Yuriy Povstenko

Linear Fractional Diffusion-Wave Equation for Scientists and Engineers

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Yuriy Povstenko
Jan Długosz University in Częstochowa
Częstochowa, Poland

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Preface

What we know is a drop, what we don't know is an ocean.

Isaac Newton

The diffusion-wave equation is a mathematical model of important physical phenomena ranging from amorphous, colloid, glassy and porous materials through fractals, percolation clusters, random and disordered media to comb structures, dielectrics, semiconductors, polymers and biological systems.

Currently, there is no publishing treatise devoted to systematic presentations of solutions to the diffusion-wave equation, except some papers that are scattered in the literature. This book, which for the major part is based on author's investigations, fills in such a blank. The book, which aims at presenting a comprehensive view of the state-of-the-art, is organized as follows. In Introduction (Chapter 1) we show the place and importance of fractional partial differential equations, especially the time-fractional diffusion-wave equation, in describing different physical phenomena. As the integral transform technique is used throughout the book, the material needed for application of this technique is presented in Chapter 2, where we describe the integral transforms used for solving the considered problems and discuss the properties of the Mittag-Leffler, Wright, and Mainardi functions appearing in the solutions. In Chapter 3 we consider the time-nonlocal generalizations of classical Fourier's, Fick's and Darcy's laws. We show that nonlocal dependence between the heat flux and the temperature gradient with a "long-tail" power kernel results in the time-fractional diffusion-wave equation with the Caputo fractional derivative. Different kinds of boundary conditions for this equation are discussed (Dirichlet, Neumann, Robin, perfect thermal contact). Chapter 4, Chapter 5 and Chapter 6 are devoted to the solutions of equations with one space variable in Cartesian coordinates, in the case of axial and central symmetry, respectively. Equations with two space variables are studied in Chapter 7, Chapter 8 and Chapter 9 in Cartesian, polar and cylindrical coordinate systems, respectively. In Chapter 10, Chapter 11 and Chapter 12 the solutions to equations with three space variables are obtained in Cartesian, cylindrical and spherical coordinates.

In the Conclusions chapter we summarize the obtained results and briefly discuss general properties of the solutions. For convenience of the reader, the Appendix contains integrals used to obtain different solutions presented in the book.

The list of References cannot be considered as a compete bibliography, and the interested reader is referred to bibliographies mentioned in the books in the bibliography, where additional references can be found.

The corresponding sections of the book may be used by university lecturers of courses in fractional calculus, heat and mass transfer, transport processes in porous media and fractals for graduate and postgraduate students. The book can also serve as a reference textbook for specialists in applied mathematics, physics, geophysics and engineering science.

Yuriy Povstenko
Częstochowa, 2015

Chapter 1

Introduction

The world-known specialist on parabolic equations Professor Marian Krzyżański should present a talk at "Professor Seminar" at the Jagello University of Cracaw.

The well-known specialist on elliptic equations Professor Frantiszek Leja came to a lecture hall and asked "What will be your talk about?"

"About parabolic equations," the speaker answered.

"It is not interesting," claimed the Dean Leja and went out of the auditorium.

I laughed loudly and, of course, was asked why? "In fact, only hyperbolic equations are of great interest."

Tadeusz Trajdos

Partial differential equations arise in various fields of science. Today, the literature on these equations is unbounded. Usually partial differential equations are divided into three basic types – elliptic, parabolic and hyperbolic. The simplest example of an elliptic equation is the Laplace equation, but we begin our consideration from another representative of this type equations – the Helmholtz equation

$$T = a\Delta T. \tag{1.1}$$

The simplest example of a parabolic equation is the diffusion equation (heat conduction equation)

$$\frac{\partial T}{\partial t} = a\Delta T. \tag{1.2}$$

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The well-known example of a hyperbolic equation is the wave equation

$$\frac{\partial^2 T}{\partial t^2} = a\Delta T. \tag{1.3}$$

It should be emphasized that the solutions of the equations belonging to each of the above-mentioned types have their own characteristic features. For example, dissipation is common to parabolic equations, wave fronts and finite speed of propagation are specific for hyperbolic equations. There are many excellent books and textbooks devoted to classical partial differential equations, some of them ([2, 39, 63, 74, 75, 216, 219]) are quoted in the References.

Elliptic partial differential equations are studied in [50, 64, 65, 88], among others. Solutions to the parabolic heat conduction equation in various spatial domains are presented in the books [26, 31, 98, 140]. The books [5, 16, 86, 199] are devoted to investigation of hyperbolic partial differential equations. The fundamental solutions to standard partial differential equations were considered in [21, 38, 85]. Many solutions have also been given in collections of problems [20, 87, 211].

In the last few decades, considerable research efforts have been expended to study fractional differential and integral equations – equations with operators of differentiation and integration of fractional (not integer) order [36, 56, 77, 82, 99, 118, 132, 133, 143, 202, 206].

The notion of the Riemann–Liouville fractional integral is introduced as a natural generalization of the repeated integral written in a convolution type form:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad \alpha > 0, \tag{1.4}$$

where $\Gamma(\alpha)$ is the gamma function.

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^α:

$$D^\alpha_{RL} f(t) = D^m I^{m-\alpha} f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\,\mathrm{d}\tau\right], \quad m-1 < \alpha < m. \tag{1.5}$$

The Caputo fractional derivative is defined as

$$\begin{aligned} D^\alpha_C f(t) &\equiv \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = I^{m-\alpha} D^m f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\mathrm{d}^m f(\tau)}{\mathrm{d}\tau^m}\,\mathrm{d}\tau, \quad m-1 < \alpha < m. \end{aligned} \tag{1.6}$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions [57]. The major utility of the Caputo fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the governing equation is of fractional order [105, 143]. If care is taken, the results obtained using the Caputo formulation can be recast to the Riemann–Liouville version and vice versa according to the following relation [56]:

$$D_{RL}^{\alpha} f(t) = D_C^{\alpha} f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+), \quad m-1 < \alpha < m. \tag{1.7}$$

It should be also emphasized that [56]

$$\frac{\mathrm{d}^{\alpha} 1}{\mathrm{d}t^{\alpha}} = 0, \qquad \alpha > 0, \tag{1.8}$$

whereas

$$D_{RL}^{\alpha}\, 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \qquad \alpha \geq 0. \tag{1.9}$$

Fractional differential equations have many applications in physics, geophysics, geology, chemistry, rheology, engineering, bioengineering, robotics, medicine and finance (see, for example, the books [23, 99, 104, 132, 143, 198, 221, 230, 235]; the monographs [11, 70, 121, 142, 204]; the extensive surveys [102, 103, 114, 115, 200, 201, 217, 234)]; and several papers [9, 10, 22, 24, 32, 46, 51, 81]). The interested reader is also referred to a historical survey [218] and a survey of useful formulas [223].

The time-fractional diffusion-wave equation

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a \Delta T, \qquad 0 < \alpha \leq 2, \tag{1.10}$$

describes many important physical phenomena in different media. The limiting case $\alpha = 0$ in (1.10) corresponding to the Helmholtz equation (1.1) and is associated with localized diffusion (localized heat conduction). The subdiffusion regime is characterized by the value $0 < \alpha < 1$. Of course, the standard diffusion equation (standard heat conduction equation) (1.2) corresponds to $\alpha = 1$. The superdiffusion regime is associated with $1 < \alpha < 2$. The limiting case $\alpha = 2$ corresponding to the wave equation (1.3) is also known as ballistic diffusion (ballistic heat conduction).

Various mathematical aspects relating to existence, uniqueness, correctness, well-posedness of solutions to fractional diffusion-wave equations were considered by many authors. Here we refer to [6, 17, 40–42, 49, 66–68, 76, 78–80, 83, 91, 92, 124, 143, 205, 222, 224–229, 233], among others.

Starting from the pioneering papers [45, 100, 101, 209, 231], considerable interest has been shown in finding solutions to time-fractional diffusion-wave equations (1.10). Fujita [45] treated integrodifferential equations which interpolate the diffusion equation and the wave equation. The fundamental solution for the fractional diffusion-wave equation in one space dimension was obtained by Mainardi [100, 101], who also considered the signaling problem and the evolution of the initial box-signal. Schneider and Wyss [209] converted the diffusion-wave equation with appropriate initial conditions into the integrodifferential equation and found the corresponding Green functions in terms of Fox functions. Wyss [231] obtained solutions to the Cauchy problem in terms of H-functions using the Mellin transform. The studies mentioned above do not consider solutions to the two-dimensional and three-dimensional diffusion-wave equation in finite domains. Presently, in the literature there exists no book devoted to the diffusion-wave equation. This book, which in large part is based on the author's investigations [97, 145–193], bridges the gaps in this field. Presenting the solutions to the time-fractional diffusion-wave equation, we follow the encyclopedical book of Polyanin [144], where the corresponding results for standard partial differential equations are given.

Chapter 2

Mathematical Preliminaries

Да это ж математика богов
Владимир Высоцкий[1]

2.1 Integral transforms

The integral transform technique allows us to remove partial derivatives from the considered equations and to obtain the algebraic equation in a transform domain. Here we briefly recall the integral transforms which are used in this book to reduce the differential operators to an algebraic form. The Laplace transform with respect to time is marked by an asterisk, the Fourier transforms are denoted by a tilde, the Hankel transforms are indicated by a hat and the Legendre transform is designated by a star. Additional information concerning integral transforms can be found in [34, 37, 48, 140, 212], among others.

2.1.1 Laplace transform

The Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = f^*(s) = \int_0^\infty f(t)\, \mathrm{e}^{-st}\, \mathrm{d}t, \tag{2.1}$$

where s is the transform variable.

[1] This is mathematics of gods.
Vladimir Vysotsky

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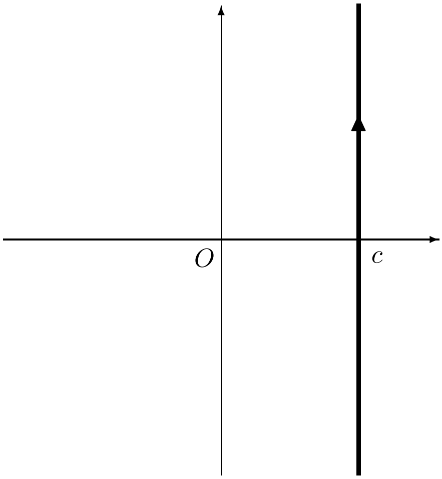

Figure 2.1: The Bromwich path of integration in the complex s-plane

The inverse Laplace transform is carried out according to the Fourier–Mellin formula

$$\mathcal{L}^{-1}\left\{f^*(s)\right\} = f(t) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} f^*(s)\,\mathrm{e}^{st}\,\mathrm{d}s, \quad t > 0, \tag{2.2}$$

where c is a positive fixed number. The transform $f^*(s)$ is assumed analytical for $\mathfrak{Re}\, s > c$, all the singularities of $f^*(s)$ must lie to the left of the vertical line known as the Bromwich path of integration (see Fig. 2.1).

For the primitive of a function $f(t)$

$$If(t) = \int\limits_0^t f(\tau)\,\mathrm{d}\tau \tag{2.3}$$

we have

$$\mathcal{L}\left\{If(t)\right\} = \frac{1}{s}\,f^*(s), \tag{2.4}$$

whereas in the case of the m-fold primitive of a function $f(t)$,

$$I^m f(t) = \int\limits_0^t \mathrm{d}t_1 \int\limits_0^{t_1} \mathrm{d}t_2 \cdots \int\limits_0^{t_{m-1}} f(t_m)\,\mathrm{d}t_m, \tag{2.5}$$

the Laplace transform rule reads

$$\mathcal{L}\left\{I^m f(t)\right\} = \frac{1}{s^m}\,f^*(s). \tag{2.6}$$

The Laplace transform of the first derivative $f'(t)$ is easily obtained integrating the appropriate formula by parts which leads to

$$\mathcal{L}\{f'(t)\} = sf^*(s) - f(0^+), \tag{2.7}$$

and for the mth derivative $f^{(m)}(t)$

$$\mathcal{L}\left\{f^{(m)}(t)\right\} = s^m f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)s^{m-1-k}. \tag{2.8}$$

The Laplace transform rule for the fractional integral (1.4) is similar to the rule (2.6):

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha} f^*(s). \tag{2.9}$$

The Riemann–Liouville derivative of the fractional order α (1.5) for its Laplace transform rule requires knowledge of the initial values of the fractional integral $I^{m-\alpha} f(t)$ and its derivatives of the order $k = 1, 2, \ldots, m-1$

$$\mathcal{L}\{D^\alpha_{RL} f(t)\} = s^\alpha f^*(s) \; - \sum_{k=0}^{m-1} D^k I^{m-\alpha} f(0^+) s^{m-1-k}, \quad m-1 < \alpha < m. \tag{2.10}$$

The Laplace transform rule for the Caputo derivative (1.6) has the following form

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha < m. \tag{2.11}$$

The convolution theorem, often used for inversion of the Laplace transform, reads as

$$\mathcal{L}^{-1}\{f^*(s)\, g^*(s)\} = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau = \int_0^t f(\tau)\, g(t-\tau)\, \mathrm{d}\tau. \tag{2.12}$$

If the transform $f^*(s)$ can be expanded into the absolutely convergent series

$$f^*(s) = \sum_{k=0}^{\infty} \frac{c_k}{s^{\lambda_k}} \tag{2.13}$$

with arbitrary powers $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ (need not be integers), then the inverse transform $f(t)$ has the expansion

$$f(t) = \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(\lambda_k)}\, t^{\lambda_k - 1}. \tag{2.14}$$

If the transform $f^*(s)$ can be expanded into the absolutely convergent series

$$f^*(s) = \sum_{k=0}^{\infty} c_k s^{\lambda_k} \tag{2.15}$$

with arbitrary powers $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ (need not be integers), then the inverse transform $f(t)$ for $t \to \infty$ has the asymptotic expansion

$$f(t) \sim \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(-\lambda_k)}\, t^{-\lambda_k - 1}. \tag{2.16}$$

To calculate the inverse Laplace transform the Cauchy residue theorem is of fundamental importance.

Cauchy residue theorem. If $f(z)$ is analytic within and on a simple, closed contour $\mathfrak{C}$ except at finitely many points $z_1, z_2, \ldots, z_m$ lying in the interior of $\mathfrak{C}$, then

$$\frac{1}{2\pi i} \int_{\mathfrak{C}} f(z)\,\mathrm{d}z = \sum_{k=1}^{m} \mathrm{Res}_{\,z_k}\, f(z), \tag{2.17}$$

where integration is carried out in the positive direction.

Now choose the integration contour $\mathfrak{C}$ shown in Fig. 2.2 containing the portion of the vertical line $\mathfrak{Re}\, s = c$, two parts of the circle of radius R (designating as $\mathfrak{C}_R$), and a loop which starts from $-\infty$ along the upper side of the negative real axis, encircles a small circle of the radius ε in the negative direction and ends at $-\infty$ along the lower side of the negative real axis.

For a sufficiently "good" function $f^*(s)$

$$\lim_{R\to\infty} \int_{\mathfrak{C}_R} f^*(s)\,\mathrm{e}^{st}\,\mathrm{d}s = 0. \tag{2.18}$$

Hence

$$f(t) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{Ha_\varepsilon} f^*(s)\,\mathrm{e}^{st}\,\mathrm{d}s + \sum_{k=1}^{m} \mathrm{Res}_{\,s_k}\, f^*(s)\,\mathrm{e}^{st}, \tag{2.19}$$

where the Hankel path of integration Ha_ϵ is a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles a small circle in the positive direction and ends at $-\infty$ along the upper side of the negative real axis (see Fig. 2.3). It should be noted that multiplying $f^*(s)$ by e^{st} does not affect the poles of $f^*(s)$.

2.1.2 Exponential Fourier transform

The exponential Fourier transform

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)\mathrm{e}^{ix\xi}\mathrm{d}x \tag{2.20}$$

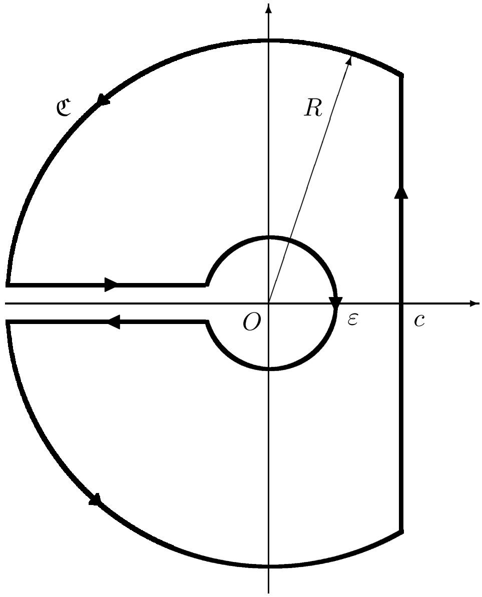

Figure 2.2: The closed path of integration in the complex s-plane

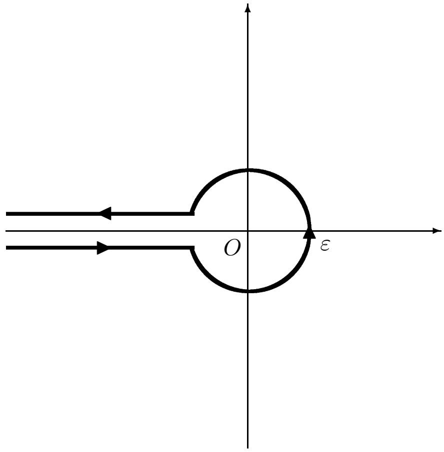

Figure 2.3: The Hankel path of integration in the complex s-plane

is used in the domain $-\infty < x < \infty$ and has the inverse

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(\xi) \mathrm{e}^{-ix\xi} \mathrm{d}\xi. \tag{2.21}$$

The Fourier transform of the mth derivative of a function has the form

$$\mathcal{F}\left\{\frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m}\right\} = (-i\xi)^m \widetilde{f}(\xi), \tag{2.22}$$

in particular,

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2 \widetilde{f}(\xi). \tag{2.23}$$

The convolution theorem for the exponential Fourier transform reads:

$$\begin{aligned}\mathcal{F}^{-1}\left\{\widetilde{f}(\xi)\,\widetilde{g}(\xi)\right\} &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x-u)\,g(u)\,\mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(u)\,g(x-u)\,\mathrm{d}u. \end{aligned} \tag{2.24}$$

2.1.3 Sin-Fourier transform

The sin-Fourier transform is defined as

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \int\limits_0^{\infty} f(x)\sin(x\xi)\,\mathrm{d}x \tag{2.25}$$

with the inverse

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi)\} = f(x) = \frac{2}{\pi}\int\limits_0^{\infty} \widetilde{f}(\xi)\sin(x\xi)\,\mathrm{d}\xi. \tag{2.26}$$

The sin-Fourier transform is used in the domain $0 \le x < \infty$ for Dirichlet boundary condition with the prescribed boundary value of a function, since for the second derivative of a function we get

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2 \widetilde{f}(\xi) + \xi\, f(x)\bigg|_{x=0}. \tag{2.27}$$

2.1.4 Three-fold Fourier transform in the case of spherical symmetry

If the considered function $f(x,y,z)$ depends only on the radial coordinate $r = \left(x^2+y^2+z^2\right)^{1/2}$, then the three-fold Fourier transform (2.20) can be simplified. Introducing the spherical coordinates

$$\begin{aligned} &x = r\sin\varphi\cos\vartheta, \quad y = r\sin\varphi\sin\vartheta, \quad z = r\cos\varphi, \\ &\xi = \varrho\sin\phi\cos\theta, \quad \eta = \varrho\sin\phi\sin\theta, \quad \zeta = \varrho\cos\phi, \end{aligned} \tag{2.28}$$

we have

$$\begin{aligned}\widetilde{f}(\xi,\eta,\zeta) &= \frac{1}{(2\pi)^{3/2}} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y,z)\,\mathrm{e}^{i(x\xi+y\eta+z\zeta)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{1}{(2\pi)^{3/2}} \int\limits_{0}^{\infty} r^2 f(r)\,\mathrm{d}r \int\limits_{0}^{\pi} \mathrm{e}^{ir\varrho\cos\varphi\cos\phi}\,\sin\varphi\,\mathrm{d}\varphi \\ &\times \int\limits_{0}^{2\pi} \mathrm{e}^{ir\varrho\sin\varphi\sin\phi\cos(\vartheta-\theta)}\,\mathrm{d}\vartheta. \end{aligned} \tag{2.29}$$

Due to the periodic character of the third integrand

$$\int\limits_{0}^{2\pi} \mathrm{e}^{ir\varrho\sin\varphi\sin\phi\cos(\vartheta-\theta)}\,\mathrm{d}\vartheta = \int\limits_{0}^{2\pi} \mathrm{e}^{ir\varrho\sin\varphi\sin\phi\cos\vartheta}\,\mathrm{d}\vartheta.$$

Using the integral representation of the Bessel function of the first kind of the zeroth order [1]

$$\int\limits_{0}^{2\pi} \mathrm{e}^{iz\cos\vartheta}\mathrm{d}\vartheta = 2\pi J_0(z), \tag{2.30}$$

we get

$$\begin{aligned}\widetilde{f}(\xi,\eta,\zeta) &= \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} r^2 f(r)\,\mathrm{d}r \\ &\times \int\limits_{0}^{\pi} \sin\varphi\cos\left(r\varrho\cos\varphi\cos\phi\right)\, J_0\left(r\varrho\sin\varphi\sin\phi\right)\,\mathrm{d}\varphi \\ &= \sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} r^2 f(r)\,\mathrm{d}r \int\limits_{0}^{1} \cos\left(r\varrho v\cos\phi\right) J_0\left(r\varrho\sqrt{1-v^2}\sin\phi\right)\,\mathrm{d}v. \end{aligned}$$

Next, we use the integral [196]

$$\int\limits_{0}^{1} \cos\left(av\right) J_0\left(b\sqrt{1-v^2}\right)\,\mathrm{d}v = \frac{1}{\sqrt{a^2+b^2}}\,\sin\sqrt{a^2+b^2},$$

and for the three-fold Fourier transform in the central symmetric case we obtain the following pair of equations:

$$\mathcal{F}\left\{f(x,y,z)\right\} = \mathcal{F}\left\{f(r)\right\} = \widetilde{f}(\varrho) = \sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} r\,f(r)\,\frac{\sin(r\varrho)}{\varrho}\,\mathrm{d}r, \tag{2.31}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\varrho)\right\} = f(r) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty \varrho\,\widetilde{f}(\varrho)\,\frac{\sin(r\varrho)}{r}\,\mathrm{d}\varrho. \tag{2.32}$$

This result coincides with the particular case of the m-fold Fourier transform in the central symmetric case obtained by another method in [212]:

$$\varrho^{\frac{1}{2}m-1}\widetilde{f}(\varrho) = \int\limits_0^\infty r^{\frac{1}{2}m-1}\,f(r)\,J_{\frac{1}{2}m-1}(r\varrho)\,r\,\mathrm{d}r, \tag{2.33}$$

$$r^{\frac{1}{2}m-1}f(r) = \int\limits_0^\infty \varrho^{\frac{1}{2}m-1}\,\widetilde{f}(\varrho)\,J_{\frac{1}{2}m-1}(r\varrho)\,\varrho\,\mathrm{d}\varrho, \tag{2.34}$$

where $J_\nu(r)$ is the Bessel function.

For $m = 3$, taking into account that the Bessel function of the order one-half is represented as [1]

$$J_{1/2}(z) = \sqrt{\frac{2z}{\pi}}\,\frac{\sin z}{z}, \tag{2.35}$$

from (2.33) and (2.34) we get (2.31) and (2.32).

In this case

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\varrho^2\widetilde{f}(\varrho). \tag{2.36}$$

The pair of transform equations (2.31) and (2.32) seems like the pair of sin-Fourier transform equations (2.25) and (2.26) for the function $rf(r)$ (accurate to constant multipliers), but Eq. (2.31) does not need the value of a function at $r = 0$ as in Eq. (2.27). This allows us to consider also functions with singularities at $r = 0$ on condition that the integral in (2.31) is convergent.

2.1.5 Cos-Fourier transform

For the cos-Fourier transform we have

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \int\limits_0^\infty f(x)\cos(x\xi)\mathrm{d}x, \tag{2.37}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi)\} = f(x) = \frac{2}{\pi}\int\limits_0^\infty \widetilde{f}(\xi)\cos(x\xi)\mathrm{d}\xi. \tag{2.38}$$

The cos-Fourier transform is used in the domain $0 \le x < \infty$ in the case of Neumann boundary condition with the prescribed boundary value of the normal derivative of a function, since for the second derivative of a function it leads to the following formula:

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\widetilde{f}(\xi) - \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0}. \tag{2.39}$$

2.1.6 Sin-cos-Fourier transform

In the case of the Robin boundary conditions with the prescribed boundary value of linear combination of a function and its normal derivative, the sin-cos-Fourier transform is employed:

$$\mathcal{F}\left\{f(x)\right\} = \widetilde{f}(\xi) = \int_0^\infty K(x,\xi)\, f(x)\, \mathrm{d}x, \tag{2.40}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi)\right\} = f(x) = \frac{2}{\pi}\int_0^\infty K(x,\xi)\, \widetilde{f}(\xi)\, \mathrm{d}\xi \tag{2.41}$$

with the kernel

$$K(x,\xi) = \frac{\xi\cos(x\xi) + H\sin(x\xi)}{\sqrt{\xi^2+H^2}}. \tag{2.42}$$

In classical heat conduction the quantity H is usually connected with the heat transfer coefficient, in the case of spherical coordinates the quantity $1/R$ often stands in place of H.

Application of sin-cos-Fourier transform to the second derivative of a function gives

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\widetilde{f}(\xi) + \frac{\xi}{\sqrt{\xi^2+H^2}}\left[-\frac{\mathrm{d}f(x)}{\mathrm{d}x} + Hf(x)\right]\Bigg|_{x=0}. \tag{2.43}$$

It is obvious that for $H \to \infty$ the sin-cos-Fourier transform turns into the standard sin-Fourier transform, while for $H \to 0$ it turns into the standard cos-Fourier transform.

2.1.7 Finite sin-Fourier transform

The finite sin-Fourier transform is the convenient reformulation of the sin-Fourier series in the domain $0 \le x \le L$:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_k) = \int_0^L f(x)\sin(x\xi_k)\,\mathrm{d}x, \tag{2.44}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(x) = \frac{2}{L}\sum_{k=1}^{\infty}\widetilde{f}(\xi_k)\sin(x\xi_k), \tag{2.45}$$

where

$$\xi_k = \frac{k\pi}{L}. \tag{2.46}$$

The finite sin-Fourier transform is used in the case of the Dirichlet boundary condition as for the second derivative of a function we have

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_k^2 \widetilde{f}(\xi_k) + \xi_k \left[f(0) - (-1)^k f(L)\right]. \tag{2.47}$$

2.1.8 Finite cos-Fourier transform

The finite cos-Fourier transform is the convenient reformulation of the cos-Fourier series in the domain $0 \leq x \leq L$:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_k) = \int_0^L f(x) \cos(x\xi_k)\, \mathrm{d}x, \tag{2.48}$$

$$\begin{aligned}\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(x) &= \frac{1}{L}\widetilde{f}(0) + \frac{2}{L}\sum_{k=1}^{\infty} \widetilde{f}(\xi_k) \cos(x\xi_k) \\ &= \frac{2}{L}\sum_{k=0}^{\infty}{}' \, \widetilde{f}(\xi_k) \cos(x\xi_k), \end{aligned} \tag{2.49}$$

where the prime near the sum denotes that the term corresponding to $k = 0$ should be multiplied by 1/2 and as in (2.46)

$$\xi_k = \frac{k\pi}{L}. \tag{2.50}$$

The finite cos-Fourier transform is used in the case of Neumann boundary condition as

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_k^2 \, \widetilde{f}(\xi_k) - \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0} + (-1)^k \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=L}. \tag{2.51}$$

2.1.9 Finite sin-cos-Fourier transform

The finite sin-cos-Fourier transform is used in the case of the Robin boundary condition:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_k) = \int_0^L f(x) \frac{\xi_k \cos(x\xi_k) + H \sin(x\xi_k)}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}}\, \mathrm{d}x, \tag{2.52}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(x) = \frac{2}{L}\sum_{k=1}^{\infty} \widetilde{f}(\xi_k) \frac{\xi_k \cos(x\xi_k) + H \sin(x\xi_k)}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}}, \tag{2.53}$$

where ξ_k are the positive roots of the transcendental equation

$$\tan(L\xi_k) = \frac{2H\xi_k}{\xi_k^2 - H^2} \tag{2.54}$$

and

$$\begin{aligned}\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} &= -\xi_k^2 \widetilde{f}(\xi_k) \\ &+ \frac{\xi_k}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}}\left[-\frac{\mathrm{d}f(x)}{\mathrm{d}x} + Hf(x)\right]\Bigg|_{x=0} \\ &+ \frac{\xi_k}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}}\,\frac{\xi_k^2 + H^2}{\xi_k^2 - H^2}\cos(L\xi_k)\left[\frac{\mathrm{d}f(x)}{\mathrm{d}x} + Hf(x)\right]\Bigg|_{x=L}.\end{aligned} \tag{2.55}$$

We have restricted ourselves to the case of the same H in the Robin boundary conditions at $x = 0$ and $x = L$. The general case of different coefficients H_1 and H_2 is considered in [48].

