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Studies in Contemporary Economics

Evert van Imhoff

Optimal Economic Growth and Non-Stable Population

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PREFACE AND ACKNOWLEDGEMENTS

When I joined the Erasmus University of Rotterdam, in January 1985, Jo Ritzen did not waste much time in rousing my interest for growth theory and population economics. It was he who suggested the subject for the research reported in this book (which is a slightly revised version of my Ph.D. dissertation), thereby sending me on an adventurous journey into the unknown. This journey has not been an easy one, with ups and downs changing places at regular intervals. Especially during those downs, having Jo Ritzen as my supervisor proved to be a wonderful experience. I am deeply indebted to him for his never wavering encouragement and assistance.

Several people made valuable comments on earlier drafts of (parts of) the manuscript. They are Harry van Dalen, Onno van Hilten, Miriam King, Joop Odink and Ton Vorst. Chapter 4.2, "A model with exogenous embodied technical change", was written in the summer of 1987 during a stay at the International Institute for Applied Systems Analysis in Laxenburg, Austria. I am grateful for the opportunities it provided me with.

Mrs. Amy Gons-Daugherty agreed to teach me one final lesson in English and corrected my errors with an almost frustrating eye for even the smallest detail. She has been a great help.

Finally, I have to thank those who contributed to the necessary atmosphere which enabled me to finish this work. At the university, the pleasant surroundings were created by my colleagues at the Department of Public Finance and Fiscal Law. At home, most of the burden was taken by Annelies. Without her, life would have been much more difficult and much less interesting.

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1 INTRODUCTION

This book studies optimal economic growth in a closed economy which experiences non-stable population growth. The economy is described by means of a neoclassical growth model which distinguishes overlapping generations within the population. The basic neoclassical growth model is extended to include various types of technical change, as well as investment in human capital or education.

The research described in this book connects the analytical tools of traditional growth theory with the actual demographic experience of most industrialized countries. The role of demographic processes in the growth-theoretical literature is discussed in the next section. The discussion will show that growth theory needs to extend its scope through the construction of growth models which explicitly recognize demographic forces as a potential source of non-stationarities. This book constitutes a first attempt at such a demographic extension.

1.1 Growth theory and demographic change

The theory of economic growth (e.g. Solow, 1970; Burmeister & Dobell, 1970; Wan, 1971) attempts to describe and to explain the long-run development of an economic system (or, in short, economy). An economic system is essentially dynamic in nature. Among the most important sources of dynamics in economics are the following:

- accumulation of capital (investment);
- technical change;
- population growth.

Some of these dynamic forces are, at least in part, endogenous to the economic system (i.e. determined by economic variables).

The order in which these three sources of dynamics have been given roughly corresponds to the chronological order in which they have been studied by growth theorists. With the notable exception of Malthus (1798/1970), who was a demographic economist *avant-la-lettre*, the economics profession has for a long time almost exclusively directed its attention to the accumulation of physical capital as the principal driving force behind economic growth. These efforts culminated in the seminal formulation of the neoclassical growth model by Solow (1956) and Swan (1956) simultaneously.

The neoclassical growth model suffers from one major drawback: it cannot explain the observed secular rise in consumption per capita by mere long-run accumulation of physical capital. Triggered off by the pioneering work of Solow (1957), which earned him the 1987 Nobel Prize in Economic Sciences, and Salter (1960/1966) growth theory broadened its view to include technical change in its analytical models.

Growth theory became a field of considerable interest in the 1960s and early 1970s. A vast literature developed, in which authors derived new results for existing models or old results for new models.

A central concept within growth theory is the concept of the steady state. A steady state is a situation of economic development in which all variables grow at a constant rate. These rates can be different for different variables; some can also be zero, so that the corresponding variable is a constant in steady state.

It should be pointed out that a necessary, though by no means sufficient, condition for a steady state to occur is that the relevant exogenous variables (such as population) grow at a constant rate. Demographers use the term stable population for a population which grows at a constant rate (Keyfitz, 1977). The main characteristic of a stable population is that it has a constant age composition.

The bulk of the publications on growth theory of the 1960s and 1970s have in common that they are very heavily concentrated on steady states (cf. the illustrative title of Von Weizsäcker's (1971) book). They are particularly restricted to the case of stable population. Population plays only a minor role. Some attention was paid to the long-run effect of the population growth rate (Samuelson, 1975a; 1975b). Also it was recognized that the growth rate of the population could well be partly dependent on economic conditions, leading to growth models with endogenous population (Merton, 1969; Sato & Davis, 1971; Strigens, 1975). But the obvious fact that, first, demographic changes exist, and second, that these changes, once they occur, by their very nature have long-lasting effects, has not been taken into account.

The sharp drop in the number of births experienced by virtually all industrialized countries around 1970 has started a deviation from stable population which will remain until all affected cohorts have died. The decline in the number of births causes the population to age, and this in two respects. First, when the growth rate of the number of births is constant for a period long enough for the population to be stable again (i.e. to have a constant age-structure), the proportion of the elderly in the total population is permanently higher than before the start of the fertility decline. This is the long-run ageing effect. Second, during the transition phase there is a period in which the elderly stem from higher growth-rate cohorts than the younger generations, rendering the proportion of the elderly in the total population higher than it is in the final stable population. This is the transitory ageing effect. The transitory ageing effect of the fertility decline is, of course, larger than the long-run ageing effect.

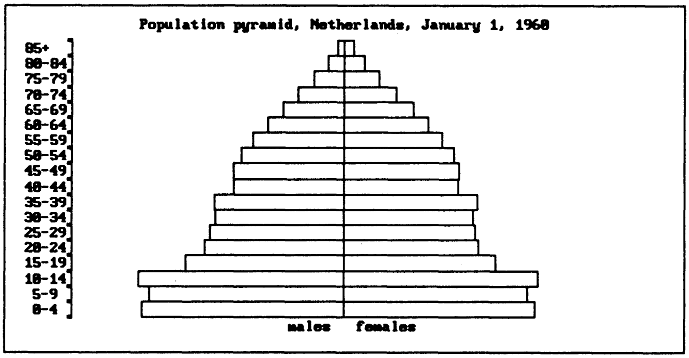
From the previous paragraph it follows that it takes quite a long time for a change in the fertility pattern to work its way all through the age structure. Thus the ageing of the population that accompanies the present fertility decline will be a major social and economic problem until at least the year 2060. These observations illustrate that population can be an important source of dynamics in economic development.

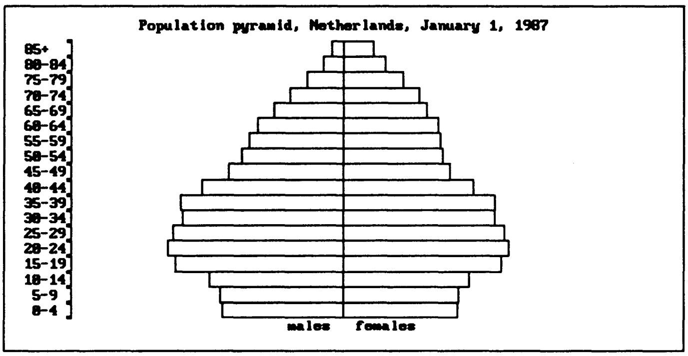
There are several reasons why interest in growth theory gradually diminished in the mid 1970s. For one thing, the field by that time had been analysed quite thoroughly and the set of pretty and elegant theoretical results had become more or less exhausted. Also, the relevance of growth theory and its "stylized facts" was dealt a severe blow by the actual economic circumstances of that period, characterized by continued capital accumulation on the one hand and at the same time, sharp drops in economic growth rates on the other.

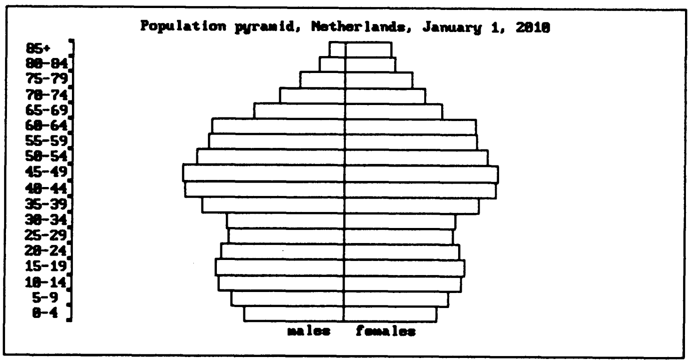
Probably the main reason for the wavering interest in growth theory stems from the simple observation that many countries began to experience a considerable change in the size and age composition of the population. The percentage of youngsters (often sharply) decreases while that of the elderly increases. Figure 1.1 illustrates this ageing of the population in the case of the Netherlands. Europe, as a whole, is an ageing continent. Ageing of the population is not limited to western countries but is also pronounced in countries like Japan, China, South Korea, and Singapore.

Traditional growth theory with its emphasis on the steady state is not capable of handling these dramatic deviations from exponential population growth. Understanding the economic consequences of demographic change requires the construction of growth models that explicitly recognize demographic forces as a potential source of non-stationarities in economic development. A natural way of incorporating the demographic structure into growth models is to make the model one of overlapping generations (Samuelson,

Figure 1.1: The age-composition of the population in the Netherlands







Source: Dutch Central Bureau of Statistics (CBS), Population Statistics.
For 2010: CBS Statistisch Bulletin, 1985/50, p. 6 (Medium projection).

1958). Such an extension of the traditional growth models moves growth theory into the realm of demographic or population economics (Arthur & McNicoll, 1977; 1978).

Although the present significant demographic changes have greatly intensified the interest in demographic economics much work remains to be done. Most research in this field is concerned with the effect of ageing on government expenditures (Clark e.a., 1978; Steinmann, 1984). Recently Ritzen (1986b) studied the revenue side of the public sector in the context of demographic change. Studies of the plain long-run economic effects of a declining population growth rate are relatively scarce. Some results can be found in Auerbach & Kotlikoff (1985), and Van Imhoff & Ritzen (1987). These studies conclude that the long-run consequences of ageing can be highly significant.

1.2 Optimal economic growth

In this book we are concerned with the consequences of demographic change on optimal economic growth, which may be somewhat different from actual economic growth. The theory of optimal economic growth assumes that one or more variables in the economic system can be controlled. The theory is concerned with determining these control variables in such a way that the resulting economic development is optimal in respect to some objective or welfare function. Optimal economic growth theory was pioneered by Ramsey in his seminal (1928) article. Here the control variable is the savings (investment) rate which has to be determined by a central planning agency.

Probably the most famous result of optimal economic growth theory is the so-called Golden Rule of Capital Accumulation (Phelps, 1961; Robinson, 1962). This concept was originally derived within the context of comparative statics (i.e. comparing steady states). The Golden Rule states that the steady state with the highest level of consumption per capita is characterized by the equality between the marginal productivity of capital and the growth rate of the population. Cass (1965) has shown that the Golden Rule can alternatively be defined as the equilibrium position (singular solution) of an optimal control problem, with the integral of consumption per capita as the objective function. Since its first appearance in economic literature, numerous generalizations and extensions of the Golden Rule have been established by various authors.

The main limitation of the Golden Rule concept is its tight link with the notion of the steady state. Although the latter is very appealing from a theoretical standpoint, it is hardly relevant to actual economic development. At noted above, steady states can come about only if the exogenous variables grow at a constant rate. Clearly, this condition is not satisfied in reality. This is most obvious for population, the growth rate of which fluctuates quite strongly; currently, the population growth rate is falling in almost all industrialized countries. Another example of an exogenous variable growing at a non-constant rate is technical change, although one should add that it is not immediately clear whether this is a truly exogenous variable. The latter, in fact, also applies to population; throughout this study I will assume that demographic variables are completely exogenous to the economic system.

A population has a constant age-structure (is stable) if the age-specific mortality rates are constant and if the growth rate of the number of births has been constant for at least n years, where n is the maximum lifespan (say 100 years). Obviously, since the age-structure of the population is an important economic variable, a temporary change in the birth growth rate during m years results in a departure of the economic growth path from steady state for at least m+n years. The period during which the age-structure of the population is non-constant (when the population itself is non-stable) can be labelled a period of <u>demographic transition</u>. The economic growth path during such periods of demographic transition I denote by the term <u>non-stationary</u>. This term should be understood to indicate any economic growth path that is not a steady state.

The study of the optimal development of an economy in periods of demographic transition, using neoclassical growth models, was started some years ago by Ritzen & Van Praag (1985). In their study, savings are considered as being determined by a central planning authority which aims at maximizing some intertemporal social welfare function in discrete time. The effect on capital formation and consumption per capita are explored for a sequence of time periods in which the rate of population growth declines sharply. This book continues the analysis of Ritzen & Van Praag, but also extends it in several directions.

1.3 Population dynamics and economic development

The effects of population dynamics on the performance of the economy have several distinct causal sequences. Among the most important variables that are directly affected by changes in the population growth rate, are the following:

1. the sheer size of the population, that is, the number of consumers and size of the labour force;
2. the labour force/population-ratio and its complement, the dependency-ratio;
3. more generally, the age-structure of the population.

These and related demographic variables, in turn, affect many of the economic variables. In particular, they affect the optimal values of the control variables.

There are numerous mechanisms which are responsible for these indirect effects of demographic change on economic development. The most important of these are savings (e.g. through pension schemes) and public expenditures (e.g. for education, public pensions).

In the 1980s our country (as well as many other, mostly industrialized countries) can be considered to be in the middle of the transition phase between two situations of stable population growth. Since 1970 the yearly number of children born alive has decreased dramatically. Today's population projections indicate that the number of births will remain approximately constant until about 1995 and will decrease even further in the years after. The most significant, and, for economic performance, most important aspect of such a transition phase is the continuous shifting of the population's age structure.

An important control variable which determines the pattern of economic growth is education. Education is very tightly linked to the age-structure of the population, since the bulk of education is imparted to the young. From the investment point of view this fact is easily explained: investment in human capital takes place in the beginning of the life cycle; in the later stages of a person's life these investments are repaid in the form of a larger contribution to production.

When the number of live births decreases, the fraction of the labour force in the total population is diminished with a certain delay (the population ages). This effect, by creating a relative scarcity of human capital as compared to physical capital, as well as by increasing the dependency-ratio, tends to lower consumption per capita. The question arises whether this pressure could be relieved by increasing the amount invested in education, thus improving the productivity of the labour force. The book aims at answering this question (Chapter 6).

A second effect of the population's ageing is that the introduction of technological innovations in the production process could be hampered. In "normal" times this introduction can be assumed to be primarily achieved by means of the constant influx into the labour force of recently educated young people. When the relative share of this influx is reduced, it might well be the case that education for adults becomes necessary in order to prevent the rate of technological progress being hampered by the ageing of the labour force. Consequently, there is an important relation between technological development and educational policy which will receive further elaboration in this book (Chapter 7.2).

1.4 Research goals and outline of the book

The purpose of this book is to analyse the effects of changes in the growth rate of population on the optimal economic growth path. The goals of the research are twofold. At the theoretical level I will construct models of economic growth with which the effects on social welfare of demography, investment in physical and human capital, and technical progress can be analysed. Such models allow for the derivation of rules for optimal policy, given the behaviour over time of the exogenous variables. The second goal of the project is to formulate general guidelines for long-run economic and educational policy in the decades before us, given the available demographic projections.

Four broad groups of growth models are analysed. These are as follows:

1. the simple neoclassical one-sector model of Solow (1956). This analysis can be regarded as a non-stationary generalization of the classical Golden Rule case (Chapter 2);
2. one-sector models with technical change (Chapter 4);
3. one-sector models with education (Chapter 6);
4. one-sector models with education and technical change (Chapter 7).

The Chapters 3 and 5 are of an introductory nature. They discuss the concepts, as well as issues in modelling these concepts, of technical change and education, respectively. From these chapters it will be seen that both technical change and education are quite complex phenomena for which no unambiguous way of modelling is available. Rather then committing myself to one single model, with its unavoidable limitations, I have studied several models which can be viewed as alternatives for each other. Each model concentrates on one particular aspect of the complex interplay between demography, economic growth, technical change, and education.

The analysis consists of four steps for each model:

1. formulation of the model in mathematical terms;
2. derivation of the necessary conditions for optimal economic growth, using the Maximum Principle of control theory (e.g. Tu, 1984; Kamien & Schwartz, 1983);
3. characterization of steady states as equilibrium points (singular solutions) of the optimal control problem, and comparative statics analysis (which is assessing the effects of changes in the long-run growth rate of the population on the steady-state values of the economic variables);
4. analysis of the non-stationary optimal economic growth path, which is the optimal growth path moving the economy from its initial steady state to its new steady state, after the period of demographic transition has come to an end.

The following assumptions will be made throughout:

1. demographic development is exogenous and known in advance;
2. there is one single production sector that produces an aggregate commodity. Production can be either consumed or added to the stock of physical capital (investment);
3. economic decisions are made by a central planning agency which seeks to maximize some social welfare function in terms of consumption per capita. Decentralized decision-making in the market is ignored (see e.g. Ritzen (1977), ch. 6).

Throughout the book the social welfare function to be maximized by the central planning agency will be assumed to be given by consumption per capita, discounted over an infinitely long time period. Consequently, the Hamiltonian of the corresponding optimal-control problem becomes linear in the savings rate. Thus, the equilibrium of the optimal trajectory is a singular solution. When the growth rate of population changes this singular equilibrium will move. It will be shown in Chapter 2 that the optimal policy is to determine the control variables in such a way that the economy remains in the singular equilibrium, even when this equilibrium itself moves. This result justifies the book's concentration on singular trajectories.

Chapter 8 provides a rather detailed application of the theoretical models to real-life problems. The consequences of the present fertility decline will be analysed in the case of the Netherlands, in order to get some insight into the order of magnitude of the policy adjustments involved. Chapter 9 gives a summary and evaluates the advantages and limitations of the approach pursued in this book.

2 OPTIMAL ECONOMIC GROWTH IN THE SIMPLE ONE-SECTOR GROWTH MODEL

In this chapter I analyse the optimal economic growth path for the simple one-sector growth model of Solow (1956) and Swan (1956). Attention is directed particularly to the effect of demographic development on the nature of this optimal growth path. Unlike previous authors (e.g. Cass, 1965; Samuelson, 1975a) I do not restrict attention to comparative statics effects of population growth but explicitly consider optimal growth paths along which the rate of population growth is changing.

The following assumptions will be made throughout this chapter:

1. There is no time-dependence in the aggregate production function. This assumption will be relaxed in Chapters 3 and 4 where technical change is introduced into the model.
2. Human capital of an individual is a constant function of age. Investment in education will not be considered until Chapter 5.

The plan of this chapter is as follows. Section 2.1 describes the simple one-sector model. In section 2.2 a condition for optimal economic growth will be derived. This condition turns out to be a generalized version of the well-known Golden Rule of Accumulation (e.g. Phelps, 1961). Section 2.3 gives some comparative statics results. These results allow for a generalization of Samuelson's (1975a) analysis of the optimal rate of population growth (section 2.4). Section 2.5 investigates some properties of the non-stationary optimal growth path. The final section summarizes the main results.

2.1 The simple one-sector growth model

The simple one-sector growth model consists of three building blocks: population and labour; production and investment; and the social welfare function to be maximized by the central planning agency. Each block will be described in turn.

2.1.1 Population and labour

The model recognizes overlapping generations in continuous time. The density of the number of individuals born at time t is denoted by $B(t)$. This density develops over time according to the time-dependent growth rate of births, denoted by $g^B(t)$:

$$\hat{B}(t) = \dot{B}(t)/B(t) \equiv g^B(t) \tag{1}$$

Here the notation $\dot{x}$ is used to denote the total derivative of variable x with respect to time; $\hat{x}$ denotes the logarithmic derivative of x.

The dynamic path of $g^B(t)$ will be assumed to be completely exogenously determined.

People die according to some fixed age-specific survival schedule $\mu(\cdot)$. Maximum age is denoted by n. Total population at time t, P(t), can then be written as

$$P(t) = \int_0^n \mu(v) \cdot B(t-v)\, dv \tag{2}$$

where

$$\mu(0) = 1 \qquad \mu(n) = 0 \qquad \mu'(v) \le 0 \tag{3}$$

Each individual is endowed with a stock of human capital, h(v). The stock of human capital is a function of age, v, only, so that all individuals of a given age are equal in their labour-efficiency. Typically the function $h(\cdot)$ is assumed to be unimodal, with low values for v close to zero and for v close to n, and higher values for ages in the intermediate range. The labour force at time t, L(t), measured in units of human capital can now be written as

$$L(t) = \int_0^n h(v) \cdot \mu(v) \cdot B(t-v)\, dv \tag{4}$$

The rate of population growth $g^P(t)$ is defined by

$$g^P(t) = \hat{P}(t) = \dot{P}(t)/P(t) \tag{5}$$

Similarly, the rate of growth of the labour force $g^L(t)$ is defined by:

$$g^L(t) = \hat{L}(t) = \dot{L}(t)/L(t) \tag{6}$$

In general, given the fixed survival schedule $\mu(\cdot)$ and the age-ability profile $h(\cdot)$, the growth rates $g^P(t)$ and $g^L(t)$ are completely determined by the dynamic path of $g^B(t)$. When $g^B(t)$ is constant for an interval of at least length n, then the age structure of the population is also constant and we have:

$$g^B(t) = g^P(t) = g^L(t) = g \tag{7}$$

In this case the population is said to be <u>stable</u>.

2.1.2 Production and investment

The production of the aggregate commodity is described by the following neoclassical production function:

$$Y(t) = F[K(t),\ L(t)] \tag{8}$$

in which Y(t) is aggregate output and K(t) is the stock of physical capital. With respect to the production function F[·] we make the usual neoclassical assumptions:

$$\text{constant returns to scale: } F[\lambda\cdot K(t),\lambda\cdot L(t)] = \lambda\cdot F[K(t),L(t)] \tag{9.1}$$

$$\text{positive marginal products: } F_K[\cdot] > 0 \ ;\ F_L[\cdot] > 0 \tag{9.2}$$

$$\text{concavity: } F_{KK}[\cdot] < 0 \ ;\ F_{LL}[\cdot] < 0 \ ;\ F_{KL}[\cdot] > 0 \tag{9.3}$$

Here F_x and F_{xx} denote the first and second derivatives, respectively, of the production function F[·] with respect to input x.

By virtue of constant returns to scale (9.1) (linear homogeneity) we can write the production function in terms of quantities per unit of human capital:

$$y(t) = f[k(t)] \tag{10}$$

where

$$y(t) = Y(t)/L(t) \tag{11}$$

$$k(t) = K(t)/L(t) \tag{12}$$

$$f[\cdot] \equiv F[\cdot,\ 1] \tag{13}$$

with

$$f_k[\cdot] > 0 \ ; \ f_{kk}[\cdot] < 0 \tag{14}$$

Physical capital is subject to depreciation at a constant rate δ. In each period a fraction of total output is saved and added to the capital stock. The (gross) rate of savings s(t) cannot exceed one: the economy is closed. It will be assumed that physical capital, once installed, is not fit for consumption which implies that the rate of savings cannot become negative. Thus the development over time of the stock of physical capital can be described as follows:

$$\dot{K} = s(t)\cdot Y(t) - \delta\cdot K(t) \qquad \text{with} \qquad 0 \le s(t) \le 1 \tag{15}$$

or, in terms of physical capital per unit of human capital:

$$\dot{k}(t) = s(t)\cdot y(t) - \{\delta + g^L(t)\}\cdot k(t) \tag{16}$$

Output not invested in physical capital is consumed. Total consumption equals:

$$C(t) = \{1-s(t)\}\cdot Y(t) = \{1-s(t)\}\cdot f[k(t)]\cdot L(t) \tag{17}$$

2.1.3 Social welfare

We will take the social welfare function, of which the maximization is the object of the central planning agency, to be simply the discounted sum of per capita consumption:

$$W = \int_0^\infty e^{-\rho t}\cdot\frac{C(t)}{P(t)}\, dt \tag{18}$$

where ρ is the social rate of time preference. For a discussion of this and related social welfare functions see Burmeister & Dobell (1970), pp. 398-400. One reason for choosing specification (18) is that it corresponds closely to the social welfare function in the earlier writings on the steady-state Golden Rule, maximizing long-run sustainable consumption per head.

2.2 The Non-Stationary Golden Rule

The problem confronting the central planning agency of the preceding section can be formulated as follows:

$$\underset{s(\cdot)}{\text{Maximize}} \int_0^\infty e^{-\rho t}\cdot\{1-s(t)\}\cdot f[k(t)]\cdot\frac{L(t)}{P(t)}\,dt \tag{19}$$

subject to

$$\dot{k}(t) = s(t)\cdot f[k(t)] - \{\delta+g^L(t)\}\cdot k(t) \tag{20}$$

$$k(0) = k_0 \tag{21}$$

$$0 \leq s(t) \leq 1 \tag{22}$$

This problem can be solved by straightforward application of Pontryagin's Maximum Principle (e.g. Cass, 1965; Shell, 1967). The corresponding Hamiltonian is given by:

$$H(t) = e^{-\rho t}\cdot\{1-s(t)\}\cdot f[k(t)]\cdot\frac{L(t)}{P(t)} + \psi(t)\cdot\left[s(t)\cdot f[k(t)] - \{\delta+g^L(t)\}\cdot k(t)\right] \tag{23}$$

The adjoint or costate variable $\psi(t)$ satisfies:

$$\dot{\psi}(t) = -\frac{\partial H(t)}{\partial k(t)} = -f_k[k(t)]\cdot\left[e^{-\rho t}\cdot\{1-s(t)\}\cdot\frac{L(t)}{P(t)} + s(t)\cdot\psi(t)\right] + \\ + \psi(t)\cdot\{\delta+g^L(t)\} \tag{24}$$

This costate can be interpreted as the shadow-price of capital (Dorfman, 1969). The transversality conditions for the two-point boundary-value problem are:

$$k(0) = k_0 \quad \text{and} \quad \lim_{t\to\infty} \psi(t)\cdot k(t) = 0 \tag{25}$$

A necessary condition for an optimum is that the control s(t) be chosen so that the Hamiltonian is maximized at any point in time. The partial derivative of H(t) with respect to s(t) is given by:

$$\frac{\partial H(t)}{\partial s(t)} = f[k(t)] \cdot \{- e^{-\rho t} \cdot \frac{L(t)}{P(t)} + \psi(t)\} \tag{26}$$

This expression is independent of s(t). Consequently, the optimal control depends on the successive values of the switch function

$$S(t) = - e^{-\rho t} \cdot \frac{L(t)}{P(t)} + \psi(t) \tag{27}$$

Define the extended shadow-price

$$\varphi(t) = \psi(t) \cdot e^{\rho t} \cdot \frac{P(t)}{L(t)} \tag{28}$$

Then the optimal control is:

$$s(t) = 1 \qquad \text{if} \qquad \varphi(t) > 1 \tag{29.a}$$

$$s(t) = 0 \qquad \text{if} \qquad \varphi(t) < 1 \tag{29.b}$$

$$0 \le s(t) \le 1 \qquad \text{if} \qquad \varphi(t) = 1 \tag{29.c}$$

If $\varphi(t)=1$, a singular solution arises, while cases (29.a) and (29.b) refer to the "bang-bang" parts of the solution. For cases (b) and (c) we have, from (24) and (28):

$$\dot{\varphi}(t) = \varphi(t) \cdot \{\rho + g^P(t) + \delta\} - f_k[k(t)] \tag{30}$$

For case (a):

$$\dot{\varphi}(t) = \varphi(t) \cdot \{\rho + g^P(t) + \delta - f_k[k(t)]\} \tag{31}$$

A <u>singular trajectory</u> occurs when case (c) lasts longer than a single instant. In that case we have as an additional condition, from (29.c):

$$\dot{\varphi}(t) = 0 \tag{32}$$

In combination with (30) this yields:

$$f_k[k(t)] = \rho + \delta + g^P(t) \tag{33}$$

Condition (33) can be interpreted as a generalized Golden Rule. It states that the marginal product of physical capital should equal the rate of population growth, corrected for depreciation and social impatience.

The Non-Stationary Golden Rule (33) has been derived for a discrete-time version of the present model by Ritzen & van Praag (1985). Its most striking property is its close similarity to the traditional Golden Rule for optimal steady-state economic growth:

$$f_k = \rho + \delta + g \tag{34}$$

(cf. Phelps, 1961; Burmeister & Dobell, 1970, p. 397). Indeed, it turns out that the steady-state Golden Rule applies equally well to situations in which population is non-stable.

Not all of the properties of the Golden Rule can be generalized to hold true for its non-stationary counterpart. For example, if $\rho=0$ (i.e. no social time preference), the steady-state condition (34) is equivalent to saying that the savings rate should equal the imputed factor share of capital:

$$s = (g+\delta)\cdot k/y = f_k\cdot k/y \tag{35}$$

This property no longer applies when the economy moves along a non-stationary singular trajectory.

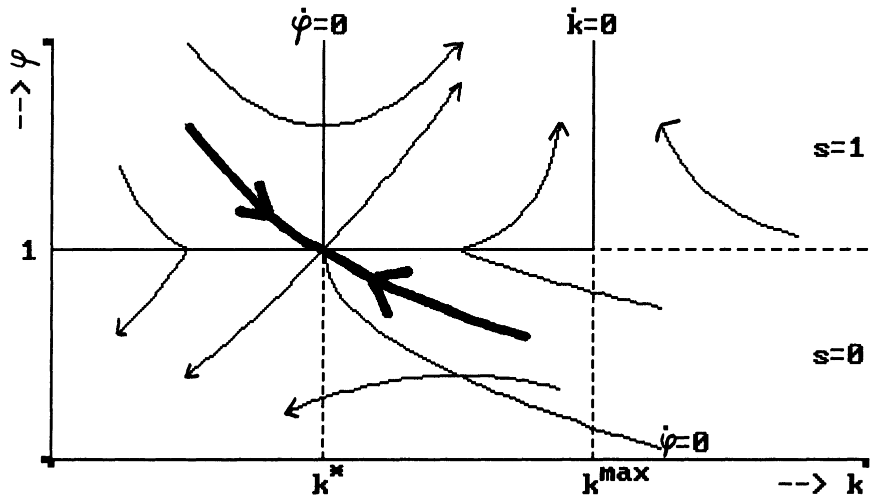
Disregarding for the moment the possible time-dependence of g^L and g^P we can summarize our findings in a phase diagram in (k,φ)- space, see Figure 2.1. Here k^* and k^{max} are implicitly defined by:

$$f_k[k^*] = \rho + \delta + g^P \tag{36}$$

$$0 = f[k^{max}] - (\delta+g^L)\cdot k^{max} \tag{37}$$

From Figure 2.1 it is evident that the only trajectories satisfying the necessary conditions and yielding optimal growth paths are the two leading towards point $(k^*,1)$. Depending on the initial value of k the optimal policy consists of an initial period of either maximum or minimum saving in order to reach the equilibrium point $(k^*,1)$ as quickly as possible. Once this equilibrium point has been reached, the savings rate s(t) should be chosen so that the economy remains forever at the equilibrium point. The optimal savings

Figure 2.1: Phase diagram in (k,φ)-space



rate is obtained by solving (20) subject to the non-stationary Golden Rule (33).

The fact that $g^P(t)$ in (33) is time-dependent does not affect the equilibrium property of point $(k^*,1)$. A change in g^P is equivalent to a shift of the complete phase diagram along the k-axis. The only policy that is clearly optimal is to choose the savings rate s so that the economy remains at the equilibrium point $(k^*,1)$, even though the equilibrium point itself changes position. In other words, once the stage of singular control has been reached, the optimal policy consists of determining the control so that the control remains singular forever after.

Throughout the analysis it will be assumed that the constraint (22) is not binding. If this assumption turns out to be invalid the problem of determining the optimal savings rate can only be solved by explicitly computing the value of the costate variable. This possibility will be ignored from now on.

2.3 Comparative statics and the Golden Rule

In this section we shall consider optimal equilibrium growth in the case of stable population. If population is stable and the economy is in its equilibrium position $(k^*,1)$, then the economy is said to be in steady state.

In comparing steady states three endogenous variables are of special interest:

- the physical capital/human capital ratio k, for simplicity termed the capital/labour ratio;
- the savings rate s;
- consumption per capita C/P.

The steady-state values of these endogenous variables are dependent on the form of the production function f[·] and the following exogenous variables:

- the growth rate of population g;
- the rate of depreciation δ;
- the social rate of time preference ρ.

Comparative statics consists of determining the nature of the relationship between the exogenous and endogenous variables.

The following three equations serve as the starting point for the analysis:

the Golden Rule: $f_k = \rho + \delta + g$ (38)

the steady-state condition: $\dot{k} = 0 = s \cdot f - (\delta + g) \cdot k$ (39)

the definition of consumption per capita: $C/P = (1-s) \cdot f \cdot L/P$ (40)

Total differentiation of (38)-(40) yields:

$$\begin{bmatrix} f_{kk} & 0 & 0 \\ sf_k - (\delta+g) & f & 0 \\ -(1-s)f_k \cdot L/P & f \cdot L/P & 1 \end{bmatrix} \begin{bmatrix} dk \\ ds \\ dC/P \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ k & k & 0 \\ (1-s)f\frac{\partial(L/P)}{\partial g} & 0 & 0 \end{bmatrix} \begin{bmatrix} dg \\ d\delta \\ d\rho \end{bmatrix} \tag{41}$$

For a linearly homogenous production function the elasticity of substitution can be written as follows (e.g. Allen, 1938):

$$\sigma \equiv \frac{d \log (K/L)}{d \log(F_L/F_K)} = - \frac{f_k(f-kf_k)}{kff_{kk}} \tag{42}$$

By using (38), (39) and (42), inversion of the matrix on the left-hand side of (41) yields:

$$\begin{bmatrix} f_{kk} & 0 & 0 \\ sf_k - (\delta+g) & f & 0 \\ -(1-s)f_k \cdot \frac{L}{P} & f \cdot \frac{L}{P} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{f_{kk}} & 0 & 0 \\ -\frac{k}{f} \cdot \sigma \cdot \frac{\delta+g}{\rho+\delta+g} & \frac{1}{f} & 0 \\ \frac{L}{P} \cdot \rho \cdot \frac{1}{f_{kk}} & -\frac{L}{P} & 1 \end{bmatrix} \tag{43}$$

so that:

$$\frac{dk}{dg} = \frac{dk}{d\delta} = \frac{dk}{d\rho} = \frac{1}{f_{kk}} < 0 \tag{44}$$

$$\frac{ds}{dg} = \frac{ds}{d\delta} = \frac{k}{f}\cdot\left[1 - \sigma\cdot\frac{\delta+g}{\rho+\delta+g} \right] \gtreqless 0 \tag{45}$$

$$\frac{ds}{d\rho} = - \frac{k}{f}\cdot\sigma\cdot\frac{\delta+g}{\rho+\delta+g} < 0 \tag{46}$$

$$\frac{d(C/P)}{dg} = \frac{L}{P}\cdot\left[\rho\cdot\frac{1}{f_{kk}} - k \right] + (1-s)\cdot f\cdot\frac{\partial(L/P)}{\partial g} \gtreqless 0 \tag{47}$$

$$\frac{d(C/P)}{d\delta} = \frac{L}{P}\cdot\left[\rho\cdot\frac{1}{f_{kk}} - k \right] < 0 \quad (\text{for} \quad \rho\geq 0) \tag{48}$$

$$\frac{d(C/P)}{d\rho} = \frac{L}{P}\cdot\rho\cdot\frac{1}{f_{kk}} \leq 0 \quad (\text{for } \rho\geq 0) \tag{49}$$

The definitely negative signs of dk/dg, $dk/d\delta$, $dk/d\rho$, $ds/d\rho$, $d(C/P)/d\delta$ and $d(C/P)/d\rho$ are plausible. Since $d(C/P)/d\rho$ in (49) changes sign whenever ρ does, it follows that long-run optimal and sustainable consumption per head is maximum for $\rho=0$, as one would expect.

The effect of both g and δ on s cannot be signed without additional information on ρ and σ, the social rate of time preference and the elasticity of substitution, respectively. For $\rho=0$ and $\sigma<1$, the effect of both g and δ on the optimal savings rate is positive. For positive ρ such a positive effect also prevails for σ somewhat larger than unity. These observations suggest that the effect of g and δ on s is generally positive.

2.4 Optimal population growth

The effect of the steady-state rate of population growth on long-run consumption per capita in equation (47) is expressed in two terms. The first term on the right of the equality sign is negative. Thus, if the ratio L/P does not change with g, i.e. if the age/labour-efficiency profile is uniform, then consumption per capita is inversely related to the rate of population growth. This would suggest that an optimal population policy consists of choosing g as small as possible, preferably negative.

Samuelson considered the problem of optimum population growth in (1975a). He argued that the not very appealing result of the previous paragraph can be overcome by extending the model with life-cycle saving behaviour. If the period during which individuals are active in the labour market is followed by a period of retirement, the retired persons must be supported by the working population. The lower the growth rate of population, the less the number of working individuals available to support a single retired person (cf. Samuelson, 1958). Samuelson subsequently showed that some optimal growth rate of population, for which these two counter-acting effects of population growth on consumption per capita cancel out exactly, might well exist (see, however, Deardorff, 1976; Samuelson, 1976).

The presence of the second term in our equation (47) provides a generalization of Samuelson's argument (see also Arthur & McNicoll, 1978). From (2) and (4) we have, for stable population:

$$\frac{L}{P} = \frac{\int_0^n h(v)\cdot\mu(v)\cdot e^{-gv}\, dv}{\int_0^n \mu(v)\cdot e^{-gv}\, dv} \tag{50}$$

Let us define:

$$\ell(v) = \frac{h(v)\cdot\mu(v)\cdot e^{-gv}}{\int_0^n h(v)\cdot\mu(v)\cdot e^{-gv}\, dv} \qquad 0 \le v \le n \tag{51}$$

$$p(v) = \frac{\mu(v)\cdot e^{-gv}}{\int_0^n \mu(v)\cdot e^{-gv}\, dv} \qquad 0 \le v \le n \tag{52}$$

Both $\ell(\cdot)$ and $p(\cdot)$ can be interpreted as proper probability density functions. Consequently we can define the mean and variance of these probability density functions. The mean age of the labour force and its variance are, respectively:

$$m^L(v) = \int_0^n v\cdot\ell(v)\, dv \tag{53}$$

$$var^L(v) = \int_0^n \{v-m^L(v)\}^2\cdot\ell(v)\, dv \tag{54}$$

while the mean age of the population and its variance are given by:

$$m^P(v) = \int_0^n v\cdot p(v)\, dv \tag{55}$$

$$\mathrm{var}^P(v) = \int_0^n \{v-m^P(v)\}^2 \cdot p(v)\, dv \tag{56}$$

All these quantities are functions of the age-efficiency profile, $h(\cdot)$, the survival schedule, $\mu(\cdot)$, and the long-run population growth rate, g.

Using these definitions we have, on differentiating (50) with respect to g:

$$\frac{d}{dg}\left[\frac{L}{P}\right] = \frac{L}{P}\cdot\{m^P(v) - m^L(v)\} \tag{57}$$

$$\frac{d^2}{dg^2}\left[\frac{L}{P}\right] = \frac{L}{P}\cdot\{m^P(v) - m^L(v)\}^2 + \frac{L}{P}\cdot\{\mathrm{var}^L(v) - \mathrm{var}^P(v)\} \tag{58}$$

Equation (57) states that a higher growth rate of population increases (decreases) the labour force/population ratio if and only if the mean age of the population is higher (lower) than the mean age of the labour force. This result makes sense: since a fast-growing population has a relatively young age-structure, the relative productivity of the population will increase with g if labour efficiency is concentrated in the younger age groups, and vice versa.

Equation (58) shows how g affects the difference between the mean ages of the population and of the labour force. If these mean ages are equal, implying that the labour force/population ratio reaches an extremum for the particular value of g, then the sign of the effect depends on the relative variances of the two age distributions. The unimodality of the age-efficiency profile $h(\cdot)$ guarantees that the variance of the age of the population ($\mathrm{var}^P(v)$) is larger than the variance of the age of the labour force ($\mathrm{var}^L(v)$). Thus, the expression on the right-hand side (RHS) of (58) is negative for $m^L(v)=m^P(v)$. This implies that an extremum of L/P is a maximum.

From the foregoing analysis we conclude that for very low values of g the second term on the RHS of (47) could well become positive. We cannot exclude the possibility that there exists an adverse effect of g on L/P strong enough to impose an effective lower bound to the extent to which the rate of population growth can be profitably reduced.

The two-generation model of Samuelson (1975a) emerges as a special case of our model by putting:

$$\mu(v) = 1 \qquad \text{for all } v,\ 0 \le v \le n \tag{59}$$

$$h(v) = \begin{cases} 1 & \text{for all } v,\ 0 \le v \le n/2 \\ 0 & \text{for all } v,\ n/2 < v \le n \end{cases} \tag{60}$$

It is not hard to see that in this special case the RHS of (57) is always positive.

The comparative-statics results derived in this and the previous section are summarized in Table 2.1.

Table 2.1: Comparative-statics results for the simple one-sector growth model

effect of: on	g	δ	ρ
k	-	-	-
s	+ 1)	+ 1)	-
C/P	- 2)	-	sign(ρ)

1) for $\sigma < 1 + \frac{\rho}{\delta+g}$

2) except possibly for very low values of g

2.5 The non-stationary optimal growth path

In this section I will explore the nature of the optimal growth path during periods of demographic transition. It will be clear that the issues involved are so complex that it is very difficult, if not impossible to obtain any definite general results. As a "second-best" solution to the problem I will indicate some plausible properties of the non-stationary optimal growth path. Some numerical examples are included to illustrate the main points.

The steady state will serve as the starting point for the analysis. The non-stationarity of the population is induced by a gradual change in the dynamic path of the growth rate of births $g^B(t)$, causing $g^P(t)$ and $g^L(t)$ to change as well. The economy will only return to a new steady-state growth path if the growth rate of births stops changing and remains constant for at least n periods. Typically one should think of $g^B(t)$ gradually decreasing over time, this being the present demographic experience in most industrialized countries.

As in the previous section I concentrate on the dynamic path of k, s and C/P.

For k we have, from the non-stationary Golden Rule (33):

$$\dot{k} = \frac{1}{f_{kk}} \cdot \dot{g}^P \tag{61}$$

Thus k changes over time in the opposite direction of g^P.

For s we have, from (20):

$$s = \frac{\dot{k} + (\delta + g^L) \cdot k}{f[k]} \tag{62}$$

Differentiation with respect to time yields:

$$\dot{s} = \frac{1}{f} \cdot \left[\ddot{k} - \frac{f_k}{f} \cdot (\dot{k})^2 + k \cdot \left[\dot{g}^L - \dot{g}^P \cdot \sigma \cdot \frac{\delta + g^L}{\rho + \delta + g^P} \right] \right] \tag{63}$$

using (61), (33) and (42). Here the notation $\ddot{x}$ is used to denote the second derivative of variable x with respect to time. For $\ddot{k}$ we have, from (61):

$$\ddot{k} = \frac{1}{f_{kk}} \cdot \ddot{g}^P - \frac{f_{kkk}}{f_{kk}^2} \cdot \dot{k} \cdot \dot{g}^P = \frac{1}{f_{kk}} \cdot \ddot{g}^P - \frac{f_{kkk}}{f_{kk}^3} \cdot (\dot{g}^P)^2 \tag{64}$$

Now (63) can be written as:

$$\dot{s} = \frac{1}{f} \cdot \left[\frac{\ddot{g}^P}{f_{kk}} - \left[\frac{\dot{g}^P}{f_{kk}} \right]^2 \cdot \left[\frac{f_k}{f} + \frac{f_{kkk}}{f_{kk}} \right] + k \cdot \left[\dot{g}^L - \dot{g}^P \cdot \sigma \cdot \frac{\delta + g^L}{\rho + \delta + g^P} \right] \right] \tag{65}$$

The term f_{kkk}/f_{kk} in (65) is not recognized as some familiar characteristic of the production function (Miller (1976), however, discusses the role of the third derivative of utility functions). Its algebraic sign can only be determined for special classes of production functions. For CD functions the sign is negative. For CES functions the sign can be ascertained by differentiating the elasticity of substitution (42) with respect to k:

$$\frac{d\sigma}{dk} = -\frac{d}{dk}\frac{f_k(f-f_k k)}{kff_{kk}} = -\frac{1}{kff_{kk}}\cdot\{f_{kk}(f-2kf_k)+f_{kk}(f+kf_k+kf\frac{f_{kkk}}{f_{kk}})\cdot\sigma\} \tag{66}$$

If we assume that $d\sigma/dk=0$, as is the case for the class of CES production functions, then after some rearranging of terms (66) yields:

$$\frac{f_{kkk}}{f_{kk}} = \frac{1}{\sigma k}\cdot\left[\ (2-\sigma)\cdot\frac{kf_k}{f} - (1+\sigma)\ \right] = \frac{1}{\sigma k}\cdot\{(2-\sigma)\cdot\pi_K - (1+\sigma)\} \tag{67}$$

where

$$\pi_K = \frac{kf_k}{f} \tag{68}$$

denotes the imputed factor share of physical capital. Since π_K is between zero and unity, the RHS of (67) is negative for all $\sigma>1/2$. In addition, for $\pi_K<1/2$ the RHS of (67) is negative for all $\sigma>0$. Thus we should expect f_{kkk}/f_{kk} to be negative. This property boils down to assuming the third derivative of the production function to be positive, implying that the marginal product of capital, f_k, is downward-sloping and convex to the origin.

Furthermore we have, from (67) and (68):

$$\frac{f_k}{f} + \frac{f_{kkk}}{f_{kk}} = \frac{\pi_K}{k} + \frac{1}{\sigma k}\cdot\{(2-\sigma)\cdot\pi_K - (1+\sigma)\} = \frac{1}{\sigma k}\cdot\{2\pi_K - (1+\sigma)\} \tag{69}$$

Again, the RHS of (69) is negative for all $\sigma>0$ if $\pi_K<1/2$ and is negative for $\sigma>1$ for any π_K. Barring unrealistic values for π_K and σ we thus expect $(f_k/f)+(f_{kkk}/f_{kk})$ to be negative. This property implies that the curvature of the marginal product function (kf_{kkk}/f_{kk}) exceeds the elasticity of the production function (kf_k/f).

Now we return to equation (65) and sign the various terms on its right-hand side. In the initial phases of a period of demographic transition with the growth rate of births falling, $\dot{g}^P$ will be negative while $\dot{g}^L$ will be almost zero. Furthermore, $\ddot{g}^P$ will also be negative. Then all terms on the RHS of (65) are positive, yielding the conclusion that the savings rate <u>rises</u> on an initial time interval of the non-stationary optimal growth path, as the population growth rate falls.

After some time the decreasing growth rate of births will have pushed itself further and further into the age-structure of the population. Gradually $\ddot{g}^P$ will return towards zero, while at the same time $\dot{g}^L$ becomes negative and approaches $\dot{g}^P$ in absolute value. For positive ρ and/or σ smaller than unity this will result in the third term becoming negative. The net effect on s is ambiguous: it might be negative, zero or positive during the middle phases of the transition period. Whatever may be the case, it is certain that the <u>rise of s decreases over time</u>.

Finally, the decrease in the birth growth rate comes to a halt. $\dot{g}^P$ becomes gradually zero, $\ddot{g}^P$ being positive and $\dot{g}^L$ lagging behind $\dot{g}^P$. Ultimately the economy will reach a phase in which the first and third term on the RHS of (65) are negative and more than offset the positive second term which becomes less important as $\dot{g}^P$ approaches zero. Thus before returning to steady state the optimal growth path will exhibit a <u>falling</u> savings rate.

I therefore conclude that during a period of demographic transition with the growth rate of births falling, the curve of the optimal savings rate follows an inverted U-shaped pattern.

The presence of the second term on the RHS of (65), expected to be generally positive regardless of the direction of the demographic change, suggests that the preceding analysis can <u>not</u> simply be inverted for application to the case of a rising growth rate of population. The initial path of s cannot unambiguously be signed in the latter case. Either there is some interval over which s initially falls, in which case the complete curve is more J-shaped than U-shaped, or s is rising all the time.

The development of consumption per capita along the non-stationary Golden Rule-path can be derived from (40) and (20):

$$C/P = (L/P)\cdot\{f[k] - \dot{k} - (\delta+g^L)\cdot k\} \tag{70}$$

Differentiating (70) with respect to time, using (33), yields:

$$\frac{d}{dt}\left[\frac{C}{P}\right] = \frac{C}{P}\cdot(g^L - g^P) + \frac{L}{P}\cdot\{(\rho+g^P-g^L)\cdot\dot{k} - \ddot{k} - \dot{g}^L\cdot k\} \tag{71}$$

Consider first the terms within the second pair of brackets. For g^B falling, the third term is positive during the complete period of demographic transition, and for reasonable values of ρ so is the first term; the second term is initially negative, finally positive but except for a short initial interval probably relatively small as compared to the two other terms. Thus the complete expression $(L/P)\cdot\{(\rho+g^P-g^L)\cdot\dot{k}-\ddot{k}-\dot{g}^L\cdot k\}$ is expected to be positive along almost the full non-stationary growth path.

The first expression in (71), on the other hand, is initially positive, ultimately negative, while its sign during the middle phases of the demographic transition depends on whether the respective growth rates are above or below some critical value. For the reasons discussed in the previous section, under "normal" circumstances these growth rates are not sufficiently low to cause any significant adverse effect on the ratio L/P. As a consequence the term g^L-g^P is expected to be positive for the main part of the transition period, except for some relatively short final interval when $\dot{g}^L$ is close to zero and $\dot{g}^P$ slightly negative.

I conclude that a falling birth growth rate generally causes a steady rise in consumption per capita, except for a possible slight decline during a short initial interval as well as just before the economy settles back to its new steady-state growth path.

Table 2.2 summarizes the findings of the analysis of the non-stationary optimal growth path.

Some numerical illustrations are given in Figures 2.2 and 2.3, for falling and rising birth growth rates, respectively. The production function used is of the constant elasticity of substitution variety, with σ taking values of 0.5 and 0.9. The values of the parameters used in the computations are given in Table 2.3.

The illustrations highlight the conclusions of the theoretical analysis. The optimal savings rate follows an inverted U-shaped or U-shaped pattern, according to whether the growth rate of population is falling or rising. Consumption per capita is almost invariably inversely related to the growth rate of population.

