie in the support of the measure $\dot{V}$. When $\xi \in \mathcal{W}$, then

$$\lambda_i(V(\xi)) = \xi, \tag{9.1.9}$$

for some $i \in \{1, \cdots, n\}$. Indeed, if $\xi$ is the limit of a sequence $\{\xi_m\}$ in $\mathcal{S}$, then $V(\xi_m+) - V(\xi_m-) \to 0$, as $m \to \infty$, and (9.1.9) follows from the Rankine-Hugoniot condition (9.1.8). On the other hand, if $\xi$ is in the interior of the set of points of continuity of $V$, and (9.1.9) fails for $i = 1, \cdots, n$, then $\lambda_i(V(\zeta)) \neq \zeta$ for $\zeta \in (\xi - \varepsilon, \xi + \varepsilon)$ and $i = 1, \cdots, n$, in which case, by virtue of (9.1.6), the measure $\dot{V}$ would vanish on $(\xi - \varepsilon, \xi + \varepsilon)$, contrary to our hypothesis that $\xi \in \operatorname{spt} \dot{V}$. If $\xi$ is a point of differentiability of $V$, (9.1.6) implies

$$\dot{V}(\xi) = b(\xi) R_i(V(\xi)), \tag{9.1.10}$$

where the scalar $b(\xi)$ is determined by combining (9.1.9) with (9.1.10):

$$[\mathrm{D}\lambda_i(V(\xi)) R_i(V(\xi))] b(\xi) = 1. \tag{9.1.11}$$

In particular, $V(\xi)$ is a point of genuine nonlinearity of the $i$-characteristic family.

We have thus shown that self-similar solutions are composites of constant states, shocks, and centered simple waves. The simple waves will be centered rarefaction waves, when $V$ is an outgoing wave fan, or centered compression waves, when $V$ depicts a focusing collection of waves. The two configurations are differentiated by time irreversibility, induced by admissibility conditions on weak solutions. More stringent conditions are imposed on outgoing wave fans, so these are generally simpler.

Of central importance will be to understand how a jump discontinuity at the origin, introduced by the initial data, is resolved into an outgoing wave fan. This is the object of the

**9.1.1 Riemann Problem:** Determine a self-similar (generally weak) solution $U$ of (9.1.1) on $(-\infty, \infty) \times (0, \infty)$, with initial condition

$$U(x,0) = \begin{cases} U_L\,, & \text{for } x < 0 \\ U_R\,, & \text{for } x > 0, \end{cases} \tag{9.1.12}$$

where $U_L$ and $U_R$ are given states in $\mathscr{O}$.

Following our discussion, above, we shall seek a solution of the Riemann problem in the form (9.1.2), where $V$ satisfies the ordinary differential equation (9.1.3), on $(-\infty, \infty)$, together with boundary conditions

$$V(-\infty) = U_L\,, \quad V(\infty) = U_R\,. \tag{9.1.13}$$

The specter of nonuniqueness raises again the issue of admissibility, which will be the subject of discussion in the following sections.

## 9.2 Wave Fan Admissibility Criteria

Various aspects of admissibility have already been discussed, for general weak solutions, in Chapter IV, and for single shocks, in Chapter VIII. We have thus encountered a number of admissibility criteria and we have seen that they are strongly interrelated but not quite equivalent. As we shall see later, the most discriminating among these criteria, namely viscous shock profiles and the Liu $E$-condition, are sufficiently powerful to weed out all spurious solutions, so long as we are confined to strictly hyperbolic systems and shocks of moderate strength. However, once one moves to systems that are not strictly hyperbolic and/or to solutions with strong shocks, the situation becomes murky. The question of admissibility is still open.

Any rational new admissibility criterion should adhere to certain basic principles, the fruits of the long experience with the subject. They include:

**9.2.1 Localization:** The test of admissibility of a solution should apply individually to each point $(\bar{x}, \bar{t})$ in the domain and should thus involve only the restriction of the solution to an arbitrarily small neighborhood of $(\bar{x}, \bar{t})$, say the circle $\{(x,t) : |x-\bar{x}|^2 + |t-\bar{t}|^2 < r^2\}$ where $r$ is fixed but arbitrarily small. This is compatible with the general principle that solutions of hyperbolic systems should have the local dependence property.

**9.2.2 Evolutionarity:** The test of admissibility should be forward-looking, without regard for the past. Thus, admissibility of a solution at the point $(\bar{x}, \bar{t})$ should depend solely on its restriction to the semicircle $\{(x,t) : |x-\bar{x}|^2+|t-\bar{t}|^2 < r^2\,,\ t \geq \bar{t}\}$. This is in line with the principle of time irreversibility, which pervades the admissibility criteria we have encountered thus far, such as entropy, viscosity, etc.

**9.2.3 Invariance under translations:** A solution $U$ will be admissible at $(\bar{x}, \bar{t})$ if and only if the translated solution $\bar{U}$, $\bar{U}(x,t) = U(x+\bar{x}, t+\bar{t})$, is admissible at the origin $(0, 0)$.

**9.2.4 Invariance under dilations:** A solution $U$ will be admissible at $(0,0)$ if and only if, for each $\alpha > 0$, the dilated solution $\bar{U}_\alpha$, $\bar{U}_\alpha(x,t) = U(\alpha x, \alpha t)$, is admissible at $(0,0)$.

Let us focus attention to weak solutions $U$ with the property that, for each fixed point $(\bar{x}, \bar{t})$ in the domain, the limit

$$\bar{U}(x,t) = \lim_{\alpha \downarrow 0} U(\bar{x} + \alpha x, \bar{t} + \alpha t) \tag{9.2.1}$$

exists in $L^1_{loc}((-\infty,\infty) \times (0,\infty))$. Notice that in that case $\bar{U}$ is necessarily a self-similar solution of (9.1.1). In the spirit of the principles listed above, one may use the admissibility of $\bar{U}$ at the origin as a test for the admissibility of $U$ at the point $(\bar{x}, \bar{t})$. Since $\bar{U}$ depicts a fan of waves radiating from the origin, such tests constitute *wave fan admissibility criteria*.

Passing to the limit in (9.2.1) amounts to observing, so to say, the solution $U$ under a microscope focused at the point $(\bar{x}, \bar{t})$. When $U \in BV_{loc}$, the limit exists, by virtue of Theorem 1.7.4, except possibly on the set of irregular points $(\bar{x}, \bar{t})$, which has one-dimensional Hausdorff measure zero. In particular, if $(\bar{x}, \bar{t})$ is a point of approximate continuity of $U$, then $\bar{U}(x,t)$ will be a constant state $U_0$, while if $(\bar{x}, \bar{t})$ is a point of approximate jump discontinuity, then $\bar{U}(x,t) = U_-$, for $x < st$, and $\bar{U}(x,t) = U_+$, for $x > st$, where $U_\pm$ are the approximate one-sided limits of $U$, and $s$ is the slope of the jump discontinuity at $(\bar{x}, \bar{t})$. Whether the limit will also exist at the irregular points $(\bar{x}, \bar{t})$ of $U$, where the resulting wave fan $\bar{U}$ should be more complex, will be investigated in Chapter XI, for genuinely nonlinear scalar conservation laws, in Chapter XII, for genuinely nonlinear systems of two conservation laws, and in Chapter XIV, for general genuinely nonlinear systems of conservation laws.

As we saw in Section 9.1, the wave fan $\bar{U}$ is generally a composite of constant states, shocks, and centered rarefaction waves. The simplest wave fan admissibility criterion postulates that the fan is admissible if each one of its shocks, individually, satisfies the shock admissibility conditions discussed in Chapter VIII. As we shall see in the following section, this turns out to be adequate in many cases. Other fan admissibility criteria, which regard the wave fan as an entity rather than as a collection of individual waves, include the entropy rate condition and the viscous fan profile test. These will be discussed later.

## 9.3 Solution of the Riemann Problem via Wave Fan Curves

The aim here is to construct a solution of the Riemann problem by piecing together constant states, centered rarefaction waves, and admissible shocks. We limit our investigation to the case where wave speeds of different characteristic families are strictly separated. This will cover waves of small amplitude in general strictly hyperbolic systems as well as waves of any amplitude in special systems such as (7.1.8), in which all 1-waves travel to the left and all 2-waves travel to the right.

Let us then consider an outgoing wave fan (9.1.2), of bounded variation. Following the discussion in Section 9.1, $(-\infty, \infty)$ is decomposed into the union of the shock set $\mathcal{S}$, the rarefaction wave set $\mathcal{W}$ and the constant state set $\mathcal{C}$. Since the wave speeds of distinct characteristic families are strictly separated, $\mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i$ and $\mathcal{W} = \bigcup_{i=1}^{n} \mathcal{W}_i$ , where $\mathcal{S}_i$ is the (at most countable) set of points of jump discontinuity of $V$ that are $i$-shocks and $\mathcal{W}_i$ is the (possibly empty) set of points of continuity of $V$ in the support of the measure $\dot{V}$ that satisfy (9.1.9). The set $\mathcal{S}_i \bigcup \mathcal{W}_i$ is closed and contains points in the range of wave speeds of the $i$-characteristic family.

We now assume that the shocks satisfy the Lax $E$-condition, i.e., for all $\xi \in \mathcal{S}_i$ ,

$$\lambda_i(V(\xi-)) \geq \xi \geq \lambda_i(V(\xi+)). \tag{9.3.1}$$

Then $\mathcal{S}_i \bigcup \mathcal{W}_i$ is necessarily a closed interval $[\alpha_i, \beta_i]$. Indeed, suppose $\mathcal{S}_i \bigcup \mathcal{W}_i$ is disconnected. Then there is an open interval $(\xi_1, \xi_2) \subset \mathcal{C}$ with endpoints $\xi_1$ and $\xi_2$ contained in $\mathcal{S}_i \bigcup \mathcal{W}_i$ . In particular, $V(\xi_1+) = V(\xi_2-)$. On the other hand, by virtue of (9.1.9) and (9.3.1), $\xi_1 \geq \lambda_i(V(\xi_1+))$, $\xi_2 \leq \lambda_i(V(\xi_2-))$, which is a contradiction to $\xi_1 < \xi_2$ . Notice further that any $\xi \in \mathcal{S}_i$ with $\xi > \alpha_i$ (or $\xi < \beta_i$) is the limit of an increasing (or decreasing) sequence of points of $\mathcal{W}_i$ and so $\lambda_i(V(\xi-)) = \xi$ (or $\lambda_i(V(\xi+)) = \xi$). We have thus established the following

**9.3.1 Theorem.** *Assume the wave speeds of distinct characteristic families are strictly separated. Any self-similar solution* (9.1.2) *of the Riemann Problem* (9.1.1), (9.1.12), *with shocks satisfying the Lax E-condition, comprises* $n + 1$ *constant states* $U_L = U_0, U_1, \cdots, U_{n-1}, U_n = U_R$ *. For* $i = 1, \cdots, n$ , $U_{i-1}$ *is joined to* $U_i$ *by an i-wave fan, namely a composite of centered i-rarefaction waves and/or i-shocks with the property that i-shocks bordered from the left (and/or the right) by i-rarefaction waves are left (and/or right) i-contact discontinuities* (*Fig.* 9.3.1).

It will be shown in the following two sections that the locus of states that may be joined on the right (or left) of a fixed state $\bar{U} \in \mathcal{O}$ by an admissible $i$-wave fan, composed of $i$-rarefaction waves and admissible $i$-shocks, is a Lipschitz curve $\Phi_i(\tau; \bar{U})$ (or $\Psi_i(\tau; \bar{U})$), called the *forward* (or *backward*) *i-wave fan curve through* $\bar{U}$, which may be parametrized so that

$$\Phi_i(\tau; \bar{U}) = \bar{U} + \tau R_i(\bar{U}) + P_i(\tau; \bar{U}), \tag{9.3.2}$$

$$\Psi_i(\tau; \bar{U}) = \bar{U} + \tau R_i(\bar{U}) + Q_i(\tau; \bar{U}), \tag{9.3.3}$$

where $P_i$ and $Q_i$ are Lipschitz continuous functions of $(\tau, U)$ that vanish at $\tau = 0$, and their Lipschitz constant becomes arbitrarily small if $\tau$ is restricted to a sufficiently small neighborhood of the origin.

Taking, for the time being, the existence of wave fan curves with the above properties for granted, we note that to solve the Riemann problem we have to determine an $n$-tuple $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, realized as a vector in $I\!R^n$, such that, starting out from $U_0 = U_L$ and computing successively $U_i = \Phi_i(\varepsilon_i; U_{i-1})$, $i = 1, \cdots, n$, we end up with $U_n = U_R$ . Accordingly, we define the function

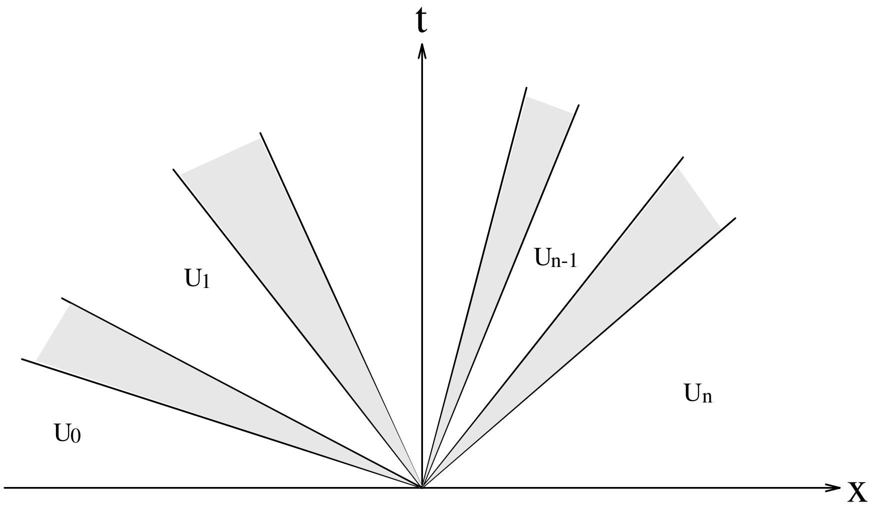


**Fig. 9.3.1**

$$\Omega(\varepsilon; \bar{U}) = \Phi_n(\varepsilon_n; \Phi_{n-1}(\varepsilon_{n-1}; \cdots \Phi_1(\varepsilon_1; \bar{U}) \cdots)). \tag{9.3.4}$$

Clearly,

$$\Omega(\varepsilon; \bar{U}) = \bar{U} + \sum_{i=1}^{n} \varepsilon_i R_i(\bar{U}) + G(\varepsilon; \bar{U}), \tag{9.3.5}$$

where $G$ is a Lipschitz function that vanishes at $\varepsilon = 0$ and whose Lipschitz constant becomes arbitrarily small when $\varepsilon$ is confined to a sufficiently small neighborhood of the origin. When $U_R$ is sufficiently close to $U_L$, there exists a unique $\varepsilon$ near 0 such that $\Omega(\varepsilon; U_L) = U_R$. Indeed, this $\varepsilon$ may be constructed through the iteration scheme: $\varepsilon^{(0)} = 0$ and for $m = 1, 2, \dots$

$$\varepsilon_i^{(m)} = L_i(U_L)\big[U_R - U_L\big] - L_i(U_L)G\big(\varepsilon^{(m-1)}; U_L\big), \quad i = 1, \dots, n, \tag{9.3.6}$$

which converges by an obvious contraction argument. This generates a solution to the Riemann problem that is unique within the class of self-similar solutions with waves of moderate strength. The wave fan joining $U_L$ with $U_R$ is conveniently identified by its left state $U_L$ and the $n$-tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. The value of $\varepsilon_i$ determines the $i$-*wave amplitude* and $|\varepsilon_i|$ measures the $i$-*wave strength*.

In the special case where the $\Phi_i$ are $C^{2,1}$, we shall see that

$$\dot{\Phi}_i(0; \bar{U}) = R_i(\bar{U}), \qquad \ddot{\Phi}_i(0; \bar{U}) = \mathrm{D}R_i(\bar{U})R_i(\bar{U}). \tag{9.3.7}$$

Then $\Omega$ is also $C^{2,1}$. Since $\Omega(0, \cdots, 0, \varepsilon_i, 0, \cdots, 0; \bar{U}) = \Phi_i(\varepsilon_i; \bar{U})$,

$$\frac{\partial \Omega}{\partial \varepsilon_i}(0; \bar{U}) = R_i(\bar{U}), \qquad 1 \le i \le n, \tag{9.3.8}$$

$$\frac{\partial^2 \Omega}{\partial \varepsilon_i^2}(0;\bar{U}) = \mathrm{D}R_i(\bar{U})R_i(\bar{U}), \qquad 1 \le i \le n. \tag{9.3.9}$$

Moreover, for $j<k$, $\Omega(0,\cdots,0,\varepsilon_j,0,\cdots,0,\varepsilon_k,0,\cdots,0;\bar{U}) = \Phi_k\big(\varepsilon_k;\Phi_j\big(\varepsilon_j;\bar{U}\big)\big)$ and so

$$\frac{\partial^2 \Omega}{\partial \varepsilon_j \partial \varepsilon_k}(0;\bar{U}) = \mathrm{D}R_k(\bar{U})R_j(\bar{U}), \qquad 1 \le j < k \le n. \tag{9.3.10}$$

By virtue of (9.3.8), (9.3.9) and (9.3.10),

$$\begin{aligned} U_R = U_L &+ \sum_{i=1}^{n} \varepsilon_i R_i(U_L) + \tfrac{1}{2}\sum_{i=1}^{n} \varepsilon_i^2 \mathrm{D}R_i(U_L)R_i(U_L) \\ &+ \sum_{j=1}^{n}\sum_{k=j+1}^{n} \varepsilon_j\varepsilon_k \mathrm{D}R_k(U_L)R_j(U_L) + O\big(|\varepsilon|^3\big). \end{aligned} \tag{9.3.11}$$

Clearly, we may also synthesize the solution of the Riemann problem in the reverse order, starting out from $U_n = U_R$ and computing successively the states $U_{i-1} = \Psi_i(\varepsilon_i;U_i)$ , $i = n,\cdots,1$, until we reach $U_0 = U_L$. Under certain circumstances, a mixed strategy may be advantageous. For example, the most efficient procedure for solving the Riemann problem for a system of two conservation laws, $n = 2$, is to draw the forward 1-wave curve $\Phi_1(\varepsilon_1;U_L)$ through the left state $U_L$ and the backward 2-wave curve $\Psi_2(\varepsilon_2;U_R)$ through the right state $U_R$. The intersection of these two curves will determine the intermediate constant state: $U_M = \Phi_1(\varepsilon_1;U_L) = \Psi_2(\varepsilon_2;U_R)$.

## 9.4 Systems with Genuinely Nonlinear or Linearly Degenerate Characteristic Families

Our project here is to construct the wave fan curves for systems in which wave fans are particularly simple. When the $i$-characteristic family is linearly degenerate, no centered $i$-rarefaction waves exist and hence, by Theorem 8.2.5, any $i$-wave fan is necessarily an $i$-contact discontinuity. In that case the forward and backward $i$-wave fan curves coincide with the shock curve $W_i$ in Theorem 8.2.5, namely, we have $\Phi_i(\tau;\bar{U}) = \Psi_i(\tau;\bar{U}) = W_i(\tau;\bar{U})$.

When the $i$-characteristic family is genuinely nonlinear, $i$-contact discontinuities are ruled out by Theorem 8.2.1, and so any $i$-wave fan of small amplitude must be either a single centered $i$-rarefaction wave or a single compressive $i$-shock. Let us normalize the field $R_i$ so that (7.6.13) holds, $\mathrm{D}\lambda_i R_i = 1$. The states that may be joined to $\bar{U}$ by a weak $i$-shock lie on the $i$-shock curve $W_i(\tau;\bar{U})$ described by Theorem 8.2.1. On account of Theorem 8.3.1, the shock that joins $\bar{U}$, on the left, with $W_i(\tau;\bar{U})$, on the right, is compressive if and only if $\tau < 0$. On the other hand, by Theorem 7.6.5, the state $\bar{U}$ may be joined on the right (or left) by centered $i$-rarefaction waves to states $V_i(\tau;\bar{U})$ for $\tau > 0$ (or $\tau < 0$). It then follows that we may construct the forward $i$-wave fan curve by $\Phi_i(\tau; \bar{U}) = W_i(\tau; \bar{U})$, for $\tau < 0$, and $\Phi_i(\tau; \bar{U}) = V_i(\tau; \bar{U})$, for $\tau > 0$. Similarly, the backward $i$-wave fan curve is defined by $\Psi_i(\tau; \bar{U}) = V_i(\tau; \bar{U})$, for $\tau < 0$, and $\Psi_i(\tau; \bar{U}) = W_i(\tau; \bar{U})$, for $\tau > 0$. These curves are $C^{2,1}$, by account of Theorem 8.2.2, and satisfy (9.3.2), (9.3.3) and (9.3.7), by Theorem 8.2.1.

In view of the above discussion, we have now established the existence of solution to the Riemann problem for systems with characteristic families that are either genuinely nonlinear or linearly degenerate:

**9.4.1 Theorem.** *Assume the system* (9.1.1) *is strictly hyperbolic and each characteristic family is either genuinely nonlinear or linearly degenerate. For $|U_R - U_L|$ sufficiently small, there exists a unique self-similar solution* (9.1.2) *of the Riemann problem* (9.1.1), (9.1.12), *with small total variation. This solution comprises $n+1$ constant states $U_L = U_0, U_1, \cdots, U_{n-1}, U_n = U_R$. When the $i$-characteristic family is linearly degenerate, $U_i$ is joined to $U_{i-1}$ by an $i$-contact discontinuity, while when the $i$-characteristic family is genuinely nonlinear, $U_i$ is joined to $U_{i-1}$ by either a centered $i$-rarefaction wave or a compressive $i$-shock.*

In particular, Theorem 9.4.1 establishes the existence of solutions, with small total variation, to the Riemann problem for the system (7.1.9) of isentropic gas dynamics, when $2p'(\rho) + \rho p''(\rho) > 0$, so that both characteristic families are genuinely nonlinear; also for the system (7.1.5) of adiabatic thermoelasticity, under the assumption $\sigma_{uu}(u, s) \neq 0$, in which case the 1- and the 3-characteristic families are genuinely nonlinear while the 2-characteristic family is linearly degenerate. As noted earlier, shock and rarefaction wave curves for the above systems exist even in the range of strong shocks, and thus one may attempt to construct solutions of the Riemann problem even when $U_L$ and $U_R$ are far apart. The range of $U_L$ and $U_R$ for which the construction is possible depends on the asymptotic behavior of shock and rarefaction wave curves as the state variables $\rho$ and $u$ approach the boundary points of their physical range, namely zero and infinity. This will be discussed in Section 9.6. For the time being, in order to illustrate the above ideas by means of a simple example, let us consider the system (7.1.8), assuming that $\sigma(u)$ is defined on $(-\infty, \infty)$ and $0 < a \le \sigma'(u) \le b < \infty$, $\sigma''(u) < 0$. It is convenient to reparametrize the wave curves, employing $u$ as the new parameter. In that case, the forward 1-wave curve $\Phi_1$ and the backward 2-wave curve $\Psi_2$ through the typical point $(\bar{u}, \bar{v})$ of the state space may be represented as $v = \varphi(u; \bar{u}, \bar{v})$ and $v = \psi(u; \bar{u}, \bar{v})$, respectively. Recalling the form of the Hugoniot locus (8.2.11) and rarefaction wave curves (7.6.15) for this system, we deduce that

$$\varphi(u; \bar{u}, \bar{v}) = \begin{cases} \bar{v} - \sqrt{[\sigma(u) - \sigma(\bar{u})](u - \bar{u})}\ , & u \le \bar{u} \\ \bar{v} + \displaystyle\int_{\bar{u}}^{u} \sqrt{\sigma'(\omega)}\, d\omega, & u > \bar{u} \end{cases} \tag{9.4.1}$$

$$\psi(u;\bar{u},\bar{v}) = \begin{cases} \bar{v} + \sqrt{[\sigma(u)-\sigma(\bar{u})](u-\bar{u})}\ , & u \le \bar{u} \\ \bar{v} - \int_{\bar{u}}^{u} \sqrt{\sigma'(\omega)}d\omega, & u > \bar{u}. \end{cases} \tag{9.4.2}$$

Figure 9.4.1 depicts a solution of the Riemann problem that comprises a compressive 1-shock and a centered 2-rarefaction wave. The intermediate constant state $(u_M, v_M)$ is determined on the $u$-$v$ plane as the intersection of the forward 1-wave fan curve $\Phi_1$ through $(u_L, v_L)$ with the backward 2-wave fan curve $\Psi_2$ through $(u_R, v_R)$, namely by solving the equation

$$v_M = \varphi(u_M; u_L, v_L) = \psi(u_M; u_R, v_R). \tag{9.4.3}$$

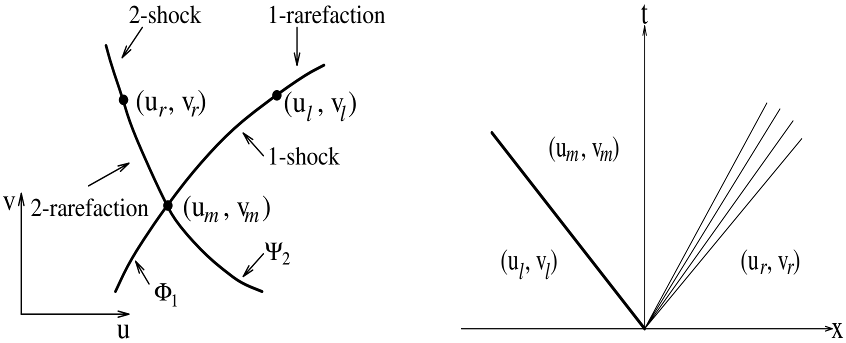


**Fig. 9.4.1(a,b)**

For systems of two conservation laws it is often expedient to perform the construction of the intermediate constant state on the plane of Riemann invariants rather than in the original state space. The reason is that, as noted in Section 7.6, in the plane of Riemann invariants rarefaction wave curves become straight lines parallel to the coordinate axes. This facilitates considerably the task of locating the intersection of wave curves of different characteristic families. Figure 9.4.2 depicts the configuration of the wave curves of Fig. 9.4.1 in the plane $w$-$z$ of Riemann invariants.

## 9.5 General Strictly Hyperbolic Systems

Our next task is to describe admissible wave fans, and construct the corresponding wave fan curves, for systems with characteristic families that are neither genuinely nonlinear nor linearly degenerate. In that case, the Lax $E$-condition is no longer sufficiently selective to single out a unique solution to the Riemann problem so the more stringent Liu $E$-condition will be imposed on shocks.

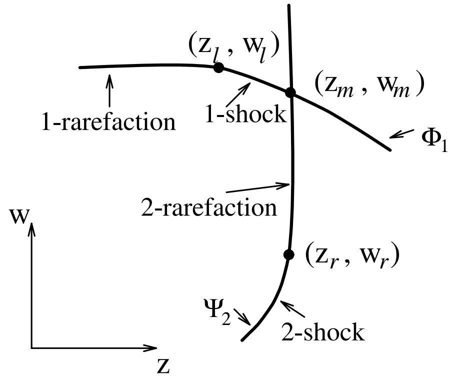


**Fig. 9.4.2**

We begin with the scalar conservation law (7.1.2), where $f(u)$ may have inflection points. The Liu $E$-condition is now expressed by the Oleinik $E$-condition (8.4.3). By Theorem 9.3.1, the solution of the Riemann problem comprises two constant states $u_L$ and $u_R$ joined by a wave fan that is a composite of shocks and/or centered rarefaction waves. There exists precisely one such wave fan with shocks satisfying Oleinik's $E$-condition, and it is constructed by the following procedure: When $u_L < u_R$ (or $u_L > u_R$), we let $g$ denote the *convex* (or *concave*) *envelope* of $f$ over the interval $[u_L, u_R]$ (or $[u_R, u_L]$); namely, $g(u)$ is the infimum (or supremum) of all convex combinations $\theta_1 f(u_1) + \theta_2 f(u_2)$, with $\theta_1 \geq 0$, $\theta_2 \geq 0, \theta_1 + \theta_2 = 1, u_1, u_2 \in [u_L, u_R]$ (or $[u_R, u_L]$) and $\theta_1 u_1 + \theta_2 u_2 = u$. Thus the graph of $g$ may be visualized as the configuration of a flexible string anchored at the points $(u_L, f(u_L))$, $(u_R, f(u_R))$ and stretched under (or over) the "obstacle" $\{(u, v) : u_L \leq u \leq u_R, v \geq f(u)\}$ (or $\{(u, v) : u_R \leq u \leq u_L, v \leq f(u)\}$). The slope $\xi = g'(u)$ is a continuous nondecreasing (or nonincreasing) function whose inverse $u = \omega(\xi)$ generates the wave fan $u = \omega(x/t)$. In particular, the flat parts of $g'(u)$ give rise to the shocks while the intervals over which $g'(u)$ is strictly monotone generate the rarefaction waves. Figure 9.5.1 depicts an example in which the resulting wave fan consists of a centered rarefaction wave bordered by one-sided contact discontinuities.

To prepare the ground for the investigation of systems, we construct wave fans, and corresponding wave fan curves, for the simple system (7.1.8), where $\sigma(u)$ may have inflection points. The Liu $E$-condition here reduces to the Wendroff $E$-condition (8.4.4). Similar to the genuinely nonlinear case, we shall employ $u$ as parameter and determine the forward 1-wave fan curve $\Phi_1$ and the backward 2-wave fan curve $\Psi_2$, through the state $(\bar{u}, \bar{v})$, in the form $v = \varphi(u; \bar{u}, \bar{v})$ and $v = \psi(u; \bar{u}, \bar{v})$, respectively. Recalling the equations (8.2.11) for the Hugoniot locus, the equations (7.6.15) for the rarefaction wave curves, and (8.4.4), one easily verifies that

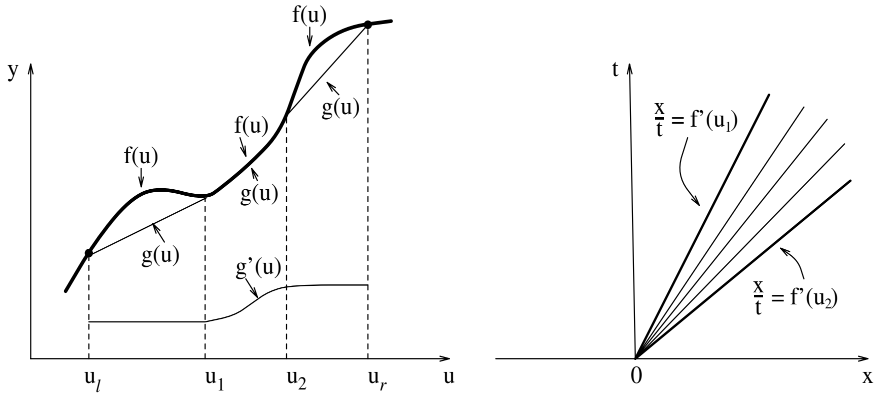


**Fig. 9.5.1**

(9.5.1) $$\varphi(u;\bar{u},\bar{v}) = \bar{v} + \int_{\bar{u}}^{u} \sqrt{g'(\omega;u,\bar{u})}d\omega,$$

(9.5.2) $$\psi(u;\bar{u},\bar{v}) = \bar{v} - \int_{\bar{u}}^{u} \sqrt{g'(\omega;u,\bar{u})}d\omega,$$

where $g'(\omega;u,\bar{u})$ is the derivative, with respect to $\omega$, of the monotone increasing, continuously differentiable function $g(\omega;u,\bar{u})$ which is constructed by the following procedure: For fixed $u \leq \bar{u}$ (or $u \geq \bar{u}$), $g(\cdot,u,\bar{u})$ is the convex (or concave) envelope of $\sigma(\cdot)$ over the interval $[u,\bar{u}]$ (or $[\bar{u},u]$). Indeed, as in the case of the scalar conservation law discussed above, the states $(\bar{u},\bar{v})$ and $(u,v)$, $v = \phi(u;\bar{u},\bar{v})$, are joined by a 1-wave fan $(\omega(x/t), v(x/t))$, where $\omega(\xi)$ is the inverse of the function $\xi = \sqrt{g'(\omega;u,\bar{u})}$ and

(9.5.3) $$v(\xi) = \bar{v} + \int_{\bar{u}}^{\omega(\xi)} \sqrt{g'(\omega;u,\bar{u})}d\omega.$$

Again, the flat parts of $g'$ give rise to shocks while the intervals over which $g'$ is strictly monotone generate the rarefaction waves. In the genuinely nonlinear case, $\sigma''(u) < 0$, (9.5.1) and (9.5.2) reduce to (9.4.1) and (9.4.2). Once $\phi$ and $\psi$ have been determined, the Riemann problem is readily solved, as in the genuinely nonlinear case, by locating the intermediate constant state $(u_M, v_M)$ through the equation (9.4.3).

After this preparation, we continue with a somewhat sketchy and informal description of the construction of wave fan curves for general systems. To avoid aggravating complications induced by various degeneracies, we limit the investigation to $i$-characteristic families that are *piecewise genuinely nonlinear* in the sense that if $U$ is a state of linear degeneracy, $\mathrm{D}\lambda_i(U)R_i(U) = 0$, then $\mathrm{D}(\mathrm{D}\lambda_i(U)R_i(U))R_i(U) \neq 0$. This implies, in particular, that the set of states of linear degeneracy of the $i$-characteristic family is locally a smooth manifold of codimension one, which is transversal to the vector field $R_i$. The scalar conservation law (7.1.2) and the system (7.1.8) of isentropic elasticity will satisfy this assumption when the functions $f(u)$ and $\sigma(u)$ have isolated, nondegenerate inflection points, i.e., $f'''(u)$ and $\sigma'''(u)$ are nonzero at any point $u$ where $f''(u)$ and $\sigma''(u)$ vanish. Even after these simplifications, the construction is complicated. The ideas may become more transparent if the reader refers back to the model system (7.1.8) to illustrate each step. Familiarity with Lemma 8.2.4 and the remarks following its statement will also prove helpful.

Assuming the $i$-characteristic family is piecewise genuinely nonlinear, we consider the forward $i$-wave fan curve $\Phi_i(\tau; \bar{U})$ through a point $\bar{U}$ of genuine nonlinearity, say $\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U}) = 1$. Then $\Phi_i$ starts out as in the genuinely nonlinear case, namely, for $\tau$ positive small it coincides with the $i$-rarefaction wave curve $V_i(\tau; \bar{U})$ through $\bar{U}$, while for $\tau$ negative, near zero, it coincides with the $i$-shock curve $W_i(\tau; \bar{U})$ through $\bar{U}$. In particular, (9.3.7) holds. We shall follow $\Phi_i$ along the positive $\tau$-direction; the description for $\tau < 0$ is quite analogous.

For $\tau > 0$, $\Phi_i(\tau; \bar{U})$ will stay with the $i$-rarefaction wave curve $V_i(\tau; \bar{U})$ for as long as the latter sojourns in the region of genuine nonlinearity: $\mathrm{D}\lambda_i(V_i)R_i(V_i) > 0$. Suppose now $V_i(\tau; \bar{U})$ first encounters the set of states of linear degeneracy of the $i$-characteristic family at the state $\tilde{U} = V_i(\tilde{\tau}; \bar{U}) : \mathrm{D}\lambda_i(\tilde{U})R_i(\tilde{U}) = 0$. The set of states of linear degeneracy in the vicinity of $\tilde{U}$ forms a manifold $\mathcal{M}$ of codimension 1, transversal to the vector field $R_i$; see Fig. 9.5.2 (a,b).

The extension of $\Phi_i$ beyond $\tilde{U}$ is constructed as follows: For $\tau^* < \tilde{\tau}$, with $\tilde{\tau} - \tau^*$ small, we draw the $i$-shock curve $W_i(\zeta; U^*)$ through the state $U^* = V_i(\tau^*, \bar{U})$. By account of (8.2.1), $s_i(0; U^*) = \lambda_i(U^*)$ and since $\mathrm{D}\lambda_i(U^*)R_i(U^*) > 0$, (8.2.2) implies that for $\zeta$ negative, near 0, $\dot{s}_i(\zeta; U^*) > 0$ and $s_i(\zeta; U^*) < \lambda_i(W_i(\zeta; U^*))$. However, after crossing $\mathcal{M}$, $W_i(\zeta; U^*)$ enters the region where $\mathrm{D}\lambda_i(U)R_i(U) < 0$ and thus $\lambda_i(W_i(\zeta; U^*))$ will become decreasing. Eventually, $\zeta^*$ will be reached where $s_i(\zeta^*; U^*) = \lambda_i(W_i(\zeta^*; U^*))$. For $\zeta < \zeta^*$, by virtue of Lemma 8.2.4, $s_i(\zeta, U^*) > \lambda_i(W_i(\zeta; U^*))$ and $\dot{s}_i(\zeta, U^*) < 0$. Finally, a value $\zeta^\sharp$ will be attained with $s_i(\zeta^\sharp; U^*) = \lambda_i(U^*)$. Then the state $U^\sharp = W_i(\zeta^\sharp; U^*)$, on the right, is joined to $U^*$, on the left, by a left $i$-contact discontinuity with speed $\lambda_i(U^*)$. This shock satisfies the Liu $E$-condition, since $s_i(\zeta; U^*) > \lambda_i(U^*)$ for $\zeta < \zeta^\sharp$. In particular, $\lambda_i(U^*) = s_i(\zeta^\sharp; U^*) > \lambda_i(U^\sharp)$. Consequently, $\bar{U}$, on the left, is joined to $U^\sharp$, on the right, by an admissible $i$-wave fan, comprising the $i$-rarefaction wave that joins $U^*$ to $\bar{U}$ and the admissible left $i$-contact discontinuity that joins $U^\sharp$ to $U^*$. It can be shown that as $U^*$ moves along the curve $V_i(\tau; \bar{U})$ from $\tilde{U}$ towards $\bar{U}$, the corresponding $U^\sharp$ traces a curve, say $\Gamma$. If $U^* = \tilde{U}$, then $U^\sharp = \tilde{U}$ so $\Gamma$ starts out from $\tilde{U}$. Also $\Gamma$ at $\tilde{U}$ is tangential to $R_i(\tilde{U})$. We adjoin $\Gamma$ to $V_i(\tau; \bar{U})$ and consider it as the continuation of $\Phi_i(\tau; \bar{U})$ beyond $\tilde{U}$, with the proper parametrization.

$\Phi_i(\tau; \bar{U})$ will stay with $\Gamma$ up until a state $\hat{U}$ is reached at which one of the following two alternatives first occurs:

One possibility is depicted in Fig. 9.5.2(a): $\Gamma$ crosses another manifold $\mathcal{N}$ of states of linear degeneracy of the $i$-characteristic family, entering the region $\mathrm{D}\lambda_i(U)R_i(U) > 0$, and eventually $U^*$ backs up to a position $U^0$ so that the corresponding $U^\sharp$, denoted by $\hat{U}$, satisfies $\lambda_i(\hat{U}) = \lambda_i(U^0)$. In that case, $\Phi_i(\tau; \bar{U})$ is extended beyond $\hat{U}$ as the $i$-rarefaction curve $V_i(\zeta; \hat{U})$ through $\hat{U}$, properly

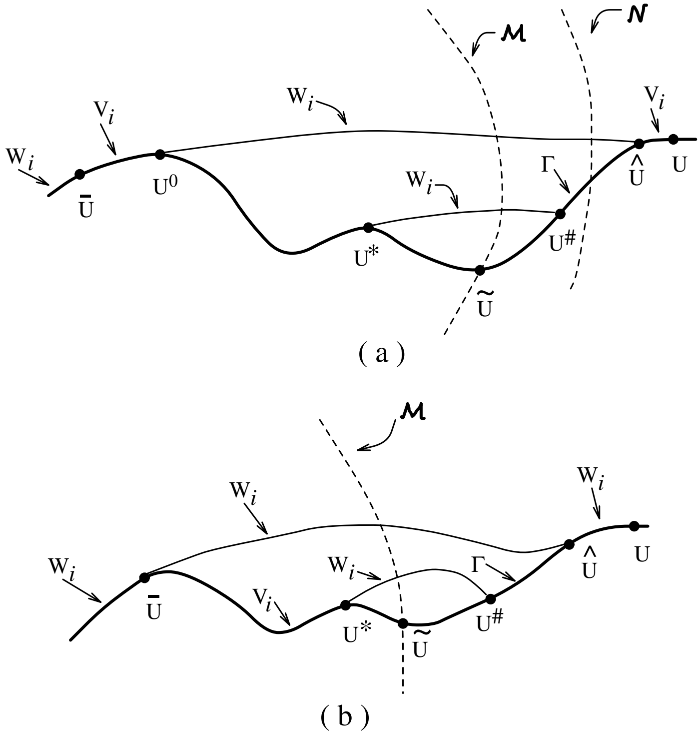


**Fig. 9.5.2(a,b)**

reparametrized. Any state $U$ on that curve is joined, on the right, to $\bar{U}$ by a wave fan comprising an $i$-rarefaction wave that joins $U^0$ to $\bar{U}$, an $i$-contact discontinuity that joins $\hat{U}$ to $U^0$ and a second $i$-rarefaction wave that joins $U$ to $\hat{U}$.

The alternative is depicted in Fig. 9.5.2(b): $U^*$ backs up all the way to $\bar{U}$ and the corresponding $U^\sharp$, denoted by $\hat{U}$, satisfies $\lambda_i(\hat{U}) < \lambda_i(\bar{U})$. In that case $\hat{U}$ lies on the $i$-shock curve through $\bar{U}$, say $\hat{U} = W_i(\hat{\tau}; \bar{U})$. As $s_i(\hat{\tau}; \bar{U}) = \lambda_i(\bar{U}) > \lambda_i(\hat{U})$, Lemma 8.2.4 implies $\dot{s}_i(\hat{\tau}; \bar{U}) < 0$. Then $\Phi_i(\tau; \bar{U})$ is extended beyond $\hat{U}$ along the $i$-shock curve $W_i(\tau; \bar{U})$. Any state $U$ on this arc of the curve is joined, on the right, to $\bar{U}$ by a single shock that satisfies the Liu $E$-condition.

By continuing this process we complete the construction of $\Phi_i(\tau; \bar{U})$ within the range of waves of moderate strength, and for certain systems even for strong waves. Furthermore, careful review of the construction verifies that the graph of $\Phi_i$ contains all states in a small neighborhood of $\bar{U}$ that may be joined to $\bar{U}$ by an admissible $i$-wave fan.

As we saw earlier, before crossing any manifold of states of linear degeneracy, $\Phi_i$ is $C^{2,1}$. Its regularity may be reduced to $C^{1,1}$ after the first crossing with such a manifold, and it may become merely Lipschitz beyond a second crossing (references in Section 9.12). Nevertheless, (9.3.2) will still hold, within the realm of waves of moderate strength so that the range of $\tau$ for which the Lipschitz constant of $P_i(\tau; U)$

is small transcends the manifolds of states of linear degeneracy, and does not depend on their number. Consequently, one may trace wave fan curves for any strictly hyperbolic system whose flux may be realized as the $C^1$ limit of a sequence of fluxes with characteristic families that are piecewise genuinely nonlinear.

Later on, in Section 9.8, we will encounter an alternative construction of wave fan curves, for general strictly hyperbolic systems, without any requirement of piecewise genuine nonlinearity, which resembles the construction for the scalar conservation law described earlier in this section.

Once wave fan curves satisfying (9.3.2) are in place, one may employ the construction of the solution to the Riemann problem, described above, thus arriving at the following generalization of Theorem 9.4.1:

**9.5.1 Theorem.** *Assume the system* (9.1.1) *is strictly hyperbolic. For* $|U_R - U_L|$ *sufficiently small, there exists a unique self-similar solution* (9.1.2) *of the Riemann problem* (9.1.1), (9.1.12)*, with small total variation. This solution comprises* $n+1$ *constant states* $U_L = U_0, U_1, \cdots, U_{n-1}, U_n = U_R$*, and* $U_i$ *is joined to* $U_{i-1}$ *by an admissible* $i$*-wave fan, composed of* $i$*-rarefaction waves and (at most countable)* $i$*-shocks which satisfy the Liu* $E$*-condition.*

## 9.6 Failure of Existence or Uniqueness; Delta Shocks and Transitional Waves

The orderly picture painted by Theorem 9.5.1 breaks down when one leaves the realm of strictly hyperbolic systems and waves of small amplitude.

The following exemplifies the difficulties that may be encountered in the construction of solutions. We consider the isentropic flow of an infinitely long column of a polytropic gas, with equation of state $p = \frac{1}{\gamma}\rho^\gamma$, $\gamma > 1$, under the following initial conditions. The density is constant $\bar{\rho} > 0$, throughout the length of the column. The right half of the column is subjected to a uniform impulse $\bar{\rho}\bar{v} > 0$, while the left half is subjected to an equal and opposite impulse $-\bar{\rho}\bar{v}$. Thus, in Lagrangian coordinates, we have to solve the Riemann problem for the system (7.1.8), with $\sigma(u) = -\frac{1}{\gamma}u^{-\gamma}$ for initial data $\big(u_L, v_L\big) = \big(\bar{u}, -\bar{v}\big)$ and $\big(u_R, v_R\big) = \big(\bar{u}, \bar{v}\big)$, where $\bar{u} = 1/\bar{\rho}$.

With reference to (9.4.1) and (9.4.2), it is clear that any intersection of the forward 1-wave curve through $(\bar{u}, -\bar{v})$ with the backward 2-wave curve through $(\bar{u}, \bar{v})$ will take place at $u_M > \bar{u}$, so that the jump discontinuity at the origin will be resolved into two rarefaction waves. In that range,

$$\begin{cases} \varphi\big(u; \bar{u}, \bar{v}\big) = \frac{2}{1-\gamma}\big(u^{\frac{1-\gamma}{2}} - w\big), & u \ge \bar{u}, \\ \psi\big(u; \bar{u}, \bar{v}\big) = -\frac{2}{1-\gamma}\big(u^{\frac{1-\gamma}{2}} - w\big), & u \ge \bar{u}, \end{cases} \tag{9.6.1}$$

where we have set

$$w = \bar{u}^{\frac{1-\gamma}{2}} + \frac{1-\gamma}{2}\bar{v}. \tag{9.6.2}$$

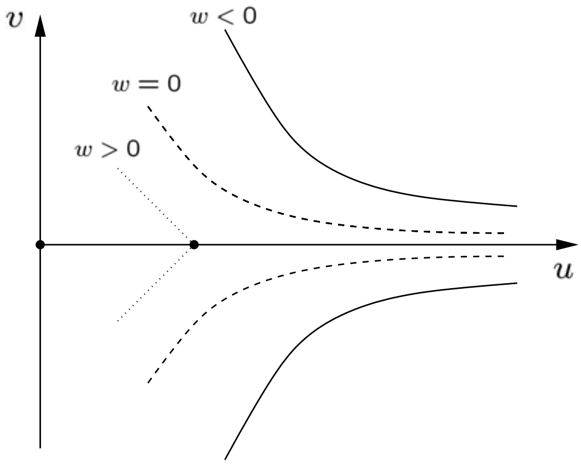


**Fig. 9.6.1**

The form of the solution will depend on the sign of $w$. Figure 9.6.1 depicts the wave curves when $w > 0$, $w = 0$ or $w < 0$.

When $w > 0$, the wave curves intersect at $u_M = w^{\frac{2}{1-\gamma}}$, $v_M = 0$, and the Riemann problem admits the solution

$$
(u(\xi), v(\xi)) = \begin{cases} \left(\bar{u}, -\bar{v}\right), & -\infty < \xi \le -\xi_F \\ \left(|\xi|^{-\frac{2}{\gamma+1}},\ -\frac{2}{\gamma-1}|\xi|^{\frac{\gamma-1}{\gamma+1}} + \frac{2}{\gamma-1}|\xi_0|^{\frac{\gamma-1}{\gamma+1}}\right), & -\xi_F < \xi \le -\xi_S \\ \left(u_M, 0\right), & -\xi_S < \xi < \xi_S \\ \left(|\xi|^{-\frac{2}{\gamma+1}},\ \frac{2}{\gamma-1}|\xi|^{\frac{\gamma-1}{\gamma+1}} - \frac{2}{\gamma-1}|\xi_0|^{\frac{\gamma-1}{\gamma+1}}\right), & \xi_S \le \xi < \xi_F \\ \left(\bar{u}, \bar{v}\right), & \xi_F \le \xi < \infty, \end{cases} \tag{9.6.3}
$$

where $\xi_F = \bar{u}^{-\frac{\gamma+1}{2}}$ and $\xi_S = w^{\frac{\gamma+1}{\gamma-1}}$.

When $w = 0$, the two wave curves intersect at infinity. As $w \downarrow 0$, $(u(\xi), v(\xi))$ of (9.6.3) reduces to

$$
(u(\xi), v(\xi)) = \begin{cases} \left(\bar{u}, -\bar{v}\right), & -\infty < \xi \le -\xi_F \\ \left(|\xi|^{-\frac{2}{\gamma+1}},\ \frac{2}{\gamma-1}\operatorname{sgn}\xi\,|\xi|^{\frac{\gamma-1}{\gamma+1}}\right), & -\xi_F < \xi < \xi_F \\ \left(\bar{u}, \bar{v}\right), & \xi_F \le \xi < \infty. \end{cases} \tag{9.6.4}
$$

Notice that the singularity of $u$ at $\xi = 0$ is integrable while $v$ is continuous. It is then easy to check that $(u(\xi), v(\xi))$, defined by (9.6.4), satisfies, in the sense of distributions, (9.1.3) for the system (7.1.8), namely

$$\begin{cases} (-v - \xi u)^{\cdot} + u = 0 \\ \left(\frac{1}{\gamma}u^{-\gamma} - \xi v\right)^{\cdot} + v = 0, \end{cases} \tag{9.6.5}$$

and thus solves the Riemann problem for $w = 0$. That $u(0) = \infty$ simply means that the density $\rho$ vanishes along the line $x = 0$.

When $w < 0$, the two wave curves fail to intersect, even at infinity, and no standard solution to the Riemann problem may thus be constructed. The physical problem is of course still solvable. Indeed, we may reformulate and solve it, in Eulerian coordinates, as a Riemann problem for the system (7.1.10), where $\kappa = 1/\gamma$, with data: $\rho(x, 0) = \bar{\rho}$, $x \in (-\infty, \infty)$; $v(x, 0) = -\bar{v}$, $x \in (-\infty, 0)$; and $v(x, 0) = \bar{v}$, $x \in (0, \infty)$. The solution comprises two rarefaction waves whose tail ends recede from each other with respective speeds $\pm\frac{2}{\gamma-1}w$, leaving in the wake between them a *vacuum state* where $\rho$ vanishes. The discussion of Section 2.2 does not cover the mapping of this flow to Lagrangian coordinates, because the determinant $\rho^{-1}$ of the deformation gradient is unbounded on the vacuum state. In fact, under this change of coordinates the full sector in physical space-time occupied by the vacuum state is mapped to the single line $x = 0$ in the reference space-time. Consequently, $u = \rho^{-1}$ becomes very singular at $\xi = 0$:

$$(u(\xi), v(\xi)) = \begin{cases} (\bar{u}, -\bar{v}), & -\infty < \xi \le -\xi_F \\ \left(|\xi|^{-\frac{2}{\gamma+1}} - \frac{4}{\gamma-1}w\delta_0,\ \frac{2}{\gamma-1}\operatorname{sgn}\xi\left[|\xi|^{\frac{\gamma-1}{\gamma+1}} - w\right]\right), & -\xi_F < \xi < \xi_F \\ (\bar{u}, \bar{v}), & \xi_F \le \xi < \infty, \end{cases} \tag{9.6.6}$$

where $\delta_0$ denotes the Dirac delta function at the origin. In fact, $(u(\xi), v(\xi))$ of (9.6.6) is a distributional solution of (9.6.5), providing one regards $u^{-\gamma}(\xi)$ as a continuous function which vanishes at the origin $\xi = 0$. In the $(x, t)$ coordinates, (9.6.6) induces the stationary singularity $-\frac{4}{\gamma-1}wt\delta_0$ on $u$, along the $t$-axis. This new type of singularity that supports point masses is called a *delta shock.*

One might argue that the delta shock appeared here because we employed Lagrangian coordinates, which are ill suited for this problem. It turns out, however, that delta shocks are often present in solutions of the Riemann problem, especially for systems that are not strictly hyperbolic. Relevant references are cited in Section 9.12. As we shall see in Section 9.8, the method of vanishing viscosity contributes insight into the formation of delta shocks.

Failure of strict hyperbolicity is also a source of difficulties in regard to uniqueness of solutions to the Riemann problem. The Liu $E$-condition is no longer sufficiently discriminating to single out a unique solution. To illustrate this, let us consider the Riemann problem for the model system (7.2.11), with data $(u_L, v_L) = (1, 0)$ and $(u_R, v_R) = (a, 0)$, where $a \in (-\frac{1}{2}, 0)$. One solution comprises the two constant states $(1, 0)$ and $(a, 0)$ joined by an overcompressive shock, of speed $s = 1+a+a^2$, which satisfies the Liu $E$-condition. There is, however, another solution comprising three constant states, $(1, 0)$, $(-1, 0)$ and $(a, 0)$, where $(-1, 0)$ is joined to $(1, 0)$ by a 1-contact discontinuity of speed 1 and $(a, 0)$ is joined to $(-1, 0)$ by a 2-shock of speed $s = 1 - a + a^2$. Both shocks satisfy the Liu $E$-condition. Following the discussion on this system in Section 8.6, one may be inclined to disqualify overcompressive shocks, in which case the second solution of the Riemann problem emerges as the admissible one. This of course hinges on the premise that (8.6.4) is the proper dissipative form of (7.2.11).

The issue of nonuniqueness also arises in the context of (usually not strictly hyperbolic) systems that admit undercompressive shocks. Consider, for definiteness, such a system of two conservation laws. Any undercompressive shock is crossed by both 1-characteristics, from right to left, and 2-characteristics, from left to right. Consequently, such a shock may be incorporated into a wave fan that contains a compressive 1-shock, or 1-rarefaction wave, on its left, and a compressive 2-shock, or 2-rarefaction wave, on its right. In that capacity, the undercompressive shock serves as a "bridge" joining the two characteristic families and so is dubbed a *transitional wave*. It is also possible to have rarefaction transitional waves that are composites of a 1-rarefaction and a 2-rarefaction and may occur when the 1-rarefaction wave curve and the 2-rarefaction wave curve meet tangentially on the line along which strict hyperbolicity fails, $\lambda_1(U) = \lambda_2(U)$. The possibility of including transitional waves renders the family of solutions to the Riemann problem richer and thereby the issue of uniqueness thornier. As pointed out in Chapter VIII, viscosity or viscosity-capillarity conditions, as well as kinetic relations are being used as admissibility criteria for these undercompressive shocks. The importance of working with genuine physical systems cannot be overemphasized at this point.

## 9.7 The Entropy Rate Admissibility Criterion

According to the entropy shock admissibility criterion, the entropy production across shocks, defined by the left-hand side of (8.5.1), must be negative, so in particular the total entropy shall be decreasing. We have seen, however, that this requirement is generally insufficiently discriminating to rule out all spurious solutions. A wave fan admissibility criterion will be introduced here, which is a strengthened version of the entropy admissibility condition, as it stipulates that the combined entropy production of all shocks in the fan is not just negative but as small as possible, or equivalently, as it turns out, that the total entropy is not just decreasing but actually decreasing at the highest allowable rate.

We assume that our system (9.1.1) is endowed with a designated entropy-entropy flux pair $(\eta(U), q(U))$, and consider the admissibility of wave fans $U(x,t) = V(x/t)$, with prescribed end-states $V(-\infty) = U_L$ and $V(\infty) = U_R$. The combined entropy production of the shocks in $V$ is given by

(9.7.1) $$\mathcal{P}_V = \sum_{\xi} \{q(V(\xi+)) - q(V(\xi-)) - \xi\,[\eta(V(\xi+)) - \eta(V(\xi-))]\}\,,$$

where the summation runs over the at most countable set of points $\xi$ of jump discontinuity of $V$.

Because of the Rankine-Hugoniot jump condition (9.1.8), for any $A \in \mathbb{M}^{1\times n}$ and $a \in \mathbb{R}$, the entropy-entropy flux pair $(\eta(U) + AU + a, q(U) + AF(U))$ yields the same value for the combined entropy production as $(\eta(U), q(U))$. One may thus assume, without loss of generality, that

(9.7.2) $$\eta\big(U_L\big) = \eta\big(U_R\big) = 0.$$

After this normalization, the rate of change of the total entropy in the wave fan is given by

(9.7.3) $$\dot{\mathcal{H}}_V = \frac{d}{dt}\int_{-\infty}^{\infty} \eta(U(x,t))dx = \frac{d}{dt}\int_{-\infty}^{\infty} \eta\Big(V\Big(\frac{x}{t}\Big)\Big)dx = \int_{-\infty}^{\infty} \eta(V(\xi))d\xi.$$

Actually, $\mathcal{P}_V$ and $\dot{\mathcal{H}}_V$ are related through

(9.7.4) $$\dot{\mathcal{H}}_V = \mathcal{P}_V + q\big(U_\ell\big) - q\big(U_r\big).$$

To verify this, begin with the identity

(9.7.5) $$\eta(V(\xi)) = [\xi\eta(V(\xi)) - q(V(\xi))]^{\cdot} + \dot{q}(V(\xi)) - \xi\dot{\eta}(V(\xi)),$$

which holds in the sense of measures, and note that the generalized chain rule, Theorem 1.7.5, yields

(9.7.6) $$\dot{q}(V) - \xi\dot{\eta}(V) = \big[\widetilde{\mathrm{D}q}(V) - \xi\widetilde{\mathrm{D}\eta}(V)\big]\dot{V}.$$

From (9.7.6),(7.4.1) and (9.1.6) it follows that the measure $\dot{q}(V) - \xi\dot{\eta}(V)$ is concentrated in the set of points of jump discontinuity of $V$. Therefore, combining (9.7.1), (9.7.2), (9.7.3) and (9.7.5), one arrives at (9.7.4).

**9.7.1 Definition.** A wave fan $U(x,t) = V(x/t)$, with end-states $V(-\infty) = U_L$, $V(\infty) = U_R$, satisfies the *entropy rate admissibility criterion* if $\mathcal{P}_V \le \mathcal{P}_{\bar{V}}$, or equivalently $\dot{\mathcal{H}}_V \le \dot{\mathcal{H}}_{\bar{V}}$, holds for any other wave fan $\bar{U}(x,t) = \bar{V}(x/t)$ with the same end-states $\bar{V}(-\infty) = U_L$, $\bar{V}(\infty) = U_R$.

In its connection to continuum physics, the entropy rate admissibility criterion is a more stringent version of the Second Law of thermodynamics: Not only should the physical entropy increase, but it should be increasing at the maximum rate allowed by the balance laws of mass, momentum and energy. The kinetic theory seems to lend some credence to that thesis, at least for waves of small amplitude (references in Section 9.12). However, the status of the entropy rate principle shall ultimately be judged on the basis of its implications in the context of familiar systems, and its comparison to other, firmly established, admissibility conditions. This will be our next task.

We begin our investigation by testing the entropy rate criterion on the scalar conservation law:

**9.7.2 Theorem.** *For the scalar conservation law* (7.1.2), *with designated entropy-entropy flux pair* (8.5.3), *a wave fan satisfies the entropy rate admissibility criterion if and only if every shock satisfies the Oleinik E-condition.*

**Proof.** Let us fix some wave fan $u(x,t) = \omega(x/t)$, with end-states $\omega(-\infty) = u_L$, $\omega(\infty) = u_R$. As $\xi$ runs from $-\infty$ to $+\infty$, $y = \omega(\xi)$ traces, on the graph of $y = f(u)$, a (finite or infinite) number of "arcs", separated by gaps induced by the shocks: When $\xi$ is a point of jump discontinuity of $\omega$, $f(\omega)$ jumps from $f(\omega(\xi-))$ to $f(\omega(\xi+))$. We produce a continuous curve by filling these gaps with the chord that connects $(\omega(\xi-), f(\omega(\xi-))$ with $(\omega(\xi+), f(\omega(\xi+)))$. This may be effected by the following procedure: We let $v(\xi)$ denote the variation of $\omega$ over the interval $(-\infty, \xi)$. Note that $v$ is a left-continuous nondecreasing function. We now construct the curve $y = \gamma_\omega(\tau)$, $\tau \in [0, v(\infty)]$, as follows: If $\tau = v(\xi)$, for some $\xi \in (-\infty, \infty)$, then $\gamma_\omega(\tau) = f(\omega(\xi))$. On the other hand, if $v(\xi-) < \tau < v(\xi+)$, for some $\xi \in (-\infty, \infty)$, then

$$\gamma_\omega(\tau) = \frac{v(\xi+) - \tau}{v(\xi+) - v(\xi-)} f(\omega(\xi-)) + \frac{\tau - v(\xi-)}{v(\xi+) - v(\xi-)} f(\omega(\xi+)). \tag{9.7.7}$$

Notice that $\gamma_\omega$ is a (possibly self-intersecting) curve with endpoints $(u_L, f(u_L))$ and $(u_R, f(u_R))$ having the property that, as $\tau$ runs from 0 to $v(\infty)$, the $u$-slope $d^+\gamma_\omega/du = (d^+\gamma_\omega/d\tau)(du/d\tau)^{-1}$ is nondecreasing.

We recall, from Section 8.5, that the entropy production of a shock that joins $u_-$, on the left, to $u_+$, on the right, is given by the left-hand side of (8.5.4), which measures the signed area of the domain bordered by the arc of the graph of $f$ with endpoints $(u_-, f(u_-))$, $(u_+, f(u_+))$, and the chord that connects $(u_-, f(u_-))$, $(u_+, f(u_+))$. It follows that the entropy production $\mathcal{P}_\omega$ of the wave fan $\omega$ is measured by the signed area of the domain bordered by the arc of the graph of $f$ with endpoints $(u_L, f(u_L))$, $(u_R, f(u_R))$ and the graph of the curve $\gamma_\omega$. Consequently, the difference $\mathcal{P}_\omega - \mathcal{P}_{\bar{\omega}}$ in the entropy production of two wave fans $\omega$ and $\bar{\omega}$ with the same end-states, $u_L$ and $u_R$, is measured by the signed area of the domain bordered by the corresponding curves $\gamma_\omega$ and $\gamma_{\bar{\omega}}$. We conclude that a wave fan $u(x,t) = \omega(x/t)$ with given end-states $u_L$ and $u_R$, such that $u_L < u_R$ (or $u_L > u_R$), minimizes the total entropy production if and only if the curve $\gamma_\omega$ is the convex (or concave) envelope of $f$ over the interval $[u_L, u_R]$ (or $[u_R, u_L]$). As we saw already in Section 9.5, this is the unique wave fan whose shocks satisfy the Oleinik $E$-condition. The proof is complete.

It is interesting that just one entropy suffices to rule out all spurious solutions. The situation is similar with the system (7.1.8) of isentropic elasticity:

**9.7.3 Theorem.** *For the system* (7.1.8), *with designated entropy-entropy flux pair* (7.4.10), *a wave fan satisfies the entropy rate admissibility criterion if and only if every shock satisfies the Wendroff E-condition* (8.4.4).

The proof of the above theorem, which can be found in the references cited in Section 9.12, is based on the observation that the entropy production of a shock is given by the left-hand side of (8.5.5) and thus, as in the case of the scalar conservation law, may be interpreted as an area.

We now turn to general strictly hyperbolic systems but limit our investigation to shocks with small amplitude:

**9.7.4 Theorem.** *For any strictly hyperbolic system* (9.1.1) *of conservation laws, with designated entropy-entropy flux pair* $(\eta, q)$, *where* $\eta$ *is (locally) uniformly convex, a wave fan with waves of moderate strength may satisfy the entropy rate admissibility criterion only if every shock satisfies the Liu E-condition.*

**Proof.** The assertion is established by contradiction: Assuming some shock in a wave fan $U(x,t) = V(x/t)$ violates the Liu $E$-condition, one constructs another wave fan $\bar{U}(x,t) = \bar{V}(x/t)$, with the same end-states, $U_L$ , $U_R$ , but lower entropy production, $\mathcal{P}_{\bar{V}} < \mathcal{P}_V$. Here it will suffice to illustrate the idea in the special case where $V$ consists of just a single $i$-shock joining constant states $U_L$ , on the left, and $U_R$ , on the right. The general proof is found in the literature cited in Section 9.12.

Let $W_i(\cdot)$ denote the $i$-shock curve through $U_L$ and let $s_i(\cdot)$ be the corresponding shock speed function, with properties listed in Theorem 8.2.1. In particular, $U_L = W_i(0)$, $U_R = W_i(\tau)$ and the speed of the shock is $s = s_i(\tau)$. For definiteness, assume $\tau > 0$. When the shock violates the Liu $E$-condition, there are $\xi$ in $(0, \tau)$ with $s_i(\xi) < s$. The case where $s_i(\xi) < s$ for all $\xi \in (0, \tau)$ is simpler; so let us consider the more interesting situation where there is $\xi_0 \in (0, \tau)$ such that $s_i(\xi) > s$ for $\xi \in (0, \xi_0)$, $s_i(\xi_0) = s$, and $\dot{s}_i(\xi_0) < 0$. We identify the state $U_M = W_i(\xi_0)$. Since $U_L$ may be joined to both $U_M$ and $U_R$ by shocks of speed $s$, it follows that $U_M$ and $U_R$ can also be joined by a shock of speed $s$. Consequently, one may visualize the shock that joins $U_L$ and $U_R$ as a composite of two shocks, one that joins $U_L$ and $U_M$ and one that joins $U_M$ and $U_R$ , both propagating with the same speed $s$. The plan of the proof is to perform a perturbation that splits the original shock into two shocks, one with speed slightly lower than $s$, the other with speed slightly higher than $s$, and to show that the resulting wave fan has lower entropy production.

To that end, we construct $n + 2$ families of constant states $U_0(\varepsilon) = U_L$, $U_1(\varepsilon), \cdots, U_{i-1}(\varepsilon), U_*(\varepsilon), U_i(\varepsilon), \cdots, U_n(\varepsilon) = U_R$ , depending smoothly on the parameter $\varepsilon$ that takes values in a small neighborhood $(-a, a)$ of 0, having the following properties: $U_j(0) = U_L$ , for $j = 0, \cdots, i - 1$; $U_*(0) = U_M$ ; $U_j(0) = U_R$ , for $j = i, \cdots, n$. For $j = 1, \cdots, i - 1, i + 1, \cdots, n$, $U_{j-1}(\varepsilon)$ is joined to $U_j(\varepsilon)$ by a (not necessarily admissible) $j$-shock with speed $\sigma_j(\varepsilon)$; $U_{i-1}(\varepsilon)$ is joined to $U_*(\varepsilon)$ by an $i$-shock with speed $s_-(\varepsilon)$; and $U_*(\varepsilon)$ is joined to $U_i(\varepsilon)$ by an $i$-shock with speed $s_+(\varepsilon)$. The corresponding Rankine-Hugoniot conditions read

(9.7.8)

$$F(U_j(\varepsilon))-F(U_{j-1}(\varepsilon)) = \sigma_j(\varepsilon)[U_j(\varepsilon)-U_{j-1}(\varepsilon)]\,,\ j=1,\cdots,i-1,i+1,\cdots,n,$$

$$F(U_*(\varepsilon)) - F(U_{i-1}(\varepsilon)) = s_-(\varepsilon)[U_*(\varepsilon) - U_{i-1}(\varepsilon)], \tag{9.7.9}$$

$$F(U_i(\varepsilon)) - F(U_*(\varepsilon)) = s_+(\varepsilon)[U_i(\varepsilon) - U_*(\varepsilon)]. \tag{9.7.10}$$

In particular, $\sigma_j(0) = \lambda_j(U_L)$, for $j = 1, \cdots, i-1$; $s_-(0) = s_+(0) = s$; and $\sigma_j(0) = \lambda_j(U_R)$, for $j = i, \cdots, n$. The construction may be effected by the method employed in Section 9.3 for constructing solutions to the Riemann problem, with $j$-shock curves playing here the role of the wave fan curves $\Phi_j$ used there. The implicit function theorem here yields a one-parameter family of states (rather than a single state, as in Section 9.3), because of the additional degree of freedom, namely the intermediate state $U_*$. We choose the parametrization so that $s'_-(0) = -1$. Here and below the prime denotes differentiation with respect to $\varepsilon$.

Our first task is to show that, for $\varepsilon$ positive small, the constant states $U_0(\varepsilon), \cdots, U_{i-1}(\varepsilon), U_*(\varepsilon), U_i(\varepsilon), \cdots, U_n(\varepsilon)$, together with the connecting shocks, may be assembled into a wave fan $V_\varepsilon$. For that purpose it suffices to prove that $s_-(\varepsilon) < s_+(\varepsilon)$. We differentiate (9.7.8), (9.7.9), (9.7.10) with respect to $\varepsilon$ and set $\varepsilon = 0$ to get

$$[\mathrm{D}F(U_-) - \lambda_j(U_-)I][U_j'(0) - U_{j-1}'(0)] = 0, \quad j = 1, \cdots, i-1, \tag{9.7.11}$$

$$[\mathrm{D}F(U_+) - \lambda_j(U_+)I][U_j'(0) - U_{j-1}'(0)] = 0, \quad j = i+1, \cdots, n, \tag{9.7.12}$$

$$[\mathrm{D}F(U_M) - sI]U_*'(0) - [\mathrm{D}F(U_L) - sI]U_{i-1}'(0) = s_-'(0)[U_M - U_L], \tag{9.7.13}$$

$$[\mathrm{D}F(U_R) - sI]U_i'(0) - [\mathrm{D}F(U_M) - sI]U_*'(0) = s_+'(0)[U_R - U_M]. \tag{9.7.14}$$

Upon combining (9.7.13) with (9.7.14) we deduce

$$\begin{aligned} s_-'(0)[U_M - U_L] + s_+'(0)[U_R - U_M] \\ = [\mathrm{D}F(U_R) - sI]U_i'(0) - [\mathrm{D}F(U_L) - sI]U_{i-1}'(0). \end{aligned} \tag{9.7.15}$$

Both vectors on the left-hand side of (9.7.15) are nearly collinear to $R_i(U_L)$ and $R_i(U_R)$. On the other hand, by virtue of (9.7.11), $U_{i-1}'(0)$ lies in the span of $\{R_1(U_L), \cdots, R_{i-1}(U_L)\}$, while by account of (9.7.12), $U_i'(0)$ lies in the span of $\{R_{i+1}(U_R), \cdots, R_n(U_R)\}$. Therefore, the right-hand side of (9.7.15) is nearly orthogonal to $L_i(U_L)$ and $L_i(U_R)$. Let us set $\ell = |U_M - U_L|$. Recalling that $s_-'(0) = -1$, we deduce that in (9.7.15) both terms on the right-hand side are $o(\ell)$ and thus the two terms on the left-hand side must cancel each other out to leading order. In particular, $s_+'(0) > 0$ so that, for $\varepsilon$ positive small, $s_-(\varepsilon) < s < s_+(\varepsilon)$, which establishes the desired separation of shocks.

The total entropy production of the wave fan $V_\varepsilon$ is

$$\begin{aligned}\mathcal{P}(\varepsilon) = \sum_{j\neq i}\{q(U_j(\varepsilon)) - q(U_{j-1}(\varepsilon)) - \sigma_j(\varepsilon)[\eta(U_j(\varepsilon)) - \eta(U_{j-1}(\varepsilon))]\} \\ + q(U_*(\varepsilon)) - q(U_{i-1}(\varepsilon)) - s_-(\varepsilon)[\eta(U_*(\varepsilon)) - \eta(U_{i-1}(\varepsilon))] \\ + q(U_i(\varepsilon)) - q(U_*(\varepsilon)) - s_+(\varepsilon)[\eta(U_i(\varepsilon)) - \eta(U_*(\varepsilon))].\end{aligned} \tag{9.7.16}$$

To establish that for $\varepsilon$ positive small the wave fan $V_\varepsilon$ dissipates entropy at a higher rate than $V$, it suffices to show that $\mathcal{P}'(0) < 0$. The derivative, with respect to $\varepsilon$, of the summation term on the right-hand side of (9.7.16), evaluated at $\varepsilon = 0$, reduces to

$$\begin{aligned}\sum_{j=1}^{i-1}[\mathrm{D}q(U_L) - \lambda_j(U_L)\mathrm{D}\eta(U_L)][U_j'(0) - U_{j-1}'(0)] \\ + \sum_{j=i+1}^{n}[\mathrm{D}q(U_R) - \lambda_j(U_R)\mathrm{D}\eta(U_R)][U_j'(0) - U_{j-1}'(0)],\end{aligned} \tag{9.7.17}$$

which vanishes by virtue of (7.4.1), (9.7.11) and (9.7.12). We evaluate, at $\varepsilon = 0$, the derivative of the remaining terms on the right-hand side of (9.7.16). After a straightforward calculation, making use of (7.4.1), (9.7.13) and (9.7.14), we conclude

$$\begin{aligned}\mathcal{P}'(0) = -s_-'(0)[\eta(U_M) - \eta(U_L) - \mathrm{D}\eta(U_M)(U_M - U_L)] \\ - s_+'(0)[\eta(U_R) - \eta(U_M) - \mathrm{D}\eta(U_M)(U_R - U_M)] \\ + [\mathrm{D}\eta(U_M) - \mathrm{D}\eta(U_L)][\mathrm{D}F(U_L) - sI]U_{i-1}'(0) \\ + [\mathrm{D}\eta(U_R) - \mathrm{D}\eta(U_M)][\mathrm{D}F(U_R) - sI]U_i'(0).\end{aligned} \tag{9.7.18}$$

We examine the four terms on the right-hand side of (9.7.18), in the light of the scaling analysis of (9.7.15), discussed earlier in the proof. Considering that $\eta$ is convex, $s_-'(0) = -1$, $s_+'(0) > 0$ and $|U_M - U_L| = \ell$, it follows that the first term is majorized by $-\beta\ell^2$ and the second term is majorized by $-\beta\ell|U_R - U_M|$, for some $\beta > 0$. On the other hand, the third term is $o(\ell)\ell$ and the fourth term is $o(\ell)|U_R - U_M|$. Consequently, for $\ell$ sufficiently small, $\mathcal{P}'(0) < 0$. This completes the proof.

Beyond the range of shocks of moderate strength, the entropy rate admissibility criterion is no longer generally equivalent to the Liu $E$-condition. The issue has been discussed in detail (references in Section 9.12) in the context of the system (7.1.5) of adiabatic thermoelasticity for a polytropic gas with internal energy $\varepsilon = e^s u^{1-\gamma}$, which induces, by (7.1.6), pressure $p = -\sigma = (\gamma - 1)e^s u^{-\gamma}$. The designated entropy-entropy flux pair is given by (7.4.9), namely $(-s, 0)$. For this system, with 1- and 3-characteristic families that are genuinely nonlinear and 2-characteristic family that is linearly degenerate, the Lax $E$-condition and the Liu $E$-condition are equivalent.

It has been shown that when $\gamma \geq 5/3$ a wave fan, of arbitrary strength, satisfies the entropy rate admissibility criterion if and only if its shocks satisfy the Lax $E$-condition. The reader should note that 5/3 is the value for the adiabatic exponent $\gamma$ predicted by the kinetic theory in the case of a monatomic ideal gas.

When $\gamma < 5/3$ (polyatomic gases), the situation is different. Consider a wave fan comprising three constant states $(u_L, v_L, s_L)$, $(u_M, v_M, s_M)$ and $(u_R, v_R, s_R)$, where the first two are joined by a stationary 2-contact discontinuity, while the second and the third are joined by a 3-rarefaction wave. In particular, we must have $v_M = v_L,\ p(u_M, s_M) = p(u_L, s_L),\ s_M = s_R$, and $z(u_M, v_M, s_M) = z(u_R, v_R, s_R)$, where $z(u, v, s)$ denotes the second 3-Riemann invariant listed in (7.3.4). The total entropy production of this wave fan is of course zero. For $u_R/u_L$ in a certain range, there is a second wave fan with the same end-states, which comprises four constant states $(u_L, v_L, s_L)$, $(u_1, v_1, s_1)$, $(u_2, v_2, s_2)$ and $(u_R, v_R, s_R)$, where the first two are joined by a 1-shock that satisfies the Lax $E$-condition, the second is joined to the third by a 2-contact discontinuity, while the last two are joined by a 3-shock that violates the Lax $E$-condition. It turns out that when $u_M/u_L$ is not too large, i.e., the contact discontinuity is not too strong, the total entropy production of the second wave fan is positive, and hence the first wave fan has lower entropy rate. By contrast, when $u_M/u_L$ is sufficiently large, the total entropy production of the second wave fan is negative and so the first wave fan no longer satisfies the entropy rate criterion.

Similar issues arise for systems that are not strictly hyperbolic. Let us consider our model system (7.2.11). Recall the two wave fans with the same end-states $(1, 0)$ and $(a, 0)$, $a \in (-\frac{1}{2}, 0)$, described in Section 9.6: The first one comprises the states $(1, 0)$ and $(a, 0)$, joined by an overcompressive shock of speed $1+a+a^2$. The second comprises three states, $(1, 0)$, $(-1, 0)$ and $(a, 0)$, where the first two are joined by a 1-contact discontinuity of speed 1, while the second is joined to the third by a 2-shock of speed $1 - a + a^2$. If we designate the entropy-entropy flux pair

$$\eta = \tfrac{1}{2}(u^2 + v^2), \qquad q = \tfrac{3}{4}(u^2 + v^2)^2, \tag{9.7.19}$$

the entropy production of the overcompressive shock is $\frac{1}{4}(a^2 - 1)(1 - a)^2$ while the entropy production of the second wave fan is $\frac{1}{4}(a^2 - 1)(1 + a)^2$. Thus the entropy rate criterion favors the overcompressive shock, even though, as we saw in Section 8.6, this is incompatible with the stable shock profile condition. The reader should bear in mind, however, that these conclusions are tied to our selections for artificial viscosity and entropy. Whether (8.6.4) is the proper dissipative form and (9.7.19) is the natural entropy-entropy flux pair for (7.2.11) may only be decided when this system is considered in the context of some physical model.

In addition to providing a wave fan admissibility criterion, the entropy rate principle, Definition 9.7.1, suggests an alternative method for constructing solutions to the Riemann problem (9.1.1), (9.1.12), namely by minimizing the functional $\dot{\mathcal{H}}_V$, defined by (9.7.3), over the set of $BV$ solutions $V(\xi)$ of the boundary-value problem

(9.1.3), (9.1.13). In order to simplify the analysis, we shall perform the minimization here over the more narrow class of $BV$ solutions $V(\xi)$ of (9.1.3), (9.1.13) with *monotone $i$-wave fans;* that is, for any $i = 1, \cdots, n$, if $[\zeta_i, \xi_i]$ is the interval corresponding to the $i$-wave fan, then $[V(\xi+) - V(\zeta-)] \cdot R_i(U_L)$ does not change sign, for all $\zeta_i \leq \zeta \leq \xi \leq \xi_i$. This additional restriction is actually superfluous, as it can be shown that any minimizer of $\dot{\mathcal{H}}_V$, over all $BV$ solutions of (9.1.3), (9.1.13), has necessarily monotone $i$-wave fans. It should also be noted that all solutions to the Riemann problem constructed in Sections 9.4 and 9.5 have monotone $i$-wave fans. The same holds true for solutions involved in the proof of Theorem 9.7.4.

For arbitrary strictly hyperbolic systems (9.1.1), when $|U_R - U_L|$ is sufficiently small, there are solutions to the Riemann problem (9.1.1), (9.1.12) with monotone $i$-wave fans. For example, one may easily synthesize such a solution, in which each $i$-wave fan is just a single (not necessarily admissible) $i$-shock, by simply repeating the construction described in Section 9.3, with $i$-shock curves $W_i$ in the place of the $i$-wave fan curve $\Phi_i$ used there. One should also recall that if $V(\xi)$ is a solution of (9.1.3), (9.1.13) with small oscillation and if $[\zeta_i, \xi_i]$ is the interval corresponding to the $i$-wave fan, then for any $\zeta_i \leq \zeta \leq \xi \leq \xi_i$, the vector $V(\xi+) - V(\zeta-)$ is nearly parallel to $R_i(U_L)$. Hence, when $|U_R - U_L|$ is sufficiently small, the family of solutions to (9.1.3), (9.1.13) with monotone $i$-wave fans have uniformly bounded total variation over $(-\infty, \infty)$. Then Helly's theorem yields the compactness that induces the existence of a minimizer to the functional $\dot{\mathcal{H}}_V = \int \eta(V(\xi))d\xi$. This together with Theorem 9.7.4 implies

**9.7.5 Theorem.** *Consider any strictly hyperbolic system* (9.1.1) *that is endowed with a uniformly convex entropy $\eta(U)$. When $|U_R - U_L|$ is sufficiently small, there exists a solution $U(x,t) = V(x/t)$ of the Riemann problem* (9.1.1), (9.1.12) *where $V(\xi)$ minimizes the entropy rate $\dot{\mathcal{H}}_V$, or equivalently the total entropy production $\mathcal{P}_V$, over all wave fans with monotone $i$-wave fans and end-states $U_L$, $U_R$. Furthermore, this solution coincides with the unique solution established by Theorem* 9.5.1.

## 9.8 Viscous Wave Fans

The viscous shock admissibility criterion, introduced in Section 8.6, characterizes admissible shocks for the hyperbolic system of conservation laws (9.1.1) as $\mu \downarrow 0$ limits of traveling wave solutions of the associated dissipative system (8.6.1). The aim here is to extend this principle from single shocks to general wave fans. The difficulty is that, in contrast to (9.1.1), the system (8.6.1) is not invariant under uniform stretching of the space-time coordinates and thus it does not possess traveling wave fans as solutions. As a remedy, it has been proposed that in the place of (8.6.1) one should employ a system with time-varying viscosity,

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = \mu t \partial_x^2 U(x,t), \tag{9.8.1}$$

which is invariant under the transformation $(x,t) \mapsto (\alpha x, \alpha t)$. It is easily seen that $U = V_\mu(x/t)$ is a self-similar solution of (9.8.1) if and only if $V_\mu(\xi)$ satisfies the ordinary differential equation

$$\mu \ddot{V}_\mu(\xi) = \dot{F}(V_\mu(\xi)) - \xi \dot{V}_\mu(\xi). \tag{9.8.2}$$

A self-similar solution $U = V(x/t)$ of (9.1.1) is said to satisfy the *viscous wave fan admissibility criterion* if $V$ is the almost everywhere limit, as $\mu \downarrow 0$, of a uniformly bounded family of solutions $V_\mu$ of (9.8.2).

In addition to serving as a test of admissibility, the viscous wave fan criterion suggests an alternative approach for constructing solutions to the Riemann problem (9.1.1), (9.1.12). Towards that end, one has to show that for any fixed $\mu > 0$ there exists some solution $V_\mu(\xi)$ of (9.8.2) on $(-\infty, \infty)$, with boundary conditions

$$V_\mu(-\infty) = U_L\,, \quad V_\mu(+\infty) = U_R\,, \tag{9.8.3}$$

and then prove that the family $\{V_\mu(\xi) : 0 < \mu < 1\}$ has uniformly bounded variation on $(-\infty, \infty)$. In that case, by Helly's theorem (cf. Section 1.7), a convergent sequence $\{V_{\mu_m}\}$ may be extracted, with $\mu_m \downarrow 0$ as $m \to \infty$, whose limit $V$ induces the solution $U(x, t) = V(x/t)$ to the Riemann problem.

The above program has been implemented successfully under a variety of conditions. One may solve the Riemann problem under quite general data $U_L$ and $U_R$ albeit for special systems, most notably for pairs of conservation laws. Alternatively, one may treat general systems but only in the context of weak waves, requiring that $|U_R - U_L|$ be sufficiently small. Let us consider this last situation first. The analysis is lengthy and technical so only the main ideas shall be outlined. For the details, the reader may consult the references cited in Section 9.12.

The crucial step is to establish a priori bounds on the total variation of $V_\mu(\xi)$ over $(-\infty, \infty)$, independent of $\mu$. To prepare the ground for systems, let us begin with the scalar conservation law (7.1.2). Setting $\lambda(u) = f'(u)$ and $\dot{V}_\mu(\xi) = a(\xi)$, we write (9.8.2) in the form

$$\mu \dot{a} + [\xi - \lambda(V_\mu(\xi))]a = 0. \tag{9.8.4}$$

The solution of (9.8.4) is $a(\xi) = \tau\phi(\xi)$, where

$$\phi(\xi) = \frac{\exp[-\frac{1}{\mu}g(\xi)]}{\int_{-\infty}^{\infty} \exp[-\frac{1}{\mu}g(\zeta)]d\zeta}\,, \tag{9.8.5}$$

$$g(\xi) = \int_s^\xi [\zeta - \lambda(V_\mu(\zeta))]d\zeta. \tag{9.8.6}$$

The lower limit of integration $s$ is selected so that $g(\xi) \geq 0$ for all $\xi$ in $(-\infty, \infty)$. The amplitude $\tau$ is determined with the help of the boundary conditions, that is $V_\mu(-\infty) = u_L$, $V_\mu(\infty) = u_R$, $\tau = u_R - u_L$. From (9.8.5) it follows that the $L^1$ norm of $a(\xi)$ is bounded, uniformly in $\mu$, and so the family $\{V_\mu : 0 < \mu < 1\}$ has uniformly bounded variation on $(-\infty, \infty)$.

Turning now to general strictly hyperbolic systems (9.1.1), we realize $V_\mu(\xi)$ as the composition of wave fans associated with distinct characteristic families, by writing

(9.8.7) $$\dot{V}_\mu(\xi) = \sum_{j=1}^{n} a_j(\xi) R_j(V_\mu(\xi)).$$

We substitute $\dot{V}_\mu$ from (9.8.7) into (9.8.2). Upon multiplying the resulting equation, from the left, by $L_i(V_\mu(\xi))$, we deduce

(9.8.8) $$\mu \dot{a}_i + [\xi - \lambda_i(V_\mu(\xi))]a_i = \mu \sum_{j,k=1}^{n} \beta_{ijk}(V_\mu(\xi)) a_j a_k \ ,$$

where

(9.8.9) $$\beta_{ijk}(U) = -L_i(U) \mathrm{D} R_j(U) R_k(U).$$

In (9.8.8), the left-hand side coincides with the left-hand side of (9.8.4), for the scalar conservation law, while the right-hand side accounts for the interactions of distinct characteristic families. The reader should notice the analogy between (9.8.8) and (7.8.6). It should also be noted that when our system is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants, $a_i(\xi) = \dot{w}_i(V_\mu(\xi))$. In that case, as shown in Section 7.3, for $j \neq k$, $\mathrm{D}R_j R_k$ lies in the span of $\{R_j, R_k\}$ and so (9.8.9) implies $\beta_{ijk} = 0$ when $i \neq j \neq k \neq i$. For special systems, such as (7.3.18), with coinciding shock and rarefaction wave curves, $\mathrm{D}R_j R_j$ is collinear to $R_j$ and so $\beta_{ijk} = 0$ even when $i \neq j = k$ so that the equations in (9.8.8) decouple. In general, the thrust of the analysis is to demonstrate that in the context of solutions with small oscillation, i.e., $a_i$ small, the effect of interactions, of quadratic order, will be even smaller.

The solution of (9.8.8) may be partitioned into

(9.8.10) $$a_i(\xi) = \tau_i \phi_i(\xi) + \theta_i(\xi),$$

where

(9.8.11) $$\phi_i(\xi) = \frac{\exp[-\frac{1}{\mu} g_i(\xi)]}{\int_{-\infty}^{\infty} \exp[-\frac{1}{\mu} g_i(\zeta)] d\zeta} \ ,$$

(9.8.12) $$g_i(\xi) = \int_{s_i}^{\xi} [\zeta - \lambda_i(V_\mu(\zeta))] d\zeta,$$

and $\theta_i(\xi)$ satisfies the equation

(9.8.13) $$\mu \dot{\theta}_i + [\xi - \lambda_i(V_\mu(\xi))]\theta_i = \mu \sum_{j,k=1}^{n} \beta_{ijk}(V_\mu(\xi))[\tau_j \phi_j(\xi) + \theta_j][\tau_k \phi_k(\xi) + \theta_k].$$

The differential equations (9.8.13) may be transformed into an equivalent system of integral equations by means of the variation of parameters formula:

(9.8.14)
$$\theta_i(\xi) = \phi_i(\xi) \int_{c_i}^{\xi} \phi_i^{-1}(\zeta)\beta_{ijk}(V_\mu(\zeta))[\tau_j\phi_j(\zeta) + \theta_j(\zeta)][\tau_k\phi_k(\zeta) + \theta_k(\zeta)]d\zeta .$$

Careful estimation shows that

(9.8.15)
$$|\theta_i(\xi)| \le c(\tau_1^2 + \cdots + \tau_n^2) \sum_{j=1}^{n} \phi_j(\xi),$$

which verifies that, in (9.8.10), $\theta_i$ is subordinate to $\tau_i\phi_i$ , i.e., the characteristic families decouple to leading order.

It can be shown, by means of a contraction argument, that for any fixed $(\tau_1, \cdots, \tau_n)$ in a small neighborhood of the origin, there exists some solution $V_\mu(\xi)$ of (9.8.2) on $(-\infty, \infty)$, which satisfies (9.8.7), (9.8.10) and (9.8.15). To solve the boundary-value problem (9.8.2), (9.8.3), the $(\tau_1, \cdots, \tau_n)$ have to be selected so that

(9.8.16)
$$\sum_{j=1}^{n} \int_{-\infty}^{\infty} [\tau_j\phi_j(\xi) + \theta_j(\xi)]R_j(V_\mu(\xi))d\xi = U_R - U_L .$$

It has been proved that (9.8.16) admits a unique solution $(\tau_1, \cdots, \tau_n)$, at least when $|U_R - U_L|$ is sufficiently small. The result is summarized in the following

**9.8.1 Theorem.** *Assume the system* (9.1.1) *is strictly hyperbolic on* $\mathcal{O}$ *and fix any state* $U_L \in \mathcal{O}$. *There is* $\delta > 0$ *such that for any* $U_R \in \mathcal{O}$ *with* $|U_R - U_L| < \delta$ *and every* $\mu > 0$, *the boundary-value problem* (9.8.2), (9.8.3) *possesses a solution* $V_\mu(\xi)$, *which admits the representation* (9.8.7), (9.8.10) *with* $(\tau_1, \cdots, \tau_n)$ *close to the origin and* $\theta_i$ *obeying* (9.8.15). *Moreover, the family* $\{V_\mu(\xi) : 0 < \mu < 1\}$ *of solutions has uniformly bounded (and small) total variation on* $(-\infty, \infty)$. *In particular, one may extract a sequence* $\{V_{\mu_m}(\xi)\}$, *with* $\mu_m \downarrow 0$ *as* $m \to \infty$, *which converges, boundedly almost everywhere, to a function* $V(\xi)$ *such that the wave fan* $U = V(x/t)$ *solves the Riemann problem* (9.1.1), (9.1.12).

Careful analysis of the process that generates $V(\xi)$ as the limit of the sequence $\{V_{\mu_m}(\xi)\}$ reveals that $V(\xi)$ has the structure described in Theorem 9.3.1. Furthermore, for any point $\xi$ of jump discontinuity of $V$, $V(\xi-)$, on the left, is connected to $V(\xi+)$, on the right, by a viscous shock profile, and so the viscous shock admissibility criterion is satisfied (with $B = I$), as discussed in Section 8.6. In particular, any shock of $V$ satisfies the Liu $E$-condition and thus $V$ coincides with the unique solution established by Theorem 9.5.1.

The construction of the solution $V_\mu(\xi)$ to the boundary-value problem (9.8.2), (9.8.3) and the derivation of the bound on the total variation of the family $\{V_\mu\}$, asserted by Theorem 9.8.1, do not depend on the fact that the system (9.8.1) is conservative but apply equally well to any system

(9.8.17) $$\mu \ddot{V}_\mu(\xi) = A(V_\mu(\xi))\dot{V}_\mu(\xi) - \xi \dot{V}_\mu(\xi),$$

so long as the matrix $A(U)$ has real distinct eigenvalues. If $V(\xi)$ is the $\mu \downarrow 0$ limit of $V_\mu(\xi)$, the function $U(x,t) = V(x/t)$ may be interpreted as a solution of the Riemann problem for the strictly hyperbolic, nonconservative system

(9.8.18) $$\partial_t U + A(U)\partial_x U = 0,$$

even though it does not necessarily satisfy that system in the sense of distributions.

Viscous wave fans induce an alternative, implicit construction of wave fan curves for general strictly hyperbolic systems (9.1.1), without any requirement of piecewise genuine nonlinearity.

To trace the forward $i$-wave curve that emanates from some fixed state $\bar{U}$, assume that a state $\hat{U}$, on the right, is connected to $\bar{U}$, on the left, by an $i$-wave fan of moderate strength. Suppose this wave fan is the $\mu \downarrow 0$ limit of a family of viscous wave fans $V_\mu(\xi)$. Thus $V_\mu$ is defined for $\xi$ in a small neighborhood of $\bar{\xi} = \lambda_i(\bar{U})$, it takes values near $\bar{U}$, and $\mu \dot{V}_\mu(\xi)$ is small. We stretch the domain by rescaling the variable, $\xi = \mu\zeta$. We also rescale the $a_j$ in the expansion (9.8.7) by setting $w_j = \mu a_j$, and assemble the vector $W = (w_1, \cdots, w_n)$. Then we may recast (9.8.7), (9.8.8) into an autonomous first order system

(9.8.19) $$\begin{cases} V' = \sum_{j=1}^{n} w_j R_j(V) \\ w_j' = [\lambda_j(V) - \xi]w_j + \sum_{k,\ell=1}^{n} \beta_{jk\ell}(V) w_k w_\ell\,, \qquad j = 1, \cdots, n \\ \xi' = \mu \\ \mu' = 0, \end{cases}$$

where the prime denotes differentiation with respect to $\zeta$.

Linearization of (9.8.19) about the equilibrium point $V = \bar{U}$, $W = 0$, $\xi = \lambda_i(\bar{U})$, $\mu = 0$ yields the system

(9.8.20) $$\begin{cases} V' = \sum_{j=1}^{n} w_j R_j(\bar{U}) \\ w_j' = [\lambda_j(\bar{U}) - \xi]w_j\,, \qquad j = 1, \cdots, n \\ \xi' = \mu \\ \mu' = 0. \end{cases}$$

The center subspace $\mathcal{N}$ of this system consists of all vectors $(V, W, \xi, \mu) \in I\!R^{2n+2}$ with $w_j = 0$ for $j \neq i$, and therefore has dimension $n+3$. By the center manifold theorem, any solution of (9.8.19) that sojourns in the vicinity of the above equilibrium point must lie on a $(n+3)$-dimensional manifold $\mathcal{M}$, which is tangential to $\mathcal{N}$ at the equilibrium point, is invariant under the flow generated by (9.8.19), and admits the local representation

$$w_j = \varphi_j(V, \omega, \xi; \mu), \qquad j \neq i, \tag{9.8.21}$$

where $\omega$ stands for $w_i$. By the theory of skew-product flows, the functions $\varphi_j$ can be selected so that $\varphi_j(V, 0, \xi; \mu) = 0$, for all $V$, $\xi$ and $\mu$ close to $\bar{U}$, $\lambda_i(\bar{U})$ and 0. We may thus set

$$w_j = \omega\psi_j(V, \omega, \xi; \mu), \qquad j \neq i, \tag{9.8.22}$$

where $\psi_j(\bar{U}, 0, \lambda_i(\bar{u}); 0) = 0$, since $\mathcal{M}$ is tangential to $\mathcal{N}$ at the equilibrium point. We also introduce a new variable $\tau$ such that

$$\frac{d}{d\tau} = \frac{1}{\omega}\frac{d}{d\zeta}\,. \tag{9.8.23}$$

In order to see how the components $(V_\mu, \omega_\mu, \xi_\mu)$ of our solution evolve on $\mathcal{M}$ as functions of $\tau$, we combine (9.8.19), (9.8.22) and (9.8.23) to deduce

$$\frac{dV_\mu}{d\tau} = P_\mu(V_\mu, \omega_\mu, \xi_\mu), \tag{9.8.24}$$

$$\frac{d\omega_\mu}{d\tau} = p_\mu(V_\mu, \omega_\mu, \xi_\mu) - \xi_\mu\,, \tag{9.8.25}$$

where we have set

$$P_\mu(V, \omega, \xi) = R_i(V) + \sum_{j\neq i}\psi_j(V, \omega, \xi; \mu)R_j(V), \tag{9.8.26}$$

$$p_\mu(V, \omega, \xi) = \lambda_i(V) + \sum_{k,\ell=1}^{n}\omega\beta_{ik\ell}(V)\psi_k(V, \omega, \xi; \mu)\psi_\ell(V, \omega, \xi; \mu). \tag{9.8.27}$$

In particular, $P_0(\bar{U}, 0, \lambda_i(\bar{U})) = R_i(\bar{U}),\ p_0(\bar{U}, 0, \lambda_i(\bar{U})) = \lambda_i(\bar{U})$.

To derive an equation for $\xi_\mu(\tau)$, we note that (9.8.23) together with $(9.8.19)_3$ yield $d\xi_\mu/d\tau = \xi'_\mu/\omega = \mu/\omega$. We differentiate this relation with respect to $\tau$ and use (9.8.25) to get

$$\mu\frac{d^2\xi_\mu}{d\tau^2} = -\left(\frac{d\xi_\mu}{d\tau}\right)^2[p_\mu(V_\mu, \omega_\mu, \xi_\mu) - \xi_\mu]. \tag{9.8.28}$$

As $\mu \downarrow 0$, $(V_\mu, \omega_\mu, \xi_\mu)$ converge uniformly to $(V, \omega, \xi)$. In particular, we have $V(0) = \bar{U},\ \omega(0) = 0$ and $V(s) = \hat{U}$, for some, say positive, small $s$. By virtue of

(9.8.28), $[0, s]$ is the union of an at most countable family of $\tau$-intervals, associated with shocks, over which $d\xi/d\tau = 0$, and $\tau$-intervals, associated with rarefaction waves, over which $\xi = p_0(V, \omega, \xi)$. Furthermore, at points of transition from shock to rarefaction (or rarefaction to shock) $d^2\xi_\mu/d\tau^2$ should be nonnegative (or nonpositive). It then follows that

$$\xi(\tau) = \frac{dg}{d\tau}(\tau), \qquad 0 \le \tau \le s, \tag{9.8.29}$$

where $g$ is the convex envelope, over $[0, s]$, of the function

$$f(\tau) = \int_0^\tau p_0(V(\sigma), \omega(\sigma), \xi(\sigma))d\sigma, \qquad 0 \le \tau \le s, \tag{9.8.30}$$

i.e., $g(\tau) = \inf\{\theta_1 f(\tau_1) + \theta_2 f(\tau_2) : \theta_1 \ge 0,\ \theta_2 \ge 0,\ \theta_1 + \theta_2 = 1, 0 \le \tau_1 \le \tau_2 \le s,$ $\theta_1\tau_1 + \theta_2\tau_2 = \tau\}$. Then (9.8.24) and (9.8.25) yield

$$V(t) = \bar{U} + \int_0^\tau P_0(V(\sigma), \omega(\sigma), \xi(\sigma))d\sigma, \qquad 0 \le \tau \le s, \tag{9.8.31}$$

$$\omega(\tau) = f(\tau) - g(\tau), \qquad 0 \le \tau \le s. \tag{9.8.32}$$

It can be shown that, once $P_0(V, \omega, \xi)$ and $p_0(V, \omega, \xi)$ are specified, the system of equations (9.8.29), (9.8.31) and (9.8.32) can be solved by Picard iteration to yield the functions $V(\tau)$, $\omega(\tau)$ and $\xi(\tau)$, over $[0, s]$, for any small positive $s$. The treatment of negative $s$ is similar, except that now $g$ is the concave envelope of $f$ over $[s, 0]$. Hence, these equations provide an implicit representation of the $i$-wave fan curve $\Phi_i$ emanating from $\bar{U}$, by setting $\Phi_i(s; \bar{U}) = V(s)$. By its definition through (9.8.23), $\tau$ is nearly equal to the projection of $V - \bar{U}$ on $R_i$. Thus, the above construction of the $i$-wave fan curve closely resembles the construction of the wave fan for the scalar conservation law described at the opening of Section 9.5.

Our next project is to construct, by the method of viscous wave fans, solutions to the Riemann problem for systems of just two conservation laws,

$$\begin{cases} \partial_t u + \partial_x f(u, v) = 0 \\ \partial_t v + \partial_x g(u, v) = 0, \end{cases} \tag{9.8.33}$$

albeit under unrestricted initial data

$$\big(u(x, 0), v(x, 0)\big) = \begin{cases} \big(u_L, v_L\big), & x < 0 \\ \big(u_R, v_R\big), & x > 0. \end{cases} \tag{9.8.34}$$

The crucial restriction will be that $f_v$ and $g_u$ have the same sign, say for definiteness

$$f_v(u, v) < 0, \quad g_u(u, v) < 0, \qquad \text{for all } (u, v). \tag{9.8.35}$$

In particular, the system is strictly hyperbolic. Coupled symmetric systems and the system (7.1.8) of isentropic elastodynamics are typical representatives of this class. The analysis will demonstrate how delta shocks may emerge as "concentrations" in the limit of viscous profiles.

Equations (9.8.2), (9.8.3) here take the form

$$\begin{cases} \mu \ddot{u}_\mu(\xi) = \dot{f}\big(u_\mu(\xi), v_\mu(\xi)\big) - \xi \dot{u}_\mu(\xi) \\ \mu \ddot{v}_\mu(\xi) = \dot{g}\big(u_\mu(\xi), v_\mu(\xi)\big) - \xi \dot{v}_\mu(\xi), \end{cases} \tag{9.8.36}$$

$$\big(u_\mu(-\infty), v_\mu(-\infty)\big) = \big(u_L, v_L\big), \quad \big(u_\mu(\infty), v_\mu(\infty)\big) = \big(u_R, v_R\big). \tag{9.8.37}$$

The importance of the assumption (9.8.35) stems from the following

**9.8.2 Lemma.** *Let* $\big(u_\mu(\xi), v_\mu(\xi)\big)$ *be a solution of* (9.8.36),(9.8.37) *on* $(-\infty, \infty)$. *Then one of the following holds:*

*(a) Both* $u_\mu(\xi)$ *and* $v_\mu(\xi)$ *are constant on* $(-\infty, \infty)$.
*(b)* $u_\mu(\xi)$ *is strictly increasing (or decreasing), with no critical points on* $(-\infty, \infty)$*;* $v_\mu(\xi)$ *has at most one critical point on* $(-\infty, \infty)$*, which is necessarily a maximum (or minimum).*
*(c)* $v_\mu(\xi)$ *is strictly increasing (or decreasing), with no critical points on* $(-\infty, \infty)$*;* $u_\mu(\xi)$ *has at most one critical point on* $(-\infty, \infty)$*, which is necessarily a maximum (or minimum).*

**Proof.** Notice that $\dot{u}_\mu(\xi_0) = 0$ and $\ddot{u}_\mu(\xi_0) = 0$ imply $\dot{v}_\mu(\xi_0) = 0$; similarly $\dot{v}_\mu(\xi_0) = 0$ and $\ddot{v}_\mu(\xi_0) = 0$ imply $\dot{u}_\mu(\xi_0) = 0$. Therefore, by uniqueness of solutions to the initial-value problem for ordinary differential equations, if either one of $u_\mu(\xi)$, $v_\mu(\xi)$ has degenerate critical points, then both these functions must be constant on $(-\infty, \infty)$.

Turning to nondegenerate critical points, note that $\dot{u}_\mu(\xi_0) = 0$ and $\ddot{u}_\mu(\xi_0) < 0$ (or $\ddot{u}_\mu(\xi_0) > 0$) imply $\dot{v}_\mu(\xi_0) > 0$ (or $\dot{v}_\mu(\xi_0) < 0$); similarly, $\dot{v}_\mu(\xi_0) = 0$ and $\ddot{v}_\mu(\xi_0) < 0$ (or $\ddot{v}_\mu(\xi_0) > 0$) imply $\dot{u}_\mu(\xi_0) > 0$ (or $\dot{u}_\mu(\xi_0) < 0$).

Suppose now $v_\mu(\xi)$ has more than one nondegenerate critical points and pick two consecutive ones, a maximum at $\xi_1$ and a minimum at $\xi_2$. For definiteness, assume $\xi_1 < \xi_2$. Then $\dot{v}_\mu(\xi_1) = 0$, $\ddot{v}_\mu(\xi_1) < 0$, $\dot{v}_\mu(\xi_2) = 0$, $\ddot{v}_\mu(\xi_2) > 0$ and $\dot{v}_\mu(\xi) < 0$ for $\xi \in (\xi_1, \xi_2)$. Hence, $\dot{u}_\mu(\xi_1) > 0$ and $\dot{u}_\mu(\xi_2) < 0$. Therefore, there exists $\xi_0$ in $(\xi_1, \xi_2)$ such that $\dot{u}_\mu(\xi_0) = 0$ and $\ddot{u}_\mu(\xi_0) < 0$. But this implies $\dot{v}_\mu(\xi_0) > 0$, which is a contradiction. The case $\xi_1 > \xi_2$ also leads to a contradiction. The same argument shows that $u_\mu(\xi)$ may have at most one nondegenerate critical point.

Finally, suppose both $u_\mu(\xi)$ and $v_\mu(\xi)$ have nondegenerate critical points, say at $\xi_1$ and $\xi_2$, respectively. For definiteness, assume $\xi_1 < \xi_2$ and $\xi_2$ is a maximum of $v_\mu(\xi)$. Then $\dot{v}_\mu(\xi) > 0$ for $\xi \in (-\infty, \xi_2)$ and $\dot{v}_\mu(\xi_2) = 0$, $\ddot{v}_\mu(\xi_2) < 0$. This implies $\dot{u}_\mu(\xi_2) > 0$. But then, $\xi_1$ is necessarily a minimum of $u_\mu(\xi)$, with $\dot{u}_\mu(\xi_1) = 0$,

$\ddot{u}_\mu(\xi_1) > 0$. This in turn implies $\dot{v}_\mu(\xi_1) < 0$, which is a contradiction. All other possible combinations lead to similar contradictions. The proof is complete.

Because of the very special configuration of the graphs of $u_\mu(\xi)$ and $v_\mu(\xi)$, it is relatively easy to establish existence of solutions to (9.8.36), (9.8.37). Indeed, it turns out that for that purpose it is sufficient to bound *a priori* the unique "peak" attained by $u_\mu(\xi)$ or $v_\mu(\xi)$, in terms of the given data $(u_L, v_L)$, $(u_R, v_R)$, and the parameter $\mu$. The reader may find the derivation of such estimates, and resulting proof of existence, in the literature cited in Section 9.12, under the assumption that either the growth of $f(u, v)$ and $g(u, v)$ is restricted by

$$|f(u,v)| \leq h(v)(1+|u|)^p, \qquad |g(u,v)| \leq h(u)(1+|v|)^p, \tag{9.8.38}$$

where $h$ is a continuous function and $p < 2$, or the system (9.8.33) is endowed with an entropy $\eta(u, v)$, with the property that the eigenvalues of the Hessian matrix $D^2\eta(u, v)$ are bounded from below by $(1 + |u|)^{-p}(1 + |v|)^{-p}$, for some $p < 3$. The first class of systems contains in particular (7.1.8), and the second class includes all symmetric systems.

Assuming $(u_\mu, v_\mu)$ exist, we pass to the limit, as $\mu \downarrow 0$, in order to obtain solutions of the Riemann problem. For that purpose, we shall need estimates independent of $\mu$. Let us consider, for definiteness, the case where $v_\mu(\xi)$ is strictly increasing on $(-\infty, \infty)$, while $u_\mu(\xi)$ is strictly increasing on $(-\infty, \xi_\mu)$, attains its maximum at $\xi_\mu$, and is strictly decreasing on $(\xi_\mu, \infty)$. All other possible configurations may be treated in a similar manner.

Let us set $\bar{u} = \max\{u_L, u_R\}$ and identify the points $\xi_\ell \in (-\infty, \xi_\mu) \cup \{-\infty\}$ and $\xi_r \in (\xi_\mu, \infty) \cup \{\infty\}$ with the property $u(\xi_\ell) = u(\xi_r) = \bar{u}$. For any interval $(a, b) \subset (-\infty, \infty)$, using $(9.8.36)_1$ and (9.8.35), we deduce

$$\begin{aligned} \int_a^b \left[u_\mu(\xi) - \bar{u}\right] d\xi &\leq \int_{\xi_\ell}^{\xi_r} \left[u_\mu(\xi) - \bar{u}\right] d\xi = -\int_{\xi_\ell}^{\xi_r} \xi \dot{u}_\mu(\xi) d\xi \\ &= \mu\dot{u}_\mu(\xi_r) - \mu\dot{u}_\mu(\xi_\ell) - f(\bar{u}, v_\mu(\xi_r)) + f(\bar{u}, v_\mu(\xi_\ell)) \\ &\leq f(\bar{u}, v_L) - f(\bar{u}, v_R). \end{aligned} \tag{9.8.39}$$

By virtue of (9.8.39), there is a sequence $\{\mu_k\}$, $\mu_k \downarrow 0$ as $k \to 0$, such that $\{\xi_{\mu_k}\}$ converges to some point $\xi_0 \in (-\infty, \infty) \cup \{-\infty, \infty\}$, $\{v_{\mu_k}(\xi)\}$ converges, pointwise on $(-\infty, \infty)$, to a monotone increasing function $v(\xi)$, and $\{u_{\mu_k}(\xi)\}$ converges, pointwise on $(-\infty, \xi_0) \cup (\xi_0, \infty)$, to a locally integrable function $u(\xi)$, which is monotone increasing on $(-\infty, \xi_0)$ and monotone decreasing on $(\xi_0, \infty)$. Furthermore, it is easily seen that $u(-\infty) = u_L, u(\infty) = u_R$, $v(-\infty) = v_L$ and $v(\infty) = v_R$.

When $\xi_0 = -\infty$ (or $\xi_0 = \infty$), $u(\xi)$ is a monotone increasing (or decreasing) function on $(-\infty, \infty)$, in which case $(u(\xi), v(\xi))$ is a standard solution to the Riemann problem. The situation becomes interesting when $\xi_0 \in (-\infty, \infty)$. In that case, as $k \to \infty$, $u_{\mu_k} \to u + \omega\delta_{\xi_0}$, in the sense of distributions, where $\delta_{\xi_0}$ denotes the Dirac delta function at $\xi_0$ and $\omega \geq 0$.

We multiply both equations in (9.8.36) by a test function $\varphi \in C_0^\infty(-\infty, \infty)$, integrate the resulting equations over $(-\infty, \infty)$ and integrate by parts to get

$$\begin{cases} \int_{-\infty}^{\infty} \{\mu u_\mu \ddot{\varphi} + [f(u_\mu, v_\mu) - \xi u_\mu]\dot{\varphi} - u_\mu \varphi\}\, d\xi = 0, \\ \\ \int_{-\infty}^{\infty} \{\mu v_\mu \ddot{\varphi} + [g(u_\mu, v_\mu) - \xi v_\mu]\dot{\varphi} - v_\mu \varphi\}\, d\xi = 0. \end{cases} \tag{9.8.40}$$

We apply (9.8.40) for test functions that are constant over some open interval containing $\xi_0$ and let $\mu \downarrow 0$ along the sequence $\{\mu_k\}$ thus obtaining

$$\begin{cases} \int_{-\infty}^{\infty} [f(u(\xi), v(\xi)) - \xi u(\xi)]\dot{\varphi}(\xi) d\xi = \int_{-\infty}^{\infty} u(\xi)\varphi(\xi) d\xi + \omega\varphi(\xi_0), \\ \\ \int_{-\infty}^{\infty} [g(u(\xi), v(\xi)) - \xi v(\xi)]\dot{\varphi}(\xi) d\xi = \int_{-\infty}^{\infty} v(\xi)\varphi(\xi) d\xi. \end{cases} \tag{9.8.41}$$

By shrinking the support of $\varphi$ around $\xi_0$, one deduces that

$$\begin{cases} \lim_{\xi \uparrow \xi_0}[f(u(\xi), v(\xi)) - \xi u(\xi)] - \lim_{\xi \downarrow \xi_0}[f(u(\xi), v(\xi)) - \xi u(\xi)] = \omega, \\ \\ \lim_{\xi \uparrow \xi_0}[g(u(\xi), v(\xi)) - \xi v(\xi)] - \lim_{\xi \downarrow \xi_0}[g(u(\xi), v(\xi)) - \xi v(\xi)] = 0, \end{cases} \tag{9.8.42}$$

where all four limits exist (finite). In particular, this implies that the functions $f(u(\xi), v(\xi))$ and $g(u(\xi), v(\xi))$ are locally integrable on $(-\infty, \infty)$, and (9.8.41) holds for arbitrary $\varphi \in C_0^\infty(-\infty, \infty)$. Equivalently,

$$\begin{cases} [f(u, v) - \xi u]^{\cdot} + u + \omega\delta_{\xi_0} = 0, \\ \\ [g(u, v) - \xi v]^{\cdot} + v = 0, \end{cases} \tag{9.8.43}$$

in the sense of distributions. We thus conclude that when $\omega = 0$ then $(u(\xi), v(\xi))$ is a standard solution of the Riemann problem, possibly with $u(\xi_0) = \infty$, just like the solution (9.6.4) for the system (7.1.8). Whereas, when $\omega > 0$, $(u(\xi) + \omega\delta_{\xi_0}, v(\xi))$ may be interpreted as a nonstandard solution to the Riemann problem, containing a delta shock at $\xi_0$, like the solution (9.6.6) for the system (7.1.8).

We now assume that the system is endowed with an entropy-entropy flux pair $(\eta, q)$, where $\eta(u, v)$ is convex, with superlinear growth,

$$\text{(9.8.44)}\qquad \frac{\eta(u,v)}{|u|+|v|} \to \infty, \quad \text{as } |u|+|v| \to \infty,$$

and show that $(u(\xi), v(\xi))$ is a standard solution to the Riemann problem, i.e., $\omega = 0$. We multiply $(9.8.36)_1$ by $\eta_u(u_\mu, v_\mu)$, $(9.8.36)_2$ by $\eta_v(u_\mu, v_\mu)$, and add the resulting two equations to get

$$\text{(9.8.45)}\qquad \mu\ddot{\eta}(u_\mu, v_\mu) - \mu\left[\eta_{uu}\dot{u}_\mu^2 + 2\eta_{uv}\dot{u}_\mu\dot{v}_\mu + \eta_{vv}\dot{v}_\mu^2\right] = \dot{q}(u_\mu, v_\mu) - \xi\dot{\eta}(u_\mu, v_\mu).$$

We let $\bar{\eta} = \max\{\eta(u_L, v_L), \eta(u_R, v_R)\}$ and then identify the greatest number $\xi_L$ in $(-\infty, \xi_0) \cup \{-\infty\}$ and the smallest number $\xi_R$ in $(\xi_0, \infty) \cup \{\infty\}$ with the property that $\eta(u_\mu(\xi_L), v_\mu(\xi_L)) = \eta\left(u_\mu(\xi_R), v_\mu(\xi_R)\right) = \bar{\eta}$. Using (9.8.45),

$$\text{(9.8.46)}\qquad \int_{\xi_L}^{\xi_R} \left[\eta(u_\mu, v_\mu) - \bar{\eta}\right] d\xi = -\int_{\xi_L}^{\xi_R} \xi\dot{\eta}(u_\mu, v_\mu) d\xi \le q\left(u_\mu(\xi_L), v_\mu(\xi_L)\right) - q\left(u_\mu(\xi_R), v_\mu(\xi_R)\right).$$

The right-hand side of (9.8.46) is bounded, uniformly in $\mu > 0$. Therefore, combining (9.8.46) with (9.8.44) yields

$$\text{(9.8.47)}\qquad \int_{\{u_\mu \ge \bar{u}\}} u_\mu(\xi) d\xi \to 0, \quad \text{as } \bar{u} \to \infty,$$

uniformly in $\mu > 0$, and hence $\omega = 0$.

It is clear that the same argument applies to all possible configurations of $\left(u_\mu(\xi), v_\mu(\xi)\right)$. We have thus established

**9.8.3 Theorem.** *Assume that the system* (9.8.33)*, where* $f_v g_u > 0$*, is endowed with a convex entropy* $\eta(u, v)$*, exhibiting superlinear growth* (9.8.44)*. Then sequences* $\{(u_{\mu_k}, v_{\mu_k})\}$ *of solutions to* (9.8.36), (9.8.37)*, with* $\mu_k \to 0$ *as* $k \to \infty$*, converge pointwise, as well as in the sense of distributions, to standard solutions* $(u, v)$ *of the Riemann problem* (9.8.33), (9.8.34)*. At least one of the functions* $u(\xi), v(\xi)$ *is monotone on* $(-\infty, \infty)$*, while the other may have at most one extremum, which may be bounded or unbounded.*

In particular, any symmetric system of two conservation laws, with $f_v = g_u \neq 0$, satisfies the assumptions of the above theorem. In the literature cited in Section 9.12, the reader will find assumptions on $f$ and $g$ under which the resulting solution to the Riemann problem is necessarily bounded. It has also been shown that any shock in these solutions satisfies the viscous shock admissibility criterion and thereby the Liu $E$-condition.

Following up on the discussion in Section 8.6, one may argue that wave fan solutions of the Riemann problem, with end-states $U_L$ and $U_R$, should not be termed admissible unless they are captured through the $t \to \infty$ asymptotics of solutions of parabolic systems (8.6.1), under initial data $U_0(x)$ which decay sufficiently fast to $U_L$ and $U_R$, as $x \to \mp\infty$. In fact, the results reported in Section 8.6 on the asymptotic stability of viscous shock profiles address a special case of the above issue. The complementary special case, the asymptotic stability of rarefaction waves, has also been studied extensively (references in Section 9.12). The task of combining the above two ingredients so as to synthesize the full solution of the Riemann problem, has not yet been accomplished in a definitive manner.

## 9.9 Interaction of Wave Fans

Up to this point, we have exploited the invariance of systems of conservation laws under uniform rescaling of the space-time coordinates in order to perform stretchings that reveal the local structure of solutions. However, one may also operate at the opposite end of the scale by performing contractions of the space-time coordinates that will provide a view of solutions from a large distance from the origin. It is plausible that initial data $U_0(x)$ which converge sufficiently fast to states $U_L$ and $U_R$, as $x \to -\infty$ and $x \to \infty$, generate solutions that look from afar like centered wave fans joining the state $U_L$, on the left, with the state $U_R$, on the right. Actually, as we shall see in later chapters, this turns out to be true. Indeed, it seems that the quintessential property of hyperbolic systems of conservation laws in one-space dimension is that the Riemann problem describes the asymptotics of solutions at both ends of the time scale: instantaneous and long-term.

The purpose here is to discuss a related question, which, as we shall see in Chapter XIII, is of central importance in the construction of solutions by the random choice method. We consider three wave fans: the first, joining a state $U_L$, on the left, with a state $U_M$, on the right; the second, joining the state $U_M$, on the left, with a state $U_R$, on the right; and the third, joining the state $U_L$, on the left, with the state $U_R$, on the right. These may be identified by their left states $U_L$, $U_M$ and $U_L$, together with the respective $n$-tuples $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ of wave amplitudes. Based on the arguments presented above, it is natural to regard the wave fan $\varepsilon$ as the result of the interaction of the wave fan $\alpha$, on the left, with the wave fan $\beta$, on the right. Recalling (9.3.4), $U_M = \Omega(\alpha; U_L)$, $U_R = \Omega(\beta; U_M)$ and $U_R = \Omega(\varepsilon; U_L)$ whence we deduce

$$\Omega(\varepsilon; U_L) = \Omega(\beta; \Omega(\alpha; U_L)). \tag{9.9.1}$$

This determines implicitly the relation

$$\varepsilon = E(\alpha; \beta; U_L). \tag{9.9.2}$$

Our task is to study the properties of the function $E$ in the vicinity of $(0; 0; U_L)$.

Let us first consider systems with characteristic families that are either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2), in which case the wave fan curves $\Phi_i$, and thereby $\Omega$ and $E$, are all $C^{2,1}$ functions. Since $\Omega(0; \bar{U}) = \bar{U}$,

$$E(\alpha; 0; U_L) = \alpha, \quad E(0; \beta; U_L) = \beta, \tag{9.9.3}$$

whence

$$\frac{\partial E_k}{\partial \alpha_i}(0; 0; U_L) = \delta_{ik}\ , \quad \frac{\partial E_k}{\partial \beta_j}(0; 0; U_L) = \delta_{jk}\ , \tag{9.9.4}$$

namely, the Kronecker delta.