2.1.10 Finite sin-Fourier transform for a sphere

This type of finite sin-Fourier transform is convenient for central symmetric problems for a sphere $0 \le r \le R$. In the case of the Dirichlet boundary condition:

$$\mathcal{F}\{f(r)\} = \widetilde{f}(\xi_k) = \int_0^R r\,f(r)\,\frac{\sin(r\xi_k)}{\xi_k}\,\mathrm{d}r, \tag{2.56}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(r) = \frac{2}{R}\sum_{k=1}^{\infty} \xi_k\,\widetilde{f}(\xi_k)\,\frac{\sin(r\xi_k)}{r}, \tag{2.57}$$

where

$$\xi_k = \frac{k\pi}{R} \tag{2.58}$$

and

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\xi_k^2\,\widetilde{f}(\xi_k) + (-1)^{k+1} R f(R). \tag{2.59}$$

For the Neumann boundary condition

$$\mathcal{F}\{f(r)\} = \widetilde{f}(\xi_k) = \int_0^R r\,f(r)\,\frac{\sin(r\xi_k)}{\xi_k}\,\mathrm{d}r, \tag{2.60}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(r) = \frac{3}{R^3}\widetilde{f}(0) + \frac{2}{R}\sum_{k=1}^{\infty}\widetilde{f}(\xi_k)\frac{\xi_k}{\sin^2(R\xi_k)}\,\frac{\sin(r\xi_k)}{r}, \tag{2.61}$$

where ξ_k are the positive roots of the transcendental equation

$$\tan(R\xi_k) = R\xi_k \tag{2.62}$$

and

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\xi_k^2\,\widetilde{f}(\xi_k) + \frac{R\sin(R\xi_k)}{\xi_k}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\bigg|_{r=R}. \tag{2.63}$$

For the Robin boundary condition

$$\mathcal{F}\{f(r)\} = \widetilde{f}(\xi_k) = \int\limits_0^R f(r)\,\frac{\sin(r\xi_k)}{\xi_k}\,r\,\mathrm{d}r, \tag{2.64}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(r) = 2\sum_{k=1}^{\infty}\widetilde{f}(\xi_k)\frac{\xi_k^2}{R\xi_k - \sin(R\xi_k)\cos(R\xi_k)}\,\frac{\sin(r\xi_k)}{r}, \tag{2.65}$$

where ξ_k are the positive roots of the transcendental equation

$$\tan(R\xi_k) = \frac{R\xi_k}{1 - RH}, \tag{2.66}$$

and for the Laplace operator in the case of central symmetric problem we obtain

$$\begin{aligned}\mathcal{F}&\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}\\ &= -\xi_k^2\,\widetilde{f}(\xi_k) + \frac{R\sin(R\xi_k)}{\xi_k}\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r} + Hf(r)\right]\bigg|_{r=R}.\end{aligned} \tag{2.67}$$

2.1.11 Finite Fourier transform for 2π-periodic functions

Consider series development of the 2π-periodic function in the interval $[0, 2\pi]$

$$f(\varphi) = \frac{1}{2}\,a_0 + \sum_{m=1}^{\infty}\left[a_m\cos(m\varphi) + b_m\sin(m\varphi)\right], \tag{2.68}$$

where

$$\begin{aligned}a_m &= \frac{1}{\pi}\int\limits_0^{2\pi} f(\eta)\cos(m\eta)\,\mathrm{d}\eta, \qquad m = 0, 1, 2, \ldots\\ b_m &= \frac{1}{\pi}\int\limits_0^{2\pi} f(\eta)\sin(m\eta)\,\mathrm{d}\eta, \qquad m = 1, 2, \ldots\end{aligned} \tag{2.69}$$

Now we insert the coefficients (2.69) into the equality (2.68), thus obtaining

$$f(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\,\mathrm{d}\eta + \frac{1}{\pi}\sum_{m=1}^{\infty}\int_0^{2\pi} f(\eta)\,\cos[m(\varphi-\eta)]\,\mathrm{d}\eta \tag{2.70}$$

or

$$f(\varphi) = \frac{1}{\pi}\sum_{m=0}^{\infty}{}'\int_0^{2\pi} f(\eta)\,\cos[m(\varphi-\eta)]\,\mathrm{d}\eta, \tag{2.71}$$

where the prime near the sum denotes that the term corresponding to $m = 0$ should be multiplied by 1/2.

Formula (2.71) can be considered as the integral transform

$$\mathcal{F}\{f(\varphi)\} = \widetilde{f}(\varphi, m) = \int_0^{2\pi} f(\eta)\,\cos[m(\varphi-\eta)]\,\mathrm{d}\eta \tag{2.72}$$

with the inverse

$$\mathcal{F}^{-1}\{\widetilde{f}(\varphi, m)\} = f(\varphi) = \frac{1}{\pi}\sum_{m=0}^{\infty}{}'\widetilde{f}(\varphi, m). \tag{2.73}$$

This transform is used for solving equations in polar, cylindrical and spherical coordinates as the following equation is fulfilled:

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(\varphi)}{\mathrm{d}\varphi^2}\right\} = -m^2\widetilde{f}(\varphi, m). \tag{2.74}$$

2.1.12 Legendre transform

The Legendre transform is applied to solve equations in spherical coordinates and reads:

$$\mathcal{P}\{f(\mu, m)\} = f^{\star}(n, m) = \int_{-1}^{1} f(\mu, m)\,P_n^m(\mu)\,\mathrm{d}\mu, \tag{2.75}$$

where $P_n^m(\mu)$ is the associated Legendre function of the first kind of degree n and order m. The inverse Legendre transform has the form

$$\begin{aligned}\mathcal{P}^{-1}\{f^{\star}(n, m)\} &= f(\mu, m)\\ &= \sum_{n=0}^{\infty}\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)\,f^{\star}(n, m), \quad n \geq m.\end{aligned} \tag{2.76}$$

The importance of this integral transform results from the following formula:

$$\mathcal{P}\left\{\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial f}{\partial\mu}\right] - \frac{m^2}{1-\mu^2}\,f\right\} = -n(n+1)f^{\star}(n, m). \tag{2.77}$$

2.1.13 Hankel transform

The Hankel transform is used to solve problems in cylindrical coordinates in the domain $0 \leq r < \infty$ and is defined as

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi) = \int_0^\infty f(r)\, J_\nu(r\xi)\, r\, \mathrm{d}r, \tag{2.78}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi)\} = f(r) = \int_0^\infty \widehat{f}(\xi)\, J_\nu(r\xi)\, \xi\, \mathrm{d}\xi, \tag{2.79}$$

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\xi^2 \widehat{f}(\xi), \tag{2.80}$$

where $J_\nu(r)$ is the Bessel function of the order ν.

2.1.14 Two-fold Fourier transform in the case of axial symmetry

If the considered function $f(x,y)$ depends only on the radial coordinate

$$r = \left(x^2 + y^2\right)^{1/2},$$

then the two-fold Fourier transform (2.20) can be simplified. Introducing the polar coordinates

$$\begin{aligned} x &= r\sin\varphi, \qquad & y &= r\cos\varphi, \\ \xi &= \varrho\sin\phi, \qquad & \eta &= \varrho\cos\phi, \end{aligned} \tag{2.81}$$

we have

$$\widetilde{f}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, \mathrm{e}^{i(x\xi + y\eta)}\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi}\int_0^\infty r f(r)\mathrm{d}r \int_0^{2\pi} \mathrm{e}^{ir\varrho\cos(\varphi-\phi)}\mathrm{d}\varphi.$$

Due to the periodic character of the second integrand

$$\int_0^{2\pi} \mathrm{e}^{ir\varrho\cos(\varphi-\phi)}\, \mathrm{d}\varphi = \int_0^{2\pi} \mathrm{e}^{ir\varrho\cos\varphi}\, \mathrm{d}\varphi.$$

Using the integral representation of the Bessel function of the first kind of the zeroth order (2.30) we get

$$\mathcal{F}\left\{f(x,y)\right\} = \widetilde{f}(\xi,\eta) = \mathcal{H}\left\{f(r)\right\} = \widehat{f}(\varrho) = \int\limits_0^\infty r f(r)\, J_0(r\varrho)\,\mathrm{d}r, \tag{2.82}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi,\eta)\right\} = f(x,y) = \mathcal{H}^{-1}\left\{\widehat{f}(\varrho)\right\} = f(r) = \int\limits_0^\infty \varrho \widehat{f}(\varrho)\, J_0(r\varrho)\,\mathrm{d}\varrho. \tag{2.83}$$

Hence, in the case of axial symmetry the two-fold Fourier transform with respect to the Cartesian coordinates is reduced to the Hankel transform with respect to the radial coordinate. Formulae (2.82) and (2.83) can also be obtained from the general formulae (2.33) and (2.34) for $m = 2$ [212].

2.1.15 Finite Hankel transform

The Fourier–Bessel and Dini series can be interpreted in terms of finite Hankel transform used in cylindrical coordinates in the domain $0 \le r \le R$. The specific form of the finite Hankel transform depends on the type of boundary conditions at $r = R$. For the Dirichlet boundary condition with the given boundary value of a function at $r = R$ we have

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu k}) = \int\limits_0^R f(r)\, J_\nu(r\xi_{\nu k})\, r\,\mathrm{d}r, \tag{2.84}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu k})\} = f(r) = \frac{2}{R^2}\sum_{k=1}^{\infty} \widehat{f}(\xi_{\nu k}) \frac{J_\nu(r\xi_{\nu k})}{[J'_\nu(R\xi_{\nu k})]^2}, \tag{2.85}$$

where $\xi_{\nu k}$ are the positive roots of the transcendental equation

$$J_\nu(R\xi_{\nu k}) = 0 \tag{2.86}$$

and

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2}\, f(r)\right\} = -\,\xi_{\nu k}^2 \widehat{f}(\xi_{\nu k}) - R\xi_{\nu k} J'_\nu(R\xi_{\nu k}) f(R). \tag{2.87}$$

In the case of the Neumann boundary conditions with the given boundary value of a normal derivative of a function we have

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu k}) = \int\limits_0^R f(r)\, J_\nu(r\xi_{\nu k})\, r\,\mathrm{d}r, \tag{2.88}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu k})\} = f(r) = \frac{2}{R^2}\sum_{k=0}^{\infty} \widehat{f}(\xi_{\nu k})\, \frac{R^2\xi_{\nu k}^2}{R^2\xi_{\nu k}^2 - \nu^2}\, \frac{J_\nu(r\xi_{\nu k})}{[J_\nu(R\xi_{\nu k})]^2}, \tag{2.89}$$

where $\xi_{\nu k}$ are positive roots of the transcendental equation

$$J_\nu'(R\xi_{\nu k}) = 0. \tag{2.90}$$

It should be noted that for $\nu = 0$ there also appears the zero root $\xi_{00} = 0$, which must be taken into account in (2.89).

The basic equation for this integral transform reads:

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\xi_{\nu k}^2 \widehat{f}(\xi_{\nu k}) + R J_\nu(R\xi_{\nu k}) \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)\Bigg|_{r=R}. \tag{2.91}$$

For the Robin boundary condition with the given linear combination of values of function and its normal derivative at the boundary the corresponding finite Hankel transform has the following form:

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu k}) = \int_0^R f(r)\, J_\nu(r\xi_{\nu k})\, r\, \mathrm{d}r, \tag{2.92}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu k})\} = f(r) = \frac{2}{R^2}\sum_{k=1}^{\infty} \widehat{f}(\xi_{\nu k})\, \frac{R^2\xi_{\nu k}^2}{R^2H^2 + (R^2\xi_{\nu k}^2 - \nu^2)}\, \frac{J_\nu(r\xi_{\nu k})}{[J_\nu(R\xi_{\nu k})]^2}, \tag{2.93}$$

where $\xi_{\nu k}$ are positive roots of the transcendental equation

$$\xi_{\nu k} J_\nu'(R\xi_{\nu k}) + H J_\nu(R\xi_{\nu k}) = 0 \tag{2.94}$$

and

$$\begin{aligned}\mathcal{H}&\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} \\ &= -\,\xi_{\nu k}^2 \widehat{f}(\xi_{\nu k}) + R J_\nu(R\xi_{\nu k}) \left[\frac{\mathrm{d}f(r)}{\mathrm{d}r} + H f(r)\right]\Bigg|_{r=R}.\end{aligned} \tag{2.95}$$

Now we consider the particular cases of the finite Hankel transform of the zeroth order. For the Dirichlet boundary condition we get

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_k) = \int_0^R f(r)\, J_0(r\xi_k)\, r\, \mathrm{d}r, \tag{2.96}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_k)\} = f(r) = \frac{2}{R^2}\sum_{k=1}^{\infty} \widehat{f}(\xi_k) \frac{J_0(r\xi_k)}{[J_1(R\xi_k)]^2} \tag{2.97}$$

with the sum over all positive roots of the zeroth-order Bessel function

$$J_0(R\xi_k) = 0, \tag{2.98}$$

and

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\xi_k^2\widehat{f}(\xi_k)+R\xi_k J_1(R\xi_k)f(R). \tag{2.99}$$

In the case of the Neumann boundary condition

$$\mathcal{H}\{f(r)\}=\widehat{f}(\xi_k)=\int_0^R r f(r)J_0(r\xi_k)\,\mathrm{d}r, \tag{2.100}$$

$$\begin{aligned}\mathcal{H}^{-1}\{\widehat{f}(\xi_k)\}=f(r)&=\frac{2}{R^2}\sum_{k=0}^{\infty}\widehat{f}(\xi_k)\frac{J_0(r\xi_k)}{[J_0(R\xi_k)]^2}\\&=\frac{2}{R^2}\,\widehat{f}(0)+\frac{2}{R^2}\sum_{k=1}^{\infty}\widehat{f}(\xi_k)\frac{J_0(r\xi_k)}{[J_0(R\xi_k)]^2},\end{aligned} \tag{2.101}$$

where ξ_k are nonnegative roots of the equation

$$J_1(R\xi_k)=0. \tag{2.102}$$

To obtain the correct results it should be emphasized that this equation also has the root $\xi_0=0$. This root should be taken into consideration, and sometimes it is convenient to treat it separately (see Eq. (2.101)).

The fundamental equation for this transform has the form

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\xi_k^2\widehat{f}(\xi_k)+RJ_0(R\xi_k)\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)\bigg|_{r=R}. \tag{2.103}$$

For the Robin boundary condition we have

$$\mathcal{H}\{f(r)\}=\widehat{f}(\xi_k)=\int_0^R r f(r)J_0(r\xi_k)\,\mathrm{d}r \tag{2.104}$$

with the inverse

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_k)\}=f(r)=\frac{2}{R^2}\sum_{k=1}^{\infty}\widehat{f}(\xi_k)\,\frac{\xi_k^2}{H^2+\xi_k^2}\,\frac{J_0(r\xi_k)}{[J_0(R\xi_k)]^2}, \tag{2.105}$$

where ξ_k are the positive roots of the transcendental equation

$$\xi_k J_1(R\xi_k)=HJ_0(R\xi_k). \tag{2.106}$$

In this instance

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\xi_k^2\widehat{f}(\xi_k)+\;RJ_0(R\xi_k)\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r}+Hf(r)\right]\bigg|_{r=R}. \tag{2.107}$$

2.1.16 Weber transform

The Weber integral transform of order ν is defined as

$$\mathcal{W}\{f(r)\} = \widehat{f}(\xi) = \int_R^\infty K_\nu(r, R, \xi) f(r)\, r\, \mathrm{d}r \tag{2.108}$$

having the inverse

$$\mathcal{W}^{-1}\{\widehat{f}(\xi)\} = f(r) = \int_0^\infty K_\nu(r, R, \xi) \widehat{f}(\xi)\, \xi\, \mathrm{d}\xi. \tag{2.109}$$

The significance of the Weber transform for problems in the domain $R \le r < \infty$ is due to the formula

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\}$$
$$= -\xi^2 \widehat{f}(\xi) + Rf(R)\left.\frac{\partial K_\nu(r, R, \xi)}{\partial r}\right|_{r=R} - RK_\nu(R, R, \xi)\left.\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right|_{r=R}. \tag{2.110}$$

The specific expression of the kernel $K_\nu(r, R, \xi)$ depends on the boundary conditions at $r = R$.

For the Dirichlet boundary condition the kernel is chosen as

$$K_\nu^{(\mathrm{D})}(r, R, \xi) = \frac{J_\nu(r\xi)Y_\nu(R\xi) - Y_\nu(r\xi)J_\nu(R\xi)}{\sqrt{J_\nu^2(R\xi) + Y_\nu^2(R\xi)}}, \tag{2.111}$$

where $J_\nu(r)$ and $Y_\nu(r)$ are the Bessel functions of the first and second kind, respectively.

Since

$$K_\nu^{(\mathrm{D})}(R, R, \xi) = 0,$$

$$\frac{\partial K_\nu^{(\mathrm{D})}(r, R, \xi)}{\partial r} = \frac{J_\nu'(r\xi)Y_\nu(R\xi) - Y_\nu'(r\xi)J_\nu(R\xi)}{\sqrt{J_\nu^2(R\xi) + Y_\nu^2(R\xi)}}\, \xi,$$

and [1]

$$J_\nu(z)Y_\nu'(z) - Y_\nu(z)J_\nu'(z) = \frac{2}{\pi z},$$

then

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\xi^2 \widehat{f}(\xi) - \frac{2}{\pi}\frac{1}{\sqrt{J_\nu^2(R\xi) + Y_\nu^2(R\xi)}} f(R). \tag{2.112}$$

Similarly, in the case of the Neumann boundary condition

$$K_\nu^{(\mathrm{N})}(r,R,\xi) = \frac{J_\nu(r\xi)Y'_\nu(R\xi) - Y_\nu(r\xi)J'_\nu(R\xi)}{\sqrt{[J'_\nu(R\xi)]^2 + [Y'_\nu(R\xi)]^2}}, \tag{2.113}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\,\xi^2 \widehat{f}(\xi) - \frac{2}{\pi\xi}\,\frac{1}{\sqrt{[J'_\nu(R\xi)]^2 + [Y'_\nu(R\xi)]^2}}\,\left.\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right]\right|_{r=R}. \tag{2.114}$$

For the Robin boundary condition

$$K_\nu^{(\mathrm{R})}(r,R,\xi) = \frac{J_\nu(r\xi)[\xi Y'_\nu(R\xi) - HY_\nu(R\xi)] - Y_\nu(r\xi)[\xi J'_\nu(R\xi) - HJ_\nu(R\xi)]}{\sqrt{[\xi J'_\nu(R\xi) - HJ_\nu(R\xi)]^2 + [\xi Y'_\nu(R\xi) - HY_\nu(R\xi)]^2}}, \tag{2.115}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\,\xi^2 \widehat{f}(\xi) + \frac{2}{\pi}\,\frac{1}{\sqrt{[\xi J'_\nu(R\xi) - HJ_\nu(R\xi)]^2 + [\xi Y'_\nu(R\xi) - HJ_\nu(R\xi)]^2}} \times \left.\left[-\frac{\mathrm{d}f(r)}{\mathrm{d}r} + Hf(r)\right]\right|_{r=R}. \tag{2.116}$$

The formulae above simplify considerably in the case $\nu = 0$. For the Dirichlet boundary condition

$$K_0^{(\mathrm{D})}(r,R,\xi) = \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{\sqrt{J_0^2(R\xi) + Y_0^2(R\xi)}} \tag{2.117}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\,\xi^2 \widehat{f}(\xi) - \frac{2}{\pi}\,\frac{1}{\sqrt{J_0^2(R\xi) + Y_0^2(R\xi)}}\, f(R). \tag{2.118}$$

For the Neumann boundary condition

$$K_0^{(\mathrm{N})}(r,R,\xi) = -\frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{\sqrt{J_1^2(R\xi) + Y_1^2(R\xi)}} \tag{2.119}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}$$
$$=-\xi^2\widehat{f}(\xi)-\frac{2}{\pi\xi}\frac{1}{\sqrt{J_1^2(R\xi)+Y_1^2(R\xi)}}\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right]\Bigg|_{r=R}. \tag{2.120}$$

In the case of the Robin boundary condition

$$K_0^{(\mathrm{R})}(r,R,\xi)$$
$$=\frac{Y_0(r\xi)[\xi J_1(R\xi)+HJ_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+HY_0(R\xi)]}{\sqrt{[\xi J_1(R\xi)+HJ_0(R\xi)]^2+[\xi Y_1(R\xi)+HY_0(R\xi)]^2}} \tag{2.121}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\,\xi^2\widehat{f}(\xi)$$
$$+\frac{2}{\pi}\frac{1}{\sqrt{[\xi J_1(R\xi)+HJ_0(R\xi)]^2+[\xi Y_1(R\xi)+HJ_0(R\xi)]^2}}$$
$$\times\left[-\frac{\mathrm{d}f(r)}{\mathrm{d}r}+Hf(r)\right]\Bigg|_{r=R}. \tag{2.122}$$

2.2 Mittag-Leffler function

The Mittag-Leffler function in one parameter α [119, 120] (see also [43, 56, 59, 77, 143])

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\qquad \alpha>0,\ \ z\in C, \tag{2.123}$$

provides a generalization of the exponential function

$$\mathrm{e}^z=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)},\qquad z\in C. \tag{2.124}$$

The generalized Mittag-Leffler function in two parameters α and β [43, 56, 59, 71, 72, 77, 143] is described by the series representation

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad \alpha>0,\ \beta>0,\ z\in C. \tag{2.125}$$

Here we recall several particular cases of the Mittag-Leffler functions for negative real values of argument:

$$E_0(-x)\;=\;\frac{1}{1+x}, \tag{2.126}$$

$$E_{1/2}(-x) = e^{x^2}\operatorname{erfc}(x), \tag{2.127}$$

$$E_1(-x) = e^{-x}, \tag{2.128}$$

$$E_2(-x) = \cos\sqrt{x}, \tag{2.129}$$

$$E_{1/2,\,1/2}(-x) = \frac{1}{\sqrt{\pi}} - x e^{x^2}\operatorname{erfc}(x), \tag{2.130}$$

$$E_{0,2}(-x) = \frac{1}{1+x}, \tag{2.131}$$

$$E_{1/2,\,3/2}(-x) = \frac{1}{x}\left[1 - e^{x^2}\operatorname{erfc}(x)\right], \tag{2.132}$$

$$E_{1/2\,,\,2}(-x) = \frac{1}{x^2}\left[\frac{2x}{\sqrt{\pi}} + e^{x^2}\operatorname{erfc}(x) - 1\right], \tag{2.133}$$

$$E_{1,2}(-x) = \frac{1-e^{-x}}{x}, \tag{2.134}$$

$$E_{2,2}(-x) = \frac{\sin\sqrt{x}}{\sqrt{x}}. \tag{2.135}$$

The Mittag-Leffler functions with the index 1/2 often appear in applications. It is convenient to obtain the helpful integral representations of these functions. For example, we have

$$E_{1/2}(-x) = e^{x^2}\operatorname{erfc}(x) = e^{x^2}\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt.$$

Substitution $t = u + x$ leads to

$$E_{1/2}(-x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2-2ux}\,du. \tag{2.136}$$

Similarly,

$$E_{1/2\,,\,1/2}(-x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2-2ux}u\,du. \tag{2.137}$$

Several functional relations between different Mittag-Leffler functions can be found in [43]. We present the relations which will be used in the following:

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z), \tag{2.138}$$

$$\frac{d\left[z^{\beta-1}E_{\alpha,\beta}\left(z^\alpha\right)\right]}{dz} = z^{\beta-2}E_{\alpha,\beta-1}\left(z^\alpha\right), \tag{2.139}$$

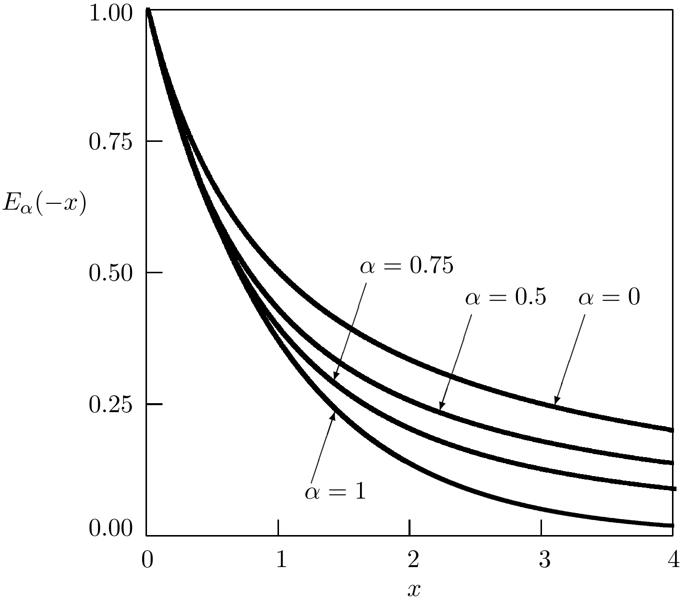

Figure 2.4: The Mittag-Leffler functions $E_\alpha(-x)$ for $0 \leq \alpha \leq 1$

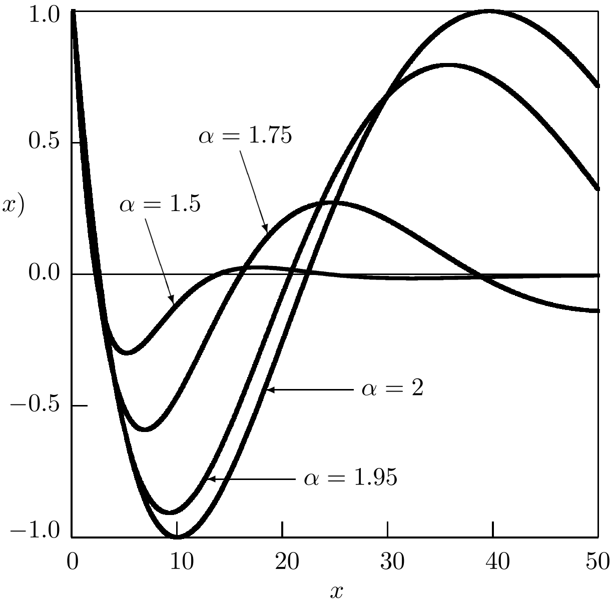

Figure 2.5: The Mittag-Leffler functions $E_\alpha(-x)$ for $1 < \alpha \leq 2$

$$\frac{\mathrm{d}\left[z^{\beta-1}E_{1,\beta}\left(z\right)\right]}{\mathrm{d}z} = z^{\beta-2}E_{1,\beta-1}\left(z\right). \tag{2.140}$$

The essential role of the Mittag-Leffler functions in fractional calculus results from the formula for the inverse Laplace transform [56, 77, 143]:

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha+b}\right\} = t^{\beta-1}\,E_{\alpha,\beta}(-bt^\alpha). \tag{2.141}$$

For three important particular cases $\beta = 1$, $\beta = 2$ and $\beta = \alpha$, respectively, we get

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha+b}\right\} = E_\alpha(-bt^\alpha), \tag{2.142}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha+b}\right\} = t\,E_{\alpha,2}(-bt^\alpha), \tag{2.143}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha+b}\right\} = t^{\alpha-1}\,E_{\alpha,\alpha}(-bt^\alpha). \tag{2.144}$$

The series representation of the Mittag-Leffler functions is inconvenient for numerical calculation. The integral representations of these functions suitable for such calculation were obtained in [52, 56]. In the subsequent discussion we restrict ourselves to the case of negative real values of argument. We have

$$\begin{aligned} E_\alpha(-bt^\alpha) &= \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha+b}\right\} = \frac{1}{2\pi i}\int_{Br}\mathrm{e}^{st}\frac{s^{\alpha-1}}{s^\alpha+b}\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{Ha}\mathrm{e}^{st}\frac{s^{\alpha-1}}{s^\alpha+b}\,\mathrm{d}s + \sum_k \mathrm{Res}_{\,s_k}\left(\mathrm{e}^{st}\frac{s^{\alpha-1}}{s^\alpha+b}\right) \\ &= f_\alpha(b,t) + g_\alpha(b,t). \end{aligned} \tag{2.145}$$

On the upper and lower sides of the Hankel path

$$s = r\mathrm{e}^{\pm i\pi} \pm i\varepsilon. \tag{2.146}$$

If $\varepsilon \to 0$, then

$$\mathrm{d}r = -\mathrm{d}s, \qquad s^\alpha = r^\alpha\left[\cos(\alpha\pi) \pm i\sin(\alpha\pi)\right]$$

and

$$\begin{aligned} f_\alpha(b,t) &= \frac{1}{\pi}\,\mathfrak{Im}\int_0^\infty \mathrm{e}^{-tr}\frac{r^{\alpha-1}[\cos(\alpha\pi)+i\sin(\alpha\pi)]}{r^\alpha[\cos(\alpha\pi)+i\sin(\alpha\pi)]+b}\,\mathrm{d}r \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \mathrm{e}^{-tr}\frac{br^{\alpha-1}}{r^{2\alpha}+2r^\alpha b\cos(\alpha\pi)+b^2}\,\mathrm{d}r. \end{aligned} \tag{2.147}$$

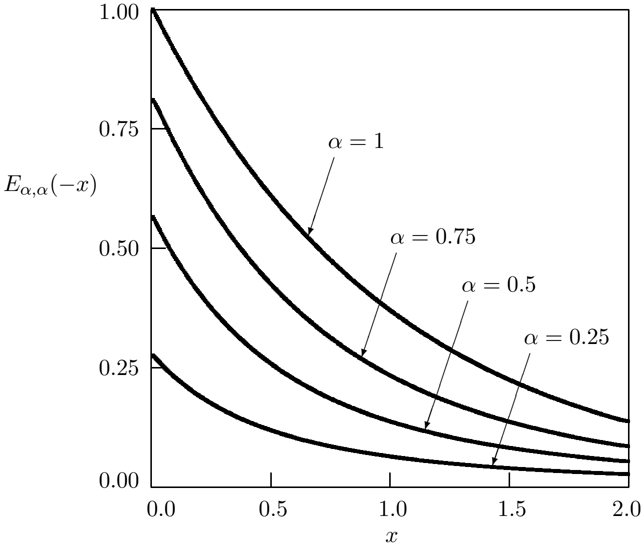

Figure 2.6: The Mittag-Leffler functions $E_{\alpha,\alpha}(-x)$ for $0 \leq \alpha \leq 1$

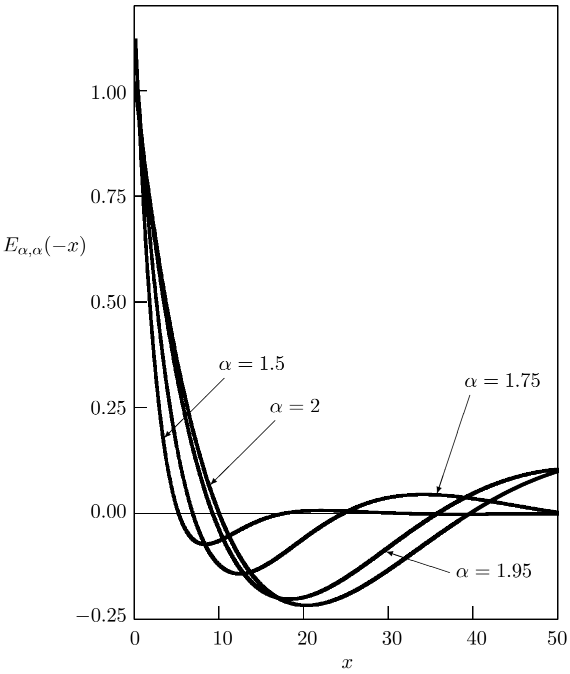

Figure 2.7: The Mittag-Leffler functions $E_{\alpha,\alpha}(-x)$ for $1 < \alpha \leq 2$

It is worthwhile introducing the substitution $r = b^{1/\alpha}u$ which leads to

$$f_\alpha(b,t) = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \mathrm{e}^{-tb^{1/\alpha}u}\,\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u. \tag{2.148}$$

To investigate the poles of $\dfrac{s^{\alpha-1}}{s^\alpha+b}$ it should be mentioned that

$$s_k = b^{1/\alpha}\left[\cos\frac{(2k+1)\pi}{\alpha} + i\sin\frac{(2k+1)\pi}{\alpha}\right], \tag{2.149}$$

but only the poles situated in the main Riemann sheet are relevant, i.e., those s_k for which

$$-\pi < \frac{(2k+1)\pi}{\alpha} < \pi.$$

For $0<\alpha<1$ there are no such poles and

$$g_\alpha(b,t) = 0. \tag{2.150}$$

For $1<\alpha<2$ there are two such poles:

$$b^{1/\alpha}\left[\cos\left(\frac{\pi}{\alpha}\right) \pm i\sin\left(\frac{\pi}{\alpha}\right)\right],$$

and

$$g_\alpha(b,t) = \frac{2}{\alpha}\exp\left[tb^{1/\alpha}\cos\left(\frac{\pi}{\alpha}\right)\right]\,\cos\left[tb^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right)\right]. \tag{2.151}$$

Finally we arrive at the following result [52, 56]:

$$E_\alpha(-x) = \begin{cases} \dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u, \\ \hfill 0<\alpha<1; \\ \dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u \\ \quad +\dfrac{2}{\alpha}\exp\left[x^{1/\alpha}\cos\left(\dfrac{\pi}{\alpha}\right)\right]\,\cos\left[x^{1/\alpha}\sin\left(\dfrac{\pi}{\alpha}\right)\right], \\ \hfill 1<\alpha<2. \end{cases} \tag{2.152}$$

Similarly, for $1<\alpha<2$ we obtain

$$E_{\alpha,2}(-x) = -\frac{\sin(\alpha\pi)}{\pi\,x^{1/\alpha}}\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\frac{u^{\alpha-2}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u$$

$$+\frac{2}{\alpha x^{1/\alpha}}\exp\left[x^{1/\alpha}\cos\left(\frac{\pi}{\alpha}\right)\right]\,\cos\left[x^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right)-\frac{\pi}{\alpha}\right] \tag{2.153}$$

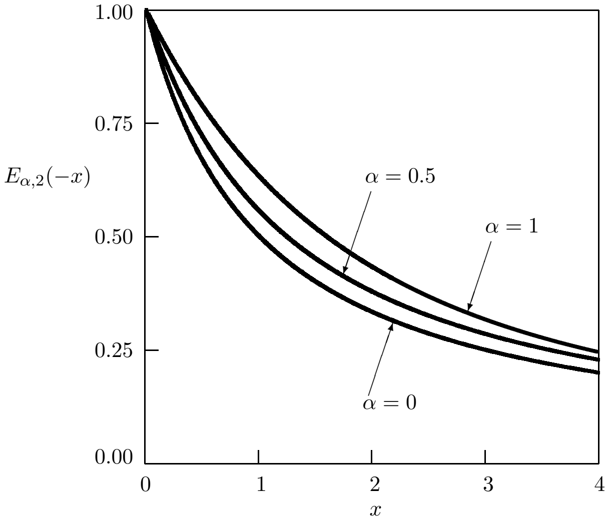

Figure 2.8: The Mittag-Leffler functions $E_{\alpha,2}(-x)$ for $0 \leq \alpha \leq 1$

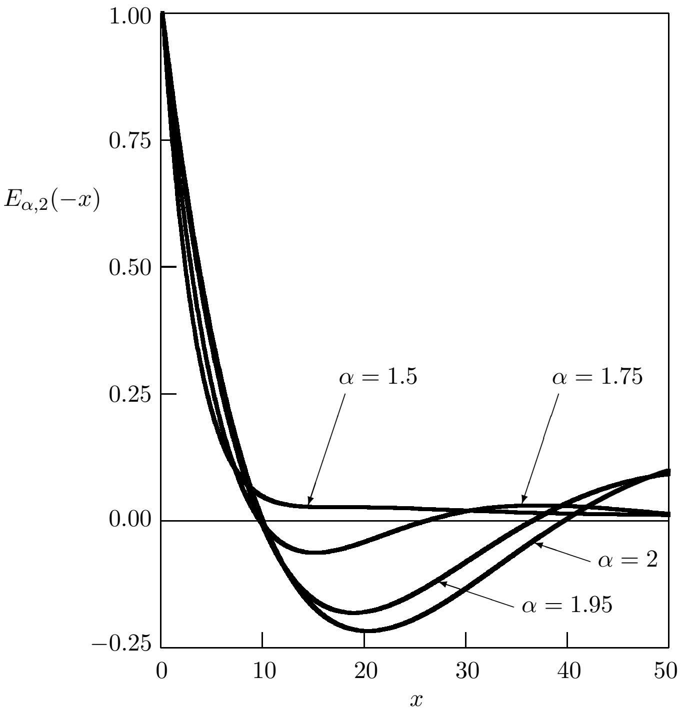

Figure 2.9: The Mittag-Leffler functions $E_{\alpha,2}(-x)$ for $1 < \alpha \leq 2$

and

$$E_{\alpha,\alpha}(-x) = \begin{cases} \dfrac{\sin(\alpha\pi)}{\pi x^{(\alpha-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u} \frac{u^\alpha}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u, \\ \hfill 0 < \alpha < 1; \\ \dfrac{\sin(\alpha\pi)}{\pi x^{(\alpha-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u} \frac{u^\alpha}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u \\ \quad - \dfrac{2}{\alpha x^{(\alpha-1)/\alpha}} \exp\left[x^{1/\alpha} \cos\left(\dfrac{\pi}{\alpha}\right)\right] \cos\left[x^{1/\alpha} \sin\left(\dfrac{\pi}{\alpha}\right) + \dfrac{\pi}{\alpha}\right], \\ \hfill 1 < \alpha < 2. \end{cases} \tag{2.154}$$

Typical curves for $E_\alpha(-x)$ are presented in Figs. 2.4 and 2.5; for $E_{\alpha,\alpha}(-x)$ are shown in Figs. 2.6 and 2.7; $E_{\alpha,2}(-x)$ are depicted in Figs. 2.8 and 2.9 for various values of α.