Table 2.2: Characteristics of the non-stationary optimal growth path

	$\dot{k}$	$\dot{s}$	$(\dot{C/P})$
1. g^B falling			
- i nitial phases	> 0	> 0	?
- midd le phases	> 0	?	> 0
- final p hases	> 0	< 0	?
2. g^B rising			
- initial phases	< 0	?	?
- middle phases	< 0	> 0	< 0
- final phases	< 0	> 0	?

Table 2.3: Parameters used for illustrations in Figures 2.2 and 2.3

Number of generations	4
Survival schedule	$\mu(1)=1$; $\mu(2)=0.9$; $\mu(3)=0.7$; $\mu(4)=0.4$
Age-ability profile	$h(1)=0$; $h(2)=1$; $h(3)=1$; $h(4)=0$
Social rate of impatience	$\rho=0$
Depreciation rate	$\delta=0.3$
Growth rate of births	0.30 0.25 0.20 0.15 0.10 0.05 0.00
Production function	$y=\{\alpha k^{-\frac{1-\sigma}{\sigma}} + (1-\alpha)\}^{-\frac{\sigma}{1-\sigma}}$
	$\alpha=0.25$
	$\sigma=0.5$ / 0.9 (upper / lower panel)

Figure 2.2: Optimal economic growth with g^B falling

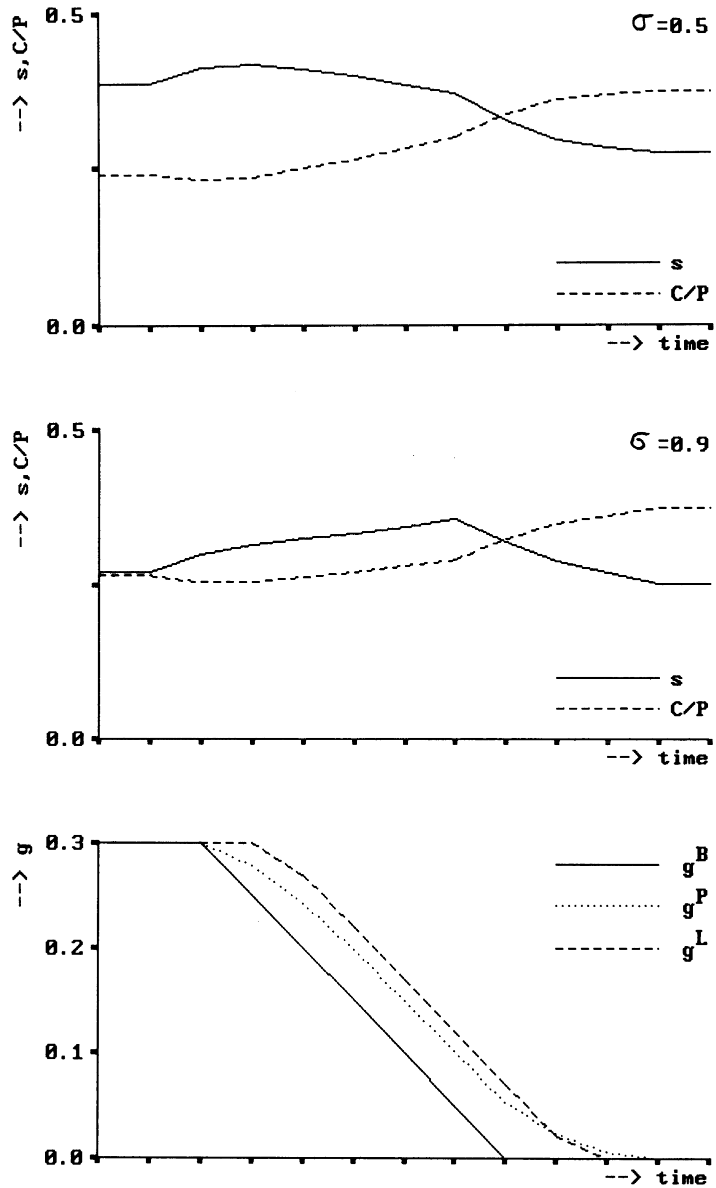
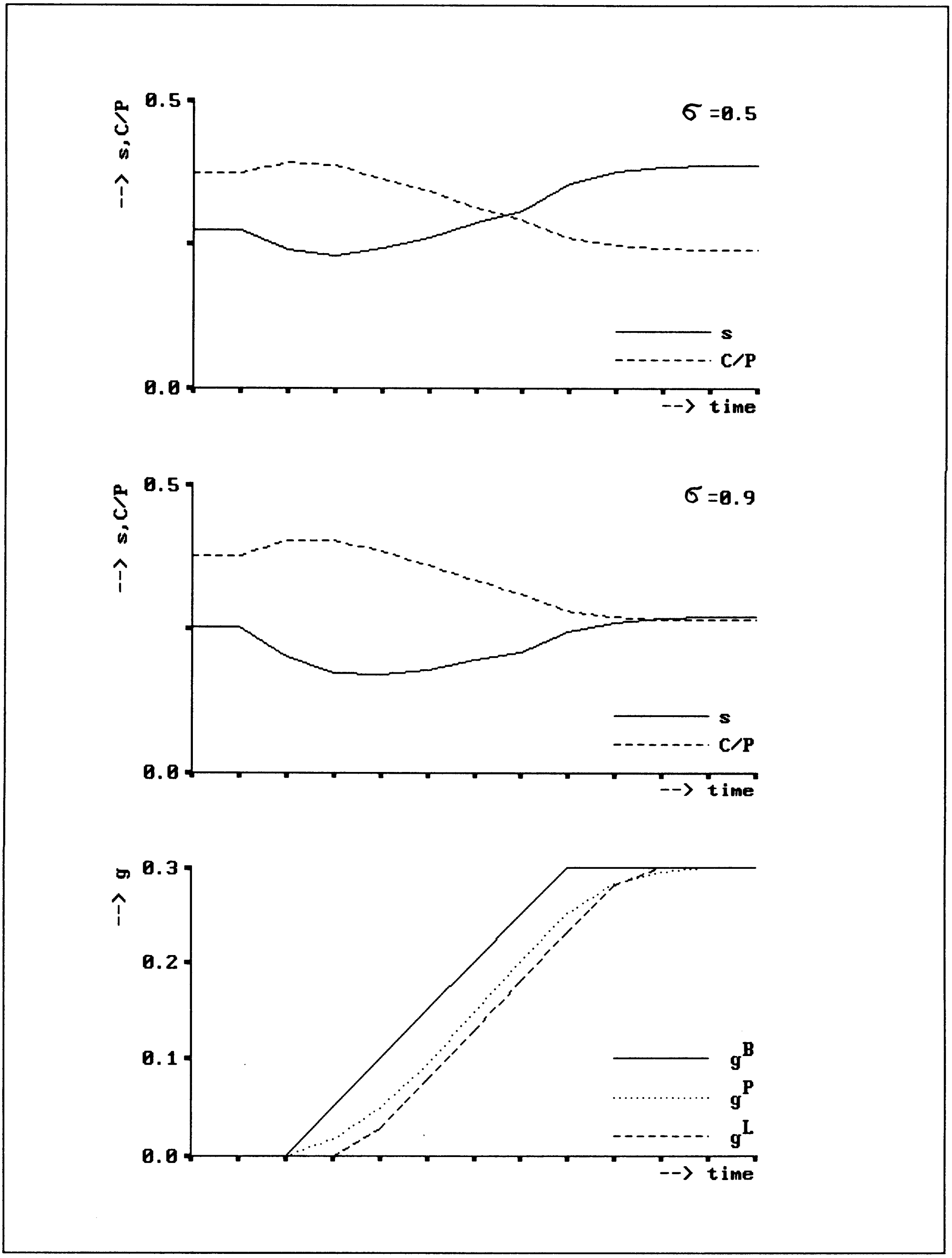


Figure 2.3: Optimal economic growth with g^B rising



2.6 Summary

In this chapter I have analysed optimal economic growth in the simple one-sector neoclassical growth model.

The optimal growth path under non-stable population is characterized by a Golden Rule which is a strikingly straightforward generalization of the traditional steady-state Golden Rule of Accumulation.

Comparative statics analysis shows that a rise in the long-run growth rate of population should generally be accompanied by an increase of the optimal rate of savings and results in a decrease in consumption per capita, the latter possibly with the exception for very low values of g. The results allow for a generalization of Samuelson's (1975a) analysis of optimal population policy. My conclusion is that there could be an effective lower-bound to the extent to which the rate of population growth can be profitably reduced. When the mean age of the population exceeds that of the labour force, then it is no longer a certainty that a reduction of the rate of population growth increases per capita consumption along optimal trajectories of economic growth.

The nature of the optimal growth path during periods of demographic transition is so complex that definite general results cannot be obtained. I have shown that under plausible assumptions the optimal savings rate follows an inverted U-shaped or U-shaped pattern, according to whether the growth rate of births is falling or rising. Consumption per capita, on the other hand, is for most of the time inversely related to the growth rate of population.

3 TECHNICAL CHANGE AND ECONOMIC GROWTH

The one-sector growth model presented in Chapter 2 suffers from one major drawback: it does not allow for a rise in income or consumption per capita in the long run. Except for obviously unrealistic functional forms of the production functions there is a definite upper limit to the long-run rate at which physical capital can be advantageously accumulated. As soon as the economy has reached this long-run upper limit, marking the beginning of the Golden Age, consumption per capita remains at its highest possible level which is constant over time.

If the accumulation of capital cannot explain the secular rise of consumption per capita, then there must be some other determinant of economic growth (always in per capita terms) at work. This determinant of long-run growth of income per head is generally termed technical change.

Since the concept of technical change is rather vague I will devote some space to a precise statement of what technical change exactly is and how it can be introduced into models of economic growth. These introductory issues are dealt with in the present chapter. In the sub-chapters of Chapter 4, a number of different growth models with technical change will be analysed with special reference to the case of non-stable population. There the analysis of Chapter 2 will be generalized for the more realistic case in which technical change prevails.

The plan of this chapter is as follows. In Sub-chapter 3.1 a definition of technical change is given. In Sub-chapter 3.2 I discuss three major topics in modelling technical change: the endogeneity issue, the neutrality issue and the embodiment issue. Sub-chapter 3.3 gives an overview of one-sector growth models with technical change ordered according to the classifications developed in Sub-chapter 3.2. This overview serves to clarify the selection of models that will form the object of analysis in Chapter 4.

3.1 On the concept of technical change

The literature on technical change, both in growth theory and in other areas of economic research, has reached vast proportions during the last thirty years. Still, the ongoing debate does not seem to have even produced a generally accepted definition of the concept itself. Typical for this confusing state of affairs is the survey article by Kennedy & Thirlwall (1972). In their introduction the authors announce that they will "use the term [technical progress] in two main senses" (p. 12), thus carefully circumventing the problem of defining the subject matter of their article but at the same time missing the opportunity to settle, once and for all, the debate's terminological confusion.

In the early days Solow (1957), in his typically uncomplicated style, defined technical change simply as "any kind of shift in the production function" (p. 312). This characterization quickly conquered growth theory, permitting the discussion to focus on higher-level topics, like the exogeneity/endogeneity, neutrality and embodiment issues. In this general spirit Wan (1971) states that technical change "implies a ... shifting of the entire isoquant map" (p. 147), in order to subsequently concentrate his attention upon the nature of these shifts.

The snag is, of course, that this view of technical change merely transfers the problem of definition to the production function itself, or for that matter to the corresponding isoquant map. A production function is generally believed to describe the relationship between factor quantities and the maximum output that can be produced from those factor quantities. But most (if not all) concepts included in this "definition" are not at all unambiguously defined. Consequently, the question whether shifts in the production function have occurred, let alone what kind of shifts, hinges on more fundamental questions, of which the most relevant are the following:

- What are factors of production? If by these all variables are meant that affect maximum output then by definition no shifts in the production function occur.

- What is exactly meant by quantities or, more specifically, in what units are quantities measured. If "technical change" merely affects units of measurement then the relationship between factor quantities and maximum output remains unchanged.

This second question is of course related to the first one.

With regard to the first fundamental question, let us consider knowledge. Kennedy & Thirlwall (1972) define technology as "useful knowledge pertaining to the art of production" (p. 12) and denote changes in technology by the term technical change. Such "useful knowledge" can be increased by allocating resources to research and development, just as the stock of physical capital can be increased by building new "useful machines". It is not immediately clear why the former type of investment should be labelled "shifting the production function" while the latter is simply labelled "accumulation of physical capital". Shell (1967b) constructs a growth model in which "technical knowledge" and physical capital are treated in a practically identical way. Yet, when in the same volume the model is slightly reformulated it appears that he interprets increases in technical knowledge as shifting the production function (Shell, 1967a).

The second fundamental question can be illustrated by considering the role of education, which will be discussed more fully in Chapter 5. Few economists nowadays will question the positive impact of education on the productivity of the labour force, convinced as they are by the pioneering work of Schultz (1961a; 1961b; 1962) and Becker (1964/1975). If we measure the labour force in biological units, e.g. man-hours, then increased education could well be labelled "technical progress" (e.g. Uzawa, 1965; Wan, 1971; Kennedy & Thirlwall, 1972). Alternatively, the very name of the Human-Capital school suggests that the labour force should be measured in units of human capital (e.g. Hu, 1976; Ritzen, 1977). Skipping for the moment the question whether "human capital" should be interpreted as a homogeneous or a heterogeneous concept (cf. Chapter 5), this view implies that education merely increases the quantity of some factor of production (or, alternatively, transforms one factor of production into another factor of production) and has not necessarily anything to do with technical change.

In my view the two fundamental questions formulated above can only be answered after agreement has been reached over a third one:

- How do we visualize the process by which output is produced? Who or what directs the production process, what elements of the process are under his/its control and what elements are not? How does the time dimension enter the production process?

This third fundamental question is in a way the most fundamental of all and a proper answer to it would probably solve all other problems associated with the "tenuous concept of the production function" (Kennedy, 1966, p. 442). Unfortunately, the problems raised are of such infinite complexity that a reconciliation between completeness and manageability of the answer seems fairly remote. Thus, even if it were my ambition (which it is not) to settle two centuries of controversy over production theory I would not have the ability to satisfy it. This being so I shall merely seek to sketch a framework which is sufficiently consistent to allow a satisfactory characterization of technical change.

Even for centrally planned economies it is realistic to describe production as being carried out by decentralized production units or firms. The "resources" available for production are distributed over these firms. The production process within the firm is assumed to be directed by an entrepreneur who is so closely linked to the firm that he might be identified with it. It is assumed that the time span between the start of the production process and the moment at which the output produced becomes available for further use is "short", in the sense that both the distribution of resources over firms and all other variables affecting maximum output remain constant during this time span.

The economy as a whole is characterized by either a set of competitive markets or a central authority (or some combination of the two). Imagine that just before the start of the production process either the market or the central authority can "threaten" to change the variables affecting a firm's maximum output in such a direction that the change would damage the firm's achievement.

I define any variable for which the threat might conceivably be executed by the term "factor of production". All other variables, of which the values could not conceivably be changed in a direction disadvantageous to the firm even by the most malicious authority, are not factors of production; they determine the position and shape of the production function but do not enter the production function as arguments, only as parameters. Finally, whenever either the marginal rate of substitution between two factors of production is constant or two factors are always used in fixed proportions, they are treated as a single factor of production after suitable redefinition of their respective units of measurement (cf. Hicks, 1939/1946).

Now the production function can be defined as describing the relationship between the quantities of the factors of production (as defined in the previous paragraph) and the maximum output that can be produced with these factor quantities. Any shift in this relationship is defined as technical change.

The preceding characterization of the production process and the corresponding definition of a factor of production may appear far-fetched and needlessly complicated. It should be kept in mind, however, that its only purpose is to allow a satisfactory distinction between technical change, on the one hand, and changes in factor inputs on the other, the two being, by definition, the only determinants of maximum output. Let us take some examples and find out whether this purpose has been served.

Consider knowledge. Knowledge is information that is to be found in entrepreneurial brains, books, blueprints, computer tapes or similar articles. As far as knowledge lives in the entrepreneurial brain is does not fit our definition of factor of production: it cannot be removed by external forces without seriously injuring the entrepreneur himself and this has already been ruled out by identifying the entrepreneur with the firm. When knowledge is in books, not brains, the definitional test is somewhat more troublesome to apply. If the book is not in print (imagine Alfred Nobel suffering a fatal stroke while putting the last hand to his treatise on the art of dynamite production) it certainly should be labelled a factor of production. Apart from these bizarre special cases it is hard to imagine a society in which an entrepreneur is not able to obtain new copies of books recently confiscated by secret police if he really wanted to.

When knowledge is distributed by means of a patent system the answer depends on the degree of compliance with the patent laws. With 100 percent compliance such knowledge would be a factor of production. Although it is not so much the knowledge itself as it is the patent warranting its use that fits the definitional test, both variables are treated as one single factor of production, being always used in fixed proportions under full compliance. However, examples like Rubick's Cube being freely Made in Taiwan indicate that full compliance is not generally achievable.

As a consequence of the preceding discussion the diffusion of knowledge should be labelled technical change (or progress). Here diffusion of knowledge is understood to indicate an increase in the number of entrepreneurial brains equipped with the knowledge in question; it should not be confused with the embodiment of technical change, which will be discussed more fully in Chapter 3.2, Section 3.2.3. The characterization of diffusion of knowledge as technical progress is in accordance with authors like Hicks (1965) and Heertje (1977). On the other hand, Kennedy & Thirlwall (1972) specifically state that "technical progress should not be confused with the diffusion of existing technical knowledge, which does nothing to change production possibilities" (p. 12). But, indeed, when production is decentralized, diffusion of knowledge does change production possibilities. The view of Kennedy and Thirlwall seems to be shared by Nelson & Phelps (1966), although their formulation is less explicit and somewhat ambiguous.

Next, consider education. For one thing, education increases the productive potential of individuals enjoying it. It is clear that the labour services of an educated person can be denied to an individual firm, either through the market (a competitor offering a higher wage) or by the central authority (decreeing transfer of the person involved). Thus, according to my definition, education is not technical change but increases the stock of some factor of production. Moreover, if educated people perform the same task as non-educated people, only performing better, then all labour services derived from both educated and non-educated people are lumped together into one single factor of production called, say, "human capital"; if, on the other hand, the marginal rate of substitution between educated and non-educated labour is not constant, implying that the two types of labour perform qualitatively different tasks, then at least two factors of production should be distinguished. This discussion will be continued in Chapter 5.

Carrying the implications of my definition to their limits, a change of weather might be labelled a - probably temporary - technical change. Thus, rising world prices for coffee when frost spoils most of Brazil's coffee crop can be related directly to exogenous disembodied Hicks-neutral technical regress. While such a statement is admittedly not very meaningful I believe that the case of weather-induced technical change illustrates the conceptual clarity and consistency of the definitions given above.

A concept of technical change based on the fundamentals of production theory is to be preferred to inaccurate and ad-hoc definitions like Binswanger's (1978a): "technical change ... refer[s] to changes in techniques of production at the firm or industry level that result both from research and development and from learning by doing" (pp. 18-19).

3.2 Classifications of technical change

Using the concept of the production function developed in the previous section, technical change in period t is defined as any shift which transforms the production function prevailing in period t-1 into the production function prevailing in period t. In general the nature of the shift will depend on the arguments of the production function, i.e. on the quantities of the factors of production. If we make the standard (though by no means innocent; cf. Samuelson, 1947/1983, pp. 83-87) neoclassical assumption that the production function is homogeneous of degree one in all of its arguments, then the magnitude of the shift is proportional to the scale of production and we can equivalently define technical change in terms of the unit-isoquants in periods t-1 and t, respectively.

The rate of technical change at time t will be defined as the magnitude of the shift that transforms the production function $F[\cdot;t-1]$ into $F[\cdot;t]$ after correction for the scale of output. The bias or direction of technical change will be defined as the change in the categoral income distribution that results from the technical change. These two components of the concept of technical change will be discussed more fully in Section 3.2.2.

Most of the time I will restrict my attention to the hyper-aggregative case of two factors of production, "capital" K and "labour" L, as in the neoclassical one-sector model of Chapter 2.

In this chapter I will discuss some issues on the nature of technical change. At the same time I will try to give an overview of the literature on technical change in models of economic growth.

Three major classifications are common in the literature:

- exogenous vs. endogenous technical change;
- neutral vs. biased technical change;
- embodied vs. disembodied technical change.

Each classification will be discussed in turn.

3.2.1 Exogenous vs. endogenous technical change

A variable is called exogenous with respect to a particular model if its value is not affected by forces explained in that model. For those economists who dislike thinking in terms of models, an exogenous variable is completely determined by non-economic forces, i.e. is not affected by people's management of scarce resources that have alternative uses. All variables that are not exogenous are endogenous.

Thus, when Solow writes some antilog of time in front of his production function he clearly treats technical change as an exogenous variable. The justification for doing so can be found in one of several arguments, given in decreasing order of implausibility:

- blueprints for new techniques of production are falling like manna from heaven;
- changing the figures in Edison's dictum somewhat, technical change is for 100 percent the result of inspiration and has nothing to do with economic forces ("perspiration");
- technical change *does* require perspiration but the extent to which perspiration is shed does not depend on any variables explained in the growth model under consideration.

Although modelling technical change as a purely exogenous variable can certainly yield some fruitful insights, the original figures mentioned by Edison indicate that it should serve only as a first step towards a more satisfactory, i.e. endogenous treatment.

The general way in which technical change has been defined suggests that it is not just a single variable but rather a whole complex of variables that economists should try to explain. Numerous authors have constructed models in which one or several variables out of this complex total have been made endogenous. As far as technical change is concerned it is not so much a question of exogeneity versus endogeneity; but rather that exogeneity is only the beginning of the scale which ranges through increasing degrees of endogeneity to finally disappear into the infinity of complete understanding.

An initial modest degree of endogeneity is obtained with models in which some kind of research sector is discerned. In such models some aggregate called "stock of technical knowledge" is introduced of which the rate of growth is an increasing function of research effort (e.g. Uzawa, 1965; Phelps, 1966a; Shell, 1967b). However, the nature of this function as well as the direction of the resulting technical change (usually postulated to be Harrod-neutral) remain completely exogenous.

Similarly, in the early models on the diffusion of technical knowledge through e.g. education, the nature and development of the "theoretical level of technology" is left unspecified (Nelson & Phelps, 1966).

A third group of models elaborate on the "learning-by-doing" hypothesis (Arrow, 1962; Levhari, 1966a and 1966b; Sheshinski, 1967). Here some learning function provides a causal link between the activity of production (or, alternatively, investment) and technical progress. Although some theoretical and empirical support for the learning hypothesis exists, the functional form of the learning function and the direction of the technical progress thus induced are generally left unexplained. Incidentally, these learning-models illustrate that "endogenous" is not necessarily equivalent to "resulting from deliberate economic decisions". Contrary to research and education, learning-by-doing is generally believed to proceed without conscious recognition (Wan, 1971, p. 215-216).

A somewhat higher degree of endogeneity is achieved by the so-called "induced invention"-models. These models were originally intended to explain not so much the rate as the direction of technical change. However, as we will see in the next section the rate and direction of technical change are closely related. By postulating the existence of a technical progress function and assuming that firms maximize the instantaneous rate of technical progress subject to the restrictions implied by this frontier, the "induced invention"-models yield both rate and direction of technical change as endogenous variables (e.g. Kennedy, 1964; Samuelson, 1965; Ahmad, 1966; Drandakis & Phelps, 1966; Chang, 1972).

These early writers on induced invention were able to obtain their remarkable results at the expense of having to introduce an alternative exogenous concept, viz. that of the technical progress function or "innovation possibility curve" (Ahmad, 1966). Binswanger (1974; 1978b; 1978c) has tried to remedy possible embarrassment about this exogenous cuckoo in the economists' nest by formulating a micro-economic theory of induced invention. Thus the technical progress frontier itself becomes an endogenous relationship. Its building blocks consist of a number of purely "technological" (i.e. non-economic) relations on the one hand and a couple of behavioural relations on the other, describing the agents' economic choices with respect to these technological (exogenous) variables.

The whole exogeneity-versus-endogeneity issue is not one of the simple "either-or" type. As is so often the case in economic model-building, the answer depends on the stage of the model-building process at which one is content to consider one's own job as finished and prepared to leave the remainder of the task of explaining things to fellow disciplines. In this respect one degree of endogeneity is not just better or worse than another. The bad thing starts only when one tries to draw conclusions that reach beyond the borders of generality incorporated in the explicit and implicit assumptions of the model.

3.2.2 Neutral vs. biased technical change

The bias or direction of technical change has been defined in terms of the effects of technical change upon the categoral income distribution, i.e. the relative shares of the factors of production in total output. If technical change leaves the categoral income distribution unaffected then it is said to be neutral, otherwise it is said to be biased.

However the categoral income distribution is not only affected by technical change but also by changes in the quantities of the factor inputs, either autonomous or induced by the technical change itself. As a consequence any statement concerning the bias of some technical change is conditional to the changes in the factor quantities one is prepared to allow for. Thus there is an infinite number of senses in which technical change could be termed neutral.

In the case of two factor inputs, capital and labour, there are three obvious ways in which a particular kind of change in factor quantities might be an interesting benchmark in defining neutral technical change (cf. Burmeister & Dobell, 1970, pp. 67-77):

1. The most obvious case is the one in which factor quantities do not change at all or, possibly (since the production function exhibits constant returns to scale anyway), change in the same proportion such that their ratio remains constant. This case, in which the capital-labour ratio remains constant, corresponds to Hicks' measure of the bias of technical change.

2. The second case is the one in which factor quantities both change in such a way that the capital-output ratio remains constant. It corresponds to Harrod's measure of the bias of technical change.

3. The third case is the reverse of the second and leaves the labour-output ratio constant. It corresponds to Solow's measure of the bias of technical change.

The Hicksian measure is particularly suited for short-run analysis when factor quantities are fixed. It describes the effect of technical change upon the categoral income distribution for the case in which the factor proportions are held fixed.

Harrod's measure is especially concerned with steady-state economic growth. In steady state the major economic variables either remain constant or grow steadily at some fixed rate. The relevance of a constant capital-output ratio stems from it corresponding to a constant savings rate.

Solow's measure has its roots in the early models with capital vintages. Capital vintages arise when technical change is embodied in capital equipment. The embodiment issue will be dealt with more fully in the next sub-chapter. Here it suffices to note that, since the maximization of output at any time requires that labour be allocated in such a way that its marginal product is equalized across all vintages actually used, and since labour's share is obtained by multiplying its marginal product by the labour-output ratio, Solow's measure of the bias of technical change can alternatively be interpreted in terms of differences in labour-output ratios across capital vintages.

In the case of two factor inputs the bias of technical change can be defined unidimensionally in terms of either factor share. Technical change is termed capital-using, capital-saving or neutral according to whether it increases, decreases or leaves unchanged capital's relative share, given the particular measure used.

Let us formulate each measure of the bias of technical change in terms of the production function (cf. Chang, 1972). Write the production function at time t as follows:

$$Q = F[K,L;\ t] \tag{1}$$

where Q denotes output, K and L denote the quantities of the factors of production capital and labour, respectively, and t stands for (continuous) time, at the same time determining position and shape of the production function $F[\cdot]$. The <u>rate of technical change</u> at time t is defined as the magnitude of the shift that transforms $F[\cdot;t]$ into $F[\cdot;t+dt]$ correcting for the scale of output, i.e.

$$R(t) = \frac{1}{Q}\cdot\frac{\partial}{\partial t} F[K,L;\ t] = \frac{1}{Q}\cdot\frac{\partial Q}{\partial t} = Q_t/Q \tag{2}$$

Here the notation x_t is used to denote the <u>partial</u> derivative of x with respect to t; it should not be confused with the notation $\dot{x}$ which denotes the <u>total</u> time-derivative of x.

Equation (2) makes it clear that R(t) is a function of the factor inputs K and L.

By virtue of the assumed linear homogeneity of $F[\cdot]$ in K and L we have:

$$Q = K\cdot\frac{\partial Q}{\partial K} + L\cdot\frac{\partial Q}{\partial L} = K\cdot F_K + L\cdot F_L \tag{3}$$

Partial differentiation with respect to t yields:

$$Q_t = K\cdot F_{Kt} + L\cdot F_{Lt} = (K\cdot F_K)\cdot\frac{F_{Kt}}{F_K} + (L\cdot F_L)\cdot\frac{F_{Lt}}{F_L} \tag{4}$$

or, using (2):

$$R(t) = \pi_K\cdot m_K + \pi_L\cdot m_L \tag{5}$$

where

$$\pi_K = \frac{K\cdot F_K}{Q} \quad ; \quad \pi_L = \frac{L\cdot F_L}{Q} \quad ; \quad \pi_K + \pi_L = 1 \tag{6}$$

denote the relative shares of capital and labour, and

$$m_K = \frac{F_{Kt}}{F_K} \quad ; \quad m_L = \frac{F_{Lt}}{F_L} \tag{7}$$

denote the ceteris-paribus rates of change in the factors' marginal products.

Now we have, from (6):

$$\hat{\pi}_K = \hat{F}_K + \hat{K} - \hat{Q} \tag{8}$$

Using homogeneity:

$$\hat{F}_K = \frac{1}{F_K}\cdot[F_{Kt} + F_{KK}\cdot\dot{K} + F_{KL}\cdot\dot{L}] = \frac{1}{F_K}\cdot[F_{Kt} - L\cdot F_{KL}\cdot(\hat{K}-\hat{L})] =$$

$$= \frac{F_{Kt}}{F_K} - \frac{L\cdot F_L}{Q}\cdot\frac{F\cdot F_{KL}}{F_K\cdot F_L}\cdot(\hat{K}-\hat{L}) = m_K - \frac{\pi_L}{\sigma}\cdot(\hat{K}-\hat{L}) \tag{9}$$

where

$$\sigma = \frac{d\ \log\ (K/L)}{d\ \log\ (F_L/F_K)} = \frac{F_K\cdot F_L}{F\cdot F_{KL}} \tag{10}$$

is the elasticity of substitution (e.g. Burmeister & Dobell, 1970, p. 11).

From (3) and (5):

$$\hat{Q} = \pi_K\cdot\hat{K} + \pi_L\cdot\hat{L} + R \tag{11}$$

which yields

$$\hat{Q}-\hat{K} = -\pi_L\cdot(\hat{K}-\hat{L}) + R \tag{12}$$

$$\hat{K}-\hat{L} = (1/\pi_K)\cdot(\hat{Q}-\hat{L}-R) = \tag{13}$$

$$= -\ (1/\pi_L)\cdot(\hat{Q}-\hat{K}-R) \tag{14}$$

Using these equations the three measures of the bias of technical change can be written as follows:

$$D^{Hicks} \equiv \hat{\pi}_K\Big|_{\hat{K}=\hat{L}} = m_K - R = \pi_L\cdot(m_K-m_L) \tag{15}$$

$$D^{Harrod} \equiv \hat{\pi}_K\Big|_{\hat{Q}=\hat{K}} = m_K - (\pi_L/\sigma)\cdot(-1/\pi_L)\cdot(-R) - 0 = m_K - R/\sigma \tag{16}$$

$$D^{Solow} \equiv \hat{\pi}_K\Big|_{\hat{Q}=\hat{L}} = m_K + \pi_L\cdot(1-1/\sigma)\cdot(1/\pi_K)\cdot(-R) - R =$$

$$= \frac{\pi_L}{\pi_K}\cdot\left[\frac{\pi_K}{\pi_L}\cdot m_K - R + \frac{R}{\sigma} - \frac{\pi_K}{\pi_L}\cdot R\right] = (\pi_L/\pi_K)\cdot\{(R/\sigma) - m_L\} \tag{17}$$

From (5) and (15)-(17) we can deduce the following conclusions:

- Technical change simultaneously Hicks-neutral, Harrod-neutral and Solow-neutral is possible *if and only if* the elasticity of substitution equals unity (cf. Uzawa, 1961); if technical change is <u>always</u> simultaneously

Hicks-neutral, Harrod-neutral and Solow-neutral, then the production function is Cobb-Douglas.

- If technical change is simultaneously Hicks-neutral and Harrod-neutral, then it is also Solow-neutral; if it is simultaneously Harrod-neutral and Solow-neutral, then it is also Hicks-neutral; if it is simultaneously Solow-neutral and Hicks-neutral, then it is also Harrod-neutral.

- The three measures (15)-(17) are of the same sign if and only if the elasticity of substitution equals unity.

If one requires that technical change be neutral in some sense for <u>any</u> input-output combination (i.e. for all points of the production function) one has to impose certain restrictions on the parametrization of the production function (cf. Burmeister & Dobell, 1969). For the three measures (15)-(17) these specific parametrizations take the following form:

Hicks: $\frac{\partial}{\partial t}\frac{F_K}{F_L} = 0$, i.e. $F[K,L;\ t] = a_1[t]\cdot G_1[K,L]$ (18)

Harrod $\frac{\partial}{\partial K}\frac{F_t}{F_L} = 0$, i.e. $F[K,L;\ t] = G_2[K,a_2[t]\cdot L]$ (19)

Solow: $\frac{\partial}{\partial L}\frac{F_t}{F_K} = 0$, i.e. $F[K,L;\ t] = G_3[a_3[t]\cdot K,L]$ (20)

where $G[x,y]$ is homogeneous of degree one in x and y. These formulas follow directly from (15)-(17), (5), (10) and the requirement that $F[\cdot]$ be homogeneous of degree one in K and L.

The special character of technical change implied by the parametrizations (18), (19) or (20) is termed output-augmenting, labour-augmenting or capital-augmenting, respectively. All three are special cases of the more general factor-augmenting technical change:

$$F[K,L;\ t] = G[a[t]\cdot K,b[t]\cdot L] \tag{21}$$

Burmeister & Dobell (1969) have shown that the parametrization (21) is equivalent to requiring that technical change be neutral for all input-output combinations along growth paths on which the capital-labour ratio is a function of time alone. They also show how (18)-(20) can be obtained as special cases of (21).

It should be stressed that any one of the parametrizations (18)-(21) severely restricts the nature of the shifts in the production function. This restrictiveness is not so much caused by the assumed neutrality of technical change as by the additional requirement that neutrality should prevail for <u>all</u> points of the production function. A similar case, in which the apparently harmless additional assumption that some relationship holds <u>irrespective</u> of one's initial position turned out to be responsible for quite unsuspected and indeed unwelcome results (the assumption in question turned out to imply that the production function is Cobb-Douglas), has been the subject of an intricate discussion between Ahmad and Kennedy (Ahmad, 1966; Kennedy, 1967; Ahmad, 1967). As Samuelson (1965) put it:

> "Once I write down a transformation function ... I forget to make it so general as to be compatible with all behaviour. It would be all right, indeed salutary, to put restrictions on behavior if such restrictions grew out of empirical observations, but if they grow merely out of the happenstance of definition, the result can be harmful" (p. 351).

Since there is not a priori any convincing reason why technical change should be inherently restricted in a fashion as implied by any of the specifications (18) through (21), extreme care must be taken before committing oneself to factor-augmenting or similarly restrictive parametrizations.

Still, however true these observations may be for empirical or microeconomic work, there are good reasons why growth theorists need not be too concerned about the warnings raised. As long as one believes in steady-state growth - despite the constant influx of exogenous shocks battering the economic system - representing a reasonable approximation to actual economic development and at the same time does not seriously question the famous "stylized facts" of economic history (e.g. Solow, 1970, p. 2-3), only technical change that is always Harrod-neutral makes sense.

This special status of Harrod-neutrality derives from the simple fact that commodities are produced by means of commodities. Anyone acquainted with Hofstadter's (1979) "eternal golden braid" will appreciate that any system producing its own origins is extremely liable to explosive behaviour. Steady-state economic growth serving as a razor's edge, the slightest deviation from long-run Harrod-neutrality would result in capital becoming either increasingly or decreasingly efficient in producing output *and consequently itself*, thus inducing an ever widening distance between actual and stylized economic development. This is obviously the case for the one-sector model but can be easily shown to hold for any disaggregated multi-sector model as well. Thus, as long as capital plays a role in producing output, growth theory and Harrod-neutrality of technical change (if any) will be closely linked.

3.2.3 Embodied vs. disembodied technical change

Although the distinction between "embodied" and "disembodied" technical change is quite familiar in the literature it is not easily defined in terms of shifts of the production function.

The founding father of the embodiment hypothesis Solow (1960) formulated its essence as follows: "Improvements in [embodied] technology affect output only to the extent that they are carried into practice either by net capital formation or by the replacement of old-fashioned equipment by the latest models, with a consequent shift in the distribution of equipment by date of birth" (p. 91). Wan (1971) defines embodied technical progress as "an 'inward shift' of the entire isoquant map ... due to the use of newer equipment" (p. 147). And Bliss (1968): "[embodied] technical progress takes the form of a flow of new ideas for the construction of investments, but it does not include any new ideas for the more efficient employment of existing machines" (p. 105).

On closer inspection these and similar definitions run into serious trouble. Sticking to Bliss' terminology, consider an entrepreneur with such a new idea for the construction of some machine. According to Solow's statement cited above his output in the present period (t) is not affected by this new idea since the factor inputs at the firm's disposal are exactly the same as they were before.

In the next period (t+1) the entrepreneur has constructed a number of new machines. According to Wan's statement the isoquant map has consequently shifted. This observation, however, is inconsistent with the definition of an isoquant. Isoquants are functions of factor inputs and the new machine "embodying" the entrepreneur's new idea is clearly a factor of production. Then the effect of the introduction of the new machine should be measured in terms of movements along the isoquants, not in terms of shifts of the isoquants. In this respect the fact that the new type of machine is a factor of production of a kind that did not exist before is not an essential issue.

Thus we conclude that the technical change does not occur in period t+1. Then it must have occurred in period t. Somehow the statement that output in period t is not affected by the entrepreneur's new idea must be false.

The two seemingly contradictory conclusions that embodied technical change occurring at time t does not affect output at time t but at the same time shifts the production function (by definition of technical change) can be reconciled only if we allow technical change to affect the rates at which aggregate output can be transformed into final commodities, i.e. consumption goods and investment goods. This interpretation of embodied technical change has been formulated very clearly by Layard & Walters (1978):

> "There is the assumption that technical progress somehow increases outputs from given inputs. This may be possible if organisation is improved and the like. But in most people's minds technical progress has to be embodied in new inputs. A given person with a given calculator cannot suddenly do more sums per hour, but he may be able to do more sums with an electronic calculator costing the same to make as the mechanical calculator. This leads us to think of technical progress as occurring because of improvements in the quality of machines created *at a given cost in terms of consumption foregone*. Thus we are no longer assuming that consumption and machines are the same thing" (p. 297, my italics).

In other words, the change in production possibilities due to embodied technical change depends on the allocation of output. In the quotation cited above, Bliss explicitly stated that embodied technical progress does not include any new ideas for the more efficient employment of existing investments. This view is inconsistent with the notion that new investments are produced by means of existing investments. Thus Bliss seems to imply that embodied technical change affects the efficiency of the employment of existing investments according to the final commodity that is being produced.

It should be pointed out that I have never assumed that consumption and machines (or any produced factor of production) are one and the same thing. However, in referring to aggregate output I have implicitly assumed that the marginal rate of substitution in production between consumption goods and investment goods, as well as that between any two different kinds of investment goods, is constant. It is precisely this marginal rate of substitution that is affected by embodied technical change.

Summing up, embodied and disembodied technical change can be defined in two alternative ways. I define disembodied technical change as follows:

1. Disembodied technical change is any shift of the production function that leaves the marginal rate of substitution in production between any two final commodities unchanged.

2. Disembodied technical change is any shift of the production function that in relative terms is independent of the final commodity in terms of which output is measured.

These two definitions of disembodied technical change are clearly equivalent. Embodied technical change is defined accordingly as any technical change that is not disembodied.

Using the definitions it is clear why our entrepreneur's new idea for the construction of some machine is a form of embodied technical change. In terms of consumption goods the new idea leaves the production function unaffected. In terms of the new machine, however, the production function is shifted, having been identically zero before.

Since embodied technical change can by definition occur only if multiple final commodities are distinguished the concept is especially difficult to interpret in terms of a one-sector model. Restricting attention to technical change embodied in physical capital (embodiment in human capital will be discussed in Chapter 7.2), it is typically assumed that equipment of different dates of manufacture (vintages) is operated independently so that the aggregate production function can be written as the sum of all vintage production functions (e.g. Solow, 1960). By definition of a one-sector model, consumption goods and investment goods are perfect substitutes in production.

If technical change is completely embodied and does not shift the production function in terms of consumption goods, then for equipment of a fixed vintage, the number of consumption goods attainable from given quantities of capital and labour is the same, at any point of time, while the number of investment goods attainable varies over time. According to Jorgenson (1966), these observations imply that a one-sector model with completely embodied technical change is equivalent to a two-sector model with completely disembodied technical change that is confined to the investment-goods sector and with identical production functions (except for a time-dependent scale factor) in the two sectors (p. 10). But then we could, by appropriate change of units of measurement, reformulate the model as a one-sector model with disembodied technical change.

If this were true then there would be no need for a distinction between embodied and disembodied technical change on theoretical grounds. On empirical grounds the relevance of the embodiment hypothesis has been questioned by various authors. Denison (1964) argues that the hypothesis is of little importance since its factual implications are hardly different from those of the disembodiment hypothesis. Jorgenson (1966) goes even further, showing that when technical change is measured by index numbers of total factor productivity the two hypotheses are empirically indistinguishable (i.e. the embodied and disembodied components of technical change are unidentified).

Still, Jorgenson's theoretical argument against the embodiment hypothesis is not complete. His conclusion is warranted only under the additional assumption that the investment goods embodying the new technology are only quantitatively, not qualitatively different from the investment goods of previous vintages. Surely, an economy endowed with production units exhibiting the most bizarre varieties of production techniques (i.e. vintage production functions) will behave quite differently from an economy with only two production units and one single stock of physical capital.

The assumption that investment goods embodying subsequent levels of technology are not qualitatively different is intuitively seen to be equivalent to assuming that embodied technical change is purely capital-augmenting, i.e. Solow-neutral for all points of the vintage production function (cf. Section 3.2.2). This is precisely the condition under which the investment goods of various vintages can be aggregated into one single aggregate capital stock, the existence of which is implicitly assumed by Jorgenson.

The equivalence of purely capital-augmenting embodied technical change and the existence of an aggregate capital stock has been proven by Fisher (1965) and Diamond (1965). A slightly different proof will be given below.

In Chapter 3.1, I defined as a condition for the aggregation of two factors of production that their marginal rate of substitution be constant. The marginal rate of substitution between two capital goods of different vintages v_1 and v_2, say, equals:

$$MRS^{K(v_2)}_{K(v_1)} = \frac{\partial Q}{\partial K(v_2)} : \frac{\partial Q}{\partial K(v_1)} = \frac{\partial Q(v_2)}{\partial K(v_2)} : \frac{\partial Q(v_1)}{\partial K(v_1)} = \frac{F_K(v_2)}{F_K(v_1)} \qquad (22)$$

Putting $v_1=v$, $v_2=v+dv$ and letting $dv \to 0$ the condition for aggregation becomes:

$$\frac{d}{dv} \log F_K(v) = \hat{F}_K(v) \text{ is a function of v only} \tag{23}$$

with the ^ notation now denoting relative changes across vintages, keeping calendar time fixed.

Maximization of output given past investments requires that the marginal product of labour be equalized across capital vintages, i.e.

$$\hat{F}_L(v) = 0 \tag{24}$$

In a way analogous to equation (9) we can write:

$$\hat{F}_L = m_L + (\pi_K/\sigma)\cdot(\hat{K}-\hat{L}) \tag{25}$$

which implies, in combination with (24):

$$\hat{K}-\hat{L} = -\sigma\cdot m_L/\pi_K \tag{26}$$

Substitution of (26) into (9) yields, using (5) and (6):

$$\hat{F}_K = m_K + (\pi_L/\pi_K)\cdot m_L = R/\pi_K = F_v/(K\cdot F_K) \tag{27}$$

$\hat{F}_K$ is a function of K, L and v. Since F_K and consequently $\hat{F}_K$ is homogeneous of degree zero in K and L, a necessary and sufficient condition for (23) to hold is:

$$\frac{\partial}{\partial K}\hat{F}_K = 0 \tag{28}$$

Straightforward manipulation of (28), using definition (10) and homogeneity, yields the following:

$$0 = \frac{\partial}{\partial K}\hat{F}_K = \frac{\partial}{\partial K}\frac{F_v}{K\cdot F_K} = \frac{F_{Kv}\cdot K\cdot F_K - F_v\cdot F_K - F_v\cdot K\cdot F_{KK}}{(K\cdot F_K)^2} \Rightarrow$$

$$\Rightarrow F_{Kv}\cdot K\cdot F_K - F_v\cdot F_K - F_v\cdot K\cdot F_{KK} = (F_v - L\cdot F_{Lv})\cdot F_K - F_v\cdot F_K + F_v\cdot L\cdot F_{KL} = 0$$

$$\Rightarrow F_v\cdot L\cdot F_{KL} = F_K\cdot L\cdot F_{Lv} \Rightarrow \frac{F_v}{F}\cdot\frac{F\cdot F_{KL}}{F_K\cdot F_L} = \frac{F_{Lv}}{F_L} \Rightarrow R/\sigma = m_L \tag{29}$$

which is equivalent to putting (17) equal to zero. Thus there exists an aggregate capital stock under embodied technical change if and only if embodied technical change is Solow-neutral for all input-output combinations.

If technical change is not purely capital-augmenting we expect growth models with embodied technical change to yield different conclusions than models with disembodied technical change. These differences should become marked particularly along paths of transition between steady states, with which we are especially concerned when population is non-stable.

In concluding this section some final remarks are in order. First, although embodied technical change is usually thought of as increasing the economy's efficiency in producing investment goods, our definition equally allows embodied technical change to increase the economy's efficiency in producing consumption goods while leaving its efficiency in producing capital equipment unaffected. Product innovation is an example of such a type of embodied technical change.

Second, Bliss (1968) seems to suggest that complete embodiment of technical change is equivalent to the vintage production function being of the "putty-clay" type, i.e. the capital-labour ratio being variable before an investment is realized but fixed thereafter. I do not see why this should be so. Ex-post substitution is quite compatible with the embodiment of technical change. On the other hand, even a production unit with completely fixed factor inputs could become more productive with the passage of time. Thus neither feature of the production process necessarily implies the other.

Finally, it should be stressed that there is no connection whatsoever between the embodiment issue and any factor-augmenting character of technical change. In particular, technical change embodied in one factor of production does not necessarily augment that factor, nor does it necessarily <u>not</u> augment other factors (cf. Binswanger, 1978c, p. 129).

3.3 Technical change in one-sector growth models

The 1960s formed the Golden Age of growth theory. Stimulated by the pioneering work of Solow (1956) and Swan (1956), by the discovery of the Golden Rule (Phelps, 1961; Robinson, 1962) and by the development of the powerful tools of optimal control theory (Pontryagin e.a., 1962; first applied to the problem of economic growth by Cass, 1965), numerous authors embarked on formulating endless variations on the basic neoclassical growth model, deriving new theorems and investigating conditions for optimal economic growth.

The general striving for product differentiation is reflected in the variety of ways in which technical change has been incorporated into these models. Using the three major criteria for classification discussed in the previous section and restricting the attention to one-sector growth models, the most important contributions are summarized in Table 3.3.1.

Table 3.3.1 shows a number of unconquered areas. Except for the otherwise quite specific learning models, embodied technical change has been invariably modelled as a completely exogenous variable. This state of affairs may be largely ascribed to the great analytical complexity of models with embodied technical change that is not Solow-neutral. Furthermore, it is doubtful whether rendering embodied technical change endogenous would add significantly to our economic insights.

Rather than fill the gaps in Table 3.3.1, I propose to apply some models from the more crowded cells to the problem of non-stable population. In Chapter 4.1 exogenous disembodied labour-augmenting technical change will be added to the elementary one-sector model of Chapter 2. The introduction of this type of technical change requires a trivial modification of the steady-state Golden Rule (cf. Burmeister & Dobell, 1970, p. 406) but its implications for the nature of the non-stationary optimal growth path is not immediately clear.

In Chapter 4.2 the model of Solow e.a. (1966) is analysed for the case of non-stable population. Due to the existence of heterogenous non-aggregable investment goods, the application of the Maximum Principle in this model is by no means straightforward. The system to be controlled has a memory and time lags, being themselves endogenous, play an important role in the derivation of the optimal non-stationary growth path.

Returning to the mathematically more manageable case of disembodied technical change, Chapter 4.3 introduces endogenous technical change. The model

Table 3.3.1: Classifications of technical change in one-sector growth models

	Disembodied		Embodied	
	Model	Remarks	Model	Remarks
exogenous 1)	Solow, 1956	Cobb-Douglas	Solow, 1960	Cobb-Douglas
	Uzawa, 1961		Massell, 1962	id. + Putty-Clay
	Mirrlees, 1967	Rate not constant	Solow e.a., 1966	Clay-Clay
	Shell, 1967a	Hicks-neutral	Levhari, 1966c	
			Bliss, 1968	Putty-Clay
rate endogenous bias exogenous 1)				
1. research	Uzawa, 1965			
	Phelps, 1966a			
	Shell, 1967b	Bias not specified		
2. diffusion	Nelson & Phelps, 1966			
	Phelps, 1966b			
3. learning	Sheshinski, 1967		Arrow, 1962	Clay-Clay
	Cigno, 1984	Harrod-neutrality is implicit	Levhari, 1966a	Clay-Clay
			Levhari, 1966b	
			Black, 1969	Clay-Clay
bias endogenous	Samuelson, 1965	Factor-augmenting		
	Samuelson, 1966	id.		
	Drandakis & Phelps, 1966	id.		
	Nordhaus, 1967	id. + research		
	Chang, 1972	Various parametriz.		

1) Technical change is Harrod-neutral unless stated otherwise.

will be extended to include a research sector that is engaged in the task of increasing the level of technology, along the lines of Shell (1967b) and Sato & Suzawa (1983); the resulting model is the model of Uzawa (1965). The optimal growth path, both steady-state and non-stationary, is now characterized by two Golden Rules: one for investment in physical capital and one for investment in research.

4 OPTIMAL ECONOMIC GROWTH UNDER CONDITIONS OF TECHNICAL CHANGE

This chapter analyses optimal economic growth under non-stable population and conditions of technical change. Three models, each with a different type of technical change, will be investigated.

In Chapter 4.1 exogenous disembodied labour-augmenting technical change will be added to the simple one-sector model of Chapter 2. The introduction of this type of technical change requires a trivial modification of the Golden Rule but its implications for the nature of the non-stationary optimal growth path are not immediately clear.