Starting from the identity

$$\begin{aligned}
&E(\alpha; \beta; U_L) - E(\alpha; 0; U_L) - E(0; \beta; U_L) + E(0; 0; U_L) \\
&= \sum_{i,j=1}^{n} \{E(\alpha_1, \cdots, \alpha_i, 0, \cdots, 0; 0, \cdots, 0, \beta_{j+1}, \cdots, \beta_n; U_L) \\
&\quad -E(\alpha_1, \cdots, \alpha_{i-1}, 0, \cdots, 0; 0, \cdots, 0, \beta_{j+1}, \cdots, \beta_n; U_L) \\
&\quad -E(\alpha_1, \cdots, \alpha_i, 0, \cdots, 0; 0, \cdots, 0, \beta_j, \cdots, \beta_n; U_L) \\
&\quad +E(\alpha_1, \cdots, \alpha_{i-1}, 0, \cdots, 0; 0, \cdots, 0, \beta_j, \cdots, \beta_n; U_L)\},
\end{aligned} \tag{9.9.5}$$

one immediately deduces

$$\begin{aligned}
E(\alpha; \beta; U_L) &= \alpha + \beta \\
&+ \sum_{i,j=1}^{n} \alpha_i \beta_j \int_0^1 \int_0^1 \frac{\partial^2 E}{\partial \alpha_i \partial \beta_j}(\alpha_1, \cdots, \alpha_{i-1}, \rho\alpha_i, 0, \cdots, 0; \\
&\qquad 0, \cdots, 0, \sigma\beta_j, \beta_{j+1}, \cdots, \beta_n; U_L) d\rho d\sigma.
\end{aligned} \tag{9.9.6}$$

We say the $i$-wave of the wave fan $\alpha$ and the $j$-wave of the wave fan $\beta$ are *approaching* when either (a) $i > j$ or (b) $i = j$, the $i$-characteristic family is genuinely nonlinear, and at least one of $\alpha_i$, $\beta_i$ is negative, i.e., corresponds to a shock. The *amount of wave interaction* of the fans $\alpha$ and $\beta$ will be measured by the quantity

$$D(\alpha, \beta) = \sum_{\text{app}} |\alpha_i||\beta_j|, \tag{9.9.7}$$

where $\sum_{\text{app}}$ denotes summation over all pairs of approaching waves. The crucial observation is that when the wave fans $\alpha$ and $\beta$ do not include any approaching waves, i.e., $D(\alpha, \beta) = 0$, then the wave fan $\varepsilon$ is synthesized by "glueing together" the wave fan $\alpha$, on the left, and the wave fan $\beta$, on the right; that is, $\varepsilon = \alpha + \beta$. In particular, whenever the $i$-wave of $\alpha$ and the $j$-wave of $\beta$ are not approaching, either because $i < j$ or because $i = j$ and both $\alpha_i$ and $\beta_i$ are positive (i.e., they correspond to rarefaction waves) then

$$E(\alpha_1, \cdots, \alpha_i, 0, \cdots, 0; 0, \cdots, 0, \beta_j, \cdots, \beta_n; U_L) = (\alpha_1, \cdots, \alpha_i, 0, \cdots, 0) + (0, \cdots, 0, \beta_j, \cdots, \beta_n), \tag{9.9.8}$$

whence it follows that the corresponding $(i, j)$-term in the summation on the right-hand side of (9.9.6) vanishes. Thus (9.9.6) reduces to

$$\varepsilon = \alpha + \beta + \sum_{\text{app}} \alpha_i \beta_j \frac{\partial^2 E}{\partial \alpha_i \partial \beta_j}(0; 0; U_L) + D(\alpha, \beta) O(|\alpha| + |\beta|). \tag{9.9.9}$$

The salient feature of (9.9.9), which will play a key role in Chapter XIII, is that the effect of wave interaction is induced solely by pairs of approaching waves and vanishes in the absence of such pairs. In order to determine the leading interaction term, of quadratic order, we first differentiate (9.9.1) with respect to $\beta_j$ and set $\beta = 0$. Upon using (9.3.8), this yields

$$\sum_{k=1}^{n} \frac{\partial E_k}{\partial \beta_j}(\alpha; 0; U_L) \frac{\partial \Omega}{\partial \varepsilon_k}(E(\alpha; 0; U_L); U_L) = R_j(\Omega(\alpha; U_L)). \tag{9.9.10}$$

Next we differentiate (9.9.10) with respect to $\alpha_i$ and set $\alpha = 0$. Recall that we are interested only in the case where the $i$-wave of $\alpha$ and the $j$-wave of $\beta$ are approaching, so in particular $i \geq j$. Therefore, upon using (9.9.3), (9.9.4), (9.3.8), (9.3.9), (9.3.10) and (7.2.15), we conclude

$$\sum_{k=1}^{n} \frac{\partial^2 E_k}{\partial \alpha_i \partial \beta_j}(0; 0; U_L) R_k(U_L) = -[R_i(U_L), R_j(U_L)], \tag{9.9.11}$$

whence

$$\frac{\partial^2 E_k}{\partial \alpha_i \partial \beta_j}(0; 0; U_L) = -L_k(U_L)[R_i(U_L), R_j(U_L)]. \tag{9.9.12}$$

In particular, when the system is endowed with a coordinate system of Riemann invariants, under the normalization (7.3.8) the Lie brackets $[R_i, R_j]$ vanish (cf. (7.3.10)), and hence the quadratic term in (9.9.9) drops out.

Upon combining (9.9.9) with (9.9.12) we arrive at

**9.9.1 Theorem.** *In a system with characteristic families that are either genuinely nonlinear or linearly degenerate, let* $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ *be the wave fan generated by the interaction of the wave fan* $\alpha = (\alpha_1, \cdots, \alpha_n)$*, on the left, with the wave fan* $\beta = (\beta_1, \cdots, \beta_n)$*, on the right. Then*

$$\varepsilon = \alpha + \beta - \sum_{i>j} \alpha_i \beta_j L[R_i, R_j] + D(\alpha, \beta) O(|\alpha| + |\beta|), \tag{9.9.13}$$

*where* $L$ *denotes the* $n \times n$ *matrix with* $k$*-row vector the left eigenvector* $L_k$*, and* $D(\alpha, \beta)$ *is the amount of wave interaction of* $\alpha$ *and* $\beta$*. When the system is endowed with a coordinate system of Riemann invariants, the quadratic term vanishes.*

We now consider wave interactions for systems with characteristic families that may be merely piecewise genuinely nonlinear, so that the incoming and outgoing wave fans will be composed of $i$-wave fans, each one comprising a finite sequence of $i$-shocks and $i$-rarefactions, which henceforth will be dubbed the *elementary waves.* There are two obstacles to overcome. The first is technical: As noted in Section 9.5, the wave fan curves $\Phi_i$ , and thereby the functions $\Omega$ and $E$, may now be merely Lipschitz continuous. Thus, the derivation, above, of (9.9.13) is no longer valid, as it relies on Taylor expansion. The most serious difficulty, however, is how to identify approaching waves. It is clear that an $i$-wave, on the left, and a $j$-wave, on the right, will be approaching if $i > j$ and not approaching if $i < j$. The situation is more delicate when both incoming waves belong to the same characteristic family. Recall that in the genuinely nonlinear case two incoming $i$-waves always approach when at least one of them is a shock and never approach when both are rarefactions. By contrast, here two incoming $i$-wave fans may include pairs of non-approaching $i$-shocks as well as pairs of approaching $i$-rarefaction waves. Consequently, the analog of Theorem 9.9.1 for such systems is quite involved:

**9.9.2 Theorem.** *In a system with characteristic families that are either piecewise genuinely nonlinear or linearly degenerate, let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ be the wave fan generated by the interaction of the wave fan $\alpha = (\alpha_1, \dots, \alpha_n)$, on the left, with the wave fan $\beta = (\beta_1, \dots, \beta_n)$, on the right. Then*

$$\varepsilon = \alpha + \beta + O(1)D(\alpha, \beta), \tag{9.9.14}$$

*where*

$$D(\alpha, \beta) = \sum \theta|\gamma||\delta|, \tag{9.9.15}$$

*with the summation running over all pairs of elementary waves, such that the first one, with amplitude $\gamma$, is part of an $i$-wave fan incoming from the left, while the second one, with amplitude $\delta$, is part of a $j$-wave fan incoming from the right; and the weighting factor $\theta$ is selected according to the following rules.*

*(a) When $i < j$, then $\theta = 0$.*
*(b) When either $i > j$ or $i = j$ and $\gamma\delta < 0$, then $\theta = 1$.*
*(c) When $i = j$ and $\gamma\delta > 0$, then $\theta$ is determined as follows:*
*$(c)_1$ If both incoming elementary waves are $i$-shocks, with respective speeds $\sigma_L$ and $\sigma_R$ , then*

$$\theta = (\sigma_L - \sigma_R)^+ . \tag{9.9.16$_1$}$$

*$(c)_2$ If the elementary wave incoming from the left is an $i$-shock with speed $\sigma_L$ , while the elementary wave incoming from the right is an $i$-rarefaction, joining states $U_R$ and $V_i(\tau_R; U_R)$, then*

$$\theta = \frac{1}{\tau_R} \int_0^{\tau_R} [\sigma_L - \lambda_i(V_i(\tau; U_R))]^+ d\tau. \tag{9.9.16$_2$}$$

*(c)$_3$ If the elementary wave incoming from the left is an $i$-rarefaction, joining states $U_L$ and $V_i(\tau_L; U_L)$, while the elementary wave incoming from the right is an $i$-shock with speed $\sigma_R$, then*

$$\theta = \frac{1}{\tau_L}\int_0^{\tau_L}[\lambda_i(V_i(\tau'; U_L)) - \sigma_R]^+ d\tau'. \tag{9.9.16$_3$}$$

*(c)$_4$ If, finally, both incoming elementary waves are $i$-rarefactions, with the one on the right joining $U_R$ and $V_i(\tau_R; U_R)$ and the one on the left joining $U_L$ and $V_i(\tau_L; U_L)$, then*

$$\theta = \frac{1}{\tau_L \tau_R}\int_0^{\tau_L}\int_0^{\tau_R}[\lambda_i(V_i(\tau'; U_L)) - \lambda_i(V_i(\tau; U_R))]^+ d\tau d\tau'. \tag{9.9.16$_4$}$$

**Sketch of Proof.** The objective here is to explain why and how the weighting factor $\theta$ comes into play. For the case where an $i$-elementary wave, incoming from the left, is interacting with a $j$-elementary wave, incoming from the right, it is easy to understand, based on our earlier discussions in this section, why it should be $\theta = 0$ when $i < j$ and $\theta = 1$ when $i > j$; the real difficulty arises when $i = j$.

It should be noted that if one accepts (9.9.16)$_1$ as the correct value for the weighting factor $\theta$ in the case of interacting shocks, then (9.9.16)$_2$, (9.9.16)$_3$ and (9.9.16)$_4$, which concern rarefaction waves, may be derived as follows. Any $i$-rarefaction wave is visualized as a fan of infinitely many (nonadmissible) $i$-rarefaction shocks, each with infinitesimal amplitude and characteristic speed, and then its contribution to the amount of wave interaction is evaluated by tallying the contributions of these infinitesimal shocks, using (9.9.16)$_1$.

In what follows, it will be shown that (9.9.16)$_1$ does indeed provide the correct value for the weighting factor when each incoming wave fan consists of a single $i$-shock. The proof for general incoming wave fans, which can be found in the references cited in Section 9.12, is long and technical.

Assume the $i$-shock incoming from the left joins $U_L$ with $U_M$ and has amplitude $\gamma$ and speed $\sigma_L$, while the $i$-shock incoming from the right joins $U_M$ with $U_R$ and has amplitude $\delta$ and speed $\sigma_R$. By the Lax $E$-condition, $\sigma_L \geq \lambda_i(U_M) \geq \sigma_R$, so that the relative speed $\theta = \sigma_L - \sigma_R$ of the two incoming shocks is nonnegative. Notice that $\theta$ essentially measures the angle between these two shocks; accordingly, $\theta$ is dubbed the *incidence angle.*

The collision of the two incoming shocks will generate an outgoing wave fan $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, which is determined by solving the Riemann problem with end-states $U_L$ and $U_R$. For simplicity, we assume that the $i$-wave fan of $\varepsilon$ consists of a single $i$-shock, joining $\bar{U}_L$ with $\bar{U}_R$, having amplitude $\varepsilon_i$ and speed $\sigma$.

There are two distinct possible wave configurations, as depicted in Fig. 9.9.1 (a) and (b), depending on whether $\gamma$ and $\delta$ have the same or opposite signs. In either case

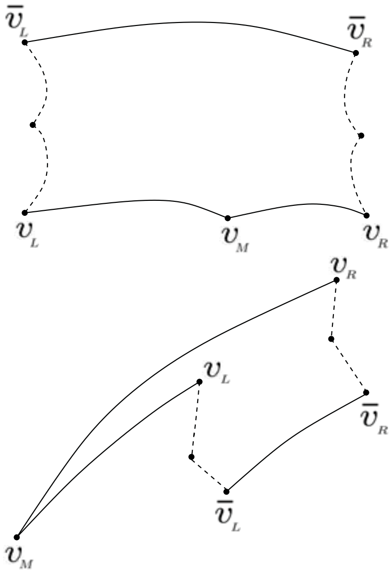

**Fig. 9.9.1 (a)** **Fig. 9.9.1 (b)**

$\varepsilon = \varepsilon(\gamma, \delta; U_L)$, where $\varepsilon(0, \delta; U_L) = (0, \dots, 0, \delta, 0, \dots, 0)$ and $\varepsilon(\gamma, 0; U_L) = (0, \dots, 0, \gamma, 0, \dots, 0)$. Therefore, $\varepsilon_i = \gamma + \delta + O(\gamma\delta)$ and $\varepsilon_j = O(\gamma\delta)$, for $j \neq i$. This relatively crude bound, $O(\gamma\delta)$, for the amount of wave interaction will suffice for the intended applications, in Chapter XIII, when $\gamma\delta < 0$, as in that case the cancellation in the linear term dominates. By contrast, when $\gamma\delta > 0$ a more refined estimate is needed. It is at this point that the incidence angle $\theta$ will come into play, as a measure of the rate the shock speed varies along the shock curve.

Recalling the discussion in Section 9.3,

$$\bar{U}_L = U_L + \sum_{j<i} \varepsilon_j R_j(U_M) + O(|\varepsilon_L|)\gamma + o(|\varepsilon_L|), \tag{9.9.17}$$

$$\bar{U}_R = U_R + \sum_{j>i} \varepsilon_j R_j(U_M) + O(|\varepsilon_R|)\delta + o(|\varepsilon_R|), \tag{9.9.18}$$

$$F(\bar{U}_L) = F(U_L) + \sum_{j<i} \varepsilon_j \lambda_j(U_M) R_j(U_M) + O(|\varepsilon_L|)\delta + o(|\varepsilon_L|), \tag{9.9.19}$$

$$F(\bar{U}_R) = F(U_R) + \sum_{j>i} \varepsilon_j \lambda_j(U_M) R_j(U_M) + O(|\varepsilon_R|)\delta + o(|\varepsilon_R|), \tag{9.9.20}$$

where $\varepsilon_L$ and $\varepsilon_R$ stand for $(\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \dots, 0)$ and $(0, \dots, 0, \varepsilon_{i+1}, \dots, \varepsilon_n)$, respectively.

For convenience, we measure the amplitude of $i$-shocks by the projection of their jump on the left eigenvector $L_i(U_M)$. Thus

(9.9.21)

$$\gamma = L_i(U_M)[U_M - U_L], \quad \delta = L_i(U_M)[U_R - U_M], \quad \varepsilon_i = L_i(U_M)[\bar{U}_R - \bar{U}_L].$$

Starting out from the equation

$$[\bar{U}_R - \bar{U}_L] - [U_M - U_L] - [U_R - U_M] = [\bar{U}_R - U_R] - [\bar{U}_L - U_L], \tag{9.9.22}$$

multiplying it from the left by $L_i(U_M)$, and using (9.9.17), (9.9.18) and (9.9.21), we deduce

$$\varepsilon_i = \gamma + \delta + O(|\varepsilon_L| + |\varepsilon_R|)(\gamma + \delta) + o(|\varepsilon_L| + |\varepsilon_R|). \tag{9.9.23}$$

Similarly, we consider the equation

(9.9.24)

$$\sigma[\bar{U}_R - \bar{U}_L] - \sigma_L[U_M - U_L] - \sigma_R[U_R - U_M] = [F(\bar{U}_R) - F(U_R)] - [F(\bar{U}_L) - F(U_L)],$$

which we get by combining the Rankine-Hugoniot jump conditions for the three shocks; we multiply it from the left by $L_i(U_M)$ and use (9.9.19) and (9.9.20) to get

$$\sigma\varepsilon_i = \sigma_L\gamma + \sigma_R\delta + O(|\varepsilon_L| + |\varepsilon_R|)(\gamma + \delta) + o(|\varepsilon_L| + |\varepsilon_R|). \tag{9.9.25}$$

Recalling that $\sigma_L - \sigma_R = \theta$ (9.9.25) together with (9.9.23) yield

$$\begin{cases} \sigma_L = \sigma + \dfrac{\delta\theta}{\gamma + \delta} + O(|\varepsilon_L| + |\varepsilon_R|) + o(|\varepsilon_L| + |\varepsilon_R|)(\gamma + \delta)^{-1} \\[2ex] \sigma_R = \sigma - \dfrac{\gamma\theta}{\gamma + \delta} + O(|\varepsilon_L| + |\varepsilon_R|) + o(|\varepsilon_L| + |\varepsilon_R|)(\gamma + \delta)^{-1}. \end{cases} \tag{9.9.26}$$

Substituting $\sigma_L$ and $\sigma_R$ from (9.9.26) into (9.9.24) and using (9.9.22), (9.9.17), (9.9.18), (9.9.19) and (9.9.20), we obtain

$$\begin{aligned} &\sum_{j \neq i} \varepsilon_j |\lambda_j(U_M) - \sigma| R_j(U_M) \\ &= \frac{-\delta\theta}{\gamma + \delta}[U_M - U_L] + \frac{\gamma\theta}{\gamma + \delta}[U_R - U_M] + O(|\varepsilon_L| + |\varepsilon_R|)(\gamma + \delta) + o(|\varepsilon_L| + |\varepsilon_R|). \end{aligned} \tag{9.9.27}$$

Multiplying the above equation, from the left, by $L_j(U_M)$, $j \neq i$, and noting that

$$L_j(U_M)[U_M - U_L] = O(\gamma^2), \qquad L_j(U_M)[U_R - U_M] = O(\delta^2), \tag{9.9.28}$$

we deduce

$$\varepsilon_j = O(1)\theta\gamma\delta, \qquad j \neq i. \tag{9.9.29}$$

Equations (9.9.23) and (9.9.25) then give

$$\varepsilon_i = \gamma + \delta + O(1)\theta\gamma\delta, \tag{9.9.30}$$

$$\sigma\varepsilon_i = \sigma_L\gamma + \sigma_R\delta + O(1)\theta\gamma\delta. \tag{9.9.31}$$

We have thus established the assertion of the theorem, for the special case considered here.

One may regard the amplitude of a shock as its "mass" and the product of the amplitude with the speed of a shock as its "momentum". Thus, one may interpret (9.9.30) as balance of "mass" and (9.9.31) as balance of "momentum" under collision of two shocks. Equation (9.9.14) may then be interpreted as balance of "mass" under collision of wave fans. Similarly, one may define the "momentum" of an $i$-wave fan comprising, say, $M$ $i$-shocks with amplitude $\gamma_I$ and speed $\sigma_I$ , $I = 1, \dots, M$, and $N$ $i$-rarefaction waves, joining states $U_J$ and $V_i(\tau_J; U_J)$ by tallying the "momenta" of its constituent elementary waves:

$$\Gamma_i = \sum_{I=1}^{M} \sigma_I\gamma_I + \sum_{J=1}^{N} \int_0^{\tau_J} \lambda_i(V_i(\tau; U_J))d\tau. \tag{9.9.32}$$

Then (9.9.31) admits the following extension. When two incoming wave fans $\alpha$ and $\beta$ interact, the "momentum" $\Gamma_i$ of the outgoing $i$-wave fan is related to the "momenta" $\Gamma_i^-$ and $\Gamma_i^+$ of the incoming $i$-wave fans by

$$\Gamma_i = \Gamma_i^- + \Gamma_i^+ + O(1)D(\alpha, \beta), \qquad i = 1, \dots, n. \tag{9.9.33}$$

## 9.10 Breakdown of Weak Solutions

As we saw in the previous section, wave collisions may induce wave amplification. The following example shows that, as a result, there exist resonating wave patterns that drive the oscillation and/or total variation of weak solutions to infinity, in finite time.

Consider the system

$$\begin{cases} \partial_t u + \partial_x(uv + w) = 0 \\ \partial_t v + \partial_x(\frac{1}{16}v^2) = 0 \\ \partial_t w + \partial_x(u - uv^2 - vw) = 0. \end{cases} \tag{9.10.1}$$

The characteristic speeds are $\lambda_1 = -1$, $\lambda_2 = \frac{1}{8}v$, $\lambda_3 = 1$, so that strict hyperbolicity holds for $-8 < v < 8$. The first and third characteristic families are linearly degenerate, while the second characteristic family is genuinely nonlinear. Clearly, the system is partially decoupled: The second, Burgers-like, equation by itself determines $v$.

The Rankine-Hugoniot jump conditions for a shock of speed $s$, joining the state $(u_-, v_-, w_-)$, on the left, with the state $(u_+, v_+, w_+)$, on the right, here read

$$\begin{cases} u_+v_+ - u_-v_- + w_+ - w_- = s(u_+ - u_-) \\ \frac{1}{16}v_+^2 - \frac{1}{16}v_-^2 = s(v_+ - v_-) \\ u_+ - u_- - u_+v_+^2 + u_-v_-^2 - v_+w_+ + v_-w_- = s(w_+ - w_-). \end{cases} \tag{9.10.2}$$

One easily sees that 1-shocks are 1-contact discontinuities, with $s = -1$, $v_- = v_+$ and

$$w_+ - w_- = -(v_\pm + 1)(u_+ - u_-). \tag{9.10.3$_1$}$$

Similarly, 3-shocks are 3-contact discontinuities, with $s = 1$, $v_- = v_+$ and

$$w_+ - w_- = -(v_\pm - 1)(u_+ - u_-). \tag{9.10.3$_3$}$$

Finally, for 2-shocks, $s = \frac{1}{16}(v_- + v_+)$, and $v_+ < v_-$, in order to satisfy the Lax $E$-condition.

Collisions between any two shocks, joining constant states, induce a jump discontinuity, which can be resolved by solving simple Riemann problems. In particular, when a 1-shock or a 3-shock collides with a 2-shock, the 2-shock remains undisturbed, as $(9.10.1)_2$ is decoupled from the other two equations of the system. This collision, however, produces both a 1- and a 2-outgoing shock, which may be interpreted as the "transmitted" and the "reflected" part of the incident 1- or 2-shock.

We now construct a piecewise constant, admissible solution of (9.10.1) with wave pattern depicted in Fig. 9.10.1: Two 2-shocks issue from the points $(-1, 0)$ and $(1, 0)$, with respective speeds $\frac{1}{4}$ and $-\frac{1}{4}$. On the left of the left 2-shock, $v = 4$; on the right of the right 2-shock, $v = -4$; and $v = 0$ between the two 2-shocks. A 1-shock issues from the origin $(0, 0)$, and upon colliding with the left 2-shock it is partly transmitted as a 1-shock and partly reflected as a 3-shock. This 3-shock, upon

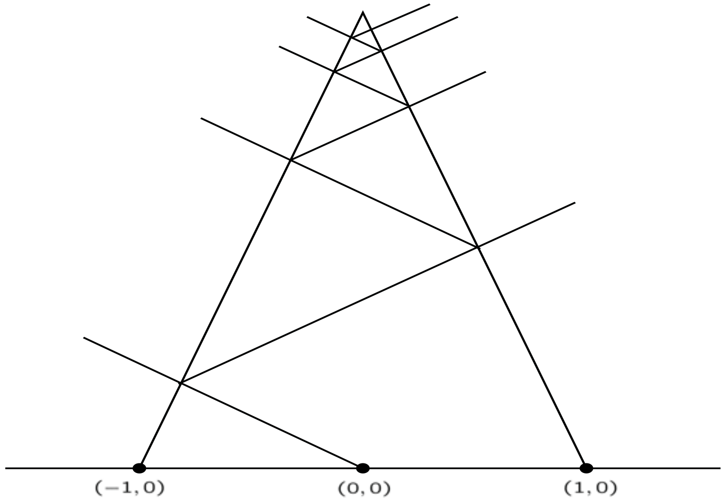


**Fig. 9.10.1**

impinging on the right 2-shock, is in turn partly transmitted as a 3-shock and partly reflected as a 1-shock, and the process is repeated *ad infinitum.*

By checking the Rankine-Hugoniot conditions (9.10.2), one readily verifies that, for instance, initial data

$$(u(x,0), v(x,0), w(x,0)) = \begin{cases} (-65, 4, 225), & -\infty < x < -1 \\ (15, 0, -15), & -1 < x < 0 \\ (-15, 0, 15), & 0 < x < 1 \\ (-63, -4, -225), & 1 < x < \infty \end{cases} \tag{9.10.4}$$

generate a solution with the above structure.

The aim is to demonstrate that each reflection increases the strength of the shock by a constant factor. With collisions becoming progressively more frequent as the distance between the two 2-shocks is decreasing, until finally vanishing at $t = 4$, the conclusion will then be that the oscillation of the solution explodes as $t \uparrow 4$. It will be convenient to measure the strength of 1- and 3-shocks by the size of the jump of $u$ across them.

Let us first examine the interaction depicted in Fig. 9.10.2, where a 1-shock hits the left 2-shock, from the right.

We need to compare the strength $|u_3 - u_2|$ of the reflected 3-shock with the strength $|u_3 - u_4|$ of the incident 1-shock. We write the Rankine-Hugoniot conditions, (9.10.2) or (9.10.3), as applicable, for the five shocks involved in the interaction:

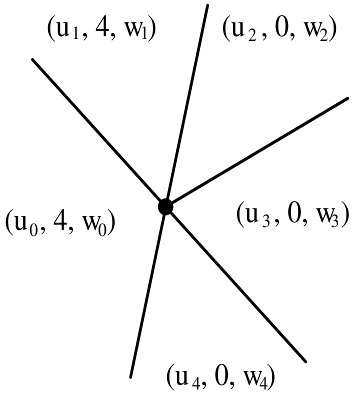


**Fig. 9.10.2**

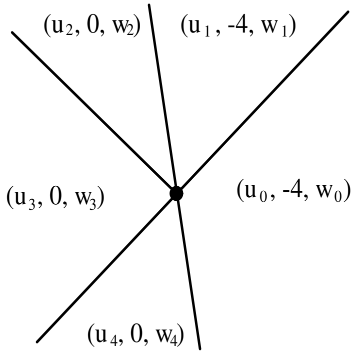


**Fig. 9.10.3**

$$
\left\{
\begin{aligned}
& w_3 - w_4 = -(u_3 - u_4) \\
& w_1 - w_0 = -5(u_1 - u_0) \\
& w_3 - w_2 = u_3 - u_2 \\
& -4u_0 + w_4 - w_0 = \frac{1}{4}(u_4 - u_0) \\
& u_4 - u_0 + 16u_0 + 4w_0 = \frac{1}{4}(w_4 - w_0) \\
& -4u_1 + w_2 - w_1 = \frac{1}{4}(u_2 - u_1) \\
& u_2 - u_1 + 16u_1 + 4w_1 = \frac{1}{4}(w_2 - w_1).
\end{aligned}
\right.
\tag{9.10.5}
$$

After elementary eliminations, one arrives at

$$u_3 - u_2 = -\frac{10}{9}(u_3 - u_4), \tag{9.10.6}$$

which shows that as the 1-shock is reflected into a 3-shock the strength increases by a factor 10/9.

Next we examine the interaction depicted in Fig. 9.10.3, where a 3-shock hits the right 2-shock from the left. By writing again the Rankine-Hugoniot conditions, completely analogous to (9.10.5), and after straightforward eliminations, one ends up once more with (9.10.6). Thus, the strength $|u_2 - u_3|$ of the reflected 1-shock exceeds the strength $|u_4 - u_3|$ of the incident 3-shock by a factor 10/9.

We have now confirmed that the oscillation of the solution blows up as $t \uparrow 4$. The above setting, which renders the calculation particularly simple, may appear at first as a singular, isolated example. However, after some reflection one realizes that the wave resonance persists under small perturbations of the equations and/or initial data, i.e., this kind of catastrophe is sort of generic.

Catastrophes of a different nature may occur as well: The total variation may blow up even though the oscillation remains bounded. This may be demonstrated in the context of the system

$$\begin{cases} \partial_t u + \partial_x(uv^2 + w) = 0 \\ \partial_t v + \partial_x(\frac{1}{16}v^2) = 0 \\ \partial_t w + \partial_x(u - uv^4 - v^2 w) = 0, \end{cases} \tag{9.10.7}$$

which has the same characteristic speeds as (9.10.1), and similarly admits piecewise constant solutions with the wave pattern depicted in Fig. 9.10.1. It is possible to adjust the speeds of the two 2-shocks in such a manner that after any two successive reflections 1- and 3-shocks regain their original left and right states, i.e., the solution takes values in a finite set of states. On the other hand, as $t$ approaches from below the time $t^*$ of collision of the two 2-shocks, the number of shocks, of fixed strength, that cross the $t$-time line grows without bound thus driving the total variation to infinity. Details may be found in the references cited in Section 9.12.

In view of the above, it is hopeless to expect global existence of weak solutions to the Cauchy problem, for general systems of conservation laws and general initial data. Consequently, the aim of the theory should be to establish existence in the large, either for general systems under "small" initial data, or for special systems under general initial data. The hope is that this special class will include the systems arising in continuum physics, which are endowed with special features.

## 9.11 Self-similar Solutions for Multidimensional Conservation Laws

A vehicle for probing the behavior of hyperbolic systems of conservation laws in several space dimensions is the study of self-similar solutions, in which the number of variables is reduced. Self-similarity may be induced by the invariance of the system under uniform stretching of the space-time coordinates and/or by symmetries reflecting isotropy of the underlying continuous medium. The theory of self-similar solutions is currently in a stage of active development, and voluminous literature is already available, including a number of specialized monographs, cited in Section 9.12. The aim here is to offer the reader a glimpse of that area, by outlining a small sample of relevant results.

Let us begin with a scalar conservation law in two space dimensions,

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0, \tag{9.11.1}$$

and seek self-similar solutions $u(x, y, t) = v(x/t, y/t)$. If we define $\xi = x/t$ and $\zeta = y/t$, $v(\xi, \zeta)$ satisfies the equation

$$-\xi v_\xi + f(v)_\xi - \zeta v_\zeta + g(v)_\zeta = 0. \tag{9.11.2}$$

The characteristics of (9.11.2), along which $v$ is constant, are determined by the ordinary differential equation

$$[f'(v(\xi, \zeta)) - \xi]d\zeta - [g'(v(\xi, \zeta)) - \zeta]d\xi = 0. \tag{9.11.3}$$

Notice the set of singular points

$$\mathcal{B} = \{(\xi, \zeta) : \ \xi = f'(v), \ \zeta = g'(v)\}, \tag{9.11.4}$$

parametrized by $v$. For simplicity, we make the assumption

$$f''(u) > 0, \quad g''(u) > 0, \quad [f''(u)/g''(u)]' > 0, \tag{9.11.5}$$

in which case $\mathcal{B}$ is a strictly increasing, concave curve $\zeta = \zeta(\xi)$.

The Rankine-Hugoniot jump condition across a shock curve reads

$$[\lambda(u_-, u_+) - \xi]d\zeta - [\mu(u_-, u_+) - \zeta]d\xi = 0, \tag{9.11.6}$$

where

$$\lambda(u_-, u_+) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \qquad \mu(u_-, u_+) = \frac{g(u_+) - g(u_-)}{u_+ - u_-}. \tag{9.11.7}$$

Notice that any shock curve joining two fixed states $u_-$ and $u_+$ lies on some straight line emanating from the nodal point $\xi = \lambda(u_-, u_+), \zeta = \mu(u_-, u_+)$. Under the convention that the normal vector $(d\zeta, -d\xi)$ is pointing towards the $(+)$ side of shock curves, admissible shocks should satisfy Oleinik's $E$-condition, namely

$$[\lambda(u_-, u_0) - \lambda(u_-, u_+)]d\zeta - [\mu(u_-, u_0) - \mu(u_-, u_+)]d\xi \geq 0, \tag{9.11.8}$$

for any $u_0$ between $u_-$ and $u_+$.

The objective is to construct $BV$ solutions of (9.11.2) on $\mathbb{R}^2$ that satisfy assigned boundary conditions at infinity: $v(r\cos\theta, r\sin\theta) \to h(\theta)$, as $r \to \infty$. In particular, a natural extension of the classical Riemann problem to two space dimensions is to determine a self-similar solution of (9.11.1) with initial values

$$u(x, y, 0) = \begin{cases} u_{NE} & 0 < x < \infty, \quad 0 < y < \infty \\ u_{SE} & 0 < x < \infty, \ -\infty < y < 0 \\ u_{NW} & -\infty < x < 0, \quad 0 < y < \infty \\ u_{SW} & -\infty < x < 0, \ -\infty < y < 0, \end{cases} \tag{9.11.9}$$

where $u_{NE}$, $u_{SE}$, $u_{NW}$ and $u_{SW}$ are given constants.

If $u = v(x/t, y/t)$ is the solution of (9.11.1), (9.11.9), then for large $\xi$ (or $-\xi$), $v$ depends solely on $\zeta$ and depicts an admissible shock or rarefaction wave that joins the states $u_{NE}$ and $u_{SE}$ (or $u_{SW}$ and $u_{NW}$) and propagates in the $y$-direction. Similarly, for large $\zeta$ (or $-\zeta$), $v$ depends solely on $\xi$ and depicts an admissible shock or rarefaction wave that joins the states $u_{NW}$ and $u_{NE}$ (or $u_{SW}$ and $u_{SE}$) and propagates in the $x$-direction. An interesting wave pattern emerges in the region of the $\xi$-$\zeta$ plane where the above four waves interact. In fact, depending on the relative positions of $u_{NE}$, $u_{SE}$, $u_{NW}$ and $u_{SW}$ on the real axis, there are 32 distinct wave configurations, which are described and classified in the literature cited in Section 9.12. For illustration, the two simplest cases will be recorded below.

Assume first $u_{SW} < u_{NW} < u_{SE} < u_{NE}$. In that case the solution is Lipschitz continuous on $\mathbb{R}^2$, with level curves depicted in Figure 9.11.1. Indeed, the pairs of states $(u_{NW}, u_{NE})$, $(u_{SW}, u_{SE})$, $(u_{NE}, u_{SE})$ and $(u_{NW}, u_{SW})$ are all connected by rarefaction waves. The line $\mathcal{B}$ of singular points, defined by (9.11.4), marks the border between these rarefaction waves, and serves as a "roof valley" allowing for Lipschitz continuous transition of the solution across it.

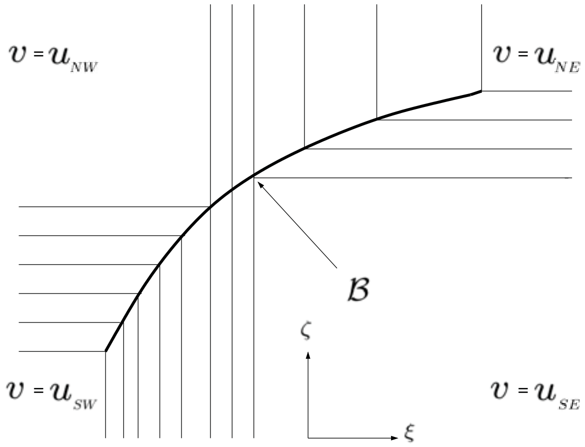


**Fig. 9.11.1**

Assume next $u_{SW} > u_{SE} > u_{NW} > u_{NE}$. In that case the solution comprises constant states joined by admissible shocks, as depicted in Figure 9.11.2. Indeed, the pairs of states $(u_{NW}, u_{NE})$ and $(u_{NW}, u_{SW})$ are connected by two shocks which collide at the point $A = (\lambda(u_{NE}, u_{NW}), \mu(u_{NW}, u_{SW}))$; and the pairs of states $(u_{SW}, u_{SE})$ and $(u_{SE}, u_{NE})$ are similarly connected by two shocks which collide at the point $B = (\lambda(u_{SW}, u_{SE}), \mu(u_{NE}, u_{SE}))$. The wave pattern is completed by two shocks joining $u_{NE}$ with $u_{SW}$. Both emanate from the node $O = (\lambda(u_{NE}, u_{SW}), \mu(u_{NE}, u_{SW}))$; one terminates at the point $A$ and the other at the point $B$.

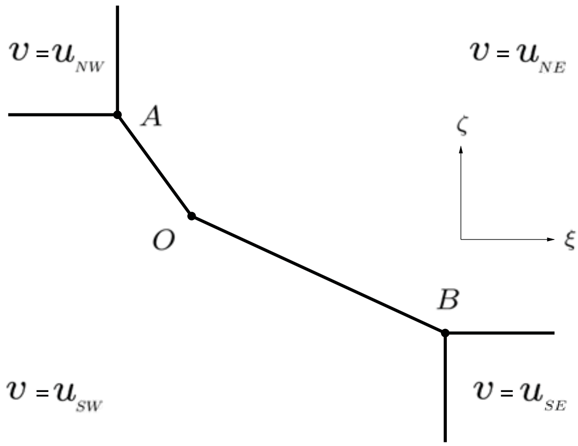


**Fig. 9.11.2**

The remaining cases involve combinations of shocks and rarefaction waves, which may interact to generate more complex wave patterns. In the absence of conditions (9.11.5), the wave configuration is even more intricate. See the references cited in Section 9.12.

It is not to be expected that multi-dimensional Riemann problems will play as pivotal a role as their one-dimensional counterparts. Nevertheless, they are valuable, as they provide a graphic illustration of the geometric complexity of solutions of systems of conservation laws in several space dimensions.

The difficulty in dealing with self-similar solutions to systems of hyperbolic conservation laws in two space variables stems from the fact that the resulting equations on the $\xi$-$\zeta$ plane are no longer hyperbolic, but they are of mixed elliptic-hyperbolic type. A typical example is the system that governs self-similar solutions of the Euler equations (3.3.17) for isentropic flow, with zero body force, which reads

$$\text{(9.11.10)} \qquad \begin{cases} (\rho u)_\xi + (\rho w)_\zeta = -2\rho \\ (\rho u^2 + p(\rho))_\xi + (\rho u w)_\zeta = -3\rho u \\ (\rho u w)_\xi + (\rho w^2 + p(\rho))_\zeta = -3\rho w, \end{cases}$$

where $u$ and $w$ are the components of the velocity relative to the moving frame $x = \xi t$, $y = \zeta t$ of spatial coordinates, i.e., $u = v_1 - \xi$, $w = v_2 - \zeta$. The characteristic speeds of this system are $\lambda_0 = w/u$ and

$$\text{(9.11.11)} \qquad \lambda_\pm = \frac{uw \pm c\sqrt{u^2 + w^2 - c^2}}{u^2 - c^2},$$

where $c$ is the sonic speed, $c^2 = p'(\rho)$. Thus, the system is hyperbolic in the region $u^2 + w^2 > c^2$ (supersonic flow) and elliptic-hyperbolic in the region $u^2 + w^2 < c^2$ (subsonic flow). The equation $u^2 + w^2 = c^2$ determines the *sonic curve* of the flow.

Because of the analytical difficulties posed by (9.11.10), there has been experimentation with simpler systems, such as the system of *pressureless gas dynamics* and the so-called *pressure gradient system,* which exhibit similar features. Both these systems derive from the Euler equations (3.3.21), the former by setting $p \equiv 0$ and the latter by dropping the convective terms $\mathrm{div}(\rho v^\top)$ and $\mathrm{div}(\rho v v^\top)$. The even simpler system of unsteady transonic *small disturbance equations* has also been extensively investigated.

A two-pronged attack is currently under way on problems involving self-similar solutions of the system (9.11.10), its simplified versions, and their extensions to non-isentropic flow. On the one hand, a research program is in progress which aims at obtaining a detailed description of the geometric structure of solutions to the Riemann problem, by a combination of analytical and numerical methods. In parallel, the issue of existence of solutions is being probed by qualitative methods from the theory of partial differential equations of mixed type. The principal objective of the analysis is to locate the sonic curve, which appears as a free boundary. Similar issues arise, and similar techniques are used, for treating multi-dimensional steady transonic flows.

## 9.12 Notes

The Riemann problem was originally formulated, and solved, by Riemann [1], in the context of the system (7.1.9) of isentropic gas dynamics. The method of shock and rarefaction wave curves was gradually developed in order to solve special Riemann problems, for the system of isentropic or adiabatic gas dynamics, describing wave interactions and shock tube experiments. This early research is surveyed in Courant and Friedrichs [1]. The distillation of that work led to the solution, by Lax [2], of the Riemann problem, with weak waves, for strictly hyperbolic systems of conservation laws with characteristic families that are genuinely nonlinear or linearly degenerate (Theorem 9.4.1). Detailed expositions of the solution to the Riemann problem for the system of adiabatic (nonisentropic) gas dynamics are found in the texts by Smoller [3], Serre [11], Godlewski and Raviart [2], and especially in the monograph by Chang and Hsiao [3]. Early references addressing the issue of large data include Smoller [1,2], Smith [1] and Sever [1,2].

Dealing with systems that are not genuinely nonlinear required additional effort. Following the prescription of the Oleinik $E$-condition, the form of the solution of the Riemann problem for the general scalar conservation law was described by Gelfand [1], through an example. Subsequently, Wendroff [1] solved the Riemann problem for the systems (7.1.8) and (7.1.5), when $\sigma_{uu}$ may change sign. The construction of the solution for (7.1.2) and (7.1.8) described in this section, which employs the convex or concave envelope of $f$ and $\sigma$, is found in Dafermos [2] and Leibovich [1]. The above results were apparently anticipated by research in China, in the 1960's, which did not circulate in the international scientific community until much later, e.g. Chang and Hsiao [1,2] and Hsiao and Zhang [1]. The monograph by Chang and Hsiao [3] provides a detailed exposition and many references. The study of special systems motivated the solution by Liu [1] of the Riemann problem for arbitrary piecewise genuinely nonlinear, strictly hyperbolic systems of conservation laws. A more thorough analysis, by Iguchi and LeFloch [1], of the structure of wave fan curves associated with piecewise genuinely nonlinear characteristic families has led to the solution of the Riemann problem for more general strictly hyperbolic systems of conservation laws, not necessarily piecewise genuinely nonlinear. See also Liu and Yang [6]. The observation that wave fan curves may be merely Lipschitz is due to Bianchini [6].

For systems with flux functions that are merely Lipschitz continuous, the Riemann problem is solved by Correia, LeFloch and Thanh [1]. The Riemann problem for balance laws is discussed in Goatin and LeFloch [4].

For early references on delta shocks, consult Keyfitz and Kranzer [2,3,4]. A detailed discussion of the theory of delta shocks in the context of the system of pressureless gas dynamics, in one or two space dimensions, is contained in Li, Zhang and Yang [1]. See also Ercole [1], Joseph [1], Hayes and LeFloch [1], Tan [1], Tan, Zhang and Zheng [1], Sheng and Zhang [1], Li and Yang [1] and Sever [4]. Chen and Hailiang Liu [1,2] demonstrate that in the limit, as the response of the pressure on the density relaxes to zero, solutions of the Riemann problem for the system of isentropic or nonisentropic gas dynamics reduce to solutions of the equations of pressureless gas dynamics with delta shocks.

The entropy rate admissibility criterion was proposed by Dafermos [3]. For motivation from the kinetic theory, see Ferziger and Kaper [1, §5.5] and Kohler [1]. Additional motivation is provided by the vanishing viscosity approach; see Bethuel, Despres and Smets [1]. Theorems 9.7.2 and 9.7.3 are taken from Dafermos [3], while Theorem 9.7.4 is found in Dafermos [15]. For a detailed proof of Theorem 9.7.5, see Dafermos [23]. In the context of the system of adiabatic (nonisentropic) gas dynamics, the entropy rate criterion is discussed by Hsiao [1]. The efficacy of the entropy rate criterion has also been tested on systems that change type, modeling phase transitions (Hattori [1,2,3,4,5,6,7], Pence [2]). See also Sever [3] and Krejčí and Straskraba [1]. An alternative characterization of the solution of the Riemann problem by means of an entropy inequality is due to Heibig and Serre [1]. For an interesting entropy minimization property in (multidimensional) gas dynamics, see Tadmor [1].

The study of self-similar solutions of hyperbolic systems of conservation laws as limits of self-similar solutions of dissipative systems with time-dependent artificial viscosity was initiated, independently, by Kalasnikov [1], Tupciev [1,2], and Dafermos [4]. This approach has been employed to solve the Riemann problem for systems of two conservation laws that may be strictly hyperbolic (Dafermos [4,5], Dafermos and DiPerna [1], Yong Jung Kim [1], Slemrod and Tzavaras [1,2], Tzavaras [1,2]), nonstrictly hyperbolic with delta shocks (Ercole [1], Keyfitz and Kranzer [3,4], Joseph [1], Tan [1], Tan, Zhang and Zheng [1], Sheng and Zhang [1], and Li and Yang [1]), or of mixed type (Slemrod [4], Fan [1,2]). See also Slemrod [5], for solutions with spherical symmetry to the system of isentropic gas dynamics. For recent results indicating that any structurally stable solution of a Riemann problem, even in the presence of strong or undercompressive shocks, may be approximated by viscous wave fans, see Schecter [1,2], Lin and Schecter [1], Schecter and Szmolyan

[1], and Weishi Liu [1]. For numerical computations, see Schecter, Marchesin and Plohr [3]. The method has also been applied successfully to the system of pressureless gas dynamics in two space dimensions. See the book by Li, Zhang and Yang [1]. The treatment of general strictly hyperbolic systems of conservation laws outlined in Section 9.8 follows Tzavaras [2]; the earliest complete construction of solutions to the Riemann problem without any assumption of piecewise genuine nonlinearity (Theorem 9.8.1) appeared in that paper. See also Tzavaras [4]. Joseph and LeFloch [1,2,3,4] use a similar approach for solving the Riemann problem on a half-plane, as well as on a quarter-plane, even for more general viscosity matrices. These authors also apply a variant of the method in which time-dependent viscosity is replaced by time-dependent relaxation, while LeFloch and Rohde [1] replace time-dependent viscosity with time-dependent viscosity-capillarity. The construction of the wave fan curves by the method of viscous wave fans, in Section 9.8, imitates a similar construction, by the standard vanishing viscosity approach, due to Bianchini and Bressan [5] and Bianchini [6]. In particular, the derivation of (9.8.22) was explained to the author by John Mallet-Paret. An alternative construction of the wave fan curves by the method of viscous wave fans is found in Joseph and LeFloch [4]. The discussion of systems (9.8.33) of two conservation laws is adapted from Dafermos and DiPerna [1]. Single rarefaction waves were constructed by the standard vanishing viscosity approach in Lin and Yang [1].

The current status of the theory of the Riemann problem for systems that are not strictly hyperbolic is far from definitive. Both existence and admissibility of solutions raise thorny issues, as wave fans may comprise a great variety of exotic waves such as overcompressive or undercompressive shocks, delta shocks and oscillations. It is futile to aim for an all-encompassing theory; one should focus, instead, on specific systems arising in continuum physics, most notably elasticity and multi-phase flows. Progress has been made on the classification of such systems and on the existence and uniqueness of admissible solutions; see Glimm [2], Azevedo and Marchesin [1], Azevedo, Marchesin, Plohr and Zumbrun [1,2], Freistühler [3], Isaacson, Marchesin and Plohr [1], Isaacson, Marchesin, Plohr and Temple [1], Isaacson and Temple [2], Schaeffer and Shearer [2], Schecter, Marchesin and Plohr [1,2], M. Shearer [4,5], Shearer, Schaeffer, Marchesin and Paes-Leme [1], Schecter and Shearer [1], Schulze and Shearer [1], Tang and Ting [1], Zhu and Ting [1], Čanić [1,2], Čanić and Peters [1], Peters and Čanić [1], Ercole [1], Keyfitz and Kranzer [1,2,3], Tan [1], and Tan, Zhang and Zheng [1].

The solution of the Riemann problem for systems of mixed type, employed to model phase transitions, comprises phase boundaries, in addition to classical shocks and rarefaction waves. As already noted in Section 8.8, the admissibility of phase boundaries is dictated by kinetic relations. Solutions of Riemann problems of this type are found in Fan [1,2,5,6], Frid and Liu [1], Hattori [1,2,3,4,7], Holden [1], Hsiao [2], Hsiao and DeMottoni [1], Keyfitz [1], LeFloch and Thanh [1], Mercier and Piccoli [1], Pence [2], M. Shearer [1,3], Shearer and Yang [1], and Slemrod [4]. For an informative discussion and additional references see the monograph by LeFloch [5].

The Riemann problem has also been posed for quasilinear hyperbolic systems (9.8.18) that are not in conservative form, and solved either by piecing together rarefaction waves and shocks defined by the approach outlined in Section 8.7 (see LeFloch and Liu [1] and LeFloch and Tzavaras [1]) or via the vanishing viscosity approach (Bianchini and Bressan [5], Bianchini [6]). Equally well, one may employ a construction via viscous wave fans, as explained in Section 9.8. For applications, see Andrianov and Warnecke [1,2], LeFloch and Thanh [2] and LeFloch and Shearer [1].

The asymptotic stability of viscous rarefaction waves is discussed in Liu, Matsumura and Nishihara [1], Liu and Xin [1], Szepessy and Zumbrun [1], and Xin [2]. The asymptotic stability of viscous wave fans, containing both shocks and rarefaction waves, is under investigation.

The study of interactions of wave fans and the original proof of Theorem 9.9.1, for genuinely nonlinear systems, is due to Glimm [1]. The derivation presented here is taken from Yong [1]. For systems that are not genuinely nonlinear, wave interaction estimates were originally obtained by Liu [15], who was the first to realize the key role of the incidence angle. For recent detailed and rigorous expositions see Iguchi and LeFloch [1] and Liu and Yang [6]. For a description of actual wave interactions, see Greenberg [1,2] for the system of isentropic elasticity, Liu and Zhang [1] for a scalar combustion model, and Luo and Yang [1] for the Euler equations of isentropic gas flow with frictional damping.

The example of breakdown of weak solutions presented in Section 9.10 is taken from Jenssen [1]. Additional examples were constructed by Baiti and Jenssen [3], R. Young [5,6], and Young and Szeliga [1]. In particular, it is shown that even solutions starting out from initial data with arbitrarily small total variation may blow up in finite time when the characteristic speeds of distinct families are not uniformly separated on the range of the solution. For earlier work indicating rapid magnification, or even blowing up, in the supremum or the total variation of solutions; see Jeffrey [2], R. Young [2] and Joly, Métivier and Rauch [2]. No instability has been detected thus far in solutions of systems with physical interest. It is conceivable that the special structure of these systems may offset the agents of instability.

Detailed constructions of solutions to the Riemann problem (and other self-similar solutions) for the scalar conservation law, Euler's equations, the pressure gradient system and the equations of pressureless gas dynamics, in two space dimensions, are found in the monographs by Chang and Hsiao [3], Li, Zhang and Yang [1] and Yuxi Zheng [1]. Research papers in that area include Guckenheimer [2], Wagner [1], Lindquist [1], Zhang and Zhang [1], Zhang, Li and Zhang [1], Zhang and Zheng [1,2], Tan and Zhang [1], Chen, Li and Tan [1], Chang, Chen and Yang [1], Shuxin Chen [1], Schulz-Rinne [1], Yang and Huang [1], Xiaozhou Yang [1], Serre [9,10] and Lien and Liu [1].

For construction of transonic flows via qualitative methods for partial differential equations that change type, see Čanić and Keyfitz [1,2], Čanić, Keyfitz and Lieberman [1], Čanić, Keyfitz and Kim [1,2,3], and Chen and Feldman [1,2,3]. An informative survey of the current status of this research program, together with an extensive list of references, are found in Keyfitz [3].

There is voluminous literature on transonic steady flow. A classical reference is Courant and Friedrichs [1]. The relevant issue of oblique shock reflexion, and in particular of Mach reflexion, was discussed early by von Neumann [1,2,3]. For more recent references on these topics, which lie beyond the scope of this book, see Schaeffer [2], Shuxin Chen [2,3,4], Yongqian Zhang [1,2], Chen and Feldman [1,2,3], and Morawetz [1].

Self-similar, spherically symmetric solutions in gas dynamics are discussed in the books by Courant and Friedrichs [1] and Yuxi Zheng [1]. For self-similar, radially symmetric solutions representing cavitation in elastodynamics and gas dynamics, see Pericak-Spector and Spector [1,2] and Yan [1]. Radially symmetric solutions for the complex Burgers equation are constructed in Noelle [2].

# X

# Generalized Characteristics

As already noted in Section 7.9, the function space of choice for weak solutions of hyperbolic systems of conservation laws in one-space dimension is $BV$, since it is within its confines that one may discern shocks and study their propagation and interactions. The notion of characteristic, introduced in Section 7.2 for classical solutions, will here be extended to the framework of $BV$ weak solutions. It will be established that generalized characteristics propagate with either classical characteristic speed or with shock speed. In particular, it will be shown that the extremal backward characteristics, emanating from any point in the domain of an admissible solution, always propagate with classical characteristic speed. The implications of these properties to the theory of weak solutions will be demonstrated in following chapters.

## 10.1 $BV$ Solutions

We consider the strictly hyperbolic system

$$\partial_t U + \partial_x F(U) = 0 \tag{10.1.1}$$

of conservation laws. Throughout this chapter, $U$ will denote a bounded measurable function on $(-\infty, \infty) \times (0, \infty)$, of class $BV_{\text{loc}}$, which is a weak solution of (10.1.1). Following the general theory of $BV$ functions in Section 1.7, we infer that $(-\infty, \infty) \times (0, \infty) = \mathcal{C} \bigcup \mathcal{J} \bigcup \mathcal{I}$ where $\mathcal{C}$ is the set of points of approximate continuity of $U$, $\mathcal{J}$ denotes the set of points of approximate jump discontinuity (shock set) of $U$, and $\mathcal{I}$ stands for the set of irregular points of $U$. The one-dimensional Hausdorff measure of $\mathcal{I}$ is zero : $\mathcal{H}^1(\mathcal{I}) = 0$. The shock set $\mathcal{J}$ is essentially covered by the (at most) countable union of $C^1$ arcs. With any $(\bar{x}, \bar{t}) \in \mathcal{J}$ are associated one-sided approximate limits $U_\pm$ and a "tangent" line of slope (shock speed) $s$ which, as shown in Section 1.8, are related by the Rankine-Hugoniot jump condition (8.1.2).

We shall be assuming throughout that the Lax $E$-condition, introduced in Section 8.3, holds here in a strong sense: each shock is compressive but not overcompressive. That is, if $U_\pm$ are the one-sided limits and $s$ is the corresponding shock speed associated with any point of the shock set, then there is $i \in \{1, \cdots, n\}$ such that

(10.1.2) $$\lambda_{i-1}(U_\pm) < \lambda_i(U_+) \le s \le \lambda_i(U_-) < \lambda_{i+1}(U_\pm).$$

In (10.1.2), the first inequality is not needed when $i = 1$ and the last inequality is unnecessary when $i = n$. Moreover, since (10.1.1) is strictly hyperbolic, the first and the last inequalities will hold automatically whenever the oscillation of $U$ is sufficiently small.

For convenience, we normalize $U$ as explained in Section 1.7. In particular, at every point $(\bar{x}, \bar{t}) \in \mathcal{C}$, $U(\bar{x}, \bar{t})$ equals the corresponding approximate limit $U_0$. Recalling that $\mathcal{H}^1(\mathcal{I}) = 0$ and using Theorem 1.7.2, we easily conclude that there is a subset $\mathcal{N}$ of $(0, \infty)$, of measure zero, having the following properties. For any fixed $\bar{t} \notin \mathcal{N}$, the function $U(\cdot, \bar{t})$ has locally bounded variation on $(-\infty, \infty)$, and $(\bar{x}, \bar{t}) \in \mathcal{C}$ if and only if $U(\bar{x}-, \bar{t}) = U(\bar{x}+, \bar{t})$, while $(\bar{x}, \bar{t}) \in \mathcal{J}$ if and only if $U(\bar{x}-, \bar{t}) \neq U(\bar{x}+, \bar{t})$. In the latter case, $U_- = U(\bar{x}-, \bar{t})$ and $U_+ = U(\bar{x}+, \bar{t})$.

The above properties of $U$ follow just from membership in $BV$. The fact that $U$ is also a solution of (10.1.1) should induce additional structure. Based on experience with special systems, to be discussed in later chapters, it seems plausible to expect the following: $U$ should be (classically) continuous on $\mathcal{C}$ and the one-sided limits $U_\pm$ at points of $\mathcal{J}$ should be attained in the classical sense. Moreover, $\mathcal{I}$ should be the (at most) countable set of endpoints of the arcs that comprise $\mathcal{J}$. Uniform stretching of the $(x, t)$ coordinates about any point of $\mathcal{I}$ should yield, in the limit, a wave fan with the properties described in Section 9.1, i.e., $\mathcal{I}$ should consist of shock generation and shock interaction points. To what extent the picture painted above accurately describes the structure of solutions of general hyperbolic systems of conservation laws will be discussed in later chapters.

## 10.2 Generalized Characteristics

Characteristics associated with classical, Lipschitz continuous, solutions were introduced in Section 7.2, through Definition 7.2.1. They provide one of the principal tools of the classical theory for the study of analytical and geometric properties of solutions. It is thus natural to attempt to extend the notion to the framework of weak solutions.

Here we opt to define characteristics of the $i$-characteristic family, associated with the weak solution $U$, exactly as in the classical case, namely as integral curves of the ordinary differential equation (7.2.7), in the sense of Filippov:

**10.2.1 Definition.** A *generalized $i$-characteristic* for the system (10.1.1), associated with the (generally weak) solution $U$, on the time interval $[\sigma, \tau] \subset [0, \infty)$, is a Lipschitz function $\xi : [\sigma, \tau] \to (-\infty, \infty)$ which satisfies the differential inclusion

(10.2.1) $$\dot{\xi}(t) \in \Lambda_i(\xi(t), t), \qquad \text{a.e. on } [\sigma, \tau],$$

where

(10.2.2) $$\Lambda_i(\bar{x}, \bar{t}) := \bigcap_{\varepsilon > 0} \Big[ \operatorname*{ess\,inf}_{[\bar{x}-\varepsilon, \bar{x}+\varepsilon]} \lambda_i(U(x, \bar{t})) \, , \, \operatorname*{ess\,sup}_{[\bar{x}-\varepsilon, \bar{x}+\varepsilon]} \lambda_i(U(x, \bar{t})) \Big] .$$

From the general theory of contingent equations like (10.2.1), one immediately infers the following

**10.2.2 Theorem.** *Through any fixed point* $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times [0, \infty)$ *pass two (not necessarily distinct) generalized* $i$*-characteristics, associated with* $U$ *and defined on* $[0, \infty)$, *namely the minimal* $\xi_-(\cdot)$ *and the maximal* $\xi_+(\cdot)$, *with* $\xi_-(t) \le \xi_+(t)$ *for* $t \in [0, \infty)$. *The funnel-shaped region confined between the graphs of* $\xi_-(\cdot)$ *and* $\xi_+(\cdot)$ *comprises the set of points* $(x, t)$ *that may be connected to* $(\bar{x}, \bar{t})$ *by a generalized* $i$*-characteristic associated with* $U$.

Other standard properties of solutions of differential inclusions also have useful implications for the theory of generalized characteristics: If $\{\xi_m(\cdot)\}$ is a sequence of generalized $i$-characteristics, associated with $U$ and defined on $[\sigma, \tau]$, which converges to some Lipschitz function $\xi(\cdot)$, uniformly on $[\sigma, \tau]$, then $\xi(\cdot)$ is necessarily a generalized $i$-characteristic associated with $U$. In particular, if $\xi_m(\cdot)$ is the minimal (or maximal) generalized $i$-characteristic through a point $(x_m, \bar{t})$ and $x_m \uparrow \bar{x}$ (or $x_m \downarrow \bar{x}$), as $m \to \infty$, then $\{\xi_m(\cdot)\}$ converges to the minimal (or maximal) generalized $i$-characteristic $\xi_-(\cdot)$ (or $\xi_+(\cdot)$) through the point $(\bar{x}, \bar{t})$.

In addition to classical $i$-characteristics, $i$-shocks that satisfy the Lax $E$-condition are obvious examples of generalized $i$-characteristics. In fact, it turns out that these are the only possibilities. Indeed, even though Definition 10.2.1 would seemingly allow $\dot{\xi}$ to select any value in the interval $\Lambda_i$, the fact that $U$ is a solution of (10.1.1) constrains generalized $i$-characteristics associated with $U$ to propagate either with classical $i$-characteristic speed or with $i$-shock speed:

**10.2.3 Theorem.** *Let* $\xi(\cdot)$ *be a generalized* $i$*-characteristic, associated with* $U$ *and defined on* $[\sigma, \tau]$. *The following holds for almost all* $t \in [\sigma, \tau]$ : *When* $(\xi(t), t) \in \mathcal{C}$, *then* $\dot{\xi}(t) = \lambda_i(U_0)$ *with* $U_0 = U(\xi(t)\pm, t)$. *When* $(\xi(t), t) \in \mathcal{J}$, *then* $\dot{\xi}(t) = s$, *where* $s$ *is the speed of the* $i$*-shock that joins* $U_-$, *on the left, to* $U_+$, *on the right, with* $U_\pm = U(\xi(t)\pm, t)$. *In particular,* $s$ *satisfies the Rankine-Hugoniot condition* (8.1.2) *as well as the Lax* $E$*-condition* (10.1.2).

**Proof.** Let us recall the properties of $BV$ solutions recorded in Section 10.1. It is then clear that for almost all $t \in [\sigma, \tau]$ with $(\xi(t), t) \in \mathcal{C}$ the interval $\Lambda_i(\xi(t), t)$ reduces to the single point $\lambda_i(U(\xi(t)\pm, t))$ and so $\dot{\xi}(t) = \lambda_i(U(\xi(t)\pm, t))$, by virtue of (10.2.1).

Applying the measure equality (10.1.1) to arbitrary subarcs of the graph of $\xi$, and using Theorem 1.7.8 (in particular Equation (1.7.12)), yields

$$F(U(\xi(t)+, t)) - F(U(\xi(t)-, t)) = \dot{\xi}(t)[U(\xi(t)+, t) - U(\xi(t)-, t)], \tag{10.2.3}$$

almost everywhere on $[\sigma, \tau]$. Consequently, for almost all $t \in [\sigma, \tau]$ with $(\xi(t), t)$ in $\mathcal{J}$, we have $\dot{\xi}(t) = s$, where $s$ is the speed of a shock that joins the states $U_- = U(\xi(t)-, t)$ and $U_+ = U(\xi(t)+, t)$. Because of the structure of solutions, there is $j \in \{1, \cdots, n\}$ such that $\lambda_{j-1}(U_\pm) < \lambda_j(U_+) \le s \le \lambda_j(U_-) < \lambda_{j+1}(U_\pm)$. On the other hand, (10.2.1) implies that $s$ lies in the interval with endpoints $\lambda_i(U_-)$ and $\lambda_i(U_+)$. Therefore, $j = i$ and (10.1.2) holds. This completes the proof.

The above theorem motivates the following terminology:

**10.2.4 Definition.** A generalized $i$-characteristic $\xi(\cdot)$, associated with $U$ and defined on $[\sigma, \tau]$, is called *shock-free* if $U(\xi(t)-, t) = U(\xi(t)+, t)$, for almost all $t$ in $[\sigma, \tau]$.

A consequence of the proof of Theorem 10.2.3 is that (10.2.1) is equivalent to

$$\dot{\xi}(t) \in [\lambda_i(U(\xi(t)+, t))\ ,\ \lambda_i(U(\xi(t)-, t))], \qquad \text{a.e. on } [\sigma, \tau]. \tag{10.2.4}$$

In what follows, an important role will be played by the special generalized characteristics that manage to propagate at the maximum or minimum allowable speed:

**10.2.5 Definition.** A generalized $i$-characteristic $\xi(\cdot)$, associated with $U$ and defined on $[\sigma, \tau]$, is called a *left $i$-contact* if

$$\dot{\xi}(t) = \lambda_i(U(\xi(t)-, t)), \qquad \text{a.e. on } [\sigma, \tau], \tag{10.2.5}$$

and/or a *right $i$-contact* if

$$\dot{\xi}(t) = \lambda_i(U(\xi(t)+, t)), \qquad \text{a.e. on } [\sigma, \tau]. \tag{10.2.6}$$

Clearly, shock-free $i$-characteristics are left and right $i$-contacts. Note that, since they are generalized $i$-characteristics, left (or right) $i$-contacts should also satisfy the assertion of Theorem 10.2.3, namely $\dot{\xi}(t) = s$ for almost all $t \in [\sigma, \tau]$ with $(\xi(t), t) \in \mathcal{J}$. Of course this is impossible in systems that do not admit left (or right) contact discontinuities. In any such system, left (or right) contacts are necessarily shock-free. In particular, recalling Theorem 8.2.1, we conclude that when the $i$-characteristic family for the system (10.1.1) is genuinely nonlinear and the oscillation of $U$ is sufficiently small, then any left or right $i$-contact is necessarily shock-free.

## 10.3 Extremal Backward Characteristics

With reference to some point $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times [0, \infty)$, a generalized characteristic through $(\bar{x}, \bar{t})$ is dubbed *backward* when defined on $[0, \bar{t}]$, or *forward* when defined on $[\bar{t}, \infty)$. The extremal, minimal and maximal, backward and forward generalized characteristics through $(\bar{x}, \bar{t})$ propagate at extremal speeds and are thus natural candidates for being contacts. This turns out to be true, at least for the backward extremal characteristics, in consequence of the Lax $E$-condition:

**10.3.1 Theorem.** *The minimal (or maximal) backward $i$-characteristic, associated with $U$, emanating from any point $(\bar{x}, \bar{t})$ of the upper half-plane is a left (or right) $i$-contact.*

**Proof.** Let $\xi(\cdot)$ denote the minimal backward $i$-characteristic emanating from $(\bar{x}, \bar{t})$ and defined on $[0, \bar{t}]$. We fix $\varepsilon > 0$ and select $\bar{t} = \tau_0 > \tau_1 > \cdots > \tau_k = 0$, for some $k \geq 1$, through the following algorithm: We start out with $\tau_0 = \bar{t}$. Assuming $\tau_m > 0$ has been determined, we let $\xi_m(\cdot)$ denote the minimal backward $i$-characteristic emanating from $(\xi(\tau_m) - \varepsilon, \tau_m)$. If $\xi_m(t) < \xi(t)$ for $0 < t \leq \tau_m$, we set $\tau_{m+1} = 0, m + 1 = k$ and terminate. Otherwise, we locate $\tau_{m+1} \in (0, \tau_m)$ with the property $\xi_m(t) < \xi(t)$ for $\tau_{m+1} < t \leq \tau_m$ and $\xi_m(\tau_{m+1}) = \xi(\tau_{m+1})$. Clearly, this algorithm will terminate after a finite number of steps. Next we construct a left-continuous, piecewise Lipschitz function $\xi_\varepsilon(\cdot)$ on $[0, \bar{t}]$, with jump discontinuities (when $k \geq 2$) at $\tau_1, \cdots, \tau_{k-1}$, by setting $\xi_\varepsilon(t) = \xi_m(t)$ for $\tau_{m+1} < t \leq \tau_m$, with $m = 0, 1, \cdots, k - 1$, and $\xi_\varepsilon(0) = \xi_{k-1}(0)$. Then

$$(10.3.1) \quad \xi_\varepsilon(\bar{t}) - \xi_\varepsilon(0) = (k-1)\varepsilon + \sum_{m=0}^{k-1} \int_{\tau_{m+1}}^{\tau_m} \dot{\xi}_m(t)dt \geq \int_0^{\bar{t}} \lambda_i(U(\xi_\varepsilon(t)+, t))dt.$$

By standard theory of contingent equations like (10.2.1), $\xi_\varepsilon(t) \uparrow \xi(t)$ as $\varepsilon \downarrow 0$, uniformly on $[0, \bar{t}]$. Therefore, letting $\varepsilon \downarrow 0$, (10.3.1) yields

$$(10.3.2) \qquad \xi(\bar{t}) - \xi(0) \geq \int_0^{\bar{t}} \lambda_i(U(\xi(t)-, t))dt.$$

On the other hand, $\dot{\xi}(t) \leq \lambda_i(U(\xi(t)-, t))$, almost everywhere on $[0, \bar{t}]$, and so $\dot{\xi}(t) = \lambda_i(U(\xi(t)-, t))$ for almost all $t \in [0, \bar{t}]$, i.e., $\xi(\cdot)$ is a left $i$-contact.

Similarly one shows that the maximal backward $i$-characteristic emanating from $(\bar{x}, \bar{t})$ is a right $i$-contact. This completes the proof.

In view of the closing remarks in Section 10.2, Theorem 10.3.1 has the following corollary:

**10.3.2 Theorem.** *Assume the $i$-characteristic family for the system* (10.1.1) *is genuinely nonlinear and the oscillation of $U$ is sufficiently small. Then the minimal and the maximal backward $i$-characteristics, emanating from any point $(\bar{x}, \bar{t})$ of the upper half-plane, are shock-free.*

The implications of the above theorem will be seen in following chapters.

For future use, it will be expedient to introduce here a special class of backward characteristics emanating from infinity:

**10.3.3 Definition.** A *minimal* (or *maximal*) *$i$-separatrix*, associated with the solution $U$, is a Lipschitz function $\xi : [0, \bar{t}) \to (-\infty, \infty)$ such that $\xi(t) = \lim_{m\to\infty} \xi_m(t)$, uniformly on compact time intervals, where $\xi_m(\cdot)$ is the minimal (or maximal) backward $i$-characteristic emanating from a point $(x_m, t_m)$, with $t_m \to \bar{t}$, as $m \to \infty$. In particular, when $\bar{t} = \infty$, the $i$-separatrix $\xi(\cdot)$ is called a *minimal* (or *maximal*) *$i$-divide*.

Note that the graphs of any two minimal (or maximal) $i$-characteristics may run into each other but they cannot cross. Consequently, the graph of a minimal (or maximal) backward $i$-characteristic cannot cross the graph of any minimal (or maximal)

$i$-separatrix. Similarly, the graphs of any two minimal (or maximal) $i$-separatrices cannot cross. In particular, any minimal (or maximal) $i$-divide divides the upper half-plane into two parts in such a way that no forward $i$-characteristic may cross from the left to the right (or from the right to the left).

Minimal or maximal $i$-separatrices are necessarily generalized $i$-characteristics, which, by virtue of Theorem 10.3.1, are left or right $i$-contacts. In particular, when the $i$-characteristic family is genuinely nonlinear and the oscillation of $U$ is sufficiently small, Theorem 10.3.2 implies that minimal or maximal $i$-separatrices are shock-free.

One should not expect that all solutions possess $i$-divides. An important class that always do, are solutions that are periodic in $x$, $U(x+L,t) = U(x,t)$ for some $L > 0$ and all $(x,t) \in (-\infty,\infty) \times (0,\infty)$. Indeed, in that case, given any sequence $\{t_m\}$, with $t_m \to \infty$ as $m \to \infty$, it is always possible to locate $\{x_m\}$ with the property that the minimal or maximal backward $i$-characteristic $\xi_m(\cdot)$ emanating from $(x_m, t_m)$ will be intercepted by the $x$-axis at a point lying inside any fixed interval of length $L$, say $\xi_m(0) \in [0, L)$ , $m = 1, 2, \cdots$. The Arzela theorem then implies that $\{\xi_m(\cdot)\}$ contains convergent subsequences whose limits are necessarily $i$-divides.

This chapter will close with the following remark: Generalized characteristics were introduced here in connection to $BV$ solutions of (10.1.1) defined on the entire upper half-plane. It is clear, however, that the notion and many of its properties are of purely local nature and thus apply to $BV$ solutions defined on arbitrary open subsets of $\mathbb{R}^2$.

## 10.4 Notes

The presentation of the theory of generalized characteristics in this chapter follows Dafermos [16]. An exposition of the general theory of differential inclusions is found in the monograph by Filippov [1]. An early paper introducing generalized characteristics (for scalar conservation laws) as solutions of the classical characteristic equations, in the sense of Filippov, is Wu [1]. See also Hörmander [1]. Glimm and Lax [1] employ an alternative definition of generalized characteristics, namely Lipschitz curves propagating either with classical characteristic speed or with shock speed, constructed as limits of a family of "approximate characteristics". In view of Theorem 10.2.3, the two notions are closely related. This will be discussed in Chapter XIII.

The notion of divide was introduced in Dafermos [18].

Generalized characteristics in several space dimensions are considered by Poupaud and Rascle [1], in the context of linear transport equations with discontinuous coefficients.

# XI

# Genuinely Nonlinear Scalar Conservation Laws

Despite its apparent simplicity, the genuinely nonlinear scalar conservation law in one-space dimension possesses a surprisingly rich theory, which deserves attention not only for its intrinsic interest but also because it provides valuable insight in the behavior of systems. The discussion here will employ the theory of generalized characteristics developed in Chapter X. From the standpoint of this approach, the special feature of genuinely nonlinear scalar conservation laws is that the extremal backward generalized characteristics are essentially classical characteristics, that is, straight lines along which the solution is constant. This property induces such a heavy constraint that one is able to derive very precise information on regularity and large time behavior of solutions.

Solutions are (classically) continuous at points of approximate continuity and locally Lipschitz continuous in the interior of the set of points of continuity. Points of approximate jump discontinuity lie on classical shocks. The remaining, irregular, points are at most countable and are formed by the collision of shocks and/or the focussing of compression waves. Generically, solutions with smooth initial data are piecewise smooth.

Genuine nonlinearity gives rise to a host of dissipative mechanisms that affect the large time behavior of solutions. Entropy dissipation induces $O(t^{-\frac{p}{p+1}})$ decay of solutions with initial data in $L^p(-\infty, \infty)$. When the initial data have compact support, spreading of characteristics generates $N$-wave profiles. Confinement of characteristics under periodic initial data induces $O(t^{-1})$ decay in the total variation per period and the formation of sawtoothed profiles.

Another important feature of admissible weak solutions of the Cauchy problem for the genuinely nonlinear scalar conservation law is that they are related explicitly to their initial values, through the Lax function. This property, which will be established here by the method of generalized characteristics, may serve alternatively as the starting point for developing the general theory of solutions to the Cauchy problem.

Additional insight is gained from comparison theorems on solutions. It will be shown that the lap number of any admissible solution is nonincreasing with time.

Moreover, the $L^1$ distance of any two solutions is generally nonincreasing, but typically conserved, whereas a properly weighted $L^1$ distance is strictly decreasing.

One of the advantages of the method of generalized characteristics is that it readily extends to inhomogeneous, genuinely nonlinear balance laws. This theory will be outlined here and two examples will be presented in order to demonstrate how inhomogeneity and source terms may affect the large time behavior of solutions.

## 11.1 Admissible *BV* Solutions and Generalized Characteristics

We consider the scalar conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \tag{11.1.1}$$

which is genuinely nonlinear, $f''(u) > 0$, $-\infty < u < \infty$. Throughout this chapter we shall be dealing with admissible weak solutions $u$ on $(-\infty, \infty) \times [0, \infty)$ whose initial values $u_0$ are bounded and have locally bounded variation on $(-\infty, \infty)$. By virtue of Theorem 6.2.6, $u$ is in $BV_{\text{loc}}$ and for any $t \in [0, \infty)$ the function $u(\cdot, t)$ has locally bounded variation on $(-\infty, \infty)$.

As noted in Section 8.5, the entropy shock admissibility criterion will be satisfied almost everywhere (with respect to one-dimensional Hausdorff measure) on the shock set $\mathcal{J}$ of the solution $u$, for any entropy-entropy flux pair $(\eta, q)$ with $\eta$ convex. This in turn implies that the Lax $E$-condition will also hold almost everywhere on $\mathcal{J}$. Consequently, we have

$$u(x+, t) \leq u(x-, t), \tag{11.1.2}$$

for almost all $t \in (0, \infty)$ and all $x \in (-\infty, \infty)$.

By account of Theorem 10.2.3, a Lipschitz curve $\xi(\cdot)$, defined on the time interval $[\sigma, \tau] \subset [0, \infty)$, will be a generalized characteristic, associated with the solution $u$, if for almost all $t \in [\sigma, \tau]$

$$\dot{\xi}(t) = \begin{cases} f'(u(\xi(t)\pm, t)), & \text{when } u(\xi(t)+, t) = u(\xi(t)-, t) \\ \dfrac{f(u(\xi(t)+, t)) - f(u(\xi(t)-, t))}{u(\xi(t)+, t) - u(\xi(t)-, t)}, & \text{when } u(\xi(t)+, t) < u(\xi(t)-, t). \end{cases} \tag{11.1.3}$$

The special feature of genuinely nonlinear scalar conservation laws is that generalized characteristics that are shock-free are essentially classical characteristics:

**11.1.1 Theorem.** *Let $\xi(\cdot)$ be a generalized characteristic for* (11.1.1), *associated with the admissible solution $u$, shock-free on the time interval* $[\sigma, \tau]$. *Then there is a constant $\bar{u}$ such that*

$$u(\xi(\tau)+, \tau) \leq \bar{u} \leq u(\xi(\tau)-, \tau), \tag{11.1.4}$$

$$u(\xi(t)+,t)=\bar{u}=u(\xi(t)-,t), \qquad \sigma<t<\tau\,, \tag{11.1.5}$$

$$u(\xi(\sigma)-,\sigma)\le\bar{u}\le u(\xi(\sigma)+,\sigma). \tag{11.1.6}$$

*In particular, the graph of* $\xi(\cdot)$ *is a straight line with slope* $f'(\bar{u})$.

**Proof.** Fix $r$ and $s$, $\sigma\le r<s\le\tau$. For $\varepsilon>0$, we integrate the measure equality (11.1.1) over the set $\{(x,t): r<t<s,\ \xi(t)-\varepsilon<x<\xi(t)\}$ and use Green's theorem to get

$$\int_{\xi(s)-\varepsilon}^{\xi(s)}u(x,s)dx-\int_{\xi(r)-\varepsilon}^{\xi(r)}u(x,r)dx \tag{11.1.7}$$

$$=\int_r^s\{f(u(\xi(t)-\varepsilon+,t))-f(u(\xi(t)-,t))-\dot{\xi}(t)[u(\xi(t)-\varepsilon+,t)-u(\xi(t)-,t)]\}dt.$$

By virtue of Definition 10.2.4, $\dot{\xi}(t)=f'(u(\xi(t)-,t))$ , a.e. on $[r,s]$. Since $f$ is convex, this implies that the right-hand side of (11.1.7) is nonnegative. Consequently, multiplying (11.1.7) by $1/\varepsilon$ and letting $\varepsilon\downarrow 0$ yields

$$u(\xi(s)-,s)\ge u(\xi(r)-,r), \qquad \sigma\le r<s\le\tau. \tag{11.1.8}$$

Next we apply (11.1.1) to the set $\{(x,t): r<t<s,\ \ \xi(t)<x<\xi(t)+\varepsilon\}$ and repeat the above procedure to deduce

$$u(\xi(s)+,s)\le u(\xi(r)+,r), \qquad \sigma\le r<s\le\tau. \tag{11.1.9}$$

We now fix $t_1$ and $t_2$, $\sigma<t_1<t_2<r$, such that $u(\xi(t_1)-,t_1)=u(\xi(t_1)+,t_1)$, $u(\xi(t_2)-,t_2)=u(\xi(t_2)+,t_2)$; then fix any $t\in(t_1,t_2)$. We apply (11.1.8) and (11.1.9) first with $r=t_1$, $s=t_2$, then with $r=t_1$, $s=t$, and finally with $r=t$, $s=t_2$. This yields (11.1.5). To complete the proof, we apply (11.1.8), (11.1.9) for $s=\tau$, $r\in(\sigma,\tau)$, to obtain (11.1.4), and for $r=\sigma$, $s\in(\sigma,\tau)$, to deduce (11.1.6).

**11.1.2 Corollary.** *Assume* $\xi(\cdot)$ *and* $\zeta(\cdot)$ *are distinct generalized characteristics for* (11.1.1), *associated with the admissible weak solution* $u$, *which are shock-free on the time interval* $[\sigma,\tau]$. *Then* $\xi(\cdot)$ *and* $\zeta(\cdot)$ *cannot intersect for any* $t\in(\sigma,\tau)$.

The above two propositions have significant implications on extremal backward characteristics:

**11.1.3 Theorem.** *Let* $\xi_-(\cdot)$ *and* $\xi_+(\cdot)$ *denote the minimal and maximal backward characteristics, associated with some admissible solution* $u$, *emanating from any point* $(\bar{x},\bar{t})\in(-\infty,\infty)\times(0,\infty)$. *Then*

$$\begin{cases} u(\xi_-(t)-,t)=u(\bar{x}-,\bar{t})=u(\xi_-(t)+,t) \\ u(\xi_+(t)-,t)=u(\bar{x}+,\bar{t})=u(\xi_+(t)+,t) \end{cases} \qquad 0<t<\bar{t}, \tag{11.1.10}$$

$$\begin{cases} u_0(\xi_-(0)-) \le u(\bar{x}-,\bar{t}) \le u_0(\xi_-(0)+) \\ \\ u_0(\xi_+(0)-) \le u(\bar{x}+,\bar{t}) \le u_0(\xi_+(0)+). \end{cases} \tag{11.1.11}$$

*In particular,* $u(\bar{x}+,\bar{t}) \le u(\bar{x}-,\bar{t})$ *holds for all* $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$ *and* $\xi_-(\cdot), \xi_+(\cdot)$ *coincide if and only if* $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$.

**Proof.** By virtue of Theorem 10.3.2, both $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are shock-free on $[0,\bar{t}]$. We may then apply Theorem 11.1.1, with $\sigma = 0$ and $\tau = \bar{t}$. When $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$, $\bar{u} = u(\bar{x}\pm,\bar{t})$, by account of (11.1.4), and thus $\xi_-(\cdot), \xi_+(\cdot)$ coincide. In the general case, consider an increasing (or decreasing) sequence $\{x_n\}$, converging to $\bar{x}$, such that $u(x_n+,\bar{t}) = u(x_n-,\bar{t})$, $n = 1, 2, \cdots$. Let $\xi_n(\cdot)$ denote the unique backward characteristic emanating from $(x_n,\bar{t})$. Then $u(\xi_n(t)\pm,t) = u(x_n\pm,\bar{t})$ for all $t \in (0,\bar{t})$. As noted in Section 10.2, the sequence $\{\xi_n(\cdot)\}$ converges from below (or above) to $\xi_-(\cdot)$ (or $\xi_+(\cdot)$). Consequently, $u(\xi_-(t)-,t) = \lim u(x_n\pm,\bar{t}) = u(\bar{x}-,\bar{t})$ (or $u(\xi_+(t)+,t) = \lim u(x_n\pm,\bar{t}) = u(\bar{x}+,\bar{t})$). The proof is complete.

We now turn to the properties of forward characteristics:

**11.1.4 Theorem.** *A unique forward generalized characteristic, associated with an admissible solution* $u$, *issues from any point* $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$.

**Proof.** Suppose two distinct forward characteristics $\phi(\cdot)$ and $\psi(\cdot)$ issue from $(\bar{x},\bar{t})$, such that $\phi(s) < \psi(s)$ for some $s > \bar{t}$. Let $\xi(\cdot)$ denote the maximal backward characteristic emanating from $(\phi(s),s)$ and $\zeta(\cdot)$ denote the minimal backward characteristic emanating from $(\psi(s),s)$, both being shock-free on $[0,s]$. For $t \in [\bar{t},s]$, $\xi(t) \ge \phi(t)$ and $\zeta(t) \le \psi(t)$; hence $\xi(\cdot)$ and $\zeta(\cdot)$ must intersect at some $t \in [\bar{t},s)$, in contradiction to Corollary 11.1.2. This completes the proof.

Note that, by contrast, multiple forward characteristics may issue from points lying on the $x$-axis. In particular, the focus of any centered rarefaction wave must necessarily lie on the $x$-axis.

The next proposition demonstrates that, once they form, jump discontinuities propagate as shock waves for eternity:

**11.1.5 Theorem.** *Let* $\chi(\cdot)$ *denote the unique forward generalized characteristic, associated with the admissible solution* $u$, *issuing from a point* $(\bar{x},\bar{t})$ *such that* $\bar{t} > 0$ *and* $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$. *Then* $u(\chi(s)+,s) < u(\chi(s)-,s)$ *for all* $s \in [\bar{t},\infty)$.

**Proof.** Let $\xi_-(\cdot)$ and $\xi_+(\cdot)$ denote the minimal and maximal backward characteristics emanating from $(\bar{x},\bar{t})$. Since $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$, $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are distinct: $\xi_-(0) < \xi_+(0)$.

Fix any $s \in [\bar{t},\infty)$ and consider the minimal and maximal backward characteristics $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ emanating from $(\chi(s),s)$. For $t \in [0,\bar{t}]$, necessarily $\zeta_-(t) \le \xi_-(t)$ and $\zeta_+(t) \ge \xi_+(t)$. Thus $\zeta_-(0) < \zeta_+(0)$ so that $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ are distinct. Consequently, $u(\chi(s)+,s) < u(\chi(s)-,s)$. This completes the proof.

In view of the above, it is possible to identify the points from which shocks originate:

**11.1.6 Definition.** We call $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times [0, \infty)$ a *shock generation point* if some forward generalized characteristic $\chi(\cdot)$ issuing from $(\bar{x}, \bar{t})$ is a shock, i.e., $u(\chi(t)+, t) < u(\chi(t)-, t)$, for all $t > \bar{t}$, while every backward characteristic emanating from $(\bar{x}, \bar{t})$ is shock-free.

When $(\bar{x}, \bar{t})$ is a shock generation point with $\bar{t} > 0$, there are two possibilities: $u(\bar{x}+, \bar{t}) = u(\bar{x}-, \bar{t})$ or $u(\bar{x}+, \bar{t}) < u(\bar{x}-, \bar{t})$. In the former case, the shock starts out at $(\bar{x}, \bar{t})$ with zero strength and develops as it evolves. In the latter case, distinct minimal and maximal backward characteristics $\xi_-(\cdot)$ and $\xi_+(\cdot)$ emanate from $(\bar{x}, \bar{t})$. The sector confined between the graphs of $\xi_-(\cdot)$ and $\xi_+(\cdot)$ must be filled by characteristics, connecting $(\bar{x}, \bar{t})$ with the $x$-axis, which, by definition, are shock-free and hence are straight lines. Thus in that case the shock is generated at the focus of a *centered compression wave*, so it starts out with positive strength.

## 11.2 The Spreading of Rarefaction Waves

We are already familiar with the destabilizing role of genuine nonlinearity: Compression wave fronts get steeper and eventually break, generating shocks. It turns out, however, that at the same time genuine nonlinearity also exerts a regularizing influence by inducing the spreading of rarefaction wave fronts. It is remarkable that this effect is purely geometric and is totally unrelated to the regularity of the initial data:

**11.2.1 Theorem.** *For any admissible solution $u$,*

$$\frac{f'(u(y\pm, t)) - f'(u(x\pm, t))}{y - x} \le \frac{1}{t}, \quad -\infty < x < y < \infty, \quad 0 < t < \infty. \tag{11.2.1}$$

**Proof.** Fix $x$, $y$ and $t$ with $x < y$ and $t > 0$. Let $\xi(\cdot)$ and $\zeta(\cdot)$ denote the maximal or minimal backward characteristics emanating from $(x, t)$ and $(y, t)$, respectively. By virtue of Theorem 11.1.3, $\xi(0) = x - tf'(u(x\pm, t))$, $\zeta(0) = y - tf'(u(y\pm, t))$. Furthermore, $\xi(0) \le \zeta(0)$, on account of Corollary 11.1.2. This immediately implies (11.2.1). The proof is complete.

Notice that (11.2.1) establishes a one-sided Lipschitz condition for $f'(u(\cdot, t))$, with Lipschitz constant independent of the initial data. By the general theory of scalar conservation laws, presented in Chapter VI, admissible solutions of (11.1.1) with initial data in $L^\infty(-\infty, \infty)$ may be realized as a.e. limits of sequences of solutions with initial data of locally bounded variation on $(-\infty, \infty)$. Consequently, (11.2.1) should hold even for admissible solutions with initial data that are merely in $L^\infty(-\infty, \infty)$. Clearly, (11.2.1) implies that, for fixed $t > 0$, $f'(u(\cdot, t))$, and thereby also $u(\cdot, t)$, have bounded variation over any bounded interval of $(-\infty, \infty)$. We have thus shown that, due to genuine nonlinearity, solutions are generally smoother than their initial data:

**11.2.2 Theorem.** *Admissible solutions of* (11.1.1), *with initial data in* $L^\infty(-\infty, \infty)$, *are in* $BV_{\text{loc}}$ *on* $(-\infty, \infty) \times (0, \infty)$ *and satisfy the one-sided Lipschitz condition* (11.2.1).

## 11.3 Regularity of Solutions

The properties of generalized characteristics established in the previous section lead to a precise description of the structure and regularity of admissible weak solutions.

**11.3.1 Theorem.** *Let* $\chi(\cdot)$ *be the unique forward generalized characteristic and* $\xi_-(\cdot)$, $\xi_+(\cdot)$ *the extremal backward characteristics, associated with an admissible solution* $u$, *emanating from any point* $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$. *Then* $(\bar{x}, \bar{t})$ *is a point of continuity of the function* $u(x-, t)$ *relative to the set* $\{(x, t) : 0 \le t \le \bar{t},\ x \le \xi_-(t)$ *or* $\bar{t} < t < \infty,\ x \le \chi(t)\}$ *and also a point of continuity of the function* $u(x+, t)$ *relative to the set* $\{(x, t) : 0 \le t \le \bar{t},\ x \ge \xi_+(t)$ or $\bar{t} < t < \infty,\ x \ge \chi(t)\}$. *Furthermore,* $\chi(\cdot)$ *is differentiable from the right at* $\bar{t}$ *and*

$$\frac{d^+}{dt}\chi(\bar{t}) = \begin{cases} f'(u(\bar{x}\pm, \bar{t})), & if\ u(\bar{x}+, \bar{t}) = u(\bar{x}-, \bar{t}) \\[2ex] \dfrac{f(u(\bar{x}+, \bar{t})) - f(u(\bar{x}-, \bar{t}))}{u(\bar{x}+, \bar{t}) - u(\bar{x}-, \bar{t})}, & if\ u(\bar{x}+, \bar{t}) < u(\bar{x}-, \bar{t}). \end{cases} \tag{11.3.1}$$

**Proof.** Take any sequence $\{(x_n, t_n)\}$ in the set $\{(x, t) : 0 \le t < \bar{t},\ x \le \xi_-(t)$ or $\bar{t} < t < \infty,\ x \le \chi(t)\}$, which converges to $(\bar{x}, \bar{t})$ as $n \to \infty$. Let $\xi_n(\cdot)$ denote the minimal backward characteristic emanating from $(x_n, t_n)$. Clearly, $\xi_n(t) \le \xi_-(t)$ for $t \le \bar{t}$. Thus, as $n \to \infty$, $\{\xi_n(\cdot)\}$ converges from below to $\xi_-(\cdot)$. Consequently, $\{u(x_n-, t_n)\}$ converges to $u(\bar{x}-, \bar{t})$.

Similarly, for any sequence $\{(x_n, t_n)\}$ in the set $\{(x, t) : 0 \le t < \bar{t},\ x \ge \xi_+(t)$ or $\bar{t} < t < \infty,\ x \ge \chi(t)\}$, converging to $(\bar{x}, \bar{t})$, the sequence $\{u(x_n+, t_n)\}$ converges to $u(\bar{x}+, \bar{t})$.

For $\varepsilon > 0$,

$$\frac{1}{\varepsilon}[\chi(\bar{t} + \varepsilon) - \chi(\bar{t})] = \frac{1}{\varepsilon}\int_{\bar{t}}^{\bar{t}+\varepsilon} \dot{\chi}(t)dt, \tag{11.3.2}$$

where $\dot{\chi}(t)$ is determined through (11.1.3), with $\xi \equiv \chi$. As shown above, $\dot{\chi}(t)$ is continuous from the right at $\bar{t}$ and so, letting $\varepsilon \downarrow 0$ in (11.3.2), we arrive at (11.3.1). This completes the proof.

The above theorem has the following corollary:

**11.3.2 Theorem.** *Let $u$ be an admissible solution and assume $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$, for some $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. Then $(\bar{x},\bar{t})$ is a point of continuity of $u$. A unique generalized characteristic $\chi(\cdot)$, associated with $u$, defined on $[0,\infty)$, passes through $(\bar{x},\bar{t})$. Furthermore, $\chi(\cdot)$ is differentiable at $\bar{t}$ and $\dot{\chi}(\bar{t}) = f'(u(\bar{x}\pm,\bar{t}))$.*

Next we focus attention on points of discontinuity.

**11.3.3 Theorem.** *Let $u$ be an admissible solution and assume $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$, for some $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. When the extremal backward characteristics $\xi_-(\cdot)$, $\xi_+(\cdot)$ are the only backward generalized characteristics emanating from $(\bar{x},\bar{t})$ that are shock-free on $(0,\bar{t})$, then $(\bar{x},\bar{t})$ is a point of jump discontinuity of $u$ in the following sense: There is a generalized characteristic $\chi(\cdot)$, associated with $u$, defined on $[0,\infty)$ and passing through $(\bar{x},\bar{t})$, such that $(\bar{x},\bar{t})$ is a point of continuity of the function $u(x-,t)$ relative to $\{(x,t) : 0 < t < \infty,\ x \le \chi(t)\}$ and also a point of continuity of the function $u(x+,t)$ relative to $\{(x,t) : 0 < t < \infty,\ x \ge \chi(t)\}$. Furthermore, $\chi(\cdot)$ is differentiable at $\bar{t}$ and*

$$\dot{\chi}(\bar{t}) = \frac{f(u(\bar{x}+,\bar{t})) - f(u(\bar{x}-,\bar{t}))}{u(\bar{x}+,\bar{t}) - u(\bar{x}-,\bar{t})}. \tag{11.3.3}$$

**Proof.** Fix any point on the $x$-axis, in the interval $(\xi_-(0),\xi_+(0))$, and connect it to $(\bar{x},\bar{t})$ by a characteristic $\chi(\cdot)$. Extend $\chi(\cdot)$ to $[\bar{t},\infty)$ as the unique forward characteristic issuing from $(\bar{x},\bar{t})$.

Take any sequence $\{(x_n,t_n)\}$ in the set $\{(x,t) : 0 < t < \infty,\ x \le \chi(t)\}$, that converges to $(\bar{x},\bar{t})$, as $n\to\infty$. Let $\xi_n(\cdot)$ denote the minimal backward characteristic emanating from $(x_n,t_n)$. As $n\to\infty$, $\{\xi_n(\cdot)\}$, or a subsequence thereof, will converge to some backward characteristic emanating from $(\bar{x},\bar{t})$, which is a straight line and shock-free. Since $\xi_n(t) \le \chi(t)$, this implies that $\{\xi_n(\cdot)\}$ must necessarily converge to $\xi_-(\cdot)$. Consequently, $\{u(x_n-,t_n)\}$ converges to $u(\bar{x}-,\bar{t})$, as $n\to\infty$.

Similarly, for any sequence $\{(x_n,t_n)\}$ in $\{(x,t) : 0 < t < \infty,\ x \ge \chi(t)\}$, converging to $(\bar{x},\bar{t})$, the sequence $\{u(x_n+,t_n)\}$ converges to $u(\bar{x}+,\bar{t})$.

To verify (11.3.3), we start out again from (11.3.2), where now $\varepsilon$ may be positive or negative. As shown above, $\bar{t}$ is a point of continuity of $\dot{\chi}(t)$ and so, letting $\varepsilon\to 0$, we arrive at (11.3.3). This completes the proof.

**11.3.4 Theorem.** *The set of irregular points of any admissible solution $u$ is (at most) countable. $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$ is an irregular point if and only if $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$ and, in addition to the extremal backward characteristics $\xi_-(\cdot)$, $\xi_+(\cdot)$, there is at least another, distinct, backward characteristic $\xi(\cdot)$, associated with $u$, emanating from $(\bar{x},\bar{t})$, which is shock-free on $(0,\bar{t})$. Irregular points are generated by the collision of shocks and/or by the focusing of centered compression waves.*

**Proof.** Necessity follows from Theorems 11.3.2 and 11.3.3. To show sufficiency, consider the subset $\mathcal{X}$ of the interval $[\xi_-(0),\xi_+(0)]$ with the property that, for $x \in \mathcal{X}$, the straight line segment connecting the points $(x, 0)$ and $(\bar{x}, \bar{t})$ is a characteristic associated with $u$, which is shock-free on $(0, \bar{t})$.

When $\mathcal{X} \equiv [\xi_-(0), \xi_+(0)]$ , $(\bar{x}, \bar{t})$ is the focus of a centered compression wave and the assertion of the theorem is clearly valid. In general, however, $\mathcal{X}$ will be a closed proper subset of $[\xi_-(0), \xi_+(0)]$, containing at least the three points $\xi_-(0), \xi(0)$ and $\xi_+(0)$. The complement of $\mathcal{X}$ relative to $[\xi_-(0), \xi_+(0)]$ will then be the (at most) countable union of disjoint open intervals. Let $(\alpha_-, \alpha_+)$ be one of these intervals, contained, say in $(\xi_-(0), \xi(0))$. The straight line segments connecting the points $(\alpha_-, 0)$ and $(\alpha_+, 0)$ with $(\bar{x}, \bar{t})$ will be shock-free characteristics $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ along which $u$ is constant, say $u_-$ and $u_+$. Necessarily, $u(\bar{x}-, \bar{t}) \geq u_- > u_+ > u(\bar{x}+, \bar{t})$. Consider a characteristic $\chi(\cdot)$ connecting a point of $(\alpha_-, \alpha_+)$ with $(\bar{x}, \bar{t})$. Then $\zeta_-(t) < \chi(t) < \zeta_+(t)$, $0 \leq t < \bar{t}$. Take any sequence $\{(x_n, t_n)\}$ in the set $\{(x, t) : 0 \leq t < \bar{t},\ \zeta_-(t) \leq x \leq \chi(t)\}$, converging to $(\bar{x}, \bar{t})$, as $n \to \infty$. If $\xi_n(\cdot)$ denotes the minimal backward characteristic emanating from $(x_n, t_n)$, the sequence $\{\xi_n(\cdot)\}$ will necessarily converge to $\zeta_-(\cdot)$. In particular, this implies $u(x_n-, t_n) \longrightarrow u_-$, as $n \to \infty$. Similarly one shows that if $\{(x_n, t_n)\}$ is any sequence in the set $\{(x, t) : 0 \leq t < \bar{t},\ \chi(t) \leq x \leq \zeta_+(t)\}$ converging to $(\bar{x}, \bar{t})$, then $u(x_n+, t_n) \longrightarrow u_+$, as $n \to \infty$. Thus, near $\bar{t}$ $\chi(\cdot)$ is a shock, which is differentiable from the left at $\bar{t}$ with

$$\frac{d^-}{dt}\chi(\bar{t}) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{11.3.4}$$

Since $f'(u_-) > \frac{d^-}{dt}\chi(\bar{t}) > f'(u_+)$, we conclude that $(\bar{x}, \bar{t})$ is an irregular point of $u$.

We have thus shown that $(\bar{x}, \bar{t})$ is a point of collision of shocks, one for each open interval of the complement of $\mathcal{X}$, and centered compression waves, when the measure of $\mathcal{X}$ is positive.

For fixed positive $\varepsilon$, we consider irregular points $(\bar{x}, \bar{t})$, as above, with the additional property $\xi_+(0) - \xi(0) > \varepsilon$, $\xi(0) - \xi_-(0) > \varepsilon$. It is easy to see that one may fit an at most finite set of such points in any bounded subset of the upper half-plane. This in turn implies that the set of irregular points of any admissible solution is (at most) countable. The proof is complete.

The effect of genuine nonlinearity, reflected in the properties of characteristics, is either to smooth out solutions by rarefaction or to form jump discontinuities through compression. Aspects of this polarizing influence, which inhibits the existence of solutions with "intermediate" regularity, are manifested in the following Theorems 11.3.5, 11.3.6 and 11.3.10.

To begin with, every admissible $BV$ solution is necessarily a special function of bounded variation, in the sense of Definition 1.7.9:

**11.3.5 Theorem.** *There is an (at most) countable set $\mathcal{T} \subset [0, \infty)$ such that, for any $t \in [0, \infty)\backslash\mathcal{T}$, $u(\cdot, t)$ belongs to $SBV_{\mathrm{loc}}(-\infty, \infty)$. Furthermore, $u$ belongs to $SBV_{\mathrm{loc}}\big((-\infty, \infty) \times [0, \infty)\big)$.*

The proof of the above proposition is found in the literature cited in Section 11.12.

Next, we shall see that continuity is automatically upgraded to Lipschitz continuity:

**11.3.6 Theorem.** *Assume the set $\mathcal{C}$ of points of continuity of an admissible solution u has nonempty interior $\mathcal{C}^0$. Then u is locally Lipschitz on $\mathcal{C}^0$.*

**Proof.** Fix any point $(\bar{x}, \bar{t}) \in \mathcal{C}^0$ and assume that the circle $\mathcal{B}_r$ of radius $r$, centered at $(\bar{x}, \bar{t})$, is contained in $\mathcal{C}^0$. Consider any point $(x, t)$ at a distance $\rho < r$ from $(\bar{x}, \bar{t})$. The (unique) characteristics, associated with $u$, passing through $(\bar{x}, \bar{t})$ and $(x, t)$ are straight lines with slopes $f'(u(\bar{x}, \bar{t}))$ and $f'(u(x, t))$, respectively, which cannot intersect inside the circle $\mathcal{B}_r$. Elementary trigonometric estimations then imply that $|f'(u(x, t)) - f'(u(\bar{x}, \bar{t}))|$ cannot exceed $c\rho/r$, where $c$ is any upper bound of $1 + f'(u)^2$ over $\mathcal{B}_r$. Hence, if $a > 0$ is a lower bound of $f''(u)$ over $\mathcal{B}_r$, $|u(x, t) - u(\bar{x}, \bar{t})| \leq \frac{c}{ar}\rho$. This completes the proof.

The reader should be aware that admissible solutions have been constructed whose set of points of continuity has empty interior.

We now investigate the regularity of admissible solutions with smooth initial data. In what follows, it shall be assumed that $f$ is $C^{k+1}$ and $u$ is the admissible solution with $C^k$ initial data $u_0$, for some $k \in \{1, 2, \cdots, \infty\}$.

For $(x, t) \in (-\infty, \infty) \times (0, \infty)$, we let $y_-(x, t)$ and $y_+(x, t)$ denote the interceptors on the $x$-axis of the minimal and maximal backward characteristics, associated with $u$, emanating from the point $(x, t)$. In particular,

$$x = y_-(x, t) + tf'(u_0(y_-(x, t)) = y_+(x, t) + tf'(u_0(y_+(x, t)), \tag{11.3.5}$$

$$u(x-, t) = u_0(y_-(x, t)), \quad u(x+, t) = u_0(y_+(x, t)). \tag{11.3.6}$$

For fixed $t > 0$, both $y_-(\cdot, t)$ and $y_+(\cdot, t)$ are monotone nondecreasing and the first one is continuous from the left while the second is continuous from the right. Consequently,

$$1 + t\frac{d}{dy}f'(u_0(y)) \geq 0, \quad y = y_\pm(x, t), \tag{11.3.7}$$

holds for all $(x, t) \in (-\infty, \infty) \times (0, \infty)$.

Any point $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$ of continuity of $u$ is necessarily also a point of continuity of $y_\pm(x, t)$ and $y_-(\bar{x}, \bar{t}) = y_+(\bar{x}, \bar{t})$. Therefore, by virtue of (11.3.5), (11.3.6) and the implicit function theorem we deduce

**11.3.7 Theorem.** *If $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$ is a point of continuity of u and*

$$1 + \bar{t}\frac{d}{dy}f'(u_0(y)) > 0, \quad y = y_\pm(\bar{x}, \bar{t}), \tag{11.3.8}$$

*then u is $C^k$ on a neighborhood of $(\bar{x}, \bar{t})$.*

With reference to Theorem 11.3.3, if $(\bar{x}, \bar{t})$ is a point of jump discontinuity of $u$, then $(\bar{x}, \bar{t})$ is a point of continuity of $y_-(x,t)$ and $y_+(x,t)$ relative to the sets $\{(x,t) : 0 < t < \infty, \ \ x \leq \chi(t)\}$ and $\{(x,t) : 0 < t < \infty, \ \ x \geq \chi(t)\}$, respectively. Consequently, the implicit function theorem together with (11.3.5) and (11.3.6) yields

**11.3.8 Theorem.** *If* (11.3.8) *holds at a point* $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$ *of jump discontinuity of* $u$, *then, in a neighborhood of* $(\bar{x}, \bar{t})$, *the shock* $\chi(\cdot)$ *passing through* $(\bar{x}, \bar{t})$ *is* $C^{k+1}$ *and* $u$ *is* $C^k$ *on either side of the graph of* $\chi(\cdot)$.

Next we consider shock generation points, introduced by Definition 11.1.6.

**11.3.9 Theorem.** *If* $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$ *is a shock generation point, then*

$$1 + \bar{t}\frac{d}{dy}f'(u_0(y)) = 0, \qquad y_-(\bar{x}, \bar{t}) \leq y \leq y_+(\bar{x}, \bar{t}). \tag{11.3.9}$$

*Furthermore, when* $k \geq 2$,

$$\frac{d^2}{dy^2}f'(u_0(y)) = 0, \qquad y_-(\bar{x}, \bar{t}) \leq y \leq y_+(\bar{x}, \bar{t}). \tag{11.3.10}$$

**Proof.** Recall that there are two types of shock generation points: points of continuity, in which case $y_-(\bar{x}, \bar{t}) = y_+(\bar{x}, \bar{t})$, and foci of centered compression waves, with $y_-(\bar{x}, \bar{t}) < y_+(\bar{x}, \bar{t})$. When $(\bar{x}, \bar{t})$ is a point of continuity, (11.3.9) is a consequence of (11.3.7) and Theorem 11.3.7. When $(\bar{x}, \bar{t})$ is the focus of a compression wave, $\bar{x} = y + \bar{t}f'(u_0(y))$ for any $y \in [y_-(\bar{x}, \bar{t}), y_+(\bar{x}, \bar{t})]$ and this implies (11.3.9).

When $y_-(\bar{x}, \bar{t}) < y_+(\bar{x}, \bar{t})$, differentiation of (11.3.9) with respect to $y$ yields (11.3.10). To establish (11.3.10) for the case $(\bar{x}, \bar{t})$ is a point of continuity, we take any sequence $\{x_n\}$ that converges from below (or above) to $\bar{x}$. Then $\{y_-(x_n, \bar{t})\}$ will approach from below (or above) $y_\pm(\bar{x}, \bar{t})$. Because of (11.3.7), $1+\bar{t}\frac{d}{dy}f'(u_0(y)) \geq 0$ for $y = y_-(x_n, \bar{t})$; and this together with (11.3.9) imply that $y_\pm(\bar{x}, \bar{t})$ is a critical point of $\frac{d}{dy}f'(u_0(y))$. The proof is complete.

For $k \geq 3$, the set of functions $u_0$ in $C^k$ with the property that $\frac{d}{dy}f'(u_0(y))$ has infinitely many critical points in a bounded interval is of the first category. Therefore, generically, initial data $u_0 \in C^k$, with $k \geq 3$, induce solutions with a locally finite set of shock generation points and thereby with a locally finite set of shocks. In other words,

**11.3.10 Theorem.** *Generically, admissible solutions of* (11.1.1) *with initial data in* $C^k$, $k \geq 3$, *are piecewise* $C^k$ *smooth functions and do not contain centered compression waves. In particular, solutions with analytic initial data are always piecewise analytic.*

## 11.4 Divides, Invariants and the Lax Formula

The theory of generalized characteristics will be used here to establish interesting and fundamental properties of admissible solutions of (11.1.1). The starting point will be a simple but, as we shall see, very useful identity.

Let us consider two admissible solutions $u$ and $u^*$, with corresponding initial values $u_0$ and $u_0^*$, and trace one of the extremal backward characteristics $\xi(\cdot)$, associated with $u$, and one of the extremal backward characteristics $\xi^*(\cdot)$, associated with $u^*$, that emanate from any fixed point $(x,t) \in (-\infty,\infty) \times (0,\infty)$. Thus, $\xi(\cdot)$ and $\xi^*(\cdot)$ will be straight lines, and along $\xi(\cdot)$ $u$ will be constant, equal to $u(x-,t)$ or $u(x+,t)$, while along $\xi^*(\cdot)$ $u^*$ will be constant, equal to $u^*(x-,t)$ or $u^*(x+,t)$. In particular, $\dot{\xi}(\tau) = f'(u(x\pm,t))$ and $\dot{\xi}^*(\tau) = f'(u^*(x\pm,t))$ , $0 < \tau < t$.

We write (11.1.1), first for $u$ then for $u^*$, we subtract the resulting two equations, we integrate over the triangle with vertices $(x,t)$, $(\xi(0),0)$, $(\xi^*(0),0)$, and apply Green's theorem thus arriving at the identity

(11.4.1)
$$\int_0^t \{f(u(x\pm,t)) - f(u^*(\xi(\tau)-,\tau)) - f'(u(x\pm,t))[u(x\pm,t) - u^*(\xi(\tau)-,\tau)]\}d\tau$$
$$+\int_0^t \{f(u^*(x\pm,t)) - f(u(\xi^*(\tau)-,\tau)) - f'(u^*(x\pm,t))[u^*(x\pm,t) - u(\xi^*(\tau)-,\tau)]\}d\tau$$
$$= \int_{\xi^*(0)}^{\xi(0)} [u_0(y) - u_0^*(y)]dy.$$

The usefulness of (11.4.1) lies in that, because of the convexity of $f$, both integrals on the left-hand side are nonpositive.

As a first application of (11.4.1), we use it to locate divides associated with an admissible solution $u$. The notion of divide was introduced by Definition 10.3.3. In the context of the genuinely nonlinear scalar conservation law, following the discussion in Section 10.3, divides are shock-free and hence, by virtue of Theorem 11.1.1, straight lines along which $u$ is constant.

**11.4.1 Theorem.** *A divide, associated with the admissible solution $u$, with initial data $u_0$, along which $u$ is constant $\bar{u}$, issues from the point $(\bar{x}, 0)$ of the $x$-axis if and only if*

(11.4.2)
$$\int_{\bar{x}}^{z} [u_0(y) - \bar{u}]dy \geq 0, \qquad -\infty < z < \infty.$$

**Proof.** Assume that (11.4.2) holds. We apply (11.4.1) with $u^* \equiv \bar{u}$, $t \in (0,\infty)$, and $x = \bar{x} + tf'(\bar{u})$. In particular, $\xi^*(\tau) = \bar{x} + \tau f'(\bar{u})$ and $\xi^*(0) = \bar{x}$. Hence the right-hand side of (11.4.1) is nonnegative, on account of (11.4.2). But then both integrals on the left-hand side must vanish, so that $u(x\pm,t) = \bar{u}$. We have thus established that the straight line $x = \bar{x} + tf'(\bar{u})$ is a shock-free characteristic on $[0,\infty)$, which is a divide associated with $u$.

Conversely, assume the straight line $x = \bar{x} + tf'(\bar{u})$ is a divide associated with $u$. Take any $z \in (-\infty, \infty)$ and fix $\tilde{u}$ such that $\tilde{u} < \bar{u}$ if $z > \bar{x}$ and $\tilde{u} > \bar{u}$ if $z < \bar{x}$. The straight lines $z + tf'(\tilde{u})$ and $\bar{x} + tf'(\bar{u})$ will then intersect at a point $(x, t)$ with $t > 0$. We apply (11.4.1) with $u^* \equiv \tilde{u}$, in which case $\xi(0) = \bar{x}$, $\xi^*(0) = z$. The left-hand side is nonpositive and so

$$\int_z^{\bar{x}} [u_0(y) - \tilde{u}]dy \leq 0. \tag{11.4.3}$$

Letting $\tilde{u} \to \bar{u}$ we arrive at (11.4.2). This completes the proof.

The above proposition has implications for the existence of important time invariants of solutions:

**11.4.2 Theorem.** *Assume $u_0$ is integrable over $(-\infty, \infty)$ and the maxima*

$$\max_x \int_x^{-\infty} u_0(y)dy = q_- , \qquad \max_x \int_x^{\infty} u_0(y)dy = q_+ \tag{11.4.4}$$

*exist. If $u$ is the admissible solution with initial data $u_0$, then, for any $t > 0$,*

$$\max_x \int_x^{-\infty} u(y,t)dy = q_- , \qquad \max_x \int_x^{\infty} u(y,t)dy = q_+ . \tag{11.4.5}$$

**Proof.** Notice that $q_-$ exists if and only if $q_+$ exists and in fact, by virtue of Theorem 11.4.1, both maxima are attained on the set of $\bar{x}$ with the property that the straight line $x = \bar{x} + tf'(0)$ is a divide associated with $u$, along which $u$ is constant, equal to zero. But then, again by Theorem 11.4.1, both maxima in (11.4.5) will be attained at $\hat{x} = \bar{x} + tf'(0)$.

We now normalize $f$ by $f(0) = 0$ and take the integral of (11.1.1), first over the domain $\{(y, \tau) : 0 < \tau < t,\ -\infty < y < \bar{x} + \tau f'(0)\}$ and then also over the domain $\{(y, \tau) : 0 < \tau < t,\ \bar{x} + \tau f'(0) < y < \infty\}$. Applying Green's theorem, and since $u$ vanishes along the straight line $x = \bar{x} + \tau f'(0)$,

$$\int_{\hat{x}}^{-\infty} u(y,t)dy = \int_{\bar{x}}^{-\infty} u_0(y)dy, \quad \int_{\hat{x}}^{\infty} u(y,t)dy = \int_{\bar{x}}^{\infty} u_0(y)dy, \tag{11.4.6}$$

which verifies (11.4.5). The proof is complete.

One of the most striking features of genuinely nonlinear scalar conservation laws is that admissible solutions may be determined explicitly from the initial data by the following procedure. We start out with the Legendre transform

$$g(v) = \max_u [uv - f(u)], \tag{11.4.7}$$

noting that the maximum is attained at $u = [f']^{-1}(v)$. With given initial data $u_0(\cdot)$ we associate the *Lax function*

(11.4.8)
$$G(y;x,t)=\int_0^y u_0(z)dz+tg\left(\frac{x-y}{t}\right),$$

defined for $(x,t)\in(-\infty,\infty)\times(0,\infty)$ and $y\in(-\infty,\infty)$.

**11.4.3 Theorem.** *For fixed* $(x,t)\in(-\infty,\infty)\times(0,\infty)$, *the Lax function* $G(y;x,t)$ *is minimized at a point* $\bar{y}\in(-\infty,\infty)$ *if and only if the straight line segment that connects the points* $(x,t)$ *and* $(\bar{y},0)$ *is a generalized characteristic associated with the admissible solution u with initial data* $u_0$, *which is shock-free on* $(0,t)$.

**Proof.** We fix $y$ and $\bar{y}$ in $(-\infty,\infty)$, integrate (11.1.1) over the triangle with vertices $(x,t)$, $(y,0)$, $(\bar{y},0)$, and apply Green's theorem to get

(11.4.9)
$$\int_0^{\bar{y}} u_0(z)dz+\int_0^t\left[\frac{x-\bar{y}}{t}u(\bar{y}+\tau\frac{x-\bar{y}}{t}\pm,\tau)-f(u(\bar{y}+\tau\frac{x-\bar{y}}{t}\pm,\tau))\right]d\tau$$
$$=\int_0^{y} u_0(z)dz+\int_0^t\left[\frac{x-y}{t}u(y+\tau\frac{x-y}{t}\pm,\tau)-f(u(y+\tau\frac{x-y}{t}\pm,\tau))\right]d\tau.$$

By virtue of (11.4.7) and (11.4.8), the left-hand side of (11.4.9) is less than or equal to $G(\bar{y};x,t)$, with equality holding if and only if $f'(u(\bar{y}+\tau\frac{x-\bar{y}}{t}\pm,\tau))=\frac{x-\bar{y}}{t}$, almost everywhere on $(0,t)$, i.e., if and only if the straight line segment that connects the points $(x,t)$ and $(\bar{y},0)$ is a shock-free characteristic. Similarly, the right-hand side of (11.4.9) is less than or equal to $G(y;x,t)$, with equality holding if and only if the straight line segment that connects the points $(x,t)$ and $(y,0)$ is a shock-free characteristic. Assuming then that the straight line segment connecting $(x,t)$ with $(\bar{y},0)$ is indeed a shock-free characteristic, we deduce from (11.4.9) that $G(\bar{y};x,t)\leq G(y;x,t)$ for any $y\in(-\infty,\infty)$.

Conversely, assume $G(\bar{y};x,t)\leq G(y;x,t)$, for all $y\in(-\infty,\infty)$. In particular, pick $y$ so that $(y,0)$ is the intercept by the $x$-axis of the minimal backward characteristic emanating from $(x,t)$. As shown above, $y$ is a minimizer of $G(\cdot;x,t)$ and so $G(y;x,t)=G(\bar{y};x,t)$. Moreover, the right-hand side of (11.4.9) equals $G(y;x,t)$ and so the left-hand side equals $G(y;x,t)$. As explained above, this implies that the straight line segment connecting $(x,t)$ with $(\bar{y},0)$ is a shock-free characteristic. The proof is complete.

The above proposition may be used to determine the admissible solution $u$ from the initial data $u_0$: For fixed $(x,t)\in(-\infty,\infty)\times(0,\infty)$, we let $y_-$ and $y_+$ denote the smallest and the largest minimizer of $G(\cdot\,;x,t)$ over $(-\infty,\infty)$. We then have

(11.4.10)
$$u(x\pm,t)=[f']^{-1}\left(\frac{x-y_\pm}{t}\right).$$

By account of Theorems 11.3.2, 11.3.3 and 11.3.4, we conclude that $(x,t)$ is a point of continuity of $u$ if and only if $y_-=y_+$; a point of jump discontinuity of $u$ if and only if $y_- < y_+$ and $y_-, y_+$ are the only minimizers of $G(\cdot; x, t)$; or an irregular point of $u$ if and only if $y_- < y_+$ and there exist additional minimizers of $G(\cdot\,; x, t)$ in the interval $(y_-, y_+)$. One may develop the entire theory of the Cauchy problem for genuinely nonlinear scalar conservation laws on the basis of the above construction of admissible solutions, in lieu of the approach via generalized characteristics. It should be noted, however, that the method of generalized characteristics affords greater flexibility, as it applies to solutions defined on arbitrary open subsets of $\mathbb{R}^2$, not necessarily on the entire upper half-plane.

The change of variables $u = \partial_x v$, reduces the conservation law (11.1.1) to the *Hamilton-Jacobi equation*

$$\partial_t v(x,t) + f(\partial_x v(x,t)) = 0. \tag{11.4.11}$$

In that context, $u$ is an admissible weak solution of (11.1.1) if and only if $v$ is a *viscosity solution* of (11.4.11); (references in Section 11.12). In fact, Theorems 11.4.2 and 11.4.3 reflect properties of solutions of Hamilton-Jacobi equations rather than of hyperbolic conservation laws, in that they readily extend to the multi-space dimensional versions of the former though not of the latter.

## 11.5 Decay of Solutions Induced by Entropy Dissipation

Genuine nonlinearity gives rise to a multitude of dissipative mechanisms which, acting individually or collectively, affect the large time behavior of solutions. In this section we shall get acquainted with examples in which the principal agent of damping is entropy dissipation.

**11.5.1 Theorem.** *Let $u$ be the admissible solution with initial data $u_0$ such that*

$$\int_x^{x+\ell} u_0(y)dy = O(\ell^r), \qquad \text{as } \ell \to \infty, \tag{11.5.1}$$

*for some $r \in [0,1)$, uniformly in $x$ on $(-\infty,\infty)$. Then*

$$u(x\pm,t) = O\left(t^{-\frac{1-r}{2-r}}\right), \qquad \text{as } t \to \infty, \tag{11.5.2}$$

*uniformly in $x$ on $(-\infty,\infty)$.*

**Proof.** We fix $(x,t) \in (-\infty,\infty) \times (0,\infty)$ and write (11.4.1) for $u^* \equiv 0$. Notice that $\xi(0) - \xi^*(0) = t[f'(u(x\pm,t)) - f'(0)]$. Also recall that both integrals on the left-hand side are nonpositive. Consequently, using (11.5.1), we deduce

$$\Phi(u(x\pm,t)) = O(t^{r-1}), \quad \text{as } t \to \infty\,, \tag{11.5.3}$$

uniformly in $x$ on $(-\infty,\infty)$, where we have set

$$\Phi(u) = \frac{f(0) - f(u) + uf'(u)}{|f'(u) - f'(0)|^r} = \frac{\int_0^u vf''(v)dv}{|\int_0^u f''(v)dv|^r} . \tag{11.5.4}$$

A simple estimation yields $\Phi(u) \geq K|u|^{2-r}$, with $K > 0$, and so (11.5.3) implies (11.5.2). This completes the proof.