In the general case, the integral representation of the generalized Mittag-Leffler function $E_{\alpha,\beta}$ can be obtained for $\alpha > 0$, $\beta > 0$, $\beta < \alpha + 1$:

$$E_{\alpha,\beta}(-x) = \begin{cases} \dfrac{1}{\pi x^{(\beta-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\, u^{\alpha-\beta}\, \frac{u^\alpha \sin(\beta\pi) + \sin[(\beta-\alpha)\pi]}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u, \\ \hfill 0 < \alpha < 1; \\ \dfrac{1}{\pi x^{(\beta-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\, u^{\alpha-\beta}\, \frac{u^\alpha \sin(\beta\pi) + \sin[(\beta-\alpha)\pi]}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u \\ \quad + \dfrac{2}{\alpha x^{(\beta-1)/\alpha}} \exp\left[x^{1/\alpha} \cos\left(\dfrac{\pi}{\alpha}\right)\right] \cos\left[x^{1/\alpha} \sin\left(\dfrac{\pi}{\alpha}\right) + (1-\beta)\dfrac{\pi}{\alpha}\right] \\ \hfill 1 < \alpha < 2. \end{cases} \tag{2.155}$$

To investigate convergence of integral containing the Mittag-Leffler function it may be useful to have their asymptotic representations for large negative values of argument. Such a representation can be obtained expanding $(s^{\alpha-\beta})/(s^\alpha + b)$ in series for small s taking into account that

$$\frac{1}{s^\alpha + b} = \frac{1}{b}\left[1 - \frac{s^\alpha}{b} + \frac{s^{2\alpha}}{b^2} - \frac{s^{3\alpha}}{b^3} + \cdots\right]. \tag{2.156}$$

For $t \to \infty$ (see (2.15) and (2.16)) we have

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + b}\right\} \sim \frac{1}{\Gamma(1-\alpha) b t^\alpha} - \frac{1}{\Gamma(1-2\alpha) b^2 t^{2\alpha}}, \tag{2.157}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha+b}\right\} \sim \frac{1}{\Gamma(2-\alpha)bt^{\alpha-1}} - \frac{1}{\Gamma(2-2\alpha)b^2t^{2\alpha-1}}, \tag{2.158}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha+b}\right\} \sim -\frac{1}{\Gamma(-\alpha)b^2t^{\alpha+1}}, \tag{2.159}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha+b}\right\} \sim \frac{1}{\Gamma(\beta-\alpha)bt^{\alpha-\beta+1}} - \frac{1}{\Gamma(\beta-2\alpha)b^2t^{2\alpha-\beta+1}}. \tag{2.160}$$

Hence, for $x \to \infty$ the desired results read as follows:

$$E_\alpha(-x) \sim \frac{1}{\Gamma(1-\alpha)x} - \frac{1}{\Gamma(1-2\alpha)x^2}, \tag{2.161}$$

$$E_{\alpha,2}(-x) \sim \frac{1}{\Gamma(2-\alpha)x} - \frac{1}{\Gamma(2-2\alpha)x^2}, \tag{2.162}$$

$$E_{\alpha,\alpha}(-x) \sim -\frac{1}{\Gamma(-\alpha)x^2}, \tag{2.163}$$

$$E_{\alpha,\beta}(-x) \sim \frac{1}{\Gamma(\beta-\alpha)x} - \frac{1}{\Gamma(\beta-2\alpha)x^2}. \tag{2.164}$$

2.3 Wright function and Mainardi function

The Wright function is defined as [43, 53, 54, 77, 90, 100, 101, 107, 143]

$$W(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k+\beta)}, \qquad \alpha > -1, \quad z \in C. \tag{2.165}$$

Its integral representation has the following form [100, 143]:

$$W(\alpha,\beta;z) = \frac{1}{2\pi i}\int_{Ha} s^{-\beta} e^{s+zs^{-\alpha}}\,\mathrm{d}s, \qquad \alpha > -1, \quad z \in C, \tag{2.166}$$

where Ha denotes the Hankel path of integration in the complex s-plane.

The Wright function is a generalization of the exponential function and the Bessel functions (see [100, 143]):

$$W(0,1;z) = e^z, \tag{2.167}$$

$$W\left(-\frac{1}{2},\frac{1}{2};-z\right) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right), \tag{2.168}$$

$$W\left(1,\nu+1;-\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^\nu J_\nu(z), \tag{2.169}$$

$$W\left(1, \nu+1; \frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^{\nu} I_{\nu}(z). \tag{2.170}$$

Comparison of the definition of the Wright function (2.165) and the series expansion of the complementary error function [1]

$$\operatorname{erfc}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (2z)^k}{k!\,\Gamma\left(-\frac{1}{2}k+1\right)} \tag{2.171}$$

allows us to obtain the additional relation

$$W\left(-\frac{1}{2}, 1; -z\right) = \operatorname{erfc}\left(\frac{z}{2}\right). \tag{2.172}$$

The Wright function satisfies the equations

$$\alpha z W(\alpha, \alpha+\beta; z) = W(\alpha, \beta-1; z) + (1-\beta) W(\alpha, \beta; z), \tag{2.173}$$

$$\frac{\mathrm{d}W(\alpha, \beta; z)}{\mathrm{d}z} = W(\alpha, \alpha+\beta; z). \tag{2.174}$$

The Mainardi function $M(\alpha; z)$ [100, 101, 143] is the particular case of the Wright function

$$M(\alpha; z) = W(-\alpha, 1-\alpha; -z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma[-\alpha k + (1-\alpha)]},$$
$$0 < \alpha < 1, \qquad z \in C, \tag{2.175}$$

and also

$$M(\alpha; z) = \frac{1}{\alpha z} W(-\alpha, 0; -z), \qquad 0 < \alpha < 1. \tag{2.176}$$

For $\alpha = 1/q$, where $q \geq 2$ is a positive integer, the Mainardi function can be expressed in terms of simpler functions, for example [100, 101]:

$$M\left(\frac{1}{2}; z\right) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right), \tag{2.177}$$

$$M\left(\frac{1}{3}; z\right) = 3^{2/3} \operatorname{Ai}\left(\frac{z}{3^{1/3}}\right). \tag{2.178}$$

Similarly (see [67]):

$$M\left(\frac{2}{3}; z\right) = \exp\left(-\frac{2z^3}{27}\right)\left[3^{-1/3} z \operatorname{Ai}\left(\frac{z^2}{3^{4/3}}\right) - 3^{1/3} \operatorname{Ai}'\left(\frac{z^2}{3^{4/3}}\right)\right], \tag{2.179}$$

where Ai(z) is the Airy function, the prime denotes the derivative.

The Mainardi and Wright functions appear in the formulae for the inverse Laplace transform [100, 101]

$$\mathcal{L}^{-1}\left\{\exp\left(-\lambda s^{\alpha}\right)\right\} = \frac{\alpha\lambda}{t^{\alpha+1}} M\left(\alpha; \lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{2.180}$$

$$\mathcal{L}^{-1}\left\{s^{\alpha-1}\exp\left(-\lambda s^{\alpha}\right)\right\} = \frac{1}{t^{\alpha}} M\left(\alpha; \lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{2.181}$$

and [214] (see also [47, 100, 101, 117])

$$\mathcal{L}^{-1}\left\{s^{-\beta}\exp\left(-\lambda s^{\alpha}\right)\right\} = t^{\beta-1} W\left(-\alpha, \beta; -\lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0. \tag{2.182}$$

The Laplace transform of the Wright function is expressed in terms of the Mittag-Leffler function [43, 77, 143]

$$\mathcal{L}\left\{W(\alpha, \beta; t)\right\} = \frac{1}{s}\, E_{\alpha,\beta}\left(\frac{1}{s}\right). \tag{2.183}$$

Integration of (2.174) gives

$$\int\limits_0^{\infty} W(\alpha, \beta; -x)\, \mathrm{d}x = \frac{1}{\Gamma(\beta-\alpha)}, \tag{2.184}$$

in particular

$$\int\limits_0^{\infty} M(\alpha; x)\, \mathrm{d}x = 1. \tag{2.185}$$

The Mittag-Leffler function and the Mainardi function are related by the cos-Fourier transform:

$$M\left(\frac{\alpha}{2}; x\right) = \frac{2}{\pi}\int\limits_0^{\infty} E_{\alpha}(-\xi^2)\, \cos(x\xi)\, \mathrm{d}\xi, \qquad 0 < \alpha < 2. \tag{2.186}$$

Similar relations are valid for the following Wright functions:

$$W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -x\right) = \frac{2}{\pi}\int\limits_0^{\infty} E_{\alpha,2}(-\xi^2)\, \cos(x\xi)\, \mathrm{d}\xi, \quad 0 < \alpha < 2, \tag{2.187}$$

$$W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -x\right) = \frac{2}{\pi}\int\limits_0^{\infty} E_{\alpha,\alpha}(-\xi^2)\, \cos(x\xi)\, \mathrm{d}\xi, \quad 0 < \alpha < 2. \tag{2.188}$$

The relation above are proved in Chapter 4 (see (4.11) and (4.14), (4.27) and (4.28), (4.34) and (4.35)).

Chapter 3

Physical Backgrounds

It's a poor sort of memory that only works backwards.

Lewis Carroll
"Through the Looking-Glass"

Who controls the past controls the future.
Who controls the present controls the past.

George Orwell
"Nineteen Eighty-Four"

3.1 Nonlocal generalizations of the Fourier law

The conventional theory of heat conduction is based on the classical (local) Fourier law, which relates the heat flux vector $\mathbf{q}$ to the temperature gradient

$$\mathbf{q} = -k \,\mathrm{grad}\, T, \tag{3.1}$$

where k is the thermal conductivity of a solid. In combination with a law of conservation of energy

$$\rho\, C\, \frac{\partial T}{\partial t} = -\mathrm{div}\, \mathbf{q} \tag{3.2}$$

with ρ being the mass density, C specific heat capacity, the Fourier law (3.1) leads to the parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a\, \Delta T, \tag{3.3}$$

where $a = k/(\rho\, C)$ is the thermal diffusivity coefficient.

Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*,
DOI 10.1007/978-3-319-17954-4_3

It should be noted that (3.1) is a phenomenological law which states the proportionality of the flux to the gradient of the transported quantity. It is met in several physical contexts with different names. It is well known that from a mathematical viewpoint, the Fourier law (3.1) in the theory of heat conduction, and the Fick law in the theory of diffusion,

$$\mathbf{J} = -k_c \operatorname{grad} c, \tag{3.4}$$

where $\mathbf{J}$ is the matter flux, c is the concentration, k_c is the diffusion conductivity, are identical.

In combination with the balance equation for mass,

$$\rho \frac{\partial c}{\partial t} = -\operatorname{div} \mathbf{J}, \tag{3.5}$$

the Fick law leads to the classical diffusion equation:

$$\frac{\partial c}{\partial t} = a_c \, \Delta c. \tag{3.6}$$

Here $a_c = k_c/\rho$ is the diffusivity coefficient, ρ is the mass density.

Similarly, the classical empirical Darcy law, describing the flow of fluid through a porous medium, states proportionality between the fluid mass flux $\mathbf{J}$ and the gradient of the pore pressure p [12–14, 30, 207]:

$$\mathbf{J} = -k_p \operatorname{grad} p, \tag{3.7}$$

where k_p is the hydraulic conductivity, and in combination with the balance equation

$$\beta\phi \frac{\partial p}{\partial t} = -\operatorname{div} \mathbf{J}, \tag{3.8}$$

leads to the parabolic equation for the pressure

$$\frac{\partial p}{\partial t} = a_p \, \Delta p. \tag{3.9}$$

Here β is the compressibility coefficient characterizing the fluid and skeleton, ϕ is porosity, $a_p = k_p/(\beta\phi)$ is the hydraulic diffusivity coefficient.

In this book we discuss heat conduction, but it is obvious that the discussion concerns also diffusion as well as the theory of fluid flow through a porous solid.

Nonclassical theories, in which the Fourier law and the standard heat conduction equations are replaced by more general equations, constantly attract the attention of researchers.

The general time-nonlocal constitutive equations for heat flux were considered in [33, 62, 123, 127, 128, 130, 131]. Choosing 0 as a “starting point”, we can write

$$\mathbf{q}(t) = -k \int_0^t K(t-\tau) \operatorname{grad} T(\tau) \, \mathrm{d}\tau \tag{3.10}$$

and the heat conduction equation with memory [127, 128]

$$\frac{\partial T}{\partial t} = a \int_0^t K(t-\tau)\, \Delta T(\tau)\, \mathrm{d}\tau. \tag{3.11}$$

Here $K(t-\tau)$ is the weight function (the time-nonlocality kernel).

The classical Fourier law (3.1) and the standard heat conduction equation (3.3) are obtained for "instantaneous memory" with the kernel being Dirac's delta.

"Full sclerosis" corresponds to the choice of the kernel as the time derivative of Dirac's delta or

$$\mathbf{q}(t) = -k \frac{\partial}{\partial t} \operatorname{grad} T(t), \tag{3.12}$$

thus leading to the Helmholtz equation for temperature

$$T = a \Delta T. \tag{3.13}$$

"Full memory" [61, 127] means that there is no fading of memory, the kernel is constant and

$$\mathbf{q}(t) = -k \int_0^t \operatorname{grad} T(\tau)\, \mathrm{d}\tau. \tag{3.14}$$

As a result we have the wave equation for temperature

$$\frac{\partial^2 T}{\partial t^2} = a \Delta T. \tag{3.15}$$

The time-nonlocal dependence between the heat flux vector and the temperature gradient with the "long-tail" power kernel [145, 158, 159, 165, 166]

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{\alpha-1} \operatorname{grad} T(\tau)\, \mathrm{d}\tau, \qquad 0 < \alpha \le 1, \tag{3.16}$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \operatorname{grad} T(\tau)\, \mathrm{d}\tau, \qquad 1 < \alpha \le 2, \tag{3.17}$$

can be interpreted in terms of fractional integrals and derivatives

$$\mathbf{q}(t) = -k D_{RL}^{1-\alpha} \operatorname{grad} T(t), \qquad 0 < \alpha \le 1, \tag{3.18}$$

$$\mathbf{q}(t) = -k I^{\alpha-1} \operatorname{grad} T(t), \qquad 1 < \alpha \le 2. \tag{3.19}$$

In the case $0 < \alpha \le 1$, as a consequence of (1.5), (3.2) and (3.18), we have

$$\frac{\partial T}{\partial t} = a \frac{\partial}{\partial t} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \Delta T(\tau)\, \mathrm{d}\tau \right]. \tag{3.20}$$

Integrating (3.20) with respect to time, we obtain

$$T(t) - T(0) = a\,\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\Delta T(\tau)\,\mathrm{d}\tau \tag{3.21}$$

or

$$T(t) - T(0) = aI^{\alpha}\,\Delta T\,. \tag{3.22}$$

It should be emphasized that the Caputo fractional derivative of constant is zero (see (1.8)). Hence, applying to both sides of (3.22) the Caputo derivative $\frac{\partial^\alpha}{\partial t^\alpha}$ and taking into account that for $\alpha > 0$ [77]

$$\frac{\partial^\alpha}{\partial t^\alpha}\,I^{\alpha}T(t) = T(t), \tag{3.23}$$

we obtain

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta T, \qquad 0 < \alpha \le 1. \tag{3.24}$$

Similarly, for $1 < \alpha \le 2$ we get

$$\frac{\partial T}{\partial t} = a\,I^{\alpha-1}\Delta T \tag{3.25}$$

or after applying $\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}$ to both sides of (3.25)

$$\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\,\frac{\partial T}{\partial t} = a\,\Delta T. \tag{3.26}$$

In the general case,

$$\frac{\partial^\alpha}{\partial t^\alpha}\,\frac{\partial^\beta T}{\partial t^\beta} \ne \frac{\partial^{\alpha+\beta} T}{\partial t^{\alpha+\beta}}, \tag{3.27}$$

but for integer $\beta = m$ [143]

$$\frac{\partial^\alpha}{\partial t^\alpha}\,\frac{\partial^m T}{\partial t^m} = \frac{\partial^{\alpha+m} T}{\partial t^{\alpha+m}}. \tag{3.28}$$

Therefore, (3.26) can be rewritten as

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta T, \qquad 1 < \alpha \le 2. \tag{3.29}$$

Hence, the constitutive equations (3.18) and (3.19) yield the time-fractional heat conduction (diffusion) equation with Caputo derivative:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T, \qquad 0 < \alpha \le 2. \tag{3.30}$$

Equation (3.30) describes the whole spectrum from local heat conduction (the Helmholtz equation (3.13) when $\alpha \to 0$) through the standard heat conduction equation (3.3) ($\alpha = 1$) to the ballistic heat conduction (the wave equation (3.15) when $\alpha = 2$).

3.2 Boundary conditions

If the time-fractional diffusion-wave equation is considered in a bounded domain, the corresponding boundary conditions should be imposed. Different kinds of boundary conditions for equation (3.30) were analyzed in [180, 183]. It should be emphasized that due to the generalized constitutive equations for the heat flux (3.18) and (3.19) the boundary conditions for the time-fractional heat conduction equation have their traits in comparison with those for the standard heat conduction equation.

The Dirichlet boundary condition (the boundary condition of the first kind) specifies the temperature over the surface of the body

$$T\Big|_S = g(\mathbf{x}_S, t), \tag{3.31}$$

where $\mathbf{x}_S$ is a point at a surface S.

For the fractional heat conduction equation, two types of Neumann boundary condition (the boundary condition of the second kind) can be considered: the mathematical condition with the prescribed boundary value of the normal derivative

$$\frac{\partial T}{\partial n}\bigg|_S = g(\mathbf{x}_S, t) \tag{3.32}$$

and the physical condition with the prescribed boundary value of the heat flux

$$D_{RL}^{1-\alpha}\frac{\partial T}{\partial n}\bigg|_S = g(\mathbf{x}_S, t), \qquad 0 < \alpha \leq 1, \tag{3.33}$$

$$I^{\alpha-1}\frac{\partial T}{\partial n}\bigg|_S = g(\mathbf{x}_S, t), \qquad 1 < \alpha \leq 2, \tag{3.34}$$

where $\partial/\partial n$ denotes differentiation along the outward-drawn normal at the boundary surface S.

Similarly, the mathematical Robin boundary condition (the boundary condition of the third kind) is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain

$$\left(c_1 T + c_2\frac{\partial T}{\partial n}\right)\bigg|_S = g(\mathbf{x}_S, t) \tag{3.35}$$

with some nonzero constants c_1 and c_2, while the physical Robin boundary condition specifies a linear combination of the values of temperature and the values of the heat flux at the boundary. For example, the Newton condition of convective heat exchange between a body and the environment with the temperature T_e

$$\mathbf{q}\cdot\mathbf{n}\Big|_S = H\left(T\Big|_S - T_e\right), \tag{3.36}$$

where H is the convective heat transfer coefficient, leads to

$$\left(HT + kD_{RL}^{1-\alpha}\frac{\partial T}{\partial n}\right)\bigg|_S = g(\mathbf{x}_S, t), \qquad 0 < \alpha \le 1, \tag{3.37}$$

$$\left(HT + kI^{\alpha-1}\frac{\partial T}{\partial n}\right)\bigg|_S = g(\mathbf{x}_S, t), \qquad 1 < \alpha \le 2, \tag{3.38}$$

with $g(\mathbf{x}_S, t) = HT_e(\mathbf{x}_S, t)$.

In the case of the classical heat conduction equation ($\alpha = 1$) the mathematical and physical Neumann boundary conditions coincide as well as the mathematical and physical Robin boundary conditions, but for fractional heat conduction equation ($\alpha \neq 1$) they are essentially different.

It should be noted that in fractional calculus, where integrals and derivatives of arbitrary (not integer) order are considered, there is no sharp boundary between integration and differentiation. For this reason, some authors [99, 143] do not use a separate notation for the fractional integral $I^\alpha f(t)$. The fractional integral of the order $\alpha > 0$ is denoted as

$$I^\alpha f(t) = D_{RL}^{-\alpha} f(t), \qquad \alpha > 0.$$

Using this notation, Eqs. (3.18) and (3.19) can be rewritten as one dependence

$$\mathbf{q}(t) = -kD_{RL}^{1-\alpha}\operatorname{grad} T(t), \qquad 0 < \alpha \le 2. \tag{3.39}$$

If the surfaces of two solids are in perfect thermal contact, the temperatures on the contact surface and the heat fluxes through the contact surface are the same for both solids, and we obtain the boundary conditions of the fourth kind:

$$T_1\Big|_S = T_2\Big|_S, \tag{3.40}$$

$$k_1 D_{RL}^{1-\alpha}\frac{\partial T_1}{\partial n}\bigg|_S = k_2 D_{RL}^{1-\beta}\frac{\partial T_2}{\partial n}\bigg|_S, \quad 0 < \alpha \le 2, \quad 0 < \beta \le 2, \tag{3.41}$$

where subscripts 1 and 2 refer to solids 1 and 2, respectively, and n is the common normal at the contact surface.

Chapter 4

Equations with One Space Variable in Cartesian Coordinates

I am among those who think that science has great beauty.

Maria Curie-Skłodowska

4.1 Domain $-\infty < x < \infty$

4.1.1 Statement of the problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2} + \Phi(x,t), \tag{4.1}$$

$$t = 0: \quad T = f(x), \qquad 0 < \alpha \leq 2, \tag{4.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x), \quad 1 < \alpha \leq 2, \tag{4.3}$$

$$\lim_{x \to \pm\infty} T(x,t) = 0. \tag{4.4}$$

The solution:

$$T(x,t) = \int\limits_{-\infty}^{\infty} f(\xi)\,\mathcal{G}_f(x-\xi,t)\,\mathrm{d}\xi + \int\limits_{-\infty}^{\infty} F(\xi)\,\mathcal{G}_F(x-\xi,t)\,\mathrm{d}\xi$$

$$+ \int\limits_0^t \int\limits_{-\infty}^{\infty} \Phi(\xi,\tau)\,\mathcal{G}_\Phi(x-\xi,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau, \tag{4.5}$$

where $\mathcal{G}_f(x,t)$ is the fundamental solution to the first Cauchy problem, $\mathcal{G}_F(x,t)$ is the fundamental solution to the second Cauchy problem, $\mathcal{G}_\Phi(x,t)$ is the fundamental solution to the source problem.

Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*,
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4.1.2 Fundamental solution to the first Cauchy problem

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a\,\frac{\partial^2 \mathcal{G}_f}{\partial x^2}, \tag{4.6}$$

$$t = 0: \quad \mathcal{G}_f = p_0\,\delta(x), \qquad 0 < \alpha \le 2, \tag{4.7}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{4.8}$$

For the sake of convenience and to obtain the nondimensional quantities used in calculations we have introduced the constant multiplier p_0 in (4.7). In the subsequent text we will use constant multipliers in the fundamental solutions: p_0 for the first Cauchy problem, w_0 for the second Cauchy problem, q_0 for the source problem, and g_0 for the boundary-value problem, respectively.

Using the Laplace transform (2.1) with respect to time t and the exponential Fourier transform (2.20) with respect to the space coordinate x, we obtain

$$\widetilde{\mathcal{G}}_f^*(\xi, s) = \frac{p_0}{\sqrt{2\pi}}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}, \tag{4.9}$$

where s is the Laplace transform variable, ξ is the Fourier transform variable.

The inverse Laplace transform, taking into account (2.142), gives

$$\widetilde{\mathcal{G}}_f(\xi, t) = \frac{p_0}{\sqrt{2\pi}}\,E_\alpha\left(-a\xi^2 t^\alpha\right), \tag{4.10}$$

and the inverse Fourier transform results in [106, 155]

$$\mathcal{G}_f(x, t) = \frac{p_0}{2\pi}\int_{-\infty}^{\infty} E_\alpha(-a\xi^2 t^\alpha)\,\cos(x\xi)\,d\xi. \tag{4.11}$$

We may change the order of the inverse integral transforms starting from the inverse Fourier transform, thus obtaining

$$\mathcal{G}_f^*(x, s) = \frac{p_0}{2\pi}\int_{-\infty}^{+\infty} \frac{s^{\alpha-1}}{s^\alpha + a\xi^2}\,\cos(x\xi)\,\mathrm{d}\xi. \tag{4.12}$$

Evaluating the integral in (4.12) (see equation (A.7) from Appendix), we get

$$\mathcal{G}_f^*(x, s) = \frac{p_0}{2\sqrt{a}}\,s^{\alpha/2-1}\exp\left(-\frac{|x|}{\sqrt{a}}\,s^{\alpha/2}\right). \tag{4.13}$$

Using (2.181), we arrive at [100]

$$\mathcal{G}_f(x, t) = \frac{p_0}{2\sqrt{a}t^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a}t^{\alpha/2}}\right). \tag{4.14}$$

A comparison of (4.11) and (4.14) proves (2.186).

Let us now look at the particular cases – solutions of classical equations.

Helmholtz equation ($\alpha \to 0$)

$$\mathcal{G}_f = \frac{p_0}{2\sqrt{a}}\, \mathrm{e}^{-|x|/\sqrt{a}}. \tag{4.15}$$

Classical diffusion equation ($\alpha = 1$)

$$\mathcal{G}_f = \frac{p_0}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at}\right). \tag{4.16}$$

Wave equation ($\alpha = 2$)

$$\mathcal{G}_f = \frac{p_0}{2}\left[\delta(x-\sqrt{a}t)+\delta(x+\sqrt{a}t)\right]. \tag{4.17}$$

Subdiffusion with $\alpha = 1/2$

$$\mathcal{G}_f = \frac{p_0}{\sqrt{2a}\pi t^{1/4}} \int\limits_0^\infty \frac{1}{\sqrt{u}} \exp\left(-u^2 - \frac{x^2}{8a\sqrt{t}u}\right) \mathrm{d}u. \tag{4.18}$$

Subdiffusion with $\alpha = 2/3$

$$\mathcal{G}_f = \frac{3^{2/3}p_0}{2\sqrt{a}t^{1/3}} \mathrm{Ai}\left(\frac{|x|}{3^{1/3}\sqrt{a}t^{1/3}}\right). \tag{4.19}$$

Fast diffusion with $\alpha = 4/3$

$$\begin{aligned}\mathcal{G}_f \;=\; & \frac{3^{1/3}p_0}{2\sqrt{a}t^{2/3}}\left[\frac{|x|}{3^{2/3}\sqrt{a}t^{2/3}}\mathrm{Ai}\left(\frac{x^2}{3^{4/3}at^{4/3}}\right) - \mathrm{Ai}'\left(\frac{x^2}{3^{4/3}at^{4/3}}\right)\right] \\ & \times \exp\left(-\frac{2|x|^3}{27a^{2/3}t^2}\right),\end{aligned} \tag{4.20}$$

where $\mathrm{Ai}(x)$ is the Airy function; the prime denotes the derivative.