Chapter 4.2 analyses a model with technical progress that is embodied in physical capital. The production function is characterized by fixed factor proportions (Solow e.a., 1966). Due to the existence of heterogenous non-aggregable investment goods, the application of the Maximum Principle in this model is by no means straightforward. The system to be controlled has a memory and time lags, being themselves endogenous, play an important role in the derivation of the optimal non-stationary growth path.

Chapter 4.3 introduces endogenous technical change. The model will be extended to include a research sector that is engaged in the task of increasing the level of technology, along the lines of Uzawa (1965). The optimal growth path, both steady-state and non-stationary, is now characterized by two Golden Rules: one for investment in physical capital and one for investment in research.

4.1 A model with exogenous disembodied technical change

This Chapter extends the analysis of Chapter 2 to the case of completely exogenous, disembodied, and labour-augmenting (i.e. Harrod-neutral everywhere) technical change. First I consider the Generalized Golden Rule for the extended model. Section 4.1.2 gives some comparative statics results. In Section 4.1.3 the nature of the non-stationary optimal growth path in the presence of technical change is analysed. The final section summarizes the main results.

4.1.1 The Generalized Golden Rule

According to equation (3.2.19), the production function shifts over time in the following fashion:

$$Y(t) = F[K(t),L(t);\ t] = F[K(t),\lambda(t)\cdot L(t)] \tag{1}$$

Using the notation of Chapter 2 we now have the following maximization problem:

$$\text{Maximize} \quad \int_0^\infty e^{-\rho t}\cdot\{1-s(t)\}\cdot\lambda(t)\cdot f[k(t)/\lambda(t)]\cdot\frac{L(t)}{P(t)}\,dt \tag{2}$$

subject to

$$\dot{k}(t) = s(t)\cdot\lambda(t)\cdot f[k(t)/\lambda(t)] - \{\delta+g^L(t)\}\cdot k(t) \tag{3}$$

$$k(0) = k_0 \tag{4}$$

$$0 \le s(t) \le 1 \tag{5}$$

The only difference between this problem and the maximization problem of Chapter 2 is that we now have the expression $\lambda(t)\cdot f[k(t)/\lambda(t)]$ instead of $f[k(t)]$ for output per unit of human capital. Making this substitution into the Generalized Golden Rule (2.33) we find:

$$\frac{d}{dk}\,\{\lambda(t)\cdot f[k(t)/\lambda(t)]\} = f'[k(t)/\lambda(t)] = \frac{dY(t)}{dK(t)} = \rho + \delta + g^P(t) \tag{6}$$

where $f'[\cdot]$ denotes the first derivative of $f[\cdot]$. Thus we reach the perhaps surprising conclusion that technical change does not affect the condition for optimal economic growth (cf. Nordhaus, 1967). In particular, this conclusion is contrary to the common textbook knowledge that the marginal product of physical capital should equal the sum of the population growth rate and the rate of labour augmentation (e.g. Burmeister & Dobell, 1970, pp. 406-407; Wan, 1971, p. 305; cf. Phelps, 1966b, pp. 137-157).

The argument leading to the textbook conclusion runs as follows. Labour-augmenting technical progress is equivalent to an increase in population measured in "efficiency units". Thus technical progress can be introduced into the model simply by replacing g everywhere by the growth rate of population measured in efficiency units, which is $g+\hat{\lambda}$. This argument is not valid because it implicitly affects the social welfare function. Efficiency units do not consume, only natural persons do. Even if we wish to measure the labour force in efficiency units this does not warrant measuring the "consuming" population in efficiency units as well. Indeed, since the Generalized Golden Rule refers to g^P, not g^L, our result that technical change does not affect the Golden Rule is not so surprising after all.

Phelps' (1966b) analysis, on the other hand, consists of showing that for $f'=\hat{\lambda}+g$ consumption per head is maximized at any point in time. Our result (6) does <u>not</u> imply that Phelps' argument is wrong. Rather is says that <u>given the social welfare function</u> (2) it is not optimal to maintain Phelps' Golden Rule path. This observation is similar to one of Pearce's (1962) criticisms of Phelps' (1961) original work on the Golden Rule (see also Phelps, 1962).

From the Generalized Golden Rule (6) we have that in equilibrium under stationary population the ratio k/λ is constant, or

$$\hat{k} = \hat{\lambda} \Leftrightarrow \hat{K} = g + \hat{\lambda} \tag{7}$$

Physical capital grows at a rate $g+\hat{\lambda}$. Since the quantity $\lambda \cdot L$ grows also at a rate $g+\hat{\lambda}$ and since the production function is homogeneous, output, too, grows at a rate $g+\hat{\lambda}$, implying a constant savings rate. This is of course what we should have in steady state.

<u>4.1.2 Comparative statics</u>

For comparative statics the following three equations are relevant:

the Golden Rule: $$f'[k/\lambda] = \rho + \delta + g \tag{8}$$

the steady-state condition: $$\dot{k} = \hat{\lambda}\cdot k = s\cdot\lambda\cdot f[k/\lambda] - (\delta+g)\cdot k \tag{9}$$

the definition of C/P: $$C/P = (1-s)\cdot\lambda\cdot f[k/\lambda]\cdot(L/P) \tag{10}$$

The relevant exogenous variables are:

- the growth rate of population g;
- the rate of depreciation δ;
- the social rate of time preference ρ;
- the rate of labour augmentation $\hat{\lambda}$.

Since in steady state consumption per capita C/P grows at a rate $\hat{\lambda}$ it is somewhat troublesome to speak of comparative statics effects of the exogenous variables on C/P. Consequently, I will measure the steady-state effect of a change in any exogenous variable by the difference between the long-run optimal growth paths of two economies that differ in the exogenous variable under consideration only. In particular the two economies to be compared are taken to start from a common level of labour-augmenting technology λ.

Keeping λ constant, total differentiation of (8)-(10) yields:

$$\begin{bmatrix} f'' & 0 & 0 \\ sf'-(\delta+g+\hat{\lambda}) & f & 0 \\ -(1-s)\lambda f'\frac{L}{P} & \lambda f\frac{L}{P} & 1 \end{bmatrix} \begin{bmatrix} d(k/\lambda) \\ ds \\ d(C/P) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ k/\lambda & k/\lambda & 0 & k/\lambda \\ (1-s)\lambda f\frac{\partial(L/P)}{\partial g} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dg \\ d\delta \\ d\rho \\ d\hat{\lambda} \end{bmatrix} \tag{11}$$

Inverting the matrix on the left-hand side of (11):

$$\begin{bmatrix} f'' & 0 & 0 \\ sf'-(\delta+g+\hat{\lambda}) & f & 0 \\ -(1-s)\lambda f'\frac{L}{P} & \lambda f\frac{L}{P} & 1 \end{bmatrix}^{-1} = \frac{1}{ff''}\begin{bmatrix} f & 0 & 0 \\ -sf'+(\delta+g+\hat{\lambda}) & f'' & 0 \\ \lambda f\frac{L}{P}(f'-\delta-g-\hat{\lambda}) & -\lambda ff''\frac{L}{P} & ff'' \end{bmatrix} =$$

$$= \begin{bmatrix} 1/f'' & 0 & 0 \\ -\frac{k/\lambda}{f}\cdot\sigma\cdot\frac{\delta+g+\hat{\lambda}}{\delta+g+\rho} & 1/f & 0 \\ \frac{L}{P}\cdot\lambda\cdot\frac{1}{f''}\cdot(\rho-\hat{\lambda}) & -\lambda\cdot\frac{L}{P} & 1 \end{bmatrix} \tag{12}$$

Combination of (11) and (12) gives:

$$\frac{d(k/\lambda)}{dg} = \frac{d(k/\lambda)}{d\delta} = \frac{d(k/\lambda)}{d\rho} = 1/f'' < 0 \tag{13}$$

$$\frac{d(k/\lambda)}{d\hat{\lambda}} = 0 \tag{14}$$

$$\frac{ds}{dg} = \frac{ds}{d\delta} = \frac{k/\lambda}{f}\cdot\left[1 - \sigma\cdot\frac{\delta+g+\hat{\lambda}}{\delta+g+\rho} \right] \gtreqless 0 \tag{15}$$

$$\frac{ds}{d\rho} = -\frac{k/\lambda}{f}\cdot\sigma\cdot\frac{\delta+g+\hat{\lambda}}{\delta+g+\rho} < 0 \tag{16}$$

$$\frac{ds}{d\hat{\lambda}} = \frac{k/\lambda}{f} > 0 \tag{17}$$

$$\frac{d(C/P)}{dg} = \frac{L}{P}\cdot\lambda\cdot\left[\frac{\rho-\hat{\lambda}}{f''} - (k/\lambda) \right] + (1-s)\cdot\lambda\cdot f\cdot\frac{\partial(L/P)}{\partial g} \gtreqless 0 \tag{18}$$

$$\frac{d(C/P)}{d\delta} = \frac{L}{P}\cdot\lambda\cdot\left[\frac{\rho-\hat{\lambda}}{f''} - (k/\lambda) \right] \gtreqless 0 \tag{19}$$

$$\frac{d(C/P)}{d\rho} = \frac{L}{P}\cdot\lambda\cdot\frac{\rho-\hat{\lambda}}{f''} \tag{20}$$

$$\frac{d(C/P)}{d\hat{\lambda}} = -\lambda\cdot\frac{L}{P}\cdot(k/\lambda) < 0 \tag{21}$$

Now a necessary condition for the integral (2) to converge is that

$$\rho > \hat{\lambda} \tag{22}$$

Given condition (22), the derivatives in (19) and (20) are negative. Furthermore, if we assume that the elasticity of substitution σ does not exceed unity, condition (22) ensures that the derivatives in (15) are also negative.

For the effect of g on <u>initial</u> consumption per capita the analysis of Chapter 2 remains valid: the effect is generally negative except possibly for values of g so low that the adverse effect of lowering g on the labour-population ratio L/P outweighs the positive effect given by the first term on the RHS of equation (18).

Comparing the expressions for the comparative-statics effects of g, δ and ρ to the corresponding equations in Chapter 2, we find that the presence of a positive rate of labour augmentation changes the effects on s and C/P in absolute terms but not in sign.

We now turn to the long-run effect of $\hat{\lambda}$ itself. A change in $\hat{\lambda}$ does not affect the ratio k/λ as should be obvious from the Generalized Golden Rule (6). For s we have from (17) a positive effect of $\hat{\lambda}$. Since s cannot exceed unity, this implies an upper bound to the ratio of labour-augmentation compatible with steady-state optimal economic growth:

$$\hat{\lambda} < \frac{f[(k/\lambda)^*]}{(k/\lambda)^*} - (\delta+g) \tag{23}$$

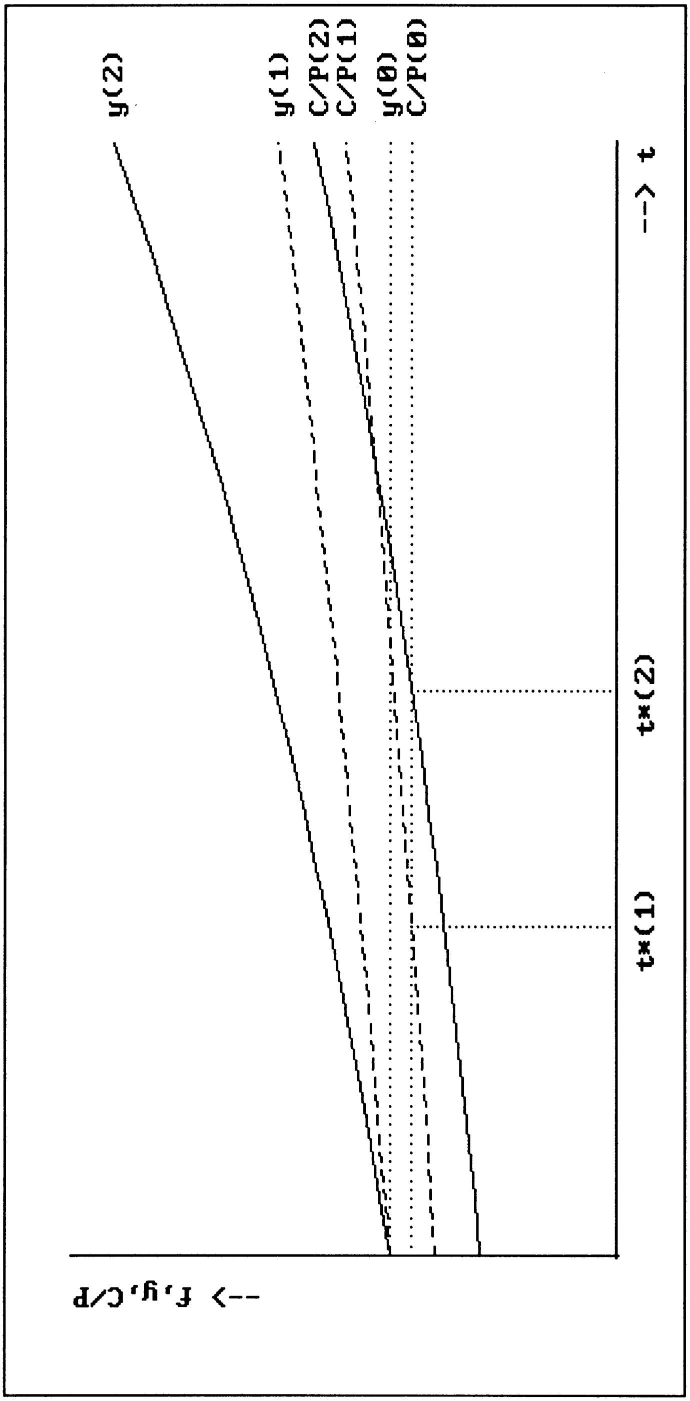
From (21) we see that an increase in $\hat{\lambda}$ (keeping initial λ fixed) lowers the <u>initial</u> level of consumption per capita. This effect is solely due to the decrease of the optimal propensity to consume (1-s). On the other hand an increase in $\hat{\lambda}$ steepens the long-run growth path of output and consumption per capita. This implies that the C/P-path of an economy with a high $\hat{\lambda}$ is initially below the C/P-path of an economy with a lower $\hat{\lambda}$ but will eventually overtake this alternative path and dominate it ever after. These effects of $\hat{\lambda}$ on C/P are illustrated in Figure 4.1.1.

Taking the case without labour-augmenting technical change ($\hat{\lambda}$=0) as the benchmark and normalizing $\lambda(0)$=1, this overtaking is easily seen to take place at time

$$t^*(\hat{\lambda}) = (1/\hat{\lambda})\cdot\log\frac{f[k^*(0)] - k^*(0)\cdot(\delta+g)}{f[k^*(0)] - k^*(0)\cdot(\delta+g+\hat{\lambda})} \tag{24}$$

Then

Figure 4.1.1: The effect of labour-augmentation on the long-run optimal consumption path



$$\lim_{\hat{\lambda} \to \frac{f[k^*(0)]}{k^*(0)} - (\delta+g)} t^*(\hat{\lambda}) = +\infty \tag{25}$$

which illustrates once more that there is an upper limit to the feasible rate of labour-augmentation.

The various comparative-statics results are summarized in Table 4.1.1. This table is essentially the same as its counterpart in Chapter 2, except for the last column, which is new.

Table 4.1.1: Comparative-statics results for the one-sector model with exogenous disembodied labour-augmenting technical change

effect of: on:	g	δ	ρ	$\hat{\lambda}$ 1)
k	-	-	-	0
s	+ 2)	+ 2)	-	+
C/P (level)	- 3)	-	-	-

1) $\hat{\lambda}$ restricted by:

a) $\hat{\lambda} < \rho$

b) $\hat{\lambda} < \frac{f[(k/\lambda)^*]}{(k/\lambda)^*} - (\delta+g)$

2) For $\sigma < \frac{\delta+g+\rho}{\delta+g+\hat{\lambda}}$

3) Except possibly for very low values of g

4.1.3 The non-stationary optimal growth path

This section extends the analysis of Section 2.5 to the case of exogenous disembodied labour-augmenting technical change.

For notational convenience I define:

$$x \equiv k/\lambda \tag{26}$$

Since the role of x is virtually the same as was the role of k in Section 2.5, I concentrate on the dynamic paths of s and C/P.

For s we have, using (9) and (26):

$$s = \frac{\dot{k} + (\delta+g^L)\cdot k}{\lambda\cdot f[k/\lambda]} = \frac{\dot{x} + (\delta+g^L+\hat{\lambda})\cdot x}{f[x]} \tag{27}$$

Differentiation with respect to time yields:

$$\dot{s} = \frac{1}{f}\cdot\left[\ddot{x} + x\dot{g}^L + x\dot{\hat{\lambda}} + (\delta+g^L+\hat{\lambda})\cdot\dot{x} - \frac{f'}{f}\cdot\dot{x}\cdot\{\dot{x}+(\delta+g^L+\hat{\lambda})\cdot x\}\right] =$$

$$= \frac{1}{f}\cdot\left\{\frac{\ddot{g}^P}{f''} - \left[\frac{\dot{g}^P}{f''}\right]^2\cdot\left[\frac{f'}{f} + \frac{f'''}{f''}\right] + x\cdot\left[\dot{g}^L + \dot{\hat{\lambda}} - \dot{g}^P\cdot\sigma\cdot\frac{\delta+g^L+\hat{\lambda}}{\delta+g^P+\rho}\right]\right\} \tag{28}$$

The nature of the dynamic path of s during periods of demographic transition depends, among other things, on the way in which the rate of labour augmentation $\hat{\lambda}$ varies with the growth rates of population g^P and/or labour g^L. Here I will investigate two cases:

1. The rate of labour augmentation is independent of demographic forces. In order to concentrate on the effect of demographic transition I will assume that $\hat{\lambda}$ is constant in this case.
2. The rate of labour augmentation is positively correlated with the growth rate of labour. This view has been advocated by e.g. Simon (1977 and 1986). In this case $\hat{\lambda}$ falls and $\dot{\hat{\lambda}}$ is negative during periods in which the economy experiences a secular decline of the growth rate of births, population, and labour.

The properties of the dynamic path of s in the absence of technical change have been discussed in Section 2.5 on the basis of equation (2.63). If $\dot{\hat{\lambda}}=0$ (case 1) the only difference between that equation and the present (28) is the presence of the term $\hat{\lambda}$ in the numerator of the last term on the RHS of (28), increasing the positive effect of lowering the growth rate of population on the optimal savings rate. Thus, other things being equal, a higher rate of labour augmentation implies a rising portion of the s(t)-curve (for negative $\dot{g}^B$) that is initially steeper and lasts longer. Also the negative net effect of lowering g^B on the savings rate is smaller under conditions of technical progress. The latter conclusion can alternatively be obtained from equation (15), observing that

$$\frac{d}{d\hat{\lambda}}\left[\frac{ds}{dg}\right] = -\frac{k/\lambda}{f}\cdot\sigma\cdot\frac{1}{\delta+g+\rho} < 0 \tag{29}$$

If, on the other hand, $\hat{\lambda}$ is correlated with g^L (case 2), then the presence of the term $\dot{\hat{\lambda}}$ on the RHS of (28) reinforces the negative effect on s of the decreasing growth rate of labour. In this case the decreasing portion of the s(t)-curve is steeper than in the absence of such correlation. However, without further information on the form of the production function, the interaction between g^L and $\hat{\lambda}$, and other crucial parameters of the system, any more definite conclusions cannot be obtained.

Next, let us briefly consider the dynamic path of C/P. From (10) and (27) we have:

$$C/P = (L/P)\cdot\lambda\cdot\{f - \dot{x} - (\delta+g^L+\hat{\lambda})\cdot x\} \tag{30}$$

Differentiating with respect to time and substituting previous results:

$$(\dot{C/P}) = \frac{C}{P}\cdot\hat{\lambda} + \frac{C}{P}\cdot(g^L-g^P) + \frac{L}{P}\cdot\lambda\cdot\{(\rho-\hat{\lambda}+g^P-g^L)\cdot\dot{x} - \ddot{x} - (\dot{g}^L+\dot{\hat{\lambda}})\cdot x\} \tag{31}$$

This expression corresponds to equation (2.71). The first term on its RHS describes the secular rise in consumption per capita as a result of technical progress. Only relatively large offsetting effects of the remaining terms could (temporarily) change the sign of the overall rate of change of C/P, being positive most of the time.

The results of the analysis in Section 2.5 can in the present context be interpreted as applying to transitional deviations around the long-run trend in consumption per capita. There I concluded that such deviations are mostly positive for falling g^B, corresponding to a positive net effect of lowering g^B on consumption per capita.

In so far as exceptions to this general tendency occur they are reinforced by the multiplicative presence of the factor λ in the third group of terms on the RHS of (31). In this respect technical change renders the direction of the dynamic path of C/P during periods of demographic transition somewhat less predictable. If $\hat{\lambda}$ is positively correlated with g^L, however, the probability of such exceptions to the general trend occurring is diminished.

However, regardless of the sign of the net effect of the second and third terms on the RHS of (31), it is highly improbable that any negative influence would be so strong as to outweigh the secular positive effect of the first term. Thus it seems reasonably safe to conclude that consumption per capita rises steadily in the presence of technical progress, even though slight deviations in its rate of change may occur during periods of demographic transition.

4.1.4 Summary

The main results of our analysis of exogenous disembodied labour-augmenting technical change in the one-sector growth model can be summarized as follows:

1. Contrary to what many authors suggest, technical change does not affect the Generalized Golden Rule.
2. The signs of the previously derived comparative-statics effects are not affected by the presence of technical change.
3. Technical change increases the optimal steady-state savings rate, implying an upper bound to the rate of labour-augmentation compatible with steady-state optimal economic growth.
4. Technical change lowers the initial level of consumption per capita but increases its rate of growth.
5. If the rate of technical change is independent of demographic forces, the s(t)-curve for a falling birth growth rate rises faster and longer, the higher the rate of labour augmentation. Also, technical change implies a smaller negative net effect of lower population growth on the optimal savings rate.
6. Consumption per capita rises steadily in the presence of technical progress, even though slight deviations in its rate of change may occur during periods of demographic transition.

4.2 A model with exogenous embodied technical change

This chapter analyses optimal economic growth in a model with technical change that is embodied in physical capital. If technical change is embodied in capital the model becomes one of capital vintages, i.e. capital goods (machines) are distinguished by their date of construction. Thus the development of the economy is an explicit function of its history, at least of its most recent history. This feature should lead one to expect that non-stationarities in the economic development, triggered off by the occurrence of demographic change, are particularly severe and persistent in this model.

Section 4.2.1 outlines the model. In Section 4.2.2 a condition for optimal economic growth will be established. This condition turns out to be in many respects similar to the well-known Golden Rule of Accumulation. Section 4.2.3 gives some comparative statics results. In Section 4.2.4 the stability of steady states is analysed. It is shown that if the condition for stability of steady states is satisfied, the social welfare function becomes a divergent integral. Therefore, Section 4.2.5 checks the second-order conditions of the maximization problem and confirms that the generalized Golden Rule yields a true maximum for the social welfare function. Section 4.2.6 investigates some properties of the non-stationary optimal growth path. The final section summarizes the main results.

4.2.1 The fixed-coefficients capital vintage model

The aggregate commodity is produced from labour and (physical) capital where capital goods are distinguished by their date of construction. Production obtained from capital of a certain vintage is described by a so-called vintage production function:

$$Q(v,t) = F[K(v,t),\ L(v,t);\ v] \qquad (1)$$

Here v is the time at which the capital goods under consideration were constructed; K(v,t) is the size of the capital stock installed at time v and still in existence at time t (this could be less than the amount originally invested as a result of depreciation); L(v,t) is the amount of labour allocated to work with the capital goods in question; and Q(v,t) is the resulting output. The fact that the vintage production function $F[\cdot]$ is parametrized with an index v reflects the presence of capital-embodied technical change: the productivity of given amounts of factor inputs K and L depends on the date on which the capital goods were installed.

Total production at time t is given by the sum of all outputs produced from the different capital vintages, i.e.

$$Q(t) = \int_{-\infty}^{t} Q(v,t)\, dv \tag{2}$$

Physical capital is subject to depreciation at a constant rate δ:

$$K(v,t) = K(v,v)\cdot e^{\delta(v-t)} \tag{3}$$

In each period a fraction of total output is saved and added to the capital stock (invested):

$$K(t,t) \equiv I(t) = s(t)\cdot Q(t) \tag{4}$$

Output not invested in physical capital is consumed. Total consumption equals:

$$C(t) = Q(t) - I(t) \tag{5}$$

We are left with the specification of the production function (1). I assume the vintage production function to be characterized by fixed factor proportions ("clay-clay"):

$$Q(v,t) = \min\{\kappa(v)\cdot K(v,t),\ \lambda(v)\cdot L(v,t)\} \quad \text{for all } v \le t \tag{6}$$

This model has been investigated extensively by Solow e.a. (1966). $\kappa(\cdot)$ and $\lambda(\cdot)$ are indexes of capital-augmenting and labour-augmenting technology, respectively. The development over time of these indexes is assumed to satisfy:

$$\kappa'(v) \ge 0 \quad ; \quad \lambda'(v) \ge 0 \tag{7}$$

Most of the time I will assume that $\kappa(\cdot)$ is constant and that $\lambda(\cdot)$ grows exponentially over time, i.e. technical change is exponential and Harrod-neutral everywhere.

From (2), (6) and (7) it is evident that, given the stocks of physical capital of all different vintages, production at time t is maximized by allocating labour across capital vintages such that:

$$L(v,t) = \begin{cases} \frac{\kappa(v)}{\lambda(v)} \cdot K(v,t) & \text{for all } v \geq t - T(t) \\ 0 & \text{for all } v < t - T(t) \end{cases} \tag{8}$$

where T(t) denotes the <u>age</u> of the oldest capital vintage in use at time t. T(t) is restricted by the size of the labour force:

$$L(t) = \int_{t-T(t)}^{t} L(v,t)\, dv \tag{9}$$

(2)-(4), (6), (8) and (9) together imply:

$$L(t) = \int_{t-T(t)}^{t} \frac{\kappa(v)}{\lambda(v)} \cdot e^{\delta(v-t)} \cdot I(v)\, dv \tag{10}$$

$$Q(t) = \int_{t-T(t)}^{t} \kappa(v) \cdot e^{\delta(v-t)} \cdot I(v)\, dv \tag{11}$$

It should be stressed that in (10), L(t) is <u>exogenous</u> and T(t) <u>endogenous</u>, not the other way round.

4.2.2 Optimal economic growth

The central planning agency maximizes the social welfare function

$$W = \int_{0}^{\infty} e^{-\rho t} \cdot \frac{Q(t)-I(t)}{L(t)}\, dt \tag{12}$$

subject to (10), (11), and

$$0 \leq I(t) \leq Q(t) \qquad \text{(boundary restriction on the control)} \tag{13}$$

$$I(v) = I_v \quad \text{for all } v<0 \qquad \text{(initial conditions)} \tag{14}$$

The control variable is I(t). Although I(t) determines T(t) via (10) and T(t) determines Q(t) via (11), I treat I(t), T(t) and Q(t) as three independent control variables that are restricted by (10) and (11).

In the analysis that follows I have made use of some very valuable advice given to me by Onno van Hilten of Limburg University (cf. Malcomson, 1975; Nickell, 1975; Verheyen & van Lieshout, 1978).

Linking restrictions (10) and (11) to the maximand (12) with the use of the Lagrange multipliers $\theta_L(t)$ and $\theta_Q(t)$ yields:

$$W = \int_0^\infty \left[e^{-\rho t} \cdot \frac{Q(t)-I(t)}{P(t)} + \theta_L(t) \cdot \left\{ \int_{t-T(t)}^{t} \frac{\kappa(v)}{\lambda(v)} \cdot e^{\delta(v-t)} \cdot I(v)\, dv - L(t) \right\} + \right.$$

$$\left. + \theta_Q(t) \cdot \left\{ \int_{t-T(t)}^{t} \kappa(v) \cdot e^{\delta(v-t)} \cdot I(v)\, dv - Q(t) \right\} \right] dt =$$

$$= \int_0^\infty \left[e^{-\rho t} \cdot \frac{Q(t)-I(t)}{P(t)} - \theta_L(t) \cdot L(t) - \theta_Q(t) \cdot Q(t) \right] dt \ +$$

$$+ \int_0^\infty \left[\int_{t-T(t)}^{t} \kappa(v) \cdot e^{\delta(v-t)} \cdot I(v) \cdot \left[\theta_Q(t) + \theta_L(t)/\lambda(v)\right] dv \right] dt \qquad (15)$$

The last term on the RHS of (15) is a double integral. The area over which the integration is performed is the shaded area in Figure 4.2.1.

If the function T(t) is such that

$$T'(t) < 1 \quad \text{for all } t \qquad (16)$$

(i.e. capital once out of use remains out of use forever), then the following inverse funcion of t-T(t) exists:

$$t+Z(t) = INV[t-T(t)] \qquad (17)$$

From (17):

$$t = t + Z(t) - T[t+Z(t)] \ \Rightarrow \ Z(t) = T[t+Z(t)] \qquad (18)$$

Thus, Z(t) is the age at which capital installed at time t will become obsolete. From (16) and (18) we find that:

$$Z'(t) = T'[t+Z(t)].\{1+Z'(t)\} \ \Rightarrow \ Z'(t) > -1 \quad \text{for all } t \qquad (19)$$

Using the definition of Z(t), a double integral of some function f(v,t) over

Figure 4.2.1: The double integral before changing the order of integration

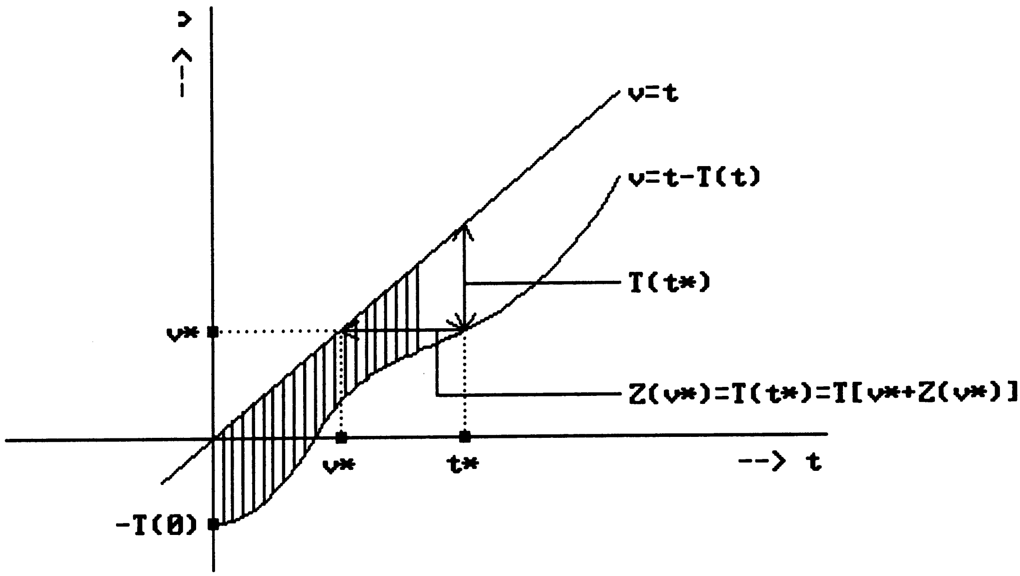
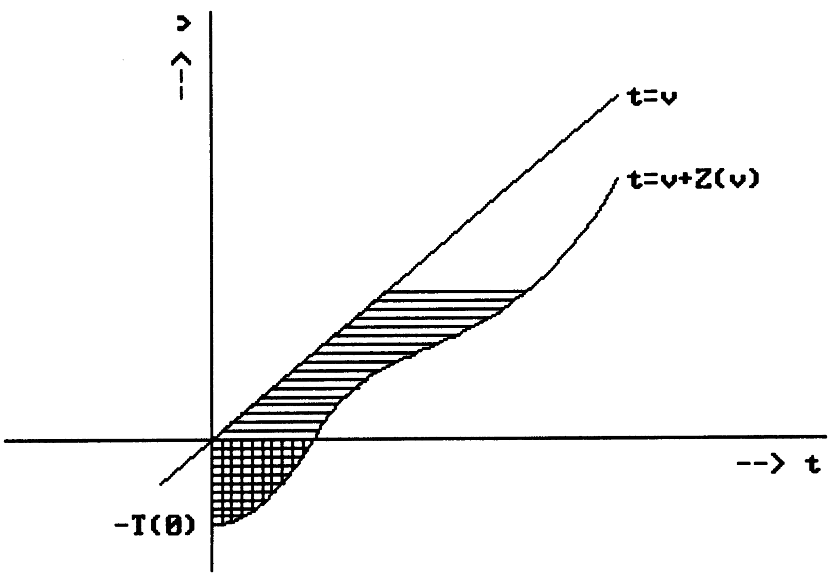


Figure 4.2.2: The double integral after changing the order of integration



the shaded area in Figure 4.2.1 can be rewritten by changing the order of integration as follows:

$$\int_0^\infty \left[\int_{t-T(t)}^{t} f(v,t)\, dv \right] dt =$$

$$= \int_0^\infty \left[\int_v^{v+Z(v)} f(v,t)\, dt \right] dv + \int_{-T(0)}^{0} \left[\int_0^{v+Z(v)} f(v,t)\, dt \right] dv \tag{20}$$

In Figure 4.2.2 the shaded area corresponds to the first integral on the RHS of (20) while the cross-hatched area corresponds to the second integral.

Using (20) after interchanging the symbols v and t, the integral in (15) can be written as:

$$W = \int_0^\infty \left[e^{-\rho t}.\frac{Q(t)-I(t)}{P(t)} - \theta_L(t)\cdot L(t) - \theta_Q(t)\cdot Q(t) \right] dt +$$

$$+ \int_0^\infty \kappa(t)\cdot e^{\delta t}\cdot I(t)\cdot \left[\int_t^{t+Z(t)} e^{-\delta v}\cdot \left[\theta_Q(v) + \theta_L(v)/\lambda(t)\right] dv \right] dt +$$

$$+ \int_{-T(0)}^{0} \kappa(t)\cdot e^{\delta t}\cdot I_t\cdot \left[\int_0^{t+Z(t)} e^{-\delta v}\cdot \left[\theta_Q(v)+\theta_L(v)/\lambda(t)\right] dv\right] dt \tag{21}$$

In writing the third integral in (21) use has been made of the initial conditions in (14).

Necessary conditions for the maximization of W are that the integrand in (21), to be denoted by w(t), be maximized with respect to the controls $I(\cdot)$, $Q(\cdot)$ and $Z(\cdot)$, as well as the multipliers $\theta_Q(\cdot)$ and $\theta_L(\cdot)$, at each point in time. If attention is restricted to time periods later than Z(0), then the third integral in (21) vanishes and the necessary conditions are the following:

$$\frac{\partial w(t)}{\partial Q(t)} = \frac{e^{-\rho t}}{P(t)} - \theta_Q(t) = 0 \tag{22}$$

$$\frac{\partial w(t)}{\partial I(t)} = - \frac{e^{-\rho t}}{P(t)} + \kappa(t)\cdot e^{\delta t}\cdot \int_t^{t+Z(t)} e^{-\delta v}\cdot \{\theta_Q(v)+\theta_L(v)/\lambda(t)\}\, dv =$$

$$= \begin{cases} \geq 0 & \text{if} \quad I(t) = Q(t) \\ = 0 & \text{if} \quad 0 < I(t) < Q(t) \\ \leq 0 & \text{if} \quad I(t) = 0 \end{cases} \tag{23}$$

$$\frac{\partial w(t)}{\partial Z(t)} = \kappa(t)\cdot e^{\delta\cdot Z(t)}\cdot I(t)\cdot\left[\theta_Q[t+Z(t)] + \theta_L[t+Z(t)]/\lambda(t)\right] = 0 \tag{24}$$

$$\frac{\partial w(t)}{\partial \theta_Q(t)} = -\ Q(t) \quad + \int_{t-T(t)}^{t} \kappa(v)\cdot e^{\delta(v-t)}\cdot I(v)\ dv = 0 \tag{25}$$

$$\frac{\partial w(t)}{\partial \theta_L(t)} = -\ L(t) \quad + \int_{t-T(t)}^{t} \frac{\kappa(v)}{\lambda(v)}\cdot e^{\delta(v-t)}\cdot I(v)\ dv = 0 \tag{26}$$

From now on I will concentrate on <u>singular arcs</u>. In other words: I will assume an <u>interior solution</u> to optimal investment, such that $0<I(t)<Q(t)$ and the RHS of (23) is identically zero.

Under this assumption (24) implies:

$$\theta_Q[t+Z(t)] + \theta_L[t+Z(t)]/\lambda(t) = 0 \tag{27}$$

or, equivalently, lagging (27) by Z(t) periods and using (17):

$$\theta_Q(t) + \theta_L(t)/\lambda[t-T(t)] = 0 \tag{28}$$

On the other hand we have from (22) and the observation that the conditions (22)-(26) must hold for longer than a single instant along a singular arc:

$$\frac{d}{dt}\,\frac{\partial w(t)}{\partial Q(t)} = -\ \rho\cdot\frac{e^{-\rho t}}{P(t)} - \frac{\dot{P}(t)}{P(t)}\cdot\frac{e^{-\rho t}}{P(t)} - \dot{\theta}_Q(t) = 0 \tag{29}$$

from which, using (22):

$$\dot{\theta}_Q(t) = -\ \{\rho + g^P(t)\}\cdot\theta_Q(t) \tag{30}$$

Integrating (30):

$$\theta_Q(v) = \theta_Q(t)\cdot\exp\left[\ -\int_t^v \{\rho + g^P(u)\}\ du\ \right] \tag{31}$$

Substitution of (28) and (31) into (23), using (22), yields:

$$0 = \frac{e^{-\rho t}}{P(t)}\cdot\left[\ -\ 1 + \int_t^{t+Z(t)} \exp\left[\ -\int_t^v \{\rho+g^P(u)\}\ du\ \right]\cdot\kappa(t)\cdot e^{\delta(t-v)}\cdot \right.$$
$$\left. \cdot\left[\ 1 - \frac{\lambda[v-T(v)]}{\lambda(t)}\ \right]\ dv\ \right] \tag{32}$$

Bearing in mind the inverse relationship between Z(·) and T(·), equation (32) is a condition for the occurrence of a singular arc in terms of the lifetimes of subsequent capital vintages.

I will now show that condition (32) is equivalent to the Non-Stationary Golden Rule for the simple (non-vintage) neoclassical model as analysed in Chapter 2. The marginal productivity of capital of some vintage v can be obtained from (11):

$$\frac{dQ(t)}{dI(v)} = J_{[t-T(t),t]}(v)\cdot\kappa(v)\cdot e^{\delta(v-t)} + \kappa[t-T(t)]\cdot e^{-\delta\cdot T(t)}\cdot I[t-T(t)]\cdot\frac{dT(t)}{dI(v)} \quad (33)$$

where the <u>indicator function</u> $J_A(x)$ is defined by

$$J_A(x) = \begin{cases} 1 & \Leftrightarrow \quad x \in A \\ 0 & \Leftrightarrow \quad x \notin A \end{cases} \quad (34)$$

From (17) we have:

$$0 = J_{[t-T(t),t]}(v)\cdot\frac{\kappa(v)}{\lambda(v)}\cdot e^{\delta(v-t)}\cdot d[I(v)] + $$
$$+ \frac{\kappa[t-T(t)]}{\lambda[t-T(t)]}\cdot e^{-\delta\cdot T(t)}\cdot I[t-T(t)]\cdot d[T(t)] \quad (35)$$

from which

$$\frac{dT(t)}{dI(v)} = -J_{[t-T(t),t]}(v)\cdot\frac{\kappa(v)}{\lambda(v)}\cdot\frac{\lambda[t-T(t)]}{\kappa[t-T(t)]}\cdot e^{\delta\cdot(v-t+T(t))}\cdot\frac{1}{I[t-T(t)]} \quad (36)$$

and thus, from (33) and (36):

$$\frac{dQ(t)}{dI(v)} = J_{[t-T(t),t]}(v)\cdot\kappa(v)\cdot e^{\delta(v-t)}\cdot\left[1 - \frac{\lambda[t-T(t)]}{\lambda(v)}\right] \quad (37)$$

From (37) and (17) it follows that:

$$\frac{dQ(v)}{dI(t)} = J_{[t,t+Z(t)]}(v)\cdot\kappa(t)\cdot e^{\delta(t-v)}\cdot\left[1 - \frac{\lambda[v-T(v)]}{\lambda(t)}\right] \quad (38)$$

(cf. Solow e.a., 1966).

Thus it can be seen that the integral in the RHS of (32) is equal to the present value of all future <u>returns</u> to investment made at time t, discounted at a rate equal to the sum of the rates of social impatience (ρ) and population growth (g^P). On the other hand, the marginal <u>costs</u> of investment (in terms of consumption foregone) equal unity. Thus condition (32) simply says that the singular arc is characterized by the familiar equality of marginal costs of and returns to investment.

The condition discussed in the previous paragraph is easily seen to be the finite-lifetime equivalent of the Non-Stationary Golden Rule of Chapter 2. With infinite lifetime of capital (no obsolescence) the condition becomes:

$$1 = \int_t^\infty \exp\left[- \int_t^v \{\rho + g^P(u)\}\, du \right] \cdot \frac{\partial Q(v)}{\partial I(t)}\, dv =$$

$$= \int_t^\infty \exp\left[- \int_t^v \{\rho + \delta + g^P(u)\}\, du \right] \cdot \frac{\partial Q(v)}{\partial K(t)}\, dv \qquad (39)$$

Differentiating (39) with respect to time t:

$$0 = - \frac{\partial Q(t)}{\partial K(t)} + \{\rho + \delta + g^P(t)\} \cdot \int_t^\infty \exp\left[- \int_t^v \{\rho + \delta + g^P(u)\}\, du \right] \cdot \frac{\partial Q(v)}{\partial K(t)}\, dv \qquad (40)$$

from which, using (39):

$$\frac{\partial Q(t)}{\partial K(t)} = \rho + \delta + g^P(t) \qquad (41)$$

which is the Non-Stationary Golden Rule.

4.2.3 Comparative statics

If population grows at a constant rate g, and if technical progress is exponential and Harrod-neutral everywhere, i.e.

$$\kappa(t) = \kappa_0 \quad ; \quad \lambda(t) = \lambda_0 \cdot e^{\lambda t} \quad \text{with } \kappa_0,\ \lambda_0,\ \lambda \text{ constant} \qquad (42)$$

then the optimal growth path could well lead the economy into a steady state.

In steady state the optimal savings rate is constant. As Solow e.a. (1966) have shown a constant savings rate for this model implies the maximum age of capital to be constant too, i.e. $Z(t) = T(t) = T^*$, say.

The value of T^* can be obtained by solving (32). Carrying out the integration yields:

$$1 = \int_t^{t+T^*} e^{\alpha \cdot (t-v)} \cdot \kappa_0 \cdot \left[1 - e^{\lambda \cdot (v - T^* - t)} \right] dv =$$

$$= \kappa_0 \cdot (1/\alpha) \cdot [\, 1 - e^{-\alpha \cdot T^*} \,] - \kappa_0 \cdot \frac{e^{-\lambda \cdot T^*}}{\alpha - \lambda} \cdot [\, 1 - e^{-(\alpha-\lambda) \cdot T^*} \,]$$

$$\Rightarrow \{(\alpha/\kappa_0) - 1\} \cdot (\alpha - \lambda) + \alpha \cdot e^{-\lambda \cdot T^*} - \lambda \cdot e^{-\alpha \cdot T^*} = 0 \tag{43}$$

where I write

$$\alpha = \rho + \delta + g \tag{44}$$

for notational convenience.

Equation (43) cannot be explicitly solved for T^*. If we define

$$G(T^*) = \{(\alpha/\kappa_0) - 1\} \cdot (\alpha - \lambda) + \alpha \cdot e^{-\lambda \cdot T^*} - \lambda \cdot e^{-\alpha \cdot T^*} \tag{45}$$

a steady-state value for T^* exists if $G(\cdot)$ has a finite positive root. Since

$$G'(T^*) = \alpha \cdot \lambda \cdot [\, e^{-\alpha \cdot T^*} - e^{-\lambda \cdot T^*} \,] \tag{46}$$

is monotonous on R^+, a positive root of $G(\cdot)$ is unique if it exists. The existence of such a root depends on the values of α and λ. Analysis of the function $G(\cdot)$ yields the following table:

parameter values	number of positive roots of $G(\cdot)$
$\alpha = 0$, $\lambda \neq 0$	infinite (identity)
$\lambda = 0$, $\alpha \neq 0$	none
$\alpha = \lambda$	infinite (identity)
$\alpha > \lambda > 0$	one root (if $\alpha < \kappa_0$)
$\lambda > \alpha > 0$	one root (if $\alpha < \kappa_0$)
$\alpha > 0 > \lambda$	none
$\lambda > 0 > \alpha$	one root
$0 > \alpha > \lambda$	none
$0 > \lambda > \alpha$	none

Thus we have the following <u>existence condition</u>:

an optimal steady-state value for T^* exists only if
$\lambda > 0$, $\alpha < \kappa_0$, $\alpha \neq 0$ and $\alpha \neq \lambda$ (47)

However, a steady state must also be feasible. That is, the savings rate $s^* = I^*/Q^*$ corresponding to the steady-state value of T^* must be between zero and unity (cf. condition (13)). From (10) and (42) it is seen that in steady state investment $I(\cdot)$ grows at an exponential rate $g+\lambda$; and from (11) so does production $Q(\cdot)$. Then we have from (11):

$$Q(t) = \kappa_0 \cdot \int_{t-T^*}^{t} e^{\delta \cdot (v-t)} \cdot I(v)\, dv = \kappa_0 \cdot \int_{t-T^*}^{t} s^* \cdot Q(t) \cdot e^{(\delta+g+\lambda)\cdot(v-t)}\, dv$$

$$\Rightarrow \quad 0 \le s^* = (1/\kappa_0) \cdot \frac{\delta + g + \lambda}{1 - e^{-(\delta+g+\lambda)\cdot T^*}} \le 1 \tag{48}$$

Given the form of condition (43) and expression (48) it is very difficult to obtain general comparative-statics results, that is to sign the partial derivatives of s^* and T^* with respect to the parameters g, λ, ρ and δ. Some numerical calculations of steady states are given in Table 4.2.1. These results suggest that for reasonable values of the parameters the signs of the partial derivatives are as in Table 4.2.2.

It is interesting to note that these comparative-statics results, as far as the savings rate is concerned, are essentially the same as for the simple neoclassical model with disembodied technical change (see Chapter 4.1). Moreover, if one is prepared to interpret an increase in T as a decrease in the "capital/labour-ratio", then (with the exception of the effect of λ) the results of the two models are similar too for the capital variable.

4.2.4 Stability of steady states

Along a singular arc the endogenous variable $T(t)$, being the age of the oldest capital vintage in operation at time t, develops over time according to equation (32). The question can now be raised: does the optimal economic growth path under suitable external conditions converge towards a steady state? Particularly, if population grows at a constant rate g and if technical progress is exponential and Harrod-neutral everywhere, does a trajectory $T(\cdot)$ satisfying (32) then converge towards the constant value T^*? This question is important as it relates to the stability of the steady state.

From (32) and (42) we have:

$$1/\kappa_0 = \int_{t}^{t+Z(t)} \exp\left[- \int_{t}^{v} \{\rho+\delta+g^P(u)\}\, du\right] \cdot \left[1 - e^{\lambda \cdot \{v-t-T(v)\}}\right] dv \tag{49}$$

Table 4.2.1: Selected numerical steady-state values

ρ	g	δ	λ	T^*	s^*
0.02	0.04	0.02	0.04	8.3115	0.1772
0.02	0.04	0.02	0.00	n.a.	n.a.
0.02	0.04	0.02	0.02	12.6149	0.1289
0.02	0.04	0.02	0.04	8.3115	0.1772
0.02	0.04	0.02	0.06	6.7333	0.2165
0.02	0.04	0.00	0.04	8.0682	0.1682
0.02	0.04	0.02	0.04	8.3115	0.1772
0.02	0.04	0.04	0.04	8.5706	0.1868
0.02	0.04	0.06	0.04	8.8468	0.1971
0.02	0.00	0.02	0.04	n.a.	n.a.
0.02	0.02	0.02	0.04	8.0682	0.1682
0.02	0.04	0.02	0.04	8.3115	0.1772
0.02	0.06	0.02	0.04	8.5706	0.1868
0.00	0.04	0.02	0.04	8.0682	0.1806
0.02	0.04	0.02	0.04	8.3115	0.1772
0.04	0.04	0.02	0.04	8.5706	0.1737
0.06	0.04	0.02	0.04	8.8468	0.1703

Table 4.2.2: Comparative-statics effects

effect on: \ of:	ρ	g	δ	λ
T^*	+	+	+	-
s^*	-	+	+	+

Differentiation of (49) with respect to t yields:

$$0 = \{1+\dot{Z}(t)\}\cdot\exp\left[-\int_t^{t+Z(t)} \{\rho+\delta+g^P(u)\}\, du\right].\left[1-e^{\lambda\cdot\{Z(t)-T[t+Z(t)]\}}\right] +$$

$$- \left[1 - e^{-\lambda\cdot T(t)} \right] +$$

$$+ \{\rho+\delta+g^P(t)\}\cdot\int_t^{t+Z(t)} \exp\left[-\int_t^v \{\rho+\delta+g^P(u)\}\, du\right]\cdot\left[1-e^{\lambda\cdot\{v-t-T(v)\}}\right] dv +$$

$$+ \lambda\cdot \int_t^{t+Z(t)} \exp\left[-\int_t^v \{\rho+\delta+g^P(u)\}\, du\right]\cdot e^{\lambda\cdot\{v-t-T(v)\}}\, dv \qquad (50)$$

The first term in (50) is equal to zero because of (18). The third term is equal to $\{\rho+\delta+g^P(t)\}/\kappa_0$, using (49). The fourth term equals

$$\lambda\cdot\left[- 1/\kappa_0 + \int_t^{t+Z(t)} \exp\left[- \int_t^v \{\rho+\delta+g^P(u)\}\, du \right] dv \right] \qquad (51)$$

also using (49). Thus, equation (50) can be written as:

$$\int_t^{t+Z(t)} \exp\left[- \int_t^v \{\rho+\delta+g^P(u)\}\, du \right] dv = \frac{1 - e^{-\lambda\cdot T(t)} - \{\rho+\delta+g^P(t)-\lambda\}/\kappa_0}{\lambda} \qquad (52)$$

Equation (52), together with definition (18), is an interesting type of difference equation linking T(t) and Z(t): it gives a relationship along the singular arc between the oldest age of capital <u>in use</u> at time t, on the one hand, and the oldest age that capital <u>installed</u> at time t will ever reach, on the other hand.