In particular, when $u_0 \in L^p$ (11.5.1) holds with $r = 1 - \frac{1}{p}$, by virtue of Hölder's inequality. Therefore, Theorem 11.5.1 has the following corollary:

**11.5.2 Theorem.** *Let $u$ be the admissible solution with initial data $u_0$ in $L^p(-\infty, \infty)$, $1 \leq p < \infty$. Then*

$$u(x\pm, t) = O\left(t^{-\frac{p}{p+1}}\right), \qquad as \;\; t \to \infty, \tag{11.5.5}$$

*uniformly in $x$ on $(-\infty, \infty)$.*

In the above examples, the comparison function was the solution $u^* \equiv 0$. Next we consider the case where the comparison function is the solution of a Riemann problem comprising two constant states $u_-$ and $u_+$, $u_- > u_+$, joined by a shock, namely,

$$u^*(x, t) = \begin{cases} u_- , \; x < st \\ u_+ , \; x > st, \end{cases} \tag{11.5.6}$$

where

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} . \tag{11.5.7}$$

**11.5.3 Theorem.** *Let $u$ denote the admissible solution with intial data $u_0$ such that the improper integrals $\int_{-\infty}^0 [u_0(y) - u_-]dy$ and $\int_0^\infty [u_0(y) - u_+]dy$ exist, for $u_-$ and $u_+$ with $u_- > u_+$. Normalize the origin $x = 0$ so that*

$$\int_{-\infty}^0 [u_0(y) - u_-]dy + \int_0^\infty [u_0(y) - u_+]dy = 0. \tag{11.5.8}$$

*Consider any forward characteristic $\chi(\cdot)$ issuing from $(0, 0)$. Then, as $t \to \infty$,*

$$\chi(t) = st + o(1), \tag{11.5.9}$$

*with $s$ given by (11.5.7), and*

$$u(x\pm, t) = \begin{cases} u_- + o(t^{-1/2}), \; uniformly \;\; for \; x < \chi(t) \\ u_+ + o(t^{-1/2}), \; uniformly \;\; for \; x > \chi(t). \end{cases} \tag{11.5.10}$$

**Proof.** Fix any $(x,t) \in (-\infty,\infty) \times (0,\infty)$ and write (11.4.1) for the solution $u$, with initial data $u_0$, and the comparison solution $u^*$ given by (11.5.6). By virtue of $f'(u_-) > s > f'(u_+)$, as $t \to \infty$, $\xi^*(0) \to -\infty$, uniformly in $x$ on $(-\infty, st)$, and $\xi^*(0) \to \infty$, uniformly in $x$ on $(st,\infty)$. Similarly, as $t \to \infty$, $\xi(0) \to -\infty$, uniformly in $x$ on $(-\infty, \chi(t))$, and $\xi(0) \to \infty$ uniformly in $x$ on $(\chi(t), \infty)$. Indeed, in the opposite case one would be able to find a sequence $\{(x_n, t_n)\}$, with $t_n \to \infty$ as $n \to \infty$, such that the intercepts $\xi_n(0)$ of the minimal backward characteristics $\xi_n(\cdot)$ emanating from $(x_n, t_n)$ are confined in a bounded set. But then some subsequence of $\{\xi_n(\cdot)\}$ would converge to a divide issuing from some point $(\bar{x}, 0)$. However, this is impossible, because, since $u_- > u_+$, (11.5.8) is incompatible with (11.4.2), for any $\bar{x} \in (-\infty,\infty)$ and every $\bar{u} \in (-\infty,\infty)$.

In view of the above, (11.5.8) implies that the right-hand side of (11.4.1) is $o(1)$, as $t \to \infty$, uniformly in $x$ on $(-\infty,\infty)$. The same will then be true for each integral on the left-hand side of (11.4.1), because they are of the same sign (nonpositive).

Consider first points $(x,t) \in (-\infty,\infty) \times (0,\infty)$ with $x < \min\{\chi(t), st\}$. Then $\xi(\tau) < s\tau$, $0 < \tau < t$, and so the first integral on the left-hand side of (11.4.1) yields

$$t\{f(u(x\pm,t)) - f(u_-) - f'(u(x\pm,t))[u(x\pm,t) - u_-]\} = o(1). \tag{11.5.11}$$

Since $f$ is uniformly convex, (11.5.11) implies $u(x\pm,t) - u_- = o(t^{-1/2})$.

A similar argument demonstrates that for points $(x,t) \in (-\infty,\infty) \times (0,\infty)$ with $x > \max\{\chi(t), st\}$ we have $u(x\pm,t) - u_+ = o(t^{-1/2})$.

Next, consider points $(x,t) \in (-\infty,\infty) \times (0,\infty)$ with $st \le x < \chi(t)$. Then $\xi(\cdot)$ will have to intersect the straight line $x = s\tau$, say at $\tau = r$, $r \in [0,t]$, in which case the first integral on the left-hand side of (11.4.1) gives

$$(t-r)\{f(u(x\pm,t)) - f(u_+) - f'(u(x\pm,t))[u(x\pm,t) - u_+]\} = o(1), \tag{11.5.12}$$

$$r\{f(u(x\pm,t)) - f(u_-) - f'(u(x\pm,t))[u(x\pm,t) - u_-]\} = o(1). \tag{11.5.13}$$

For $x < \chi(t)$, it was shown above that $\xi(0) \to -\infty$, as $t \to \infty$, and this in turn implies $r \to \infty$. It then follows from (11.5.13) and the convexity of $f$ that $u(x\pm,t) = u_- + o(1)$. Then (11.5.12) implies that $t - r = o(1)$ so that $\chi(t) - st = o(1)$ and (11.5.13) yields (11.5.11). From (11.5.11) and the convexity of $f$ we deduce, as before, $u(x\pm,t) - u_- = o(t^{-1/2})$.

A similar argument establishes that for points $(x,t) \in (-\infty,\infty) \times (0,\infty)$ with $\chi(t) < x \le st$ we have $u(x\pm,t) - u_+ = o(t^{-1/2})$ and also $\chi(t) - st = o(1)$. This completes the proof.

## 11.6 Spreading of Characteristics and Development of $N$-Waves

Another feature of genuine nonlinearity, affecting the large time behavior of solutions, is spreading of characteristics. In order to see the effects of this mechanism, we shall study the asymptotic behavior of solutions with initial data of compact support. We already know, by account of Theorem 11.5.2, that the amplitude decays to zero as $O(t^{-1/2})$. The closer examination here will reveal that asymptotically the solution attains the profile of an *N-wave*, namely, a centered rarefaction wave flanked from both sides by shocks whose amplitudes decay like $O(t^{-1/2})$.

**11.6.1 Theorem.** *Let $u$ be the admissible solution with initial data $u_0$, such that $u_0(x) = 0$ for $|x| > \ell$. Consider the minimal forward characteristic $\chi_-(\cdot)$ issuing from $(-\ell, 0)$ and the maximal forward characteristic $\chi_+(\cdot)$ issuing from $(\ell, 0)$. Then*

$$u(x\pm, t) = 0, \quad for\ \ t > 0\ \ and\ \ x < \chi_-(t)\ \ or\ \ x > \chi_+(t). \tag{11.6.1}$$

*As $t \to \infty$,*

$$f'(u(x\pm, t)) = \frac{x}{t} + O(\frac{1}{t}), \quad uniformly\ \ for\ \ \chi_-(t) < x < \chi_+(t), \tag{11.6.2}$$

$$u(x\pm, t) = \frac{1}{f''(0)}[\frac{x}{t} - f'(0)] + O(\frac{1}{t}), \quad uniformly\ \ for\ \ \chi_-(t) < x < \chi_+(t), \tag{11.6.3}$$

$$\begin{cases} \chi_-(t) = tf'(0) - [2q_- tf''(0)]^{1/2} + O(1) \\ \chi_+(t) = tf'(0) + [2q_+ tf''(0)]^{1/2} + O(1), \end{cases} \tag{11.6.4}$$

*with $q_-$ and $q_+$ given by* (11.4.4). *Moreover, the decreasing variation of $u(\cdot, t)$ over the interval $[\chi_-(t), \chi_+(t)]$ is $O(t^{-1})$.*

**Proof.** Since $\chi_-(\cdot)$ is minimal and $\chi_+(\cdot)$ is maximal, the extremal backward characteristics emanating from any point $(x, t)$ with $t > 0$ and $x < \chi_-(t)$ or $x > \chi_+(t)$ will be intercepted by the $x$-axis outside the support of $u_0$. This establishes (11.6.1).

On the other hand, the minimal or maximal backward characteristic $\xi(\cdot)$ emanating from a point $(x, t)$ with $t > 0$ and $\chi_-(t) < x < \chi_+(t)$ will be intercepted by the $x$-axis inside the interval $[-\ell, \ell]$, i.e., $\xi(0) \in [-\ell, \ell]$. Consequently, as $t \to \infty$, $x - tf'(u(x\pm, t)) = \xi(0) = O(1)$, which yields (11.6.2).[1]

By account of Theorem 11.5.2, $u$ is $O(t^{-1/2})$, as $t \to \infty$, and thus, assuming $f$ is $C^3$, $f'(u) = f'(0) + f''(0)u + O(t^{-1})$. Therefore, (11.6.3) follows from (11.6.2).

To derive the asymptotics of $\chi_\pm(t)$, as $t \to \infty$, we first note that on account of $0 \geq \dot{\chi}_-(t) - f'(0) \geq O(t^{-1/2})$, $0 \leq \dot{\chi}_+(t) - f'(0) \leq O(t^{-1/2})$ and this in turn yields $0 \geq \chi_-(t) - tf'(0) \geq O(t^{1/2})$, $0 \leq \chi_+(t) - tf'(0) \leq O(t^{1/2})$. Next

[1] As $t \to \infty$, the $\xi(0)$ accumulate at the set of points from which divides originate. In the generic case where (11.4.2) holds, with $\bar{u} = 0$, at a single point $\bar{x}$, which we normalize so that $\bar{x} = 0$, the $\xi(0)$ accumulate at the origin and hence in (11.6.2) $O(t^{-1})$ is upgraded to $o(t^{-1})$. When, in addition, $u_0$ is $C^1$ and $u_0'(0) > 0$, then in (11.6.2) $O(t^{-1})$ is improved to $O(t^{-2})$ and, for $t$ large, the profile $u(\cdot, t)$ is $C^1$ on the interval $(\chi_-(t), \chi_+(t))$.

we appeal to Theorem 11.4.2: A divide $x = \bar{x} + tf'(0)$ originates from some point $(\bar{x}, 0)$, with $\bar{x} \in [-\ell, \ell]$, along which $u$ is zero, and for any $t > 0$,

$$\int_{\bar{x}+tf'(0)}^{\chi_-(t)} u(y,t)dy = q_- , \qquad \int_{\bar{x}+tf'(0)}^{\chi_+(t)} u(y,t)dy = q_+ . \tag{11.6.5}$$

In (11.6.5) we insert $u$ from its asymptotic form (11.6.3), and after performing the simple integration we deduce

$$\frac{1}{2q_\pm tf''(0)}[\chi_\pm(t) - tf'(0)]^2 = 1 + O(t^{-1/2}) \tag{11.6.6}$$

whence (11.6.4) follows. The proof is complete.

## 11.7 Confinement of Characteristics and Formation of Saw-toothed Profiles

The confinement of the intercepts of extremal backward characteristics in a bounded interval of the $x$-axis induces bounds on the decreasing variation of characteristic speeds and thereby, by virtue of genuine nonlinearity, on the decreasing variation of the solution itself.

**11.7.1 Theorem.** *Let $\chi_-(\cdot)$ and $\chi_+(\cdot)$ be generalized characteristics on* $[0, \infty)$, *associated with an admissible solution $u$, and $\chi_-(t) < \chi_+(t)$ for $t \in [0, \infty)$. Then, for any $t > 0$, the decreasing variation of the function $f'(u(\cdot, t))$ over the interval $(\chi_-(t), \chi_+(t))$ cannot exceed $[\chi_+(0) - \chi_-(0)]t^{-1}$. Thus the decreasing variation of $u(\cdot, t)$ over the interval $(\chi_-(t), \chi_+(t))$ is $O(t^{-1})$ as $t \to \infty$.*

**Proof.** Fix $t > 0$ and consider any mesh $\chi_-(t) < x_1 < x_2 < \cdots < x_{2m} < \chi_+(t)$ such that $(x_i, t)$ is a point of continuity of $u$ and also $u(x_{2k-1}, t) > u(x_{2k}, t)$, $k = 1, \cdots, m$. Let $\xi_i(\cdot)$ denote the (unique) backward characteristic emanating from $(x_i, t)$. Then $\chi_-(0) \leq \xi_1(0) \leq \cdots \leq \xi_{2m}(0) \leq \chi_+(0)$. Furthermore, we have that $\xi_i(0) = x_i - tf'(u(x_i, t))$ and so

$$\sum_{k=1}^{m} t[f'(u(x_{2k-1}, t)) - f'(u(x_{2k}, t))] \leq \chi_+(0) - \chi_-(0) \tag{11.7.1}$$

whence the assertion of the theorem follows. This completes the proof.

In particular, referring to the setting of Theorem 11.6.1, we deduce that the decreasing variation of the $N$-wave profile $u(\cdot, t)$ over the interval $(\chi_-(t), \chi_+(t))$ is $O(t^{-1})$, as $t \to \infty$.

Another corollary of Theorem 11.7.1 is that when the initial data $u_0$, and thereby the solution $u$, are periodic in $x$, then the decreasing variation, and hence also the total variation, of $u(\cdot, t)$ over any period interval is $O(t^{-1})$ as $t \to \infty$. We may achieve finer resolution than $O(t^{-1})$ by paying closer attention to the initial data:

**11.7.2 Theorem.** *Let $u$ be an admissible solution with initial data $u_0$. Assume $\chi_-(t) = x_- + tf'(\bar{u})$ and $\chi_+(t) = x_+ + tf'(\bar{u})$, $x_- < x_+$, are adjacent divides associated with $u$, that is* (11.4.2) *holds for $\bar{x} = x_-$ and $\bar{x} = x_+$ but for no $\bar{x}$ in the interval $(x_-, x_+)$. Then*

(11.7.2)
$$\int_{\chi_-(t)}^{\chi_+(t)} u(x,t)dx = \int_{x_-}^{x_+} u_0(y)dy = (x_+ - x_-)\bar{u}, \qquad t \in [0, \infty).$$

*Consider any forward characteristic $\psi(\cdot)$ issuing from the point $(\frac{x_- + x_+}{2}, 0)$. Then, as $t \to \infty$,*

(11.7.3)
$$\psi(t) = \tfrac{1}{2}[\chi_-(t) + \chi_+(t)] + o(1),$$

(11.7.4)
$$u(x\pm, t) = \begin{cases} \bar{u} + \dfrac{1}{f''(\bar{u})} \dfrac{x - \chi_-(t)}{t} + o\left(\dfrac{1}{t}\right), \ \textit{uniformly for } \chi_-(t) < x < \psi(t) \\[2ex] \bar{u} + \dfrac{1}{f''(\bar{u})} \dfrac{x - \chi_+(t)}{t} + o\left(\dfrac{1}{t}\right), \ \textit{uniformly for } \psi(t) < x < \chi_+(t). \end{cases}$$

*Moreover, the decreasing variation of $u(\cdot, t)$ over the intervals $(\chi_-(t), \psi(t))$ and $(\psi(t), \chi_+(t))$ is $o(t^{-1})$ as $t \to \infty$.*

**Proof.** To verify the first equality in (11.7.2), it suffices to integrate (11.1.1) over the parallelogram $\{(x, \tau) : 0 < \tau < t,\ \chi_-(\tau) < x < \chi_+(\tau)\}$ and then apply Green's theorem. The second equality in (11.7.2) follows because (11.4.2) holds for both $\bar{x} = x_-$ and $\bar{x} = x_+$.

For $t > 0$, we let $\xi_-^t(\cdot)$ and $\xi_+^t(\cdot)$ denote the minimal and the maximal backward characteristics emanating from the point $(\chi(t), t)$. As $t \uparrow \infty$, $\xi_-^t(0) \downarrow x_-$ and $\xi_+^t(0) \uparrow x_+$, because otherwise there would exist divides originating at points $(\bar{x}, 0)$ with $\bar{x} \in (x_-, x_+)$, contrary to our assumptions. It then follows from Theorem 11.7.1 that the decreasing variation of $f'(u(\cdot, t))$, and thereby also the decreasing variation of $u(\cdot, t)$ itself, over the intervals $(\chi_-(t), \psi(t))$ and $(\psi(t), \chi_+(t))$ is $o(t^{-1})$ as $t \to \infty$.

The extremal backward characteristics emanating from any point $(x, t)$ with $\chi_-(t) < x < \psi(t)$ (or $\psi(t) < x < \chi_+(t)$) will be intercepted by the $x$-axis inside the interval $[x_-, \xi_-^t(0)]$ (or $[\xi_+^t(0), x_+]$) and thus

(11.7.5)
$$x - tf'(u(x\pm, t)) = \begin{cases} x_- + o(t^{-1}), \ \text{uniformly for } \chi_-(t) < x < \psi(t) \\[2ex] x_+ + o(t^{-1}), \ \text{uniformly for } \psi(t) < x < \chi_+(t). \end{cases}$$

Since $u(\chi_-(t),t) = u(\chi_+(t),t) = \bar{u}$, Theorem 11.7.1 implies $u - \bar{u} = O(t^{-1})$ and so, as $t \to \infty$, $f'(u) = f'(\bar{u}) + f''(\bar{u})(u-\bar{u}) + O(t^{-2})$. This together with (11.7.5) yield (11.7.4).

Finally, introducing $u$ from (11.7.4) into (11.7.2) we arrive at (11.7.3). The proof is complete.

We shall employ the above proposition to describe the asymptotics of periodic solutions:

**11.7.3 Theorem.** *When the initial data $u_0$ are periodic, with mean $\bar{u}$, then, as the time $t \to \infty$, the admissible solution $u$ tends, at the rate $o(t^{-1})$, to a periodic serrated profile consisting of wavelets of the form* (11.7.4). *The number of wavelets (or teeth) per period equals the number of divides per period or, equivalently, the number of points on any interval of the $x$-axis of period length at which the primitive of the function $u_0 - \bar{u}$ attains its minimum. In particular, in the generic case where the minimum of the primitive of $u_0 - \bar{u}$ is attained at a single point on each period interval, $u$ tends to a sawtooth shaped profile with a single tooth per period.*

**Proof.** It is an immediate corollary of Theorems 11.4.1 and 11.7.2. If $u_0$ is periodic, (11.4.2) may hold only when $\bar{u}$ is the mean of $u_0$ and is attained at points $\bar{x}$ where the primitive of $u_0 - \bar{u}$ is minimized. The set of such points is obviously invariant under period translations and contains at least one (generically precisely one) point in each interval of period length.

## 11.8 Comparison Theorems and $L^1$ Stability

The assertions of Theorem 6.2.3 will be reestablished here, in sharper form, for the special case of genuinely nonlinear scalar conservation laws (11.1.1), in one-space dimension. The key factor will be the properties of the function

$$Q(u,v,w) = \begin{cases} f(v) - f(u) - \dfrac{f(u)-f(w)}{u-w}[v-u], & \text{if } u \neq w \\ f(v) - f(u) - f'(u)[v-u], & \text{if } u = w, \end{cases} \tag{11.8.1}$$

defined for $u$, $v$ and $w$ in $\mathbb{R}$. Clearly, $Q(u,v,w) = Q(w,v,u)$. Since $f$ is uniformly convex, $Q(u,v,w)$ will be negative when $v$ lies between $u$ and $w$, and positive when $v$ lies outside the interval with endpoints $u$ and $w$. In particular, for the Burgers equation (4.2.1), $Q(u,v,w) = \frac{1}{2}(v-u)(v-w)$.

The first step is to refine the ordering property:

**11.8.1 Theorem.** *Let $u$ and $\bar{u}$ be admissible solutions of* (11.1.1), *on the upper half-plane, with respective initial data $u_0$ and $\bar{u}_0$ such that*

$$u_0(x) \le \bar{u}_0(x), \quad \textit{for all } x \in (y, \bar{y}). \tag{11.8.2}$$

*Let $\psi(\cdot)$ be any forward characteristic, associated with the solution $u$, issuing from the point $(y, 0)$, and let $\bar{\psi}(\cdot)$ be any forward characteristic, associated with $\bar{u}$, issuing from $(\bar{y}, 0)$. Then, for any $t > 0$ with $\psi(t) < \bar{\psi}(t)$,*

$$u(x,t) \le \bar{u}(x,t), \quad \textit{for all } x \in (\psi(t), \bar{\psi}(t)). \tag{11.8.3}$$

**Proof.** We fix any interval $(z, \bar{z})$ with $\psi(t) < z < \bar{z} < \bar{\psi}(t)$ and consider the maximal backward characteristic $\xi(\cdot)$, associated with the solution $u$, emanating from the point $(z, t)$, and the minimal backward characteristic $\bar{\zeta}(\cdot)$, associated with $\bar{u}$, emanating from the point $(\bar{z}, t)$. Thus, $\xi(0) \ge y$ and $\bar{\zeta}(0) \le \bar{y}$.

Suppose first $\xi(0) < \bar{\zeta}(0)$. We integrate the equation

$$\partial_t[u - \bar{u}] + \partial_x[f(u) - f(\bar{u})] = 0 \tag{11.8.4}$$

over the trapezoid $\{(x,\tau) : 0 < \tau < t, \ \xi(\tau) < x < \bar{\zeta}(\tau)\}$ and apply Green's theorem to get

$$\begin{aligned}\int_z^{\bar{z}} [u(x,t) - \bar{u}(x,t)]dx - \int_{\xi(0)}^{\bar{\zeta}(0)} [u_0(x) - \bar{u}_0(x)]dx \\ = -\int_0^t Q(u(\xi(\tau),\tau), \bar{u}(\xi(\tau),\tau), u(\xi(\tau),\tau))d\tau \\ -\int_0^t Q(\bar{u}(\bar{\zeta}(\tau),\tau), u(\bar{\zeta}(\tau),\tau), \bar{u}(\bar{\zeta}(\tau),\tau))d\tau.\end{aligned} \tag{11.8.5}$$

Both integrals on the right-hand side of (11.8.5) are nonnegative. Hence, by virtue of (11.8.2), the integral of $u(\cdot,t) - \bar{u}(\cdot,t)$ over $(z, \bar{z})$ is nonpositive.

Supppose now $\xi(0) \ge \bar{\zeta}(0)$. Then the straight lines $\xi(\cdot)$ and $\bar{\zeta}(\cdot)$ must intersect at some time $s \in [0, t)$. In that case we integrate (11.8.4) over the triangle $\{(x,\tau) : s < \tau < t, \ \xi(\tau) < x < \bar{\zeta}(\tau)\}$ and employ the same argument as above to deduce that the integral of $u(\cdot,t) - \bar{u}(\cdot,t)$ over $(z, \bar{z})$ is again nonpositive.

Since $(z, \bar{z})$ is an arbitrary subinterval of $(\psi(t), \bar{\psi}(t))$, we conclude (11.8.2). The proof is complete.

As a corollary of the above theorem, we infer that the number of sign changes of the function $u(\cdot,t) - \bar{u}(\cdot,t)$ over $(-\infty, \infty)$ is nonincreasing with time. Indeed, assume there are points $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$ such that, on each interval $(y_i, y_{i+1})$, $u_0(\cdot) - \bar{u}_0(\cdot)$ is nonnegative when $i$ is even and nonpositive when $i$ is odd. Let $\psi_i(\cdot)$ be any forward characteristic, associated with the solution $u$, issuing from the point $(y_i, 0)$ with $i$ odd, and $\bar{\psi}_i(\cdot)$ any forward characteristic, associated with $\bar{u}$, issuing from $(y_i, 0)$ with $i$ even. These curves are generally assigned finite life spans, according to the following prescription. At the time $t_1$ of the earliest collision between some $\psi_i$ and some $\bar{\psi}_j$, these two curves are terminated. Then, at the time $t_2$ of the next collision between any (surviving) $\psi_k$ and $\bar{\psi}_\ell$, these two curves are likewise terminated; and so on. By virtue of Theorem 11.8.1, $u(\cdot,t) - \bar{u}(\cdot,t)$ undergoes $n$ sign changes for any $t \in [0, t_1)$, $n - 2$ sign changes for any $t \in [t_1, t_2)$, and so on. In particular, the so-called *lap number*, which counts the crossings of the graph of the solution $u(\cdot, t)$ with any fixed constant $\bar{u}$, is nonincreasing with time.

By Theorem 6.2.3, the spatial $L^1$ distance of any pair of admissible solutions of a scalar conservation law is nonincreasing with time. In the present setting, it will be shown that it is actually possible to determine under what conditions is the $L^1$ distance strictly decreasing and at what rate:

**11.8.2 Theorem.** *Let $u$ and $\bar{u}$ be admissible solutions of* (11.1.1) *with initial data $u_0$ and $\bar{u}_0$ in $L^1(-\infty, \infty)$. Thus $\|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1(-\infty,\infty)}$ is a nonincreasing function of $t$ which is locally Lipschitz on $(0, \infty)$. For any fixed $t \in (0, \infty)$, consider the (possibly empty and at most countable) sets*

$$\begin{cases} \mathcal{J} = \{y \in (-\infty, \infty) : u_+ < \bar{u}_+ \leq \bar{u}_- < u_-\}, \\ \bar{\mathcal{J}} = \{y \in (-\infty, \infty) : \bar{u}_+ < u_+ \leq u_- < \bar{u}_-\}, \end{cases} \tag{11.8.6}$$

*where $u_\pm$ and $\bar{u}_\pm$ stand for $u(y\pm, t)$ and $\bar{u}(y\pm, t)$, respectively. Let*

$$u_* = \begin{cases} u_\pm \text{ if } u_+ = u_-\,, \\ u_- \text{ if } u_+ < u_- \text{ and } \dfrac{f(u_+) - f(u_-)}{u_+ - u_-} \geq \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-}\,, \\ u_+ \text{ if } u_+ < u_- \text{ and } \dfrac{f(u_+) - f(u_-)}{u_+ - u_-} < \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-}\,, \end{cases} \tag{11.8.7$_1$}$$

$$\bar{u}_* = \begin{cases} \bar{u}_\pm \text{ if } \bar{u}_+ = \bar{u}_-\,, \\ \bar{u}_- \text{ if } \bar{u}_+ < \bar{u}_- \text{ and } \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-} \geq \dfrac{f(u_+) - f(u_-)}{u_+ - u_-}\,, \\ \bar{u}_+ \text{ if } \bar{u}_+ < \bar{u}_- \text{ and } \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-} < \dfrac{f(u_+) - f(u_-)}{u_+ - u_-}\,. \end{cases} \tag{11.8.7$_2$}$$

*Then*

$$\frac{d^+}{dt}\|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1(-\infty,\infty)} = 2\sum_{y\in\mathcal{J}} Q(u_-, \bar{u}_*, u_+) + 2\sum_{y\in\bar{\mathcal{J}}} Q(\bar{u}_-, u_*, \bar{u}_+). \tag{11.8.8}$$

**Proof.** First we establish (11.8.8) for the special case where $u(\cdot, t) - \bar{u}(\cdot, t)$ undergoes a finite number of sign changes on $(-\infty, \infty)$, i.e., there are points $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$ such that, on each interval $(y_i, y_{i+1})$, $u(\cdot, t) - \bar{u}(\cdot, t)$ is nonnegative when $i$ is even and nonpositive when $i$ is odd. In particular, any $y \in \mathcal{J}$ must be one of the $y_i$, with $i$ odd, and any $y \in \bar{\mathcal{J}}$ must be one of the $y_i$, with $i$ even.

Let $\psi_i(\cdot)$ be the (unique) forward characteristic, associated with the solution $u$, issuing from the point $(y_i, t)$ with $i$ odd, and let $\bar{\psi}_i(\cdot)$ be the forward characteristic, associated with $\bar{u}$, issuing from $(y_i, t)$ with $i$ even. We fix $s > t$ with $s - t$ so small that no collisions of the above curves may occur on $[t, s]$, and integrate (11.8.4) over the domains $\{(x, \tau) : t < \tau < s,\ \psi_i(\tau) < x < \bar{\psi}_{i+1}(\tau)\}$, for $i$ odd, and $\{(x, \tau) : t < \tau < s\ ,\ \bar{\psi}_i(\tau) < x < \psi_{i+1}(\tau)\}$, for $i$ even. We apply Green's theorem and employ Theorem 11.8.1, to deduce

(11.8.9)
$$
\begin{aligned}
&\|u(\cdot, s) - \bar{u}(\cdot, s)\|_{L^1(-\infty,\infty)} - \|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1(-\infty,\infty)}\\
&= \sum_{i \text{ even}} \int_{\bar{\psi}_i(s)}^{\psi_{i+1}(s)} [u(x, s) - \bar{u}(x, s)]dx + \sum_{i \text{ odd}} \int_{\psi_i(s)}^{\bar{\psi}_{i+1}(s)} [\bar{u}(x, s) - u(x, s)]dx\\
&- \sum_{i \text{ even}} \int_{y_i}^{y_{i+1}} [u(x, t) - \bar{u}(x, t)]dx - \sum_{i \text{ odd}} \int_{y_i}^{y_{i+1}} [\bar{u}(x, t) - u(x, t)]dx\\
&= \sum_{i \text{ odd}} \int_t^s \{Q(u(\psi_i(\tau)-, \tau), \bar{u}(\psi_i(\tau)-, \tau), u(\psi_i(\tau)+, \tau))\\
&\qquad + Q(u(\psi_i(\tau)+, \tau), \bar{u}(\psi_i(\tau)+, \tau), u(\psi_i(\tau)-, \tau))\}d\tau\\
&+ \sum_{i \text{ even}} \int_t^s \{Q(\bar{u}(\bar{\psi}_i(\tau)-, \tau), u(\bar{\psi}_i(\tau)-, \tau), \bar{u}(\bar{\psi}_i(\tau)+, \tau))\\
&\qquad + Q(\bar{u}(\bar{\psi}_i(\tau)+, \tau), u(\bar{\psi}_i(\tau)+, \tau), \bar{u}(\bar{\psi}_i(\tau)-, \tau))\}d\tau.
\end{aligned}
$$

By virtue of Theorem 11.3.1, as $s \downarrow t$ the integrand in the first integral on the right-hand side of (11.8.9) tends to zero, if $y_i \notin \mathcal{J}$, or to $2Q(u_-, \bar{u}_*, u_+)$, if $y_i \in \mathcal{J}$. Similarly, the integrand in the second integral on the right-hand side of (11.8.9) tends to zero, if $y_i \notin \bar{\mathcal{J}}$, or to $2Q(\bar{u}_-, u_*, \bar{u}_+)$, if $y_i \in \bar{\mathcal{J}}$. Therefore, upon dividing (11.8.9) by $s - t$ and letting $s \downarrow t$, we arrive at (11.8.8).

We now turn to the general situation, where $u(\cdot, t) - \bar{u}(\cdot, t)$ may undergo infinitely many sign changes over $(-\infty, \infty)$. In that case, the open set $\{x \in (-\infty, \infty) : u(x\pm, t) - \bar{u}(x\pm, t) < 0\}$ is the countable union of disjoint open intervals $(y_i, \bar{y}_i)$. For $m = 1, 2, \cdots$, we let $u_m$ denote the admissible solution of our conservation law (11.1.1) on $(-\infty, \infty) \times [t, \infty)$, with

(11.8.10)
$$
u_m(x, t) = \begin{cases} \bar{u}(x, t), & x \in \bigcup_{i=m}^{\infty} (y_i, \bar{y}_i) \\ u(x, t), & \text{otherwise.} \end{cases}
$$

Thus $u_m(\cdot, t) - \bar{u}(\cdot, t)$ undergoes a finite number of sign changes over $(-\infty, \infty)$ and so, for $\tau \geq t$, $\frac{d^+}{d\tau}\|u_m(\cdot, \tau) - \bar{u}(\cdot, \tau)\|_{L^1}$ is evaluated by the analog of (11.8.8).

Moreover, the function $\tau \mapsto \frac{d^+}{d\tau}\|u_m(\cdot,\tau) - \bar{u}(\cdot,\tau)\|_{L^1}$ is right-continuous at $t$ and the modulus of right continuity is independent of $m$. To verify this, note that the total contribution of small jumps to the rate of change of $\|u_m(\cdot,\tau) - \bar{u}(\cdot,\tau)\|_{L^1}$ is small, controlled by the total variation of $u(\cdot,t)$ and $\bar{u}(\cdot,t)$ over $(-\infty,\infty)$, while the contribution of the (finite number of) large jumps is right-continuous, by account of Theorem 11.3.1. Therefore, by passing to the limit, as $m \to \infty$, we establish (11.8.8) for general solutions $u$ and $\bar{u}$. The proof is complete.

According to the above theorem, the $L^1$ distance of $u(\cdot,t)$ and $\bar{u}(\cdot,t)$ may decrease only when the graph of either of these functions happens to cross the graph of the other at a point of jump discontinuity. More robust contraction is realized in terms of a new metric which weighs the $L^1$ distance of two solutions by a weight specially tailored to them.

For $v$ and $\bar{v}$ in $BV(-\infty,\infty)$, let

$$\rho(v,\bar{v}) = \int_{-\infty}^{\infty} \{(V(x) + \bar{V}(\infty) - \bar{V}(x))[v(x) - \bar{v}(x)]^+ + (\bar{V}(x) + V(\infty) - V(x))[\bar{v}(x) - v(x)]^+\}dx, \tag{11.8.11}$$

where the superscript $+$ denotes "positive part", $w^+ = \max\{w, 0\}$, and $V$ or $\bar{V}$ denotes the variation function of $v$ or $\bar{v}$, defined by $V(x) = TV_{(-\infty,x)}v(\cdot)$, and $\bar{V}(x) = TV_{(-\infty,x)}\bar{v}(\cdot)$.

**11.8.3 Theorem.** *Let $u$ and $\bar{u}$ be admissible solutions of* (11.1.1) *with initial data $u_0$ and $\bar{u}_0$ in $BV(-\infty,\infty)$. Then, for any fixed $t \in (0,\infty)$,*

$$\begin{aligned} \frac{d^+}{dt}\rho(u(\cdot,t),\bar{u}(\cdot,t)) \le &-\int_{-\infty}^{\infty} Q(u(x,t),\bar{u}(x,t),u(x,t))dV_t^c(x) \\ &-\int_{-\infty}^{\infty} Q(\bar{u}(x,t),u(x,t),\bar{u}(x,t))d\bar{V}_t^c(x) \\ &-\sum_{y\in\mathcal{K}}(u_- - u_+)Q(u_-,\bar{u}_*,u_+) - \sum_{y\in\bar{\mathcal{K}}}(\bar{u}_- - \bar{u}_+)Q(\bar{u}_-,u_*,\bar{u}_+) \\ &+(V_t(\infty) + \bar{V}_t(\infty))\{\sum_{y\in\mathcal{J}} Q(u_-,\bar{u}_*,u_+) + \sum_{y\in\bar{\mathcal{J}}} Q(\bar{u}_-,u_*,\bar{u}_+)\}, \end{aligned} \tag{11.8.12}$$

*where $V_t$ or $\bar{V}_t$ is the variation function of $u(\cdot,t)$ or $\bar{u}(\cdot,t)$; $V_t^c$ or $\bar{V}_t^c$ denotes the continuous part of $V_t$ or $\bar{V}_t$; $u_\pm$ or $\bar{u}_\pm$ stand for $u(y\pm,t)$ or $\bar{u}(y\pm,t)$, $u_*$ and $\bar{u}_*$ are again determined through* $(11.8.7)_1$ *and* $(11.8.7)_2$; *the sets $\mathcal{J}$ and $\bar{\mathcal{J}}$ are defined by* (11.8.6) *and $\mathcal{K}$ or $\bar{\mathcal{K}}$ denotes the set of jump points of $u(\cdot,t)$ or $\bar{u}(\cdot,t)$:*

$$\begin{cases} \mathcal{K} = \{y \in (-\infty,\infty) : u_+ < u_-\} \\ \bar{\mathcal{K}} = \{y \in (-\infty,\infty) : \bar{u}_+ < \bar{u}_-\}. \end{cases} \tag{11.8.13}$$

**Proof.** We begin as in the proof of Theorem 11.8.2: We assume there are points $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$ such that, on each interval $(y_i, y_{i+1})$, $u(\cdot, t) - \bar{u}(\cdot, t)$ is nonnegative when $i$ is even and nonpositive when $i$ is odd. We consider the forward characteristic $\psi_i(\cdot)$, associated with $u$, issuing from each point $(y_i, t)$, with $i$ odd, and the forward characteristic $\bar{\psi}_i(\cdot)$, associated with $\bar{u}$, issuing from each $(y_i, t)$, with $i$ even.

We focus our attention on some $(y_i, y_{i+1})$ with $i$ even. We shall discuss only the case $-\infty < y_i < y_{i+1} < \infty$, as the other cases are simpler. With the exception of $\bar{\psi}_i(\cdot)$, all characteristics to be considered below will be associated with the solution $u$. The argument varies somewhat, depending on whether the forward characteristic $\chi_0$ issuing from $(y_i, t)$ lies to the left or to the right of $\bar{\psi}_i(\cdot)$; for definiteness, we shall treat the latter case, which is slightly more complicated.

We fix $\varepsilon$ positive small and identify all $z_1, \cdots, z_N$, $y_i < z_1 < \cdots < z_N < y_{i+1}$, such that $u(z_I-, t) - u(z_I+, t) \geq \varepsilon$, $I = 1, \cdots, N$. We consider the forward characteristic $\chi_I(\cdot)$ issuing from the point $(z_I, t)$, $I = 1, \cdots, N$. Then we select $s > t$ with $s - t$ so small that the following hold: (a) No intersection of any two of the characteristics $\chi_0, \chi_1, \cdots, \chi_N, \psi_{i+1}$ may occur on the time interval $[t, s]$. (b) For $I = 1, \cdots, N$, if $\zeta_I(\cdot)$ and $\xi_I(\cdot)$ denote the minimal and the maximal backward characteristics emanating from the point $(\chi_I(s), s)$, then the total variation of $u(\cdot, t)$ over the intervals $(\zeta_I(t), z_I)$ and $(z_I, \xi_I(t))$ does not exceed $\varepsilon/N$. (c) If $\zeta(\cdot)$ denotes the minimal backward characteristic emanating from $(\psi_{i+1}(s), s)$, then the total variation of $u(\cdot, t)$ over the interval $(\zeta(t), y_{i+1})$ does not exceed $\varepsilon$. (d) If $\zeta_0(\cdot)$ is the minimal backward characteristic emanating from $(\psi_i(s), s)$ and $\xi_0(\cdot)$ is the maximal backward characteristic emanating from $(\chi_0(s), s)$, then the total variation of $u(\cdot, t)$ over the intervals $(\zeta_0(t), y_i)$ and $(y_i, \xi_0(t))$ does not exceed $\varepsilon$.

For $I = 0, \cdots, N-1$, and some $k$ to be fixed later, we pick a mesh on the interval $[\chi_I(s), \chi_{I+1}(s)] : \chi_I(s) = x_I^0 < x_I^1 < \cdots < x_I^k < x_I^{k+1} = \chi_{I+1}(s)$; and likewise for $[\chi_N(s), \psi_{i+1}(s)] : \chi_N(s) = x_N^0 < x_N^1 < \cdots < x_N^k < x_N^{k+1} = \psi_{i+1}(s)$. For $I = 0, \cdots, N$ and $j = 1, \cdots, k$, we consider the maximal backward characteristic $\xi_I^j(\cdot)$ emanating from the point $(x_I^j, s)$ and identify its intercept $z_I^j = \xi_I^j(t)$ by the $t$-time line. We also set $z_0^0 = y_i$, $z_N^{k+1} = y_{i+1}$ and $z_{I-1}^{k+1} = z_I^0 = z_I$, $I = 1, \cdots, N$.

We now note the identity

$$R - S = -D, \tag{11.8.14}$$

where

$$R = \int_{\bar{\psi}_i(s)}^{\chi_0(s)} V_t(y_i)[u(x,s) - \bar{u}(x,s)]dx + \sum_{I=0}^{N} \sum_{j=0}^{k} \int_{x_I^j}^{x_I^{j+1}} V_t(z_I^i+)[u(x,s) - \bar{u}(x,s)]dx, \tag{11.8.15}$$

$$S = \sum_{I=0}^{N} \sum_{j=0}^{k} \int_{z_I^j}^{z_I^{j+1}} V_t(z_I^j+)[u(x,t) - \bar{u}(x,t)]dx, \tag{11.8.16}$$

(11.8.17)

$$D=\sum_{I=0}^{N}\sum_{j=1}^{k}\int_t^s [V_t(z_I^j+)-V_t(z_I^{j-1}+)]Q(u(\xi_I^j(\tau),\tau),\bar{u}(\xi_I^j(\tau)-,\tau),u(\xi_I^j(\tau),\tau))d\tau$$

$$+\sum_{I=1}^{N}\int_t^s [V_t(z_I+)-V_t(z_{I-1}^k+)]Q(u(\chi_I(\tau)-,\tau),\bar{u}(\chi_I(\tau)-,\tau),u(\chi_I(\tau)+,\tau))d\tau$$

$$+\int_t^s [V_t(y_i+)-V_t(y_i)]Q(u(\chi_0(\tau)-,\tau),\bar{u}(\chi_0(\tau)-,\tau),u(\chi_0(\tau)+,\tau))d\tau$$

$$-\int_t^s V_t(y_i)Q(\bar{u}(\bar{\psi}_i(\tau)-,\tau),u(\bar{\psi}_i(\tau)+,\tau),\bar{u}(\bar{\psi}_i(\tau)+,\tau))d\tau$$

$$-\int_t^s V_t(z_N^k+)Q(u(\psi_{i+1}(\tau)-,\tau),\bar{u}(\psi_{i+1}(\tau)-,\tau),u(\psi_{i+1}(\tau)+,\tau))d\tau.$$

To verify (11.8.14), one first integrates (11.8.4) over the following four domains: $\{(x,\tau) : t<\tau<s,\ \bar{\psi}_i(\tau)<x<\chi_0(\tau)\}$, $\{(x,\tau) : t<\tau<s,\ \xi_I^j(\tau)<x<\xi_I^{j+1}(\tau)\}$, $\{(x,\tau) : t<\tau<s,\ \chi_I(\tau)<x<\xi_I^1(\tau)\}$, $\{(x,\tau) : t<\tau<s,\ \xi_I^k(\tau)<x<\chi_{I+1}(\tau)\}$, $\{(x,\tau) : t<\tau<s,\ \xi_N^k(\tau)<x<\psi_{i+1}(\tau)\}$ and applies Green's theorem; then forms the weighted sum of the resulting equations, with respective weights $V_t(y_i)$, $V_t(z_I^j+)$, $V_t(z_I+)$, $V_t(z_I^k+)$, $V_t(z_N^k+)$.

To estimate $R$, we note that $V_t(y_i)\ \geq\ V_s(\chi_0(s))$, and $V_t(z_I^j+)\ \geq\ V_s(x_I^j+)$, $I=0,\cdots,N,\ j=0,\cdots,k$. Hence, if we pick the $x_I^{j+1}-x_I^j$ sufficiently small, we can guarantee

$$R\geq\int_{\bar{\psi}_i(s)}^{\psi_{i+1}(s)} V_s(x)[u(x,s)-\bar{u}(x,s)]dx-(s-t)\varepsilon. \tag{11.8.18}$$

To estimate $S$, it suffices to observe that $V_t(\cdot)$ is nondecreasing, and so

$$S\leq\int_{\bar{\psi}_i(t)}^{\psi_{i+1}(t)} V_t(x)[u(x,t)-\bar{u}(x,t)]dx. \tag{11.8.19}$$

To estimate $D$, the first observation is that, due to the properties of $Q$, all five terms are nonnegative. For $I=0,\cdots,N$ and $j=1,\cdots,k$, $V_t(z_I^j+)-V_t(z_I^{j-1}+)\geq V_t^c(z_I^j)-V_t^c(z_I^{j-1})$. Furthermore,

$$Q(u(\xi_I^j(\tau),\tau),\bar{u}(\xi_I^j(\tau)-,\tau),u(\xi_I^j(\tau),\tau)) = Q(u(z_I^j,t),\bar{u}(p_\tau(z_I^j),t),u(z_I^j,t)), \tag{11.8.20}$$

where the monotone increasing function $p_\tau$ is determined through

$$p_\tau(x)=x+(\tau-t)[f'(u(x,t))-f'(\bar{u}(p_\tau(x),t))]. \tag{11.8.21}$$

Upon choosing the $x_I^{j+1} - x_I^j$ so small that the oscillation of $V_t^c(\cdot)$ over each one of the intervals $(z_I^j, z_I^{j+1})$ does not exceed $\varepsilon$, the standard estimates on Stieltjes integrals imply

(11.8.22)
$$\sum_{I=0}^{N}\sum_{j=1}^{k}[V_t(z_I^j+) - V_t(z_I^{j-1}+)]Q(u(\xi_I^j(\tau),\tau), \bar{u}(\xi_I^j(\tau)-,\tau), u(\xi_I^j(\tau),\tau))$$
$$\geq \int_{y_i}^{y_{i+1}} Q(u(x,t), \bar{u}(p_\tau(x),t), u(x,t))dV_t^c(x) - c\varepsilon.$$

We now combine (11.8.14) with (11.8.18), (11.8.19), (11.8.17) and (11.8.22), then we divide the resulting inequality by $s-t$, we let $s \downarrow t$, and finally we let $\varepsilon \downarrow 0$. This yields

$$\frac{d^+}{dt}\int_{\bar{\psi}_i(t)}^{\psi_{i+1}(t)} V_t(x)[u(x,t) - \bar{u}(x,t)]dx \tag{11.8.23}$$
$$\leq -\int_{y_i}^{y_{i+1}} Q(u(x,t), \bar{u}(x,t), u(x,t))dV_t^c(x)$$
$$-\sum(u_- - u_+)Q(u_-, \bar{u}_*, u_+)$$
$$+V_t(y_i)Q(\bar{u}_-, u_*, \bar{u}_+) + V_t(y_{i+1})Q(u_-, \bar{u}_*, u_+),$$

where the summation runs over all $y$ in $\mathcal{K}\bigcap(y_i, y_{i+1})$ and also over $y_i$ if $y_i \in \mathcal{K}$ and $\chi_0$ lies to the right of $\bar{\psi}_i$. The $u_\pm$, $\bar{u}_\pm$, $u_*$ and $\bar{u}_*$ are of course evaluated at the corresponding $y$.

Next we focus attention on intervals $(y_i, y_{i+1})$ with $i$ odd. A completely symmetrical argument yields, in the place of (11.8.23),

$$\frac{d^+}{dt}\int_{\psi_i(t)}^{\bar{\psi}_{i+1}(t)} (V_t(\infty) - V_t(x))[\bar{u}(x,t) - u(x,t)]dx \tag{11.8.24}$$
$$\leq -\int_{y_i}^{y_{i+1}} Q(u(x,t), \bar{u}(x,t), u(x,t))dV_t^c(x)$$
$$-\sum(u_- - u_+)Q(u_-, \bar{u}_*, u_+)$$
$$+ (V_t(\infty) - V_t(y_i+))Q(u_-, \bar{u}_*, u_+)$$
$$+ (V_t(\infty) - V_t(y_{i+1}))Q(\bar{u}_-, u_*, \bar{u}_+),$$

where the summation runs over all $y$ in $\mathcal{K}\bigcap(y_i, y_{i+1})$, and also over $y_{i+1}$ if $y_{i+1} \in \mathcal{K}$ and the forward characteristic, associated with $u$, issuing from the point $(y_{i+1}, t)$ lies to the left of $\bar{\psi}_{i+1}$.

We thus write (11.8.23), for all $i$ even, then (11.8.24), for all $i$ odd, and sum over $i = 0, \cdots, n$. This yields

(11.8.25)

$$\frac{d^+}{dt}\int_{-\infty}^{\infty}\{V_t(x)[u(x,t)-\bar{u}(x,t)]^+ + (V_t(\infty)-V_t(x))[\bar{u}(x,t)-u(x,t)]^+\}dx$$

$$\leq -\int_{-\infty}^{\infty} Q(u(x,t),\bar{u}(x,t),u(x,t))dV_t^c(x) - \sum_{y\in\mathcal{K}}(u_- - u_+)Q(u_-,\bar{u}_*,u_+)$$

$$+ V_t(\infty)\{\sum_{y\in\mathcal{J}} Q(u_-,\bar{u}_*,u_+) + \sum_{y\in\bar{\mathcal{J}}} Q(\bar{u}_-,u_*,\bar{u}_+)\}.$$

By employing a technical argument, as in the proof of Theorem 11.8.2, one shows that (11.8.25) remains valid even when $u(\cdot,t)-\bar{u}(\cdot,t)$ is allowed to undergo infinitely many sign changes on $(-\infty,\infty)$.

We write the inequality resulting from (11.8.25) by interchanging the roles of $u$ and $\bar{u}$, and then combine it with (11.8.25). This yields (11.8.12). The proof is complete.

The estimate (11.8.12) is sharp, in that it holds as equality, at least for piecewise smooth solutions. All terms on the right-hand side of (11.8.12) are negative, with the exception of $-(u_- - u_+)Q(u_-,\bar{u}_*,u_+)$, for $y\in\mathcal{J}$, and $-(\bar{u}_- - \bar{u}_+)Q(\bar{u}_-,u_*,\bar{u}_+)$, for $y\in\bar{\mathcal{J}}$. However, even these positive terms are offset by the negative terms $V_t(\infty)Q(u_-,\bar{u}_*,u_+)$ and $\bar{V}_t(\infty)Q(\bar{u}_-,u_*,\bar{u}_+)$. Thus, $\rho(u(\cdot,t),\bar{u}(\cdot,t))$ is generally strictly decreasing.

An analog of the functional $\rho$ will be employed in Chapter XIV for establishing $L^1$ stability of solutions for systems of conservation laws.

## 11.9 Genuinely Nonlinear Scalar Balance Laws

The notion of generalized characteristic may be extended in a natural way to general systems of balance laws, and may be used, in particular, for deriving a precise description of the structure of solutions of genuinely nonlinear, scalar balance laws

$$\partial_t u(x,t) + \partial_x f(u(x,t),x,t) + g(u(x,t),x,t) = 0. \tag{11.9.1}$$

Extending the analysis from (11.1.1) to (11.9.1) is rather straightforward, so it will suffice to outline here the main steps, with few proofs.

We assume that $f$ and $g$ are, respectively, $C^2$ and $C^1$ given functions, defined on $(-\infty,\infty)\times(-\infty,\infty)\times[0,\infty)$, and the genuine nonlinearity condition, namely $f_{uu}(u,x,t) > 0$ holds for all $(u,x,t)$. We will be dealing with solutions $u(x,t)$ of (11.9.1), of class $BV_{loc}$ on the upper half-plane $(-\infty,\infty)\times[0,\infty)$, such that $u(\cdot,t)$ has locally bounded variation in $x$ on $(-\infty,\infty)$, for any fixed $t\in[0,\infty)$, and the Lax $E$-condition (11.1.2) holds for almost all $t\in[0,\infty)$ and all $x\in(-\infty,\infty)$. Solutions in this class may be constructed by solving the Cauchy problem with initial data that are bounded and have locally bounded variation on $(-\infty,\infty)$, for instance by the vanishing viscosity method expounded in Chapter VI. Restrictions have to be imposed on $f$ and $g$ in order to prevent the blowing up of the solution in finite time. For that purpose, it is sufficient to assume $|f_u| \le A$, for $u$ in bounded intervals, and $f_x + g_u \le B$, for all $u$, uniformly on the upper half-plane. The reader may find details in the references cited in Section 11.12.

Similar to Definition 10.2.1, a generalized characteristic of (11.9.1), associated with the solution $u$, is a Lipschitz curve $\xi(\cdot)$, defined on some closed time interval $[\sigma, \tau] \subset [0, \infty)$, and satisfying the differential inclusion

$$\dot{\xi}(t) \in [f_u(u(\xi(t)+, t), \xi(t), t),\ \ f_u(u(\xi(t)-, t), \xi(t), t)]\,, \tag{11.9.2}$$

for almost all $t \in [\sigma, \tau]$. As in Section 11.1, it can be shown that (11.9.2) is actually equivalent to

(11.9.3)

$$\dot{\xi}(t) = \begin{cases} f_u(u(\xi(t)\pm, t), \xi(t), t), & \text{if } u(\xi(t)+, t) = u(\xi(t)-, t) \\ \dfrac{f(u(\xi(t)+, t), \xi(t), t) - f(u(\xi(t)-, t), \xi(t), t)}{u(\xi(t)+, t) - u(\xi(t)-, t)}\,, & \text{if } u(\xi(t)+, t) < u(\xi(t)-, t) \end{cases}$$

for almost all $t \in [\sigma, \tau]$; compare with (11.1.3).

Similar to Definition 10.2.4, the characteristic $\xi(\cdot)$ is called *shock-free* on $[\sigma, \tau]$ if $u(\xi(t)-, t) = u(\xi(t)+, t)$, almost everywhere on $[\sigma, \tau]$. The key result is the following generalization of Theorem 11.1.1.

**11.9.1 Theorem.** *Let $\xi(\cdot)$ be a generalized characteristic for* (11.9.1)*, associated with the admissible solution $u$, which is shock-free on* $[\sigma, \tau]$*. Then there is a $C^1$ function $v$ on* $[\sigma, \tau]$ *such that*

$$u(\xi(\tau)+, \tau) \le v(\tau) \le u(\xi(\tau)-, \tau), \tag{11.9.4}$$

$$u(\xi(t)+, t) = v(t) = u(\xi(t)-, t), \qquad \sigma < t < \tau, \tag{11.9.5}$$

$$u(\xi(\sigma)-, \sigma) \le v(\sigma) \le u(\xi(\sigma)+, \sigma). \tag{11.9.6}$$

*Furthermore,* $(\xi(\cdot), v(\cdot))$ *satisfy the classical characteristic equations*

$$\begin{cases} \dot{\xi} = f_u(v, \xi, t) \\ \dot{v} = -f_x(v, \xi, t) - g(v, \xi, t) \end{cases} \tag{11.9.7}$$

*on* $(\sigma, \tau)$*. In particular, $\xi(\cdot)$ is $C^1$ on* $[\sigma, \tau]$.

**Proof.** Let $I = \{t \in (\sigma, \tau) : u(\xi(t)-, t) = u(\xi(t)+, t)\}$. For any $t \in I$, let us set $v(t) = u(\xi(t)\pm, t)$. In particular, (11.9.3) implies

$$\dot{\xi}(t) = f_u(v(t), \xi(t), t)\,, \quad \text{a.e. on } (\sigma, \tau). \tag{11.9.8}$$

Fix $r$ and $s$, $\sigma \le r < s \le \tau$. For $\varepsilon > 0$, we integrate the measure equality (11.9.1) over the set $\{(x,t) : r < t < s, \xi(t) - \varepsilon < x < \xi(t)\}$, apply Green's theorem, and use (11.9.8) and $f_{uu} > 0$ to get

(11.9.9)

$$\int_{\xi(s)-\varepsilon}^{\xi(s)} u(x,s)dx - \int_{\xi(r)-\varepsilon}^{\xi(r)} u(x,r)dx + \int_r^s \int_{\xi(t)-\varepsilon}^{\xi(t)} g(u(x,t),x,t)dx\,dt$$

$$= \int_r^s \{f(u(\xi(t)-\varepsilon+,t),\xi(t)-\varepsilon,t) - f(v(t),\xi(t),t)$$

$$-f_u(v(t),\xi(t),t)[u(\xi(t)-\varepsilon+,t) - v(t)]\}\,dt$$

$$\ge \int_r^s \{f(u(\xi(t)-\varepsilon+,t),\xi(t)-\varepsilon,t) - f(u(\xi(t)-\varepsilon+,t),\xi(t),t)\}\,dt.$$

Multiplying (11.9.9) by $1/\varepsilon$ and letting $\varepsilon \downarrow 0$ yields

(11.9.10)

$$u(\xi(s)-,s) \ge u(\xi(r)-,r) - \int_r^s \{f_x(v(t),\xi(t),t) + g(v(t),\xi(t),t)\}\,dt.$$

Next we integrate (11.9.1) over the set $\{(x,t) : r < t < s,\ \xi(t) < x < \xi(t) + \varepsilon\}$ and follow the same procedure, as above, to deduce

(11.9.11)

$$u(\xi(s)+,s) \le u(\xi(r)+,r) - \int_r^s \{f_x(v(t),\xi(t),t) + g(v(t),\xi(t),t)\}\,dt.$$

For any $t \in (\sigma,\tau)$, we apply (11.9.10) and (11.9.11), first for $r = t,\ s \in I \cap (t,\tau)$, then for $s = t,\ r \in I \cap (\sigma,t)$. This yields $u(\xi(t)-,t) = u(\xi(t)+,t)$. Therefore, $I = (\sigma,\tau)$ and (11.9.5) holds. For any $r$ and $s$ in $(\sigma,\tau)$, (11.9.10) and (11.9.11) combine into

(11.9.12)
$$v(s) = v(r) - \int_r^s \{f_x(v(t),\xi(t),t) + g(v(t),\xi(t),t)\}\,dt.$$

In conjunction with (11.9.8), (11.9.12) implies that $(\xi(\cdot), v(\cdot))$ are $C^1$ functions on $[\sigma,\tau]$ which satisfy the system (11.9.7).

To verify (11.9.4) and (11.9.6), it suffices to write (11.9.10), (11.9.11), first for $s = \tau$, $r \in (\sigma, \tau)$ and then for $r = \sigma$, $s \in (\sigma, \tau)$. This completes the proof.

**11.9.2 Remark.** When the balance law is a conservation law, $g \equiv 0$, and $f$ does not depend explicitly on $t$, (11.9.7) implies $\dot{f}(v, \xi) = 0$, that is, $f$ stays constant along shock-free characteristics.

The family of backward generalized characteristics emanating from any point $(\bar{x}, \bar{t})$ of the upper half-plane span a funnel bordered by the minimal backward characteristic $\xi_-(\cdot)$ and the maximal backward characteristic $\xi_+(\cdot)$. Theorem 10.3.2 is readily extended to systems of balance laws, and in the present context yields that both $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are shock-free on $(0, \bar{t})$. Thus, upon substituting Theorem 11.9.1 for Theorem 11.1.1, one easily derives the following generalization of Theorem 11.1.3:

**11.9.3 Theorem.** *Let $u$ be an admissible solution of* (11.9.1) *with initial data $u_0$. Given any point $(\bar{x}, \bar{t})$ on the upper half-plane, consider the solutions $(\xi_-(\cdot), v_-(\cdot))$ and $(\xi_+(\cdot), v_+(\cdot))$ of the system* (11.9.7)*, satisfying initial conditions $\xi_-(\bar{t}) = \bar{x}$, $v_-(\bar{t}) = u(\bar{x}-, \bar{t})$ and $\xi_+(\bar{t}) = \bar{x}$, $v_+(\bar{t}) = u(\bar{x}+, \bar{t})$. Then $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are respectively the minimal and the maximal backward characteristics emanating from $(\bar{x}, \bar{t})$. Furthermore,*

$$(11.9.13) \qquad \begin{cases} u(\xi_-(t)-, t) = v_-(t) = u(\xi_-(t)+, t) \\ \\ u(\xi_+(t)-, t) = v_+(t) = u(\xi_+(t)+, t) \end{cases} \qquad 0 < t < \bar{t},$$

$$(11.9.14) \qquad \begin{cases} u_0(\xi_-(0)-) \geq v_-(0) \geq u_0(\xi_-(0)+) \\ \\ u_0(\xi_+(0)-) \geq v_+(0) \geq u_0(\xi_+(0)+). \end{cases}$$

*In particular, $u(\bar{x}+, \bar{t}) \leq u(\bar{x}-, \bar{t})$ holds for all $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$ and $\xi_-(\cdot)$, $\xi_+(\cdot)$ coincide if and only if $u(\bar{x}+, \bar{t}) = u(\bar{x}-, \bar{t})$.*

The Theorems 11.1.4 and 11.1.5 which describe properties of forward characteristics for homogeneous conservation laws can also be readily extended to nonhomogeneous balance laws:

**11.9.4 Theorem.** *A unique forward generalized characteristic $\chi(\cdot)$, associated with an admissible solution $u$, issues from any point $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$. Furthermore, if $u(\bar{x}+, \bar{t}) < u(\bar{x}-, \bar{t})$, then $u(\chi(s)+, s) < u(\chi(s)-, s)$ for all $s \in [\bar{t}, \infty)$.*

Solutions of the inhomogeneous balance law (11.9.1) have similar structure, and enjoy similar regularity properties with the solutions of the homogeneous conservation law (11.1.1), described in Section 11.3. A number of relevant propositions are stated below. The reader may find the proofs in the literature cited in Section 11.12.

**11.9.5 Theorem.** *Let $u$ be an admissible solution and assume $u(\bar{x}+, \bar{t}) = u(\bar{x}-, \bar{t})$, for some $(\bar{x}, \bar{t}) \in (-\infty, \infty) \times (0, \infty)$. Then $(\bar{x}, \bar{t})$ is a point of continuity of $u$. A* *unique generalized characteristic* $\chi(\cdot)$*, associated with* $u$*, defined on* $[0,\infty)$*, passes through* $(\bar{x},\bar{t})$*. Furthermore,* $\chi(\cdot)$ *is differentiable at* $\bar{t}$ *and* $\dot{\chi}(\bar{t}) = f_u(u(\bar{x}\pm,\bar{t}),\bar{x},\bar{t})$*.*

**11.9.6 Theorem.** *Let* $u$ *be an admissible solution and assume* $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$*, for some* $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$*. When the extremal backward characteristics* $\xi_-(\cdot)$*,* $\xi_+(\cdot)$ *are the only backward generalized characteristics emanating from* $(\bar{x},\bar{t})$ *that are shock-free, then* $(\bar{x},\bar{t})$ *is a point of jump discontinuity of* $u$ *in the following sense: There is a generalized characteristic* $\chi(\cdot)$*, associated with* $u$*, defined on* $[0,\infty)$ *and passing through* $(\bar{x},\bar{t})$*, such that* $(\bar{x},\bar{t})$ *is a point of continuity of the function* $u(x-,t)$ *relative to* $\{(x,t) : 0<t<\infty,\ x\le\chi(t)\}$ *and also a point of continuity of the function* $u(x+,t)$ *relative to* $\{(x,t) : 0<t<\infty,\ x\ge\chi(t)\}$*. Furthermore,* $\chi(\cdot)$ *is differentiable at* $\bar{t}$ *and*

$$\dot{\chi}(\bar{t}) = \frac{f(u(\bar{x}+,\bar{t}),\bar{x},\bar{t}) - f(u(\bar{x}-,\bar{t}),\bar{x},\bar{t})}{u(\bar{x}+,\bar{t}) - u(\bar{x}-,\bar{t})}\,. \tag{11.9.15}$$

**11.9.7 Theorem.** *The set or irregular points of any admissible solution* $u$ *is (at most) countable.* $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$ *is an irregular point if and only if* $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$ *and, in addition to the extremal backward characteristics* $\xi_-(\cdot)$ *,* $\xi_+(\cdot)$*, there is at least another, distinct, backward characteristic* $\xi(\cdot)$*, associated with* $u$*, emanating from* $(\bar{x},\bar{t})$*, which is shock-free. Irregular points are generated by the collision of shocks and/or by the focusing of centered compression waves.*

**11.9.8 Theorem.** *Assume that* $f$ *and* $g$ *are, respectively,* $C^{k+1}$ *and* $C^k$ *functions on* $(-\infty,\infty)\times(-\infty,\infty)\times[0,\infty)$*, for some* $3\ge k\ge\infty$*. Let* $u$ *be an admissible solution with initial data* $u_0$ *in* $C^k$*. Then* $u(x,t)$ *is* $C^k$ *on the complement of the closure of the shock set. Furthermore, generically,* $u$ *is piecewise smooth and does not contain centered compression waves.*

The large time behavior of solutions of inhomogeneous, genuinely nonlinear balance laws can be widely varied, and the method of generalized characteristics provides an efficient tool for determining the asymptotic profile. Two typical, very simple, examples will be presented in the following two sections to demonstrate the effect of source terms or inhomogeneity on the asymptotics of solutions with periodic initial data.

## 11.10 Balance Laws with Linear Excitation

We consider the balance law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) - u(x,t) = 0, \tag{11.10.1}$$

with $f''(u) > 0,\ -\infty < u < \infty$. For convenience, we normalize $f$ and the space-time frame so that $f(0) = 0$ and $f'(0) = 0$.

The aim is to demonstrate that, as a result of the competition between the destabilizing action of the source term and the damping effect of genuine nonlinearity, periodic solutions with zero mean become asymptotically standing waves of finite amplitude.

In what follows, $u(x,t)$ will denote an admissible solution of (11.10.1), of locally bounded variation on the upper half-plane, with initial values $u(x,0) = u_0(x)$ of locally bounded variation on $(-\infty, \infty)$.

The system (11.9.7) for shock-free characteristics here takes the form

$$\begin{cases} \dot{\xi} = f'(v) \\ \\ \dot{v} = v. \end{cases} \tag{11.10.2}$$

In particular, divides are characteristics that are shock-free on $[0, \infty)$. Clearly, $u$ grows exponentially along divides, with the exception of stationary ones, $x = \bar{x}$, along which $u$ vanishes. The following proposition, which identifies the points of origin of stationary divides, should be compared with Theorem 11.4.1.

**11.10.1 Lemma.** *The line $x = \bar{x}$ is a stationary divide, associated with the solution $u$, if and only if*

$$\int_{\bar{x}}^{z} u_0(x)dx \geq 0, \qquad -\infty < z < \infty, \tag{11.10.3}$$

*i.e., $\bar{x}$ is a minimizer of the primitive of $u_0(\cdot)$.*

**Proof.** The reason it is here possible to locate the point of origin of divides with such precision is that the homogeneous balance law (11.10.1) may be regarded equally well as an inhomogeneous conservation law:

$$\partial_t \left[ e^{-t} u(x,t) \right] + \partial_x \left[ e^{-t} f(u(x,t)) \right] = 0. \tag{11.10.4}$$

Assume first (11.10.3) holds. Fix any $\bar{t} > 0$ and consider the minimal backward characteristic $\xi(\cdot)$ emanating from the point $(\bar{x}, \bar{t})$. We integrate (11.10.4) over the set bordered by the graph of $\xi(\cdot)$, the line $x = \bar{x}$ and the $x$-axis. Applying Green's theorem and using Theorem 11.9.3 and (11.10.2) yields

$$\begin{aligned} &\int_0^{\bar{t}} e^{-t} \left\{ f'(u(\xi(t),t)u(\xi(t),t) - f(u(\xi(t),t)) \right\} dt \\ &\qquad + \int_0^{\bar{t}} e^{-t} f(u(\bar{x}\pm,t))dt + \int_{\bar{x}}^{z} u_0(x)dx = 0. \end{aligned} \tag{11.10.5}$$

All three terms on the left-hand side of (11.10.5) are nonnegative and hence they should all vanish. Thus $u(\bar{x}\pm, t) = 0,\ t \in (0, \infty)$, and $x = \bar{x}$ is a stationary divide.

Conversely, assume $x = \bar{x}$ is a stationary divide, along which $u$ vanishes. Fix any $z < \bar{x}$. For $\varepsilon > 0$, let $\chi(\cdot)$ be the curve issuing form the point $(z, 0)$ and having slope $\dot{\chi}(t) = f'\left(\varepsilon e^t\right)$. Suppose $\chi(\cdot)$ intersects the line $x = \bar{x}$ at time $\bar{t}$. We integrate (11.10.4) over the set $\{(x,t) : 0 \le t \le \bar{t},\ \chi(t) \le x \le \bar{x}\}$ and apply Green's theorem. Upon adding and subtracting terms that depend solely on $t$, we end up with

$$(11.10.6)\qquad \int_0^{\bar{t}} e^{-t}\left\{f\left(\varepsilon e^t\right) - f(u(\chi(t)+,t)) - f'\left(\varepsilon e^t\right)\left[\varepsilon e^t - u(\chi(t)+,t)\right]\right\}dt$$

$$-\int_0^{\bar{t}} e^{-t} f\left(\varepsilon e^t\right)dt = \int_z^{\bar{x}} [u_0(x) - \varepsilon]\,dx.$$

Both terms on the left-hand side of the above equation are nonpositive, and hence so also is the right-hand side. Letting $\varepsilon \downarrow 0$, we arrive at (11.10.3), for any $z < \bar{x}$. The case $z > \bar{x}$ is handled by the same method. This completes the proof.

Next we show that between adjacent stationary divides the solution attains asymptotically a standing wave profile of finite amplitude. The following proposition should be compared with Theorem 11.7.2.

**11.10.2 Lemma.** *Assume $x = x_-$ and $x = x_+$, $x_- < x_+$, are adjacent divides, associated with the solution $u$, i.e.,* (11.10.3) *holds for $\bar{x} = x_-$ and $\bar{x} = x_+$, but not for any $\bar{x}$ in the interval $\left(x_-, x_+\right)$. Consider any forward characteristic $\psi(\cdot)$ issuing from the point $\left(\frac{x_- + x_+}{2}, 0\right)$. Then, as $t \to \infty$,*

$$(11.10.7)\qquad u(x\pm, t) = \begin{cases} v_-(x) + o(1), & uniformly\ for\ \ x_- < x < \psi(t) \\ v_+(x) + o(1), & uniformly\ for\ \ \psi(t) < x < x_+, \end{cases}$$

*where $v_-(x)$ and $v_+(x)$ are solutions of the differential equation $\partial_x f(v) = v$, with $v_-(x_-) = 0$ and $v_+(x_+) = 0$. Furthermore,*

$$(11.10.8)\qquad \psi(t) = x_0 + o(1),$$

*where $x_0$ is determined by the condition*

$$(11.10.9)\qquad \int_{x_-}^{x_0} v_-(y)dy + \int_{x_0}^{x_+} v_+(y)dy = 0.$$

*In particular, if $u_0$ is differentiable at $x_\pm$ and $u_0'(x_\pm) > 0$, then the order $o(1)$ in* (11.10.7) *and* (11.10.8) *is upgraded to exponential: $O\left(e^{-t}\right)$.*

**Proof.** As $t \to \infty$, the minimal backward characteristic $\zeta(\cdot)$ emanating from the point $(\psi(t), t)$ converges to a divide which is trapped inside the interval $[x_-, x_+)$

and thus is stationary. Since $x_-$ and $x_+$ are adjacent, $\zeta(\cdot)$ must converge to $x_-$. In particular, $\zeta(0) = x_- + o(1)$, as $t \to \infty$.

We fix $t > 0$ and pick any $x \in (x_-, \psi(t)]$. Let $\zeta(\cdot)$ denote the minimal backward characteristic emanating from $(x, t)$; it is intercepted by the $x$-axis at $\xi(0) = \xi_0$, with $x_- \leq \xi_0 \leq \zeta(0)$. In particular, $\xi_0 = x_- + o(1)$, as $t \to \infty$. Recalling Theorem 11.9.3, we integrate the system (11.10.2) to get $v(\tau) = \bar{u}e^{\tau}, 0 \leq \tau \leq t$, where $u_0(\xi_0-) \leq \bar{u} \leq u_0(\xi_0+)$, and

$$x - \xi_0 = \int_0^t f'(v(\tau))d\tau = \int_{\bar{u}}^{u(x-,t)} \frac{f'(v)}{v} dv. \tag{11.10.10}$$

Now, (11.10.10) implies $u(x-,t) = O(1)$ whence $\bar{u} = e^{-t}u(x-,t) = O(e^{-t})$. In turn, by virtue of $\xi_0 = x_- + o(1)$ and $\bar{u} = O\left(e^{-t}\right)$, (11.10.10) yields the upper half of (11.10.7). When $u_0'(x_-) > 0$, then $\bar{u} = O\left(e^{-t}\right)$ implies in particular that $\xi_0 = x_- + O\left(e^{-t}\right)$ and so $o(1)$ is upgraded to $O\left(e^{-t}\right)$. The lower half of (11.10.7) is treated by the same method.

Integrating (11.10.1) over $[x_-, x_+] \times [0, t]$, we deduce $\int_{x_-}^{x_+} u(x,t)dx = 0$, so that (11.10.7) yields (11.10.8), (11.10.9). The proof is complete.

When the initial data $u_0(\cdot)$ are periodic, with mean $M$, then the solution $u(\cdot, t)$, at time $t$, is also periodic, with mean $Me^t$, and thus blows up as $t \to \infty$, unless $M = 0$. If $M = 0$, (11.10.3) is satisfied for at least one $\bar{x}$ in each period interval. Therefore, Lemma 11.10.2 has the following corollary, akin to Theorem 11.7.3.

**11.10.3 Theorem.** *When the initial data $u_0$ are periodic, with mean zero, then, as $t \to \infty$, the solution $u$ tends to a periodic serrated profile consisting of wavelets of the form* (11.10.7). *The number of wavelets per period equals the number of points $\bar{x}$ in any period interval for which* (11.10.3) *holds. In the generic case where* (11.10.3) *is satisfied at a single point $\bar{x}$ in each period interval, $u$ tends to a sawtooth profile with a single tooth per period.*

## 11.11 An Inhomogeneous Conservation Law

Here we discuss the large time behavior of periodic solutions of an inhomogeneous conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t), x) = 0, \tag{11.11.1}$$

where $f$ is a $C^2$ function with the following properties:

(a) Periodicity in $x$: $f(u, x+1) = f(u, x), \ -\infty < u < \infty, \ -\infty < x < \infty.$

(b) Genuine nonlinearity: $f_{uu}(u,x) \geq \mu > 0,\ -\infty < u < \infty,\ -\infty < x < \infty$.

(c) The set of critical points consists of minima and saddles. For some $\bar{u}$ in $(-\infty,\infty)$, $b$ in $(0,1)$, $x_k = b + k$ and $k = 0, \pm 1, \pm 2, \dots$ the following hold: $f_u(\bar{u},x_k) = f_x(\bar{u},x_k) = 0$, $f_{uu}(\bar{u},x_k) f_{xx}(\bar{u},x_k) - f_{ux}^2(\bar{u},x_k) > 0$, and also $f_u(0,k) = f_x(0,k) = 0$, $f_{uu}(0,k) f_{xx}(0,k) - f_{ux}^2(0,k) < 0$.

(d) Normalization: $f(0,k) = 0$, hence $f(\bar{u},x_k) < 0$.

A typical example of such a function is $f(u,x) = u^2 - \sin^2(\pi x)$.

The system (11.9.7), for shock-free characteristics, here takes the form

(11.11.2)
$$\begin{cases} \dot{\xi} = f_u(v,\xi) \\ \dot{v} = -f_x(v,\xi). \end{cases}$$

As noted in Remark 11.9.2, orbits of (11.11.2) are level curves of the function $f(u,x)$. By virtue of the properties of $f$, the phase portrait of (11.11.2) has the form depicted in Fig. 11.11.1. Orbits dwelling on level curves $f = p$, with $p > 0$, are unidirectional, from left to right or from right to left. By contrast, orbits dwelling on level curves $f = p$, with $p < 0$, are periodic. Finally, orbits dwelling on the level curves $f = 0$ are heteroclinic, joining neighboring saddle points; and in particular those dwelling on the nonnegative branch, $v = v_+(x)$, join $(k+1,0)$ to $(k,0)$, while those dwelling on the nonpositive branch, $v = v_-(x)$, join $(k,0)$ to $(k+1,0)$, $k = 0, \pm 1, \pm 2, \dots$.

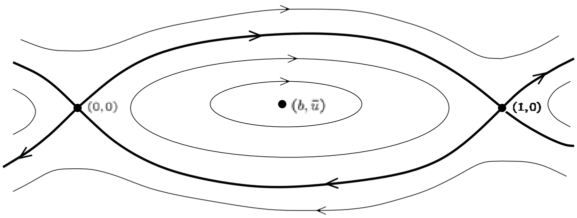


**Fig. 11.11.1**

For $p \in [f(\bar{u},b),\infty)\backslash\{0\}$, we define $T(p)$ as follows: If $p < 0$, $T(p)$ is the period around the level curve $f = p$. If $p > 0$, $T(p)$ is the time it takes to traverse a $\xi$-interval of length two, along the level curve $f = p$. The flow along any orbit moves at a swift pace, except near the equilibrium points $(k,0)$, where it slows down. In the linearized system about $(k,0)$, the sojourn in the vicinity of the equilibrium point, along the orbit on the level $p$, lasts for $-\lambda_0^{-1} \log|p|$ time units, where $\pm\lambda_0$ are the eigenvalues of the Jacobian matrix of the vector field $(f_u, -f_x)$, evaluated at the saddle point $(0,k)$, i.e., $\lambda_0 = \left[f_{ux}^2(k,0) - f_{uu}(k,0) f_{xx}(k,0)\right]^{1/2}$. Therefore,

$$\frac{T(p)}{\log|p|} = -\frac{2}{\lambda_0} + o(1), \qquad \text{as } p \to 0. \tag{11.11.3}$$

For any $M \in (-\infty, \infty)$, the equation (11.11.1) admits a unique admissible periodic stationary solution $u_M(x)$, with mean $M$. Let

$$M_\pm = \int_0^1 v_\pm(x)dx. \tag{11.11.4}$$

For $M \geq M_+$ or $M \leq M_-$, $u_M(x)$ is just the unique level curve $f = p \geq 0$ with mean $M$. By contrast, for $M_- < M < M_+$, $u_M$ is a weak solution containing a single admissible stationary shock per period:

$$u_M(x) = \begin{cases} v_+(x), \; k \leq x < k+a \\ \\ v_-(x), \; k+a < x < k+1, \end{cases} \qquad k = 0, \pm 1, \pm 2, \dots \tag{11.11.5}$$

where $a \in (0, 1)$ is determined by

$$\int_0^a v_+(x)dx + \int_a^1 v_-(x)dx = M. \tag{11.11.6}$$

The aim is to show that, as $t \to \infty$, 1-periodic solutions of (11.11.1), with mean $M$, converge to $u_M$. We shall only discuss the interesting case $M_- < M < M_+$.

**11.11.1 Theorem.** *Let $u(x,t)$ be the admissible solution of* (11.11.1)*, on the upper half-plane, with initial data $u_0(x)$ which are 1-periodic functions with mean $M$ in $(M_-, M_+)$. Then, as $t \to \infty$, for any $\lambda < \lambda_0$,*

$$f\left(u(x\pm,t),x\right) = o\left(e^{-\lambda t}\right), \;\; \textit{uniformly on } \; (-\infty,\infty), \tag{11.11.7}$$

(11.11.8)

$$u(x\pm,t) = \begin{cases} v_+(x) + o\left(e^{-\frac{1}{2}\lambda t}\right), \; k \leq x < \chi_k(t) \\ \\ v_-(x) + o\left(e^{-\frac{1}{2}\lambda t}\right), \; \chi_k(t) < x \leq k+1, \end{cases} \qquad k = 0, \pm 1, \pm 2, \dots$$

*where*

$$\chi_k(t) = k + a + o\left(e^{-\frac{1}{2}\lambda t}\right), \tag{11.11.9}$$

*with $a$ determined through* (11.11.6).

**Proof.** We fix any $k = 0, \pm 1, \pm 2, \dots$, and note that

(11.11.10)
$$\int_k^{k+1} u(x,t)dx = M.$$

Since $M \in (M_-, M_+)$, (11.10.10) implies that there are $x \in (k, k+1)$ such that $f(u(x-,t),x) = p < 0$. For such an $x$, the minimal backward characteristic $\zeta(\cdot)$ emanating from $(x,t)$ is the restriction to $[0,t]$ of the $T(p)$-periodic orbit that dwells on the level curve $f = p$. The minimal backward characteristic $\bar{\zeta}(\cdot)$ emanating from the point $(\bar{x},t)$, where $\bar{x} = x - \varepsilon$ with $\varepsilon$ positive and small, will likewise be the restriction to $[0,t]$ of a periodic orbit dwelling on some level curve $f = \bar{p}$, with $|p - \bar{p}|$ small. It is now clear from the phase portrait, Fig. 11.11.1, that if $t$ is larger than the period $T(p)$ the graphs of $\zeta(\cdot)$ and $\bar{\zeta}(\cdot)$ must intersect at some time $\tau \in (0,t)$, in contradiction to Theorem 11.9.4. Thus $t \le T(p)$ and hence $f(u(x-,t),x) \to 0$, at $t \to \infty$ by virtue of (11.11.3).

Suppose next there is $x \in [k, k+1]$ with $f(u(x-,t),x) = p > 0$. We fix $\bar{x}$ such that $\bar{x} < x < \bar{x} + 1$ and $f(u(\bar{x}-,t),\bar{x}) < 0$. If $\zeta(\cdot)$ and $\xi(\cdot)$ denote the minimal backward characteristics emanating from the points $(\bar{x},t)$ and $(x,t)$, respectively, then $\zeta(\tau) < \xi(\tau) < \zeta(\tau) + 1$, for $0 < \tau < t$. Hence, $|x - \xi(0)| < 2$. But then $t \le T(p)$ and hence $f(u(x-,t),x) \to 0$, as $t \to \infty$, in this case as well.

By genuine nonlinearity and $f(u(x-,t),x) = o(1)$, for $t$ large, $u(x-,t)$ must be close to either $v_-(x)$ or $v_+(x)$. Since admissible solutions are allowed to jump only downwards, there exists a characteristic $\chi_k(\cdot)$, with $\chi_k(t) \in (k, k+1)$ for $t \in [0,\infty)$, such that, for $t$ large, $u(x-,t)$ is close to $v_-(x)$ if $k \le x < \chi_k(t)$, and close to $v_+(x)$ if $\chi_k(t) < x \le k+1$. Minimal backward characteristics emanating from points $(x,t)$, with $\chi_{k-1}(t) < x < \chi_k(t)$ and $f(u(x-,t)) = p \gtrless 0$, are trapped between $\chi_{k-1}(\cdot)$ and $\chi_k(\cdot)$, so that our earlier estimate $t \le T(p)$ becomes sharper: $t \le \frac{1}{2}T(p) + O(1)$. This together with (11.11.3) imply (11.11.7), which in turn yields (11.11.8). Finally, by combining (11.11.8) with (11.11.10) we arrive at (11.11.9), where $a$ is determined through (11.11.6). The proof is complete.

A more detailed picture of the asymptotic behavior of the above solution $u(x,t)$ emerges by locating its divides. By account of (11.11.7), any divide must be dwelling on the level curve $f = 0$. We shall see that the point of origin of any divide within the period intrval $[\chi_{k-1}(0), \chi_k(0)]$ may be determined explicitly from the initial data. For that purpose we introduce the function

(11.11.11)
$$v_k(x) = \begin{cases} v_+(x), & -\infty < x \le k \\ v_-(x), & k < x < \infty, \end{cases}$$

which is a steady-state solution of (11.11.1):

(11.11.12)
$$\partial_t v_k(x) + \partial_x f(v_k(x), x) = 0.$$

**11.11.2 Theorem.** *Under the assumptions of Theorem* 11.11.1, *a divide associated with the solution $u(x,t)$ issues from the point $(\bar{x}, 0)$, with $\chi_{k-1}(0) \le \bar{x} \le \chi_k(0)$, if and only if $\bar{x}$ is a minimizer of the function*

(11.11.13)
$$\Phi_k(z) = \int_k^z [u_0(x) - v_k(x)]\, dx,$$

*over* $(-\infty, \infty)$.

**Proof.** Assume first $\bar{x} \in \left[\chi_{k-1}(0), \chi_k(0)\right]$ minimizes $\Phi_k$ over $(-\infty, \infty)$. We construct the characteristic $\xi(\cdot)$, associated with the solution $v_k$, issuing from the point $(\bar{x}, 0)$. Thus, $\xi(\cdot)$ will be determined by solving the system (11.11.2) with initial conditions $\xi(0) = \bar{x}$ and $v(0) = v_-(\bar{x})$ if $\bar{x} \geq k$, or $v(0) = v_+(\bar{x})$ if $\bar{x} < k$. In either case, $\dot{\xi}(t) = f_u\,(v_k(\xi(t)), \xi(t))$. We fix any $\bar{t} > 0$ and consider the minimal backward characteristic $\zeta(\cdot)$, associated with the solution $u(x,t)$, emanating from the point $(\xi(\bar{t}), \bar{t})$ and intercepted by the $x$-axis at $\zeta(0) = z \in \left[\chi_{k-1}(0), \chi_k(0)\right]$. Thus, $\dot{\zeta}(t) = f_u\,(u(\zeta(t)-, t), \zeta(t))$. We subtract (11.11.12) from (11.11.1) and integrate the resulting equation over the set bordered by the $x$-axis and the graphs of $\xi(\cdot)$ and $\zeta(\cdot)$ over $[0, \bar{t}]$. Applying Green's theorem yields

(11.11.14)
$$\begin{aligned}
&\int_0^{\bar{t}} \{f(u(\xi(t)-, t), \xi(t)) - f\,(v_k(\xi(t)), \xi(t)) \\
&\qquad - f_u\,(v_k(\xi(t)), \xi(t))\,[u(\xi(t)-, t) - v_k(\xi(t))]\}\, dt \\
&- \int_0^{\bar{t}} \{f(u(\zeta(t)-, t), \zeta(t)) - f\,(v_k(\zeta(t)), \zeta(t)) \\
&\qquad - f_u(u(\zeta(t)-, t), \zeta(t))\,[u(\zeta(t)-, t) - v_k(\zeta(t))]\}\, dt \\
&= \int_z^{\bar{x}} [u_0(x) - v_k(x)]\, dx = \Phi_k(\bar{x}) - \Phi_k(z).
\end{aligned}$$

Both terms on the left-hand side of the above equation are nonnegative, while the right-hand side is nonpositive. Thus, all three terms must vanish and $\xi(\cdot)$ is indeed a divide associated with $u(x,t)$.

Conversely, assume $(\bar{x}, 0)$ is the point of origin of a divide $\xi(\cdot)$ associated with the solution $u(x,t)$. Thus $\xi(\cdot)$ will solve the system (11.11.2) with initial conditions $\xi(0) = \bar{x}$ and $v(0) = v_-(\bar{x})$ if $\bar{x} \geq k$, or $v(0) = v_+(\bar{x})$ if $\bar{x} < k$. In either case, $u(\xi(t)\pm, t) = v_k(\xi(t))$, $t \in (0, \infty)$. We fix any $z \in (k-1, k+1)$ and construct the characteristic $\zeta(\cdot)$, associated with the solution $v_k$, that issues from the point $(z, 0)$. Thus, $v_k(\zeta(t)) = v_+(\zeta(t))$ if $z \leq k$, or $v_k(\zeta(t)) = v_-(\zeta(t))$ if $z \geq k$. In either case, $\dot{\zeta}(t) = f_u\,(v_k(\zeta(t)), \zeta(t))$ and $\zeta(t) - \xi(t) \to 0$, as $t \to \infty$. We subtract (11.11.12) from (11.11.1) and integrate the resulting equation over the set bordered by the $x$-axis and the graphs of $\xi(\cdot)$ and $\zeta(\cdot)$ on $[0, \infty)$. Applying Green's theorem yields

$$\int_0^\infty \{f(u(\zeta(t)-,t),\zeta(t)) - f(v_k(\zeta(t)),\zeta(t))$$

$$-f_u(v_k(\zeta(t)),\zeta(t))[u(\zeta(t)-,t) - v_k(\zeta(t))]\}\,dt \tag{11.11.15}$$

$$= \int_{\bar{x}}^{z} [u_0(x) - v_k(x)]\,dx = \Phi_k(z) - \Phi_k(\bar{x}).$$

The left-hand side, and thereby also the right-hand side, of (11.11.15) is nonnegative. Therefore, $\bar{x}$ minimizes $\Phi_k$ over $(k-1, k+1)$, and hence even over $(-\infty,\infty)$, as $M \in (M_-, M_+)$. The proof is complete.

As $t \to \infty$, the family of minimal backward characteristics emanating from points $(\chi_k(t), t)$ converges monotonically to the divide that issues from $(x_+, 0)$, where $x_+$ is the largest of the minimizers of $\Phi_k$. Similarly, the family of maximal backward characteristics emanating from the points $(\chi_{k-1}(t), t)$ converges monotonically to the divide that issues from $(x_-, 0)$, where $x_-$ is the smallest of the minimizers of $\Phi_k$. Generically, $\Phi_k$ should attain its minimum at a single point, in which case $x_- = x_+$.

## 11.12 Notes

There is voluminous literature on the scalar conservation law in one-space dimension, especially the genuinely nonlinear case, beginning with the seminal paper of Hopf [1], on the Burgers equation, already cited in earlier chapters.

In the 1950's, the qualitative theory was developed by the Russian school, headed by Oleinik [1,2,4], based on the vanishing viscosity approach as well as on the Lax-Friedrichs finite difference scheme (Lax [1]). It is in that context that Theorem 11.2.2 was originally established. The reader may find an exposition in the text by Smoller [3]. The culmination of that approach was the development of the theory of scalar conservation laws in several space dimensions, discussed in Chapter VI.

In a different direction, Lax [2] discovered the explicit representation (11.4.10) for solutions to the Cauchy problem and employed it to establish the existence of invariants (Theorem 11.4.2), the development of $N$-waves under initial data of compact support (Theorem 11.6.1) as well as the formation of sawtooth profiles under periodic initial data (Theorem 11.7.3). The original proof, by Schaeffer [1], that generically solutions are piecewise smooth was also based on the same method and so is the proof of Theorem 11.3.5, by Ambrosio and De Lellis [2]. This approach readily extends (Oleinik [1]) to inhomogeneous, genuinely nonlinear scalar conservation laws, which may also be casted as Hamilton-Jacobi equations, but cannot handle balance laws. A thorough presentation of the theory of viscosity solutions for Hamilton-Jacobi equations is found in the monograph by Lions [1].

The approach via generalized characteristics, pursued in this chapter, is taken from Dafermos [7], for the homogeneous conservation law, and Dafermos[8], for the inhomogeneous balance law. In fact, these papers consider the more general situation where $f_{uu} \geq 0$, and one-sided limits $u(x\pm, t)$ exist for all $x \in (-\infty, \infty)$ and almost all $t \in (0, \infty)$, even though $u(\cdot, t)$ may not be a function of bounded variation.

The property that the lap number of solutions of conservation laws (8.6.2) with viscosity is nonincreasing with time was discovered independently by Nickel [1] and Matano [1]. The $L^1$ contraction property for piecewise smooth solutions in one-space dimension was noted by Quinn [1]. The functional (11.8.11), in alternative, albeit completely equivalent, form was designed by Liu and Yang [3], who employ it to establish Theorem 11.8.3, for piecewise smooth solutions. For an alternative derivation, see Goatin and LeFloch [1].

Section 11.10 improves on an earlier result of Lyberopoulos [1], while the example discussed in Section 11.11 is new. The effects of inhomogeneity and source terms on the large time behavior of solutions are also discussed, by the method of generalized characteristics, in Dafermos [14], Lyberopoulos [2], Fan and Hale [1,2], Härterich [1], Ehrt and Härterich [1], Mascia and Sinestrari [1] and Fan, Jin and Teng [1]. Problems of this type are also treated by different methods in Liu [23], Dias and LeFloch [1] and Sinestrari [1].

So much is known about the scalar conservation and balance law in one-space dimension that it would be hopeless to attempt to provide comprehensive coverage. What follows is just a sample of relevant results.

Let us begin with the genuinely nonlinear case. For a probabilistic interpretation of generalized characteristics, see Rezakhanlou [1]. For an interesting application of the method of generalized characteristics in elastostatics, under incompressibility and inextensibility constraints, see Choksi [1].

The optimal convergence rate to $N$-waves is established by Yong Jung Kim [1]. The interesting, metastable status of $N$-waves for the Burgers equation with viscosity is demonstrated in Kim and Tzavaras [1].

An explicit representation of admissible solutions on the quarter-plane, analogous to Lax's formula for the upper half-plane, is presented in LeFloch [1] and LeFloch and Nédélec [1]. An analog of Lax's formula has also been derived for the special systems with coinciding shock and rarefaction wave curves; see Benzoni-Gavage [1].

The analog of (11.2.1) holds for scalar conservation laws (6.1.1), in several space variables, if $g_\alpha(u) = f(u)v_\alpha$ , where $v$ is a constant vector (Hoff [1]).

For a Chapman-Enskog type regularization of the scalar conservation law, see Shochet and Tadmor [1].

A kinetic formulation, different from the one discussed in Section 6.7, is presented in Brenier and Corrias [1].

Panov [1] and, independently, De Lellis, Otto and Westickenberg [2] show that the entropy inequality for just one uniformly convex entropy suffices for singling out the unique admissible weak solution in $L^\infty$.

Regularity of solutions in Besov spaces is established in Lucier [2]. For the rate of convergence of numerical schemes see e.g. Nessyahu and Tadmor [1] and Osher and Tadmor [1].

The connection of the scalar conservation law with the system of "pressureless gas" (7.1.11) and the related model of "sticky particles" is investigated in E, Rykov and Sinai [1], Brenier and Grenier [1], and Bouchut and James [1]. The interesting theory of the pressureless gas is developed in Wang and Ding [1], Wang, Huang and Ding [1], Huang and Wang [1], Li and Warnecke [1] and Ding and Huang [1]. See also Huang [1] and Sever [5,6].

Homogenization effects under random periodic forcing are demonstrated in E [2,3], E and Serre [1] and E, Khanin, Mazel and Sinai [1].

For boundary control problems associated with the scalar conservation law, see Ancona and Marson [1,2].

A stochastic scalar conservation law is discussed by Jong Uhn Kim [1].

The case where $f(u, x)$ is piecewise constant in $x$ is discussed in Lyons [1], Klingenberg and Risebro [1] and Diehl [1].

When $f$ has inflection points, the structure of solutions is considerably more intricate, as a result of the formation of contact discontinuities, which become sources of signals propagating into the future. The method of generalized characteristics extends to this case, as well, but the analysis becomes considerably more complicated (Dafermos[11], Jenssen and Sinestrari [1]). See also Marson [1]. For the construction of solutions, see Ballou [1]. Regularity is discussed in Ballou [2], Guckenheimer [1], Dafermos [11] and Cheverry [4]. The large time behavior is investigated in Dafermos [1,11], Greenberg and Tong [1], Conlon [1], Cheng [1,2,3], Weinberger [1], Sinestrari [2] and Mascia [1]. See also Baiti and Jenssen [1].

In the special case $f(u) = u^m$, the properties of solutions may be studied effectively with the help of the underlying self-similarity transformation; see Bénilan and Crandall [1] and Liu and Pierre [1]. This last paper also considers initial data that are merely measures. For recent developments in that direction, see Chasseigne [1]. The limit behavior as $m \to \infty$ is discussed in Xu [1].

# XII

# Genuinely Nonlinear Systems of Two Conservation Laws

The theory of solutions of genuinely nonlinear, strictly hyperbolic systems of two conservation laws will be developed in this chapter at a level of precision comparable to that for genuinely nonlinear scalar conservation laws, expounded in Chapter XI. This will be achieved by exploiting the presence of coordinate systems of Riemann invariants and the induced rich family of entropy-entropy flux pairs. The principal tools in the investigation will be generalized characteristics and entropy estimates.

The analysis will reveal a close similarity in the structure of solutions of scalar conservation laws and pairs of conservation laws. Thus, as in the scalar case, jump discontinuities are generally generated by the collision of shocks and/or the focussing of compression waves, and are then resolved into wave fans approximated locally by the solution of associated Riemann problems.

The total variation of the trace of solutions along space-like curves is controlled by the total variation of the initial data, and spreading of rarefaction waves affects total variation, as in the scalar case.

The dissipative mechanisms encountered in the scalar case are work here at as well, and have similar effects on the large time behavior of solutions. Entropy dissipation induces $O(t^{-1/2})$ decay of solutions with initial data in $L^1(-\infty, \infty)$. When the initial data have compact support, the two characteristic families asymptotically decouple, the characteristics spread and form a single $N$-wave profile for each family. Finally, as in the scalar case, confinement of characteristics under periodic initial data induces $O(t^{-1})$ decay in the total variation per period and formation of saw-toothed profiles, one for each characteristic family.

## 12.1 Notation and Assumptions

We consider a genuinely nonlinear, strictly hyperbolic system of two conservation laws,

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0, \tag{12.1.1}$$

on some disk $\mathcal{O}$ centered at the origin. The eigenvalues of D$F$ (characteristic speeds) will here be denoted by $\lambda$ and $\mu$, with $\lambda(U) < 0 < \mu(U)$ for $U \in \mathcal{O}$, and the associated eigenvectors will be denoted by $R$ and $S$.

The system is endowed with a coordinate system $(z, w)$ of Riemann invariants, vanishing at the origin $U = 0$, and normalized according to (7.3.8):

$$\text{(12.1.2)} \qquad \mathrm{D}zR = 1, \quad \mathrm{D}zS = 0, \quad \mathrm{D}wR = 0, \quad \mathrm{D}wS = 1.$$

The condition of genuine nonlinearity is now expressed by (7.5.4), which here reads

$$\text{(12.1.3)} \qquad \lambda_z < 0, \qquad \mu_w > 0.$$

The direction in the inequalities (12.1.3) has been selected so that $z$ increases across admissible weak 1-shocks while $w$ decreases across admissible weak 2-shocks.

For definiteness, we will consider systems with the property that the interaction of any two shocks of the same characteristic family produces a shock of the same family and a rarefaction wave of the opposite family. Note that this condition is here expressed by

$$\text{(12.1.4)} \qquad S^\top \mathrm{D}^2 z S > 0, \qquad R^\top \mathrm{D}^2 w R > 0.$$

Indeed, in conjunction with (8.2.19), (12.1.3) and Theorem 8.3.1, the inequalities (12.1.4) imply that $z$ increases across admissible weak 2-shocks while $w$ decreases across admissible weak 1-shocks. Therefore, the admissible shock and rarefaction wave curves emanating from the state $(\bar{z}, \bar{w})$ have the shape depicted in Fig. 12.1.1. Consequently, as seen in Fig. 12.1.2(a), a 2-shock that joins the state $(z_\ell, w_\ell)$, on the left, with the state $(z_m, w_m)$, on the right, interacts with a 2-shock that joins $(z_m, w_m)$, on the left, with the state $(z_r, w_r)$, on the right, to produce a 1-rarefaction wave, joining $(z_\ell, w_\ell)$, on the left, with a state $(z_0, w_\ell)$, on the right, and a 2-shock joining $(z_0, w_\ell)$, on the left, with $(z_r, w_r)$, on the right, as depicted in Fig. 12.1.2(b). Similarly, the interaction of two 1-shocks produces a 1-shock and a 2-rarefaction wave.

Also for definiteness, we assume

$$\text{(12.1.5)} \qquad \lambda_w < 0, \qquad \mu_z > 0,$$

or equivalently, by virtue of (7.3.14) and (7.4.15),

$$\text{(12.1.6)} \qquad R^\top \mathrm{D}^2 z S > 0, \qquad S^\top \mathrm{D}^2 w R > 0.$$

The prototypical example is the system (7.1.8) of isentropic thermoelasticity, which satisfies all three assumptions (12.1.3), (12.1.4) and (12.1.6), with Riemann invariants (7.3.2), provided $\sigma''(u) < 0$, i.e., the elastic medium is a soft spring or a gas. When the medium is a hard spring, i.e., $\sigma''(u) > 0$, the sign of the Riemann invariants in (7.3.2) has to be reversed.

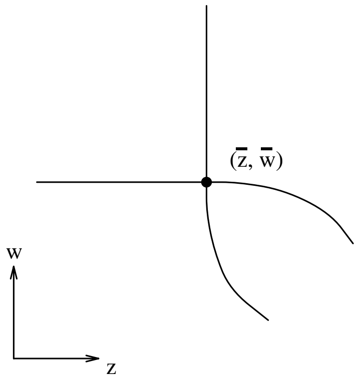


**Fig. 12.1.1**

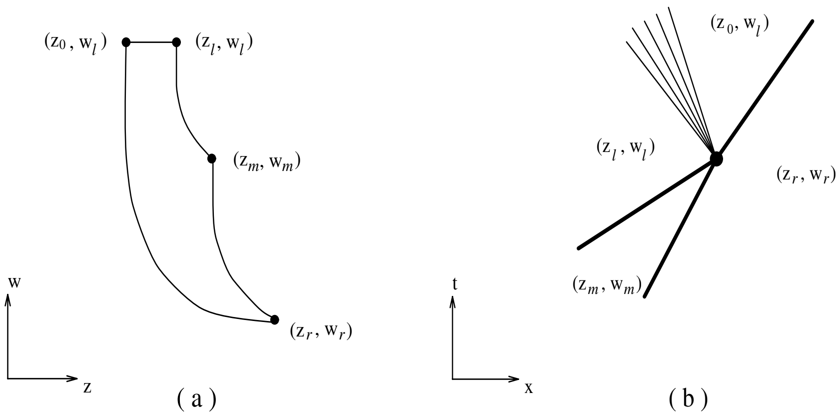


**Fig. 12.1.2 (a,b)**

## 12.2 Entropy-Entropy Flux Pairs and the Hodograph Transformation

As explained in Section 7.4, our system is endowed with a rich family of entropy-entropy flux pairs $(\eta, q)$, which may be determined as functions of the Riemann invariants $(z, w)$ by solving the system (7.4.12), namely

$$q_z = \lambda \eta_z\,, \qquad q_w = \mu \eta_w\,. \tag{12.2.1}$$

The integrability condition (7.4.13) now takes the form

$$\eta_{zw} + \frac{\lambda_w}{\lambda - \mu}\eta_z + \frac{\mu_z}{\mu - \lambda}\eta_w = 0. \tag{12.2.2}$$

The entropy $\eta(z, w)$ will be a convex function of the original state variable $U$ when the inequalities (7.4.16) hold, that is,

(12.2.3)
$$\begin{cases} \eta_{zz} + (R^\top \mathrm{D}^2 z R)\eta_z + (R^\top \mathrm{D}^2 w R)\eta_w \geq 0 \\ \eta_{ww} + (S^\top \mathrm{D}^2 z S)\eta_z + (S^\top \mathrm{D}^2 w S)\eta_w \geq 0 \,. \end{cases}$$

In the course of our investigation, we shall face the need to construct entropy-entropy flux pairs with prescribed specifications, by solving (12.2.1) or (12.2.2) under assigned side conditions. To verify that the constructed entropy satisfies the condition (12.2.3), for convexity, it usually becomes necessary to estimate the second derivatives $\eta_{zz}$ and $\eta_{ww}$ in terms of the first derivatives $\eta_z$ and $\eta_w$ . For that purpose, one may employ the equations obtained by differentiating (12.2.2) with respect to $z$ and $w$:

(12.2.4)
$$\begin{cases} \eta_{zzw} + \dfrac{\lambda_w}{\lambda - \mu}\eta_{zz} = \dfrac{(\mu-\lambda)\lambda_{zw} + \lambda_z\lambda_w - 2\lambda_w\mu_z}{(\lambda-\mu)^2}\eta_z + \dfrac{(\lambda-\mu)\mu_{zz} - \lambda_z\mu_z + 2\mu_z^2}{(\lambda-\mu)^2}\eta_w \\ \eta_{wwz} + \dfrac{\mu_z}{\mu - \lambda}\eta_{ww} = \dfrac{(\mu-\lambda)\lambda_{ww} - \lambda_w\mu_w + 2\lambda_w^2}{(\mu-\lambda)^2}\eta_z + \dfrac{(\lambda-\mu)\mu_{zw} + \mu_z\mu_w - 2\lambda_w\mu_z}{(\mu-\lambda)^2}\eta_w \,. \end{cases}$$

As an illustration, we consider the important family of *Lax entropy-entropy flux pairs*

(12.2.5)
$$\begin{cases} \eta(z,w) = e^{kz}\left[\phi(z,w) + \dfrac{1}{k}\chi(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \\ q(z,w) = e^{kz}\lambda(z,w)\left[\psi(z,w) + \dfrac{1}{k}\theta(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \end{cases}$$

(12.2.6)
$$\begin{cases} \eta(z,w) = e^{kw}\left[\alpha(z,w) + \dfrac{1}{k}\beta(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \\ q(z,w) = e^{kw}\mu(z,w)\left[\gamma(z,w) + \dfrac{1}{k}\delta(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \end{cases}$$

where $k$ is a parameter. These are designed to vary stiffly with one of the two Riemann invariants so as to be employed for decoupling the two characteristic families. To construct them, one substitutes $\eta$ and $q$ from (12.2.5) or (12.2.6) into the system (12.2.1), thus deriving recurrence relations for the coefficients, and then shows that the remainder is $O(k^{-2})$. The recurrence relations for the coefficients of the family (12.2.5), read as follows:

(12.2.7)
$$\psi = \phi,$$

(12.2.8)
$$\lambda\theta + (\lambda\psi)_z = \lambda\chi + \lambda\phi_z \,,$$

(12.2.9) $$(\lambda\psi)_w = \mu\phi_w \, .$$

Combining (12.2.7) with (12.2.9) yields

(12.2.10) $$(\mu - \lambda)\phi_w = \lambda_w \phi,$$

which may be satisfied by selecting

(12.2.11) $$\phi(z, w) = \exp \int_0^w \frac{\lambda_w(z, \omega)}{\mu(z, \omega) - \lambda(z, \omega)} d\omega.$$

In particular, this $\phi$ is positive, uniformly bounded away from zero on compact sets. Hence, for $k$ sufficiently large, the inequalities (12.2.3) will hold, the second one by virtue of (12.1.4). Consequently, for $k$ large the Lax entropy is a convex function of $U$.

Important implications of (12.2.7) and (12.2.8) are the estimates

(12.2.12) $$q - \lambda\eta = \frac{1}{k} e^{kz} \left[ -\lambda_z \phi + O\left(\frac{1}{k}\right) \right],$$

(12.2.13) $$q - (\lambda + \varepsilon)\eta = -e^{kz} \left[ \varepsilon\phi + O\left(\frac{1}{k}\right) \right],$$

whose usefulness will become clear later.

There is a curious formal analogy between maps $(z, w) \mapsto (\eta, q)$, that carry pairs of Riemann invariants into entropy-entropy flux pairs, and *hodograph transformations* $(z, w) \mapsto (x, t)$, constructed by the following procedure: Suppose $(z(x, t), w(x, t))$ are the Riemann invariants of a $C^1$ solution of (12.1.1), on some domain $\mathcal{D}$ of the $x$-$t$ plane. In the vicinity of any point of $\mathcal{D}$ where the Jacobian determinant $J = z_x w_t - w_x z_t$ does not vanish, the map $(x, t) \mapsto (z, w)$ admits a $C^1$ inverse $(z, w) \mapsto (x, t)$; with derivatives $x_z = J^{-1} w_t$ , $t_z = -J^{-1} w_x$ , $x_w = -J^{-1} z_t$ , and $t_w = J^{-1} z_x$ . Since $z_t + \lambda z_x = 0$ and $w_t + \mu w_x = 0$ on $\mathcal{D}$, we deduce $J = (\lambda - \mu) z_x w_x$ and

(12.2.14) $$x_z = \mu t_z \, , \quad x_w = \lambda t_w \, ,$$

which should be compared and contrasted to (12.2.1). Elimination of $x$ between the two equations in (12.2.14) yields

(12.2.15) $$t_{zw} + \frac{\mu_w}{\mu - \lambda} t_z + \frac{\lambda_z}{\lambda - \mu} t_w = 0,$$

namely the analog of (12.2.2). One may thus construct (classical) solutions of the nonlinear system (12.1.1) of two conservation laws by solving the linear system (12.2.14), or equivalently the linear second order hyperbolic equation (12.2.15). Numerous important special solutions of the system of isentropic gas dynamics, and other systems of two conservation laws arising in mathematical physics, have been derived through that process.

## 12.3 Local Structure of Solutions

Throughout this chapter, $U$ will denote a function of locally bounded variation, defined on $(-\infty, \infty) \times [0, \infty)$ and taking values in a disk of small radius, centered at the origin, which is a weak solution of (12.1.1) satisfying the Lax $E$-condition, in the sense described in Section 10.1. In particular,

$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) \leq 0 \tag{12.3.1}$$

will hold, in the sense of measures, for any entropy-entropy flux pair $(\eta, q)$, with $\eta$ convex.

The notion of generalized characteristic, developed in Chapter X, will play a pivotal role in the discussion.

**12.3.1 Definition.** A Lipschitz curve, with graph $\mathcal{A}$ embedded in the upper half-plane, is called *space-like* relative to $U$ when every point $(\bar{x}, \bar{t}) \in \mathcal{A}$ has the following property: The set $\{(x,t) : 0 \leq t < \bar{t}\,,\;\; \zeta(t) < x < \xi(t)\}$ of points confined between the graphs of the maximal backward 2-characteristic $\zeta(\cdot)$ and the minimal backward 1-characteristic $\xi(\cdot)$, emanating from $(\bar{x}, \bar{t})$, has empty intersection with $\mathcal{A}$.

Clearly, any generalized characteristic, of either family, associated with $U$, is space-like relative to $U$. Similarly, all time lines, $t =$ constant, are space-like.

The solution $U$ will be conveniently monitored through its induced Riemann invariant coordinates $(z, w)$. In Section 12.5, it is shown that the total variation of the trace of $z$ and $w$ along space-like curves is controlled by the total variation of their initial data. In anticipation of that result, we shall be assuming henceforth that, for any space-like curve $t = t^*(x)$, $z(x\pm, t^*(x))$ and $w(x\pm, t^*(x))$ are functions of bounded variation, with total variation bounded by a positive constant $\theta$. Since the oscillation of the solution is small and all arguments will be local, we may assume without further loss of generality that $\theta$ is small.

In order to describe the local structure of the solution, we associate with the generic point $(\bar{x}, \bar{t})$ of the upper half-plane eight, not necessarily distinct, curves (see Fig. 12.3.1) determined as follows:

For $t < \bar{t}$ : $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are the minimal and the maximal backward 1-characteristics emanating from $(\bar{x}, \bar{t})$; similarly, $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ are the minimal and the maximal backward 2-characteristics emanating from $(\bar{x}, \bar{t})$.

For $t > \bar{t}$: $\phi_+(\cdot)$ is the maximal forward 1-characteristic and $\psi_-(\cdot)$ is the minimal forward 2-characteristic issuing from $(\bar{x}, \bar{t})$. To determine the remaining two curves $\phi_-(\cdot)$ and $\psi_+(\cdot)$, we consider the minimal backward 1-characteristic $\xi(\cdot)$ and the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from the generic point $(x,t)$ and locate the points $\xi(\bar{t})$ and $\zeta(\bar{t})$ where these characteristics are intercepted by the $\bar{t}$-time line. Then $\phi_-(t)$ is determined by the property that $\xi(\bar{t}) < \bar{x}$ when $x < \phi_-(t)$ and $\xi(\bar{t}) \geq \bar{x}$ when $x > \phi_-(t)$. Similarly, $\psi_+(t)$ is characterized by the property that $\zeta(\bar{t}) \leq \bar{x}$ when $x < \psi_+(t)$ and $\zeta(\bar{t}) > \bar{x}$ when $x > \psi_+(t)$. In particular, $\phi_-(t) \leq \phi_+(t)$ and if $\phi_-(t) < x < \phi_+(t)$ then $\xi(\bar{t}) = \bar{x}$. Similarly, $\psi_-(t) \leq \psi_+(t)$ and $\psi_-(t) < x < \psi_+(t)$ implies $\zeta(\bar{t}) = \bar{x}$.

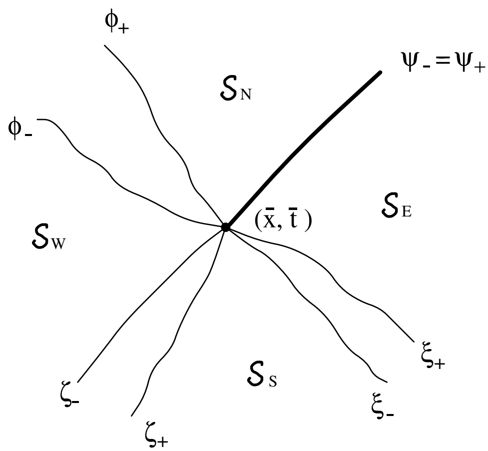


**Fig. 12.3.1**

We fix $\tau > \bar{t}$ and let $\xi_\tau(\cdot)$ denote the minimal backward 1-characteristic emanating from the point $(\phi_-(\tau), \tau)$. We also consider any sequence $\{x_m\}$ converging from above to $\phi_-(\tau)$ and let $\xi_m(\cdot)$ denote the minimal backward 1-characteristic emanating from $(x_m, \tau)$. Then the sequence $\{\xi_m(\cdot)\}$, or some subsequence thereof, will converge to some backward 1-characteristic $\hat{\xi}_\tau(\cdot)$ emanating from $(\phi_-(\tau), \tau)$. Moreover, for any $\bar{t} \le t \le \tau$, it is $\xi_\tau(t) \le \phi_-(t) \le \hat{\xi}_\tau(t)$. In particular, this implies that $\phi_-(\cdot)$ is a Lipschitz continuous space-like curve, with slope in the range of λ. Similarly, $\psi_+(\cdot)$ is a Lipschitz continuous space-like curve, with slope in the range of $\mu$.

Referring again to Fig. 12.3.1, we see that the aforementioned curves border regions:

(12.3.2) $$\mathcal{S}_W = \{(x,t) : x < \bar{x},\ \zeta_-^{-1}(x) < t < \phi_-^{-1}(x)\},$$

(12.3.3) $$\mathcal{S}_E = \{(x,t) : x > \bar{x},\ \xi_+^{-1}(x) < t < \psi_+^{-1}(x)\},$$

(12.3.4) $$\mathcal{S}_N = \{(x,t) : t > \bar{t},\ \phi_+(t) < x < \psi_-(t)\},$$

(12.3.5) $$\mathcal{S}_S = \{(x,t) : t < \bar{t},\ \zeta_+(t) < x < \xi_-(t)\}.$$

**12.3.2 Definition.** The solution is called *locally regular* at the point $(\bar{x}, \bar{t})$ of the upper half-plane when the following hold:

(a) As $(x,t)$ tends to $(\bar{x}, \bar{t})$ through any one of the regions $\mathcal{S}_W$, $\mathcal{S}_E$, $\mathcal{S}_N$ or $\mathcal{S}_S$, $(z(x\pm,t), w(x\pm,t))$ tend to respective limits $(z_W, w_W)$, $(z_E, w_E)$, $(z_N, w_N)$ or $(z_S, w_S)$, where, in particular, it is $z_W = z(\bar{x}-, \bar{t})$, $w_W = w(\bar{x}-, \bar{t})$, $z_E = z(\bar{x}+, \bar{t})$, $w_E = w(\bar{x}+, \bar{t})$.

(b)$_1$ If $p_\ell(\cdot)$ and $p_r(\cdot)$ are any two backward 1-characteristics emanating from $(\bar{x}, \bar{t})$, with $\xi_-(t) \le p_\ell(t) < p_r(t) \le \xi_+(t)$, for $t < \bar{t}$, then

$$(12.3.6)_1 \qquad z_S = \lim_{t\uparrow\bar{t}} z(\xi_-(t)\pm, t) \le \lim_{t\uparrow\bar{t}} z(p_\ell(t)-, t) \le \lim_{t\uparrow\bar{t}} z(p_\ell(t)+, t)$$
$$\le \lim_{t\uparrow\bar{t}} z(p_r(t)-, t) \le \lim_{t\uparrow\bar{t}} z(p_r(t)+, t) \le \lim_{t\uparrow\bar{t}} z(\xi_+(t)\pm, t) = z_E \,,$$

$$(12.3.7)_1 \qquad w_S = \lim_{t\uparrow\bar{t}} w(\xi_-(t)\pm, t) \ge \lim_{t\uparrow\bar{t}} w(p_\ell(t)-, t) \ge \lim_{t\uparrow\bar{t}} w(p_\ell(t)+, t)$$
$$\ge \lim_{t\uparrow\bar{t}} w(p_r(t)-, t) \ge \lim_{t\uparrow\bar{t}} w(p_r(t)+, t) \ge \lim_{t\uparrow\bar{t}} w(\xi_+(t)\pm, t) = w_E \,.$$

(b)$_2$ If $q_\ell(\cdot)$ and $q_r(\cdot)$ are any two backward 2-characteristics emanating from $(\bar{x}, \bar{t})$, with $\zeta_-(t) \le q_\ell(t) < q_r(t) \le \zeta_+(t)$, for $t < \bar{t}$, then

$$(12.3.6)_2 \qquad w_W = \lim_{t\uparrow\bar{t}} w(\zeta_-(t)\pm, t) \ge \lim_{t\uparrow\bar{t}} w(q_\ell(t)-, t) \ge \lim_{t\uparrow\bar{t}} w(q_\ell(t)+, t)$$
$$\ge \lim_{t\uparrow\bar{t}} w(q_r(t)-, t) \ge \lim_{t\uparrow\bar{t}} w(q_r(t)+, t) \ge \lim_{t\uparrow\bar{t}} w(\zeta_+(t)\pm, t) = w_S \,,$$

$$(12.3.7)_2 \qquad z_W = \lim_{t\uparrow\bar{t}} z(\zeta_-(t)\pm, t) \le \lim_{t\uparrow\bar{t}} z(q_\ell(t)-, t) \le \lim_{t\uparrow\bar{t}} z(q_\ell(t)+, t)$$
$$\le \lim_{t\uparrow\bar{t}} z(q_r(t)-, t) \le \lim_{t\uparrow\bar{t}} z(q_r(t)+, t) \le \lim_{t\uparrow\bar{t}} z(\zeta_+(t)\pm, t) = z_S \,.$$

(c)$_1$ If $\phi_-(t) = \phi_+(t)$, for $\bar{t} < t < \bar{t} + s$, then $z_W \le z_N$, $w_W \ge w_N$. On the other hand, if $\phi_-(t) < \phi_+(t)$, for $\bar{t} < t < \bar{t}+s$, then $w_W = w_N$ and as $(x, t)$ tends to $(\bar{x}, \bar{t})$ through the region $\{(x, t) : t > \bar{t},\ \phi_-(t) < x < \phi_+(t)\}$, $w(x\pm, t)$ tends to $w_W$. Furthermore, if $p_\ell(\cdot)$ and $p_r(\cdot)$ are any two forward 1-characteristics issuing from $(\bar{x}, \bar{t})$, with $\phi_-(t) \le p_\ell(t) \le p_r(t) \le \phi_+(t)$, for $\bar{t} < t < \bar{t} + s$, then

$$(12.3.8)_1 \qquad z_W = \lim_{t\downarrow\bar{t}} z(\phi_-(t)\pm, t) \ge \lim_{t\downarrow\bar{t}} z(p_\ell(t)-, t) = \lim_{t\downarrow\bar{t}} z(p_\ell(t)+, t)$$
$$\ge \lim_{t\downarrow\bar{t}} z(p_r(t)-, t) = \lim_{t\downarrow\bar{t}} z(p_r(t)+, t) \ge \lim_{t\downarrow\bar{t}} z(\phi_+(t)\pm, t) = z_N \,.$$

(c)$_2$ If $\psi_-(t) = \psi_+(t)$, for $\bar{t} < t < \bar{t} + s$, then $w_N \ge w_E$, $z_N \le z_E$. On the other hand, if $\psi_-(t) < \psi_+(t)$, for $\bar{t} < t < \bar{t} + s$, then $z_N = z_E$ and as $(x, t)$ tends to $(\bar{x}, \bar{t})$ through the region $\{(x, t) : t > \bar{t},\ \psi_-(t) < x < \psi_+(t)\}$, $z(x\pm, t)$ tends to $z_E$. Furthermore, if $q_\ell(\cdot)$ and $q_r(\cdot)$ are any two forward 2-characteristics issuing from $(\bar{x}, \bar{t})$, with $\psi_-(t) \le q_\ell(t) \le q_r(t) \le \psi_+(t)$, for $\bar{t} < t < \bar{t} + s$, then

$$(12.3.8)_2 \qquad w_N = \lim_{t\downarrow\bar{t}} w(\psi_-(t)\pm, t) \le \lim_{t\downarrow\bar{t}} w(q_\ell(t)-, t) = \lim_{t\downarrow\bar{t}} w(q_\ell(t)+, t)$$
$$\le \lim_{t\downarrow\bar{t}} w(q_r(t)-, t) = \lim_{t\downarrow\bar{t}} w(q_r(t)+, t) \le \lim_{t\downarrow\bar{t}} w(\psi_+(t)\pm, t) = w_E \,.$$

The motivation for the above definition lies in

**12.3.3 Theorem.** *For $\theta$ sufficiently small, the solution is locally regular at any point of the upper half-plane.*

The proof will be provided in the next section. However, the following remarks are in order here. Definition 12.3.2 is motivated by experience with piecewise smooth solutions. Indeed, at points of local regularity incoming waves of the two characteristic families collide to generate a jump discontinuity, which is then resolved into an outgoing wave fan. Statements $(b)_1$ and $(b)_2$ regulate the incoming waves, allowing for any combination of admissible shocks and focussing compression waves. Statements $(c)_1$ and $(c)_2$ characterize the outgoing wave fan. In particular, $(c)_1$ implies that the state $(z_W, w_W)$, on the left, may be joined with the state $(z_N, w_N)$, on the right, by a 1-rarefaction wave or admissible 1-shock; while $(c)_2$ implies that the state $(z_N, w_N)$, on the left, may be joined with the state $(z_E, w_E)$, on the right, by a 2-rarefaction wave or admissible 2-shock. Thus, the outgoing wave fan is locally approximated by the solution of the Riemann problem with end-states $(z(\bar{x}-, \bar{t}), w(\bar{x}-, \bar{t}))$ and $(z(\bar{x}+, \bar{t}), w(\bar{x}+, \bar{t}))$.

A simple corollary of Theorem 12.3.3 is that $\phi_-(\cdot)$ is a 1-characteristic while $\psi_+(\cdot)$ is a 2-characteristic.