Dependence of the nondimensional solution

$$\overline{\mathcal{G}}_f = \frac{\sqrt{a}t^{\alpha/2}}{p_0}\mathcal{G}_f \tag{4.21}$$

on the nondimensional distance (similarity variable)

$$\bar{x} = \frac{x}{\sqrt{a}t^{\alpha/2}} \tag{4.22}$$

is shown in Fig. 4.1 for various values of α. In Fig. 4.2 we depict the typical curves for $1 \le \alpha \le 2$ and $\bar{x} > 0$. The vertical lines in Figs. 4.1 and 4.2 represent the Dirac delta function.

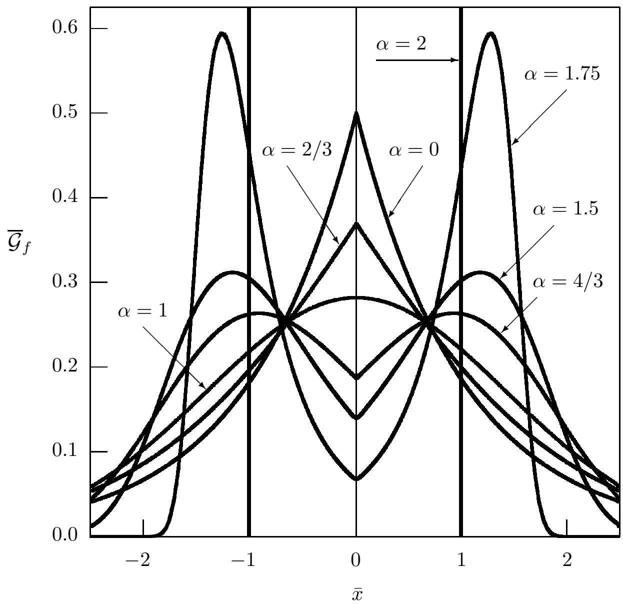


Figure 4.1: Dependence of the fundamental solution to the first Cauchy problem on distance

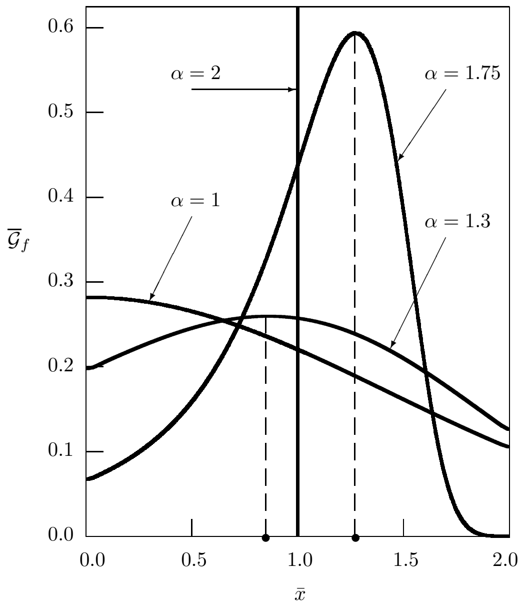


Figure 4.2: Representative curves for $1 \le \alpha \le 2$ (the fundamental solution to the first Cauchy problem) [155]

4.1.3 Fundamental solution to the second Cauchy problem

$$\frac{\partial^\alpha \mathcal{G}_F}{\partial t^\alpha} = a\,\frac{\partial^2 \mathcal{G}_F}{\partial x^2}, \tag{4.23}$$

$$t = 0: \quad \mathcal{G}_F = 0, \qquad 1 < \alpha \le 2, \tag{4.24}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_F}{\partial t} = w_0\,\delta(x), \quad 1 < \alpha \le 2. \tag{4.25}$$

The Laplace transform with respect to time t and the exponential Fourier transform with respect to the space coordinate x lead to

$$\widetilde{\mathcal{G}}_F^*(\xi, s) = \frac{w_0}{\sqrt{2\pi}}\,\frac{s^{\alpha-2}}{s^\alpha + a\xi^2}. \tag{4.26}$$

First, we invert the Laplace transform, next the Fourier transform, thus obtaining

$$\mathcal{G}_F(x,t) = \frac{w_0 t}{2\pi}\int_{-\infty}^{\infty} E_{\alpha,2}(-a\xi^2 t^\alpha)\,\cos(x\xi)\,d\xi. \tag{4.27}$$

Changing the order of the inverse transforms, we get the solution in terms of the Wright function (see [67]):

$$\mathcal{G}_F(x,t) = \frac{w_0 t^{1-\alpha/2}}{2\sqrt{a}}\,W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|x|}{\sqrt{a}t^{\alpha/2}}\right). \tag{4.28}$$

A comparison of (4.27) and (4.28) proves the relation (2.187).

The particular case of the obtained fundamental solution corresponding to the wave equation ($\alpha = 2$) has the following form:

$$\mathcal{G}_F(x,t) = \begin{cases} \dfrac{w_0}{2\sqrt{a}}, & |x| < \sqrt{a}t, \\ 0, & |x| > \sqrt{a}t. \end{cases} \tag{4.29}$$

4.1.4 Fundamental solution to the source problem

$$\frac{\partial^\alpha \mathcal{G}_\Phi}{\partial t^\alpha} = a\,\frac{\partial^2 \mathcal{G}_\Phi}{\partial x^2} + q_0\,\delta(x)\,\delta(t), \tag{4.30}$$

$$t = 0: \quad \mathcal{G}_\Phi = 0, \qquad 0 < \alpha \le 2, \tag{4.31}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_\Phi}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{4.32}$$

The solution in the transform domain:

$$\widetilde{\mathcal{G}}_\Phi^*(\xi, s) = \frac{q_0}{\sqrt{2\pi}}\,\frac{1}{s^\alpha + a\xi^2}. \tag{4.33}$$

First, inverting the Laplace transform, next the Fourier transform, we get

$$\mathcal{G}_\Phi(x,t) = \frac{q_0 t^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \cos(x\xi)\, d\xi. \tag{4.34}$$

Changing the order of the inverse transforms, we obtain the solution in terms of the Wright function (see [67]):

$$\mathcal{G}_\Phi(x,t) = \frac{q_0}{2\sqrt{a}t^{1-\alpha/2}} W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|x|}{\sqrt{a}t^{\alpha/2}}\right). \tag{4.35}$$

A comparison of (4.34) and (4.35) proves the relation (2.188).

In the case of the standard diffusion equation the solution (4.34) coincides with (4.16); for the wave equation the solution (4.34) coincides with the solution (4.29).

Subdiffusion with $\alpha = 1/2$

$$\mathcal{G}_\Phi = \frac{q_0}{\sqrt{2a\pi}t^{3/4}} \int_0^\infty \sqrt{u}\, \exp\left(-u^2 - \frac{x^2}{8a\sqrt{t}u}\right) \mathrm{d}u. \tag{4.36}$$

Subdiffusion with $\alpha = 2/3$

$$\mathcal{G}_\Phi = \frac{3^{1/3} q_0}{2\sqrt{a}t^{2/3}} \mathrm{Ai}'\left(\frac{|x|}{3^{1/3}\sqrt{a}t^{1/3}}\right). \tag{4.37}$$

Fast diffusion with $\alpha = 4/3$

$$\mathcal{G}_\Phi = \frac{3^{2/3} q_0}{2\sqrt{a}t^{1/3}} \mathrm{Ai}\left(\frac{x^2}{3^{4/3} a t^{4/3}}\right) \exp\left(-\frac{2}{27}\frac{x^3}{a^{3/2}t^2}\right). \tag{4.38}$$

The solutions (4.37) and (4.38) were obtained in [67].

Dependence of the nondimensional fundamental solutions

$$\overline{\mathcal{G}}_F = \frac{\sqrt{a}}{w_0 t^{1-\alpha/2}} \mathcal{G}_F, \qquad \overline{\mathcal{G}}_\Phi = \frac{\sqrt{a}t^{1-\alpha/2}}{q_0} \mathcal{G}_\Phi \tag{4.39}$$

on the similarity variable is presented in Figs. 4.3 and 4.4, respectively. The similarity variable $\bar{x}$ is the same as in (4.22). Comparison of solutions (4.27) and (4.34) presented in Figs. 4.3 and 4.4 shows that the step curve corresponding to the limiting case $\alpha = 2$ is approximated by solution (4.27) and (4.34) in different ways. It also should be mentioned that the solution (4.27) is considered only for $1 < \alpha \le 2$, whereas the solution (4.34) is valid for $0 < \alpha \le 2$.

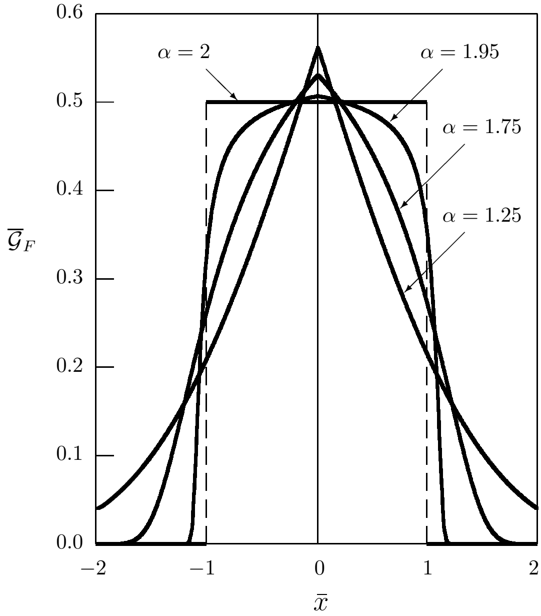


Figure 4.3: The fundamental solution to the second Cauchy problem

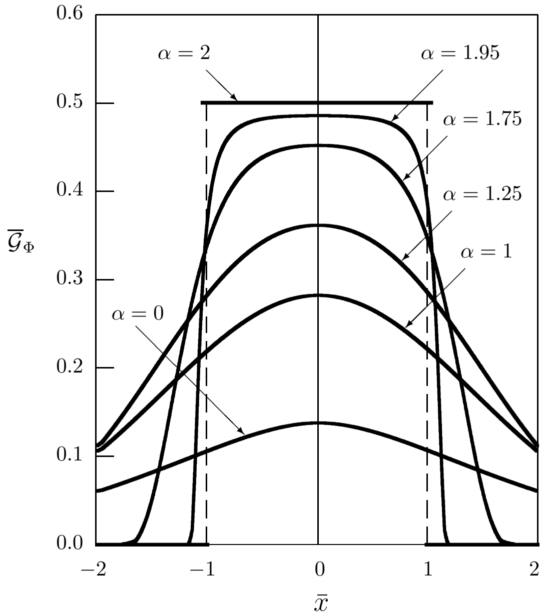


Figure 4.4: The fundamental solution to the source problem

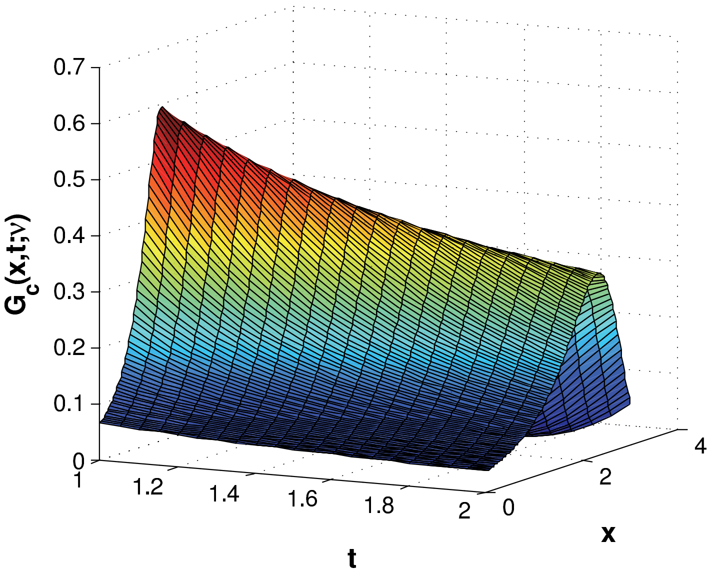
4.2 Distinguishing features of the fundamental solution to the first Cauchy problem

It is well known that the diffusion and wave equations behave quite differently regarding their response to a localized disturbance: whereas the diffusion equation describes a process, where a disturbance spreads infinitely fast, the propagation speed of the disturbance is constant for the wave equation. In a certain sense, the time-fractional diffusion-wave equation interpolates between these two different responses. On the one hand, the support of the solution to this equation is not compact on the real line for each $t > 0$ for a non-negative disturbance that is not identically equal to zero, i.e., its response to a localized disturbance spreads infinitely fast (see [45]). On the other hand, the fundamental solution to the time-fractional diffusion-wave equation possesses a maximum that disperses with a finite speed similar to the behavior of the fundamental solution of the wave equation. The problem to describe location of the maximum of the fundamental solution to the first Cauchy problem for the one-dimensional time-fractional diffusion-wave equation of order $1 < \alpha < 2$ was considered for the first time in [45]. Fujita proved that the fundamental solution takes its maximum at the point $x_* = \pm c_\alpha t^{\alpha/2}$ for each $t > 0$, where $c_\alpha > 0$ is a constant determined by α. Another proof of this formula for the maximum location along with numerical results for the constant c_α for $1 < \alpha < 2$ were presented in [155]. An extension and consolidation of these results was provided in [97].

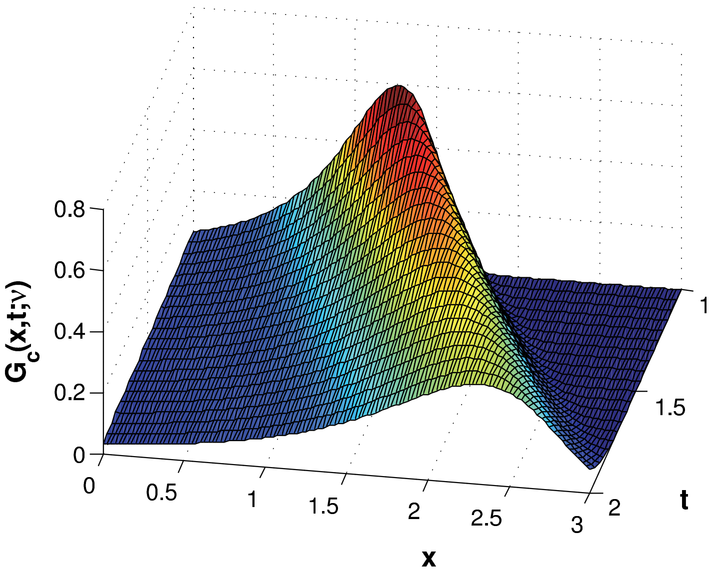
It should be noted that the similarity variable (4.22) plays a very important role in the analysis of the maximum point location of the fundamental solution $\mathcal{G}_f$. The form of the similarity variable can be explained by the Lie group analysis of the time-fractional diffusion-wave equation (4.6) [19, 54, 95]. In particular, it has been proved in [19] that the only invariant of the symmetry group of scaling transformations of the time-fractional diffusion-wave equation (4.6) has the form $\frac{x}{\sqrt{a}t^{\alpha/2}}$ that explains the form of the scaling variable.

In Fig. 4.2 several plots of the fundamental solution $\mathcal{G}_f$ for different values of the parameter $1 < \alpha < 2$ are presented. It can be seen that each fundamental solution has only one maximum and that location of the maximum point changes with the value of α. In Figs. 4.5a and 4.5b, the fundamental solution $\mathcal{G}_f$ is plotted for $\alpha = 1.75$ from different perspectives. In Fig. 4.6 we compare the shape of the curves describing the solution for two different times. In this Figure, the similarity variable is introduced as $\frac{x}{\sqrt{a}t_0^{\alpha/2}}$, and the solutions are presented for $t/t_0 = 1$ (the firm curves in Fig. 4.6) and $t/t_0 = 1.25$ (the dashed-line curves in Fig. 4.6) The plots show that both the location of maximum and the maximum value depend on the time $t > 0$: whereas the maximum value decreases with the time (Fig. 4.5a), the x-coordinate of the maximum location becomes even larger (Fig. 4.5b).

Because the Mainardi function $M(\alpha/2; x)$ is an analytical function for $0 < \alpha < 2$, there exist partial derivatives of the fundamental solution $\mathcal{G}_f$ (4.14) of



a)



b)

Figure 4.5: The fundamental solution for $\alpha = 1.75$ ($\nu = \alpha/2$). Plots from different perspectives [97]

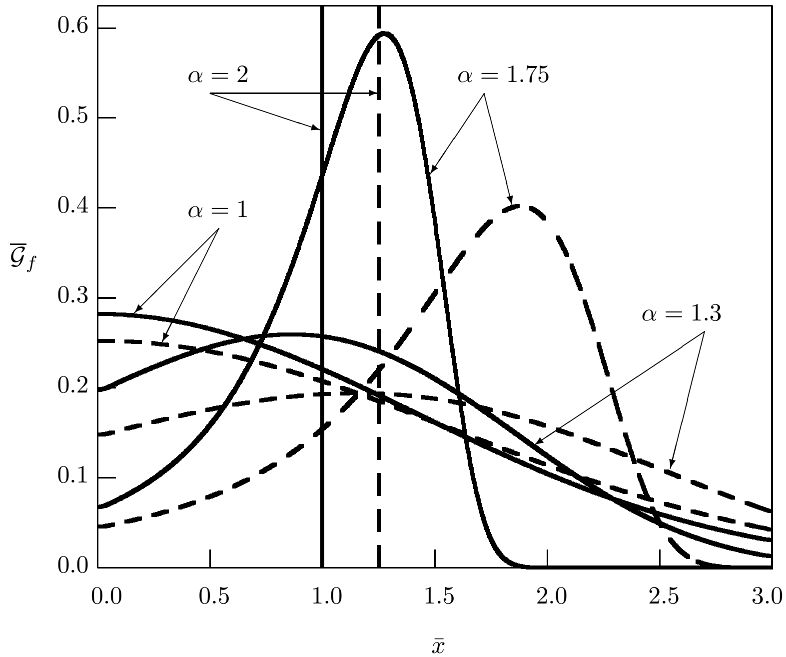


Figure 4.6: Representative curves for $1 \le \alpha \le 2$ and different times (the fundamental solution to the first Cauchy problem) [155]

arbitrary orders for $t > 0$, $x > 0$ and we can use the standard analytical method for finding its extremum points. We first fix a value $t > 0$ and look for the critical points of the fundamental solution $\mathcal{G}_f$ that are determined as solutions to the equation

$$\frac{\partial \mathcal{G}_f(x,t)}{\partial x} = \frac{p_0}{2at^\alpha}\, M'\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a}t^{\alpha/2}}\right) = 0$$

or to the equation

$$M'\left(\frac{\alpha}{2}; \frac{x_*}{\sqrt{a}t^{\alpha/2}}\right) = 0. \tag{4.40}$$

Now we try to find a function $x_* = x_*(t)$ that gives a solution to the equation (4.40) for each $t > 0$. The equation (4.40) can be interpreted as an implicit function that determines the function $x_* = x_*(t)$. Let us find the time-derivative of x_* as a derivative of an implicit function:

$$\frac{\mathrm{d}x_*(t)}{\mathrm{d}t} = -\frac{\dfrac{\partial}{\partial t} M'\left(\dfrac{\alpha}{2}; \dfrac{x}{\sqrt{a}t^{\alpha/2}}\right)\Big|_{x=x_*}}{\dfrac{\partial}{\partial x} M'\left(\dfrac{\alpha}{2}; \dfrac{x}{\sqrt{a}t^{\alpha/2}}\right)\Big|_{x=x_*}}$$

$$= -\frac{M''\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a}t^{\alpha/2}}\right)\left(-\frac{\alpha}{2}\,\frac{x}{\sqrt{a}}t^{-\alpha/2-1}\right)}{M''\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a}t^{\alpha/2}}\right)\frac{1}{\sqrt{a}t^{\alpha/2}}}\Bigg|_{x=x_*},$$

from which we obtain a simple differential equation for $x_*(t)$

$$\frac{\mathrm{d}x_*(t)}{\mathrm{d}t} = \frac{\alpha}{2}\,\frac{x_*}{t} \tag{4.41}$$

with the solution

$$x_*(t) = c_\alpha t^{\alpha/2}, \tag{4.42}$$

where c_α is a constant of integration. Formula (4.42) is in accordance with the Fujita result [45].

As mentioned in [45], the maximum point of the fundamental solution $\mathcal{G}_f$ propagates for $t > 0$ with a finite speed $v(t, \alpha)$ that is determined by

$$v(t, \alpha) = x_*'(t) = \frac{\alpha}{2}\,c_\alpha t^{\alpha/2-1}. \tag{4.43}$$

This formula shows that for every $1 < \alpha < 2$ the propagation speed of the maximum point of the fundamental solution $\mathcal{G}_f$ is a decreasing function in t that varies from $+\infty$ at time $t = 0^+$ to zero as $t \to +\infty$. For $\alpha = 1$ (diffusion) the propagation speed is equal to zero, whereas for $\alpha = 2$ (wave propagation) it remains constant and is equal to $\sqrt{a}$.

Dependence of $\bar{c}_\alpha = c_\alpha/\sqrt{a}$ on the order of fractional derivative α is plotted in Fig. 4.7, whereas Fig. 4.8 presents the dependence of the maximum value of the solution on α. The fundamental solution $\mathcal{G}_f$ has a unique maximum for each $1 < \alpha \leq 2$ and the maximum location changes with α. Surprisingly, the maximum location does not always lay between zero (maximum location for the diffusion equation, $\alpha = 1$) and one (maximum location for the wave equation, $\alpha = 2$). Figure 4.7 shows that the curve $\bar{c}_\alpha$ has a maximum located at the point $\alpha \simeq 1.69$. The value of the maximum is approximately equal to 1.28. It is interesting to note that for $1.36 \leq \alpha \leq 2$ the value of $\bar{c}_\alpha$ is greater than or equal to one. In this sense, one can say that the fractional diffusion-wave equation behaves more like the diffusion equation for $\alpha < 1.36$ and more like the wave equation for $1.36 \leq \alpha < 2$.

As expected, the maximum value of the fundamental solution tends to infinity as α tends to 2 that corresponds to the case of the wave equation. Another interesting feature of the curve $\max \overline{\mathcal{G}}_f(x_*) = \bar{m}_\alpha$ that can be seen in Fig. 4.8 is that the maximum is first monotonically decreasing and then starts to increase. The minimum location is at $\alpha \simeq 1.22$ and the minimum value is nearly equal to 0.25. Whereas the maximum value changes very slow on the interval $1 < \alpha < 1.9$, it starts to rapidly grow in a small neighborhood of the point $\alpha = 2$. It should be noted that despite the fact that the curves $\bar{c}_\alpha$ and $\bar{m}_\alpha$ are not monotone and possess a minimum and a maximum, respectively, the product $\bar{c}_\alpha \bar{m}_\alpha$ is a monotone increasing function for all $1 \leq \alpha \leq 2$ [97].

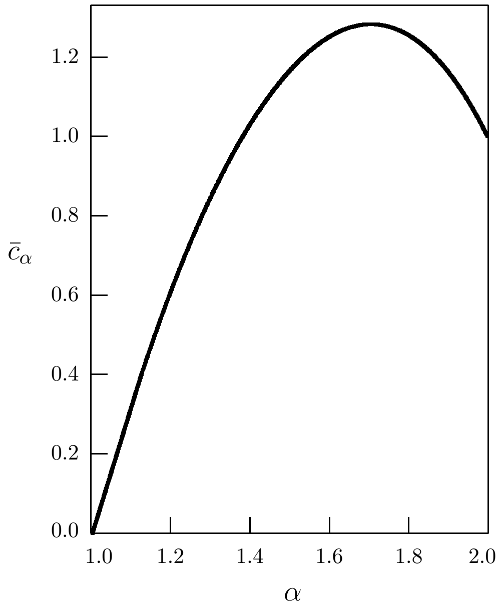


Figure 4.7: Maximum locations of the fundamental solution to the first Cauchy problem ($1 \leq \alpha \leq 2$) [155]

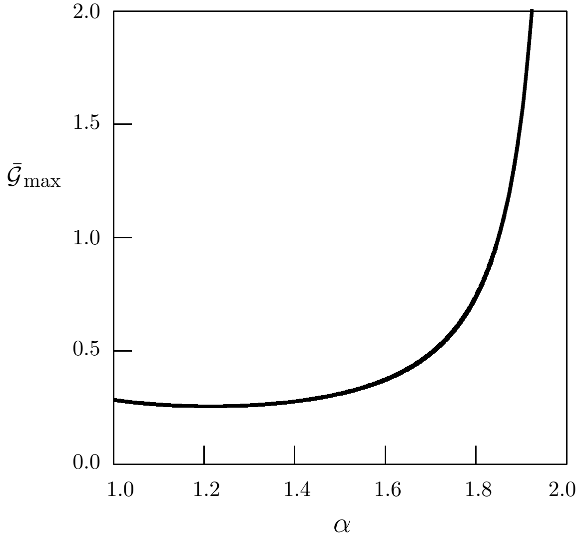


Figure 4.8: Maximum values of the fundamental solution to the first Cauchy problem ($1 \leq \alpha \leq 2$) [155]

4.3 Evolution of the unit step

In order to gain a better insight into the characteristic features of solutions to the time-fractional diffusion-wave equation we investigate evolution of the unit step. The first Cauchy problem with the unit step initial condition was considered in [101]. Mainardi's results were supplemented with additional numerical calculations in [162]. The corresponding results for the second Cauchy problem and for the source problem were obtained in [182].

4.3.1 First Cauchy problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2}\,, \tag{4.44}$$

$$t = 0: \quad T = \begin{cases} T_0, & 0 \le |x| < l, \\ 0, & l < |x| < \infty, \end{cases} \qquad 0 < \alpha \le 2, \tag{4.45}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{4.46}$$

The solution:

$$T = \frac{2T_0}{\pi} \int_0^\infty E_\alpha(-a\xi^2 t^\alpha)\,\frac{\cos(x\xi)\,\sin(l\xi)}{\xi}\,\mathrm{d}\xi. \tag{4.47}$$

We introduce the nondimensional quantities

$$\bar{x} = \frac{x}{l}, \qquad \kappa = \frac{\sqrt{a}\,t^{\alpha/2}}{l}, \qquad \overline{T} = \frac{T}{T_0} \tag{4.48}$$

and present a few particular cases of solution (4.47).

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \begin{cases} 1 - \dfrac{1}{2}\left[\exp\left(-\dfrac{1-|\bar{x}|}{\kappa}\right) + \exp\left(\dfrac{1+|\bar{x}|}{\kappa}\right)\right], & 0 \le |\bar{x}| < 1, \\[2ex] \dfrac{1}{2}\left[\exp\left(-\dfrac{|\bar{x}|-1}{\kappa}\right) - \exp\left(\dfrac{1+|\bar{x}|}{\kappa}\right)\right], & 1 \le |\bar{x}| < \infty. \end{cases} \tag{4.49}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{T} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-u^2} \left[\mathrm{erf}\left(\frac{1+|\bar{x}|}{2\sqrt{2}\kappa\sqrt{u}}\right) + \mathrm{erf}\left(\frac{1-|\bar{x}|}{2\sqrt{2}\kappa\sqrt{u}}\right)\right] \mathrm{d}u. \tag{4.50}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{2}\left[\operatorname{erf}\left(\frac{1+|\bar{x}|}{2\kappa}\right) + \operatorname{erf}\left(\frac{1-|\bar{x}|}{2\kappa}\right)\right]. \tag{4.51}$$

Wave equation ($\alpha = 2$)

a) $0 < \kappa < 1$

$$\overline{T} = \begin{cases} 1, & 0 \le |\bar{x}| < 1-\kappa, \\ \frac{1}{2}, & 1-\kappa < |\bar{x}| < 1+\kappa, \\ 0, & 1+\kappa < |\bar{x}| < \infty. \end{cases} \tag{4.52}$$

b) $\kappa = 1$

$$\overline{T} = \begin{cases} \frac{1}{2}, & 0 \le |\bar{x}| < 2, \\ 0, & 2 < |\bar{x}| < \infty. \end{cases} \tag{4.53}$$

c) $1 < \kappa < \infty$

$$\overline{T} = \begin{cases} 0, & 0 \le |\bar{x}| < \kappa - 1, \\ \frac{1}{2}, & \kappa - 1 < |\bar{x}| < 1+\kappa, \\ 0, & 1+\kappa < |\bar{x}| < \infty. \end{cases} \tag{4.54}$$

The solution described by Eq. (4.47) is presented in Figs. 4.9–4.11 for $\kappa = 0.75$, $\kappa = 1$ and $\kappa = 1.5$, respectively.