It is easily seen that a necessary and sufficient condition for the difference equation (52) to converge is given by

$$\left| \frac{dZ(t)}{dT(t)} \right| < 1 \qquad (53)$$

Carrying out the differentiation yields:

$$\frac{dZ(t)}{dT(t)} = \exp\left[-\lambda\cdot T(t) + \int_t^{t+Z(t)} \{\rho+\delta+g^P(u)\}\, du \right] \qquad (54)$$

which in the case of stable population reduces to:

$$\frac{dZ(t)}{dT(t)} = e^{(\rho+\delta+g)\cdot Z(t) - \lambda\cdot T(t)} \qquad (55)$$

This expression is always positive. In the neighbourhood of the steady state we have $Z(t) \approx T(t)$, so that a necessary condition for local convergence (local stability of the steady state) is:

$$\lambda > \rho+\delta+g \tag{56}$$

This is quite an uncomfortable result as it implies that the integral of the social welfare function (12) diverges for a locally stable singular arc (with $g+\delta>0$, as will generally be the case). This raises the question whether the social welfare function is capable of ranking alternative paths, and indeed casts doubt on the relevance of the analysis of steady states. In the next section it will be shown that, even though the objective function is divergent under the stability condition (56), the Golden-Rule path defined by (32) is optimal in the sense that it dominates its neighbouring paths.

A difference equation similar in kind to (52) can also be derived for $s(\cdot)$. From (10) and (42) we have:

$$L(t) = \int_{t-T(t)}^{t} (\kappa_0/\lambda_0)\cdot e^{-\lambda\cdot v}\cdot e^{\delta\cdot(v-t)}\cdot I(v)\, dv \tag{57}$$

Differentiation of (57) with respect to time yields:

$$\dot{L}(t) = -\delta\cdot\int_{t-T(t)}^{t}(\kappa_0/\lambda_0)\cdot e^{-\lambda v}\cdot e^{\delta(v-t)}\cdot I(v)\, dv + (\kappa_0/\lambda_0)\cdot e^{-\lambda t}\cdot I(t) + \\ - \{1-\dot{T}(t)\}\cdot(\kappa_0/\lambda_0)\cdot e^{-\lambda\cdot t}\cdot e^{(\lambda-\delta)\cdot T(t)}\cdot I[t-T(t)] \tag{58}$$

After substitution of (57) and some rearranging (58) reduces to:

$$I(t) = (\lambda_0/\kappa_0)\cdot\{g^L(t)+\delta\}\cdot e^{\lambda t}\cdot L(t)+\{1-\dot{T}(t)\}\cdot e^{(\lambda-\delta)T(t)}\cdot I[t-T(t)] \tag{59}$$

or, equivalently:

$$s(t) = (\lambda_0/\kappa_0)\cdot\{g^L(t)+\delta\}\cdot\frac{e^{\lambda\cdot t}\cdot L(t)}{Q(t)} + \\ + \{1-\dot{T}(t)\}\cdot e^{(\lambda-\delta)\cdot T(t)}\cdot s[t-T(t)]\cdot\frac{Q[t-T(t)]}{Q(t)} \tag{60}$$

Expression (60) is a kind of difference equation linking the savings rate at time t to the savings rate at the time at which the oldest capital in use at time t was installed.

In the neighbourhood of the steady state we have:

$$\frac{e^{\lambda \cdot t} \cdot L(t)}{Q(t)} \approx \text{constant}$$

$$\dot{T}(t) \approx 0$$

$$\frac{Q[t-T(t)]}{Q(t)} \approx e^{-(g+\lambda)\cdot T(t)}$$

so that

$$0 < \frac{d\ s(t)}{d\ s[t-T(t)]} \approx e^{-(g+\delta)\cdot T(t)} < 1 \qquad (\text{if } g+\delta>0) \tag{61}$$

Thus the non-stationary time path of $s(\cdot)$ is locally convergent as the economy approaches its new steady-state growth path.

4.2.5 Second-order conditions

In the previous section it was concluded that, if the parameters are such that the Golden-Rule path converges towards a steady state, the objective function is a divergent integral. This raises the question whether the Golden-Rule path can be called truly optimal.

In order to investigate this issue, I will calculate the effect on the value of the social welfare function of small variations in the Golden-Rule trajectory for the control variable $I(\cdot)$. In other words, I will investigate whether the Golden-Rule path is optimal in the sense that it dominates its neighbouring paths.

The social welfare function W is a functional of $I(\cdot)$ and $Q(\cdot)$, where $Q(t)$ is a function of investments made in the past. If, starting from any control trajectory $I(\cdot)$, we apply a small variation $\delta I(\cdot)$ to this control trajectory, then the first-order change in the value of W can be written as follows:

$$\delta W = \int_0^\infty \left\{ -\frac{e^{-\rho t}}{P(t)} + \int_0^\infty \frac{e^{-\rho v}}{P(v)} \cdot \frac{dQ(v)}{dI(t)}\, dv \right\} \cdot \delta I(t)\, dt \tag{62}$$

Substitution of (38) into (62) yields:

$$\delta W = \int_0^\infty \left[-\frac{e^{-\rho t}}{P(t)} + \int_t^{t+Z(t)} \frac{e^{-\rho v}}{P(v)} \cdot \kappa(t) \cdot e^{\delta(t-v)} \cdot \left\{ 1 - \frac{\lambda[v-T(v)]}{\lambda(t)} \right\} dv \right] \cdot \delta I(t)\, dt$$

$$= \int_0^\infty \frac{e^{-\rho t}}{P(t)} \cdot \left[-1 + \int_t^{t+Z(t)} \exp\left\{ -\int_t^v \{\rho + g^P(\tau)\}\, d\tau \right\} \cdot \kappa(t) \cdot \right.$$

$$\left. \cdot e^{\delta(t-v)} \cdot \left\{ 1 - \frac{\lambda[v-T(v)]}{\lambda(t)} \right\} dv \right] \cdot \delta I(t)\, dt \tag{63}$$

Since (32) holds for all t, the RHS of (63) vanishes along the Golden-Rule path. This is of course what we would expect, as a singular trajectory by definition makes the first variation of the objective function become zero.

The second-order change in W can be obtained from (62):

$$\delta^2 W = \int_0^\infty \left\{ \int_0^\infty \frac{e^{-\rho v}}{P(v)} \cdot \frac{d^2 Q(v)}{dI^2(t)}\, dv \right\} \cdot \delta^2 I(t)\, dt \tag{64}$$

From (38) and (37):

$$\frac{d^2 Q(v)}{dI^2(t)} = J_{[t,t+Z(t)]}(v) \cdot \kappa(t) \cdot e^{\delta(t-v)} \cdot (-1) \cdot \frac{\lambda'[v-T(v)]}{\lambda(t)} \cdot \left\{ -\frac{dT(v)}{dI(t)} \right\} =$$

$$= -J_{[t,t+Z(t)]}(v) \cdot e^{\delta\{T(t)-2(t-v)\}} \cdot \left\{ \frac{\kappa(t)}{\lambda(t)} \right\}^2 \cdot \frac{\lambda[v-T(v)]}{\kappa[v-T(v)]} \cdot \frac{\lambda'[v-T(v)]}{I[v-T(v)]} \le 0 \tag{65}$$

where the last step is based on (7). From (64) and (65) it is immediately seen that the second-order change in W is nonpositive:

$$\delta^2 W \le 0 \quad , \tag{66}$$

with strict inequality for positive labour-augmenting technical progess.

Thus we can conclude that, even though the objective function may be a divergent integral, the Golden-Rule path is optimal in the sense that it dominates its neighbouring paths.

4.2.6 The non-stationary optimal growth path

Along the singular arc the time path of the lifetime of subsequent capital vintages is governed by condition (32). Clearly this condition is too complicated to allow the derivation of general characteristics of the non-stationary optimal economic growth path (as in Chapter 2 or Chapter 4.1). For this particular vintage-model, therefore, I must be content with the more modest target of trying to simulate an optimal economic growth path given some fixed values for the external parameters.

The simulation problem can be described as follows. Specified values for:

- ρ, the social rate of impatience;
- δ, the rate of capital depreciation; and:
- λ, the rate of labour-augmenting technical progress,

are given. It is assumed that initially the growth rate of population has been constant (g_0) for a long time, and that the economy is in its optimal steady state corresponding to this population growth rate g_0. For the sake of convenience, the labour force and the population are assumed to be identical. At a certain point in time (t_a) the growth rate of population begins to fall (linearly for the sake of convenience) until at time t_b it reaches a new level g_1 which it will keep forever afterwards. Now what is the optimal economic growth path for these external conditions? The parameters used in the simulation problem are listed in Table 4.2.3.

Table 4.2.3: Parameters used in the simulation

parameter	ρ	δ	λ	g_0	g_1	t_a	t_b	κ_0	λ_0
value	0.0	0.0	0.05	0.03	0.02	20	30	1.0	1.0

In order to solve the dynamic optimization problem described in the previous paragraph I have simulated the difference equation (52). A simple check of this method is to run a simulation with $g_1 = g_0$. If everything goes well the simulation method should find that the optimal policy is to remain in the initial steady state forever.

The computation consists of two steps: computation of $T(\cdot)$ from (52); and computation of $s(\cdot)$ given $T(\cdot)$. The computation of $T(\cdot)$ starts from the initial steady state given by

$$T(t) = T_0^* \quad ; \quad s(t) = s_0^* \qquad t = t_0 \, , \, \ldots \, , \, t_0 + \mathrm{int}(T_0^*/dt) + 1 \tag{67}$$

where dt is the discretization parameter. Now for $t = t_0$, $t_0 + dt$, ... the variable $Z(t)$ is obtained by numerically solving (52); the integral is approximated using the Trapezium Rule followed by Romberg Integration (e.g. Churchhouse, 1981). The result is stored as T[t+Z(t)]. Since the index t is necessarily discrete and the solution Z(t) is generally not, the values of T[t+dt·int(Z(t)/dt)] are approximated by parabolic interpolation between three consecutive values of t+Z(t).

The second step involves computing the time path of I(·) corresponding to the simulated path of T(·), using the difference equation (59). Here the time derivative of T(·) was approximated by central differences while I[t-T(t)] was obtained by exponential interpolation between two consecutive values of I(·). Once I(t) has been found Q(t) and s(t) follow easily. The results of this approach are summarized in Table 4.2.4 and Figure 4.2.3.

The results of the simulation confirm the point raised in the introduction, viz. that because of the fact that the state of the economy is a function of its history non-stationarities are particularly severe and persistent. The oscillations in the optimal trajectories of T(·) and s(·) are quite strong (taking into account that the demographical disturbance of the original steady state is relatively small) and take a very long time to dampen out. However, the optimally growing economy gradually converges to a new stationary growth path, in which both the optimal savings rate s and the optimal lifetime of capital equipment T are once more constant.

4.2.7 Summary

In this chapter I have investigated optimal economic growth in a model with technical progress that is embodied in physical capital. The production function corresponding to each capital vintage has been taken to be of the fixed-coefficients type, as in Solow e.a. (1966).

A suitable transformation of the Lagrangean allows the derivation of necessary conditions for optimal economic growth. These necessary conditions are in terms of two key variables which are inversely related to each other, viz.:

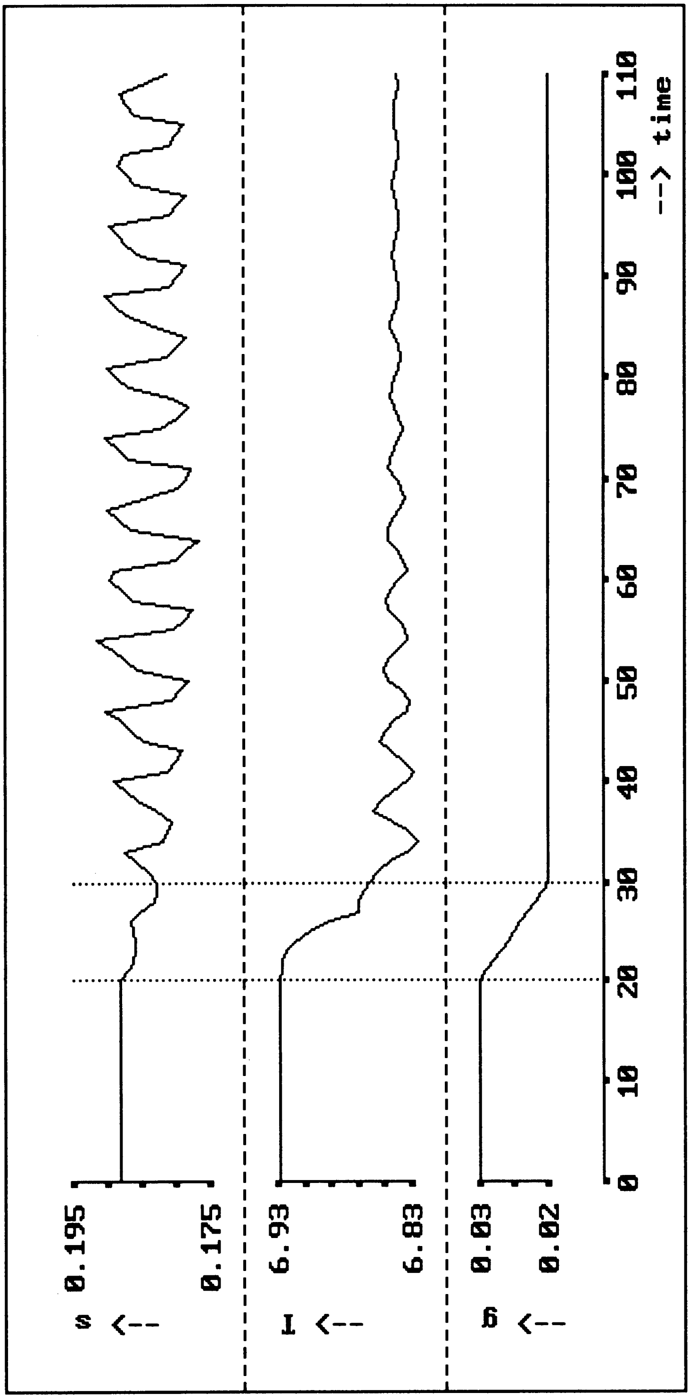
- T(t), the age of the oldest capital in use at time t; and
- Z(t), the age at which capital installed at time t will become obsolete.

Along a singular trajectory the necessary conditions reduce to a Generalized Golden Rule. It is shown that this Generalized Golden Rule is nothing more

Table 4.2.4: The simulated non-stationary optimal economic growth path

t	g(t)	$T^*(t)$	$s^*(t)$
0	0.03	6.9278	0.1880
10	0.03	6.9278	0.1880
20	0.03	6.9278	0.1880
21	0.029	6.9276	0.1870
22	0.028	6.9264	0.1863
23	0.027	6.9232	0.1860
24	0.026	6.9167	0.1859
25	0.025	6.9060	0.1862
26	0.024	6.8899	0.1867
27	0.023	6.8700	0.1852
28	0.022	6.8690	0.1830
29	0.021	6.8666	0.1827
30	0.02	6.8624	0.1827
31	0.02	6.8554	0.1842
32	0.02	6.8457	0.1858
33	0.02	6.8334	0.1875
34	0.02	6.8248	0.1819
35	0.02	6.8327	0.1812
36	0.02	6.8447	0.1806
37	0.02	6.8581	0.1829
38	0.02	6.8518	0.1856
39	0.02	6.8436	0.1873
40	0.02	6.8330	0.1892
41	0.02	6.8289	0.1811
42	0.02	6.8358	0.1801
43	0.02	6.8460	0.1791
44	0.02	6.8543	0.1847
45	0.02	6.8490	0.1866
46	0.02	6.8420	0.1880
47	0.02	6.8330	0.1903
48	0.02	6.8320	0.1806
49	0.02	6.8380	0.1794
50	0.02	6.8467	0.1781
51	0.02	6.8514	0.1857
52	0.02	6.8469	0.1874
53	0.02	6.8408	0.1892
54	0.02	6.8331	0.1917
55	0.02	6.8345	0.1802
56	0.02	6.8397	0.1788
57	0.02	6.8470	0.1775
58	0.02	6.8491	0.1863
59	0.02	6.8452	0.1879
60	0.02	6.8401	0.1897
61	0.02	6.8339	0.1890
62	0.02	6.8364	0.1799
63	0.02	6.8409	0.1785
64	0.02	6.8472	0.1767
65	0.02	6.8473	0.1866
66	0.02	6.8440	0.1883
67	0.02	6.8396	0.1900
68	0.02	6.8353	0.1847
69	0.02	6.8379	0.1797
70	0.02	6.8417	0.1783
71	0.02	6.8470	0.1777
72	0.02	6.8459	0.1869
73	0.02	6.8431	0.1885
74	0.02	6.8393	0.1902
75	0.02	6.8367	0.1821
76	0.02	6.8390	0.1796
77	0.02	6.8423	0.1782
78	0.02	6.8464	0.1804
79	0.02	6.8448	0.1870
80	0.02	6.8424	0.1886
81	0.02	6.8392	0.1901
82	0.02	6.8379	0.1811
83	0.02	6.8399	0.1796
84	0.02	6.8427	0.1783
85	0.02	6.8455	0.1832
86	0.02	6.8439	0.1871
87	0.02	6.8419	0.1886
88	0.02	6.8392	0.1903
89	0.02	6.8388	0.1809
90	0.02	6.8405	0.1797
91	0.02	6.8429	0.1784
92	0.02	6.8446	0.1851
93	0.02	6.8433	0.1872
94	0.02	6.8415	0.1885
95	0.02	6.8392	0.1896
96	0.02	6.8395	0.1809
97	0.02	6.8410	0.1798
98	0.02	6.8431	0.1785
99	0.02	6.8439	0.1858
100	0.02	6.8428	0.1872
101	0.02	6.8413	0.1884
102	0.02	6.8394	0.1875
103	0.02	6.8401	0.1810
104	0.02	6.8414	0.1799
105	0.02	6.8431	0.1788
106	0.02	6.8434	0.1860
107	0.02	6.8424	0.1871
108	0.02	6.8411	0.1882
109	0.02	6.8398	0.1851
110	0.02	6.8405	0.1811
∞	0.02	6.8417	0.1839

Figure 4.2.3: The simulated non-stationary optimal economic growth path



than a disguised version of the Golden Rule for more traditional growth models.

A comparative-statics analysis bears out that the optimal savings rate in steady state varies positively with the growth rate of population (g), the rate of labour-augmenting technical progress (λ), and the rate of depreciation (δ); and negatively with the social rate of impatience (ρ). These results are essentially the same as for models with disembodied technical change.

Investigation into the stability of steady states yields the conclusion that a necessary condition for the optimal economic growth path to converge is that $\lambda > \rho+g+\delta$. This is a puzzling result, as the integral in the social welfare function is divergent if this stability condition is satisfied. For T and s two difference equations have been derived which describe the dynamics of the optimally controlled economy.

A method of numerically simulating the non-stationary optimal growth path, which simply integrates the two difference equations referred to above, yields a plausible and theoretically satisfying result. The results of the simulation show that non-stationarities are particularly severe and persistent in this model, as a result of the fact that the state of the economy is a function of its history.

4.3 A model with endogenous technical change

In this chapter we are concerned with optimal economic growth when technical change is the result of deliberate economic decisions. Just as investment in physical capital increases the stock of physical capital available for productive activities, so research effort increases the quantity of output that can be produced with given factor inputs (cf. Section 3.2.1).

It is obvious that the properties of a growth model with endogenous technical change depend critically on the specification of the relationship between research effort and technical change. There are numerous ways in which this can be done. Many specifications, however, are incompatible with steady-state technically progressive economic growth.

The problems involved are very similar to those mentioned in Section 3.2.2 where I discussed the necessity of Harrod-neutrality of technical change for steady-state economic growth. As soon as a research sector is introduced, the level of technology becomes a produced determinant of output just as physical capital is a produced factor of production. If there is a long-run change in the rate at which technology produces itself, either directly or indirectly via a produced input in the research sector, steady-state economic growth breaks down.

These observations can explain some puzzling results in the literature. For example, Phelps (1966a) arrives at a Golden Rule of Research for a model in which the research sector employs labour only. When subsequently physical capital is introduced as an additional factor input in the research sector Phelps finds that Golden Age growth is possible only if the production function of the research sector is Cobb-Douglas (Phelps, 1966a, p. 143).

Shell treats the level of technology as a special kind of physical capital. He concludes, for both a one-sector (1966) and a two-sector (1967b) model, that optimal technical change is zero in the long run. This result is obviously unsatisfactory. Some criticisms of Shell's model have been put forward by Sato (1966). Sato particularly emphasizes the need for Harrod-neutrality of technical change as opposed to Shell's Hicks-neutral specification. It can be shown, however, that even then optimal steady-state economic growth with continuous technical progress is not generally possible.

Similar conclusions apply to related models (e.g. Conlisk, 1969; Suzuki, 1976). I have analysed numerous variants and each time arrived at one of the following conclusions:

- either optimal technical change is zero in the long run;
- or steady-state growth with positive technical change is not optimal.

All of the above problems have been neatly circumvented by Uzawa (1965). In his model technical change is produced by an "educational sector" that employs labour only. The rate of change in technology, inducing a labour-augmenting shift in the production function of the "productive sector", is taken to be a function of the proportion of the labour force employed in the educational sector. Since Uzawa's "educational sector" plays no other role in the model it could equally well be labelled a "research sector" and I will do so from now on (cf. Wan, 1971, p. 234).

Although Uzawa's model of endogenous technical change is far from general I prefer it to the more restrictive research model of Phelps (1966a) and Gomulka (1970). Its attractive properties are its relative analytical simplicity and its capability of generating steady-state growth paths. For these reasons is has become quite popular in the literature. For example, Takayama (1980) employs it to establish conditions for optimal technical change in the context of exhaustible natural resources.

In the remainder of this chapter I will generalize Uzawa's model of endogenous technical change to the case of non-stable population growth. In Section 4.3.1 the Generalized Golden Rule as well as a Generalized Golden Rule of Research are derived. Section 4.3.2 gives some comparative statics results; and Section 4.3.3 deals with the non-stationary optimal growth path in the Uzawa model. The final section summarizes the main results.

4.3.1 The Golden Rule of Research

In the Uzawa model research effort is measured by the proportion of the labour force employed in the research sector:

$$\frac{L_R(t)}{L(t)} = 1-u(t) \tag{1}$$

Here u(t) denotes the proportion of the labour force employed in the productive sector, i.e. engaged in the production of the aggregate commodity. Total output is given by:

$$Y(t) = F[K(t), \lambda(t)\cdot u(t)\cdot L(t)] \tag{2}$$

The research activities result in labour-augmenting technical progress. It is assumed that the relationship between research effort and the rate of labour-augmentation in the productive sector can be written as follows:

$$\hat{\lambda}(t) = h[1-u(t)] - \delta_\lambda \tag{3}$$

where the function $h[\cdot]$ satisfies the following properties:

$$h' > 0 \qquad h'' < 0 \qquad h[0] = 0 \tag{4}$$

The parameter δ_λ measures the rate of instantaneous decay of technical knowledge, caused e.g. by imperfect transmission of technical knowledge to subsequent generations (Shell, 1966).

The stock of physical capital develops as usual according to:

$$\dot{K}(t) = s(t)\cdot Y(t) - \delta_K\cdot K(t) \tag{5}$$

Let us define:

$$k(t) = \frac{K(t)}{L(t)} \tag{6}$$

From (2), (5), (6), and constant returns to scale:

$$\dot{k}(t) = s(t)\cdot u(t)\cdot\lambda(t)\cdot f\left[\frac{k(t)}{u(t)\cdot\lambda(t)}\right] - \{\delta_K + g^L(t)\}\cdot k(t) \tag{7}$$

where $f[\cdot] = F[\cdot,1]$. Throughout the analysis it will be assumed that $u(t)$ is between 0 and 1 for all t.

The maximization problem can now be formulated as follows: choose $s(t)$ and $u(t)$ to

$$\text{maximize} \quad \int_0^\infty e^{-\rho t}\cdot\{1-s(t)\}\cdot\frac{u(t)\cdot\lambda(t)\cdot L(t)}{P(t)}\cdot f\left[\frac{k(t)}{u(t)\cdot\lambda(t)}\right]\,dt \tag{8}$$

subject to:

$$\dot{k}(t) = s(t)\cdot u(t)\cdot\lambda(t)\cdot f\left[\frac{k(t)}{u(t)\cdot\lambda(t)}\right] - \{\delta_K+g^L(t)\}\cdot k(t) \tag{9}$$

$$\dot{\lambda}(t) = \lambda(t)\cdot\{h[1-u(t)] - \delta_\lambda\} \tag{10}$$

$$0 \leq s(t) \leq 1 \tag{11}$$

$$0 \leq u(t) \leq 1 \tag{12}$$

The Hamiltionian is (dropping the time-arguments):

$$H(t) = e^{-\rho t} \cdot (1-s) \cdot \frac{u\lambda L}{P} \cdot f[k/u\lambda] + \psi_K \cdot \{su\lambda \cdot f[k/u\lambda] - (\delta_K + g^L) \cdot k\} +$$

$$+ \psi_\kappa \cdot \lambda \cdot \{h[1-u] - \delta_\lambda\} \tag{13}$$

The costate variables ψ_K and ψ_λ satisfy:

$$-\dot{\psi}_K = \frac{\partial H}{\partial k} = e^{-\rho t} \cdot (1-s) \cdot (L/P) \cdot f' + \psi_K \cdot sf' - \psi_K \cdot (\delta_K + g^L) \tag{14}$$

$$-\dot{\psi}_\lambda = \frac{\partial H}{\partial \lambda} = \{f - (k/u\lambda) \cdot f'\} \cdot u \cdot \{e^{-\rho t} \cdot (1-s) \cdot (L/P) + \psi_K \cdot s\} + \psi_\lambda \cdot \{h[1-u] - \delta_\lambda\} \tag{15}$$

The partial derivatives of the Hamiltonian with respect to the controls are, respectively:

$$\frac{\partial H}{\partial s} = u\lambda f \cdot \{-e^{-\rho t} \cdot (L/P) + \psi_K\} \tag{16}$$

$$\frac{\partial H}{\partial u} = \{f - (k/u\lambda) \cdot f'\} \cdot \lambda \cdot \{e^{-\rho t} \cdot (1-s) \cdot (L/P) + \psi_K \cdot s\} - h'[1-u] \cdot \psi_\lambda \cdot \lambda \tag{17}$$

If we define the extended shadow-prices:

$$\varphi_K(t) = \psi_K(t) \cdot e^{\rho t} \cdot \frac{P(t)}{L(t)} \tag{18}$$

$$\varphi_\lambda(t) = \psi_\lambda(t) \cdot e^{\rho t} \cdot \frac{P(t)}{L(t)} \tag{19}$$

then we have from (16):

$$s(t) = 1 \quad \text{if} \quad \varphi_K(t) > 1 \tag{20.a}$$

$$s(t) = 0 \quad \text{if} \quad \varphi_K(t) < 1 \tag{20.b}$$

$$0 \leq s(t) \leq 1 \quad \text{if} \quad \varphi_K(t) = 1 \tag{20.c}$$

For cases (b) and (c) we find, from (14) and (18):

$$\dot{\varphi}_K(t) = \varphi_K(t)\cdot\{\rho + g^P(t) + \delta_K\} - f'[k/u\lambda] \qquad (21)$$

For case (a):

$$\dot{\varphi}_K(t) = \varphi_K(t)\cdot\{\rho + g^P(t) + \delta_K - f'[k/u\lambda]\} \qquad (22)$$

It should be observed that these formulas are essentially the same as their counterparts in Chapter 2.

From (17) and (19) we have:

$$e^{\rho t}\cdot\frac{P(t)}{L(t)}\cdot\frac{\partial H}{\partial u} = \lambda\cdot\{f - f'\cdot(k/u\lambda)\}\cdot\{1 + s\cdot(\lambda_K-1)\} - h'[1-u]\cdot\varphi_\lambda\lambda \qquad (23)$$

and from (15):

$$\dot{\varphi}_\lambda(t) = \varphi_\lambda\cdot\{\rho+g^P(t)-g^L(t)-h[1-u]+\delta_\lambda\} - \{f-f'\cdot(k/u\lambda)\}\cdot\{1+s\cdot(\varphi_K-1)\}\cdot u \qquad (24)$$

Assume that the optimal value of u is an interior solution to

$$\frac{\partial H}{\partial u} = 0 \qquad (25)$$

Then from (23):

$$\{f - f'\cdot(k/u\lambda)\}\cdot\{1 + s\cdot(\varphi_K - 1)\} = h'[1-u]\cdot\varphi_\lambda \qquad (26)$$

Substitution of (26) into (24) yields:

$$\dot{\varphi}_\lambda(t) = \varphi_\lambda\cdot\left[\{\rho+g^P(t)-g^L(t)+\delta_\lambda\} - \{h[1-u]+u\cdot h'[1-u]\}\right] \qquad (27)$$

Inspection of the equations (20), (21), (26) and (27) above reveals that an equilibrium point $(x^*,u^*,\varphi_K^*,\varphi_\lambda^*)$ exists, defined by:

$$f'[x^*/u^*] = \rho + g^P(t) + \delta_K \qquad (28)$$

$$h[1-u^*] + u^*\cdot h'[1-u^*] = \rho + g^P(t) - g^L(t) + \delta_\lambda \qquad (29)$$

$$\varphi_K^* = 1 \qquad (30)$$

$$\varphi_\lambda^* \cdot h'[1-u^*] = f[x^*/u^*] - (x^*/u^*) \cdot f'[x^*/u^*] \tag{31}$$

where

$$x = k/\lambda \tag{32}$$

Uzawa (1965) shows that the only trajectories $\{x(t), u(t), \varphi_K(t), \varphi_\lambda(t)\}$ satisfying all the necessary conditions and yielding true optimal growth paths are the ones leading to the equilibrium point $(x^*, u^*, \varphi_K^*, \varphi_\lambda^*)$ defined by (28) through (31). Unfortunately his analysis is incomplete. More specifically, the phase diagram in his Figure 3 for the case in which $s=1$ is not consistent with other properties of the system. In the Appendix to this chapter I correct Uzawa's analysis. The conclusion concerning the long-run optimality of the equilibrium point $(x^*, u^*, \varphi_K^*, \varphi_\lambda^*)$, however, remains valid.

Equation (28) is the now quite familiar Generalized Golden Rule. Equation (29) gives the optimal long-run allocation of the labour force over the productive sector and the research sector. In steady state, with population stable, this Generalized Golden Rule of Research reduces to:

$$h[1-u^*] + u^* \cdot h'[1-u^*] = \rho + \delta_\lambda \tag{33}$$

or, equivalently:

$$u^* \cdot \frac{d\hat{\lambda}[u^*]}{d(1-u^*)} = \rho - \hat{\lambda}[u^*] \tag{34}$$

Equation (34) states that the marginal product of the research sector (measured in terms of increases in the rate of labour-augmentation), modified by the labour leakage, should equal the social rate of time preference minus the rate of labour-augmentation.

It should be observed that the positivity of the LHS of (34) implies that

$$\rho > \hat{\lambda}[u^*] \tag{35}$$

thus ensuring the convergence of the integral (8) (cf. condition (4.1.22)).

Condition (29) implies that the relative size of the research sector depends on demographic forces in the short run only. Any departure of u from its equilibrium value u^* is temporary. As soon as the process of demographic transition is finished the optimal u returns to its previous steady-state value. Thus one should expect that the differences between the present model and the model of Chapter 4.1 are more pronounced with respect to the non-stationary growth path than with respect to the comparative statics results.

4.3.2 Comparative statics

I start from the following four equations:

the Golden Rule: $f'[z] = \rho + \delta_K + g$ (36)

the Golden Rule of Research: $h[1-u] + u \cdot h'[1-u] = \rho + \delta_\lambda$ (37)

the steady-state condition: $s = \dfrac{(\delta_K + g + h[1-u] - \delta_\lambda) \cdot z}{f[z]}$ (38)

the definition of C/P: $C/P = (1-s) \cdot u \cdot \lambda \cdot f[z] \cdot (L/P)$ (39)

where

$z = x/u$ (40)

is constant along the optimal steady-state growth path. As in Section 4.1.2 I will keep the initial level of λ in equation (39) constant.

The exogenous variables are:

- the growth rate of population g;
- the rate of depreciation of physical capital δ_K;
- the social rate of time preference ρ;
- the rate of depreciation of technology δ_λ.

Keeping λ constant, total differentiation of (36)-(39) yields:

$$\begin{bmatrix} f'' & 0 & 0 & 0 \\ 0 & -uh'' & 0 & 0 \\ (\delta_K+g+h-\delta_\lambda)\left[\frac{zf'-f}{f^2}\right] & \frac{h'z}{f} & 1 & 0 \\ -(1-s)\cdot\frac{L}{P}\cdot u\lambda f' & -(1-s)\cdot\frac{L}{P}\cdot\lambda f & \frac{L}{P}\cdot u\lambda f & 1 \end{bmatrix} \begin{bmatrix} dz \\ du \\ ds \\ d(C/P) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{z}{f} & \frac{z}{f} & 0 & -\frac{z}{f} \\ (1-s)u\lambda f\cdot\frac{\partial(L/P)}{\partial g} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dg \\ d\delta_K \\ d\rho \\ d\delta_\lambda \end{bmatrix} \tag{41}$$

Inverting the matrix on the LHS of (41):

$$\begin{bmatrix} f'' & 0 & 0 & 0 \\ 0 & -uh'' & 0 & 0 \\ (\delta_K+g+h-\delta_\lambda)\left[\frac{zf'-f}{f^2}\right] & \frac{h'z}{f} & 1 & 0 \\ -(1-s)\cdot\frac{L}{P}\cdot u\lambda f' & -(1-s)\cdot\frac{L}{P}\cdot\lambda f & \frac{L}{P}\cdot u\lambda f & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} \frac{1}{f''} & 0 & 0 & 0 \\ 0 & -\frac{1}{uh''} & 0 & 0 \\ -\frac{z}{f}\cdot\sigma\cdot\frac{\delta_K+g+h-\delta_\lambda}{\delta_K+g+\rho} & \frac{z}{f}\cdot\frac{h'}{uh''} & 1 & 0 \\ \frac{1}{f''}\cdot\frac{L}{P}\cdot u\lambda\cdot\{\rho-h+\delta_\lambda\} & -\frac{1}{uh''}\cdot\frac{L}{P}\cdot\lambda\cdot\{uh'z+(1-s)f\} & -\frac{L}{P}\cdot u\lambda f & 1 \end{bmatrix} \tag{42}$$

Combination of (42) and (41) finally gives:

$$\frac{dz}{dg} = \frac{dz}{d\delta_K} = \frac{dz}{d\rho} = \frac{1}{f''} < 0 \tag{43}$$

$$\frac{dz}{d\delta_\lambda} = 0 \tag{44}$$

$$\frac{du}{dg} = \frac{du}{d\delta_K} = 0 \tag{45}$$

$$\frac{du}{d\rho} = \frac{du}{d\delta_\lambda} = -\frac{1}{uh''} > 0 \tag{46}$$

$$\frac{ds}{dg} = \frac{ds}{d\delta_K} = \frac{z}{f}\cdot\left[1 - \sigma\cdot\frac{\delta_K+g+h[1-u]-\delta_\lambda}{\delta_K+g+\rho}\right] \gtreqless 0 \tag{47}$$

$$\frac{ds}{d\rho} = \frac{z}{f}\cdot\left[\frac{h'}{uh''} - \sigma\cdot\frac{\delta_K+g+h[1-u]-\delta_\lambda}{\delta_K+g+\rho}\right] < 0 \tag{48}$$

$$\frac{ds}{d\delta_\lambda} = \frac{z}{f}\cdot\left[\frac{h'}{uh''} - 1\right] < 0 \tag{49}$$

$$\frac{d(C/P)}{dg} = \frac{L}{P}\cdot u\lambda\cdot\left[\frac{\rho-h[1-u]+\delta_\lambda}{f''} - z\right] + (1-s)\cdot u\lambda f\cdot\frac{\partial(L/P)}{\partial g} \gtreqless 0 \tag{50}$$

$$\frac{d(C/P)}{d\delta_K} = \frac{L}{P}\cdot u\lambda\cdot\left[\frac{\rho-h[1-u]+\delta_\lambda}{f''} - z\right] < 0 \tag{51}$$

$$\frac{d(C/P)}{d\rho} = \frac{L}{P}\cdot\lambda\cdot\left[\frac{\rho-h[1-u]+\delta_\lambda}{f''}\cdot u - \frac{h'z}{h''} - \frac{(1-s)f}{uh''}\right] \gtreqless 0 \tag{52}$$

$$\frac{d(C/P)}{d\delta_\lambda} = \frac{L}{P}\cdot\lambda\cdot\left[-\frac{h'z}{h''} - \frac{(1-s)f}{uh''} + uz\right] > 0 \tag{53}$$

Most of these comparative statics results are similar to those derived in Section 4.1.2 for the case of exogenous technical change. Their discussion will not be repeated here.

The derivatives for u given in (45) and (46) are of course obvious from the expression for the Golden Rule of Research (37). Optimal research effort is negatively related to both ρ and δ_λ. A higher social rate of time preference (ρ) makes society more myopic and less willing to invest in research which is advantageous only in the long run. A higher rate of instantaneous decay of technical knowledge (δ_λ) may be thought of as rendering research activities less worthwhile. This argument is, however, not necessarily valid as is shown by the positive effect (for $\sigma<1$) for δ_K on the optimal savings rate. Thus we have another aspect in which physical capital and the stock of technical knowledge are fundamentally different in this model: a higher rate of depreciation warrants more investment effort in the case of physical capital but less investment effort in the case of technical knowledge.

From (3) and (46) we find:

$$\frac{d\hat{\lambda}}{d\rho} = \frac{h'}{uh''} < 0 \tag{54}$$

$$\frac{d\hat{\lambda}}{d\delta_\lambda} = \frac{h'}{uh''} - 1 < 0 \tag{55}$$

These equations help explain the results (48) and (49). A rise in δ_λ decreases the rate of labour augmentation and consequently the optimal rate of savings (cf. equation (4.1.17)). A rise in ρ affects both the rate of labour augmentation and the "capital-labour" ratio z, both effects resulting in a lower optimal savings rate.

The net effect of ρ on the initial level of consumption per capita (equation (52)) cannot be signed without specific knowledge of the functions f[·] and h[·]. Of the three terms in brackets in (52), the first gives the effect of ρ via z, both direct and indirect via the effect of z on s (negative); the second term gives the indirect effect of ρ via u and s (positive); and the third term gives the direct effect of ρ via u (also positive). The relative magnitudes of these three terms depend on the functional form of the production functions in the productive and research sector.

The various comparative statics results are summarized in Table 4.3.1. It can be concluded that in steady state there are only minor differences between the two models with endogenous and exogenous technical change, respectively.

4.3.3 The non-stationary optimal growth path

In this section I will analyse optimal investment in physical capital and research during periods of demographic transition.

First let us investigate optimal research effort (1-u). Differentiation of (29) with respect to time gives:

$$-u \cdot \dot{u} \cdot h''[1-u] = \dot{g}^P - \dot{g}^L \tag{56}$$

Table 4.3.1: Comparative-statics results for the one-sector model with endogenous disembodied labour-augmenting technical change

effect of: on:	g	δ_K	ρ	δ_λ
z	< 0	< 0	< 0	0
u	0	0	> 0	> 0
s	> 0 1)	> 0 1)	< 0	< 0
C/P (level)	< 0 2)	< 0	?	> 0
$\hat{\lambda}$ 3)	0	0	< 0	< 0

1) For $\sigma < \dfrac{\delta_K+g+\rho}{\delta_K+g+h[1-u]-\delta_\lambda}$

2) Except possibly for very low values of g

3) Restricted by: $\hat{\lambda} = h[1-u] - \delta_\lambda < \rho$

which together with (4) implies:

$$\text{sign}\ (\dot{u}) = \text{sign}\ (\dot{g}^P - \dot{g}^L) \tag{57}$$

If population grows relative to the labour force, a higher proportion of the labour force is allocated to the productive sector; if on the other hand the labour force grows relative to the population, then research activities are increased. This result can also be interpreted in scarcity terms: a relative increase in the population temporarily increases the demand for consumption goods, causing a shift towards the (short-term oriented) production of consumption goods (u is increased).

Consider a period of demographic transition with the growth rate of births falling. As the age-ability profile $h(\cdot)$ is single-peaked as well as zero for both very young and very old ages, $\dot{g}^P$ will be negative while $\dot{g}^L$ will be almost zero in the initial phases of the transition process. The same will be true near the end of the transition process, shortly before the economy settles into its new steady state. Thus the optimal non-stationary growth path of $u(\cdot)$ will be falling both in the beginning and in the end. And, since the net change in u must be zero, we conclude that u must be rising in the middle phases of the transition period.

From (28) we have:

$$f''[x/u]\cdot(x/u)\cdot\{\hat{x}-\hat{u}\} = \dot{g}^P \tag{58}$$

from which:

$$\text{sign}\ (\hat{x}-\hat{u}) = -\ \text{sign}\ (\dot{g}^P) \tag{59}$$

Here x is the capital/labour-ratio (in efficiency units) for the economy as a whole; x/u is the capital/labour-ratio for the productive sector. Thus, for g^B falling, the growth rate of the total capital/labour-ratio x must exceed the growth rate of the employment share of the productive sector during the full period of demographic transition.

Unfortunately, the latter observation does not take us very far in determining the shape of the optimal trajectory for the savings rate s. Both u and $\hat{\lambda}$ change along the non-stationary growth path and assessing the net impact of these changes together with (59) on the direction of $\dot{s}$ is not possible in general. If the gradual decline in g^B is relatively gentle, then $\dot{g}^P \approx \dot{g}^L$ during the middle phases of the transition process; in that case $\dot{u}$ and $\dot{\lambda}$ are almost zero and the analysis of Section 4.1.3 applies. But this is only a special case.

Therefore I have to rely on some numerical simulations again. The specifications for the functions and the values of the parameters used for these simulations are summarized in Table 4.3.2. The results of the simulations themselves are given in Tables 4.3.3 and 4.3.4, and Figures 4.3.1 and 4.3.2.

The plot of the optimal savings rate is again roughly U-shaped or inversely U-shaped, depending on the direction of the demographic change. The oscillations near the endpoints of the non-stationary trajectory accompany the changes in optimal research effort u.

Research effort itself behaves rather unevenly, with most of the dynamics occurring near the endpoints of the transition period. This picture can partly be ascribed to the rather crude discretization (only 4 generations have been discerned), but even with a finer discretization the fact remains that for most of the transition period g^P and g^L change at the same rate, rendering the RHS of (57) almost equal to zero.

The rate of technical progress, which is a function of research effort, changes only mildly along the non-stationary growth path. This result is of course highly sensitive to the specification of the research production function h[·]. Without any empirical work to guide me, I did some trial and error specifications before deciding to use the present one. The very low rate of technical progress that it produces indicates that some more trial and error specifications are needed, preferably in combination with empirical research, before the model can be used for the description of real-life economies.

Table 4.3.2: Parameters used for illustrations in Figures 4.3.1 and 4.3.2

Number of generations	4
Survival schedule	$\mu(1)=1$; $\mu(2)=0.9$; $\mu(3)=0.7$; $\mu(4)=0.4$
Age-ability profile	$h(1)=0$; $h(2)=1$; $h(3)=1$; $h(4)=0$
Social rate of impatience	$\rho=0.1$
Depreciation rates	$\delta_K=0.1$; $\delta_\lambda=0.05$
Growth rate of births	0.30 ··· 0.00
Production function	$y=\{\alpha k^{-\frac{1-\sigma}{\sigma}}+(1-\alpha)\}^{-\frac{\sigma}{1-\sigma}}$
	$\alpha=0.25$
	$\sigma=0.5$
Research prod. function	$h[1-u]=\gamma\cdot(1-u)^\beta$
	$\gamma=0.12$
	$\beta=0.3$

Table 4.3.3: Optimal economic growth path with g^B falling

t	g^B	g^P	g^L	u	s	$\hat{\lambda}$
1	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
2	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
3	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
4	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
5	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
6	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
7	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
8	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
9	0.2875	0.3000	0.3000	0.7479	0.2949	0.0294
10	0.2750	0.2945	0.3000	0.7164	0.2832	0.0322
11	0.2625	0.2853	0.2922	0.7072	0.2960	0.0330
12	0.2500	0.2739	0.2797	0.7144	0.3021	0.0324
13	0.2375	0.2615	0.2672	0.7149	0.2942	0.0324
14	0.2250	0.2492	0.2548	0.7155	0.2896	0.0323
15	0.2125	0.2368	0.2423	0.7161	0.2849	0.0323
16	0.2000	0.2244	0.2298	0.7167	0.2801	0.0322
17	0.1875	0.2120	0.2174	0.7173	0.2753	0.0321
18	0.1750	0.1996	0.2049	0.7179	0.2705	0.0321
19	0.1625	0.1873	0.1924	0.7185	0.2655	0.0320
20	0.1500	0.1749	0.1799	0.7191	0.2605	0.0320
21	0.1375	0.1625	0.1675	0.7197	0.2555	0.0319
22	0.1250	0.1502	0.1550	0.7203	0.2504	0.0319
23	0.1125	0.1378	0.1425	0.7210	0.2452	0.0318
24	0.1000	0.1254	0.1301	0.7216	0.2399	0.0318
25	0.0875	0.1131	0.1176	0.7223	0.2346	0.0317
26	0.0750	0.1007	0.1051	0.7230	0.2292	0.0316
27	0.0625	0.0884	0.0927	0.7237	0.2238	0.0316
28	0.0500	0.0760	0.0802	0.7243	0.2183	0.0315
29	0.0375	0.0637	0.0677	0.7250	0.2127	0.0315
30	0.0250	0.0513	0.0553	0.7258	0.2071	0.0314
31	0.0125	0.0390	0.0428	0.7265	0.2015	0.0313
32	0.0000	0.0266	0.0304	0.7272	0.1959	0.0313
33	0.0000	0.0143	0.0179	0.7279	0.1903	0.0312
34	0.0000	0.0061	0.0054	0.7514	0.1968	0.0290
35	0.0000	0.0016	0.0000	0.7560	0.1588	0.0286
36	0.0000	0.0000	0.0000	0.7479	0.1351	0.0294
37	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
38	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
39	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
40	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294

Table 4.3.4: Optimal economic growth path with g^B rising

t	g^B	g^P	g^L	u	s	$\hat{\lambda}$
1	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
2	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
3	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
4	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
5	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
6	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
7	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
8	0.0000	0.0000	0.0000	0.7479	0.1405	0.0294
9	0.0125	0.0000	0.0000	0.7479	0.1405	0.0294
10	0.0250	0.0042	0.0000	0.7674	0.1494	0.0275
11	0.0375	0.0121	0.0070	0.7714	0.1230	0.0271
12	0.0500	0.0231	0.0196	0.7647	0.1131	0.0277
13	0.0625	0.0358	0.0321	0.7652	0.1286	0.0277
14	0.0750	0.0484	0.0446	0.7658	0.1393	0.0276
15	0.0875	0.0611	0.0572	0.7663	0.1494	0.0276
16	0.1000	0.0737	0.0697	0.7668	0.1589	0.0275
17	0.1125	0.0864	0.0823	0.7673	0.1680	0.0275
18	0.1250	0.0990	0.0948	0.7678	0.1767	0.0274
19	0.1375	0.1117	0.1073	0.7683	0.1850	0.0274
20	0.1500	0.1243	0.1199	0.7688	0.1930	0.0273
21	0.1625	0.1370	0.1324	0.7693	0.2006	0.0273
22	0.1750	0.1496	0.1449	0.7697	0.2080	0.0272
23	0.1875	0.1623	0.1575	0.7702	0.2152	0.0272
24	0.2000	0.1749	0.1700	0.7706	0.2221	0.0272
25	0.2125	0.1875	0.1825	0.7710	0.2288	0.0271
26	0.2250	0.2002	0.1951	0.7714	0.2354	0.0271
27	0.2375	0.2128	0.2076	0.7718	0.2417	0.0270
28	0.2500	0.2254	0.2201	0.7722	0.2479	0.0270
29	0.2625	0.2380	0.2326	0.7726	0.2540	0.0270
30	0.2750	0.2507	0.2452	0.7730	0.2599	0.0269
31	0.2875	0.2633	0.2577	0.7734	0.2657	0.0269
32	0.3000	0.2759	0.2702	0.7737	0.2713	0.0268
33	0.3000	0.2885	0.2828	0.7741	0.2768	0.0268
34	0.3000	0.2957	0.2953	0.7498	0.2746	0.0292
35	0.3000	0.2990	0.3000	0.7426	0.2893	0.0299
36	0.3000	0.3000	0.3000	0.7479	0.2972	0.0294
37	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
38	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
39	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294
40	0.3000	0.3000	0.3000	0.7479	0.2949	0.0294

Figure 4.3.1: Optimal economic growth path with g^B falling

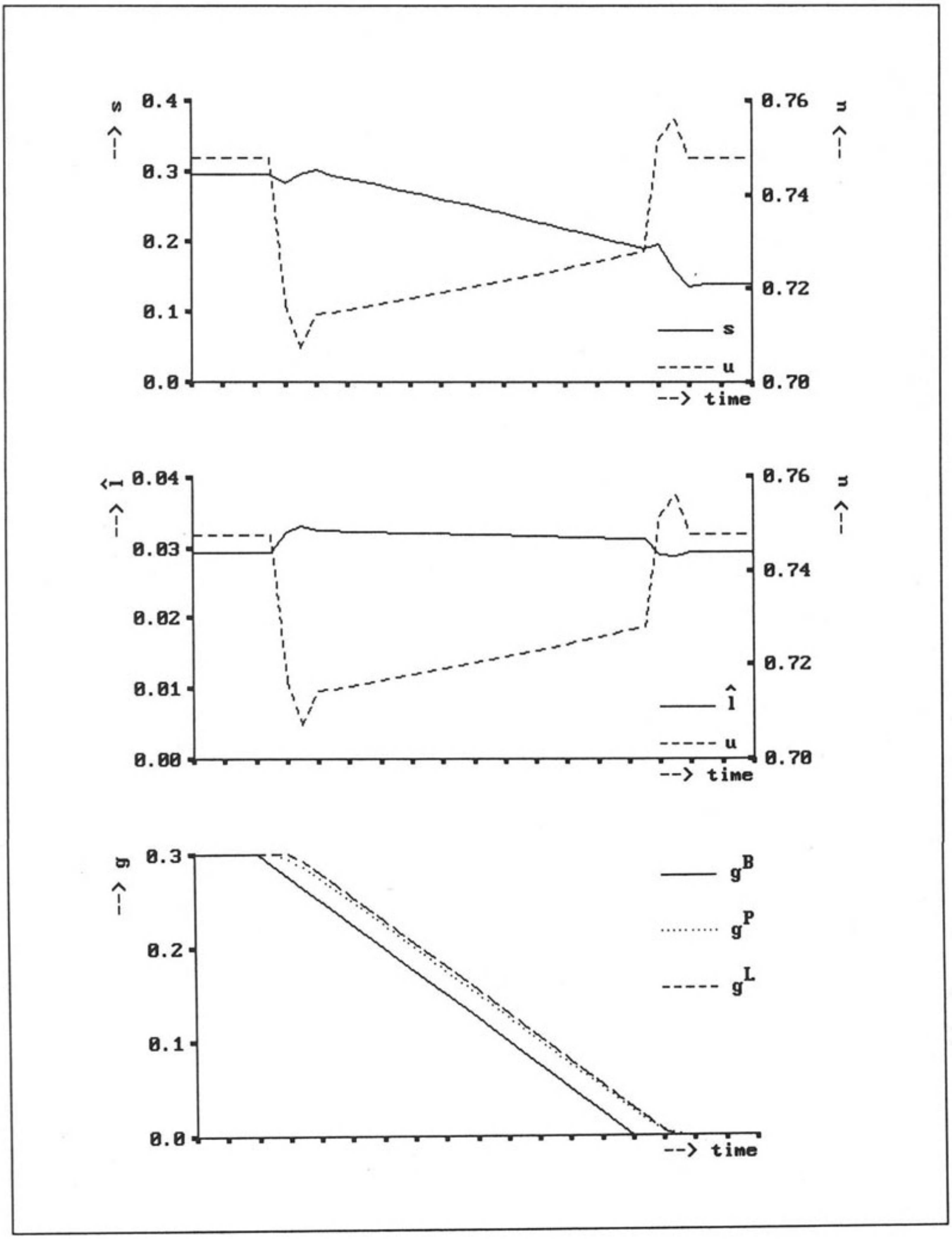
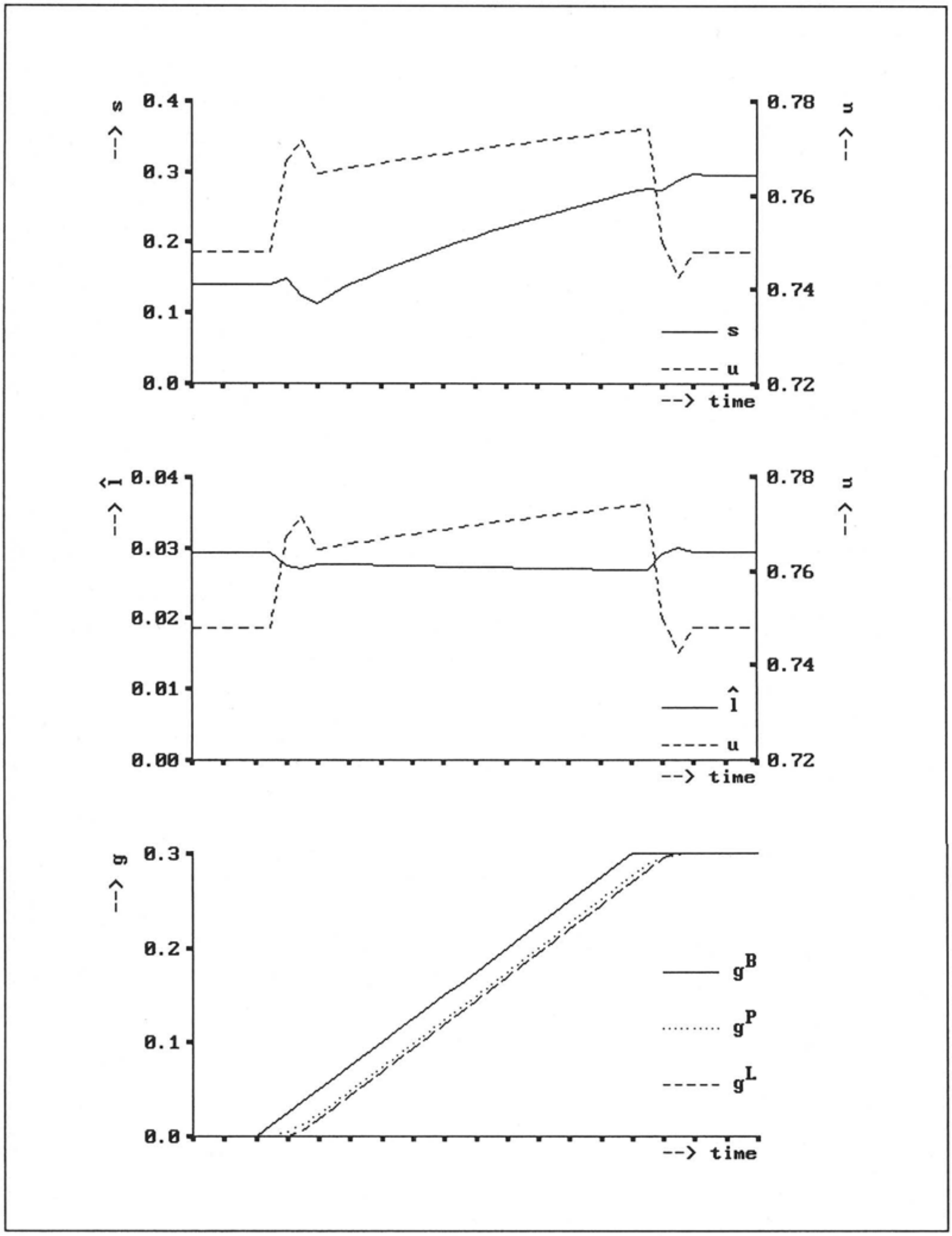


Figure 4.3.2: Optimal economic growth path with g^B rising



4.3.4 Summary

The main results of the analysis of Uzawa's (1965) model of endogenous technical change can be summarized as follows:

1. A singular trajectory is characterized by two Golden Rules: the Non-Stationary Golden Rule of Accumulation, and the Generalized Golden Rule of Research.
2. A higher rate of depreciation warrants more investment effort in the case of physical capital, but less investment effort in the case of technical knowledge.
3. The optimal relative size of the research sector depends on demographic forces in the short run only. In steady state there are only minor differences between the models with endogenous and exogenous technical change, respectively.
4. During periods of demographic transition the plot of the optimal savings rate is roughly U-shaped or inversely U-shaped, depending on whether the growth rate of births is falling or rising. There are some small oscillations near the endpoints of the non-stationary trajectory accompanying the changes in optimal research effort.
5. Most of the changes in optimal research effort occur near the endpoints of the transition period.
6. It is hard to find a specification of the research production function that yields plausible numerical results.