Definition 12.3.2 and Theorem 12.3.3 apply even to points on the initial line, $\bar{t} = 0$, after discarding the irrelevant parts of the statements, pertaining to $t < \bar{t}$. It should be noted, however, that there is an important difference between $\bar{t} = 0$ and $\bar{t} > 0$. In the former case, $(z(\bar{x}\pm, 0), w(\bar{x}\pm, 0))$ are unrestricted, being induced arbitrarily by the initial data, and hence the outgoing wave fan may comprise any combination of shocks and rarefaction waves. By contrast, when $\bar{t} > 0$, statements $(b)_1$ and $(b)_2$ in Definition 12.3.2 induce the restrictions $z_W \leq z_E$ and $w_W \geq w_E$. This, combined with statements $(c)_1$ and $(c)_2$, rules out the possibility that both outgoing waves may be rarefactions.

## 12.4 Propagation of Riemann Invariants Along Extremal Backward Characteristics

The theory of the genuinely nonlinear scalar conservation law, expounded in Chapter XI, owes its simplicity to the observation that extremal backward generalized characteristics are essentially classical characteristics, namely straight lines along which the solution stays constant. It is thus natural to investigate whether solutions $U$ of systems (12.1.1) exhibit similar behavior. When $U$ is Lipschitz continuous, the Riemann invariants $z$ and $w$ stay constant along 1-characteristics and 2-characteristics, respectively, by virtue of Theorem 7.3.4. One should not expect, however, that this will hold for weak solutions, because Riemann invariants generally jump across shocks of both characteristic families. In the context of piecewise smooth solutions, Theorem 8.2.3 implies that, under the current normalization conditions, the trace of $z$ (or $w$) along shock-free 1-characteristics (or 2-characteristics) is a nonincreasing step function. The jumps of $z$ (or $w$) occur at the points where the characteristic crosses a shock of the opposite family, and are of cubic order in the strength of the crossed shock. It is remarkable that this property essentially carries over to general weak solutions:

**12.4.1 Theorem.** *Let $\xi(\cdot)$ be the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from any fixed point $(\bar{x}, \bar{t})$ of the upper half-plane. Set*

$$\bar{z}(t) = z(\xi(t)-, t), \qquad \bar{w}(t) = w(\xi(t)+, t), \qquad 0 \le t \le \bar{t}. \tag{12.4.1}$$

*Then $\bar{z}(\cdot)$ (or $\bar{w}(\cdot)$) is a nonincreasing saltus function whose variation is concentrated in the set of points of jump discontinuity of $\bar{w}(\cdot)$ (or $\bar{z}(\cdot)$). Furthermore, if $\tau \in (0, \bar{t})$ is any point of jump discontinuity of $\bar{z}(\cdot)$ (or $\bar{w}(\cdot)$), then*

$$\bar{z}(\tau-) - \bar{z}(\tau+) \le a[\bar{w}(\tau+) - \bar{w}(\tau)]^3, \tag{12.4.2$_1$}$$

*or*

$$\bar{w}(\tau-) - \bar{w}(\tau+) \le a[\bar{z}(\tau+) - \bar{z}(\tau)]^3, \tag{12.4.2$_2$}$$

*where a is a positive constant depending solely on $F$.*

The proof of the above proposition will be intermingled with the proof of Theorem 12.3.3, on local regularity of the solution, and will be partitioned into several steps. The assumption that the trace of $(z, w)$ along space-like curves has bounded variation will be employed only for special space-like curves, namely, generalized characteristics and time lines, $t$ =constant.

**12.4.2 Lemma.** *When $\xi(\cdot)$ is the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from $(\bar{x}, \bar{t})$, $\bar{z}(\cdot)$ (or $\bar{w}(\cdot)$) is nonincreasing on $[0, \bar{t}]$.*

**Proof.** The two cases are quite similar, so it will suffice to discuss the first one, namely where $\xi(\cdot)$ is a 1-characteristic. Then, by virtue of Theorem 10.3.2, $\xi(\cdot)$ is shock-free and hence

$$\dot{\xi}(t) = \lambda(U(\xi(t)\pm, t)), \quad \text{a.e. on } [0, \bar{t}]. \tag{12.4.3}$$

We fix numbers $\tau$ and $s$, with $0 \le \tau < s \le \bar{t}$. For $\varepsilon$ positive and small, we let $\xi_\varepsilon(\cdot)$ denote the minimal Filippov solution of the ordinary differential equation

$$\frac{dx}{dt} = \lambda(U(x, t)) + \varepsilon, \tag{12.4.4}$$

on $[\tau, s]$, with initial condition $\xi_\varepsilon(s) = \xi(s) - \varepsilon$. Applying (12.1.1), as equality of measures, to arcs of the graph of $\xi_\varepsilon(\cdot)$ and using Theorem 1.7.8, we deduce

(12.4.5)

$$F(U(\xi_\varepsilon(t)+,t)) - F(U(\xi_\varepsilon(t)-,t)) - \dot{\xi}_\varepsilon(t)[U(\xi_\varepsilon(t)+,t) - U(\xi_\varepsilon(t)-,t)] = 0,$$

a.e. on $[\tau, s]$. Therefore, $\xi_\varepsilon(\cdot)$ propagates with speed $\lambda(U(\xi_\varepsilon(t)\pm,t)) + \varepsilon$, at points of approximate continuity, or with 1-shock speed, at points of approximate jump discontinuity. In particular, $\lambda(U(\xi_\varepsilon(t)+,t)) \leq \lambda(U(\xi_\varepsilon(t)-,t))$, almost everywhere on $[\tau, s]$, and so, by the definition of Filippov solutions of (12.4.4),

$$\dot{\xi}_\varepsilon(t) \geq \lambda(U(\xi_\varepsilon(t)+,t)) + \varepsilon, \quad \text{a.e. on } [\tau, s]. \tag{12.4.6}$$

For any entropy-entropy flux pair $(\eta, q)$, with $\eta$ convex, integrating (12.3.1) over the region $\{(x,t) : \tau < t < s,\ \xi_\varepsilon(t) < x < \xi(t)\}$ and applying Green's theorem yields

$$\begin{aligned} \int_{\xi_\varepsilon(s)}^{\xi(s)} \eta(U(x,s))dx - \int_{\xi_\varepsilon(\tau)}^{\xi(\tau)} \eta(U(x,\tau))dx \\ \leq -\int_\tau^s \{q(U(\xi(t)-,t)) - \dot{\xi}(t)\eta(U(\xi(t)-,t))\}dt \\ + \int_\tau^s \{q(U(\xi_\varepsilon(t)+,t)) - \dot{\xi}_\varepsilon(t)\eta(U(\xi_\varepsilon(t)+,t))\}dt. \end{aligned} \tag{12.4.7}$$

In particular, we write (12.4.7) for the Lax entropy-entropy flux pair (12.2.5). For $k$ large, the right-hand side of (12.4.7) is nonpositive, by virtue of (12.4.3), (12.4.6), (12.2.12), (12.1.3) and (12.2.13). Hence

$$\int_{\xi_\varepsilon(s)}^{\xi(s)} \eta(z(x,s), w(x,s))dx \leq \int_{\xi_\varepsilon(\tau)}^{\xi(\tau)} \eta(z(x,\tau), w(x,\tau))dx. \tag{12.4.8}$$

We raise (12.4.8) to the power $1/k$ and then let $k \to \infty$. This yields

$$\text{ess sup}_{(\xi_\varepsilon(s),\xi(s))}\, z(\cdot, s) \leq \text{ess sup}_{(\xi_\varepsilon(\tau),\xi(\tau))}\, z(\cdot, \tau). \tag{12.4.9}$$

Finally, we let $\varepsilon \downarrow 0$. By standard theory of Filippov solutions, the family $\{\xi_\varepsilon(\cdot)\}$ contains a sequence that converges, uniformly on $[\tau, s]$, to some Filippov solution $\xi_0(\cdot)$ of the equation $dx/dt = \lambda(U(x,t))$, with initial condition $\xi_0(s) = \xi(s)$. But then $\xi_0(\cdot)$ is a backward 1-characteristic emanating from the point $(\xi(s), s)$. Moreover, $\xi_0(t) \leq \xi(t)$, for $\tau \leq t \leq s$. Since $\xi(\cdot)$ is minimal, $\xi_0(\cdot)$ must coincide with $\xi(\cdot)$ on $[\tau, s]$. Thus (12.4.9) implies $\bar{z}(s) \leq \bar{z}(\tau)$ and so $\bar{z}(\cdot)$ is nonincreasing on $[\tau, s]$. The proof is complete.

**12.4.3 Lemma.** *Let $\xi(\cdot)$ be the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from $(\bar{x}, \bar{t})$. Then, for any $\tau \in (0, \bar{t}]$,*

$$z(\xi(\tau)-,\tau) \leq \bar{z}(\tau-) \leq z(\xi(\tau)+,\tau), \tag{12.4.10$_1$}$$

*or*

$$w(\xi(\tau)-,\tau) \geq \bar{w}(\tau-) \geq w(\xi(\tau)+,\tau). \tag{12.4.10$_2$}$$

*In particular,*

(12.4.11)

$$z(x-,t) \leq z(x+,t), \quad w(x-,t) \geq w(x+,t), \qquad -\infty < x < \infty, \;\; 0 < t < \infty.$$

This will be established in conjunction with

**12.4.4 Lemma.** *Let $\xi(\cdot)$ be the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from $(\bar{x},\bar{t})$. For any $\tau$ and $s$ with $0 < \tau < s \leq \bar{t}$,*

$$z(\xi(\tau)+,\tau) - z(\xi(s)+,s) \leq b\,\mathrm{osc}_{[\tau,s]}\bar{w}(\cdot)\; TV_{[\tau,s]}\bar{w}(\cdot), \tag{12.4.12$_1$}$$

*or*

$$w(\xi(\tau)-,\tau) - w(\xi(s)-,s) \leq b\,\mathrm{osc}_{[\tau,s]}\bar{z}(\cdot)\; TV_{[\tau,s]}\bar{z}(\cdot), \tag{12.4.12$_2$}$$

*where $b$ is a positive constant depending on $F$. Furthermore, if $\bar{w}(\tau+) > \bar{w}(\tau)$ (or $\bar{z}(\tau+) > \bar{z}(\tau)$), then* (12.4.2)$_1$ *(or* (12.4.2)$_2$*) holds.*

**Proof.** It suffices to discuss the case where $\xi(\cdot)$ is a 1-characteristic. Consider any convex entropy $\eta$ with associated entropy flux $q$. We fix $\varepsilon$ positive and small and integrate (12.3.1) over the region $\{(x,t) : \tau < t < s, \;\; \xi(t) < x < \xi(t)+\varepsilon\}$. Notice that both curves $x = \xi(t)$ and $x = \xi(t)+\varepsilon$ have slope $\lambda(\bar{z}(t),\bar{w}(t))$, almost everywhere on $(\tau,s)$. Therefore, Green's theorem yields

$$\int_{\xi(s)}^{\xi(s)+\varepsilon} \eta(z(x,s),w(x,s))dx - \int_{\xi(\tau)}^{\xi(\tau)+\varepsilon} \eta(z(x,\tau),w(x,\tau))dx \tag{12.4.13}$$
$$\leq -\int_{\tau}^{s} H(z(\xi(t)+\varepsilon+,t), w(\xi(t)+\varepsilon+,t), \bar{z}(t), \bar{w}(t))dt,$$

under the notation

$$H(z,w,\bar{z},\bar{w}) = q(z,w) - q(\bar{z},\bar{w}) - \lambda(\bar{z},\bar{w})[\eta(z,w) - \eta(\bar{z},\bar{w})]. \tag{12.4.14}$$

One easily verifies, with the help of (12.2.1), that

$$H_z(z,w,\bar{z},\bar{w}) = [\lambda(z,w) - \lambda(\bar{z},\bar{w})]\eta_z(z,w), \tag{12.4.15}$$

$$H_w(z,w,\bar{z},\bar{w}) = [\mu(z,w) - \lambda(\bar{z},\bar{w})]\eta_w(z,w), \tag{12.4.16}$$

$$H_{zz}(z,w,\bar{z},\bar{w}) = \lambda_z(z,w)\eta_z(z,w) + [\lambda(z,w) - \lambda(\bar{z},\bar{w})]\eta_{zz}(z,w), \tag{12.4.17}$$

(12.4.18) $H_{zw}(z, w, \bar{z}, \bar{w}) = \lambda_w(z, w)\eta_z(z, w) + [\lambda(z, w) - \lambda(\bar{z}, \bar{w})]\eta_{zw}(z, w),$

(12.4.19) $H_{ww}(z, w, \bar{z}, \bar{w}) = \mu_w(z, w)\eta_w(z, w) + [\mu(z, w) - \lambda(\bar{z}, \bar{w})]\eta_{ww}(z, w).$

Let us introduce the notation $z_0 = z(\xi(\tau)+, \tau)$, $w_0 = w(\xi(\tau)+, \tau) = \bar{w}(\tau)$, $z_1 = z(\xi(s)+, s)$, $w_1 = w(\xi(s)+, s) = \bar{w}(s)$ and set $\delta = \text{osc}_{[\tau,s]}\bar{w}(\cdot)$. We then apply (12.4.13) for the entropy $\eta$ constructed by solving the Goursat problem for (12.2.2), with data

(12.4.20)
$$\begin{cases} \eta(z, w_0) = -(z - z_0) + \beta(z - z_0)^2 \,, \\ \eta(z_0, w) = -3\beta\delta(w - w_0) + \beta(w - w_0)^2 \,, \end{cases}$$

where $\beta$ is a positive constant, sufficiently large for the following to hold on a small neighborhood of the point $(z_0, w_0)$:

(12.4.21) $\eta$ is a convex function of $U$,

(12.4.22) $\eta(z, w)$ is a convex function of $(z, w)$,

(12.4.23) $H(z, w, \bar{z}, \bar{w})$ is a convex function of $(z, w)$.

It is possible to satisfy the above requirements when $|z - z_0|$, $|w - w_0|$ and $\delta$ are sufficiently small. In particular, (12.4.21) will hold by virtue of (12.2.3), (12.1.4), (12.4.20), (12.2.2) and (12.2.4). Similarly, (12.4.22) follows from (12.4.20), (12.2.2) and (12.2.4). Finally, (12.4.23) is verified by combining (12.4.17), (12.4.18), (12.4.19), (12.4.20), (12.2.2) and (12.2.4).

By virtue of (12.4.23), (12.4.15) and (12.4.16),

(12.4.24) $H(z, w, \bar{z}, \bar{w}) \geq [\mu(\bar{z}, \bar{w}) - \lambda(\bar{z}, \bar{w})]\eta_w(\bar{z}, \bar{w})[w - \bar{w}].$

One may estimate $\eta_w(\bar{z}(t), \bar{w}(t))$ by integrating (12.2.2), as an ordinary differential equation for $\eta_w$, along the line $w = \bar{w}(t)$, starting out from the initial value $\eta_w(z_0, \bar{w}(t))$ at $z = z_0$. Because $|\bar{w}(t) - w_0| \leq \delta$, (12.4.20) gives $-5\beta\delta \leq \eta_w(z_0, \bar{w}(t)) \leq -\beta\delta < 0$. Since $\lambda_w < 0$ and $\eta_z < 0$, (12.2.2) then implies $\eta_w(z, \bar{w}(t)) < 0$, for $z \leq z_0$. In anticipation of $(12.4.10)_1$, we now assume $z_0 \geq \bar{z}(\tau)$, which we already know will apply for almost all choices of $\tau$ in $(0, s)$, namely when $z(\xi(\tau)-, \tau) = z(\xi(\tau)+, \tau)$. By Lemma 12.4.2, $\bar{z}(t) \leq \bar{z}(\tau)$ and so $\eta_w(\bar{z}(\tau), \bar{w}(\tau)) < 0$, for $\tau \leq t \leq s$.

For $t \in [\tau, s]$, let $\zeta_t(\cdot)$ denote the maximal backward 2-characteristic emanating from the point $(\xi(t) + \varepsilon, t)$ (Fig. 12.4.1). We also draw the maximal forward 2-characteristic $\psi(\cdot)$, issuing from the point $(\xi(\tau), \tau)$, which collides with the curve $x = \xi(t) + \varepsilon$ at time $r$, where $0 < r - \tau < c_0\varepsilon$.

For $t \in (r, s)$, the graph of $\zeta_t(\cdot)$ intersects the graph of $\xi(\cdot)$ at time $\sigma_t$. By Lemma 12.4.2,

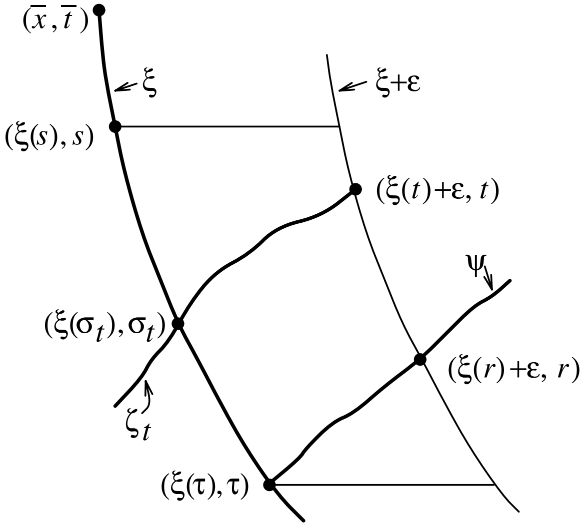


**Fig. 12.4.1**

(12.4.25)
$$w(\xi(t)+\varepsilon+,t) = w(\zeta_t(t)+,t) \leq w(\zeta_t(\sigma_t)+,\sigma_t) = w(\xi(\sigma_t)+,\sigma_t) = \bar{w}(\sigma_t).$$

Since $\eta_w(\bar{z}(t), \bar{w}(t)) < 0$, (12.4.24) and (12.4.25) together imply

(12.4.26)
$$H(z(\xi(t)+\varepsilon+,t), w(\xi(t)+\varepsilon+,t), \bar{z}(t), \bar{w}(t))$$
$$\geq [\mu(\bar{z}(t), \bar{w}(t)) - \lambda(\bar{z}(t), \bar{w}(t))]\eta_w(\bar{z}(t), \bar{w}(t))[\bar{w}(\sigma_t) - \bar{w}(t)].$$

Because the two characteristic speeds $\lambda$ and $\mu$ are strictly separated, $0 < t - \sigma_t < c_1\varepsilon$ and so (12.4.26) yields

(12.4.27)
$$-\int_r^s H(z(\xi(t)+\varepsilon+,t), w(\xi(t)+\varepsilon+,t), \bar{z}(t), \bar{w}(t))dt$$
$$\leq c_2\varepsilon \sup_{(\tau,s)}|\eta_w(\bar{z}(\cdot), \bar{w}(\cdot))|\, NV_{(\tau,s)}\bar{w}(\cdot),$$

with $NV$ denoting negative (i.e., decreasing) variation.

Next, we restrict $t$ to the interval $(\tau, r)$. Then, $\zeta_t(\cdot)$ is intercepted by the $\tau$-time line at $\zeta_t(\tau) \in [\xi(\tau), \xi(\tau)+\varepsilon)$. By virtue of Lemma 12.4.2,

(12.4.28)
$$w(\xi(t)+\varepsilon+,t) = w(\zeta_t(t)+,t) \leq w(\zeta_t(\tau)+,\tau) = w_0 + o(1)\,, \text{ as } \varepsilon \downarrow 0.$$

On the other hand, upon setting $z_+ = \bar{z}(\tau+)$, $w_+ = \bar{w}(\tau+)$, we readily observe that $\bar{z}(t) = z_+ + o(1)$, $\bar{w}(t) = w_+ + o(1)$, as $\varepsilon \downarrow 0$. Therefore, combining (12.4.24) with (12.4.28) yields

(12.4.29)
$$-\int_\tau^r H(z(\xi(t)+\varepsilon+,t), w(\xi(t)+\varepsilon+,t), \bar{z}(t), \bar{w}(t))dt$$
$$\leq -[\mu(z_+,w_+)-\lambda(z_+,w_+)]\eta_w(z_+,w_+)[w_0-w_+](r-\tau)+o(\varepsilon).$$

We now multiply (12.4.13) by $1/\varepsilon$ and then let $\varepsilon \downarrow 0$. Using (12.4.27), (12.4.28) and recalling that $0 < r-\tau < c_0\varepsilon$, we deduce

(12.4.30)
$$\eta(z_1,w_1)-\eta(z_0,w_0) \leq c_3 \sup_{(\tau,s)}|\eta_w(\bar{z}(\cdot),\bar{w}(\cdot))|\, NV_{[\tau,s)}\bar{w}(\cdot).$$

In particular, $s$ is the limit of an increasing sequence of $\tau$ with the property $z(\xi(\tau)-,\tau) = z(\xi(\tau)+,\tau)$, for which (12.4.30) is valid. It follows that $\eta(z_1,w_1) \leq \eta(\bar{z}(s-),\bar{w}(s-))$. Now applying (12.4.25) for $t = s$, and letting $\varepsilon \downarrow 0$, yields $w_1 \leq \bar{w}(s-)$. Also, $\eta_w < 0$, $\eta_z < 0$. Hence, $\bar{z}(s-) \leq z_1$. By Lemma 12.4.2, $\bar{z}(s) \leq \bar{z}(s-)$ and so $z(\xi(s)-,s) = \bar{z}(s) \leq \bar{z}(s-) \leq z_1 = z(\xi(s)+,s)$. Since $s$ is arbitrary, we may write these inequalities for $s = \tau$ and this verifies $(12.4.10)_1$. Lemma 12.4.3 has now been proved. Furthermore, $z_0 \geq \bar{z}(\tau)$ has been established and hence (12.4.30) is valid for all $\tau$ and $s$ with $0 < \tau < s \leq \bar{t}$.

From (12.4.22) and (12.4.20) it follows

(12.4.31)
$$\eta(z_1,w_1)-\eta(z_0,w_0) \geq z_0 - z_1 - 3\beta\delta(w_1-w_0).$$

Combining (12.4.30) with (12.4.31),

(12.4.32)
$$z_0 - z_1 \leq 3\beta\delta(w_1-w_0) + c_3 \sup_{(\tau,s)}|\eta_w(\bar{z}(\cdot),\bar{w}(\cdot))|\, NV_{[\tau,s)}\bar{w}(\cdot).$$

To establish $(12.4.12)_1$ for general $\tau$ and $s$, it would suffice to verify it just for $\tau$ and $s$ with $s-\tau$ so small that $TV_{[\tau,s]}\bar{w}(\cdot) < 2\delta$. For such $\tau$ and $s$, (12.4.32) gives the preliminary estimate $z_0 - z_1 \leq c_4\delta$, and in fact $z_0 - \bar{z}(t) \leq c_4\delta$, for all $t \in (\tau,s)$. But then, since $|\eta_w(z_0,\bar{w}(t))| \leq 5\beta\delta$, (12.2.2) implies $\sup_{(\tau,s)}|\eta_w(\bar{z}(\cdot),\bar{w}(\cdot))| \leq c_5\delta$. Inserting this estimate into (12.4.32), we arrive at $(12.4.12)_1$, with $b = 3\beta + c_3c_5$.

Finally, we assume $\bar{w}(\tau+) > \bar{w}(\tau)$, say $w_+ - w_0 = \delta_0 > 0$, and proceed to verify $(12.4.2)_1$. Keeping $\tau$ fixed, we choose $s-\tau$ so small that $TV_{[\tau,s]}\bar{w}(\cdot) < 2\delta_0$ and hence $\delta < 2\delta_0$. We need to improve the estimate (12.4.29) and thus we restrict $t$ to the interval $[\tau, r]$.

By account of (12.4.22), (12.4.15) and (12.4.16),

(12.4.33)
$$H(z(\xi(t)+\varepsilon+,t), w(\xi(t)+\varepsilon+,t), \bar{z}(t), \bar{w}(t))$$
$$\geq H(z_+, w_0, \bar{z}(t), \bar{w}(t))$$
$$-[\lambda(z_+,w_0)-\lambda(\bar{z}(t),\bar{w}(t))][z(\xi(t)+\varepsilon+,t)-z_+]$$
$$-3\beta\delta[\mu(z_+,w_0)-\lambda(\bar{z}(t),\bar{w}(t))][w(\xi(t)+\varepsilon+,t)-w_0].$$

We have already seen that, as $\varepsilon \downarrow 0$, $\bar{z}(t) = z_+ + o(1)$, $\bar{w}(t) = w_+ + o(1)$. In particular, for $\varepsilon$ small, $\lambda(z_+,w_0)-\lambda(\bar{z}(t),\bar{w}(t)) > 0$, by virtue of (12.1.5). Furthermore, if $\hat{\xi}(\cdot)$ denotes the minimal backward 1-characteristic emanating from any point $(x,t)$ with $\xi(t) < x < \xi(t)+2\varepsilon$, by Lemma 12.4.2 $z(x-,t) \leq z(\hat{\xi}(\tau)-,\tau) = z_0+o(1)$, as

$\varepsilon \downarrow 0$. On the other hand, $(12.4.12)_1$, with $s \downarrow \tau$, implies $z_0 - z_+ \leq b\delta_0^2$. Therefore, as $\varepsilon \downarrow 0$, $z(\xi(t) + \varepsilon+, t) \leq z_+ + b\delta_0^2 + o(1)$. Finally, we recall (12.4.28). Collecting the above, we deduce from (12.4.33):

$$H(z(\xi(t) + \varepsilon+, t), w(\xi(t) + \varepsilon+, t), \bar{z}(t), \bar{w}(t)) \geq H(z_+, w_0, z_+, w_+) - c_6\delta_0^3 + o(1), \quad \text{as } \varepsilon \downarrow 0. \tag{12.4.34}$$

To estimate the right-hand side of (12.4.34), let us visualize $q$ as a function of $(z, \eta)$. By the chain rule and (12.2.1), we deduce $q_\eta = \mu$, $q_{\eta\eta} = \mu_w/\eta_w$. For $w \in [w_0, w_+]$, $q_{\eta\eta} < 0$. Hence

$$H(z_+, w_0, z_+, w_+) \geq [\mu(z_+, w_0) - \lambda(z_+, w_+)][\eta(z_+, w_0) - \eta(z_+, w_+)]. \tag{12.4.35}$$

The next step is to show

$$\frac{r - \tau}{\varepsilon} \geq \frac{1}{\mu(z_0, w_+) - \lambda(z_+, w_+)} + o(1), \quad \text{as } \varepsilon \downarrow 0. \tag{12.4.36}$$

To see this, let us begin with

$$\varepsilon = \psi(r) - \xi(r) = \int_\tau^r [\dot{\psi}(t) - \dot{\xi}(t)]dt \leq \int_\tau^r [\mu(z(\psi(t)-, t), w(\psi(t)-, t)) - \lambda(\bar{z}(t), \bar{w}(t))]dt. \tag{12.4.37}$$

As shown above, $z(\psi(t)-, t) \leq z_0 + o(1)$, as $\varepsilon \downarrow 0$. On the other hand, the maximal backward 2-characteristic $\zeta(\cdot)$, emanating from a point $(x, t)$ with $\xi(t) < x < \psi(t)$, will intersect the graph of $\xi(\cdot)$ at time $\sigma \in (\tau, r]$ and hence, by Lemma 12.4.2, $w(x+, t) \leq \bar{w}(\sigma)$. In particular, $w(\psi(t)-, t) \leq w_+ + o(1)$, as $\varepsilon \downarrow 0$. Since $\mu_z > 0$ and $\mu_w > 0$, (12.4.37) implies $\varepsilon \leq (r - \tau)[\mu(z_0, w_+) - \lambda(z_+, w_+) + o(1)]$ whence (12.4.36) immediately follows.

Once again we multiply (12.4.13) by $1/\varepsilon$, let $\varepsilon \downarrow 0$ and then also let $s \downarrow \tau$. Combining (12.4.27), (12.4.34), (12.4.35) and (12.4.36), we conclude:

$$\eta(z_+, w_0) - \eta(z_0, w_0) \leq \frac{\mu(z_0, w_+) - \mu(z_+, w_0)}{\mu(z_0, w_+) - \lambda(z_+, w_+)}[\eta(z_+, w_0) - \eta(z_+, w_+)] + c_7\delta_0^3. \tag{12.4.38}$$

By virtue of (12.4.20), $\eta(z_+, w_0) - \eta(z_0, w_0) \geq z_0 - z_+$. The right-hand side of (12.4.38) is bounded by $a\delta_0^3$, because $\eta_w = O(\delta_0)$. Therefore, $z_0 - z_+ \leq a\delta_0^3$. Now $\bar{z}(\tau-) \leq z_0$, by account of $(12.4.10)_1$. Hence $\bar{z}(\tau-) - \bar{z}(\tau+) \leq a\delta_0^3$, which establishes $(12.4.2)_1$.

Since total variation is additive, we deduce immediately

**12.4.5 Corollary.** *In* (12.4.12)$_1$ *(or* (12.4.12)$_2$*),* $\mathrm{osc}_{[\tau,s]}\bar{w}(\cdot)$ *(or* $\mathrm{osc}_{[\tau,s]}\bar{z}(\cdot)$*) may be replaced by the local oscillation of* $\bar{w}(\cdot)$ *(or* $\bar{z}(\cdot)$*) in the interval* $[\tau, s]$*, which is measured by the maximum jump of* $\bar{w}(\cdot)$ *(or* $\bar{z}(\cdot)$*) in* $[\tau, s]$*. In particular,* $\bar{z}(\cdot)$ *(or* $\bar{w}(\cdot)$*) is a saltus function whose variation is concentrated in the set of points of jump discontinuity of* $\bar{w}(\cdot)$ *(or* $\bar{z}(\cdot)$*).*

We have thus verified all the assertions of Theorem 12.4.1, except that (12.4.2) has been established under the extraneous assumption $\bar{w}(\tau) < \bar{w}(\tau+)$. By Lemma 12.4.3, $\bar{w}(\tau) = w(\xi(\tau)+, \tau) \le w(\xi(\tau)-, \tau)$. On the other hand, when $(\xi(\tau), \tau)$ is a point of local regularity of the solution, Condition $(c)_1$ of Definition 12.3.2 implies $\bar{w}(\tau+) = w(\xi(\tau)-, \tau)$. Hence, by establishing Theorem 12.3.3, we will justify, in particular, the assumption $\bar{w}(\tau) < \bar{w}(\tau+)$.

We thus turn to the proof of Theorem 12.3.3. Our main tool will be the estimate (12.4.12). In what follows, $\delta$ will denote an upper bound of the oscillation of $z$ and $w$ on the upper half-plane. We fix any point $(\bar{x}, \bar{t})$ of the upper half-plane and construct the curves $\xi_\pm(\cdot)$, $\zeta_\pm(\cdot)$, $\phi_\pm(\cdot)$ and $\psi_\pm(\cdot)$, as described in Section 12.3 and sketched in Fig. 12.3.1. The first step is to verify the part of Condition (a) of Definition 12.3.2 pertaining to the "western" sector $\mathcal{S}_W$.

**12.4.6 Lemma.** *For* $\theta$ *sufficiently small, as* $(x, t)$ *tends to* $(\bar{x}, \bar{t})$ *through the region* $\mathcal{S}_W$*, defined by* (12.3.2)*,* $(z(x\pm, t), w(x\pm, t))$ *converge to* $(z_W, w_W)$*, where we set* $z_W = z(\bar{x}-, \bar{t})$*,* $w_W = w(\bar{x}-, \bar{t})$*.*

**Proof.** We shall construct a sequence $x_0 < x_1 < x_2 < \cdots < \bar{x}$ such that, for $m = 0, 1, 2, \cdots$,

$$\mathrm{osc}_{\mathcal{S}_W \cap \{x > x_m\}} z \le (3b\theta)^m \delta, \qquad \mathrm{osc}_{\mathcal{S}_W \cap \{x > x_m\}} w \le (3b\theta)^m \delta, \tag{12.4.39}$$

where $b$ is the constant appearing in (12.4.12). Clearly, (12.4.39) will readily imply the assertion of the proposition, provided $3b\theta < 1$.

For $m = 0$, (12.4.39) is satisfied with $x_0 = -\infty$. Arguing by induction, let us assume $x_0 < x_1 < \cdots < x_{k-1} < \bar{x}$ have already been fixed so that (12.4.39) holds for $m = 0, \cdots, k-1$. We proceed to determine $x_k$. We fix $\hat{t} \in (0, \bar{t})$ with $\bar{t} - \hat{t}$ so small that $\zeta_-(\hat{t}) > x_{k-1}$ and the oscillation of $z(\zeta_-(\tau)\pm, \tau)$ over the interval $[\hat{t}, \bar{t})$ does not exceed $\frac{1}{3}(3b\theta)^k\delta$. Next we locate $\hat{x} \in (x_{k-1}, \zeta_-(\hat{t}))$ with $\zeta_-(\hat{t}) - \hat{x}$ so small that the oscillation of $w(y-, \hat{t})$ over the interval $(\hat{x}, \zeta_-(\hat{t})]$ is similarly bounded by $\frac{1}{3}(3b\theta)^k\delta$.

By the construction of $\phi_-(\cdot)$, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from any point $(x, t)$ in $\mathcal{S}_W \cap \{x > x_k\}$ stays to the left of the graph of $\phi_-(\cdot)$. At the same time, as $(x, t)$ tends to $(\bar{x}, \bar{t})$ through $\mathcal{S}_W$, the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from it will tend to some backward 2-characteristic emanating from $(\bar{x}, \bar{t})$, which necessarily lies to the right of the minimal characteristic $\zeta_-(\cdot)$ or coincides with $\zeta_-(\cdot)$. It follows that when $\bar{x} - x_k$ is sufficiently small,

$\xi(\cdot)$ will have to cross the graph of $\zeta_-(\cdot)$ at some time $t^* \in (\hat{t}, \bar{t})$, while $\zeta(\cdot)$ must intersect either the graph of $\zeta_-(\cdot)$ at some time $\tilde{t} \in (\hat{t}, \bar{t})$ or the $\hat{t}$-time line at some $\tilde{x} \in (\hat{x}, \zeta_-(\hat{t})]$.

By virtue of Lemmas 12.4.2 and 12.4.3,

$$z(x-, t) \le z(\xi(t^*)-, t^*) = z(\zeta_-(t^*)-, t^*) \le z(\zeta_-(t^*)+, t^*). \tag{12.4.40}$$

By account of (12.4.39), for $m = k - 1$, and the construction of $\hat{t}$, the oscillation of $w(\xi(\tau)+, \tau)$ over the interval $[t^*, t]$ does not exceed $(3b\theta)^{k-1}\delta + \frac{1}{3}(3b\theta)^k\delta$, which in turn is majorized by $2(3b\theta)^{k-1}\delta$. Then $(12.4.12)_1$ yields

(12.4.41)

$$z(x+, t) \ge z(\xi(t^*)+, t^*) - 2b\theta(3b\theta)^{k-1}\delta = z(\zeta_-(t^*)+, t^*) - \frac{2}{3}(3b\theta)^k\delta.$$

Recalling that the oscillation of $z(\zeta_-(\tau)+, \tau)$ over $[\hat{t}, \bar{t})$ is bounded by $\frac{1}{3}(3b\theta)^k\delta$, (12.4.40) and (12.4.41) together imply the bound (12.4.39) on the oscillation of $z$, for $m = k$.

The argument for $w$ is similar: Assume, for example, that $\xi(\cdot)$ intersects the $\hat{t}$-time line, rather than the graph of $\zeta_-(\cdot)$. By virtue of Lemmas 12.4.2 and 12.4.3,

$$w(x+, t) \le w(\zeta(\hat{t})+, \hat{t}) = w(\tilde{x}+, \hat{t}) \le w(\tilde{x}-, \hat{t}). \tag{12.4.42}$$

The oscillation of $z(\zeta(\tau)-, \tau)$ over the interval $[\hat{t}, t]$ does not exceed $(3b\theta)^{k-1}\delta$, by account of (12.4.39), for $m = k - 1$. Then $(12.4.12)_2$ implies

$$w(x-, t) \ge w(\zeta(\hat{t})-, \hat{t}) - b\theta(3b\theta)^{k-1}\delta = w(\tilde{x}-, \hat{t}) - \frac{1}{3}(3b\theta)^k\delta. \tag{12.4.43}$$

The bound (12.4.39) on the oscillation of $w$, for $m = k$, now easily follows from (12.4.42), (12.4.43) and the construction of $\hat{t}$ and $\hat{x}$. The proof is complete.

The part of Condition (a) of Definition 12.3.2 pertaining to the "eastern" sector $\mathcal{S}_E$ is validated by a completely symmetrical argument. The next step is to check the part of Condition (a) that pertains to the "southern" sector $\mathcal{S}_S$.

**12.4.7 Lemma.** *For $\theta$ sufficiently small, as $(x, t)$ tends to $(\bar{x}, \bar{t})$ through the region $\mathcal{S}_S$, defined by* (12.3.5), $(z(x\pm, t), w(x\pm, t))$ *tend to a constant state* $(z_S, w_S)$.

**Proof.** Similar to the proof of Lemma 12.4.6, the aim is to find $t_0 < t_1 < \cdots < \bar{t}$ such that

$$\operatorname{osc}_{\mathcal{S}_S \cap \{t > t_m\}} z \le (4b\theta)^m\delta, \qquad \operatorname{osc}_{\mathcal{S}_S \cap \{t > t_m\}} w \le (4b\theta)^m\delta, \tag{12.4.44}$$

for $m = 0, 1, 2, \cdots$. For $m = 0$, (12.4.44) is satisfied with $t_0 = 0$. Arguing by induction, we assume $t_0 < t_1 < \cdots < t_{k-1} < \bar{t}$ have already been fixed so that (12.4.44) holds for $m = 0, \cdots, k-1$, and proceed to locate $t_k$. We fix $\hat{t} \in (t_{k-1}, \bar{t})$ with $\bar{t} - \hat{t}$ sufficiently small that the oscillation of $z(\zeta_+(\tau)-, \tau)$, $w(\zeta_+(\tau)+, \tau)$, $z(\xi_-(\tau)-, \tau)$, $w(\xi_-(\tau)+, \tau)$ over the interval $[\hat{t}, \bar{t})$ does not exceed $\frac{1}{4}(4b\theta)^k\delta$. Next we locate $\hat{x}$ and $\tilde{x}$ in the interval $(\zeta_+(\hat{t}), \xi_-(\hat{t}))$ with $\hat{x} - \zeta_+(\hat{t})$ and $\xi_-(\hat{t}) - \tilde{x}$ so small that the oscillation of $z(y-, \hat{t})$ over the interval $(\tilde{x}, \xi_-(\hat{t})]$ and the oscillation of $w(y+, \hat{t})$ over the interval $[\zeta_+(\hat{t}), \hat{x})$ do not exceed $\frac{1}{4}(4b\theta)^k\delta$.

Since $\xi_-(\cdot)$ is the minimal backward 1-characteristic and $\zeta_+(\cdot)$ is the maximal backward 2-characteristic emanating from $(\bar{x}, \bar{t})$, we can find $t_k \in (\hat{t}, \bar{t})$ with $\bar{t} - t_k$ so small that the following holds for any $(x, t)$ in $\mathcal{S}_S \cap \{t > t_k\}$: (a) the minimal backward 1-characteristic $\xi(\cdot)$ emanating from $(x, t)$ must intersect either the $\hat{t}$-time line at $x' \in (\tilde{x}, \xi_-(\hat{t})]$ or the graph of $\xi_-(\cdot)$ at time $t' \in (\hat{t}, \bar{t})$; and (b) the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from $(x, t)$ must intersect either the $\hat{t}$-time line at $x^* \in [\zeta_+(\hat{t}), \hat{x})$ or the graph of $\zeta_+(\cdot)$ at some time $t^* \in (\hat{t}, \bar{t})$. One then repeats the argument employed in the proof of Lemma 12.4.6 to verify the (12.4.44) is indeed satisfied for $m = k$, with $t_k$ determined as above. The proof is complete.

To conclude the validation of Condition (a) of Definition 12.3.2, it remains to check the part pertaining to the "northern" sector $\mathcal{S}_N$.

**12.4.8 Lemma.** *For $\theta$ sufficiently small, as $(x, t)$ tends to $(\bar{x}, \bar{t})$ through the region $\mathcal{S}_N$, defined by* (12.3.4), $(z(x\pm, t), w(x\pm, t))$ *tend to a constant state* $(z_N, w_N)$.

**Proof.** For definiteness, we treat the typical configuration depicted in Fig. 12.3.1, where $\psi_- \equiv \psi_+$, so that $\psi_-(\cdot)$ is a 2-shock of generally positive strength at $t = \bar{t}$, while $\phi_-(t) < \phi_+(t)$, for $t > \bar{t}$, in which case, as we shall see in Lemma 12.4.10, it is $\lim_{t \downarrow \bar{t}} z(\phi_+(t)-, t) = \lim_{t \downarrow \bar{t}} z(\phi_+(t)+, t)$ and $\lim_{t \downarrow \bar{t}} w(\phi_+(t)-, t) = \lim_{t \downarrow \bar{t}} w(\phi_-(t)+, t)$. Only slight modifications in the argument are needed for the case of alternative feasible configurations.

The aim is to find $t_0 > t_1 > \cdots > \bar{t}$ such that

$$\text{(12.4.45)} \qquad \operatorname{osc}_{\mathcal{S}_N \cap \{t < t_m\}} z \le a(ab\theta)^m\delta, \qquad \operatorname{osc}_{\mathcal{S}_N \cap \{t < t_m\}} w \le 3(ab\theta)^m\delta,$$

for $m = 0, 1, 2, \cdots$, where $a \ge 1$ is a constant, independent of $m$ and $\theta$, to be specified below. Clearly, (12.4.45) is satisfied for $m = 0$, with $t_0 = \infty$. Arguing by induction, we assume $t_0 > t_1 > \cdots > t_{k-1} > \bar{t}$ have already been fixed so that (12.4.45) holds for $m = 0, \cdots, k-1$, and proceed to determine $t_k$.

We select $t_k \in (\bar{t}, t_{k-1})$ with $t_k - \bar{t}$ so small that the oscillation of $z(\phi_+(\tau)-, \tau)$ over the interval $(\bar{t}, t_k)$ does not exceed $a(ab\theta)^{k-1}\delta$, the oscillation of $w(\phi_+(\tau)-, \tau)$ over $(\bar{t}, t_k)$ is bounded by $(ab\theta)^k\delta$, and the oscillation of $U(\psi_-(\tau)-, \tau)$ over $(\bar{t}, t_k)$ is majorized by $(ab\theta)^{2k}\delta^2$.

The bound (12.4.45) on the oscillation of $w$, for $m = k$, will be established by the procedure employed in the proof of Lemmas 12.4.6 and 12.4.7. We thus fix any $(x, t)$ in $\mathcal{S}_N \cap \{t < t_k\}$ and consider the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from it, which intersects the graph of $\phi_+(\cdot)$ at some time $\tilde{t} \in (\bar{t}, t_k)$. By virtue of Lemmas 12.4.2 and 12.4.3:

(12.4.46) $$w(x+,t) \le w(\zeta(\tilde{t})+,\tilde{t}) = w(\phi_+(\tilde{t})+,\tilde{t}) \le w(\phi_+(\tilde{t})-,\tilde{t}).$$

By account of (12.4.45), for $m = k-1$, and the construction of $t_k$, the oscillation of $z(\zeta(\tau)-,\tau)$ over the interval $[\tilde{t},t]$ does not exceed $2a(ab\theta)^{k-1}\delta$. Then $(12.4.12)_2$ implies

(12.4.47) $$w(x-,t) \ge w(\zeta(\tilde{t})-,\tilde{t}) - 2(ab\theta)^k\delta = w(\phi_+(\tilde{t})-,\tilde{t}) - 2(ab\theta)^k\delta.$$

The inequalities (12.4.46), (12.4.47) coupled with the condition that the oscillation of $w(\phi_+(\tau)-,\tau)$ over $(\bar{t},t_k)$ is majorized by $(ab\theta)^k\delta$ readily yield the bound (12.4.45) on the oscillation of $w$, for $m = k$.

To derive the corresponding bound on the oscillation of $z$ requires an entirely different argument. Let us define $\bar{U} = \lim_{t\downarrow\bar{t}} U(\psi_-(t)-,t)$, with induced values $(\bar{z},\bar{w})$ for the Riemann invariants, and then set $\Delta z = z - \bar{z}$, $\Delta w = w - \bar{w}$. On $\mathcal{S}_N \cap \{t < t_k\}$, as shown above,

(12.4.48) $$|\Delta w| \le 3(ab\theta)^k\delta.$$

We construct the minimal backward 1-characteristic $\xi(\cdot)$, emanating from any point $(y,t)$ of approximate continuity in $\mathcal{S}_N \cap \{t < t_k\}$, which is intercepted by the graph of $\psi_-(\cdot)$ at time $t^* \in (\bar{t},t_k)$. Lemma 12.4.2 implies $z(y,t) \le z(\xi(t^*)-,t^*) = z(\psi_-(t^*)-,t^*)$ and this in conjunction with the selection of $t_k$ yields

(12.4.49) $$\Delta z(y,t) \le c_1(ab\theta)^{2k}\delta^2,$$

for some constant $c_1$ independent of $k$ and $\theta$.

We now fix any point of approximate continuity $(x,t)$ in $\mathcal{S}_N \cap \{t < t_k\}$. We consider, as above, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from $(x,t)$, which is intercepted by the graph of $\psi_-(\cdot)$ at time $t^* \in (\bar{t},t_k)$, and integrate the conservation law (12.1.1) over the region $\{(y,\tau) : t^* < \tau < t,\ \xi(\tau) < y < \psi_-(\tau)\}$. By Green's theorem,

(12.4.50)
$$\begin{aligned}
&\int_x^{\psi_-(t)} [U(y,t) - \bar{U}]dy \\
&\quad + \int_{t^*}^{t} \{F(U(\psi_-(\tau)-,\tau)) - F(\bar{U}) - \dot{\psi}_-(\tau)[U(\psi_-(\tau)-,\tau) - \bar{U}]\}d\tau \\
&- \int_{t^*}^{t} \{F(U(\xi(\tau)+,\tau)) - F(\bar{U}) - \lambda(U(\xi(\tau)+,\tau))[U(\xi(\tau)+,\tau) - \bar{U}]\}d\tau = 0.
\end{aligned}$$

Applying repeatedly (7.3.12), we obtain, for $U = U(z,w)$,

(12.4.51) $$U = \bar{U} + \Delta z R(\bar{U}) + \Delta w S(\bar{U}) + O(\Delta z^2 + \Delta w^2),$$

(12.4.52)
$$\begin{aligned}
F(U) - F(\bar{U}) - \lambda(U)[U - \bar{U}] &= \Delta w[\mu(\bar{U}) - \lambda(\bar{U})]S(\bar{U}) \\
&- \frac{1}{2}\Delta z^2 \lambda_z(\bar{U})R(\bar{U}) - \Delta z\Delta w \lambda_z(\bar{U})S(\bar{U}) + O(\Delta w^2 + |\Delta z|^3).
\end{aligned}$$

We also note that the oscillation of $w(\xi(\tau)+, \tau)$ over the interval $(t^*, t]$ is bounded by $3(ab\theta)^k\delta$ and so, by account of $(12.4.12)_1$ and Lemma 12.4.3, we have

$$0 \le \Delta z(\xi(\tau)+, \tau) - \Delta z(x, t) \le 3b\theta(ab\theta)^k\delta \le 3(ab\theta)^k\delta, \tag{12.4.53}$$

for any $\tau \in (t^*, t)$.

We substitute from (12.4.51), (12.4.52) into (12.4.50) and then multiply the resulting equation, from the left, by $Dz(\bar{U})$. By using (12.1.2), (12.4.49), (12.4.48), (12.1.3), (12.4.53) and the properties of $t_k$, we end up with

$$\Delta z^2(x, t) \le c(ab\theta)^{2k}\delta^2, \tag{12.4.54}$$

where $c$ is a constant independent of $(x, t)$, $k$ and $\theta$. Consequently, upon selecting $a = \max\{1, 2\sqrt{c}\}$, we arrive at the desired bound (12.4.45) on the oscillation of $z$, for $m = k$. This completes the proof.

To establish Condition (b) of Definition 12.3.2, we demonstrate

**12.4.9 Lemma.** *Let $p_\ell(\cdot)$ and $p_r(\cdot)$ be any backward 1-characteristics emanating from $(\bar{x}, \bar{t})$, with $p_\ell(t) < p_r(t)$, for $t < \bar{t}$. If $\theta$ is sufficiently small, then*

$$\lim_{t\uparrow\bar{t}} z(p_\ell(t)+, t) \le \lim_{t\uparrow\bar{t}} z(p_r(t)-, t), \tag{12.4.55}$$

$$\lim_{t\uparrow\bar{t}} w(p_\ell(t)+, t) \ge \lim_{t\uparrow\bar{t}} w(p_r(t)-, t). \tag{12.4.56}$$

**Proof.** Consider any sequence $\{(x_n, t_n)\}$ with $t_n \uparrow \bar{t}$, as $n \to \infty$, and $x_n$ in $(p_\ell(t_n), p_r(t_n))$ so close to $p_r(t_n)$ that $\lim_{n\to\infty} [w(x_n+, t_n) - w(p_r(t_n)-, t_n)] = 0$. Let $\zeta_n(\cdot)$ denote the maximal backward 2-characteristic emanating from $(x_n, t_n)$, which intersects the graph of $p_\ell(\cdot)$ at time $t_n^*$. By virtue of Lemma 12.4.2, it follows that $w(x_n+, t_n) \le w(\zeta_n(t_n^*)+, t_n^*) = w(p_\ell(t_n^*)+, t_n^*)$. Since $t_n^* \uparrow \bar{t}$, as $n \to \infty$, this establishes (12.4.56).

To verify (12.4.55), we begin with another sequence $\{(x_n, t_n)\}$, with $t_n \uparrow \bar{t}$, as $n \to \infty$, and $x_n \in (p_\ell(t_n), p_r(t_n))$ such that $\lim_{n\to\infty} [z(x_n-, t_n) - z(p_\ell(t_n)+, t_n)] = 0$. We construct the minimal backward 1-characteristics $\xi_n(\cdot)$ and $\xi_n^*(\cdot)$, emanating from the points $(x_n, t_n)$ and $(p_r(t_n), t_n)$, respectively. Because of minimality, we now have $\xi_n(t) \le \xi_n^*(t) \le p_r(t)$, for $t \le t_n$. As $n \to \infty$, $\{\xi_n(\cdot)\}$ and $\{\xi_n^*(\cdot)\}$ will converge, uniformly, to shock-free minimal 1-separatrices (in the sense of Definition 10.3.3) $\chi(\cdot)$ and $\chi^*(\cdot)$, emanating from $(\bar{x}, \bar{t})$, such that $\chi(t) \le \chi^*(t) \le p_r(t)$, for $t \le \bar{t}$. In particular, $\dot{\chi}(\bar{t}-) \ge \dot{\chi}^*(\bar{t}-)$ and so

$$\lim_{t\uparrow\bar{t}} \lambda(z(\chi(t)\pm, t), w(\chi(t)\pm, t)) \ge \lim_{t\uparrow\bar{t}} \lambda(z(\chi^*(t)\pm, t), w(\chi^*(t)\pm, t)). \tag{12.4.57}$$

Applying (12.4.56) with $\chi(\cdot)$ and $\chi^*(\cdot)$ in the roles of $p_\ell(\cdot)$ and $p_r(\cdot)$ yields

$$\lim_{t\uparrow\bar{t}} w(\chi(t)+,t) \geq \lim_{t\uparrow\bar{t}} w(\chi^*(t)-,t). \tag{12.4.58}$$

Since $\lambda_z < 0$ and $\lambda_w < 0$, (12.4.57) and (12.4.58) together imply

$$\lim_{t\uparrow\bar{t}} z(\chi(t)\pm,t) \leq \lim_{t\uparrow\bar{t}} z(\chi^*(t)\pm,t). \tag{12.4.59}$$

By virtue of Lemma 12.4.2, $z(\xi_n(t)-,t)$ and $z(\xi_n^*(t)-,t)$ are nonincreasing functions on $[0,t_n]$ and so

$$\lim_{t\uparrow\bar{t}} z(\chi(t)\pm,t) \geq \lim_{t\uparrow\bar{t}} z(p_\ell(t)+,t), \tag{12.4.60}$$

$$\lim_{t\uparrow\bar{t}} z(\chi^*(t)\pm,t) \geq \lim_{t\uparrow\bar{t}} z(p_r(t)-,t). \tag{12.4.61}$$

Thus, to complete the proof of (12.4.55), one has to show

$$\lim_{t\uparrow\bar{t}} z(\chi^*(t)\pm,t) = \lim_{t\uparrow\bar{t}} z(p_r(t)-,t). \tag{12.4.62}$$

Since (12.4.62) is trivially true when $\chi^* \equiv p_r$, we take up the case where $\chi^*(t) < p_r(t)$, for $t < \bar{t}$. We set $\mathcal{S} = \{(x,t) : 0 \leq t < \bar{t}, \ \ \chi^*(t) < x < p_r(t)\}$. We shall verify (12.4.62) by constructing $t_0 < t_1 < \cdots < \bar{t}$ such that

$$\operatorname{osc}_{\mathcal{S}\cap\{t>t_m\}} z \leq (3b\theta)^m\delta, \qquad \operatorname{osc}_{\mathcal{S}\cap\{t>t_m\}} w \leq (3b\theta)^m\delta, \tag{12.4.63}$$

for $m = 0, 1, 2, \cdots$.

For $m = 0$, (12.4.63) is satisfied with $t_0 = 0$. Arguing by induction, we assume $t_0 < t_1 < \cdots < t_{k-1} < \bar{t}$ have already been fixed so that (12.4.63) holds for $m = 0, \cdots, k-1$, and proceed to determine $t_k$. We fix $\hat{t} \in (t_{k-1}, \bar{t})$ with $\bar{t} - \hat{t}$ so small that the oscillation of $z(\chi^*(\tau)\pm,\tau)$ and $w(\chi^*(\tau)-,\tau)$ over the interval $[\hat{t},\bar{t})$ does not exceed $\frac{1}{3}(3b\theta)^k\delta$. Next we locate $\hat{x} \in (\chi^*(\hat{t}), p_r(\hat{t}))$ with $\hat{x} - \chi^*(\hat{t})$ so small that the oscillation of $z(y+,\hat{t})$ over the interval $[\chi^*(\hat{t}),\hat{x})$ is similarly bounded by $\frac{1}{3}(3b\theta)^k\delta$.

By the construction of $\chi^*(\cdot)$, if we fix $t_k \in (\hat{t},\bar{t})$ with $\bar{t} - t_k$ sufficiently small, then the minimal backward 1-characteristic $\xi(\cdot)$, emanating from any point $(x,t)$ in $\mathcal{S} \cap \{t > t_k\}$, will intersect either the graph of $\chi^*(\cdot)$ at some time $t^* \in (\hat{t},\bar{t})$ or the $\hat{t}$-time line at some $x^* \in (\chi^*(\hat{t}),\hat{x})$; while the maximal backward 2-characteristic $\zeta(\cdot)$, emanating from $(x,t)$, will intersect the graph of $\chi^*(\cdot)$ at some time $\tilde{t} \in (\hat{t},\bar{t})$.

Assume, for definiteness, that $\xi(\cdot)$ intersects the $\hat{t}$-time line. By virtue of Lemmas 12.4.2 and 12.4.3,

$$z(x-,t) \leq z(\xi(\hat{t})-,\hat{t}) = z(x^*-,\hat{t}) \leq z(x^*+,\hat{t}). \tag{12.4.64}$$

By account of (12.4.63), for $m = k-1$, the oscillation of $w(\xi(\tau)+,\tau)$ over the interval $[\hat{t},t]$ does not exceed $(3b\theta)^{k-1}\delta$. It then follows from $(12.4.12)_1$

$$z(x+,t) \geq z(\xi(\hat{t})+,\hat{t}) - b\theta(3b\theta)^{k-1}\delta = z(x^*+,\hat{t}) - \frac{1}{3}(3b\theta)^k\delta. \tag{12.4.65}$$

Recalling that the oscillation of $z(y+,\hat{t})$ over $[\chi^*(\hat{t}),\hat{x})$ and the oscillation of $z(\chi^*(\tau)+,\tau)$ over $[\hat{t},\bar{t})$ are bounded by $\frac{1}{3}(3b\theta)^k\delta$, (12.4.64) and (12.4.65) together imply the bound (12.4.63) on the oscillation of $z$, for $m = k$.

The argument for $w$ is similar: On the one hand, Lemmas 12.4.2 and 12.4.3 give

$$w(x+,t) \leq w(\zeta(\tilde{t})+,\tilde{t}) = w(\chi^*(\tilde{t})+,\tilde{t}) \leq w(\chi^*(\tilde{t})-,\tilde{t}). \tag{12.4.66}$$

On the other hand, considering that the oscillation of $z(\zeta(\tau)-,\tau)$ over the interval $[\tilde{t},t]$ is bounded by $(3b\theta)^{k-1}\delta + \frac{1}{3}(3b\theta)^k\delta$, which in turn is smaller than $2(3b\theta)^{k-1}\delta$, $(12.4.12)_2$ yields

$$w(x-,t) \geq w(\zeta(\tilde{t})-,\tilde{t}) - 2b\theta(3b\theta)^{k-1}\delta = w(\chi^*(\tilde{t})-,\tilde{t}) - \frac{2}{3}(3b\theta)^k\delta. \tag{12.4.67}$$

Since the oscillation of $w(\chi^*(\tau)-,\tau)$ over $[\hat{t},\bar{t})$ does not exceed $\frac{1}{3}(3b\theta)^k\delta$, the inequalities (12.4.66) and (12.4.67) together imply the bound (12.4.63) on the oscillation of $w$, for $m = k$. The proof of the proposition is now complete.

In particular, one may apply Lemma 12.4.9 with $\xi(\cdot)$ and/or $\xi^*(\cdot)$ in the role of $p_\ell(\cdot)$ or $p_r(\cdot)$, so that, by virtue of Lemma 12.4.3, the inequalities $(12.3.6)_1$ and $(12.3.7)_1$ follow from (12.4.55) and (12.4.56). We have thus verified condition $(b)_1$ of Definition 12.3.2. Condition $(b)_2$ may be validated by a completely symmetrical argument.

It remains to check Condition (c) of Definition 12.3.2. It will suffice to verify $(c)_1$, because then $(c)_2$ will readily follow by a similar argument. In the shock case, $\phi_- \equiv \phi_+$, the required inequalities $z_W \leq z_N$ and $w_W \geq w_N$ are immediate corollaries of Lemma 12.4.3. Thus, one need only consider the rarefaction wave case.

**12.4.10 Lemma.** *Let $\phi_-(t) < \phi_+(t)$, for $t > \bar{t}$. For $\theta$ sufficiently small, as $(x,t)$ tends to $(\bar{x},\bar{t})$ in the region $\mathcal{W} = \{(x,t) : t > \bar{t},\ \phi_-(t) < x < \phi_+(t)\}$, $w(x\pm,t)$ tend to $w_W$. Furthermore,* $(12.3.8)_1$ *holds for any* 1*-characteristics $p_\ell(\cdot)$ and $p_r(\cdot)$, with $\phi_-(t) \leq p_\ell(t) \leq p_r(t) \leq \phi_+(t)$, for $t > \bar{t}$.*

**Proof.** Consider $(x,t)$ that tend to $(\bar{x},\bar{t})$ through $\mathcal{W}$. The maximal backward 2-characteristic $\zeta(\cdot)$ emanating from $(x,t)$ is intercepted by the $\bar{t}$-time line at $\zeta(\bar{t})$, which tends from below to $\bar{x}$. It then readily follows on account of Lemma 12.4.2 that $\limsup w(x\pm,t) \leq w_W$. To verify the assertion of the proposition, one needs to show that $\liminf w(x\pm,t) = w_W$. The plan is to argue by contradiction and so we make the hypothesis $\liminf w(x\pm,t) = w_W - \beta$, with $\beta > 0$.

We fix $\hat{t} > \bar{t}$ with $\hat{t} - \bar{t}$ so small that

$$w_W - 2\beta < w(x\pm,t) \leq w_W + \beta, \quad \bar{t} < t < \hat{t}\,,\ \phi_-(t) < x < \phi_+(t) \tag{12.4.68}$$

and also the oscillation of the functions $z(\phi_-(t)\pm,t)$ and $w(\phi_-(t)\pm,t)$ over the interval $(\bar{t},\hat{t})$ does not exceed $\frac{1}{2}\beta$.

We consider the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from any point $(\tilde{x}, \tilde{t})$, with $\bar{t} < \tilde{t} < \hat{t}$, $\phi_-(\tilde{t}) < x < \phi_+(\tilde{t})$, and intersecting the graph of $\phi_-(\cdot)$ at time $t^* \in (\bar{t}, \hat{t})$. We demonstrate that when $\theta$ is sufficiently small, independent of $\beta$, then

(12.4.69) $$w(\zeta(t)-, t) - w(\tilde{x}-, \tilde{t}) \le \frac{\beta}{4}\,, \qquad t^* < t < \tilde{t}.$$

Indeed, if (12.4.69) were false, one may find $t_1$, $t_2$, with $t^* < t_1 < t_2 \le \tilde{t}$ and $t_2 - t_1$ arbitrarily small, such that

(12.4.70) $$|z(\zeta(t_1)\pm, t_1) - z(\zeta(t_2)\pm, t_2)| > \frac{\beta}{4b\theta}\,.$$

In particular, if $\xi_1(\cdot)$ and $\xi_2(\cdot)$ denote the minimal backward 1-characteristics that emanate from the points $(\zeta(t_1), t_1)$ and $(\zeta(t_2), t_2)$, respectively, and thus necessarily pass through the point $(\bar{x}, \bar{t})$, then $t_1$ and $t_2$ may be fixed so close that

(12.4.71) $$0 \le \int_0^{t_1} \lambda(z(\xi_2(t)-, t), w(\xi_2(t)-, t))dt - \int_0^{t_1} \lambda(z(\xi_1(t)-, t), w(\xi_1(t)-, t))dt \le \beta t_0\,.$$

By virtue of (12.4.68), $|w(\xi_2(t)-, t) - w(\xi_1(t)-, t)| < 3\beta$, for all $t$ in $(\bar{t}, t_1)$. Also, on account of Lemma 12.4.2, $(12.4.12)_1$ and (12.4.68), we have

(12.4.72) $$\begin{cases} z(\zeta(t_1)-, t_1) \le z(\xi_1(t)-, t) = z(\xi_1(t)+, t) \le z(\zeta(t_1)+, t_1) + 3\beta b\theta, \\ z(\zeta(t_2)-, t_2) \le z(\xi_2(t)-, t) = z(\xi_2(t)+, t) \le z(\zeta(t_2)+, t_2) + 3\beta b\theta, \end{cases}$$

for almost all $t$ in $(\bar{t}, t_1)$. It is now clear that, for $\theta$ sufficiently small, (12.4.72) renders the inequalities (12.4.70) and (12.4.71) incompatible. This provides the desired contradiction that verifies (12.4.69).

By Lemma 12.4.6, and the construction of $\hat{t}$,

(12.4.73) $$\lim_{t\downarrow\bar{t}} z(\phi_-(t)-, t) = z_W\,, \quad \lim_{t\downarrow\bar{t}} w(\phi_-(t)-, t) = w_W\,,$$

(12.4.74) $$|z(\phi_-(t^*)-, t^*) - z_W| \le \frac{\beta}{2}\,, \qquad |w(\phi_-(t^*)-, t^*) - w_W| \le \frac{\beta}{2}\,.$$

The next step is to establish an estimate

(12.4.75) $$|z(\phi_-(t^*)-, t^*) - \lim_{t\downarrow t^*} z(\zeta(t)-, t)| \le a\beta,$$

for some constant $a$ independent of $\theta$ and $\beta$. Let

(12.4.76)
$$\lim_{t\downarrow\bar{t}} z(\phi_-(t)+,t) = z_W + \gamma,$$

with $\gamma \geq 0$. We fix $t_3 \in (t^*, \hat{t})$ and $x_3 \in (\phi_-(t_3), \phi_+(t_3))$, with $x_3 - \phi_-(t_3)$ so small that

(12.4.77)
$$|z(x_3\pm, t_3) - z_W - \gamma| \leq \beta.$$

By also choosing $t_3 - t^*$ small, the minimal backward 1-characteristic $\xi(\cdot)$, emanating from the point $(x_3, t_3)$, will intersect the graph of $\zeta(\cdot)$ at time $t_4$, arbitrarily close to $t^*$. By Lemma 12.4.2, $z(\zeta(t_4)-, t_4) \geq z(x_3-, t_3)$. On the other hand, by (12.4.68), Lemma 12.4.4 implies $z(\zeta(t_4)+, t_4) \leq z(x_3+, t_3) + 3b\theta\beta$. Hence, for $\theta$ so small that $6b\theta \leq 1$, we have $|z_W + \gamma - \lim_{t\downarrow t^*} z(\zeta(t)-,t)| \leq \frac{3}{2}\beta$. In conjunction with (12.4.74), this yields

(12.4.78)
$$|z(\phi_-(t^*)-, t^*) - \lim_{t\downarrow t^*} z(\zeta(t)-,t)| \leq 2\beta + \gamma.$$

Thus, to verify (12.4.75), one has to show $\gamma \leq c\beta$.

The characteristic $\xi(\cdot)$ lies to the right of $\phi_-(\cdot)$ and passes through the point $(\bar{x}, \bar{t})$, so $\dot{\phi}_-(\bar{t}+) \leq \dot{\xi}(\bar{t}+)$. On account of (12.4.73), (12.4.76), (8.2.1), (8.2.2), (7.3.12), (8.2.3), and (12.1.2), we conclude

(12.4.79)
$$\dot{\phi}_-(\bar{t}+) = \lambda(z_W, w_W) + \tfrac{1}{2}\lambda_z(z_W, w_W)\gamma + O(\gamma^2).$$

To estimate $\dot{\xi}(\bar{t}+) = \lim_{t\downarrow\bar{t}} \lambda(z(\xi(t)-,t), w(\xi(t)-,t))$, we recall $\lambda_z < 0$, $\lambda_w < 0$, $z(\xi(t)-,t) \geq z(x_3-, t_3) \geq z_W + \gamma - \beta$, $w(\xi(t)-,t) \geq w_W - 2\beta$, and so

(12.4.80)
$$\dot{\xi}(\bar{t}+) \leq \lambda(z_W + \gamma - \beta, w_W - 2\beta) = \lambda(z_W, w_W) + \lambda_z(z_W, w_W)\gamma + O(\beta + \gamma^2).$$

Therefore, $\gamma = O(\beta)$ and (12.4.75) follows from (12.4.78).

By virtue of Lemma 12.4.4, (12.4.75) yields

(12.4.81)
$$w(\phi_-(t^*)-, t^*) - \lim_{t\downarrow t^*} w(\zeta(t)-,t) \leq ab\theta\beta.$$

Hence, if $\theta \leq (8ab)^{-1}$, then (12.4.69), (12.4.81) and (12.4.74) together imply $w_W - w(\tilde{x}-, \tilde{t}) \leq \frac{7}{8}\beta$, for all $(\tilde{x}, \tilde{t})$ in $\mathcal{W} \cap \{t < \hat{t}\}$. This provides the desired contradiction to the hypothesis $\liminf w(x\pm, t) = w_W - \beta$, with $\beta > 0$, thus verifying the assertion that, as $(x,t)$ tends to $(\bar{x}, \bar{t})$ through $\mathcal{W}$, $w(x\pm, t)$ tend to $w_W$.

We now focus attention on $\phi_+(\cdot)$. We already have $\lim_{t\downarrow\bar{t}} w(\phi_+(t)-,t) = w_W$, $\lim_{t\downarrow\bar{t}} z(\phi_+(t)+,t) = z_N$, $\lim_{t\downarrow\bar{t}} w(\phi_+(t)+,t) = w_N$. We set $z_0 = \lim_{t\downarrow\bar{t}} z(\phi_+(t)-,t)$. Then $\lambda(z_0, w_W) \geq \dot{\phi}_+(\bar{t}+) \geq \lambda(z_N, w_N)$. The aim is to show that $\dot{\phi}_+(\bar{t}+) = \lambda(z_0, w_W)$ so as to infer $z_N = z_0$, $w_N = w_W$. We consider the minimal backward

1-characteristic $\xi(\cdot)$ emanating from the point $(\phi_+(t_5), t_5)$, where $t_5 - \bar{t}$ is very small. The assertion $z_N = z_0$, $w_N = w_W$ is obviously true when $\xi \equiv \phi_+$, so let us assume that $\xi(t) < \phi_+(t)$ for $t \in (\bar{t}, t_5)$. Then $|w(\xi(t)+, t) - w_W|$ is very small on $(\bar{t}, t_5)$. Moreover, by Lemma 12.4.4, the oscillation of $z(\xi(t)+, t)$ over the interval $(\bar{t}, t_5)$ is very small so this function takes values near $z_0$. Hence, $t_5 - \bar{t}$ sufficiently small renders $\dot{\xi}(\bar{t}+)$ arbitrarily close to $\lambda(z_0, w_W)$. Since $\dot{\xi}(\bar{t}+) \leq \dot{\phi}_+(\bar{t}+)$, we conclude that $\dot{\phi}_+(\bar{t}+) \geq \lambda(z_0, w_W)$ and thus necessarily $\dot{\phi}_+(\bar{t}_+) = \lambda(z_0, w_W)$.

Consider now any forward 1-characteristic $\chi(\cdot)$ issuing from $(\bar{x}, \bar{t})$, such that $\phi_-(t) \leq \chi(t) \leq \phi_+(t)$, for $t > \bar{t}$. Since $\lim_{t\downarrow\bar{t}} w(\chi(t)-, t)$ and $\lim_{t\downarrow\bar{t}} w(\chi(t)+, t)$ take the same value, namely $w_W$, $\lim_{t\downarrow\bar{t}} z(\chi(t)-, t)$ and $\lim_{t\downarrow\bar{t}} z(\chi(t)+, t)$ must also take the same value, say $z_\chi$. In particular, $\dot{\chi}(\bar{t}+) = \lambda(z_\chi, w_W)$. Therefore, if $p_\ell(\cdot)$ and $p_r(\cdot)$ are any 1-characteristics, with $\phi_-(t) \leq p_\ell(t) \leq p_r(t) \leq \phi_+(t)$, for $t > \bar{t}$, the inequalities $\dot{\phi}_-(\bar{t}+) \leq \dot{p}_\ell(\bar{t}+) \leq \dot{p}_r(\bar{t}+) \leq \dot{\phi}_+(\bar{t}+)$, ordering the speeds of propagation at $\bar{t}$, together with $\lambda_z < 0$, imply $(12.3.8)_1$. The proof is complete.

We have now completed the proof of Theorem 12.3.3, on local regularity, as well as of Theorem 12.4.1, on the laws of propagation of Riemann invariants along extremal backward characteristics. These will serve as the principal tools for deriving a priori estimates leading to a description of the long time behavior of solutions.

Henceforth, our solutions will be normalized on $(-\infty, \infty) \times (0, \infty)$ by defining $(z(x,t), w(x,t)) = (z_S, w_S)$, namely the "southern" limit at $(x, t)$. The trace of the solution on any space-like curve is then defined as the restriction of the normalized $(z, w)$ to this curve. In particular, this renders the trace of $(z, w)$ along the minimal backward 1-characteristic and the maximal backward 2-characteristic, emanating from any point $(\bar{x}, \bar{t})$, continuous from the left on $(0, \bar{t}]$.

## 12.5 Bounds on Solutions

We consider a solution, normalized as above, bounded by

$$|z(x,t)| + |w(x,t)| < 2\delta, \qquad -\infty < x < \infty\,, \quad 0 < t < \infty, \tag{12.5.1}$$

where $\delta$ is a small positive constant. It is convenient to regard the initial data as multivalued functions, allowing $(z(x,0), w(x,0))$ to take as values any state in the range of the solution of the Riemann problem with end-states $(z(x\pm, 0), w(x\pm, 0))$. The supremum and total variation are measured for the selection that maximizes these quantities. We then assume

$$\sup_{(-\infty,\infty)} |z(\cdot, 0)| + \sup_{(-\infty,\infty)} |w(\cdot, 0)| \leq \delta, \tag{12.5.2}$$

$$TV_{(-\infty,\infty)} z(\cdot, 0) + TV_{(-\infty,\infty)} w(\cdot, 0) < a\delta^{-1}\,, \tag{12.5.3}$$

where $a$ is a small constant, to be fixed later, independently of $\delta$. Thus, there is a tradeoff, allowing for arbitrarily large total variation at the expense of keeping the oscillation sufficiently small. The aim is to establish bounds on the solution. In what follows, $c$ will stand for a generic constant that depends solely on $F$. The principal result is

**12.5.1 Theorem.** *Consider any space-like curve $t = t^*(x)$, $x_\ell \leq x \leq x_r$, in the upper half-plane, along which the trace of $(z, w)$ is denoted by $(z^*, w^*)$. Then*

$$(12.5.4)_1 \qquad TV_{[x_\ell,x_r]}z^*(\cdot) \leq TV_{[\xi_\ell(0),\xi_r(0)]}z(\cdot,0) + c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)\},$$

$$(12.5.4)_2 \qquad TV_{[x_\ell,x_r]}w^*(\cdot) \leq TV_{[\zeta_\ell(0),\zeta_r(0)]}w(\cdot,0) + c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)\},$$

*where $\xi_\ell(\cdot)$, $\xi_r(\cdot)$ are the minimal backward 1-characteristics and $\zeta_\ell(\cdot)$, $\zeta_r(\cdot)$ are the maximal backward 2-characteristics emanating from the endpoints $(x_\ell, t_\ell)$ and $(x_r, t_r)$ of the graph of $t^*(\cdot)$.*

Since generalized characteristics are space-like curves, one may combine the above proposition with Theorem 12.4.1 and the assumptions (12.5.1), (12.5.3) to deduce the following corollary:

**12.5.2 Theorem.** *For any point $(x, t)$ of the upper half-plane:*

$$(12.5.5)_1 \qquad \sup_{(-\infty,\infty)} z(\cdot,0) \geq z(x,t) \geq \inf_{(-\infty,\infty)} z(\cdot,0) - ca\delta,$$

$$(12.5.5)_2 \qquad \sup_{(-\infty,\infty)} w(\cdot,0) \geq w(x,t) \geq \inf_{(-\infty,\infty)} w(\cdot,0) - ca\delta.$$

Thus, on account of our assumption (12.5.2) and by selecting $a$ sufficiently small, we secure a posteriori that the solution will satisfy (12.5.1).

The task of proving Theorem 12.5.1 is quite laborious and will require extensive preparation. In the course of the proof we shall verify that certain quantities measuring the total amount of wave interaction are also bounded.

Consider a 1-shock joining the state $(z_-, w_-)$, on the left, with the state $(z_+, w_+)$, on the right. The jumps $\Delta z = z_+ - z_-$ and $\Delta w = w_+ - w_-$ are related through an equation

$$(12.5.6)_1 \qquad \Delta w = f(\Delta z; z_-, w_-)$$

resulting from the reparametrization of the 1-shock curve emanating from the state $(z_-, w_-)$. In particular, $f$ and its first two derivatives with respect to $\Delta z$ vanish at $\Delta z = 0$ and hence $f$ as well as $\partial f/\partial z_-$ and $\partial f/\partial w_-$ are $O(\Delta z^3)$ as $\Delta z \to 0$.

Similarly, the jumps $\Delta w = w_+ - w_-$ and $\Delta z = z_+ - z_-$ of the Riemann invariants across a 2-shock joining the state $(z_-, w_-)$, on the left, with the state $(z_+, w_+)$, on the right, are related through an equation

$$(12.5.6)_2 \qquad \Delta z = g(\Delta w; z_+, w_+)$$

resulting from the reparametrization of the backward 2-shock curve (see Section 9.3) that emanates from the state $(z_+, w_+)$. Furthermore, $g$ together with $\partial g/\partial z_+$ and $\partial g/\partial w_+$ are $O(\Delta w^3)$ as $\Delta w \to 0$.

For convenience, points of the upper half-plane will be labelled by single capital letters $I$, $J$, etc. With any point $I = (\bar{x}, \bar{t})$ we associate the special characteristics $\phi_\pm^I, \psi_\pm^I, \xi_\pm^I, \zeta_\pm^I$ emanating from it, as discussed in Section 12.3 and depicted in Fig. 12.3.1, and identify the limits $(z_W^I, w_W^I)$, $(z_E^I, w_E^I)$, $(z_N^I, w_N^I)$, $(z_S^I, w_S^I)$ as $I$ is approached through the sectors $\mathcal{S}_W^I$, $\mathcal{S}_E^I$, $\mathcal{S}_N^I$, $\mathcal{S}_S^I$. From $I$ emanate minimal 1-separatrices $p_\pm^I$ and maximal 2-separatrices $q_\pm^I$ constructed as follows: $p_-^I$ (or $q_+^I$) is simply the minimal (or maximal) backward 1-characteristic $\xi_-^I$ (or 2-characteristic $\zeta_+^I$) emanating from $I$; while $p_+^I$ (or $q_-^I$) is the limit of a sequence of minimal (or maximal) backward 1-characteristics $\xi_n$ (or 2-characteristics $\zeta_n$) emanating from points $(x_n, t_n)$ in $\mathcal{S}_E^I$ (or $\mathcal{S}_W^I$), where $(x_n, t_n) \to (\bar{x}, \bar{t})$, as $n \to \infty$. We introduce the notation

$$(12.5.7)_1 \qquad \mathcal{F}_I = \{(x,t) : 0 \le t < \bar{t}, \quad p_-^I(t) \le x \le p_+^I(t)\},$$

$$(12.5.7)_2 \qquad \mathcal{G}_I = \{(x,t) : 0 \le t < \bar{t}, \quad q_-^I(t) \le x \le q_+^I(t)\}.$$

By virtue of Theorems 12.3.3 and 12.4.1,

$$(12.5.8)_1 \qquad \lim_{t \uparrow \bar{t}} z(p_-^I(t), t) = z_S^I\,, \qquad \lim_{t \uparrow \bar{t}} z(p_+^I(t), t) = z_E^I\,,$$

$$(12.5.8)_2 \qquad \lim_{t \uparrow \bar{t}} w(q_-^I(t), t) = w_W^I\,, \qquad \lim_{t \uparrow \bar{t}} w(q_+^I(t), t) = w_S^I\,.$$

The cumulative strength of 1-waves and 2-waves, incoming at $I$, is respectively measured by

$$(12.5.9) \qquad \Delta z^I = z_E^I - z_S^I\,, \qquad \Delta w^I = w_S^I - w_W^I\,.$$

If the incoming 1-waves alone were allowed to interact, they would produce an outgoing 1-shock with $w$-amplitude

$$(12.5.10)_1 \qquad \Delta w_*^I = f(\Delta z^I; z_S^I, w_S^I),$$

together with an outgoing 2-rarefaction wave. Consequently, $|\Delta w_*^I|$ exceeds the cumulative $w$-strength $|w_E^I - w_S^I|$ of incoming 1-waves. Similarly, the interaction of incoming 2-waves alone would produce an outgoing 2-shock with $z$-amplitude

$$(12.5.10)_2 \qquad \Delta z_*^I = g(\Delta w^I; z_S^I, w_S^I),$$

exceeding their cumulative $z$-strength $z_S^I - z_W^I$. Note that if $z_S^I = z_W^I$, $w_S^I = w_W^I$ then $\Delta w_*^I = w_N^I - w_W^I$, while if $z_S^I = z_E^I$, $w_S^I = w_E^I$ then $\Delta z_*^I = z_E^I - z_N^I$.

We visualize the upper half-plane as a partially ordered set under the relation induced by the rule $I < J$ whenever $J$ is confined between the graphs of the minimal 1-separatrices $p_-^I$ and $p_+^I$ emanating from $I$. In particular, when $J$ lies strictly to the right of the graph of $p_-^I$, then $I$ lies on the graph of the 1-characteristic $\phi_-^J$ emanating from $J$. Thus $I < J$ implies that $I$ always lies on the graph of a forward 1-characteristic issuing from $J$, that is either $\phi_-^J$ or $p_-^I$. This special characteristic will be denoted by $\chi_-^J$.

We consider 1-*characteristic trees* $\mathcal{M}$ consisting of a finite set of points of the upper half-plane, called *nodes*, with the following properties: $\mathcal{M}$ contains a unique minimal node $I_0$, namely the *root* of the tree. Furthermore, if $J$ and $K$ are any two nodes, then the point $I$ of confluence of the forward 1-characteristics $\chi_-^J$ and $\chi_-^K$, which pass through the root $I_0$, is also a node of $\mathcal{M}$. In general, $\mathcal{M}$ will contain several maximal nodes (Fig. 12.5.1).

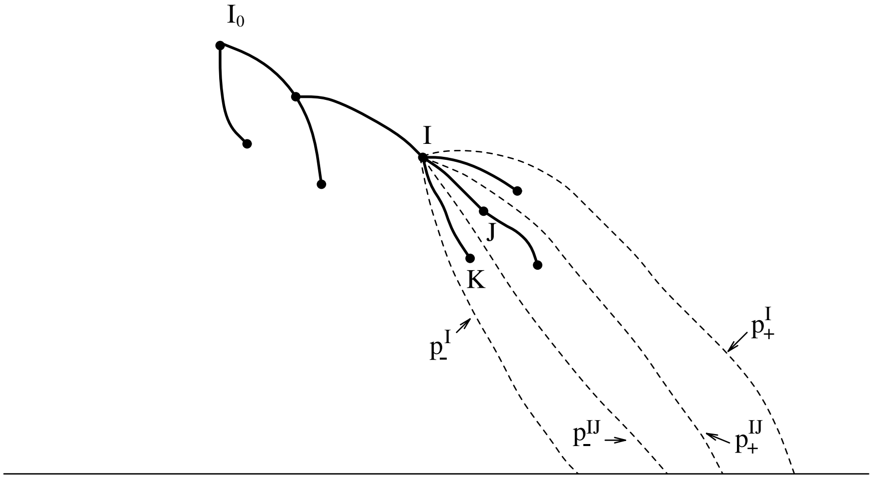


**Fig. 12.5.1**

Every node $J \neq I_0$ is *consecutive* to some node $I$, namely, its strict greatest lower bound relative to $\mathcal{M}$. The set of nodes that are consecutive to a node $I$ is denoted by $\mathcal{C}_I$. When $J$ is consecutive to $I$, the pair $(I, J)$ is called a *link*. A finite sequence $\{I_0, I_1, \cdots, I_m\}$ of nodes such that $I_{j+1}$ is consecutive to $I_j$, for $j = 0, \cdots, m-1$, which connects the root $I_0$ with some maximal node $I_m$, constitutes a *chain* of $\mathcal{M}$.

If $(I, J)$ is a link of $\mathcal{M}$, so that $I = (\chi_-^J(\bar{t}), \bar{t})$, we set

$$z_\pm^{IJ} = \lim_{t \uparrow \bar{t}} z(\chi_-^J(t)\pm, t), \qquad w_\pm^{IJ} = \lim_{t \uparrow \bar{t}} w(\chi_-^J(t)\pm, t), \tag{12.5.11$_1$}$$

$$\Delta z^{IJ} = z_+^{IJ} - z_-^{IJ}, \qquad \Delta w^{IJ} = w_+^{IJ} - w_-^{IJ}. \tag{12.5.12$_1$}$$

In particular,

$$\Delta w^{IJ} = f(\Delta z^{IJ}; z_-^{IJ}, w_-^{IJ}). \tag{12.5.13$_1$}$$

With $(I, J)$ we associate minimal 1-separatrices $p_\pm^{IJ}$, emanating from $I$, constructed as follows: $p_-^{IJ}$ is the $t \uparrow \bar{t}$ limit of the family $\xi_t$ of minimal backward 1-characteristics emanating from the point $(\chi_-^J(t), t)$; while $p_+^{IJ}$ is the limit of a sequence of minimal backward 1-characteristics $\xi_n$ emanating from points $(x_n, t_n)$ such that, as $n \to \infty$, $t_n \uparrow \bar{t}$, $x_n - \chi_-^J(t_n) \downarrow 0$ and $z(x_n-, t_n) \to z_+^{IJ}$, $w(x_n-, t_n) \to w_+^{IJ}$. Notice that the graphs of $p_\pm^{IJ}$ are confined between the graph of $p_-^I$ and the graph of $p_+^I$; see Fig. 12.5.1. In turn, the graphs of $p_\pm^J$, as well as the graphs of $p_\pm^K$, for any $K > J$, are confined between the graph of $p_-^{IJ}$ and the graph of $p_+^{IJ}$. Furthermore,

$$\lim_{t\uparrow\bar{t}} z(p_-^{IJ}(t), t) = z_-^{IJ}, \qquad \lim_{t\uparrow\bar{t}} z(p_+^{IJ}(t), t) = z_+^{IJ}. \tag{12.5.14$_1$}$$

Indeed, the first of the above two equations has already been established in the context of the proof of Lemma 12.4.9 (under different notation; see (12.4.62)); while the second may be verified by a similar argument.

We now set

$$\mathcal{P}_1(\mathcal{M}) = -\sum_{I\in\mathcal{M}} [\Delta w_*^I - \sum_{J\in\mathcal{C}_I} \Delta w^{IJ}], \tag{12.5.15$_1$}$$

$$\mathcal{Q}_1(\mathcal{M}) = \sum_{I\in\mathcal{M}} \sum_{J\in\mathcal{C}_I} |\Delta w^{IJ} - \Delta w_*^J|. \tag{12.5.16$_1$}$$

By virtue of $(12.3.7)_1$,

$$\sum_{J\in\mathcal{C}_I} \Delta w^{IJ} \geq w_E^I - w_S^I \geq \Delta w_*^I, \tag{12.5.17$_1$}$$

so that both $\mathcal{P}_1$ and $\mathcal{Q}_1$ are nonnegative.

With subsets $\mathcal{F}$ of the upper half-plane, we associate functionals

$$\mathcal{P}_1(\mathcal{F}) = \sup_{\mathcal{J}} \sum_{\mathcal{M}\in\mathcal{J}} \mathcal{P}_1(\mathcal{M}), \tag{12.5.18$_1$}$$

$$\mathcal{Q}_1(\mathcal{F}) = \sup_{\mathcal{J}} \sum_{\mathcal{M}\in\mathcal{J}} \mathcal{Q}_1(\mathcal{M}), \tag{12.5.19$_1$}$$

where $\mathcal{J}$ denotes any (finite) collection of 1-characteristic trees $\mathcal{M}$ contained in $\mathcal{F}$, which are disjoint, in the sense that the roots of any pair of them are non-comparable. One may view $\mathcal{P}_1(\mathcal{F})$ as a measure of the amount of 1-wave interactions inside $\mathcal{F}$, and $\mathcal{Q}_1(\mathcal{F})$ as a measure of strengthening of 1-shocks induced by interaction with 2-waves.

We introduce corresponding notions for the 2-characteristic family: $I < J$ whenever $J$ is confined between the graphs of the maximal 2-separatrices $q_-^I$ and $q_+^I$ emanating from $I$. In that case, $I$ lies on the graph of a forward 2-characteristic $\chi_+^J$ issuing from $J$, namely either $\psi_+^J$ or $q_+^I$. One may then construct 2-*characteristic* trees $\mathcal{N}$, with *nodes*, *root*, *links* and *chains* defined as above. In the place of $(12.5.11)_1$, $(12.5.12)_1$ and $(12.5.13)_1$, we now have

$$(12.5.11)_2 \qquad z_\pm^{IJ} = \lim_{t\uparrow\bar{t}} z(\chi_+^J(t)\pm, t), \qquad w_\pm^{IJ} = \lim_{t\uparrow\bar{t}} w(\chi_+^J(t)\pm, t),$$

$$(12.5.12)_2 \qquad \Delta z^{IJ} = z_+^{IJ} - z_-^{IJ}, \qquad \Delta w^{IJ} = w_+^{IJ} - w_-^{IJ},$$

$$(12.5.13)_2 \qquad \Delta z^{IJ} = g(\Delta w^{IJ}; z_+^{IJ}, w_+^{IJ}).$$

With links $(I, J)$ we associate maximal 2-separatrices $q_\pm^{IJ}$, emanating from $I$, in analogy to $p_\pm^{IJ}$. The graphs of $q_\pm^{IJ}$ are confined between the graphs of $q_-^I$ and $q_+^I$. On the other hand, the graphs of $q_\pm^J$ are confined between the graphs of $q_-^{IJ}$ and $q_+^{IJ}$. In the place of $(12.5.14)_1$,

$$(12.5.14)_2 \qquad \lim_{t\uparrow\bar{t}} w(q_-^{IJ}(t), t) = w_-^{IJ}, \qquad \lim_{t\uparrow\bar{t}} w(q_+^{IJ}(t), t) = w_+^{IJ}.$$

Analogs of $(12.5.15)_1$ and $(12.5.16)_1$ are also defined:

$$(12.5.15)_2 \qquad \mathcal{P}_2(\mathcal{N}) = \sum_{I\in\mathcal{N}} [\Delta z_*^I - \sum_{J\in\mathcal{C}_I} \Delta z^{IJ}],$$

$$(12.5.16)_2 \qquad \mathcal{Q}_2(\mathcal{N}) = \sum_{I\in\mathcal{N}} \sum_{J\in\mathcal{C}_I} |\Delta z^{IJ} - \Delta z_*^J|,$$

which are nonnegative since

$$(12.5.17)_2 \qquad \sum_{J\in\mathcal{C}_I} \Delta z^{IJ} \le z_S^I - z_W^I \le \Delta z_*^I.$$

This induces functionals analogous to $\mathcal{P}_1$ and $\mathcal{Q}_1$:

$$(12.5.18)_2 \qquad \mathcal{P}_2(\mathcal{F}) = \sup_{\mathcal{J}} \sum_{\mathcal{N}\in\mathcal{J}} \mathcal{P}_2(\mathcal{N}),$$

$$(12.5.19)_2 \qquad \mathcal{Q}_2(\mathcal{F}) = \sup_{\mathcal{J}} \sum_{\mathcal{N}\in\mathcal{J}} \mathcal{Q}_2(\mathcal{N}).$$

**12.5.3 Lemma.** *Let $\mathcal{F}_1, \cdots, \mathcal{F}_m$ be a collection of subsets of a set $\mathcal{F}$ contained in the upper half-plane. Suppose that for any $I \in \mathcal{F}_i$ and $J \in \mathcal{F}_j$ that are comparable, say*

*$I < J$, the arc of the characteristic $\chi_-^J$ (or $\chi_+^J$) which connects $J$ to $I$ is contained in $\mathcal{F}$. Then*

$$(12.5.20)_1 \qquad \sum_{i=1}^{m} \{\mathcal{P}_1(\mathcal{F}_i) + \mathcal{Q}_1(\mathcal{F}_i)\} \leq k\{\mathcal{P}_1(\mathcal{F}) + \mathcal{Q}_1(\mathcal{F})\},$$

*or*

$$(12.5.20)_2 \qquad \sum_{i=1}^{m} \{\mathcal{P}_2(\mathcal{F}_i) + \mathcal{Q}_2(\mathcal{F}_i)\} \leq k\{\mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F})\},$$

*where $k$ is the smallest positive integer with the property that any $k+1$ of $\mathcal{F}_1, \cdots, \mathcal{F}_m$ have empty intersection.*

**Proof.** It will suffice to verify $(12.5.20)_1$. With each $i = 1, \cdots, m$, we associate a family $\mathcal{J}_i$ of disjoint 1-characteristic trees $\mathcal{M}$ contained in $\mathcal{F}_i$. Clearly, by adjoining if necessary additional nodes contained in $\mathcal{F}$, one may extend the collection of the $\mathcal{J}_i$ into a single family $\mathcal{J}$ of disjoint trees contained in $\mathcal{F}$. The contribution of the additional nodes may only increase the value of $\mathcal{P}_1$ and $\mathcal{Q}_1$. Therefore,

$$(12.5.21) \qquad \sum_{i=1}^{m} \sum_{\mathcal{M}\in\mathcal{J}_i} \{\mathcal{P}_1(\mathcal{M}) + \mathcal{Q}_1(\mathcal{M})\} \leq k \sum_{\mathcal{M}\in\mathcal{J}} \{\mathcal{P}_1(\mathcal{M}) + \mathcal{Q}_1(\mathcal{M})\},$$

where the factor $k$ appears on the right-hand side because the same node or link may be counted up to $k$ times on the left-hand side. Recalling $(12.5.18)_1$ and $(12.5.19)_1$, we arrive at $(12.5.20)_1$. The proof is complete.

**12.5.4 Lemma.** *Consider a space-like curve $t = \bar{t}(x)$, $\hat{x} \leq x \leq \tilde{x}$, in the upper half-plane. The trace of $(z, w)$ along $\bar{t}$ is denoted by $(\bar{z}, \bar{w})$. Let $\hat{p}(\cdot)$ and $\tilde{p}(\cdot)$ (or $\hat{q}(\cdot)$ and $\tilde{q}(\cdot)$) be minimal (or maximal) 1-separatrices (or 2-separatrices) emanating from the left endpoint $(\hat{x}, \hat{t})$ and the right endpoint $(\tilde{x}, \tilde{t})$ of the graph of $\bar{t}$. The trace of $z$ (or $w$) along $\hat{p}$ and $\tilde{p}$ (or $\hat{q}$ and $\tilde{q}$) is denoted by $\hat{z}$ and $\tilde{z}$ (or $\hat{w}$ and $\tilde{w}$). Let $\mathcal{F}$ (or $\mathcal{G}$) stand for the region bordered by the graphs of $\hat{p}$, $\tilde{p}$ (or $\hat{q}$, $\tilde{q}$), $\bar{t}$ and the $x$-axis. Then*

$(12.5.22)_1$

$$|\tilde{z}(\tilde{t}-) - \hat{z}(\hat{t}-)| \leq |\tilde{z}(0+) - \hat{z}(0+)| + c\delta^2 TV_{[\hat{x},\tilde{x}]}\bar{w}(\cdot) + \mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F}),$$

*or*

$(12.5.22)_2$

$$|\tilde{w}(\tilde{t}-) - \hat{w}(\hat{t}-)| \leq |\tilde{w}(0+) - \hat{w}(0+)| + c\delta^2 TV_{[\hat{x},\tilde{x}]}\bar{z}(\cdot) + \mathcal{P}_1(\mathcal{G}) + \mathcal{Q}_1(\mathcal{G}).$$

**Proof.** It will suffice to verify $(12.5.22)_1$. We write

(12.5.23) $\tilde{z}(\tilde{t}-)-\hat{z}(\hat{t}-)=[\tilde{z}(0+)-\hat{z}(0+)]+[\tilde{z}(\tilde{t}-)-\tilde{z}(0+)]-[\hat{z}(\hat{t}-)-\hat{z}(0+)].$

By virtue of Theorem 12.4.1,

$$\begin{cases} \hat{z}(\hat{t}-) - \hat{z}(0+) = \sum[\hat{z}(\tau+) - \hat{z}(\tau-)], \\ \tilde{z}(\tilde{t}-) - \tilde{z}(0+) = \sum[\tilde{z}(\tau+) - \tilde{z}(\tau-)], \end{cases} \tag{12.5.24}$$

where the summations run over the countable set of jump discontinuities of $\hat{z}(\cdot)$ and $\tilde{z}(\cdot)$.

By account of Theorem 12.3.3, if $\bar{z}(\cdot)$ is the trace of $z$ along any minimal 1-separatrix which passes through some point $K = (x, \tau)$, then

$$z_S^K - z_W^K \leq \bar{z}(\tau-) - \bar{z}(\tau+) \leq \Delta z_*^K. \tag{12.5.25}$$

Starting out from points $K$ of jump discontinuity of $\hat{z}(\cdot)$ on the graph of $\hat{p}$, we construct the characteristic $\phi_-^K$ until it intersects the graph of either $\tilde{p}$ or $\bar{t}$. This generates families of disjoint 2-characteristic trees $\mathcal{N}$, with maximal nodes, say $K_1 = (x_1, \tau_1), \cdots, K_m = (x_m, \tau_m)$ lying on the graph of $\hat{p}$ and root $K_0 = (x_0, \tau_0)$ lying on the graph of either $\tilde{p}$ or $\bar{t}$. In the former case, on account of (12.5.25), $(12.5.15)_2$ and $(12.5.16)_2$,

$$|\tilde{z}(\tau_0+) - \tilde{z}(\tau_0-) - \sum_{\ell=1}^{m}[\hat{z}(\tau_\ell+) - \hat{z}(\tau_\ell-)]| \leq \mathcal{P}_2(\mathcal{N}) + \mathcal{Q}_2(\mathcal{N}). \tag{12.5.26}$$

On the other hand, if $K_0$ lies on the graph of $\bar{t}$,

$$\sum_{J \in \mathcal{C}_{K_0}} \Delta z^{K_0 J} \leq z_S^{K_0} - z_W^{K_0} \leq c\delta^2 |w_S^{K_0} - w_W^{K_0}|, \tag{12.5.27}$$

and so

$$| - \sum_{\ell=1}^{m}[\hat{z}(\tau_\ell+) - \hat{z}(\tau_\ell-)]| \leq c\delta^2 |w_S^{K_0} - w_W^{K_0}| + \mathcal{P}_2(\mathcal{N}) + \mathcal{Q}_2(\mathcal{N}). \tag{12.5.28}$$

Suppose that on the graph of $\tilde{p}$ there still remain points $K_0$ of jump discontinuity of $\tilde{z}(\cdot)$ which cannot be realized as roots of trees with maximal nodes on the graph of $\hat{p}$. We then adjoin (trivial) 2-characteristic trees $\mathcal{N}$ that contain a single node, namely such a $K_0 = (x_0, \tau_0)$, in which case

$$|\tilde{z}(\tau_0+) - \tilde{z}(\tau_0-)| \leq \mathcal{P}_2(\mathcal{N}) + \mathcal{Q}_2(\mathcal{N}). \tag{12.5.29}$$

Recalling (12.5.23) and tallying the jump discontinuities of $\bar{z}_1(\cdot)$ and $\bar{z}_2(\cdot)$, as indicated in (12.5.24), according to (12.5.26), (12.5.28) or (12.5.29), we arrive at $(12.5.22)_1$ . The proof is complete.

**12.5.5 Lemma.** *Under the assumptions of Theorem* 12.5.1, (12.5.30)$_1$

$$TV_{[x_\ell,x_r]}z^*(\cdot) \le TV_{[\xi_\ell(0),\xi_r(0)]}z(\cdot,0) + c\delta^2 TV_{[x_\ell,x_r]}w^*(\cdot) + 2\{\mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F})\},$$

(12.5.30)$_2$

$$TV_{[x_\ell,x_r]}w^*(\cdot) \le TV_{[\zeta_\ell(0),\zeta_r(0)]}w(\cdot,0) + c\delta^2 TV_{[x_\ell,x_r]}z^*(\cdot) + 2\{\mathcal{P}_1(\mathcal{G}) + \mathcal{Q}_1(\mathcal{G})\},$$

*where $\mathcal{F}$ denotes the region bordered by the graphs of $\xi_\ell$ , $\xi_r$ , $t^*$, and the $x$-axis, while $\mathcal{G}$ stands for the region bordered by the graphs of $\zeta_\ell$ , $\zeta_r$ , $t^*$, and the $x$-axis.*

**Proof.** It will suffice to establish (12.5.30)$_1$ . We have to estimate

(12.5.31) $$TV_{[x_\ell,x_r]}z^*(\cdot) = \sup \sum_{i=1}^{m} |z_S^{L_i} - z_S^{L_{i-1}}|,$$

where the supremum is taken over all finite sequences $\{L_0, \cdots, L_m\}$ of points along $t^*$ (Fig. 12.5.2).

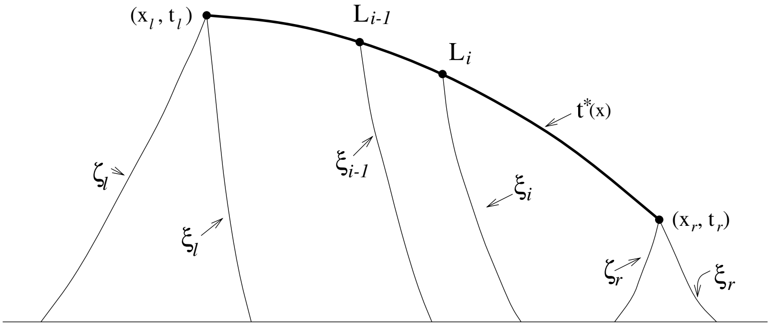


**Fig. 12.5.2**

We construct the minimal backward 1-characteristics $\xi_i$ emanating from $L_i = (x_i, t_i)$, $i = 0, \cdots, m$, and let $z_i(\cdot)$ denote the trace of $z$ along $\xi_i(\cdot)$. We apply Lemma 12.5.4 with $\bar{t}$ the arc of $t^*$ with endpoints $L_{i-1}$ and $L_i$; $\hat{x} = x_{i-1}$ ; $\tilde{x} = x_i$ ; $\hat{p} = \xi_{i-1}$ ; $\tilde{p} = \xi_i$ ; and $\mathcal{F} = \mathcal{F}_i$ , namely the region bordered by the graphs of $\xi_{i-1}$ , $\xi_i$ , $t^*$, and the $x$-axis. The estimate (12.5.22)$_1$ then yields

(12.5.32)

$$|z_S^{L_i} - z_S^{L_{i-1}}| \le |z_i(0+) - z_{i-1}(0+)| + c\delta^2 TV_{[x_{i-1},x_i]}w^*(\cdot) + \mathcal{P}_2(\mathcal{F}_i) + \mathcal{Q}_2(\mathcal{F}_i).$$

Combining (12.5.31), (12.5.32) and Lemma 12.5.3, we arrive at (12.5.30)$_1$. The proof is complete.

**12.5.6 Lemma.** *Let $\mathcal{M}$ (or $\mathcal{N}$) be a 1-characteristic (or 2-characteristic) tree rooted at $I_0$ . Then*

$(12.5.33)_1$

$$\mathcal{P}_1(\mathcal{M}) + \mathcal{Q}_1(\mathcal{M}) \le c\delta^2(1 + V_{\mathcal{M}})\{TV_{[p_-^{I_0}(0),p_+^{I_0}(0)]}z(\cdot,0) + \mathcal{P}_2(\mathcal{F}_{I_0}) + \mathcal{Q}_2(\mathcal{F}_{I_0})\},$$

*or*

$(12.5.33)_2$

$$\mathcal{P}_2(\mathcal{N}) + \mathcal{Q}_2(\mathcal{N}) \le c\delta^2(1 + W_{\mathcal{N}})\{TV_{[q_-^{I_0}(0),q_+^{I_0}(0)]}w(\cdot,0) + \mathcal{P}_1(\mathcal{G}_{I_0}) + \mathcal{Q}_1(\mathcal{G}_{I_0})\},$$

*where* $V_{\mathcal{M}}$ *(or* $W_{\mathcal{N}}$*) denotes the maximum of*

$$(12.5.34)_1 \qquad \sum_{i=0}^{m-1}\{|z_-^{I_i I_{i+1}} - z_S^{I_{i+1}}| + |w_-^{I_i I_{i+1}} - w_S^{I_{i+1}}|\}$$

*or*

$$(12.5.34)_2 \qquad \sum_{i=0}^{m-1}\{|z_+^{I_i I_{i+1}} - z_S^{I_{i+1}}| + |w_+^{I_i I_{i+1}} - w_S^{I_{i+1}}|\}$$

*over all chains* $\{I_0, \cdots, I_m\}$ *of* $\mathcal{M}$ *(or* $\mathcal{N}$*).*

**Proof.** It will suffice to validate $(12.5.33)_1$, the other case being completely analogous. By virtue of $(12.5.15)_1$ and $(12.5.16)_1$,

$$(12.5.35) \qquad \mathcal{P}_1(\mathcal{M}) \le -\sum_{I\in\mathcal{M}}[\Delta w_*^I - \sum_{J\in\mathcal{C}_I}\Delta w_*^J] + \mathcal{Q}_1(\mathcal{M})$$
$$= \sum_{\substack{\text{maximal}\\ \text{nodes}}} \Delta w_*^K - \Delta w_*^{I_0} + \mathcal{Q}_1(\mathcal{M}).$$

Since $\Delta w_*^K \le 0$, to establish $(12.5.33)_1$ it is sufficient to show

$$(12.5.36) \qquad -\Delta w_*^{I_0} \le c\delta^2\{TV_{[p_-^{I_0}(0),p_+^{I_0}(0)]}z(\cdot,0) + \mathcal{P}_2(\mathcal{F}_{I_0}) + \mathcal{Q}_2(\mathcal{F}_{I_0})\},$$

$$(12.5.37) \quad \mathcal{Q}_1(\mathcal{M}) \le c\delta^2(1 + V_{\mathcal{M}})\{TV_{[p_-^{I_0}(0),p_+^{I_0}(0)]}z(\cdot,0) + \mathcal{P}_2(\mathcal{F}_{I_0}) + \mathcal{Q}_2(\mathcal{F}_{I_0})\}.$$

To demonstrate (12.5.36), we first employ $(12.5.10)_1$ to get

$$(12.5.38) \qquad -\Delta w_*^{I_0} = -f(\Delta z^{I_0}; z_S^{I_0}, w_S^{I_0}) \le c\delta^2 \Delta z^{I_0},$$

and then, to estimate $\Delta z^{I_0}$, we apply Lemma 12.5.4, with $(\hat{x},\hat{t}) = (\tilde{x},\tilde{t}) = I_0$, $\hat{p} = p_-^{I_0}$ and $\tilde{p} = p_+^{I_0}$.

We now turn to the proof of (12.5.37), recalling the definition $(12.5.16)_1$ of $\mathcal{Q}_1(\mathcal{M})$. For any nodes $I \in \mathcal{M}$ and $J \in \mathcal{C}_I$, we use $(12.5.10)_1$ and $(12.5.13)_1$ to get

(12.5.39)
$$\Delta w^{IJ} - \Delta w_*^J = f(\Delta z^{IJ}; z_-^{IJ}, w_-^{IJ}) - f(\Delta z^{IJ}; z_S^J, w_S^J) + f(\Delta z^{IJ}; z_S^J, w_S^J) - f(\Delta z^J; z_S^J, w_S^J).$$

By account of the properties of the function $f$,

(12.5.40)
$$|f(\Delta z^{IJ}; z_-^{IJ}, w_-^{IJ}) - f(\Delta z^{IJ}; z_S^J, w_S^J)| \le c\delta^2 \Delta z^{IJ}\{|z_-^{IJ} - z_S^J| + |w_-^{IJ} - w_S^J|\},$$

(12.5.41)
$$|f(\Delta z^{IJ}; z_S^J, w_S^J) - f(\Delta z^J; z_S^J, w_S^J)| \le c\delta^2 |\Delta z^{IJ} - \Delta z^J|.$$

Thus, to verify (12.5.37) we have to show

(12.5.42)
$$\sum_{I\in\mathcal{M}} \sum_{J\in\mathcal{C}_I} \Delta z^{IJ}\{|z_-^{IJ} - z_S^J| + |w_-^{IJ} - w_S^J|\} \le V_{\mathcal{M}}\{\Delta z^{I_0} + \sum_{I\in\mathcal{M}} \sum_{J\in\mathcal{C}_I} |\Delta z^{IJ} - \Delta z^J|\},$$

(12.5.43)
$$\sum_{I\in\mathcal{M}} \sum_{J\in\mathcal{C}_I} |\Delta z^{IJ} - \Delta z^J| \le c\{TV_{[p_-^{I_0}(0), p_+^{I_0}(0)]} z(\cdot, 0) + \mathcal{P}_2(\mathcal{F}_{I_0}) + \mathcal{Q}_2(\mathcal{F}_{I_0})\}.$$

We tackle (12.5.42) first. We perform the summation starting out from the maximal nodes and moving down towards the root of $\mathcal{M}$. For $L \in \mathcal{M}$, we let $\mathcal{M}_L$ denote the subtree of $\mathcal{M}$ which is rooted at $L$ and contains all $I \in \mathcal{M}$ with $L < I$. For some $K \in \mathcal{M}$, assume

(12.5.44)
$$\sum_{I\in\mathcal{M}_L} \sum_{J\in\mathcal{C}_I} \Delta z^{IJ}\{|z_-^{IJ} - z_S^J| + |w_-^{IJ} - w_S^J|\} \le V_{\mathcal{M}_L}\{\Delta z^L + \sum_{I\in\mathcal{M}_L} \sum_{J\in\mathcal{C}_I} |\Delta z^{IJ} - \Delta z^J|\}$$

holds for every $L \in \mathcal{C}_K$. Since $\Delta z^L \le \Delta z^{KL} + |\Delta z^{KL} - \Delta z^L|$ and

(12.5.45)
$$\sum_{L\in\mathcal{C}_K} \Delta z^{KL} \le \Delta z^K,$$

(12.5.44) implies

(12.5.46)
$$\begin{aligned}\sum_{I\in\mathcal{M}_K}\sum_{J\in\mathcal{C}_I}\Delta z^{IJ}\{|z_-^{IJ}-z_S^J|+|w_-^{IJ}-w_S^J|\}\\ \le\sum_{L\in\mathcal{C}_K}\Delta z^{KL}\{|z_-^{KL}-z_S^L|+|w_-^{KL}-w_S^L|+V_{\mathcal{M}_L}\}\\ +\sum_{L\in\mathcal{C}_K}V_{\mathcal{M}_L}\{|\Delta z^{KL}-\Delta z^L|+\sum_{I\in\mathcal{M}_L}\sum_{J\in\mathcal{C}_I}|\Delta z^{IJ}-\Delta z^J|\}\\ \le V_{\mathcal{M}_K}\{\Delta z^K+\sum_{I\in\mathcal{M}_K}\sum_{J\in\mathcal{C}_I}|\Delta z^{IJ}-\Delta z^J|\}.\end{aligned}$$

Thus, proceeding step by step, we arrive at (12.5.42).

It remains to show (12.5.43). We note that

(12.5.47)
$$\Delta z^{IJ}-\Delta z^J=[z_+^{IJ}-z_E^J]+[z_S^J-z_-^{IJ}].$$

We bound the right-hand side by applying Lemma 12.5.4 twice: First with $(\hat{x},\hat{t})=J$, $(\tilde{x},\tilde{t})=I$, $\hat{p}=p_+^J$, $\tilde{p}=p_+^{IJ}$, and then with $(\hat{x},\hat{t})=I$, $(\tilde{x},\tilde{t})=J$, $\hat{p}=p_-^{IJ}$, $\tilde{p}=p_-^J$. In either case, the arc of $\chi_J^-$ joining $I$ to $J$ serves as $\bar{t}$. We combine the derivation of $(12.5.22)_1$ for the two cases: The characteristic $\phi_-^K$ issuing from any point $K$ on the graph of $p_+^J$ is always intercepted by the graph of $p_+^{IJ}$; never by the graph of $\chi_J^-$. On the other hand, $\phi_-^K$ issuing from points $K$ on the graph of $p_-^{IJ}$ and crossing the graph of $\chi_J^-$, may be prolonged until they intersect the graph of $p_+^{IJ}$. Consequently, the contribution of the common $\bar{t}$ drops out and we are left with the estimate

(12.5.48)
$$\begin{aligned}|\Delta z^{IJ}-\Delta z^J|\le TV_{[p_-^{IJ}(0),p_-^J(0)]}z(\cdot,0)+TV_{[p_+^J(0),p_+^{IJ}(0)]}z(\cdot,0)\\ +\mathcal{P}_2(\mathcal{F}_{IJ})+\mathcal{Q}_2(\mathcal{F}_{IJ}),\end{aligned}$$

with $\mathcal{F}_{IJ}$ defined through

(12.5.49)
$$\mathcal{F}_{IJ}=\{(x,t):0\le t<t_I,\ \ p_-^{IJ}(t)\le x\le p_+^{IJ}\}\cap\bar{\mathcal{F}}_J^C.$$

When $(I,J)$ and $(K,L)$ are any two distinct links (possibly with $I=K$), the intervals $(p_-^{IJ}(0),p_-^J(0))$, $(p_+^J(0),p_+^{IJ}(0))$, $(p_-^{KL}(0),p_-^L(0))$ and $(p_+^L(0),p_+^{KL}(0))$ are pairwise disjoint; likewise, the interiors of the sets $\mathcal{F}_{IJ}$ and $\mathcal{F}_{KL}$ are disjoint. Therefore, by virtue of Lemma 12.5.3, tallying (12.5.48) over $J\in\mathcal{C}_I$ and then over $I\in\mathcal{M}$ yields (12.5.43). The proof is complete.

**12.5.7 Lemma.** *Under the assumptions of Theorem* 12.5.1, *if $\mathcal{H}$ denotes the region bordered by the graphs of $\zeta_\ell$, $\xi_r$, $t^*$, and the $x$-axis, then*

(12.5.50) $$\mathcal{P}_1(\mathcal{H}) + \mathcal{Q}_1(\mathcal{H}) + \mathcal{P}_2(\mathcal{H}) + \mathcal{Q}_2(\mathcal{H}) \le c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)\}.$$

**Proof.** Consider any family $\mathcal{J}$ of disjoint 1-characteristic trees $\mathcal{M}$ contained in $\mathcal{H}$. If $I$ and $J$ are the roots of any two trees in $\mathcal{J}$, $(p_-^I(0), p_+^I(0))$ and $(p_-^J(0), p_+^J(0))$ are disjoint intervals contained in $(\zeta_\ell(0), \xi_\ell(0))$; also $\mathcal{F}_I$ and $\mathcal{F}_J$ are subsets of $\mathcal{H}$ with disjoint interiors. Consequently, by combining Lemmas 12.5.3 and 12.5.6 we deduce

$(12.5.51)_1$

$$\mathcal{P}_1(\mathcal{H}) + \mathcal{Q}_1(\mathcal{H}) \le c\delta^2(1 + V_{\mathcal{H}})\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + \mathcal{P}_2(\mathcal{H}) + \mathcal{Q}_2(\mathcal{H})\},$$

where $V_{\mathcal{H}}$ denotes the supremum of the total variation of the trace of $(z, w)$ over all 2-characteristics with graph contained in $\mathcal{H}$.

Similarly,

$(12.5.51)_2$

$$\mathcal{P}_2(\mathcal{H}) + \mathcal{Q}_2(\mathcal{H}) \le c\delta^2(1 + W_{\mathcal{H}})\{TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0) + \mathcal{P}_1(\mathcal{H}) + \mathcal{Q}_1(\mathcal{H})\},$$

where $W_{\mathcal{H}}$ stands for the supremum of the total variation of the trace of $(z, w)$ over all 2-characteristics with graph contained in $\mathcal{H}$.

The constants in $(12.5.30)_1$ and $(12.5.30)_2$ do not depend on the particular $t^*$, so long as $\mathcal{H}$ remains fixed. In particular, we may apply these estimates for $t^*$ any 1-characteristic or 2-characteristic, contained in $\mathcal{H}$. Therefore,

$(12.5.52)_1$ $$(1 - c\delta^2)V_{\mathcal{H}} \le TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0) + 2\{\mathcal{P}_1(\mathcal{H}) + \mathcal{Q}_1(\mathcal{H}) + \mathcal{P}_2(\mathcal{H}) + \mathcal{Q}_2(\mathcal{H})\},$$

$(12.5.52)_2$ $$(1 - c\delta^2)W_{\mathcal{H}} \le TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0) + 2\{\mathcal{P}_1(\mathcal{H}) + \mathcal{Q}_1(\mathcal{H}) + \mathcal{P}_2(\mathcal{H}) + \mathcal{Q}_2(\mathcal{H})\}.$$

Combining $(12.5.51)_1$, $(12.5.51)_2$, $(12.5.52)_1$, $(12.5.52)_2$ and recalling (12.5.3), we deduce (12.5.50), provided $\delta$ is sufficiently small. This completes the proof.

We now combine Lemmas 12.5.5 and 12.5.7. Since $\mathcal{F}$ and $\mathcal{G}$ are subsets of $\mathcal{H}$, $(12.5.30)_1$, $(12.5.30)_2$ and (12.5.50) together imply $(12.5.4)_1$ and $(12.5.4)_2$. The assertion of Theorem 12.5.1 has thus been established.

In addition to serving as a stepping stone in the proof of Theorem 12.5.1, Lemma 12.5.7 reveals that the amount of self-interaction of waves of the first and second characteristic family, measured by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, as well as the amount of mutual interaction of waves of opposite families, measured by $\mathcal{Q}_1$ and $\mathcal{Q}_2$, are bounded and controlled by the total variation of the initial data.

In our derivation of (12.5.4), the initial data were regarded as multi-valued and their total variation was evaluated for the "most unfavorable" selection of allowable values. According to this convention, the set of values of $z(x,0)$ is either confined between $z(x-,0)$ and $z(x+,0)$ or else it lies within $c|w(x+,0) - w(x-,0)|^3$ distance from $z(x+,0)$; and an analogous property holds for $w(x,0)$. Consequently, (12.5.4) will still hold, with readjusted constant $c$, when $(z(\cdot,0), w(\cdot,0))$ are renormalized to be single-valued, for example continuous from the right at $\xi_r(0)$ and at $\zeta_r(0)$ and continuous from the left at any other point.

## 12.6 Spreading of Rarefaction Waves

In Section 11.2 we saw that the spreading of rarefaction waves induces one-sided Lipschitz conditions on solutions of genuinely nonlinear scalar conservation laws. Here we shall encounter a similar effect in the context of our system (12.1.1) of two conservation laws. We shall see that the spreading of 1- (or 2-) rarefaction waves acts to reduce the falling (or rising) slope of the corresponding Riemann invariant $z$ (or $w$). Because of intervening wave interactions, this mechanism is no longer capable of sustaining one-sided Lipschitz conditions, as in the scalar case; it still manages, however, to keep the total variation of solutions bounded, independently of the initial data.

Let us consider again the solution $(z, w)$ discussed in the previous section, with small oscillation (12.5.1). The principal result is

**12.6.1 Theorem.** *For any* $-\infty < x < y < \infty$ *and* $t > 0$,

$$TV_{[x,y]}z(\cdot,t) + TV_{[x,y]}w(\cdot,t) \le b\,\frac{y-x}{t} + \beta\delta, \tag{12.6.1}$$

*where* $b$ *and* $\beta$ *are constants that may depend on* $F$ *but are independent of the initial data.*

The proof of the above theorem will be partitioned into several steps. The notation introduced in Section 12.5 will be used here freely. In particular, as before, $c$ will stand for a generic constant that may depend on $F$ but is independent of $\delta$.

**12.6.2 Lemma.** *Fix* $\bar{t} > 0$ *and pick any* $-\infty < x_\ell < x_r < \infty$, *with* $x_r - x_\ell$ *small compared to* $\bar{t}$. *Construct the minimal (or maximal) backward* 1-*(or* 2-*) characteristics* $\xi_\ell(\cdot)$, $\xi_r(\cdot)$ *(or* $\zeta_\ell(\cdot)$, $\zeta_r(\cdot)$*) emanating from* $(x_\ell,\bar{t})$, $(x_r,\bar{t})$, *and let* $\mathcal{F}$ *(or* $\mathcal{G}$*) denote the region bordered by the graphs of* $\xi_\ell$, $\xi_r$ *(or* $\zeta_\ell$, $\zeta_r$*) and the time lines* $t=\bar{t}$ *and* $t=\bar{t}/2$. *Then*

$$z(x_\ell,\bar{t}) - z(x_r,\bar{t}) \le \hat{c}\exp(\bar{c}\delta\bar{V})\frac{x_r - x_\ell}{\bar{t}} + \mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F}), \tag*{(12.6.2)$_1$}$$

*or*

$$w(x_r,\bar{t}) - w(x_\ell,\bar{t}) \le \hat{c}\exp(\bar{c}\delta\bar{V})\frac{x_r - x_\ell}{\bar{t}} + \mathcal{P}_1(\mathcal{G}) + \mathcal{Q}_1(\mathcal{G}), \tag*{(12.6.2)$_2$}$$

*where $\bar{V}$ denotes the total variation of the trace of $w$ (or $z$) along $\xi_\ell(\cdot)$ (or $\zeta_r(\cdot)$) over the interval $[\frac{1}{2}\bar{t}, \bar{t}]$.*

**Proof.** It will suffice to show $(12.6.2)_1$. Let $(z_\ell(\cdot), w_\ell(\cdot))$ and $(z_r(\cdot), w_r(\cdot))$ denote the trace of $(z, w)$ along $\xi_\ell(\cdot)$ and $\xi_r(\cdot)$, respectively.