4.3.2 Second Cauchy problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2}, \qquad 1 < \alpha \le 2, \tag{4.55}$$

$$t = 0: \quad T = 0, \qquad 1 < \alpha \le 2, \tag{4.56}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = \begin{cases} w_0, & 0 \le |x| < l, \\ 0, & l < |x| < \infty, \end{cases} \qquad 1 < \alpha \le 2. \tag{4.57}$$

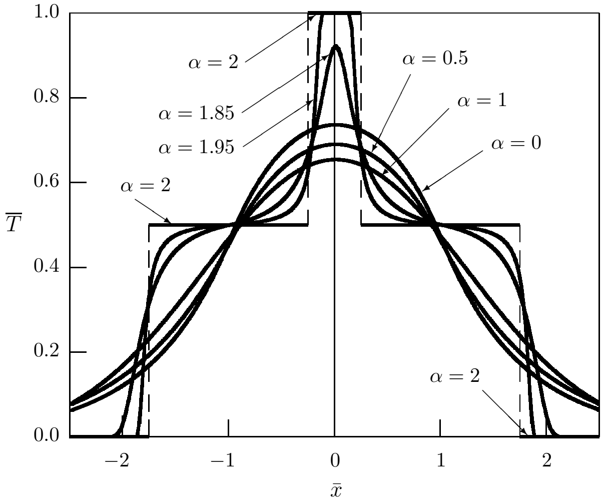


Figure 4.9: Evolution of the unit box (the first Cauchy problem; $\kappa = 0.75$) [162]

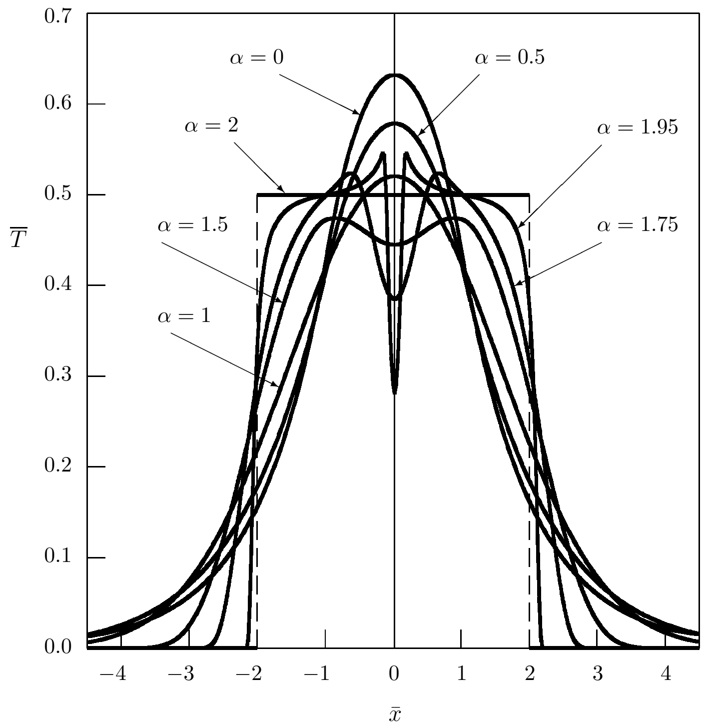


Figure 4.10: Evolution of the unit box (the first Cauchy problem; $\kappa = 1$) [162]

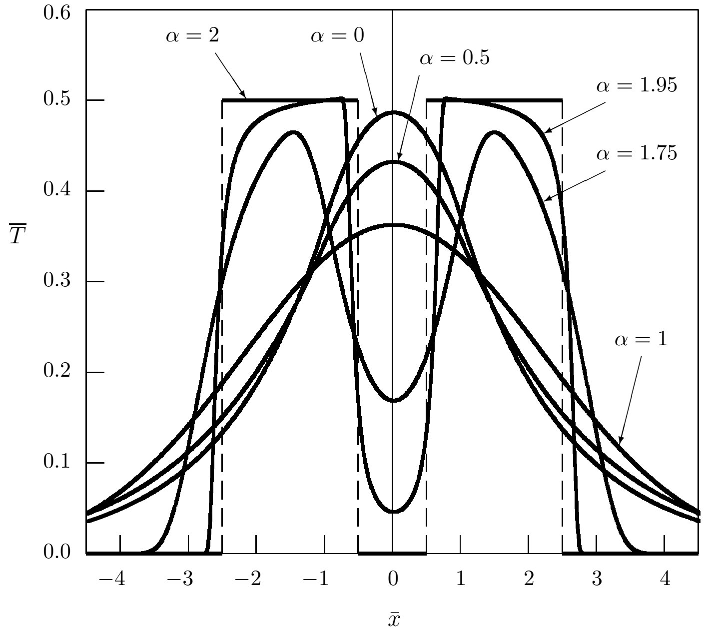


Figure 4.11: Evolution of the unit box (the first Cauchy problem; $\kappa = 1.5$) [162]

The solution:

$$T = \frac{2w_0 t}{\pi} \int_0^{\infty} E_{\alpha,2}(-a\xi^2 t^\alpha) \, \frac{\cos(x\xi)\, \sin(l\xi)}{\xi} \, d\xi. \tag{4.58}$$

Wave equation ($\alpha = 2$)

a) $0 < \kappa < 1$

$$\overline{T} = \begin{cases} 1, & 0 \le |\bar{x}| < 1 - \kappa, \\ \dfrac{1 + \kappa - |\bar{x}|}{2\kappa}, & 1 - \kappa < |\bar{x}| < 1 + \kappa, \\ 0, & 1 + \kappa < |\bar{x}| < \infty. \end{cases} \tag{4.59}$$

b) $\kappa = 1$

$$\overline{T} = \begin{cases} 1 - \dfrac{|\bar{x}|}{2}, & 0 \le |\bar{x}| < 2, \\ 0, & 2 < |\bar{x}| < \infty. \end{cases} \tag{4.60}$$

c) $1 < \kappa < \infty$

$$\overline{T} = \begin{cases} \dfrac{1}{\kappa}, & 0 \le |\bar{x}| < \kappa - 1, \\ \dfrac{1+\kappa-|\bar{x}|}{2\kappa}, & \kappa - 1 < |\bar{x}| < 1+\kappa, \\ 0, & 1+\kappa < |\bar{x}| < \infty. \end{cases} \tag{4.61}$$

Here $\overline{T} = T/(w_0 t)$.

The solution (4.58) is presented in Figs. 4.12 and 4.13 for $\kappa = 0.5$ and $\kappa = 1.25$, respectively.

4.3.3 Source problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2} + \delta(t) \begin{cases} q_0, & 0 \le |x| < l, \\ 0, & l < |x| < \infty, \end{cases} \tag{4.62}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{4.63}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{4.64}$$

The solution:

$$T = \frac{2q_0 t^{\alpha-1}}{\pi} \int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\,\frac{\cos(x\xi)\,\sin(l\xi)}{\xi}\,\mathrm{d}\xi \tag{4.65}$$

is presented in Figs. 4.14 and 4.15 for $\kappa = 0.5$ and $\kappa = 1.25$, respectively; $\overline{T} = T/(q_0 t^{\alpha-1})$.

4.4 Domain $0 < x < \infty$

4.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2} + \Phi(x,t), \tag{4.66}$$

$$t = 0: \quad T = f(x), \qquad 0 < \alpha \le 2, \tag{4.67}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x), \quad 1 < \alpha \le 2, \tag{4.68}$$

$$x = 0: \quad T = g(t), \tag{4.69}$$

$$\lim_{x\to\infty} T(x,t) = 0. \tag{4.70}$$

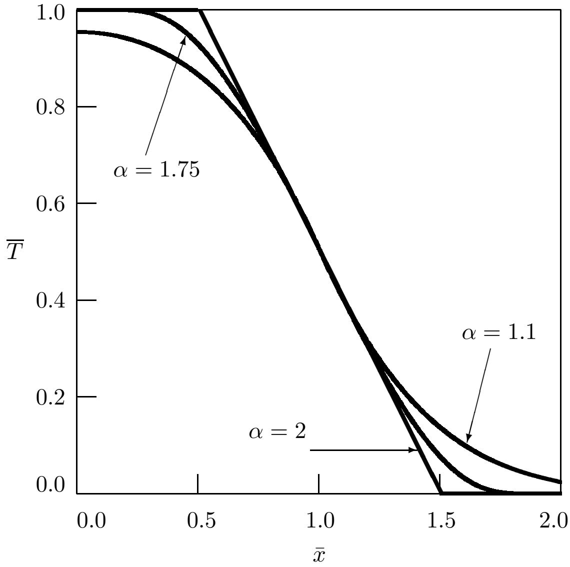


Figure 4.12: Evolution of the unit box (the second Cauchy problem; $\kappa = 0.5$) [182]

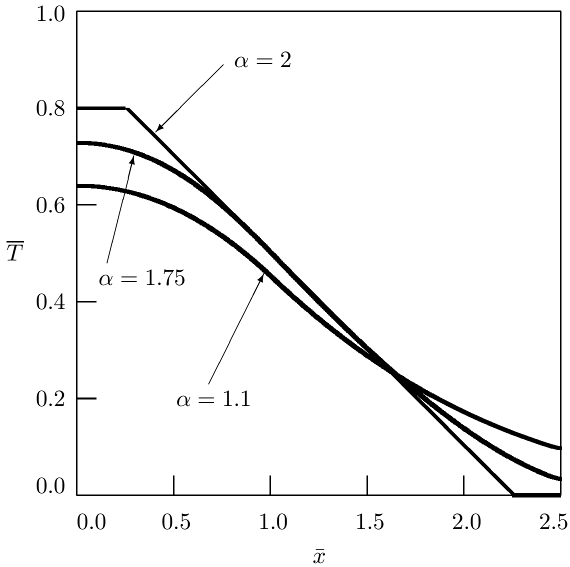


Figure 4.13: Evolution of the unit box (the second Cauchy problem; $\kappa = 1.25$) [182]

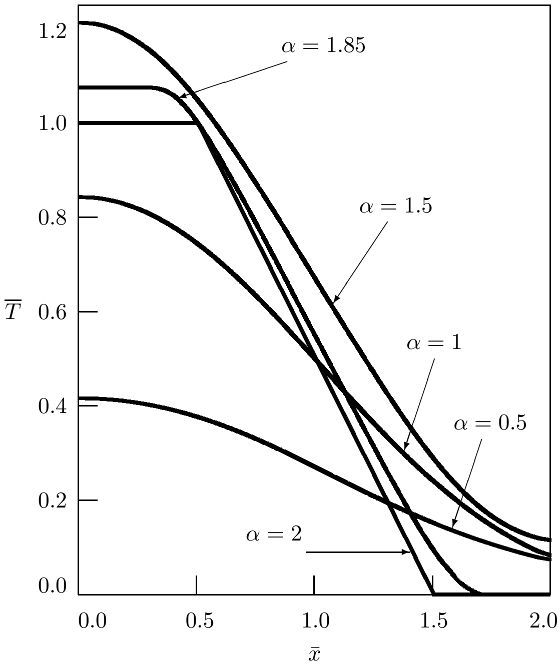


Figure 4.14: Evolution of the unit box (the source problem; $\kappa = 0.5$) [182]

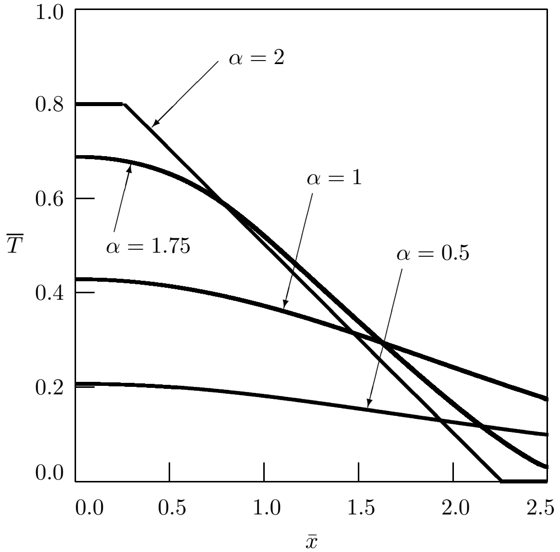


Figure 4.15: Evolution of the unit box (the source problem; $\kappa = 1.25$) [182]

The solution:

$$T(x,t) = \int_0^\infty f(\rho)\,\mathcal{G}_f(x,\varrho,t)\,\mathrm{d}\varrho + \int_0^\infty F(\rho)\,\mathcal{G}_F(x,\varrho,t)\,\mathrm{d}\varrho$$
$$+ \int_0^t \int_0^\infty \Phi(\rho,\tau)\,\mathcal{G}_\Phi(x,\varrho,t-\tau)\,\mathrm{d}\varrho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(x,t-\tau)\,\mathrm{d}\tau. \tag{4.71}$$

The fundamental solutions are obtained using the Laplace transform (2.1) with respect to time t and the sin-Fourier transform (2.25) with respect to the space coordinate x:

$$\begin{pmatrix} \mathcal{G}_f(x,\varrho,t) \\ \mathcal{G}_F(x,\varrho,t) \\ \mathcal{G}_\Phi(x,\varrho,t) \end{pmatrix} = \frac{2}{\pi}\int_0^\infty \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \sin(x\xi)\,\sin(\varrho\xi)\,\mathrm{d}\xi. \tag{4.72}$$

The fundamental solution to the Dirichlet problem $\mathcal{G}_g(x,t)$ is calculated as

$$\mathcal{G}_g(x,t) = \frac{a g_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,\varrho,t)}{\partial \varrho}\bigg|_{\varrho=0}. \tag{4.73}$$

Let us analyze the fundamental solution to the first Cauchy problem introducing the following nondimensional quantities:

$$\bar{x} = \frac{x}{\varrho}, \qquad \kappa = \frac{\sqrt{a}\,t^{\alpha/2}}{\varrho}, \qquad \overline{\mathcal{G}}_f = \frac{\varrho}{p_0}\,\mathcal{G}_f. \tag{4.74}$$

Helmholtz equation ($\alpha \to 0$)

$$\overline{\mathcal{G}}_f = \frac{1}{2\kappa}\left[\exp\left(-\frac{|\bar{x}-1|}{\kappa}\right) - \exp\left(-\frac{\bar{x}+1}{\kappa}\right)\right]. \tag{4.75}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{\mathcal{G}}_f = \frac{1}{\pi\kappa\sqrt{2}}\int_0^\infty \frac{1}{\sqrt{u}}\,\mathrm{e}^{-u^2}\left\{\exp\left[-\frac{(\bar{x}-1)^2}{8\kappa^2 u}\right] - \exp\left[-\frac{(\bar{x}+1)^2}{8\kappa^2 u}\right]\right\}\mathrm{d}u. \tag{4.76}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{\mathcal{G}}_f = \frac{1}{2\sqrt{\pi}\kappa}\left\{\exp\left[-\frac{(\bar{x}-1)^2}{4\kappa^2}\right] - \exp\left[-\frac{(\bar{x}+1)^2}{4\kappa^2}\right]\right\}. \tag{4.77}$$

Wave equation ($\alpha = 2$)

$$\overline{\mathcal{G}}_f = \frac{1}{2}\left[\delta(\bar{x}+\kappa-1)+\delta(\bar{x}-\kappa-1)-\delta(\bar{x}-\kappa+1)\right]. \tag{4.78}$$

The fundamental solution to the Dirichlet problem (the signaling problem) was obtained by Mainardi [101].

4.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2} + \Phi(x,t), \tag{4.79}$$

$$t = 0: \quad T = f(x), \qquad 0 < \alpha \le 2, \tag{4.80}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x), \quad 1 < \alpha \le 2, \tag{4.81}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} = g(t), \tag{4.82}$$

$$\lim_{x\to\infty} T(x,t) = 0. \tag{4.83}$$

The solution:

$$\begin{aligned} T(x,t) \; &= \int_0^\infty f(\rho)\,\mathcal{G}_f(x,\varrho,t)\,\mathrm{d}\varrho + \int_0^\infty F(\rho)\,\mathcal{G}_F(x,\varrho,t)\,\mathrm{d}\varrho \\ &+ \int_0^t\int_0^\infty \Phi(\rho,\tau)\,\mathcal{G}_\Phi(x,\varrho,t-\tau)\,\mathrm{d}\varrho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_m(x,t-\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.84}$$

The fundamental solutions are obtained using the Laplace transform (2.1) with respect to time t and the cos-Fourier transform (2.37) with respect to the space coordinate x:

$$\begin{pmatrix} \mathcal{G}_f(x,\varrho,t) \\ \mathcal{G}_F(x,\varrho,t) \\ \mathcal{G}_\Phi(x,\varrho,t) \end{pmatrix} = \frac{2}{\pi}\int_0^\infty \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \cos(x\xi)\,\cos(\varrho\xi)\,\mathrm{d}\xi. \tag{4.85}$$

The fundamental solution to the mathematical Neumann problem $\mathcal{G}_m(x,t)$ with the boundary condition

$$x = 0: \quad -\frac{\partial \mathcal{G}_m}{\partial x} = g_0\,\delta(t) \tag{4.86}$$

is calculated as

$$\mathcal{G}_m(x,t) = \frac{a g_0}{q_0}\,\mathcal{G}_\Phi(x,\varrho,t)\Big|_{\varrho=0}, \tag{4.87}$$

whereas the fundamental solution to the physical Neumann problem $\mathcal{G}_p(x,t)$ with the boundary condition

$$x = 0: \quad -D_{RL}^{1-\alpha} \frac{\partial \mathcal{G}_p}{\partial x} = g_0\, \delta(t), \qquad 0 < \alpha \le 1, \tag{4.88}$$

$$x = 0: \quad -I^{\alpha-1} \frac{\partial \mathcal{G}_p}{\partial x} = g_0\, \delta(t), \qquad 1 < \alpha \le 2, \tag{4.89}$$

is calculated as

$$\mathcal{G}_{g_p}(x,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x, \varrho, t)\bigg|_{\varrho=0}. \tag{4.90}$$

4.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\, \frac{\partial^2 T}{\partial x^2} + \Phi(x,t), \tag{4.91}$$

$$t = 0: \quad T = f(x), \qquad 0 < \alpha \le 2, \tag{4.92}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x), \quad 1 < \alpha \le 2, \tag{4.93}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} + HT = g(t), \tag{4.94}$$

$$\lim_{x \to \infty} T(x,t) = 0. \tag{4.95}$$

The solution:

$$T(x,t) = \int_0^\infty f(\rho)\, \mathcal{G}_f(x, \varrho, t)\, \mathrm{d}\varrho + \int_0^\infty F(\rho)\, \mathcal{G}_F(x, \varrho, t)\, \mathrm{d}\varrho$$
$$+ \int_0^t \int_0^\infty \Phi(\rho, \tau)\, \mathcal{G}_\Phi(x, \varrho, t-\tau)\, \mathrm{d}\varrho\, \mathrm{d}\tau + \int_0^t g(\tau)\, \mathcal{G}_m(x, t-\tau)\, \mathrm{d}\tau. \tag{4.96}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t and the sin-cos-Fourier transform (2.40), (2.42) with respect to the space coordinate x:

$$\begin{pmatrix} \mathcal{G}_f(x,\varrho,t) \\ \mathcal{G}_F(x,\varrho,t) \\ \mathcal{G}_\Phi(x,\varrho,t) \end{pmatrix} = \frac{2}{\pi} \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}$$
$$\times \frac{\xi \cos(x\xi) + H \sin(x\xi)}{\xi^2 + H^2} \big[\xi \cos(\varrho\xi) + H \sin(\varrho\xi)\big]\, \mathrm{d}\xi. \tag{4.97}$$

The fundamental solution to the mathematical Robin problem $\mathcal{G}_m(x,t)$ with the boundary condition

$$x = 0: \quad -\frac{\partial \mathcal{G}_m}{\partial x} + H\mathcal{G}_m = g_0\,\delta(t) \tag{4.98}$$

is calculated as

$$\mathcal{G}_m(x,t) = \frac{a g_0}{q_0}\,\mathcal{G}_\Phi(x,\varrho,t)\bigg|_{\varrho=0} \tag{4.99}$$

and has the following form [181]

$$\mathcal{G}_m(x,t) = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\,\frac{\xi^2\cos(x\xi) + H\xi\sin(x\xi)}{\xi^2 + H^2}\,\mathrm{d}\xi \tag{4.100}$$

with the following particular cases:

Classical diffusion equation ($\alpha = 1$)

$$\begin{aligned}\mathcal{G}_m(x,t) \;&=\; ag_0\left[\frac{1}{\sqrt{\pi a t}}\exp\left(-\frac{x^2}{4at}\right)\right.\\ &\left.-\; H\exp\left(Hx + H^2 at\right)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{at}} + H\sqrt{at}\right)\right].\end{aligned} \tag{4.101}$$

Wave equation ($\alpha = 2$)

$$\mathcal{G}_m(x,t = \begin{cases}\sqrt{a}g_0\mathrm{e}^{-H(\sqrt{a}t - x)}, & 0 < x < \sqrt{a}t,\\ 0, & \sqrt{a}t < x < \infty.\end{cases} \tag{4.102}$$

Now we examine the fundamental solution $\mathcal{G}_p(x,t)$ to the time-fractional diffusion-wave equation under physical Robin boundary condition [181]

$$x = 0: \quad -D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial x} + H\mathcal{G}_p = g_0\,\delta(t), \quad 0 < \alpha \le 1, \tag{4.103}$$

$$x = 0: \quad -I^{\alpha-1}\frac{\partial \mathcal{G}_p}{\partial x} + H\mathcal{G}_p = g_0\,\delta(t), \quad 1 < \alpha \le 2. \tag{4.104}$$

The Laplace transform with respect to time t leads to the following equation

$$s^\alpha \mathcal{G}_p^* = a\,\frac{\partial^2 \mathcal{G}_p^*}{\partial x^2} \tag{4.105}$$

and the boundary condition

$$x = 0: \quad -\frac{\partial \mathcal{G}_p^*}{\partial x} + s^{\alpha-1}H\mathcal{G}_p^* = g_0 s^{\alpha-1}, \quad 0 < \alpha \le 2. \tag{4.106}$$

In this case the kernel (2.42) of the sin-cos-Fourier transform (2.40) with respect to the space coordinate x has the more complicated form

$$K(x,\xi) = \frac{\xi\cos(x\xi) + s^{\alpha-1}H\sin(x\xi)}{\sqrt{\xi^2 + (s^{\alpha-1}H)^2}}, \tag{4.107}$$

and in the transform domain we get

$$\widetilde{\mathcal{G}}_p^* = a g_0 \frac{\xi}{\sqrt{\xi^2 + (s^{\alpha-1}H)^2}}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{4.108}$$

Inversion of the Laplace transform in (4.108) depends on the value of α. For $0 < \alpha < 1$ we have [181]

$$\begin{aligned}
\mathcal{G}_p &= \frac{2ag_0}{\pi}\int_0^\infty E_\alpha(-a\xi^2 t^\alpha)\,\cos(x\xi)\,\mathrm{d}\xi - \frac{2ag_0}{\pi}\int_0^\infty \frac{H^2}{\xi^2}\,\cos(x\xi)\,\mathrm{d}\xi \\
&\times \int_0^t (t-\tau)^{1-2\alpha} E_{2-2\alpha,2-2\alpha}\left[-\frac{H^2}{\xi^2}(t-\tau)^{2-2\alpha}\right] E_\alpha(-a\xi^2\tau^\alpha)\,\mathrm{d}\tau \\
&+ \frac{2ag_0}{\pi}\int_0^\infty \frac{H}{\xi}\,\sin(x\xi)\,\mathrm{d}\xi \int_0^t (t-\tau)^{1-2\alpha}\tau^{\alpha-1} \\
&\times E_{2-2\alpha,2-2\alpha}\left[-\frac{H^2}{\xi^2}(t-\tau)^{2-2\alpha}\right] E_{\alpha,\alpha}(-a\xi^2\tau^\alpha)\,\mathrm{d}\tau,
\end{aligned} \tag{4.109}$$

whereas for $1 < \alpha < 2$ we get

$$\begin{aligned}
\mathcal{G}_p &= \frac{2ag_0}{\pi}\int_0^\infty \frac{\xi}{H}\,t^{\alpha-1}E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\,\sin(x\xi)\,\mathrm{d}\xi + \frac{2ag_0}{\pi}\int_0^\infty \frac{\xi^2}{H^2}\,\cos(x\xi)\,\mathrm{d}\xi \\
&\times \int_0^t (t-\tau)^{2\alpha-3} E_{2\alpha-2,2\alpha-2}\left[-\frac{\xi^2}{H^2}(t-\tau)^{2\alpha-2}\right] E_\alpha(-a\xi^2\tau^\alpha)\,\mathrm{d}\tau \\
&- \frac{2ag_0}{\pi}\int_0^\infty \frac{\xi^3}{H^3}\,\sin(x\xi)\,\mathrm{d}\xi \int_0^t (t-\tau)^{2\alpha-3}\tau^{\alpha-1} \\
&\times E_{2\alpha-2,2\alpha-2}\left[-\frac{\xi^2}{H^2}(t-\tau)^{2\alpha-2}\right] E_{\alpha,\alpha}(-a\xi^2\tau^\alpha)\,\mathrm{d}\tau.
\end{aligned} \tag{4.110}$$

The particular case of the ballistic diffusion has the following form.

Wave equation ($\alpha = 2$)

$$\mathcal{G}_p = \frac{a g_0}{1+\sqrt{a}H}\,\delta(x - \sqrt{a}t). \tag{4.111}$$

4.5 Domain $0 < x < L$

4.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2} + \Phi(x,t), \tag{4.112}$$

$$t = 0: \quad T = f(x), \qquad 0 < \alpha \le 2, \tag{4.113}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x), \quad 1 < \alpha \le 2, \tag{4.114}$$

$$x = 0: \quad T = g_1(t), \tag{4.115}$$

$$x = L: \quad T = g_2(t). \tag{4.116}$$

The solution:

$$\begin{aligned} T(x,t) =& \int_0^L f(\rho)\,\mathcal{G}_f(x,\varrho,t)\,\mathrm{d}\varrho + \int_0^L F(\rho)\,\mathcal{G}_F(x,\varrho,t)\,\mathrm{d}\varrho \\ &+ \int_0^t \int_0^L \Phi(\rho,\tau)\,\mathcal{G}_\Phi(x,\varrho,t-\tau)\,\mathrm{d}\varrho\,\mathrm{d}\tau \\ &+ \int_0^t g_1(\tau)\,\mathcal{G}_{g_1}(x,t-\tau)\,\mathrm{d}\tau + \int_0^t g_2(\tau)\,\mathcal{G}_{g_2}(x,t-\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.117}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t and the finite sin-Fourier transform (2.44) with respect to the space coordinate x:

$$\begin{pmatrix} \mathcal{G}_f(x,\varrho,t) \\ \mathcal{G}_F(x,\varrho,t) \\ \mathcal{G}_\Phi(x,\varrho,t) \end{pmatrix} = \frac{2}{L}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix} \sin(x\xi_k)\,\sin(\varrho\xi_k), \tag{4.118}$$

where $\xi_k = k\pi/L$. The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g_1}(x,t) = \frac{a g_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,\varrho,t)}{\partial \varrho}\bigg|_{\varrho=0}, \tag{4.119}$$

$$\mathcal{G}_{g_2}(x,t) = -\frac{a g_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,\varrho,t)}{\partial \varrho}\bigg|_{\varrho=L}. \tag{4.120}$$

Several problems for the time-fractional diffusion-wave equation in a domain $0 < x < L$ under Dirichlet boundary condition were considered by Agrawal [3].

4.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2} + \Phi(x,t), \tag{4.121}$$

$$t=0: \quad T = f(x), \qquad 0<\alpha\le 2, \tag{4.122}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x), \quad 1<\alpha\le 2, \tag{4.123}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(t), \tag{4.124}$$

$$x=L: \quad \frac{\partial T}{\partial x} = g_2(t). \tag{4.125}$$

The solution:

$$\begin{aligned} T(x,t) &= \int_0^L f(\rho)\,\mathcal{G}_f(x,\varrho,t)\,\mathrm{d}\varrho + \int_0^L F(\rho)\,\mathcal{G}_F(x,\varrho,t)\,\mathrm{d}\varrho \\ &+ \int_0^t\int_0^L \Phi(\rho,\tau)\,\mathcal{G}_\Phi(x,\varrho,t-\tau)\,\mathrm{d}\varrho\,\mathrm{d}\tau \\ &+ \int_0^t g_1(\tau)\,\mathcal{G}_{g_1}(x,t-\tau)\,\mathrm{d}\tau + \int_0^t g_2(\tau)\,\mathcal{G}_{g_2}(x,t-\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.126}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t and the finite cos-Fourier transform (2.48) with respect to the space coordinate x:

$$\begin{pmatrix} \mathcal{G}_f(x,\varrho,t) \\ \mathcal{G}_F(x,\varrho,t) \\ \mathcal{G}_\Phi(x,\varrho,t) \end{pmatrix} = \frac{2}{L}\sum_{k=0}^{\infty}{}' \begin{pmatrix} p_0\,E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix} \cos(x\xi_k)\cos(\varrho\xi_k), \tag{4.127}$$

where $\xi_k = k\pi/L$. The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,t) = \frac{ag_0}{q_0}\mathcal{G}_\Phi(x,\varrho,t)\bigg|_{\varrho=0}, \qquad \mathcal{G}_{m2}(x,t) = \frac{ag_0}{q_0}\mathcal{G}_\Phi(x,\varrho,t)\bigg|_{\varrho=L}, \tag{4.128}$$

$$\mathcal{G}_{p1}(x,t) = \frac{ag_0}{p_0}\mathcal{G}_f(x,\varrho,t)\bigg|_{\varrho=0}, \qquad \mathcal{G}_{p2}(x,t) = \frac{ag_0}{p_0}\mathcal{G}_f(x,\varrho,t)\bigg|_{\varrho=L}. \tag{4.129}$$

4.5.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\frac{\partial^2 T}{\partial x^2} + \Phi(x,t), \tag{4.130}$$

$$t = 0: \quad T = f(x), \qquad 0 < \alpha \le 2, \tag{4.131}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x), \quad 1 < \alpha \le 2, \tag{4.132}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} + HT = g_1(t), \tag{4.133}$$

$$x = L: \quad \frac{\partial T}{\partial x} + HT = g_2(t). \tag{4.134}$$

The solution:

$$\begin{aligned} T(x,t) \;=\; & \int_0^L f(\rho)\,\mathcal{G}_f(x,\varrho,t)\,\mathrm{d}\varrho + \int_0^L F(\rho)\,\mathcal{G}_F(x,\varrho,t)\,\mathrm{d}\varrho \\ & + \int_0^t\int_0^L \Phi(\rho,\tau)\,\mathcal{G}_\Phi(x,\varrho,t-\tau)\,\mathrm{d}\varrho\,\mathrm{d}\tau \\ & + \int_0^t g_1(\tau)\,\mathcal{G}_{g_1}(x,t-\tau)\,\mathrm{d}\tau + \int_0^t g_2(\tau)\,\mathcal{G}_{g_2}(x,t-\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.135}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t and the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate x:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(x,\varrho,t) \\ \mathcal{G}_F(x,\varrho,t) \\ \mathcal{G}_\Phi(x,\varrho,t) \end{pmatrix} &= \frac{2}{L}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix} \\ & \times \frac{\xi_k\cos(x\xi_k) + H\sin(x\xi_k)}{\xi_k^2 + H^2 + 2H/L}\Big[\xi_k\cos(\varrho\xi_k) + H\sin(\varrho\xi_k)\Big], \end{aligned} \tag{4.136}$$

where ξ_k are the positive roots of the transcendental equation

$$\tan(L\xi_k) = \frac{2H\xi_k}{\xi_k^2 - H^2}\,.$$

The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(x,t) = \frac{a g_0}{q_0}\mathcal{G}_\Phi(x,\varrho,t)\Big|_{\varrho=0}, \qquad \mathcal{G}_{m2}(x,t) = \frac{a g_0}{q_0}\mathcal{G}_\Phi(x,\varrho,t)\Big|_{\varrho=L}. \tag{4.137}$$

4.6 Two joint half-lines

4.6.1 Statement of the problem

Time-fractional heat conduction in two joint half-lines was considered in [183, 187]. The general mathematical formulation of the problem is stated as follows: to solve the time-fractional heat conduction equations

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2} + \Phi_1(x,t), \quad x > 0, \quad 0 < \alpha \le 2, \tag{4.138}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2} + \Phi_2(x,t), \quad x < 0, \quad 0 < \beta \le 2, \tag{4.139}$$

under the initial conditions

$$t = 0: \quad T_1 = f_1(x), \quad x > 0, \quad 0 < \alpha \le 2, \tag{4.140}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = F_1(x), \quad x > 0, \quad 1 < \alpha \le 2, \tag{4.141}$$

$$t = 0: \quad T_2 = f_2(x), \quad x < 0, \quad 0 < \beta \le 2, \tag{4.142}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = F_2(x), \quad x < 0, \quad 1 < \beta \le 2, \tag{4.143}$$

and the boundary conditions of perfect thermal contact

$$T_1(x,t)\Big|_{x=0^+} = T_2(x,t)\Big|_{x=0^-}, \tag{4.144}$$

$$k_1 D_{RL}^{1-\alpha} \frac{\partial T_1(x,t)}{\partial x}\bigg|_{x=0^+} = k_2 D_{RL}^{1-\beta} \frac{\partial T_2(x,t)}{\partial x}\bigg|_{x=0^-}, \quad 0 < \alpha \le 2, \quad 0 < \beta \le 2, \tag{4.145}$$

which state that two bodies in contact must have the same temperature at the contact point and the heat fluxes through the contact point must be the same.

