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Wolfgang Rautenberg

A Concise Introduction to Mathematical Logic

Springer

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Wolfgang Rautenberg

A Concise Introduction to Mathematical Logic

Textbook

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Foreword

by Lev Beklemishev, Utrecht

The field of mathematical logic—evolving around the notions of logical validity, provability, and computation—was created in the first half of the previous century by a cohort of brilliant mathematicians and philosophers such as Frege, Hilbert, Gödel, Turing, Tarski, Malcev, Gentzen, and some others. The development of this discipline is arguably among the highest achievements of science in the twentieth century: it expanded mathematics into a novel area of applications, subjected logical reasoning and computability to rigorous analysis, and eventually led to the creation of computers.

The textbook by Professor Wolfgang Rautenberg is a well-written introduction to this beautiful and coherent subject. It contains classical material such as logical calculi, beginnings of model theory, and Gödel's incompleteness theorems, as well as some topics motivated by applications, such as a chapter on logic programming. The author has taken great care to make the exposition readable and concise; each section is accompanied by a good selection of exercises.

A special word of praise is due for the author's presentation of Gödel's second incompleteness theorem in which the author has succeeded in giving an accurate and simple proof of the derivability conditions and the provable Σ_1-completeness, a technically difficult point that is usually omitted in textbooks of comparable level. This textbook can be recommended to all students who want to learn the foundations of mathematical logic.

Preface

This book is based on the second edition of my *Einführung in die Mathematische Logik* whose favorable reception facilitated the preparation of this English version. The book is aimed at students of mathematics, computer science, or linguistics. Because of the epistemological applications of Gödel's incompleteness theorems, this book may also be of interest to students of philosophy with an adequate mathematical background. Although the book is primarily designed to accompany lectures on a graduate level, most of the first three chapters are also readable by undergraduates. These first hundred pages cover sufficient material for an undergraduate course on mathematical logic, combined with a due portion of set theory. Some of the sections of Chapter **3** are partly descriptive, providing a perspective on decision problems, automated theorem proving, nonstandard models, and related topics.

Using this book for independent and individual study depends less on the reader's mathematical background than on his (or her) ambition to master the technical details. Suitable examples accompany the theorems and new notions throughout. To support a private study, the indexes have been prepared carefully. We always try to portray simple things simply and concisely and to avoid excessive notation, which could divert the reader's mind from the essentials. Linebreaks in formulas have been avoided. A special section at the end provides solution hints to most exercises, and complete solutions of exercises that are relevant for the text.

Starting from Chapter **4**, the demands on the reader begin to grow. The challenge can best be met by attempting to solve the exercises without recourse to the hints. The density of information in the text is pretty high; a newcomer may need one hour for one page. Make sure to have paper and pencil at hand when reading the text. Apart from a sufficient training in logical (or mathematical) deduction, additional prerequisites are assumed only for parts of Chapter **5**, namely some knowledge of classical algebra, and at the very end of the last chapter some acquaintance with models of axiomatic set theory.

On top of the material for a one-semester lecture course on mathematical logic, basic material for a course in logic for computer scientists is included in Chapter **4** on logic programming. An effort has been made to capture some of the interesting aspects of this discipline's logical foundations. The resolution theorem is proved constructively. Since all recursive functions are computable in PROLOG, it is not hard to get the undecidability of the existence problem for successful resolutions.

Chapter **5** concerns applications of mathematical logic in various methods of model construction and contains enough material for an introductory course on model theory. It presents in particular a proof of quantifier eliminability in the theory of real closed fields, a basic result with a broad range of applications.

A special aspect of the book is the thorough treatment of Gödel's incompleteness theorems. Since these require a closer look at recursive predicates, Chapter **6** starts with basic recursion theory. One also needs it for solving questions about decidability and undecidability. Defining formulas for arithmetical predicates are classified early, in order to elucidate the close relationship between logic and recursion theory. Along these lines, in **6.4** we obtain in one sweep Gödel's first incompleteness theorem, the undecidability of the tautology problem by Church, and Tarski's result on the nondefinability of truth. Decidability and undecidability are dealt with in **6.5**, and **6.6** includes a sketch of the solution to Hilbert's tenth problem.