APPENDIX TO CHAPTER 4.3

In this Appendix I show that for Uzawa's (1965) model of endogenous technical change all optimal paths must finally converge towards the long-run equilibrium point $(x^*, u^*, \varphi_K^*, \varphi_\lambda^*)$ defined by equations (28)-(31).

case a: $\varphi_K > 1$

The system is characterized by the following equations:

From (20.a):

$$s = 1 \tag{A.1}$$

From (32), (9), (10) and (A.1):

$$\hat{x} = f[x/u]/(x/u) - (\delta_K + g^L - \delta_\lambda) - h[1-u] \equiv \alpha[x,\varphi] \tag{A.2}$$

From (26) and (A.1):

$$f[x/u] - (x/u) \cdot f'[x/u] = h'[1-u] \cdot \varphi \tag{A.3}$$

where

$$\varphi \equiv \varphi_\lambda / \varphi_K \tag{A.4}$$

From (A.4), (27) and (22):

$$\hat{\varphi} = -(g^L + \delta_K - \delta_\lambda) + f'[x/u] - \{h[1-u] + u \cdot h'[1-u]\} \equiv \gamma[x,\varphi] \tag{A.5}$$

Equation (A.3) determines the optimal value of u, given x and φ. For convenience I assume that (A.3) always yields an interior solution for u. This assumption does not affect the analysis in any fundamental way.

Total differentiation of (A.3) yields:

$$-(x/u) \cdot f''[x/u] \cdot \{(1/u) \cdot dx - (x/u^2) \cdot du\} = -h'' \cdot \varphi \cdot du + h' \cdot d\varphi \tag{A.6}$$

from which:

$$0 < \frac{du}{dx} = \frac{u\cdot x\cdot f''[x/u]}{x^2\cdot f''[x/u] + \varphi\cdot u^3\cdot h''[1-u]} < \frac{u\cdot x\cdot f''[x/u]}{x^2\cdot f''[x/u]} = u/x \tag{A.7}$$

$$0 > \frac{du}{d\varphi} = \frac{u^3\cdot h'[1-u]}{x^2\cdot f''[x/u] + \varphi\cdot u^3\cdot h''[1-u]} \tag{A.8}$$

Differentiating (A.2):

$$\frac{d\hat{x}}{dx} = \frac{d\alpha}{dx} = \frac{\partial\alpha}{\partial x} + \frac{\partial\alpha}{\partial u}\cdot\frac{du}{dx} =$$

$$= -(1/u)\cdot\frac{f-(x/u)\cdot f'}{(x/u)^2} + \left[\ h' + (x/u^2)\cdot\frac{f-(x/u)\cdot f'}{(x/u)^2}\ \right]\cdot\frac{du}{dx} \gtreqless 0 \tag{A.9}$$

$$\frac{d\hat{x}}{d\varphi} = \frac{d\alpha}{d\varphi} = \frac{\partial\alpha}{\partial u}\cdot\frac{du}{d\varphi} = \left[\ h' + (x/u^2)\cdot\frac{f-(x/u)\cdot f'}{(x/u)^2}\ \right]\cdot\frac{du}{d\varphi} < 0 \tag{A.10}$$

Differentiating (A.5), and using the second inequality in (A.7):

$$\frac{d\hat{\varphi}}{dx} = \frac{d\gamma}{dx} = \frac{\partial\gamma}{\partial x} + \frac{\partial\gamma}{\partial u}\cdot\frac{du}{dx} = f''/u + \{u\cdot h'' - (x/u^2)\cdot f''\}\cdot\frac{du}{dx} <$$

$$< u\cdot h''\cdot\frac{du}{dx} + f''/u - (x/u^2)\cdot f''\cdot(u/x) = u\cdot h''\cdot\frac{du}{dx} < 0 \tag{A.11}$$

$$\frac{d\hat{\varphi}}{d\varphi} = \frac{d\gamma}{dx} = \frac{\partial\gamma}{\partial u}\cdot\frac{du}{d\varphi} = \{u\cdot h'' - (x/u^2)\cdot f''\}\cdot\frac{du}{d\varphi} \gtreqless 0 \tag{A.12}$$

Disregarding the possibility of derivatives being equal to zero (except perhaps at some isolated points) and restricting attention to a neighbourhood of the equilibrium point (x^*,φ^*) sufficiently small for the derivatives not to change sign, there are four possible combinations of signs of the derivatives in (A.9)-(A.12). Of these four combinations, two can be ruled out as will be demonstrated below.

From (A.2) and (A.5) it follows that

$$\alpha[x,\varphi] - \gamma[x,\varphi] = \hat{x} - \hat{\varphi} = \frac{f - (x/u)\cdot f'}{x/u} + u\cdot h' > 0 \tag{A.13}$$

Now for some fixed $x=\bar{x}$ let us define:

$$\varphi_\alpha[\bar{x}] \equiv \varphi \mid \alpha[\bar{x},\varphi] = 0 \tag{A.14}$$

$$\varphi_\gamma[\bar{x}] \equiv \varphi \mid \gamma[\bar{x},\varphi] = 0 \tag{A.15}$$

From (A.13)-(A.15) we find:

$$\gamma[\bar{x},\varphi_\alpha[\bar{x}]] < 0 \quad ; \quad \alpha[\bar{x},\varphi_\alpha[\bar{x}]] = 0 \tag{A.16}$$

$$\gamma[\bar{x},\varphi_\gamma[\bar{x}]] = 0 \quad ; \quad \alpha[\bar{x},\varphi_\gamma[\bar{x}]] > 0 \tag{A.17}$$

Finally, from (A.16) and (A.17) it follows that, regardless of whether $\varphi_\alpha[\bar{x}]>\varphi_\gamma[\bar{x}]$ or $\varphi_\alpha[\bar{x}]<\varphi_\gamma[\bar{x}]$:

$$\text{sign}\left[\frac{d\alpha}{d\varphi}\right] = \text{sign}\left[\frac{d\gamma}{d\varphi}\right] \tag{A.18}$$

so that, using (A.10)

$$\frac{d\hat{\varphi}}{d\varphi} = \frac{d\gamma}{d\varphi} < 0 \tag{A.19}$$

We can summarize our finding in the following table:

case	sign of:				slope of:		sign of
	$\frac{d\alpha}{dx}$	$\frac{d\alpha}{d\varphi}$	$\frac{d\gamma}{dx}$	$\frac{d\gamma}{d\varphi}$	$\varphi_\alpha[x]$	$\varphi_\gamma[x]$	$\varphi_\alpha[x]-\varphi_\gamma[x]$
1	>0	<0	<0	<0	rising	falling	>0
2	<0	<0	<0	<0	falling	falling	>0

For each of these cases we can draw phase diagrams in (x,φ)-space, depicted in Figures 4.3.A1 and 4.3.A2. Neither of these corresponds to Uzawa's Figure 3, reproduced here as Figure 4.3.A3. Indeed, it is not difficult to see that Uzawa's phase diagram for the case s=1 is not consistent with some of the system's properties.

Figures 4.3.A1 and 4.3.A2 each yield the same conclusion: starting from the situation in which $\varphi_K>1$ and s=1, the only trajectories yielding true optimal growth paths point towards the equilibrium point (x^*,φ^*).

case b: $\varphi_K < 1$

Now we have the following equations:
From (20.c):

$$s = 0 \tag{A.20}$$

From (32), (9), (10) and (A.20):

$$\hat{x} = -(\delta_K + g^L - \delta_\lambda) - h[1-u] < 0 \tag{A.21}$$

From (26) and (A.20):

$$f[x/u] - (x/u)\cdot f'[x/u] = h'[1-u]\cdot\varphi_\lambda \tag{A.22}$$

From (27):

$$\hat{\varphi}_\lambda = (\rho + g^P - g^L + \delta_\lambda) - \{h[1-u] + u\cdot h'[1-u]\} \tag{A.23}$$

Again, from (A.22):

$$\frac{du}{dx} > 0 \tag{A.24}$$

$$\frac{du}{d\varphi_\lambda} < 0 \tag{A.25}$$

Differentiating (A.23):

$$\frac{d\hat{\varphi}_\lambda}{dx} = \frac{\partial\hat{\varphi}_\lambda}{\partial u}\cdot\frac{du}{dx} = uh''\cdot\frac{du}{dx} < 0 \tag{A.26}$$

$$\frac{d\hat{\varphi}_\lambda}{d\varphi_\lambda} = \frac{\partial\hat{\varphi}_\lambda}{\partial u}\cdot\frac{du}{d\varphi_\lambda} = uh''\cdot\frac{du}{d\varphi_\lambda} > 0 \tag{A.27}$$

The phase diagram corresponding to equations (A.21), (A.26) and (A.27) is given in Figure 4.3.A4. This diagram shows that, starting from the situation in which $\varphi_K<1$ and $s=0$, the only trajectory yielding a true optimal growth path points towards the equilibrium point (x^*,φ^*).

This concludes the proof of the long-run optimality of the equilibrium point $(x^*,u^*,\varphi_K^*,\varphi_\lambda^*)$ defined by equations (28) through (31).

Figure 4.3.A1: Phase diagram for s=1 - case 1

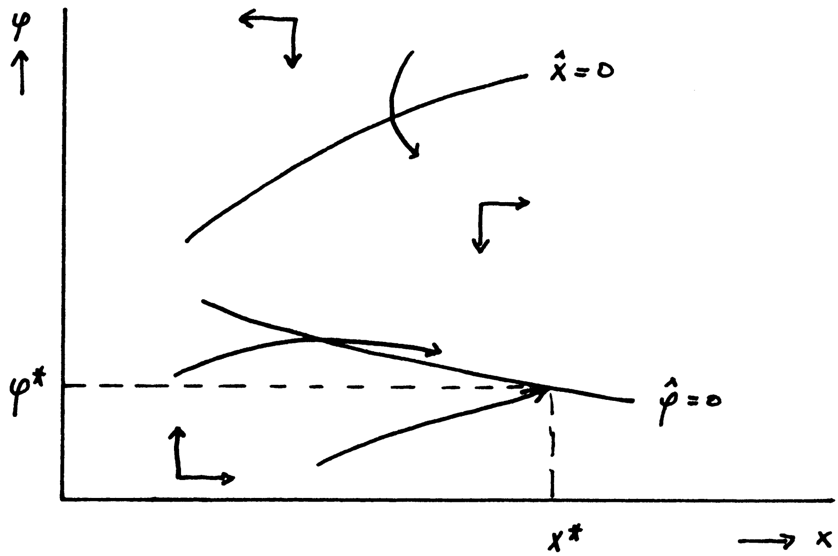


Figure 4.3.A2: Phase diagram for s=1 - case 2

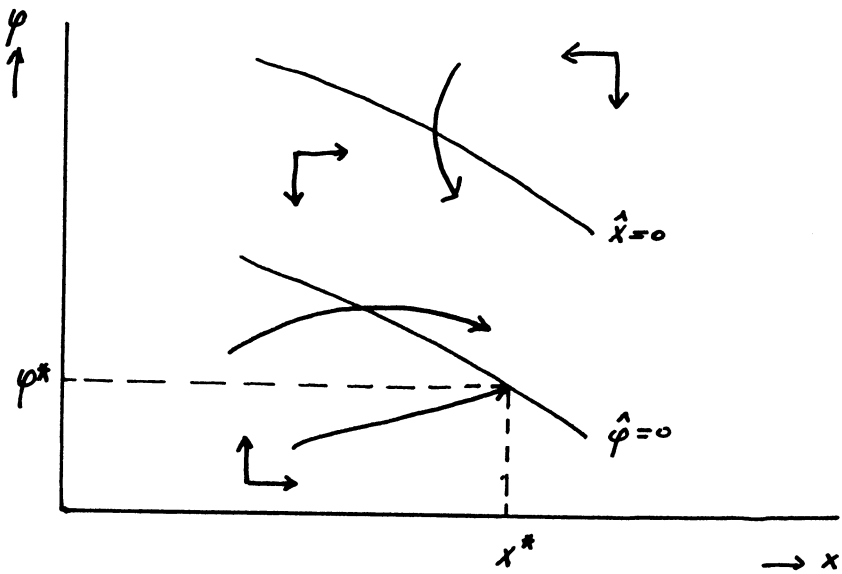


Figure 4.3.A3: Uzawa's phase diagram for s=1

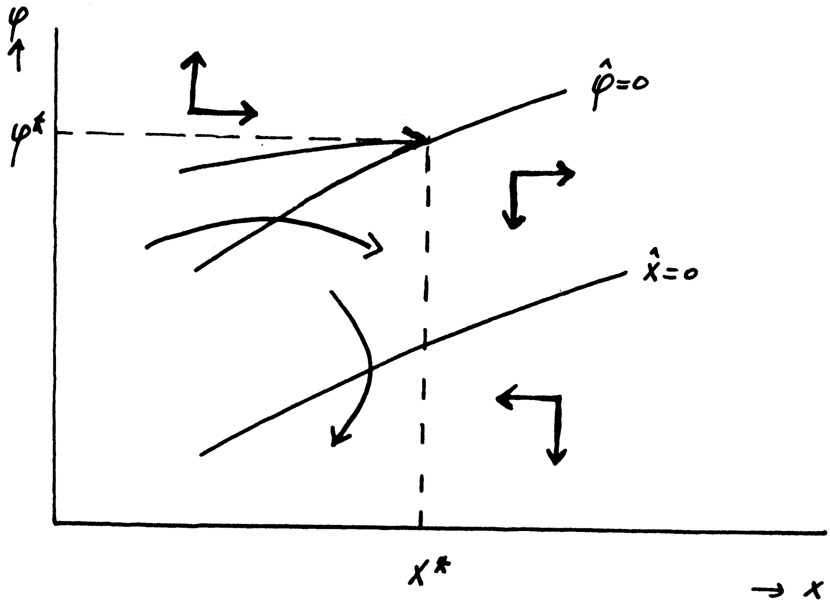
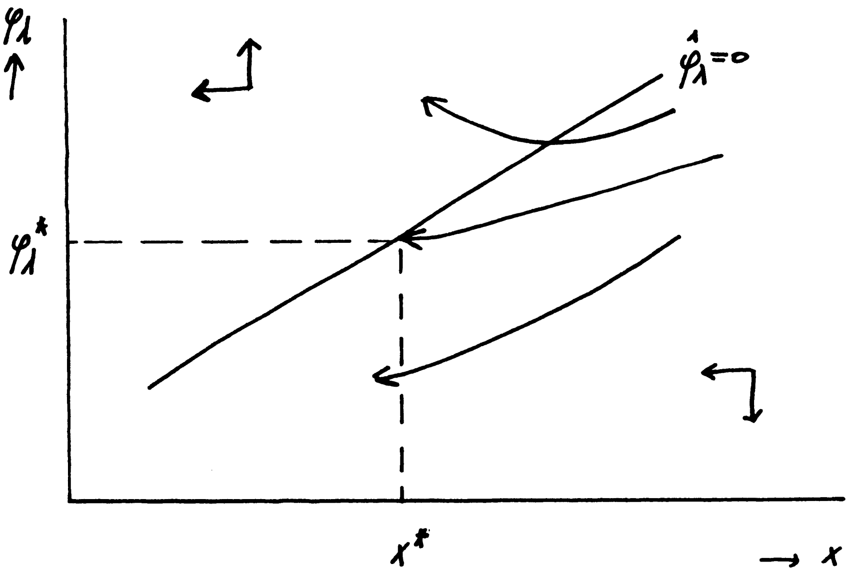


Figure 4.3.A4: Phase diagram for s=0



5 EDUCATION AND ECONOMIC GROWTH

This chapter discusses the relationship between education and economic growth. After a short historical introduction the attention focusses on the form and economic significance of the educational production process. A major conclusion that emerges from the discussion is that education is a very complex human activity and that there is no consensus among economists on the way in which this activity should be modelled. This chapter concludes with a short characterization of the models that form the basis of the analyses in the subsequent chapters.

5.1 Economics and education

Economics is the science which studies human behaviour as a relationship between ends and scarce means that have alternative uses (Robbins, 1932/1962, p. 16). If this definition is applied, one finds that a vast proportion of human activities belongs to the domain of economics. The fact that this domain is steadily expanding (Hirshleifer, 1985) merely illustrates that economists regularly discover new fields of human behaviour which involve the allocation of scarce resources.

Education is such a relatively new field. This is not to say that the classical economists did not recognize the economic significance of education. For example Adam Smith (1776/1974, pp. 201 ff.) explicitly stressed the importance of education for the society's capacity to produce. Just as investments in machines increase the stock of the factor of production physical capital, so education results in a higher productivity of the factor labour. After Smith, however, the profession directed its attention almost exclusively to the factors land and particularly physical capital ("machines"). Increasing the quantity and quality of the capital stock was considered as the driving force behind economic growth. Education was primarily viewed as a consumption good and analysed on the same footing as potatoes, soap, and holiday trips.

A number of circumstances caused a dramatic change in the economists' attitude towards education. The more or less explosive economic development of the fifties and sixties resulted in a highly strained labour market, especially for high-level educated personnel. The educational level of the labour force was clearly operating more and more as an effective constraint on society's strive for economic growth. This feeling was strengthened by the launching of the first Sputnik-missile in 1957 wich caused quite a shock in the Western world. The general opinion was that the Soviet-Union's lead in the aerospace race was directly related to educational matters (Ritzen, 1986a).

Since the Human Capital school started its activities at the beginning of the sixties (e.g. Schultz, 1961a, 1961b and 1962; Mincer, 1958; Becker, 1964/1975) it has been explicitly recognized that not only the stock of physical capital but also the labour force, i.e. the stock of human capital, can be increased by investment and saving behaviour. Education was drawn into the domain of economics. The central insight of this new branch of economics (the "economics of education") is that the quality and quantity of labour are not exogenously determined variables: rather they are the outcomes of deliberate economic decision-making processes.

5.2 Education as a production process

Education is a very complex human activity. Concentrating on its economic aspects, it is obvious that education uses scarce resources and produces useful outputs. In this respect education is just a special kind of a production process. However, contrary to processes yielding physical outputs, the nature of the educational production process is not easily characterized (cf. Hanushek, 1979). There is no general agreement on any of the following three central characteristics of the educational production process:

- what useful outputs does education produce?
- what scarce resources does it use as inputs?
- what is the technical relationship between inputs and outputs?

These questions are discussed below.

5.2.1 Educational outputs

Education does not produce something out of nothing. Rather, it transforms individuals. This implies that the output of education should be measured in terms of value added: after education the individuals are more useful (in one way or another) than they were before, the difference constituting the contribution of education to economic welfare (cf. Hanushek, 1986).

The list of useful transformations of individuals performed by education is long and comprises such diverse items as the following (cf. Haveman & Wolfe, 1984):

- consumption: education is a pleasant way of spending one's time, both in the short run (piano lessons) and in the long run (enjoying playing the piano) (see, however, Lazear, 1977);
- baby-sitting facilities: education allows the parents to participate in the production process;
- individual character development;
- information: this role of education is stressed by the advocates of the screening and related theories (Arrow, 1973);
- redistribution of economic opportunities;
- socialization, national integration, political awareness: these are essential ingredients for the social structure of the community;
- knowledge and skills useful for future use in production processes (both educational and other): this output is generally termed as "human capital".

Most of these educational outputs are essentially micro-economic in nature. On the macro level, with which we are concerned in this study, these multiple outputs can be grouped into four main categories:

- consumption;
- infrastructure;
- factor inputs;
- technical knowledge.

As far as education produces consumption goods it does not differ essentially from the traditional productive sector. Thus, in the aggregative growth model the consumptive aspects of education can be safely ignored.

The infrastructure output of education is in a way the most important category. It consists of those requirements that are essential for the functioning of a developed society: general social and political awareness of the population, universal ability to read, write and communicate, etcetera. The relationship between education and economic growth is primarily concentrated at this basic level for developing countries, where the infrastructure is typically weak. However, for the developed industrial (and post-industrial) societies this mechanism through which education affects growth is of relatively little importance. Once the basic educational infrastructure has been completed it is there and will be maintained. Thus, by restricting attention to developed societies I implicitly assume that education performs its most basic tasks in a satisfactory way and that it keeps on doing so, irrespective of economic and demographic developments.

The Human Capital school concentrates on the third group of educational outputs: the productivity of the labour force or, alternatively, "human capital". However, the concept of human capital is not without its ambiguities (nor is physical capital, for that matter).

The concept of human capital was originally introduced in order to overcome the obviously unrealistic traditional assumption of homogeneous labour. Surely, people differ in their capacity to produce and these differences are at least partly due to differences in educational achievement. But it is not at all evident that these differences can be measured by a uni-dimensional unit of measurement like human capital. Such an approach, as is prevalent especially in the initial human capital literature, implies that the productivity differences between individuals are purely quantitative, not qualitative in nature. In the language of Chapter 3.1, the use of a single-valued concept of human capital implies that the marginal rate of substitution between two types of labour is constant.

In fact the use of such a homogeneous concept of human capital is just as difficult to imagine as the concept of homogeneous labour. An alternative way of expressing the influence of education on individual labour productivity is to distinguish different levels of labour and to interpret education as transforming one kind of labour into another. This approach has been followed by e.g. Tu (1969), Ritzen (1977) and Sethi & McGuire (1977). There are various terms by which these different kinds of labour can be denoted, some of which are "skills", "capabilities" and "educational levels". They have in common that human capital is not measured by some unit of measurement but by some qualitative index.

A further step is to combine the cardinal measurability of human capital with its multi-dimensionality. This approach has been pioneered by Hartog (1981). In his model the productive capacity of an individual is described by a vector of numbers, each number indicating the degree to which the individual has the skill in question. The advantages of this so-called "multi-capability theory" are its generality and its flexibility. A major disadvantage, however, is that its application raises a number of conceptual and analytical difficulties.

In the preceding chapters I have made no attempt to defend the highly aggregative level of the models used. For the same reasons I am satisfied with the use of the homogeneous human capital concept in the major parts of the remainder of this study. It is only in Section 6.2 that I will analyse a model in which different "skills" are distinguished. The reasons for doing so are that such a model is especially suited for analysing the role of time lags in the educational sector and that the results can serve to check the robustness of the conclusions obtained from the more aggregative model.

The fourth group of educational outputs that was mentioned above has been termed as technical change. As should be apparent from the discussion in Chapter 3 the way in which economists have interpreted and modelled this aspect of education is far from uniform. The confusion centres around the crucial (at least for modelling purposes) distinction between human capital as a factor of production and human capital as an indicator of the level of technology. This distinction has been discussed very clearly in e.g. Mincer (1984).

For the reasons mentioned in Chapter 3.1, I see no reason why the capacity to produce, embodied in an educated individual who is for hire in the labour market, should be termed anything else but a factor of production. In my view, the bulk of educational performance consists of teaching well-known useful knowledge or manual capabilities to new generations and has nothing to do with technical change. For this reason we can dismiss the practice of authors like Razin (1972a; 1972b) - probably misguided by the unfortunate terminology of Uzawa (1965) - who model education as producing technical knowledge only.

However, even if the main role of education is to increase the stock of factor inputs (human capital), there are various ways in which one could well imagine education affecting also the level of technology. One approach is to concentrate on education as the mechanism by which new technical knowledge is transmitted to entrepreneurs. This leads to models in which education affects the rate of technological diffusion, as in Nelson & Phelps (1966) and in Stephens (1971).

Another possibility is that education creates, not merely transmits, new technical knowledge. In a somewhat disguised form this assumption can be found in the learning-by-doing models (Arrow, 1962; Sheshinski, 1967; Fellner, 1969). From a more empirical point of view, the frequently encountered institutional linkage between education and research (as e.g. in universities) could lead one to interpret new technical knowledge and productive capabilities (human capital) as joint products of the educational production process. This insight might be modelled by assuming a positive relationship between the education attendance rate and the rate of technical progress.

5.2.2 Educational inputs

On the input side of the educational production process the diversity in the number and nature of relevant variables is equally prevalent as it is on the output side. The degree to which education realizes its objectives is not only dependent on the organization of the process itself ("school inputs"), but also on the characteristics of the individual students and peer-group composition, serving in fact as the "raw materials" of the production process, and on external inputs, like social environment (see e.g. Hanushek, 1979 and 1986). Since these and similar variables are essentially micro in nature, they can easily be disregarded in the highly aggregative context of this study.

Still, even at the macro level there are various educational inputs to consider. Probably the most important are the following:

- students' time;
- students' human capital (quality);
- teachers' time (and quality);
- parents' time (and quality);
- physical commodities.

If one were asked to give a single characterization of the educational production process virtually everyone would answer that it takes time. However diverse the various specifications of the production function for human capital that can be found in the literature, almost all of them have in common that students' time is one of the leading factor inputs (e.g. Ritzen & Winkler, 1977).

The students' human capital as an educational input can take various forms. Among other things this argument of the production function can be used to take into account the effect of innate ability: individuals are at the time of their birth endowed with some stock of human capital that determines to a certain extent the success of their schooling career. Another aspect of the human capital variable is that it allows one to model the output of one phase of the educational production process serving as the input for a subsequent phase, or, alternatively, to take into account the possibility that the efficiency of learning is an increasing function of educational achievement.

The value of these two inputs, students' time and students' human capital, constitute the so-called indirect costs of education, frequently also labelled as "earnings foregone". For most types of education these indirect costs form the bulk of the total costs as far as the individual student is concerned, since the other inputs are typically heavily subsidized or even supplied freely by the government. In many micro-economic studies of demand for education the direct costs are therefore completely neglected (e.g. Haley, 1976; Rosen, 1976; Theeuwes e.a., 1985), although other studies do allow for market inputs like tuition fees and the like (e.g. Ben-Porath, 1967; Kodde, 1985).

At the macro level the direct costs of education are of course significant (cf. Hanushek, 1986), which has led some authors to incorporate such inputs in their macro-educational production functions (e.g. Ritzen, 1977). Still, there are several grounds on which one could defend the exclusion of non-student related factor inputs. According to one argument, in extending the aggregative model to include an educational sector one should concentrate on the essential characteristics of education. Alternatively, one should realize that education is not confined to formal full-time schooling. A large proportion of human capital formation is done in the form of training "on the job", where direct factor inputs are almost impossible to measure but are probably much smaller than in regular public schools (Mincer, 1962). If, as is invariably done, the full-time and part-time types of education are lumped together into a single aggregate educational sector, every specification of factor inputs like teachers' services and physical commodities becomes rather arbitrary, while it does not seem to add significantly to the model's descriptive power.

5.2.3 Functional form

Given the wide range of choice for both factor inputs and educational outputs, it is not surprising that even some moderate degree of consensus on the functional form of the educational production function is remote. Especially in the context of macro-economic models the functional form varies from author to author and sometimes even between different studies. The choice for a particular functional form is partly governed by the specification of inputs and outputs. From the remaining degrees of freedom the choice is greatly restricted by the necessity for analytical convenience and often by the wish for explicit solutions to the complete model. For lack of a well-established theoretical foundation of the learning process, it is considerations of convenience rather than of plausibility or empirical support that govern the choice for the specification of the educational production function.

5.3 Selection of models

For lack of some generally accepted practice to guide me, I have simply selected some specifications of the educational production process for the analyses in the chapters to follow.

In Chapter 6, I concentrate on the factor-input producing effect of education, i.e. the human-capital approach. Chapter 6.1 treats human capital as a homogeneous concept that needs time and human capital to produce. The corresponding production function is postulated to be of the form proposed by Haley (1976). Chapter 6.2 distinguishes two qualitatively different levels of labour. Education is assumed to require some fixed period of time, as in Tu (1969) and Ritzen (1977, ch. 5); a generalization of this model to time periods varying with innate ability has been provided by Sethi & McGuire (1977).

Chapter 7 deals with education under conditions of technical progress. Two aspects of the interaction between education and technical progress are considered. Chapter 7.1 analyses the effect of exogenous technical change on optimal investment in education. Chapter 7.2 investigates the role of education for the rate of diffusion of new technical knowledge. Both models of Chapter 7 are generalizations of the model of Chapter 6.1, i.e. the model with homogeneous human capital.

6 OPTIMAL ECONOMIC GROWTH AND INVESTMENT IN EDUCATION

In the simple one-sector growth model of Chapter 2 it was assumed that an individual's stock of human capital was some exogenously given function of the individual's age. The purpose of this chapter is to extend the simple model by making human capital a function of investment in education.

Chapter 6.1 examines an economic growth model with homogeneous human capital. That is, the labour services of two individuals with a different level of educational achievement are perfectly substitutable in the production. (Because of the maintained assumption that all individuals of the same generation are treated alike, such differences occur only between, not within generations). Consequently, the amount of human capital available in the labour force can be measured as one single factor of production (cf. Chapter 3.1).

Chapter 6.2 analyses a model in which two types of labour are distinguished, viz. educated and non-educated labour. Here the rate of substitution between the two types of labour can vary according to the economic circumstances. Hence the model can be termed as one with heterogeneous human capital.

6.1 A model with homogeneous human capital

A typical feature of human capital is that it is embodied in people who, by the laws of nature, have only a finite lifetime. While physical capital might, at least in principle, last forever, human capital inevitably dies some day. In this respect human capital is fundamentally different from physical capital. It also follows that, in addition to the quantity of human capital, another important aspect of human capital is the way in which it is distributed over the generations at present alive. For this reason it is essential to choose the individual as the starting point for the analysis of the role of human capital in the economy. This is particularly important if the focus is on the economic consequences of demographic change (cf. Arthur & McNicoll, 1977).

The organization of this chapter is as follows. Section 6.1.1 discusses the model as far as it differs from the model of Chapter 2. Section 6.1.2 establishes the conditions for optimal economic growth. Sections 6.1.3 and 6.1.4 discuss the comparative statics and dynamic properties of the optimal growth path, respectively.

6.1.1 Homogeneous human capital in a model of overlapping generations

Each newly-born person is endowed with an initial stock of human capital equal to h_0. Without loss of generality this initial stock can be put equal to one. In the course of his lifetime an individual can accumulate human capital by allocating resources to education. Here there is no qualitative distinction between formal education and training on the job.

Following Haley (1976) I postulate a production function of human capital in which both the stock of human capital and the fraction of time spent training enter as arguments, in the following way:

$$\dot{h}(t) = E[u(t)\cdot h(t)] - \delta_H \cdot h(t) \tag{1}$$

Here h(t) denotes the stock of human capital, u(t) the fraction of time spent training which is restricted by

$$0 \leq u(t) \leq 1 \tag{2}$$

and δ_H denotes the rate of depreciation of human capital, assumed to be constant. The educational production function or schooling function $E[\cdot]$ is assumed to satisfy:

$$E[0] = 0 \quad ; \quad E'[\cdot] > 0 \quad ; \quad E''[\cdot] < 0 \tag{3}$$

If we define

$$\epsilon(t) \equiv u(t)\cdot h(t) \tag{4}$$

i.e. $\epsilon(t)$ is the amount of human capital allocated to education, equations (1)-(2) reduce to

$$\dot{h}(t) = E[\epsilon(t)] - \delta_H \cdot h(t) \ , \qquad 0 \le \epsilon(t) \le h(t) \tag{5}$$

The greater the stock of human capital of an individual, the greater his efficiency in the production of the aggregate commodity. By measuring the stock of human capital in efficiency units the potential efficiency of an individual at time t is equal to $h(t)$. I maintain the initial assumption that all individuals of the same generations are treated alike, i.e. $u(t)$ and $h(t)$ are always the same <u>within</u> generations.

Let us now introduce differences <u>between</u> generations. It will be convenient to index each generation by its date of birth rather than by its age since the former is invariant over time. Thus generation v consists of those individuals born at time v and aged t-v at time t. The stock of human capital embodied in an individual of generation v at time t is denoted by $h_v(t)$ and the corresponding training effort by $u_v(t)$ or, alternatively, $\epsilon_v(t)$. Equation (5) can now be written as follows:

$$\dot{h}_v(t) = E[\epsilon_v(t)] - \delta_H \cdot h_v(t) \ , \qquad t-n \le v \le t \tag{6a}$$

$$0 \le \epsilon_v(t) \le h_v(t) \ , \qquad t-n \le v \le t \tag{6b}$$

$$h_v(v) = 1 \tag{6c}$$

where n is maximum age, as before. The actual efficiency of an individual is equal to his potential efficiency multiplied by the fraction of time devoted to the production of the aggregate commodity, i.e. one minus the fraction of time spent training. Now the labour force at time t measured in efficiency units (units of human capital) equals

$$L(t) = \int_{t-n}^{t} \{h_v(t) - \epsilon_v(t)\} \cdot \mu(t-v) \cdot B(v) \, dv \tag{7}$$

Here B(v) denotes the number of births at time v and $\mu(\cdot)$ some fixed age-specific survival schedule, as before.

6.1.2 Optimal investment in education

The central planning agency faces the following optimization problem:

$$\text{Maximize} \quad \int_0^\infty e^{-\rho t} \{1-s(t)\} \cdot \frac{F[K(t),L(t)]}{P(t)} \, dt \tag{8}$$

subject to (6a)-(6c), (7), the capital accumulation equation

$$\dot{K}(t) = s(t) \cdot F[K(t),L(t)] - \delta_K \cdot K(t) \;, \qquad 0 \le s(t) \le 1 \tag{9}$$

and initial conditions. The controls are the savings rate s(t) and the generation-specific training efforts $\epsilon_v(t)$, $t-n \le v \le t$. Since v is continuous, we have in fact a continuum of control variables (and of dynamic restrictions and costate variables as well), corresponding to the continuum of active generations.

A heuristic method of solving such optimal control problems proceeds as follows. The integral (7), giving the labour force as a function of this continuum of control variables $\epsilon_v(t)$ and state variables $h_v(t)$, can be interpreted as a summation of discrete variables over intervals of widht dv, where dv approaches zero (dv→0). As long as we keep this interpretation in mind, there is no fundamental difference between ordinary optimal control with a finite number of controls and the present continuous optimal control problem.

The Hamiltonian can now be written as follows:

$$H(t) = e^{-\epsilon t} \{1-s(t)\} \cdot \frac{F[K(t),L(t)]}{P(t)} + \sum_{v=t-n;\,dv\to 0}^{v=t} \psi_v(t) \cdot \{E[\epsilon_v(t)] - \delta_H \cdot h_v(t)\} +$$

$$+ \psi_K(t) \cdot \{s(t) \cdot F[K(t),L(t)] - \delta_K \cdot K(t)\} \tag{10}$$

The costate variables $\psi_K(t)$ and $\psi_v(t)$, $t-n \le v \le t$, satisfy:

$$-\dot{\psi}_v(t) = \frac{\partial H(t)}{\partial h_v(t)} + \frac{\partial H(t)}{\partial L(t)} \cdot \frac{\partial L(t)}{\partial h_v(t)} =$$

$$= \mu(t-v) \cdot B(v) \cdot \left[e^{-\rho t}\{1-s(t)\} \cdot \frac{1}{P(t)} + s(t) \cdot \psi_K(t)\right] \cdot F_L(t) \cdot dv +$$

$$- \delta_H \cdot \psi_v(t) \;, \; t-n \le v \le t \tag{11}$$

$$-\dot{\psi}_K(t) = \frac{\partial H(t)}{\partial K(t)} = \left[e^{-\rho t}\{1-s(t)\} \cdot \frac{1}{P(t)} + s(t) \cdot \psi_K(t)\right] \cdot F_K(t) - \delta_K \cdot \psi_K(t) \tag{12}$$

Note the presence of the interval width dv in (11).

Necessary conditions for an optimum are - dropping the time index for notational convenience - :

$$\frac{\partial H}{\partial s} = F[K,L] \cdot \left[\psi_K - \frac{e^{-\rho t}}{P} \right] \quad \begin{cases} \ge 0 & \text{if } s=1 \\ = 0 & \text{if } 0 < s < 1 \\ \le 0 & \text{if } s=0 \end{cases} \tag{13}$$

$$\frac{\partial H}{\partial \epsilon_v} + \frac{\partial H}{\partial L} \cdot \frac{\partial L}{\partial h_v} = -\mu(t-v) \cdot B(v) \cdot \left[e^{-\rho t}(1-s) \cdot \frac{1}{P} + s \cdot \psi_K\right] \cdot F_L \cdot dv + \psi_v \cdot E'[\epsilon_v] =$$

$$= \begin{cases} \ge 0 & \text{if } \epsilon_v = h_v \\ = 0 & \text{if } 0 < \epsilon_v < h_v \qquad t-n \le v \le t \\ \le 0 & \text{if } \epsilon_v = 0 \end{cases} \tag{14}$$

In addition we have the following transversality condition for human capital of individuals about to die:

$$\psi_{t-n}(t) \cdot h_{t-n}(t) = 0 \tag{15}$$

which, in view of (3) and (6), reduces to:

$$\psi_{t-n}(t) = 0 \tag{16}$$

Let us concentrate on singular solutions for the optimal savings rate. From (13) we find the conditions necessary for such a singular solution to occur:

$$\psi_K = \frac{e^{-\rho t}}{P} \tag{17}$$

$$\dot{\psi}_K = -\psi_K\cdot(\rho+g^P) \tag{18}$$

Substitution of (17) and (18) into (12) yields

$$f'[k(t)] = F_K(t) = \rho + \delta_K + g^P(t) \tag{19}$$

which is the, by now, quite familiar Generalized Golden Rule.

Condition (19) determines the optimal capital/labour ratio (or rather the optimal physical/human capital ratio), conditional to the pattern of population growth. Then the marginal productivity of labour $F_L(t)$ is determined as well:

$$w(t) \equiv F_L(t) = f[k(t)] - k(t)\cdot f'[k(t)] \tag{20}$$

using the linear homogeneity of the production function.

From (11), (17) and (20):

$$\delta_H\cdot\psi_v(t) - \dot{\psi}_v(t) = B(v)\cdot\mu(t-v)\cdot\frac{e^{-\rho t}}{P(t)}\cdot w(t)\cdot dv \;, \qquad t-n \le v \le t \tag{21}$$

Integrating (21) with repect to t:

$$-e^{-\delta_H t}\cdot\psi_v(t) = B(v)\cdot dv\cdot \int_A^t e^{-\delta_H \tau}\cdot\mu(\tau-v)\cdot\frac{e^{-\rho\tau}}{P(\tau)}\cdot w(\tau)\; d\tau \;, \quad t-n\le v\le t \tag{22}$$

The constant of integration A can be obtained from the transversality condition (16):

$$\psi_v(v+n) = 0 \quad \Rightarrow \quad A = v+n \tag{23}$$

Subtitution of (22) and (23) into (14) yields:

$$\frac{\partial H(t)}{\partial \epsilon_v(t)} + \frac{\partial H}{\partial L}\cdot\frac{\partial L}{\partial h_v} = B(v)\cdot dv\cdot\Bigg[-\mu(t-v)\cdot\frac{e^{-\rho t}}{P(t)}\cdot w(t) +$$

$$+ E'[\epsilon_v(t)]\cdot e^{\delta_H t}\cdot \int_t^{v+n} e^{-\delta_H \tau}\cdot\mu(\tau-v)\cdot\frac{e^{-\rho\tau}}{P(\tau)}\cdot w(\tau)\; d\tau\Bigg] =$$

$$= \begin{cases} \ge 0 & \text{if} \quad \epsilon_v(t) = h_v(t) \\ = 0 & \text{if} \quad 0 < \epsilon_v(t) < h_v(t) \\ \le 0 & \text{if} \quad \epsilon_v(t) = 0 \end{cases} \;, \qquad t-n \le v \le t \tag{24}$$

The interpretation of condition (24) is as follows. The first term in brackets denotes the "present value" of production foregone per unit of human capital allocated to education. Here "present value" involves discounting at a rate equal to the sum of the rate of impatience, the rate of population growth, and the mortality rate. The second term in brackets denotes the present value of the production in the future that can be derived from allocating one additional unit of human capital to education, after correction for depreciation of human capital. Thus, optimal (interior) training decisions are characterized by the familiar equality of opportunity costs and discounted returns. This decision criterion is well-known from the micro-economic theory of demand for education (cf. Kodde, 1985).

The equations implied by condition (24) are relatively easily solved. The optimal training profile for one generation can be determined independently of the training profiles of the other generations. Moreover this optimal training profile is completely determined by the sequence of growth rates of population $g^P(t)$ over the lifetime of the generation under consideration, given that the capital-labour ratio k(t) is optimal in every period (as determined by equations (19) and (20)).

6.1.3 Comparative statics

If the economy is in steady state, i.e. w and k are constant and labour and population grow at a constant rate g, the condition for optimal training effort (24) can be simplified as follows:

$$E'[\epsilon_v(t)] \gtreqless \frac{\mu(t-v)}{\int_t^{v+n} \mu(\tau-v)\cdot e^{-(\rho+g+\delta_H)\cdot(\tau-t)}d\tau} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \epsilon_v(t) = h_v(t) \\ 0 < \epsilon_v(t) < h_v(t), \quad t-n \le v \le t \\ \epsilon_v(t) = 0 \end{cases} \tag{25}$$

By virtue of assumption (3) the LHS of (25) is a strictly decreasing function of $\epsilon_v(t)$.

Let us denote the quotient in (25) by S(t-v) and let us write S(t-v) as a function of the individual's age a=t-v:

$$S(a) = \frac{\mu(a)}{\int_a^n \mu(z)\cdot e^{-(\rho+g+\delta_H)\cdot(z-a)}dz} \quad , \quad 0 \le a \le n \tag{26}$$

Differentiating (26) logarithmically with respect to a:

$$\frac{d\log S(a)}{da} = S(a) + \hat{\mu}(a) - (\rho+g+\delta_H) \tag{27}$$

This expression cannot be signed in general. However, two special cases yield a positive derivative. If the survival schedule $\mu(a)$ is exponential, i.e. $\mu(a)=e^{-\mu a}$ with μ constant, then (27) reduces to:

$$\frac{d\log S(a)}{da} = \frac{\mu+\rho+g+\delta_H}{1 - e^{-(\mu+\rho+g+\delta_H)\cdot(n-a)}} - (\mu+\rho+g+\delta_H) > 0 \quad (\text{ for } \mu+\rho+g+\delta_H>0) \tag{28}$$

The other special case is the one in which no mortality occurs for ages below n. This case is equivalent to putting μ equal to zero in (28) and again a non-negative derivative emerges. These two examples suggest that one should generally expect the first term on the RHS of (27), which is positive, to outweigh the negative two other terms.

The above analysis implies that (at least in steady state) the optimal age-training profile is non-increasing, which is perfectly consistent with both the theoretical and empirical literature on the subject (e.g. Ben-Porath, 1967; Von Weizsäcker, 1967; Theeuwes e.a., 1985). Typically, individual investment in education over the life-cycle consists of an initial period of maximum investment (full-time education), then a period of partial investment with the training effort diminishing gradually over time (on-the-job training), and finally a period without training (cf. Blinder & Weiss, 1976). Such an age-training profile corresponds to the familiar inverted U-shaped form of the age-income profile (e.g. Hartog, 1976).

Next we investigate the effect of changes in the exogenous parameters on the steady-state values of the human capital variables.

From (26) it is immediately seen that S(a) is an increasing function of both ρ, g, and δ_H. This together with (3) and (25) implies that

$$\frac{d\epsilon(a)}{d\rho} = \frac{d\epsilon(a)}{dg} = \frac{d\epsilon(a)}{d\delta_H} \le 0 \tag{29}$$

where we use $\epsilon(a)$ as a shorthand notation for $\epsilon_v(t)\big|_{t-v=a}$, training effort at age a.