4.6.2 Fundamental solution to the first Cauchy problem

In this case the following initial-boundary-value problem is solved:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \quad x > 0, \quad 0 < \alpha \le 2, \tag{4.146}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad x < 0, \quad 0 < \beta \le 2, \tag{4.147}$$

under the initial conditions

$$t=0: \quad T_1 = p_0\delta(x-\varrho), \quad x>0, \qquad 0<\alpha\le 2, \tag{4.148}$$

$$t=0: \qquad \frac{\partial T_1}{\partial t}=0, \qquad x>0, \qquad 1<\alpha\le 2, \tag{4.149}$$

$$t=0: \qquad T_2=0, \qquad x<0, \qquad 0<\beta\le 2, \tag{4.150}$$

$$t=0: \qquad \frac{\partial T_2}{\partial t}=0, \qquad x<0, \qquad 1<\beta\le 2. \tag{4.151}$$

The boundary condition of perfect thermal contact (4.145) is rewritten as

$$k_1 D_{RL}^{1-\alpha}\frac{\partial T_1(x,t)}{\partial x}\bigg|_{x=0^+} = \varphi(t), \quad 0<\alpha\le 2, \tag{4.152}$$

$$k_2 D_{RL}^{1-\beta}\frac{\partial T_2(x,t)}{\partial x}\bigg|_{x=0^-} = \varphi(t), \quad 0<\beta\le 2, \tag{4.153}$$

where $\varphi(t)$ is the unknown function which should be found from the condition (4.144).

The Laplace transform with respect to time t (for simplicity neglecting the initial value of temperature gradient) and the cos-Fourier transforms with respect to the spatial coordinates $x>0$ and $x<0$ give

$$\widetilde{T}_1^*(\xi,s) = \frac{s^{\alpha-1}}{s^\alpha + a_1\xi^2}\left[p_0\cos(\varrho\xi) - \frac{a_1}{k_1}\,\varphi^*(s)\right], \tag{4.154}$$

$$\widetilde{T}_2^*(\xi,s) = \frac{a_2}{k_2}\,\frac{s^{\beta-1}}{s^\beta + a_2\xi^2}\,\varphi^*(s). \tag{4.155}$$

Inversion of the cos-Fourier transform with taking into account (A.7) results in

$$\begin{aligned} T_1^*(x,s) \;=\; & \frac{p_0}{2\sqrt{a_1}}\,s^{\alpha/2-1}\left[\exp\left(-\frac{x+\varrho}{\sqrt{a_1}}s^{\alpha/2}\right) + \exp\left(-\frac{|x-\varrho|}{\sqrt{a_1}}s^{\alpha/2}\right)\right] \\ & - \frac{\sqrt{a_1}}{k_1}\,\varphi^*(s)\,s^{\alpha/2-1}\exp\left(-\frac{x}{\sqrt{a_1}}s^{\alpha/2}\right), \qquad x\ge 0, \end{aligned} \tag{4.156}$$

$$T_2^*(x,s) = \frac{\sqrt{a_2}}{k_2}\,\varphi^*(s)\,s^{\beta/2-1}\exp\left(-\frac{|x|}{\sqrt{a_2}}s^{\beta/2}\right), \quad x\le 0. \tag{4.157}$$

The requirement that the temperatures at the two sides of contact are the same ($T_1^*(0,s) = T_2^*(0,s)$) allows us to find the function $\varphi^*(s)$:

$$\varphi^*(s) = \frac{p_0 k_1 k_2}{\sqrt{a}_1}\,\frac{s^{\alpha/2}}{k_2\sqrt{a_1}s^{\alpha/2} + k_1\sqrt{a_2}s^{\beta/2}}\exp\left(-\frac{\varrho}{\sqrt{a}_1}s^{\alpha/2}\right). \tag{4.158}$$

Inversion of the Laplace transform in (4.158) depends on the relation between the orders α and β. For $\alpha < \beta$ we have

$$\varphi(t) = \frac{\alpha \varrho p_0 k_2}{2a_1\sqrt{a_2}} \int_0^t \frac{(t-\tau)^{\beta/2-\alpha/2-1}}{\tau^{1+\alpha/2}} M\left(\frac{\alpha}{2}; \frac{\varrho}{\sqrt{a_1}\tau^{\alpha/2}}\right) \times E_{\beta/2-\alpha/2,\,\beta/2-\alpha/2}\left[-\frac{(t-\tau)^{\beta/2-\alpha/2}}{\gamma}\right] \mathrm{d}\tau, \tag{4.159}$$

where $\gamma = k_1\sqrt{a}_2/(k_2\sqrt{a}_1)$.

For $\alpha > \beta$ we obtain

$$\varphi(t) = \frac{\alpha \varrho p_0 k_1}{2a_1^{3/2}} \left[\frac{1}{t^{1+\alpha/2}} M\left(\frac{\alpha}{2}; \frac{\varrho}{\sqrt{a_1}t^{\alpha/2}}\right) - \gamma \int_0^t \frac{(t-\tau)^{\alpha/2-\beta/2-1}}{\tau^{1+\alpha/2}} \times M\left(\frac{\alpha}{2}; \frac{\varrho}{\sqrt{a_1}\tau^{\alpha/2}}\right) E_{\alpha/2-\beta/2,\alpha/2-\beta/2}\left[-\gamma(t-\tau)^{\alpha/2-\beta/2}\right] \mathrm{d}\tau\right]. \tag{4.160}$$

Inversion of the Laplace transform with taking into account (2.181) reads:

$$T_1(x,t) = \frac{p_0}{2\sqrt{a_1}t^{\alpha/2}} \left[M\left(\frac{\alpha}{2}; \frac{x+\varrho}{\sqrt{a}_1 t^{\alpha/2}}\right) + M\left(\frac{\alpha}{2}; \frac{|x-\varrho|}{\sqrt{a}_1 t^{\alpha/2}}\right)\right] - \frac{\sqrt{a}_1}{k_1} \int_0^t \frac{\varphi(t-\tau)}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a}_1 \tau^{\alpha/2}}\right) \mathrm{d}\tau, \quad x \geq 0, \tag{4.161}$$

$$T_2(x,t) = \frac{\sqrt{a}_2}{k_2} \int_0^t \frac{\varphi(t-\tau)}{\tau^{\beta/2}} M\left(\frac{\beta}{2}; \frac{|x|}{\sqrt{a}_2 \tau^{\beta/2}}\right) \mathrm{d}\tau, \quad x \leq 0. \tag{4.162}$$

Let us consider several particular cases of the obtained solution. For $\alpha = \beta$ we have

$$T_1(x,t) = \frac{p_0}{2\sqrt{a_1}t^{\alpha/2}} \left[M\left(\frac{\alpha}{2}; \frac{|x-\varrho|}{\sqrt{a_1}t^{\alpha/2}}\right) + \frac{\gamma-1}{\gamma+1} M\left(\frac{\alpha}{2}; \frac{x+\varrho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x \geq 0, \tag{4.163}$$

$$T_2(x,t) = \frac{p_0\gamma}{(\gamma+1)\sqrt{a_1}t^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a_2}t^{\alpha/2}} + \frac{\rho}{\sqrt{a_1}t^{\alpha/2}}\right), \quad x \leq 0, \tag{4.164}$$

in particular, for $\alpha = \beta = 2$:

$$T_1(x,t) = \frac{p_0}{2}\left[\delta(x-\varrho+\sqrt{a_1}t)+\delta(x-\varrho-\sqrt{a_1}t)\right.$$
$$\left. + \frac{\gamma-1}{\gamma+1}\,\delta(x+\varrho-\sqrt{a_1}t)\right], \quad x \geq 0, \tag{4.165}$$

$$T_2(x,t) = \frac{p_0\gamma}{\gamma+1}\delta\left(\frac{\sqrt{a_1}}{\sqrt{a_2}}|x|+\varrho-\sqrt{a_1}t\right), \quad x \leq 0. \tag{4.166}$$

For $\alpha = 1$, $\beta = 2$, we get

$$T_1(x,t) = \frac{p_0}{2\sqrt{\pi a_1 t}}\left\{\exp\left[-\frac{(x+\rho)^2}{4a_1 t}\right]+\exp\left[-\frac{(x-\rho)^2}{4a_1 t}\right]\right\}$$
$$- \frac{p_0}{\gamma\sqrt{a_1}}\exp\left(\frac{x+\rho}{\gamma\sqrt{a_1}}+\frac{t}{\gamma^2}\right)\operatorname{erfc}\left(\frac{x+\rho}{2\sqrt{a_1 t}}+\frac{\sqrt{t}}{\gamma}\right), \quad x \geq 0, \tag{4.167}$$

$$T_2(x,t) = \begin{cases} \dfrac{p_0}{\sqrt{a_1}}\left\{-\dfrac{1}{\gamma}\exp\left(\dfrac{\rho}{\sqrt{a}_1\gamma}+\dfrac{t+x/\sqrt{a_2}}{\gamma^2}\right)\right. \\ \times\operatorname{erfc}\left[\dfrac{\rho}{2\sqrt{a_1(t+x/\sqrt{a_2})}}+\dfrac{\sqrt{t+x/\sqrt{a_2}}}{\gamma}\right] \\ \left.+\dfrac{1}{\sqrt{\pi(t+x/\sqrt{a}_2)}}\exp\left[-\dfrac{\rho^2}{4a_1(t+x/\sqrt{a_2})}\right]\right\}, \\ \qquad\qquad -\sqrt{a_2}t < x \leq 0, \\ 0, \qquad\qquad -\infty < x < -\sqrt{a_2}t. \end{cases} \tag{4.168}$$

When $\alpha = 2$, $\beta = 1$, we arrive at

$$T_1(x,t) = \frac{p_0}{2}[\delta(x-\rho-\sqrt{a_1}t)+\delta(x-\rho+\sqrt{a_1}t)-\delta(x+\rho-\sqrt{a_1}t)]$$
$$+ \begin{cases} \dfrac{p_0\gamma}{\sqrt{a_1}}\left\{\dfrac{1}{\sqrt{\pi[t-(x+\rho)/\sqrt{a_1}]}}\right. \\ \left.-\gamma\exp\left[\gamma^2\left(t-\dfrac{x+\rho}{\sqrt{a_1}}\right)\right]\operatorname{erfc}\left(\gamma\sqrt{t-\dfrac{x+\rho}{\sqrt{a_1}}}\right)\right\}, \\ \qquad 0 \leq x < \sqrt{a_1}t-\rho, \\ 0, \qquad \sqrt{a_1}t-\rho < x < \infty, \end{cases} \tag{4.169}$$

$$T_2(x,t) = \begin{cases} \dfrac{p_0\gamma}{\sqrt{a_1}}\left\{\dfrac{1}{\sqrt{\pi(t-\rho/\sqrt{a_1})}}\exp\left[-\dfrac{x^2}{4a_2(t-\rho/\sqrt{a_1})}\right]\right. \\ -\gamma\exp\left[\dfrac{\gamma|x|}{\sqrt{a_2}}+\gamma^2\left(t-\dfrac{\rho}{\sqrt{a_1}}\right)\right] \\ \left.\times\operatorname{erfc}\left[\dfrac{|x|}{2\sqrt{a_2(t-\rho/\sqrt{a_1})}}+\gamma\sqrt{t-\dfrac{\rho}{\sqrt{a_1}}}\right]\right\}, & \sqrt{a_1}t>\rho, \\ 0, & \sqrt{a_1}t<\rho. \end{cases} \quad (4.170)$$

The results of numerical calculations of the fundamental solution to the first Cauchy problem are shown in Figs. 4.16–4.18. We have introduced the following nondimensional quantities:

$$\bar{x}=\frac{x}{\varrho},\quad \kappa=\frac{\sqrt{a_1}t^{\alpha/2}}{\varrho},\quad \epsilon=\frac{\sqrt{a_1}}{\sqrt{a_2}}t^{\alpha/2-\beta/2},\quad \bar{\gamma}=\gamma t^{\alpha/2-\beta/2},\quad \overline{T}=\frac{\varrho T}{p_0}. \quad (4.171)$$

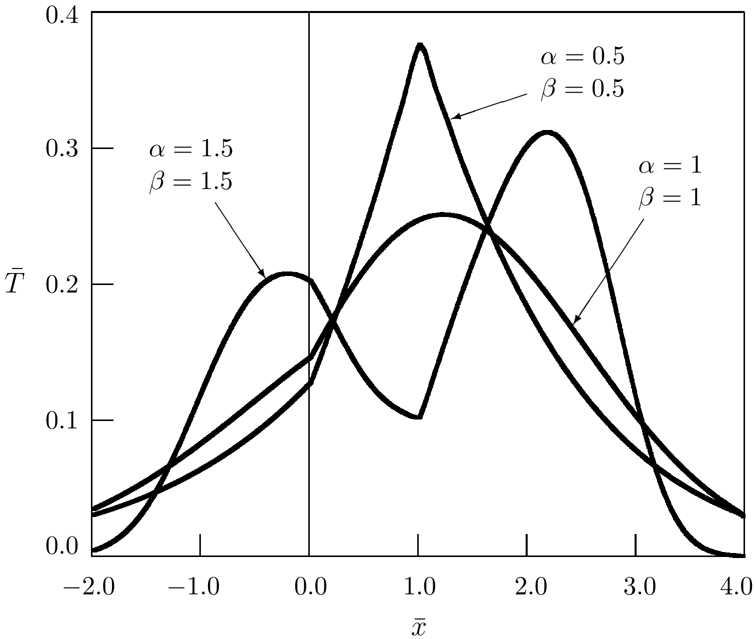


Figure 4.16: Dependence of the fundamental solution to the first Cauchy problem for two joint half-lines on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$ [187]

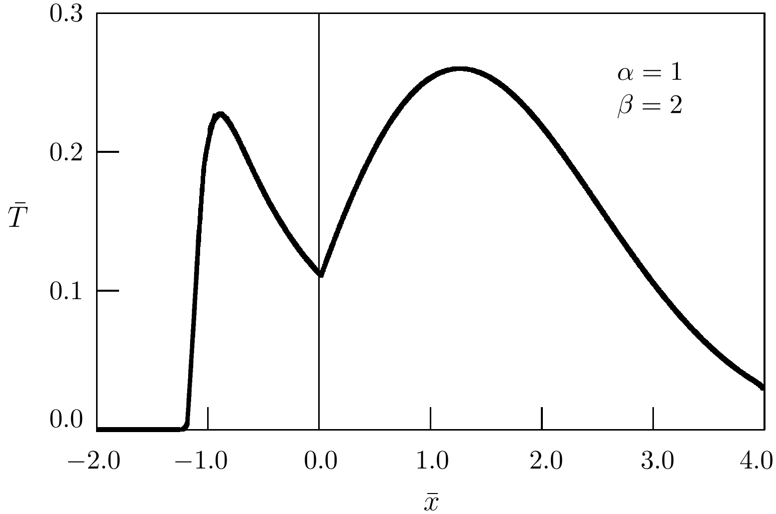


Figure 4.17: Dependence of the fundamental solution to the first Cauchy problem for two joint half-lines on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$ [187]

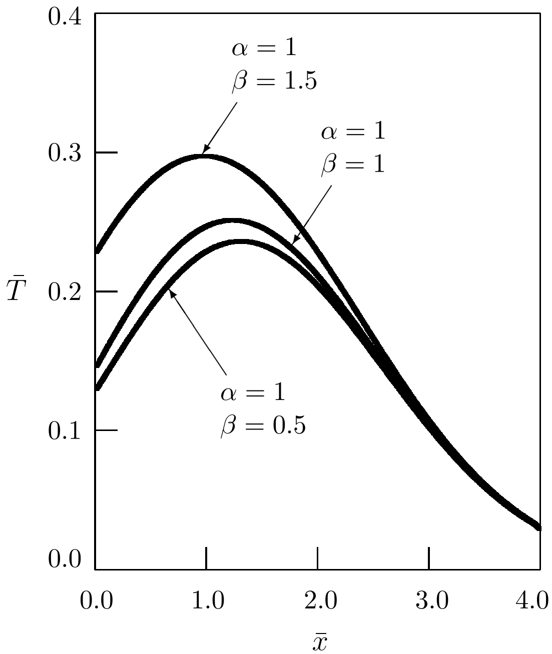


Figure 4.18: Dependence of the fundamental solution to the first Cauchy problem for two joint half-lines on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$ [187]

4.6.3 Fundamental solution to the second Cauchy problem

Now we solve the following initial-boundary-value problem:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \quad x > 0, \quad 1 < \alpha \le 2, \tag{4.172}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad x < 0, \quad 0 < \beta \le 2, \tag{4.173}$$

under the initial conditions

$$t = 0: \quad T_1 = 0, \quad x > 0, \quad 1 < \alpha \le 2, \tag{4.174}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = w_0\, \delta(x - \rho), \quad x > 0, \quad 1 < \alpha \le 2, \tag{4.175}$$

$$t = 0: \quad T_2 = 0, \quad x < 0, \quad 0 < \beta \le 2, \tag{4.176}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = 0, \quad x < 0, \quad 1 < \beta \le 2, \tag{4.177}$$

and the conditions of perfect contact (4.144) and (4.145) stating the equality of temperatures and fluxes at the contact point.

It should be emphasized that the second Cauchy problem for the fractional heat conduction equation (4.172) in the domain $x > 0$ is formulated for $1 < \alpha \le 2$, whereas in the general case heat conduction in the domain $x < 0$ can occur not only for $1 < \beta \le 2$, but also for $0 < \beta \le 1$.

The solution is obtained in the similar manner and reads:

$$\begin{aligned} T_1(x,t) &= \frac{w_0}{2\sqrt{a_1}\, t^{\alpha/2-1}} \left[W\left(-\frac{\alpha}{2}, 2 - \frac{\alpha}{2}; -\frac{x+\varrho}{\sqrt{a_1}\, t^{\alpha/2}}\right) \right. \\ &\quad \left. + W\left(-\frac{\alpha}{2}, 2 - \frac{\alpha}{2}; -\frac{|x-\varrho|}{\sqrt{a_1}\, t^{\alpha/2}}\right) \right] \\ &\quad - \frac{\sqrt{a_1}}{k_1} \int_0^t \frac{\varphi(t-\tau)}{\tau^{\alpha/2}}\, M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1}\, \tau^{\alpha/2}}\right) \mathrm{d}\tau, \quad x \ge 0, \end{aligned} \tag{4.178}$$

$$T_2(x,t) = \frac{\sqrt{a_2}}{k_2} \int_0^t \frac{\varphi(t-\tau)}{\tau^{\beta/2}}\, M\left(\frac{\beta}{2}; \frac{|x|}{\sqrt{a_2}\, \tau^{\beta/2}}\right) \mathrm{d}\tau, \quad x \le 0, \tag{4.179}$$

where

$$\begin{aligned} \varphi(t) &= \frac{\alpha \rho w_0 k_1}{2 a_1^{3/2}} \int_0^t \frac{1}{\tau^{1+\alpha/2}}\, M\left(\frac{\alpha}{2}; \frac{\varrho}{\sqrt{a_1}\, \tau^{\alpha/2}}\right) \\ &\quad \times \left\{ 1 - E_{\beta/2-\alpha/2}\left[-\frac{(t-\tau)^{\beta/2-\alpha/2}}{\gamma} \right] \right\} \mathrm{d}\tau \end{aligned} \tag{4.180}$$

for $\alpha < \beta$ and

$$\varphi(t) = \frac{\alpha w_0 k_1 \rho}{2a_1^{3/2}} \int\limits_0^t \frac{1}{\tau^{1+\alpha/2}} M\left(\frac{\alpha}{2}; \frac{\varrho}{\sqrt{a_1}\tau^{\alpha/2}}\right) \times E_{\alpha/2-\beta/2}\left[-\gamma(t-\tau)^{\alpha/2-\beta/2}\right] \mathrm{d}\tau \tag{4.181}$$

for $\alpha > \beta$.

In particular, if $\alpha = \beta$, then

$$T_1(x,t) = \frac{w_0}{2\sqrt{a_1}t^{\alpha/2-1}}\left[W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|x-\varrho|}{\sqrt{a_1}t^{\alpha/2}}\right) + \frac{\gamma-1}{\gamma+1} W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{x+\varrho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x \geq 0, \tag{4.182}$$

$$T_2(x,t) = \frac{w_0\gamma}{(\gamma+1)\sqrt{a_1}t^{\alpha/2-1}} \times W\left[-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\left(\frac{|x|}{\sqrt{a_2}t^{\alpha/2}} + \frac{\varrho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x \leq 0. \tag{4.183}$$

If $\alpha = \beta = 2$, then

$$T_1(x,t) = \frac{w_0}{4\sqrt{a_1}}\left\{\operatorname{sign}(x-\varrho+\sqrt{a_1}t) - \operatorname{sign}(x-\varrho-\sqrt{a_1}t) + \frac{\gamma-1}{\gamma+1}\left[1-\operatorname{sign}(x+\varrho-\sqrt{a_1}t)\right]\right\}, \quad x \geq 0, \tag{4.184}$$

$$T_2(x,t) = \frac{\gamma w_0}{2(\gamma+1)\sqrt{a_1}}\left[1-\operatorname{sign}\left(\frac{\sqrt{a_1}}{\sqrt{a_2}}|x| + \varrho - \sqrt{a_1}t\right)\right], \quad x \leq 0. \tag{4.185}$$

Figures 4.19 and 4.20 present the dependence of the fundamental solution to the second Cauchy problem on distance. In this case $\overline{T} = \rho T/(w_0 t)$, other nondimensional quantities are the same as in (4.171).

4.6.4 Fundamental solution to the source problem

Consider the time-fractional heat conduction equation with the source term

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2} + q_0\,\delta(x-\varrho)\,\delta(t), \quad x > 0, \quad 0 < \alpha \leq 2, \tag{4.186}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad x < 0, \qquad 0 < \beta \leq 2, \tag{4.187}$$

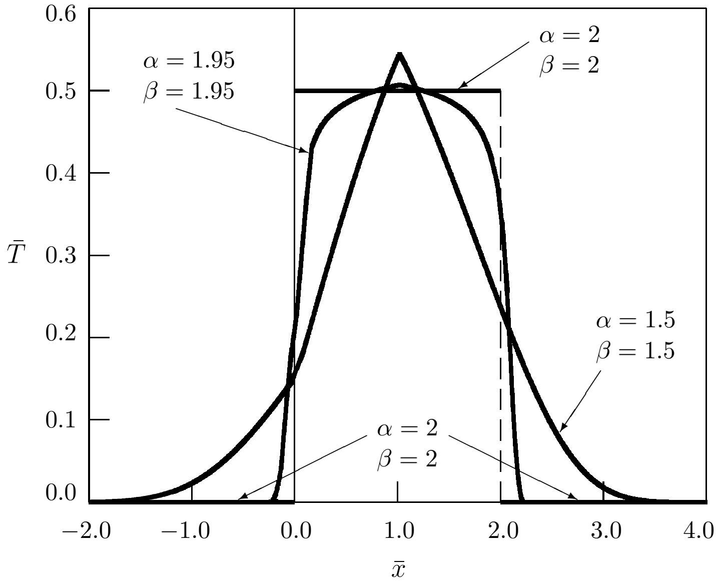


Figure 4.19: Dependence of the fundamental solution to the second Cauchy problem for two joint half-lines on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$ [187]

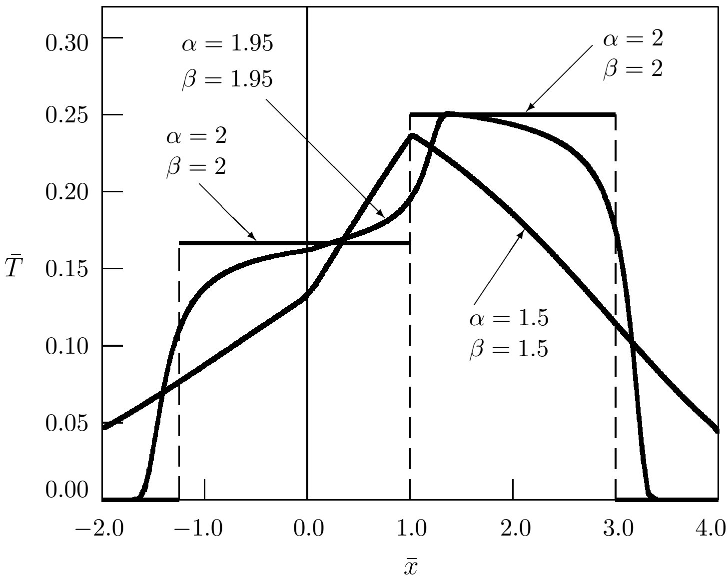


Figure 4.20: Dependence of the fundamental solution to the second Cauchy problem for two joint half-lines on distance; $\kappa = 2$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$ [187]

under the zero initial conditions

$$t=0: \quad T_1=0, \quad x>0, \quad 0<\alpha\le 2, \tag{4.188}$$

$$t=0: \quad \frac{\partial T_1}{\partial t}=0, \quad x>0, \quad 1<\alpha\le 2, \tag{4.189}$$

$$t=0: \quad T_2=0, \quad x<0, \quad 0<\beta\le 2, \tag{4.190}$$

$$t=0: \quad \frac{\partial T_2}{\partial t}=0, \quad x<0, \quad 1<\beta\le 2, \tag{4.191}$$

and the conditions of perfect contact (4.144) and (4.145).

The solution has the following form:

$$\begin{aligned} T_1(x,t) \;=\; & \frac{q_0 t^{\alpha/2-1}}{2\sqrt{a_1}}\left[W\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{x+\varrho}{\sqrt{a_1}t^{\alpha/2}}\right)+W\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\varrho|}{\sqrt{a_1}t^{\alpha/2}}\right)\right] \\ & -\frac{\sqrt{a_1}}{k_1}\int_0^t \frac{\varphi(t-\tau)}{\tau^{\alpha/2}}\,M\left(\frac{\alpha}{2};\frac{x}{\sqrt{a_1}\tau^{\alpha/2}}\right)\mathrm{d}\tau, \quad x\ge 0, \end{aligned} \tag{4.192}$$

$$T_2(x,t)=\frac{\sqrt{a_2}}{k_2}\int_0^t \frac{\varphi(t-\tau)}{\tau^{\beta/2}}\,M\left(\frac{\beta}{2};\frac{|x|}{\sqrt{a_2}\tau^{\beta/2}}\right)\mathrm{d}\tau, \quad x\le 0, \tag{4.193}$$

where

$$\begin{aligned} \varphi(t) \;=\; & \frac{q_0 k_2}{\sqrt{a_1 a_2}}\int_0^t \frac{(t-\tau)^{\beta/2-\alpha/2-1}}{\tau^{2-\alpha}}\,W\left(-\frac{\alpha}{2},\alpha-1;-\frac{\varrho}{\sqrt{a_1}\tau^{\alpha/2}}\right) \\ & \times\, E_{\beta/2-\alpha/2,\,\beta/2-\alpha/2}\left[-\frac{(t-\tau)^{\beta/2-\alpha/2}}{\gamma}\right]\mathrm{d}\tau \end{aligned} \tag{4.194}$$

for $\alpha<\beta$ and

$$\begin{aligned} \varphi(t) \;=\; & \frac{q_0 k_1}{a_1}\int_0^t \frac{(t-\tau)^{\alpha/2-\beta/2-1}}{\tau^{2-\alpha/2-\beta/2}}\,W\left(-\frac{\alpha}{2},\frac{\alpha}{2}+\frac{\beta}{2}-1;-\frac{\varrho}{\sqrt{a_1}\tau^{\alpha/2}}\right) \\ & \times\, E_{\alpha/2-\beta/2,\,\alpha/2-\beta/2}\left[-\gamma(t-\tau)^{\alpha/2-\beta/2}\right]\mathrm{d}\tau \end{aligned} \tag{4.195}$$

for $\alpha>\beta$.

Consider several particular cases of the solution. For $\alpha=\beta$

$$\begin{aligned} T_1(x,t) \;=\; & \frac{q_0 t^{\alpha/2-1}}{2\sqrt{a_1}}\left[W\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\varrho|}{\sqrt{a_1}t^{\alpha/2}}\right)\right. \\ & \left.+\,\frac{\gamma-1}{\gamma+1}\,W\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{x+\varrho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x\ge 0, \end{aligned} \tag{4.196}$$

$$T_2(x,t) = \frac{\gamma q_0 t^{\alpha/2-1}}{(\gamma+1)\sqrt{a_1}} W\left[-\frac{\alpha}{2}, \frac{\alpha}{2}; -\left(\frac{|x|}{\sqrt{a_2}t^{\alpha/2}} + \frac{\varrho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x \le 0. \tag{4.197}$$

It is evident that in the case $\alpha = \beta = 2$ the solutions to the second Cauchy problem and to the source problem coincide and are described by (4.184) and (4.185).

For $\alpha = 1$ and $\beta = 2$ the solution to the source problem coincides with the corresponding solution (4.167) and (4.168) to the first Cauchy problem.

In the case $\alpha = 2$, $\beta = 1$, we get:

$$\begin{aligned} T_1 &= \frac{q_0}{4\sqrt{a_1}}\Big[1 - \operatorname{sign}(x - \sqrt{a_1}t + \varrho) - \operatorname{sign}(x - \sqrt{a_1}t - \varrho) \\ &\quad + \operatorname{sign}(x + \sqrt{a_1}t - \varrho)\Big] - \frac{q_0}{2\sqrt{a_1}} \exp\left[\gamma^2\left(t - \frac{x+\varrho}{\sqrt{a_1}}\right)\right] \\ &\quad \times \operatorname{erfc}\left(\gamma\sqrt{t - \frac{x+\varrho}{\sqrt{a_1}}}\right)\left[1 - \operatorname{sign}(x - \sqrt{a_1}t + \varrho)\right], \quad x \ge 0, \end{aligned} \tag{4.198}$$

$$T_2(x,t) = \begin{cases} \dfrac{q_0}{\sqrt{a_1}}\Bigg\{ - \exp\left[\dfrac{\gamma|x|}{\sqrt{a_2}} + \gamma^2\left(t - \dfrac{\varrho}{\sqrt{a_1}}\right)\right] \\[2ex] \times \operatorname{erfc}\left[\dfrac{|x|}{2\sqrt{a_2(t - \varrho/\sqrt{a_1})}} + \gamma\sqrt{t - \dfrac{\varrho}{\sqrt{a_1}}}\right] \\[2ex] + \operatorname{erfc}\left[\dfrac{|x|}{2\sqrt{a_2(t - \varrho/\sqrt{a_1})}}\right]\Bigg\}, & t > \varrho/\sqrt{a_1}, \quad x \le 0, \\ 0, & t < \varrho/\sqrt{a_1}, \quad x \le 0. \end{cases} \tag{4.199}$$

Dependence of the fundamental solution to the source problem $\overline{T} = \rho t^{1-\alpha} T/q_0$ on distance $\bar{x} = x/\rho$ is depicted in Figs. 4.21 and 4.22.