Chapter **7** is devoted exclusively to Gödel's second incompleteness theorem and some of its generalizations. Of particular interest thereby is the fact that questions about self-referential arithmetical statements are algorithmically decidable due to Solovay's completeness theorem. Here and elsewhere, Peano arithmetic PA plays a key role, a basic theory for the foundations of mathematics and computer science, introduced already in **3.3**. The chapter includes some of the latest results in the area of self-reference not yet covered by other textbooks.

Remarks in small print refer occasionally to notions that are undefined or will be introduced later, or direct the reader toward the bibliography, which represents an incomplete selection only. It lists most English textbooks on mathematical logic and, in addition, some original papers, mainly for historical reasons. This book contains only material that will remain the subject of lectures in the future. The material is treated in a rather streamlined fashion, which has allowed us to cover many different topics. Nonetheless, the book provides only a selection of results and can at most accentuate certain topics. This concerns above all the Chapters **4**, **5**, **6**, and **7**, which go a step beyond the elementary. Philosophical and foundational problems of mathematics are not systematically discussed within the constraints of this book, but are to some extent considered when appropriate.

The seven chapters of the book consist of numbered sections. A reference like Theorem 5.4 is to mean Theorem 4 in Section **5** of a given chapter. In cross-referencing from another chapter, the chapter number will be adjoined. For instance, Theorem 6.5.4 is Theorem 5.4 in Chapter **6**. You may find additional information about the book or contact me on my website *www.math.fu-berlin.de/∼raut* .

I would like to thank the colleagues who offered me helpful criticism along the way; their names are too numerous to list here. Particularly useful for Chapter **7** were hints from Lev Beklemishev (Moscow) and Wilfried Buchholz (Munich). Thanks also to the publisher, in particular Martin Peters, Mark Spencer, and David Kramer.

Wolfgang Rautenberg
December 2005

Contents

Introduction

Traditional logic as a part of philosophy is one of the oldest scientific disciplines. It can be traced back to the Stoics and to Aristotle.[1] It is one of the roots of what is nowadays called philosophical logic. Mathematical logic, however, is a relatively young discipline, having arisen from the endeavors of Peano, Frege and Russell to reduce mathematics entirely to logic. It steadily developed during the twentieth century into a broad discipline with several subareas and numerous applications in mathematics, computer science, linguistics, and philosophy.

One of the features of modern logic is a clear distinction between object language and metalanguage. The latter is normally a kind of a colloquial language, although it differs from author to author and depends also on the audience the author has in mind. In any case, it is mixed up with semiformal elements, most of which have their origin in set theory. The amount of set theory involved depends on one's objectives. General semantics and model theory use stronger set-theoretical tools than does proof theory. But on average, little more is assumed than knowledge of the most common set-theoretical terminology, presented in almost every mathematical course for beginners. Much of it is used only as a façon de parler.

Since this book concerns mathematical logic, its language is similar to the language common to all mathematical disciplines. There is one essential difference though. In mathematics, metalanguage and object language strongly interact with each other and the latter is semiformalized in the best of cases. This method has proved successful. Separating object language and metalanguage is relevant only in special context, for example in axiomatic set theory, where formalization is needed to specify how certain axioms look like. Strictly formal languages are met more often in computer science. In analysing complex software or a programming language, like in logic, formal linguistic entities are the objects of consideration.

The way of arguing about formal languages and theories is traditionally called the *metatheory*. An important task of a metatheoretical analysis is to specify procedures of logical inference by so-called *logical calculi*, which operate purely syntactical. There are many different logical calculi. The choice may depend on the formalized language, on the logical basis, and on certain aims of the formalization. Basic metatheoretical tools are in any case the naive natural numbers and inductive proof procedures. We will sometimes call them proofs by *metainduction*, in particular when talking about formalized theories that may speak about natural numbers and induction themselves. Induction can likewise be carried out on certain sets of strings over a fixed alphabet, or on the system of rules of a logical calculus.