We consider the infimum $\tilde{\mu}$ and the supremum $\bar{\mu}$ of the characteristic speed $\mu(z, w)$ over the range of the solution. The straight lines with slope $\tilde{\mu}$ and $\bar{\mu}$ emanating from the point $(\xi_r(t), t)$, $t \in [\frac{1}{2}\bar{t}, \bar{t}]$, are intercepted by $\xi_\ell(\cdot)$ at time $f(t)$ and $g(t)$, respectively. Both functions $f$ and $g$ are Lipschitz with slope $1 + O(\delta)$, and

$$0 \le g(t) - f(t) \le c_1\delta[\xi_r(t) - \xi_\ell(f(t))]. \tag{12.6.3}$$

The map that carries $(\xi_r(t), t)$ to $(\xi_\ell(f(t)), f(t))$ induces a pairing of points of the graphs of $\xi_\ell$ and $\xi_r$. From

$$\xi_r(t) - \xi_\ell(f(t)) = \tilde{\mu}[t - f(t)], \tag{12.6.4}$$

we obtain

$$\dot{f}(t) = 1 - \frac{1}{\tilde{\mu} - \dot{\xi}_\ell(f(t))}[\dot{\xi}_r(t) - \dot{\xi}_\ell(f(t))], \tag{12.6.5}$$

$$\frac{d}{dt}[\xi_r(t) - \xi_\ell(f(t))] = \frac{\tilde{\mu}}{\tilde{\mu} - \dot{\xi}_\ell(f(t))}[\dot{\xi}_r(t) - \dot{\xi}_\ell(f(t))], \tag{12.6.6}$$

almost everywhere on $[\frac{1}{2}\bar{t}, \bar{t}]$. In order to bound the right-hand side of (12.6.6) from below, we begin with

$$\begin{aligned} \dot{\xi}_r(t) - \dot{\xi}_\ell(f(t)) &= \lambda(z_r(t), w_r(t)) - \lambda(z_\ell(f(t)), w_\ell(f(t))) \\ &= \bar{\lambda}_z[z_r(t) - z_\ell(f(t))] + \bar{\lambda}_w[w_r(t) - w_\ell(f(t))]. \end{aligned} \tag{12.6.7}$$

By virtue of Theorem 12.4.1,

$$z_r(t) - z_\ell(f(t)) \le z(x_r, \bar{t}) - z(x_\ell, \bar{t}) - \sum[z_r(\tau+) - z_r(\tau-)], \tag{12.6.8}$$

where the summation runs over the set of jump points of $z_r(\cdot)$ inside the interval $(t, \bar{t})$. As in the proof of Lemma 12.5.4, with each one of these jump points $\tau$ one may associate the trivial 2-characteristic tree $\mathcal{N}$ which consists of the single node $(\xi_r(\tau), \tau)$ so as to deduce

$$-\sum[z_r(\tau+) - z_r(\tau-)] \le \mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F}). \tag{12.6.9}$$

For $t \in [\frac{1}{2}\bar{t}, \bar{t}]$, we construct the maximal backward 2-characteristic emanating from $(\xi_r(t), t)$, which is intercepted by $\xi_\ell(\cdot)$ at time $h(t)$; $f(t) \le h(t) \le g(t)$. By account of Theorem 12.4.1, $w_\ell(h(t)) \ge w_r(t)$ and so

$$w_r(t) - w_\ell(f(t)) \le w_\ell(h(t)) - w_\ell(f(t)) \le V(f(t)) - V(g(t)), \tag{12.6.10}$$

where $V(\tau)$ measures the total variation of $w_\ell(\cdot)$ over the interval $[\tau, \bar{t})$.

We now integrate (12.6.6) over the interval $(s, \bar{t})$. Recalling that $\bar{\lambda}_z < 0$, $\bar{\lambda}_w < 0$, upon combining (12.6.7), (12.6.8), (12.6.9) and (12.6.10), we deduce

$$(12.6.11)\quad \xi_r(s) - \xi_\ell(f(s)) \le \xi_r(\bar t) - \xi_\ell(f(\bar t)) + c_2^{-1}(\bar t - s)[z(x_r,\bar t) - z(x_\ell,\bar t) + \mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F})] + c_3\int_s^{\bar t}[V(f(t)) - V(g(t))]dt.$$

By interchanging the order of integration,

$$(12.6.12)\quad \int_s^{\bar t}[V(f(t)) - V(g(t))]dt = -\int_s^{\bar t}\int_{f(t)}^{g(t)} dV(\tau)dt$$

$$\le -\int_{f(s)}^{f(\bar t)}[f^{-1}(\tau) - g^{-1}(\tau)]dV(\tau) - \int_{f(\bar t)}^{g(\bar t)}[\bar t - g^{-1}(\tau)]dV(\tau)$$

$$= -\int_s^{\bar t}[t - g^{-1}(f(t))]dV(f(t)) - \int_{f(\bar t)}^{g(\bar t)}[\bar t - g^{-1}(\tau)]dV(\tau).$$

On account of (12.6.3),

$$(12.6.13)\qquad t - g^{-1}(f(t)) \le c_4\delta[\xi_r(t) - \xi_\ell(f(t))], \qquad \frac{\bar t}{2} \le t \le \bar t,$$

$$(12.6.14)\qquad \bar t - g^{-1}(\tau) \le c_4\delta[\xi_r(\bar t) - \xi_\ell(f(\bar t))], \qquad f(\bar t) \le \tau \le g(\bar t),$$

and hence (12.6.11) yields

$$(12.6.15)\quad \xi_r(s) - \xi_\ell(f(s)) \le \exp(c_3c_4\delta\bar V)[\xi_r(\bar t) - \xi_\ell(f(\bar t))] + c_2^{-1}(\bar t - s)[z(x_r,\bar t) - z(x_\ell,\bar t) + \mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F})] - c_3c_4\delta\int_s^{\bar t}[\xi_r(t) - \xi_\ell(f(t))]dV(f(t)),$$

for any $s \in [\frac{3}{4}\bar t, \bar t]$. Integrating the above, Gronwall-type, inequality, we obtain

$$(12.6.16)\qquad \xi_r(s) - \xi_\ell(f(s)) \le \exp(2c_3c_4\delta\bar V)[\xi_r(\bar t) - \xi_\ell(f(\bar t))]$$

$$+c_2^{-1}\Big[\int_s^{\bar t}\exp\{c_3c_4\delta[V(f(s)) - V(f(\tau))]\}d\tau\Big][z(x_r,\bar t) - z(x_\ell,\bar t) + \mathcal{P}_2(\mathcal{F}) + \mathcal{Q}_2(\mathcal{F})].$$

We apply (12.6.16) for $s = \frac{3}{4}\bar t$. The left-hand side of (12.6.16) is nonnegative. Also, $\xi_r(\bar t) - \xi_\ell(f(\bar t)) \le c_5(x_r - x_\ell)$. Therefore, (12.6.16) implies $(12.6.2)_1$ with constants $\bar c = 2c_3c_4$ , $\hat c = 4c_2c_5$ . The proof is complete.

In what follows, we shall be operating under the assumption that the constants $\bar V$ appearing in $(12.6.2)_1$ and $(12.6.2)_2$ satisfy

(12.6.17) $$\bar{c}\delta\bar{V} \le \log 2.$$

This will certainly be the case, by virtue of Theorem 12.5.1, when the initial data satisfy (12.5.3) with $a$ sufficiently small. Furthermore, because of the finite domain of dependence property, (12.6.17) shall hold for $\bar{t}$ sufficiently small, even when the initial data have only locally bounded variation and satisfy (12.5.2) with $\delta$ sufficiently small. It will be shown below that (12.6.17) actually holds for any $\bar{t} > 0$, provided only the initial data have sufficiently small oscillation, i.e., $\delta$ is small.

**12.6.3 Lemma.** *For any $-\infty < \bar{x} < \bar{y} < \infty$ and $\bar{t} > 0$,*

(12.6.18)
$$NV_{[\bar{x},\bar{y}]}z(\cdot,\bar{t}) + PV_{[\bar{x},\bar{y}]}w(\cdot,\bar{t}) \le 4\hat{c}\,\frac{\bar{y}-\bar{x}}{\bar{t}} + c\delta^2\left\{TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}z\left(\cdot,\frac{\bar{t}}{2}\right) + TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}w\left(\cdot,\frac{\bar{t}}{2}\right)\right\},$$

(12.6.19)
$$TV_{[\bar{x},\bar{y}]}z(\cdot,\bar{t}) + TV_{[\bar{x},\bar{y}]}w(\cdot,\bar{t}) \le 8\hat{c}\,\frac{\bar{y}-\bar{x}}{\bar{t}} + 8\delta + c\delta^2\left\{TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}z\left(\cdot,\frac{\bar{t}}{2}\right) + TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}w\left(\cdot,\frac{\bar{t}}{2}\right)\right\},$$

*where $\bar{\lambda}$ is the infimum of $\lambda(z,w)$ and $\bar{\mu}$ is the supremum of $\mu(z,w)$ over the range of the solution.*

**Proof.** By combining $(12.6.2)_1$, $(12.6.2)_2$, (12.6.17) and Lemma 12.5.3, we immediately infer

(12.6.20)
$$NV_{[\bar{x},\bar{y}]}z(\cdot,\bar{t}) + PV_{[\bar{x},\bar{y}]}w(\cdot,\bar{t}) \le 4\hat{c}\,\frac{\bar{y}-\bar{x}}{\bar{t}} + 2[\mathcal{P}_1(\mathcal{H}) + \mathcal{Q}_1(\mathcal{H}) + \mathcal{P}_2(\mathcal{H}) + \mathcal{Q}_2(\mathcal{H})],$$

where $\mathcal{H}$ denotes the region bordered by the graph of the minimal backward 1-characteristic $\xi(\cdot)$ emanating from $(\bar{y},\bar{t})$, the graph of the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from $(\bar{x},\bar{t})$, and the time lines $t=\bar{t}$ and $t=\bar{t}/2$.

We estimate $\mathcal{P}_1(\mathcal{H}) + \mathcal{Q}_1(\mathcal{H}) + \mathcal{P}_2(\mathcal{H}) + \mathcal{Q}_2(\mathcal{H})$ by applying Lemma 12.5.7, with the time origin shifted from $t=0$ to $t=\bar{t}/2$. This yields (12.6.18).

Since total variation is the sum of negative variation and positive variation, while the difference of negative variation and positive variation is majorized by the oscillation, (12.6.18) together with (12.5.1) yield (12.6.19). The proof is complete.

**Proof of Theorem 12.6.1** In order to establish (12.6.1), we first write (12.6.19) with $\bar{t}=t$, $\bar{x}=x$ and $\bar{y}=y$. To estimate the right-hand side of the resulting inequality, we reapply (12.6.19), for $\bar{t}=\frac{1}{2}t$, $\bar{x}=x-\frac{1}{2}\bar{\mu}t$ and $\bar{y}=x-\frac{1}{2}\bar{\lambda}t$. This yields

$$(12.6.21)\qquad TV_{[x-\frac{1}{2}\bar{\mu}t,y-\frac{1}{2}\bar{\lambda}t]}z\left(\cdot\,,\frac{t}{2}\right)+TV_{[x-\frac{1}{2}\bar{\mu}t,y-\frac{1}{2}\bar{\lambda}t]}w\left(\cdot\,,\frac{t}{2}\right)$$
$$\leq 16\hat{c}\,\frac{y-x}{t}+8\hat{c}(\bar{\mu}-\bar{\lambda})+8\delta$$
$$+\,c\delta^2\left\{TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}z\left(\cdot\,,\frac{t}{4}\right)+TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}w\left(\cdot\,,\frac{t}{4}\right)\right\}.$$

Similarly, in order to estimate the right-hand side of (12.6.21), we apply (12.6.19) with $\bar{t}=\frac{1}{4}t,\ \bar{x}=x-\frac{3}{4}\bar{\mu}t$ and $\bar{y}=y-\frac{3}{4}\bar{\lambda}t$. We thus obtain

$$(12.6.22)\qquad TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}z\left(\cdot\,,\frac{t}{4}\right)+TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}w\left(\cdot\,,\frac{t}{4}\right)$$
$$\leq 32\hat{c}\,\frac{y-x}{t}+24\hat{c}(\bar{\mu}-\bar{\lambda})+8\delta$$
$$+\,c\delta^2\left\{TV_{[x-\frac{7}{8}\bar{\mu}t,y-\frac{7}{8}\bar{\lambda}t]}z\left(\cdot\,,\frac{t}{8}\right)+TV_{[x-\frac{7}{8}\bar{\mu}t,y-\frac{7}{8}\bar{\lambda}t]}w\left(\cdot\,,\frac{t}{8}\right)\right\}.$$

Continuing on and passing to the limit, we arrive at (12.6.1) with

$$(12.6.23)\qquad b=\frac{8\hat{c}}{1-2c\delta^2}\ ,\qquad \beta=\frac{8}{1-c\delta^2}+\frac{8c\hat{c}\delta(\bar{\mu}-\bar{\lambda})}{(1-c\delta^2)(1-2c\delta^2)}\ .$$

The above derivations hinge on the assumption that (12.6.17) holds; hence, in order to complete the proof, we now have to verify this condition. Recalling the definition of $\bar{V}$ in Lemma 12.6.2 and applying Theorem 12.5.1, with time origin shifted from 0 to $\frac{1}{2}t$, we deduce

$$(12.6.24)\qquad \bar{V}\leq c\sup_{\bar{x}}\{TV_{[\bar{x}-\frac{1}{2}\bar{\mu}t,\bar{x}-\frac{1}{2}\bar{\lambda}t]}z(\cdot\,,\frac{t}{2})+TV_{[\bar{x}-\frac{1}{2}\bar{\mu}t,\bar{x}-\frac{1}{2}\bar{\lambda}t]}w(\cdot\,,\frac{t}{2})\}.$$

We estimate the right-hand side of (12.6.24) by means of (12.6.1), which yields

$$(12.6.25)\qquad \bar{V}\leq cb(\bar{\mu}-\bar{\lambda})+c\beta\delta,$$

so that (12.6.17) is indeed satisfied, provided $\delta$ is sufficiently small. The proof is complete.

We now show that initial data of sufficiently small oscillation, but arbitrarily large total variation, induce the $L^\infty$ bound (12.5.1), which has been assumed throughout this section.

**12.6.4 Theorem.** *There is a positive constant $\gamma$, depending solely on $F$, such that solutions generated by initial data with small oscillation*

$$(12.6.26)\qquad |z(x,0)|+|w(x,0)|<\gamma\delta^2,\qquad -\infty<x<\infty,$$

*but unrestricted total variation, satisfy* (12.5.1).

**Proof.** Assuming (12.6.26) holds, with $\gamma$ sufficiently small, we will demonstrate that $-\delta < z(x,t) < \delta$ and $-\delta < w(x,t) < \delta$ on the upper half-plane. Arguing by contradiction, suppose any one of the above four inequalities is violated at some point, say for example $z(\bar{x},\bar{t}) \geq \delta$.

We determine $\bar{y}$ through $8\hat{c}(\bar{y}-\bar{x}) = \delta\bar{t}$, where $\hat{c}$ is the constant appearing in $(12.6.2)_1$ , and apply (12.6.18). The first term on the right-hand side of (12.6.18) is here bounded by $\frac{1}{4}\delta$; the second term is bounded by $\tilde{c}\delta^2$, on account of (12.6.1). Consequently, for $\delta$ sufficiently small, the negative (decreasing) variation of $z(\cdot,\bar{t})$ over the interval $[\bar{x},\bar{y}]$ does not exceed $\frac{1}{2}\delta$. It follows that $z(x,\bar{t}) \geq \frac{1}{2}\delta$, for all $x \in [\bar{x},\bar{y}]$. In particular,

$$\int_{\bar{x}}^{\bar{y}} [|z(x,\bar{t})| + |w(x,\bar{t})|]dx \geq (\bar{y}-\bar{x})\frac{\delta}{2} = \frac{1}{16\hat{c}}\delta^2\bar{t}. \tag{12.6.27}$$

We now appeal to the $L^1$ estimate (12.8.3), which will be established in Section 12.8, Lemma 12.8.2, and combine it with (12.6.26) to deduce

$$\int_{\bar{x}}^{\bar{y}} [|z(x,\bar{t})| + |w(x,\bar{t}|]dx \leq 4[(\bar{y}-\bar{x}) + 2c\bar{t}]\gamma\delta^2 = \gamma[\frac{\delta}{2\hat{c}} + 8c]\delta^2\bar{t}. \tag{12.6.28}$$

It is clear that for $\gamma$ sufficiently small (12.6.27) is inconsistent with (12.6.28), and this provides the desired contradiction. The proof is complete.

In conjunction with the compactness properties of $BV$ functions, recounted in Section 1.7, the estimate (12.6.1) indicates that, starting out with solutions with initial data of locally bounded variation, one may construct, via completion, $BV_{\text{loc}}$ solutions under initial data that are merely in $L^\infty$, with sufficiently small oscillation. Thus, the solution operator of genuinely nonlinear systems of two consesrvation laws regularizes the initial data by the mechanism already encountered in the context of the genuinely nonlinear scalar conservation law (Theorem 11.2.2).

## 12.7 Regularity of Solutions

The information collected thus far paints the following picture for the regularity of solutions:

**12.7.1 Theorem.** *Let $U(x,t)$ be an admissible $BV$ solution of the genuinely nonlinear system* (12.1.1) *of two conservation laws, with the properties recounted in the previous sections. Then*

(a) *Any point $(\bar{x},\bar{t})$ of approximate continuity is a point of continuity of $U$.*
(b) *Any point $(\bar{x},\bar{t})$ of approximate jump discontinuity is a point of (classical) jump discontinuity of $U$.*
(c) *Any irregular point $(\bar{x},\bar{t})$ is the focus of a centered compression wave of either, or both, characteristic families, and/or a point of interaction of shocks of the same or opposite characteristic families.*

(d) *The set of irregular points is (at most) countable.*

**Proof.** Assertions (a), (b) and (c) are corollaries of Theorem 12.3.3. In particular, $(\bar{x}, \bar{t})$ is a point of approximate continuity if and only if $(z_W, w_W) = (z_E, w_E)$, in which case all four limits $(z_W, w_W)$, $(z_E, w_E)$, $(z_N, w_N)$ and $(z_S, w_S)$ coincide. When $(z_W, w_W) \neq (z_E, w_E)$, then $(\bar{x}, \bar{t})$ is a point of approximate jump discontinuity in the 1-shock set if $(z_W, w_W) = (z_S, w_S)$, $(z_E, w_E) = (z_N, w_N)$; or a point of approximate jump discontinuity in the 2-shock set if $(z_W, w_W) = (z_N, w_N)$, $(z_E, w_E) = (z_S, w_S)$; and an irregular point in all other cases.

To verify asssertion (d), assume the irregular point $I = (\bar{x}, \bar{t})$ is a node of some 1-characteristic tree $\mathcal{M}$ or a 2-characteristic tree $\mathcal{N}$. If $I$ is the focusing point of a centered 1-compression wave and/or point of interaction of 1-shocks, then, by virtue of $(12.5.15)_1$, $I$ will register a positive contribution to $\mathcal{P}_1(\mathcal{M})$. Similarly, if $I$ is the focusing point of a centered 2-compression wave and/or point of interaction of 2-shocks, then, by account of $(12.5.15)_2$, $I$ will register a positive contribution to $\mathcal{P}_2(\mathcal{N})$. Finally, suppose $I$ is a point of interaction of a 1-shock with a 2-shock. We adjoin to $\mathcal{M}$ an additional node $K$ lying on the graph of $\chi_-^I$ very close to $I$. Then $|\Delta w^{KI} - \Delta w_*^I| > 0$ and so, by $(12.5.16)_1$, we get a positive contribution to $\mathcal{Q}_1(\mathcal{M})$. Since the total amount of wave interaction is bounded, by virtue of Lemma 12.5.7, we conclude that the set of irregular points is necessarily (at most) countable. This completes the proof.

An analog of Theorem 11.3.5 is also in force here:

**12.7.2 Theorem.** *Assume the set $\mathcal{C}$ of points of continuity of the solution $U$ has nonempty interior $\mathcal{C}^0$. Then $U$ is locally Lipschitz continuous on $\mathcal{C}^0$.*

**Proof.** We verify that $z$ is locally Lipschitz continuous on $\mathcal{C}^0$. Assume $(\bar{x}, \bar{t}) \in \mathcal{C}^0$ and $\mathcal{C}$ contains a rectangle $\{(x,t) : |x - \bar{x}| < kp,\ \ |t - \bar{t}| < p\}$, with $p > 0$ and $k$ large compared to $|\lambda|$ and $\mu$. By shifting the axes, we may assume, without loss of generality, that $\bar{t} = p$. We fix $\bar{y} > \bar{x}$, where $\bar{y} - \bar{x}$ is small compared to $p$, and apply $(12.6.2)_1$, with $x_\ell = \bar{x}$, $x_r = \bar{y}$. Since the solution is continuous in the rectangle, both $\mathcal{P}_2(\mathcal{F})$ and $\mathcal{Q}_2(\mathcal{F})$ vanish and so, recalling (12.6.17),

$$z(\bar{x}, \bar{t}) - z(\bar{y}, \bar{t}) \leq \frac{2\hat{c}}{p}(\bar{y} - \bar{x}). \tag{12.7.1}$$

The functions $(\hat{z}, \hat{w})(x,t) = (z,w)(\bar{x} + \bar{y} - x, 2p - t)$ are Riemann invariants of another solution $\hat{U}$ which is continuous, and thereby admissible, on the rectangle $\{(x,t) : |x - \bar{y}| < kp,\ \ |t - \bar{t}| < p\}$. Applying (12.7.1) to $\hat{z}$ yields

$$z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t}) = \hat{z}(\bar{x}, \bar{t}) - \hat{z}(\bar{y}, \bar{t}) \leq \frac{2\hat{c}}{p}(\bar{y} - \bar{x}). \tag{12.7.2}$$

We now fix $\bar{s} > \bar{t}$, with $\bar{s} - \bar{t}$ small compared to $p$. We construct the minimal backward 1-characteristic $\xi$ emanating from $(\bar{x}, \bar{s})$, which is intercepted by the $\bar{t}$-time line at the point $\bar{y} = \xi(\bar{t})$, where $0 < \bar{y} - \bar{x} \leq -\bar{\lambda}(\bar{s} - \bar{t})$. By Theorem 12.4.1, $z(\bar{x}, \bar{s}) = z(\bar{y}, \bar{t})$ and so, by virtue of (12.7.1) and (12.7.2),

$$|z(\bar{x},\bar{s}) - z(\bar{x},\bar{t})| \le \frac{2\hat{c}}{p}(\bar{y}-\bar{x}) \le -\frac{2\bar{\lambda}\hat{c}}{p}(\bar{s}-\bar{t}). \tag{12.7.3}$$

Thus $z$ is Lipschitz.

A similar argument shows that $w$ is also Lipschitz in $\mathcal{C}^0$. This completes the proof.

## 12.8 Initial Data in $L^1$

Recall that, by virtue of Theorem 11.5.2, initial data in $L^1$ induce decay of solutions of genuinely nonlinear scalar conservation laws, as $t \to \infty$, at the rate $O(t^{-\frac{1}{2}})$. The aim here is to establish an analogous property for solutions of genuinely nonlinear systems of two conservation laws. Accordingly, we consider a solution $(z(x,t), w(x,t))$ of small oscillation (12.5.1), with initial values of unrestricted total variation lying in $L^1(-\infty,\infty)$:

$$L = \int_{-\infty}^{\infty} [|z(x,0)| + |w(x,0)|]dx \quad < \infty. \tag{12.8.1}$$

The principal result is

**12.8.1 Theorem.** *As $t \to \infty$,*

$$(z(x,t), w(x,t)) = O(t^{-\frac{1}{2}}), \tag{12.8.2}$$

*uniformly in $x$ on $(-\infty,\infty)$.*

The proof will be partitioned into several steps.

**12.8.2 Lemma.** *For any $\bar{t} \in [0,\infty)$, and $-\infty < \bar{x} < \bar{y} < \infty$,*

$$\int_{\bar{x}}^{\bar{y}} [|z(x,\bar{t})| + |w(x,\bar{t})|]dx \le 4\int_{\bar{x}-c\bar{t}}^{\bar{y}+c\bar{t}} [|z(x,0)| + |w(x,0)|]dx. \tag{12.8.3}$$

*In particular, $(z(\cdot,\bar{t}), w(\cdot,\bar{t}))$ are in $L^1(-\infty,\infty)$.*

**Proof.** We construct a Lipschitz continuous entropy $\eta$ by solving the Goursat problem for (12.2.2) with prescribed data

$$\begin{cases} \eta(z,0) = |z| + \alpha z^2, & -\infty < z < \infty, \\ \eta(0,w) = |w| + \alpha z^2, & -\infty < w < \infty, \end{cases} \tag{12.8.4}$$

where $\alpha$ is a positive constant. From (12.2.3) it follows that, for $\alpha$ sufficiently large, $\eta$ is a convex function of $U$ on some neighborhood of the origin containing the range of the solution.

Combining (12.2.2) and (12.8.4), one easily deduces, for $\delta$ small,

$$\frac{1}{2}(|z|+|w|) \leq \eta(z,w) \leq 2(|z|+|w|), \quad -2\delta < z < 2\delta, -2\delta < w < 2\delta. \tag{12.8.5}$$

Furthermore, if $q$ is the entropy flux associated with $\eta$, normalized by $q(0,0)=0$, (12.2.1) and (12.8.5) imply

$$|q(z,w)| \leq c\eta(z,w), \qquad -2\delta < z < 2\delta, \ -2\delta < w < 2\delta. \tag{12.8.6}$$

We now fix $\bar{t} > 0$, $-\infty < \bar{x} < \bar{y} < \infty$ and integrate (12.3.1), for the entropy-entropy flux pair $(\eta, q)$ constructed above, on the trapezoid $\{(x,t) : 0 < t < \bar{t}, \bar{x} - c(\bar{t}-t) < x < \bar{y} + c(\bar{t}-t)\}$. Upon using (12.8.6), this yields

$$\int_{\bar{x}}^{\bar{y}} \eta(z(x,\bar{t}), w(x,\bar{t}))dx \leq \int_{\bar{x}-c\bar{t}}^{\bar{y}+c\bar{t}} \eta(z(x,0), w(x,0))dx. \tag{12.8.7}$$

By virtue of (12.8.5), (12.8.7) implies (12.8.3). The proof is complete.

**12.8.3 Lemma.** *Let* $(\bar{z}(\cdot), \bar{w}(\cdot))$ *denote the trace of* $(z,w)$ *along the minimal (or maximal) backward* 1-(*or* 2-) *characteristic* $\xi(\cdot)$ (*or* $\zeta(\cdot)$) *emanating from any point* $(\bar{y}, \bar{t})$ *of the upper half-plane. Then*

$$\int_0^{\bar{t}} [\bar{z}^2(t) + |\bar{w}(t)|]dt \leq \tilde{c}L, \tag{12.8.8$_1$}$$

*or*

$$\int_0^{\bar{t}} [|\bar{z}(t)| + \bar{w}^2(t)]dt \leq \tilde{c}L. \tag{12.8.8$_2$}$$

**Proof.** It will suffice to verify (12.8.8)$_1$ . Suppose $\eta$ is any Lipschitz continuous convex entropy associated with entropy flux $q$, normalized so that $\eta(0,0) = 0$, $q(0,0) = 0$. We fix $\bar{x} < \bar{y}$ and integrate the inequality (12.3.1) over the region $\{(x,t) : 0 < t < \bar{t}, \ \bar{x} < x < \xi(t)\}$ to get

$$\begin{aligned} &\int_{\bar{x}}^{\bar{y}} \eta(z(x,\bar{t}), w(x,\bar{t}))dx - \int_{\bar{x}}^{\xi(0)} \eta(z(x,0), w(x,0))dx \\ &\qquad + \int_0^{\bar{t}} G(\bar{z}(t), \bar{w}(t))dt - \int_0^{\bar{t}} q(z(\bar{x}+,t), w(\bar{x}+,t))dt \leq 0, \end{aligned} \tag{12.8.9}$$

where $G$ is defined by

$$G(z,w) = q(z,w) - \lambda(z,w)\eta(z,w). \tag{12.8.10}$$

We seek an entropy-entropy flux pair that renders $G(z,w)$ positive definite on $(-2\delta, 2\delta) \times (-2\delta, 2\delta)$. On account of (12.2.1),

$$G_z = -\lambda_z \eta, \tag{12.8.11}$$

$$G_w = [(\mu - \lambda)\eta]_w - \mu_w \eta, \tag{12.8.12}$$

which indicate that $G$ decays fast, at least quadratically, as $z \to 0$, but it may decay more slowly, even linearly, as $w \to 0$.

We construct an entropy $\eta$ by solving the Goursat problem for (12.2.2) with data

$$\begin{cases} \eta(z,0) = 2z + \alpha z^2, & -\infty < z < \infty, \\ \eta(0,w) = |w| + \alpha w^2, & -\infty < w < \infty. \end{cases} \tag{12.8.13}$$

For $\alpha$ sufficiently large, it follows from (12.2.3) that $\eta$ is a convex function of $U$ on some neighborhood of the origin containing the range of the solution. From (12.8.12), (12.2.2) and (12.8.13) we deduce

$$G(0,w) = [\mu(0,0) - \lambda(0,0)]|w| + O(w^2), \tag{12.8.14}$$

$$\eta(z,w) = 2z + |w| + O(z^2 + w^2), \tag{12.8.15}$$

for $(z,w)$ near the origin. Combining (12.8.14) with (12.8.11) and (12.8.15), we conclude

$$G(z,w) = [\mu(0,0) - \lambda(0,0)]|w| - \lambda_z(0,0)z^2 + O(w^2 + |zw| + |z|^3). \tag{12.8.16}$$

We now return to (12.8.9). By account of Lemma 12.8.2, $(z(\cdot,t), w(\cdot,t))$ are in $L^1(-\infty,\infty)$, for all $t \in [0,\bar{t}]$, and hence

$$\liminf_{\bar{x}\to-\infty} \left| \int_0^{\bar{t}} q(z(\bar{x}+,t), w(\bar{x}+,t))dt \right| = 0. \tag{12.8.17}$$

Therefore, (12.8.9), (12.8.17), (12.8.15), (12.8.3) and (12.8.1) together imply

$$\int_0^{\bar{t}} G(\bar{z}(t), \bar{w}(t))dt \le 12L, \tag{12.8.18}$$

provided (12.5.1) holds, with $\delta$ sufficiently small. The assertion $(12.8.8)_1$ now follows easily from (12.8.18), (12.8.16) and (12.1.3). This completes the proof.

Lemma 12.8.3 indicates that along minimal backward 1-characteristics $z$ is $O(t^{-\frac{1}{2}})$ and $w$ is $O(t^{-1})$, while along maximal backward 2-characteristics $z$ is $O(t^{-1})$ and $w$ is $O(t^{-\frac{1}{2}})$. In fact, recalling that $\bar{z}(\cdot)$ and $\bar{w}(\cdot)$ are nonincreasing along minimal and maximal backward 1- and 2-characteristics, respectively, we infer directly from $(12.8.8)_1$ and $(12.8.8)_2$ that the positive parts of $z(x,t)$ and $w(x,t)$ are $O(t^{-\frac{1}{2}})$, as $t \to \infty$. The proof of Theorem 12.8.1 will now be completed by establishing $O(t^{-\frac{1}{2}})$ decay on both sides:

**12.8.4 Lemma.** *For $\delta$ sufficiently small,*

$$z^2(x,t) \le \frac{8\tilde{c}L}{t}\,, \tag{12.8.19$_1$}$$

$$w^2(x,t) \le \frac{8\tilde{c}L}{t}\,, \tag{12.8.19$_2$}$$

*hold, for all* $-\infty < x < \infty\,, 0 < t < \infty$, *where* $\tilde{c}$ *is the constant in* (12.8.8)$_1$ *and* (12.8.8)$_2$ .

**Proof.** Arguing by contradiction, suppose the assertion is false and let $\bar{t} > 0$ be the greatest lower bound of the set of points $t$ on which (12.8.19)$_1$ and/or (12.8.19)$_2$ is violated for some $x$. According to Theorem 12.3.3, the continuation of the solution beyond $\bar{t}$ is initiated by solving Riemann problems along the $\bar{t}$-time line. Consequently, since (12.8.19)$_1$ and/or (12.8.19)$_2$ fail for $t > \bar{t}$, one can find $\bar{y} \in (-\infty, \infty)$ such that

$$z^2(\bar{y},\bar{t}) > \frac{4\tilde{c}L}{\bar{t}}\,, \tag{12.8.20$_1$}$$

and/or

$$w^2(\bar{y},\bar{t}) > \frac{4\tilde{c}L}{\bar{t}}\,. \tag{12.8.20$_2$}$$

For definiteness, assume (12.8.20)$_1$ holds.

Let $(\bar{z}(\cdot), \bar{w}(\cdot))$ denote the trace of $(z, w)$ along the minimal backward 1-characteristic $\xi(\cdot)$ emanating from $(\bar{y}, \bar{t})$. By applying Theorem 12.5.1, with the time origin shifted from $t = 0$ to $t = \bar{t}/2$, we deduce

$$TV_{[\frac{1}{2}\bar{t},\bar{t}]}\bar{w}(\cdot) \le \hat{c}\{TV_{[\bar{y}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}\,z(\cdot\,,\frac{\bar{t}}{2}) + TV_{[\bar{y}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}\,w(\cdot\,,\frac{\bar{t}}{2})\}, \tag{12.8.21}$$

where $\bar{\lambda}$ stands for the infimum of $\lambda(z, w)$ and $\bar{\mu}$ denotes the supremum of $\mu(z, w)$ over the range of the solution. We estimate the right-hand side of (12.8.21) with the help of Theorem 12.6.1, thus obtaining

$$TV_{[\frac{1}{2}\bar{t},\bar{t}]}\bar{w}(\cdot) \le \hat{c}[b(\bar{\mu}-\bar{\lambda}) + \beta\delta]. \tag{12.8.22}$$

By hypothesis,

$$\bar{w}^2(t) \le \frac{16\tilde{c}L}{\bar{t}}\,, \qquad \frac{\bar{t}}{2} \le t < \bar{t}. \tag{12.8.23}$$

We also have $|\bar{z}(t)| \le 2\delta$. Therefore, by applying (12.4.2)$_1$ we deduce

$$\bar{z}^2(\bar{t}-) - \bar{z}^2(t) \le \bar{c}\delta\,\frac{4\tilde{c}L}{\bar{t}}\,, \tag{12.8.24}$$

with $\bar{c} = 64a\hat{c}[b(\bar{\mu}-\bar{\lambda}) + \beta\delta]$.

Since $\bar{z}(\bar{t}-) = z(\bar{y}, \bar{t})$, combining $(12.8.20)_1$ with (12.8.24) yields

$$\bar{z}^2(t) \geq \frac{4\tilde{c}L}{\bar{t}}(1 - \bar{c}\delta), \qquad \frac{\bar{t}}{2} \leq t < \bar{t}. \tag{12.8.25}$$

From (12.8.25),

$$\int_{\frac{\bar{t}}{2}}^{\bar{t}} \bar{z}^2(t)dt \geq 2\tilde{c}L(1 - \bar{c}\delta), \tag{12.8.26}$$

which provides the desired contradiction to $(12.8.8)_1$, when $\delta$ is sufficiently small. The proof is complete.

## 12.9 Initial Data with Compact Support

Here we consider the large time behavior of solutions, with small oscillation (12.5.1), to our genuinely nonlinear system (12.1.1) of two conservation laws under initial data $(z(x,0), w(x,0))$ that vanish outside a bounded interval $[-\ell, \ell]$. We already know, from Section 12.8, that $(z(x,t), w(x,t)) = O(t^{-\frac{1}{2}})$. The aim is to examine the asymptotics in finer scale, establishing the analog of Theorem 11.6.1 on the genuinely nonlinear scalar conservation law. **12.9.1 Theorem.** *Employing the notation introduced in Section* 12.3, *consider the special forward characteristics* $\phi_-(\cdot)$, $\psi_-(\cdot)$ *issuing from* $(-\ell, 0)$ *and* $\phi_+(\cdot)$, $\psi_+(\cdot)$ *issuing from* $(\ell, 0)$. *Then*

(a) *For t large,* $\phi_-$, $\psi_-$, $\phi_+$ *and* $\psi_+$ *propagate according to*

$$\phi_-(t) = \lambda(0,0)t - (p_-t)^{\frac{1}{2}} + O(1), \tag{12.9.1$_1$}$$

$$\psi_+(t) = \mu(0,0)t + (q_+t)^{\frac{1}{2}} + O(1), \tag{12.9.1$_2$}$$

$$\phi_+(t) = \lambda(0,0)t + (p_+t)^{\frac{1}{2}} + O\left(t^{\frac{1}{4}}\right), \tag{12.9.2$_1$}$$

$$\psi_-(t) = \mu(0,0)t - (q_-t)^{\frac{1}{2}} + O\left(t^{\frac{1}{4}}\right), \tag{12.9.2$_2$}$$

*where* $p_-$, $p_+$, $q_-$ *and* $q_+$ *are nonnegative constants.*

(b) *For* $t > 0$ *and either* $x < \phi_-(t)$ *or* $x > \psi_+(t)$,

$$z(x,t) = 0, \quad w(x,t) = 0. \tag{12.9.3}$$

(c) *For t large,*

$$TV_{[\phi_-(t),\psi_+(t)]}\, z(\cdot,t) + TV_{[\phi_-(t),\psi_+(t)]}\, w(\cdot,t) = O\left(t^{-\frac{1}{2}}\right). \tag{12.9.4}$$

(d) *For $t$ large and $\phi_-(t) < x < \phi_+(t)$,*

$$\lambda(z(x,t),0) = \frac{x}{t} + O\left(\frac{1}{t}\right), \tag{12.9.5$_1$}$$

*while for $\psi_-(t) < x < \psi_+(t)$,*

$$\mu(0,w(x,t)) = \frac{x}{t} + O\left(\frac{1}{t}\right). \tag{12.9.5$_2$}$$

(e) *For $t$ large and $x > \phi_+(t)$, if $p_+ > 0$ then*

$$0 \le -z(x,t) \le c[x - \lambda(0,0)t]^{-\frac{3}{2}}, \tag{12.9.6$_1$}$$

*while for $x < \psi_-(t)$, if $q_- > 0$ then*

$$0 \le -w(x,t) \le c[\mu(0,0)t - x]^{-\frac{3}{2}}. \tag{12.9.6$_2$}$$

According to the above proposition, as $t \to \infty$ the two characteristic families decouple and each one develops an $N$-wave profile, of width $O(t^{\frac{1}{2}})$ and strength $O(t^{-\frac{1}{2}})$, which propagates into the rest state at characteristic speed. When one of $p_-$, $p_+$ (or $q_-$, $q_+$) vanishes, the 1- (or 2-) $N$-wave is one-sided, of triangular profile. If both $p_-$, $p_+$ (or $q_-$, $q_+$) vanish, the 1- (or 2-) $N$-wave is absent altogether. In the wake of the $N$-waves, the solution decays at the rate $O(t^{-\frac{3}{4}})$, so long as $p_+ > 0$ and $q_- > 0$. In cones properly contained in the wake, the decay is even faster, $O(t^{-\frac{3}{2}})$.

Statement (b) of Theorem 12.9.1 is an immediate corollary of Theorem 12.5.1. The remaining assertions will be established in several steps.

**12.9.2 Lemma.** *As $t \to \infty$, the total variation decays according to* (12.9.4).

**Proof.** We fix $t$ large and construct the maximal forward 1-characteristic $\chi_-(\cdot)$ issuing from $(\psi_+(t^{\frac{1}{2}}), t^{\frac{1}{2}})$ and the minimal forward 2-characteristic $\chi_+(\cdot)$ issuing from $(\phi_-(t^{\frac{1}{2}}), t^{\frac{1}{2}})$.

In order to estimate the total variation over the interval $(\chi_-(t), \chi_+(t))$, we apply Theorem 12.5.1, shifting the time origin from 0 to $t^{\frac{1}{2}}$. The minimal backward 1-characteristics as well as the maximal backward 2-characteristics emanating from points $(x,t)$ with $\chi_-(t) < x < \chi_+(t)$ are intercepted by the $t^{\frac{1}{2}}$-time line outside the support of the solution. Furthermore, the oscillation of $(z,w)$ along the $t^{\frac{1}{2}}$-time line is $O(t^{-\frac{1}{4}})$ so that in (12.5.4)$_1$ and (12.5.4)$_2$ one may take $\delta = O(t^{-\frac{1}{4}})$. Therefore,

$$TV_{(\chi_-(t),\chi_+(t))}\, z(\cdot,t) + TV_{(\chi_-(t),\chi_+(t))}\, w(\cdot,t) = O(t^{-\frac{1}{2}}). \tag{12.9.7}$$

In order to estimate the total variation over the intervals $[\phi_-(t), \chi_-(t)]$ and $[\chi_+(t), \psi_+(t)]$, we apply Theorem 12.6.1, shifting the time origin from 0 to $\frac{1}{2}t$.

The oscillation of $(z, w)$ along the $\frac{1}{2}t$-time line is $O(t^{-\frac{1}{2}})$ so that in (12.6.1) we may take $\delta = O(t^{-\frac{1}{2}})$. Since $\chi_-(t) - \phi_-(t)$ and $\psi_+(t) - \chi_+(t)$ are $O(t^{\frac{1}{2}})$,

$$\begin{cases} TV_{[\phi_-(t),\chi_-(t)]}\, z(\cdot, t) + TV_{[\phi_-(t),\chi_-(t)]}\, w(\cdot, t) = O(t^{-\frac{1}{2}}), \\ TV_{[\chi_+(t),\psi_+(t)]}\, z(\cdot, t) + TV_{[\chi_+(t),\psi_+(t)]}\, w(\cdot, t) = O(t^{-\frac{1}{2}}). \end{cases} \tag{12.9.8}$$

Combining (12.9.7) with (12.9.8), we arrive at (12.9.4). This completes the proof.

**12.9.3 Lemma.** *Let $\bar{\lambda}$ be any fixed strict upper bound of $\lambda(z, w)$ and $\bar{\mu}$ any fixed strict lower bound of $\mu(z, w)$, over the range of the solution. Then, for $t$ large and $x > \bar{\lambda}t$,*

$$z(x, t) = O\big(t^{-\frac{3}{2}}\big), \tag{12.9.9$_1$}$$

*while for $x < \bar{\mu}t$,*

$$w(x, t) = O\big(t^{-\frac{3}{2}}\big). \tag{12.9.9$_2$}$$

**Proof.** We fix $t$ large and $x > \bar{\lambda}t$. Since $\bar{\lambda}$ is a strict upper bound of $\lambda(z, w)$, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from $(x, t)$ will be intercepted by the graph of $\psi_+$ at time $t_1 \geq \kappa t$, where $\kappa$ is a positive constant depending solely on $\bar{\lambda}$. If $(\bar{z}(\cdot), \bar{w}(\cdot))$ denotes the trace of $(z, w)$ along $\xi(\cdot)$, then the oscillation of $\bar{w}(\cdot)$ over $[t_1, t]$ is $O(t^{-\frac{1}{2}})$. Applying Theorem 12.5.1, with time origin shifted to $t_1$, and using Lemma 12.9.2, we deduce that the total variation of $\bar{w}(\cdot)$ over $[t_1, t]$ is likewise $O(t^{-\frac{1}{2}})$. It then follows from Theorem 12.4.1 that $\bar{z}(t-) = O(t^{-\frac{3}{2}})$. Since $z(x, t) = \bar{z}(t-)$, we arrive at (12.9.9)$_1$.

In a similar fashion, one establishes (12.9.9)$_2$, for $x < \bar{\mu}t$. The proof is complete.

**12.9.4 Lemma.** *Assertion* (d) *of Theorem* 12.9.1 *holds.*

**Proof.** By the construction of $\phi_-$ and $\phi_+$, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from any point $(x, t)$ with $\phi_-(t) < x < \phi_+(t)$ will be intercepted by the $x$-axis on the interval $[-\ell, \ell]$. Therefore, if $(\bar{z}(\cdot), \bar{w}(\cdot))$ denotes the trace of $(z, w)$ along $\xi(\cdot)$,

$$\begin{aligned} x &= \int_1^t \lambda(\bar{z}(\tau), \bar{w}(\tau))d\tau + \xi(1) \\ &= t\lambda(z(x, t), 0) + \int_1^t \{\bar{\lambda}_z[\bar{z}(\tau) - \bar{z}(t-)] + \bar{\lambda}_w \bar{w}(\tau)\}d\tau + O(1). \end{aligned} \tag{12.9.10}$$

By account of Lemma 12.9.3, $\bar{w}(\tau) = O(\tau^{-\frac{3}{2}})$. Applying Theorem 12.5.1, with time origin shifted to $\tau$, and using Lemma 12.9.2, we deduce that the total variation of $\bar{w}(\cdot)$ over $[\tau, t]$ is $O(\tau^{-\frac{1}{2}})$. It then follows from Theorem 12.4.1 that $\bar{z}(\tau) - \bar{z}(t-)$

is $O(\tau^{-\frac{7}{2}})$. In particular, the integral on the right-hand side of (12.9.10) is $O(1)$ and this establishes $(12.9.5)_1$.

A similar argument shows $(12.9.5)_2$. The proof is complete.

**12.9.5 Lemma.** *For $t$ large, $\phi_-(t)$ and $\psi_+(t)$ satisfy* $(12.9.1)_1$ *and* $(12.9.1)_2$.

**Proof.** For $t$ large, $\phi_-(t)$ joins the state $(z(\phi_-(t)-, t), w(\phi_-(t)-, t)) = (0, 0)$, on the left, to the state $(z(\phi_-(t)+, t), w(\phi_-(t)+, t))$, on the right, where $w(\phi_-(t)+, t)$ is $O(t^{-\frac{3}{2}})$, while $z(\phi_-(t)+, t)$ satisfies $(12.9.5)_1$ for $x = \phi_-(t)$. The jump accross $\phi_-(t)$ is $O(t^{-\frac{1}{2}})$. Consequently, by use of (8.1.9) we infer

$$\dot{\phi}_-(t) = \frac{1}{2}\lambda(0, 0) + \frac{1}{2t}\phi_-(t) + O\left(\frac{1}{t}\right), \tag{12.9.11}$$

almost everywhere.

We set $\phi_-(t) = \lambda(0, 0)t - v(t)$. By the admissibility condition $\dot{\phi}_-(t) \le \lambda(0, 0)$, we deduce that $\dot{v}(t) \ge 0$. Substituting into (12.9.11) yields

$$\dot{v}(t) = \frac{1}{2t}v(t) + O\left(\frac{1}{t}\right). \tag{12.9.12}$$

If $v(t) = O(1)$, as $t \to \infty$, we obtain $(12.9.1)_1$ with $p_- = 0$. On the other hand, if $v(t) \uparrow \infty$, as $t \to \infty$, then (12.9.12) implies $v(t) = (p_- t)^{\frac{1}{2}} + O(1)$, which establishes $(12.9.1)_1$ with $p_- > 0$.

One validates $(12.9.1)_2$ by a similar argument. The proof is complete.

**12.9.6 Lemma.** *For $t$ large, $\phi_+(t)$ and $\psi_-(t)$ satisfy* $(12.9.2)_1$ *and* $(12.9.2)_2$. *Furthermore, Assertion* (e) *of Theorem* 12.9.1 *holds.*

**Proof.** For $t$ large, $\phi_+(t)$ joins the state $(z(\phi_+(t)-, t), w(\phi_+(t)-, t))$, on the left, to the state $(z(\phi_+(t)+, t), w(\phi_+(t)+, t))$, on the right, where both $w(\phi_+(t)\pm, t)$ are $O(t^{-\frac{3}{2}})$, while $z(\phi_+(t)-, t)$ satisfies $(12.9.5)_1$ for $x = \phi_+(t)$. The jump across $\phi_+(t)$ is $O(t^{-\frac{1}{2}})$. Hence, by use of (8.1.9) we obtain

$$\dot{\phi}_+(t) = \frac{1}{2}\lambda(z(\phi_+(t)+, t), 0) + \frac{1}{2t}\phi_+(t) + O\left(\frac{1}{t}\right). \tag{12.9.13}$$

Since $\phi_+$ is maximal, minimal backward 1-characteristics $\zeta(\cdot)$ emanating from points $(x, t)$ with $x > \phi_+(t)$ stay strictly to the right of $\phi_+(\cdot)$ on $[0, t]$ and are thus intercepted by the $x$-axis at $\zeta(0) > \ell$. By virtue of Theorem 12.4.1, it follows that $z(\phi_+(t)+, t) \le 0$ and so $\lambda(z(\phi_+(t)+, t), 0) \ge \lambda(0, 0)$.

We now set $\phi_+(t) = \lambda(0, 0)t + v(t)$, $\lambda(z(\phi_+(t)+, t), 0) = \lambda(0, 0) + g(t)$. As shown above, $g(t) \ge 0$. Furthermore, notice that the admissibility condition $\dot{\phi}_+(t) \ge \lambda(z(\phi_+(t)+, t), w(\phi_+(t)+, t))$ implies $\dot{v}(t) \ge g(t) + O(t^{-\frac{3}{2}})$. When $v(t)$ is bounded, as $t \to \infty$, we obtain $(12.9.2)_1$, with $p_+ = 0$, corresponding to the case of one-sided $N$-wave. This case is delicate and will not be discussed here, so let us assume $v(t) \to \infty$, as $t \to \infty$.

Substituting $\phi_+(t)$ into (12.9.13), we obtain

$$\dot{v}(t) = \frac{1}{2t}v(t) + \frac{1}{2}g(t) + O(\frac{1}{t}). \tag{12.9.14}$$

Since $g(t) \geq 0$, (12.9.14) yields $v(t) \geq \alpha t^{\frac{1}{2}}$, with $\alpha > 0$. On the other hand, we know that $v(t) = O(t^{\frac{1}{2}})$ and so (12.9.14) implies

$$\frac{\dot{v}}{v} \geq \frac{1}{2t} + \beta g(t) t^{-\frac{1}{2}} + O(t^{-\frac{3}{2}}). \tag{12.9.15}$$

It is clear that (12.9.15) induces a contradiction to $v(t) = O(t^{\frac{1}{2}})$ unless

$$\int_1^\infty g(\tau)\tau^{-\frac{1}{2}} d\tau < \infty. \tag{12.9.16}$$

We now demonstrate that, in consequence of (12.9.16), there is $T > 0$ with the property that

$$\inf\{\tau^{\frac{1}{2}} g(\tau) : \frac{t}{2} \leq \tau \leq t\} < \frac{\alpha}{2}, \qquad \text{for all } t > T. \tag{12.9.17}$$

Indeed, if this assertion is false, we can find a sequence $\{t_m\}$, with $t_{m+1} \geq 2t_m$, $m = 1, 2, \cdots$, along which (12.9.17) is violated. But then

$$\int_1^\infty g(\tau)\tau^{-\frac{1}{2}} d\tau \geq \tfrac{1}{2}\alpha \sum_m \int_{\frac{1}{2}t_m}^{t_m} \frac{dt}{t} = \infty, \tag{12.9.18}$$

in contradiction to (12.9.16).

Let us fix $(x, t)$, with $t > T$ and $x > \phi_+(t)$. The minimal backward 1-characteristic $\zeta(\cdot)$ emanating from $(x, t)$ stays strictly to the right of $\phi_+(\cdot)$. We locate $\bar{t} \in [\frac{1}{2}t, t]$ such that

$$\lambda(z(\phi_+(\bar{t})+, \bar{t}), 0) - \lambda(0, 0) = g(\bar{t}) < \tfrac{1}{4}\alpha \bar{t}^{-\frac{1}{2}} \tag{12.9.19}$$

and consider the minimal backward 1-characteristic $\xi(\cdot)$ emanating from a point $(\bar{x}, \bar{t})$, where $\bar{x}$ lies between $\phi_+(\bar{t})$ and $\zeta(\bar{t})$ and is so close to $\phi_+(\bar{t})$ that

$$\lambda(z(\bar{x}, \bar{t}), 0) - \lambda(0, 0) < \tfrac{1}{4}\alpha \bar{t}^{-\frac{1}{2}}. \tag{12.9.20}$$

Let $(\bar{z}(\cdot), \bar{w}(\cdot))$ denote the trace of $(z, w)$ along $\xi(\cdot)$. By virtue of Theorem 12.4.1, $\bar{z}(\cdot)$ is a nonincreasing function on $(0, \bar{t})$ so that $\bar{z}(\tau) \leq \bar{z}(\bar{t}-) = z(\bar{x}, \bar{t})$. Consequently, on account of (12.9.20),

$$\begin{aligned} \dot{\xi}(\tau) = \lambda(\bar{z}(\tau), \bar{w}(\tau)) &\leq \lambda(z(\bar{x}, \bar{t}), \bar{w}(\tau)) \\ &\leq \lambda(0, 0) + \tfrac{1}{2}\alpha \bar{t}^{-\frac{1}{2}} + \bar{c}|\bar{w}(\tau)|. \end{aligned} \tag{12.9.21}$$

The integral of $|\bar{w}(\cdot)|$ over $[0, \bar{t}]$ is $O(1)$, by virtue of Lemma 12.8.3. Moreover,

(12.9.22) $$\xi(\bar{t}) = \bar{x} > \phi_+(\bar{t}) \geq \lambda(0,0)\bar{t} + \alpha\bar{t}^{\frac{1}{2}}.$$

Therefore, integrating (12.9.21) over $[0, \bar{t}]$ yields

(12.9.23) $$\xi(0) \geq \tfrac{1}{2}\alpha\bar{t}^{\frac{1}{2}} + O(1) \geq \tfrac{\sqrt{2}}{4}\alpha t^{\frac{1}{2}} + O(1).$$

Since $\zeta(\cdot)$ stays to the right of $\xi(\cdot)$, (12.9.23) implies, in particular, that the graph of $\zeta(\cdot)$ will intersect the graph of $\psi_+(\cdot)$ at time $\hat{t} = O(t^{\frac{1}{2}})$.

Let $(\hat{z}(\cdot), \hat{w}(\cdot))$ denote the trace of $(z, w)$ along $\zeta(\cdot)$. The oscillation of $\hat{w}(\cdot)$ over $[\hat{t}, t)$ is $O(t^{-\frac{1}{4}})$. Furthermore, on account of Theorem 12.5.1, with time origin shifted to $\hat{t}$, and Lemma 12.9.2, we deduce that the total variation of $\hat{w}(\cdot)$ over $[\hat{t}, t)$ is also $O(t^{-\frac{1}{4}})$. It then follows from Theorem 12.4.1 that $\hat{z}(t-) = O(t^{-\frac{3}{4}})$.

By virtue of the above result, (12.9.21) now implies

(12.9.24) $$\dot{\xi}(\tau) \leq \lambda(0,0) + O(t^{-\frac{3}{4}}) + \bar{c}|\bar{w}(\tau)|,$$

which, upon integrating over $[0, t]$, yields

(12.9.25) $$\xi(0) \geq x - \lambda(0,0)t + O(t^{\frac{1}{4}}) \geq \tfrac{1}{2}[x - \lambda(0,0)t]\,.$$

Thus, $\hat{t} \geq c'[x - \lambda(0,0)t]$. But then the oscillation and total variation of $\hat{w}(\cdot)$ over $[\hat{t}, t]$ is bounded by $\hat{c}[x-\lambda(0,0)t]^{-\frac{1}{2}}$, in which case $(12.9.6)_1$ follows from Theorem 12.4.1.

Finally, we return to (12.9.14). Since $z(\phi_+(t)+, t)$ is $O(t^{-\frac{3}{4}})$, we deduce that $g(t) = O(t^{-\frac{3}{4}})$, and this in turn yields $v(t) = (p_+t)^{\frac{1}{2}} + O(t^{\frac{1}{4}})$, with $p_+ > 0$. We have thus verified $(12.9.2)_1$ .

A similar argument establishes $(12.9.6)_2$ , for $x < \psi_-(t)$, and validates $(12.9.2)_2$ . This completes the proof of Lemma 12.9.6 and thereby the proof of Theorem 12.9.1.

It is now easy to determine the large time asymptotics of the solution $U(x, t)$ in $L^1(-\infty, \infty)$. Starting out from the (finite) Taylor expansion

(12.9.26) $$U(z, w) = zR(0,0) + wS(0,0) + O(z^2 + w^2),$$

and using Theorem 12.9.1, we conclude

**12.9.7 Theorem.** *Assume $p_+ > 0$ and $q_- > 0$. Then, as $t \to \infty$,*

(12.9.27)

$$\|U(x,t) - M(x,t;p_-,p_+)R(0,0) - N(x,t;q_-,q_+)S(0,0)\|_{L^1(-\infty,\infty)} = O(t^{-\frac{1}{4}}),$$

*where M and N denote the N-wave profiles*:

(12.9.28)$_1$

$$M(x,t;p_-,p_+) = \begin{cases} \dfrac{x-\lambda(0,0)t}{\lambda_z(0,0)t}\,, & for \quad -(p_-t)^{\frac{1}{2}} \le x - \lambda(0,0)t \le (p_+t)^{\frac{1}{2}} \\ 0 & otherwise, \end{cases}$$

(12.9.28)$_2$

$$N(x,t;q_-,q_+) = \begin{cases} \dfrac{x-\mu(0,0)t}{\mu_w(0,0)t}\,, & for \quad -(q_-t)^{\frac{1}{2}} \le x - \mu(0,0)t \le (q_+t)^{\frac{1}{2}} \\ 0 & otherwise. \end{cases}$$

## 12.10 Periodic Solutions

The study of genuinely nonlinear hyperbolic systems (12.1.1) of two conservation laws will be completed with a discussion of the large time behavior of solutions with small oscillation that are periodic,

(12.10.1) $$U(x+\ell,t) = U(x,t), \qquad -\infty < x < \infty,\ t > 0,$$

and have zero mean[1]:

(12.10.2) $$\int_y^{y+\ell} U(x,t)dx = 0, \qquad -\infty < y < \infty,\ t > 0.$$

The confinement of waves resulting from periodicity induces active interactions and cancellation. As a result, the total variation per period decays at the rate $O(t^{-1})$:

**12.10.1 Theorem.** *For any* $x \in (\infty, \infty)$, *and* $t > 0$,

(12.10.3) $$TV_{[x,x+\ell]}\, z(\cdot,t) + TV_{[x,x+\ell]}\, w(\cdot,t) \le \frac{b\ell}{t}\,.$$

**Proof.** Apply (12.6.1) with $y = x + n\ell$; then divide by $n$ and let $n \to \infty$. This completes the proof.

We now resolve the asymptotics at the scale $O(t^{-1})$. The mechanism encountered in Section 11.7, in the context of genuinely nonlinear scalar conservation laws, namely the confinement of the intercepts of extremal backward characteristics in intervals of the $x$-axis of period length, is here in force as well and generates similar,

[1] If the Cauchy problem has unique solution, initial data that are periodic with zero mean necessarily generate solutions with the same property.

serrated asymptotic profiles. The nodes of the profiles are again tracked by divides, in the sense of Definition 10.3.3.

**12.10.2 Theorem.** *The upper half-plane is partitioned by minimal (or maximal)* 1- (*or* 2-) *divides along which* $z$ (*or* $w$) *decays rapidly to zero,* $O(t^{-2})$, *as* $t \to \infty$. *Let* $\chi_-(\cdot)$ *and* $\chi_+(\cdot)$ *be any two adjacent* 1- (*or* 2-) *divides, with* $\chi_-(t) < \chi_+(t)$. *Then* $\chi_+(t) - \chi_-(t)$ *approaches a constant at the rate* $O(t^{-1})$, *as* $t \to \infty$. *Furthermore, between* $\chi_-$ *and* $\chi_+$ *lies a* 1- (*or* 2-) *characteristic* $\psi$ *such that, as* $t \to \infty$,

$$\psi(t) = \tfrac{1}{2}[\chi_-(t) + \chi_+(t)] + o(1), \tag{12.10.4}$$

$$\lambda_z(0,0)z(x,t) = \begin{cases} \dfrac{x - \chi_-(t)}{t} + o\left(\dfrac{1}{t}\right), & \chi_-(t) < x < \psi(t), \\[2ex] \dfrac{x - \chi_+(t)}{t} + o\left(\dfrac{1}{t}\right), & \psi(t) < x < \chi_+(t), \end{cases} \tag{12.10.5$_1$}$$

*or*

$$\mu_w(0,0)w(x,t) = \begin{cases} \dfrac{x - \chi_-(t)}{t} + o\left(\dfrac{1}{t}\right), & \chi_-(t) < x < \psi(t), \\[2ex] \dfrac{x - \chi_+(t)}{t} + o\left(\dfrac{1}{t}\right), & \psi(t) < x < \chi_+(t). \end{cases} \tag{12.10.5$_2$}$$

The first step towards proving the above proposition is to investigate the large time behavior of divides:

**12.10.3 Lemma.** *Along minimal (or maximal)* 1- (*or* 2-) *divides,* $z$ (*or* $w$) *decays at the rate* $O(t^{-2})$, *as* $t \to \infty$. *Furthermore, if* $\chi_-(\cdot)$ *and* $\chi_+(\cdot)$ *are any two minimal (or maximal)* 1- (*or* 2-) *divides, then, as* $t \to \infty$,

$$\chi_+(t) - \chi_-(t) = h_\infty + O\left(\frac{1}{t}\right), \tag{12.10.6}$$

$$\int_{\chi_-(t)}^{\chi_+(t)} z(x,t)dx = O\left(\frac{1}{t^2}\right), \tag{12.10.7$_1$}$$

*or*

$$\int_{\chi_-(t)}^{\chi_+(t)} w(x,t)dx = O\left(\frac{1}{t^2}\right). \tag{12.10.7$_2$}$$

**Proof.** Assume $\chi(\cdot)$ is a minimal 1-divide, say the limit of a sequence $\{\xi_n(\cdot)\}$ of minimal backward 1-characteristics emanating from points $\{(x_n, t_n)\}$, with $t_n \to \infty$, as $n \to \infty$. Let $(z_n(\cdot), w_n(\cdot))$ denote the trace of $(z, w)$ along $\xi_n(\cdot)$. Applying Theorem 12.5.1, with time origin shifted to $\tau$, and using Theorem 12.10.1, we deduce that the total variation of $w_n(\cdot)$ over any interval $[\tau, \tau + 1] \subset [0, t_n]$ is $O(\tau^{-1})$, uniformly in $n$. Therefore, by virtue of Theorem 12.4.1, $z_n(\cdot)$ is a nonincreasing function on $[0, t_n]$ whose oscillation over $[\tau, \tau + 1]$ is $O(\tau^{-3})$, uniformly in $n$. It follows that the trace $\bar{z}(\cdot)$ of $z$ along $\chi(\cdot)$ is likewise a nonincreasing function with $O(\tau^{-3})$ oscillation over $[\tau, \tau + 1]$. By tallying the oscillation of $\bar{z}(\cdot)$ over intervals of unit length, from $t$ to infinity, we verify the assertion $\bar{z}(t) = O(t^{-2})$.

A similar argument shows that the trace $\bar{w}(\cdot)$ of $w$ along maximal 2-divides is likewise $O(t^{-2})$, as $t \to \infty$.

Let $\chi_-(\cdot)$ and $\chi_+(\cdot)$ be minimal 1-divides with $h(t) = \chi_+(t) - \chi_-(t) \geq 0$, for $0 \leq t < \infty$. Note that, because of periodicity, $h(0) < k\ell$, for some integer $k$, implies $h(t) \leq k\ell,\ 0 \leq t < \infty$. Letting $(z_-(\cdot), w_-(\cdot))$ and $(z_+(\cdot), w_+(\cdot))$ denote the trace of $(z, w)$ along $\chi_-(\cdot)$ and $\chi_+(\cdot)$, respectively, we have

$$\dot{h}(\tau) = \lambda(z_+(\tau), w_+(\tau)) - \lambda(z_-(\tau), w_-(\tau)), \tag{12.10.8}$$

for almost all $\tau$ in $[0, \infty)$.

The maximal backward 2-characteristic $\zeta_\tau(\cdot)$ emanating from the point $(\chi_+(\tau), \tau)$ is intercepted by the graph of $\chi_-(\cdot)$ at time $\tau - f(\tau)$. If $(\hat{z}(\cdot), \hat{w}(\cdot))$ denotes the trace of $(z, w)$ along $\zeta_\tau(\cdot)$, Theorems 12.5.1 and 12.10.1 together imply that the total variation of $\hat{z}(\cdot)$ over the interval $[\tau - f(\tau), \tau]$ is $O(\tau^{-1})$, as $\tau \to \infty$. It then follows from Theorem 12.4.1 that the oscillation of $\hat{w}(\cdot)$ over $[\tau - f(\tau), \tau]$ is $O(\tau^{-3})$. Hence

$$w_+(\tau) = w_-(\tau - f(\tau)) + O(\tau^{-3}). \tag{12.10.9}$$

Since $z_\pm(\tau) = O(\tau^{-2})$, (12.10.8) yields

$$\dot{h}(\tau) = \lambda(0, w_-(\tau - f(\tau))) - \lambda(0, w_-(\tau)) + O(\frac{1}{\tau^2}). \tag{12.10.10}$$

From $\dot{h}(\tau) = O(\tau^{-1})$ and $\dot{\zeta}_\tau = \mu(0, 0) + O(\tau^{-1})$, we infer that the oscillation of $f(\cdot)$ over the interval $[\tau, \tau + 1]$ is $O(\tau^{-1})$. The total variation of $w_-(\cdot)$ over $[\tau, \tau + 1]$ is likewise $O(\tau^{-1})$. Then, for any $t < t' < \infty$,

$$|\int_t^{t'} \{\lambda(0, w_-(\tau - f(\tau))) - \lambda(0, w_-(\tau))\} d\tau| \leq \frac{c}{t}\,. \tag{12.10.11}$$

Upon combining (12.10.10) with (12.10.11), one arrives at (12.10.6).

Let $U_-(\cdot)$ and $U_+(\cdot)$ denote the trace of $U$ along $\chi_-(\cdot)$ and $\chi_+(\cdot)$, respectively. Integration of (12.1.1) over $\{(x, \tau) : t < \tau < \infty,\ \chi_-(\tau) < x < \chi_+(\tau)\}$ yields the equation

$$\text{(12.10.12)} \qquad \int_{\chi_-(t)}^{\chi_+(t)} U(x,t)dx = \int_t^\infty \{F(U_+(\tau)) - \lambda(U_+(\tau))U_+(\tau) - F(U_-(\tau)) + \lambda(U_-(\tau))U_-(\tau)\}d\tau.$$

We multiply (12.10.12), from the left, by the row vector $\mathrm{D}z(0)$. On account of (7.3.12), $U_z = R$ and $U_w = S$ so that, using (12.1.2), we deduce

$$\text{(12.10.13)} \qquad \mathrm{D}z(0)U = z + O(z^2 + w^2),$$

$$\text{(12.10.14)} \quad \mathrm{D}z(0)[F(U) - \lambda(U)U] = \mathrm{D}z(0)F(0) + aw^2 + O(z^2 + |zw| + |w|^3),$$

where the constant $a$ is the value of $\frac{1}{2}(\lambda - \mu)S^\top \mathrm{D}^2 z S$ at $U = 0$. By virtue of $z_\pm(\tau) = O(\tau^{-2}),\ w_\pm(\tau) = O(\tau^{-1})$ and (12.10.9), we conclude

$$\text{(12.10.15)} \quad \int_{\chi_-(t)}^{\chi_+(t)} z(x,t)dx = a\int_t^\infty [w_-^2(\tau - f(\tau)) - w_-^2(\tau)]d\tau + O\left(\frac{1}{t^2}\right).$$

As explained above, over the interval $[\tau, \tau+1]$ the oscillation of $f(\cdot)$ is $O(\tau^{-1})$ and the total variation of $w_-^2(\cdot)$ is $O(\tau^{-2})$. Then, the integral on the right-hand side of (12.10.15) is $O(t^{-2})$, as $t \to \infty$, which establishes $(12.10.7)_1$ .

When $\chi_-(\cdot)$ and $\chi_+(\cdot)$ are maximal 2-divides, a similar argument verifies (12.10.6) and $(12.10.7)_2$ . The proof is complete.

The remaining assertions of Theorem 12.10.2 will be established through the following

**12.10.4 Lemma.** *Consider any two adjacent minimal (or maximal)* 1- (*or* 2-) *divides* $\chi_-(\cdot), \chi_+(\cdot)$, *with* $\chi_-(t) < \chi_+(t),\ 0 \le t < \infty$. *The special forward* 1- (*or* 2-) *characteristic* $\phi_-(\cdot)$ (*or* $\psi_+(\cdot)$), *in the notation of Section* 12.3, *issuing from any fixed point* $(\bar{x}, 0)$, *where* $\chi_-(0) < \bar{x} < \chi_+(0)$, *is denoted by* $\psi(\cdot)$. *Then* $\psi(\cdot)$ *satisfies* (12.10.4). *Furthermore,* $(12.10.5)_1$ (*or* $(12.10.5)_2$) *holds.*

**Proof.** It will suffice to discuss the case where $\chi_-$ and $\chi_+$ are 1-divides. We consider minimal backward 1-characteristics $\xi(\cdot)$ emanating from points $(x,t)$, with $t > 0$ and $\chi_-(t) < x < \chi_+(t)$. Their graphs are trapped between the graphs of $\chi_-$ and $\chi_+$. The intercepts $\xi(0)$ of such $\xi$, by the $x$-axis, cannot accumulate to any $\hat{x}$ in the open interval $(\chi_-(0), \chi_+(0))$, because in that case a minimal 1-divide would issue from the point $(\hat{x}, 0)$, contrary to our assumption that $\chi_-$ , $\chi_+$ are adjacent. Therefore, by the construction of $\psi(\cdot)$ we infer that, as $t \to \infty$, $\xi(\tau) \to \chi_-(\tau)$, when $x$ is in $(\chi_-(t), \psi(t)]$, or $\xi(\tau) \to \chi_+(\tau)$, when $x$ is in $(\psi(t), \chi_+(t)]$, the convergence being uniform on compact subsets of $[0, \infty)$.

Let us now fix $\xi(\cdot)$ that emanates from some point $(x,t)$, with $\chi_-(t) < x \le \psi(t)$, and set $h(\tau) = \xi(\tau) - \chi_-(\tau),\ 0 \le \tau \le t$. Then, for almost all $\tau \in [0,t]$ we have

$$\text{(12.10.16)} \qquad \dot{h}(\tau) = \lambda(\bar{z}(\tau), \bar{w}(\tau)) - \lambda(z_-(\tau), w_-(\tau)),$$

where $(\bar{z}(\cdot), \bar{w}(\cdot))$ denotes the trace of $(z, w)$ along $\xi(\cdot)$, while $(z_-(\cdot), w_-(\cdot))$ stands for the trace of $(z, w)$ along $\chi_-(\cdot)$.

By virtue of Theorems 12.5.1 and 12.10.1, the total variation of $\bar{w}(\cdot)$ on any interval $[s, s+1] \subset [0, t]$ is $O(s^{-1})$. It then follows from Theorem 12.4.1 that the oscillation of $\bar{z}(\cdot)$ over $[s, s+1]$ is $O(s^{-3})$ and hence

$$\bar{z}(\tau) = z(x, t) + O\left(\frac{1}{\tau^2}\right). \tag{12.10.17}$$

Furthermore, by Lemma 12.10.3, $z_-(\tau) = O(\tau^{-2})$. Also, $z(x, t) = O(t^{-1})$ so, *a fortiori*, $z(x, t) = O(\tau^{-1})$. By account of these observations, (12.10.16) yields

$$\dot{h}(\tau) = \lambda_z(0, 0)z(x, t) + \lambda(0, \bar{w}(\tau)) - \lambda(0, w_-(\tau)) + O\left(\frac{1}{\tau^2}\right). \tag{12.10.18}$$

For any fixed $\tau >> 0$, we consider the maximal backward 2-characteristic $\zeta_\tau(\cdot)$ emanating from the point $(\xi(\tau), \tau)$, which is intercepted by the graph of $\chi_-(\cdot)$ at time $\tau - f(\tau)$. If $(\hat{z}(\cdot), \hat{w}(\cdot))$ denotes the trace of $(z, w)$ along $\zeta_\tau(\cdot)$, Theorems 12.5.1 and 12.10.1 together imply that the total variation of $\hat{z}(\cdot)$ over the interval $[\tau - f(\tau), \tau]$ is $O(\tau^{-1})$. It then follows from Theorem 12.4.1 that the oscillation of $\hat{w}(\cdot)$ over $[\tau - f(\tau), \tau]$ is $O(\tau^{-3})$. Hence

$$\bar{w}(\tau) = w_-(\tau - f(\tau)) + O\left(\frac{1}{\tau^3}\right), \tag{12.10.19}$$

and so (12.10.18) implies

$$\dot{h}(\tau) = \lambda_z(0, 0)z(x, t) + \lambda(0, w_-(\tau - f(\tau))) - \lambda(0, w_-(\tau)) + O\left(\frac{1}{\tau^2}\right). \tag{12.10.20}$$

As in the proof of Lemma 12.10.3, on any interval $[\tau, \tau+1] \subset [0, t]$ the oscillation of $f(\cdot)$ is $O(\tau^{-1})$ and the total variation of $w_-(\cdot)$ is also $O(\tau^{-1})$. Therefore, upon integrating (12.10.20) over the interval $[s, t]$, $0 < s < t$, we deduce

$$x - \chi_-(t) - \lambda_z(0, 0)z(x, t)t = \xi(s) - \chi_-(s) + O\left(\frac{1}{s}\right) + sO\left(\frac{1}{t}\right). \tag{12.10.21}$$

With reference to the right-hand side of (12.10.21), given $\varepsilon > 0$, we first fix $s$ so large that $O(s^{-1})$ is less than $\frac{1}{3}\varepsilon$. With $s$ thus fixed, we determine $\hat{t}$ such that, for $t \geq \hat{t}$, $sO(t^{-1})$ does not exceed $\frac{1}{3}\varepsilon$, while at the same time $\xi(s) - \chi_-(s) < \frac{1}{3}\varepsilon$, for all $x \in (\chi_-(t), \psi(t)]$. Clearly, it is sufficient to check this last condition for $t = \hat{t}$, $x = \psi(\hat{t})$. We have thus verified that the left-hand side of (12.10.21) is $o(1)$, as $t \to \infty$, uniformly in $x$ on $(\chi_-(t), \psi(t))$, which verifies the upper half of $(12.10.5)_1$. The lower half of $(12.10.5)_1$ is established by a similar argument. This completes the proof.

## 12.11 Notes

There is voluminous literature addressing various aspects of the theory of genuinely nonlinear systems of two conservation laws. The approach in this chapter, via the theory of generalized characteristics, is principally due to the author, and some of the proofs are recorded here in print for the first time. Most of the results were derived earlier in the framework of solutions constructed by the random choice method, which will be presented in Chapter XIII. The seminal contribution in that direction is Glimm and Lax [1].

The Lax entropies, discussed in Section 12.2, were first introduced in Lax [4]. The hodograph transformation was discovered by Riemann [1] and by Helmholtz [1]. For detailed discussions and applications to aerodynamics, see Courant and Friedrichs [1] and Von Mises [1]. For applications to other areas of mathematical physics, see Fusco [1].

A somewhat stronger version of Theorem 12.3.3 was established by DiPerna [3], for solutions constructed by the random choice method. Theorem 12.4.1 improves a proposition in Dafermos [16].

Theorems 12.5.1, 12.6.1 and 12.6.4 were originally established in Glimm and Lax [1], for solutions constructed by the random choice method, by use of the theory of approximate conservation laws and approximate characteristics, which will be outlined in Section 13.3. The treatment here employs and refines methodology developed by Dafermos and Geng [1,2], for special systems, and Trivisa [1], for general systems, albeit when solutions are "countably regular". Trivisa [2] extends these results to genuine nonlinar systems of $n$ conservation laws endowed with a coordinate system of Riemann invariants.

The results of Section 12.7 were established earlier by DiPerna [3], for solutions constructed by the random choice method.

For solutions with initial data in $L^1$, Temple [5] derives decay at the rate $O(1/\sqrt{\log t})$. The $O(t^{-\frac{1}{2}})$ decay rate established in Theorem 12.8.1, which is taken from Dafermos [16], is sharp. Similarly, Lemma 12.8.2 improves an earlier result of Temple [8]. $L^1$ stability has now been established for general systems; see Chapter XIV.

The mechanism that generates $N$-wave profiles was understood quite early, through formal asymptotics (see Courant and Friedrichs [1]), even though a rigorous proof was lacking (Lax [2]). In a series of papers by DiPerna [4,6] and Liu [8,9.22], decay to $N$-waves of solutions with initial data of compact support, constructed by the random choice method, was established at progressively sharper rates, not only for genuinely nonlinear sytems of two conservation laws but even for systems of $n$ conservation laws with characteristic families that are either genuinely nonlinear or linearly degenerate. The decay rates recorded in Theorem 12.9.1 are sharp. When the initial data do not have compact support but instead approach distinct limits $U_L$ and $U_R$, as $x \to \pm\infty$, then the solution $U$ converges, as $t \to \infty$, to the solution of the Riemann problem with initial data (9.1.12); see Liu [6] and compare with the scalar case discussed in Section 11.5. Relatively little is known for systems that are not genuinely nonlinear; see Zumbrun [1,2].

Theorem 12.10.1 is due to Glimm and Lax [1], while Theorem 12.10.2 is taken from Dafermos [18]. Decay of solutions with periodic initial data may be peculiar to systems of two conservation laws. Indeed, the work of R. Young [3,4] indicates that, for the system of nonisentropic gas dynamics, solutions with periodic initial data remain bounded but do not necessarily decay.

For applications of the theory of characteristics to investigating uniqueness, regularity and large time behavior of solutions of special systems with coinciding shock and rarefaction wave curves (Temple [3]), see Serre [7,11], Dafermos and Geng [1,2], Heibig [2], Heibig and Sahel [1] and Ostrov [1]. $BV$ solutions for such systems have been constructed by the Godunov difference scheme (LeVeque and Temple [1]) as well as by the method of vanishing viscosity (Serre [1,11]).

# XIII

# The Random Choice Method

This chapter introduces the celebrated random choice method, which has provided the earliest, but still very effective, scheme for constructing globally defined, admissible $BV$ solutions to the Cauchy problem for strictly hyperbolic systems of conservation laws, under initial data with small total variation. The solution is obtained as the limit of a sequence of approximate solutions that do not smear shocks. Solutions to the Riemann problem, discussed at length in Chapter IX, serve as building blocks for constructing the approximate solutions to the Cauchy problem. Striving to preserve the sharpness of shocks may be in conflict with the requirement of consistency of the algorithm. The "randomness" feature of the method is employed in order to strike the delicate balance of safeguarding consistency without smearing the sharpness of propagating shock fronts. By paying the price of delineating the global wave pattern, the device of wave tracing, which will be discussed here only briefly, renders the algorithm deterministic.

A detailed presentation of the random choice method will be given for systems with characteristic families that are either genuinely nonlinear or linearly degenerate. The case of more general systems, which involves substantial technical complication, will be touched on rather briefly here.

The chapter will close with a discussion on how the algorithm may be adapted for handling inhomogeneity and source terms encountered in hyperbolic systems of balance laws.

## 13.1 The Construction Scheme

We consider the initial-value problem for a strictly hyperbolic system of conservation laws, defined on a ball $\mathcal{O}$ centered at the origin:

$$\begin{cases} \partial_t U(x,t) + \partial_x F(U(x,t)) = 0, \quad -\infty < x < \infty, \ \ 0 \le t < \infty, \\ U(x,0) = U_0(x), \quad -\infty < x < \infty. \end{cases} \tag{13.1.1}$$

The initial data $U_0$ are functions of bounded variation on $(-\infty, \infty)$. The ultimate goal is to establish the following

**13.1.1 Theorem.** *There are positive constants* $\delta_0$ *and* $\delta_1$ *such that if*

$$\sup_{(-\infty,\infty)} |U_0(\cdot)| < \delta_0\,, \tag{13.1.2}$$

$$TV_{(-\infty,\infty)} U_0(\cdot) < \delta_1\,, \tag{13.1.3}$$

*then there exists a solution* $U$ *of* (13.1.1), *which is a function of locally bounded variation on* $(-\infty, \infty) \times [0, \infty)$, *taking values in* $\mathcal{O}$. *This solution satisfies the entropy admissibility criterion for any entropy-entropy flux pair* $(\eta, q)$ *of the system, with* $\eta(U)$ *convex. Furthermore, for each fixed* $t \in [0, \infty)$, $U(\cdot, t)$ *is a function of bounded variation on* $(-\infty, \infty)$ *and*

$$\sup_{(-\infty,\infty)} |U(\cdot, t)| \le c_0 \sup_{(-\infty,\infty)} |U_0(\cdot)|, \qquad 0 \le t < \infty, \tag{13.1.4}$$

$$TV_{(-\infty,\infty)} U(\cdot, t) \le c_1 TV_{(-\infty,\infty)} U_0(\cdot), \qquad 0 \le t < \infty, \tag{13.1.5}$$

$$\int_{-\infty}^{\infty} |U(x,t) - U(x,\tau)| dx \le c_2 |t - \tau| TV_{(-\infty,\infty)} U_0(\cdot),\ 0 \le \tau < t < \infty, \tag{13.1.6}$$

*where* $c_0, c_1$ *and* $c_2$ *are constants depending solely on* $F$. *When the system is endowed with a coordinate system of Riemann invariants,* $\delta_1$ *in* (13.1.3) *may be fixed arbitrarily large, so long as*

$$(\sup_{(-\infty,\infty)} |U_0(\cdot)|)(TV_{(-\infty,\infty)} U_0(\cdot)) < \delta_2\,, \tag{13.1.7}$$

*with* $\delta_2$ *sufficiently small, depending on* $\delta_1$.

The proof of the above proposition is quite lengthy and shall occupy the entire chapter. Even though the assertion holds at the level of generality stated above, certain steps in the proof (Sections 13.3, 13.4, 13.5 and 13.6) will be carried out under the simplifying assumption that each characteristic family of the system is either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2). The case of general systems will be touched on in Sections 13.7 and 13.8.

The solution $U$ will be attained as the $h \downarrow 0$ limit of a family of approximate solutions $U_h$ constructed by the following process.

We fix a spatial mesh-length $h$, which will serve as parameter, and an associated temporal mesh-length $\lambda^{-1}h$, where $\lambda$ is a fixed upper bound of the characteristic speeds $|\lambda_i(U)|$, for $U \in \mathcal{O}$ and $i = 1, \cdots, n$. Setting $x_r = rh$ , $r = 0, \pm 1, \pm 2, \cdots$ and $t_s = s\lambda^{-1}h$ , $s = 0, 1, 2, \cdots$, we build the staggered grid of *mesh-points* $(x_r, t_s)$, with $s = 0, 1, 2, \cdots$, and $r + s$ even.

Assuming now $U_h$ has been defined on $\{(x,t) : -\infty < x < \infty,\ 0 \le t < t_s\}$, we determine $U_h(\cdot, t_s)$ as a step function that is constant on intervals defined by neighboring mesh-points along the line $t = t_s$ ,

$$(13.1.8) \qquad U_h(x, t_s) = U_s^r\,, \qquad x_{r-1} < x < x_{r+1}\,, \quad r+s \ \text{odd},$$

and approximates the function $U_h(\cdot, t_s-)$. The major issue of selecting judiciously the constant states $U_s^r$ will be addressed in Section 13.2.

Next we determine $U_h$ on the strip $\{(x,t) : -\infty < x < \infty\,, \ \ t_s \leq t < t_{s+1}\}$ as a solution of our system, namely,

$$(13.1.9) \qquad \partial_t U_h(x,t) + \partial_x F(U_h(x,t)) = 0, \quad -\infty < x < \infty, \quad t_s \leq t < t_{s+1}\,,$$

under the initial condition (13.1.8), along the line $t = t_s$ . Notice that the solution of (13.1.9), (13.1.8) consists of centered wave fans emanating from the mesh-points lying on the $t_s$-time line (Fig. 13.1.1). The wave fan centered at the mesh point $(x_r, t_s)$, $r+s$ even, is constructed by solving the Riemann problem for our system, with left state $U_s^{r-1}$ and right state $U_s^{r+1}$. We employ admissible solutions, with shocks satisfying the viscous shock admissibility condition (cf. Chapter IX). The resulting outgoing waves from neighboring mesh-points do not interact on the time interval $[t_s, t_{s+1})$, because of our selection of the ratio $\lambda$ of spatial and temporal mesh-lengths.

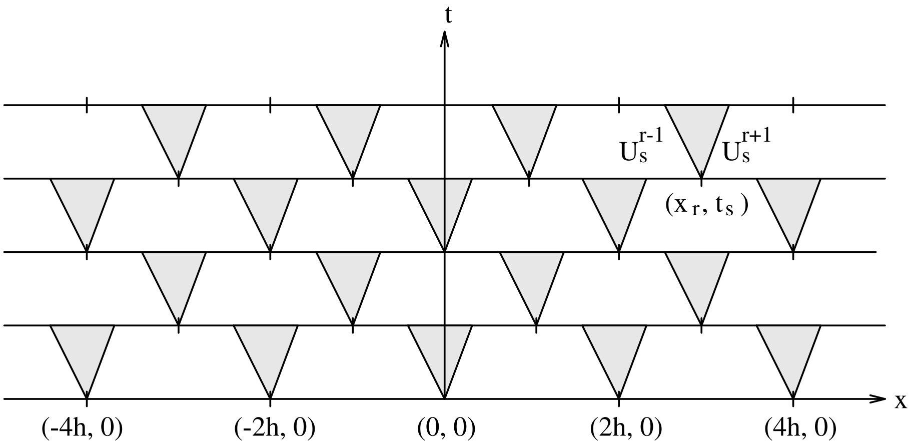


**Fig. 13.1.1**

To initiate the algorithm, at $s = 0$, we employ the initial data:

$$(13.1.10) \qquad U_h(x, 0-) = U_0(x), \qquad -\infty < x < \infty.$$

The construction of $U_h$ may proceed for as long as one can solve the resulting Riemann problems. As we saw in Chapter IX, this can be effected, in general, so long as the jumps $|U_s^{r+1} - U_s^{r-1}|$ stay sufficiently small.

After considerable preparation, we shall demonstrate, in Sections 13.5 and 13.6, that the $U_h$ satisfy estimates

$$(13.1.11) \qquad \sup_{(-\infty,\infty)}|U_h(\cdot, t)| \leq c_0 \ \sup_{(-\infty,\infty)}|U_0(\cdot)|, \qquad 0 \leq t < \infty,$$

(13.1.12) $$TV_{(-\infty,\infty)}U_h(\cdot,t) \le c_1 TV_{(-\infty,\infty)}U_0(\cdot), \qquad 0 \le t < \infty,$$

(13.1.13)

$$\int_{-\infty}^{\infty} |U_h(x,t) - U_h(x,\tau)|dx \le c_2(|t-\tau| + h)TV_{(-\infty,\infty)}U_0(\cdot),\ 0 \le \tau < t < \infty.$$

In particular, (13.1.11) guarantees that when (13.1.2) holds with $\delta_0$ sufficiently small, $U_h$ may be constructed on the entire upper half-plane.

## 13.2 Compactness and Consistency

Deferring the proof of (13.1.11), (13.1.12) and (13.1.13) to Sections 13.5 and 13.6, here we shall take these stability estimates for granted and will examine their implications. By virtue of (13.1.12), Helly's theorem and the Cantor diagonal process, there is a sequence $\{h_m\}$, with $h_m \to 0$ as $m \to \infty$, such that $\{U_{h_m}(\cdot,\tau)\}$ is Cauchy in $L^1_{loc}(-\infty,\infty)$, for each positive rational number $\tau$. Since the rationals are dense in $[0,\infty)$, (13.1.13) implies that $\{U_{h_m}(\cdot,t)\}$ must be Cauchy in $L^1_{loc}(-\infty,\infty)$, for any $t \ge 0$. Thus

(13.2.1) $$U_{h_m}(x,t) \to U(x,t), \quad \text{as } m \to \infty, \quad \text{in } L^1_{loc}((-\infty,\infty)\times[0,\infty)),$$

where, for each fixed $t \in [0,\infty)$, $U(\cdot,t)$ is a function of bounded variation on $(-\infty,\infty)$, which satisfies (13.1.4), (13.1.5) and (13.1.6). In particular, $U$ is in $BV_{loc}$.

We now turn to the question of *consistency* of the algorithm, investigating whether $U$ is a solution of the initial-value problem (13.1.1). By its construction, $U_h$ satisfies the system inside each strip $\{(x,t) : -\infty < x < \infty,\ t_s \le t < t_{s+1}\}$. Consequently, the errors are induced by the jumps of $U_h$ across the dividing time lines $t = t_s$. To estimate the cumulative effect of these errors, we fix any $C^\infty$ test function $\phi$, with compact support on $(-\infty,\infty)\times[0,\infty)$, we apply the measure (13.1.9) to $\phi$ on the rectangle $\{(x,t) : x_{r-1} < x < x_{r+1},\ t_s \le t < t_{s+1},\ r+s \text{ odd}\}$ and sum over all such rectangles in the upper half-plane. After an integration by parts, and upon using (13.1.8) and (13.1.10), we obtain

(13.2.2) $$\int_0^\infty \int_{-\infty}^\infty [\partial_t\phi U_h + \partial_x\phi F(U_h)]dxdt + \int_{-\infty}^\infty \phi(x,0)U_0(x)dx$$
$$= \sum_{s=0}^\infty \sum_{r+s \text{ odd}} \int_{x_{r-1}}^{x_{r+1}} \phi(x,t_s)[U_h(x,t_s-) - U_s^r]dx.$$

Therefore, $U$ will be a weak solution of (13.1.1), i.e., the algorithm will be consistent, if $U_s^r$ approximates the function $U_h(\cdot,t_s-)$, over the interval $(x_{r-1},x_{r+1})$, in such a manner that the right-hand side of (13.2.2) tends to zero, as $h \downarrow 0$.

One may attain consistency via the *Lax-Friedrichs scheme*:

(13.2.3) $$U_s^r = \frac{1}{2h}\int_{x_{r-1}}^{x_{r+1}} U_h(x, t_s-)dx, \qquad r+s \text{ odd.}$$

Indeed, with that choice, each integral on the right-hand side of (13.2.2) is majorized by $h^2 \max|\partial_x\phi|\,\text{osc}_{(x_{r-1},x_{r+1})}U_h(\cdot, t_s-)$. The sum of these integrals over $r$ is then majorized by $h^2 \max|\partial_x\phi|\,TV_{(-\infty,\infty)}U_h(\cdot, t_s-)$, which, in turn, is bounded by $c_1\delta_1 h^2 \max|\partial_x\phi|$, on account of (13.1.12) and (13.1.3). The summation over $s$, within the support of $\phi$, involves $O(h^{-1})$ terms, and so finally the right-hand side of (13.2.2) is $O(h)$, as $h \downarrow 0$.

Even though it passes the test of consistency, the Lax-Friedrichs scheme stumbles on the issue of stability: It is at present unknown whether estimates (13.1.12) and (13.1.13) hold within its framework.[1] One of the drawbacks of this scheme is that it smears, through averaging, the shocks of the exact solution. This feature may be vividly illustrated in the context of the Riemann problem for the linear, scalar conservation law,

(13.2.4) $$\begin{cases} \partial_t u(x,t) + a\lambda\partial_x u(x,t) = 0, \quad -\infty < x < \infty,\ 0 \le t < \infty \\ u(x,0) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 < x < \infty, \end{cases} \end{cases}$$

where $a$ is a constant in $(-1, 1)$ (recall that $\lambda$ denotes the ratio of the spatial and temporal mesh-lengths). The solution of (13.2.4) comprises, of course, the constant states $u = 0$, on the left, and $u = 1$, on the right, joined by the shock $x = a\lambda t$. The first four steps of the construction of the approximate solution $u_h$ according to the Lax-Friedrichs scheme are depicted in Fig. 13.2.1. The smearing of the shock is clear.

In order to prevent the smearing of shocks, we try a different policy for evaluating the $U_s^r$. We start out with some sequence $\wp = \{a_0, a_1, a_2, \cdots\}$, where $a_s \in (-1, 1)$, we set $y_s^r = x_r + a_s h$, and build, on the upper half-plane, another staggered grid of points $(y_s^r, t_s)$, with $s = 0, 1, 2, \cdots$ and $r+s$ odd. We employ $(y_s^r, t_s)$ as a *sampling point* for the interval $(x_{r-1}, x_{r+1})$, on the $t_s$-time line, by selecting

(13.2.5) $$U_s^r = \lim_{t\uparrow t_s} U_h(y_s^r-, t), \qquad r+s \text{ odd.}$$

To test this approach, we consider again the Riemann problem (13.2.4). The first few steps of the construction of the approximate solution $U_h$ are depicted in Fig. 13.2.2. We observe that according to the rule (13.2.5), as one passes from $t = t_s$ to $t = t_{s+1}$, the shock is preserved but its location is shifted by $h$, to the left when $a_s > a$, or to the right when $a_s < a$. Consequently, in the limit $h \downarrow 0$ the shock will

[1] In fact, it has been recently demonstrated, in the context of the closely related Godunov scheme, that selecting $\lambda$ to be an irrational number, but very close to a rational, induces resonance generating spurious oscillations in the approximate solutions, which drives the total variation to infinity.

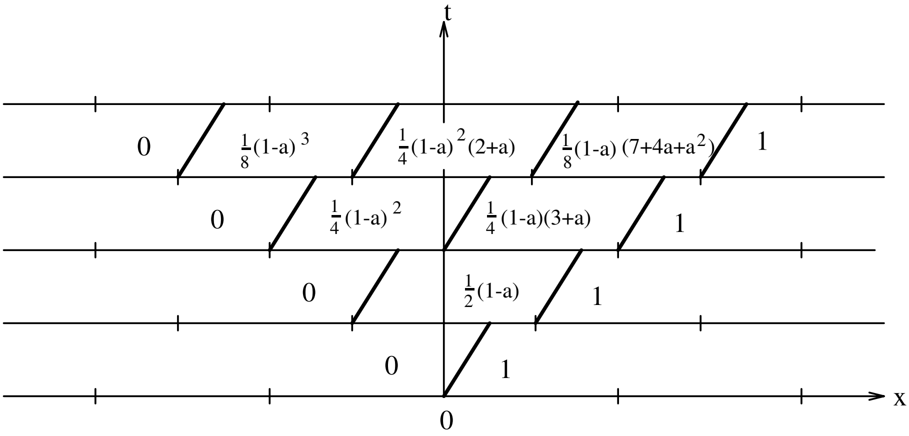


**Fig. 13.2.1**

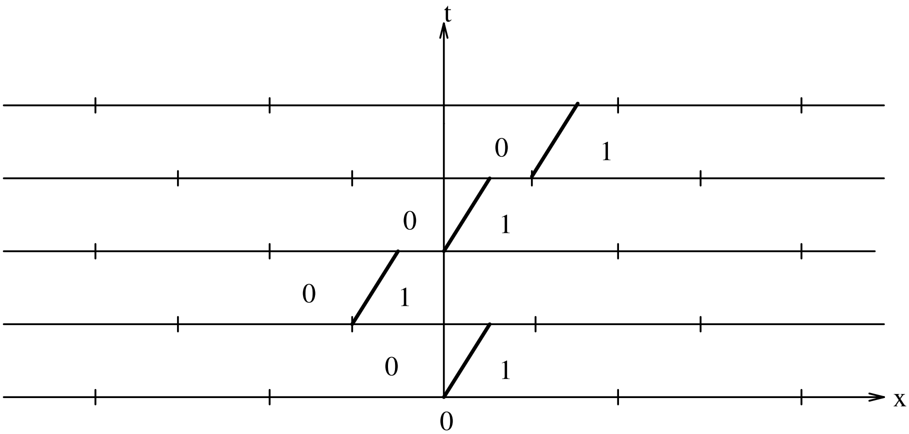


**Fig. 13.2.2**

be thrown off course, unless the number $m_-$ of indices $s \leq m$ with $a_s < a$ and the number $m_+$ of indices $s \leq m$ with $a_s > a$ are related through $m_- - m_+ \sim am$, as $m \to \infty$. Combining this with $m_- + m_+ = m$, we conclude that $u_h$ will converge to the solution of (13.2.4) if and only if $m_-/m \to \frac{1}{2}(1+a)$ and $m_+/m \to \frac{1}{2}(1-a)$, as $m \to \infty$. For consistency of the algorithm, it will be necessary that the above condition hold for arbitrary $a \in (-1, 1)$. Clearly, this will be the case only when the sequence $\wp$ is *equidistributed* on the interval $(-1, 1)$, that is, for any subinterval $I \subset (-1, 1)$ of length $\mu(I)$:

(13.2.6) $$\lim_{m\to\infty} \frac{2}{m}[\text{number of indices } s \leq m \text{ with } a_s \in I] = \mu(I),$$

uniformly with respect to $I$.

Later on, in Section 13.8, we shall see that the algorithm based on (13.2.5), with any sequence $\wp$ which is equidistributed in $(-1, 1)$, is indeed consistent, for the general initial-value problem (13.1.1); but this may be established only by paying the price of tracking the global wave pattern. The objective here is to demonstrate a slightly weaker result, whose proof however relies solely on the stability estimate (13.1.5). Roughly, it will be shown that if one picks the sequence $\wp$ at random, then the resulting algorithm will be consistent, with probability one. It is from this feature that the method derives its name: *random choice*.

We realize sequences $\wp$ as points in the Cartesian product space $\mathcal{A} = \prod_{s=0}^{\infty}(-1, 1)$. Each factor $(-1, 1)$ is regarded as a probability space, under Lebesgue measure rescaled by a factor $1/2$, and this induces a probability measure $\nu$ on $\mathcal{A}$ as well. In connection to our earlier discussions on consistency, it may be shown (references in Section 13.10) that almost all sequences $\wp \in \mathcal{A}$ are equidistributed in $(-1, 1)$. The main result is

**13.2.1 Theorem.** *There is a null subset $\mathcal{N}$ of $\mathcal{A}$ with the property that the algorithm induced by any sequence $\wp \in \mathcal{A}\backslash\mathcal{N}$ is consistent. That is, when the $U_s^r$ are evaluated through* (13.2.5), *with $y_s^r = x_r + a_s h$, then the limit $U$ in* (13.2.1) *is a solution of the initial-value problem* (13.1.1).

**Proof.** The right-hand side of (13.2.2) is completely determined by the spatial mesh-length $h$, the sequence $\wp$ and the test function $\phi$, so it shall be denoted by $e(\wp; \phi, h)$. By virtue of (13.2.5),

(13.2.7) $$e(\wp; \phi, h) = \sum_{s=0}^{\infty} e_s(\wp; \phi, h),$$

where

(13.2.8) $$e_s(\wp; \phi, h) = \sum_{r+s \text{ odd}} \int_{x_{r-1}}^{x_{r+1}} \phi(x, t_s)[U_h(x, t_s-) - U_h(y_s^r, t_s-)]dx.$$

The integral on the right-hand side of (13.2.8) is bounded from above by $2h \max|\phi| \operatorname{osc}_{(x_{r-1},x_{r+1})} U_h(\cdot, t_s-)$ and hence $e_s(\wp; \phi, h)$ is in turn majorized by $2h \max|\phi|\, TV_{(-\infty,\infty)} U_h(\cdot, t_s-)$. By (13.1.12) and (13.1.3), we conclude

(13.2.9) $$|e_s(\wp; \phi, h)| \le 2c_1\delta_1 h \max|\phi|, \qquad s = 0, 1, 2, \cdots.$$

In the summation (13.2.7), the number of nonzero terms, lying inside the support of $\phi$, is $O(h^{-1})$, and so the most one may generally extract from (13.2.9) is $e(\wp; \phi, h) = O(1)$, as $h \downarrow 0$. This again indicates that one should not expect consistency for an arbitrary sequence $\wp$. The success of the random choice method stems from the fact that, as $h \downarrow 0$, the average of $e_s(\wp; \phi, h)$ decays to zero faster than $e_s(\wp; \phi, h)$ itself. Indeed,

$$\text{(13.2.10)} \qquad \int_{-1}^{1} \int_{x_{r-1}}^{x_{r+1}} \phi(x, t_s)[U_h(x, t_s-) - U_h(y_s^r, t_s-)]dxda_s$$

$$= \frac{1}{h} \int_{x_{r-1}}^{x_{r+1}} \int_{x_{r-1}}^{x_{r+1}} \phi(x, t_s)[U_h(x, t_s-) - U_h(y, t_s-)]dxdy$$

is majorized by $2h^2 \max |\partial_x \phi| \operatorname{osc}_{(x_{r-1}x_{r+1})} U_h(\cdot, t_s-)$. The sum over $r$ of these integrals is then majorized by $2h^2 \max |\partial_x \phi| \, TV_{(-\infty,\infty)} U_h(\cdot, t_s-)$. Recalling (13.1.12) and (13.1.3), we finally conclude

$$\text{(13.2.11)} \qquad |\int_{-1}^{1} e_s(\wp; \phi, h)da_s| \leq 2c_1\delta_1 h^2 \max |\partial_x \phi|, \qquad s = 0, 1, 2, \cdots.$$

Next we demonstrate that, for $0 \leq s < \sigma < \infty$, $e_s(\wp; \phi, h)$ and $e_\sigma(\wp; \phi, h)$ are "weakly correlated" in that their inner product in $\mathcal{A}$ decays to zero very rapidly, $O(h^3)$, as $h \downarrow 0$. In the first place, $e_s(\wp; \phi, h)$ depends on $\wp$ solely through the first $s + 1$ components $(a_0, \cdots, a_s)$ and, similarly, $e_\sigma(\wp; \phi, h)$ depends on $\wp$ only through $(a_0, \cdots, a_\sigma)$. Hence, upon using (13.2.9) and (13.2.11),

$$\text{(13.2.12)} \qquad |\int_{\mathcal{A}} e_s(\wp; \phi, h)e_\sigma(\wp; \phi, h)d\nu(\wp)|$$

$$= |2^{-\sigma-1} \int_{-1}^{1} \cdots \int_{-1}^{1} e_s(\int_{-1}^{1} e_\sigma da_\sigma)da_0 \cdots da_{\sigma-1}|$$

$$\leq 2c_1^2\delta_1^2 h^3 \max |\phi| \max |\partial_x \phi|.$$

By virtue of (13.2.7),

$$\text{(13.2.13)} \qquad |e|^2 = \sum_{s=0}^{\infty} |e_s|^2 + 2 \sum_{s=0}^{\infty} \sum_{\sigma=s+1}^{\infty} e_s e_\sigma \,.$$

Since $\phi$ has compact support, on the right-hand side of (13.2.13) the first summation contains $O(h^{-1})$ nonzero terms and the second summation contains $O(h^{-2})$ nonzero terms. Consequently, on account of (13.2.9) and (13.2.11),

$$\text{(13.2.14)} \qquad \int_{\mathcal{A}} |e(\wp; \phi, h)|^2 d\nu(\wp) = O(h), \qquad \text{as } h \downarrow 0.$$

Thus there exists a null subset $\mathcal{N}_\phi$ of $\mathcal{A}$ such that $e(\wp; \phi, h_m) \to 0$, as $m \to \infty$, for any $\wp \in \mathcal{A} \backslash \mathcal{N}_\phi$. If $\{\phi_k\}$ is any countable set of test functions, which is $C^1$-dense in the set of all test functions with compact support in $(-\infty, \infty) \times [0, \infty)$, the null subset $\mathcal{N} = \bigcup_k \mathcal{N}_{\phi_k}$ of $\mathcal{A}$ will obviously satisfy the asssertion of the theorem. The proof is complete.

To conclude this section, we discuss the admissibility of the constructed solution.

**13.2.2 Theorem.** *Assume the system is endowed with an entropy-entropy flux pair* $(\eta, q)$, *where* $\eta(U)$ *is convex in* $\mathcal{O}$. *Then there is a null subset* $\mathcal{N}$ *of* $\mathcal{A}$ *with the following property: When the* $U_s^r$ *are evaluated via* (13.2.5), *with* $y_s^r = x_r + a_s h$, *then for any* $\wp \in \mathcal{A} \backslash \mathcal{N}$, *the limit* $U$ *in* (13.2.1) *is a solution of* (13.1.1) *which satisfies the entropy admissibility criterion.*

**Proof.** Inside each strip $\{(x,t) : -\infty < x < \infty,\ t_s \leq t < t_{s+1}\}$, $U_h$ is a solution of (13.1.9), with shocks that satisfy the viscous shock admissibility condition and thereby also the entropy shock admissibility criterion, relative to the entropy-entropy flux pair $(\eta, q)$ (cf. Theorem 8.6.2). Therefore, we have

$$\partial_t \eta(U_h(x,t)) + \partial_x q(U_h(x,t)) \leq 0, \quad -\infty < x < \infty, \quad t_s \leq t < t_{s+1}\,, \tag{13.2.15}$$

in the sense of measures.

Consider any nonnegative $C^\infty$ test function $\phi$ with compact support on $(-\infty, \infty) \times [0, \infty)$. We apply the measure (13.2.15) to the function $\phi$ on the rectangle $\{(x,t) : x_{r-1} < x < x_{r+1}\,,\ t_s \leq t < t_{s+1}\,,\ r+s \text{ odd}\}$ and sum over all such rectangles in the upper half-plane. After an integration by parts, and upon using (13.1.8) and (13.1.10), this yields

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty [\partial_t \phi\, \eta(U_h) + \partial_x \phi\, q(U_h)] dx dt + \int_{-\infty}^\infty \phi(x,0) \eta(U_0(x)) dx \\ \geq \sum_{s=0}^\infty \sum_{r+s \text{ odd}} \int_{x_{r-1}}^{x_{r+1}} \phi(x, t_s)[\eta(U_h(x, t_s-)) - \eta(U_s^r)] dx. \end{aligned} \tag{13.2.16}$$

Retracing the steps of the proof of Theorem 13.2.1, we deduce that there is a null subset $\mathcal{N}_\phi$ of $\mathcal{A}$ with the property that, when $\wp \in \mathcal{A} \backslash \mathcal{N}_\phi$, the right-hand side of (13.2.16) tends to zero, along the sequence $\{h_m\}$, as $m \to \infty$. Consequently, the limit $U$ in (13.2.1) satisfies the inequality

$$\int_0^\infty \int_{-\infty}^\infty [\partial_t \phi\, \eta(U) + \partial_x \phi\, q(U)] dx dt + \int_{-\infty}^\infty \phi(x,0) \eta(U_0(x)) dx \geq 0. \tag{13.2.17}$$

We now consider any countable set $\{\phi_k\}$ of nonnegative test functions that is $C^1$-dense in the set of all nonnegative test functions with compact support in the upper half-plane $(-\infty, \infty) \times [0, \infty)$, and define $\mathcal{N} = \bigcup_k \mathcal{N}_{\phi_k}$. It is clear that if one selects any $\wp \in \mathcal{A} \backslash \mathcal{N}$ then (13.2.17) will hold for all nonnegative test functions $\phi$ and hence $U$ will satisfy the entropy admissibility condition. This completes the proof.

In the absence of entropy-entropy flux pairs, or whenever the entropy admissibility criterion is not sufficiently discriminating to rule out all spurious solutions (cf. Chapter VIII), the question of admissibility of solutions constructed by the random choice method is subtle. It is plausible that the requisite shock admissibility conditions will hold at points of approximate jump discontinuity of the solution $U$, so long as they are satisfied by the shocks of the approximate solutions $U_h$ . Proving this, however, requires a more refined treatment of the limit process that yields $U$ from $U_h$ which may be attained by the method of wave partitioning outlined in Section 13.8.

## 13.3 Wave Interactions, Approximate Conservation Laws and Approximate Characteristics in Genuinely Nonlinear Systems

We now embark on the long journey that will eventually lead to the stability estimates (13.1.11), (13.1.12) and (13.1.13). The first step is to estimate local changes in the total variation of the approximate solutions $U_h$ . For simplicity, we limit the discussion to systems with characteristic families that are either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2). The general case is considerably more complicated and will be discussed briefly in Section 13.7.

According to the construction scheme, a portion of the wave fan emanating from the mesh-point $(x_{r-1}, t_{s-1})$, $r + s$ even, combines with a portion of the wave fan emanating from the mesh-point $(x_{r+1}, t_{s-1})$ to produce the wave fan that emanates from the mesh-point $(x_r, t_s)$. This is conveniently illustrated by enclosing the mesh-point $(x_r, t_s)$ in a diamond-shaped region $\Delta_s^r$ with vertices at the four surrounding sampling points, $(y_s^{r-1}, t_s)$, $(y_{s-1}^r, t_{s-1})$, $(y_s^{r+1}, t_s)$ and $(y_{s+1}^r, t_{s+1})$; see Fig. 13.3.1.

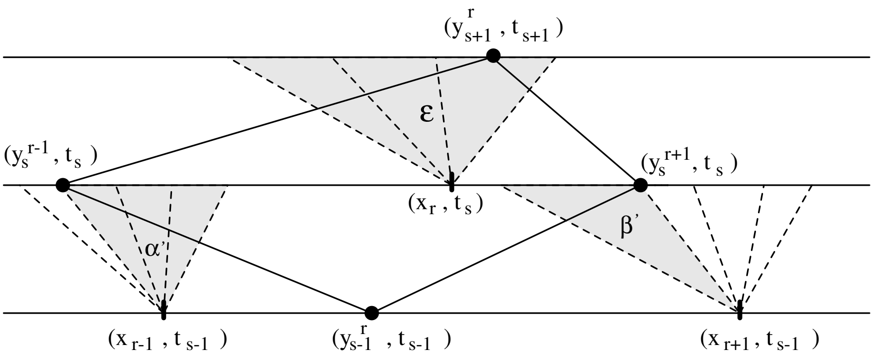


**Fig. 13.3.1**

A wave fan emanating from $(x_{r-1}, t_{s-1})$ and joining the state $U_s^{r-1}$, on the left, with the state $U_{s-1}^r$ , on the right, enters $\Delta_s^r$ through its "southwestern" edge. It may be represented, as explained in Sections 9.3 and 9.9, by the $n$-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of its wave amplitudes. A second wave fan, emanating from $(x_{r+1}, t_{s-1})$, joining the state $U_{s-1}^r$ , on the left, with the state $U_s^{r+1}$, on the right, and similarly represented by the $n$-tuple $\beta = (\beta_1, \cdots, \beta_n)$ of its wave amplitudes, enters $\Delta_s^r$ through its "southeastern" edge.

The output from $\Delta_s^r$ consists of the full wave fan which emanates from $(x_r, t_s)$, joins the state $U_s^{r-1}$, on the left, with the state $U_s^{r+1}$, on the right, and is represented by the $n$-tuple $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ of its wave amplitudes. A portion $\beta' = (\beta_1', \cdots, \beta_n')$ of $\varepsilon$ exits through the "northwestern" edge of $\Delta_s^r$ and enters the diamond $\Delta_{s+1}^{r-1}$, while the balance $\alpha' = (\alpha_1', \cdots, \alpha_n')$ exits through the "northeastern" edge of $\Delta_s^r$ and enters the diamond $\Delta_{s+1}^{r+1}$. Clearly, $\varepsilon_i = \alpha_i' + \beta_i'$, $i = 1, \cdots, n$. As explained in Section 9.4, for genuinely nonlinear characteristic families, a positive amplitude indicates a rarefaction wave and a negative amplitude indicates a compressive shock. Needless to say, a zero amplitude indicates that the wave of that family is missing from the wave fan in question. In particular, there is $j = 1, \cdots, n$ such that $\alpha_i' = 0$ for $i = 1, \cdots, j-1$ and $\beta_i' = 0$ for $i = j+1, \cdots, n$. Both $\alpha_j'$ and $\beta_j'$ may be nonzero, but then both must be positive, associated with rarefaction waves.

If the incoming wave fans $\alpha$ and $\beta$ were allowed to propagate freely, beyond the $t_s$-time line, the resulting wave interactions would generate a very intricate wave pattern. Nevertheless, following the discussion in Section 9.9, it should be expected that as $t \to \infty$ this wave pattern will reduce to a centered wave fan which is none other than $\varepsilon$. Thus the essense of our construction scheme is that it replaces actual, complex, wave patterns by their time-asymptotic, simpler, forms. In that connection, the role of "random choice" is to arrange the relative position of the wave fans in such a manner that "on the average" the law of "mass" conservation holds.

According to the terminology of Section 9.9, the wave fan $\varepsilon$ shall be regarded as the result of the interaction of the wave fan $\alpha$, on the left, with the wave fan $\beta$, on the right. It is convenient to realize $\varepsilon$, $\alpha$ and $\beta$ as $n$-vectors normed by the $\ell_1^n$ norm, in which case Theorem 9.9.1 yields the estimate

$$|\varepsilon - (\alpha + \beta)| \le [c_3 + c_4(|\alpha| + |\beta|)]\mathcal{D}(\Delta_s^r), \tag{13.3.1}$$

with $c_3$ and $c_4$ depending solely on $F$. In particular, $c_3 = 0$ when the system is endowed with a coordinate system of Riemann invariants. Here the symbol $\mathcal{D}(\Delta_s^r)$ is being used, in the place of $D(\alpha, \beta)$ in Section 9.9, to denote the *amount of wave interaction in the diamond* $\Delta_s^r$, namely,

$$\mathcal{D}(\Delta_s^r) = \sum_{\text{app}} |\alpha_k||\beta_j|. \tag{13.3.2}$$

The summation runs over all pairs of approaching waves, i.e. over all $(k, j)$ such that either $k > j$, or $k = j$ and at least one of $\alpha_j$, $\beta_j$ is negative, corresponding to a shock.

Formula (13.3.1) will serve as the vehicle for estimating how the total variation and the supremum of the approximate solutions $U_h$ change with time, as a result of wave interactions.

By (13.3.1), when $\alpha_i$ and $\beta_i$ have the same sign, the total strength $|\alpha_i'| + |\beta_i'|$ of $i$-waves leaving the diamond $\Delta_s^r$ is nearly equal to the total strength $|\alpha_i| + |\beta_i|$ of entering $i$-waves. However, when $\alpha_i$ and $\beta_i$ have opposite signs, cancellation of

$i$-waves takes place. To account for this phenomenon, which greatly affects the behavior of solutions, certain notions will now be introduced.

The *amount of i-wave cancellation in the diamond* $\Delta_s^r$ is conveniently measured by the quantity

$$\mathcal{C}_i(\Delta_s^r) = \tfrac{1}{2}(|\alpha_i| + |\beta_i| - |\alpha_i + \beta_i|). \tag{13.3.3}$$

In order to account separately for shocks and rarefaction waves, we rewrite (13.3.1) in the form

$$\varepsilon_i^{\pm} = \alpha_i^{\pm} + \beta_i^{\pm} - \mathcal{C}_i(\Delta_s^r) + [c_3 O(1) + O(\tau)]\mathcal{D}(\Delta_s^r), \tag{13.3.4}$$

where the superscript plus or minus denotes positive or negative part of the amplitude, and $\tau$ is the oscillation of $U_h$ .

Upon summing (13.3.4) over any collection of diamonds, whose union forms a domain $\Lambda$ in the upper half-plane, we end up with equations

$$L_i^{\pm}(\Lambda) = E_i^{\pm}(\Lambda) - \mathcal{C}_i(\Lambda) + [c_3 O(1) + O(\tau)]\mathcal{D}(\Lambda), \tag{13.3.5}$$

where $E_i^-$ (or $E_i^+$) denotes the total amount of $i$-shock (or $i$-rarefaction wave) that enters $\Lambda$ , $L_i^-$ (or $L_i^+$) denotes the total amount of $i$-shock (or $i$-rarefaction wave) that leaves $\Lambda$ , $\mathcal{C}_i(\Lambda)$ is the amount of $i$-wave cancellation inside $\Lambda$, and $\mathcal{D}(\Lambda)$ is the amount of wave interaction inside $\Lambda$. The equations (13.3.5) express the balance of $i$-waves relative to $\Lambda$ and, accordingly, are called *approximate conservation laws* for $i$-shocks (with minus sign) or $i$-rarefaction waves (with plus sign).

The total amount of wave cancellation in the diamond $\Delta_s^r$ is naturally measured by

$$\mathcal{C}(\Delta_s^r) = \sum_{i=1}^{n} \mathcal{C}_i(\Delta_s^r). \tag{13.3.6}$$

Notice that (13.3.1) implies

$$|\alpha'| + |\beta'| = |\varepsilon| \leq |\alpha| + |\beta| - 2\mathcal{C}(\Delta_s^r) + [c_3 + c_4(|\alpha| + |\beta|)]\mathcal{D}(\Delta_s^r). \tag{13.3.7}$$

An *approximate i-characteristic* associated with the approximate solution $U_h$ , and defined on the time interval $[t_\ell, t_m)$, is a sequence $\chi^{(\ell)}, \cdots, \chi^{(m-1)}$ of straight line segments, such that, for $s = \ell, \cdots, m-1$, $\chi^{(s)}$ is either a classical $i$-characteristic or an $i$-shock for $U_h$ , emanating from some mesh-point $(x_r, t_s)$, $r+s$ even, and defined on the time interval $[t_s, t_{s+1})$. Moreover, for $s = \ell+1, \cdots, m-1$, $\chi^{(s)}$ is a proper sequel to $\chi^{(s-1)}$, according to the following rules: $\chi^{(s-1)}$ must enter the diamond $\Delta_s^r$ centered at $(x_r, t_s)$. Whenever the interaction of $i$-waves entering $\Delta_s^r$ produces an $i$-shock, $\chi^{(s)}$ is that shock. On the other hand, when the interaction of the $i$-waves entering $\Delta_s^r$ produces an $i$-rarefaction wave, then $\chi^{(s)}$ is a classical $i$-characteristic identified by the requirement that the amount of $i$-rarefaction wave that leaves $\Delta_s^r$ on the left (right) of $\chi^{(s)}$ does not exceed the amount of $i$-rarefaction

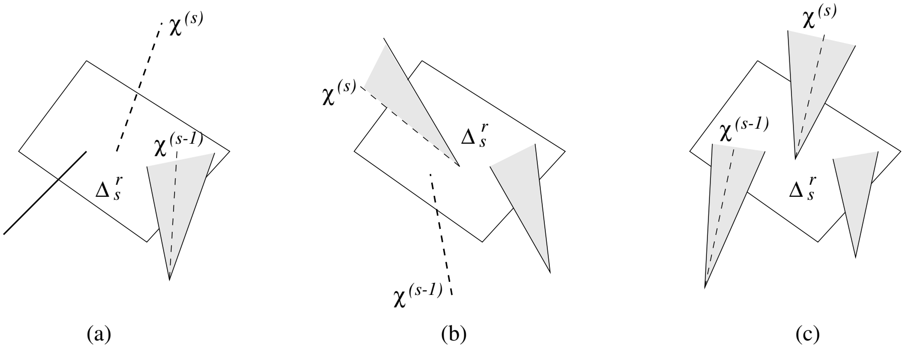


**Fig. 13.3.2**

wave that enters $\Delta_s^r$ on the left (right) of $\chi^{(s-1)}$. In applying the above rule, we tacitly assume that $\varepsilon_i = \alpha_i + \beta_i$ , disregarding the potential (small) contribution to $i$-rarefaction wave by wave interactions.

Figure 13.3.2 depicts three representative configurations. Only $i$-waves are illustrated and the approximate $i$-characteristic is drawn as a dotted line. In case (a), an $i$-shock interacts with an $i$-rarefaction wave to produce an $i$-shock. $\chi^{(s-1)}$ is a classical $i$-characteristic but $\chi^{(s)}$ will be the outgoing $i$-shock. In case (b), $\chi^{(s-1)}$ is an $i$-shock whose interaction with an $i$-rarefaction wave produces an $i$-rarefaction wave. Since the amount of $i$-rarefaction wave that enters $\Delta_s^r$ on the left of $\chi^{(s-1)}$ is nil, $\chi^{(s)}$ must be the left edge of the outgoing rarefaction wave. Finally, in case (c) two $i$-rarefaction waves interact to produce an $i$-rarefaction wave. Then $\chi^{(s)}$ is selected so that the amount of $i$-rarefaction wave on its left equals the amount of $i$-rarefaction wave that enters $\Delta_s^r$ on the left of $\chi^{(s-1)}$. This will automatically assure that the amount of $i$-rarefaction wave that leaves $\Delta_s^r$ on the right of $\chi^{(s)}$ equals the amount of $i$-rarefaction wave that enters $\Delta_s^r$ on the right of $\chi^{(s-1)}$, provided one neglects potential contribution to $i$-rarefaction wave by wave interactions.

The above construction of approximate characteristics has been designed so that the following principle holds: Rarefaction waves cannot cross approximate characteristics of their own family. Consequently, approximate conservation laws

$$(13.3.8) \qquad L_i^+(\Lambda_\pm) = E_i^+(\Lambda_\pm) - \mathcal{C}_i(\Lambda_\pm) + [c_3 O(1) + O(\tau)]\mathcal{D}(\Lambda_\pm),$$

for $i$-rarefaction waves, hold for the domains $\Lambda_\pm$ in which the diamond $\Delta_s^r$ is divided by any approximate $i$-characteristic (Fig. 13.3.3).

The corresponding approximate conservation laws for $i$-shocks assume a more complicated form, depending on how one apportions between $\Lambda_-$ and $\Lambda_+$ the strength of $i$-shocks that lie on the dividing boundary of $\Lambda_-$ and $\Lambda_+$ .

One may immediately extend the approximate conservation laws for $i$-rarefaction waves from the single diamond to any domain $\Lambda$ formed by the union of a collection of diamonds and thus write (13.3.8) for the domains $\Lambda_\pm$ into which $\Lambda$ is divided by any approximate $i$-characteristic.

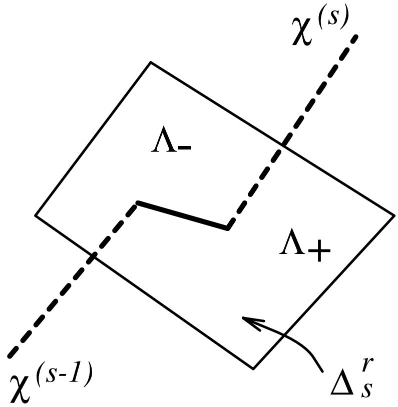


**Fig. 13.3.3**

Approximate conservation laws may be employed for deriving fine properties of approximate solutions, at least for systems of two conservation laws, which yield, in the limit, properties of solutions comparable to those established in Chapter XII by the method of generalized characteristics. Indeed, the $h \downarrow 0$ limit of any convergent sequence of approximate $i$-characteristics is necessarily a generalized $i$-characteristic, in the sense of Chapter X.