Human capital as a function of age is obtained by integrating (5):

$$h(a) = e^{-\delta_H \cdot a} + \int_0^a e^{\delta_H \cdot (z-a)} \cdot E[\epsilon(z)]\, dz \tag{30}$$

(using (6c)). From (30) and (29):

$$\frac{dh(a)}{d\rho} = \int_0^a e^{\delta_H \cdot (z-a)} \cdot E'[\epsilon(z)] \cdot \frac{d\epsilon(z)}{d\rho}\, dz \leq 0 \tag{31}$$

$$\frac{dh(a)}{dg} = \int_0^a e^{\delta_H \cdot (z-a)} \cdot E'[\epsilon(z)] \cdot \frac{d\epsilon(z)}{dg}\, dz \leq 0 \tag{32}$$

$$\frac{dh(a)}{d\delta_H} = \int_0^a e^{\delta_H \cdot (z-a)} \cdot E'[\epsilon(z)] \cdot \frac{d\epsilon(z)}{d\delta_H}\, dz - a \cdot e^{-\delta_H \cdot a} + \\ + \int_0^a (z-a) \cdot e^{\delta_H \cdot (z-a)} \cdot E[\epsilon(z)]\, dz \leq 0 \tag{33}$$

All these comparative statics effects are of course very plausible: since optimal training effort is governed by the equality of earnings foregone and discounted returns, and since ρ, g, and δ_H all three enter this discount rate positively, increasing any one of these parameters should be expected to decrease both the training effort and the stock of human capital for all generations.

The third human-capital variable of interest, since it directly enters the expression for steady-state consumption per capita, is the labour/population-ratio L/P. In steady state this ratio equals, using (7):

$$\frac{L}{P} = \frac{\int_0^n \{h(a) - \epsilon(a)\} \cdot \mu(a) \cdot e^{-ga}\, da}{\int_0^n \mu(a) \cdot e^{-ga}\, da} \tag{34}$$

As in Section 2.4 we can define the mean age of the labour force and of the total population, respectively:

$$m^L = \frac{\int_0^n a \cdot \{h(a) - \epsilon(a)\} \cdot \mu(a) \cdot e^{-ga}\, da}{\int_0^n \{h(a) - \epsilon(a)\} \cdot \mu(a) \cdot e^{-ga}\, da} \tag{35}$$

$$m^P = \frac{\int_0^n a \cdot \mu(a) \cdot e^{-ga}\, da}{\int_0^n \mu(a) \cdot e^{-ga}\, da} \tag{36}$$

Using (35) and (36), we have from (34):

$$\frac{d}{dg}\frac{L}{P} = \frac{L}{P}\cdot(m^P - m^L) + \frac{\int_0^n \left[\frac{dh(a)}{dg} - \frac{d\epsilon(a)}{dg}\right]\cdot\mu(a)\cdot e^{-ga}\, da}{\int_0^n \mu(a)\cdot e^{-ga}\, da} \tag{37}$$

The first term on the RHS of (37) is recognized from Section 2.4. There it was concluded that this term is positive for g below some critical level and negative above this critical level. The existence of such a critical level was shown to open the possibility that there might exist an effective lower bound to the extent to which the rate of population growth can profitably be reduced (in the sense that reducing g increases consumption per capita).

With endogenous training policy this analysis has to be slightly modified. The second term on the RHS of (37) cannot be unambiguously signed. It seems reasonable, however, that the effect of g on the average human capital stock $h(\cdot)$ outweighs the effect of g on the average training effort $\epsilon(\cdot)$. If this is indeed the case then the second term on the RHS of (37) is negative, thus <u>lowering</u> the critical level of g (if it exists) at which the labour/population-ratio reaches its maximum. Thus, the introduction of an educational sector into the growth model <u>increases</u> the extent to which the rate of population growth can be profitably reduced. This is so, because investment in education is more attractive the lower the rate of population growth, thus offsetting the adverse effect of a lower g on the dependency ratio.

The effect of ρ and δ_H on L/P are given by, respectively:

$$\frac{d}{d\rho}\frac{L}{P} = \frac{\int_0^n \left[\frac{dh(a)}{d\rho} - \frac{d\epsilon(a)}{d\rho}\right]\cdot\mu(a)\cdot e^{-ga}\, da}{\int_0^n \mu(a)\cdot e^{-ga}\, da} \tag{38}$$

$$\frac{d}{d\delta_H}\frac{L}{P} = \frac{\int_0^n \left[\frac{dh(a)}{d\delta_H} - \frac{d\epsilon(a)}{d\delta_H}\right]\cdot\mu(a)\cdot e^{-ga}\, da}{\int_0^n \mu(a)\cdot e^{-ga}\, da} \tag{39}$$

As argued in the previous paragraph, it is reasonable to assume both these derivatives to be negative.

Since the remaining comparative statics results are essentially the same as for the model of Chapter 2 without educational sector, they will not be discussed here again. All comparative statics effects are summarized in Table 6.1.1.

Table 6.1.1: Comparative-statics results for the one-sector model with homogeneous human capital

effect of: on:	g	δ_K	δ_H	ρ
k	-	-	0	-
s	+ 1)	+ 1)	0	-
$\epsilon(\cdot)$	-	0	-	-
$h(\cdot)$	-	0	-	-
C/P	- 2)	-	-	-

1) for $\sigma < 1 + \frac{\rho}{\delta_K + g}$

2) except possibly for very low value of g

6.1.4 The non-stationary optimal growth path

In this section I will analyse the effect of a changing rate of population growth on optimal investment in both physical and human capital.

As for the model of Chapter 2, the physical/human-capital ratio k is negatively related to the growth rate of population. That is, if $g^B(\cdot)$ gradually decreases over time, then $k(\cdot)$ should gradually increase over time, and vice versa. This follows immediately from condition (19).

Now consider the case of $g^B(\cdot)$ falling, being the present demographic experience in most industrialized countries. Since $w[k]$ is an increasing function of k, this implies that $w(\cdot)$ rises continuously. In terms of the condition for optimal training effort (24) this means that the future returns to education rise relative to the opportunity cost, implying an increase in optimal investment in education. As a second effect, the rate at which these returns are discounted falls, making investment in education even more attractive.

In the case of $g^B(\cdot)$ rising over time the story is reversed. Thus we should expect the optimal training effort to generally move in the opposite direction of the growth rate of population. There could be some exceptions to this general picture, viz. when the growth in the labour productivity $w(\cdot)$ flattens down towards the end of the demographic transition process; this flattening down partly reverses the change in the pattern of future returns to education.

With respect to the optimal savings rate $s(\cdot)$ the analysis is slightly more complicated than in Chapter 2. This is because the demographic development now affects k not only directly via the Golden Rule (19), but also through the size of the labour force via induced changes in the optimal educational policy. For instance, if $g^B(\cdot)$ starts to fall this will have two effects on the optimal savings rate s:

- a direct effect via the Golden Rule, as well as through the labour force / population ratio L/P at given age-ability profiles. As was demonstrated in Chapter 2, this effect is initially positive and finally negative.
- an indirect effect via changes in the growth rate of the labour force due to adjustments in educational policy. This indirect effect will actually occur before the direct effect, as can be easily verified from (19) and (24). The initial adjustment will consist of an increase in the training effort by those generations constituting the older part of the labour force by the time the fall in the birth growth rate comes into effect. The result is that there will be a drop in the growth rate of the labour force before the initial drop in the optimal capital/ labour-ratio. This means that the optimal savings rate will fall before it starts to rise in accordance with the analysis of Chapter 2. Thus the time path of $s(\cdot)$ will still be inversely U-shaped, but with "wiggles" at both ends.

Some numerical illustrations are presented in Tables 6.1.3 and 6.1.4, and Figures 6.1.1 and 6.1.2. In order to keep the presentation manageable the number of overlapping generations that are distinguished has been limited to five. This number implies that one period should be interpreted as a time span of something like 15 years. The values of the parameters ρ, δ_K, δ_H and $g(\cdot)$ have been scaled so as to bear a reasonably realistic correspondence to such a time span.

The values of the parameters used are summarized in Table 6.1.2. The educational production function has been specified to be

$$E[\epsilon] = \gamma \cdot \epsilon^{\beta}$$

Table 6.1.2: Parameters used for illustrations in Figures 6.1.1 and 6.1.2

Number of generations	5
Survival schedule	$\mu(1)=\mu(2)=\mu(3)=\mu(4)=\mu(5)=1$
Social rate of impatience	$\rho=0.25$
Depreciation rates	$\delta_K=0.3$; $\delta_H=0.3$
Growth rate of births	0.30 0.25 0.20 0.15 0.10 0.05 0.00
Production function	$y=\{\alpha k^{-\frac{1-\sigma}{\sigma}} + (1-\alpha)\}^{-\frac{\sigma}{1-\sigma}}$
	$\alpha=0.25$
	$\sigma=0.5$
Educational prod. function	$E[\epsilon] = \gamma\cdot\epsilon^{\beta}$
	$\gamma=2.0$
	$\beta=0.8$

Table 6.1.3: Optimal economic growth path with g^B falling

t	g	$\hat{g}$	u_1	u_2	u_3	u_4	u_5	s	C/P
0	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
1	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
2	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
3	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
4	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
5	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
6	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
7	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
8	0.30	0.30	1	1	0.878	0.075	0	0.303	1.418
9	0.30	0.30	1	1	0.912	0.075	0	0.292	1.421
10	0.30	0.30	1	1	1.000	0.084	0	0.352	1.262
11	0.25	0.28	1	1	1.000	0.093	0	0.351	1.339
12	0.20	0.26	1	1	1.000	0.111	0	0.354	1.424
13	0.15	0.22	1	1	1.000	0.136	0	0.354	1.547
14	0.10	0.18	1	1	1.000	0.168	0	0.346	1.724
15	0.05	0.14	1	1	1.000	0.210	0	0.341	1.929
16	0.00	0.09	1	1	1.000	0.237	0	0.307	2.285
17	0.00	0.05	1	1	1.000	0.259	0	0.277	2.618
18	0.00	0.03	1	1	1.000	0.273	0	0.243	2.946
19	0.00	0.01	1	1	1.000	0.274	0	0.217	3.174
20	0.00	0.00	1	1	1.000	0.261	0	0.196	3.307
21	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
22	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
23	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
24	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
25	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
26	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
27	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
28	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
29	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
30	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253

Table 6.1.4: Optimal economic growth path with g^B rising

t	g	$\hat{g}$	u_1	u_2	u_3	u_4	u_5	s	C/P
0	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
1	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
2	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
3	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
4	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
5	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
6	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
7	0.00	0.00	1	1	1.000	0.261	0	0.202	3.253
8	0.00	0.00	1	1	1.000	0.261	0	0.211	3.216
9	0.00	0.00	1	1	1.000	0.236	0	0.191	3.344
10	0.05	0.01	1	1	1.000	0.202	0	0.174	3.360
11	0.10	0.03	1	1	1.000	0.164	0	0.156	3.276
12	0.15	0.06	1	1	1.000	0.127	0	0.154	3.003
13	0.20	0.11	1	1	1.000	0.097	0	0.173	2.611
14	0.25	0.16	1	1	0.973	0.078	0	0.205	2.231
15	0.30	0.22	1	1	0.881	0.072	0	0.215	1.994
16	0.30	0.26	1	1	0.847	0.073	0	0.245	1.716
17	0.30	0.28	1	1	0.843	0.074	0	0.284	1.497
18	0.30	0.29	1	1	0.858	0.075	0	0.306	1.402
19	0.30	0.30	1	1	0.878	0.076	0	0.317	1.363
20	0.30	0.30	1	1	0.878	0.075	0	0.315	1.389
21	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
22	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
23	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
24	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
25	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
26	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
27	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
28	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
29	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400
30	0.30	0.30	1	1	0.878	0.075	0	0.312	1.400

Figure 6.1.1: Optimal economic growth path with g^B falling

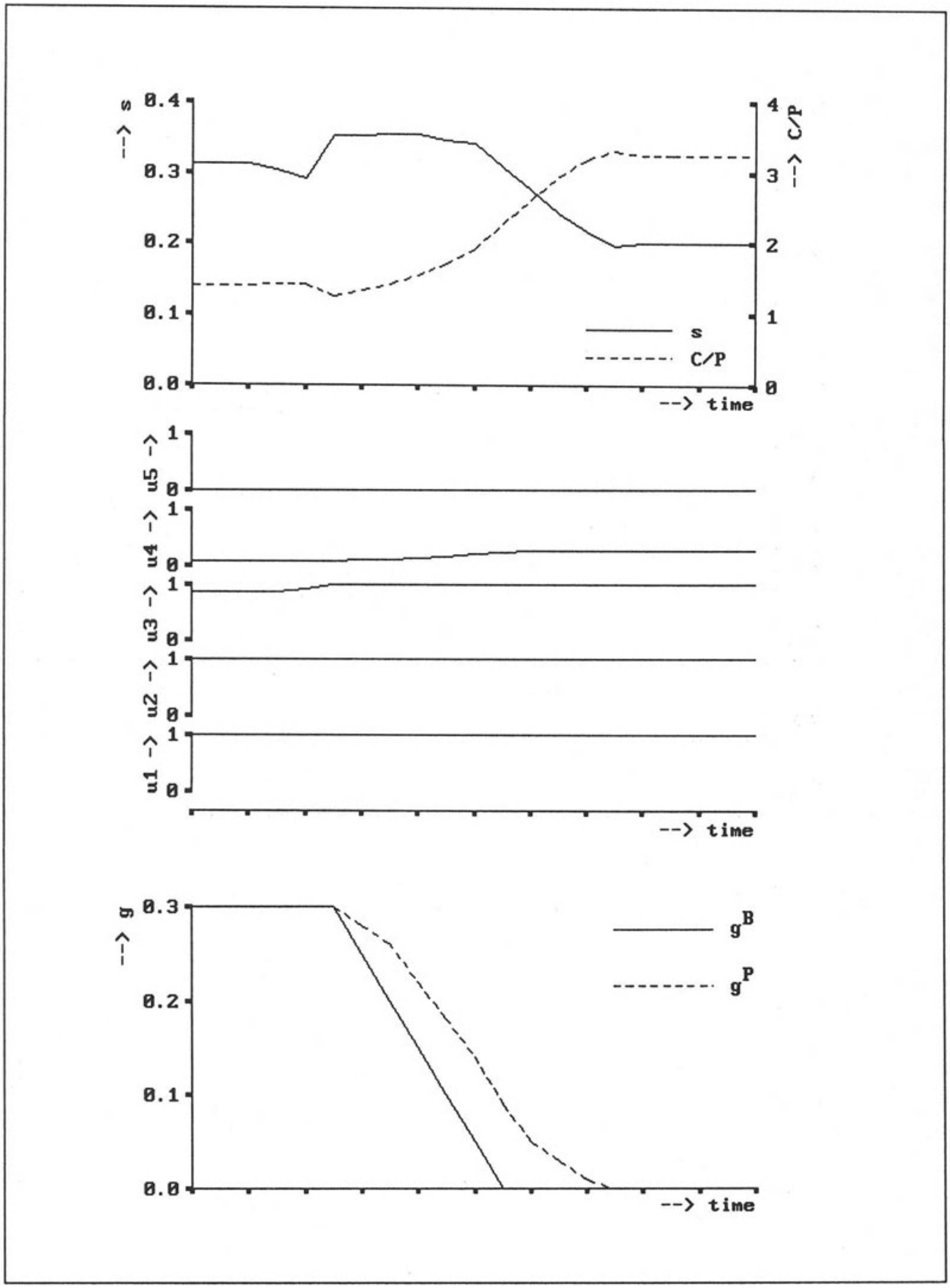
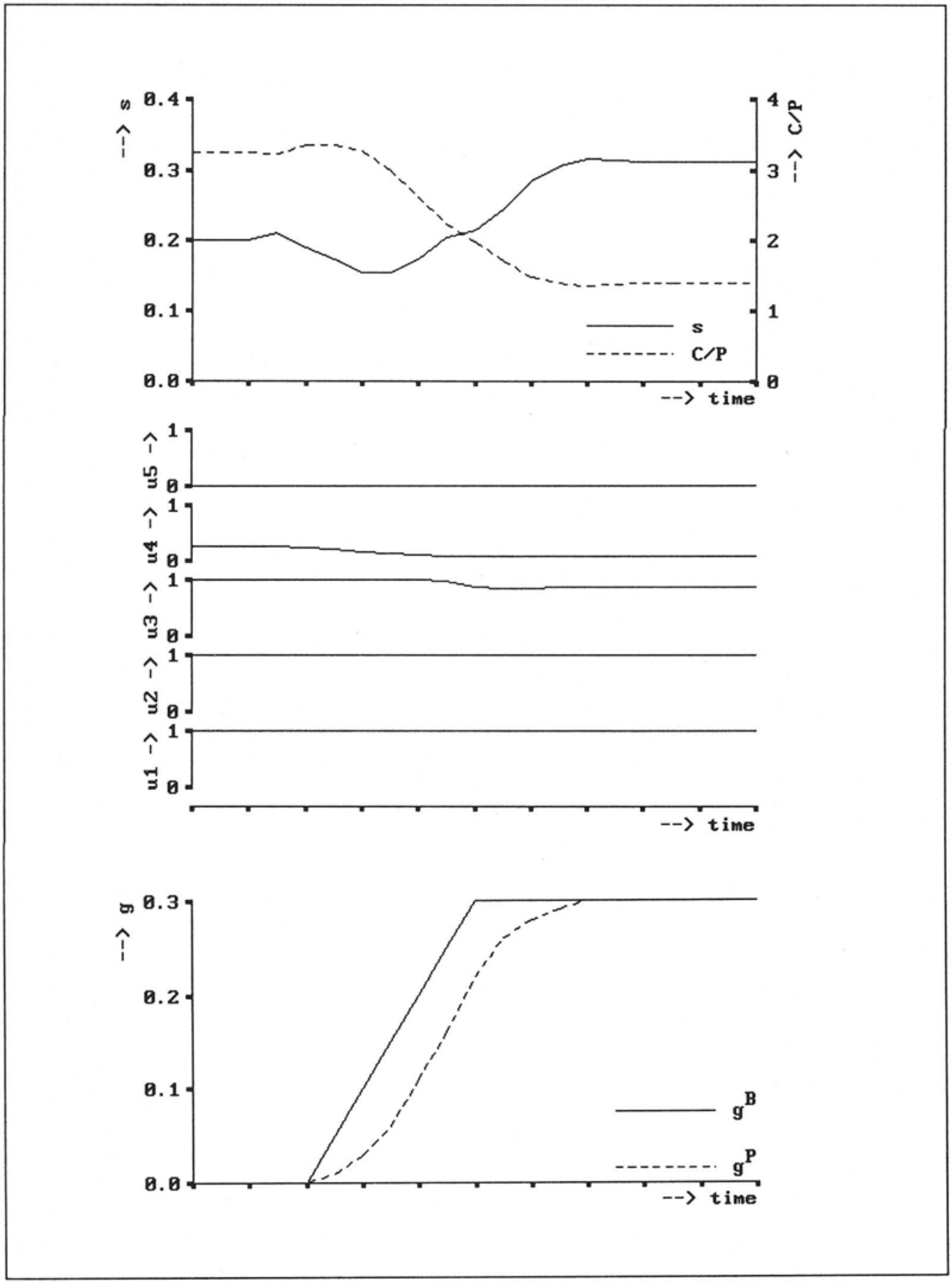


Figure 6.1.2: Optimal economic growth path with g^B rising



following Haley (1976). The values for γ and β have been obtained by "fitting" the function to some crude measure of mean gross incomes classified by educational achievement. The aggregate production function has been taken to be of the CES type.

The plot of the optimal savings rate has the familiar U-shape or inverted U-shape during periods of demographic transition, depending on whether the rate of population growth is rising or falling. The small oscillations near the endpoints of the non-stationary trajectory accompany the changes in optimal training policy for the varous age groups, especially the third one.

The simulated optimal training <u>profiles</u> are at rather high levels: individuals spend roughly 60% of their life training and only 40% working. For the first and second age group optimal training effort is maximum all the time. The fall in $g^B(\cdot)$ results in a modest rise of optimal education for the third and fourth age group; the reverse relationship holds true for the case of a rise in the population growth rate.

Finally, consumption per capita is more or less inversely related to the rate of population growth during the full period of demographic transition.

6.1.5 Summary

In this chapter I have analysed a model of overlapping generations in which the age-ability profile of an individual is a function of investment in education. The educational production function has the individual's stock of human capital and the fraction of time spent training, as arguments.

The conclusions of the analysis can be summarized as follows:

1. Along the singular trajectory optimal training decisions are characterized by the equality of opportunity costs and discounted returns.
2. The optimal age-training profile is non-increasing.
3. When the growth rate of population falls, investment in education becomes more attractive. The introduction of an educational sector increases the extent to which the rate of population growth can profitably be reduced.
4. In general, training effort moves in the opposite direction to the growth rate of population along the non-stationary economic growth path.
5. The plot of the optimal savings rate has the familiar U-shape or inverted U-shape during periods of demographic transition, depending on whether the

rate of population growth is rising or falling. Some small oscillations occur near the endpoints of the non-stationary trajectory; these accompany the changes in optimal training policy for the various age-groups.

6.2 A model with heterogeneous human capital

This chapter investigates a model originally developed by Tu (1969) with two types of labour: skilled and unskilled. Unskilled labour can be transformed into skilled labour through education.

A similar model, with multiple types of labour (but without overlapping generations), is analysed in Ritzen (1977, ch. 5). A generalization of the model to include schooling periods varying with innate ability has been developed by Sethi & McGuire (1977).

The specification of the educational production function gives rise to an optimal control problem with integral state equations. In Section 6.2.1 the necessary conditions for the solution to this type of problem will be formulated. These conditions constitute a generalization of a theorem of Kamien & Muller (1976) to include the case of bounded controls.

Section 6.2.2 outlines the model with two types of labour. In Section 6.2.3 the conditions for optimal economic growth are established. The comparative-statics properties of the model are discussed in Section 6.2.4. Section 6.2.5 focusses on the dynamic properties of the optimal economic growth path. The chapter concludes with a summary.

6.2.1 Optimal control with integral state equations and bounded controls

Kamien & Muller (1976) prove a simple theorem, stating the necessary conditions for optimal control of a system governed by integral state equations. Their theorem can be stated as follows: in order to maximize

$$\int_a^b f[x(t),u(t),t]\ dt \tag{1}$$

with respect to the control u(t), subject to

$$x(t) = x(a) + \int_a^t g[x(v),u(v),v,t]\ dv \tag{2}$$

the following conditions must be satisfied:

$$\frac{\partial H}{\partial u} = 0 = f_u[x(t),u(t),t] + \int_t^b \lambda(v)\cdot g_u[x(t),u(t),t,v]\ dv \tag{3}$$

$$\frac{\partial H}{\partial x} = \lambda(t) = f_x[x(t),u(t),t] + \int_t^b \lambda(v)\cdot g_x[x(t),u(t),t,v]\ dv \tag{4}$$

where the Hamiltonian H(t) is given by

$$H(t) = f[x(t),u(t),t] + \int_t^b \lambda(v)\cdot g[x(t),u(t),t,v]\, dv \tag{5}$$

Kamien & Muller provide a heuristic proof of their theorem based on the results of the calculus of variations. Thus their theorem cannot automatically be generalized to apply to problems with constraints on the state variables and/or the controls. Such constraints do play a role in the models of this study, where controls like the savings rate s and the enrollment rate h (to be introduced in the next section) must lie within the interval [0,1]. Specifically, such constraints require that condition (3) be replaced by the more general condition:

H(t) must be maximized with respect to u(t), (6)

thus allowing for corner solutions.

However, using the results of Sethi (1974), I will prove that the above theorem can be generalized to include cases in which the control space is restricted. Sethi, extending the earlier work of Vinokurov (1969) and Bakke (1974), establishes the necessary conditions for optimal control if the dynamic restrictions are integro-differential equations:

$$\dot{x}(t) = h[x(t),u(t),t] + \int_a^t g[x(v),u(v),v,t]\, dv \tag{7}$$

In this case the Hamiltonian reads:

$$H(t) = f[x(t),u(t),t] + p(t)\cdot h[x(t),u(t),t] + \int_t^b p(v)\cdot g[x(t),u(t),t,v]\, dv \tag{8}$$

and the necessary conditions are given by (6) and the costate equations

$$\dot{p}(t) = -\frac{\partial H}{\partial x} \tag{9}$$

Now the integral equation (2) is a special case of the integro-differential equation (7). Differentiate (2) with respect to time:

$$\dot{x}(t) = g[x(t),u(t),t,t] + \int_a^t g_4[x(v),u(v),v,t]\, dv \tag{10}$$

Here $g_4[\cdot]$ denotes the partial derivative of $g[\cdot]$ with respect to its fourth argument. We form the Hamiltonian (8) for restriction (10):

$$H(t) = f[x(t),u(t),t] + p(t)\cdot g[x(t),u(t),t,t] + \\ + \int_t^b p(v)\cdot g_4[x(t),u(t),t,v]\,dv \tag{11}$$

Now integrate the third term by parts:

$$\int_t^b p(v)\cdot g_4[x(t),u(t),t,v]\,dv = p(b)\cdot g[x(t),u(t),t,b] + \\ - p(t)\cdot g[x(t),u(t),t,t] - \int_t^b \dot{p}(v)\cdot g[x(t),u(t),t,v]\,dv \tag{12}$$

The first term on the RHS of (12) is equal to zero by virtue of the transversality conditions in the absence of constraints on x(b). Then the Hamiltonian in (11) reduces to:

$$H(t) = f[x(t),u(t),t] - \int_t^b \dot{p}(v)\cdot g[x(t),u(t),t,v]\,dv \equiv \\ \equiv f[x(t),u(t),t] + \int_t^b \lambda(v)\cdot g[x(t),u(t),t,v]\,dv \tag{13}$$

which is the Hamiltonian of Kamien & Muller (5) if the shadow price $\lambda(v)$ is defined as

$$\lambda(v) \equiv - \dot{p}(v) \tag{14}$$

Thus, application of the theorem of Sethi (1974) to the problem of optimal control with integral state equations yields (4) and (6) as necessary conditions. This completes the proof of the generalizability of the theorem of Kamien & Muller (1976) to problems with a constrained control space.

6.2.2 The model with two types of labour

In the growth model of Tu (1969) two types of labour are distinguished: skilled and unskilled. The educational production function is such that it takes a fixed number of years to provide an uneducated person with the training necessary to become a skilled labourer.

During an individual's stay at school he is not available for productive purposes. Thus total population consists of three categories: skilled labour, unskilled labour, and students.

Provided that it is possible to make skilled people perform unskilled jobs, it is clearly not an optimal policy to educate people of a positive age as long as not all newly-born children are enrolled in the educational sector. For this reason I have concentrated on the proportion of newly-born children entering the educational sector as the control variable of educational policy. This proportion will be termed the <u>enrollment rate</u> and is denoted by h(v) for generation v.

If we write m for the fixed length of the training period, the number of people in each category of the population can be written as follows:

unskilled labour: $$P_1(t) = \int_{t-n}^{t} \{1-h(v)\}\cdot\mu(t-v)\cdot B(v)\, dv \qquad (15)$$

skilled labour: $$P_2(t) = \int_{t-n}^{t-m} h(v)\cdot\mu(t-v)\cdot B(v)\, dv \qquad (16)$$

students: $$P_0(t) = \int_{t-m}^{t} h(v)\cdot\mu(t-v)\cdot B(v)\, dv \qquad (17)$$

total population: $$P(t) = \int_{t-n}^{t} \mu(t-v)\cdot B(v)\, dv = P_0(t) + P_1(t) + P_2(t) \qquad (18)$$

If we ignore the possibility of there being so many trained people along some segments of the optimal growth path that some of them have to be temporarily placed in unskilled jobs, the actual and potential quantity of the labour force are equal for both types op labour. The production function is then given by:

$$Y(t) = F[K(t),L_1(t),L_2(t)] = F[K(t),P_1(t),P_2(t)] \qquad (19)$$

The capital stock accumulates as usual according to:

$$\dot{K}(t) = s(t)\cdot Y(t) - \delta\cdot K(t) \qquad (20)$$

where s(t) is the gross savings rate and the second control variable.

6.2.3 Optimal economic growth

The problem confronting the central planning agency is to choose s(t) and h(t) such that

$$W = \int_0^\infty e^{-\rho t} \cdot \frac{\{1-s(t)\} \cdot Y(t)}{P(t)} \, dt \tag{21}$$

is maximized, subject to the integral equations (15) and (16), the production function (19), the differential equation (20), and given the exogenous dynamic path of (18).

In order to be able to apply the theorem of the previous section to this maximization problem, the dynamic restrictions must be written in the form (2). For the stock of physical capital integration of (20) yields:

$$K(t) = K(-\infty) + \int_{-\infty}^{t} \{s(v) \cdot Y(v) - \delta \cdot K(v)\} \, dv \tag{22}$$

For skilled and unskilled labour the required form can be obtained by introducing the indicator function $J_A(t)$, defined by:

$$J_A(t) = \begin{cases} 1 \Leftrightarrow t \in A \\ 0 \Leftrightarrow t \notin A \end{cases} \tag{23}$$

Now (15) and (16) can be written as, respectively:

$$P_1(t) = \int_{-\infty}^{t} J_{[t-n,t]}(v) \cdot \{1-h(v)\} \cdot \mu(t-v) \cdot B(v) \, dv \tag{24}$$

$$P_2(t) = \int_{-\infty}^{t} J_{[t-n,t-m]}(v) \cdot h(v) \cdot \mu(t-v) \cdot B(v) \, dv \tag{25}$$

The Hamiltonian is:

$$\begin{aligned} H(t) = {} & e^{-\rho t} \cdot \frac{\{1-s(t)\} \cdot Y(t)}{P(t)} + \\ & + \int_t^\infty \psi_1(v) \cdot J_{[v-n,v]}(t) \cdot \{1-h(t)\} \cdot \mu(v-t) \cdot B(t) \, dv + \\ & + \int_t^\infty \psi_2(v) \cdot J_{[v-n,v-m]}(t) \cdot h(t) \cdot \mu(v-t) \cdot B(t) \, dv + \\ & + \int_t^\infty \psi_K(v) \cdot \{s(t) \cdot Y(t) - \delta \cdot K(t)\} \, dv \end{aligned} \tag{26}$$

The costate variables $\psi_1(t)$, $\psi_2(t)$ and $\psi_K(t)$ satisfy:

$$\psi_1(t) = \frac{\partial H(t)}{\partial P_1(t)} = F_1(t) \cdot \left[e^{-\rho t} \cdot \frac{1-s(t)}{P(t)} + s(t) \cdot \int_t^\infty \psi_K(v)\, dv \right] \tag{27}$$

$$\psi_2(t) = \frac{\partial H(t)}{\partial P_2(t)} = F_2(t) \cdot \left[e^{-\rho t} \cdot \frac{1-s(t)}{P(t)} \div s(t) \cdot \int_t^\infty \psi_K(v)\, dv \right] \tag{28}$$

$$\psi_K(t) = \frac{\partial H(t)}{\partial K(t)} = F_K(t) \cdot \left[e^{-\rho t} \cdot \frac{1-s(t)}{P(t)} + s(t) \cdot \int_t^\infty \psi_K(v)\, dv \right] + \\ - \delta \cdot \int_t^\infty \psi_K(v)\, dv \tag{29}$$

where

$$F_i(t) \equiv \frac{\partial Y(t)}{\partial P_i(t)} = \frac{\partial Y(t)}{\partial L_i(t)}, \quad i=1,2 \quad ; \quad F_K(t) \equiv \frac{\partial Y(t)}{\partial K(t)} \tag{30}$$

denote the partial derivatives of the production function (19).

Necessary conditions for an optimum are:

$$\frac{\partial H(t)}{\partial s(t)} = Y(t) \cdot \left[- \frac{e^{-\rho t}}{P(t)} + \int_t^\infty \psi_K(v)\, dv \right] = \begin{cases} \geq 0 \text{ if } s(t) = 1 \\ = 0 \text{ if } 0 < s(t) < 1 \\ \leq 0 \text{ if } s(t) = 0 \end{cases} \tag{31}$$

$$\frac{\partial H(t)}{\partial h(t)} = - \int_t^\infty \psi_1(v) \cdot J_{[v-n,v]}(t) \cdot \mu(v-t) \cdot B(t)\, dv + \\ + \int_t^\infty \psi_2(v) \cdot J_{[v-n,v-m]}(t) \cdot \mu(v-t) \cdot B(t)\, dv = \\ = B(t) \cdot \left[- \int_t^{t+n} \psi_1(v) \cdot \mu(v-t)\, dv + \int_{t+m}^{t+n} \psi_2(v) \cdot \mu(v-t)\, dv \right] = \\ = \begin{cases} \geq 0 & \text{if } h(t) = 1 \\ = 0 & \text{if } 0 < h(t) < 1 \\ \leq 0 & \text{if } h(t) = 0 \end{cases} \tag{32}$$

From (31) the conditions for a singular solution for the optimal savings rate can be derived:

$$\int_t^\infty \psi_K(v)\, dv = \frac{e^{-\rho t}}{P(t)} \tag{33}$$

$$\frac{d}{dt} \left[\int_t^\infty \psi_K(v)\, dv \right] = \frac{d}{dt} \left[\frac{e^{-\rho t}}{P(t)} \right] \Leftrightarrow \psi_K(t) = \{\rho + g^P(t)\} \cdot \frac{e^{-\rho t}}{P(t)} \tag{34}$$

Combination of (29), (33) and (34) yields the Non-Stationary Golden Rule of Accumulation:

$$F_K(t) = \rho + \delta + g^P(t) \tag{35}$$

On the other hand, substitution of (27), (28), (33) and (34) into (32) gives:

$$\frac{\partial H(t)}{\partial h(t)} = B(t)\cdot\left[-\int_t^{t+n} F_1(v)\cdot\frac{e^{-\rho v}}{P(v)}\cdot\mu(v-t)\ dv + \int_{t+m}^{t+n} F_2(v)\cdot\frac{e^{-\rho v}}{P(v)}\cdot\mu(v-t)\ dv\right]$$

$$= B(t)\cdot\left[\ -\ \int_t^{t+m} F_1(v)\cdot\frac{e^{-\rho v}}{P(v)}\cdot\mu(v-t)\ dv\ +\right.$$

$$\left. +\ \int_{t+m}^{t+n} \{F_2(v)-F_1(v)\}\cdot\frac{e^{-\rho v}}{P(v)}\cdot\mu(v-t)\ dv\ \right] =$$

$$= \begin{cases} \geq 0 & \text{if} \quad h(t) = 1 \\ = 0 & \text{if} \quad 0 < h(t) < 1 \\ \leq 0 & \text{if} \quad h(t) = 0 \end{cases} \tag{36}$$

The interpretation of condition (36) for the optimal enrollment ratio h is rather similar to that of condition (6.1.24). The first term in brackets is the present value (discounted at the sum of the rates of impatience, population growth, and mortality) of production foregone per student, or, equivalently, the (indirect) cost of education per student. The second term is discounted returns to education. An interior solution to the optimal enrollment rate is thus characterized by the equality of costs and returns. In this respect the model with heterogeneous human capital is not fundamentally different from the model with homogeneous human capital.

6.2.4 Comparative statics

In steady state, with population growing at a constant rate g, the variables K, L_1 and L_2 also grow at this same constant rate g. Due to the assumed linear homogeneity of the production function, total production and all marginal factor productivities can be written in terms of the two factor ratios $k \equiv K/L_2$ and $\ell \equiv L_1/L_2$:

$$F[K,L_1,L_2] = L_2\cdot F[K/L_2,L_1/L_2,1] \equiv L_2\cdot f[k,\ell] \tag{37}$$

$$F_K = f_k[k,\ell] \equiv r \tag{38}$$

$$F_1 = f_\ell[k,\ell] = w_1 \tag{39}$$

$$F_2 = f[k,\ell] - k \cdot f_k[k,\ell] - \ell \cdot f_\ell[k,\ell] = w_2 \tag{40}$$

As both factor ratios k and ℓ are constant in steady state, so are the marginal products r, w_1 and w_2.

For simplicity it is assumed, from now on, that the survival schedule $\mu(\cdot)$ is exponential, i.e.

$$\mu(a) = \begin{cases} e^{-\mu a} & \text{for} \quad a \leq n \\ 0 & \text{for} \quad a > n \end{cases} \tag{41}$$

Using this simplification, the singularity condition for the optimal enrollment ratio (36) reduces to:

$$w_1 \cdot \int_t^{t+n} e^{-(\rho+g+\mu)\cdot v} \, dv = w_2 \cdot \int_{t+m}^{t+n} e^{-(\rho+g+\mu)\cdot v} \, dv \tag{42}$$

which implies

$$\frac{w_2[k,\ell]}{w_1[k,\ell]} = \begin{cases} \dfrac{1 - e^{-(\rho+g+\mu)\cdot n}}{e^{-(\rho+g+\mu)\cdot m} - e^{-(\rho+g+\mu)\cdot n}} & \text{if} \quad \rho+g+\mu \neq 0 \\ \dfrac{n}{n-m} & \text{if} \quad \rho+g+\mu = 0 \end{cases} \tag{43}$$

On the other hand we have the Golden Rule (35) which in steady state can be written as:

$$r[k,\ell] = \rho + g + \delta \tag{44}$$

Equations (43)-(44) constitute a system in the two state variables k and ℓ. The steady-state solution $[k^*,\ell^*]$ to this system is of course a function of the exogenous parameters n, m, ρ, δ, g, and μ.

Once k^* and ℓ^* have been determined, the steady-state values of the controls h and s follow easily. From (15), (16) and (41):

$$\ell^* = P_1/P_2 = \begin{cases} \dfrac{1-h^*}{h^*}\cdot\dfrac{1 - e^{-(g+\mu)\cdot n}}{e^{-(g+\mu)\cdot m} - e^{-(g+\mu)\cdot n}} & \text{if} \quad g+\mu \neq 0 \\ \dfrac{1-h^*}{h^*}\cdot\dfrac{n}{n-m} & \text{if} \quad g+\mu = 0 \end{cases} \tag{45}$$

Similarly, from (20):

$$s^* = \frac{(g+\delta)\cdot k^*}{f[k^*,\ell^*]} \tag{46}$$

Finally, consumption per capita is given by:

$$\begin{aligned} (C/P)^* &= (1-s^*)\cdot f[k^*,\ell^*]\cdot P_2/P = \\ &= (1-s^*)\cdot f[k^*,\ell^*]\cdot h^*\cdot\frac{e^{-(g+\mu)\cdot m} - e^{-(g+\mu)\cdot n}}{1 - e^{-(g+\mu)\cdot n}} \end{aligned} \tag{47}$$

Equations (43)-(47) can be used to derive comparative-statics results, i.e. the effect of the exogenous parameters on the endogenous variables k^*, ℓ^*, h^*, s^*, and $(C/P)^*$. However, it should be clear that most of the signs of these effects depend crucially on the way in which k and ℓ interact in the production function. That is, the comparative statics for this model depend on the degree to which capital, unskilled labour, and skilled labour are mutually substitutable or complementary in the production process. Thus it is quite difficult to obtain general comparative statics results.

Table 6.2.1 contains some numerical calculations of steady states for the illustrative case of the following Cobb-Douglas production function:

$$Y = K^{0.3}\cdot L_1^{0.5}\cdot L_2^{0.2}$$

These results suggest that for reasonable values of the parameters the signs of the partial derivatives are as summarized in Table 6.2.2.

Comparing Table 6.2.2 with Table 6.1.1 of Section 6.1 again yields the conclusion that the two models with heterogeneous and homogeneous human capital, respectively, have many properties in common. E.g. just as for the model of Section 6.1, here too both g, ρ, μ and m have a negative impact on the optimal training effort h. This result is plausible, since an increase in any of these parameters reduces the returns to (profitability of) education, according to the Golden Rule of Education (36).

Table 6.2.1: Selected numerical steady-state values

changing parameter	ℓ	k	h	s	C/P
reference point*	9.449	4.974	0.226	0.200	0.326
g = 0.00	6.884	7.080	0.233	0.150	0.447
0.05	8.030	5.745	0.229	0.180	0.377
0.10	9.449	4.974	0.226	0.200	0.326
0.15	11.204	4.507	0.223	0.214	0.287
0.20	13.369	4.225	0.220	0.225	0.255
δ = 0.00	9.449	8.877	0.226	0.150	0.412
0.05	9.449	6.454	0.226	0.180	0.361
0.10	9.449	4.974	0.226	0.200	0.326
0.15	9.449	3.991	0.226	0.214	0.300
0.20	9.449	3.298	0.226	0.225	0.279
ρ = 0.00	6.884	7.080	0.286	0.300	0.343
0.05	8.030	5.745	0.255	0.240	0.337
0.10	9.449	4.974	0.226	0.200	0.326
0.15	11.204	4.507	0.197	0.171	0.312
0.20	13.369	4.225	0.171	0.150	0.297
μ = 0.00	6.884	3.967	0.233	0.200	0.354
0.05	8.030	4.428	0.229	0.200	0.340
0.10	9.449	4.974	0.226	0.200	0.326
0.15	11.204	5.618	0.223	0.200	0.312
0.20	13.369	6.373	0.220	0.200	0.297
m = 2	4.760	3.048	0.253	0.200	0.382
3	6.658	3.874	0.239	0.200	0.354
4	9.449	4.974	0.226	0.200	0.326
5	13.704	6.487	0.213	0.200	0.298
6	20.565	8.669	0.202	0.200	0.269

*) The values of the parameters in the reference point are:

$g=0.10$; $\delta=0.10$; $\rho=0.10$; $\mu=0.10$; $m=4$; $n=10$.

The production function is taken to be:

$$Y = K^{0.3} \cdot L_1^{0.5} \cdot L_2^{0.2}$$

Table 6.2.2: Comparative-statics results for the one-sector model with heterogeneous human capital

effect of: on	g	δ	ρ	μ	m
ℓ	+	0	+	+	+
k	-	-	-	+	+
h	-	0	-	-	-
s	+	+	-	0	0
C/P	-	-	-	-	-

6.2.5 Dynamics

In this section I will briefly explore the nature of the optimal growth path during periods of demographic transition.

As for most of the models which are analysed in this study, the dynamics of the non-stationary growth path are quite complex. For the particular model under consideration, however, the situation is even worse, as it turns out that the singular trajectory is unstable. Thus, singular control breaks down as soon as the economy, triggered off by demographic changes, leaves its steady state.

I have not been able to prove that this instability is an inherent property of the present model. In carrying out several numerical simulations the results that were obtained exhibited again and again explosive behaviour of the optimal enrollment rate (h). Closer inspection of the mathematical equations involved then indicated that for plausible values of the parameters the model is indeed unstable.

The analysis of the non-stationary optimal growth path starts from the two Golden Rules (35) and (36). Differentiation of (36) with respect to time yields, using the simplifying assumption on the survival schedule (41):

$$0 = \mu \cdot \int_{t}^{t+n} w_1(v) \cdot \frac{e^{-\rho v}}{P(v)} \cdot e^{-\mu(v-t)}\, dv + w_1(t+n) \cdot \frac{e^{-\rho(t+n)}}{P(t+n)} \cdot e^{-\mu n} +$$

$$- w_1(t) \cdot \frac{e^{-\rho t}}{P(t)} - \mu \cdot \int_{t+m}^{t+n} w_2(v) \cdot \frac{e^{-\rho v}}{P(v)} \cdot e^{-\mu(v-t)}\, dv +$$

$$- w_2(t+n)\cdot\frac{e^{-\rho(t+n)}}{P(t+n)}\cdot e^{-\mu n} + w_2(t+m)\cdot\frac{e^{-\rho(t+m)}}{P(t+m)}\cdot e^{-\mu m} \tag{48}$$

From (48) and (36) we have, after some rearranging:

$$w_2(t) - w_1(t) =$$

$$= w_2(t+m-n)\cdot\frac{P(t)}{P(t+m-n)}\cdot e^{(\rho+\mu)\cdot(n-m)} - w_1(t-n)\cdot\frac{P(t)}{P(t-n)}\cdot e^{(\rho+\mu)\cdot n} \tag{49}$$

Together with (35), (49) constitutes a system of higher-order difference equations in k and ℓ. This system can be solved in a recursive fashion: at time t all variables on the RHS of (49) are known, allowing the solution of the system (35) and (49) in a straightforward manner.

Some numerical simulations that I tried for specific forms of the production function $f[k,\ell]$ showed that the trajectories $k(\cdot)$, $\ell(\cdot)$, $w_1(\cdot)$ and $w_2(\cdot)$ thus obtained approach their final steady-state values in an oscillatory but nevertheless converging fashion.

Once values for the state variables $k(\cdot)$ and $\ell(\cdot)$ have thus been obtained, the next step is to find the corresponding time paths of the controls $h(\cdot)$ and $s(\cdot)$. It is here that insurmountable problems arise. From the definition of $\ell(t)$ we have:

$$\ell(t)\cdot P_2(t) = P_1(t) \quad \Rightarrow \quad \dot{\ell}(t)\cdot P_2(t) + \ell(t)\cdot\dot{P}_2(t) = \dot{P}_1(t) \tag{50}$$

On substitution of (15), (16) and (41) into (50), carrying out the differentiation and rearranging terms yields:

$$1 - \dot{\ell}(t)\cdot\frac{P_2(t)}{B(t)} - e^{-\mu n}.\frac{B(t-n)}{B(t)} =$$

$$= h(t) + h(t-m)\cdot\ell(t)\cdot e^{-\mu m}.\frac{B(t-m)}{B(t)} - h(t-n)\cdot\{1+\ell(t)\}\cdot e^{-\mu n}.\frac{B(t-n)}{B(t)} \tag{51}$$

Equation (51) is a higher-order difference equation in $h(\cdot)$. In the neighbourhood of the steady state it reduces to:

$$1 - e^{-\beta n} = h(t) + h(t-m)\cdot\ell^*\cdot e^{-\beta m} - h(t-n)\cdot\{1+\ell^*\}\cdot e^{-\beta n} \tag{52}$$

where

$$\beta = g + \mu \tag{53}$$

for notational convenience.

Difference equation (52) has an equilibrium solution which in implicit form is given by the steady-state expression (45). Now let us take $n = 2 \cdot m$ and investigate the characteristic equation of (52). The two roots of the characteristic equation are given by:

$$\lambda_1 = -e^{-\beta m} \cdot (1+\ell^*) \quad ; \quad \lambda_2 = e^{-\beta m} \tag{54}$$

The difference equation (52) diverges if the dominant root (which is λ_1) is larger than unity in absolute value. It is, however, not hard to see that this is almost always the case. If one substitutes reasonable values for $e^{-\beta m}$ (e.g. 1/2) and h^* (e.g. 1/2) into (45) and subsequently puts the corresponding value for ℓ^* (here: 3) into (54), the resulting value for the dominant root λ_1 will invariably be below minus one.

That is, for reasonable values of the parameters the optimal singular solution of $h(\cdot)$ is unstable, and its non-stationary singular trajectory is explosive.

The instability of the steady state and the breaking down of singular control for the model with heterogeneous human capital renders much of the analysis in previous sections highly suspicious. It is not clear on intuitive grounds why singular control does not work here. One could argue that the way in which the educational policy variable $h(\cdot)$ affects the composition of the labour force is too inflexible and makes the state of the economy too rigidly dependent on decisions made in the near past. However, the same history-dependence was also a property of the capital-vintage model of Chapter 4.2 which *did* turn out to be singularly controllable.

Further investigation into the dynamic properties of the optimal growth path for the present model is necessary to answer the many questions that are left open here. The present analysis is admittedly very unsatisfactory in that respect. What the results of this section <u>do</u> make clear is that it is quite dangerous to restrict one's attention to the comparative statics of optimal solutions, as many authors (like Tu himself (1969)) do. It is dangerous because it is not clear what meaning one should attach to an optimal steady state to reach which is not optimal.

6.2.6 Summary

This chapter has analysed optimal economic growth for a model with heterogeneous human capital originally developed by Tu (1969). A simple theorem has been proved that gives the necessary conditions for optimal control of a system with integral state equations and bounded controls.

Optimal educational policy consists of selecting the optimal enrollment rate, being the proportion of newly-born individuals entering the educational sector. The length of the training period has been taken to be fixed. An interior solution to the optimal enrollment rate is characterized by the equality of costs of and discounted returns to education.

The comparative-statics properties of the model depend on the way in which physical capital and the two types of labour interact in the production function. Some numerical calculations indicate that the comparative-statics results of the present model are similar to those for the more traditional model with homogeneous human capital (Chapter 6.1).

Analysis of the non-stationary optimal growth path bears out that the singular trajectory is unstable. Thus, singular control breaks down as soon as the economy, triggered off by demographic changes, leaves its steady-state growth path. This very unsatisfactory result makes it clear that it is quite dangerous to restrict one's attention to the comparative statics of optimal growth paths.

7 OPTIMAL EDUCATIONAL POLICY UNDER CONDITIONS OF TECHNICAL CHANGE

The simple one-sector growth model of Chapter 2, for which optimal economic growth has been analysed under conditions of demographic change, has until now been extended in two directions. Chapter 4 has introduced technical change; and Chapter 6 has introduced education as a second instrument of economic policy.

This chapter combines these two extensions. It deals with optimal educational policy under conditions of technical progress. Two aspects of the interaction between education and technical progress are considered. Chapter 7.1 analyses the effect of (exogenous) technical change on optimal investment in education. Chapter 7.2 investigates the role of education for the rate of adoption of new technical knowledge. While technical progress is disembodied in the model of Chapter 7.1, it is embodied in human capital in Chapter 7.2.

7.1 Education and exogenous disembodied technical change

This chapter extends the analysis of Chapter 6.1 to include the case of completely exogenous, disembodied, and labour-augmenting technical change. Most of the model's properties are identical to the combined properties of the models analysed in Chapters 4.1 and 6.1, respectively. These will not be discussed again. For the main part of this - consequently brief - section I will concentrate on the effect of technical change on optimal educational policy.

7.1.1 Optimal investment in education

As in Chapter 4.1 the production function shifts over time according to

$$Y(t) = F[K(t), L(t); t] = F[K(t), \lambda(t) \cdot L(t)] \tag{1}$$

The only implication of parametrization (1) for the formulation of the central planning agency's maximization problem is that the expression for output per unit of human capital becomes $\lambda(t) \cdot f[k(t)/\lambda(t)]$ instead of $f[k(t)]$, as it was in Chapter 6.1. Making this substitution into the Non-Stationary Golden Rule (6.1.19) yields:

$$\frac{d}{dk} \{\lambda(t) \cdot f[k(t)/\lambda(t)]\} = f'[k(t)/\lambda(t)] = F_K(t) = \rho + \delta_K + g^P(t) \tag{2}$$

Thus, as in Chapter 4.1, technical change does not affect the marginal productivity condition for <u>physical</u> capital.