[1] The Aristotelian syllogisms are useful examples for inferences in a first-order language with unary predicate symbols. One of these serves as an example in Section **4.4** on logic programming.

The logical means of the metatheory are sometimes allowed or even explicitly required to be different from those of the object language. But in this book the logic of object languages, as well as that of the metalanguage, are classical, two-valued logic. There are good reasons to argue that classical logic is the logic of common sense. Mathematicians, computer scientists, linguists, philosophers, physicists, and others are using it as a common platform for communication.

It should be noticed that logic used in the sciences differs essentially from logic used in everyday language, where logic is more an art than a serious task of saying what follows from what. In everyday life, nearly every utterance depends on the context. In most cases logical relations are only alluded to and rarely explicitly expressed. Some basic assumptions of two-valued logic mostly fail, for instance, a context-free use of the logical connectives. Problems of this type are not dealt with in this book. To some extent, many-valued logic or Kripke semantics can help to clarify the situation, and sometimes intrinsic mathematical methods must be used in order to analyze and solve such problems. We shall use Kripke semantics here for a different goal though, the analysis of self-referential sentences in Chapter **7**.

Let us add some historical remarks, which, of course, a newcomer may find easier to understand *after* and not *before* reading at least parts of this book. In the relatively short period of development of modern mathematical logic in the last century, some highlights may be distinguished, of which we mention just a few.

The first was the axiomatization of set theory in various ways. The most important approaches are the ones of Zermelo (improved by Fraenkel and von Neumann) and the theory of types by Whitehead and Russell. The latter was to become the sole remnant of Frege's attempt to reduce mathematics to logic. Instead it turned out that mathematics can be based entirely on set theory as a first-order theory. Actually, this became more salient after the rest of the hidden assumptions by Russell and others were removed from axiomatic set theory[2] around 1915; see [Hej].

Right after these axiomatizations were completed, Skolem discovered that there are countable models of the set-theoretic axioms, a drawback for the hope for an axiomatic definition of a set. Just then, two distinguished mathematicians, Hilbert and Brouwer, entered the scene and started their famous quarrel on the foundations of mathematics. It is described in an excellent manner in [Kl2, Chapter IV] and need therefore not be repeated here.

As a next highlight, Gödel proved the completeness of Hilbert's rules for predicate logic, presented in the first modern textbook on mathematical logic, [HA]. Thus, to some extent, a dream of Leibniz became real, namely to create an *ars inveniendi* for mathematical truth. Meanwhile, Hilbert had developed his view on a foundation of

[2] For instance, the notion of an ordered pair is indeed a set-theoretical and not a logical one.

mathematics into a program. It aimed at proving the consistency of arithmetic and perhaps the whole of mathematics including its nonfinitistic set-theoretic methods by finitary means. But Gödel showed by his incompleteness theorems in 1931 that Hilbert's original program fails or at least needs thorough revision.

Many logicians consider these theorems to be the top highlights of mathematical logic in the twentieth century. A consequence of these theorems is the existence of consistent extensions of Peano arithmetic in which true and false sentences live in peaceful coexistence with each other, called "dream theories" in Section **7.2**. It is an intellectual adventure of holistic beauty to see wisdoms from number theory known for ages, like the Chinese remainder theorem, or simple properties of prime numbers and Euclid's characterization of coprimeness (page 193) unexpectedly assuming pivotal positions within the architecture of Gödel's proofs.

The methods Gödel developed in his paper were also basic for the creation of recursion theory around 1936. Church's proof of the undecidability of the tautology problem marks another distinctive achievement. After having collected sufficient evidence by his own investigations and by those of Turing, Kleene, and some others, Church formulated his famous thesis (Section **6.1**), although in 1936 no computers in the modern sense existed nor was it foreseeable that computability would ever play the basic role it does today.

As already mentioned, Hilbert's program had to be revised. A decisive step was undertaken by Gentzen, considered to be another groundbreaking achievement of mathematical logic and the starting point of contemporary proof theory. The logical calculi in **1.2** and **3.1** are akin to Gentzen's calculi of natural deduction.