4.6.5 Uniform initial temperature in one of half-lines

Consider the time-fractional heat conduction equations in two half-lines

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \qquad x > 0, \qquad 0 < \alpha \le 2, \tag{4.200}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \qquad x < 0, \qquad 0 < \beta \le 2, \tag{4.201}$$

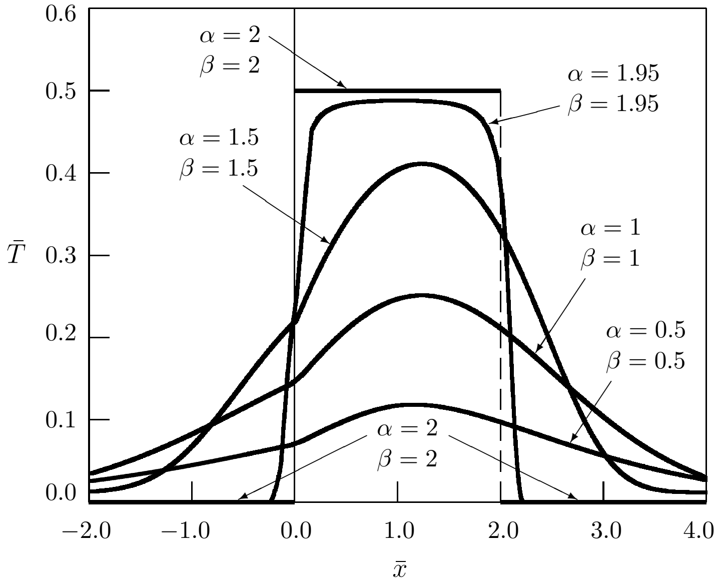


Figure 4.21: Dependence of the fundamental solution to the source problem for two joint half-lines on distance; $\kappa = 1$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$ [187]

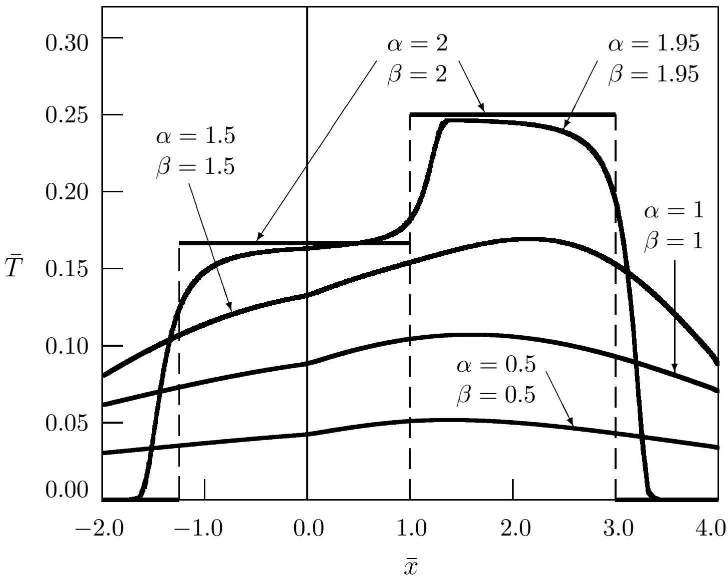


Figure 4.22: Dependence of the fundamental solution to the source problem for two joint half-lines on distance; $\kappa = 2$, $\bar{\gamma} = 0.5$, $\epsilon = 0.8$ [187]

under the initial conditions

$$t = 0: \quad T_1 = T_0, \quad x > 0, \quad 0 < \alpha \le 2, \tag{4.202}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = 0, \quad x > 0, \quad 1 < \alpha \le 2, \tag{4.203}$$

$$t = 0: \quad T_2 = 0, \quad x < 0, \quad 0 < \beta \le 2, \tag{4.204}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = 0, \quad x < 0, \quad 1 < \beta \le 2, \tag{4.205}$$

and the boundary conditions of perfect thermal contact (4.144) and (4.145). The conditions at infinity are also assumed

$$\left.\frac{\partial T_1(x,t)}{\partial x}\right|_{x\to\infty} = 0, \qquad \left.\frac{\partial T_2(x,t)}{\partial x}\right|_{x\to-\infty} = 0. \tag{4.206}$$

The solution has the following form [183]:

$$\begin{aligned} T_1 &= T_0 - \frac{T_0}{\gamma} \int_0^t \frac{(t-\tau)^{\beta/2-1}}{\tau^{\alpha/2}} E_{\beta/2-\alpha/2,\,\beta/2}\left[-\frac{1}{\gamma}(t-\tau)^{\beta/2-\alpha/2}\right] \\ &\quad \times M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1}\tau^{\alpha/2}}\right) \mathrm{d}\tau, \qquad x > 0, \end{aligned} \tag{4.207}$$

$$\begin{aligned} T_2 &= T_0 \int_0^t \frac{(t-\tau)^{\beta/2-1}}{\tau^{\beta/2}} E_{\beta/2-\alpha/2,\,\beta/2}\left[-\frac{1}{\gamma}(t-\tau)^{\beta/2-\alpha/2}\right] \\ &\quad \times M\left(\frac{\beta}{2}; \frac{|x|}{\sqrt{a_2}\tau^{\beta/2}}\right) \mathrm{d}\tau, \qquad x < 0, \end{aligned} \tag{4.208}$$

for $\alpha < \beta$, where $\gamma = k_1\sqrt{a_2}/(k_2\sqrt{a_1})$.

Similarly, for $\alpha > \beta$ we obtain

$$\begin{aligned} T_1 &= T_0 - T_0 \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} E_{\alpha/2-\beta/2,\,\alpha/2}\left[-\gamma(t-\tau)^{\alpha/2-\beta/2}\right] \\ &\quad \times M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1}\tau^{\alpha/2}}\right) \mathrm{d}\tau, \qquad x > 0, \end{aligned} \tag{4.209}$$

$$\begin{aligned} T_2 &= \gamma T_0 \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\beta/2}} E_{\alpha/2-\beta/2,\,\alpha/2}\left[-\gamma(t-\tau)^{\alpha/2-\beta/2}\right] \\ &\quad \times M\left(\frac{\beta}{2}; \frac{|x|}{\sqrt{a_2}\tau^{\beta/2}}\right) \mathrm{d}\tau, \qquad x < 0. \end{aligned} \tag{4.210}$$

The equations given above simplify for $\alpha = \beta$:

$$T_1 = T_0 - \frac{T_0}{(1+\gamma)} W\left(-\frac{\alpha}{2}, 1; -\frac{x}{\sqrt{a_1} t^{\alpha/2}}\right), \qquad x > 0, \tag{4.211}$$

$$T_2 = \frac{\gamma T_0}{(1+\gamma)} W\left(-\frac{\alpha}{2}, 1; -\frac{|x|}{\sqrt{a_1} t^{\alpha/2}}\right), \qquad x < 0, \tag{4.212}$$

where $W(\alpha, \beta; x)$ is the Wright function.

For the standard diffusion (heat conduction) equation ($\alpha = 1$) we arrive at the well-known solution [98, 140]

$$T_1 = T_0 - \frac{T_0}{1+\gamma} \operatorname{erfc}\left(\frac{x}{2\sqrt{a_1 t}}\right), \qquad x > 0, \tag{4.213}$$

$$T_2 = \frac{\gamma T_0}{1+\gamma} \operatorname{erfc}\left(\frac{|x|}{2\sqrt{a_2 t}}\right), \qquad x < 0. \tag{4.214}$$

In the case of the wave equation ($\alpha = 2$):

$$T_1 = \begin{cases} \dfrac{\gamma}{1+\gamma} T_0, & 0 \le x < \sqrt{a_1} t, \\ T_0, & \sqrt{a_1} t < x < \infty, \end{cases} \tag{4.215}$$

$$T_2 = \begin{cases} \dfrac{\gamma}{1+\gamma} T_0, & -\sqrt{a_2} t < x \le 0, \\ 0, & -\infty < x < -\sqrt{a_2} t. \end{cases} \tag{4.216}$$

We also present the solutions for two limiting cases:

$\alpha = 1$, $\beta = 2$

$$T_1 = T_0 \left[\operatorname{erf}\left(\frac{x}{2\sqrt{a_1 t}}\right) + \exp\left(\frac{x}{\sqrt{a_1}\gamma} + \frac{t}{\gamma^2}\right) \times \operatorname{erfc}\left(\frac{x}{2\sqrt{a_1 t}} + \frac{\sqrt{t}}{\gamma}\right)\right], \qquad x > 0, \tag{4.217}$$

$$T_2 = \begin{cases} T_0 \exp\left[\dfrac{1}{\gamma^2}\left(t - \dfrac{|x|}{\sqrt{a_2}}\right)\right] \operatorname{erfc}\left(\dfrac{1}{\gamma}\sqrt{t - \dfrac{|x|}{\sqrt{a_2}}}\right), & \\ & -\sqrt{a_2} t < x \le 0, \\ 0, & -\infty < x < -\sqrt{a_2} t\,. \end{cases} \tag{4.218}$$

$\alpha = 2$, $\beta = 1$

$$T_1 = \begin{cases} T_0 \left\{1 - \exp\left[\gamma^2\left(t - \frac{x}{\sqrt{a_1}}\right)\right] \operatorname{erfc}\left(\gamma\sqrt{t - \frac{x}{\sqrt{a_1}}}\right)\right\}, & \\ & 0 \le x < \sqrt{a_1}t\,, \\ T_0, & \sqrt{a_1}t < x < \infty\,, \end{cases} \tag{4.219}$$

$$T_2 = T_0\left[\operatorname{erfc}\left(\frac{|x|}{2\sqrt{a_2 t}}\right) - \exp\left(\frac{\gamma|x|}{\sqrt{a_2}} + \gamma^2 t\right)\right.$$
$$\left. \times \operatorname{erfc}\left(\frac{|x|}{2\sqrt{a_2 t}} + \gamma\sqrt{t}\right)\right], \qquad x < 0. \tag{4.220}$$

Figures 4.23–4.26 show the variation of the nondimensional temperature $\overline{T} = T/T_0$ with distance for typical values of the orders of fractional derivatives. In Figs. 4.23 and 4.24, the nondimensional distance is introduced as $\bar{x} = \frac{x}{\sqrt{a_1} t^{\alpha/2}}$ for $x > 0$ and $\bar{x} = \frac{x}{\sqrt{a_2} t^{\alpha/2}}$ for $x < 0$. In Fig. 4.25, $\bar{x} = \frac{x}{\sqrt{a_1 t}}$ for $x > 0$ and $\bar{x} = \frac{x}{\sqrt{a_2 t}}$ for $x < 0$, whereas in Fig. 4.26 $\bar{x} = \frac{x}{\sqrt{a_1 t}}$ for $x > 0$ and $\bar{x} = \frac{x}{\sqrt{a_2 t}}$ for $x < 0$.

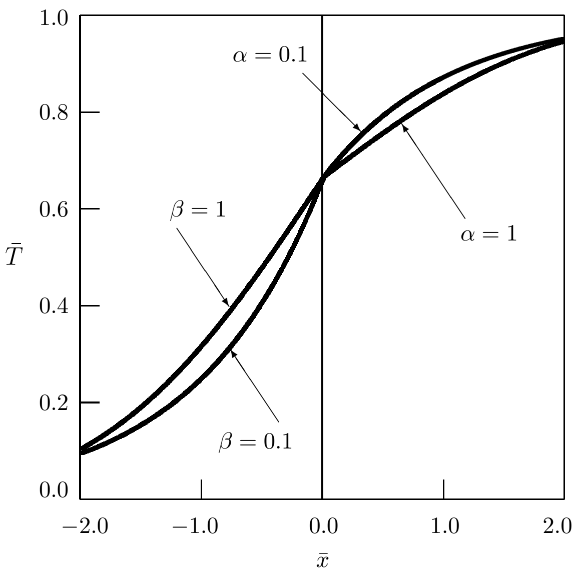


Figure 4.23: Dependence of the solution in two joint half-lines on distance for $0 < \alpha = \beta \le 1$ ($\gamma = 2$) [183]

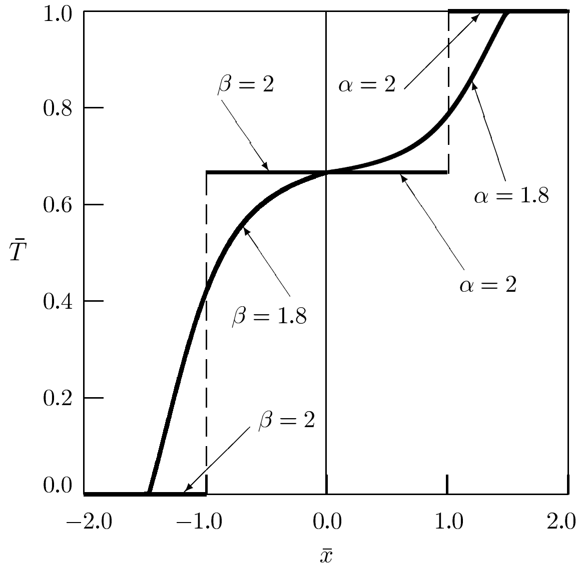


Figure 4.24: Dependence of the solution in two joint half-lines on distance for $1 < \alpha = \beta \leq 2$ ($\gamma = 2$) [183]

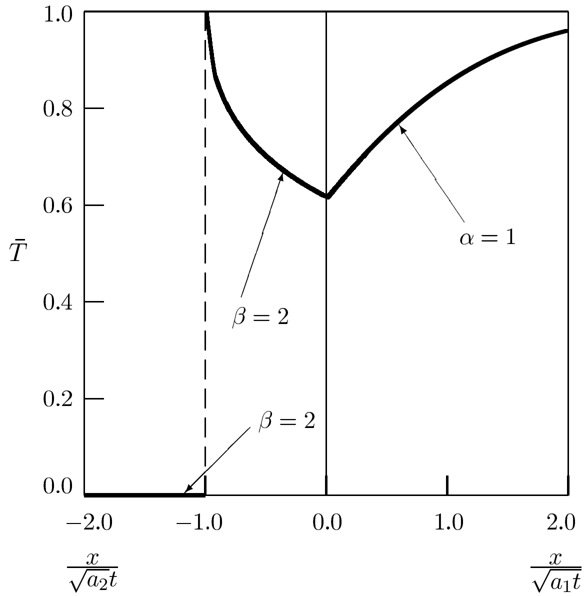


Figure 4.25: Dependence of the solution in two joint half-lines on distance for $\alpha = 1, \ \beta = 2 \quad (\bar{\gamma} = \gamma/\sqrt{t} = 2)$ [183]

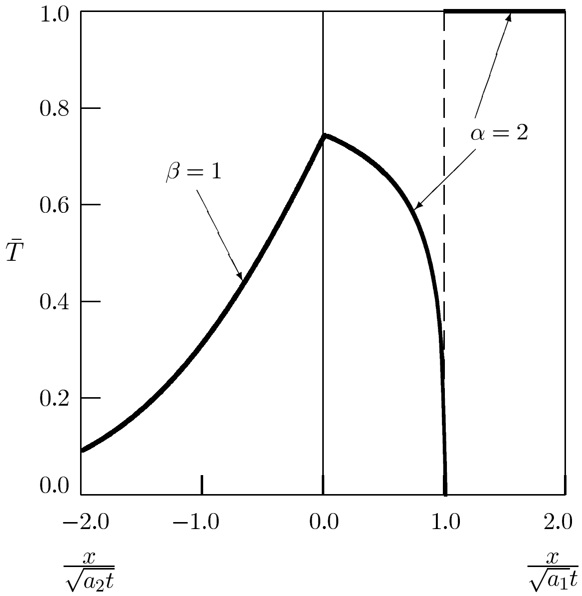


Figure 4.26: Dependence of the solution in two joint half-lines on distance for $\alpha = 2$, $\beta = 1$ $(\bar{\gamma} = \gamma\sqrt{t} = 2)$ [183]

4.7 Semi-infinite composed body

4.7.1 Statement of the problem

Consider the time-fractional heat conduction equations with the Caputo derivative in a two-layer medium composed of a region $0 < x < L$ and a region $L < x < \infty$ [192]:

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2} + \Phi_1(x,t), \quad 0 < x < L, \quad 0 < \alpha \le 2, \tag{4.221}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2} + \Phi_2(x,t), \quad L < x < \infty, \quad 0 < \beta \le 2, \tag{4.222}$$

under the initial conditions

$$t = 0: \quad T_1 = f_1(x), \quad 0 < x < L, \quad 0 < \alpha \le 2, \tag{4.223}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = F_1(x), \quad 0 < x < L, \quad 1 < \alpha \le 2, \tag{4.224}$$

$$t = 0: \quad T_2 = f_2(x), \quad L < x < \infty, \quad 0 < \beta \le 2, \tag{4.225}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = F_2(x), \quad L < x < \infty, \quad 1 < \beta \le 2, \tag{4.226}$$

and the boundary conditions of perfect thermal contact

$$x = L: \quad T_1(x,t) = T_2(x,t), \tag{4.227}$$

$$x = L: \quad k_1 D_{RL}^{1-\alpha} \frac{\partial T_1(x,t)}{\partial x} = k_2 D_{RL}^{1-\beta} \frac{\partial T_2(x,t)}{\partial x},$$
$$0 < \alpha \le 2, \qquad 0 < \beta \le 2. \tag{4.228}$$

The boundary surface $x = 0$ is kept insulated:

$$x = 0: \quad \frac{\partial T(x,t)}{\partial x} = 0. \tag{4.229}$$

In addition, the boundedness condition at infinity is assumed

$$\lim_{x \to \infty} T(x,t) = 0. \tag{4.230}$$

4.7.2 Uniform initial temperature in the layer

In this case we consider the time-fractional heat conduction equations

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \quad 0 < x < L, \quad 0 < \alpha \le 2, \tag{4.231}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \frac{\partial^2 T_2}{\partial x^2}, \quad L < x < \infty, \quad 0 < \beta \le 2, \tag{4.232}$$

under the initial conditions

$$t = 0: \quad T_1 = T_0, \quad 0 < x < L, \qquad 0 < \alpha \le 2, \tag{4.233}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = 0, \quad 0 < x < L, \qquad 1 < \alpha \le 2, \tag{4.234}$$

$$t = 0: \quad T_2 = 0, \quad L < x < \infty, \qquad 0 < \beta \le 2, \tag{4.235}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = 0, \quad L < x < \infty, \qquad 1 < \beta \le 2, \tag{4.236}$$

and the boundary conditions (4.227)–(4.230).

The Laplace transform with respect time gives two ordinary differential equations

$$s^\alpha T_1^* - s^{\alpha-1} T_0 = a_1 \frac{\mathrm{d}^2 T_1^*}{\mathrm{d}x^2}, \qquad 0 < x < L, \tag{4.237}$$

$$s^\beta T_2^* = a_2 \frac{\mathrm{d}^2 T_2^*}{\mathrm{d}x^2}, \qquad L < x < \infty, \tag{4.238}$$

and the boundary conditions

$$x = 0: \quad \frac{\mathrm{d}T_1^*}{\mathrm{d}x} = 0, \tag{4.239}$$

$$x = L: \quad T_1^* = T_2^*, \tag{4.240}$$

$$x = L: \quad k_1 s^{1-\alpha} \frac{\mathrm{d}T_1^*}{\mathrm{d}x} = k_2 s^{1-\beta} \frac{\mathrm{d}T_2^*}{\mathrm{d}x}, \tag{4.241}$$

$$x \to \infty: \quad T_2^* = 0. \tag{4.242}$$

The solutions of equations (4.237) and (4.238) have the following form:

$$T_1^* = \frac{T_0}{s} + A \sinh\left(\sqrt{\frac{s^\alpha}{a_1}} x\right) + B \cosh\left(\sqrt{\frac{s^{\alpha/2}}{a_1}} x\right), \quad 0 < x < L, \tag{4.243}$$

$$T_2^* = C \exp\left(-\sqrt{\frac{s^\beta}{a_2}} x\right) + D \exp\left(\sqrt{\frac{s^\beta}{a_2}} x\right), \quad L < x < \infty. \tag{4.244}$$

From conditions (4.239) and (4.242) it follows that

$$B = 0, \qquad D = 0, \tag{4.245}$$

whereas the conditions of the perfect thermal contact (4.240) and (4.241) give

$$A = -\frac{T_0}{s} \frac{1}{\cosh\left(\sqrt{\frac{s^\alpha}{a_1}} L\right) + \gamma s^{\beta/2-\alpha/2} \sinh\left(\sqrt{\frac{s^\alpha}{a_1}} L\right)}, \tag{4.246}$$

$$\begin{aligned} C &= \frac{T_0}{s} \exp\left(\sqrt{\frac{s^\beta}{a_2}} L\right) \\ &- \frac{T_0}{s} \frac{\cosh\left(\sqrt{\frac{s^\alpha}{a_1}} L\right) \exp\left(\sqrt{\frac{s^\beta}{a_2}} L\right)}{\cosh\left(\sqrt{\frac{s^\alpha}{a_1}} L\right) + \gamma s^{\beta/2-\alpha/2} \sinh\left(\sqrt{\frac{s^\alpha}{a_1}} L\right)}, \end{aligned} \tag{4.247}$$

where

$$\gamma = \frac{k_1 \sqrt{a_2}}{k_2 \sqrt{a_1}}.$$

Hence,

$$T_1^* = \frac{T_0}{s} - \frac{T_0}{s} \frac{\cosh\left(\sqrt{\frac{s^\alpha}{a_1}} x\right)}{\cosh\left(\sqrt{\frac{s^\alpha}{a_1}} L\right) + \gamma s^{\beta/2-\alpha/2} \sinh\left(\sqrt{\frac{s^\alpha}{a_1}} L\right)} \tag{4.248}$$

$$T_2^* = \frac{T_0}{s} \exp\left[-\sqrt{\frac{s^\beta}{a_2}}(x-L)\right]$$
$$- \frac{T_0}{s} \frac{\cosh\left(\sqrt{\frac{s^\alpha}{a_1}}L\right) \exp\left[-\sqrt{\frac{s^\beta}{a_2}}(x-L)\right]}{\cosh\left(\sqrt{\frac{s^\alpha}{a_1}}L\right) + \gamma s^{\beta/2-\alpha/2} \sinh\left(\sqrt{\frac{s^\alpha}{a_1}}L\right)}. \tag{4.249}$$

Now we will investigate the approximate solution of the considered problem for small values of time. In the case of classical heat conduction equation this method was described in [98, 140]. Based on Tauberian theorems for the Laplace transform, for small values of time t (the large values of the transform variable s) we can neglect the exponential term in comparison with 1:

$$1 \pm \exp\left(-2\sqrt{\frac{s^\alpha}{a_1}}L\right) \simeq 1, \tag{4.250}$$

thus obtaining

$$T_1^* \simeq \frac{T_0}{s}\left\{1 - \frac{\exp\left[-\sqrt{\frac{s^\alpha}{a_1}}(L-x)\right]}{1+\gamma s^{\beta/2-\alpha/2}}\right\}, \tag{4.251}$$

$$T_2^* \simeq \frac{T_0}{s} \frac{\gamma s^{\beta/2-\alpha/2}}{1+\gamma s^{\beta/2-\alpha/2}} \exp\left[-\sqrt{\frac{s^\beta}{a_2}}(x-L)\right]. \tag{4.252}$$

Inverting the Laplace transform, we get:

a) $\beta > \alpha$

$$T_1(x,t) \simeq T_0 - \frac{T_0}{\gamma} \int_0^t \frac{(t-\tau)^{\beta/2-1}}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{L-x}{\sqrt{a_1}\tau^{\alpha/2}}\right)$$
$$\times E_{\beta/2-\alpha/2\,,\,\beta/2}\left[-\frac{1}{\gamma}(t-\tau)^{\beta/2-\alpha/2}\right] \mathrm{d}\tau, \quad 0 \le x \le L, \tag{4.253}$$

$$T_2(x,t) \simeq T_0 \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\beta/2}} M\left(\frac{\beta}{2}; \frac{x-L}{\sqrt{a_2}\tau^{\beta/2}}\right)$$
$$\times E_{\beta/2-\alpha/2\,,\,\alpha/2}\left[-\frac{1}{\gamma}(t-\tau)^{\beta/2-\alpha/2}\right] \mathrm{d}\tau, \quad L \le x < \infty. \tag{4.254}$$

b) $\alpha > \beta$

$$T_1(x,t) \simeq T_0 - T_0 \int\limits_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{L-x}{\sqrt{a_1}\tau^{\alpha/2}}\right) \times E_{\alpha/2-\beta/2\,,\,\alpha/2}\left[-\gamma\,(t-\tau)^{\alpha/2-\beta/2}\right] \mathrm{d}\tau, \qquad 0 \le x \le L, \tag{4.255}$$

$$T_2(x,t) \simeq T_0\gamma \int\limits_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\beta/2}} M\left(\frac{\beta}{2}; \frac{x-L}{\sqrt{a_2}\tau^{\beta/2}}\right) \times E_{\alpha/2-\beta/2\,,\,\alpha/2}\left[-\gamma\,(t-\tau)^{\alpha/2-\beta/2}\right] \mathrm{d}\tau, \qquad L \le x < \infty. \tag{4.256}$$

For example, for $\alpha = 2$, $\beta = 1$

$$T_1 \simeq \begin{cases} T_0, & 0 \le x < L - \sqrt{a_1}t, \\ T_0 \left\{1 - \exp\left[\gamma^2\left(t - \dfrac{L-x}{\sqrt{a_1}}\right)\right] \operatorname{erfc}\left(\gamma\sqrt{t - \dfrac{L-x}{\sqrt{a_1}}}\right)\right\}, & \\ & L - \sqrt{a_1}t < x < L\,; \end{cases} \tag{4.257}$$

$$T_2 \simeq T_0\left[\operatorname{erfc}\left(\frac{x-L}{2\sqrt{a_2 t}}\right) - \exp\left(\gamma\,\frac{x-L}{\sqrt{a_2}} + \gamma^2 t\right) \times \operatorname{erfc}\left(\frac{x-L}{2\sqrt{a_2 t}} + \gamma\sqrt{t}\right)\right], \qquad L < x < \infty. \tag{4.258}$$

For $\alpha = 1$, $\beta = 2$

$$T_1 \simeq T_0\left[\operatorname{erf}\left(\frac{L-x}{2\sqrt{a_1 t}}\right) + \exp\left(\frac{L-x}{\gamma\sqrt{a_1}} + \frac{t}{\gamma^2}\right) \times \operatorname{erfc}\left(\frac{L-x}{2\sqrt{a_1 t}} + \frac{\sqrt{t}}{\gamma}\right)\right], \qquad 0 \le x < L, \tag{4.259}$$

$$T_2 \simeq \begin{cases} T_0 \exp\left[\dfrac{1}{\gamma^2}\left(t - \dfrac{x-L}{\sqrt{a_2}}\right)\right] \operatorname{erfc}\left(\dfrac{1}{\gamma}\sqrt{t - \dfrac{x-L}{\sqrt{a_2}}}\right), & \\ & L < x < L + \sqrt{a_2}t\,, \\ 0, & L + \sqrt{a_1}t < x < \infty\,. \end{cases} \tag{4.260}$$

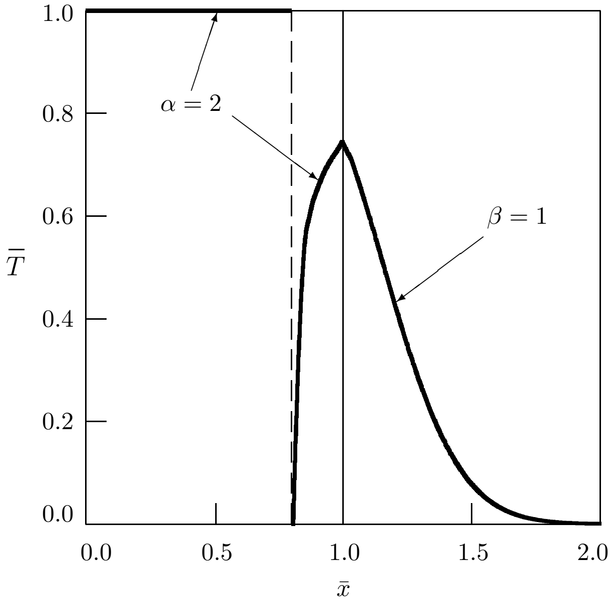


Figure 4.27: Dependence of the solution in a semi-infinite composed body on distance [192]

In particular, if $\alpha = \beta$, then

$$T_1 \simeq T_0 - \frac{T_0}{1+\gamma} W\left(-\frac{\alpha}{2}, 1; -\frac{L-x}{\sqrt{a_1} t^{\alpha/2}}\right), \qquad 0 \leq x \leq L, \tag{4.261}$$

$$T_2 \simeq \frac{T_0 \gamma}{1+\gamma} W\left(-\frac{\alpha}{2}, 1; -\frac{x-L}{\sqrt{a_2} t^{\alpha/2}}\right), \qquad L \leq x < \infty. \tag{4.262}$$

Several results of numerical calculations are presented in Figs. 4.27–4.29 with the following nondimensional quantities:

$$\overline{T} = \frac{T}{T_0}, \quad \bar{x} = \frac{x}{L}, \quad \kappa = \frac{\sqrt{a_1} t^{\alpha/2}}{L}, \quad \bar{\gamma} = \gamma t^{\alpha/2 - \beta/2}, \quad \epsilon = \frac{\sqrt{a_1}}{\sqrt{a_2}} t^{\alpha/2 - \beta/2}.$$

In calculations we have taken $\kappa = 0.2$, $\bar{\gamma} = 2$ and $\epsilon = 0.6$. Such values of nondimensional parameters show the typical features of the solution.

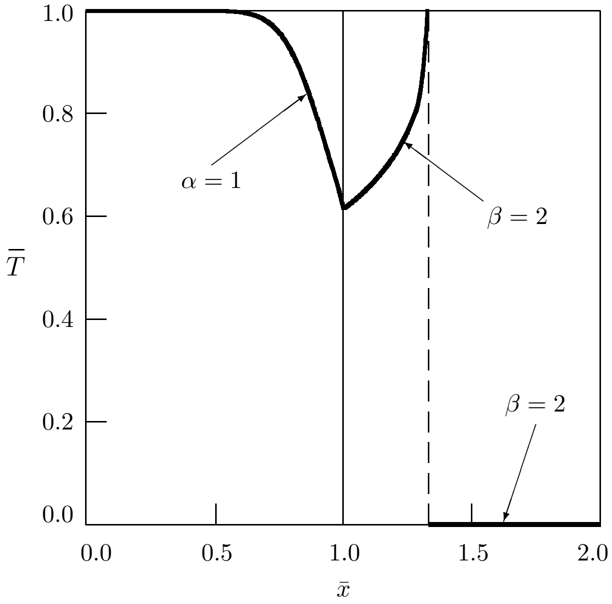


Figure 4.28: Dependence of the solution in a semi-infinite composed body on distance [192]

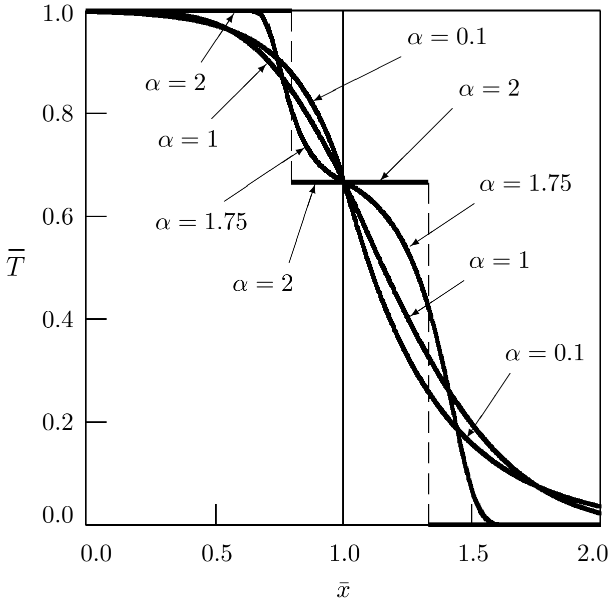


Figure 4.29: Dependence of the solution in a semi-infinite composed body on distance [192]

Chapter 5

Equations with One Space Variable in Polar Coordinates

You're either part of the solution or you're part of the problem.