## 13.4 The Glimm Functional for Genuinely Nonlinear Systems

The aim here is to establish bounds on the total variation of approximate solutions $U_h$ along certain curves. We are still operating under the assumption that each characteristic family is either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2).

A *mesh curve*, associated with $U_h$, is a polygonal graph with vertices that form a finite sequence of sample points $(y_{s_1}^{r_1}, t_{s_1}), \cdots, (y_{s_m}^{r_m}, t_{s_m})$, where $r_{\ell+1} = r_\ell + 1$ and $s_{\ell+1} = s_\ell - 1$ or $s_{\ell+1} = s_\ell + 1$ (Fig. 13.4.1). Thus the edges of any mesh curve $I$ are also edges of diamond-shaped regions considered in the previous section. Any wave entering into a diamond through an edge shared with the mesh curve $I$ is said to be *crossing* $I$.

A mesh curve $J$ is called an *immediate successor* of the mesh curve $I$ when $J \backslash I$ is the upper (i.e., "northwestern" and "northeastern") boundary of some diamond, say $\Delta_s^r$, and $I \backslash J$ is the lower (i.e., "southwestern" and "southeastern") boundary of $\Delta_s^r$. Thus $J$ has the same vertices as $I$, save for one, $(y_{s-1}^r, t_{s-1})$, which is replaced by $(y_{s+1}^r, t_{s+1})$. This induces a natural partial ordering in the family of mesh curves: $J$ is a *successor* of $I$, denoted $I < J$, whenever there is a finite sequence, say $I = I_0, I_1, \cdots, I_m = J$, of mesh curves such that $I_\ell$ is an immediate successor of $I_{\ell-1}$, for $\ell = 1, \cdots, m$.

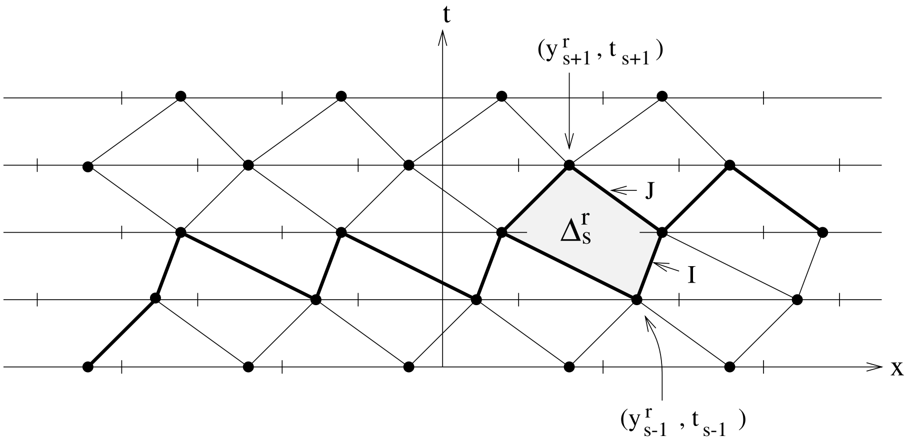


**Fig. 13.4.1**

With mesh curves $I$ we associate the functionals

$$\mathcal{S}(I) = \max |\xi|, \tag{13.4.1}$$

$$\mathcal{L}(I) = \sum |\xi|, \tag{13.4.2}$$

where both the maximum and the summation are taken over the amplitudes $\xi$ of all waves that are crossing $I$. Clearly, $\mathcal{S}(I)$ measures the oscillation and $\mathcal{L}(I)$ measures the total variation of $U_h$ along the curve $I$. We shall estimate the supremum and total variation of $U_h$ by monitoring how $\mathcal{S}$ and $\mathcal{L}$ change as one passes from $I$ to its successors.

Assume $J$ is an immediate successor of $I$, as depicted in Fig. 13.4.1. Wave fans $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ enter the diamond $\Delta_s^r$ through its "southwestern" and "southeastern" edge, respectively, and interact to generate, as discussed in Section 13.3, the wave fan $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, which exits $\Delta_s^r$ through its "northwestern" and "northeastern" edge. By virtue of (13.3.1) we deduce

$$\mathcal{S}(J) \leq \mathcal{S}(I) + [c_3 + c_4\mathcal{S}(I)]\mathcal{D}(\Delta_s^r), \tag{13.4.3}$$

$$\mathcal{L}(J) \leq \mathcal{L}(I) + [c_3 + c_4\mathcal{S}(I)]\mathcal{D}(\Delta_s^r), \tag{13.4.4}$$

where $c_3$ and $c_4$ are the constants that appear also in (13.3.1). In particular, when the system is endowed with a coordinate system of Riemann invariants, $c_3 = 0$. Clearly, $\mathcal{S}$ and $\mathcal{L}$ may increase as one passes from $I$ to $J$ and thus (13.4.3), (13.4.4) alone are insufficient to render the desired bounds (13.1.11), (13.1.12).

What saves the day is the realization that $\mathcal{L}$ may increase only as a result of interaction by approaching waves, which, after crossing paths, separate and move away from each other, never to meet again. Consequently, the potential for future interactions is embodied in the initial arrangement of waves and may thus be anticipated and estimated in advance. To formalize the above heuristic arguments, we shall associate with mesh curves $I$ a functional $\mathcal{Q}(I)$ which measures the *potential for future interactions of waves that are crossing* $I$.

An $i$-wave and a $j$-wave, crossing the mesh curve $I$, are said to be *approaching* if (a) $i > j$ and the $i$-wave is crossing on the left of the $j$-wave; or (b) $i < j$ and the $i$-wave is crossing on the right of the $j$-wave; or (c) $i = j$, the $i$-characteristic family is genuinely nonlinear and at least one of the waves is a shock. The reader should note the analogy with the notion of approaching waves in two interacting wave fans, introduced in Section 9.9. After this preparation, we set

$$\mathcal{Q}(I) = \sum_{\text{app}} |\zeta||\xi|, \tag{13.4.5}$$

where the summation runs over all pairs of approaching waves that are crossing $I$ and $\zeta$, $\xi$ are their amplitudes. Clearly,

$$\mathcal{Q}(I) \le \tfrac{1}{2}[\mathcal{L}(I)]^2. \tag{13.4.6}$$

The change in the potential of future wave interactions as one passes from the mesh curve $I$ to its immediate successor $J$, depicted in Fig. 13.4.1, is controlled by the estimate

$$\mathcal{Q}(J) - \mathcal{Q}(I) \le \{[c_3 + c_4\mathcal{S}(I)]\mathcal{L}(I) - 1\}\mathcal{D}(\Delta_s^r), \tag{13.4.7}$$

where $c_3$ and $c_4$ are the same constants appearing in (13.4.3) and (13.4.4).

To verify (13.4.7), we shall distinguish between *peripheral waves,* which are crossing both $I$ and $J$ on the left of $(y_s^{r-1}, t_s)$ or on the right of $(y_s^{r+1}, t_s)$, and *principal waves,* that is, constituents of the wave fans $\alpha$, $\beta$ or $\varepsilon$, which enter or exit $\Delta_s^r$ by crossing $I$ or $J$ between $(y_s^{r-1}, t_s)$ and $(y_s^{r+1}, t_s)$.

We first observe that pairs of principal waves from the incoming wave fans $\alpha$ and $\beta$ interact to contribute the amount $\mathcal{D}(\Delta_s^r)$ to $\mathcal{Q}(I)$. By contrast, no pair of principal waves from the outgoing wave fan $\varepsilon$ is approaching so as to make a contribution to $\mathcal{Q}(J)$.

Next we note that pairs of peripheral waves contribute equally to $\mathcal{Q}(I)$ and to $\mathcal{Q}(J)$; hence their net contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ is nil.

It remains to discuss the pairing of peripheral with principal waves. Let us examine the contributions to $\mathcal{Q}(I)$ and to $\mathcal{Q}(J)$ from the pairing of some fixed peripheral $i$-wave, of amplitude $\zeta$, with the $j$-waves of $\alpha$, $\beta$ and $\varepsilon$. One must distinguish the following cases: (i) $j > i$ and the peripheral $i$-wave is crossing $I$ on the left of $(y_s^{r-1}, t_s)$; (ii) $j < i$ and the peripheral $i$-wave is crossing $I$ on the right of $(y_s^{r+1}, t_s)$; (iii) $j = i$ and the $i$-characteristic family is linearly degenerate; (iv) $j > i$ and the peripheral $i$-wave is crossing $I$ on the right of $(y_s^{r+1}, t_s)$; (v) $j < i$ and the peripheral $i$-wave is crossing $I$ on the left of $(y_s^{r-1}, t_s)$; (vi) $j = i$, the $i$-characteristic family is genuinely nonlinear, and the peripheral wave is an $i$-shock, $\zeta < 0$; and (vii)

$j = i$, the $i$-characteristic family is genuinely nonlinear, and the peripheral wave is an $i$-rarefaction, $\zeta > 0$.

In cases (i), (ii) and (iii), the peripheral $i$-wave is not approaching any of the $j$-waves of $\alpha$, $\beta$, $\varepsilon$; hence the contribution to both $\mathcal{Q}(I)$ and $\mathcal{Q}(J)$ is nil. By contrast, in cases (iv), (v) and (vi), the peripheral $i$-wave is approaching all three of the $j$-waves of $\alpha$, $\beta$, $\varepsilon$; thus the contribution to $\mathcal{Q}(I)$ and $\mathcal{Q}(J)$ is $|\zeta|(|\alpha_j| + |\beta_j|)$ and $|\zeta||\varepsilon_j|$, respectively.

In the remaining case (vii), depending on the signs of $\alpha_i$ , $\beta_i$ and $\varepsilon_i$ , the peripheral $i$-wave may be approaching all, some, or none of the $i$-waves of $\alpha$, $\beta$ and $\varepsilon$; the contribution to $\mathcal{Q}(I)$ and $\mathcal{Q}(J)$ is $\zeta(\alpha_i^- + \beta_i^-)$ and $\zeta\varepsilon_i^-$ , respectively, where the superscript "minus" denotes "negative part".

From the above and (13.3.1), the total contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ from the pairing of any peripheral wave of amplitude $\zeta$ with all principal waves cannot exceed the amount $|\zeta|[c_3 + c_4\mathcal{S}(I)]\mathcal{D}(\Delta_s^r)$. Therefore we conclude that the overall contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ from such interactions is bounded by $[c_3 + c_4\mathcal{S}(I)]\mathcal{L}(I)\mathcal{D}(\Delta_s^r)$. This establishes (13.4.7).

The key consequence of (13.4.7) is that when $\mathcal{L}(I)$ is suffiently small the potential $\mathcal{Q}$ for future wave interactions will decrease as one passes from the mesh curve $I$ to its immediate successor $J$. We shall exploit this property to compensate for the possibility that $\mathcal{S}$ and $\mathcal{L}$ may be increasing, to the extent allowed by (13.4.3) and (13.4.4). For that purpose, we associate with mesh curves $I$ the *Glimm functional*

$$\mathcal{G}(I) = \mathcal{L}(I) + 2\kappa\mathcal{Q}(I), \tag{13.4.8}$$

where $\kappa$ is some fixed upper bound of $c_3 + c_4\mathcal{S}(I)$, independent of $I$ and $h$. Even though $\mathcal{G}$ majorizes $\mathcal{L}$, it is actually equivalent to $\mathcal{L}$ by account of (13.4.6).

**13.4.1 Theorem.** *Let $I$ be a mesh curve with $4\kappa\mathcal{L}(I) \le 1$. Then, for any mesh curve $J$ that is a successor of $I$,*

$$\mathcal{G}(J) \le \mathcal{G}(I), \tag{13.4.9}$$

$$\mathcal{L}(J) \le 2\mathcal{L}(I). \tag{13.4.10}$$

*Furthermore, the amount of wave interaction and the amount of wave cancellation in the diamonds confined between the curves $I$ and $J$ are bounded:*

$$\sum \mathcal{D}(\Delta_s^r) \le [\mathcal{L}(I)]^2, \tag{13.4.11}$$

$$\sum \mathcal{C}(\Delta_s^r) \le \mathcal{L}(I). \tag{13.4.12}$$

**Proof.** Assume first that $J$ is the immediate successor of $I$ depicted in Fig. 13.4.1. Upon combining (13.4.8) with (13.4.4) and (13.4.7), we deduce

$$\mathcal{G}(J) \le \mathcal{G}(I) + \kappa[2\kappa\mathcal{G}(I) - 1]\mathcal{D}(\Delta_s^r). \tag{13.4.13}$$

By virtue of (13.4.8), (13.4.6) and $4\kappa\mathcal{L}(I) \le 1$, we obtain

$$\mathcal{G}(I) \le 2\mathcal{L}(I), \tag{13.4.14}$$

so that $2\kappa\mathcal{G}(I) \le 1$, in which case (13.4.13) yields (13.4.9).

Assume now that $J$ is any successor of $I$. Iterating the above argument, we establish (13.4.9) for that case as well. Since $\mathcal{L}(J) \le \mathcal{G}(J)$, (13.4.10) follows from (13.4.9) and (13.4.14). Summing (13.4.7) over all diamonds confined between the curves $I$ and $J$ and using (13.4.10), we obtain

$$\tfrac{1}{2}\sum \mathcal{D}(\Delta_s^r) \le \mathcal{Q}(I) - \mathcal{Q}(J), \tag{13.4.15}$$

which yields (13.4.11), by virtue of (13.4.6).

We sum (13.3.7) over all the diamonds confined between the curves $I$ and $J$, to get

$$2\sum \mathcal{C}(\Delta_s^r) \le \mathcal{L}(I) - \mathcal{L}(J) + \kappa \sum \mathcal{D}(\Delta_s^r). \tag{13.4.16}$$

Combining (13.4.16) with (13.4.10) and (13.4.11) we arrive at (13.4.12). This completes the proof.

The above theorem is of fundamental importance. In particular, the estimates (13.4.9) and (13.4.10) provide the desired bounds on the total variation while (13.4.11) and (13.4.12) embody the dissipative effects of nonlinearity and have significant implications for regularity and large time behavior of solutions.

The assumption $4\kappa\mathcal{L}(I) \le 1$ in the above theorem means that $\mathcal{L}(I)$ itself should be sufficiently small, for general systems. However, in systems endowed with a coordinate system of Riemann invariants, where $c_3 = 0$, it would suffice that $(\sup U_h)\,\mathcal{L}(I)$ be sufficiently small. For this special class of systems, $\sup U_h$ will be estimated with the help of

**13.4.2 Theorem.** *Assume that the system is endowed with a coordinate system of Riemann invariants. Let $I$ be a mesh curve with $4\kappa\mathcal{L}(I) \le 1$. Then, for any mesh curve $J$ that is a successor of $I$,*

$$\mathcal{S}(J) \le \exp[c_4\mathcal{L}(I)^2]\,\mathcal{S}(I). \tag{13.4.17}$$

**Proof.** Assume first that $J$ is the immediate successor of $I$ depicted in Fig. 13.4.1. Since $c_3 = 0$, (13.4.3) yields

$$\mathcal{S}(J) \le [1 + c_4\mathcal{D}(\Delta_s^r)]\mathcal{S}(I). \tag{13.4.18}$$

Iterating the above argument, we deduce that if $J$ is any successor of $I$, then

$$\mathcal{S}(J) \le \prod [1 + c_4\mathcal{D}(\Delta_s^r)]\mathcal{S}(I), \tag{13.4.19}$$

where the product runs over all the diamonds confined between the curves $I$ and $J$. Combining (13.4.19) with (13.4.11), we arrive at (13.4.17). This completes the proof.

For systems endowed with a coordinate system of Riemann invariants, it is expedient to measure wave strength by the jump of the corresponding Riemann invariant across the wave. In particular, for systems with coinciding shock and rarefaction wave curves (see Section 8.2) this policy renders $\mathcal{L}$ itself nonincreasing, as one passes from a mesh curve to its successor, and thus allows us to estimate the total variation of the solution without any restriction on the size of the total variation of the initial data. There is another, very special, class of systems of two conservation laws in which a suitable measurement of wave strength yields a nonincreasing $\mathcal{L}$, and thereby existence of solutions to the Cauchy problem under initial data with large total variation. An interesting representative of that class is the system

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x (u^{-1}) = 0, \end{cases} \tag{13.4.20}$$

namely the special case of (7.1.8) with $\sigma(u) = -u^{-1}$. In classical gas dynamics, this system governs the isothermal flow of a polytropic ideal gas, in Lagrangian coordinates.

## 13.5 Bounds on the Total Variation for Genuinely Nonlinear Systems

Here we prove the estimates (13.1.12) and (13.1.13), always operating under the assumption that the oscillation of $U_h$ is bounded, uniformly in $h$. The vehicle will be the following corollary of Theorem 13.4.1:

**13.5.1 Theorem.** *Fix $0 \leq \tau < t < \infty$ and $-\infty < a < b < \infty$. Assume that $\kappa$ times the total variation of $U_h(\cdot, t)$ over the interval $[a - \lambda(t-\tau) - 6h, b + \lambda(t-\tau) + 6h]$ is sufficiently small.*[2] *Then*

$$TV_{[a,b]} U_h(\cdot, t) \leq c_1 TV_{[a-\lambda(t-\tau)-6h, b+\lambda(t-\tau)+6h]} U_h(\cdot, \tau), \tag{13.5.1}$$

*where $c_1$ depends solely on $F$. Furthermore, if $x$ is a point of continuity of both $U_h(\cdot, \tau)$ and $U_h(\cdot, t)$, and $\kappa$ times the total variation of $U_h(\cdot, t)$ over the interval $[x - \lambda(t-\tau) - 6h, x + \lambda(t-\tau) + 6h]$ is sufficiently small, then*

$$|U_h(x,t) - U_h(x,\tau)| \leq c_5 TV_{[x-\lambda(t-\tau)-6h, x+\lambda(t-\tau)+6h]} U_h(\cdot, \tau), \tag{13.5.2}$$

*where $c_5$ depends solely on $F$.*

[2] As before, $\lambda$ here denotes the ratio of spatial and temporal mesh-lengths.

**Proof.** First we determine nonnegative integers $\sigma$ and $s$ such that $t_\sigma \le \tau < t_{\sigma+1}$ and $t_s \le t < t_{s+1}$. Next we identify integers $r_1$ and $r_2$ such that $y_{s+1}^{r_1+1} < a \le y_{s+1}^{r_1+3}$ and $y_{s+1}^{r_2-3} \le b < y_{s+1}^{r_2-1}$. We then set $r_3 = r_1 - (s-\sigma)$ and $r_4 = r_2 + (s-\sigma)$.

We now construct two mesh curves $I$ and $J$, as depicted in Fig. 13.5.1, by the following procedure: $I$ originates at the sampling point $(y_\sigma^{r_3}, t_\sigma)$ and zig-zags between $t_\sigma$ and $t_{\sigma+1}$ until it reaches the sampling point $(y_\sigma^{r_4}, t_\sigma)$, where it terminates. $J$ also originates at $(y_\sigma^{r_3}, t_\sigma)$, takes $s-\sigma$ steps to the "northeast", reaching the sampling point $(y_s^{r_1}, t_s)$, then it zig-zags between $t_s$ and $t_{s+1}$ until it arrives at the sampling point $(y_s^{r_2}, t_s)$, and finally takes $s-\sigma$ steps to the "southeast" terminating at $(y_\sigma^{r_4}, t_\sigma)$.

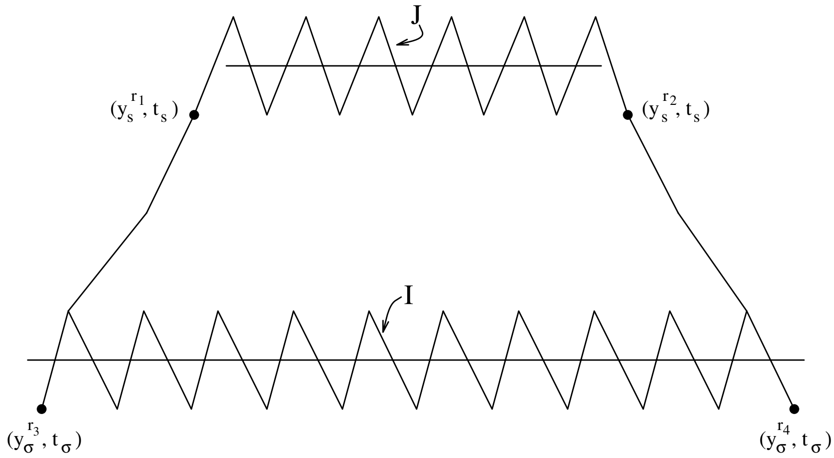


**Fig. 13.5.1**

Clearly,

$$TV_{[a,b]}U_h(\cdot, t) \le c_6 \mathcal{L}(J). \tag{13.5.3}$$

It is easy to see that $y_\sigma^{r_3} \ge a - \lambda(t-\tau) - 6h$ and $y_\sigma^{r_4} \le b + \lambda(t-\tau) + 6h$. Therefore,

$$\mathcal{L}(I) \le c_7 TV_{[a-\lambda(t-\tau)-6h, b+\lambda(t-\tau)+6h]}U_h(\cdot, \tau). \tag{13.5.4}$$

Also, $J$ is a successor of $I$ and hence, if $4\kappa\mathcal{L}(I) \le 1$, Theorem 13.4.1 implies $\mathcal{L}(J) \le 2\mathcal{L}(I)$. Combining this with (13.5.3) and (13.5.4), we arrive at (13.5.1), with $c_1 = 2c_6c_7$.

Given $x$, we repeat the above construction of $I$ and $J$ with $a = b = x$. We can identify a point $(y', \tau')$ on $I$ with $U_h(y', \tau') = U_h(x, \tau)$ as well as a point $(x', t')$ on $J$ with $U_h(x', t') = U_h(x, t)$. Hence

$$|U_h(x,t) - U_h(x,\tau)| \le c_8[\mathcal{L}(I) + \mathcal{L}(J)] \le 3c_8\mathcal{L}(I). \tag{13.5.5}$$

From (13.5.5) and (13.5.4), with $a = b = x$, we deduce (13.5.2) with $c_5 = 3c_7c_8$. This completes the proof.

Applying (13.5.1) for $\tau = 0, a \to -\infty, b \to \infty$, and taking into account that $TV_{(-\infty,\infty)}U_h(\cdot, 0) \le TV_{(-\infty,\infty)}U_0(\cdot)$, we verify (13.1.12).

Finally, we integrate (13.5.2) over $(-\infty, \infty)$, apply Fubini's theorem, and use (13.1.12) to get

(13.5.6)

$$\begin{aligned}\int_{-\infty}^{\infty} |U_h(x,t) - U_h(x,\tau)|dx &\le c_5 \int_{-\infty}^{\infty} TV_{[x-\lambda(t-\tau)-6h,x+\lambda(t-\tau)+6h]}U_h(\cdot,\tau)dx \\ &= 2c_5[\lambda(t-\tau)+6h]TV_{(-\infty,\infty)}U_h(\cdot,\tau) \\ &\le 2c_1c_5[\lambda(t-\tau)+6h]TV_{(-\infty,\infty)}U_0(\cdot),\end{aligned}$$

which establishes (13.1.13).

## 13.6 Bounds on the Supremum for Genuinely Nonlinear Systems

One may readily obtain a bound on the $L^\infty$ norm of $U_h$ from (13.5.2), with $\tau = 0$:

(13.6.1) $$\sup_{(-\infty,\infty)}|U_h(\cdot,t)| \le \sup_{(-\infty,\infty)}|U_0(\cdot)| + c_5 TV_{(-\infty,\infty)}U_0(\cdot).$$

This estimate is not as strong as the asserted (13.1.11), because, in addition to the supremum, it involves the total variation of the initial data. Even so, combining (13.6.1) with the estimates (13.1.12) and (13.1.13), established in Section 13.5, allows us to invoke the results of Section 13.2 and thus infer the existence of a solution $U$ to the initial-value problem (13.1.1), which is the limit of a sequence of approximate solutions; cf. (13.2.1). Clearly, $U$ satisfies (13.1.5) and (13.1.6), by virtue of (13.1.12) and (13.1.13). We have thus verified all the assertions of Theorem 13.1.1, save (13.1.4). Despite the fact that it is inessential for demonstrating existence of solutions, (13.1.4) has intrinsic interest, as a statement of stability, and also plays a useful role in deriving other qualitative properties of solutions. It is thus important to establish the estimate (13.1.11), which yields (13.1.4).

We first note that for systems endowed with a coordinate system of Riemann invariants, (13.1.11) is an immediate corollary of Theorem 13.4.2 and thus $\delta_1$ in (13.1.3) need not be small, so long as $\delta_2$ in (13.1.7) is. The proof in this case is so simple because terms of quadratic order are missing in the interaction estimate (13.3.1), i.e., $c_3 = 0$. By contrast, in systems devoid of this special structure, the interaction terms of quadratic order complicate the situation. The proof of (13.1.11) hinges on the special form of the quadratic terms, which, as seen in (9.9.13), involve the Lie brackets of the eigenvectors of $\mathrm{D}F$. The analysis is too laborious to be reproduced here in its entirety, so only an outline of the main ideas shall be presented. The reader may find the details in the references cited in Section 13.10.

The general strategy of the proof is motivated by the ideas expounded in Section 13.4, which culminated in the proof of Theorems 13.4.1 and 13.4.2. Two functionals, $\mathcal{R}$ and $\mathcal{P}$, will be associated with mesh curves $I$, where $\mathcal{R}(I)$ measures the oscillation of $U_h$ over $I$ while $\mathcal{P}(I)$ provides an estimate on how the oscillation may be affected by future wave interactions. For measuring the oscillation with accuracy, it becomes necessary to account for the mutual cancellation of shocks and rarefaction waves of the same characteristic family. We thus have to tally amplitudes, rather than strengths of waves. Accordingly, with any (finite) sequence of, say, $M$ waves with amplitudes $\xi = (\xi_1, \dots, \xi_M)$, we associate the number

$$|\xi| = \sum_{j=1}^{n} | \sum_{j-\text{waves}} \xi_L |, \tag{13.6.2}$$

where the second summation runs over the indices $L = 1, \cdots, M$ for which the $L$-th wave in the sequence is a $j$-wave. We then define

$$\mathcal{R}(I) = \sup |\xi|, \tag{13.6.3}$$

where the supremum is taken over all sequences of waves crossing $I$ that are *consecutive*, in the sense that any two of them occupying consecutive places in the sequence are separated by a constant state of $U_h$. After a little reflection, one sees that, as long as $\mathcal{L}(I)$ is sufficiently small, $\mathcal{R}(I)$ measures the oscillation of $U_h$ over $I$.

As one passes from $I$ to its successors, the value of $\mathcal{R}$ changes for two reasons: First, as waves travel at different speeds, crossings occur and wave sequences are reordered (notice, however, that the relative order of waves of the same characteristic family is necessarily preserved). Secondly, the amplitude of waves changes in result of wave interactions, as indicated in (9.9.13). It turns out that the effect of wave interactions of third or higher order in wave strength may be estimated grossly, as in the proof of Theorem 13.4.2. However, the effect of wave interactions of quadratic order in wave strength is more significant and thus must be estimated with higher precision. This may be accomplished in an effective manner by realizing the quadratic terms in (9.9.13) as new *virtual waves* which should be accounted for, along with the actual waves.

The aforementioned functional $\mathcal{P}$, which will help us estimate the effect of future wave interactions, is constructed by the following procedure. With any sequence of consecutive waves crossing the mesh curve $I$, one associates a family of sequences of waves, which are regarded as its "descendents". A descendent sequence of waves is derived from its "parental" one by the following two operations: (a) Admissible reorderings of the waves in the parental sequence $j$, e.g. a $k$-wave occupying the $K$-th place and an $\ell$-wave occupying the $L$-th place in the parental sequence, exchange places if $k > \ell$ and $K < L$. (b) Insertion of any virtual waves that may be generated from interactions of waves in the parental sequence. The precise construction of descendent sequences entails a major technical endeavor, which shall not be undertaken here, but can be found in the references cited in Section 13.10. For any mesh curve $I$, we set

$$\mathcal{P}(I) = \sup |\xi|, \tag{13.6.4}$$

where the supremum is now taken over the union of the descendent families of all sequences of consecutive waves that are crossing $I$.

As long as the total variation is small, $\mathcal{P}$ is actually equivalent to $\mathcal{R}$:

$$\mathcal{R}(I) \le \mathcal{P}(I) \le [1 + c_9\mathcal{L}(I)]\mathcal{R}(I). \tag{13.6.5}$$

The idea of the proof of (13.6.5) is as follows. Recall that the principal difference between $\mathcal{L}(I)$ and $\mathcal{R}(I)$ is that in the former we tally the (positive) strengths of crossing waves while in the latter we sum the (signed) amplitudes of crossing waves, thus allowing for cancellation between waves in the same characteristic family but of opposite signs (i.e., shocks and rarefaction waves). Consider the interaction of a single $j$-wave, with amplitude $\zeta$, with a number of $k$-waves. Since waves in the same characteristic family preserve their relative order, the interactions of the $k$-waves with the $j$-wave will occur consecutively and so the resulting virtual waves will also appear in the same order. Furthermore, whenever the amplitudes of the $k$-waves alternate in sign, then so do the corresponding Lie bracket terms. Consequently, the virtual waves undergo the same cancellation as their parent waves and thus the contribution to $\mathcal{P}(I)$ by the interaction of the $j$-wave with the $k$-waves will be of the order $O(1)|\zeta|\mathcal{S}(I)$. Thus the total contribution to $\mathcal{P}(I)$ from such interactions will be $O(1)\mathcal{L}(I)\mathcal{S}(I)$, whence (13.6.5) follows. The detailed proof is quite lengthy and may be found in the references.

The next step is to show that if $J$ is the immediate successor of the mesh curve $I$ depicted in Fig. 13.4.1, then

$$\mathcal{P}(J) \le \mathcal{P}(I) + c_{10}\mathcal{R}(I)\mathcal{D}(\Delta_s^r). \tag{13.6.6}$$

The idea of the proof is as follows. Sequences of waves crossing $J$ are reorderings of sequences that cross $I$, with the waves entering the diamond $\Delta_s^r$ through its "southwestern" and "southeastern" edges exchanging their relative positions as they exit $\Delta_s^r$. Furthermore, as one passes from $I$ to $J$ the virtual waves produced by the interaction of the waves that enter $\Delta_s^r$ are converted into actual waves, embodied in the waves that exit $\Delta_s^r$. Again, the detailed proof is quite lengthy and should be sought in the references.

By virtue of (13.6.5), we may substitute $\mathcal{P}(I)$ for $\mathcal{R}(I)$ on the right-hand side of (13.6.6), without violating the inequality. Therefore, upon iterating the argument, we conclude that if $J$ is any successor of $I$, then

$$\mathcal{P}(J) \le \prod [1 + c_{10}\mathcal{D}(\Delta_s^r)]\mathcal{P}(I), \tag{13.6.7}$$

where the product runs over all the diamonds $\Delta_s^r$ confined between the curves $I$ and $J$.

We now assume $4\kappa\mathcal{L}(I) \le 1$ and appeal to Theorem 13.4.1. Combining (13.6.7), (13.4.11) and (13.6.5) yields

$$\mathcal{R}(J) \le \exp[c_9\mathcal{L}(I) + c_{10}\mathcal{L}(I)^2]\mathcal{R}(I), \tag{13.6.8}$$

whence the desired estimate (13.1.11) readily follows.

## 13.7 General Systems

In this section we discuss briefly how to obtain bounds on the total variation of approximate solutions $U_h$ along mesh curves, for systems with characteristic families that are merely piecewise genuinely nonlinear. These bounds will be derived by the procedure used in Section 13.4 for genuinely nonlinear systems, except that the functional measuring the potential for future wave interactions shall be modified, as wave interactions are here governed by Theorem 9.9.2 (rather than 9.9.1). Thus, if we consider the diamond $\Delta_s^r$, with incoming wave fans $\alpha$ and $\beta$, entering through the "southwest" and the "southeast" edge, respectively, and outgoing wave fan $\varepsilon$, (9.9.14) yields

$$|\varepsilon - (\alpha + \beta)| \leq c_{11}\mathcal{D}(\Delta_s^r), \tag{13.7.1}$$

where

$$\mathcal{D}(\Delta_s^r) = \sum \theta |\gamma| |\delta|. \tag{13.7.2}$$

Recall that the above summation runs over all pairs of elementary $i$-waves, with amplitude $\gamma$, and $j$-waves, with amplitude $\delta$, entering $\Delta_s^r$ through its "southwestern" and "southeastern" edge, respectively. The weighting factor $\theta$ is determined as follows: $\theta = 0$ if $i < j$; $\theta = 1$ if either $i > j$ or $i = j$ and $\gamma\delta < 0$; finally, $\theta$ is given by (9.9.16) if $i = j$ and $\gamma\delta > 0$.

As in Section 13.4, with any mesh curve $I$ we associate the functional $\mathcal{L}(I)$, defined by (13.4.2). Assuming $J$ is the immediate successor of $I$ depicted in Fig. 13.4.1, (13.7.1) yields

$$\mathcal{L}(J) \leq \mathcal{L}(I) + c_{11}\mathcal{D}(\Delta_s^r). \tag{13.7.3}$$

The increase in $\mathcal{L}$ allowed by (13.7.3) will be offset by the decrease in a functional $\mathcal{Q}$, which monitors the potential for future wave interactions and is here defined by

$$\mathcal{Q}(I) = \sum \theta |\zeta| |\xi|. \tag{13.7.4}$$

The above summation runs over all pairs of elementary $i$-waves and $j$-waves, with respective amplitudes $\zeta$ and $\xi$, that are crossing the mesh curve $I$. When the $i$-wave is crossing $I$ on the left of the $j$-wave, then $\theta = 0$ if $i < j$ and $\theta = 1$ if $i > j$. When $i = j$ and $\zeta\xi < 0$, then $\theta = 1$. Finally, if $i = j$ and $\zeta\xi > 0$, then $\theta$ is determined by (9.9.16); and in particular by $(9.9.16)_1$ when the wave on the left is an $i$-shock with speed $\sigma_L$ and the wave on the right is an $i$-shock with speed $\sigma_R$ ; or by $(9.9.16)_2$ when the wave on the left is an $i$-shock with speed $\sigma_L$ , while the wave on the right is an $i$-rarefaction, joining $U_R$ with $V_i(\tau_R; U_R)$; or by $(9.9.16)_3$ when the wave on the left is an $i$-rarefaction, joining $U_L$ with $V_i(\tau_L; U_L)$, while the wave on the right is an $i$-shock with speed $\sigma_R$ ; or by $(9.9.16)_4$ when the wave on the left is a rarefaction, joining $U_L$ with $V_i(\tau_L; U_L)$, and the wave on the right is also a rarefaction, joining $U_R$ with $V_i(\tau_R; U_R)$.

The aim is to demonstrate the analog of (13.4.7), namely that if $J$ is the immediate successor of $I$ depicted in Fig. 13.4.1, then

$$\mathcal{Q}(J) - \mathcal{Q}(I) \leq [c_{12}\mathcal{L}(I) - 1]\mathcal{D}(\Delta_s^r). \tag{13.7.5}$$

Once (13.7.5) is established, one considers, as in Section 13.4, the Glimm functional $\mathcal{G}$, defined by (13.4.8), and shows that if $\kappa$ is selected sufficiently large and $\mathcal{L}(I)$ is small, then $\mathcal{G}(J) \leq \mathcal{G}(I)$. This in turn yields the desired estimates (13.1.12) and (13.1.13), by the arguments employed in Section 13.5.

To verify (13.7.5), let us retrace the steps in the proof of (13.4.7), making the necessary adjustments. We shall use again the terms "peripheral" and "principal" waves, to distinguish the elementary waves that are crossing both $I$ and $J$ from those which enter or exit $\Delta_s^r$, thus crossing only $I$ or only $J$.

To begin with, the interaction among principal waves of the two incoming wave fans $\alpha$ and $\beta$ contributes the amount $\mathcal{D}(\Delta_s^r)$ to $\mathcal{Q}(I)$. By contrast, pairs of principal waves from the outgoing wave fan $\varepsilon$ make no contribution to $\mathcal{Q}(J)$.

The next observation is that pairs of peripheral waves contribute equally to $\mathcal{Q}(I)$ and $\mathcal{Q}(J)$; hence their net contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ is nil.

It remains to examine the pairing of peripheral waves with principal waves. Let us estimate the contribution to $\mathcal{Q}(I)$ and to $\mathcal{Q}(J)$ from the pairing of some fixed peripheral $i$-wave, of amplitude $\zeta$, with the elementary $j$-waves of $\alpha$, $\beta$ and $\varepsilon$. As in the genuinely nonlinear situation, we must consider a number of cases: (i) $j > i$ and the peripheral $i$-wave is crossing $I$ on the left of $(y_s^{r-1}, t_s)$; (ii) $j < i$ and the peripheral $i$-wave is crossing $I$ on the right of $(y_s^{r+1}, t_s)$; (iii) $j > i$ and the peripheral $i$-wave is crossing $I$ on the right of $(y_s^{r+1}, t_s)$; (iv) $j < i$ and the peripheral $i$-wave is crossing $I$ on the left of $(y_s^{r-1}, t_s)$; (v) $j = i$, $\alpha_i\beta_i > 0$ and $\zeta(\alpha_i + \beta_i) < 0$; (vi) $j = i$, $\alpha_i\beta_i < 0$ and $\zeta(\alpha_i + \beta_i) < 0$; (vii) $j = i$, $\alpha_i\beta_i > 0$ and $\zeta(\alpha_i + \beta_i) > 0$; and (viii) $j = i$, $\alpha_i\beta_i < 0$ and $\zeta(\alpha_i + \beta_i) > 0$.

In cases (i) and (iii), the contribution to both $\mathcal{Q}(I)$ and $\mathcal{Q}(J)$ is obviously nil. By contrast, in cases (iii) and (iv), the contribution to $\mathcal{Q}(I)$ and $\mathcal{Q}(J)$ is $|\zeta|(|\alpha_j| + |\beta_j|)$ and $|\zeta||\varepsilon_j|$, respectively.

In case (v), the contribution to $\mathcal{Q}(I)$ is $|\zeta|(|\alpha_i| + |\beta_i|)$. The contribution to $\mathcal{Q}(J)$ depends on the sign of $\zeta\varepsilon_i$, but under any circumstance may not exceed the amount $|\zeta||\varepsilon_i|$. Similarly, in case (vi) the contribution to $\mathcal{Q}(I)$ is at least $|\zeta|\max\{|\alpha_i|, |\beta_i|\}$, while the contribution to $\mathcal{Q}(J)$ is at most $|\zeta||\varepsilon_i|$.

From the above and (13.7.1) it follows that the total contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ from the pairing of the peripheral $i$-wave with all the principal waves that fall under one of cases (i) through (vi) cannot exceed the amount $c_{11}|\zeta|\mathcal{D}(\Delta_s^r)$.

The remaining cases (vii) and (viii) require a more delicate treatment. In fact, it is at this point that the difference between genuinely nonlinear systems and general systems comes to the fore. For orientation, let us examine the special, albeit representative, situation considered in the proof of Theorem 9.9.2: The incoming wave fans $\alpha$ and $\beta$ consist of a single $i$-shock each, with respective amplitudes $\gamma$ and $\delta$ and respective speeds $\sigma_L$ and $\sigma_R$, $\sigma_R \leq \sigma_L$. The $i$-th wave fan of the outgoing wave fan $\varepsilon$ also consists of a single $i$-shock, with amplitude $\varepsilon_i$ and speed $\sigma$. For definiteness, it will be further assumed that the peripheral $i$-wave is likewise an $i$-shock, with amplitude $\zeta$ and speed $\sigma_0 < \sigma_R$ , which is crossing $I$ on the right of $(y_s^{r+1}, t_s)$. In accordance with case (vii), above, let $\gamma$, $\delta$, $\varepsilon_i$ and $\zeta$ be all positive. Then the contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ is

$$\zeta\{(\sigma - \sigma_0)^+\varepsilon_i - (\sigma_L - \sigma_0)^+\gamma - (\sigma_R - \sigma_0)^+\delta\}, \tag{13.7.6}$$

which is $O(1)\zeta\theta\gamma\delta$, by virtue of (9.9.30) and (9.9.31). The proof in the general case, where $\alpha$ and $\beta$ are arbitrary incoming wave fans, requires lengthy and technical analysis, but follows the same pattern, with (9.9.14) and (9.9.33) playing the role of (9.9.30) and (9.9.31); see the references cited in Section 13.10. The final conclusion is that the total contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ from the pairing of any peripheral wave of amplitude $\zeta$ with all the principal waves cannot exceed an amount $c_{12}|\zeta|\mathcal{D}(\Delta_s^r)$. Therefore, the overall contribution to $\mathcal{Q}(J) - \mathcal{Q}(I)$ from such interactions is bounded by $c_{12}\mathcal{L}(I)\mathcal{D}(\Delta_s^r)$. This establishes (13.7.5) and thereby the bounds on the total variation of $U_h$.

In the literature cited in Section 13.10, it is shown that the above estimates may even be extended to the more general class of strictly hyperbolic systems of conservation laws that can be approximated "uniformly" by systems with piecewise genuinely nonlinear characteristic families. This broader class encompasses, for example, the system (7.1.8) of isentropic elastodynamics, for arbitrary smooth, strictly increasing stress-strain curve.

## 13.8 Wave Tracing

The aim here is to track the waves of approximate solutions $U_h$ and monitor the evolution of their strength and speed of propagation. This is not an easy task, as wave interactions may induce the fusion or demise of colliding waves of the same characteristic family, while giving birth to new waves of other characteristic families.

For orientation, let us consider wave interactions in a diamond for the simple case of the Burgers equation $\partial_t u + \frac{1}{2}\partial_x u^2 = 0$, (4.2.1). The wave interaction estimate (13.3.1) now reduces to $\varepsilon = \alpha + \beta$.

In one typical situation, shocks with (negative) amplitudes $\alpha$ and $\beta$, and respective speeds $\sigma_L$ and $\sigma_R$ , enter the diamond through its "southwestern" and "southeastern" edge, respectively, and fuse into a single shock of amplitude $\varepsilon = \alpha + \beta$ and speed $\sigma$. It is instructive to regard the outgoing shock as a composite of two "virtual waves", with respective amplitudes $\alpha$ and $\beta$, so that the two incoming shocks continue on beyond the collision, with the same amplitude but altered speeds. Since $\sigma\varepsilon = \sigma_L\alpha + \sigma_R\beta$, we easily deduce

$$|\sigma - \sigma_L||\alpha| = |\sigma - \sigma_R||\beta| = \tfrac{1}{2}\alpha\beta. \tag{13.8.1}$$

Recall that $\alpha\beta$ represents the amount of wave interaction in the diamond.

In the dual situation, rarefaction waves with (positive) amplitudes $\alpha$ and $\beta$ enter the diamond through its "southwestern" and "southeastern" edge, respectively, and combine into a single rarefaction with amplitude $\varepsilon = \alpha + \beta$, which in turn splits into new rarefactions with amplitudes $\alpha'$ and $\beta'$, exiting the diamond through its "northwestern" and "northeastern" edge, respectively. Assuming, for instance, that $\alpha' < \alpha$, we visualize the left incoming wave as a composite of two rarefactions, with respective amplitudes $\alpha'$ and $\alpha - \alpha'$, and the right outgoing wave as a composite of two rarefactions, with respective amplitudes $\alpha - \alpha'$ and $\beta$. This way, all three incoming waves continue beyond the interaction with unchanged amplitudes, albeit with altered speeds.

Still another case arises when a shock of (negative) amplitude $\alpha$ and speed $\sigma_L$ enters the diamond through its "southwestern" edge and interacts with a rarefaction of (positive) amplitude $\beta$ entering through the "southeastern" edge. Assuming, for instance, that $|\alpha| > |\beta|$, the outgoing wave will be a shock with amplitude $\varepsilon = \alpha + \beta$ and speed $\sigma$. As before, we shall regard the incoming shock as a composite of two "virtual waves", with respective amplitudes $\alpha + \beta$ and $-\beta$. Then, as a result of the interaction, the second incoming virtual wave and the incoming rarefaction cancel each other out, while the first virtual wave continues on with unchanged amplitude, but with altered speed. A simple calculation shows that the change in speed is

$$|\sigma - \sigma_L| = \tfrac{1}{2}\beta. \tag{13.8.2}$$

Notice that $\beta$ represents the amount of wave cancellation in the diamond.

The waves exiting the above diamond will get involved in future collisions, in the context of which they may have to be partitioned further into finer virtual waves. These partitions should be then carried backwards in time and applied retroactively to every ancestor of the wave in question. The end result of this laborious process is that, in any specified time zone, each wave is partitioned into a number of virtual waves which fall into one of the following two categories: Those that survive all collisions, within the specified time interval, and those that are eventually extinguished by cancellation.

The situation is similar for systems of hyperbolic conservation laws, except that now one should bear in mind that collisions of any two waves generally give birth to new waves of every characteristic family. In a strictly hyperbolic system with piecewise genuinely nonlinear or linearly degenerate characteristic families, waves are partitioned into virtual waves by the following procedure.

A partitioning of an $i$-shock joining the state $U_-$, on the left, with the state $U_+$, on the right, is performed by some sequence of states $U_- = U^0, U^1, \cdots, U^\nu = U_+$, such that, for $\mu = 1, \cdots, \nu$, $U^\mu$ lies on the $i$-shock curve emanating from $U_-$, and $\lambda_i(U^\mu) \le \lambda_i(U^{\mu-1})$. Even though $U^{\mu-1}$ and $U^\mu$ are not generally joined by a shock, we regard the pair $(U^{\mu-1}, U^\mu)$ as a virtual wave, with assigned amplitude $V_i^\mu = U^\mu - U^{\mu-1}$ and speed $\lambda_i^\mu$, equal to the speed of the shock $(U_-, U_+)$.

A partitioning of an $i$-rarefaction wave joining the state $U_-$, on the left, with the state $U_+$, on the right, is similarly performed by a finite sequence of states, namely $U_- = U^0, U^1, \cdots, U^\nu = U_+$, such that, for $\mu = 1, \cdots, \nu$, $U^\mu$ lies on the $i$-rarefaction curve emanating from $U_-$ and $\lambda_i(U^\mu) > \lambda_i(U^{\mu-1})$. Even though $U^{\mu-1}$ and $U^\mu$ can now be joined by an actual $i$-rarefaction wave, $(U^{\mu-1}, U^\mu)$ will still be regarded as a virtual wave with amplitude $V_i^\mu = U^\mu - U^{\mu-1}$ and speed $\lambda_i^\mu = \lambda_i(U^{\mu-1})$.

A partitioning of a general $i$-wave, joining a state $U_-$, on the left, with a state $U_+$, on the right, by a finite sequence of $i$-shocks and $i$-rarefaction waves, is performed by combining, in an obvious way, the pure shock with the pure rarefaction case, described above.

By a laborious construction, found in the references cited in Section 13.10, the waves of the approximate solution $U_h$, over a specified time zone $\Lambda = \{(x,t) : -\infty < x < \infty,\ s_1\lambda^{-1}h \le t \le s_2\lambda^{-1}h\}$, can be partitioned into virtual waves belonging to one of the following three classes:

I. Waves, members of this class, enter $\Lambda$ at $t = s_1\lambda^{-1}h$ with positive strength, survive over the time interval $[s_1\lambda^{-1}h, s_2\lambda^{-1}h]$ and exit $\Lambda$ at $t = s_2\lambda^{-1}h$ with positive strength.

II. Waves, members of this class, enter $\Lambda$ at $t = s_1\lambda^{-1}h$ with positive strength, but are extinguished inside $\Lambda$ by mutual cancellations.

III. Waves, members of this class, are generated inside $\Lambda$, through wave interactions.

If $\mathcal{W}$ denotes the typical virtual wave in any one of the above three classes, the objective is to estimate its maximum strength, denoted by $|\mathcal{W}|$, the total variation of its amplitude, denoted by $[\mathcal{W}]$, and the total variation of its speed, denoted by $[\sigma(\mathcal{W})]$, over its life span inside $\Lambda$. The seeds for such estimations lie in the simple estimates (13.8.1) and (13.8.2), obtained in the scalar case, in conjunction with the wave interaction estimates derived in earlier sections.

For systems with genuinely nonlinear characteristic families, the requisite estimates read

$$\sum_{\mathcal{W}\in \mathrm{I}} \{[\mathcal{W}] + |\mathcal{W}|[\sigma(\mathcal{W})]\} = O(1)\mathcal{D}(\Lambda), \tag{13.8.3}$$

$$\sum_{\mathcal{W}\in \mathrm{II}} \{[\mathcal{W}] + |\mathcal{W}|\} = O(1)\mathcal{C}(\Lambda) + O(1)\mathcal{D}(\Lambda), \tag{13.8.4}$$

$$\sum_{\mathcal{W}\in \mathrm{III}} \{[\mathcal{W}] + |\mathcal{W}|\} = O(1)\mathcal{D}(\Lambda), \tag{13.8.5}$$

where $\mathcal{D}(\Lambda)$ and $\mathcal{C}(\Lambda)$ denote the total amount of wave interaction and wave cancellation inside $\Lambda$, namely

$$\mathcal{D}(\Lambda) = \sum \mathcal{D}(\Delta_s^r), \qquad \mathcal{C}(\Lambda) = \sum \mathcal{C}(\Delta_s^r), \tag{13.8.6}$$

with the summation running over all diamonds $\Delta_s^r$ contained in $\Lambda$, and $\mathcal{D}(\Delta_s^r), \mathcal{C}(\Delta_s^r)$ defined by (13.3.2), (13.3.6).

For systems with characteristic families that are merely piecewise genuinely nonlinear, the analogs of the estimates (13.8.3), (13.8.4) and (13.8.5) are considerably more complicated. The difference stems from the fact that the amount of wave interaction $\mathcal{D}(\Delta_s^r)$ is of quadratic order, (13.3.2), in the genuinely nonlinear case, but merely of cubic order, (13.7.2), in the general case. Details are given in the references cited in Section 13.10.

It is now possible to establish the following proposition, which improves Theorem 13.2.1 by removing the "randomness" hypothesis in the selection of the sequence $\wp$:

**13.8.1 Theorem.** *The algorithm induced by any sequence* $\wp = \{a_0, a_1, \cdots\}$, *which is equidistributed on the interval* $(-1, 1)$ *in the sense of* (13.2.6), *is consistent.*

In the proof, which may be found in the references cited in Section 13.10, one expresses the right-hand side of (13.2.2) in terms of the virtual waves that partition $U_h$ and proceeds to show that it tends to zero, as $h \downarrow 0$, whenever the sequence $\wp$ is equidistributed. This happens for the following reason. Recall that in Section 13.2 we did verify the consistency of the algorithm, for any equidistributed sequence $\wp$, in the context of the linear conservation law $\partial_t u + a\lambda\partial_x u = 0$, by employing the property that every wave propagates with constant amplitude and at constant speed. The partitioning of waves performed above demonstrates that even nonlinear systems have this property, albeit in an approximate sense, and this makes it possible to extend the argument for consistency to that case as well.

Though somewhat cumbersome to use, wave partitioning is an effective tool for obtaining precise information on local structure, large time behavior, and other qualitative properties of solutions; and in particular it is indispensable for deriving properties that hinge on the global wave pattern.

## 13.9 Inhomogeneous Systems of Balance Laws

It is relatively straightforward to adapt the random choice method to inhomogeneous, strictly hyperbolic systems of balance laws

$$\partial_t U(x,t) + \partial_x F(U(x,t), x, t) + G(U(x,t), x, t) = 0. \tag{13.9.1}$$

The functions $F$ and $G$ are defined on $\mathcal{O} \times (-\infty, \infty) \times [0, \infty)$, take values in $\mathbb{R}^n$, are smooth and have bounded partial derivatives. For any fixed $(x, t)$, $\mathrm{D}F(U, x, t)$ has real distinct eigenvalues $\lambda_1(U, x, t) < \cdots < \lambda_n(U, x, t)$, which are separated from each other, uniformly in $(x, t)$.

We assign initial conditions

$$U(x, 0) = U_0(x), \qquad -\infty < x < \infty, \tag{13.9.2}$$

and seek to determine the solution of the Cauchy problem as the $h \downarrow 0$ limit of approximate solutions $U_h$ constructed by a simple adaptation of the scheme described in Sections 13.1 and 13.2.

The effect of inhomogeneity and the source term is incorporated in the algorithm via *operator splitting:* At each time step, the approximate solution to the inhomogeneous balance law is obtained by concatenating solutions (or approximate solutions) of ordinary differential equations $\partial_t U + G = 0$, $\partial_x F = 0$ and homogeneous conservation laws $\partial_t U + \partial_x F(U) = 0$.

As in Section 13.2, we start out with a random sequence $\wp = \{a_0, a_1, \dots\}$, with $a_s \in (-1, 1)$. We fix the spatial mesh-length $h$, with associated time mesh-length $\lambda^{-1}h$, and build the staggered grids of mesh-points $(x_r, t_s)$, for $r+s$ even, and sampling points $(y_s^r, t_s)$, $y_s^r = x_r + a_s h$, for $r+s$ odd.

Assuming $U_h$ is already known on $\{(x,t) : -\infty < x < \infty,\ 0 \le t < t_s\}$, we define $U_s^r$, for $r+s$ odd, by means of (13.2.5), and then set

$$\hat{U}_s^r = U_s^r - \lambda^{-1} h G(U_s^r, x_r, t_s). \tag{13.9.3}$$

Next we determine $V_s^r$ and $W_s^r$, for $r+s$ odd, as solutions to the equation

$$F(V_s^r, x_{r+1}, t_s) = F(\hat{U}_s^r, x_r, t_s) = F(W_s^r, x_{r-1}, t_s). \tag{13.9.4}$$

To make (13.9.4) solvable, we may have to change coordinates $(x,t) \mapsto (y,t)$, with $y = y(x,t)$, so as to eliminate any zero characteristic speeds. Finally, we define $U_h$ on $\{(x,t) : x_{r-1} \le x <_{r+1},\ t_s \le t < t_{s+1}\}$, for $r+s$ even, as the restriction to this rectangle of the solution to the Riemann problem

$$\partial_t U_h(x,t) + \partial_x F(U_h(x,t), x_r, t_s) = 0, \qquad t \ge t_s\,, \tag{13.9.5}$$

$$U_h(x, t_s) = \begin{cases} V_s^{r-1}, & x < x_r \\ W_s^{r+1}, & x > x_r\,. \end{cases} \tag{13.9.6}$$

The algorithm is initiated, at $s = 0$, by (13.1.10).

Inhomogeneity and the source term may amplify the total variation of approximate solutions, driving it beyond the range of currently available analytical tools. In order to keep the effect of inhomogeneity under control, we impose the following restrictions on the functions $F$ and $G$: For any $U \in \mathcal{O}$, $x \in (-\infty, \infty)$ and $t \in [0, \infty)$[3]

$$|\mathrm{D}F_x(U,x,t)| < \omega, \qquad |\mathrm{D}F_t(U,x,t)| < \omega, \tag{13.9.7}$$

$$|\mathrm{D}F_x(U,x,t)| \le f(x), \qquad |G_x(U,x,t)| \le f(x), \tag{13.9.8}$$

where $f(x)$ is a $W^{1,1}(-\infty,\infty)$ function such that

$$\int_{-\infty}^{\infty} f(x)dx < \omega, \tag{13.9.9}$$

and $\omega$ is a positive number. Under these conditions, the Cauchy problem admits at least local $BV$ solutions:

[3] Throughout this section, $n$-vectors shall be regarded, and normed, as elements of $\ell_n^1$, and $n \times n$ matrices shall be regarded, and normed, as linear operators on $\ell_n^1$.

**13.9.1 Theorem.** *For sufficiently small positive numbers $\omega$ and $\delta$, there exists time $T = T(\omega, \delta)$, with $T(\omega, \delta) \to \infty$ as $(\omega, \delta) \to 0$, such that when* (13.9.7), (13.9.8), (13.9.9) *hold and*

$$TV_{(-\infty,\infty)}U_0(\cdot) < \delta, \tag{13.9.10}$$

*then there exists an admissible $BV$ solution $U$ of* (13.9.1), (13.9.2) *on the time interval* $[0, T)$. *For each fixed $t \in [0, T)$, $U(\cdot, t)$ is a function of bounded variation on* $(-\infty, \infty)$ *and*

$$TV_{(-\infty,\infty)}U(\cdot, t) \le c\, e^{\rho t}\, [TV_{(-\infty,\infty)}U_0(\cdot) + \omega], \tag{13.9.11}$$

*for some $\rho > 0$.*

The proof of the above proposition, which rests on a fairly straightforward, though tedious, adaptation of the analysis in earlier sections that culminated in the proof of Theorem 13.1.1, can be found in the references cited in Section 13.10. The reader may get a taste of the methodology from the proof of Theorem 13.9.4, below, which treats systems (13.9.1) with special structure. The exponential growth in the total variation is induced by both inhomogeneity and the source term, and the exponent $\rho$ is $O(\omega + \gamma)$, where $\gamma = \sup |\mathrm{D}G|$. Of course, the solution cannot escape as long as $TV_{(-\infty,\infty)}U_h(\cdot, t)$ stays small.

Our next project is to identify classes of systems for which the Cauchy problem admits global $BV$ solutions. The simplest mechanism that would keep the total variation small is rapid decay of the inhomogeneity and the source term as $t \to \infty$. Suppose that we replace the assumptions (13.9.7) and (13.9.8) by

$$|\mathrm{D}F_x(U, x, t)| < \omega g(t), \qquad |\mathrm{D}F_t(U, x, t)| < \omega g(t), \tag{13.9.12}$$

$$|G(U, x, t)| < \omega g(t), \qquad |\mathrm{D}G(U, x, t)| < \omega g(t), \tag{13.9.13}$$

$$|\mathrm{D}F_x(U, x, t)| \le f(x)g(t), \qquad |G_x(U, x, t)| \le f(x)g(t), \tag{13.9.14}$$

for all $U \in \mathcal{O}$, $x \in (-\infty, \infty)$, $t \in [0, \infty)$, where $f(x)$ and $\omega$ are as above, while $g(t)$ is a bounded function in $L^1(0, \infty)$. Then a simple corollary of Theorem 13.9.1, and in particular of the estimate (13.9.11), is the following

**13.9.2 Theorem.** *For sufficiently small positive numbers $\omega$ and $\delta$, when* (13.9.12), (13.9.13), (13.9.14), (13.9.9) *and* (13.9.10) *hold, then there exists a global admissible $BV$ solution $U$ of* (13.9.1),(13.9.2). *For each $t \in [0, \infty)$, $U(\cdot, t)$ is a function of bounded variation on* $(-\infty, \infty)$ *and*

$$TV_{(-\infty,\infty)}U(\cdot, t) \le c\, [TV_{(-\infty,\infty)}U_0(\cdot) + \omega]. \tag{13.9.15}$$

A considerably subtler mechanism that induces global existence to the Cauchy problem is the rapid decay of the inhomogeneity and the source term as $|x| \to \infty$, in conjunction with nonzero characteristic speeds. Indeed, when all the characteristic speeds are bounded away from zero, one should expect that as $t$ increases the bulk of the wave moves far away from the origin and eventually enters, and stays, in the region where inhomogeneity and the source term have negligible influence. To verify this conjecture requires delineating the global wave pattern and tracking the bulk of the wave. This may be effected only by the method of wave tracing, outlined in Section 13.8. A representative result in that direction is the following proposition, which is established in the references cited in Section 13.10.

**13.9.3 Theorem.** *Consider the strictly hyperbolic system of balance laws*

$$\partial_t U(x,t) + \partial_x F(U(x,t)) + G(U(x,t),x) = 0, \tag{13.9.16}$$

*with nonzero characteristic speeds, and characteristic families that are either genuinely nonlinear or linearly degenerate. Assume that for any $U$ in $\mathcal{O}$ and $x$ in $(-\infty,\infty)$,*

$$|G(U,x)| \le f(x), \quad |\mathrm{D}G(U,x)| \le f(x), \tag{13.9.17}$$

*where $f(x)$ satisfies* (13.9.9) *with $\omega$ sufficiently small. If the initial data satisfy* (13.9.10)*, with $\delta$ sufficiently small, then there exists a global admissible $BV$ solution $U$ of* (13.9.16),(13.9.2)*. For each fixed $t \in [0,\infty)$, $U(\cdot,t)$ is a function of bounded variation on* $(-\infty,\infty)$ *and*

$$TV_{(-\infty,\infty)}U(\cdot,t) \le c\,[TV_{(-\infty,\infty)}U_0(\cdot) + \omega]. \tag{13.9.18}$$

A typical application of the above proposition is to the system (7.1.18) that governs the isentropic flow of a gas through a duct of varying cross section $a(x)$. We rewrite (7.1.18) in the form (13.9.16):

$$\begin{cases} \partial_t v + \partial_x(\rho v) + a^{-1}(x)a'(x)\rho v = 0 \\ \partial_t(\rho v) + \partial_x[\rho v^2 + p(\rho)] + a^{-1}(x)a'(x)\rho v^2 = 0. \end{cases} \tag{13.9.19}$$

Clearly, in order to meet the requirement (13.9.17) of Theorem 19.9.3, one needs to assume that $a(x)$ has sufficiently small total variation on $(-\infty,\infty)$.

The remaining task is to investigate systems of balance laws with dissipative source terms. It turns out that dissipation may secure global existence of $BV$ solutions, with initial values of small total variation, even in the presence of inhomogeneity. Here, however, in order to keep the analysis as simple as possible, we shall consider only homogeneous hyperbolic systems of balance laws

$$\partial_t U(x,t) + \partial_x F(U(x,t)) + G(U(x,t)) = 0. \tag{13.9.20}$$

We assume $G(\bar{U}) = 0$, for some $\bar{U} \in \mathcal{O}$, so that $U \equiv \bar{U}$ is an equilibrium solution.

Since the analysis is in $BV$ space, we have to impose on $G$ conditions that would render it dissipative in $L^1$. In order to identify the proper assumptions, we linearize (13.9.20) about $\bar{U}$ and then set $U = R(\bar{U})V$, where $R(U)$ is the $n \times n$ matrix with column vectors a set of linearly independent right eigenvectors $R_1(U), \cdots, R_n(U)$ of $\mathrm{D}F(U)$. This yields the system

$$\partial_t V_i(x,t) + \lambda_i(\bar{U})\partial_x V_i(x,t) + \sum_{j=1}^{n} A_{ij} V_j(x,t) = 0, \qquad i = 1, \cdots, n, \tag{13.9.21}$$

where

$$A = R(\bar{U})^{-1}\mathrm{D}G(\bar{U})R(\bar{U}). \tag{13.9.22}$$

We multiply (13.2.21) by $\operatorname{sgn} V_i(x,t)$, integrate with respect to $x$ over $(-\infty, \infty)$, and sum over $i = 1, \cdots, n$, to deduce that when $A$ is column diagonally dominant, namely

$$A_{ii} - \sum_{j \neq i} |A_{ji}| \geq \nu > 0, \qquad i = 1, \cdots, n, \tag{13.9.23}$$

then, as $t \to \infty$, solutions of (13.9.21) decay exponentially to zero in $L^1(-\infty, \infty)$.

It should be noted that the diagonal dominance property (13.9.23) depends on the particular matrix $R(U)$ of right eigenvectors employed in the construction of $A$. Indeed, choosing the equivalent matrix $\hat{R}(U) = R(U)K$ of eigenvectors, where $K$ is some positive diagonal matrix, would replace $A$ with the matrix $\hat{A} = K^{-1}A\,K$; and diagonal dominance is not generally preserved under such similarity transformations. Given a matrix $A$, it is possible to find a positive diagonal matrix $K$ that renders $K^{-1}A\,K$ column diagonally dominant if and only if all eigenvalues of the matrix $\tilde{A}$, with entries $\tilde{A}_{ii} = A_{ii}$, $i = 1, \cdots, n$ and $\tilde{A}_{ij} = -|A_{ij}|$, for $i \neq j$, have positive real part (references in Section 13.10). In particular, this class of $A$ encompasses positive triangular matrices as well as row diagonally dominant matrices (by Geršgorin's theorem).

For any $\tau > 0$, multiplying the linear system $(I + \tau A)X = Y$, from the left, by the row vector $\operatorname{sgn} X^\top$, yields

$$|(I + \tau A)^{-1}| \leq (1 + \nu\tau)^{-1}. \tag{13.9.24}$$

As we shall see, it is this property that induces existence of global solutions to the Cauchy problem for (13.9.20).

**13.9.4 Theorem.** *Consider the homogeneous, strictly hyperbolic system of balance laws* (13.9.20)*, with characteristic families that are either genuinely nonlinear or linearly degenerate. Assume that for some selection of eigenvectors of* $\mathrm{D}F(U)$*, the matrix $A$, defined by* (13.9.22)*, is column diagonally dominant* (13.9.23)*. Suppose* $G(\bar{U}) = 0$*. If the initial data $U_0$ are constant $\bar{U}$ outside a bounded interval and satisfy* (13.9.10) *for $\delta$ sufficiently small, then there exists a global admissible $BV$*

*solution $U$ of* (13.9.20), (13.9.2). *For each fixed $t \in [0, \infty)$, $U(\cdot, t)$ is a function of bounded variation on $(-\infty, \infty)$ and*

$$TV_{(-\infty,\infty)}U(\cdot, t) \le c\, e^{-\mu t} TV_{(-\infty,\infty)}U_0(\cdot), \tag{13.9.25}$$

*where $\mu$ is some positive constant.*

**Sketch of Proof.** We construct the solution $U$ by means of the random choice algorithm described earlier in this section. The proof of consistency follows closely the argument used in the proof of Theorem 13.1.1 and need not be repeated here. It will suffice to establish a bound for the total variation $TV_{(-\infty,\infty)}U_h(\cdot, t)$ of the approximate solution $U_h$ that will yield in the limit $h \downarrow 0$ the asserted estimate (13.9.25).

As in Section 13.3, for $r + s$ even we consider the diamond $\Delta_s^r$ with vertices $(y_s^{r-1}, t_s)$, $(y_{s-1}^r, t_{s-1})$, $(y_s^{r+1}, t_s)$ and $(y_{s+1}^r, t_{s+1})$, depicted in Fig. 13.3.1. The aim is to estimate the strength of the outgoing wave fan $\varepsilon$, emanating from $(x_r, t_r)$, in terms of the strengths of the incoming wave fans $\alpha$ and $\beta$, which emanate from $(x_{r-1}, t_{s-1})$ and $(x_{r+1}, t_{s-1})$, respectively.

Since our system is homogeneous, (13.9.4) yields $V_s^r = \hat{U}_s^r = W_s^r$. According to the prescription of the algorithm,

$$\Omega(\alpha; U_s^{r-1}) = \hat{U}_{s-1}^r, \tag{13.9.26}$$

$$\Omega(\beta; \hat{U}_{s-1}^r) = U_s^{r+1}, \tag{13.9.27}$$

$$\Omega(\varepsilon; \hat{U}_s^{r-1}) = \hat{U}_s^{r+1}, \tag{13.9.28}$$

where $\Omega$ is the wave fan function, defined by (9.3.4).

Let us consider the wave fan $\tilde{\varepsilon}$ that would have resulted from the interaction of $\alpha$ and $\beta$ in the absence of source term, i.e.,

$$\Omega(\tilde{\varepsilon}; U_s^{r-1}) = \Omega(\beta; \Omega(\alpha; U_s^{r-1})) = U_s^{r+1}. \tag{13.9.29}$$

By virtue of Theorem 9.9.1,

$$\tilde{\varepsilon} = \alpha + \beta + O(1)\mathcal{D}(\Delta_s^r), \tag{13.9.30}$$

where the wave interaction term $\mathcal{D}(\Delta_s^r)$ is defined by (13.3.2).

We proceed to relate $\varepsilon$ to $\tilde{\varepsilon}$. Since $\Omega(0; U) = U$, for any $U \in \mathcal{O}$, (13.9.28) together with (13.9.3) and $G(\bar{U}) = 0$ yield

$$\Omega(\varepsilon; U_s^{r-1}) = U_s^{r+1} - \lambda^{-1}h[G(U_s^{r+1}) - G(U_s^{r-1})] + o(1)h|\varepsilon|, \tag{13.9.31}$$

where $o(1)$ denotes a quantity that becomes arbitrarily small when $\sup|U_h - \bar{U}|$ is sufficiently small. By virtue of (9.3.8),

$$\Omega(\varepsilon; U_s^{r-1}) - \Omega(\tilde{\varepsilon}; U_s^{r-1}) = P(\varepsilon - \tilde{\varepsilon}), \tag{13.9.32}$$

where $P$ is some matrix close to the matrix $R(\bar{U})$ of right eigenvectors of $\mathrm{D}F(\bar{U})$. Furthermore, by account of (13.9.3),

(13.9.33)

$$G(U_s^{r+1}) - G(U_s^{r-1}) = H[\hat{U}_s^{r+1} - \hat{U}_s^{r-1}] + \lambda^{-1}hH[G(U_s^{r+1}) - G(U_s^{r-1})],$$

where $H$ is some matrix close to $\mathrm{D}G(\bar{U})$. Finally, by (9.3.8) and (13.9.28),

(13.9.34) $$\hat{U}_s^{r+1} - \hat{U}_s^{r-1} = \Omega(\varepsilon; \hat{U}_s^{r-1}) - \Omega(0; \hat{U}_s^{r-1}) = Q\varepsilon,$$

where $Q$ is some matrix close to $R(\bar{U})$. We now combine (13.9.29),(13.9.31), (13.9.32),(13.9.33) and (13.9.34) to get

(13.9.35) $$\tilde{\varepsilon} = [I + \lambda^{-1}hB]\varepsilon + o(1)h|\varepsilon|,$$

where

(13.9.36) $$B = P^{-1}[I - \lambda^{-1}hH]^{-1}HQ$$

is close to the matrix $A$, defined by (13.9.22).

On account of (13.9.30),(13.9.35) and (13.9.24) we conclude that, for as long as $\sup|U_h - \bar{U}|$ stays sufficiently small,

(13.9.37) $$|\varepsilon| \leq (1 - 3\mu\lambda^{-1}h)(|\alpha| + |\beta|) + c_{13}\mathcal{D}(\Delta_s^r),$$

with $\mu = \nu/4 > 0$.

From (13.9.30),(13.9.35) and (13.9.37), we also deduce

(13.9.38) $$|\varepsilon - (\alpha + \beta)| \leq c_{14}h(|\alpha| + |\beta|) + c_{15}\mathcal{D}(\Delta_s^r).$$

As in Section 13.4, we consider mesh curves $I$ and associate with them the functionals $\mathcal{L}(I)$, $\mathcal{Q}(I)$ and $\mathcal{G}(I)$, defined by (13.4.2),(13.4.5) and (13.4.8). Assuming $J$ is the immediate successor to $I$, depicted in Fig. 13.4.1, we may retrace the analysis in Section 13.4, using (13.9.37) to get

(13.9.39) $$\mathcal{L}(J) \leq \mathcal{L}(I) - 3\mu\lambda^{-1}h(|\alpha| + |\beta|) + c_{13}\mathcal{D}(\Delta_s^r),$$

in the place of (13.4.4), and using (13.9.38) to get

(13.9.40) $$\mathcal{Q}(J) - \mathcal{Q}(I) \leq c_{14}h\mathcal{L}(I)(|\alpha| + |\beta|) + [c_{15}\mathcal{L}(I) - 1]\mathcal{D}(\Delta_s^r),$$

in the place of (13.4.7). Thus, for $\kappa$ sufficiently large and $\mathcal{L}(I)$ sufficiently small,

(13.9.41) $$\mathcal{G}(J) \leq \mathcal{G}(I) - 2\mu\lambda^{-1}h(|\alpha| + |\beta|).$$

Next, for fixed $s = 0, 1, 2, \dots$, we consider the mesh curve $J_s$ with vertices all sampling points $(y_s^{r-1}, t_s)$ and $(y_{s+1}^r, t_{s+1})$ with $r + s$ even. Then assuming that $\sup|U_h - \bar{U}|$ is so small that $\mathcal{G}(J_{s-1}) \leq 2\mathcal{L}(J_{s-1})$, (13.9.41) yields

$$\mathcal{G}(J_s) \le (1 - \mu\lambda^{-1}h)\mathcal{G}(J_{s-1}). \tag{13.9.42}$$

Thus, for any $t_s < t < t_{s+1}$,

$$TV_{(-\infty,\infty)}U_h(\cdot, t) \le (1 - \mu\lambda^{-1}h)^s \mathcal{G}(J_0), \tag{13.9.43}$$

where the total variation is measured by $\mathcal{L}(J_s)$.

Since $U_h(x, t) = \bar{U}$, for $t$ fixed and $|x|$ sufficiently large, the right-hand side of (13.9.43) also bounds $\sup_{(-\infty,\infty)}|U_h(\cdot, t) - \bar{U}|$.

On the right-hand side of (13.9.43), $\mathcal{G}(J_0)$ is bounded by $c\,TV_{(-\infty,\infty)}U_0(\cdot)$. Therefore, letting $h \downarrow 0$, (13.9.43) yields (13.9.25). The proof is complete.

## 13.10 Notes

The random choice method was developed in the fundamental paper of Glimm [1]. It is in that work that the ideas of consistency (Section 13.2), wave interactions (Section 13.3), and the Glimm functional (Section 13.4) were originally introduced, and Theorem 13.1.1 was first established, for genuinely nonlinear systems. As we shall see in the following chapter, it is Glimm-type functionals that provide the key estimates for compactness in other solution approximation schemes as well. Furthermore, the Glimm functional can be defined, and profitably employed, even in the context of general $BV$ solutions; see Section 14.11.

Resonance phenomena in Godunov's scheme that may drive the total variation of approximate solutions to infinity are discussed in Bressan and Jenssen [1], Baiti, Bressan and Jenssen [1], and Bressan, Jenssen and Baiti [1].

The construction of solutions with large variation for the special system (13.4.20) of isothermal gas dynamics is due to Nishida [1]. Related constructions of solutions with large, or at least moderately large, initial data are found in Bakhvarov [1], DiPerna [1,2], Nishida and Smoller [1], Luskin and Temple [1], Poupaud, Rascle and Vila [1], Serre [11], Ying and Wang [1], and Amadori and Guerra [2]. Existence of $BV$ solutions to the Cauchy problem for the equations of isentropic gas dynamics under initial data of arbitrarily large total variation (and even regions of vacuum) was recently announced by R. Young [7]. See also Section 14.12.

The notions of wave cancellation, approximate conservation laws and approximate characteristics (Section 13.3), which were introduced in the important memoir by Glimm and Lax [1], provide the vehicle for deriving properties of solutions of genuinely nonlinear systems of two conservation laws, constructed by the random choice method (see Section 12.11).

The derivation of bounds on the supremum, outlined in Section 13.6, is taken from the thesis of R. Young [1], where the reader may find the technical details. In fact, this work introduces a new length scale for the Cauchy problem, which, under special circumstances, may be used in order to relax the requirement of small total variation on the initial data, for certain systems of more than two conservation laws. In that direction, see Temple [7], Temple and Young [1,2], and Cheverry [3]. Local or global solutions under initial data with large total variation are also constructed by Alber [1] and Schochet [3,4].

The Glimm functional was adapted to systems that are not genuinely nonlinear by Liu [15], who was first to realize the important role played by the incidence angle between approaching waves of the same characteristic family. The outline presented here, in Section 13.7, follows the recent work by Iguchi and LeFloch [1] and Liu and Yang [6].

The method of wave partitioning is developed in Liu [7], for genuinely nonlinear systems, and in Liu [15], for general systems, and is used for establishing the deterministic consistency of the algorithm for equidistributed sequences (Theorem 13.8.1). Bressan and Marson [3] show that for "well equidistributed" sequences the rate of convergence in $L^1$ is $o(h^{1/2}|\log h|)$. For the rate of convergence in the scalar case, see Hoff and Smoller [1]. For systems with characteristic families that are either piecewise genuinely nonlinear or linearly degenerate, Liu [15] describes the local structure of solutions and shows, in particular, that any point of discontinuity of the solution is either a point of classical jump discontinuity or a point of wave interaction. Furthermore, the set of points of jump discontinuity comprise a countable family of Lipschitz curves (shocks), while the set of points of wave interaction is at most countable (compare with Theorem 12.7.1, for genuinely nonlinear systems of two conservation laws, and see also Section 14.11). As $t \to \infty$, solutions of (13.1.1) approach the solution of the Riemann problem with data (9.1.12), where $U_L = U_0(-\infty)$ and $U_R = U_0(+\infty)$; cf. Liu [9,11,15]. For a more recent exposition see Liu [28] and Liu and Yang [6].

The details of the proof of Theorems 13.9.1 and 13.9.4 are found in Dafermos and Hsiao [1]. The form of the dissipativeness condition on the matrix $A$ that does not depend on the choice of eigenvectors of $\mathrm{D}F$ was found by Amadori and Guerra [1]. For systems with weaker dissipation that may still be treated by this approach, see Amadori and Guerra [2,3] and Dafermos [20,22]. See also Crasta and Piccoli [1]. Spherically symmetric solutions of the Euler equations with damping are constructed in Hsiao, Tao and Yang [1]. See also Tong Yang [1]. For source terms induced by combustion, see Chen and Wagner [1]. Theorem 13.9.3 is taken from Liu [14]. Amadori, Gosse and Guerra [1] and Ha [1] improve this result by establishing $L^1$ stability. The effects of resonance between the waves and the source term may be seen in Liu [18], Li and Liu [1], Pego [4], Isaacson and Temple [4], Klingenberg and Risebro [2], Ha and Yang [1], Lien [1], Lan and Lin [1], and Hong and Temple [1].

There is voluminous literature on extensions and applications of the random choice method. For systems of mixed type, see Pego and Serre [1], LeFloch [3] and Corli and Sablé-Tougeron [3]. For initial-boundary-value problems, cf. Liu [11], Luskin and Temple [1], Nishida and Smoller [2], Dubroca and Gallice [1], Sablé-Tougeron [1] and Frid [1]. For solutions involving strong shocks, see Sablé-Tougeron [2], Corli and Sablé-Tougeron [1,2], Asakura [1], Corli [2] and Schochet [3,4]. For applications to gas dynamics, see Liu [4,5,12,16,17] and Temple [1]. For the effects of vacuum in gas dynamics, see Liu and Smoller [1]. For applications to the theory of relativity, see Barnes, LeFloch, Schmidt and Stewart [1]. For systems that are not in conservation form, see LeFloch [2] and LeFloch and Liu [1]. Weak $L^p$ stability is established by Temple [6]. Additional references are found in the books by Smoller [3], Serre [11] and LeFloch [5].

# XIV

# The Front Tracking Method and Standard Riemann Semigroups

A method is described in this chapter for constructing solutions of the initial-value problem for hyperbolic systems of conservation laws by tracking the waves and monitoring their interactions as they collide. Interactions between shocks are easily resolved by solving Riemann problems; this is not the case, however, with interactions involving rarefaction waves. The random choice method, expounded in Chapter XIII, side-steps this difficulty by stopping the clock before the onset of wave collisions and reapproximating the solution by step functions. In contrast, the front tracking approach circumvents the obstacle by disposing of rarefaction waves altogether and resolving all Riemann problems in terms of shocks only. Such solutions generally violate the admissibility criteria. Nevertheless, considering the close local proximity between shock and rarefaction wave curves in state space, any rarefaction wave may be approximated arbitrarily close by fans of (inadmissible) shocks of very small strength. The expectation is that in the limit, as this approximation becomes finer, one recovers admissible solutions.

The implementation of the front tracking algorithm, with proof that it converges, will be presented here, first for scalar conservation laws and then in the context of genuinely nonlinear strictly hyperbolic systems of conservation laws of any size.

By a contraction argument with respect to a suitably weighted $L^1$ distance, it will be demonstrated that solutions of genuinely nonlinear systems, constructed by the front tracking method, may be realized as orbits of the *Standard Riemann Semigroup*, which is defined on the set of functions with small total variation and is Lipschitz continuous in $L^1$. It will further be shown that any $BV$ solution that satisfies reasonable stability conditions is also identifiable with the orbit of the Standard Riemann Semigroup issuing from its initial data. This establishes, in particular, uniqueness for the initial-value problem within a broad class of $BV$ solutions, including those constructed by the random choice method, as well as those whose trace along space-like curves has bounded variation, encountered in earlier chapters.

The chapter will close with a discussion of the structural stability of the wave pattern under perturbations of the initial data.

## 14.1 Front Tracking for Scalar Conservation Laws

This section discusses the construction of the admissible solution to the initial-value problem for scalar conservation laws by a front tracking scheme that aims at eliminating rarefaction waves. The building blocks will be wave fans composed of constant states, admissible "compressive" shocks, and inadmissible "rarefaction" shocks of small strength.

The admissible solution of the Riemann problem for the scalar conservation law $\partial_t u + \partial_x f(u) = 0$, with $C^1$ flux $f$, was constructed in Section 9.5: The left end-state $u_l$ and the right end-state $u_r$ are joined by the wave fan

$$u(x,t) = [g']^{-1}\left(\frac{x}{t}\right), \tag{14.1.1}$$

where $g$ is the convex envelope of $f$ over $[u_l, u_r]$, when $u_l < u_r$, or the concave envelope of $f$ over $[u_r, u_l]$, when $u_l > u_r$. Intervals on which $g'$ is constant yield shocks, while intervals over which $g'$ is strictly monotone generate rarefaction waves. The same construction applies even when $f$ is merely Lipschitz, except that now, in addition to shocks and rarefaction waves, the ensuing wave fan may contain intermediate constant states, namely, the jump points of $g'$. In particular, when $f$, and thereby $g$, are piecewise linear, the wave fan does not contain any rarefaction waves but is composed of shocks and constant states only (Fig. 14.1.1).

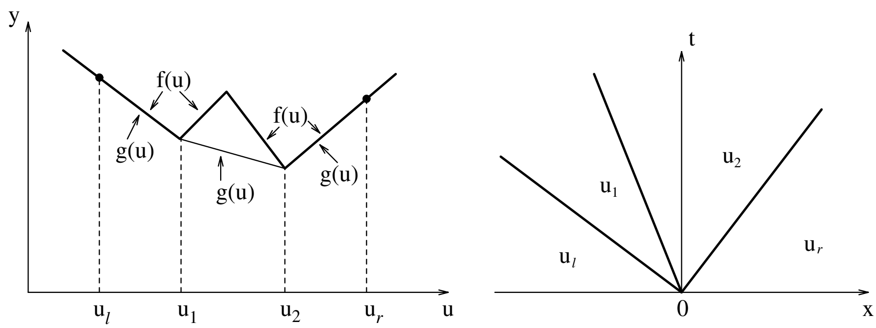


**Fig. 14.1.1**

We now consider the Cauchy problem

$$\begin{cases} \partial_t u(x,t) + \partial_x f(u(x,t)) = 0, & -\infty < x < \infty,\ \ 0 \le t < \infty, \\ u(x,0) = u_0(x), & -\infty < x < \infty, \end{cases} \tag{14.1.2}$$

for a scalar conservation law, where the flux $f$ is Lipschitz continuous on $(-\infty, \infty)$ and the initial datum $u_0$ takes values in a bounded interval $[-M, M]$ and has bounded total variation over $(-\infty, \infty)$.

To solve (14.1.2), one first approximates the flux $f$ by a sequence $\{f_m\}$ of piecewise linear functions, such that the graph of $f_m$ is a polygonal line inscribed in the graph of $f$, with vertices at the points $(\frac{k}{m}, f(\frac{k}{m}))$, $k \in \mathbb{Z}$. Next, one realizes the initial datum $u_0$ as the a.e. limit of a sequence $\{u_{0m}\}$ of step functions, where $u_{0m}$ takes values in the set $\mathcal{U}_m = \{\frac{k}{m} : k \in \mathbb{Z}, |k| \leq mM\}$, and its total variation does not exceed the total variation of $u_0$ over $(-\infty, \infty)$. Finally, one solves the initial-value problem

$$
(14.1.3) \qquad \begin{cases} \partial_t u(x,t) + \partial_x f_m(u(x,t)) = 0, & -\infty < x < \infty, \ 0 \leq t < \infty, \\ u(x,0) = u_{0m}(x), & -\infty < x < \infty, \end{cases}
$$

for $m = 1, 2, \cdots$. The aim is to show that the admissible solution $u_m$ of (14.1.3) is a piecewise constant function, taking values in $\mathcal{U}_m$, which is constructed by solving a finite number of Riemann problems for the conservation law $(14.1.3)_1$; and that the sequence $\{u_m\}$ converges to the admissible solution $u$ of (14.1.2).

The construction of $u_m$ is initiated by solving the Riemann problems that resolve the jump discontinuities of $u_{0m}$ into wave fans of shocks and constant states in $\mathcal{U}_m$. In turn, wave interactions induced by shock collisions are similarly resolved, in the order they occur, into wave fans of shocks and constant states in $\mathcal{U}_m$, resulting from the solution of Riemann problems. It should be noted that the admissible solution of the Riemann problem for $(14.1.3)_1$, with end-states in $\mathcal{U}_m$, is also a solution of $(14.1.2)_1$, albeit not necessarily an admissible one, because in that context some of the jump discontinuities may be rarefaction shocks. Thus, in addition to being the admissible solution of (14.1.3), $u_m$ is a (generally inadmissible) solution of $(14.1.2)_1$.

We demonstrate that the number of shock collisions that may be encountered in the implementation of the above algorithm is a priori bounded, and hence $u_m$ is constructed on the entire upper half-plane in finite steps. The reason is that each shock interaction simplifies the wave pattern by lowering either the number of shocks, measured by the number $j_m(t)$ of points of jump discontinuity of the step function $u_m(\cdot, t)$, or the number of "oscillations", counted by the lap number $\ell_m(t)$ of $u_m(\cdot, t)$, which is defined as follows.

For the case of a step function $v(\cdot)$ on $(-\infty, \infty)$, the *lap number* $\ell$ is set equal to 0 when $v(\cdot)$ is monotone, while when $v(\cdot)$ is nonmonotone it is defined as the largest positive integer such that there exist $\ell + 2$ points $-\infty < x_0 < \cdots < x_{\ell+1} < \infty$ of continuity of $v(\cdot)$, with $[v(x_{i+1}) - v(x_i)][v(x_i) - v(x_{i-1})] < 0$, $i = 1, \cdots, \ell$.

Clearly, both $j_m(t)$ and $\ell_m(t)$ stay constant along the open time intervals between consecutive shock collisions; they may change only across $t = 0$ and as shocks collide. When $k$ shocks, joining (left, right) states $(u_0, u_1), \cdots, (u_{k-1}, u_k)$, collide at one point, the ensuing interaction is called *monotone* if the finite sequence $\{u_0, u_1, \cdots, u_k\}$ is monotone. Such an interaction produces a single shock joining the state $u_0$, on the left, with the state $u_k$, on the right. In particular, monotone interactions leave $\ell_m(t)$ unchanged, while lowering the value of $j_m(t)$ by at least one. In contrast, across nonmonotone interactions $\ell_m(t)$ decreases by at least one, while the value of $j_m(t)$ may change in either direction, but in any case it cannot increase by more than $s_m - 1$, $s_m$ being the number of jump points of $f'_m$ over the interval $(-M, M)$; thus $s_m - 1 < 2Mm$. It follows that the integer-valued function $p_m(t) = j_m(t) + s_m \ell_m(t)$ stays constant along the open time intervals between consecutive shock collisions, while decreasing by at least one across any monotone or nonmonotone shock collision. Across $t = 0$, $\ell_m(0+) = \ell_m(0)$ and $j_m(0+) \leq (s_m + 1) j_m(0)$. Therefore, $(s_m + 1)[j_m(0) + \ell_m(0)]$ provides an upper bound for the total number of shock collisions involved in the construction of $u_m$.

As function of $t$, the total variation of $u_m(\cdot, t)$ over $(-\infty, \infty)$ stays constant along time intervals between consecutive shock collisions; it does not change across monotone shock collisions; and it decreases across nonmonotone shock collisions. Hence,

$$TV_{(-\infty,\infty)} u_m(\cdot, t) \leq TV_{(-\infty,\infty)} u_{m0}(\cdot) \leq TV_{(-\infty,\infty)} u_0(\cdot), \quad 0 \leq t < \infty. \tag{14.1.4}$$

Since the speed of any shock of $u_m$ cannot exceed the Lipschitz constant $c$ of $f$ over $[-M, M]$, (14.1.4) implies

(14.1.5)

$$\int_{-\infty}^{\infty} |u_m(x, t) - u_m(x, \tau)| dx \leq c|t - \tau| TV_{(-\infty,\infty)} u_0(\cdot), \quad 0 \leq \tau < t < \infty.$$

By virtue of (14.1.4), Helly's theorem and the Cantor diagonal process, one finds a subsequence $\{u_{m_k}\}$ such that $\{u_{m_k}(\cdot, t)\}$ is convergent in $L^1_{\text{loc}}(-\infty, \infty)$, for any rational $t \in [0, \infty)$. Then, (14.1.5) implies that $\{u_{m_k}(\cdot, t)\}$ is Cauchy in $L^1_{\text{loc}}(-\infty, \infty)$ for all $t \in [0, \infty)$, and hence $\{u_{m_k}\}$ converges in $L^1_{\text{loc}}$ to some function $u$ of locally bounded variation on $(-\infty, \infty) \times [0, \infty)$.

As discussed in Chapter VI, since $u_m$ is the admissible solution of (14.1.3),

(14.1.6)

$$\int_0^{\infty} \int_{-\infty}^{\infty} [\partial_t \psi \eta(u_m) + \partial_x \psi q_m(u_m)] dx dt + \int_{-\infty}^{\infty} \psi(x, 0) \eta(u_{0m}(x)) dx \geq 0,$$

for any convex entropy $\eta$, with associated entropy flux $q_m = \int \eta' df_m$, and all nonnegative Lipschitz test functions $\psi$ on $(-\infty, \infty) \times [0, \infty)$, with compact support. As $m \to \infty$, $\{u_{0m}\}$ converges, a.e. on $(-\infty, \infty)$, to $u_0$, and $\{q_m\}$ converges, uniformly on $[-M, M]$, to the function $q = \int \eta' df$, namely, the entropy flux associated with the entropy $\eta$ in the conservation law $(14.1.2)_1$. Upon passing to the limit in (14.1.6), along the subsequence $\{m_k\}$, we deduce

$$\int_0^{\infty} \int_{-\infty}^{\infty} [\partial_t \psi \eta(u) + \partial_x \psi q(u)] dx dt + \int_{-\infty}^{\infty} \psi(x, 0) \eta(u_0(x)) dx \geq 0, \tag{14.1.7}$$

which in turn implies that $u$ is the admissible solution of (14.1.2). By uniqueness, we infer that the entire sequence $\{u_m\}$ converges to $u$.

## 14.2 Front Tracking for Genuinely Nonlinear Systems of Conservation Laws

Consider a system of conservation laws, in canonical form

$$\partial_t U + \partial_x F(U) = 0, \tag{14.2.1}$$

which is strictly hyperbolic (7.2.8), and each characteristic family is either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2). The object of this section is to introduce a front tracking algorithm that solves the initial-value problem (13.1.1), under initial data $U_0$ with small total variation, and provides, in particular, an alternative proof of the existence Theorem 13.1.1.

The instrument of the algorithm will be special Riemann solvers, which will be employed to resolve jump discontinuities into centered wave fans composed of jump discontinuities and constant states, approximating the admissible solution of the Riemann problem. In implementing the algorithm, the initial data are approximated by step functions whose jump discontinuities are then resolved into wave fans. Interactions induced by the collision of jump discontinuities are in turn resolved, in the order they occur, into similar wave fans. It will suffice to consider the generic situation, in which no more than two jump discontinuities may collide simultaneously. The expectation is that such a construction will produce an approximate solution of the initial-value problem in the class of piecewise constant functions.

The first item on the agenda is how to design suitable Riemann solvers. The experience with the scalar conservation law, in Section 14.1, suggests that one should synthesize the centered wave fans by a combination of constant states, admissible shocks, and inadmissible rarefaction shocks with small strength.

In an *admissible $i$-shock*, the right state $U_+$ lies on the $i$-th shock curve through the left state $U_-$, that is, in the notation of Section 9.3, $U_+ = \Phi_i(\tau; U_-)$, with $\tau < 0$ when the $i$-th characteristic family is genuinely nonlinear (compressive shock) or with $\tau \lessgtr 0$ when the $i$-th characteristic family is linearly degenerate (contact discontinuity). The amplitude is $\tau$, the strength is measured by $|\tau|$, and the speed $s$ is set by the Rankine-Hugoniot jump condition (8.1.2).

Instead of actual rarefaction shocks, it is more convenient to employ "rarefaction fronts", namely jump discontinuities that join states lying on a rarefaction wave curve and propagate with characteristic speed. Thus, in an *$i$-rarefaction front* (which may arise only when the $i$-th characteristic family is genuinely nonlinear) the right state $U_+$ lies on the $i$-th rarefaction wave curve through the left state $U_-$, i.e., $U_+ = \Phi_i(\tau; U_-)$, with $\tau > 0$. Both, amplitude and strength are measured by $\tau$, and the speed is set equal to $\lambda_i(U_+)$. Clearly, these fronts violate not only the entropy admissibility criterion but even the Rankine-Hugoniot jump condition, albeit only slightly when their strength is small.

Centered rarefaction waves may be approximated by centered wave fans composed of constant states and rarefaction fronts with strength not exceeding some prescribed magnitude $\delta > 0$. Consider some $i$-rarefaction wave, centered, for definiteness, at the origin, which joins the state $U_-$, on the left, with the state $U_+$, on the right. Thus, $U_+$ lies on the $i$-rarefaction curve through $U_-$, say $U_+ = \Phi_i(\tau; U_-)$, for some $\tau > 0$. If $\nu$ is the smallest integer that is larger than $\tau/\delta$, we set $U^0 = U_-$, $U^\nu = U_+$, $U^\mu = \Phi_i(\mu\delta; U_-)$, $\mu = 1, \cdots, \nu - 1$, and approximate the rarefaction wave, inside the sector $\lambda_i(U_-) < \frac{x}{t} < \lambda_i(U_+)$, by the wave fan

$$U(x,t) = U^\mu, \qquad \lambda_i(U^{\mu-1}) < \frac{x}{t} < \lambda_i(U^\mu), \qquad \mu = 1, \cdots, \nu. \tag{14.2.2}$$

We are thus naturally lead to an *Approximate Riemann Solver*, which resolves the jump discontinuity between a state $U_l$, on the left, and $U_r$, on the right, into a wave fan composed of constant states, admissible shocks, and rarefaction fronts, by the following procedure: The starting point is the admissible solution of the Riemann problem, consisting of $n + 1$ constant states $U_l = U_0, U_1, \cdots, U_n = U_r$, where $U_{i-1}$ is joined to $U_i$ by an admissible $i$-shock or an $i$-rarefaction wave. To pass to the approximation, the domain and values of the constant states, and thereby all shocks, are retained, whereas, as described above, any rarefaction wave is replaced, within its sector, by a fan of constant states and rarefaction fronts of the same family, with strength not exceeding $\delta$ (Fig. 14.2.1).

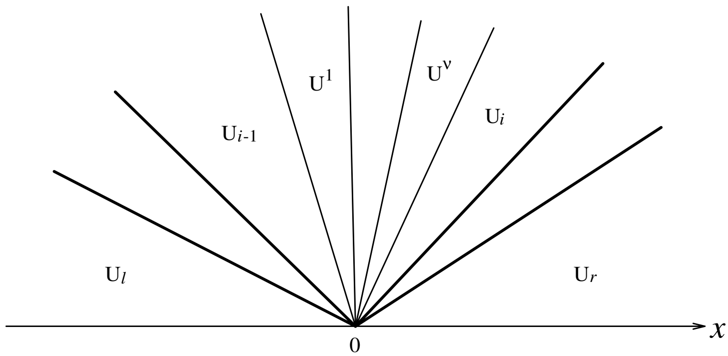


**Fig. 14.2.1**

Our earlier success with the scalar case may raise expectations that a front tracking algorithm, in which all shock interactions are resolved via the above approximate, though relatively accurate, Riemann solver, will produce an approximate solution of our system, converging to an admissible solution of the initial-value problem, as the allowable strength $\delta$ of rarefaction fronts shrinks to zero. Unfortunately, such an approach would generally fail, for the following reason: By contrast to the case for scalar conservation laws, wave interactions in systems tend to increase the complexity of the wave pattern so that collisions become progressively more frequent and the algorithm may grind to a halt in finite time. As a remedy, in order to prevent the proliferation of waves, only shocks and rarefaction fronts of substantial strength shall be tracked with relative accuracy. The rest shall not be totally disregarded but shall be treated with less accuracy: They will be lumped together to form jump discontinuities, dubbed "pseudoshocks", which propagate with artificial, supersonic speed.

A *pseudoshock* is allowed to join arbitrary states $U_-$ and $U_+$. Its strength is measured by $|U_+ - U_-|$ and its assigned speed is a fixed upper bound $\lambda_{n+1}$ of $\lambda_n(U)$, for $U$ in the range of the solution. Clearly, pseudoshocks are more serious violators of the Rankine-Hugoniot jump condition than rarefaction fronts, and may thus wreak havoc in the approximate solution, unless their combined strength is kept very small.

To streamline the exposition, $i$-rarefaction fronts and $i$-shocks (compression or contact discontinuities) together will be dubbed *$i$-fronts*. Fronts and pseudoshocks will be called collectively *waves*. Thus an $i$-front will be an *$i$-wave* and a pseudoshock will be termed *$(n+1)$-wave*. As in earlier chapters, the amplitudes of waves will be denoted by Greek letters $\alpha, \beta, \gamma, \dots$ with corresponding strengths $|\alpha|, |\beta|, |\gamma|, \cdots$.

Under circumstances to be specified below, the jump discontinuity generated by the collision of two waves shall be resolved via a *Simplified Riemann Solver*, which allows fronts to pass through the point of interaction without affecting their strength, while introducing an outgoing pseudoshock in order to bridge the resulting mismatch in the states. The following cases may arise.

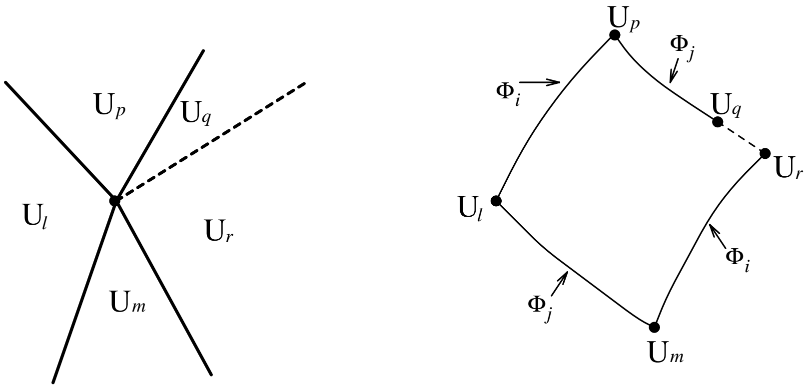


**Fig. 14.2.2**

Suppose that, for $i < j$, a $j$-front, joining the states $U_l$ and $U_m$, collides with an $i$-front, joining the states $U_m$ and $U_r$; see Fig. 14.2.2. Thus $U_m = \Phi_j(\tau_l; U_l)$ and $U_r = \Phi_i(\tau_r; U_m)$. To implement the Simplified Riemann Solver, one determines the states $U_p = \Phi_i(\tau_r; U_l)$ and $U_q = \Phi_j(\tau_l; U_p)$. Then, the outgoing wave fan will be composed of the $i$-front, joining the states $U_l$ and $U_p$, the $j$-front, joining the states $U_p$ and $U_q$, plus the pseudoshock that joins $U_q$ with $U_r$.

Suppose next that an $i$-front, joining the states $U_l$ and $U_m$, collides with another $i$-front, joining the states $U_m$ and $U_r$ (no such collision may occur unless at least one of these fronts is a compressive shock); see Fig. 14.2.3.

Thus $U_m = \Phi_i(\tau_l; U_l)$ and $U_r = \Phi_i(\tau_r; U_m)$. If $U_q = \Phi_i(\tau_l + \tau_r; U_l)$, the outgoing wave fan will be composed of the $i$-front, joining the states $U_l$ and $U_q$, plus the pseudoshock that joins $U_q$ with $U_r$.

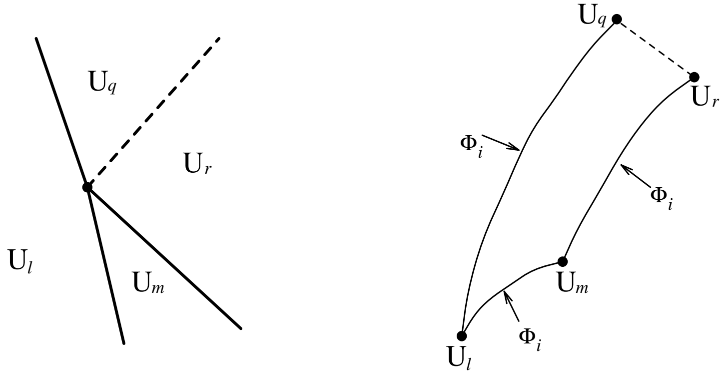


**Fig. 14.2.3**

Finally, suppose a pseudoshock, joining the states $U_l$ and $U_m$ , collides with an $i$-front, joining the states $U_m$ and $U_r$ ; see Fig. 14.2.4. Hence, $U_r = \Phi_i(\tau_m; U_m)$. We determine $U_q = \Phi_i(\tau_m; U_l)$. The outgoing wave fan will be composed of the $i$-front, joining the states $U_l$ and $U_q$ , plus the pseudoshock that joins $U_q$ with $U_r$ .

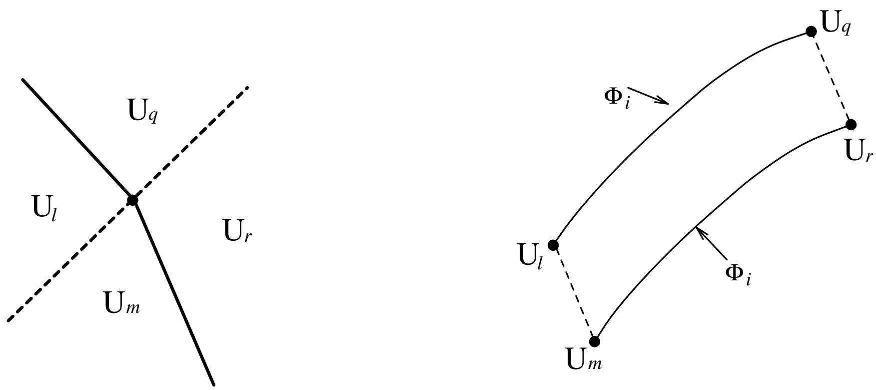


**Fig. 14.2.4**

In implementing the front tracking algorithm, one fixes, at the outset, the supersonic speed $\lambda_{n+1}$ of pseudoshocks, sets the delimiter $\delta$ for the strength of rarefaction fronts, and also specifies a third parameter $\sigma > 0$, which rules how jump discontinuities are to be resolved:

- Jump discontinuities resulting from the collision of two fronts, with respective amplitudes $\alpha$ and $\beta$, must be resolved via the Approximate Riemann Solver if $|\alpha||\beta| > \sigma$, or via the Simplified Riemann Solver if $|\alpha||\beta| \leq \sigma$.
- Jump discontinuities resulting from the collision of a pseudoshock with any front must be resolved via the Simplified Riemann Solver.

- Jump discontinuities of the step function approximating the initial data are to be resolved via the Approximate Riemann Solver.

## 14.3 The Global Wave Pattern

Starting out from some fixed initial step function, the front tracking algorithm, described in the previous section, will produce a piecewise constant function $U$ on a maximal time interval $[0, T)$. In principle, $T$ may turn out to be finite, if the number of collisions grows without bound as $t \uparrow T$, so the onus is to show that this will not happen.

To understand the structure of $U$, one has to untangle the complex wave pattern. Towards that end, waves must be tracked not just between consecutive collisions but globally, from birth to extinction or in perpetuity. The waves are granted global identity through the following convention: An $i$-wave involved in a collision does not necessarily terminate there, but generally continues on as the outgoing $i$-wave from that point of wave interaction. Any ambiguities that may arise in applying the above rule will be addressed and resolved below.

Pseudoshocks are generated by the collision of two fronts, resolved via the Simplified Riemann Solver, as depicted in Figs. 14.2.2 or 14.2.3. On the other hand, $i$-fronts may be generated either at $t = 0$, from the resolution of some jump discontinuity of the initial step function, or at $t > 0$, by the collision of a $j$-front with a $k$-front, where $j \neq i \neq k$, that is resolved via the Approximate Riemann Solver.

Every wave carries throughout its life span a number $\mu$, identifying its *generation order*, that is the maximum number of collisions predating its birth. Thus, fronts originating at $t = 0$ are assigned generation order $\mu = 0$. Any other new wave, which is necessarily generated by the collision of two waves, with respective generation orders say $\mu_1$ and $\mu_2$, is assigned generation order $\mu = \max\{\mu_1, \mu_2\} + 1$.

As postulated above, waves retain their generation order as they traverse points of interaction. Ambiguity may arise when, in a collision of an $i$-rarefaction front with a $j$-front, resolved via the Approximate Riemann Solver, the outgoing $i$-wave fan contains two $i$-rarefaction fronts. In that case, the stronger of these fronts, with strength $\delta$, is designated as the prolongation of the incoming $i$-front, while the other $i$-front, with strength $< \delta$, is regarded as a new front and is assigned a higher generation order, in accordance to the standard rule. Ambiguity may also arise when two fronts of the same family collide, since the outgoing wave fan may include (at most) one front of that family. In that situation, the convention is that the front with the lower generation order is designated the survivor, while the other one is terminated. In case both fronts are of the same generation order, either one, arbitrarily, may be designated as the survivor. Of course, both fronts may be terminated upon colliding, as depicted in Fig. 14.2.3, in the (nongeneric) case where one of them is a compression shock, the other is a rarefaction front of the same characteristic family, and both have the same strength. Pseudoshocks may also be extinguished in finite time by colliding with a front, as depicted in Fig. 14.2.4, in the (nongeneric) case $U_q = U_r$.

We now introduce the following notions, which will establish a connection with the approach pursued in Chapters X-XII.

For $i = 1, \cdots, n$, an *i-characteristic* associated with $U$ is a Lipschitz, polygonal line $x = \xi(t)$ which traverses constant states, say $\bar{U}$, at classical $i$-characteristic speed, $\dot{\xi} = \lambda_i(\bar{U})$, but upon impinging on an $i$-front, or a generation point thereof, it adheres to that front, following it throughout its lifespan. Thus, in particular, any $i$-front is an $i$-characteristic. By analogy, $(n+1)$-*characteristics* are defined as straight lines with slope $\lambda_{n+1}$. Thus, pseudoshocks are $(n+1)$-characteristics.

Consider now an oriented Lipschitz curve with graph $\mathcal{C}$, which divides the upper half-plane into its "positive" and "negative" side. We say $\mathcal{C}$ is *nonresonant* if the set $\{1, \cdots, n, n+1\}$ can be partitioned into three, pairwise disjoint, possibly empty, subsets $\mathcal{N}_-$, $\mathcal{N}_0$ and $\mathcal{N}_+$, with the following properties: $\mathcal{N}_-$ and $\mathcal{N}_+$ each consists of up to $n+1$ consecutive integers, while $\mathcal{N}_0$ may contain at most one member. For $i \in \mathcal{N}_-$ (or $i \in \mathcal{N}_+$), any $i$-characteristic impinging on $\mathcal{C}$ crosses from the positive to the negative (or from the negative to the positive) side. On the other hand, if $i \in \mathcal{N}_0$, any $i$-characteristic impinging upon $\mathcal{C}$, from either its positive or its negative side, is absorbed by $\mathcal{C}$, i.e., $\mathcal{C}$ itself is an $i$-characteristic.

Noteworthy examples of nonresonant curves include:

(a) Any $i$-characteristic, in particular any $i$-wave, so that $\mathcal{N}_- = \{1, \cdots, i-1\}$, $\mathcal{N}_0 = \{i\}$ and $\mathcal{N}_+ = \{i+1, \cdots, n+1\}$.
(b) Any *space-like curve*. Assuming $\lambda_1(U) < 0 < \lambda_{n+1}$, these may be represented by Lipschitz functions $t = \hat{t}(x)$, such that $1/\lambda_1 < d\hat{t}/dx < 1/\lambda_{n+1}$, a.e. In that case, $\mathcal{N}_+ = \{1, \cdots, n+1\}$ while both $\mathcal{N}_-$ and $\mathcal{N}_0$ are empty.

The relevance of the above will become clear in the next section.

## 14.4 Approximate Solutions

The following definition collects all the requirements on a piecewise constant function, of the type produced by the front tracking algorithm, so as to qualify as a reasonable approximation to the solution of our Cauchy problem:

**14.4.1 Definition.** For $\delta > 0$, a *$\delta$-approximate solution* of the hyperbolic system of conservation laws (14.2.1) is a piecewise constant function $U$, defined on $(-\infty, \infty) \times [0, \infty)$ and satisfying the following conditions: The domains of the constant states are bordered by jump discontinuities, called waves, each propagating with constant speed along a straight line segment $x = y(t)$. Any wave may originate either at a point of the $x$-axis, $t = 0$, or at a point of collision of other waves, and generally terminates upon colliding with another wave, unless no such collision occurs in which case it propagates all the way to infinity. Only two incoming waves may collide simultaneously, but any (finite) number of outgoing waves may originate at a point of collision. There is a finite number of points of collision, waves and constant states. The waves are of three types:

(a) *Shocks.* An (approximate) $i$-shock $x = y(t)$ borders constant states $U_-$, on the left, and $U_+$, on the right, which can be joined by an admissible $i$-shock, i.e., $U_+ = W_i(\tau; U_-)$, with $\tau < 0$ when the $i$-characteristic family is genuinely nonlinear or $\tau \lessgtr 0$ when the $i$-characteristic family is linearly degenerate, and propagates approximately at the shock speed $s = s_i(\tau; U_-)$:

$$|\dot{y}(\cdot) - s| \leq \delta. \tag{14.4.1}$$

(b) *Rarefaction Fronts.* An (approximate) $i$-rarefaction front $x = y(t)$ borders constant states $U_-$, on the left, and $U_+$, on the right, which can be joined by an $i$-rarefaction wave with strength $\leq \delta$, i.e., $U_+ = V_i(\tau; U_-)$, with $0 < \tau \leq \delta$, and propagates approximately at characteristic speed:

$$|\dot{y}(\cdot) - \lambda_i(U_+)| \leq \delta. \tag{14.4.2}$$

(c) *Pseudoshocks.* A pseudoshock $x = y(t)$ may border arbitrary states $U_-$ and $U_+$ and propagates at the specified supersonic speed:

$$\dot{y}(\cdot) = \lambda_{n+1} . \tag{14.4.3}$$

The combined strength of pseudoshocks does not exceed $\delta$:

$$\sum |U(y(t)+, t) - U(y(t)-, t)| \leq \delta, \qquad 0 < t < \infty, \tag{14.4.4}$$

where for each $t$ the summation runs over all pseudoshocks $x = y(\cdot)$ which cross the $t$-time line.

If, in addition, the step function $U(\cdot, 0)$ approximates the initial data $U_0$ in $L^1$, within distance $\delta$,

$$\int_{-\infty}^{\infty} |U(x, 0) - U_0(x)| dx \leq \delta, \tag{14.4.5}$$

then $U$ is called a *$\delta$-approximate solution of the Cauchy problem* (13.1.1).

The extra latitude afforded by the above definition in allowing the speed of (approximate) shocks and rarefaction fronts to (slightly) deviate from their more accurate values granted by the front tracking algorithm provides some flexibility which may be put to good use for ensuring that no more than two fronts may collide simultaneously.

The effectiveness of front tracking will be demonstrated through the following

**14.4.2 Theorem.** *Assume $U_0 \in BV(-\infty, \infty)$, with $TV_{(-\infty,\infty)} U_0(\cdot) \leq a << 1$. Fix any small positive $\delta$, and approximate $U_0$ by some step function $U_{0\delta}$ such that $TV_{(-\infty,\infty)} U_{0\delta}(\cdot) \leq TV_{(-\infty,\infty)} U_0(\cdot)$ and $\|U_{0\delta}(\cdot) - U_0(\cdot)\|_{L^1(-\infty,\infty)} \leq \delta$. Then the front tracking algorithm with initial data $U_{0\delta}$, fixed supersonic speed $\lambda_{n+1}$ for pseudoshocks, delimiter $\delta$ for the strength of rarefaction fronts, and sufficiently small parameter $\sigma$ (depending on $\delta$ and on the number of jump points of $U_{0\delta}$) generates a $\delta$-approximate solution $U_\delta$ of the initial-value problem* (13.1.1). *Any sequence of*

*$\delta$'s converging to zero contains a subsequence* $\{\delta_k\}$ *such that* $\{U_{\delta_k}\}$ *converges, a.e. on* $(-\infty, \infty) \times [0, \infty)$, *to a $BV$ solution $U$ of* (13.1.1), *which satisfies the entropy admissibility condition for any convex entropy-entropy flux pair* $(\eta, q)$ *of the system* (14.2.1), *together with the estimates* (13.1.5) *and* (13.1.6). *Furthermore, the trace of $U$ on any Lipschitz graph on the upper half-plane that is nonresonant relative to all $U_\delta$ has bounded variation.*

The above proposition reestablishes the assertions of Theorem 13.1.1. The property that the trace of $U$ along nonresonant curves has bounded variation establishes a connection with the class of solutions discussed in Chapter XII.

The demonstration of Theorem 14.4.2 is quite lengthy and will be presented, in installments, in the next three sections. However, the following road map may prove useful at this juncture.

As already noted in Section 14.3, once the step function $U_{0\delta}$ has been designated, the front tracking algorithm will produce $U_\delta$, at least on a time interval $[0, T)$, which as we shall see later is $[0, \infty)$. We shall be assuming throughout that the range of $U_\delta$ is contained in a ball of small radius in state space, a condition that must be verified a posteriori. The constants $c_1, c_2, \cdots, \kappa, \cdots$ which will appear in the course of the proof, all depend solely on bounds of $F$ and its derivatives in that ball.

The first step will be to establish an estimate

$$TV_{(-\infty,\infty)} U_\delta(\cdot, t) \leq c_1 TV_{(-\infty,\infty)} U_0(\cdot), \qquad 0 \leq t < T, \tag{14.4.6}$$

on the total variation, together with a bound on the total amount of wave interaction. By account of the construction of $U_\delta$, (14.4.6) will immediately imply

(14.4.7)

$$\int_{-\infty}^{\infty} |U_\delta(x, t) - U_\delta(x, \tau)| dx \leq c_2 |t - \tau| TV_{(-\infty,\infty)} U_0(\cdot), \quad 0 \leq \tau < t < T,$$

with $c_2 = cc_1$, where $c$ is any upper bound of the wave speeds; for instance $c$ is the maximum of $\lambda_{n+1}$ and $-\inf \lambda_1(U)$. The usefulness of these estimates is twofold: First, they will assist in the task of verifying that $U_\delta$ meets the requirements set by Definition 14.4.1. Secondly, they will induce compactness that makes it possible to pass to the $\delta \downarrow 0$ limit.

In verifying that $U_\delta$ is a $\delta$-approximate solution, the requirements (14.4.1), (14.4.2) and (14.4.3), on the speed of shocks, rarefaction fronts and pseudoshocks, are patently met, because of the specifications of the construction. Moreover, the selection of the delimiter entails that the strengths of rarefaction fronts will be bounded by $\delta$. The remaining requirements, namely that the combined strength of pseudoshocks is also bounded by $\delta$, as in (14.4.4), and that the number of collisions is finite, will be established by insightful analysis of the wave pattern. In particular, this will furnish the warranty that $U_\delta$ is generated, in finite steps, on the entire upper half-plane, i.e., $T = \infty$.

The final step in the proof will complete the construction of the solution to (13.1.1) by passing to the $\delta \downarrow 0$ limit in $U_\delta$, via a compactness argument relying on the estimates (14.4.6) and (14.4.7).

## 14.5 Bounds on the Total Variation

As in Section 13.4, $TV_{(-\infty,\infty)}U_\delta(\cdot, t)$ will be measured through

$$L(t) = \sum |\gamma|, \tag{14.5.1}$$

namely by the sum of the strengths of all jump discontinuities that cross the $t$-time line. Clearly, $L(\cdot)$ stays constant along time intervals between consecutive collisions of fronts and changes only across points of wave interaction. To estimate these changes, we have to investigate the various types of collisions.

Suppose a $j$-front of amplitude $\alpha$ collides with an $i$-front of amplitude $\beta$. When $|\alpha||\beta| \geq \sigma$, so that the resulting jump discontinuity is resolved, via the Approximate Riemann Solver, into a full wave fan $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, then, by virtue of Theorem 9.9.1[1],

$$|\varepsilon_j - \alpha| + |\varepsilon_i - \beta| + \sum_{k \neq i,j} |\varepsilon_k| = O(1)|\alpha||\beta|, \tag{14.5.2}$$

if $i < j$, or

$$|\varepsilon_i - \alpha - \beta| + \sum_{k \neq i} |\varepsilon_k| = O(1)|\alpha||\beta|, \tag{14.5.3}$$

if $i = j$. On the other hand, when $|\alpha||\beta| < \sigma$, in which case the resulting jump discontinuity is resolved, via the Simplified Riemann Solver, as shown in Fig. 14.2.2 or Fig. 14.2.3, the amplitude of the colliding fronts is conserved. The strength of the generated outgoing pseudoshock is easily estimated from the wave diagrams in state space:

$$|U_R - U_Q| = O(1)|\alpha||\beta|. \tag{14.5.4}$$

Consider next the case depicted in Fig. 14.2.4, where a pseudoshock collides with an $i$-front of amplitude $\beta$. Since the amplitude of the $i$-front is conserved across the collision, analysis of the wave diagram in state space, Fig. 14.2.4, yields that the strength of the outgoing pseudoshock is related to the strength of the incoming pseudoshock by

$$|U_R - U_Q| = |U_M - U_L| + O(1)|\beta||U_M - U_L|. \tag{14.5.5}$$

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[1] If the outgoing $k$-wave is a fan of $k$-rarefaction fronts, $\varepsilon_k$ denotes the cumulative amplitude and $|\varepsilon_k|$ stands for the cumulative strength of these fronts.

Let $I$ denote the set of $t \in (0, T)$ where collisions occur. We let $\Delta$ denote the "jump" operator from $t-$ to $t+$, for $t \in I$. By account of the analysis of wave interactions, above, we infer

$$\Delta L(t) \leq \kappa|\alpha||\beta|, \qquad t \in I, \tag{14.5.6}$$

where $|\alpha|$ and $|\beta|$ are the strengths of the waves that collide at $t$.

Our strategy for keeping $TV_{(-\infty,\infty)}U_\delta(\cdot, t)$ under control is to show that any increase of $L(\cdot)$ allowed by (14.5.6) is offset by the simultaneous decrease in the amount of potential wave interaction.

A $j$-wave and an $i$-wave, with the former crossing the $t$-time line to the left of the latter, are called *approaching* when either $i < j$, or $i = j$ and at least one of these waves is a compression shock.

The potential for wave interaction at $t \in (0, \tau)\backslash I$ will be measured by

$$Q(t) = \sum |\zeta||\xi|, \qquad t \in (0, T)\backslash I, \tag{14.5.7}$$

where the summation runs over all pairs of approaching waves, with strengths, say, $|\zeta|$ and $|\xi|$, which cross the $t$-time line. In particular,

$$Q(t) \leq \tfrac{1}{2}L(t)^2, \qquad t \in (0, T)\backslash I. \tag{14.5.8}$$

Clearly, $Q(\cdot)$ stays constant along time intervals between consecutive collisions. On the other hand, at any $t \in I$ where waves with strength $|\alpha|$ and $|\beta|$ collide, our analysis of wave interactions implies

$$\Delta Q(t) \leq -|\alpha||\beta| + \kappa|\alpha||\beta|L(t-), \qquad t \in I. \tag{14.5.9}$$

In analogy to the Glimm functional (13.4.8), we set

$$G(t) = L(t) + 2\kappa Q(t), \qquad t \in (0, T)\backslash I. \tag{14.5.10}$$

Combining (14.5.10) with (14.5.6) and (14.5.9), yields

$$\Delta G(t) \leq \kappa[2\kappa G(t-) - 1]|\alpha||\beta|, \qquad t \in (0, T)\backslash I. \tag{14.5.11}$$

Assume the total variation of the initial data is so small that $4\kappa L(0+) \leq 1$. Then, on account of (14.5.10) and (14.5.8), $G(0+) \leq 2L(0+) \leq (2\kappa)^{-1}$. This together with (14.5.11) and a simple induction argument yields $\Delta G(t) \leq 0$, $t \in I$, i.e., $G(\cdot)$ is nonincreasing. Hence

$$L(t) \leq G(t) \leq G(0+) \leq 2L(0+), \qquad t \in (0, T)\backslash I, \tag{14.5.12}$$

which establishes the desired estimate (14.4.6).