We further mention Gödel's discovery that it is not the axiom of choice (AC) that creates the consistency problem in set theory. Set theory with AC and the continuum hypothesis (CH) is consistent provided set theory without AC and CH is. This is a basic result of mathematical logic that would not have been obtained without the use of strictly formal methods. The same applies to the independence proof of AC and CH from the axioms of set theory by P. Cohen in 1963.

The above indicates that mathematical logic is closely connected with the aim of giving mathematics a solid foundation. Nonetheless, we confine ourself to logic and its fascinating interaction with mathematics. History shows that it is impossible to establish a programmatic view on the foundations of mathematics that pleases everybody in the mathematical community. Mathematical logic is the right tool for treating the technical problems of the foundations of mathematics, but it cannot solve its epistemological problems.

Notation

We assume that the reader is familiar with basic mathematical terminology and notation, in particular with the elementary set-theoretical operations of *union*, *intersection*, *complemention*, and *cross product*, denoted by $\cup$, $\cap$, $\setminus$, and $\times$, respectively. Here we summarize only some notation that may differ slightly from author to author, or is specific for this book.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the sets of natural numbers including 0, integers, rational, and real numbers, respectively. n, m, i, j, k denote always natural numbers unless stated otherwise. Hence, extended notation like $n \in \mathbb{N}$ is mostly omitted. $\mathbb{N}_+, \mathbb{Q}_+, \mathbb{R}_+$ denote the sets of positive members of the corresponding sets.

The *ordered pair* of elements a, b is denoted by (a, b). It should not be mixed up with the *pair set* $\{a, b\}$. Set inclusion is denoted by $M \subseteq N$, while $M \subset N$ means proper inclusion (i.e., $M \subseteq N$ and $M \neq N$). We write $M \subset N$ only if the circumstance $M \neq N$ has to be emphasized. If M is fixed in a consideration and N varies over subsets of M, then $M \setminus N$ may also be denoted by $\setminus N$ or $\neg N$. The *power set* (= set of all subsets) of M is denoted $\mathfrak{P}M$. $\emptyset$ denotes the *empty set*.

If one wants to emphasize that all elements of a set F are sets, F is also called a *family* or *system* of sets. $\bigcup F$ denotes the union of a set family F, that is, the set of elements belonging to at least one $M \in F$, and $\bigcap F$ stands for the intersection of F $(\neq \emptyset)$, which is the set of elements belonging to all $M \in F$. If $F = \{M_i \mid i \in I\}$ then $\bigcup F$ and $\bigcap F$ are mostly denoted by $\bigcup_{i \in I} M_i$ and $\bigcap_{i \in I} M_i$, respectively.

A *relation* between M and N is a subset of $M \times N$. Such a relation, call it f, is said to be a *function* (or *mapping*) from M to N if for each $a \in M$ there is precisely one $b \in N$ with $(a, b) \in f$. This b is denoted by $f(a)$ or fa or a^f and called the *value of f at a*. We denote such an f also by $f\colon M \to N$, or by $f\colon x \mapsto t(x)$ provided $f(x) = t(x)$ for some term t (terms are defined in **2.2**). $id_M\colon x \mapsto x$ denotes the *identical function* on M. $\operatorname{ran} f = \{fx \mid x \in M\}$ is called the *range* of f, while $\operatorname{dom} f = M$ is called its *domain*. $f\colon M \to N$ is *injective* if $fx = fy \Rightarrow x = y$, for all $x, y \in M$, *surjective* if $\operatorname{ran} f = N$, and *bijective* if f is both injective and surjective. The reader should basically be familiar with this terminology.

The set of all functions from M to N is denoted by N^M. The phrase "let f be a function from M to N" is sometimes shortened to "let $f\colon M \to N$." If f, g are mappings with $\operatorname{ran} g \subseteq \operatorname{dom} f$ then $h\colon x \mapsto f(g(x))$ is called their *composition*. It is sometimes denoted by $h = f \circ g$, but other notation is used as well.