Condition (2) gives the ratio

$$x(t) = \frac{k(t)}{\lambda(t)} = \frac{K(t)}{\lambda(t) \cdot L(t)} \tag{3}$$

as a function of constant parameters and exogenous demographic forces only. The marginal productivity of labour $F_L(t)$ is related to $x(t)$ as follows:

$$w(t) = F_L(t) = \lambda(t) \cdot f[x(t)] - \lambda(t) \cdot x(t) \cdot f'[x(t)] = \lambda(t) \cdot \tilde{w}(t) \tag{4}$$

Here $\tilde{w}(t)$, defined as the marginal productivity of human capital in "efficiency units", is a function of $x(t)$ only and hence completely determined by exogenous demographic forces along the singular optimal economic growth path.

Substitution of (4) into condition (6.1.24) gives the optimality condition for training policy under conditions of exogenous disembodied technical change:

$$\frac{\partial H(t)}{\partial \epsilon_v(t)} = B(v)\cdot\left[-\mu(t-v)\cdot\frac{e^{-\rho t}}{P(t)}\cdot\tilde{w}(t)\cdot\lambda(t) + \right.$$
$$\left. + E'[\epsilon_v(t)]\cdot e^{\delta_H t}\cdot \int_t^{v+n} e^{-\delta_H \tau}\cdot\mu(\tau-v)\cdot\frac{e^{-\rho\tau}}{P(\tau)}\cdot\tilde{w}(\tau)\cdot\lambda(\tau)\, d\tau\right] =$$

$$= \begin{cases} \geq 0 & \text{if} \quad \epsilon_v(t) = h_v(t) \\ = 0 & \text{if} \quad 0 < \epsilon_v(t) < h_v(t) \quad , \quad t-n \leq v \leq t \\ \leq 0 & \text{if} \quad \epsilon_v(t) = 0 \end{cases} \tag{5}$$

Contrary to physical capital, the occurrence of technical change <u>does</u> affect the optimality condition for <u>human</u> capital. Although the form of the optimality condition is not fundamentally altered - it still involves the evaluation of opportunity costs and discounted returns - the rate at which the discounting is done contains an additional component, viz. <u>minus</u> the rate of labour augmentation.

7.1.2 Comparative statics

In steady state the rate of labour augmentation $\hat{\lambda}$ should be constant. Then the function S(a) defined in equation (6.1.26) becomes:

$$S(a) = \frac{\mu(a)}{\int_a^n \mu(z)\cdot e^{-(\rho+g+\delta_H-\hat{\lambda})\cdot(z-a)} dz} \quad , \quad 0 \leq a \leq n \tag{6}$$

Extending the analysis of Section 6.1.3 to the comparative statics effects of $\hat{\lambda}$ yields the following results:

$$\frac{d\epsilon(a)}{d\hat{\lambda}} \geq 0 \tag{7}$$

$$\frac{dh(a)}{d\hat{\lambda}} = \int_0^a e^{\delta_H\cdot(z-a)}\cdot E'[\epsilon(z)]\cdot\frac{d\epsilon(z)}{d\hat{\lambda}}\, dz \geq 0 \tag{8}$$

Since the rate of labour augmentation $\hat{\lambda}$ enters the rate at which costs and returns to education are discounted with a negative sign, increasing this rate increases both the training effort and the stock of human capital for all generations.

7.1.3 Summary

This chapter, besides bringing together the results of the Chapters 4.1 and 6.1, offers one interesting new insight: technical progress makes education more attractive. The optimal growth path of an economy that experiences a secular progress of its technology is characterized by higher investments in education than the optimal growth path of an economy without such technical progress.

7.2 A model with education as transmitter of technical change

The embodiment hypothesis, discussed in Chapter 3.2, states that increases in the stock of technical knowledge increase output only to the extent that the factors of production are being "adapted" in order to be able to produce according to the newly invented technique. For example, if technical progress is embodied in physical capital new machines must be constructed if the technical improvement is to become effective. This kind of technical change - which, according to the definition of Section 3.2.3, can be characterized as shifting the production function in terms of investment goods, not in terms of consumption goods - gives rise to the so-called capital-vintage models like the model of Chapter 4.2.

In this section I will analyse a second type of embodied technical change, viz. embodied in human capital (or labour). Now the "adaptation" of the factors of production required to effectuate the technical change takes the form of education: the newly invented technique can be used only if the production unit engages labour that has been trained to produce according to this technique. Technical change of the labour-embodied type can be thought of as shifting the educational production function: its occurrence makes it possible to produce more productive human capital out of given educational inputs (time, human capital, and the like).

Just as capital-embodied technical change gives rise to capital-vintage models, so labour-embodied technical change leads to models with vintages of labour. A model of overlapping generations seems to offer a natural and very suitable context in which the economic consequences of the occurrence of labour vintages can be analysed. Of course, if one restricts one's attention exclusively to steady states many advantages of distinguishing overlapping generations disappear. This "steady-state only"-approach, in combination with some suitable additional restrictions on functional forms, has enabled authors like Nelson & Phelps (1966) and Stephens (1971) to summarize the economic consequences of labour-embodiment in two key indices: an index of "technology in theory" and an index of "technology in practice". However, the mechanism by which education affects the relationship between these two indices is largely left unspecified. It is essentially this mechanism that I will attempt to make explicit here, through a combination of overlapping generations and a specific micro-economic educational production function.

From a conceptual point of view there is a subtle difference between Nelson & Phelps' (1966) approach and my own. Nelson & Phelps regard education as a means by which technical knowledge is diffused among entrepreneurs. Thus, since according to the definition of Chapter 3.1 such diffusion should be labelled technical change, in their model education creates technical progress. On the other hand, in my model of labour-embodied technical change education diffuses technical knowledge among the labour force once the new invention has shifted the educational production function. Thus, in the diffusion model of Nelson & Phelps the technical change is endogenous while in my embodiment model it is exogenous.

However, apart from these conceptual issues, the diffusion model and the embodiment model are very much alike in the sense that both models view education as an activity that serves to bridge the ever-present gap between the potential and the actual effectiveness in production of the resources of the economy.

As indicated above, technical change of the labour-embodied type can be thought of as shifting the educational production function. The nature of this shift has not yet been specified. In the discussion of capital-embodied technical change (see Section 3.2.3) it was concluded that the theoretical implications of the embodiment hypothesis depend to a large extent on whether the vintages embodying subsequent levels of technology are qualitatively or only quantitatively different from one another. A similar question needs to be answered in the case of labour vintages.

Ideally one would like the model to be so general that it could handle all realistic combinations of both capital- and labour-embodiment, like, for example, labour requiring special up-to-date training in order to be able to use the most recently constructed equipment. Since I wish to concentrate on the role of education I will ignore the possibility of capital-embodied technical change completely. Such a degree of generality would render the model inconveniently complex anyway.

Fortunately, a great deal of complexity is removed if one is prepared to require technical change to be Harrod-neutral for all points of the production function. Such a requirement is necessary if the model is to generate realistic long-run economic growth paths. It will be recalled from Section 3.2.2 that everywhere Harrod-neutral technical change is necessarily purely labour-augmenting. But then the shift of the educational production function must be such that subsequent labour vintages are only quantitatively different, i.e. labour of different vintages can be perfectly substituted for each other in the production process.

Stephens (1971) contains an example of a model in which both capital and labour vintages are distinguished. At first sight, deriving steady-state relations for a model of such a high degree of generality appears a remarkable feat. However, much of this generality disappears again if one realizes that Stephens uses vintage production functions of the Cobb-Douglas type. Since the Cobb-Douglas function is such that technical change is at the same time purely labour-augmenting and purely capital-augmenting it allows complete aggregation of both capital and labour vintages into two aggregate factors of production.

Admittedly, the latter criticism applies equally well to my own model as it combines labour-embodiment with pure labour-augmentation. However, a major advantage of my model is that it is one of overlapping generations. As a consequence it is very much suitable for analysing the effects of both demographic forces and education on the level of technology.

It is this latter analysis that will be undertaken in the present chapter. Section 7.2.1 describes the model with education-requiring labour-embodied technical progress. In Section 7.2.2 the conditions for optimal economic growth will be established. The properties of the optimal economic growth path are investigated in Sections 7.2.3 (comparative statics) and 7.2.4 (dynamics), respectively. A summary concludes this chapter.

7.2.1 Education and labour-embodied technical progress

Each individual, indexed v by his date of birth, is endowed with a stock of human capital. Contrary to the model of Chapter 6.1, however, the productive usefulness of this stock is not dependent on its quantity as such but rather on the date on which it has been produced.

An individual who has been in school continuously from the time of his birth onward should be expected to embody human capital of the most up-to-date variety. This most up-to-date human capital will serve as a benchmark in assessing the relative up-to-dateness of the human capital of the working generations. The efficiency of the most up-to-date human capital is assumed to be an exogenous function of time, written as $\lambda(t)$. The quantity $\lambda(t)$ corresponds to Nelson & Phelps' (1966) measure of "technology in theory".

The efficiency of the human capital embodied at time t in an individual of generation v will now be indexed by the date ($\leq t$) at which the most up-to-date human capital was as effective as the human capital under consideration. Thus we can write:

$$h_v(t) = \lambda[\tau_v(t)] \tag{1}$$

where $h_v(t)$ is human capital in efficiency units (as in Chapter 6.1.) and $\tau_v(t)$ is the index of this efficiency. From the definition of $\lambda(t)$ it should be clear that

$$\tau_v(t) \le t \quad \text{for all } v \le t \text{ and all } t \tag{2}$$

I now propose to postulate an educational production process that is characterized by the following properties:

1. the output of the process is the increase in the student's efficiency index $\tau_v(t)$;
2. output is an increasing but concave function of the fraction of time spent training $u_v(t)$ (i.e. training intensity is the same as in Chapter 6.1);
3. if the human capital of an individual is of the most up-to-date variety it will remain of the most up-to-date variety as long as the individual remains a full-time student;
4. once an individual's human capital ceases to be of the most up-to-date variety it will remain less than up-to-date forever, however much time is invested in education;
5. the efficiency of the human capital embodied in a newborn individual is of the most up-to-date variety. This assumption is made for analytical convenience only and should of course not be taken too literally as its implications would then apparently be ridiculous.

An educational production function that satisfies these five conditions is the following:

$$\dot{\tau}_v(t) = E[u_v(t)] \tag{3}$$

with

$$E[0] = 0 \quad ; \quad E[1] = 1 \quad ; \quad E'[\cdot] > 0 \quad ; \quad E''[\cdot] < 0 \tag{4}$$

and

$$\tau_t(t) = t \tag{5}$$

As in Chapter 6.1 the actual efficiency of an individual is equal to his potential efficiency multiplied by the fraction of time devoted to the production of the aggregate commodity. Thus the labour force at time t measured in efficiency units equals:

$$L(t) = \int_{t-n}^{t} \{1-u_v(t)\}\cdot\lambda[\tau_v(t)]\cdot\mu(t-v)\cdot B(v)\ dv \tag{6}$$

As an analogue to Nelson & Phelps' (1966) measure of "technology in practice" we can define the average efficiency of human capital, with respect to either the labour force or the population as a whole:

$$T_L(t) = \frac{\int_{t-n}^{t} \{1-u_v(t)\}\cdot\lambda[\tau_v(t)]\cdot\mu(t-v)\cdot B(v)\ dv}{\int_{t-n}^{t} \{1-u_v(t)\}\cdot\mu(t-v)\cdot B(v)\ dv} \tag{7}$$

$$T_P(t) = \frac{\int_{t-n}^{t} \lambda[\tau_v(t)]\cdot\mu(t-v)\cdot B(v)\ dv}{\int_{t-n}^{t} \mu(t-v)\cdot B(v)\ dv} \tag{8}$$

From (2) it immediately follows that either measure of "technology in practice" can never exceed the measure of "technology in theory". It is therefore slightly more appropriate to measure the difference between "technology in theory" and "technology in practice" by the time lag between the two. These time lags $TL_L(t)$ and $TL_P(T)$, corresponding to $T_L(t)$ and $T_P(t)$, respectively, can be defined implicitly by:

$$T_L(t) = \lambda[t-TL_L(t)] \tag{9}$$

$$T_P(t) = \lambda[t-TL_P(t)] \tag{10}$$

If the optimal age-training profile is decreasing - as one would expect it to be - average efficiency of the labour force will be lower than average efficiency of the population. This obvervation illustrates an attractive property of the model, viz. that a decrease in the gap between maximum and actual efficiency of the population can be realized only at the expense of a higher dependency ratio.

7.2.2 Optimal economic growth

The Hamiltonian corresponding to the problem of maximizing the social welfare function, subject to (6), (3), and the capital accumulation equation, can be written as follows (cf. Chapter 6.1):

$$H(t) = e^{-\rho t}\cdot\frac{\{1-s(t)\}\cdot F[K(t),L(t)]}{P(t)} + \sum_{v=t-n;dv\to 0}^{v=t} \psi_v(t)\cdot E[u_v(t)]\ dv +$$

$$+ \psi_K(t)\cdot\{s(t)\cdot F[K(t),L(t)] - \delta\cdot K(t)\} \tag{11}$$

The costate variables $\psi_K(t)$ and $\psi_v(t)$, $t-n\leq v\leq t$, satisfy:

$$-\dot{\psi}_v(t) = \frac{\partial H(t)}{\partial \tau_v(t)} = \frac{\partial H(t)}{\partial L(t)}\cdot\frac{\partial L(t)}{\partial \tau_v(t)} =$$

$$= \left[e^{-\rho t}\cdot\{1-s(t)\}\cdot\frac{1}{P(t)} + s(t)\cdot\psi_K(t)\right]\cdot F_L(t)\cdot B(v)\cdot\mu(t-v)\cdot dv\cdot$$

$$\cdot\{1-u_v(t)\}\cdot\dot{\lambda}[\tau_v(t)] \tag{12}$$

$$-\dot{\psi}_K(t) = \frac{\partial H(t)}{\partial K(t)} =$$

$$= \left[e^{-\rho t}\cdot\{1-s(t)\}\cdot\frac{1}{P(t)} + s(t)\cdot\psi_K(t)\right]\cdot F_K(t) - \delta\cdot\psi_K(t) \tag{13}$$

Necessary conditions for an optimum are:

$$\frac{\partial H(t)}{\partial s(t)} = F[K(t),L(t)]\cdot\left[\psi_K(t) - \frac{e^{-\rho t}}{P(t)}\right] \begin{cases} \geq 0 & \text{if } s(t)=1 \\ = 0 & \text{if } 0 < s(t) < 1 \\ \leq 0 & \text{if } s(t)=0 \end{cases} \tag{14}$$

$$\frac{\partial H(t)}{\partial u_v(t)} + \frac{\partial H(t)}{\partial L(t)}\cdot\frac{\partial L(t)}{\partial u_v(t)} =$$

$$= - \left[e^{-\rho t}\cdot\frac{1-s(t)}{P(t)} + s(t)\cdot\psi_K(t)\right]\cdot F_L(t)\cdot B(v)\cdot\mu(t-v)\cdot dv\cdot\lambda[\tau_v(t)] +$$

$$+ \psi_v(t)\cdot E'[u_v(t)] \begin{cases} \geq 0 & \text{if } u_v(t)=1 \\ = 0 & \text{if } 0 < u_v(t) < 1 \;;\; t\text{-}n\leq v\leq t \\ \leq 0 & \text{if } u_v(t)=0 \end{cases} \tag{15}$$

as well as the transversality condition

$$\psi_{t-n}(t) = 0 \tag{16}$$

From condition (14) follow the by now familiar conditions for a singular trajectory for the optimal s(t). Substitution of these singularity conditions into (13) yields the Non-Stationary Golden Rule:

$$F_K(t) = \rho + \delta + g^P(t) \tag{17}$$

Given (17), and given the linear homogeneity of the production function, the marginal productivity of labour (in efficiency units) $w(t)\equiv F_L(t)$ is completely determined by demographic forces along the singular optimal economic growth path. Then equation (12) reduces to:

$$-\dot{\psi}_v(t) = w(t)\cdot B(v)\cdot \mu(t-v)\cdot dv\cdot \frac{e^{-\rho t}}{P(t)}\cdot \{1-u_v(t)\}\cdot \dot{\lambda}[\tau_v(t)] \tag{18}$$

Integrating (18) over t, using (16), yields:

$$\psi_v(t) = B(v)\cdot dv\cdot \int_t^{v+n} w(z)\cdot \mu(z-v)\cdot \frac{e^{-\rho z}}{P(z)}\cdot \{1-u_v(z)\}\cdot \dot{\lambda}[\tau_v(z)]\; dz \tag{19}$$

Equation (19) makes it clear that the shadow price $\psi_v(t)$ (cf. Dorfman, 1969) equals the present value - discounting at $\rho+g^P(\cdot)-\hat{\mu}(\cdot)$ as before - of the marginal returns to increasing generation v's efficiency index $\tau_v(t)$.

Combination of (15), (19), and the singularity conditions finally gives:

$$\frac{\partial H(t)}{\partial u_v(t)} + \frac{\partial H(t)}{\partial L(t)}\cdot\frac{\partial L(t)}{\partial u_v(t)} = B(v)\cdot dv\cdot \Bigg[- w(t)\cdot \mu(t-v)\cdot \frac{e^{-\rho t}}{P(t)}\cdot \lambda[\tau_v(t)] +$$

$$+ E'[u_v(t)]\cdot \int_t^{v+n} w(z)\cdot \mu(z-v)\cdot \frac{e^{-\rho z}}{P(z)}\cdot \{1-u_v(z)\}\cdot \dot{\lambda}[\tau_v(z)]\; dz \Bigg] =$$

$$= \begin{cases} \geq 0 & \text{if} \quad u_v(t)=1 \\ = 0 & \text{if} \quad 0 < u_v(t) < 1 \qquad t\text{-}n\leq v\leq t \\ \leq 0 & \text{if} \quad u_v(t)=0 \end{cases} \tag{20}$$

Again, the interpretation of condition (20) is quite similar to that of condition (6.1.24). The first term in brackets is discounted production foregone per unit of time allocated to training. The second term, bearing in mind the interpretation of the RHS of expression (19) given above, denotes the present value of the future returns to a marginal increase in training intensity. Optimal training policy can thus be determined from a straightforward evaluation of costs and returns.

Indeed, it is quite remarkable how similar condition (20) is to the optimality conditions of the previous chapter. Although the contextual setting and interpretation of the educational production functions are quite different for the two models it appears that these differences do not fundamentally alter the policy implications.

However, two differences between condition (20) above and condition (6.1.24) should be noted. First, although the optimal choice is determined by the balance between costs and returns to training, these costs and returns are not constant over time for the present model. After all, there is exogenous technical change, as should be clear from the fact that equations (5) acts as an initial condition for the system of equations implied by (20). As the index of initial efficiency changes over time, so do the costs of and returns to training for subsequent generations, albeit in the same proportions if the economy is in steady state.

Second, of a more technical nature, in evaluating condition (20) for a certain generation, all optimal training efforts for that generation in the future have to be taken into account. Thus, while equation (6.1.24) can be solved per generation at any age t-v, solving equation (20) can only be done by computing the optimal training profile over the complete life cycle of the generation under consideration. This feature of the present model significantly complicates the simulation of optimal growth paths.

7.2.3 Comparative statics

In this section I analyse the effect of the exogenous parameters of the model on the optimal training policy in steady state. The following two simplifying assumptions are made:

$$\lambda[\tau] = e^{\lambda\tau} \tag{21}$$

$$\mu(a) = \begin{cases} e^{-\mu a} & \text{for } a \le n \\ 0 & \text{for } a > n \end{cases} \tag{22}$$

i.e. technical progress and mortality are exponential. From (21) and (3) we have:

$$\lambda[\tau_v(z)] = \lambda[\tau_v(t)]\cdot\exp\left[\lambda\cdot\int_t^z E[u_v(\zeta)]\ d\zeta\right] \qquad t \le z \le v+n \tag{23}$$

In steady state w and k are constant. The condition for optimal training effort (20) then reduces to (using (22) and (23)):

$$E'[u(a)] \begin{matrix} > \\ = \\ < \end{matrix} \frac{1}{\int_a^n e^{-(\mu+\rho+g)\cdot(z-a)}\cdot\{1-u(z)\}\cdot\lambda\cdot\exp\left[\lambda\cdot\int_a^z E[u_v(\zeta)]\ d\zeta\right]\ dz}$$

$$\Leftrightarrow \begin{cases} u(a) = 1 \\ 0 < u(a) < 1 \\ u(a) = 0 \end{cases} , \qquad 0 \le a \le n \tag{24}$$

where I use u(a) as a shorthand notation for $u_v(t)\big|_{t-v=a}$, training effort at age a.

Expression (24) is quite difficult to evaluate. Without additional information on the form of the educational production function E[·], which appears both on the LHS and in the denominator on the RHS of (24), no clear-cut conclusions can be drawn. This does not only apply to the comparative-statics effects themselves, but even to the form of the age-training profile u(a). E.g. it is not possible to derive from (24) and (4) that u(a) is non-increasing.

In order to obtain some insight into the steady-state behaviour of the model I have carried out some numerical experiments. The production function F[·] has been specified as being CES, while for E[·] I have chosen the following, very tentative, specification:

$$E[u] = u^{1/2} \tag{25}$$

Table 7.2.1 gives the optimal age-training profiles for various values of the exogenous parameters. Although the optimal training intensity is in all cases a non-increasing function of age, which is quite plausible, the order of

Table 7.2.1: Selected numerical steady-state values for the optimal age-training profile

changing param.	u_1	u_2	u_3	u_4	u_5	u_6
reference point[1]	0.027	0.025	0.020	0.014	0.006	0.000
g = 0.0	0.063	0.052	0.039	0.023	0.008	0.000
0.1	0.040	0.035	0.027	0.018	0.007	0.000
0.2	0.027	0.025	0.020	0.014	0.006	0.000
0.3	0.020	0.018	0.016	0.011	0.005	0.000
0.4	0.015	0.014	0.012	0.009	0.004	0.000
δ = 0.0	0.027	0.025	0.020	0.014	0.006	0.000
0.1	0.027	0.025	0.020	0.014	0.006	0.000
0.2	0.027	0.025	0.020	0.014	0.006	0.000
0.3	0.027	0.025	0.020	0.014	0.006	0.000
0.4	0.027	0.025	0.020	0.014	0.006	0.000
ρ = 0.0	0.063	0.052	0.039	0.023	0.008	0.000
0.1	0.040	0.035	0.027	0.018	0.007	0.000
0.2	0.027	0.025	0.020	0.014	0.006	0.000
0.3	0.020	0.018	0.016	0.011	0.005	0.000
0.4	0.015	0.014	0.012	0.009	0.004	0.000
μ = 0.0	0.078	0.063	0.045	0.026	0.009	0.000
0.1	0.046	0.040	0.031	0.019	0.007	0.000
0.2	0.027	0.025	0.020	0.014	0.006	0.000
0.3	0.016	0.015	0.013	0.010	0.004	0.000
0.4	0.009	0.009	0.008	0.006	0.003	0.000
λ = 0.1	0.003	0.003	0.002	0.002	0.001	0.000
0.2	0.012	0.011	0.009	0.006	0.003	0.000
0.3	0.027	0.025	0.020	0.014	0.006	0.000
0.4	0.049	0.044	0.036	0.024	0.009	0.000
0.5	0.080	0.070	0.056	0.037	0.014	0.000

1) The values of the parameters in the reference point are:
$g=0.20$; $\delta=0.20$; $\rho=0.20$; $\mu=0.20$; $\lambda=0.30$; $n=6$.
The production function is taken to be:

$$y = \{\alpha k^{-\frac{1-\sigma}{\sigma}} + (1-\alpha)\}^{-\frac{\sigma}{1-\sigma}} \quad \text{with } \alpha=0.25 \text{ and } \sigma=0.5$$

The educational production function used is:

$$\dot{\tau}_v(t) = \{u_v(t)\}^{\gamma} \quad \text{with } \gamma=0.5$$

Table 7.2.2: Selected numerical steady-state values

changing param.	k	s	M/P	TL_L	TL_P	C/P[1]
reference point[2]	0.500	0.437	0.979	1.024	1.015	0.337
g = 0.0	0.721	0.395	0.961	1.279	1.253	0.379
0.1	0.591	0.416	0.972	1.145	1.130	0.358
0.2	0.500	0.437	0.979	1.024	1.015	0.337
0.3	0.431	0.459	0.984	0.919	0.914	0.315
0.4	0.377	0.480	0.987	0.829	0.826	0.293
δ = 0.0	0.721	0.395	0.979	1.024	1.015	0.416
0.1	0.591	0.416	0.979	1.024	1.015	0.374
0.2	0.500	0.437	0.979	1.024	1.015	0.337
0.3	0.431	0.459	0.979	1.024	1.015	0.304
0.4	0.377	0.480	0.979	1.024	1.015	0.275
ρ = 0.0	0.721	0.553	0.955	0.961	0.940	0.302
0.1	0.591	0.485	0.970	0.997	0.985	0.328
0.2	0.500	0.437	0.979	1.024	1.015	0.337
0.3	0.431	0.401	0.984	1.043	1.037	0.337
0.4	0.377	0.373	0.988	1.058	1.054	0.332
μ = 0.0	0.500	0.437	0.955	1.335	1.301	0.303
0.1	0.500	0.437	0.969	1.188	1.170	0.319
0.2	0.500	0.437	0.979	1.024	1.015	0.337
0.3	0.500	0.437	0.986	0.853	0.850	0.355
0.4	0.500	0.437	0.992	0.687	0.686	0.373
λ = 0.1	0.500	0.312	0.997	1.258	1.257	0.487
0.2	0.500	0.375	0.990	1.129	1.125	0.403
0.3	0.500	0.437	0.979	1.024	1.015	0.337
0.4	0.500	0.500	0.962	0.935	0.921	0.281
0.5	0.500	0.562	0.940	0.857	0.835	0.232

1) Level (cf. Chapter 4.1)

2) The values of the parameters in the reference point are:

$g=0.20$; $\delta=0.20$; $\rho=0.20$; $\mu=0.20$; $\lambda=0.30$; $n=6$.

The production function is taken to be:

$$y = \{\alpha k^{-\frac{1-\sigma}{\sigma}} + (1-\alpha)\}^{-\frac{\sigma}{1-\sigma}} \quad \text{with } \alpha=0.25 \text{ and } \sigma=0.5$$

The educational production function used is:

$$\dot{\tau}_v(t) = \{u_v(t)\}^{\gamma} \quad \text{with } \gamma=0.5$$

magnitude of the training variables is unrealistically small. This observation makes it clear that specification (25) is not very useful for empirical purposes.

The simulated steady-state values for some other endogenous variables are given in Table 7.2.2. The variable M/P is the labour force / population-ratio measured in <u>man-hours</u>; it is defined as:

$$M/P = \frac{\int_{t-n}^{t} \{1-u_v(t)\}\cdot\mu(t-v)\cdot B(v)\, dv}{\int_{t-n}^{t} \mu(t-v)\cdot B(v)\, dv} \tag{26}$$

M/P is a measure of the <u>average</u> working effort, i.e. one minus the average training effort.

The results of these simulations suggest that the signs of the partial derivatives are as summarized in Table 7.2.3. These comparative-statics results are in agreement with those that were obtained for previous models. As in Chapter 7.1 it is found that optimal training intensity is an increasing function of the rate of technical progress. Technical progress makes investment in education more attractive.

Table 7.2.3: Comparative-statics results for the one-sector model with labour-embodied technical change

effect on: \ of:	g	δ	ρ	μ	λ
k	-	-	-	0	0
s	+	+	-	0	+
M/P	+	0	+	+	-
TL_L	-	0	+	-	-
TL_P	-	0	+	-	-
C/P[1]	-	-	-	+	-

1) Level (cf. Chapter 4.1)

Of particular interest is the effect of the growth rate of population on optimal educational policy. As in all other models with endogenous education, the optimal training effort increases as population growth slows down. In the previous models this phenomenon could be explained by a scarcity argument: if g falls, labour (human capital) becomes relatively scarce; increasing investment in education serves to partially offset this increased scarcity of human capital.

The present model illustrates a second effect of an ageing (i.e. slower growing) population on optimal educational policy. This effect is related to the introduction of technological innovations in the production process. When the growth rate of the population is high this introduction is primarily achieved by means of the constant influx into the labour force of recently educated young people. When the relative share of this influx is reduced (g falls), increased education for adults becomes necessary in order to prevent the gap between technology in theory and technology in practice becoming too large.

It is exactly the latter mechanism that is illustrated by the numerical examples in Table 7.2.2. The effect of lowering g is that the technological time lags TL_L and TL_P increase. On the other hand the optimal training effort increases (M/P falls), partially off-setting the increase in the technological time lags.

7.2.4 The non-stationary optimal growth path

A discussion of the non-stationary optimal growth path for the model with labour-embodied technical progress can be quite brief.

As far as the savings rate and optimal training policy are concerned, the analysis is exactly the same as for the model of Chapter 6.1: depending on the direction of the change in the population growth rate, the path of s is either inversely U-shaped or U-shaped, with oscillations at both ends due to adjustments in educational policy; and the optimal training effort moves in the opposite direction of the growth rate of population.

Because of the presence of technical progress output and consumption per capita are rising all the time. As both s and training effort change during the period of demographic transition there are some deviations around this secular upward trend in consumption per capita; it is, however, highly improbable that these deviations are so large that consumption per capita could fall.

As these remarks do not add much to the findings of previous chapters one single numerical example will suffice for this model. The results of a simulation with the birth growth rate falling are presented in Table 7.2.4. It is seen that optimal training effort continuously increases (M/P decreases) and that the technological time lags (TL_L and TL_P) also continuously increase (with the exception of a very small decrease at the beginning of the educational adjustment process) during the full period of diminishing population growth.

7.2.5 Summary

In this chapter I have constructed and analysed a growth model with exogenous technical change that is embodied in people. That is, newly invented techniques can be used only if the production unit engages labour that has been trained to produce according to this new technique. If the technical progress is Harrod-neutral everywhere, or, equivalently, purely labour-augmenting, then all labour vintages can be aggregated into one single aggregate labour force.

The model thus constructed has a number of similarities to the model with homogeneous human capital discussed in Chapter 6.1. The form of the optimality conditions, the comparative-statics properties, and the form of the non-stationary optimal economic growth path are all quite similar for the two models. A major difference, apart from the obvious presence of technical change, is that the optimality condition for the training variable involves the complete optimal training profile of the generation in question. This feature of the present model complicates the simulation of optimal growth paths.

The model illustrates the negative impact of slower population growth on the rate of technology adaptation. When the growth rate of population is high, the introduction of technological innovations into the production process is primarily achieved through the constant influx of recently educated young people. When the relative share of this influx is reduced, increased education for adults becomes necessary in order to prevent the gap between technology in theory and technology in practice becoming too large.

Table 7.2.4: Optimal economic growth path with g^B falling

t	g	$\hat{g}$	s	M/P	TL_L	TL_P
0	0.20	0.200	0.475	0.979	1.024	1.015
1	0.20	0.200	0.475	0.979	1.024	1.015
2	0.20	0.200	0.475	0.979	1.024	1.015
3	0.20	0.200	0.474	0.979	1.024	1.015
4	0.20	0.200	0.474	0.978	1.023	1.014
5	0.20	0.200	0.477	0.976	1.023	1.013
6	0.18	0.193	0.476	0.975	1.027	1.016
7	0.16	0.181	0.473	0.973	1.039	1.027
8	0.14	0.166	0.468	0.971	1.056	1.043
9	0.12	0.149	0.462	0.969	1.077	1.063
10	0.10	0.130	0.456	0.967	1.101	1.084
11	0.08	0.111	0.449	0.965	1.125	1.107
12	0.06	0.092	0.443	0.963	1.150	1.129
13	0.04	0.073	0.436	0.961	1.176	1.153
14	0.02	0.054	0.431	0.960	1.202	1.177
15	0.00	0.035	0.419	0.960	1.229	1.204
16	0.00	0.021	0.411	0.960	1.252	1.226
17	0.00	0.012	0.406	0.960	1.267	1.241
18	0.00	0.006	0.402	0.960	1.276	1.249
19	0.00	0.002	0.398	0.960	1.279	1.252
20	0.00	0.000	0.398	0.960	1.279	1.252

1) The values of the parameters used are:

$\delta=0.20$; $\rho=0.20$; $\mu=0.20$; $\lambda=0.30$; $n=6$.

The production function is taken to be:

$$y = \{\alpha k^{-\frac{1-\sigma}{\sigma}} + (1-\alpha)\}^{-\frac{\sigma}{1-\sigma}} \quad \text{with } \alpha=0.25 \text{ and } \sigma=0.5$$

The educational production function used is:

$$\dot{r}_v(t) = \{u_v(t)\}^{\gamma} \quad \text{with } \gamma=0.5$$

8 SIMULATIONS FOR THE NETHERLANDS

In the previous chapters the interaction between demography, optimal economic growth, technical change, and education has been analysed from a purely theoretical point of view. On the basis of this theoretical analysis, in combination with some complementary numerical simulations, several broad guidelines for long-run economic and educational policy have been obtained.

The purpose of the present chapter is to provide a more detailed application of the theoretical models to real-life policy problems. The consequences of the present demographic changes (fertility decline and ageing of the population) for optimal investment in physical and human capital will be analysed in the case of the Netherlands. The results of the calculations will give some insight into the order of magnitude of the policy adjustments involved.

The values of the models' parameters have been specified in such a way that a realistic description of the Dutch economy and demographic structure is given by the models. Still, the results of the case study are of course very sensitive to the restrictive nature of the general neoclassical, one-sector, and closed-economy growth model itself. Thus, both the theoretical analysis of the previous chapters and the results of the case studies in the present chapter should be considered as only a first step towards a full demographic extension of traditional growth theory.

The application to the Dutch case will be undertaken using two different models. The first case study uses the model with homogeneous human capital and exogenous Harrod-neutral technical change (Chapter 7.1). The second study uses the model with heterogeneous human capital (Chapter 6.2). The specification of the latter model is quite similar to the models analysed in Ritzen (1986c) for which detailed parameter estimates for the Netherlands are available. However, as will be recalled from Chapter 6.2, the model with two types of labour suffers from the major drawback that the optimal economic growth path is non-convergent. Therefore, instead of calculating the non-stationary singular trajectory itself I use a scenario-based approach: four reasonable policies are investigated and compared, both with respect to the resulting economic development and the corresponding value of the social welfare function.

The organization of this chapter is as follows. Section 8.1 presents the demographic projections and structural relationships which underlie both case studies. Section 8.2 analyses the optimal growth path for the model of

Chapter 7.1. Section 8.3 discusses the four scenarios and presents the results of the simulations with the model of Chapter 6.2.

8.1 Demographic data

Simulation of the demographic dynamics underlying the optimal economic growth paths requires the specification of:

- the age-specific survival schedule $\mu(\cdot)$;
- the time path of the number of births $B(\cdot)$.

In order to concentrate on the economic effects of the fertility decline (as well as to avoid unnecessary complications), I have assumed that the survival schedule $\mu(\cdot)$ is fixed throughout. This is an obvious simplification of reality, as the average duration of human life has increased sharply during the last 100 years or so, and is expected to continue to increase even further in the near future.

The survival schedule actually used has been derived from life tables for the Netherlands during the first half of the 1980s (CBS, 1987). The survival probabilities have been calculated as a weighted average of the survival probabilities for males and females, using the number of births in 1986 as weights. For each 5-year age group the survival probability has been put equal to the actual survival probability for the midpoint of the age bracket. This procedure has resulted in a survival schedule for 22 age groups (corresponding to a maximum age of 110 years) given in Table 8.1.

Any numerical simulation of the non-stationary optimal growth path must start from an initial steady state. Consequently, in constructing the time path of the number of births some assumption has to be made concerning an initial stable population, even though in reality population was not stable for a considerable period. I have assumed that until the beginning of the Second World War the population was stable; the constant growth rate of the number of births for this period has been assumed to be equal to the average growth rate during the period 1900-1940 (3.66% per 5 years).

For the period 1940-1985 the number of births per 5-year period has been obtained by taking the actual population in the age group 0-5 years at the end of the 5-year period, divided by the survival probability for age group zero. A similar procedure was used for the period 1985-2010, with the

Table 8.1: Age-specific survival schedule

age group (a)	age span (years)	survivors at time t of age a from births in a 5-year period t-a
0	0 -< 5	0.9907
1	5 -< 10	0.9892
2	10 -< 15	0.9882
3	15 -< 20	0.9867
4	20 -< 25	0.9839
5	25 -< 30	0.9812
6	30 -< 35	0.9780
7	35 -< 40	0.9738
8	40 -< 45	0.9674
9	45 -< 50	0.9568
10	50 -< 55	0.9385
11	55 -< 60	0.9087
12	60 -< 65	0.8620
13	65 -< 70	0.7909
14	70 -< 75	0.6883
15	75 -< 80	0.5485
16	80 -< 85	0.3813
17	85 -< 90	0.2105
18	90 -< 95	0.0818
19	95 -< 100	0.0196
20	100 -< 105	0.0024
21	105 -< 110	0.0001
22	110 -< 115	0.0000

Source: computed from CBS (1987).

population aged 0-5 taken from the most recent Dutch CBS population forecasts (CBS, 1986, medium projection). For the years after 2010 the number of births has been assumed to remain constant at the 2010-level, eventually leading to a stable population with zero growth rate.

The resulting demographic time series are plotted in Figure 8.1 and listed in Table 8.2. In order to compare the computed time series with the actual demographic development, the table also contains some data on the size and age composition of the population in the near past and near future. The large differences between corresponding series are due mainly to the use of a constant survival schedule which *grossly* overestimates the survival probabilities for the older cohorts.

The data summarized in Tables 8.1 and 8.2 will be used to describe the demographic dynamics underlying the non-stationary optimal economic growth paths to be simulated in the next sections.

8.2 Simulation for the model with homogeneous human capital

The model used for the simulation in the present section is a slightly adapted version of the model described more fully in Chapters 6.1 and 7.1.

In order to add some more realism to the formulation of the model the following additional assumptions have been made:

- individuals follow compulsory education during the first three periods of their life (15 years). This basic educational training does not increase their productive stock of human capital; it is more like a minimal investment required to guarantee that the stock of human capital of a potential labourer is at least equal to h_0.
- there is mandatory retirement at the age of 65 years (age group 13).
- the aggregate production function is of the CES type. The elasticity of substitution between physical and human capital is fixed at 0.167; such a relatively low elasticity of substitution was found by Ritzen (1986c).
- there is exponential, exogenous, disembodied and purely labour-augmenting technical progress. The rate of technical progress is assumed to be 1% per year or about 6% per period of 5 years. In order not to let this exogenous increase in the productivity of existing human capital be completely "free", I have fixed the rate of exponential decay of human capital (δ_H) at a rather high level (20%).

Figure 8.1: Growth rates of births and total population

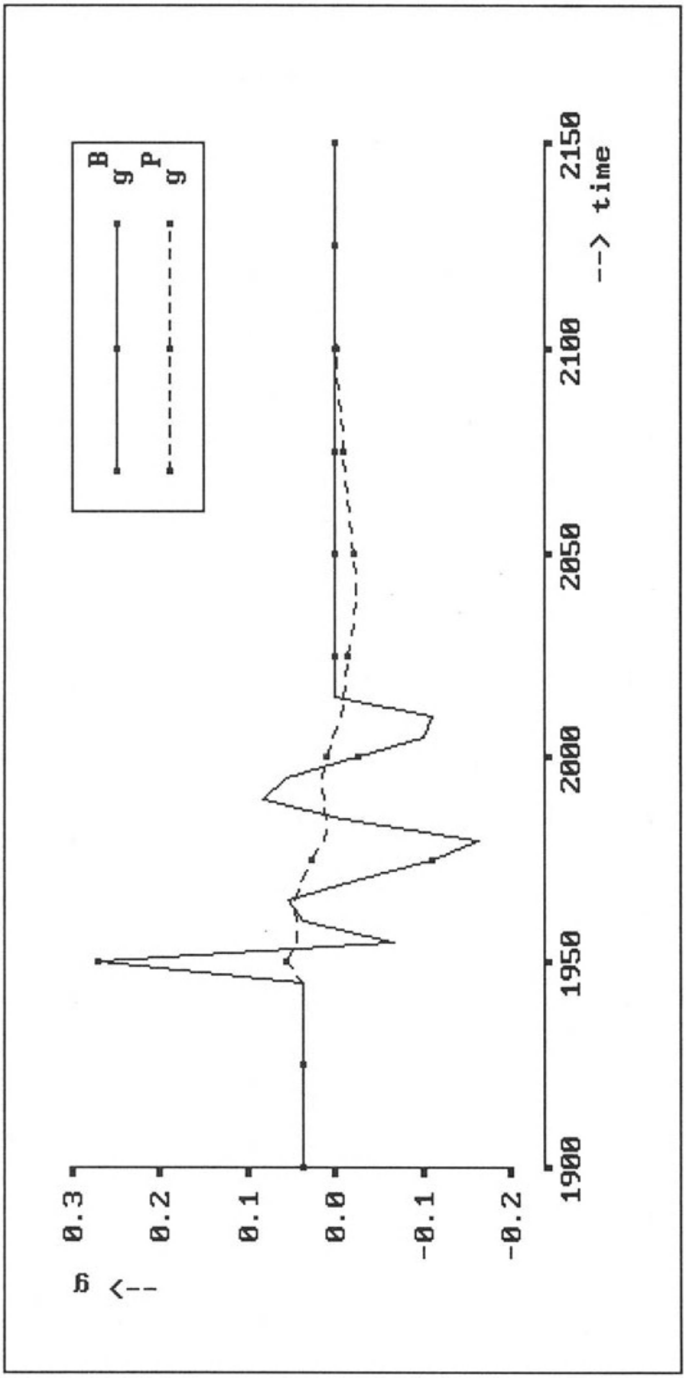


Table 8.2: Demographic time series

year	B	population * 1000 (computed)				g^P	population * 1000 (realized/projected)[1]			
		0-19	20-64	65+	total		0-19	20-64	65+	total
1900	684	2563	4417	1103	8084	.0366				
1905	709	2657	4578	1144	8379	.0366	2410	2717	334	5460
1910	735	2755	4746	1186	8686	.0366	2579	2921	359	5858
1915	761	2856	4920	1229	9004	.0366	2753	3209	378	6340
1920	789	2960	5100	1274	9334	.0366	2866	3488	402	6754
1925	818	3068	5287	1321	9676	.0366	3048	3831	430	7308
1930	848	3181	5480	1369	10030	.0366	3153	4195	477	7825
1935	879	3297	5681	1419	10397	.0366	3223	4626	543	8392
1940	911	3418	5889	1471	10778	.0366	3316	4905	613	8834
1945	945	3543	6105	1525	11173	.0366	3369	5177	674	9220
1950	1201	3882	6328	1593	11802	.0563	3742	5514	771	10027
1955	1119	4109	6560	1661	12329	.0446	4001	5789	890	10680
1960	1161	4344	6800	1731	12875	.0443	4331	6066	1020	11417
1965	1223	4608	7049	1805	13462	.0455	4631	6418	1163	12212
1970	1197	4605	7514	1880	13999	.0399	4658	6989	1311	12958
1975	1066	4552	7874	1956	14382	.0274	4646	7493	1459	13599
1980	890	4286	8246	2033	14566	.0128	4432	8044	1615	14091
1985	882	3952	8651	2113	14717	.0104	4085	8640	1730	14454
1990	956	3716	9001	2197	14914	.0134	3826	9152	1898	14876
1995	1009	3661	9192	2283	15136	.0149	3745	9518	2012	15275
2000	982	3751	9179	2371	15301	.0109	3826	9648	2113	15588
2005	883	3752	9122	2461	15335	.0023				
2010	786	3586	9100	2553	15238	-.0063	3638	9753	2358	15749
2015	786	3367	8922	2813	15102	-.0089				
2020	786	3176	8785	2962	14923	-.0119				
2025	786	3081	8519	3098	14698	-.0151				
2030	786	3081	8113	3237	14430	-.0183				
2035	786	3081	7740	3304	14124	-.0211	2990	8046	3563	14599
2040	786	3081	7486	3234	13801	-.0229				
2045	786	3081	7381	3021	13483	-.0231				
2050	786	3081	7280	2826	13187	-.0219				
2055	786	3081	7118	2731	12930	-.0195				
2060	786	3081	6915	2723	12719	-.0163				
2065	786	3081	6744	2726	12550	-.0133				
2070	786	3081	6661	2666	12408	-.0113				
2075	786	3081	6661	2538	12280	-.0103				
2080	786	3081	6661	2424	12165	-.0093				
2085	786	3081	6661	2333	12074	-.0075				
2090	786	3081	6661	2274	12016	-.0049				
2095	786	3081	6661	2246	11987	-.0024				
2100	786	3081	6661	2235	11977	-.0008				
2105	786	3081	6661	2234	11975	-.0002				
2110	786	3081	6661	2234	11974	.0000				
2115	786	3081	6661	2234	11974	.0000				

1) Source: CBS Population Statistics for 1905-1985
CBS (1986) for 1990-2035

A complete list of functional specifications and parameter values is given in Table 8.3.

The optimal economic growth path has been computed over the years 1825-2150. The first year of non-stationarity is 1905, when the cohort experiencing the first deviation from stable population during its one but last working period (1945-1950) leaves compulsory education. The first year of the final steady state is 2160 when the last generation which has to take the non-stability of the population into account when drawing up its optimal age-training profile (cohort 2100) has entered its last working period. Thus, while the birth growth rate is non-constant for only 65 years (1945-2010), the number of years for which this change affects the optimal economic growth path is about 250.

Figure 8.2 plots some key variables which determine the time path and form of the optimal age-training profile. They are the optimal factor prices (r and w) and the optimal physical/human capital ratio (k) measured in efficiency units (i.e. taking the rate of technical progress into account). The interest rate r moves, of course, parallel to the growth rate of population g^P. Although the relative factor price r/w changes quite a lot, this does not greatly affect the optimal capital/labour-ratio k; this is due to the low elasticity of substitution between physical and human capital.

Figure 8.3 gives the optimal age-training profile u(a) for selected cohorts, while Figure 8.4 gives the corresponding age-ability profile h(a). The numerical values for both profiles are quite realistic: optimal training involves full-time education until the age of 20/25, followed by on-the-job training with an intensity that decreases steadily until it becomes zero during the last period of activity in the labour force. The age-ability profile rises steeply from the beginning of voluntary education, reaches a peak around the age of 45/50 and then starts to fall.

Comparison of the profiles for the different cohorts confirms the conclusion that was reached in Chapter 6.1 on theoretical grounds, viz. that a lower growth rate of the population leads to a higher optimal training effort. The order of magnitude of this effect is moderately large: taking into account the size of the demographic change, the strength of the induced adjustment in optimal educational policy is significant although not spectacular. An individual spends 35% of his active life (15-65) training in the initial steady state versus almost 39% in the final steady state: this amounts to an increase of about 10%.