Next we estimate the total amount of wave interaction. Since $\kappa L(t-) \leq \frac{1}{2}$, (14.5.9) yields

$$\Delta Q(t) \leq -\tfrac{1}{2}|\alpha||\beta|, \qquad t \in I. \tag{14.5.13}$$

By summing (14.5.13) over all $t \in I$, and upon using (14.5.8),

$$\sum |\alpha||\beta| \le L(0+)^2\,, \tag{14.5.14}$$

where the summation runs over the set of collisions in $(-\infty, \infty) \times (0, T)$.

Let us now consider any Lipschitz graph $\mathcal{C}$ in $(-\infty, \infty) \times [0, T)$ that is nonresonant relative to $U_\delta$, as defined in Section 14.3. The aim is to estimate the total variation of the trace of $U_\delta$ on $\mathcal{C}$, measured by the sum $L_{\mathcal{C}} = \sum |\gamma|$ of the strengths of all waves that impinge on $\mathcal{C}$.

Let $J$ stand for the set of $t \in (0, T)$ where some wave impinges on $\mathcal{C}$. For $t \in (0, T)\backslash(I \bigcup J)$ we set

$$M(t) = \sum\nolimits_{-} |\gamma| + \sum\nolimits_{+} |\gamma| + \sum\nolimits_{0} |\gamma|, \tag{14.5.15}$$

where the summation $\Sigma_-$ (or $\Sigma_+$) runs over the $i$-waves, with $i \in \mathcal{N}_-$ (or $\mathcal{N}_+$), that cross the $t$-time line on the positive (or negative) side of $\mathcal{C}$; while $\Sigma_0$ runs over all $i$-waves, with $i \in \mathcal{N}_0$, that cross the $t$-time line on either side of $\mathcal{C}$. Clearly,

$$\Delta M(t) = -|\gamma|, \qquad t \in J\backslash I, \tag{14.5.16}$$

$$\Delta M(t) \le \kappa|\alpha||\beta|, \qquad t \in I\backslash J, \tag{14.5.17}$$

$$\Delta M(t) \le -|\gamma| + \kappa|\alpha||\beta|, \qquad t \in I \cap J, \tag{14.5.18}$$

where $|\alpha|$ and $|\beta|$ are the strengths of the waves colliding at $t \in I$ and $|\gamma|$ is the strength of the wave that impinges on $\mathcal{C}$ at $t \in J$. Summing the above inequalities over all $t \in I \bigcup J$ and using (14.5.14) together with $4\kappa L(0+) \le 1$, we conclude

$$L_{\mathcal{C}} \le M(0+) + \kappa \sum |\alpha||\beta| \le 2L(0+). \tag{14.5.19}$$

Another important implication of the boundedness of the amount of wave interaction is that the total number of collisions is finite and bounded, independently of $T$. Indeed, recall that the Approximate Riemann Solver is employed to resolve collisions only when the product of the strengths of the two incoming fronts exceeds $\sigma$. By virtue of (14.5.14), the number of such collisions is bounded by $L(0+)^2/\sigma$. Fronts are generated exclusively by the application of the Approximate Riemann Solver to resolve jump discontinuities of $U_{0\delta}$ or collisions of fronts. Therefore, the number of fronts is bounded. Any two fronts may collide at most once in their lifetime, so the number of collisions between fronts is also bounded. Since all pseudoshocks are generated by collisions of fronts, the number of pseudoshocks is likewise bounded. But then, even the number of collisions between fronts and pseudoshocks must be bounded. To summarize, the total number of collisions is finite, bounded solely in terms of $\delta, \sigma$, and the number of jump points of $U_{0\delta}$. Consequently, the front tracking algorithm generates $U_\delta$, in finite steps, on the entire upper half-plane. In particular, the estimates (14.4.6) and (14.4.7) will hold for $0 \le t < \infty$ and $0 \le \tau < t < \infty$, respectively.

## 14.6 Bounds on the Combined Strength of Pseudoshocks

The final task for verifying that $U_\delta$ is a $\delta$-approximate solution of (14.2.1) is to establish requirement (14.4.4). The notion of generation order was introduced in Section 14.3. Waves of high generation order are produced after a large number of collisions and so it should be expected that their strength is small. Indeed, the first step in our argument is to show that the combined strength of all waves, and thus in particular of all pseudoshocks, of sufficiently high generation order is arbitrarily small. To that end, the analysis of Section 14.5 shall be refined by sorting out and monitoring separately the waves according to their generation order.

We know by now that the total number of collisions is bounded, and hence the generation order of all waves lies in a finite range, $0 \leq \mu \leq \nu$. Note, however, that the magnitude of $\nu$ depends penultimately on $\delta$, and should be expected to grow without bounds as $\delta \downarrow 0$. For $\mu = 0, 1, \cdots, \nu$ and $t \in [0, \infty)\backslash I$, we let $L_\mu(t)$ denote the sum of the strengths of all waves with generation order $\geq \mu$ that cross the $t$-time line; and $Q_\mu(t)$ stand for the sum of the products of the strengths of all couples of approaching waves that cross the $t$-time line and have generation order $\mu_1, \mu_2$ with $\max\{\mu_1, \mu_2\} \geq \mu$. Thus, in particular, $L_0(t) = L(t)$ and $Q_0(t) = Q(t)$. Finally, we identify the set $I_\mu$ of times $t \in I$ in which a wave of generation order $\mu$ collides with a wave of generation order $\leq \mu$.

Collisions between waves of generation order $\leq \mu - 2$ cannot affect waves of generation order $\geq \mu$, and so

$$\Delta L_\mu(t) = 0, \qquad t \in I_0 \cup \cdots \cup I_{\mu-2}\,. \tag{14.6.1}$$

Any change in $L_\mu(\cdot)$ at $t \in I$ must be induced by the collision of two waves, of which at least one is of generation order $\geq \mu - 1$. These colliding waves, with strengths say $|\alpha|$ and $|\beta|$, are contributing $|\alpha||\beta|$ to $Q_{\mu-1}(t-)$ but nothing to $Q_{\mu-1}(t+)$. As in Section 14.5, the resulting drop in $Q_{\mu-1}(\cdot)$ can be used to offset the potential increment of $L_\mu(\cdot)$, which is bounded by $\kappa|\alpha||\beta|$:

$$\Delta L_\mu(t) + 2\kappa \Delta Q_{\mu-1}(t) \leq 0, \qquad t \in I_{\mu-1} \cup \cdots \cup I_\nu\,. \tag{14.6.2}$$

By similar arguments one verifies the inequalities

$$\Delta Q_\mu(t) + 2\kappa \Delta Q(t) L_\mu(t-) \leq 0, \qquad t \in I_0 \cup \cdots \cup I_{\mu-2}\,, \tag{14.6.3}$$

$$\Delta Q_\mu(t) + 2\kappa \Delta Q_{\mu-1}(t) L(t-) \leq 0, \qquad t \in I_{\mu-1}\,, \tag{14.6.4}$$

$$\Delta Q_\mu(t) \leq 0, \qquad t \in I_\mu \cup \cdots \cup I_\nu\,, \tag{14.6.5}$$

which govern the change of $Q_\mu(\cdot)$ across collisions of various orders.

A superscript $+$ or $-$ will be employed below to indicate "positive" or "negative" part: $w^+ = \max\{w, 0\}$, $w^- = \max\{-w, 0\}$. The aim is to monitor the quantities

$$\hat{L}_\mu = \sup_t L_\mu(t), \qquad \hat{Q}_\mu = \sum_{t \in I} [\Delta Q_\mu(t)]^+\,, \tag{14.6.6}$$

for $\mu = 1, \cdots, \nu$, and show

(14.6.7) $$\hat{L}_\mu \leq 2^{-\mu} c_3 a\,, \qquad \hat{Q}_\mu \leq 2^{-\mu+3} c_3^2 a^2\,,$$

where $a$ is the bound on $TV_{(-\infty,\infty)} U_0(\cdot)$.

From (14.6.1), (14.6.2) and the "initial condition" $L_\mu(0+) = 0,\ \mu = 1, \cdots, \nu$, it follows that

(14.6.8) $$\hat{L}_\mu \leq 2\kappa \sum_{t \in I} [\Delta Q_{\mu-1}(t)]^-\,, \qquad \mu = 1, \cdots, \nu.$$

Next we focus on (14.6.3), (14.6.4) and (14.6.5), with "initial condition" $Q_\mu(0+) = 0$. Recalling (14.5.8), (14.5.12) and using

(14.6.9) $$\sum_{t \in I} [\Delta Q(t)]^- = Q(0+) - Q(\infty) \leq \tfrac{1}{2} L(0+)^2\,,$$

we deduce

(14.6.10) $$\hat{Q}_\mu \leq \kappa L(0+)^2 \hat{L}_\mu + 4\kappa L(0+) \sum_{t \in I} [\Delta Q_{\mu-1}(t)]^-, \quad \mu = 1, \cdots, \nu.$$

We combine (14.6.8) with (14.6.10). Assuming the total variation of the initial data is so small that $10\kappa L(0+) \leq 1$, we deduce

(14.6.11) $$\hat{Q}_\mu \leq \frac{1}{2} \sum_{t \in I} [\Delta Q_{\mu-1}(t)]^-\ , \qquad \mu = 1, \cdots, \nu.$$

In particular, for $\mu = 1$ and by account of (14.6.9), $\hat{Q}_1 \leq \frac{1}{4} L(0+)^2$.

We finally notice that, for $\mu = 1, \cdots, \nu$, since $Q_\mu(0+) = 0$,

(14.6.12) $$\sum_{t \in I} [\Delta Q_\mu(t)]^- = \sum_{t \in I} [\Delta Q_\mu(t)]^+ - Q_\mu(\infty) \leq \hat{Q}_\mu\,.$$

Therefore, (14.6.11) yields $\hat{Q}_\mu \leq \frac{1}{2} \hat{Q}_{\mu-1}, \mu = 2, \cdots, \nu$, which in turn implies $\hat{Q}_\mu \leq 2^{-\mu-1} L(0+)^2$. This together with (14.6.9) and (14.6.10) yields the estimate $\hat{L}_\mu \leq 2^{-\mu-2} L(0+)$. We have thus established (14.6.7).

It is now clear that one can fix $\mu_0$ sufficiently large so that the combined strength of all waves of generation order $\geq \mu_0$, which is majorized by $\hat{L}_{\mu_0}$, does not exceed $\frac{1}{2}\delta$.

In order to estimate the combined strength of pseudoshocks of generation order $< \mu_0$, the first step is to estimate their number. For $\mu = 0, \cdots, \nu$, let $K_\mu$ denote the number of waves of generation order $\leq \mu$. A crude upper bound for $K_\mu$ may be derived by the following argument. The number of outgoing waves produced by resolving a jump discontinuity, via any of the two Riemann solvers, is bounded by a number $b/\delta$. Thus, $K_0 \leq \frac{b}{\delta} N$, where $N$ is the number of jump points of $U_{0\delta}$. Since any two waves may collide at most once in their lifetime, the number of collisions that may generate waves of generation order $\mu$ is bounded by $\frac{1}{2} K_{\mu-1}^2$. Therefore,

(14.6.13)
$$K_\mu \le K_{\mu-1} + \frac{b}{2\delta} K_{\mu-1}^2 \le \frac{b}{\delta} K_{\mu-1}^2 \,,$$

whence one readily deduces

(14.6.14)
$$K_\mu \le \left(\frac{b}{\delta}\right)^{2^{\mu+1}} N^{2^\mu} .$$

Next we estimate the strength of individual pseudoshocks. Any pseudoshock is generated by the collision of two fronts, with strengths $|\alpha|$ and $|\beta|$ such that $|\alpha||\beta| \le \sigma$, which is thus resolved via the Simplified Riemann Solver, as depicted in Figs. 14.2.2 and 14.2.3. It then follows from the corresponding interaction estimate (14.5.4) that the strength of any pseudoshock at birth does not exceed $c_4\sigma$. On account of (14.5.5), the collision of a pseudoshock with a front of strength $|\beta|$, as depicted in Fig. 14.2.4, may increase its strength at most by a factor $1 + \kappa|\beta|$. Consequently, the strength of a pseudoshock may ultimately grow at most by the factor $\prod(1+\kappa|\gamma|)$, where the product runs over all fronts with which the pseudoshock collides during its life span. Since pseudoshocks are nonresonant, the estimate (14.5.19) here applies and implies $\sum |\gamma| \le 2L(0+)$. Assuming $2\kappa L(0+) \le 1$, we thus conclude that the strength of each pseudoshock, at any time, does not exceed $3c_4\sigma$.

It is now clear that by employing the upper bound for $K_{\mu_0-1}$ provided by (14.6.14), and upon selecting $\sigma$ sufficiently small, one guarantees that the combined strength of pseudoshocks of generation order $< \mu_0$ is bounded by $\frac{1}{2}\delta$. In conjunction with our earlier estimate on the total strength of pseudoshocks of generation order $\ge \mu_0$, this establishes (14.4.4).

## 14.7 Compactness and Consistency

In this section, the proof of Theorem 14.4.2 will be completed by passing to the $\delta \downarrow 0$ limit. Here we will just be assuming that $\{U_\delta\}$ is any family of $\delta$-approximate solutions, in the sense of Definition 14.4.1, with $\delta$ positive and small, that satisfy estimates (14.4.6) and (14.4.7). Thus, we shall not require the special features of the particular $\delta$-approximate solutions constructed via the front tracking algorithm, for instance that shocks propagate with the correct shock speed.

Let us fix any test function $\phi$, with compact support in $(-\infty, \infty) \times [0, T)$. By applying Green's theorem,

(14.7.1)
$$\int_0^\infty \int_{-\infty}^\infty [\partial_t \phi U_\delta + \partial_x \phi F(U_\delta)] dx dt + \int_{-\infty}^\infty \phi(x,0) U_\delta(x,0) dx$$
$$= -\int_0^\infty \sum \phi(y(t),t) \{ F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t))$$
$$- \dot{y}(t)[U_\delta(y(t)+,t) - U_\delta(y(t)-,t)] \} dt,$$

where for each $t$ the summation runs over all jump discontinuities $x = y(\cdot)$ that cross the $t$-time line.

When the jump discontinuity $x = y(\cdot)$ is an (approximate) shock, then by virtue of (14.4.1),

$$(14.7.2)\quad |F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t)) - \dot{y}(t)[U_\delta(y(t)+,t) - U_\delta|y(t)-,t)]| \le \delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

Similarly, when $x = y(\cdot)$ is an (approximate) rarefaction front, with strength $\le \delta$, then by account of the proximity between shock and rarefaction wave curves, and (14.4.2),

$$(14.7.3)\quad |F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t)) - \dot{y}(t)[U_\delta(y(t)+,t) - U_\delta(y(t)-,t)]| \le c_5\delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

Finally, when $x = y(\cdot)$ is a pseudoshock,

$$(14.7.4)\quad |F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t))| \le c_6|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

By combining (14.7.2), (14.7.3), (14.7.4) with (14.4.6) and (14.4.4), we deduce that, for any fixed test function $\phi$, the right-hand side of (14.7.1) is bounded by $C_\phi[TV_{(-\infty,\infty)}U_0(\cdot) + 1]\delta$ and thus tends to zero as $\delta \downarrow 0$.

By virtue of (14.4.6), (14.4.7) and Theorem 1.7.3, any sequence of $\delta$'s converging to zero contains a subsequence $\{\delta_k\}$ such that $\{U_{\delta_k}\}$ converges a.e. to some $U$ in $BV_{\mathrm{loc}}$. Passing to the limit in (14.7.1) along the sequence $\{\delta_k\}$, and using (14.4.5), we conclude that $U$ is indeed a weak solution of (13.1.1).

By passing to the $\delta \downarrow 0$ limit in (14.4.6) and (14.4.7), one verifies that $U$ satisfies (13.1.5) and (13.1.6). Furthermore, if $\mathcal{C}$ is any Lipschitz graph that is nonresonant relative to $U_\delta$, for all $\delta$, then, as shown in Section 14.5, the trace of $U_\delta$ on $\mathcal{C}$ has bounded variation, uniformly in $\delta$, and thus, passing to the $\delta \downarrow 0$ limit, yields that the trace of $U$ on $\mathcal{C}$ will have the same property.

To conclude the proof, assume $(\eta, q)$ is an entropy-entropy flux pair for the system (14.2.1), with $\eta(U)$ convex. Let $\phi$ be any nonnegative test function, with compact support in $(-\infty,\infty) \times [0,T)$. By Green's theorem,

$$(14.7.5)\quad \int_0^\infty \int_{-\infty}^\infty [\partial_t\phi\eta(U_\delta) + \partial_x\phi q(U_\delta)]dxdt + \int_{-\infty}^\infty \phi(x,0)\eta(U_\delta(x,0))dx$$
$$= -\int_0^\infty \sum \phi(y(t),t)\{q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t)) - \dot{y}(t)[\eta(U_\delta(y(t)+,t)) - \eta(U_\delta(y(t)-,t))]\}dt,$$

where, as in (14.7.1), for each $t$ the summation runs over all jump discontinuities $x = y(\cdot)$ that cross the $t$-time line.

When $x = y(\cdot)$ is an (approximate) shock, the entropy inequality (8.5.1) together with (14.4.1) imply

(14.7.6)

$$q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t)) - \dot y(t)[\eta(U_\delta(y(t)+,t)) - \eta(U_\delta(y(t)-,t))]$$
$$\leq c_7\delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

When $x = y(\cdot)$ is an (approximate) rarefaction front, with strength $\leq \delta$, Theorem 8.5.1 together with (14.4.2) yield

(14.7.7)

$$|q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t)) - \dot y(t)[\eta(U_\delta(y(t)+,t)) - \eta(U_\delta(y(t)-,t))]|$$
$$\leq c_8\delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

Finally, when $x = y(\cdot)$ is a pseudoshock,

(14.7.8)

$$|q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t)) - \dot y(t)[\eta(U_\delta(y(t)+,t) - \eta(U_\delta(y(t)-,t))]|$$
$$\leq c_9|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

By combining (14.7.6), (14.7.7), (14.7.8) with (14.4.6) and (14.4.4), we deduce that, for fixed test function $\phi$, the right-hand side of (14.7.5) is bounded from below by $-C_\phi[TV_{(-\infty,\infty)}U_0(\cdot) + 1]\delta$. Therefore, passing to the limit along the $\{\delta_k\}$ sequence, we conclude that the solution $U$ satisfies the inequality (13.2.17), which expresses the entropy admissibility condition. The proof of Theorem 14.4.2 is now complete.

## 14.8 Continuous Dependence on Initial Data

The remainder of this chapter will address the issue of uniqueness and stability of solutions to the initial-value problem (13.1.1). The existence proofs via Theorems 13.1.1 an 14.4.2, which rely on compactness arguments, offer no clue to that question. We will approach the subject via the approximate solutions generated by the front tracking algorithm. By monitoring the time evolution of a certain functional, we will demonstrate that $\delta$-approximate solutions depend continuously on their initial data, modulo corrections of order $\delta$. This will induce stability for solutions obtained by passing to the $\delta \downarrow 0$ limit.

Our earlier experiences with the scalar conservation law strongly suggest that the $L^1$ topology should provide the proper setting for continuous dependence. However, the $L^1$ distance shall not be measured via the standard $L^1$ metric but through a functional $\rho$, specially designed for the task at hand.

Let us consider two $\delta$-approximate solutions $U$ and $\bar U$ of (14.2.1). Fixing any point $(x,t)$ of continuity for both $U$ and $\bar U$, we shall measure the distance between the vectors $U(x,t)$ and $\bar U(x,t)$ in the special curvilinear coordinate system whose coordinate curves are the shock curves, with both the admissible and the nonadmissible branches retained. To that end, the vector $\bar{U}(x,t) - U(x,t)$ is represented by curvilinear "coordinates" $p_1(x,t), \cdots, p_n(x,t)$, obtained by means of the following process: One envisages a "virtual" jump discontinuity with left state $U(x,t)$ and right state $\bar{U}(x,t)$, and resolves it into a wave fan composed of $n+1$ constant states joined exclusively by (admissible or nonadmissible) virtual shocks. For $|U(x,t) - \bar{U}(x,t)|$ sufficiently small, this resolution is unique and can be effected, via the implicit function theorem, by retracing the steps of the admissible solution to the Riemann problem, in Section 9.3, with the wave fan curves $\Phi_i$ here replaced by the shock curves $W_i$. We denote the amplitude of the resulting virtual $i$-shock by $p_i(x,t)$ and its speed by $s_i(x,t)$. The distance between $U(x,t)$ and $\bar{U}(x,t)$ will now be measured by the suitably weighted sum $\sum g_i(x,t)|p_i(x,t)|$ of the strengths of the $n$ virtual shocks, and accordingly the distance between the two approximate solutions at time $t$ will be measured through the functional

$$\rho(U(\cdot,t), \bar{U}(\cdot,t)) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} g_i(x,t)|p_i(x,t)|dx. \tag{14.8.1}$$

We proceed to introduce suitable weights $g_i$. Let $I$ and $\bar{I}$ denote the sets of collision times for $U$ and $\bar{U}$, and consider the corresponding potentials for wave interaction $Q(t)$ and $\bar{Q}(t)$, defined through (14.5.7), for $t \in (0,\infty)\backslash I$ and $t \in (0,\infty)\backslash \bar{I}$, respectively. For $t \in (0,\infty)\backslash(I \bigcup \bar{I})$ and any point of continuity $x$ of both $U(\cdot,t)$ and $\bar{U}(\cdot,t)$, we define

$$g_i(x,t) = 1 + \kappa[Q(t) + \bar{Q}(t)] + \nu A_i(x,t), \tag{14.8.2}$$

where $\kappa$ and $\nu$ are sufficiently large positive constants, to be fixed later, and

$$A_i(x,t) = \textstyle\sum_-|\gamma| + \bar{\sum}_-|\gamma| + \sum_+|\gamma| + \bar{\sum}_+|\gamma| + \sum_0|\gamma| + \bar{\sum}_0|\gamma|. \tag{14.8.3}$$

In (14.8.3), $\Sigma_-$ (or $\bar{\Sigma}_-$) sums the strengths of all $j$-fronts of $U$ (or $\bar{U}$), with $j = i+1, \cdots, n$, that cross the $t$-time line to the left of the point $x$; $\Sigma_+$ (or $\bar{\Sigma}_+$) sums the strengths of all $j$-fronts of $U$ (or $\bar{U}$), with $j = 1, \cdots, i-1$, that cross the $t$-time line to the right of the point $x$; $\Sigma_0$ (or $\bar{\Sigma}_0$) sums the strengths of all $i$-fronts of $U$ (or $\bar{U}$) that cross the $t$-time line to the left (or right) of the point $x$, when $p_i(x,t) < 0$, or to the right (or left) of the point $x$, when $p_i(x,t) > 0$. Thus, one may justifiably say that $A_i(x,t)$ represents the total strength of the fronts of $U$ and $\bar{U}$ that cross the $t$-time line and approach the virtual $i$-shock at $(x,t)$.

Once $\kappa$ and $\nu$ have been fixed, the total variation of the initial data shall be restricted to be so small that $\frac{1}{2} \leq g_i(x,t) \leq 2$. Then, $\rho(U(\cdot,t), \bar{U}(\cdot,t))$ will be equivalent to the $L^1$ distance of $U(\cdot,t)$ and $\bar{U}(\cdot,t)$:

$$\begin{aligned}\tfrac{1}{C}\|U(\cdot,t) - \bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)} &\leq \rho(U(\cdot,t), \bar{U}(\cdot,t)) \\ &\leq C\|U(\cdot,t) - \bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)}.\end{aligned} \tag{14.8.4}$$

It is easily seen that in the scalar case, $n = 1$, the functional $\rho$ introduced by (14.8.1) is closely related to the functional $\rho$, defined by (11.8.11), when the latter is restricted to step functions.

The aim is to show that $\rho(U(\cdot,t),\bar{U}(\cdot,t))$ is nonincreasing, modulo corrections of order $\delta$:

$$\rho(U(\cdot,t),\bar{U}(\cdot,t)) - \rho(U(\cdot,\tau),\bar{U}(\cdot,\tau)) \leq \omega\delta(t-\tau), \quad 0<\tau<t<\infty. \tag{14.8.5}$$

Notice that across points of $I$ or $\bar{I}$, $Q(t)$ or $\bar{Q}(t)$ decreases by an amount approximately equal to the product of the strengths of the two colliding waves, while $A_i(x,t)$ may increase at most by a quantity of the same order of magnitude. Therefore, upon fixing $\kappa/\nu$ sufficiently large, $\rho(U(\cdot,t),\bar{U}(\cdot,t))$ will be decreasing across points of $I$ or $\bar{I}$. Between consecutive points of $I\bigcup\bar{I}$, $\rho(U(\cdot,t),\bar{U}(\cdot,t))$ is continuously differentiable; hence to establish (14.8.5) it will suffice to show

$$\frac{d}{dt}\rho(U(\cdot,t),\bar{U}(\cdot,t)) \leq \omega\delta. \tag{14.8.6}$$

From (14.8.1),

$$\frac{d}{dt}\rho(U(\cdot,t),\bar{U}(\cdot,t)) = \sum_y\sum_{i=1}^n\{g_i^-|p_i^-| - g_i^+|p_i^+|\}\dot{y}, \tag{14.8.7}$$

where $\sum_y$ runs over all waves $x = y(\cdot)$ of $U$ and $\bar{U}$ that cross the $t$-time line, and $\dot{y}$, $g_i^\pm$ and $p_i^\pm$ stand for $\dot{y}(t)$, $g_i(y(t)\pm,t)$ and $p_i(y(t)\pm,t)$. By adding and subtracting, appropriately, the speed $s_i^\pm = s_i(y(t)\pm,t)$ of the virtual $i$-shocks, one may recast (14.8.7) in the form

$$\frac{d}{dt}\rho(U(\cdot,t),\bar{U}(\cdot,t)) = \sum_y\sum_{i=1}^n E_i(y(\cdot),t), \tag{14.8.8}$$

where

$$E_i(y(\cdot),t) = g_i^+(s_i^+-\dot{y})|p_i^+| - g_i^-(s_i^- - \dot{y})|p_i^-| \tag{14.8.9}$$

$$= (g_i^+-g_i^-)(s_i^+-\dot{y})|p_i^-| + g_i^-(s_i^+-s_i^-)|p_i^-| + g_i^+(s_i^+-\dot{y})(|p_i^+|-|p_i^-|).$$

Suppose first $x = y(\cdot)$ is a pseudoshock, say of $U$. Then $g_i^+ = g_i^-$ and (14.8.9) yields

$$\sum_{i=1}^n E_i(y(\cdot),t) \leq c_{10}|U(y(t)+,t) - U(y(t)-,t)|. \tag{14.8.10}$$

Thus, by virtue of (14.4.4), the portion of the sum on the right-hand side of (14.8.8) that runs over all pseudoshocks of $U$ is bounded by $c_{10}\delta$. Of course, this equally applies to the portion of the sum that runs over all pseudoshocks of $\bar{U}$.

We now turn to the case $x = y(\cdot)$ is a $j$-front of $U$ or $\bar{U}$, with amplitude $\gamma$. To complete the proof of (14.8.6), one has to show that

$$\sum_{i=1}^{n} E_i(y(\cdot), t) \leq c_{11}\delta|\gamma|. \tag{14.8.11}$$

What follows is a road map to the proof of (14.8.11), which will expose the main ideas and, in particular, will explain why the weight function $g_i(x, t)$ was designed according to (14.8.2). The detailed proof, which is quite laborious, is found in the references cited in Section 14.13.

Let us first examine the three terms on the right-hand side of (14.8.9) for $i \neq j$. By virtue of (14.8.2), $g_i^+ - g_i^-$ equals $\nu|\gamma|$ when $j > i$, or $-\nu|\gamma|$ when $j < i$. In either case, the first term

$$(g_i^+ - g_i^-)(s_i^+ - \dot{y})|p_i^-| \cong -\nu|\lambda_i - \lambda_j||p_i^-||\gamma| \tag{14.8.12}$$

is strongly negative and the idea is that this dominates the other two terms, rendering the desired inequality (14.8.6). Indeed, the second term is majorized by $c_{12}|p_i^-||\gamma|$, which is clearly dominated by (14.8.12), when $\nu$ is sufficiently large. One estimates the remaining term by the following argument. The amplitudes $(p_1^-, \cdots, p_n^-)$ or $(p_1^+, \cdots, p_n^+)$ of the virtual shocks result respectively from the resolution of the jump discontinuity between $U^-$ and $\bar{U}^-$ or $U^+$ and $\bar{U}^+$, where $U^\pm = U(y(t)\pm, t)$ and $\bar{U}^\pm = \bar{U}(y(t)\pm, t)$.

Assuming, for definiteness, that $x = y(\cdot)$ is a front of $U$, we have $\bar{U}^- = \bar{U}^+$, while the states $U^-$ and $U^+$ are connected, in state space, by a $j$-wave curve. Consequently, to leading order, $p_j^+ \cong p_j^- - \gamma$ while, for any $k \neq j$, $p_k^+ \cong p_k^-$. Indeed, a study of the wave curves easily yields the estimate

(14.8.13)
$$|p_j^+ - p_j^- + \gamma| + \sum_{k\neq j} |p_k^+ - p_k^-| = O(1)[\delta + |p_j^-|(|p_j^-| + |\gamma|) + \sum_{k\neq j} |p_k^-|]|\gamma| \ ,$$

which in turn implies

$$E_i(y(\cdot), t) \leq -a\nu|p_i^-||\gamma| + c_{12}[\delta + |p_j^-|(|p_j^-| + |\gamma|) + \sum_{k\neq j} |p_k^-|]|\gamma|, \tag{14.8.14}$$

with $a > 0$.

For $i = j$, the estimation of $E_i(y(\cdot), t)$ is more delicate, as the $j$-front may resonate with the virtual $i$-shock. The same difficulty naturally arises, and has to be addressed, even for the scalar conservation law. In fact, the scalar case was already treated, in Section 11.8, albeit under a different guise. For the system, one has to examine separately a number of cases, depending on whether $x = y(\cdot)$ is a shock or a rarefaction front, in conjunction with the signs of $p_j^-$ and $p_j^+$. The resulting estimates, which vary slightly from case to case but are essentially equivalent, are derived in the references. For example, when either $x = y(\cdot)$ is a $j$-rarefaction front and $0 < p_j^- < p_j^+$ or $x = y(\cdot)$ is a $j$-shock and $p_j^+ < p_j^- < 0$,

(14.8.15)

$$E_j(y(\cdot),t) \le -b\nu|p_j^-||\gamma|(|p_j^-|+|\gamma|)+c_{13}[\delta+|p_j^-|(|p_j^-|+|\gamma|)+\sum_{k\neq j}|p_k^-|]|\gamma|,$$

where $b > 0$.

We now sum the inequalities (14.8.14), for $i \neq j$, together with the inequality (14.8.15). Upon selecting $\nu$ sufficiently large to offset the possibly positive terms, we arrive at (14.8.11). As noted earlier, this implies (14.8.6), which in turn yields (14.8.5). Recalling (14.8.4), we conclude

(14.8.16) $$\|U(\cdot,t)-\bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)} \le C^2\|U(\cdot,0)-\bar{U}(\cdot,0)\|_{L^1(-\infty,\infty)}+C\omega\delta t,$$

which establishes that $\delta$-approximate solutions depend continuously on their initial data, modulo $\delta$. The implications for actual solutions, obtained as $\delta \downarrow 0$, will be discussed in the following section.

## 14.9 The Standard Riemann Semigroup

As a corollary of the stability properties of approximate solutions, established in the previous section, it will be shown here that any solution to our system constructed as the $\delta \downarrow 0$ limit of some sequence of $\delta$-approximate solutions is uniquely determined by its initial data and may be identified with a trajectory of a $L^1$-Lipschitz semigroup, defined on a closed subset of $L^1(-\infty,\infty)$.

The first step in our investigation is to locate the domain of the semigroup. This must be a set which is positively invariant for solutions. Motivated by the analysis in Section 14.5, with any step function $W(\cdot)$, of compact support and small total variation over $(-\infty,\infty)$, we associate a number $H(W(\cdot))$ determined by the following procedure. The jump discontinuities of $W(\cdot)$ are resolved into fans of admissible shocks and rarefaction waves, by solving classical Riemann problems. Before any wave collisions may occur, one measures the total strength $L$ and the potential for wave interaction $Q$ of these outgoing waves and then sets $H(W(\cdot)) = L + 2\kappa Q$, where $\kappa$ is a sufficiently large positive constant. Suppose a $\delta$-approximate solution $U$, with initial data $W$, is constructed by the front tracking algorithm of Section 14.2. By the rules of the construction, all jump discontinuities of $W$ will be resolved via the Approximate Riemann Solver and so, for any $\delta > 0$, $H(W(\cdot))$ will coincide with the initial value $G(0+)$ of the Glimm-type function $G(t)$ defined through (14.5.10). At a later time, as the Simplified Riemann Solver comes into play, $G(t)$ and $H(U(\cdot,t))$ may part from each other. In particular, by contrast to $G(t)$, $H(U(\cdot,t))$ will not necessarily be nonincreasing with $t$. Nevertheless, when $\kappa$ is sufficiently large, $H(U(\cdot,t)) \le H(U(\cdot,t-))$ and $H(U(\cdot,t+)) \le H(U(\cdot,t-))$. Hence $H(U(\cdot,t)) \le H(W(\cdot))$ for any $t \ge 0$ and so all sets of step functions $\{W(\cdot) : H(W(\cdot)) < r\}$ are positively invariant for $\delta$-approximate solutions constructed by the front tracking algorithm. Following this preparation, we define the set that will serve as the domain of the semigroup by

(14.9.1) $\mathcal{D} = cl\{\text{step functions } W(\cdot) \text{ with compact support} : H(W(\cdot)) < r\}$,

where cl denotes closure in $L^1(-\infty, \infty)$. By virtue of Theorem 1.7.3, the members of $\mathcal{D}$ are functions of bounded variation over $(-\infty, \infty)$, with total variation bounded by $cr$. The main result is

**14.9.1 Theorem.** *For $r$ sufficiently small, there is a family of maps $S_t : \mathcal{D} \to \mathcal{D}$,* for *$t \in [0, \infty)$, with the following properties.*

(a) *$L^1$-Lipschitz continuity on $\mathcal{D} \times [0, \infty)$: For any $W$, $\bar{W}$ in $\mathcal{D}$ and $t$, $\tau$ in* $[0, \infty)$,

(14.9.2) $\|S_t W(\cdot) - S_\tau \bar{W}(\cdot)\|_{L^1(-\infty,\infty)} \leq \kappa\{\|W(\cdot) - \bar{W}(\cdot)\|_{L^1(-\infty,\infty)} + |t - \tau|\}.$

(b) $\{S_t : t \in [0, \infty)\}$ *has the semigroup property, namely*

(14.9.3) $$S_0 = \text{identity},$$

(14.9.4) $$S_{t+\tau} = S_t S_\tau , \qquad t, \tau \in [0, \infty).$$

(c) *If $U$ is any solution of* (13.1.1), *with initial data $U_0 \in \mathcal{D}$, which is the $\delta \downarrow 0$ limit of some sequence of $\delta$-approximate solutions, then*

(14.9.5) $$U(\cdot, t) = S_t U_0(\cdot), \qquad t \in [0, \infty).$$

**Proof.** Let $U$ and $\bar{U}$ be two solutions of (13.1.1), with initial data $U_0$ and $\bar{U}_0$, which are $\delta \downarrow 0$ limits of sequences of $\delta$-approximate solutions $\{U_{\delta_n}\}$ and $\{\bar{U}_{\bar{\delta}_n}\}$, respectively. No assumption is made that these approximate solutions have necessarily been constructed by the front tracking algorithm. So long as the total variation is sufficiently small to meet the requirements of Section 14.8, we may apply (14.8.16) to get

(14.9.6) $$\|U_{\delta_n}(\cdot, t) - \bar{U}_{\bar{\delta}_n}(\cdot, t)\|_{L^1(-\infty,\infty)} \leq C^2 \|U_{\delta_n}(\cdot, 0) - \bar{U}_{\bar{\delta}_n}(\cdot, 0)\|_{L^1(-\infty,\infty)} + C\omega \max\{\delta_n, \bar{\delta}_n\} t.$$

Passing to the limit, $n \to \infty$, we deduce

(14.9.7) $$\|U(\cdot, t) - \bar{U}(\cdot, t)\|_{L^1(-\infty,\infty)} \leq C^2 \|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^1(-\infty,\infty)}.$$

When $r$ is sufficiently small, Theorem 14.4.2 asserts that for any $U_0 \in \mathcal{D}$ one can generate solutions $U$ of (13.1.1) as limits of sequences $\{U_{\delta_n}\}$ of $\delta$-approximate solutions constructed by the front tracking algorithm. Moreover, the initial values of $U_\delta$ may be selected so that $H(U_\delta(\cdot, 0)) < r$, in which case, as noted above, $H(U_\delta(\cdot, t)) < r$ and thereby $U(\cdot, t) \in \mathcal{D}$, for any $t \in [0, \infty)$. By virtue of (14.9.7), all these solutions must coincide so that $U$ is uniquely defined. In fact, (14.9.7) further implies that $U$ must even coincide with any solution, with initial value $U_0$, that is derived as the $\delta \downarrow 0$ limit of any sequence of $\delta$-approximate solutions, regardless of whether they were constructed by the front tracking algorithm.

Once $U$ has thus been identified, we define $S_t$ through (14.9.5). The Lipschitz continuity property (14.9.2) follows by combining (14.9.7) with (13.1.6), and (14.9.3) is obvious. To verify (14.9.4), it suffices to notice that for any fixed $\tau > 0$, $U(\cdot, \tau + \cdot)$ is a solution of (13.1.1), with initial data $U(\cdot, \tau)$, which is derived as the $\delta \downarrow 0$ limit of $\delta$-approximate solutions and thus, by uniqueness, must coincide with $S_t U(\cdot, \tau)$. The proof is complete.

The term Standard Riemann Semigroup is commonly used for $S_t$ , as a reminder that its building block is the solution of the Riemann problem. The question of whether this semigroup also encompasses solutions derived via alternative methods will be addressed in the next section.

## 14.10 Uniqueness of Solutions

Uniqueness for the Cauchy problem (13.1.1) shall be established here by demonstrating that any solution in a reasonable function class can be identified with the trajectory of the Standard Riemann Semigroup which emanates from the initial data. As shown in Section 14.9, this is indeed the case for solutions constructed by front tracking.

For fair comparison one should limit, at the outset, the investigation to solutions $U$ for which $U(\cdot, t)$ resides in the domain $\mathcal{D}$ of the Standard Riemann Semigroup, defined through (14.9.1). As noted earlier, this implies, in particular, that $U(\cdot, t)$ has bounded variation over $(-\infty, \infty)$:

(14.10.1) $$TV_{(-\infty,\infty)} U(\cdot, t) \leq cr.$$

It then follows from Theorem 4.3.1 that $t \mapsto U(\cdot, t)$ is $L^1$-Lipschitz,

(14.10.2) $$\int_{-\infty}^{\infty} |U(x, t) - U(x, \tau)| dx \leq c' r |t - \tau|, \qquad 0 \leq \tau < t < \infty,$$

and $U$ is in $BV_{\text{loc}}$ on $(-\infty, \infty) \times [0, \infty)$. Hence, as pointed out in Section 10.1, there is $\mathcal{N} \subset [0, \infty)$, of measure zero, such that any $(x, t)$ with $t \notin \mathcal{N}$ and $U(x-, t) = U(x+, t)$ is a point of approximate continuity of $U$ while any $(x, t)$ with $t \notin \mathcal{N}$ and $U(x-, t) \neq U(x+, t)$ is a point of approximate jump discontinuity of $U$, with one-sided approximate limits $U_\pm = U(x\pm, t)$ and associated shock speeds determined through the Rankine-Hugoniot jump condition (8.1.2).

It is presently unknown whether uniqueness prevails within the above class of solutions. Accordingly, one should endow solutions with additional structure. Here we will experiment with the

**14.10.1 Tame Oscillation Condition:** There are positive constants $\lambda$ and $\beta$ such that

(14.10.3) $$|U(x\pm, t + h) - U(x\pm, t)| \leq \beta \, TV_{(x-\lambda h, x+\lambda h)} U(\cdot, t),$$

for all $x \in (-\infty, \infty)$, $t \in [0, \infty)$ and any $h > 0$.

Clearly, solutions constructed by either the random choice method or the front tracking algorithm satisfy this condition, and so do also the solutions to systems of two conservation laws considered in Chapter XII.

The Tame Oscillation Condition induces uniqueness:

**14.10.2 Theorem.** *Any $BV$ solution $U$ of the Cauchy problem* (13.1.1), *with $U(\cdot,t)$ in $\mathcal{D}$ for all $t \in [0,\infty)$, which satisfies the Lax E-condition, at any point of approximate jump discontinuity, together with the Tame Oscillation Condition* (14.10.3), *coincides with the trajectory of the Standard Riemann Semigroup $S_t$, emanating from the initial data:*

$$U(\cdot,t) = S_t U_0(\cdot), \qquad t \in [0,\infty). \tag{14.10.4}$$

*In particular, $U$ is uniquely determined by its initial data.*

**Proof.** The demonstration will be quite lengthy. The first step is to show that at every $\tau \notin \mathcal{N}$, $U(\cdot,t)$ is tangential to the trajectory of $S_t$ emanating from $U(\cdot,\tau)$:

$$\limsup_{h\downarrow 0} \frac{1}{h} \|U(\cdot,\tau+h) - S_h U(\cdot,\tau)\|_{L^1(-\infty,\infty)} = 0. \tag{14.10.5}$$

Then we shall verify that (14.10.5), in turn, implies (14.10.4).

Fixing $\tau \notin \mathcal{N}$, we will establish (14.10.5) by the following procedure. For any fixed bounded interval $[a,b]$ and $\varepsilon > 0$, arbitrarily small, we will construct some function $U^*$ on a rectangle $[a,b]\times[\tau,\tau+\delta]$ such that

$$\limsup_{h\downarrow 0} \frac{1}{h} \|U(\cdot,\tau+h) - U^*(\cdot,h)\|_{L^1(a,b)} \leq c_{14} r\varepsilon, \tag{14.10.6}$$

$$\limsup_{h\downarrow 0} \frac{1}{h} \|S_h U(\cdot,\tau) - U^*(\cdot,h)\|_{L^1(a,b)} \leq c_{14} r\varepsilon. \tag{14.10.7}$$

Naturally, such a $U^*$ will provide a local approximation to the solution of (13.1.1) with initial data $U_0(\cdot) = U(\cdot,\tau)$, and will be constructed accordingly by patching together local approximate solutions of two types, one fit for points of strong jump discontinuity, the other suitable for regions with small local oscillation.

We begin by fixing $\lambda$ which is larger than the absolute value of all characteristic speeds and also sufficiently large for the Tame Oscillation Condition (14.10.3) to apply.

With any point $(y,\tau)$ of jump discontinuity for $U$, with limits $U_\pm = U(y\pm,\tau)$ and shock speed $s$, we associate the sector $\mathcal{K} = \{(x,\sigma) : \sigma > 0, |x-y| \leq \lambda\sigma\}$, on which we consider the solution $U^\sharp = U^\sharp_{(y,\tau)}$ defined by

$$U^\sharp(x,\sigma) = \begin{cases} U_-\,, & \text{for } x < y + s\sigma \\ U_+\,, & \text{for } x > y + s\sigma. \end{cases} \tag{14.10.8}$$

We prove that

$$\lim_{h\downarrow 0}\frac{1}{h}\int_{y-\lambda h}^{y+\lambda h}|U(x,\tau+h)-U^{\sharp}(x,h)|dx=0. \tag{14.10.9}$$

Indeed, for $0\le\sigma\le h$, let us set

$$\phi_h(\sigma)=\frac{1}{h}\int_{y-\lambda h}^{y+\lambda h}|U(x,\tau+\sigma)-U^{\sharp}(x,\sigma)|dx. \tag{14.10.10}$$

Suppose $\phi_h(h)>0$. Since $\sigma\mapsto U(\cdot,\tau+\sigma)-U^{\sharp}(\cdot,\sigma)$ is $L^1$-Lipschitz, with constant $\gamma$, we infer that, for $h<<1$, $\phi_h(h)<2\gamma$ and $\phi_h(\sigma)\ge\frac{1}{2}\phi_h(h)$, for any $\sigma$ with $h-\sigma\le\frac{h}{2\gamma}\phi_h(h)$. Then

(14.10.11)

$$\frac{1}{h^2}\int_0^h\int_{y-\lambda h}^{y+\lambda h}|U(x,\tau+\sigma)-U^{\sharp}(x,\sigma)|dxd\sigma=\frac{1}{h}\int_0^h\phi_h(\sigma)d\sigma\ge\frac{1}{4\gamma}\phi_h^2(h).$$

As $h\downarrow 0$, the left-hand side of (14.10.11) tends to zero, by virtue of Theorem 1.7.4, and this verifies (14.10.9).

Next we fix any interval $(\zeta,\xi)$, with midpoint say $z$. On the triangle $\mathcal{T}=\{(x,\sigma):\sigma>0,\ \zeta+\lambda\sigma<x<\xi-\lambda\sigma\}$, we construct the solution $U^{\flat}=U^{\flat}_{(z,\tau)}$ of the linear Cauchy problem

$$\partial_t U^{\flat}+A^{\flat}\partial_x U^{\flat}=0, \tag{14.10.12}$$

$$U^{\flat}(x,0)=U(x,\tau), \tag{14.10.13}$$

where $A^{\flat}$ is the constant matrix $\mathrm{D}F(U(z,\tau))$. The aim is to establish the estimate

$$\int_{\zeta+\lambda h}^{\xi-\lambda h}|U(x,\tau+h)-U^{\flat}(x,h)|dx \tag{14.10.14}$$

$$\le c_{15}[TV_{(\zeta,\xi)}U(\cdot,\tau)]\int_0^h TV_{(\zeta+\lambda\sigma,\xi-\lambda\sigma)}U(\cdot,\tau+\sigma)d\sigma.$$

Integrating (14.10.12) along characteristic directions and using (14.10.13) yields

$$L_i^{\flat}U^{\flat}(x,h)=L_i^{\flat}U(x-\lambda_i^{\flat}h,\tau),\qquad i=1,\cdots,n, \tag{14.10.15}$$

where $L_i^{\flat}=L_i(U(z,\tau))$ is a left eigenvector of $A^{\flat}$ associated with the eigenvalue $\lambda_i^{\flat}=\lambda_i(U(z,\tau))$. For fixed $i$, we may assume without loss of generality that $\lambda_i^{\flat}=0$, since we may change variables $x\mapsto x-\lambda_i^{\flat}t$ , $F(U)\mapsto F(U)-\lambda_i^{\flat}U$. In that case, since $U$ satisfies (14.2.1) in the sense of distributions,

$$(14.10.16)\quad \int_{\zeta+\lambda h}^{\xi-\lambda h} \phi(x) L_i^\flat [U(x,\tau+h) - U^\flat(x,h)]dx$$

$$= \int_{\zeta+\lambda h}^{\xi-\lambda h} \phi(x) L_i^\flat [U(x,\tau+h) - U(x,\tau)]dx$$

$$= \int_0^h \int_{\zeta+\lambda h}^{\xi-\lambda h} \partial_x \phi(x) L_i^\flat F(U(x,\tau+\sigma)) dx d\sigma,$$

for any test function $\phi \in C_0^\infty(\zeta+\lambda h, \xi-\lambda h)$. Taking the supremum over all such $\phi$ with $|\phi(x)| \leq 1$, yields

(14.10.17)

$$\int_{\zeta+\lambda h}^{\xi-\lambda h} |L_i^\flat [U(x,\tau+h) - U^\flat(x,h)]|dx \leq \int_0^h TV_{(\zeta+\lambda h,\xi-\lambda h)} L_i^\flat F(U(\cdot,\tau+\sigma))d\sigma.$$

Given $\zeta + \lambda h < x < y < \xi - \lambda h$, let us set, for brevity, $V = U(x,\tau+\sigma)$ and $W = U(y,\tau+\sigma)$. Recalling the notation (8.1.4), one may write

(14.10.18)

$$F(V) - F(W) = A(V,W)(V-W) = A^\flat(V-W) + [A(V,W) - A^\flat](V-W).$$

We now note that $L_i^\flat A^\flat = 0$. Furthermore, $A(V,W) - A^\flat$ is bounded in terms of the oscillation of $U$ inside the triangle $\mathcal{T}$, which is in turn bounded in terms of the total variation of $U(\cdot,\tau)$ over $(\zeta,\xi)$, by virtue of the Tame Oscillation Condition (14.10.3). Therefore, (14.10.17) yields the estimate

$$(14.10.19)\quad \int_{\zeta+\lambda h}^{\xi-\lambda h} |L_i^\flat [U(x,\tau+h) - U^\flat(x,h)]|dx$$

$$\leq c_{16}[TV_{(\zeta,\xi)}U(\cdot,\tau)] \int_0^h TV_{(\zeta+\lambda\sigma,\xi-\lambda\sigma)} U(\cdot,\tau+\sigma)d\sigma.$$

Since (14.10.19) holds for $i = 1, \cdots, n$, (14.10.14) readily follows.

We have now laid the groundwork for synthesizing a function $U^*$ that satisfies (14.10.6). We begin by identifying a finite collection of open intervals $(\zeta_j, \xi_j)$, for $j = 1, \cdots, J$, with the following properties:

(i) $[a,b] \subset \bigcup_{j=1}^{J} [\zeta_j, \xi_j]$.

(ii) The intersection of any three of these intervals is empty.

(iii) $TV_{(\zeta_j,\xi_j)} U(\cdot,\tau) < \varepsilon$, for $j = 1, \cdots, J$.

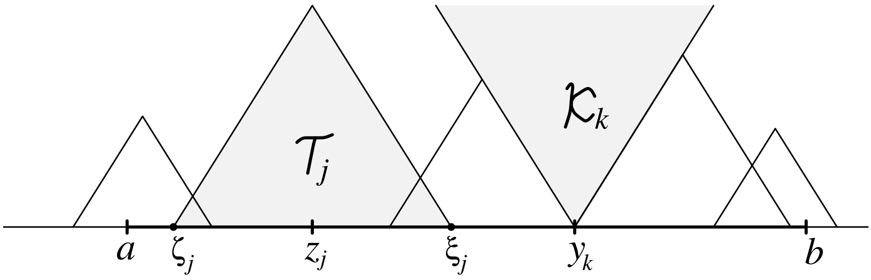


**Fig. 14.10.1**

With each $(\zeta_j, \xi_j)$, we associate, as above, the triangle $\mathcal{T}_j$ and the approximate solution $U^{\flat}_{(z_j,\tau)}$ relative to the midpoint $z_j$. We also consider $[a,b]\setminus\bigcup_{j=1}^{J}(\zeta_j,\xi_j)$, which is a finite set $\{y_1,\cdots,y_K\}$ containing the points where strong shocks cross the $\tau$-time line between $a$ and $b$. With each $y_k$ we associate the sector $\mathcal{K}_k$ and the corresponding approximate solution $U^{\sharp}_{(y_k,\tau)}$ (see Fig. 14.10.1). We then set

$$U^*(x,h) = \begin{cases} U^{\sharp}_{(y_k,\tau)}(x,h), \text{ for } (x,h) \in \mathcal{K}_k\setminus\bigcup\limits_{\ell=1}^{k-1}\mathcal{K}_\ell \\ \\ U^{\flat}_{(z_j,\tau)}(x,h), \text{ for } (x,h) \in \mathcal{T}_j\setminus\bigcup\limits_{\ell=1}^{j-1}\mathcal{T}_\ell\,. \end{cases} \tag{14.10.20}$$

Clearly, for $h$ sufficiently small $U^*(\cdot,h)$ is defined for all $x \in [a,b]$ and

$$\begin{aligned} \int_a^b |U(x,\tau+h) - U^*(x,h)|dx \\ \le \sum_{k=1}^{K}\int_{y_k-\lambda h}^{y_k+\lambda h} |U(x,\tau+h) - U^{\sharp}_{(y_k,\tau)}(x,h)|dx \\ + \sum_{j=1}^{J}\int_{\zeta_j+\lambda h}^{\xi_j-\lambda h} |U(x,\tau+h) - U^{\flat}_{(z_j,\tau)}(x,h)|dx. \end{aligned} \tag{14.10.21}$$

Upon combining (14.10.21), (14.10.9), (14.10.14) and (14.10.1), we arrive at (14.10.6), with $c_{14} = 2cc_{15}$.

We now note that $S_{t-\tau}U(\cdot,\tau)$ defines, for $t \ge \tau$, another solution of (14.2.1) which has the same properties, complies with the same bounds, and has identical restriction to $t = \tau$ with $U$. Therefore, this solution must equally satisfy the analog of (14.10.6), namely (14.10.7). Finally, (14.10.6) and (14.10.7) together yield (14.10.5).

It remains to show that (14.10.5) implies (14.10.4). To that end, we fix $t > 0$ and any, arbitrarily small, $\varepsilon > 0$. By virtue of (14.10.5) and the Vitali covering theorem, there is a finite collection of pairwise disjoint closed subintervals $[\tau_k, \tau_k + h_k]$, $k = 1, \cdots, K$, of $[0, t]$, with $0 \le \tau_1 < \cdots < \tau_K < t$, such that $\tau_k \notin \mathcal{N}$ and

(14.10.22)
$$0 \le t - \sum_{k=1}^{K} h_k < \varepsilon,$$

(14.10.23)
$$\|U(\cdot, \tau_k + h_k) - S_{h_k} U(\cdot, \tau_k)\|_{L^1(-\infty,\infty)} < \varepsilon h_k\,, \quad k = 1, \cdots, K.$$

By the triangle inequality,

(14.10.24)
$$\begin{aligned} \|U(\cdot, t) - S_t U_0(\cdot)\|_{L^1(-\infty,\infty)} \\ \le \sum_{k=0}^{K} \|S_{t-\tau_{k+1}} U(\cdot, \tau_{k+1}) - S_{t-\tau_k-h_k} U(\cdot, \tau_k + h_k)\|_{L^1(-\infty,\infty)} \\ + \sum_{k=1}^{K} \|S_{t-\tau_k-h_k} U(\cdot, \tau_k + h_k) - S_{t-\tau_k} U(\cdot, \tau_k)\|_{L^1(-\infty,\infty)}. \end{aligned}$$

In the first summation on the right-hand side of (14.10.24), $\tau_0 + h_0$ is to be interpreted as 0, and $\tau_{K+1}$ is to be interpreted as $t$. The general term in this summation is bounded by $\kappa(1 + c'r)(\tau_{k+1} - \tau_k - h_k)$, on account of (14.9.2) and (14.10.2). Hence the first sum is bounded by $\kappa(1 + c'r)\varepsilon$, because of (14.10.22). Turning now to the second summation, since $S_{t-\tau_k} = S_{t-\tau_k-h_k} S_{h_k}$,

(14.10.25)
$$\|S_{t-\tau_k-h_k} U(\cdot, \tau_k + h_k) - S_{t-\tau_k} U(\cdot, \tau_k)\|_{L^1(-\infty,\infty)} \le \kappa\varepsilon h_k\,,$$

by virtue of (14.9.2) and (14.10.23). Therefore, the second sum is bounded by $\kappa t\varepsilon$. Thus the right-hand side of (14.10.24) can be made arbitrarily small and this establishes (14.10.4). The proof is complete.

## 14.11 Continuous Glimm Functionals, Spreading of Rarefaction Waves, and Structure of Solutions

In earlier chapters we studied in great detail the structure of $BV$ solutions for scalar conservation laws as well as for systems of two conservation laws. The front tracking method, by its simplicity and explicitness, provides an appropriate vehicle for extending the investigation to genuinely nonlinear systems of arbitrary size. The aim of the study is to determine what features of piecewise constant solutions are inherited by the $BV$ solutions that are generated via the limit process. In addition to providing a fairly detailed picture of local structure and regularity, this approach exposes various stability characteristics of solutions and elucidates the issue of structural stability of the wave pattern. A sample of results will be stated below, without proofs. The reader may find a detailed exposition in the literature cited in Section 14.13.

The first step towards developing a qualitative theory is to realize within the framework of $BV$ solutions the key functionals that measure total wave strength and wave interaction potential, which were introduced earlier in the context of piecewise constant approximate solutions generated by front tracking. This will be effected by the following procedure.

Let $V$ be a function of bounded variation on $(-\infty,\infty)$ taking value in $\mathbb{R}^n$, and normalized by $V(x)=\frac{1}{2}[V(x-)+V(x+)]$. The distributional derivative $\partial_x V$ induces a signed vector-valued measure $\mu$ on $(-\infty,\infty)$, with continuous part $\mu^c$ and atomic part $\mu^a$. We represent $\mu$ by means of its "projections" $\mu_i$, $i=1,\dots,n$, on the characteristic directions, defined as follows.

The continuous part $\mu_i^c$ of $\mu_i$ is the Radon measure defined through

$$\int_{-\infty}^{\infty}\varphi(x)d\mu_i^c(x)=\int_{-\infty}^{\infty}\varphi(x)L_i(V(x))d\mu^c(x), \tag{14.11.1}$$

for all continuous functions $\varphi$ with compact support on $(-\infty,\infty)$.

The atomic part $\mu_i^a$ of $\mu_i$ is concentrated on the countable set of points of jump discontinuity of $V$. If $x$ is such a point, we set $\mu_i^a(x)=\varepsilon_i$, where $\varepsilon_i$ is the amplitude of the $i$-wave in the wave fan that solves the Riemann problem (9.1.12) with $U_L=V(x-)$, $U_R=V(x+)$. As noted in Section 9.3, $\varepsilon_i=L_i(V(x))\big[U_R-U_L\big]+O(1)|U_R-U_L|^2$. Therefore, the measure $\mu_i=\mu_i^c+\mu_i^a$ can be characterized through

$$\int_{-\infty}^{\infty}\varphi(x)d\mu_i(x)=\int_{-\infty}^{\infty}\varphi(x)\tilde{L}_i(x)d\mu(x), \tag{14.11.2}$$

where $\tilde{L}_i(x)=L_i(V(x))+O(1)|V(x+)-V(x-)|$.

We introduce the positive part $\mu_i^+$ and the negative part $\mu_i^-$ of the measure $\mu_i$, so that $\mu_i=\mu_i^+-\mu_i^-$, $|\mu_i|=\mu_i^++\mu_i^-$; and we define the functionals

$$\mathcal{L}[V]=\sum_{i=1}^{m}|\mu_i|(\mathbb{R}), \tag{14.11.3}$$

(14.11.4)

$$\mathcal{Q}[V]=\sum_{i<j}(|\mu_j|\times|\mu_i|)(\{(x,y):x<y\})+\sum_{i\in GN}(\mu_i^-\times|\mu_i|)(\{(x,y):x\neq y\}),$$

$$\mathcal{G}[V]=\mathcal{L}[V]+2\kappa\mathcal{Q}[V], \tag{14.11.5}$$

where $GN$ denotes the collection of genuinely nonlinear characteristic families of (14.2.1) and $\kappa$ is a positive constant to be specified below. These functionals enjoy the following useful semicontinuity property:

**14.11.1 Lemma.** *For $\kappa > 0$, sufficiently large, and $r > 0$, sufficiently small, the functionals $\mathcal{Q}$ and $\mathcal{G}$ are lower semicontinuous on the set*

$$\mathcal{D} = \{V \in L^1(\mathbb{R}; \mathbb{R}^n) : \mathcal{G}[V] \le r\}, \tag{14.11.6}$$

*equipped with the topology of $L^1$.*

It should be noted that even though $\mathcal{L}[V]$ is equivalent to the total variation of $V(\cdot)$, $\mathcal{L}$ is not necessarily lower semicontinuous on $\mathcal{D}$, and that $\mathcal{G}$ may fail to be lower semicontinuous if $r$ in (14.11.6) is large.

When $U$ is the solution of a Cauchy problem for (14.2.1), constructed by the front tracking algorithm, we identify its restriction $U(\cdot, t)$, to some fixed time $t$, with the function $V(\cdot)$, above. In that case, the measure $\mu_i$ encodes the $i$-waves crossing the $t$-time line, and in particular $\mu_i^+$ represents the $i$-rarefaction waves while $\mu_i^-$ represents the $i$-compression waves, including the $i$-shocks. Accordingly, this $\mu_i$ shall be dubbed the *$i$-wave measure at time $t$*. Moreover, $\mathcal{L}[U(\cdot, t)]$ and $\mathcal{Q}[U(\cdot, t)]$ respectively measure the total strength and interaction potential of all waves crossing the $t$-time line. In the particular situation where $U(\cdot, t)$ is piecewise constant on $(-\infty, \infty)$, $\mathcal{L}[U(\cdot, t)]$, $\mathcal{Q}[U(\cdot, t)]$ and $\mathcal{G}[U(\cdot, t)]$ reduce to $L(t)$, $Q(t)$ and $G(t)$ defined by (14.5.1), (14.5.7) and (14.5.10).

One may derive qualitative properties of solutions $U$ by first identifying them in the context of piecewise constant approximate solutions generated by the front tracking algorithm and then passing to the limit, taking advantage of the lower semicontinuity property of $\mathcal{Q}$ and $\mathcal{G}$ asserted by Lemma 14.11.1. In that direction, the following proposition establishes the spreading of rarefaction waves, extending to genuinely nonlinear systems of $n$ conservation laws what has already been demonstrated for scalar conservation laws and for systems of two conservation laws, in Sections 11.2 and 12.6.

**14.11.2 Theorem.** *With each genuinely nonlinear $i$-th characteristic family of the system* (14.2.1) *are associated positive numbers $c$ and $C$ with the following property. Let $U$ be the solution of the Cauchy problem for* (14.2.1)*, with initial data $U_0$, constructed by the front tracking algorithm. Fix any $t > 0$ and consider the $i$-wave measure $\mu_i$ at time $t$. Then*

$$\mu_i^+(a, b) \le c\,\frac{b-a}{t} + C\{\mathcal{Q}[U_0(\cdot)] - \mathcal{Q}[U(\cdot, t)]\} \tag{14.11.7}$$

*holds for any interval $(a, b) \subset (-\infty, \infty)$.*

In the proof, one employs the notion of generalized characteristics, introduced in Chapter X, in order to establish the corresponding estimate in the context of the piecewise constant approximate solutions that generate $U$, and then passes to the limit.

The next proposition describes the local structure of $BV$ solutions. It should be compared to Theorem 12.3.3, for systems of two conservation laws.

**14.11.3 Theorem.** *Let $U$ be the solution of a Cauchy problem for* (14.2.1)*, constructed through the front tracking algorithm. Fix any point $(\bar{x}, \bar{t})$ on the open upper half-plane and consider the rescaled function*

$$U_\alpha(x,t) = U(\bar{x} + \alpha x, \bar{t} + \alpha t), \qquad \alpha > 0. \tag{14.11.8}$$

*Then, for any $t \in (-\infty, \infty)$, as $\alpha \downarrow 0$, $U_\alpha(\cdot, t)$ converges in $L^1_{\text{loc}}$ to $\bar{U}(\cdot, t)$, where $\bar{U}$ is a self-similar solution of* (14.2.1)*. On the upper half-plane, $t \geq 0$, $\bar{U}$ coincides with the admissible solution of the Riemann problem* (9.1.1),(9.1.12)*, with end-states $U_L = U(\bar{x}-, \bar{t})$, $U_R = U(\bar{x}+, \bar{t})$. On the lower half-plane, $t < 0$, $\bar{U}$ contains only admissible shocks and/or centered compression waves. Furthermore, as $\alpha \downarrow 0$, the $i$-wave measures $\mu_i^{\pm}$ for $U_\alpha(\cdot, t)$ converge, in the weak topology of measures, to the corresponding $i$-wave measures $\bar{\mu}_i^{\pm}$ for $U(\cdot, \bar{t})$.*

The final proposition of this section provides a description of the global wave pattern, showing that admissible $BV$ solutions are more regular than general $BV$ functions. This should also be compared with the corresponding properties of solutions to scalar conservation laws and to systems of two conservation laws expounded in Sections 11.3 and 12.7.

**14.11.4 Theorem.** *Let $U$ be the solution to a Cauchy problem for* (14.2.1)*, constructed through the front tracking algorithm. Then the upper half-plane is partitioned into the union $\mathcal{C} \cup \mathcal{J} \cup \mathcal{I}$ of three subsets with the following properties:*
(a) *Any $(\bar{x}, \bar{t}) \in \mathcal{C}$ is a point of continuity of $U$.*
(b) *$\mathcal{I}$ is (at most) countable.*
(c) *$\mathcal{J}$ is the (at most) countable union of Lipschitz arcs $\{(x,t) : t \in (a_m, b_m),\ x = y_m(t)\}$, $m = 1, 2, \dots$. When $\bar{x} = y_m(\bar{t})$ and $(\bar{x}, \bar{t}) \notin \mathcal{I}$, then $(\bar{x}, \bar{t})$ is a point of continuity of $U$ relative to both sets $\{(x,t) : t \in (a_m, b_m),\ x < y_m(t)\}$ and $\{(x,t) : t \in (a_m, b_m),\ x > y_m(t)\}$, with distinct corresponding limits $U_-$ and $U_+$. Furthermore, $y_m(\cdot)$ is differentiable at $\bar{t}$, with derivative $s = \dot{y}_m(\bar{t})$, and $U_-$, $U_+$ and $s$ satisfy the Rankine-Hugoniot jump condition* (8.1.2).

The proof of the above two theorems again proceeds by examining the structure of piecewise constant approximate solutions that generate $U$, in terms of their wave measures, and then passing to the limit.

## 14.12 Stability of Strong Waves

The example of blowing up of solutions exhibited in Section 9.10 demonstrates the futility of seeking a global existence theorem for solutions to the Cauchy problem in the general class of systems considered in this chapter, under arbitrary initial data with large total variation. This raises the issue of identifying the special class of systems for which solutions with large initial data exist, and the hope that the systems of importance in continuum physics will turn out to be members. The first test for admission to membership in the above class should be that particular solutions containing waves of large amplitude, which may be explicitly known, are stable under small perturbations of their initial values. This has been achieved for the case of self-similar solutions to genuinely nonlinear systems, with strong shocks and/or strong rarefaction waves:

**14.12.1 Theorem.** *Consider the strictly hyperbolic system of conservation laws* (14.2.1) *with characteristic families that are either genuinely nonlinear or linearly degenerate. Assume that* $\bar{U}(x,t) = V(x/t)$ *is a self-similar solution, with strong compressive shocks, contact discontinuities and/or rarefaction waves, which satisfies an appropriate stability condition. For* $\delta > 0$*, define*

$$\mathcal{D}_\delta = \big\{W \in C(I\!R; I\!R^n) : \|W(\varphi(\cdot)) - V(\cdot)\|_{L^\infty(-\infty,\infty)} + TV_{(-\infty,\infty)}[W(\varphi(\cdot)) - V(\cdot)] < \delta, \text{ for some increasing } \varphi \in C^1(I\!R)\big\}. \tag{14.12.1}$$

*Then there exists a closed set* $\mathcal{D}$ *in* $L^1_{\mathrm{loc}}(I\!R; I\!R^n)$*, containing* $\mathcal{D}_\delta$ *for* $\delta$ *sufficiently small, together with a family of maps* $S_t : \mathcal{D} \to \mathcal{D}$, $t \in [0,\infty)$*, having the following properties.*

*(a)* $L^1$*-Lipschitz continuity on* $\mathcal{D} \times [0,\infty)$*: For any* $W$, $\bar{W}$ *in* $\mathcal{D}$ *and* $t$, $\tau$ *in* $[0,\infty)$,

$$\|S_t W(\cdot) - S_\tau \bar{W}(\cdot)\|_{L^1(-\infty,\infty)} \le \kappa\big\{\|W(\cdot) - \bar{W}(\cdot)\|_{L^1(-\infty,\infty)} + |t-\tau|\big\}. \tag{14.12.2}$$

*(b)* $\{S_t : t \in [0,\infty)\}$ *has the semigroup property, namely*

$$S_0 = \text{identity}, \tag{14.12.3}$$

$$S_{t+\tau} = S_t S_\tau\,, \qquad t,\tau \in [0,\infty). \tag{14.12.4}$$

*(c) For any* $U_0 \in \mathcal{D}$*,* $U(\cdot,t) = S_t U_0(\cdot)$ *is an admissible solution of the system* (14.2.1) *with initial value* $U_0$.

In establishing the above proposition, a major issue is the formulation of the "appropriate stability condition" on the self-similar solution $V$. Such a condition must ensure (a) that each elementary wave in the wave fan $V$, whether compressive shock, contact discontinuity or rarefaction, is individually stable; and (b) that the collision of weak waves with the strong waves of $V$ does not generate resonance that may lead to the breakdown of solutions exhibited in Section 9.10. Alternative, albeit equivalent, versions of stability conditions are recorded in the literature cited in Section 14.13, motivated either from analysis of wave interactions or through linearization of (14.2.1) about $V(x/t)$. Unfortunately, the statement of these conditions is complicated, technical and opaque.

To get a taste, let us consider the relatively simple special case where the wave fan $V(\cdot)$ comprises $m+1$ constant states $V_0, \dots, V_m$ connected by $m$ compressive shocks belonging to characteristic families $i_1 < \cdots < i_m$ and propagating with speeds $s_1 < \cdots < s_m$ . Each one of these shocks will be individually stable provided that the conditions (8.3.8), (8.3.13) and (8.3.14), introduced in Section 8.3, hold, namely, for $\ell = 1, \cdots, m$,

(14.12.5)
$$\lambda_{i_\ell}(V_{\ell-1}) > s_\ell > \lambda_{i_\ell}(V_\ell),$$

(14.12.6)
$$\lambda_{i_\ell+1}(V_\ell) > s_\ell > \lambda_{i_\ell-1}(V_{\ell-1}),$$

(14.12.7)
$$\det[R_1(V_{\ell-1}), \dots, R_{i_\ell-1}(V_{\ell-1}), V_\ell - V_{\ell-1}, R_{i_\ell+1}(V_\ell), \dots, R_n(V_\ell)] \neq 0.$$

In addition, one has to ensure that the collision of the above strong shocks with any weak waves produces outgoing weak waves whose weighted strength does not exceed the weighted strength of the incoming weak waves. Suppose that the strong $i_\ell$-shock is hit from the left by weak $k$-waves, $k = i_\ell, \dots, n$ with amplitude $\alpha_k$ and speed $\mu_k \sim \lambda_k(V_{\ell-1})$, and from the right by weak $k$-waves, $k = 1, \dots, i_\ell$ , with amplitude $\beta_k$ and speed $\nu_k \sim \lambda_k(V_\ell)$. These collisions will produce an outgoing strong $i_\ell$-shock together with outgoing weak $j$-waves, $j \neq i_\ell$ , with amplitude $\varepsilon_j$ and speed $\zeta_j \sim \lambda_j(V_{\ell-1})$, for $j = 1, \dots, i_\ell - 1$, and $\xi_j \sim \lambda_j(V_\ell)$ for $j = i_\ell + 1, \cdots, n$. Clearly, the $\mu_k, \nu_k, \varepsilon_j, \zeta_j$ and $\xi_j$ are all smooth functions of the $n + 1$ variables $(\beta_1, \dots, \beta_{i_\ell}, \alpha_{i_\ell}, \dots, \alpha_n)$. The wave stability condition will be satisfied if there exist positive weights $\omega_j^\ell$ , $\ell = 0, \dots, m$, $j = 1, \dots, n$, so that, for any $\ell = 1, \dots, m$,

(14.12.8)
$$\sum_{j=1}^{i_\ell-1} \omega_j^{\ell-1} \left| \frac{\partial}{\partial \alpha_k} \left[ \frac{\varepsilon_j(\zeta_j - s_\ell)}{\nu_j - s_\ell} \right] \right| + \sum_{j=i_\ell+1}^{n} \omega_j^\ell \left| \frac{\partial}{\partial \alpha_k} \left[ \frac{\varepsilon_j(\xi_j - s_\ell)}{\mu_j - s_\ell} \right] \right| < \omega_k^\ell , k = i_\ell, \dots, n,$$

(14.12.9)
$$\sum_{j=1}^{i_\ell-1} \omega_j^{\ell-1} \left| \frac{\partial}{\partial \beta_k} \left[ \frac{\varepsilon_j(\zeta_j - s_\ell)}{\nu_j - s_\ell} \right] \right| + \sum_{j=i_\ell+1}^{n} \omega_j^\ell \left| \frac{\partial}{\partial \beta_k} \left[ \frac{\varepsilon_j(\xi_j - s_\ell)}{\mu_j - s_\ell} \right] \right| < \omega_k^{\ell-1}, \quad k = 1, \dots, i_\ell ,$$

where all partial derivatives are evaluated at $\beta_i = 0$, $i = 1, \dots, i_\ell$ and $\alpha_i = 0$, $i = i_\ell, \dots, n$.

To summarize, for wave fans $V$ containing only compressive shocks, Theorem 14.12.1 applies, provided that (14.12.5), (14.12.6), (14.12.7), (14.12.8) and (14.12.9) are satisfied. The proof employs the methodology developed in earlier sections of this chapter and is quite technical. For more general wave fans $V$, which may also contain contact discontinuities and/or rarefactions, Theorem 14.12.1 holds under assumptions that are similar to, but more complicated than (14.2.8) and (14.2.9). The hope is that these conditions shall be automatically satisfied for the systems arising in continuum physics. Indeed, it has been shown that the wave stability conditions hold identically for the wave fans of the isentropic elasticity system (7.1.8), in the genuinely nonlinear case $\sigma''(u) \neq 0$. On the other hand, for the system of nonisentropic gas dynamics, for a polytropic gas (2.5.19), with adiabatic exponent $\gamma$, the wave stability conditions are verified only in the range $1.056 < \gamma < 8.757$.

## 14.13 Notes

Detailed, systematic presentation of most of the topics discussed in this chapter can be found in the texts by Bressan [9] and Holden and Risebro [2], as well as in the recent survey article by Bressan [12].

The front tracking method for scalar conservation laws was introduced by Dafermos [2] and is developed in Hedstrom [1], Holden, Holden and Høegh-Krohn [1], Holden and Holden [1], Holden and Risebro [1], Risebro and Tveito [2], Gimse and Risebro [1], Gimse [1], and Pan and Lin [1]. It has been employed, especially by the Norwegian School, as a computational tool. In fact, a similar approach had already been used for computations in the 1960's, by Barker [1]. For a detailed exposition, with applications, see Holden and Risebro [2]. The method was extended to genuinely nonlinear systems of two conservation laws by DiPerna [5] and then to genuinely nonlinear systems of any size, independently, by Bressan [2] and Risebro [1]. In Bressan's algorithm, the Approximate Riemann Solver employs pseudoshocks, while in Risebro's approach all new waves are attached to one of the two main fronts involved in the interaction. Yet another possibility, proposed by Schochet [6], is to eliminate pseudoshocks altogether, by assigning to them infinite speed, at the expense of sacrificing finite speed of propagation in the algorithm. The presentation here, Sections 14.2-14.7, follows the approach of Bressan and employs a technical simplification due to Baiti and Jenssen [2]. For a detailed treatment, see Bressan [9,12]. The notion of nonresonant curve is introduced here for the first time.

For early applications to special systems see Alber [2], Long-Wei Lin [1], Risebro and Tveito [1] and Wendroff [2].

Ancona and Marson [3,6] have recently extended the front tracking method, first to systems that are merely piecewise genuinely nonlinear and then to general strictly hyperbolic systems. A crucial role in the latter case is played by Bianchini's [6] solution of the Riemann problem; see Sections 9.12 and 15.9.

The Standard Riemann Semigroup was originally constructed by means of a very technical procedure, based on linearization, in Bressan [1,3,5], for special systems, Bressan and Colombo [1], for genuinely nonlinear systems of two conservation laws, and finally in Bressan, Crasta and Piccoli [1], for systems of $n$ conservation laws, with characteristic families that are either genuinely nonlinear or linearly degenerate. For systems with coinciding shock and rarefaction wave curves, the semigroup is defined for data with arbitrarily large total variation and may even be extended to the class of data that are merely in $L^\infty$; see Baiti and Bressan [1], Bressan and Goatin [2], Bianchini [2,3], and Colombo and Corli [2]. Similarly, for the system of isothermal gas dynamics Colombo and Risebro [1] construct the semigroup for data with arbitrarily large total variation. This approach has now been extended, by Ancona and Marson [4,5], to systems of two conservation laws that are merely piecewise genuinely nonlinear. The presentation in Sections 14.8-14.9 follows the alternative, simpler approach of Bressan, Liu and Yang [1], in which the basic estimate is derived by means of the functional $\rho$ introduced by Liu and Yang [2,3]. A detailed discussion is found in Bressan [7,9,12]. See also Bressan [8,10]. Still another method for proving continuous dependence of solutions in $L^1$ was devised, at about the same time, by Hu and LeFloch [1]. Furthermore, in the context of the Euler equations, Goatin and LeFloch [3] discuss $L^1$ continuous dependence for solutions with large total variation. Actually, $L^1$ stability has now been established, by Liu and Yang [5], even via the Glimm scheme. It should be noted that, in contrast to the scalar case, there is no standard $L^1$-contractive metric for systems (Temple [4]). The rate of decrease in the distance between two solutions (recall Theorem 11.8.3 for the scalar case) is estimated by Goatin and LeFloch [2]; see the presentation in the book by LeFloch [5].

Bianchini and Colombo [1] show that solutions to the Cauchy problem depend continuously on the flux function. The issue of "shift differentiability" of the flow generated by conservation laws, which is relevant to stability considerations, is discussed in Bressan and Guerra [1] and Bianchini [1].

Uniqueness under the Tame Oscillation Condition was established by Bressan and Goatin [1], improving an earlier theorem by Bressan and LeFloch [1] which required a Tame Variation Condition. Uniqueness also prevails when the Tame Oscillation Condition is replaced by the assumption that the trace of solutions along space-like curves has bounded variation; see Bressan and Lewicka [1]. The impetus for the above research was provided by Bressan [4], which established the unique limit of the Glimm scheme. For an alternative approach, based on Haar's method, see Hu and LeFloch [1]. Uniqueness is also discussed in Oleinik [3], Liu [3], DiPerna [5], Dafermos and Geng [2], Heibig [1], LeFloch and Xin [1], Chen and Frid [7], and Chen and Li [1].

A detailed treatment of the topics outlined in Section 14.11 is found in Bressan [9]. Continuous Glimm functionals were first introduced by Schatzman [1], in the context of piecewise Lipschitz solutions. The extension of the notion to $BV$ solutions, for genuinely nonlinear systems, and the proof of the lower semicontinuity property (Lemma 14.11.1) are due to Bressan and Colombo [2] and Baiti and Bressan [2]. Further extension, to systems that are not genuinely nonlinear, was made by LeFloch and Trivisa [1]. See also Bianchini [5].

As already noted in Sections 11.12 and 12.11, the decay of positive waves at the rate $O(1/t)$ was first discussed, for convex scalar conservation laws and genuinely nonlinear systems of two conservation laws, by Oleinik [2] and Glimm and Lax [1], respectively. The version presented here, Theorem 14.11.2, for genuinely nonlinear systems of $n$ conservation laws is taken from Bressan and Colombo [3] and Bressan [9]. A sharp decay estimate is found in Bressan and Yang [2]. See also Bressan and Coclite [1] and Bressan and Goatin [2], for special systems, and Goatin and Gosse [1], for systems of balance laws. An analogous property for piecewise genuinely nonlinear systems, originally demonstrated by Liu [15], has been reestablished, by use of continuous Glimm functionals, in LeFloch and Trivisa [1]. For implications on uniqueness, see Bressan and Goatin [1] and Goatin [1].

The local structure of $BV$ solutions was first described by DiPerna [3], for genuinely nonlinear systems, and by Liu [15], for piecewise genuinely nonlinear systems. The approach outlined here, culminating in Theorems 14.11.3 and 14.11.4, is due to Bressan and LeFloch [2]; see Bressan [9], for a detailed treatment.

The investigation of solutions that are small perturbations of a given, self-similar wave fan, with large shocks and/or rarefaction waves, was initiated by Schochet [4], who established local existence, via the random choice method, for genuinely nonlinear systems of arbitrary size, and by Bressan and Colombo [2], who first demonstrated stability (i.e., continuous dependence in $L^1$) for genuinely nonlinear systems of two conservation laws. See also Bressan and Marson [2]. The combined treatment of existence and stability, for genuinely nonlinear systems of arbitrary size, outlined in Section 14.12, is based on the work of Lewicka and Trivisa [1] and Lewicka [1,2,3,4,5].

A lot of experience has been accumulated by now on the random choice scheme and the front tracking algorithm, for constructing solutions, as well as on the linearization technique, the Liu-Yang functional and Haar's method, for establishing uniqueness and $L^1$ stability. Accordingly, the above methods have been adapted and have been employed, interchangeably or in combination, in the study of Cauchy problems for (inhomogeneous) systems of balance laws (Amadori and Guerra [1,2,3], Amadori, Gosse and Guerra [1], Crasta and Piccoli [1]), as well as initial-boundary-value problems for systems of conservation laws (Amadori [1], Amadori and Colombo [1,2]); also for nonclassical solutions, with shocks satisfying admissibility conditions dictated by some kinetic relation, possibly even for systems not in conservation form (Crasta and LeFloch [1], Baiti, LeFloch and Piccoli [1,2], Amadori, Baiti, LeFloch and Piccoli [1], Colombo and Corli [1,3]), and finally for problems in control theory (Ancona and Marson [1], Bressan and Coclite [1]).

Estimates on the rate of convergence of the front tracking algorithm have been derived by Lucier [1], in the scalar case, and by Bressan [9], for systems.

# XV

# Construction of $BV$ Solutions by the Vanishing Viscosity Method

Admissible $BV$ solutions to the Cauchy problem for general strictly hyperbolic systems of conservation laws, under initial data with small total variation, will be constructed by the vanishing viscosity method. It will be shown that these solutions may be realized as trajectories of an $L^1$-Lipschitz semigroup, which reduces to the standard Riemann semigroup, introduced in Chapter XIV, when the system is genuinely nonlinear.

## 15.1 The Main Result

Consider the Cauchy problem

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0, \qquad -\infty < x < \infty, \quad 0 < t < \infty, \tag{15.1.1}$$

$$U(x,0) = U_0(x), \qquad -\infty < x < \infty, \tag{15.1.2}$$

for a system of conservation laws which is strictly hyperbolic in a ball $\mathcal{O}$ in $\mathbb{R}^n$, centered at a certain state $U^*$, and initial data $U_0$ of bounded variation on $(-\infty,\infty)$ such that $U_0(-\infty) = U^*$.

The aim is to construct $BV$ solutions to (15.1.1), (15.1.2) as the $\mu \downarrow 0$ limit of solutions to the parabolic system

(15.1.3)

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = \mu \partial_x^2 U(x,t), \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

under the same initial condition (15.1.2). This will be effected through

**15.1.1 Theorem.** *There is* $\delta > 0$ *such that if*

$$TV_{(-\infty,\infty)} U_0(\cdot) < \delta, \tag{15.1.4}$$

*then the following hold, for some positive constants a and b:*

*(a) For any $\mu > 0$ there exists a classical solution $U_\mu$ to* (15.1.3), (15.1.2) *and*

$$TV_{(-\infty,\infty)} U_\mu(\cdot, t) \le a\, TV_{(-\infty,\infty)} U_0(\cdot), \qquad 0 < t < \infty, \tag{15.1.5}$$

$$\|U_\mu(\cdot, t) - U_\mu(\cdot, \tau)\|_{L^1(-\infty,\infty)} \le b\left(|t - \tau| + |\sqrt{\mu t} - \sqrt{\mu\tau}|\right), \quad 0 < \tau < t < \infty. \tag{15.1.6}$$

*(b) If $\bar{U}_\mu$ denotes the solution of* (15.1.3) *with initial value $\bar{U}_0$ such that $U_0 - \bar{U}_0$ is in $L^1(-\infty, \infty)$, then*

$$\|U_\mu(\cdot, t) - \bar{U}_\mu(\cdot, t)\|_{L^1(-\infty,\infty)} \le a\, \|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^1(-\infty,\infty)}, \qquad 0 < t < \infty. \tag{15.1.7}$$

*(c) As $\mu \downarrow 0$, $\{U_\mu\}$ converges in $L^1_{\mathrm{loc}}$ to a $BV$ solution $U$ of* (15.1.1), (15.1.2) *which inherits the stability properties* (15.1.5), (15.1.6) *and* (15.1.7)*, namely*

$$TV_{(-\infty,\infty)} U(\cdot, t) \le a\, TV_{(-\infty,\infty)} U_0(\cdot), \qquad 0 < t < \infty, \tag{15.1.8}$$

$$\|U(\cdot, t) - U(\cdot, \tau)\|_{L^1(-\infty,\infty)} \le b\,|t - \tau|, \qquad 0 < \tau < t < \infty, \tag{15.1.9}$$

$$\|U(\cdot, t) - \bar{U}(\cdot, t)\|_{L^1(-\infty,\infty)} \le a\, \|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^1(-\infty,\infty)}, \quad 0 < t < \infty. \tag{15.1.10}$$

*The shocks of the solution $U$ satisfy the viscosity shock admissibility criterion, and thereby all implied admissibility conditions, as described in Chapter VIII. When all characteristic families of* (15.1.1) *are either genuinely nonlinear or linearly degenerate, $U$ coincides with the solution of* (15.1.1), (15.1.2) *constructed by the random choice method of Chapter XIII or by the front tracking algorithm of Chapter XIV.*

The proof of the above proposition, which combines diverse ideas and techniques, will occupy the remainder of this chapter. For orientation, Section 15.2 will provide a road map.

It should be noted that the derivation of the estimates (15.1.5), (15.1.6) and (15.1.7) does not depend in an essential manner on the assumption that (15.1.3) is in conservative form but applies equally well to more general systems

$$\partial_t U(x, t) + A(U(x, t))\partial_x U(x, t) = \mu \partial_x^2 U(x, t), \quad -\infty < x < \infty,\ 0 < t < \infty, \tag{15.1.11}$$

provided only that $A(U)$ has real distinct eigenvalues. The $\mu \downarrow 0$ limit $U$ of the family $\{U_\mu\}$ of solutions of (15.1.11), (15.1.2) may be interpreted as a "weak" solution of

$$\partial_t U + A(U)\partial_x U = 0, \tag{15.1.12}$$

even though it does not necessarily satisfy this system in the sense of distributions.

## 15.2 Road Map to the Proof of Theorem 15.1.1

Henceforth we employ the notation $A(U) = \mathrm{D}F(U)$, with eigenvalues $\lambda_i(U)$ and right and left eigenvectors $R_i(U)$ and $L_i(U)$ normalized by $|R_i(U)| = 1$ and (7.2.3). In particular, we set $A(U^*) = A^*$, $\lambda_i(U^*) = \lambda_i^*$, $R_i(U^*) = R_i^*$ and $L_i(U^*) = L_i^*$.

The first step is to eliminate the small parameter $\mu$ from (15.1.3) by rescaling the coordinates, $(x, t) \mapsto (\mu x, \mu t)$. Indeed, if $U_\mu$ is a solution of the Cauchy problem (15.1.3), (15.1.2), then $U(x, t) = U_\mu(\mu x, \mu t)$ satisfies

(15.2.1)

$$\partial_t U(x,t) + A(U(x,t))\partial_x U(x,t) = \partial_x^2 U(x,t), \qquad -\infty < x < \infty, \ \ 0 < t < \infty$$

with initial conditions

$$U(x,0) = U_{0\mu}(x) = U_0(\mu x), \qquad -\infty < x < \infty. \tag{15.2.2}$$

Clearly, $TV_{(-\infty,\infty)}U_{0\mu}(\cdot) = TV_{(-\infty,\infty)}U_0(\cdot)$ and $TV_{(-\infty,\infty)}U_\mu(\cdot,t) = TV_{(-\infty,\infty)}\ U(\cdot,\mu^{-1}t)$, so that it will suffice to estimate the total variation of solutions $U$ of (15.2.1), in which the viscosity coefficient has been scaled to value one. The key estimate is

$$TV_{(-\infty,\infty)}U(\cdot,t) = \|\partial_x U(\cdot,t)\|_{L^1(-\infty,\infty)} < \delta_0\,, \tag{15.2.3}$$

for $t \in (0, \infty)$, where $\delta_0$ is some small positive number.

The above bound results from the synergy between the parabolic and the hyperbolic structure of (15.2.1), in the following way:
(a) There are positive constants $\alpha$ and $\kappa$ such that when $TV_{(-\infty,\infty)}U_0(\cdot) < \frac{1}{2}\kappa\delta_0$ the diffusion induces (15.2.3) for $t$ in some interval $(0, \bar{t}]$ of length $\bar{t} = (\alpha\kappa\delta_0)^{-2}$. Moreover, when (15.2.3) holds on a longer time interval $(0, T)$, with $T > \bar{t}$, then

$$\begin{gathered}\|\partial_x^2 U(\cdot,t)\|_{L^1(-\infty,\infty)} < 2\alpha\delta_0^2\,, \qquad \|\partial_x^3 U(\cdot,t)\|_{L^1(-\infty,\infty)} < 5\alpha^2\delta_0^3\,, \\ \|\partial_x^3 U(\cdot,t)\|_{L^\infty(-\infty,\infty)} < 16\alpha^3\delta_0^4\,,\end{gathered} \tag{15.2.4}$$

for any $t \in [\bar{t}, T)$. This will be established in Section 15.3.
(b) For $t > \bar{t}$, the hyperbolic structure of (15.2.1) takes charge and induces (15.2.3) for $t$ in any, bounded or unbounded, time interval $[\bar{t}, T)$ on which (15.2.4) holds. Thus (b) in conjunction with (a) establish (15.2.3) for all $t \in (0, \infty)$.

The assertion in part (b) is verified in several steps. In Section 15.4 it is explained how one employs the superposition of $n$ (viscous) traveling waves of (15.2.1) that best fits the profile of the solution $U$ in the vicinity of any point $(x, t)$ in the domain $(-\infty, \infty) \times (\bar{t}, \infty)$ so as to express $\partial_x U$ and $\partial_t U$ in a system of local coordinates

(15.2.5) $$\partial_x U = \sum_{j=1}^{n} w_j S_j \,, \qquad \partial_t U = \sum_{j=1}^{n} \omega_j S_j \,,$$

with components $w_j$ and $\omega_j$ that satisfy scalar parabolic equations of the form

(15.2.6) $$\begin{cases} \partial_t w_j + \partial_x(\sigma_j w_j) - \partial_x^2 w_j = \phi_j \\ \partial_t \omega_j + \partial_x(\sigma_j \omega_j) - \partial_x^2 \omega_j = \psi_j \,. \end{cases}$$

The next step, carried out in Sections 15.5, 15.6 and 15.7, is to demonstrate that when (15.2.3) is satisfied on a time interval $(0, T)$, with $T > \bar{t}$, and at the same time

(15.2.7) $$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} \big(|\varphi_j(x,t)| + |\psi_j(x,t)|\big)\, dx\, dt < \delta_0 \,, \qquad j = 1, \cdots, n,$$

then the sharper bound

(15.2.8) $$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} \big(|\varphi_j(x,t)| + |\psi_j(x,t)|\big)\, dx\, dt < c\delta_0^2 \,, \qquad j = 1, \cdots, n,$$

holds, for some $c$ independent of $T$.

The final ingredient is the standard estimate

(15.2.9) $$\int_{-\infty}^{\infty} |w_j(x,t)| dx \le \int_{-\infty}^{\infty} |w_j(x,\bar{t})| dx + \int_{\bar{t}}^{t} \int_{-\infty}^{\infty} |\varphi_j(x,\tau)| dx\, d\tau,$$

for solutions of (15.2.6) and $t > \bar{t}$.

One may now establish that when $\delta_0$ is sufficiently small, (15.2.3) holds for any $t \in (0, \infty)$, by means of the following argument. Assume $TV_{(-\infty,\infty)} U_0(\cdot) < \frac{1}{4}\kappa\delta_0$. Then, by (a) above, $\|\partial_x U(\cdot, t)\|_{L^1(-\infty,\infty)} < \frac{1}{2}\delta_0$, for any $t \in (0, \bar{t}]$. Suppose now that (15.2.3) holds on a bounded interval $[\bar{t}, \bar{T})$ but is violated at $t = \bar{T}$. For $c\delta_0 < 1$, as $T$ increases from $\bar{t}$ to $\bar{T}$, the left-hand side of (15.2.7) cannot assume the value $\delta_0$ unless it has already assumed the value $c\delta_0^2 < \delta_0$ at an earlier time $T$. However, this would be incompatible with the assertion, above, that (15.2.7) implies (15.2.8). Hence (15.2.7), and thereby (15.2.8), must hold for all $t \in [\bar{t}, \bar{T})$. By applying (15.2.9) for $t = \bar{T}$ and using (15.2.5), we infer that $\|\partial_x U(\cdot, \bar{T})\|_{L^1(-\infty,\infty)} < \frac{1}{2}\delta_0 + c_1\delta_0^2$, which is smaller than $\delta_0$ when $2c_1\delta_0 < 1$. We have thus arrived at a contradiction to the hypothesis that (15.2.3) is violated at $t = \bar{T}$.

The stability estimates (15.1.5), (15.1.6) and (15.1.7) will be derived in Section 15.8, with the help of (15.2.3). Clearly, once these estimates have been established, one may pass to the limit along sequences $\{\mu_k\}$, with $\mu_k \to 0$ as $k \to \infty$, and obtain solutions $U$ of (15.1.1), (15.1.2) possessing the stability properties (15.1.8), (15.1.9) and (15.1.10). It will then be shown that any solution $U = \lim_{k\to\infty} U_{\mu_k}$ satisfies the Tame Oscillation Condition 14.10.1. In turn, by virtue of Theorem 14.10.2, this will imply that when all characteristic families are either genuinely nonlinear or linearly degenerate, then $U$ must coincide with the unique solution constructed by the random choice method. Thus, for such systems, the entire family $\{U_\mu\}$ must converge to the same solution $U$, as $\mu \downarrow 0$.

For systems with general characteristic families, the issue of uniqueness has been settled in the literature cited in Section 15.9, by the following procedure.

As a first step, it is shown that, for special initial data (9.1.12), the entire family $\{U_\mu\}$ converges to the solution $V(x,t;U_L,U_R)$ of the Riemann Problem constructed by use of the wave curves identified in Section 9.8.

Next one demonstrates that any solution $U = \lim_{k\to\infty} U_{\mu_k}$ of (15.1.1), (15.1.2) is properly approximated, in the vicinity of every fixed point $(\bar{x},\bar{t})$ of the upper half-plane, by

(a) the solution $V(x-\bar{x}, t-\bar{t}; U_L, U_R)$ of the Riemann problem with end-states $U_L = U(\bar{x}-,\bar{t})$, $U_R = U(\bar{x}+,\bar{t})$, in the sense that for any $\beta > 0$,

$$\lim_{h\downarrow 0} \frac{1}{h} \int_{\bar{x}-\beta h}^{\bar{x}+\beta h} |U(x,\bar{t}+h) - V(x-\bar{x}, h; U_L, U_R)| dx = 0; \tag{15.2.10}$$

(b) the solution $W(x-\bar{x}, t-\bar{t})$ of the Cauchy problem for the linearized system

$$\begin{cases} \partial_t W(x,t) + A(U(\bar{x},\bar{t}))\partial_x W(x,t) = 0 \\ W(x,0) = U(x-\bar{x},\bar{t}), \end{cases} \tag{15.2.11}$$

in the sense that there exist positive constants $c$ and $\beta$ such that, for any $y < \bar{x} < z$,

$$\limsup_{h\downarrow 0} \int_{y+\beta h}^{z-\beta h} |U(x,\bar{t}+h) - W(x-\bar{x},h)| dx \le c[TV_{(y,z)} U(\cdot,\bar{t})]^2 . \tag{15.2.12}$$

It turns out that the above two conditions, (15.2.10) and (15.2.12), uniquely identify the solution and thus the entire family $\{U_\mu\}$ must converge to $U$, as $\mu \downarrow 0$.

## 15.3 The Effects of Diffusion

As noted in Section 15.2, the role of viscosity will be to sustain (15.2.3) on some interval $(0,\bar{t}]$, of length $\bar{t} = O(\delta_0^{-2})$, while at the same time reducing the size of the $L^1$ norms of spatial derivatives of higher order, as indicated in (15.2.4). Both objectives are met by virtue of

**15.3.1 Lemma.** *Let $U$ be the solution of* (15.2.1), (15.2.2). *There are positive constants $\alpha$ and $\kappa$ such that if* (15.2.3) *holds, for any fixed positive small $\delta_0$, on the interval* $(0,\bar{t}]$ *of length $\bar{t} = (\alpha\kappa\delta_0)^{-2}$, then*

(15.3.1) $$\|\partial_x^2 U(\cdot,t)\|_{L^1(-\infty,\infty)} < \frac{2\delta_0}{\kappa\sqrt{t}}, \qquad t \in (0,\bar{t}],$$

(15.3.2) $$\|\partial_x^3 U(\cdot,t)\|_{L^1(-\infty,\infty)} < \frac{5\delta_0}{\kappa^2 t}, \qquad t \in (0,\bar{t}],$$

(15.3.3) $$\|\partial_x^3 U(\cdot,t)\|_{L^\infty(-\infty,\infty)} < \frac{16\delta_0}{\kappa^3 t\sqrt{t}}, \qquad t \in (0,\bar{t}].$$

*Moreover, when* (15.2.3) *is satisfied on a longer interval* $(0,T)$, $\bar{t} < T \leq \infty$, *then* (15.2.4) *will hold for any* $t \in [\bar{t}, T)$. *Finally,*

(15.3.4) $$TV_{(-\infty,\infty)} U_0(\cdot) < \tfrac{1}{2}\kappa\delta_0$$

*implies* (15.2.3) *for all* $t \in (0,\bar{t}]$.

**Sketch of Proof.** We rewrite (15.2.1) in the form

(15.3.5) $$\partial_t U + A^*\partial_x U - \partial_x^2 U = [A^* - A(U)]\partial_x U.$$

The ($n \times n$ matrix-valued) Green kernel $G(x,t)$ of the linear parabolic operator on the left-hand side of (15.3.5) can be written in closed form as follows. Upon multiplying (15.3.5), from the left, by $L_i^*$, the left-hand side of this system decouples into scalar equations with operator $\partial_t + \lambda_i^*\partial_x - \partial_x^2$, whose Green function reads

(15.3.6) $$g_i(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left[-\frac{(x-\lambda_i^* t)^2}{4t}\right].$$

Therefore,

(15.3.7) $$G(x,t) = \sum_{i=1}^{n} g_i(x,t) R_i^* L_i^*.$$

A simple calculation yields

(15.3.8) $$\|G(\cdot,t)\|_{L^1(-\infty,\infty)} \leq \frac{1}{\kappa}, \quad \|\partial_x G(\cdot,t)\|_{L^{(}-\infty,\infty)} \leq \frac{1}{\kappa\sqrt{t}},$$

for some constant $\kappa$.

It will suffice to establish the desired estimates under the additional assumption $U_0 \in C^\infty$, because the general case will then follow by completion.

Differentiating (15.3.5) with respect to $x$ and applying Duhamel's principle to the resulting equation yields

(15.3.9) $$\partial_x U(\cdot,t) = G(\cdot,t) * \partial_x U_0(\cdot) + \int_0^t G(\cdot,t-\tau) * P(\cdot,\tau)\, d\tau,$$

where

(15.3.10)

$$P(x,\tau) = [A^* - A(U(x,\tau))]\partial_x^2 U(x,t) - \partial_x U^\top(x,\tau)\mathrm{D}^2 F(U(x,\tau))\partial_x U(x,\tau)$$

and $*$ denotes convolution on $(-\infty,\infty)$ with respect to the $x$-variable.

Since $\|U\|_{L^\infty} \le \|\partial_x U\|_{L^1}$ and $\|\partial_x U\|_{L^\infty} \le \|\partial_x^2 U\|_{L^1}$,

$$\|P(\cdot,\tau)\|_{L^1(-\infty,\infty)} \le \beta \|\partial_x U(\cdot,\tau)\|_{L^1(-\infty,\infty)} \|\partial_x^2 U(\cdot,\tau)\|_{L^1(-\infty,\infty)}, \tag{15.3.11}$$

where $\beta$ depends solely on $\sup |\mathrm{D}^2 F(U)|$, for $U \in \mathcal{O}$.

We now assume (15.2.3) holds on an interval $[0,\bar{t}]$ of length $\bar{t} = (\alpha\kappa\delta_0)^{-2}$, with $\alpha > 2\pi\beta\kappa^{-2}$ and proceed to verify (15.3.1) on $(0,\bar{t}]$. Differentiating (15.3.9) with respect to $x$,

$$\partial_x^2 U(\cdot,t) = \partial_x G(\cdot,t) * \partial_x U_0(\cdot) + \int_0^t \partial_x G(\cdot,t-\tau) * P(\cdot,\tau)\,d\tau. \tag{15.3.12}$$

This together with (15.3.8), (15.3.11) and (15.2.3) imply

(15.3.13)

$$\|\partial_x^2 U(\cdot,t)\|_{L^1(-\infty,\infty)} \le \frac{\delta_0}{\kappa\sqrt{t}} + \frac{\beta\delta_0}{\kappa}\int_0^t \frac{1}{\sqrt{t-\tau}} \|\partial_x^2 U(\cdot,\tau)\|_{L^1(-\infty,\infty)} d\tau.$$

Suppose (15.3.1) is false and let $t$ be the earliest time in $(0,\bar{t}]$ where it fails. Then (15.3.13) yields

(15.3.14)

$$\frac{2\delta_0}{\kappa\sqrt{t}} \le \frac{\delta_0}{\kappa\sqrt{t}} + \frac{2\beta\delta_0^2}{\kappa^2}\int_0^t \frac{1}{\sqrt{\tau(t-\tau)}} d\tau = \frac{\delta_0}{\kappa\sqrt{t}} + \frac{2\pi\beta\delta_0^2}{\kappa^2} < \frac{\delta_0}{\kappa\sqrt{t}} + \frac{\delta_0}{\kappa\sqrt{\bar{t}}},$$

which is a contradiction to $t \le \bar{t}$.

The estimates (15.3.2) and (15.3.3) are established by similar arguments. The reader may find the details in the references cited in Section 15.9.

Because of (15.3.1), (15.3.2) and (15.3.3), (15.2.4) holds at $t = \bar{t} = (\alpha\kappa\delta_0)^{-2}$. When (15.2.3) is satisfied on a longer interval $(0,T)$, $\bar{t} < T \le \infty$, then (15.2.4) will hold for any $t \in [\bar{t},T)$, because the time origin may be shifted to the point $t - \bar{t}$.

Finally, assume (15.3.4) and suppose (15.3.1) holds on some time interval $(0,\hat{t})$. Then, for any $t \in (0,\hat{t}]$, (15.3.9), (15.3.8) and (15.3.11) together imply

$$\|\partial_x U(\cdot,t)\|_{L^1(-\infty,\infty)} \le \frac{\delta_0}{2} + \frac{2\beta\delta_0}{\kappa^2}\int_0^t \frac{1}{\sqrt{\tau}} \|\partial_x U(\cdot,\tau)\|_{L^1(-\infty,\infty)} d\tau. \tag{15.3.15}$$

By Gronwall's lemma,

$$\|\partial_x U(\cdot,t)\|_{L^1(-\infty,\infty)} \leq \frac{\delta_0}{2}\exp\left[\frac{4\beta\delta_0\sqrt{t}}{\kappa^2}\right], \qquad t\in[0,\hat{t}]. \tag{15.3.16}$$

Consequently, both (15.3.1) and (15.2.3) will be satisfied on an interval $(0,\bar{t}]$ with $\bar{t}=(\alpha\kappa\delta_0)^{-2}$, provided that $\alpha$ is sufficiently small, but independent of $\delta_0$. The proof is complete.

## 15.4 Decomposition into Viscous Traveling Waves

In Section 7.8 we saw that by expressing solutions as the superposition (7.8.1) of simple waves, the hyperbolic system (15.1.1) reduces to the system (7.8.6) of weakly coupled scalar equations. Here it is shown that the analog for the parabolic system (15.2.1) is the decomposition (15.2.5) of solutions into a sum of viscous traveling waves.

A viscous wave traveling with speed $s$ is a solution $U$ of (15.2.1) in the special form $U(x,t)=V(x-st)$. The function $V$ must satisfy the second order ordinary differential equation

$$\ddot{V}=[A(V)-sI]\dot{V}, \tag{15.4.1}$$

which may be recast as the first order system

$$\begin{cases}\dot{V}=W\\[2ex] \dot{W}=[A(V)-sI]W\\[2ex] \dot{s}=0.\end{cases} \tag{15.4.2}$$

Clearly there exists a $(2n+1)$-parameter family of viscous traveling waves, parameterized by their speeds and the initial values of $V$ and $W$. However, the only ones that may serve our present purposes are those for which $W$ stays small and $s$ is close to one of the characteristic speeds $\lambda_j^*$. These will be dubbed *viscous $j$-waves.*

For the system

$$\begin{cases}\dot{V}=W\\[2ex] \dot{W}=[A^*-\lambda_j^* I]W\\[2ex] \dot{s}=0,\end{cases} \tag{15.4.3}$$

resulting from linearizing (15.4.2) about $(V=U^*,\ W=0,\ s=\lambda_j^*)$, solutions with $W$ bounded span the $(n+2)$-dimensional hyperplane

$$\mathcal{P}_j=\left\{(V,W,s):V\in I\!R^n,\ W=wR_j^*,\ w\in I\!R,\ s\in I\!R\right\} \tag{15.4.4}$$

embedded in $\mathbb{R}^{2n+1}$. It then follows from the center manifold theorem that the orbits of viscous $j$-waves span a smooth $(n+2)$-dimensional manifold $\mathcal{M}_j$ embedded in $\mathbb{R}^{2n+1}$, which is tangent to $\mathcal{P}_j$ at the point $(U^*, 0, \lambda_j^*)$. Furthermore, this *center manifold* admits the local representation

(15.4.5)

$$\mathcal{M}_j = \left\{(V, W, s) : |V - U^*| < \varepsilon,\ W = wS_j(V, v, s),\ |w| < \varepsilon,\ |s - \lambda_j^*| < \varepsilon\right\},$$

where $S_j$ is a smooth unit vector field such that $S_j(U^*, 0, \lambda_j^*) = R_j^*$. In particular, since $|S_j| = 1$,

$$S_j^\top S_j = 1, \qquad S_j^\top \dot{S}_j = 0, \qquad S_j^\top \ddot{S}_j = -\dot{S}_j^\top \dot{S}_j\,. \tag{15.4.6}$$

As $W = wS_j$ satisfies $(15.4.2)_2$,

$$\dot{w}S_j + w\dot{S}_j = w[A - sI]S_j\,. \tag{15.4.7}$$

Multiplying (15.4.7), from the left, by $S_j^\top$ and using (15.4.6) yields

$$\dot{w} = (\sigma_j - s)w, \tag{15.4.8}$$

where

$$\sigma_j(V, w, s) = S_j^\top(V, w, s)A(W)S_j(V, w, s). \tag{15.4.9}$$

Combining (15.4.8) with (15.4.7) and using $(15.4.2)_1$,

$$[A - \sigma_j I]S_j = \dot{S}_j = \mathrm{D}S_j\dot{V} + \dot{w}\partial_w S_j = w[\mathrm{D}S_jS_j + (\sigma_j - s)\partial_w S_j]. \tag{15.4.10}$$

Letting $w \to 0$ in (15.4.10), we conclude that

$$\sigma_j(V, 0, s) = \lambda_j(V), \qquad S_j(V, 0, s) = R_j(V). \tag{15.4.11}$$

It turns out that $w$ satisfies a differential equation which is derived by the following procedure. Differentiating (15.4.7),

$$\ddot{w}S_j + 2\dot{w}\dot{S}_j + w\ddot{S}_j = \dot{w}AS_j + w(AS_j)^\cdot - s\dot{w}S_j - sw\dot{S}_j\,. \tag{15.4.12}$$

By (15.4.6) and (15.4.7),

$$S_j^\top \ddot{S}_j = -\dot{S}_j^\top \dot{S}_j = -w\dot{S}_j^\top A\, S_j\,. \tag{15.4.13}$$

Therefore, multiplying (15.4.12), from the left, by $S_j^\top$ and using (15.4.6), (15.4.13) and (15.4.9) yields

$$\ddot{w} = (\sigma_j w)^\cdot - s\dot{w}. \tag{15.4.14}$$

To summarize, when $U$ is a viscous $j$-wave, so that $U(x, t) = V(x - st)$ with $s$ near $\lambda_j^*$, then

(15.4.15)
$$\begin{cases} \partial_x U = w_j S_j(U, w_j, s_j) \\ \partial_t U = \omega_j S_j(U, w_j, s_j), \end{cases}$$

where $s_j(x,t) = s$, $w_j(x,t) = w(x - st)$, $\omega_j(x,t) = -s_j(x,t) w_j(x,t)$. Furthermore, by virtue of (15.4.14),

(15.4.16)
$$\begin{cases} \partial_t w_j + \partial_x(\sigma_j w_j) - \partial_x^2 w_j = 0 \\ \partial_t \omega_j + \partial_x(\sigma_j \omega_j) - \partial_x^2 \omega_j = 0. \end{cases}$$

We now consider the possibility of representing any solution $U$ of (15.2.1), in the vicinity of each point $(x,t)$, by a superposition of $n$ viscous waves $V_1, \cdots, V_n$ in such a way that

(15.4.17)
$$U = \sum_{j=1}^n V_j\,, \qquad \partial_x U = \sum_{j=1}^n \dot{V}_j\,, \qquad \partial_x^2 U = \sum_{j=1}^n \ddot{V}_j$$

at $(x,t)$. Towards that end, motivated by (15.4.15), we try

(15.4.18)
$$\begin{cases} \partial_x U = \sum_{j=1}^n w_j S_j(U, w_j, s_j) \\ \partial_t U = \sum_{j=1}^n \omega_j S_j(U, w_j, s_j), \end{cases}$$

for appropriate coefficients $w_j(x,t)$ and $\omega_j(x,t)$. This will satisfy the first two requirements of (15.4.17). It would even satisfy the third requirement if one could select the speeds $s_j$ according to the prescription $s_j = -\omega_j / w_j$ . Indeed, in that case (15.2.1) together with (15.4.18) yield

(15.4.19)
$$\partial_x^2 U = \sum_{j=1}^n w_j \big[A(U) - s_j I\big] S_j\big(U, w_j, s_j\big),$$

which in turn implies $(15.4.17)_3$ , by virtue of (15.4.2).

Unfortunately, it is not always permissible to choose $s_j = -\omega_j / w_j$ , because the ratio $\omega_j / w_j$ may assume any value (including infinity) whereas $S_j(V, w, s)$ is solely defined for $s$ close to $\lambda_j^*$ . Nevertheless, we opt to retain (15.4.18), with $s_j$ defined by

(15.4.20)
$$s_j = \lambda_j^* - \theta\left(\lambda_j^* + \frac{\omega_j}{w_j}\right),$$

where $\theta$ is a smooth "cutoff" function such that

(15.4.21)
$$\theta(r) = \begin{cases} r \ \text{ if } \ |r| \le \delta_1 \\ 0 \ \text{ if } \ |r| > 3\delta_1 \end{cases} \qquad |\theta'(r)| \le 1, \ |\theta''(r)| \le \frac{4}{\delta_1}\,,$$

for some small positive constant $\delta_1$. Thus, $s_j = -\omega_j/w_j$ whenever $-\omega_j/w_j$ takes values near $\lambda_j^*$. On the other hand, when $-\omega_j/w_j$ is far from $\lambda_j^*$, $s_j$ is chosen constant, equal to $\lambda_j^*$.

After laborious analysis, which relies on the properties of the functions $S_j$ and is found in the literature cited in Section 15.9, one shows that as long as $|U - U^*|$, $|\partial_x U|$, $|\partial_x^2 U|$, and thereby also $|\partial_t U|$, are sufficiently small, there exists a unique set of $(w_j, \omega_j)$, $j = 1, \cdots, n$, which satisfies (15.4.18) together with (15.4.20). Moreover, with reference to the setting and notation of Lemma 15.3.1, when (15.2.3) is satisfied on an interval $(0, T)$, with $\bar{t} < T \leq \infty$, so that (15.2.4) hold for any $t \in [\bar{t}, T)$, then

$$(15.4.22)\quad \begin{cases} \sum_{j=1}^{n} \left\{ \|w_j(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\omega_j(\cdot,t)\|_{L^1(-\infty,\infty)} \right\} = O(\delta_0), \\ \sum_{j=1}^{n} \left\{ \|w_j(\cdot,t)\|_{L^\infty(-\infty,\infty)} + \|\omega_j(\cdot,t)\|_{L^\infty(-\infty,\infty)} \right\} = O(\delta_0^2), \\ \sum_{j=1}^{n} \left\{ \|\partial_x w_j(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\partial_x \omega_j(\cdot,t)\|_{L^1(-\infty,\infty)} \right\} = O(\delta_0^2), \\ \sum_{j=1}^{n} \left\{ \|\partial_x w_j(\cdot,t)\|_{L^\infty(-\infty,\infty)} + \|\partial_x \omega_j(\cdot,t)\|_{L^\infty(-\infty,\infty)} \right\} = O(\delta_0^3), \end{cases}$$

uniformly on $[\bar{t}, T)$.

As we saw above, when $U$ is just a viscous $j$-wave, $w_j$ and $\omega_j$ satisfy (15.4.16). For general solutions $U$, we have, instead, Equations (15.2.6), with source terms $\varphi_j$ and $\psi_j$. The expectation is that the approximation of $U$ by viscous waves, through (15.4.17), will be sufficiently tight to render $\varphi_j$ and $\psi_j$ "small".

After a lengthy and laborious calculation, which is found in the references cited in Section 15.9, one shows that

$$(15.4.23)\quad \begin{aligned} (\varphi_j, \psi_j) = O(1) &\sum_{i \neq k} \big( |w_i w_k| + |\omega_i \omega_k| + |w_i \omega_k| + |w_i \partial_x w_k| + |w_i \partial_x \omega_k| \big) \\ &+ O(1) \sum_i |\omega_i \partial_x w_i - w_i \partial_x \omega_i| \\ &+ O(1) \sum_i \left| w_i \partial_x \left( \frac{\omega_i}{w_i} \right) \right|^2 \chi_{\{|\lambda_i^* + \omega_i/w_i| < 3\delta_i\}} \\ &+ O(1) \sum_i \big( |\partial_x w_i| + |\partial_x \omega_i| \big) |\omega_i + s_i w_i|. \end{aligned}$$

The four terms on the right-hand side of (15.4.24) estimate the "deviation" of (15.2.6) from the single viscous $j$-wave case (15.4.16), arising for the following reasons:

(a) The first term accounts for transversal wave interactions: viscous waves belonging to distinct characteristic families, and thus propagating with distinct speeds, interact and make a contribution of quadratic order to $\varphi_j$ and $\psi_j$ .
(b) The second and third term account for interactions of waves from the same characteristic family: The viscous $i$-waves approximating the profile $U(\cdot, t)$ at two different points, say $x$ and $y$, are propagating with distinct speeds $s_i(x, t)$ and $s_i(y, t)$ and may thus interact. The key factor in the estimate is $\partial_x s_i$ which monitors the rate of change of $s_i$ .
(c) The fourth term accounts for the "error" committed by selecting $s_i$ through (15.4.20) instead of $-\omega_i/w_i$ , as would have been the case for a viscous $i$-wave. Indeed, notice that this term vanishes whenever $s_i = -\omega_i/w_i$ .

In the following three sections we will estimate the right-hand side of (15.4.23). The aim is to verify the assertion made in Section 15.2, namely that if (15.2.3) holds on $(0, T)$ then (15.2.7) implies (15.2.8).

## 15.5 Transversal Wave Interactions

The aim here is to estimate the first term on the right-hand side of (15.4.23), which accounts for the interaction between viscous waves of distinct families. Under the assumption that (15.2.3) holds for $t \in (0, T)$, which in turn yields (15.4.22) for $t \in [\bar{t}, T)$, it will be shown that (15.2.7) implies

(15.5.1)
$$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} \sum_{i \neq k} \big( |w_i w_k| + |w_i \omega_k| + |\omega_i \omega_k| + |w_i \partial_x w_k| + |w_i \partial_x \omega_k| + |\omega_i \partial_x w_k| \big) dx dt = O(\delta_0^2).$$

Towards that goal we shall compare the solutions of two parabolic equations

(15.5.2)
$$\begin{cases} \partial_t u^\flat(x,t) + \partial_x\big[\sigma^\flat(x,t) u^\flat(x,t)\big] - \partial_x^2 u^\flat(x,t) = p^\flat(x,t) \\ \partial_t u^\sharp(x,t) + \partial_x\big[\sigma^\sharp(x,t) u^\sharp(x,t)\big] - \partial_x^2 u^\sharp(x,t) = p^\sharp(x,t) \end{cases}$$

with strictly separated drifts:

(15.5.3)
$$\inf \sigma^\sharp - \sup \sigma^\flat \geq r > 0.$$

**15.5.1 Lemma.** *If* $(u^\flat, u^\sharp)$ *are solutions of* (15.5.2) *on* $(-\infty, \infty) \times [0, T)$,

(15.5.4)
$$\int_0^T \int_{-\infty}^{\infty} \left|u^\flat(x,t)\right|\left|u^\sharp(x,t)\right| dxdt \le \frac{1}{r}\left\{ \int_{-\infty}^{\infty} |u^\flat(x,0)|dx \right.$$
$$\left. + \int_0^T \int_{-\infty}^{\infty} |p^\flat(x,t)|dxdt \right\} \left\{ \int_{-\infty}^{\infty} |u^\sharp(x,0)|dx + \int_0^T \int_{-\infty}^{\infty} |p^\sharp(x,t)|dxdt \right\}.$$

**Proof.** We consider first the homogeneous case, $p^\flat = p^\sharp = 0$. We introduce the interaction potential

(15.5.5)
$$q(v^\flat, v^\sharp) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-y)|v^\flat(x)||v^\sharp(y)|dxdy,$$

for any pair of functions $v^\flat$ and $v^\sharp$ in $L^1(-\infty,\infty)$, where

(15.5.6)
$$k(z) = \begin{cases} r^{-1} & \text{if } z \ge 0 \\ r^{-1}\exp\left(\frac{1}{2}rz\right) & \text{if } z < 0. \end{cases}$$

Notice that $rk' - 2k''$ is the Dirac mass at the origin. We now have

(15.5.7)
$$\frac{d}{dt} q\big(u^\flat(\cdot,t), u^\sharp(\cdot,t)\big) = \frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-y)\,|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-y)\left\{\left[\partial_x^2 u^\flat - \partial_x\left(\sigma^\flat u^\flat\right)\right] \operatorname{sgn} u^\flat\right\}(x,t)\,|u^\sharp(y,t)|\,dxdy$$
$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-y)\left\{\left[\partial_x^2 u^\sharp - \partial_x\left(\sigma^\sharp u^\sharp\right)\right] \operatorname{sgn} u^\sharp\right\}(y,t)\,|u^\flat(x,t)|\,dxdy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k'(x-y)\left[\sigma^\flat(x,t) - \sigma^\sharp(y,t)\right]|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy$$
$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2k''(x-y)\,|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy$$
$$\le - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (rk' - 2k'')(x-y)\,|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy$$
$$= - \int_{-\infty}^{\infty} |u^\flat(x,t)|\,|u^\sharp(x,t)|\,dx.$$

Integrating (15.5.7) over $(0, T)$ and recalling (15.5.5) and (15.5.6), we deduce

$$(15.5.8)\qquad \int_0^T \int_{-\infty}^{\infty} |u^\flat(x,t)|\,|u^\sharp(x,t)|\,dxdt \le \frac{1}{r} \int_{-\infty}^{\infty} |u^\flat(x,0)|dx \int_{-\infty}^{\infty} |u^\sharp(x,0)|dx,$$

namely (15.5.4) for the special case $p^\flat = p^\sharp = 0$. In particular, if $\Gamma^\flat(x,t;y,\tau)$ and $\Gamma^\sharp(x,t;y,\tau)$ denote the Green functions for the (homogeneous form of the) equations (15.5.2),

$$(15.5.9)\qquad \int_{\max\{\tau,\tau'\}}^{T} \int_{-\infty}^{\infty} \Gamma^\flat(x,t;y,\tau)\Gamma^\sharp(x,t;y',\tau')dxdt \le \frac{1}{r},$$

for any couple of initial points $(y,\tau)$ and $(y',\tau')$.

The solutions of (15.5.2) may now be written as

(15.5.10)

$$\begin{cases} u^\flat(x,t) = \displaystyle\int_{-\infty}^{\infty} \Gamma^\flat(x,t;y,0)u^\flat(y,0)dy + \int_0^t \int_{-\infty}^{\infty} \Gamma^\flat(x,t;y,\tau)p^\flat(y,\tau)dyd\tau \\[2ex] u^\sharp(x,t) = \displaystyle\int_{-\infty}^{\infty} \Gamma^\sharp(x,t;y,0)u^\sharp(y,0)dy + \int_0^t \int_{-\infty}^{\infty} \Gamma^\sharp(x,t;y,\tau)p^\sharp(y,\tau)dyd\tau. \end{cases}$$

Combining (15.5.9) with (15.5.10), we arrive at (15.5.4). The proof is complete.