Let I and M be sets, $f\colon I \to M$, and call I the *index set*. Then f will often be denoted by $(a_i)_{i \in I}$ and is named, depending on the context, a *family*, an *I-tuple*, or a *sequence*. If 0 is identified with $\emptyset$ and $n > 0$ with $\{0, 1, \ldots, n-1\}$, as is common in set theory, then M^n can be understood as the set of finite sequences or

n-tuples $(a_i)_{i<n} = (a_0, \ldots, a_{n-1})$ *of length* n whose members are elements of M. In concatenating finite sequences which has an obvious meaning, the *empty sequence* (the only member of $M^0 = \{\emptyset\}$), plays the role of a neutral element. A sequence of the form $(a_1, \ldots, a_n)$ will frequently be denoted by $\vec{a}$. This is for $n = 0$ the empty sequence, similar to $\{a_1, \ldots, a_n\}$ for $n = 0$ being always the empty set.

If A is an *alphabet*, i.e., if the elements of A are symbols or at least called symbols, then the sequence $(a_1, \ldots, a_n)$ is written as $a_1 \cdots a_n$ and called a *string* or a *word* over the alphabet A. The empty sequence is then called the *empty string* or the *empty word*. Let $\xi\eta$ denote the concatenation of the strings ξ and η. If $\xi = \xi_1\eta\xi_2$ for some strings ξ_1, ξ_2 and $\eta \neq \emptyset$ then η is called a *substring* or *segment* of ξ. If, in addition, $\xi_1 = \emptyset$ then η is called an *initial*, and if $\xi_2 = \emptyset$, a *terminal* segment of ξ.

Subsets $P, Q, R, \ldots \subseteq M^n$ are called *n-ary predicates of* M or n-ary *relations*. A unary predicate will be identified with the corresponding subset of M. We may write $P\vec{a}$ instead of $\vec{a} \in P$, and $\neg P\vec{a}$ instead of $\vec{a} \notin P$. Metatheoretical predicates (or properties) cast in words will often be distinguished from the surrounding text by single quotes, for instance, if we speak of the syntactic predicate 'The variable x occurs in the formula α'. We can do so since quotes inside quotes will not occur. Single quoted predicates are often used in induction principles, or they are reflected in a theory, while ordinary ("double") quotes have a stylistic function only.

An n-ary *operation of* M is a function $f : M^n \to M$. Almost everywhere $f\vec{a}$ will be written instead of $f(a_1, \ldots, a_n)$. Since $M^0 = \{\emptyset\}$, a 0-ary operation of M is of the form $\{(\emptyset, c)\}$ with $c \in M$; it is denoted by c for short and called a *constant*. Each operation $f : M^n \to M$ is uniquely described by the *graph of* f,

$$\operatorname{graph} f := \{(a_1, \ldots, a_{n+1}) \in M^{n+1} \mid f(a_1, \ldots, a_n) = a_{n+1}\}.$$

Both f and $\operatorname{graph} f$ are essentially the same, but in most situations it is more convenient to distinguish between f and $\operatorname{graph} f$.

If A, B are expressions of our metalanguage, $A \Leftrightarrow B$ stands for "*A iff B*," that is, "*A if and only if B*." Similarly, $A \Rightarrow B$, $A \mathbin{\&} B$, and $A \vee B$ mean "*if A then B*," "*A and B*," and "*A or B*," respectively. This notation does not aim at formalizing the metalanguage but serves improved organization of metatheoretic statements. We agree that $\Rightarrow, \Leftrightarrow, \ldots$ separate stronger than linguistic binding particles like "there is" or "for all." Hence, in $T \vDash \alpha \Leftrightarrow \alpha \in T$, for all $\alpha \in \mathcal{L}^0$ (definition page 64) the comma should not be omitted; otherwise some serious misunderstanding may arise, since '$\alpha \in T$ for all $\alpha \in \mathcal{L}^0$' has the meaning 'the theory T is inconsistent'.

$A :\Leftrightarrow B$ means that the expression A is defined by B. Similarly, $s := t$ means that the term s is defined by the term t, or whenever s is a variable, the allocation of the value of t to s. W.l.o.g. or w.l.o.g. abbreviates "Without loss of generality."