Table 8.3: Parameters used with model 7.1

Number of generations	22 (maximum age = 110 years)
Survival schedule	See Table 8.1
Social rate of impatience	$\rho=0.025$
Depreciation rates	$\delta_K=0.05$; $\delta_H=0.20$
Birth growth rate	See Table 8.2
Production function	$y=\{\alpha k^{-\frac{1-\sigma}{\sigma}} + (1-\alpha)\}^{-\frac{\sigma}{1-\sigma}}$
	$\alpha=0.10$
	$\sigma=0.167$
Educational prod. function	$E[\epsilon] = \gamma\cdot\epsilon^{\beta}$
	$\gamma=1.00$
	$\beta=0.40$

Figure 8.2: Case study with model 7.1 : factor prices and ratio

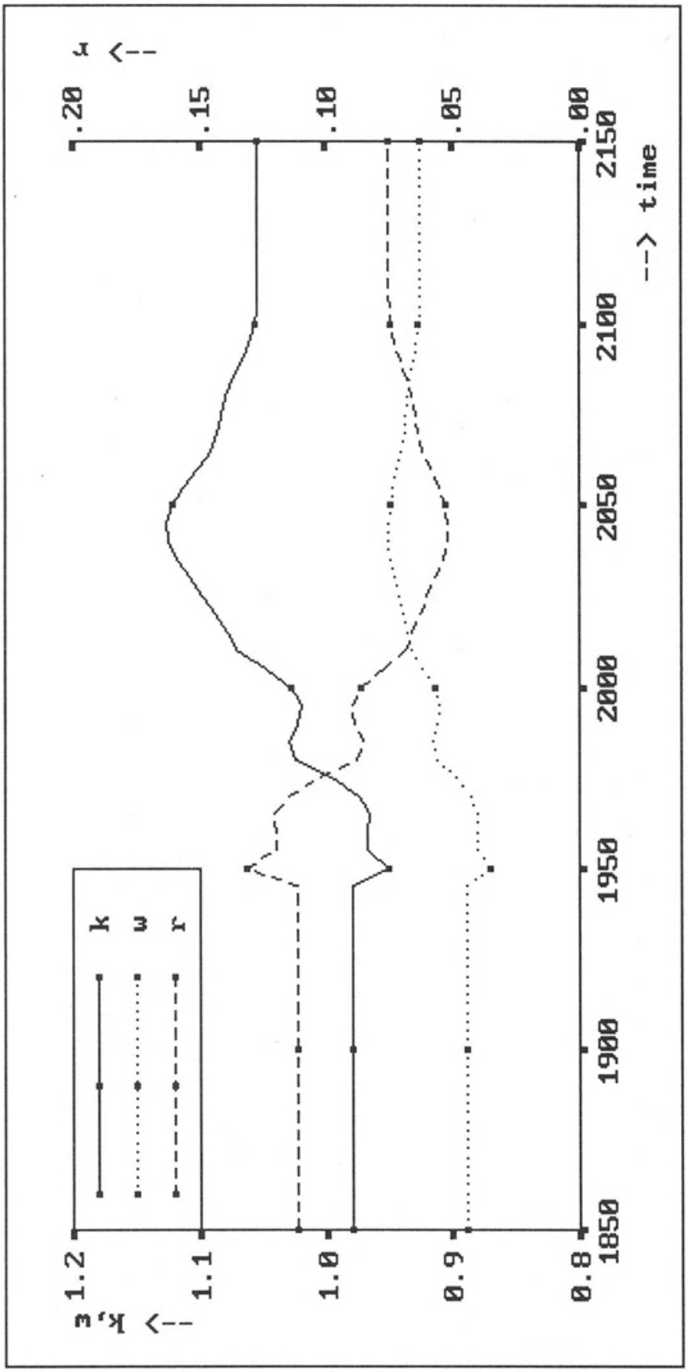


Figure 8.3: Case study with model 7.1 : age-training profiles

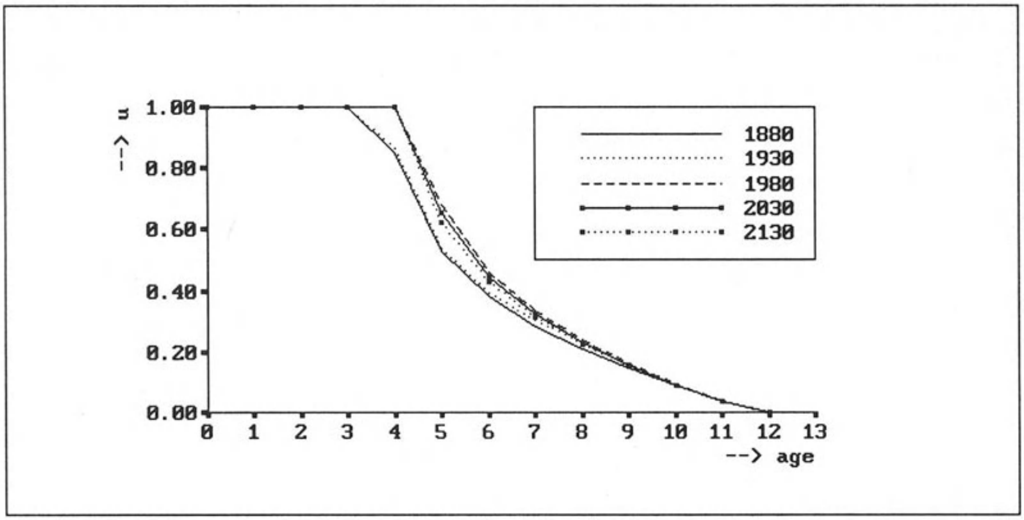
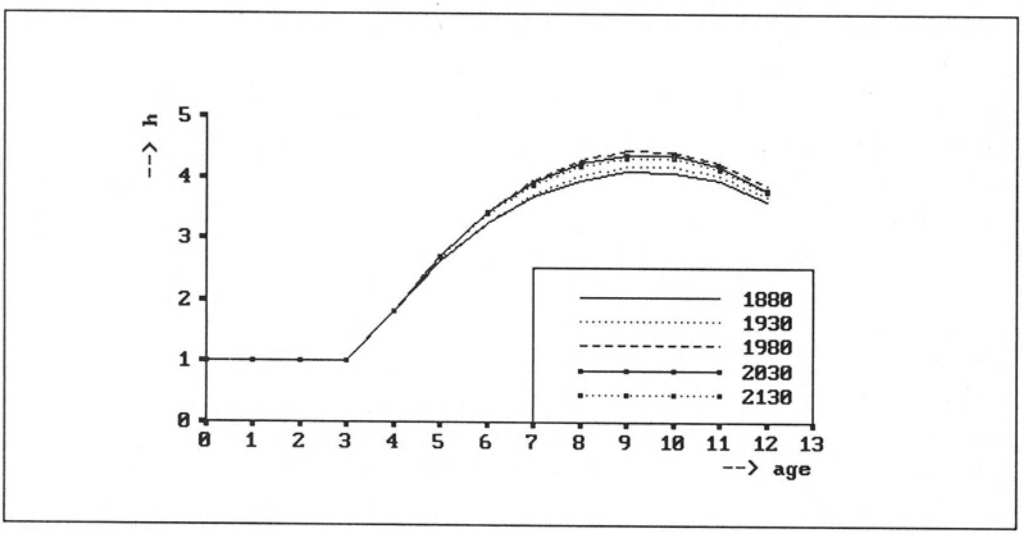


Figure 8.4: Case study with model 7.1 : age-ability profiles



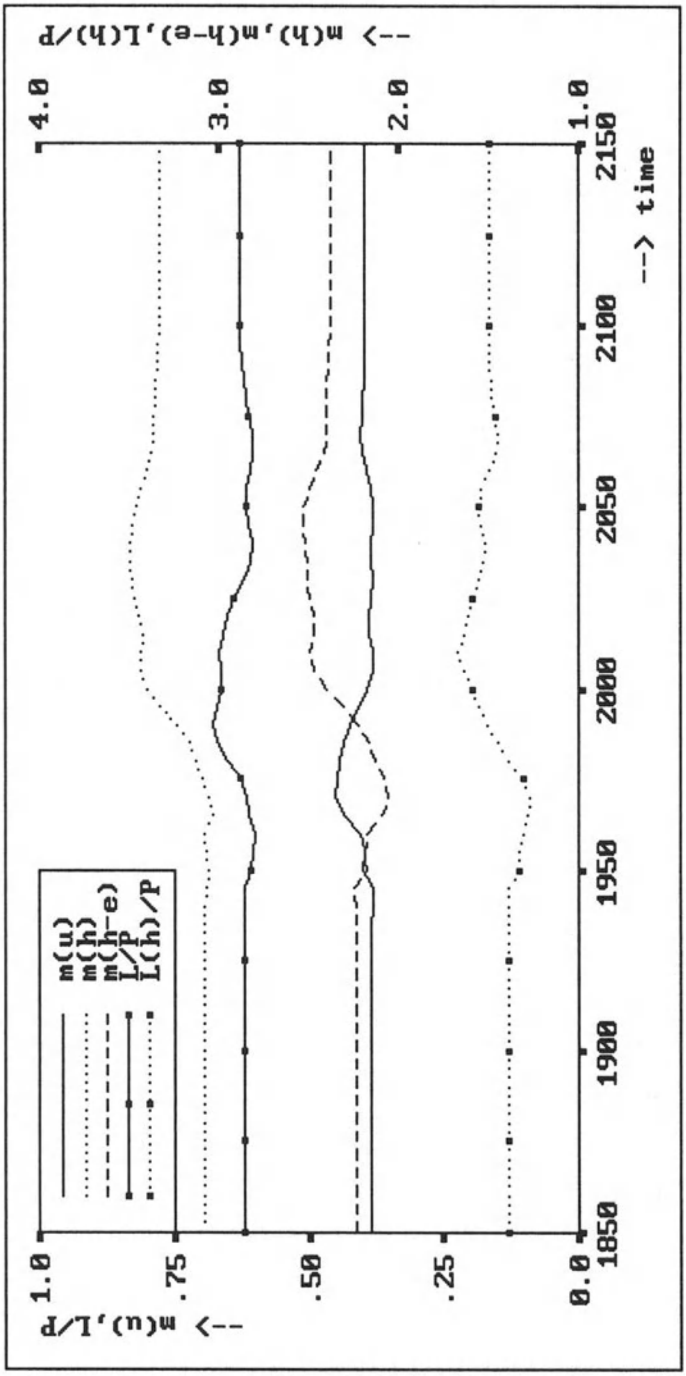
The optimal age-training profile is the highest for those cohorts that experience the strongest fall in the population growth rate during their working life (in Figure 8.3: cohort 1980). Thus optimal training effort does not increase monotonously over time but falls slightly near the end of the period of demographic transition. This seems to be in contradiction to the theoretical analysis in Chapter 6.1. There, however, attention was restricted to monotonous changes in the birth growth rate. In the Netherlands the "baby bust", which will cause the ageing of the population in the first half of the 21st century, followed the post World War II "baby boom" (cf. Figure 8.1). Such a "boom-bust" sequence causes the population growth rate to be <u>above</u> its initial steady state level at the beginning of the transition (around 1945) and <u>below</u> its final steady state level near the end of the transition (in fact for the full 21st century).

The increase in the optimal training effort is not restricted to the younger age groups but is also prevalent among the middle age groups in the labour force. This indicates that educational policy, in adjusting to the changed demographic circumstances, should not only pay attention to regular full-time education but also to part-time recurrent education, be it on the job or in the form of paid educational leave for the more mature workers.

It is hard to obtain a general picture of optimal educational policy during the full period of demographic transition from the detailed information of the age-training and age-ability profiles. Therefore I have calculated several variables which summarize the combined effects of continuous training adjustment on the one hand and continuous shifts in the age- and ability-composition of the labour force on the other hand. These variables, plotted in Figure 8.5, are the following:

- the share of the active population (15-65 years) in the total population, L/P. This is a purely demographic variable.
- the average training effort of the active population, m(u). This is a purely educational variable (although the weight attached to the various cohorts in the active population is of course demographically determined). An increase in average training effort indicates that the participation ratio of the active population falls, i.e. less time is spent on the production of the aggregate commodity and more time is spent on the production of human capital.
- the average amount of human capital in the active population, m(h). This is also a purely educational variable. It reflects the productivity of the

Figure 8.5: Case study with model 7.1 : human capital and labour force



potential labour force (active population) without recognizing that an increase in this productivity can only be obtained at the cost of a lower labour force (a higher proportion of the active population has to be engaged in training).

- the average amount of human capital in the labour force, $m(h-\epsilon)$. This educational variable gives the combined effect of changes in human capital per active person on the one hand and the changes in the participation rate per active person necessary to achieve that level of human capital on the other hand.
- the amount of human capital in the labour force per head in the population, $L(h)/P$. This variable summarizes the combined effects of demography, participation in the labour force, and educational achievement. For a given capital/labour-ratio k, it is directly proportional to output per capita.

Although the L/P ratio varies somewhat during the transition period, its final steady-state value (0.6263) is almost exactly equal to its initial steady state value (0.6213). The long-run fall in the population growth rate increases the share of the older cohorts (65+) but decreases the share of the younger cohorts (0-15) by the same amount, leaving the share of the potential labour force unchanged in the long run.

The optimal average training intensity of the active population, m(u), peaks during the second half of the 20th century. It is interesting to note that historically this peak coincides with the tremendous increase in enrollment rates. In the long run m(u) rises only negligibly (from 0.3856 to 0.3978). Here there are two offsetting forces at work: on the one hand the overall training intensity is increased (cf. Figure 8.3); on the other hand the weighting pattern of the various cohorts within the active population is shifted towards the older cohorts who have a lower training intensity (the optimal age-training profile is falling).

Because of this re-weighting effect, the positive influence of the fall in the population growth rate on optimal investment in education is illustrated by the variables m(h) and $m(h-\epsilon)$ rather than m(u). Both potential and effective human capital are concentrated amongst the more mature workers, who get a larger weight in the slower growing population. In combination with the increased general level of h and $(h-\epsilon)$ this leads to a significant increase in the average level of both variables.

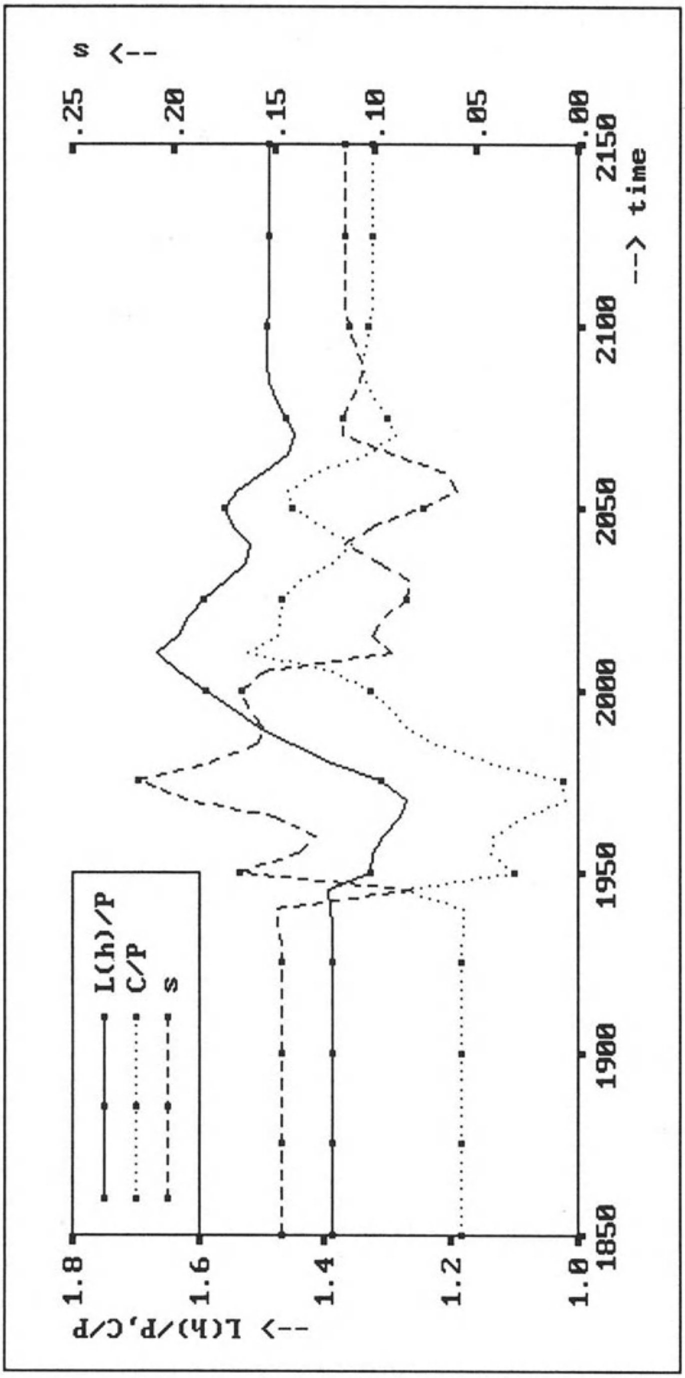
The ratio of effective human capital in the labour force and total population, L(h)/P, is the product of $m(h-\epsilon)$ and L/P. It winds its way through the demographic transition, first falling, then rising and finally falling again; in the long run the net result is a slight increase (from 1.3879 to 1.4896).

The effect of the demographic change on consumption per capita is the net result of three separate developments:

- the change in the physical/human capital ratio, k, and consequently in output per unit of human capital, f[k]. This effect on consumption is slightly positive in the long run (cf. Figure 8.1).
- the change in human capital per consumer, L(h)/P, which is also positive.
- the change in the optimal savings rate, s. As a result of the irregular pattern in the growth rate of population (due to the "boom-bust" sequence) and the thereby induced adjustments in educational policy, the savings rate follows a wildly irregular pattern before finally settling down at its terminal steady-state level which is quite below its initial level (0.1461 versus 0.1154). In the beginning of the next century there is a sharp drop in the optimal savings rate. This reduction in savings could well be realized without any active savings policy, as this is the period in which, due to the retirement of the baby-boom cohorts, the present large financial surpluses of the pension funds in the Netherlands will be greatly reduced.

These latter variables are plotted in Figure 8.6. As can be seen from this figure, consumption per capita itself has its ups and downs, but is significantly higher in the final steady state than in the initial steady state (1.3239 versus 1.1826). Apart from this, and not shown in the figures, there is a secular increase in the efficiency of human capital, production per capita and consumption due to the existence of technical progress at a rate of 6% per 5 years. With the exception of the years 1945-1950 this upward trend is everywhere strong enough to more than offset the downward sloping segments in the plot of C/P.

Figure 8.6: Case study with model 7.1 : savings rate and consumption



8.3 Simulations for the model with heterogeneous human capital

The model investigated in this section is based on the model with two types of labour described and analysed in Chapter 6.2. The following extensions, similar to those of the previous section, have been made:

- individuals follow compulsory education during the first three periods of their life (15 years). At the age of 15 they can either enter the labour force as unskilled workers or continue their education for another three periods and at the age of 30 enter the labour force as skilled workers. The educational policy variable h(t) is the proportion of the 15-year-old individuals who continue their educational career.
- there is mandatory retirement at the age of 65 years (age group 13).
- the aggregate production function is of the nested CES type (cf. the empirical work in (Ritzen, 1986c)). At the lower level of the nesting physical capital (K) and unskilled labour (L_1) are combined to produce an intermediate factor of production (H); at the higher level of the nesting this factor H is combined with skilled labour (L_2) to produce the aggregate commodity. The elasticity of substitution between K and L_1 is relatively high (0.5), between H and L_2 relatively low (0.1).

A complete list of functional specifications and parameter values is given in Table 8.4. Apart from some scaling parameters, the numerical specification of the production function follows the empirical estimates in Ritzen (1986c, p. 85).

The optimal economic growth path for the present model, as was discussed in Chapter 6.2, is unstable in two respects. First, given a converging time path for $g^P(\cdot)$, the solution $\{\ell(\cdot), k(\cdot)\}$ to the system of differential-difference equations implied by the two Non-Stationary Golden Rules (6.2.35) and (6.2.36) is diverging. Second, given a converging time path for $g^P(\cdot)$ and $\ell(\cdot)$, the trajectory of the control $h(\cdot)$ required to realize $\ell(\cdot)$ is diverging. Thus, singular control breaks down once population becomes non-stable.

Abandoning the search for an optimal way of controlling an economy described by this model, I will instead compare several alternative (non-optimal) policies. Of the numerous possibilities I have selected the following:

Table 8.4: Parameters used with model 6.2

Number of generations	22 (maximum age = 110 years)
Survival schedule	See Table 8.1
Social rate of impatience	$\rho=0.025$
Depreciation rate	$\delta=0.05$
Birth growth rate	See Table 8.2
Production function	$y=\gamma_Y\cdot\{\alpha_2 + \alpha_H\cdot H^{-\rho_Y}\}^{-1/\rho_Y}$
	$H=\gamma_H\cdot\{\beta_1\cdot\ell^{-\rho_H} + \beta_K\cdot k^{-\rho_H}\}^{-1/\rho_H}$
	$\rho_Y = 9.60$ ($= (1-\sigma_Y)/\sigma_Y$)
	$\rho_H = 1.05$ ($= (1-\sigma_H)/\sigma_H$)
	$\alpha_2 = 80.00$
	$\alpha_H = 0.01$
	$\beta_1 = 2.00$
	$\beta_K = 1.60$
	$\gamma_Y = 1.00$
	$\gamma_H = 4.00$

1. "myopic control": in each period the controls h and s are set equal to their optimal values in the steady state characterized by that period's growth rate of population.
2. "smooth control": the controls follow a linear trajectory along the full period of demographic transition, gradually moving from their initial steady-state levels to their final steady-state levels.
3. "steady-state control": the controls keep their initial optimal steady-state levels until the birth growth rate reaches its new constant level. From that period onwards the controls are set equal to their new steady-state levels.
4. "constant control": the controls remain at their initial steady-state levels forever.

All these four policies will eventually move the economy to a new steady state. Moreover, with the exception of the "constant control"-policy, this new steady state will be the optimal one.

The policies mentioned thus far have in common that the controls are determined independently of the values of the state variables k and ℓ. However, the singularity conditions (Golden Rules) are in terms of these state variables. Therefore I have also analysed two additional policies that aim at achieving some given time path of the factor ratios k and ℓ. These policies are:

5. "myopic state control": in each period the controls h and s are such that the state variables are as close as possible to their optimal values in the steady state characterized by that period's population growth rate.
6. "smooth state control": the controls are such that the state variables follow as closely as possible a linear trajectory, gradually leading them from their initial steady-state values to their final steady-state levels.

The phrase "as close as possible" is included because, due to the space constraints on the controls and to the non-convergence of the control trajectories given converging state trajectories, complete equality of actual and target state values in every period is not feasible. If in period t the target values can be achieved, they are achieved; if not, the controls are set to their maximum or minimum value, whichever is the relevant limit.

Unlike the Policies 1 to 4, Policies 5 and 6 do not lead the economy to a final steady state.

For each of these six policies the corresponding economic growth path has been computed over the time interval 1825-2320 (one hundred time periods). The key variables are plotted in Figures 8.7 to 8.11.

The optimal enrollment rate in the final steady state is only negligibly higher than it is in the initial steady state (0.8269 versus 0.8253). The drop in the long-run optimal unskilled/skilled labour ratio ℓ (from 0.3248 to 0.3032) is almost completely realized through the changed age-composition of the active population. Consequently, the differences between Policies 1 to 4 are restricted to physical variables only (s and k). These differences are relatively small in terms of consumption per capita, due to the very low elasticity of substitution between skilled labour and physical capital. The only exception is Policy 4 (constant control): as this latter policy leads the economy to the "wrong" steady state, in the long run it is clearly inferior to Policies 1 to 3 (cf. Table 8.5 below).

Policies 5 and 6, which run in terms of the state variables instead of the control variables, exhibit a growth path which is totally different from those for Policies 1 to 4. The corresponding time paths of the controls are wildly oscillatory at a constantly increasing amplitude. The occasions in which the boundaries of the control region are reached become more and more frequent; this leads to temporary deviations of the state variables from their target values, as can be seen very clearly in Figures 8.9 and 8.10.

Table 8.5 summarizes the performance of the different policies in terms of the social welfare function which is discounted consumption per capita. The major impression from this table is that the long-run social welfare implications of the various policies are quite small, the difference between the highest and lowest value for the social welfare function being about 0.5%. The small size of these differences is caused by the fact that the specification of the aggregate production function offers very little scope for factor substitution.

A striking conclusion from this table is the surprisingly high values for Policies 5 and 6. Apparently, the optimality of the non-stationary Golden Rule path (which runs in terms of the state variables) is so strong that state-variable policies, notwithstanding their instability, yield higher values for the social welfare function than the control-variable policies.

However, the stability of the growth path is an important additional objective of economic policy. This aspect has not been incorporated into the social welfare function; but given the choice between a non-stable path with high consumption and a stable path with only slightly lower consumption the

Figure 8.7: Case studies with model 6.2 : the enrollment rate

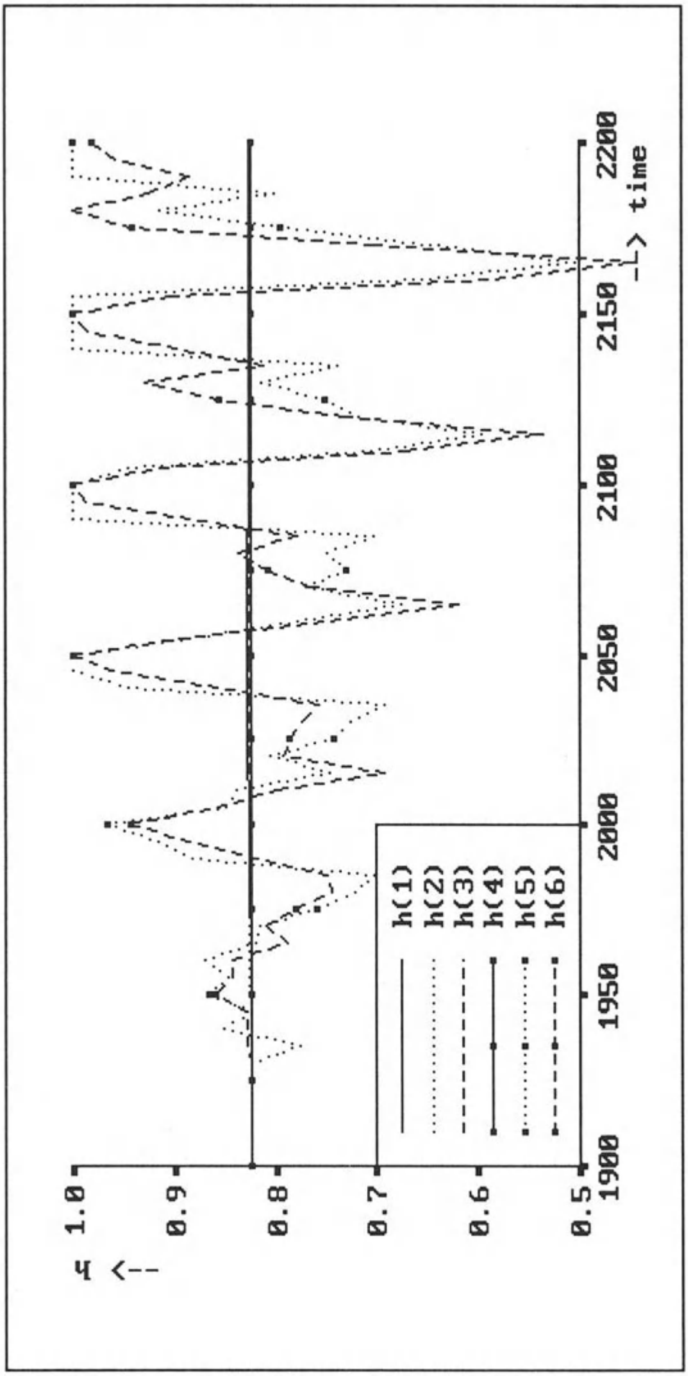


Figure 8.8: Case studies with model 6.2 : the savings rate

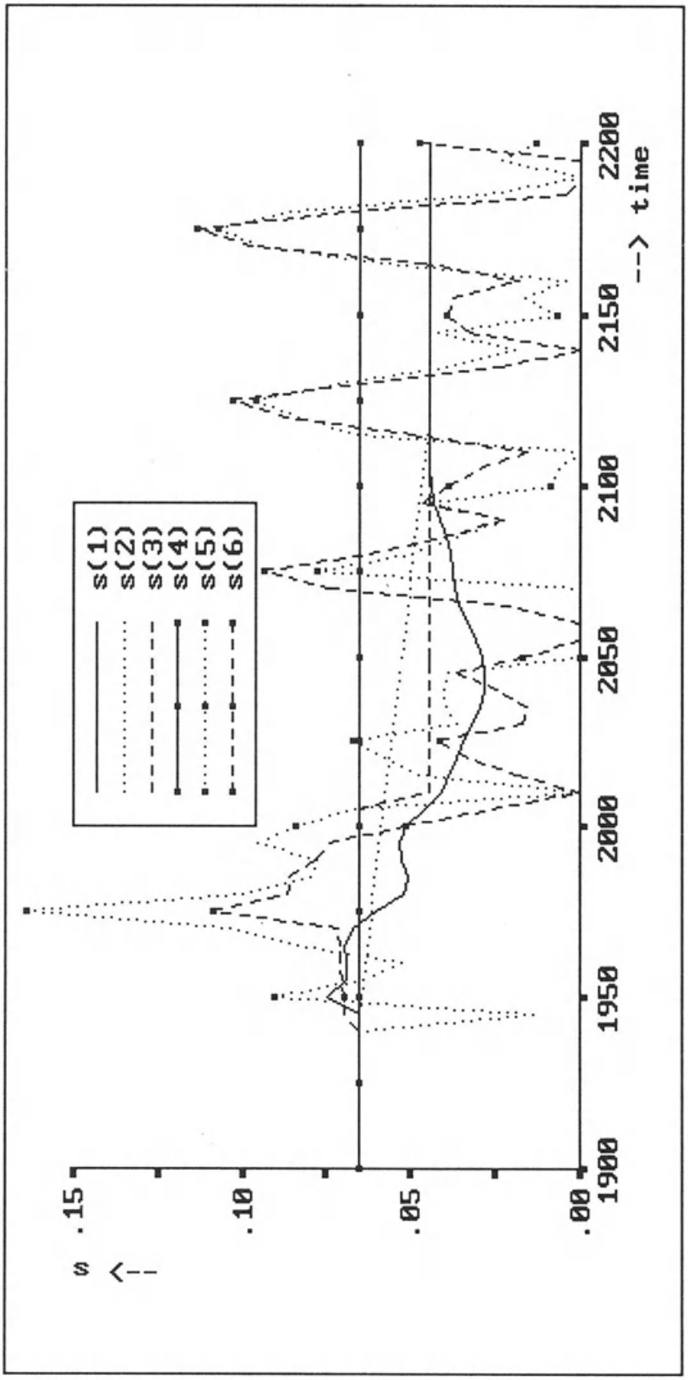


Figure 8.9: Case studies with model 6.2 : the unskilled/skilled labour ratio

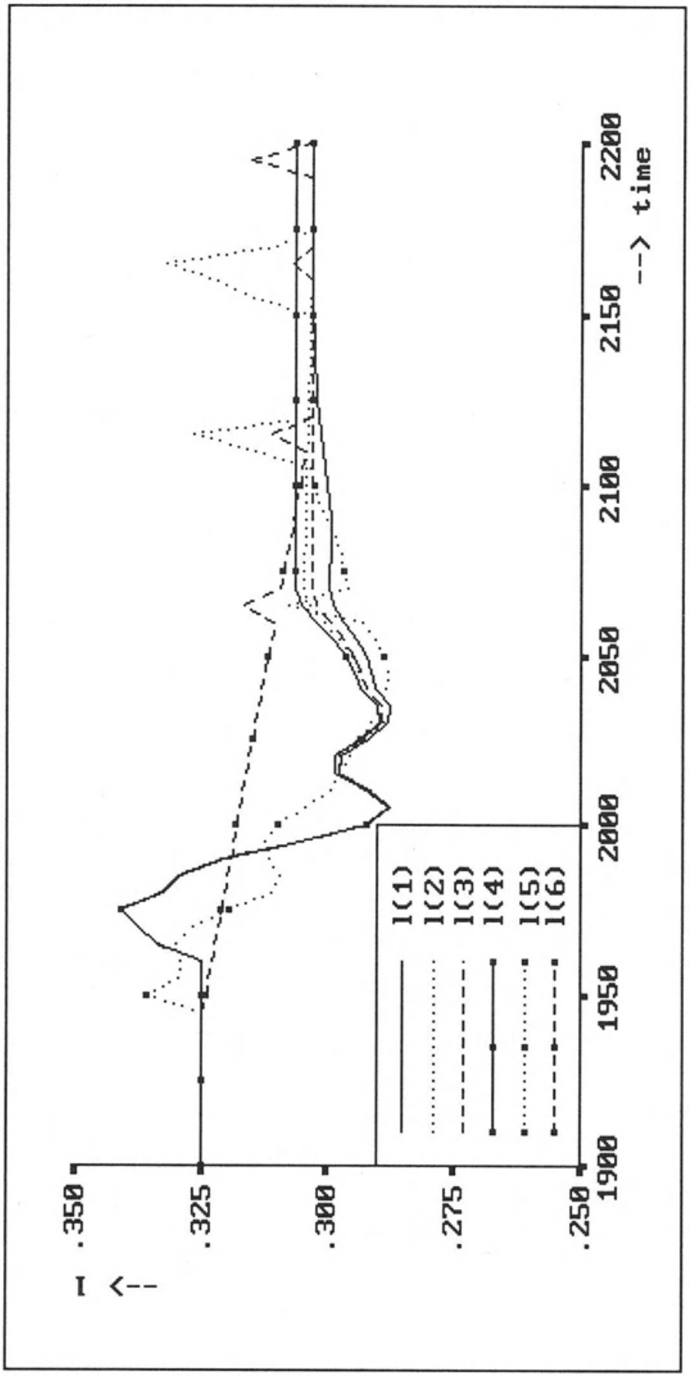


Figure 8.10: Case studies with model 6.2 : the capital/skilled-labour ratio

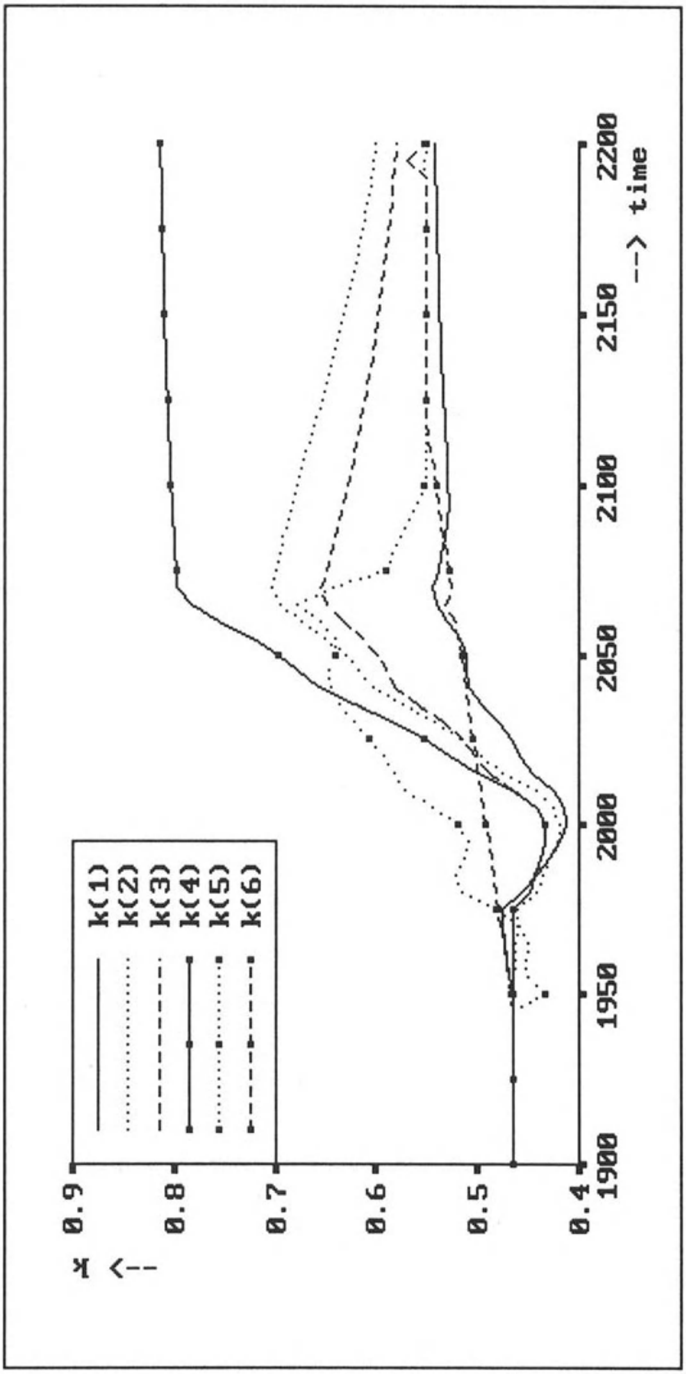


Figure 8.11: Case studies with model 6.2 : consumption per capita

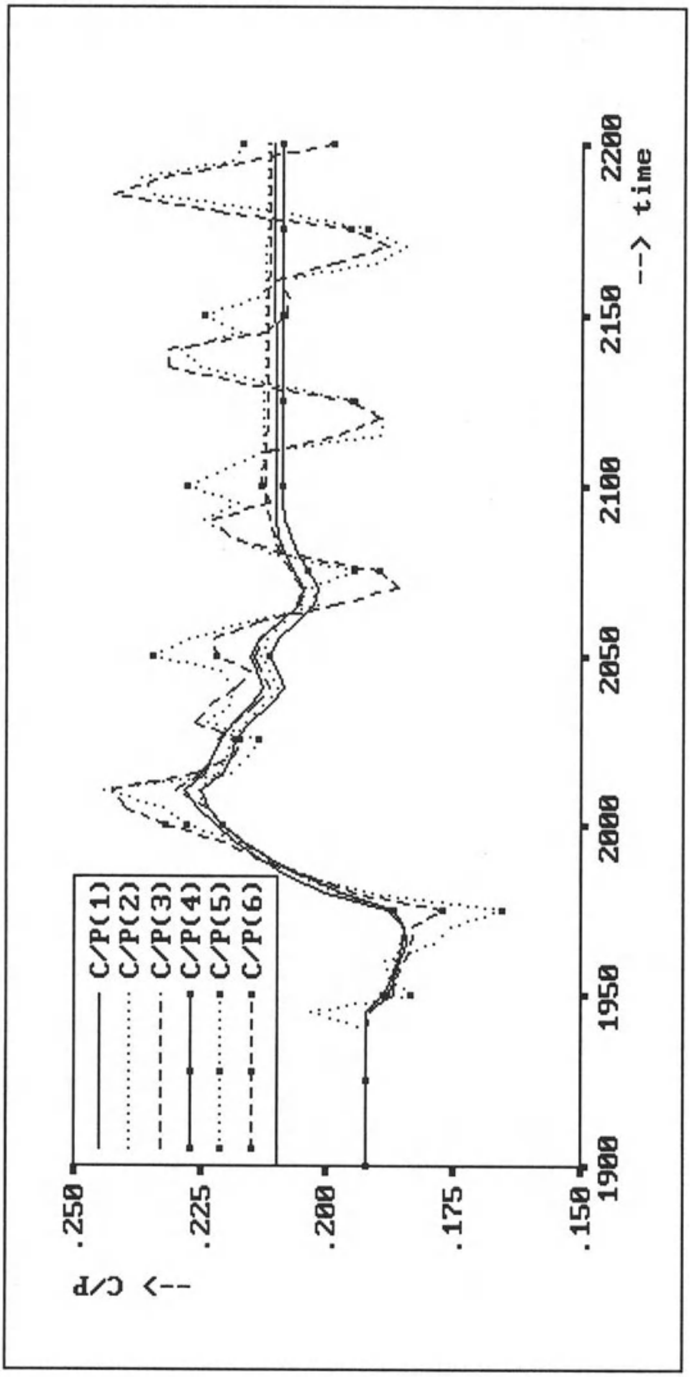


Table 8.5: Values of the social welfare function

policy	social welfare function 1825 - 2320
1. myopic control	7.5175
2. smooth control	7.5173
3. steady-state control	7.5222
4. constant control	7.4845
5. myopic state control	7.5264
6. smooth state control	7.5206

latter alternative seems much more attractive than the former. Taking this consideration into account, the values in Table 5 point towards Policy 3, "steady-state control" as the most preferable one. Discounted consumption per capita is almost as high as for the best non-converging policy (Policy 5) while it does not suffer from unreasonably large and frequent cycles in economic development, even during the period of strong demographic transition.

9 SUMMARY AND EVALUATION

In the 1980s the Netherlands (as well as many other, mostly industrialized countries) can be considered to be in the middle of a transition phase between two situations of (more or less) constant population growth. Since the end of the 1960s the annual number of children born has decreased dramatically. Today's population projections indicate that the number of births will remain approximately constant until about 1995 and will decrease even further in the years after.

This decline in the number of births causes the population to age, and this in two respects. First, when the growth rate of the number of births is constant for a period long enough for the population to be stable (i.e. to have a constant age-structure), the proportion of the elderly in the total population is permanently higher than before the start of the fertility decline. This is the long-run ageing effect. Second, during the transition phase there is a period in which the elderly stem from higher growth-rate cohorts than the younger generations, rendering the proportion of the elderly in the total population higher than it is in the final stable population. This is the transitory ageing effect. The transitory ageing effect of the fertility decline is, of course, larger than the long-run ageing effect.

Traditional growth theory, with its emphasis on the steady state, is not capable of handling such dramatic deviations from exponential population growth. Understanding the economic consequences of demographic change requires the construction of growth models that explicitly recognize demographic forces as an important potential source of non-stationarities in economic development. It is precisely such a demographic extension of traditional growth theory that has been attempted in this book.

In the preceding chapters I have studied optimal economic growth in a closed economy which experiences exogenous but non-stable population growth. The findings of this study will be summarized in Section 9.1. In Section 9.2 I will evaluate the advantages and limitations of the approach pursued in this book.

9.1 Summary

The closed economy has been described by means of an aggregative neoclassical growth model which distinguishes overlapping generations within the population. The basic neoclassical model has been extended to include technical change, as well as investment in education (human capital). In tracing the effects of demographic change on the optimal economic growth path, attention has not been restricted to steady states (comparative statics): the period of demographic transtition has also been explicitly analysed.

Throughout the book the social welfare function to be maximized has been taken to be consumption per capita, discounted over an infinitely long time period. Consequently, the Hamiltonian of the corresponding optimal-control problem becomes linear in the savings rate. Thus, the equilibrium of the optimal trajectory is a singular solution. When the growth rate of population changes this singular equilibrium will move. The optimal policy is to determine the control variables in such a way that the economy remains in the singular equilibrium even when it moves. This result justifies the book's concentration on singular trajectories.

In Chapter 2 I have studied the basic neoclassical one-sector growth model of Solow (1956). It has been shown that the optimal growth path (singular trajectory) is characterized by a strikingly straightforward generalization of the traditional steady-state Golden Rule of Capital Accumulation. The comparative statics results allow for a generalization of Samuelson's (1975a) analysis of the lower-bound for the optimal rate of population growth. The optimal savings rate in steady state varies positively with the long-run population growth rate. If certain plausible restrictions concerning the aggregate production function are satisfied, the optimal savings rate follows either an inverted U-shaped or a U-shaped pattern during the transition phase, depending on whether the birth growth rate is falling or rising.

Chapters 3 and 4 introduce technical change, which explains the secular rise in consumption and output per head. Chapter 3 offers an extensive discussion on the concept of technical change and on the various forms in which it can be modelled. Of all variables affecting a firm's maximum output, a change in those variables of which the value cannot be changed in the short run by external economic agents in such a direction that the change would damage the firm's performance, is defined as technical change. On the basis of this definition it is argued that a change in "knowledge" is technical change but a change in "human capital" (through education) is not. Three major classifications of technical change are analysed: exogenous vs. endogenous, neutral vs. biased, and embodied vs. disembodied technical change.

Chapter 4 describes and analyses one-sector models with three types of technical change: exogenous disembodied (4.1), exogenous capital-embodied (4.2), and endogenous disembodied (4.3). In each of these models the Generalized Golden Rule is exactly the same as for the model without technical change. Thus, contrary to what many authors suggest, technical change does not affect the Golden Rule. Apart from the secular rise in output and consumption per capita, the presence of technical change does not fundamentally alter the properties of the optimal economic growth path.

Chapters 5 and 6 introduce education, or investment in human capital. Education is very closely linked to the age-structure of the population since the bulk of education is imparted to the young. Also, the lifetime of the society's stock of human capital is directly related to the age-composition of the labour force, due to the simple fact that by the laws of nature human capital is embodied in people.

Chapter 5 discusses some issues in modelling investment in human capital. Education is quite a complex phenomenon for which no unambiguous way of modelling is available. Two of many possible alternative models have been studied in Chapter 6: a model with homogeneous human capital (6.1), and a model with heterogeneous human capital, leading to two types of labour (6.2).

The optimal growth paths for these models including education are characterized by two groups of optimality conditions: the Generalized Golden Rule for optimal investment in physical capital; and a continuum of Golden Rules of Education for optimal investment in human capital, corresponding to the continuum of active generations. These conditions for optimal training effort can be interpreted in terms of the familiar equality of discounted costs and returns.

For the model with homogeneous human capital it has been shown that the optimal age-training profile is non-increasing, whatever the demographic situation. When the growth rate of population falls, investment in education becomes more attractive. More generally, optimal training effort moves in the opposite direction of the growth rate of population along almost the full non-stationary economic growth path. This is the central conclusion for economic policy that follows from the theoretical analysis in this book.

Although the model with heterogeneous human capital has many properties in common with the homogeneous human-capital model, it suffers from the major drawback that it is dynamically unstable. The presence of time lags in the production of human capital (skilled labour) is largely but not completely responsible for this result. It is not completely so, because the model with capital-embodied technical change (Chapter 4.2) is also characterized by time lags but at least its optimal steady state is stable, although the oscillations in the non-stationary optimal growth path are severe and persistent.

Chapter 7 combines education and technical progress. The model of Chapter 7.1 is a straightforward extension of Chapter 6.1 to include the case of exogenous disembodied technical change. Chapter 7.2 describes and analyses a model in which technical progress is embodied in labour, i.e. it increases the productivity of labour only to the extent that labour has been trained to produce according to the most up-to-date technology. For both models optimal training effort varies positively according to the rate of technical progress and negatively according to the growth rate of population. That is, technical progress as well as ageing of the population render education more attractive. In addition, the model with labour-embodied technical change illustrates the negative impact of slower population growth on the rate of technology adaptation. When the growth rate of population is high, the introduction of technological innovations into the production process is primarily achieved through the constant influx of recently educated young people. When the relative share of this influx is reduced, increased education for adults becomes necessary in order to prevent the gap between technology in theory and technology in practice becoming too large.

Chapter 8 provides a rather detailed application of the theoretical models to real-life problems. The consequences of the present fertility decline have been analysed in the case of the Netherlands, in order to get some insight into the order of magnitude of the policy adjustments involved. For this case study it has been assumed that the number of births will remain constant (zero growth) from the year 2010 onward.

The strength of the demographically induced long-run adjustment in optimal educational policy is significant: the optimal proportion of an individual's active lifetime (15-65 years) spent training increases by about 10%. The increase in the optimal training effort is not restriced to the younger age groups but is equally prevalent among the middle age groups in the labour force. This indicates that educational policy, in adjusting to the changed demographic circumstances, should not only pay attention to regular full-time education but also to part-time recurrent education, be it on the job or in the form of paid educational leave for the more mature workers.

The optimal training intensity for successive generations peaks during the second half of the 20th century. It is interesting to note that historically this peak coincides with the tremendous increase in enrollment rates. In the long run optimal average training intensity rises only marginally. Here, there are two offsetting forces at work: on the one hand, the overall training intensity is increased; on the other hand, the weighting pattern of the various cohorts within the active population is shifted towards the older cohorts who have a lower training intensity (the optimal age-training profile is falling).

As a result of the irregular pattern in the growth rate of population (due to the "boom-bust" sequence) and the thereby induced adjustments in educational policy, the savings rate follows a wildly irregular pattern before finally settling down at its terminal steady-state level which is quite below its initial level. In the beginning of the next century there is a sharp drop in the optimal savings rate. This reduction in savings could well be realized without any active savings policy, as this is the period in which, due to the retirement of the baby-boom cohorts, the present large financial surpluses of the pension funds in the Netherlands will be greatly reduced.

9.2 Evaluation

What does the analysis in this book teach us in terms of general insights into the relationship between demographic phenomena and long-run economic development? In my view there are three major conclusions to be drawn:

1. If economic policy is adjusted to the changing demographic situation, fertility decline is beneficial in the long run - provided that the long-run population growth rate does not fall below a certain (but probably negative) critical lower bound. Even during the transition phase, when the ageing is more severe than ever, optimal consumption per capita, in general, does not fall below its initial steady-state level. On the contrary: during the full period of demographic transition it rises more or less monotonously towards its new, higher equilibrium value.

   This should be a comforting thought for those who worry about the ageing problem. The negative welfare effects on the generations bearing the heaviest burdens during the peak of the ageing phase are more than, or at worst: almost completely, offset by the positive effects of, first, a lower number of dependent children, and second, a higher efficiency of the, by then, relatively smaller labour force. Even when the net welfare effects turn out to be negative for some cohorts, a lot of consolation can be derived from the observation that all subsequent generations will benefit from the permanently higher standard of living in the periods after the demographic transition.

The conclusion on the beneficial effect of a fertility decline is in line with the classic results of Malthus, and more recently of the Club of Rome, although the analysis on which the conclusion is based is quite different. Simon (1986) takes a radically different stand by positing a positive (and causal) relationship between the rates of population growth and technical progress. His conclusion is that a fertility increase is beneficial in the long run. Of course, nothing in my own analysis precludes the existence of such a positive causal link.

2. Human capital is a good substitute for population growth. The lower the growth rate of population, the higher are the returns on investment in education. That is, while a fertility decline leads to a relative scarcity of human as compared to physical capital, an increase in overall training effort helps to reduce this scarcity while at the same time leaving everyone better off. This negative relationship between population growth and optimal investment in education reinforces the beneficial effect of fertility decline discussed above.
3. A realistic growth model is very difficult to formulate and/or to analyse. This is especially true if one wants to allow for demographic change and as a consequence has to open the door to all kinds of non-stationarities. Even with the very simple specifications used throughout this study (e.g. closed economy; one sector; homogeneous human capital; exogenous disembodied Harrod-neutral technical progress; etc.), the analysis easily gets quite complicated indeed. Given the effort required in carrying out the theoretical analysis, the returns in terms of general policy guidelines are rather low. These observations suggest that the relevance of the present study is concentrated in the theoretical insights that it provides in the properties of the specific growth models analysed, rather than in its power to offer general insight into the relationship between demographic change and economic development.

The analytical power of any model is inherently limited by the restrictiveness of the assumptions that underlie the model in question. The following are among the most important limitations of the approach adopted in this book:

1. In this study I have made several heroic assumptions concerning the Central Planning Agency's information on future demographic developments. The optimal control problem has invariably been formulated as being completely deterministic. If population is completely exogenous then clearly it is very unrealistic to assume that its future course can be perfectly predicted. Such predictions would be somewhat easier to make if population growth were endogenous, i.e. dependent on economic variables. However, the long-run causal relationship between population growth on the one hand and the economic circumstances on the other, is very fragile, not only in strength but even in the direction of the causal link (if any). This was so in Malthus' days (cf. the discussion in (Siegers, 1987)) and two centuries of scientific research have not yielded much progress in settling the issue.
2. The growth-theoretical analysis of demographic transitions puts one's faith in growth theory itself very much to the test. Because of the length of human life, the duration of a transition period is very long indeed, at least 100 years. This raises the question whether it is at all possible to describe an economic system for so long a period by means of such simple and rigid functional relationships as growth models typically assume.
3. The form of the social welfare function adopted throughout this study, being linear in average consumption per capita, is quite restrictive. There are several possible directions in which this functional specification can be generalized:
   - a nonlinear trade-off between consumption levels at different points in time by introducing an instantaneous utility function. Although such an extension could be useful it probably would not affect the results very fundamentally, given the absence of large fluctuations over time in consumption per capita.
   - the incorporation into the social welfare function of intergenerational inequality, which could be achieved by making social welfare a function of the various cohorts' life-time consumption. This procedure opens the possibility of compensating generations for a drop in consumption per capita by raising the standard of living at other stages of their life-cycle. A first step towards such an analysis was made by Bovenberg & Ritzen (1985).
   - the introduction of a penalty term into the social welfare function, reflecting the undesirability of strong adjustments in economic policy.

The incorporation of such adjustment costs (cf. Bovenberg, 1985 and 1986) would be especially relevant to those models where the demographically induced fluctuations in the optimal control variables are large, like the models with time lags analysed in Chapters 4.2 and 6.2. It might well be the case that such an extension would eliminate the non-convergence of the optimal economic growth path in the model of Chapter 6.2.

4. Nothing has been said about the feasibility of the optimal growth path. If the economy is characterized by competitive markets and decentralized decision-making, then the Central Planning Agency may not have the instruments to directly control the quantity variables. It might be able to control the prices by means of taxation and in such a way indirectly control the economy as a whole, but this would yield a second-best optimum only.
5. Finally, the analysis has been confined to a closed economy. Extending the analysis to a multi-country model is especially fruitful if demographic developments differ across countries, as they do in reality. Such a study would be able to investigate the opportunities for international cooperation in exploiting the advantages and fighting the disadvantages of fertility decline, for instance by having one country's "ageing deficit" financed by those countries which are not (yet) affected by the fertility decline.

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