**15.5.2 Lemma.** *Assume that*

$$(15.5.11)\qquad \int_0^T \int_{-\infty}^{\infty} |p^\flat(x,t)|dxdt \le \delta_0, \qquad \int_0^T \int_{-\infty}^{\infty} |p^\sharp(x,t)|dxdt \le \delta_0,$$

$$(15.5.12)\qquad \|\sigma^\flat(\cdot,t)\|_{L^\infty(-\infty,\infty)} \le c\delta_0, \qquad \|\partial_x\sigma^\flat(\cdot,t)\|_{L^\infty(-\infty,\infty)} \le c\delta_0.$$

*Let* $u^\flat, u^\sharp$ *be solutions of* (15.5.2) *such that*

$$(15.5.13)\qquad \|u^\flat(\cdot,t)\|_{L^1(-\infty,\infty)} \le \delta_0, \qquad \|u^\sharp(\cdot,t)\|_{L^1(-\infty,\infty)} \le \delta_0,$$

$$(15.5.14)\qquad \|\partial_x u^\flat(\cdot,t)\|_{L^1(-\infty,\infty)} \le c\delta_0^2, \qquad \|u^\sharp(\cdot,t)\|_{L^\infty(-\infty,\infty)} \le c\delta_0^2,$$

*for all* $t \in [0,T)$. *Then*

$$(15.5.15)\qquad \int_0^T \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t)|\,|u^\sharp(x,t)|\,dxdt = O(\delta_0^2).$$

**Proof.** The left-hand side of (15.5.15) is bounded by

$$\mathcal{J}(T) = \sup \int_0^{T-\tau} \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t) u^\sharp(x+y,t+\tau)|\, dxdt, \tag{15.5.16}$$

where the supremum is taken over all $(y,\tau) \in (-\infty,\infty) \times [0,T)$.

By account of (15.5.14),

$$\sup \int_0^1 \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t) u^\sharp(x+y,t+\tau)|\, dxdt \le c^2 \delta_0^4\,. \tag{15.5.17}$$

For $t > 1$, we write $\partial_x u^\flat$ in the form

$$\begin{aligned}\partial_x u^\flat(x,t) = &\int_{-\infty}^{\infty} \partial_x g(z,1) u^\flat(x-z,t-1) dz \\ &+ \int_0^1 \int_{-\infty}^{\infty} \partial_x g(z,s)\, [p^\flat - \partial_x(\sigma^\flat u^\flat)](x-z,t-s)\, dzds,\end{aligned} \tag{15.5.18}$$

where $g(x,t) = (4\pi t)^{-\frac{1}{2}} \exp[-x^2/4t]$ is the standard heat kernel. Hence

(15.5.19)

$$\begin{aligned}
&\int_1^{T-\tau} \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t) u^\sharp(x+y,t+\tau)|\, dxdt \\
&\le \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_x g(z,1) u^\flat(x-z,t-1) u^\sharp(x+y,t+\tau)|\, dzdxdt \\
&+ \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \|\partial_x \sigma^\flat\|_{L^\infty} |\partial_x g(z,s) u^\flat(x-z,t-s) u^\sharp(x+y,t+\tau)|\, dzdsdxdt \\
&+ \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \|\sigma^\flat\|_{L^\infty} |\partial_x g(z,s) \partial_x u^\flat(x-z,t-s) u^\sharp(x+y,t+\tau)|\, dzdsdxdt \\
&+ \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_{t-1}^{t} \int_{-\infty}^{\infty} |\partial_x g(x-z,t-s) p^\flat(z,s) u^\sharp(x+y,t+\tau)|\, dzdsdxdt.
\end{aligned}$$

Upon combining (15.5.16), (15.5.17), (15.5.19), (15.5.4), (15.5.11), (15.5.12), (15.5.13), and (15.5.14), one obtains

$$\mathcal{J}(T) \le c^2\delta_0^4 + \frac{4\delta_0^2}{\sqrt{\pi}r} + \frac{8c\delta_0^3}{\sqrt{\pi}r} + \frac{2c\delta_0}{\sqrt{\pi}}\mathcal{J}(T) + \frac{2c\delta_0^3}{\sqrt{\pi}}\,. \tag{15.5.20}$$

For $\delta_0$ sufficiently small, (15.5.20) yields $\mathcal{J}(T) = O(\delta_0^2)$ and thence (5.5.15). This completes the proof.

We have now laid the groundwork for establishing (15.5.1). Recalling (15.2.6), we apply Lemma 15.5.1 with $u^\flat = w_i$ , $\sigma^\flat = \sigma_i$ , $p^\flat = \varphi_i$ , $u^\sharp = w_k$ , $\sigma^\sharp = \sigma_k$ , $p^\sharp = \varphi_k$ , shifting the origin from $t = 0$ to $t = \bar{t}$. Using (15.2.7) and (15.4.22), we deduce that the integral of $|w_i w_k|$ over $(-\infty, \infty) \times (\bar{t}, T)$ is $O(\delta_0^2)$. The integrals of $|w_i\omega_k|$ and $|\omega_i\omega_k|$ are treated by the same argument. To estimate the integral of $|w_i\partial_x w_k|$, we apply Lemma 15.5.2 with $u^\flat = w_k$ , $\sigma^\flat = \sigma_k$ , $p^\flat = \varphi_k$ , $u^\sharp = w_i$ , $\sigma^\sharp = \sigma_i$ , $p^\sharp = \varphi_i$ . In order to meet the requirement $(15.5.12)_1$, we perform the change of variable $x \mapsto x - \lambda_k^* t$ so that the drift coefficient $\sigma_k$ is replaced by $\sigma_k - \lambda_k^*$ which is $O(\delta_0)$. The integrals of the remaining terms $|w_i\partial_x\omega_k|$ and $|\omega_i\partial_k w_k|$ are handled by the same method.

## 15.6 Interaction of Waves of the Same Family

This section provides estimates for the second and third term on the right-hand side of (15.4.23), which are induced by the interaction of viscous waves of the same family. The objective is to show that when (15.2.7) and (15.4.22) hold, for $t \in [\bar{t}, T)$, then

$$\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} |\omega_i\partial_x w_i - w_i\partial_x\omega_i|\,dxdt = O(\delta_0^2), \tag{15.6.1}$$

$$\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \left|w_i\partial_x\left(\frac{\omega_i}{w_i}\right)\right|^2 \chi_{\{|\lambda_i^*+\omega_i/w_i|<3\delta_1\}}dxdt = O(\delta_0^3). \tag{15.6.2}$$

This will be attained by monitoring the time evolution of two functionals of the solutions with very interesting geometric interpretation.

We consider solutions $(w, \omega)$ of the equations

$$\begin{cases} \partial_t w(x,t) + \partial_x[\sigma(x,t)w(x,t)] - \partial_x^2 w(x,t) = \varphi(x,t) \\ \\ \partial_t\omega(x,t) + \partial_x[\sigma(x,t)\omega(x,t)] - \partial_x^2\omega(x,t) = \psi(x,t), \end{cases} \tag{15.6.3}$$

on $[\bar{t}, T)$, where $\varphi$, $\psi$ and $\sigma$ are given, smooth functions, with $\varphi(\cdot, t)$ and $\psi(\cdot, t)$ in $L^1(-\infty, \infty)$. Hence $w(\cdot, t)$ and $\omega(\cdot, t)$ will also lie in $L^1(-\infty, \infty)$, so that one may define the functionals

$$\mathcal{L}(t) = \int_{-\infty}^{\infty} [w^2(x,t) + \omega^2(x,t)]^{1/2} dx, \tag{15.6.4}$$

$$\mathcal{A}(t) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{x<y} |w(x,t)\omega(y,t) - \omega(x,t)w(y,t)|\, dxdy. \tag{15.6.5}$$

We introduce the vector field

$$Z(x,t) = \Big( \int_{-\infty}^{x} w(y,t)dy \;,\; \int_{-\infty}^{x} \omega(y,t)dy \Big). \tag{15.6.6}$$

For fixed $t \in [\bar{t}, T)$, $Z(\cdot, t)$ defines a curve on $\mathbb{R}^2$, parametrized by $x$, and thus $Z$ represents a moving curve on $\mathbb{R}^2$. Notice that $\mathcal{L}(t)$ is the length of the curve at time $t$. Furthermore,

$$\frac{1}{2} \int_{-\infty}^{\infty} Z(y,t) \wedge \partial_x Z(y,t)\, dy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{x<y} \partial_x Z(x,t) \wedge \partial_x Z(y,t)\, dxdy \tag{15.6.7}$$

yields the sum of the areas of the regions enclosed by the curve $Z(\cdot, t)$, each multiplied by the corresponding winding number. Thus $\mathcal{A}(t)$ provides an upper bound for the area of the convex hull of $Z(\cdot, t)$.

By virtue of (15.6.3),

$$\partial_t Z(x,t) + \sigma(x,t)\partial_x Z(x,t) - \partial_x^2 Z(x,t) = \Phi(x,t), \tag{15.6.8}$$

where

$$\Phi(x,t) = \Big( \int_{-\infty}^{\infty} \varphi(y,t)dy \;,\; \int_{-\infty}^{x} \psi(y,t)dy \Big). \tag{15.6.9}$$

The plan is to show that the rate of growth of $\mathcal{L}(t)$ and $\mathcal{A}(t)$ is controlled by $\|\varphi(\cdot,t)\|_{L^1(-\infty,\infty)}$ and $\|\psi(\cdot,t)\|_{L^1(-\infty,\infty)}$, and, in particular, that these functionals are nonincreasing when $\varphi$ and $\psi$ vanish identically.

**15.6.1 Lemma.**

$$\frac{d}{dt}\mathcal{A}(t) \le - \int_{-\infty}^{\infty} \big|\omega(x,t)\partial_x w(x,t) - w(x,t)\partial_x \omega(x,t)\big|\, dx \tag{15.6.10}$$

$$+ \|w(\cdot,t)\|_{L^1(-\infty,\infty)} \|\psi(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\omega(\cdot,t)\|_{L^1(-\infty,\infty)} \|\varphi(\cdot,t)\|_{L^1(-\infty,\infty)}.$$

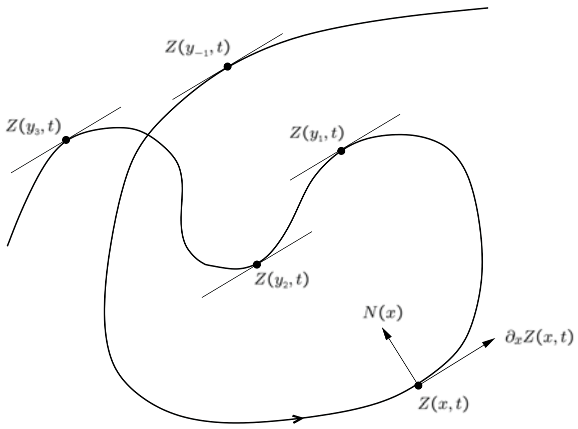


**Fig. 15.6.1**

**Proof.** Let us fix $t \in [\bar{t}, T)$ and consider the curve $Z(\cdot, t)$ in $I\!R^2$; see Fig. 15.6.1. With any $x \in (-\infty, \infty)$ we associate the unit vector $N(x)$ in $I\!R^2$ that is perpendicular to the tangent vector $\partial_x Z(x,t)$ and is oriented by

(15.6.11) $$\partial_x Z(x,t) \wedge N(x) = |\partial_x Z(x,t)|.$$

In particular, for any $W \in I\!R^2$,

(15.6.12) $$\partial_x Z(x,t) \wedge W = |\partial_x Z(x,t)|[N(x) \cdot W].$$

We now compute

(15.6.13)
$$\frac{d}{dt}\mathcal{A}(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{x<y} \operatorname{sgn}[\partial_x Z(x,t) \wedge \partial_x Z(y,t)]$$
$$\times\big[\partial_t\partial_x Z(x,t) \wedge \partial_x Z(y,t) + \partial_x Z(x,t) \wedge \partial_t\partial_x Z(y,t)\big]\,dxdy$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sgn}\big[\partial_x Z(x,t) \wedge \partial_x Z(y,t)\big]\big[\partial_x Z(x,t) \wedge \partial_t\partial_x Z(y,t)\big]\,dydx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_x Z(x,t)| \operatorname{sgn} \partial_y z(y,x,t)\, \partial_t\partial_y z(y,x,t) dydx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} |\partial_x Z(x,t)|\, \partial_t TV_{(-\infty,\infty)} z(\cdot,x,t)\,dx,$$

where we are using the notation

$$z(y, x, t) = N(x) \cdot Z(y, t). \tag{15.6.14}$$

Since $N(x) \cdot \partial_x Z(x, t) = 0$, $x$ is a critical point of $z(\cdot, x, t)$. Let us assume, for simplicity, that this function has a finite number of critical points $y_{-p} < \cdots < y_{-1} < y_0 = x < y_1 < \cdots < y_q$, and none of them is degenerate. As minima and maxima alternate,

$$\operatorname{sgn} \partial_y^2 z(y_r, x, t) = (-1)^r \operatorname{sgn} \partial_y^2 z(x, x, t). \tag{15.6.15}$$

A simple calculation yields

$$\partial_t TV_{(-\infty,\infty)} z(\cdot, x, t) = -2 \sum_{-p \le r \le q} \operatorname{sgn} \partial_y^2 z(y_r, x, t)\, \partial_t z(y_r, x, t). \tag{15.6.16}$$

We substitute into (15.6.16) $\partial_t z = N \cdot \partial_t Z$, with $\partial_t Z$ taken from (15.6.8). Since $\partial_y z(y_r, x, t) = 0$, and by virtue of (15.6.15),

(15.6.17)

$$\begin{aligned} \partial_t TV_{(-\infty,\infty)} z(\cdot, x, t) = -2 &\sum_{-p \le r \le q} |\partial_y^2 z(y_r, x, t)| \\ &-2 \operatorname{sgn} \partial_y^2 z(x, x, t) \sum_{-p \le r \le q} (-1)^r [N(x) \cdot \Phi(y_r, t)]. \end{aligned}$$

Furthermore,

$$\left| \sum_{-p \le r \le q} (-1)^r [N(x) \cdot \Phi(y_r, t)] \right| \le \int_{-\infty}^{\infty} |N(x) \cdot \partial_x \Phi(y, t)|\, dy. \tag{15.6.18}$$

By combining (15.6.13) with (15.6.17), (15.6.18), (15.6.14) and (15.6.12) we conclude that

(15.6.19)

$$\frac{d}{dt}\mathcal{A}(t) \le -\int_{-\infty}^{\infty} |\partial_x Z(x, t) \wedge \partial_x^2 Z(x, t)|\, dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_x Z(x, t) \wedge \partial_x \Phi(y, t)|\, dy dx.$$

Since $\partial_x Z = (w, \omega)$ and $\partial_x \Phi = (\varphi, \psi)$, (15.6.19) yields (15.6.10). The proof is complete.

**15.6.2 Lemma.** *Under the assumption $w^2 + \omega^2 \neq 0$,*

(15.6.20)
$$\frac{d}{dt}\mathcal{L}(t) \le \frac{1}{(1+9\delta_1^2)^{3/2}} \int_{-\infty}^{\infty} |w(x,t)| \left| \partial_x \left( \frac{\omega(x,t)}{w(x,t)} \right) \right|^2 \chi_{\{|\omega/w|<3\delta_1\}} dx$$
$$+ \|\varphi(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\psi(\cdot,t)\|_{L^1(-\infty,\infty)} .$$

**Proof.** Since $w^2 + \omega^2 \neq 0$,

(15.6.21)
$$\frac{d}{dt}\mathcal{L}(t) = \int_{-\infty}^{\infty} \frac{w\partial_t w + \omega \partial_t w}{(w^2+\omega^2)^{1/2}}\, dx.$$

We substitute $\partial_t w$ and $\partial_t \omega$ from (15.6.3) into (15.6.21). Upon using the elementary identities

(15.6.22)
$$\frac{w\partial_x(\sigma w) + \omega \partial_x(\sigma\omega)}{(w^2+\omega^2)^{1/2}} = \partial_x[\sigma(w^2+\omega^2)^{1/2}],$$

(15.6.23)
$$\frac{w\partial_x^2 w + \omega\partial_x^2 \omega}{(w^2+\omega^2)^{1/2}} = \partial_x^2 (w^2+\omega^2)^{\frac{1}{2}} - \frac{|w| \left| \partial_x \left( \frac{\omega}{w} \right) \right|^2}{\left[ 1 + \left( \frac{\omega}{w} \right)^2 \right]^{3/2}} ,$$

we deduce

(15.6.24)
$$\frac{d}{dt}\mathcal{L}(t) = -\int_{-\infty}^{\infty} \frac{|w| \left| \partial_x \left( \frac{\omega}{w} \right) \right|^2}{\left[ 1 + \left( \frac{\omega}{w} \right)^2 \right]^{3/2}} dx + \int_{-\infty}^{\infty} \frac{w\varphi + \omega\psi}{(w^2+\omega^2)^{1/2}}\, dx,$$

which easily yields (15.6.20). This completes the proof.

In order to show (15.6.1), we integrate (15.6.10) over $[\bar{t}, T)$ to get the estimate

(15.6.25)
$$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} |\omega(x,t)\partial_x w(x,t) - w(x,t)\partial_x \omega(x,t)| dx dt \le \mathcal{A}(\bar{t})$$
$$+ \sup_{[\bar{t},T)} \Big( \|w(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\omega(\cdot,t)\|_{L^1(-\infty,\infty)} \Big) \int_{\bar{t}}^{T} \int_{-\infty}^{\infty} (|\varphi(x,t)| + |\psi(x,t)|)\, dx dt.$$

Recalling (15.2.6), we apply (15.6.25) for $w = w_i$, $\omega = \omega_i$, $\varphi = \varphi_i$, $\psi = \psi_i$ and $\sigma = \sigma_i$. In that case, the right-hand side of (15.6.25) is $O(\delta_0^2)$, by virtue of (15.2.7) and (15.4.22).

To verify (15.6.2), we integrate (15.6.20) over $[\bar{t}, T)$, choosing $\delta_1 \le \frac{1}{3}$. We thus obtain

$$\text{(15.6.26)}\qquad \int_{\bar t}^{T}\int_{-\infty}^{\infty} |w(x,t)| \left|\partial_x\left(\frac{\omega(x,t)}{w(x,t)}\right)\right|^2 \chi_{\{|\omega/w|<3\delta_1\}} dxdt$$

$$\leq 4\mathcal{L}(\bar t) + 4\int_{\bar t}^{T}\int_{-\infty}^{\infty} (|\varphi(x,t)| + |\psi(x,t)|) dxdt.$$

We now apply this inequality for $w = w_i$, $\omega = \omega_i$, $\varphi = \varphi_i$, $\psi = \psi_i$ and $\sigma = \sigma_i$, after performing the change of variable $x \mapsto x - \lambda_i^* t$, which renders $\lambda_i^* = 0$. In that case the right-hand side of (15.6.26) is $O(\delta_0)$, by account of (15.2.7) and (15.4.22). Hence

$$\text{(15.6.27)}\qquad \int_{\bar t}^{T}\int_{-\infty}^{\infty} |w_i| \left|\partial_x\left(\frac{\omega_i}{w_i}\right)\right|^2 \chi_{\{|\lambda_i^*+\omega_i/w_i|<3\delta_1\}} dxdt = O(\delta_0).$$

Since $\|w_i\|_{L^\infty} = O(\delta_0^2)$, (15.6.2) follows directly from (15.6.27).

## 15.7 Energy Estimates

Here we estimate the last term on the right-hand side of (15.4.23), which stems from our fixing the speed $s_i$ according to (15.4.20). The aim is to show that

$$\text{(15.7.1)}\qquad \int_{\bar t}^{T}\int_{-\infty}^{\infty} (|\partial_x w_i| + |\partial_x \omega_i|)|\omega_i + s_i w_i|\, dxdt = O(\delta_0^2).$$

The proof, which relies on energy estimates, is technical and does not provide as much insight as the discussion in the previous two sections. Consequently, only an outline will be given here. The reader may find the details in the references cited in Section 15.9.

Since one may perform the change of variable $x \mapsto x - \lambda_i^* t$, we may assume, without loss of generality, that $\lambda_i^* = 0$ and hence $\sigma_i = O(\delta_0)$.

In addition to $\theta$, defined by (15.4.21), we will employ the "cutoff" functions

$$\text{(15.7.2)}\qquad \eta(r) = \begin{cases} 0 \text{ if } |r| \leq \frac{3}{5}\delta_1 \\ \\ 1 \text{ if } |r| \geq \frac{4}{5}\delta_1 \end{cases} \quad |\eta'(r)| \leq \frac{20}{\delta_1},\ |\eta''(r)| \leq \frac{100}{\delta_1^2},$$

and $\bar\eta(r) = \eta(|r| - \frac{1}{5}\delta_1)$. We write $\eta_i = \eta(\omega_i/w_i)$, and $\bar\eta_i = \bar\eta(\omega_i/w_i)$. We also choose $\delta_0 << \delta_1 << 1$.

**15.7.1 Lemma.** *When* $|\omega_i/w_i| \geq \frac{3}{5}\delta_1$,

$$(15.7.3)\qquad \begin{cases} |w_i| \le \dfrac{5}{2\delta_1}|\partial_x w_i| + O(\delta_0)\displaystyle\sum_{j\neq i}|w_j| \\[2ex] |\omega_i| \le 2|\partial_x w_i| + O(\delta_0)\displaystyle\sum_{j\neq i}|w_j|, \end{cases}$$

while when $|\omega_i/w_i| \le \delta_1$,

$$(15.7.4)\qquad |\partial_x w_i| \le 2\delta_1|w_i| + O(\delta_0)\sum_{j\neq i}|w_j|.$$

**Sketch of Proof.** We substitute $\partial_x U$ and $\partial_t U$ from (15.4.18) into (15.2.1) to get

$$(15.7.5)\qquad \sum_{j=1}^{n}\omega_j S_j + \sum_{j=1}^{n} w_j A S_j = \sum_{j=1}^{n}\partial_x w_j S_j + \sum_{j=1}^{n} w_j \partial_x S_j\,.$$

Multiplying, from the left, by $S_i^\top$, recalling that $S_i^\top S_i = 1$, $S_i^\top \partial_x S_i = 0$, and using (15.4.9) yields

$$(15.7.6)\quad \omega_i - \sigma_i w_i - \partial_x w_i = \sum_{j\neq i}\left\{[\partial_x w_j - \omega_j]S_i^\top S_j + w_j S_i^\top \partial_x S_j - w_j S_i^\top A S_j\right\}$$

$$= O(\delta_0)\sum_{j\neq i}(|w_j| + |\partial_x w_j - \omega_j|).$$

Assertions (15.7.3) and (15.7.4) follow from careful analysis of the above equation. This completes the proof.

Since $|\omega_i + s_i w_i|$ vanishes when $|\omega_i/w_i| \le \delta_1$ and is otherwise bounded by $|\omega_i|$, we have

$$(15.7.7)\qquad |\omega_i + s_i w_i| \le |\bar\eta_i \omega_i| \le \bar\eta_i\left[2|\partial_x w_i| + O(\delta_0)\sum_{j\neq i}|w_j|\right].$$

Therefore,

$$(15.7.8)\qquad (|\partial_x w_i| + |\partial_x \omega_i|)|\omega_i + s_i w_i|$$

$$\le 2\bar\eta_i|\partial_x w_i|^2 + 2\bar\eta_i|\partial_x w_i||\partial_x\omega_i| + \sum_{j\neq i}(|w_j\partial_x w_i| + |w_j\partial_x\omega_i|)$$

$$\le 3\eta_i|\partial_x w_i|^2 + \bar\eta_i|\partial_x\omega_i|^2 + \sum_{j\neq i}(|w_j\partial_x w_i| + |w_j\partial_x\omega_i|).$$

As shown in Section 15.5, the integral over $(-\infty,\infty)\times(\bar t, T)$ of the third term on the right-hand side of (15.7.8) is $O(\delta_0^2)$. Thus, in order to verify (15.7.1) it will suffice to show

(15.7.9)
$$\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \eta_i |\partial_x w_i|^2 dxdt = O(\delta_0^2),$$

(15.7.10)
$$\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \bar{\eta}_i |\partial_x \omega_i|^2 dxdt = O(\delta_0^2).$$

The first step towards establishing (15.7.9) is to multiply $(15.2.6)_1$ by $2\eta_i w_i$, integrate the resulting equation over $(-\infty, \infty)$, and integrate by parts. This yields

(15.7.11)
$$\int_{-\infty}^{\infty} \big\{\partial_t(\eta_i w_i^2) + \partial_x(\eta_i\sigma_i)w_i^2 - (\partial_t\eta_i + 2\sigma_i\partial_x\eta_i - \partial_x^2\eta_i)w_i^2$$
$$+ 2\eta_i(\partial_x w_i)^2 + 4(\partial_x\eta_i)w_i\partial_x w_i\big\}dx = \int_{-\infty}^{\infty} 2\eta_i w_i\varphi_i dx.$$

Hence

(15.7.12)
$$2\int_{-\infty}^{\infty} \eta_i|\partial_x w_i|^2dx = -\frac{d}{dt}\int_{-\infty}^{\infty} \eta_i w_i^2 dx + \int_{-\infty}^{\infty} (\partial_t\eta_i + 2\sigma_i\partial_x\eta_i - \partial_x^2\eta_i)w_i^2 dx$$
$$+2\int_{-\infty}^{\infty} \eta_i\sigma_i w_i\partial_x w_i dx - 4\int_{-\infty}^{\infty} (\partial_x\eta_i)w_i\partial_x w_i dx + 2\int_{-\infty}^{\infty} \eta_i w_i\varphi_i dx.$$

We proceed to estimate the right-hand side of the above equation.

Recalling the definition of $\eta_i$ and using (15.2.6), we obtain, after a short calculation,

(15.7.13)
$$(\partial_t\eta_i + 2\sigma_i\partial_x\eta_i - \partial_x^2\eta_i)w_i^2$$
$$= \eta_i'(\psi_i w_i - \varphi_i\omega_i) + 2\eta_i' w_i(\partial_x w_i)\partial_x(\omega_i/w_i) - \eta_i'' w_i^2[\partial_x(\omega_i/w_i)]^2.$$

Furthermore, using $(15.7.3)_1$ and since $\sigma_i = O(\delta_0) << \delta_1$,

(15.7.14)
$$\left|2\int_{-\infty}^{\infty} \eta_i\sigma_i w_i\partial_x w_i dx\right| \leq \int_{-\infty}^{\infty} \eta_i|\partial_x w_i|^2 dx + O(\delta_0)\int_{-\infty}^{\infty}\sum_{j\neq i}|w_j\partial_x w_i|dx.$$

On the range where $\eta_i' \neq 0$, we have $|\omega_i/w_i| < \delta_i$ and hence (15.7.4) applies. One then obtains

(15.7.15)
$$|(\partial_x\eta_i)w_i\partial_x w_i| = |\eta_i' w_i(\partial_x w_i)\partial_x(\omega_i/w_i)|$$
$$\leq O(1)|w_i\partial_x\omega_i - \omega_i\partial_x w_i| + O(\delta_0)\sum_{j\neq i}(|w_j\partial_x w_i| + |w_j\partial_x\omega_i|).$$

We now combine (15.7.12) with (15.7.13), (15.7.14), (15.7.15) and integrate the resulting inequality over $(\bar{t}, T)$. This yields an estimate of the form

$$
\begin{aligned}
(15.7.16)\qquad \int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \eta_i |\partial_x w_i|^2 dxdt \le & \int_{-\infty}^{\infty} (\eta_i w_i^2)(x,\bar{t})\, dx \\
& + O(1)\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} (|w_i\psi_i| + |w_i\varphi_i| + |\omega_i\varphi_i|) dxdt \\
& + O(1)\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} |w_i\partial_x\omega_i - \omega_i\partial_x w_i|\, dxdt \\
& + O(\delta_0)\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \sum_{j\neq i}(|w_j\partial_x w_i| + |w_j\partial_x\omega_i|) dxdt \\
& + O(1)\int_{\bar{t}}^{T}\int_{|\omega_i/w_i|<\delta_i} |w_i\partial_x(\omega_i/w_i)|^2 dxdt.
\end{aligned}
$$

By virtue of (15.4.22), (15.2.7), (15.5.1), (15.6.1) and (15.6.2), we conclude that the right-hand side of (15.7.16) is $O(\delta_0^2)$, which verifies (15.7.9).

The estimate (15.7.10) is established by a similar procedure. For the details the reader should consult the references in Section 15.9.

## 15.8 Stability Estimates

This section provides a sketch of the proof of the stability estimates (15.1.5), (15.1.6) and (15.1.7).

By account of the rescaling $U(x,t) = U_\mu(\mu x, \mu t)$, the estimates (15.1.6) and (15.1.7), for solutions of (15.1.3), (15.1.2), are respectively equivalent to

$$
(15.8.1)\quad \|U(\cdot,t)-U(\cdot,\tau)\|_{L^1(-\infty,\infty)} \le b(|t-\tau|+|\sqrt{t}-\sqrt{\tau}|), \quad 0\le\tau<t<\infty,
$$

(15.8.2)

$$
\|U(\cdot,t) - \bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)} \le a\|U_{0\mu}(\cdot) - \bar{U}_{0\mu}(\cdot)\|_{L^1(-\infty,\infty)}, \quad 0<t<\infty,
$$

for solutions of (15.2.1), (15.2.2).

The estimate (15.8.1) is obtained by integrating over $(\tau, t)$ the inequality

$$
(15.8.3)\qquad \|\partial_t U(\cdot,t)\|_{L^1(-\infty,\infty)} \le b\left(1+\frac{1}{2\sqrt{t}}\right), \qquad 0<t<\infty,
$$

which follows from (15.2.1), by virtue of (15.2.3), (15.3.1) and (15.2.4).

The estimate (15.8.2) is established by means of the following homotopy argument. We have

$$U(x,t) - \bar{U}(x,t) = \int_0^1 \frac{d}{d\xi} U_\xi(x,t) d\xi, \tag{15.8.4}$$

where $U_\xi$ denotes the solution of (15.2.1) with initial data $\xi \bar{U}_{0\mu} + (1-\xi) U_{0\mu}$. The "tangent" vector

$$W_\xi(x,t) = \frac{d}{d\xi} U_\xi(x,t) \tag{15.8.5}$$

is the solution of the linearized equation

$$\partial_t W_\xi(x,t) + \partial_x \big[ A(U_\xi(x,t)) W_\xi(x,t) \big] = \partial_x^2 W_\xi(x,t), \tag{15.8.6}$$

with initial value

$$W_\xi(\cdot,0) = \bar{U}_{0\mu}(\cdot) - U_{0\mu}(\cdot). \tag{15.8.7}$$

Equation (15.8.6) bears a close resemblance to the equation satisfied by the derivative $\partial_x U$ of solutions to (15.2.1), and may thus be treated by the methods employed in earlier sections. The analysis, which is found in the references cited in Section 15.9, shows that, as $\|\partial_x U_\xi(\cdot,t)\|_{L^2(-\infty,\infty)} < \delta_0$ on $(0,\infty)$, there exists a constant $a > 1$ such that, for any $\delta > 0$, $\|W_\xi(\cdot,0)\|_{L^1(-\infty,\infty)} < \delta/a$ implies $\|W_\xi(\cdot,t)\|_{L^1(-\infty,\infty)} < \delta$, for all $t \in (0,\infty)$. Since (15.8.6) is linear, the above assertion is equivalent to

$$\|W_\xi(\cdot,t)\|_{L^1(-\infty,\infty)} \le a \|W_\xi(\cdot,0)\|_{L^1(-\infty,\infty)}, \qquad 0 < t < \infty. \tag{15.8.8}$$

Upon combining (15.8.8) with (15.8.4), (15.8.5) and (15.8.7), we arrive at (15.8.2), thus establishing (15.1.7).

The remaining estimate (15.1.5) is an immediate corollary of (15.1.7). Indeed, we apply (15.1.7) for the two solutions $U_\mu(x,t)$ and $\bar{U}_\mu(x,t) = U_\mu(x+h,t)$, with corresponding initial values $U_0(x)$ and $\bar{U}_0(x) = U_0(x+h)$, we multiply the resulting inequality by $h^{-1}$ and then let $h \to 0$, which yields (15.1.5).

Solutions of (15.1.1) contructed by the vanishing viscosity method have the finite speed of propagation property. Indeed, by using the properties of the Green function it can be shown that when $U_0$ and $\bar{U}_0$ coincide inside an interval $(y,z)$, in which case $U_{0\mu}(x) = \bar{U}_{0\mu}(x)$ for $x \in (y/\mu, z/\mu)$, then the corresponding solutions $U$ and $\bar{U}$ of (15.2.1), (15.2.2) satisfy

(15.8.9)

$$|U(x,t) - \bar{U}(x,t)| \le c \|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^\infty(-\infty,\infty)} \left\{ \exp\left( \nu t - x + \frac{y}{\mu} \right) + \exp\left( \nu t + x - \frac{z}{\mu} \right) \right\}$$

for some positive constants $c$, $\nu$ and all $(x,t)$ in $(-\infty,\infty) \times (0,\infty)$. Upon rescaling, $(x,t) \mapsto (x/\mu, t/\mu)$, so as to return to $U_\mu$, $\bar{U}_\mu$, we conclude that the two solutions

$U = \lim_{k\to\infty} U_{\mu_k}$ and $\bar{U} = \lim_{k\to\infty} \bar{U}_{\mu_k}$ of (15.1.1), with initial values $U_0$ and $\bar{U}_0$, must coincide for all $(x, t)$ with $x \in (y + \nu t, z - \nu t)$.

It follows from the above that in the place of (15.1.10) and (15.1.8) we have the more precise estimates

$$\int_y^z |U(x,t) - \bar{U}(x,t)|dx \le a \int_{y-\nu t}^{z+\nu t} |U_0(x) - \bar{U}_0(x)|dx, \tag{15.8.10}$$

$$TV_{(y,z)}U(\cdot,t) \le aTV_{(y-\nu t,z+\nu t)}U_0(\cdot), \tag{15.8.11}$$

for any $-\infty \le y < z \le \infty$.

We next demonstrate that the finite speed of propagation property in conjunction with the stability estimate (15.1.8) imply that any solution $U$ of (15.1.1), (15.1.2) constructed by the vanishing viscosity method satisfies the Tame Oscillation Condition 14.10.1. In turn, by virtue of Theorem 14.10.2, this will imply that, when all characteristic families are either genuniely nonlinear or linearly degenerate, then $U$ must coincide with the unique solution constructed by the random choice method.

Because solutions of (15.1.1) are preserved under spatial and temporal translations, it will suffice to verify (14.10.3) at the origin, $x = 0$, $t = 0$. We fix $\lambda > \nu$ and consider the solution $\bar{U}$ of (15.1.1) with initial data

$$\bar{U}_0(x) = \begin{cases} U_0(-\lambda h+) & -\infty < x \le -\lambda h \\ U_0(x) & -\lambda h < x < \lambda h \\ U_0(\lambda h-) & \lambda h < x < \infty. \end{cases} \tag{15.8.12}$$

Then $TV_{(-\infty,\infty)}\bar{U}_0(\cdot) = TV_{(-\lambda h,\lambda h)}U_0(\cdot)$, $\bar{U}(0\pm, h) = U(0\pm, h)$ and $\bar{U}(\infty, h) = U_0(\lambda h-)$. Therefore, by account of (15.1.8),

(15.8.13)

$$\begin{aligned} |U(0\pm, h) - U_0(0\pm)| &\le |\bar{U}(0\pm, h) - \bar{U}(\infty, h)| + |U_0(\lambda h-) - U_0(0\pm)| \\ &\le (a+1)TV_{(-\lambda h,\lambda h)}U_0(\cdot), \end{aligned}$$

which establishes (14.10.3).

## 15.9 Notes

The construction of $BV$ solutions by the vanishing viscosity method had been a central open problem of long standing in the theory of hyperbolic systems of conservation laws. It has finally been solved, in a spectacular way, by Bianchini andBressan [5]. The presentation in this section abridges that fundamental paper. The ground had been prepared by the preliminary papers, Bianchini and Bressan [1,2,3,4]. For the rate of convergence, see Bressan and Yang[1]. The method has now been extended to cover initial-boundary value problems for hyperbolic conservation laws (Bianchiniand Ancona [1]), as well as the Cauchy problem for hyperbolic systems of balance laws with dissipative source (Christoforou [1]). The principal underlying ideas of this approach have been fruitfully employed for constructing solutions to the Riemann Problem (Bianchini [6]), and for establishing convergence of semidiscrete upwind schemes for hyperbolic conservation laws (Bianchini [7,8]), of Godunov's method, for special systems (Bressan and Jenssen [1]), and of the linear Jin-Xin relaxation scheme (Bianchini [9]). See also Bianchini [4] and Bressan and Shen [1]. A multitude of additional applicationsare to be expected in the near future. It should also be emphasized that these techniques apply to general quasilinear strictly hyperbolic systems, regardless of whether they are in conservation form. Of course, in the nonconservative case the constructed "solutions" do not necessarily satisfy the equations in the sense of distributions but should be interpreted in the context of the theory of nonconservative shocks by LeFloch *et al.,* outlined in Section 8.7.

There is extensive literature on alternative aspects of the vanishing viscosity approach. We have already seen, in Chapter VI, how this method applies to scalar conservation laws, in the $L^\infty$ or $BV$ setting. In Chapter XVI we shall encounter applications to certain systems of conservation laws, in the $L^p$ setting. Yet another direction is to investigate how solutions of the system with viscosity approximate given, piecewise smooth solutions of the hyperbolic system; see, for instance Goodman and Xin [1], Lin and Yang [1], Hoff and Liu [1], Serre [14], Rousset [4] and Yu [1]. One may pursue the same objective in the context of relaxation schemes; see Lattanzio and Serre [1] and Li and Pan [1].

In the vanishing viscosity approach, the approximate solutions $U_\mu$ carry information on the viscous shock profiles, which is especially valuable, when one employs genuine physical viscosity, but it is lost in the limit $\mu \to 0$. This loss of information also occurs when solutions are constructed by a vanishing capillarity or relaxation method, or even by the approach outlined in Section 8.7, in which the shock profile itself determines the notion of weak solution. As a remedy, LeFloch [6] suggests attaching the information on internal shock structure to the solution $U$ of the hyperbolic system, by means of the following interesting device. Instead of tracking $U(x,t)$ as an evolving discontinuous function of $x$, one should realize it as a moving continuous curve $(\xi(s,t), V(s,t))$, where $\xi(\cdot,t)$ is a smooth nondecreasing function of the parameter $s$, having the following properties: (a) $\xi(\pm\infty,t) = \pm\infty$. (b) $\xi(\cdot,t)$ is invertible on the set of points $x$ of continuity of $U(\cdot,t)$, and $V(\xi^{-1}(x,t),t) = U(x,t)$. (c) If $x$ is a point of discontinuity of $U(\cdot,t)$, then $\xi(s,t) = x$ for $s$ on some closed interval, say $[s_-, s_+]$, with $V(s_\pm,t) = U(x\pm,t)$ and $V(\cdot,t)$ on $(s_-,s_+)$ tracing the profile of the discontinuity that joins $U(x-,t)$ to $U(x+,t)$. The discontinuity profile will be a shock profile, when $(x,t)$ is a point of approximate jump discontinuity of $U$, or a full wave fan profile, when $(x,t)$ belongs to the set of irregular points. The above idea is conceptually pleasing and will likely find technical applications as well.

# XVI

# Compensated Compactness

Approximate solutions to hyperbolic systems of conservation laws may be generated in a variety of ways: by the method of vanishing viscosity, through difference approximations, by relaxation schemes, etc. The topic for discussion in this chapter is whether solutions may be constructed as limits of sequences of approximate solutions that are only bounded in some $L^p$ space. Since the systems are nonlinear, the difficulty lies in that the construction schemes are generally consistent only when the sequence of approximating solutions converges strongly, whereas the assumed $L^p$ bounds only guarantee weak convergence: Approximate solutions may develop high frequency oscillations of finite amplitude which play havoc with consistency. The aim is to demonstrate that entropy inequalities may save the day by quenching rapid oscillations, thus enforcing strong convergence of the approximating solutions. Some indication of this effect was alluded in Section 1.9.

The principal tools in the investigation will be the notion of Young measure and the functional analytic method of compensated compactness. The former naturally induces the very general class of measure-valued solutions and the latter is employed to verify that nonlinearity reduces measure-valued solutions to traditional ones. As it relies heavily on entropy dissipation, the approach appears to be applicable mainly to systems endowed with a rich family of entropy-entropy flux pairs, most notably the scalar conservation law and systems of just two conservation laws. Despite this limitation, the approach is quite fruitful, not only because of the abundance of important systems with such structure, but also because it provides valuable insight into the stabilizing role of entropy dissipation as well as into the "conflicted" stabilizing-destabilizing behavior of nonlinearity. Different manifestations of these factors were already encountered in earlier chapters.

Out of a host of known applications of the method, only the simplest shall be presented here, pertaining to the scalar conservation law, genuinely nonlinear systems of two conservation laws, and the system of isentropic elasticity and gas dynamics.

## 16.1 The Young Measure

The stumbling block for establishing consistency of construction schemes that generate weakly convergent sequences of approximate solutions lies in that it is not generally permissible to pass weak limits under nonlinear functions. Suppose $\Omega$ is an open subset of $\mathbb{R}^m$ and $\{U_k\}$ is a sequence in $L^\infty(\Omega; \mathbb{R}^n)$ which converges in $L^\infty$ weak* to some limit $\bar{U}$. If $g$ is any continuous real-valued function on $\mathbb{R}^n$, the sequence $\{g(U_k)\}$ will contain subsequences that converge in $L^\infty$ weak*, say to $\bar{g}$, but in general $\bar{g} \neq g(\bar{U})$. It turns out that the limit behavior of such sequences, for all continuous $g$, is encoded in a family $\{\nu_X : X \in \Omega\}$ of probability measures on $\mathbb{R}^n$, which is constructed by the following procedure.

Let $M(\mathbb{R}^n)$ denote the space of bounded Radon measures on $\mathbb{R}^n$, which is isometrically isomorphic to the dual of the space $C(\mathbb{R}^n)$ of bounded continuous functions. With $k = 1, 2, \cdots$ and any $X \in \Omega$, we associate the Dirac mass $\delta_{U_k(X)}$ in $M(\mathbb{R}^n)$, centered at the point $U_k(X)$, and realize the family $\{\delta_{U_k(X)} : X \in \Omega\}$ as an element $\nu_k$ of the space $L^\infty_w(\Omega; M(\mathbb{R}^n))$, which is isometrically isomorphic to the dual of $L^1(\Omega; C(\mathbb{R}^n))$. By virtue of standard weak compactness and separability theorems, there is a subsequence $\{\nu_j\}$ of $\{\nu_k\}$ which converges weakly* to some $\nu \in L^\infty_w(\Omega; M(\mathbb{R}^n))$. Thus, $\nu = \{\nu_X : X \in \Omega\}$ and, as $j \to \infty$,

(16.1.1)

$$\int_\Omega \psi(X, U_j(X))dX = \int_\Omega < \delta_{U_j(X)}, \psi(X, \cdot) > dX \to \int_\Omega < \nu_X, \psi(X, \cdot) > dX,$$

for any $\psi \in C(\Omega \times \mathbb{R}^n)$. The supports of the $\delta_{U_j(X)}$ are uniformly bounded and hence the $\nu_X$ must have compact support. Furthermore, since the $\delta_{U_j(X)}$ are probability measures, so are the $\nu_X$ . In particular, applying (16.1.1) for $\psi(X, U) = \phi(X)g(U)$, where $\phi \in C(\Omega)$ and $g \in C(\mathbb{R}^n)$, we arrive at the following

**16.1.1 Theorem.** *Let $\Omega$ be an open subset of $\mathbb{R}^m$. Then any bounded sequence $\{U_k\}$ in $L^\infty(\Omega; \mathbb{R}^n)$ contains a subsequence $\{U_j\}$, together with a measurable family $\{\nu_X : X \in \Omega\}$ of probability measures with compact support, such that, for any $g \in C(\mathbb{R}^n)$,*

(16.1.2) $$g(U_j) \rightharpoonup \bar{g}, \qquad \text{as } j \to \infty,$$

*in $L^\infty$ weak*, where*

(16.1.3) $$\bar{g}(X) =< \nu_X, g >= \int_{\mathbb{R}^n} g(U)d\nu_X(U).$$

The collection $\{\nu_X : X \in \Omega\}$ constitutes the family of *Young measures* associated with the subsequence $\{U_j\}$. To gain some insight, let us consider the ball $\mathcal{B}_r(X)$ in $\Omega$, with center at some $X \in \Omega$, radius $r$ and measure $|\mathcal{B}_r|$. On account of our construction of $\nu_X$ , it is easy to see that

(16.1.4) $$\nu_X = \lim_{r\downarrow 0}\lim_{j\uparrow\infty} \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r(X)} \delta_{U_j(Y)}dY, \qquad \text{a.e. on } \Omega,$$

where the limits are to be understood in the weak* sense. Notice that the averaged integral on the right-hand side of (16.1.4) may be interpreted as the probability distribution of the values of $U_j(Y)$ as $Y$ is selected uniformly at random from $\mathcal{B}_r(X)$. Thus, according to (16.1.4), $\nu_X$ represents the limiting probability distribution of the values of $U_j$ near $X$.

By virtue of (16.1.2) and (16.1.3), the subsequence $\{U_j\}$ converges, in $L^\infty$ weak*, to the mean $\bar{U} =< \nu_X, U >$ of the Young measures. The limit $\bar{g}$ of $\{g(U_j)\}$ will satisfy $\bar{g} = g(\bar{U})$, for all $g \in C(I\!R^n)$, if and only if $\nu_X$ reduces to the Dirac mass $\delta_{\bar{U}(X)}$ centered at $\bar{U}(X)$. In that case, $\{|U_j|\}$ will converge to $|\bar{U}|$, which implies that $\{U_j\}$ will converge to $\bar{U}$ strongly in $L^p_{\text{loc}}(\Omega)$, for any $1 \le p < \infty$, and some subsequence of $\{U_j\}$ will converge to $\bar{U}$ a.e. on $\Omega$. Hence, to establish strong convergence of $\{U_j\}$, one needs to verify that the support of the Young measure is confined to a point.

Certain applications require more general versions of Theorem 16.1.1. Young measures $\nu_X$ are defined even when the sequence $\{U_k\}$ is merely bounded in some $L^p(\Omega; I\!R^n)$, with $1 < p < \infty$. If $\Omega$ is bounded, the $\nu_X$ are still probability measures and (16.1.2), (16.1.3) hold for all continuous functions $g$ which satisfy a growth condition $|g(U)| \le c(1 + |U|^q)$, for some $0 < q < p$. In that case, convergence in (16.1.2) is weakly in $L^r(\Omega)$, for $1 < r < p/q$. By contrast, when $\Omega$ is unbounded, the $\nu_X$ may have mass less than one, because in the process of constructing them, as one passes to the $j \to \infty$ limit, part of the mass may leak out at infinity.

## 16.2 Compensated Compactness and the div-curl Lemma

The theory of compensated compactness strives to classify bounded (weakly compact) sets in $L^p$ space endowed with additional structure that falls short of (strong) compactness but still manages to render certain nonlinear functions weakly continuous. This is nicely illustrated by means of the following proposition, the celebrated *div-curl lemma*, which commands a surprisingly broad gamut of applications.

**16.2.1 Theorem.** *Given an open subset $\Omega$ of $I\!R^m$, let $\{G_j\}$ and $\{H_j\}$ be sequences of vector fields in $L^2(\Omega; I\!R^m)$ converging weakly to respective limits $\bar{G}$ and $\bar{H}$, as $j \to \infty$. Assume both $\{\operatorname{div} G_j\}$ and $\{\operatorname{curl} H_j\}$ lie in compact subsets of $W^{-1,2}(\Omega)$. Then*

(16.2.1) $$G_j \cdot H_j \to \bar{G}\cdot\bar{H}, \qquad \text{as } j \to \infty,$$

*in the sense of distributions.*

**Proof.** It will suffice to establish (16.2.1) for $\Omega$ bounded. Moreover, on account of $G_j \cdot \bar{H} \to \bar{G} \cdot \bar{H}$, we may assume, without loss of generality, that $\bar{H} = 0$.

Let $\Phi_j \in W_0^{1,2}(\Omega; I\!R^m) \cap W_{\text{loc}}^{2,2}(\Omega; I\!R^m)$ denote the solution of the boundary-value problem $\Delta\Phi_j = H_j$ in $\Omega$, $\Phi_j = 0$ on $\partial\Omega$. Then $\{\Phi_j\}$ converges to zero weakly in $W_{\text{loc}}^{2,2}$, and hence $\{\text{div}\,\Phi_j\}$ converges to zero weakly in $W_{\text{loc}}^{1,2}$. On the other hand, since $\Delta(\text{curl}\,\Phi_j) = \text{curl}\,H_j$, $\{\text{curl}\,\Phi_j\}$ converges to zero strongly in $W_{\text{loc}}^{1,2}$.

We now set

$$V_j = H_j - \text{grad div}\,\Phi_j \tag{16.2.2}$$

and observe that, for $\alpha = 1, \dots, m$,

$$V_{j\alpha} = \sum_{\beta=1}^{m} \partial_\beta(\partial_\beta \Phi_{j\alpha} - \partial_\alpha \Phi_{j\beta}), \tag{16.2.3}$$

so that $\{V_j\}$ converges to zero strongly in $L_{\text{loc}}^2$.

With the help of (16.2.2), we obtain

$$G_j \cdot H_j = G_j \cdot V_j + \text{div}[(\text{div}\,\Phi_j)G_j] - (\text{div}\,\Phi_j)(\text{div}\,G_j). \tag{16.2.4}$$

Each term on the right-hand side of (16.2.4) tends to zero, in the sense of distributions, as $j \to \infty$, and this establishes (16.2.1). The proof is complete.

In the applications, the following technical result is often helpful for verifying the hypotheses of Theorem 16.2.1.

**16.2.2 Lemma.** *Let $\Omega$ be an open subset of $I\!R^m$ and $\{\phi_j\}$ a bounded sequence in $W^{-1,p}(\Omega)$, for some $p > 2$. Furthermore, let $\phi_j = \chi_j + \psi_j$, where $\{\chi_j\}$ lies in a compact set of $W^{-1,2}(\Omega)$, while $\{\psi_j\}$ lies in a bounded set of the space of measures $\mathcal{M}(\Omega)$. Then $\{\phi_j\}$ lies in a compact set of $W^{-1,2}(\Omega)$.*

**Proof.** Consider the (unique) functions $g_j$ and $h_j$ in $W_0^{1,2}(\Omega)$ which solve the equations

$$\Delta g_j = \chi_j\,, \qquad \Delta h_j = \psi_j\,. \tag{16.2.5}$$

By standard elliptic theory, $\{g_j\}$ lies in a compact set of $W_0^{1,2}(\Omega)$ while $\{h_j\}$ lies in a compact set of $W_0^{1,q}(\Omega)$, for $1 < q < \frac{m}{m-1}$. Since $\phi_j = \Delta(g_j + h_j)$, $\{\phi_j\}$ is contained in a compact set of $W^{-1,q}(\Omega)$. But $\{\phi_j\}$ is bounded in $W^{-1,p}(\Omega)$, with $p > 2$, hence, by interpolation between $W^{-1,q}$ and $W^{-1,p}$, it follows that $\{\phi_j\}$ lies in a compact set of $W^{-1,2}(\Omega)$. The proof is complete.

## 16.3 Measure-Valued Solutions for Systems of Conservation Laws and Compensated Compactness

Consider a system of conservation laws,

$$\partial_t U + \partial_x F(U) = 0\,, \tag{16.3.1}$$

and suppose $\{U_k\}$ is a sequence of approximate solutions in an open subset $\Omega$ of $\mathbb{R}^2$, namely

$$\partial_t U_k + \partial_x F(U_k) \to 0\,, \qquad \text{as } k \to \infty\,, \tag{16.3.2}$$

in the sense of distributions on $\Omega$. For example, $\{U_k\}$ may have been derived via the vanishing viscosity approach, that is $U_k = U_{\mu_k}$, with $\mu_k \downarrow 0$ as $k \to \infty$, where $U_\mu$ is the solution of the parabolic system

$$\partial_t U_\mu + \partial_x F(U_\mu) = \mu \partial_x^2 U_\mu\,. \tag{16.3.3}$$

When $\{U_k\}$ lies in a bounded set of $L^\infty(\Omega; \mathbb{R}^n)$, following the discussion in Section 16.1, one may extract a subsequence $\{U_j\}$, associated with a family of Young probability measures $\{\nu_{x,t} : (x,t) \in \Omega\}$ such that $h(U_j) \rightharpoonup \langle \nu, h \rangle$, as $j \to \infty$, in $L^\infty$ weak*, for any continuous $h$. In particular, by account of (16.3.2),

$$\partial_t \langle \nu_{x,t}, U \rangle + \partial_x \langle \nu_{x,t}, F(U) \rangle = 0\,. \tag{16.3.4}$$

One may thus interpret $\nu_{x,t}$ as a new type of weak solution for (16.3.1):

**16.3.1 Definition.** A *measure-valued solution for the system of conservation laws* (16.3.1), *in an open subset* $\Omega$ *of* $\mathbb{R}^2$, *is a measurable family* $\{\nu_{x,t} : (x,t) \in \Omega\}$ *of probability measures which satisfies (16.3.4) in the sense of distributions on* $\Omega$.

Clearly, any traditional weak solution $U \in L^\infty(\Omega; \mathbb{R}^n)$ of (16.3.1) may be identified with the measure-valued solution $\nu_{x,t} = \delta_{U(x,t)}$. However, the class of measure-valued solutions is definitely broader than the class of traditional solutions. For instance, if $U$ and $V$ are any two traditional solutions of (16.3.1) in $L^\infty(\Omega; \mathbb{R}^n)$, then for any fixed $\alpha \in (0, 1)$,

$$\nu_{x,t} = \alpha \delta_{U(x,t)} + (1-\alpha)\delta_{V(x,t)} \tag{16.3.5}$$

defines a nontraditional, measure-valued solution.

At first glance, the notion of measure-valued solution may appear too broad to be relevant. However, abandoning the premise that solutions should assign at each point $(x,t)$ a specific value to the state vector provides the means for describing effectively a class of physical phenomena, such as phase transitions, where at the macroscopic level a mixture of phases may occupy the same point in space-time.

We shall not develop these ideas here, but rather regard measure-valued solutions as stepping stones towards constructing traditional solutions.

The notion of admissibility naturally extends from traditional to measure-valued solutions. The measure-valued solution $\nu_{x,t}$ on $\Omega$ is said to satisfy the *entropy admissibility condition*, relative to the entropy-entropy flux pair $(\eta, q)$ of (16.3.1), if

$$\partial_t < \nu_{x,t}, \eta(U) > + \partial_x < \nu_{x,t}, q(U) > \leq 0, \tag{16.3.6}$$

in the sense of distributions on $\Omega$.

Returning to our earlier example, suppose $\nu_{x,t}$ is generated through a sequence $\{U_{\mu_j}\}$ of solutions to the parabolic system (16.3.3). If $(\eta, q)$ is any entropy-entropy flux pair for (16.3.1), multiplying (16.3.3) by $\mathrm{D}\eta(U_\mu)$ and using (7.4.1) yields the identity

$$\partial_t \eta(U_\mu) + \partial_x q(U_\mu) = \mu \partial_x^2 \eta(U_\mu) - \mu \partial_x U_\mu^\top \mathrm{D}^2 \eta(U_\mu) \partial_x U_\mu \,. \tag{16.3.7}$$

In particular, when $\eta$ is convex the last term on the right-hand side of (16.3.7) is nonpositive. We thus conclude that any measure-valued solution $\nu_{x,t}$ of (16.3.1), constructed by the vanishing viscosity approach relative to (16.3.3), satisfies the entropy admissibility condition (16.3.6), for any entropy-entropy flux pair $(\eta, q)$ with $\eta$ convex.

Lest it be thought that admissibility suffices to reduce measure-valued solutions to traditional ones, it should be noted that when two traditional solutions $U$ and $V$ satisfy the entropy admissibility condition for an entropy-entropy flux pair $(\eta, q)$, then so does also the nontraditional measure-valued solution $\nu_{x,t}$ defined by (16.3.5). On the other hand, admissibility may be an agent for uniqueness and stability in the framework of measure-valued solutions as well. In that direction, it has been shown (references in Section 16.9) that any measure-valued solution $\nu_{x,t}$ of a scalar conservation law, on the upper half-plane, that satisfies the entropy admissibility condition for all convex entropy-entropy flux pairs, and whose initial values are Dirac masses, $\nu_{x,0} = \delta_{u_0(x)}$ for some $u_0 \in L^\infty(-\infty, \infty)$, necessarily reduces to a traditional solution, i.e., $\nu_{x,t} = \delta_{u(x,t)}$, where $u$ is the unique admissible solution of the conservation law with initial data $u(x, 0) = u_0(x)$. In particular, this implies that for scalar conservation laws any measure-valued solution constructed by the vanishing viscosity approach, with traditional initial data, reduces to a traditional solution.

Returning to the system (16.3.1), a program will be outlined for verifying that the measure-valued solution induced by the family of Young measures $\{\nu_{x,t} : (x, t) \in \Omega\}$ associated with a sequence $\{U_j\}$ of approximate solutions reduces to a traditional solution. This program will then be implemented for special systems. As already noted in Section 1.9, when (16.3.1) is hyperbolic, approximate solutions may develop sustained rapid oscillations, which prevent strong convergence of the sequence $\{U_j\}$. Thus, our enterprise is destined to fail, unless the approximate solutions somehow embody a mechanism that quenches oscillations. From the standpoint of the theory of compensated compactness, such a mechanism is manifested in the condition

(16.3.8) $$\partial_t \eta(U_j) + \partial_x q(U_j) \subset \text{ compact set in } W^{-1,2}_{\text{loc}}(\Omega),$$

for any entropy-entropy flux pair $(\eta, q)$ of (16.3.1).

To see the implications of (16.3.8), consider any two entropy-entropy flux pairs $(\eta_1, q_1)$ and $(\eta_2, q_2)$. As $j \to \infty$ , the sequences $\{\eta_1(U_j)\}$, $\{\eta_2(U_j)\}$, $\{q_1(U_j)\}$ and $\{q_2(U_j)\}$ converge to $\bar{\eta}_1 = <\nu, \eta_1>$ , $\bar{\eta}_2 = <\nu, \eta_2>$, $\bar{q}_1 = <\nu, q_1>$ and $\bar{q}_2 = <\nu, q_2>$, respectively, where for brevity we set $\nu_{x,t} = \nu$. By (16.3.8), both $\text{div}(q_2(U_j), \eta_2(U_j))$ and $\text{curl}(\eta_1(U_j), -q_1(U_j))$ lie in compact sets of $W^{-1,2}_{\text{loc}}(\Omega)$. Hence, on account of Theorem 16.2.1,

(16.3.9) $$\eta_1(U_j)q_2(U_j) - \eta_2(U_j)q_1(U_j) \rightharpoonup \bar{\eta}_1\bar{q}_2 - \bar{\eta}_2\bar{q}_1, \text{ as } j \to \infty,$$

in $L^\infty(\Omega)$ weak*, or equivalently

(16.3.10) $$< \nu, \eta_1 >< \nu, q_2 > - < \nu, \eta_2 >< \nu, q_1 > = < \nu, \eta_1 q_2 - \eta_2 q_1 > .$$

The plan is to use (16.3.10), for strategically selected entropy-entropy flux pairs, in order to demonstrate that the support of the Young measure $\nu$ is confined to a single point. Clearly, such a program may have a fair chance for success only when there is flexibility to construct entropy-entropy flux pairs with prescribed specifications. For all practical purposes, this requirement limits the applicability of the method to scalar conservation laws, systems of two conservation laws, and the special class of systems of more than two conservation laws that are endowed with a rich family of entropies (see Section 7.4). On the other hand, the method offers considerable flexibility in regard to construction scheme, as it requires only that the approximate solutions satisfy (16.3.8).

For illustration, let us verify (16.3.8) for the case of a system (16.3.1) endowed with a uniformly convex entropy, $\Omega$ is the upper half-plane, and the sequence $\{U_j\}$ of approximate solutions is generated by the vanishing viscosity approach, $U_j = U_{\mu_j}$ , where $U_\mu$ is the solution of (16.3.3) on the upper half-plane, with initial data

(16.3.11) $$U_\mu(x, 0) = U_{0\mu}(x), \qquad -\infty < x < \infty,$$

lying in a bounded set of $L^\infty(-\infty, \infty) \cap L^2(-\infty, \infty)$.

Let $\eta$ be a uniformly convex entropy, so that $\mathrm{D}^2\eta(U)$ is positive definite. We can assume $0 \le \eta(U) \le c|U|^2$, since otherwise we simply substitute $\eta$ by the entropy $\eta^*(U) = \eta(U) - \eta(0) - \mathrm{D}\eta(0)U$. Upon integrating (16.3.7) over the upper half-plane, we obtain the estimate

(16.3.12) $$\mu \int_0^\infty \int_{-\infty}^\infty |\partial_x U_\mu(x, t)|^2 dxdt \le a,$$

where $a$ is independent of $\mu$.

Consider now any, not necessarily convex, entropy-entropy flux pair $(\eta, q)$, and fix some open bounded subset $\Omega$ of the upper half-plane. Let us examine (16.3.7). The left-hand side is bounded in $W^{-1,p}(\Omega)$, for any $1 \le p < \infty$. The right-hand side is the sum of two terms: By virtue of (16.3.12), the first term tends to zero, as $\mu \downarrow 0$, in $W^{-1,2}(\Omega)$, and thus lies in a compact set of $W^{-1,2}(\Omega)$. The second term lies in a bounded set of $M(\Omega)$, again by account of (16.3.12). Therefore, (16.3.8) follows from Lemma 16.2.2.

## 16.4 Scalar Conservation Laws

Here we shall see how the program outlined in the previous section may be realized in the case of the scalar conversation law

$$\partial_t u + \partial_x f(u) = 0. \tag{16.4.1}$$

**16.4.1 Theorem.** *Let $\Omega$ be an open subset of $I\!R^2$ and $\{u_k(x,t)\}$ a bounded sequence in $L^\infty(\Omega)$ with*

$$\partial_t \eta(u_k) + \partial_x q(u_k) \subset \text{ compact set in } W^{-1,2}_{\text{loc}}(\Omega), \tag{16.4.2}$$

*for any entropy-entropy flux pair of* (16.4.1). *Then there is a subsequence $\{u_j\}$ such that*

$$u_j \rightharpoonup \bar{u}, \qquad f(u_j) \rightharpoonup f(\bar{u}), \qquad \text{as } j \to \infty, \tag{16.4.3}$$

*in $L^\infty$ weak*. Furthermore, if the set of $u$ with $f''(u) \neq 0$ in dense in $I\!R$, then $\{u_j\}$ converges almost everywhere to $\bar{u}$ on $\Omega$.*

**Proof.** By applying Theorem 16.1.1, we extract the subsequence $\{u_j\}$ and the associated family of Young measures $\nu = \nu_{x,t}$ so that $h(u_j) \rightharpoonup < \nu, h >$, for any continuous function $h$. Thus, $u_j \rightharpoonup \bar{u} =< \nu, u >$ and $f(u_j) \rightharpoonup < \nu, f >$. We thus have to show $< \nu, f >= f(\bar{u})$; and that $\nu$ reduces to the Dirac mass when there is no interval on which $f'(u)$ is constant.

We employ (16.3.10) for the particular entropy-entropy flux pairs $(u, f(u))$ and $(f(u), g(u))$, where

$$g(u) = \int_0^u [f'(v)]^2 dv, \tag{16.4.4}$$

to get

$$< \nu, u >< \nu, g > - < \nu, f >< \nu, f > = < \nu, ug - f^2 > . \tag{16.4.5}$$

From Schwarz's inequality,

$$[f(u) - f(\bar{u})]^2 \leq (u - \bar{u})[g(u) - g(\bar{u})], \tag{16.4.6}$$

we deduce

$$< \nu, [f(u) - f(\bar{u})]^2 - (u - \bar{u})[g(u) - g(\bar{u})] > \leq 0. \tag{16.4.7}$$

Upon using (16.4.5), (16.4.7) reduces to

$$[< \nu, f > - f(\bar{u})]^2 \leq 0, \tag{16.4.8}$$

whence $< \nu, f >= f(\bar{u})$. In particular, the left-hand side of (16.4.7) will vanish. Hence, (16.4.6) must hold as an equality for $u$ in the support of $\nu$. However, Schwarz's inequality (16.4.6) may hold as equality only if $f'$ is constant on the interval with endpoints $\bar{u}$ and $u$. When no such interval exists, the support of $\nu$ collapses to a single point and $\nu$ reduces to the Dirac mass $\delta_{\bar{u}}$ . The proof is complete.

As indicated in the previous section, one may generate a sequence $\{u_k\}$ that satisfies the assumptions of Theorem 16.4.1 by the method of vanishing viscosity, setting $u_k = u_{\mu_k}$ , $\mu_k \to 0$ as $k \to \infty$, where $u_\mu$ is the solution of

$$\partial_t u_\mu + \partial_x f(u_\mu) = \mu \partial_x^2 u_\mu \,, \tag{16.4.9}$$

on the upper half-plane, with initial data

$$u_\mu(x,0) = u_{0\mu}(x), \qquad -\infty < x < \infty, \tag{16.4.10}$$

that are uniformly bounded in $L^\infty(-\infty,\infty) \cap L^2(-\infty,\infty)$. Indeed, the resulting $\{u_k\}$ will be bounded in $L^\infty$, since $\|u_\mu\|_{L^\infty} \leq \|u_{0\mu}\|_{L^\infty}$ by the maximum principle. Moreover, (16.4.2) will hold for all entropy-entropy flux pairs $(\eta, q)$, by the general argument of Section 16.3, which applies here, in particular, because (16.4.1) possesses the uniformly convex entropy $u^2$. Finally, $\mu \partial_x^2 u_\mu \to 0$, as $\mu \downarrow 0$, in the sense of distributions. We thus arrive at the following

**16.4.2 Theorem.** *Suppose* $u_{0\mu} \rightharpoonup u_0$ , *as* $\mu \downarrow 0$, *in* $L^\infty(-\infty,\infty)$ *weak**. *Then there is a sequence* $\{\mu_j\}$, $\mu_j \to 0$ *as* $j \to \infty$, *such that the sequence* $\{u_{\mu_j}\}$ *of solutions of* (16.4.9), (16.4.10) *converges in* $L^\infty$ *weak* to some function* $\bar{u}$, *which is a solution of* (16.4.1), *on the upper half-plane, with initial value* $\bar{u}(x,0) = u_0(x)$ *on* $(-\infty,\infty)$. *Furthermore, if the set of* $u$ *with* $f''(u) \neq 0$ *is dense in* $I\!R$, *then* $\{u_{\mu_j}\}$, *or a subsequence thereof, converges almost everywhere to* $\bar{u}$ *on the upper half-plane.*

## 16.5 A Relaxation Scheme for Scalar Conservation Laws

The aim here is to pass to the limit, as $\mu \downarrow 0$, in the system (5.2.18), with the help of the theory of compensated compactness. Such an exercise may serve a dual purpose: For the case one is interested in (5.2.18) itself, as a model for some physical process, it will demonstrate relaxation to local equilibrium governed by the scalar conservation law (16.4.1). As a byproduct, it will establish that solutions to the Cauchy problem for (16.4.1) exist, and will suggest a method for computing them. For the latter purpose, it shall be advantageous to make the non-relaxed system (5.2.18) as simple as possible, namely semilinear,

$$\begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \\ \partial_t v(x,t) + a^2 \partial_x u(x,t) + \frac{1}{\mu}[v(x,t) - f(u(x,t))] = 0, \end{cases} \tag{16.5.1}$$

where $a$ is some positive constant. In order to simplify the analysis, we shall deal here with this semilinear system. At this point it may be helpful for the reader to review the introduction to relaxation theory presented in Section 5.2.

We assume $f'$ is bounded and select $a$ sufficiently large so that the strict subcharacteristic condition (recall (5.2.29))

$$-a + \delta < f'(u) < a - \delta, \qquad u \in (-\infty, \infty), \tag{16.5.2}$$

holds, for some $\delta > 0$. We normalize $v$ by postulating $f(0) = 0$.

Entropy-entropy flux pairs $(\eta(u,v), q(u,v))$ for (16.5.1) satisfy the linear hyperbolic system

$$\begin{cases} q_u(u,v) - a^2\eta_v(u,v) = 0 \\ q_v(u,v) - \eta_u(u,v) = 0, \end{cases} \tag{16.5.3}$$

with general solution

$$\begin{cases} \eta(u,v) = r(au+v) + s(au-v) \\ q(u,v) = ar(au+v) - as(au-v). \end{cases} \tag{16.5.4}$$

The subcharacteristic condition (16.5.2) implies that the curve $v = f(u)$ is nowhere characteristic for the system (16.5.3), and hence, given any entropy-entropy flux pair $(\hat{\eta}(u), \hat{q}(u))$ for the scalar conservation law (16.4.1), one may construct an entropy-entropy flux pair $(\eta(u,v), q(u,v))$ for (16.5.1) with Cauchy data

$$\eta(u, f(u)) = \hat{\eta}(u), \qquad q(u, f(u)) = \hat{q}(u), \qquad u \in (-\infty, \infty). \tag{16.5.5}$$

Differentiating (16.5.5) with respect to $u$ and using that $\hat{q}'(u) = \hat{\eta}'(u)f'(u)$, together with (16.5.3) and (16.5.2), we deduce that $\eta_v(u, f(u)) = 0$. This, in turn, combined with (16.5.5) and (16.5.4) yields

$$r'(au + f(u)) = s'(au - f(u)) = \frac{1}{2a}\hat{\eta}'(u), \tag{16.5.6}$$

whence one determines $r$ and $s$ on $\mathbb{R}$, and thereby $\eta$ and $q$ on $\mathbb{R}^2$. In particular, $\hat{\eta}'' \geq 0$ on $\mathbb{R}$ implies $r'' \geq 0$, $s'' \geq 0$ on $\mathbb{R}$, and hence $\eta_{vv} \geq 0$ on $\mathbb{R}^2$. Since $\eta_v(u, f(u)) = 0$, we then conclude that the dissipativeness condition (5.2.4) holds:

$$\eta_v(u,v)[v - f(u)] \geq 0, \qquad (u,v) \in \mathbb{R}^2. \tag{16.5.7}$$

Under the stronger hypothesis $\hat{\eta}''(u) \geq \beta > 0,\ u \in \mathbb{R}$, (16.5.7) becomes stricter:

$$\eta_v(u,v)[v - f(u)] \geq \gamma |v - f(u)|^2, \qquad (u,v) \in \mathbb{R}^2, \tag{16.5.8}$$

with $\gamma > 0$.

We have now laid the groundwork for establishing the existence of solutions to the Cauchy problem for (16.5.1) and for passing to the limit, as $\mu \downarrow 0$.

**16.5.1 Theorem.** *Under the subcharacteristic condition* (16.5.2)*, the Cauchy problem for the system* (16.5.1)*, with initial data*

$$(16.5.9) \qquad (u_\mu(x,0), v_\mu(x,0)) = (u_{0\mu}(x), v_{0\mu}(x)), \qquad -\infty < x < \infty,$$

*in* $L^\infty(-\infty,\infty) \cap L^2(-\infty,\infty)$*, possesses a bounded (weak) solution* $(u_\mu, v_\mu)$ *on the upper half-plane. Furthermore,*

$$(16.5.10) \qquad \frac{1}{\mu}\int_0^\infty \int_{-\infty}^\infty [v_\mu - f(u_\mu)]^2 dxdt \le b \int_{-\infty}^\infty [u_{0\mu}^2(x) + v_{0\mu}^2(x)]dx,$$

*where b is independent of* $\mu$.

**Proof.** Since (16.5.1) is semilinear hyperbolic, a local solution $(u_\mu, v_\mu)$ exists and may be continued for as long as it remains bounded in $L^\infty$. Furthermore, if $(\eta, q)$ is any entropy-entropy flux pair,

$$(16.5.11) \qquad \partial_t \eta(u_\mu, v_\mu) + \partial_x q(u_\mu, v_\mu) + \frac{1}{\mu}\eta_v(u_\mu, v_\mu)[v_\mu - f(u_\mu)] = 0.$$

We construct the entropy-entropy flux pair $(\eta_m, q_m)$, induced by (16.5.5), with $\hat{\eta}(u) = |u|^m$, $m = 2, 3, \dots$, and normalized by $\eta_m(0,0) = 0$, $q_m(0,0) = 0$. Notice that, necessarily, the first derivatives of $\eta_m$ also vanish at the origin. We integrate (16.5.11) over $(-\infty,\infty) \times [0,t]$ and use (16.5.7) to get

$$(16.5.12) \qquad \int_{-\infty}^\infty \eta_m(u_\mu(x,t), v_\mu(x,t))dx \le \int_{-\infty}^\infty \eta_m(u_{0\mu}(x), v_{0\mu}(x))dx.$$

By (16.5.6) and (16.5.2), it follows easily that $(\hat{c}|w|)^m \le r_m(w) \le (\hat{C}|w|)^m$ and $(\hat{c}|w|)^m \le s_m(w) \le (\hat{C}|w|)^m$, whence

$$(16.5.13) \qquad c^m(|u|^m + |v|^m) \le \eta_m(u,v) \le C^m(|u|^m + |v|^m), \qquad (u,v) \in I\!R^2.$$

Therefore, raising (16.5.12) to the power $\frac{1}{m}$ and letting $m \to \infty$ we conclude that $\|u_\mu(\cdot,t)\|_{L^\infty(-\infty,\infty)}$ and $\|v_\mu(\cdot,t)\|_{L^\infty(-\infty,\infty)}$ are bounded in terms of $\|u_{0\mu}(\cdot)\|_{L^\infty(-\infty,\infty)}$ and $\|v_{0\mu}(\cdot)\|_{L^\infty(-\infty,\infty)}$, uniformly in $t$ and $\mu$. Thus the solution $(u_\mu, v_\mu)$ exists on the entire upper half-plane.

Next we write (16.5.11) for the entropy-entropy flux pair $(\eta_2, q_2)$, and integrate it over $(-\infty,\infty) \times [0,\infty)$. For this case, the stronger dissipativeness inequality (16.5.8) applies and thus we deduce (16.5.10). The proof is complete.

**16.5.2 Theorem.** *Consider the family* $\{(u_\mu, v_\mu)\}$ *of solutions of the Cauchy problem* (16.5.1), (16.5.9)*, where* $\{(u_{0\mu}, v_{0\mu})\}$ *is bounded in* $L^\infty(-\infty,\infty) \cap L^2(-\infty,\infty)$ *and* $u_{0\mu} \to u_0$*, as* $\mu \downarrow 0$*, in* $L^\infty$ *weak*. Then there is a sequence* $\{\mu_j\}$*, with* $\mu_j \downarrow 0$ *as* $j \to \infty$*, such that* $\{(u_{\mu_j}, v_{\mu_j})\}$ *converges, in* $L^\infty$ *weak*, to* $(\bar{u}, f(\bar{u}))$*, where* $\bar{u}$ *is a solution of* (16.4.1)*, on the upper half-plane, with initial value* $\bar{u}(x,0) = u_0(x)$ *on* $(-\infty,\infty)$*. Furthermore, if the set of u with* $f''(u) \neq 0$ *is dense in* $I\!R$*, then* $\{(u_{\mu_j}, v_{\mu_j})\}$ *converges to* $(\bar{u}, f(\bar{u}))$*, almost everywhere on the upper half-plane.*

**Proof.** By Theorem 16.5.1, $\{(u_\mu, v_\mu)\}$ is contained in a bounded set of the space $L^\infty((-\infty,\infty) \times [0,\infty))$.

We fix any entropy-entropy flux pair $(\hat{\eta}, \hat{q})$ for (16.4.1), consider the entropy-entropy flux pair $(\eta, q)$ for (16.5.1) generated by solving the Cauchy problem (16.5.3), (16.5.5), and use (16.5.11) to write

$$\begin{aligned}(16.5.14)\quad & \partial_t \hat{\eta}(u_\mu) + \partial_x \hat{q}(u_\mu) \\ & = \partial_t[\eta(u_\mu, f(u_\mu)) - \eta(u_\mu, v_\mu)] + \partial_x[q(u_\mu, f(u_\mu)) - q(u_\mu, v_\mu)] \\ & -\frac{1}{\mu}\eta_v(u_\mu, v_\mu)[v_\mu - f(u_\mu)].\end{aligned}$$

By virtue of (16.5.10), both $\eta(u_\mu, f(u_\mu)) - \eta(u_\mu, v_\mu)$ and $q(u_\mu, f(u_\mu)) - q(u_\mu, v_\mu)$ tend to zero in $L^2$, as $\mu \downarrow 0$. Therefore, the first two terms on the right-hand side of (16.5.14) tend to zero in $W^{-1,2}$, as $\mu \downarrow 0$. On the other hand, the third term lies in a bounded set of $L^1$, again on account of (16.5.10), recalling that $\eta_v(u, f(u)) = 0$.

We now fix any sequence $\{\mu_k\}$, with $\mu_k \downarrow 0$ as $k \to \infty$, and set $(u_k, v_k) = (u_{\mu_k}, v_{\mu_k})$. In virtue of the above, Lemma 16.2.2 implies that (16.4.2) holds for any entropy-entropy flux pair $(\hat{\eta}, \hat{q})$ of (16.4.1), where $\Omega$ is the upper half-plane. Theorem 16.4.1 then yields (16.4.3), for some subsequence $\{u_j\}$. In turn, (16.4.3) together with (16.5.10) imply $v_j \to f(\bar{u})$, in $L^\infty$ weak*. In particular, $\bar{u}$ is a solution of (16.4.1), with initial values $u_0$, because of $(16.5.1)_1$.

When the set of $u$ with $f''(u) \neq 0$ is dense in $\mathbb{R}$, $\{u_j\}$ converges to $\bar{u}$ almost everywhere, on account of Theorem 16.4.1. It then follows from (16.5.10) that, likewise, $\{v_j\}$ converges to $f(\bar{u})$ almost everywhere. The proof is complete.

By combining (16.5.11), (16.5.7), (16.5.10) and (16.5.5), we infer that, at least in the case where $\{u_j\}$ converges almost everywhere, the limit $\bar{u}$ will satisfy the entropy admissibility condition, for any entropy-entropy flux pair $(\hat{\eta}, \hat{q})$, with $\hat{\eta}$ convex.

Notice that Theorem 16.5.2 places no restriction on the initial values $v_{0\mu}$ of $v_\mu$, save for the requirement that they be bounded. In particular, $v_{0\mu}$ may lie far apart from its local equilibrium value $f(u_{0\mu})$. In that situation $v_k$ must develop a boundary layer across $t = 0$.

The reader should be warned that compensated compactness is not the most efficient method for handling the simple system (16.5.1). Indeed, it has been shown (references in Section 16.9) that if $(u_\mu, v_\mu)$ and $(\bar{u}_\mu, \bar{v}_\mu)$ is any pair of solutions of (16.5.1), with corresponding initial values $(u_0, v_0)$ and $(\bar{u}_0, \bar{v}_0)$, then

$$\begin{aligned}(16.5.15)\quad & \int_{-\ell}^{\ell} \{|u_\mu(x,t) - \bar{u}_\mu(x,t)| + |v_\mu(x,t) - \bar{v}_\mu(x,t)|\} dx \\ & \leq \frac{(1+a)^2}{a} \int_{-\ell-at}^{\ell+at} \{|u_0(x) - \bar{u}_0(x)| + |v_0(x) - \bar{v}_0(x)|\} dx\end{aligned}$$

holds, for any $\ell > 0$ and $t > 0$. Armed with this estimate, one may easily establish compactness in $L^1$ as well as in $BV$, and then pass to the $\mu \downarrow 0$ relaxation limit. Nevertheless, at the time of this writing, compensated compactness is the only approach that works for the nonlinear system (5.2.18), because no analog to the estimate (16.5.15) is currently known for that case.

## 16.6 Genuinely Nonlinear Systems of Two Conservation Laws

The program outlined in Section 16.3 will here be implemented for genuinely nonlinear systems (16.3.1) of two conservation laws. In particular, our system will be endowed with a coordinate system of Riemann invariants $(z, w)$, normalized as in (12.1.2), and the condition of genuine nonlinearity will be expressed by (12.1.3), namely $\lambda_z < 0$ and $\mu_w > 0$. Moreover, the system will be equipped with a rich family of entropy-entropy flux pairs, including the Lax pairs constructed in Section 12.2, which will play a pivotal role in the analysis.

We show that the entropy conditions, in conjunction with genuine nonlinearity, quench rapid oscillations:

**16.6.1 Theorem.** *Let $\Omega$ be an open subset of $\mathbb{R}^2$ and $\{U_k(x,t)\}$ a bounded sequence in $L^\infty(\Omega; \mathbb{R}^2)$ with*

$$\partial_t \eta(U_k) + \partial_x q(U_k) \subset \textit{ compact set in } W^{-1,2}_{\mathrm{loc}}(\Omega), \tag{16.6.1}$$

*for any entropy-entropy flux pair $(\eta, q)$ of* (16.3.1). *Then there is a subsequence $\{U_j\}$ which converges almost everywhere on $\Omega$.*

**Proof.** By applying Theorem 16.1.1, we extract a subsequence $\{U_j\}$ and identify the associated family of Young measures $\nu_{x,t}$. We have to show that, for almost all $(x, t)$, the support of $\nu_{x,t}$ is confined to a single point and so this measure reduces to the Dirac mass. It will be expedient to monitor the Young measure on the plane of the Riemann invariants $(z, w)$, rather than in the original state space.

We thus let $\nu$ denote the Young measure at any fixed point $(x, t) \in \Omega$, relative to the $(z, w)$ variables, and consider the smallest rectangle $\mathcal{R} = [z^-, z^+] \times [w^-, w^+]$ that contains the support of $\nu$. We need to show $z^- = z^+$ and $w^- = w^+$. Arguing by contradiction, assume $z^- < z^+$.

We consider the Lax entropy-entropy flux pairs (12.2.5), which will be here labeled $(\eta_k, q_k)$, so as to display explicitly the dependence on the parameter $k$. We shall use the $\eta_k$ as weights for redistributing the mass of $\nu$, reallocating it near the boundary of $\mathcal{R}$. To that end, with each large positive integer $k$ we associate probability measures $\nu_k^\pm$ on $\mathcal{R}$, defined through their action on continuous functions $h(z, w)$:

$$< \nu_k^\pm, h > = \frac{< \nu, h\eta_{\pm k} >}{< \nu, \eta_{\pm k} >}. \tag{16.6.2}$$

Because of the factor $e^{kz}$ in the definition of $\eta_k$, the measure $\nu_k^-$ (or $\nu_k^+$) is concentrated near the left (or right) side of $\mathcal{R}$. As $k \to \infty$, the sequences $\{\nu_k^-\}$ and

$\{\nu_k^+\}$, or subsequences thereof, will converge, weakly* in the space of measures, to probability measures $\nu^-$ and $\nu^+$, which are respectively supported by the left edge $[z^-] \times [w^-, w^+]$ and the right edge $[z^+] \times [w^-, w^+]$ of $\mathcal{R}$.

We apply (16.3.10) for any fixed entropy-entropy flux pair $(\eta, q)$ and the Lax pairs $(\eta_{\pm k}, q_{\pm k})$ to get

$$< \nu, q > - \frac{< \nu, q_{\pm k} >}{< \nu, \eta_{\pm k} >} < \nu, \eta > = \frac{< \nu, \eta_{\pm k} q - \eta q_{\pm k} >}{< \nu, \eta_{\pm k} >} . \tag{16.6.3}$$

From (12.2.5) and (12.2.7) we infer

$$q_{\pm k} = [\lambda + O(\frac{1}{k})]\eta_{\pm k} . \tag{16.6.4}$$

Therefore, letting $k \to \infty$ in (16.6.3) yields

$$< \nu, q > - < \nu^{\pm}, \lambda >< \nu, \eta > = < \nu^{\pm}, q - \lambda\eta > . \tag{16.6.5}$$

Next, we apply (16.3.10) for the Lax pairs $(\eta_{-k}, q_{-k})$ and $(\eta_k, q_k)$, thus obtaining

$$\frac{< \nu, q_k >}{< \nu, \eta_k >} - \frac{< \nu, q_{-k} >}{< \nu, \eta_{-k} >} = \frac{< \nu, \eta_{-k} q_k - \eta_k q_{-k} >}{< \nu, \eta_{-k} >< \nu, \eta_k >} . \tag{16.6.6}$$

By (16.6.4), the left-hand side of (16.6.6) tends to $< \nu^+, \lambda > - < \nu^-, \lambda >$, as $k \to \infty$. On the other hand, the right-hand side tends to zero, because the numerator is $O(k^{-1})$ while

$$< \nu, \eta_{\pm k} > \ \geq \ c \exp[\pm\tfrac{1}{2}(z^- + z^+)]. \tag{16.6.7}$$

Hence,

$$< \nu^-, \lambda > = < \nu^+, \lambda > . \tag{16.6.8}$$

Combining (16.6.5) with (16.6.8),

$$< \nu^-, q - \lambda\eta > = < \nu^+, q - \lambda\eta > . \tag{16.6.9}$$

We apply (16.6.9) for $(\eta, q) = (\eta_k, q_k)$. On account of (12.2.12), for $k$ large,

$$\begin{cases} < \nu^-, q_k - \lambda\eta_k > \ \leq \ C\dfrac{1}{k} \exp(kz^-) \\ \\ < \nu^+, q_k - \lambda\eta_k > \ \geq \ c\dfrac{1}{k} \exp(kz^+), \end{cases} \tag{16.6.10}$$

which yields the desired contradiction to $z^- < z^+$. Similarly one shows $w^- = w^+$, so that $\mathcal{R}$ collapses to a single point. The proof is complete.

The stumbling block in employing the above theorem for constructing solutions to our system (16.3.1) is that, at the time of this writing, it has not been established that sequences of approximate solutions, produced by any of the available schemes, are bounded in $L^\infty$. Thus, boundedness has to be imposed as an extraneous (and annoying) assumption. On the other hand, once boundedness is taken for granted, it is not difficult to verify the other requirement of Theorem 16.6.1, namely (16.6.1). In particular, when the sequence of $U_k$ is generated via the vanishing viscosity approach, as solutions of the parabolic system (16.3.3), condition (16.6.1) follows directly from the discussion in Section 16.3, because genuinely nonlinear systems of two conservation laws are always endowed with uniformly convex entropies. For example, as shown in Section 12.2, under the normalization condition (12.1.4), the Lax entropy $\eta_k$ is convex, for $k$ sufficiently large. We thus have

**16.6.2 Theorem.** *For $\mu > 0$, let $U_\mu$ denote the solution on the upper half-plane of the genuinely nonlinear parabolic system of two conservation laws* (16.3.3) *with initial data* (16.3.11), *where $U_{0\mu} \rightharpoonup U_0$ in $L^\infty(-\infty,\infty)$ weak*, as $\mu \downarrow 0$. Suppose the family $\{U_\mu\}$ lies in a bounded subset of $L^\infty$. Then, there is a sequence $\{\mu_j\}$, $\mu_j \to 0$ as $j \to \infty$, such that $\{U_{\mu_j}\}$ converges, almost everywhere on the upper half-plane, to a solution $\bar{U}$ of* (16.3.1) *with initial value $\bar{U}(x,0) = U_0(x)$ , $-\infty < x < \infty$.*

One obtains entirely analogous results for sequences of approximate solutions generated by a class of one-step difference schemes with a three-point domain of dependence:

$$U(x,t+\Delta t) - U(x,t) = \frac{\alpha}{2}G(U(x,t),U(x+\Delta x,t)) - \frac{\alpha}{2}G(U(x-\Delta x,t),U(x,t)), \tag{16.6.11}$$

where $\alpha = \Delta t/\Delta x$ is the ratio of mesh-lengths and $G$, possibly depending on $\alpha$, is a function which satisfies the consistency condition $G(U,U) = F(U)$. The class includes the *Lax-Friedrichs scheme*, with

$$G(V,W) = \frac{1}{2}[F(V)+F(W)] + \frac{1}{\alpha}(V-W), \tag{16.6.12}$$

and also the *Godunov scheme*, where $G(V,W)$ denotes the state in the wake of the solution to the Riemann problem for (16.3.1), with left state $V$ and right state $W$. The condition of uniform boundedness on $L^\infty$ of the approximate solutions has to be extraneously imposed in these cases as well.

## 16.7 The System of Isentropic Elasticity

The assertion of Theorem 16.6.1 is obviously false when the system (16.3.1) is linear. On the other hand, genuine nonlinearity is far too strong a restriction: It may be allowed to fail along a finite collection of curves in state space, so long as these curves intersect transversely the level curves of the Riemann invariants. This will be demonstrated here in the context of the system (7.1.8) of conservation laws of one-dimensional, isentropic thermoelasticity,

$$(16.7.1)\qquad \begin{cases} \partial_t u - \partial_x v = 0 \\ \\ \partial_t v - \partial_x \sigma(u) = 0, \end{cases}$$

under the assumption $\sigma''(u) \neq 0$ for $u \neq 0$, but $\sigma''(0) = 0$, so that genuine nonlinearity fails along the line $u = 0$ in state space. Nevertheless, the analog of Theorem 16.6.1 still holds:

**16.7.1 Theorem.** *Let $\Omega$ be an open subset of $\mathbb{R}^2$ and $\{(u_k, v_k)\}$ a bounded sequence in $L^\infty(\Omega; \mathbb{R}^2)$ with*

$$(16.7.2)\qquad \partial_t \eta(u_k, v_k) + \partial_x q(u_k, v_k) \subset \textit{ compact set in } W^{-1,2}_{\mathrm{loc}}(\Omega),$$

*for any entropy-entropy flux pair $(\eta, q)$ of* (16.7.1). *Then there is a subsequence $\{(u_j, v_j)\}$ which converges almost everywhere on $\Omega$.*

**Proof.** As in the proof of Theorem 16.6.1, we extract a subsequence $\{(u_j, v_j)\}$ and identify the associated family of Young measures $\nu_{x,t}$. We fix $(x, t)$ in $\Omega$ and monitor the Young measure $\nu$ at $(x, t)$ relative to the Riemann invariants

$$(16.7.3)\qquad z = \int_0^u [\sigma'(\omega)]^{\frac{1}{2}} d\omega + v, \qquad w = -\int_0^u [\sigma'(\omega)]^{\frac{1}{2}} d\omega + v.$$

We need to show that the smallest rectangle $\mathcal{R} = [z^-, z^+] \times [w^-, w^+]$ that contains the support of $\nu$ collapses to a single point.

By retracing the steps in the proof of Theorem 16.6.1, that do not depend on the genuine nonlinearity of the system, we rederive (16.6.9). The remainder of the argument will depend on the relative positions of $\mathcal{R}$ and the straight line $z = w$ along which genuine nonlinearity fails.

Suppose first the line $z = w$ does not intersect the right edge of $\mathcal{R}$, that is, $z^+ \notin [w^-, w^+]$. In that case, (16.6.10) are still in force, yielding $z^- = z^+$. Hence $\mathcal{R}$ collapses to $[z^+] \times [w^-, w^+]$, which, according to our assumption, lies entirely in the genuinely nonlinear region and so by the familiar argument $w^- = w^+$, verifying the assertion of the theorem. Similar arguments apply when the line $z = w$ misses any one of the other three edges of $\mathcal{R}$.

It thus remains to examine the case where the line $z = w$ intersects all four edges of $\mathcal{R}$, i.e. $z^- = w^-$ and $z^+ = w^+$. Even in that situation, by virtue of (12.2.12), $q_k - \lambda\eta_k$ does not change sign along $[z^-] \times [w^- + \varepsilon, w^+]$ and $[z^+] \times [w^-, w^+ - \varepsilon]$, so the familiar argument still goes through, showing $z^- = z^+$, unless the measures $\nu^-$ and $\nu^+$ are respectively concentrated in the vertices $(z^-, w^-)$ and $(z^+, w^+)$. When that happens, (16.6.9) reduces to

$$(16.7.4)\quad q(z^-, w^-) - \lambda(z^-, w^-)\eta(z^-, w^-) = q(z^+, w^+) - \lambda(z^+, w^+)\eta(z^+, w^+).$$

In particular, let us apply (16.7.4) for the trivial entropy-entropy flux pair $(u, -v)$. At the "southwestern" vertex, $u^- = 0$ and $v^- = z^- = w^-$, while at the "northeastern" vertex, $u^+ = 0$ and $v^+ = z^+ = w^+$. Hence, (16.7.4) yields $z^- = z^+ = w^- = w^+$. The proof is complete.

Smoothness of $\sigma(u)$ cannot be generally relaxed as examples indicate that the assertion of the above proposition may break down when $\sigma''(u)$ is discontinuous at $u = 0$.

In particular, Theorem 16.7.1 applies when the elastic medium responds like a "hard spring", that is, $\sigma$ is concave at $u < 0$ and convex at $u > 0$:

$$u\sigma''(u) > 0, \qquad u \neq 0. \tag{16.7.5}$$

For that case, it is possible to establish $L^\infty$ bounds on the approximate solutions constructed by the vanishing viscosity method, namely, as solutions to a Cauchy problem

$$\begin{cases} \partial_t u_\mu - \partial_x v_\mu = \mu \partial_x^2 u_\mu \\ \partial_t v_\mu - \partial_x \sigma(u_\mu) = \mu \partial_x^2 v_\mu \,, \end{cases} \tag{16.7.6}$$

$$(u_\mu(x,0), v_\mu(x,0)) = (u_{0\mu}(x), v_{0\mu}(x)), \qquad -\infty < x < \infty. \tag{16.7.7}$$

**16.7.2 Theorem.** *Under the assumption* (16.7.5), *for any $M > 0$, the set $\mathcal{U}_M$, defined by*

$$\mathcal{U}_M = \{(u,v) : -M \le z(u,v) \le M, \ \ -M \le w(u,v) \le M\}, \tag{16.7.8}$$

*where $z$ and $w$ are the Riemann invariants* (16.7.3) *of* (16.7.1), *is a (positively) invariant region for solutions of* (16.7.6), (16.7.7).

**Proof.** The standard proof is based on the maximum principle. An alternative proof will be presented here, which relies on entropies and thus is closer to the spirit of the hyperbolic theory. It has the advantage of requiring less regularity for solutions of (16.7.6). Moreover, it readily extends to any other approximation scheme, which, like (16.7.6), is dissipative under convex entropies of (16.7.1).

For the system (16.7.1), the equations (7.4.1) that determine entropy-entropy flux pairs $(\eta, q)$ reduce to

$$\begin{cases} q_u(u,v) = -\sigma'(u)\eta_v(u,v) \\ q_v(u,v) = -\eta_u(u,v). \end{cases} \tag{16.7.9}$$

Notice that (16.7.9) admits the family of solutions

$$\eta_m(u,v) = Y_m(u)\cosh(mv) - 1, \tag{16.7.10}$$

$$q_m(u,v) = -\frac{1}{m} Y_m'(u)\sinh(mv), \tag{16.7.11}$$

where $m = 1, 2, \cdots$ and $Y_m$ is the solution of the ordinary differential equation

$$Y_m''(u) = m^2\sigma'(u)Y_m(u), \qquad -\infty < u < \infty, \tag{16.7.12}$$

with initial conditions

$$Y_m(0) = 1, \qquad Y_m'(0) = 0. \tag{16.7.13}$$

A simple calculation gives

$$\eta_{muu}\eta_{mvv} - \eta_{muv}^2 \geq m^2[m^2\sigma' Y_m^2 - Y_m'^2]\,. \tag{16.7.14}$$

Moreover, by virtue of (16.7.12),

$$[m^2\sigma' Y_m^2 - Y_m'^2]' = m^2\sigma'' Y_m^2\,. \tag{16.7.15}$$

Consequently, (16.7.5) implies that the right-hand side of (16.7.14) is positive and hence $\eta_m(u, v)$ is a convex function on $\mathbb{R}^2$. Furthermore, $\eta_m(0, 0) = 0$ and $\eta_{mu}(0, 0) = \eta_{mv}(0, 0) = 0$, so that $\eta_m(u, v)$ is positive definite.

Next we examine the asymptotics of $\eta_m(u, v)$ as $m \to \infty$. The change of variables $(u, Y_m) \mapsto (\xi, X_m)$:

$$\xi = \int_0^u [\sigma'(\omega)]^{\frac{1}{2}} d\omega, \tag{16.7.16}$$

$$X_m = (\sigma')^{\frac{1}{4}} Y_m\,, \tag{16.7.17}$$

transforms (16.7.12) into

$$\ddot{X}_m = m^2 X_m + [\tfrac{1}{4}(\sigma')^{-2}\sigma''' - \tfrac{5}{16}(\sigma')^{-3}(\sigma'')^2]X_m\,, \tag{16.7.18}$$

with asymptotics, derived by the variation of parameters formula,

$$X_m(\xi) = \left[\sigma'(0)^{\frac{1}{4}} + O\left(\frac{1}{m}\right)\right]\cosh(m\xi), \tag{16.7.19}$$

as $m \to \infty$, and for $\xi$ confined in any fixed bounded interval.

Upon combining (16.7.10) with (16.7.17), (16.7.19), (16.7.16) and (16.7.3), we deduce

$$\lim_{m\to\infty} \eta_m(u, v)^{\frac{1}{m}} = \begin{cases} \exp[z(u, v)], & \text{if } u > 0\,,\ v > 0, \\ \exp[w(u, v)], & \text{if } u < 0\,,\ v > 0, \\ \exp[-w(u, v)], & \text{if } u > 0\,,\ v < 0, \\ \exp[-z(u, v)], & \text{if } u < 0\,,\ v < 0. \end{cases} \tag{16.7.20}$$

We now consider the solution $(u_\mu, v_\mu)$ of (16.7.6), (16.7.7), where $(u_{0\mu}, v_{0\mu})$ lie in $L^2(-\infty,\infty)$ and take values in the region $\mathcal{U}_M$, defined by (16.7.8). We write (16.3.7), with $U_\mu = (u_\mu, v_\mu)$, $\eta = \eta_m$, $q = q_m$, and integrate it over the strip $(-\infty,\infty) \times [0, t]$, to get

$$\int_{-\infty}^{\infty} \eta_m(u_\mu(x,t), v_\mu(x,t))dx \leq \int_{-\infty}^{\infty} \eta_m(u_{0\mu}(x), v_{0\mu}(x))dx. \tag{16.7.21}$$

Raising (16.7.21) to the power $1/m$, letting $m \to \infty$ and using (16.7.20), we conclude that $(u_\mu(\cdot,t), v_\mu(\cdot,t))$ takes values in the region $\mathcal{U}_M$. The proof is complete.

The above proposition, in conjunction with Theorem 16.7.1, yields an existence theorem for the system (16.7.1), which is free from extraneous assumptions:

**16.7.3 Theorem.** *Let* $(u_\mu, v_\mu)$ *be the solution of the initial-value problem* (16.7.6), (16.7.7), *on the upper half-plane, where* $(u_{0\mu}, v_{0\mu}) \rightharpoonup (u_0, v_0)$ *in* $L^\infty(-\infty,\infty)$ *weak*. Under the condition* (16.7.5), *there is a sequence* $\{\mu_j\}$, $\mu_j \to 0$ *as* $j \to \infty$, *such that* $\{(u_{\mu_j}, v_{\mu_j})\}$ *converges almost everywhere on the upper half-plane to a solution* $(\bar{u}, \bar{v})$ *of* (16.7.1) *with initial values* $(\bar{u}(x,0), \bar{v}(x,0)) = (u_0(x), v_0(x))$, *for* $-\infty < x < \infty$.

The assumption (16.7.5) and the use of the special, artifical viscosity (16.7.6) are essential in the proof of Theorem 16.7.3, because they appear to be indispensable for establishing uniform $L^\infty$ bounds on approximate solutions. At the same time, it is interesting to know whether one may construct solutions to (16.7.1) by passing to the zero viscosity limit in the system (8.6.3) of viscoelasticity, or at least in the model system

$$\begin{cases} \partial_t u_\mu - \partial_x v_\mu = 0 \\ \partial_t v_\mu - \partial_x \sigma(u_\mu) = \mu \partial_x^2 v_\mu \,, \end{cases} \tag{16.7.22}$$

which is close to it.

Even though we do not have uniform $L^\infty$ estimates for solutions of (16.7.22), as this system is not dissipative with respect to all convex entropies of (16.7.1), we still have a number of estimates of $L^p$ type, the most prominent among them being the "energy inequality" induced by the physical entropy-entropy flux pair (7.4.10). It is thus natural to inquire whether the method of compensated compactness is applicable in conjunction with such estimates. Of course, this would force us to abandon $L^\infty$ and consider Young measures in the framework of $L^p$, a possibility already raised in Section 16.1. It turns out that this approach is effective for the problem at hand, albeit at the expense of elaborate analysis, so just the conclusion shall be recorded here. The proof is found in the references cited in Section 16.9.

**16.7.4 Theorem.** *Consider the system* (16.7.22), *where* (a) $\sigma'(u) \geq \sigma_0 > 0$, *for* $-\infty < u < \infty$; (b) $\sigma''$ *may vanish at most at one point on* $(-\infty,\infty)$; (c) $\sigma'(u)$

*grows like* $|u|^\alpha$, as $|u| \to \infty$, *for some* $\alpha \geq 0$; *and* (d) $\sigma''(u)$ *and* $\sigma'''(u)$ *grow no faster than* $|u|^{\alpha-1}$, *as* $|u| \to \infty$. *Let* $(u_\mu, v_\mu)$ *be the solution of the Cauchy problem* (16.7.22), (16.7.7), *where* $\{(u_{0\mu}, v_{0\mu})\}$ *are functions in* $W^{1,2}(-\infty, \infty)$, *which have uniformly bounded total energy,*

$$\int_{-\infty}^{\infty} [\tfrac{1}{2} v_{0\mu}^2(x) + e(u_{0\mu})] dx \leq a, \tag{16.7.23}$$

*have relatively tame oscillations,*

$$\mu \int_{-\infty}^{\infty} [v_{0\mu}'(x)]^2 dx \to 0, \qquad \text{as } \mu \to 0, \tag{16.7.24}$$

*and converge,* $u_{0\mu} \to u_0$ , $v_{0\mu} \to v_0$ , *as* $\mu \to 0$, *in the sense of distributions. Then there is a sequence* $\{\mu_j\}$, $\mu_j \to 0$ *as* $j \to \infty$, *such that* $\{(u_{\mu_j}, v_{\mu_j})\}$ *converges in* $L^p_{\text{loc}}$ , *for any* $1 < p < 2$, *to a solution* $(\bar{u}, \bar{v})$ *of* (16.7.1) *with initial values* $(\bar{u}(x,0), \bar{v}(x,0)) = (u_0(x), v_0(x))$, $-\infty < x < \infty$.

## 16.8 The System of Isentropic Gas Dynamics

The system (7.1.10) of isentropic gas dynamics, for a polytropic gas, in Eulerian coordinates, the first hyperbolic system of conservation laws ever to be derived, has served over the past two centuries as proving ground for testing the theory. It is thus fitting to conclude this work with the application of the method of compensated compactness to that system.

It is instructive to monitor the system simultaneously in its original form (7.1.10), with state variables density $\rho$ and velocity $v$, as well as in its canonical form

$$\begin{cases} \partial_t \rho + \partial_x m = 0 \\ \\ \partial_t m + \partial_x [\dfrac{m^2}{\rho} + \kappa \rho^\gamma] = 0, \end{cases} \tag{16.8.1}$$

with state variables density $\rho$ and momentum $m = \rho v$. The physical range for density is $0 \leq \rho < \infty$, while $v$ and $m$ may take any values in $(-\infty, \infty)$.

For convenience, we scale the state variables so that $\kappa = (\gamma - 1)^2/4\gamma$, and set $\theta = \frac{1}{2}(\gamma - 1)$, in which case the characteristic speeds (7.2.10) and the Riemann invariants (7.3.3) assume the form

$$\lambda = -\theta\rho^\theta + v = -\theta\rho^\theta + \frac{m}{\rho}, \qquad \mu = \theta\rho^\theta + v = \theta\rho^\theta + \frac{m}{\rho}, \tag{16.8.2}$$

$$z = -\rho^\theta + v = -\rho^\theta + \frac{m}{\rho}, \qquad w = \rho^\theta + v = \rho^\theta + \frac{m}{\rho}. \tag{16.8.3}$$

It is not difficult to construct sequences of approximate solutions taking values in compact sets of the state space $[0,\infty)\times(-\infty,\infty)$. For example, one may follow the vanishing viscosity approach relative to the system

$$\begin{cases} \partial_t\rho_\mu + \partial_x m_\mu = \mu\partial_x^2\rho_\mu \\ \partial_t m_\mu + \partial_x[\dfrac{m_\mu^2}{\rho_\mu} + \kappa\rho_\mu^\gamma] = \mu\partial_x^2 m_\mu\,, \end{cases} \tag{16.8.4}$$

which admits the family of (positively) invariant regions

$$\mathcal{U}_M = \{(\rho,m) : \rho\geq 0,\ -M\leq z(\rho,m)\leq w(\rho,m)\leq M\}. \tag{16.8.5}$$

Furthermore, solutions of (16.8.4) on the upper half-plane, with initial data that are bounded in $L^\infty(-\infty,\infty)\cap L^2(-\infty,\infty)$, satisfy

$$\partial_t\eta(\rho_\mu,m_\mu)+\partial_x q(\rho_\mu,m_\mu)\subset \text{ compact set in } W^{-1,2}_{\mathrm{loc}}\,, \tag{16.8.6}$$

for any entropy-entropy flux pair $(\eta,q)$ of (16.8.1). Approximate solutions with analogous properties are also constructed by finite difference schemes, such as the Lax-Friedrichs scheme and the Godunov scheme. They all lead to the following existence theorem:

**16.8.1 Theorem.** *For any $\gamma>1$, there exists a bounded solution $(\rho,v)$ of the system* (7.1.10) *on the upper half-plane, with assigned initial value*

$$(\rho(x,0),v(x,0)) = (\rho_0(x),v_0(x)),\qquad -\infty<x<\infty, \tag{16.8.7}$$

*where $(\rho_0,v_0)$ are in $L^\infty(-\infty,\infty)$ and $\rho_0(x)\geq 0$, for $-\infty<x<\infty$. Furthermore, the solution satisfies the entropy admissibility condition*

$$\partial_t\eta(\rho,m)+\partial_x q(\rho,m)\leq 0, \tag{16.8.8}$$

*for any entropy-entropy flux pair $(\eta,q)$ of* (16.8.1), *with $\eta(\rho,m)$ convex.*

The proof employs (16.3.10) to establish that the support of the Young measure, associated with a sequence of approximate solutions, either reduces to a single point in state space or is confined to the axis $\rho=0$ (vacuum state).

As function of $(\rho,v)$, any entropy $\eta$ of (7.1.10) satisfies the integrability condition

$$\eta_{\rho\rho} = \theta^2\rho^{\gamma-3}\eta_{vv}\,. \tag{16.8.9}$$

The above equation is singular along the axis $\rho=0$, and the nature of the singularity changes as one crosses the threshold $\gamma=3$. Accordingly, different arguments have to be used for treating the cases $\gamma<3$ and $\gamma>3$.

Of relevance here are the so-called *weak entropies*, which vanish at $\rho=0$. They admit the representation

(16.8.10) $$\eta(\rho, v) = \int_{-\infty}^{\infty} \chi(\rho, \xi - v) g(\xi) d\xi,$$

where

(16.8.11) $$\chi(\rho, v) = \begin{cases} (\rho^{2\theta} - v^2)^s \ , \text{ if } \rho^{2\theta} > v^2 \\ 0, \qquad\qquad \text{ if } \rho^{2\theta} \leq v^2 \ , \end{cases}$$

with $s = \frac{1}{2}\frac{3-\gamma}{\gamma-1}$. Thus $\chi$ is the fundamental solution of (16.8.9) under initial conditions $\eta(0, v) = 0$, $\eta_\rho(0, v) = \delta_0(v)$.

As already noted in Section 2.5, the classical kinetic theory predicts the value $\gamma = 1 + \frac{2}{n}$ for the adiabatic exponent of a gas with $n$ degrees of freedom. When the number of degrees of freedom is odd, $n = 2\ell + 1$, the exponent $s$ in (15.8.11) is the integer $\ell$. In this special situation the analysis of weak entropies and thereby the reduction of the Young measure is substantially simplified. However, even in that simpler case the proof is quite technical and shall be relegated to the references cited in Section 15.9. Only the degenerate case $\gamma = 3$ will be presented here.

For $\gamma = 3$, i.e. $\theta = 1$, (16.8.2) and (16.8.3) yield $\lambda = z$ and $\mu = w$, in which case the two characteristic families totally decouple. In particular, (12.2.1) reduce to $q_z = z\eta_z$ , $q_w = w\eta_w$ , so that there are entropy-entropy flux pairs $(\eta, q)$ which depend solely on $z$, for example $(2z, z^2)$ and $(3z^2, 2z^3)$.

Suppose now a sequence $\{(\rho_{\mu_k}, m_{\mu_k})\}$ of solutions of (16.8.4), with $\mu_k \to 0$ as $k \to \infty$, induces a weakly convergent subsequence $\{(z_j, w_j)\}$ of Riemann invariants with associated family $\nu_{x,t}$ of Young measures. We fix $(x, t)$, set $\nu_{x,t} = \nu$ and apply (16.3.10) for the two entropy-entropy flux pairs $(2z, z^2)$ and $(3z^2, 2z^3)$ to get

(16.8.12) $$4 < \nu, z >< \nu, z^3 > -3 < \nu, z^2 >< \nu, z^2 > = < \nu, z^4 > \ .$$

Next we consider the inequality

(16.8.13) $$z^4 - 4z^3\bar{z} + 6z^2\bar{z}^2 - 4z\bar{z}^3 + \bar{z}^4 = (z - \bar{z})^4 \geq 0,$$

where $\bar{z} = < \nu, z >$ , and apply the measure $\nu$ to it, thus obtaining

(16.8.14)
$$< \nu, z^4 > -4 < \nu, z^3 >< \nu, z > +6 < \nu, z^2 >< \nu, z >^2 -3 < \nu, z >^4 \geq 0.$$

Combining (16.8.14) with (16.8.12) yields

(16.8.15) $$-3[< \nu, z^2 > - < \nu, z >^2]^2 \geq 0,$$

whence $< \nu, z^2 > = < \nu, z >^2$. Therefore, $\{z_j\}$ converges strongly to $\bar{z} = < \nu, z >$.

Similarly one shows that $\{w_j\}$ converges strongly to $\bar{w} = < \nu, w >$. In particular, $(\bar{z}, \bar{w})$ induces a solution $(\bar{\rho}, \bar{v})$ of (7.1.10) by $\bar{\rho} = \frac{1}{2}(\bar{w} - \bar{z})$ and $\bar{v} = \frac{1}{2}(\bar{w} + \bar{z})$.

## 16.9 Notes

The method of compensated compactness was introduced by Murat [1] and Tartar [1,2]. The program of employing the method for constructing solutions to hyperbolic conservation laws was designed by Tartar [2,3], who laid down the fundamental condition (16.3.10) and demonstrated its use in the context of the scalar case. The first application to systems, due to DiPerna [6], provided the impetus for intensive development of these ideas, which has produced a substantial body of research. The presentation here only scratches the surface. A clear introduction is also found in the lecture notes of Evans [1], the text by Hörmander [2], the monograph by Malek, Neças, Rokyta and Růžička [1], as well as the treatise by M.E. Taylor [2]. For more detailed, and deeper development of the subject the reader is referred to the book by Serre [11] and the recent monograph by Lu [1]. An informative presentation of current research trends in the area is provided by the survey article by Gui-Qiang Chen [8].

The Young measure was introduced in L.C. Young [1]. The presentation here follows Ball [2], where the reader may find generalizations beyond the $L^\infty$ framework, as well as commentary and references to alternative constructions.

For an introduction to the theory of compensated compactness, see the lecture notes of Tartar [1,2,3]. The div-curl lemma is due to Murat and Tartar. The proof presented here is taken from Evans [1]. Lemma 16.2.2 is generally known as *Murat's lemma* (Murat [2]).

The notion of a measure-valued solution is due to DiPerna [11]. For further developments of the theory and applications to the construction of solutions to systems of conservation laws, including those of mixed type modeling phase transitions, see Chen and Frid [2], Coquel and LeFloch [1], Demengel and Serre [1], Frid [2], Poupaud and Rascle [1], Roytburd and Slemrod [1], Schochet [2], and Szepessy [1].

The scalar conservation law was first treated via the method of compensated compactness by Tartar [2]. The clever argument employed in the proof of Theorem 16.4.1 was discovered, independently, by Tartar (private communication to the author in May 1986) and by Chen and Lu [1]. See also Vecchi [1]. The scalar conservation law is treated in the $L^p$ framework by Yang, Zhu and Zhao [1].

Schonbek [2] considers a scalar balance law with singular source.

The Cauchy problem for scalar conservation laws in several spatial dimensions can also be solved in $L^\infty$ by the method of compensated compactness (DiPerna [11], Szepessy [2]). An alternative approach, combining a kinetic formulation with ideas from the theory of compensated compactness is carried out in Hwang and Tzavaras [1].

The competition between viscosity and dispersion, in scalar conservation laws, is investigated by Schonbek [1] in one-space dimension, and by Kondo and LeFloch [2], LeFloch and Natalini [1], and Hwang and Tzavaras [1] in several space dimensions.

The active investigation of relaxation for hyperbolic conservation laws, in recent years, has produced voluminous literature, so it would be impossible to include here an exhaustive list of references. The survey paper by Natalini [3] contains an extensive bibliography. A number of relevant references have already been recorded in Sections 5.6 and 6.11. A seminal role in the development of the theory was played by the work of Liu [21], motivated by Whitham [2]. The method of compensated compactness was first employed in this context by Chen and Liu [1] and by Chen, Levermore and Liu [1], for systems of two conservation laws whose relaxed form is the scalar conservation law. The particular efficacy (for theoretical and computational purposes) of the semilinear system (16.5.1) was first recognized by Jin and Xin [1]. The treatment of that system in Section 16.5 is an adaptation of the analysis in Chen, Levermore and Liu [1], Lattanzio and Marcati [1,2], and Coquel and Perthame [1]. For related results, see Collet and Rascle [1] and Klingenberg and Lu [1]. Furthermore, Lu and Klingenberg [1], Tzavaras [3,4], Gosse and Tzavaras [1], Serre [15], and Lattanzio and Serre [2] apply the method of compensated compactness to systems of three or four conservation laws whose relaxed form is a system of two conservation laws. The $L^1$-Lipschitz estimate (16.5.15) for the semilinear system (16.5.1) which leads to a treatment of the relaxation problem in the framework of the space $BV$, is due to Natalini [1]. Existence of $BV$ solutions on the upper half-plane for the nonlinear system (5.2.18) has been established by Dafermos [22], but $BV$ estimates independent of $\mu$ that would allow passing to the relaxation limit, as $\mu \downarrow 0$, are currently known only for the special case $p(u) = -u^{-1}$ (Luo, Natalini and Yang [1], Amadori and Guerra [2]). For other special systems that have been treated in $BV$, see Tveito and Winther [2], and Luo and Natalini [1]. Interesting contributions to relaxation theory also include Coquel and Perthame [1], Marcati and Natalini [1], Marcati and Rubino [1], Luo [1], Luo and Xin [1], and Luo and Yang [2].

The treatment of the genuinely nonlinear system of two conservation laws, in Section 16.6, and the system of isentropic elasticity with a single inflection point, in Section 16.7, follows the pioneering paper of DiPerna [8]. See also Gripenberg [1] and Chen, Li and Li [1]. Counterexamples to Theorem 16.7.1, when $\sigma''(u)$ is discontinuous at $u = 0$, are exhibited in Greenberg [3] and Greenberg and Rascle [1].

The system of isentropic elasticity was treated in the $L^p$ framework by J.W. Shearer [1], Peixiong Lin [1] and Serre and Shearer [1]. An alternative, original construction of solutions in $L^\infty$ (Demoulini, Stuart and Tzavaras [1]) is based on the observation that the system resulting from discretizing the time variable can be solved through a variation principle. The initial-boundary-value problem in $L^\infty$ is solved by Heidrich [1].

The theory of invariant regions via the maximum principle is due to Chueh, Conley and Smoller [1] (see also Hoff [2]). A systematic discussion, with several examples, is found in Serre [11]. The connection between stability of relaxation schemes and existence of invariant regions is discussed in Serre [15]. The proof of Theorem 16.7.2 is taken from Dafermos [13]. See also Serre [3] and Venttsel' [1].

The system of isentropic gas dynamics was first treated by the method of compensated compactness in DiPerna [9], for the special values $\gamma = 1 + \frac{2}{n}$, $n = 2\ell + 1$, of the adiabatic exponent. Subsequently, G.-Q. Chen [1] and Ding, Chen and Luo [1] extended the analysis to any $\gamma$ within the range $(1, \frac{5}{3}]$. For a survey, see Gui-Qiang Chen [2]. The case $\gamma \geq 3$ was solved by Lions, Perthame and Tadmor [1], and the full range $1 < \gamma < \infty$ is covered in Lions, Perthame and Souganidis [1]. The isothermal case, $\gamma = 1$, is singular and was treated by Huang and Wang [2], and LeFloch and Shelukhin [1]. The argument presented here, for the special case $\gamma = 3$, was communicated to the author by Gui-Qiang Chen. Extra regularity for this special value of $\gamma$ is shown by Vasseur [1]. The more general, genuinely nonlinear system (7.1.9), for a nonpolytropic gas, was treated by Chen and LeFloch [2,3] under the assumption that near the vacuum state the pressure function $p(\rho)$, together with its first four derivatives, behave like $\kappa\rho^{\gamma}$.

The approach of Serre [2,11] has rendered the method of compensated compactness sufficiently flexible to treat systems of two conservation laws even when characteristic families are linearly degenerate, strict hyperbolicity fails, etc. The construction of solutions to many interesting systems is effected in Chen [6], Chen and Glimm [1,2], Chen and Tian-Hong Li [1], Dehua Wang [1], Chen and Kan [1], Kan [1], Kan, Santos and Xin [1], Heibig [2], Lu [1], Marcati and Natalini [1,2], Rubino [1], and Zhao [1]. Since the analysis relies heavily on the availability of a rich family of entropies, the application of the method to systems of more than two conservation laws is presently limited to special systems in which the shock and rarefaction wave curves coincide for all but at most two characteristic families (Benzoni-Gavage and Serre [1]) and to the system of nonisentropic gas dynamics for a very special equation of state (Chen and Dafermos [1], Chen, Li and Li [1]).

For a variety of systems, the large time behavior of solutions with initial values that are either periodic or $L^1$ perturbations of Riemann data is established in Chen and Frid [1,3,4,6], by combining scale invariance with compactness. The method of compensated compactness has also been employed to demonstrate that the large time behavior of solutions to the Euler equations with frictional damping is governed by the porous media equation; see Serre and Xiao [1], Huang and Pan [1,2,3]. For the large time behavior of solutions to systems with relaxation, see Serre [19,20].

The kinetic formulation, which was applied effectively in Chapter VI to scalar conservation laws in several spatial dimensions, has been successfully extended to certain systems of conservation laws in one-space dimension, including the Euler equations of isentropic gas flow (Berthelin and Bouchut [1]) as well as the system of isentropic elastodynamics (Perthame and Tzavaras [1], Tzavaras [5]). A detailed discussion and a comprehensive list of references is found in the monograph by Perthame [2]. Refined properties of solutions are derived by combining the kinetic formulation with techniques from the theory of compensated compactness. In particular, for strictly hyperbolic systems of two conservation laws, Tzavaras [5] obtains an explicit formula for the coupling of oscillations between the two characteristic fields.

Valuable insight on the effects of nonlinearity in hyperbolic conservation laws is gained from the investigation of how the solution operator interacts with highly oscillatory initial data, say $U_{0\varepsilon}(x) = V(x, x/\varepsilon)$, where $V(x, \cdot)$ is periodic and $\varepsilon$ is a small positive parameter. When the system is linear, the rapid oscillations are transported along characteristics and their amplitude is not attenuated. On the opposite extreme, when the system is strictly hyperbolic and genuinely nonlinear, the results of Sections 16.4 and 16.6 indicate that, as $\varepsilon \to 0$, the resulting family of solutions $U_\varepsilon(x, t)$ contains sequences which converge strongly to solutions with initial value the weak limit of $\{U_{0\varepsilon}\}$, that is for $t > 0$ the solution operator quenches high frequency oscillations of the initial data. It is interesting to investigate intermediate situations, where some characteristic families may be linearly degenerate, strict hyperbolicity fails, etc. Following the study of many particular examples (cf. Bonnefille [1], Chen [3,4,5], E[1], Heibig [1], Rascle [1], and Serre [5,8]), a coherent theory of *propagation of oscillations* seems to be emerging (Serre [11]).

There is a well-developed theory of propagation of oscillations based on the method of *weakly nonlinear geometric optics* which derives asymptotic expansions for solutions of hyperbolic systems under initial data oscillating with high frequency and small amplitude. Following the pioneering work of Landau [1], Lighthill [1], and Whitham [1], extensive literature has emerged, of purely formal, semirigorous or rigorous nature, dealing with the cases of a single phase, or possibly resonating multiphases, etc. See, for example, Choquet-Bruhat [1], Hunter and Keller [1,2], Majda and Rosales [1], Majda, Rosales and Schonbek [1], Pego [1], Hunter [1], Joly, Métivier and Rauch [1,3], and Cheverry [1]. It is remarkable that the asymptotic expansions remain valid even after shocks develop in the solution; see DiPerna and Majda [1], Schochet [5] and Cheverry [2]. A survey is found in Majda [5] and a systematic presentation is given in Serre [11].

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