Eldridge Cleaver

5.1 Domain $0 \leq r < \infty$

5.1.1 Statement of the problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \Phi(r, t), \tag{5.1}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \leq 2, \tag{5.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \leq 2, \tag{5.3}$$

$$\lim_{r \to \infty} T(r, t) = 0. \tag{5.4}$$

The solution:

$$T(r,t) = \int_0^\infty f(\rho)\, \mathcal{G}_f(r, \rho, t)\, \rho \,\mathrm{d}\rho + \int_0^\infty F(\rho)\, \mathcal{G}_F(r, \rho, t)\, \rho \,\mathrm{d}\rho$$
$$+ \int_0^t \int_0^\infty \Phi(\rho, \tau)\, \mathcal{G}_\Phi(r, \rho, t - \tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\tau. \tag{5.5}$$

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The fundamental solutions to the first Cauchy problem $\mathcal{G}_f(r,\rho,t)$, to the second Cauchy problem $\mathcal{G}_F(r,\rho,t)$ and to the source problem $\mathcal{G}_\Phi(r,\rho,t)$ were obtained in [148].

5.1.2 Fundamental solution to the first Cauchy problem

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_f}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_f}{\partial r}\right), \tag{5.6}$$

$$t=0: \quad \mathcal{G}_f = p_0\,\frac{\delta(r-\rho)}{r}, \qquad 0<\alpha\le 2, \tag{5.7}$$

$$t=0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \qquad 1<\alpha\le 2. \tag{5.8}$$

It should be noted that the two-dimensional Dirac delta function in Cartesian coordinates $\delta(x)\,\delta(y)$ after passing to polar coordinates takes the form $\frac{1}{2\pi r}\,\delta(r)$, but for the sake of simplicity we have omitted the factor 2π in the solution (5.5) as well as the factor $\frac{1}{2\pi}$ in the delta term in (5.7). The condition at infinity (5.4) will be implied in all the problems in infinite domains considered in this chapter.

The Laplace transform with respect to time t and the Hankel transform of order 0 with respect to the radial variable r (2.78) give

$$\widehat{\mathcal{G}}_f^* = p_0 J_0(\rho\xi)\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{5.9}$$

The inverse integral transforms result in

$$\mathcal{G}_f(r,\rho,t) = p_0\int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) J_0(r\xi)\,J_0(\rho\xi)\,\xi\,d\xi. \tag{5.10}$$

It is convenient to introduce the following nondimensional quantities:

$$\bar r = \frac{r}{\rho}, \quad \eta = \rho\xi, \quad \kappa = \frac{\sqrt{a}\,t^{\alpha/2}}{\rho}, \quad \bar{\mathcal{G}}_f = \frac{\rho^2}{p_0}\,\mathcal{G}_f. \tag{5.11}$$

In this case

$$\bar{\mathcal{G}}_f = \int_0^\infty E_\alpha\left(-\kappa^2\eta^2\right) J_0(\eta)\,J_0(\bar r\eta)\,\eta\,\mathrm{d}\eta. \tag{5.12}$$

Consider several particular cases of the solution (5.12).

Helmholtz equation ($\alpha\to 0$)

$$\bar{\mathcal{G}}_f = \begin{cases} \dfrac{1}{\kappa^2}\,I_0\left(\dfrac{\bar r}{\kappa}\right)K_0\left(\dfrac{1}{\kappa}\right), & 0\le \bar r<1,\\[2ex] \dfrac{1}{\kappa^2}\,I_0\left(\dfrac{1}{\kappa}\right)K_0\left(\dfrac{\bar r}{\kappa}\right), & 1<\bar r<\infty,\end{cases} \tag{5.13}$$

where $I_0(x)$ and $K_0(x)$ are the modified Bessel functions.

Subdiffusion with $\alpha = 1/2$

$$\bar{\mathcal{G}}_f = \frac{1}{2\sqrt{\pi}\kappa^2} \int_0^\infty \exp\left(-u^2 - \frac{1+\bar{r}^2}{8\kappa^2 u}\right) I_0\left(\frac{\bar{r}}{4\kappa^2 u}\right) \frac{1}{u}\, du. \tag{5.14}$$

Classical diffusion equation ($\alpha = 1$)

$$\bar{\mathcal{G}}_f = \frac{1}{2\kappa^2} \exp\left(-\frac{1+\bar{r}^2}{4\kappa^2}\right) I_0\left(\frac{\bar{r}}{2\kappa^2}\right). \tag{5.15}$$

Wave equation ($\alpha = 2$)

a) $0 < \kappa < 1$

$$\bar{\mathcal{G}}_f = \frac{1}{2\sqrt{1-\kappa}}\,\delta(\bar{r}-1+\kappa) + \frac{1}{2\sqrt{1+\kappa}}\,\delta(\bar{r}-1-\kappa)$$
$$+\begin{cases} 0, & 0 \le \bar{r} < 1-\kappa, \\ \dfrac{\kappa}{4\pi k^2 k'^2 \bar{r}^{3/2}}\left[\mathbf{E}(k) - k'^2\mathbf{K}(k)\right], & 1-\kappa < \bar{r} < 1+\kappa, \\ 0, & 1+\kappa < \bar{r} < \infty. \end{cases} \tag{5.16}$$

b) $\kappa > 1$

$$\bar{\mathcal{G}}_f = \frac{1}{2\sqrt{1+\kappa}}\,\delta(\bar{r}-1-\kappa)$$
$$+\begin{cases} \dfrac{\kappa}{4\pi k k'^2 \bar{r}^{3/2}}\,\mathbf{E}\left(\dfrac{1}{k}\right), & 0 \le \bar{r} < \kappa-1, \\ \dfrac{\kappa}{4\pi k^2 k'^2 \bar{r}^{3/2}}\left[\mathbf{E}(k) - k'^2\mathbf{K}(k)\right], & \kappa-1 < \bar{r} < 1+\kappa, \\ 0, & 1+\kappa < \bar{r} < \infty, \end{cases} \tag{5.17}$$

where $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the complete elliptic integrals of the first and second kind, respectively,

$$k = \frac{\sqrt{\kappa^2 - (\bar{r}-1)^2}}{2\sqrt{\bar{r}}}, \qquad k' = \sqrt{1-k^2}. \tag{5.18}$$

Dependence of the fundamental solution $\bar{\mathcal{G}}_f$ on nondimensional distance $\bar{r}$ is shown in Figs. 5.1 and 5.2 for various values of κ and α. In what follows, three distinguishing values of the parameter κ are considered: $0 < \kappa < 1$, $\kappa = 1$ and $\kappa > 1$. For a wave equation these values correspond to three characteristic events: the wave front does not yet arrive at the origin, the wave front arrives at the origin, and the wave front reflects from the origin.

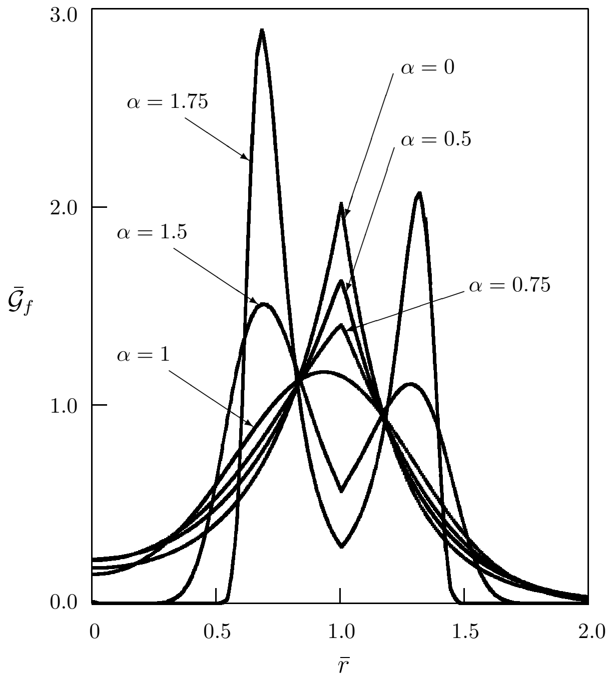

Figure 5.1: Dependence of the fundamental solution to the first Cauchy problem in a plane on distance; $\kappa = 0.25$ [148]

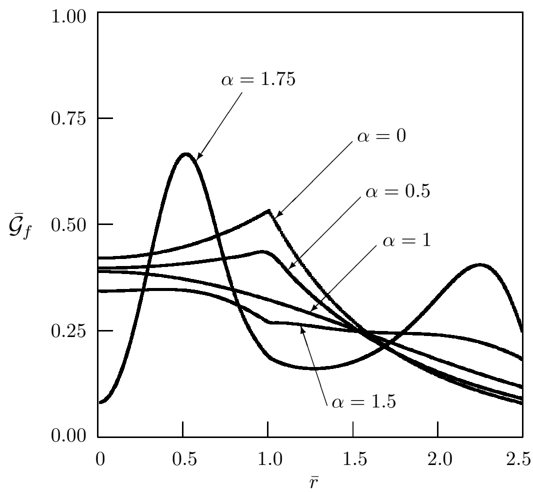

Figure 5.2: Dependence of the fundamental solution to the first Cauchy problem in a plane on distance; $\kappa = 1$

5.1.3 Fundamental solution to the second Cauchy problem

$$\frac{\partial^\alpha \mathcal{G}_F}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_F}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_F}{\partial r}\right), \tag{5.19}$$

$$t = 0: \quad \mathcal{G}_F = 0, \qquad 1 < \alpha \le 2, \tag{5.20}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_F}{\partial t} = w_0\,\frac{\delta(r-\rho)}{r}, \qquad 1 < \alpha \le 2. \tag{5.21}$$

The solution:

$$\mathcal{G}_F(r,\rho,t) = w_0 t \int_0^\infty E_{\alpha,2}\left(-a\xi^2 t^\alpha\right) J_0(r\xi)\, J_0(\rho\xi)\,\xi\, d\xi. \tag{5.22}$$

Wave equation ($\alpha = 2$)

a) $0 < \kappa < 1$

$$\bar{\mathcal{G}}_F = \begin{cases} 0, & 0 \le \bar{r} < 1-\kappa, \\ \dfrac{1}{\kappa\pi\sqrt{\bar{r}}}\,\mathbf{K}(k), & 1-\kappa < \bar{r} < 1+\kappa, \\ 0, & 1+\kappa < \bar{r} < \infty. \end{cases} \tag{5.23}$$

b) $\kappa = 1$

$$\bar{\mathcal{G}}_F = \begin{cases} \dfrac{1}{\pi\sqrt{\bar{r}}}\,\mathbf{K}(k), & 0 < \bar{r} < 2, \\ 0, & 2 < \bar{r} < \infty. \end{cases} \tag{5.24}$$

c) $\kappa > 1$

$$\bar{\mathcal{G}}_F = \begin{cases} \dfrac{1}{\kappa\pi k\sqrt{\bar{r}}}\,\mathbf{K}\left(\dfrac{1}{k}\right), & 0 \le \bar{r} < \kappa - 1, \\ \dfrac{1}{\kappa\pi\sqrt{\bar{r}}}\,\mathbf{K}(k), & \kappa - 1 < \bar{r} < 1+\kappa, \\ 0, & 1+\kappa < \bar{r} < \infty, \end{cases} \tag{5.25}$$

where $\bar{\mathcal{G}}_F = \rho^2 \mathcal{G}_F/(w_0 t)$, other nondimensional quantities are the same as in (5.11).

Dependence of the fundamental solution to the second Cauchy problem on distance is shown in Figs. 5.3–5.5.

In the case of the wave equation ($\alpha = 2$) the fundamental solution has jumps at $\bar{r} = 1-\kappa$ and at $\bar{r} = 1+\kappa$ for $0 < \kappa < 1$, has a singularity at the origin $\bar{r} = 0$ for $\kappa = 1$, and has a singularity at $\bar{r} = \kappa - 1$ and a jump at $\bar{r} = \kappa + 1$ for $\kappa > 1$.

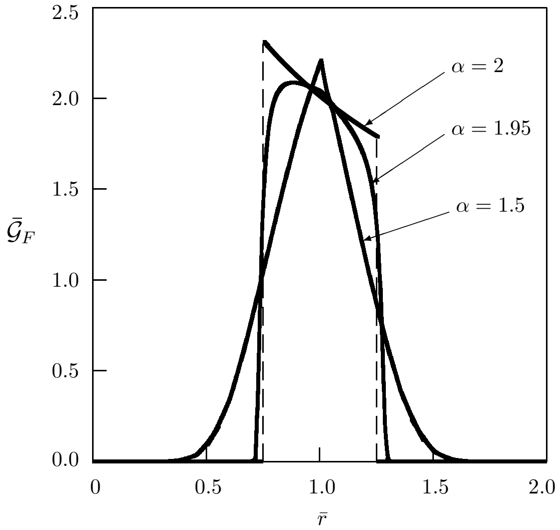

Figure 5.3: Dependence of the fundamental solution to the second Cauchy problem in a plane on distance; $\kappa = 0.25$ [148]

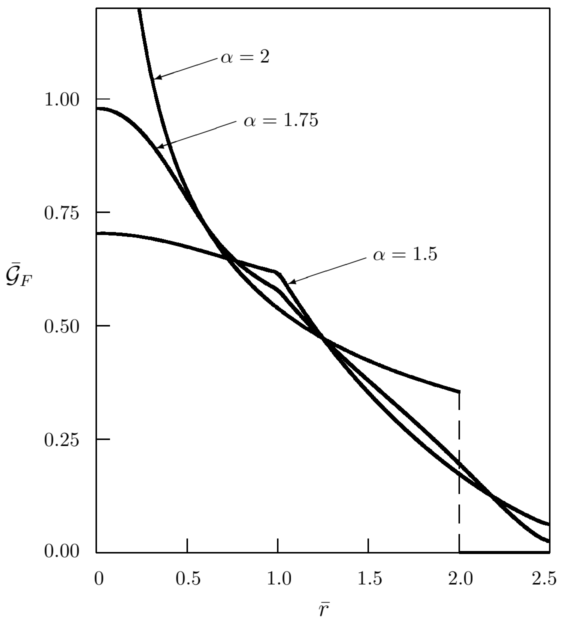

Figure 5.4: Dependence of the fundamental solution to the second Cauchy problem in a plane on distance; $\kappa = 1$

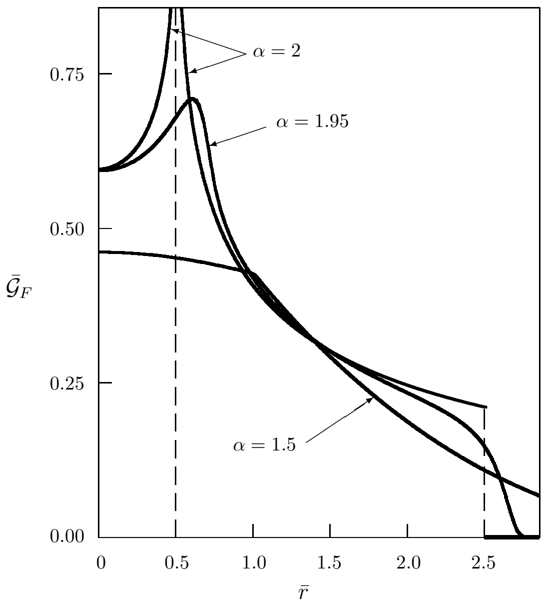

Figure 5.5: Dependence of the fundamental solution to the second Cauchy problem in a plane on distance; $\kappa = 1.5$ [148]

5.1.4 Fundamental solution to the source problem

The fundamental solution to the source problem is obtained in the similar way and has the following form:

$$\mathcal{G}_\Phi(r, \rho, t) = q_0 t^{\alpha-1} \int_0^\infty E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) J_0(r\xi)\, J_0(\rho\xi)\, \xi\, d\xi. \tag{5.26}$$

Dependence of the fundamental solution $\bar{\mathcal{G}}_\Phi = \rho^2 \mathcal{G}_\Phi/(q_0 t^{\alpha-1})$ on distance is shown in Figs. 5.6–5.8.

5.1.5 Delta-pulse at the origin

In the case of the first Cauchy problem we have

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{5.27}$$

$$t = 0: \quad T = p_0 \frac{\delta(r)}{2\pi r}, \qquad 0 < \alpha \le 2, \tag{5.28}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2\,. \tag{5.29}$$

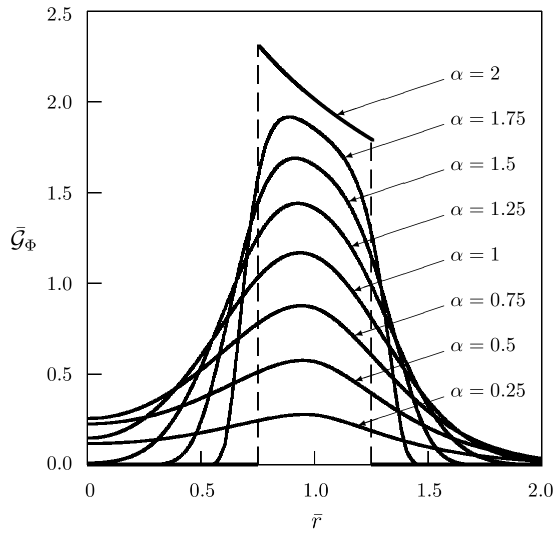

Figure 5.6: Dependence of the fundamental solution to the source problem in a plane on distance; $\kappa = 0.25$ [148]

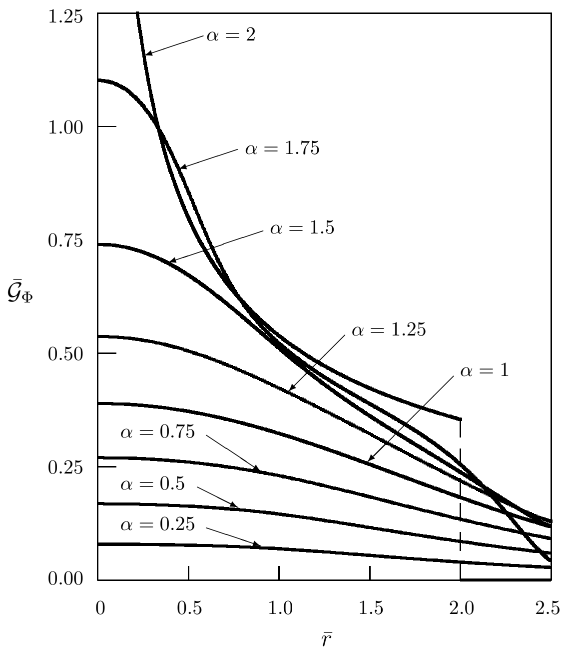

Figure 5.7: Dependence of the fundamental solution to the source problem in a plane on distance; $\kappa = 1$ [148]

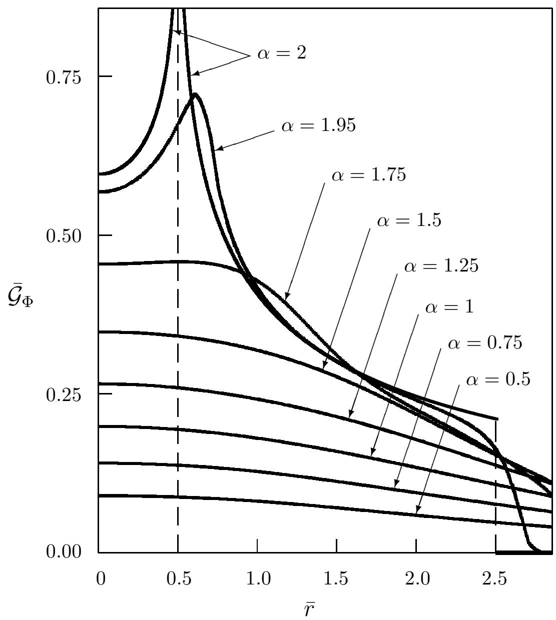

Figure 5.8: Dependence of the fundamental solution to the source problem in a plane on distance; $\kappa = 1.5$ [148]

The solution:

$$T = \frac{p_0}{2\pi} \int_0^\infty E_\alpha \left(-a\xi^2 t^\alpha\right) J_0(r\xi)\, \xi \, d\xi \tag{5.30}$$

and

$$\overline{T} = \frac{1}{2\pi} \int_0^\infty E_\alpha \left(-\eta^2\right) J_0(\bar{r}\eta)\, \eta \, \mathrm{d}\eta, \tag{5.31}$$

where

$$\bar{r} = \frac{r}{\sqrt{a} t^{\alpha/2}}, \quad \eta = \sqrt{a} t^{\alpha/2} \xi, \quad \overline{T} = \frac{a t^\alpha}{p_0} T. \tag{5.32}$$

Helmholtz equation ($\boldsymbol{\alpha \to 0}$)

$$\overline{T} = \frac{1}{2\pi} K_0(\bar{r}). \tag{5.33}$$

Here $K_0(r)$ is the modified Bessel function.

Subdiffusion with $\boldsymbol{\alpha = 1/2}$

$$\overline{T} = \frac{1}{4\pi^{3/2}} \int_0^\infty \frac{1}{u} \exp\left(-u^2 - \frac{\bar{r}^2}{8u}\right) du. \tag{5.34}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{4\pi} \exp\left(-\frac{\bar{r}^2}{4}\right). \tag{5.35}$$

Wave equation ($\alpha = 2$)

$$T = \frac{p_0}{2\pi\sqrt{a}} \frac{\partial}{\partial t} \frac{H(\sqrt{a}t - r)}{\sqrt{at^2 - r^2}}, \tag{5.36}$$

where $H(x)$ is the Heaviside step function (see also [85]).

Now we investigate the behavior of the solution (5.31) at the origin. As for large values of η we have (see (2.161)):

$$E_\alpha(-\eta^2) \sim \frac{1}{\Gamma(1-\alpha)\,\eta^2} \qquad \text{for } \eta \to \infty, \quad 0 < \alpha < 2, \tag{5.37}$$

only the fundamental solution to the classical diffusion equation has no singularity at the origin. To investigate the type of singularity we rewrite the solution (5.31) in the following form:

$$\overline{T} = \frac{1}{2\pi} \int_0^\infty \left[E_\alpha(-\eta^2) - \frac{1}{\Gamma(1-\alpha)(1+\eta^2)}\right] J_0(\bar{r}\eta)\, \eta\, d\eta$$
$$+ \frac{1}{2\pi\Gamma(1-\alpha)} \int_0^\infty \frac{1}{1+\eta^2} J_0(\bar{r}\eta)\, \eta\, d\eta. \tag{5.38}$$

The first integral in (5.38) has no singularity at the origin, while the second one can be calculated analytically (see equation (A.28) from the Appendix) and yields the logarithmic singularity at the origin

$$\overline{T} \sim \frac{1}{2\pi\Gamma(1-\alpha)} K_0(\bar{r}), \qquad 0 < \alpha < 2, \tag{5.39}$$

or

$$\overline{T} \sim -\frac{1}{2\pi\Gamma(1-\alpha)} \ln \bar{r}, \qquad 0 < \alpha < 2. \tag{5.40}$$

Comparison of (5.40) and (5.33) allows us to substitute the condition $0 < \alpha < 2$ by $0 \le \alpha < 2$. Equation (5.40), rewritten in terms of dimensional solution T,

$$T \sim -\frac{p_0}{2\pi a t^\alpha \Gamma(1-\alpha)} \ln \bar{r}, \qquad 0 \le \alpha < 2, \tag{5.41}$$

is consistent with the behavior of the solution for small values of r obtained in [208].

Dependence of the nondimensional solution $\overline{T}$ on the similarity variable $\bar{r}$ is shown in Fig. 5.9.

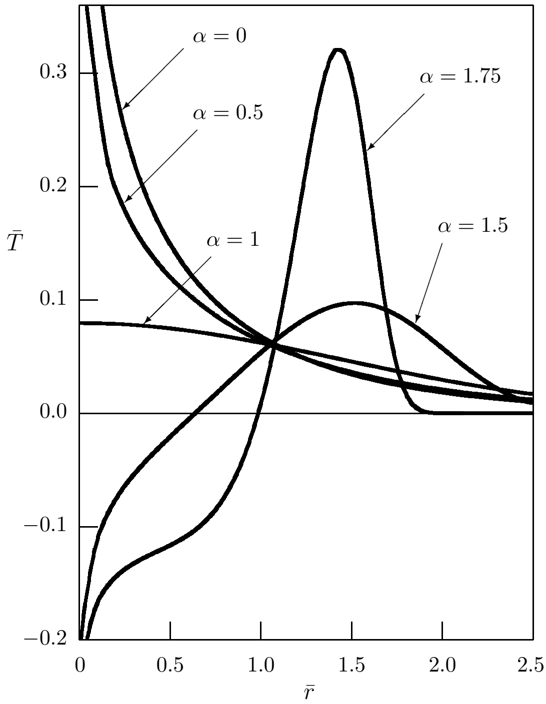

Figure 5.9: Dependence of the solution on the similarity variable $\bar{r}$ (the first Cauchy problem in a plane with the delta-pulse initial condition)

In the case of the second Cauchy problem with the delta-pulse initial condition the solution is expressed as

$$T = \frac{w_0 t}{2\pi} \int_0^\infty E_{\alpha,2}(-a\xi^2 t^\alpha)\, J_0(r\xi)\, \xi\, d\xi \tag{5.42}$$

with $\overline{T} = at^{\alpha-1}T/w_0$.

The particular case of the solution (5.42) for the wave equation ($\alpha = 2$) reads

$$\overline{T} = \begin{cases} \dfrac{1}{2\pi\sqrt{1-\bar{r}^2}}, & 0 < \bar{r} < 1, \\ 0, & 1 < \bar{r} < \infty. \end{cases} \tag{5.43}$$

To investigate behavior of the solution (5.42) at the origin we recall that for large values of η we have (see (2.162)):

$$E_{\alpha,2}(-\eta^2) \sim \frac{1}{\Gamma(2-\alpha)\,\eta^2} \quad \text{for } \eta \to \infty, \quad 1 < \alpha < 2. \tag{5.44}$$

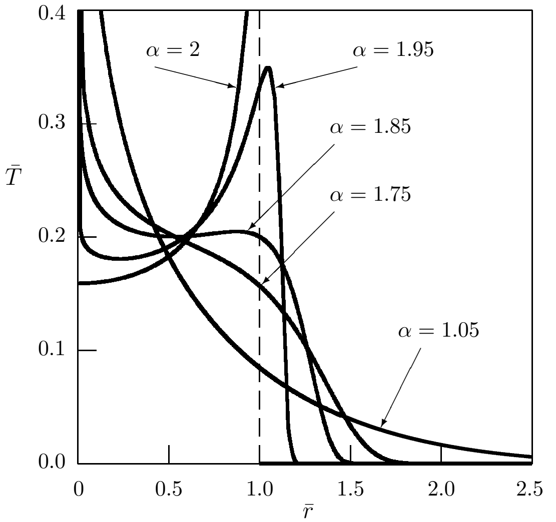

Figure 5.10: Dependence of the solution on the similarity variable $\bar{r}$ (the second Cauchy problem in a plane with the delta-pulse initial condition)

Computations similar to those carried out above lead to

$$\overline{T} \sim -\frac{1}{2\pi\Gamma(2-\alpha)} \ln \bar{r}, \qquad 1 < \alpha < 2. \tag{5.45}$$

Hence, in the case of the second Cauchy problem with the delta-pulse initial condition the solution also has its logarithmic singularity at the origin.

Dependence of nondimensional solution $\overline{T}$ on the similarity variable is depicted in Fig. 5.10.

The solution to the time-fractional diffusion wave equation with the source term $q_0 \frac{\delta(r)}{2\pi r} \delta(t)$ under zero initial conditions is expressed as

$$T = \frac{q_0 t^{\alpha-1}}{2\pi} \int_0^\infty E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) J_0(r\xi)\, \xi \, d\xi. \tag{5.46}$$

It should be noted that due to (2.163)

$$E_{\alpha,\alpha}(-\eta^2) \sim -\frac{1}{\Gamma(-\alpha)\,\eta^4} \quad \text{for } \eta \to \infty, \quad 0 < \alpha < 2. \tag{5.47}$$

Hence, the solution (5.46) has no singularity at the origin for all $0 < \alpha < 2$.

Dependence of nondimensional solution $\overline{T} = atT/q_0$ on the similarity variable is depicted in Fig. 5.11. It should be emphasized that solution (5.43), the same

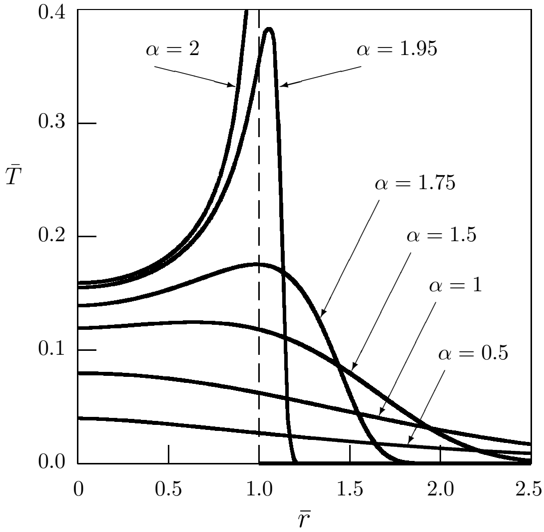

Figure 5.11: Dependence of the solution on the similarity variable $\bar{r}$ (the delta pulse source problem in a plane with zero initial conditions)

both for the source problem and the second Cauchy problem, is approximated by solutions (5.42) and (5.46) with $\alpha \to 2$ in different ways, in particular the solution (5.42) has the logarithmic singularity at the origin, whereas the solution (5.46) has no singularity.

5.2 Evolution of the unit-box signal

5.2.1 First Cauchy problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{5.48}$$

$$t = 0: \quad T = \begin{cases} T_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \qquad 0 < \alpha \le 2, \tag{5.49}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{5.50}$$

The solution [162]:

$$T = T_0 R \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) J_1(R\xi)\, J_0(r\xi)\, d\xi. \tag{5.51}$$

It is convenient to introduce the following nondimensional quantities:

$$\bar{r} = \frac{r}{R}, \qquad \kappa = \frac{\sqrt{a} t^{\alpha/2}}{R}, \qquad \overline{T} = \frac{T}{T_0}. \tag{5.52}$$

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \begin{cases} \dfrac{1}{\kappa} \left[\bar{r} I_1\left(\dfrac{\bar{r}}{\kappa}\right) K_0\left(\dfrac{\bar{r}}{\kappa}\right) + \bar{r} I_0\left(\dfrac{\bar{r}}{\kappa}\right) K_1\left(\dfrac{\bar{r}}{\kappa}\right) - I_0\left(\dfrac{\bar{r}}{\kappa}\right) K_1\left(\dfrac{1}{\kappa}\right) \right], \\ \qquad\qquad 0 \le \bar{r} < 1, \\ \dfrac{1}{\kappa} I_1\left(\dfrac{1}{\kappa}\right) K_0\left(\dfrac{\bar{r}}{\kappa}\right), \qquad 1 < \bar{r} < \infty. \end{cases} \tag{5.53}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{T} = \frac{1}{4\kappa^2 \sqrt{\pi}} \int_0^\infty \frac{1}{u} \mathrm{e}^{-u^2} \int_0^1 \exp\left(-\frac{\bar{r}^2 + x}{8\kappa^2 u}\right) I_0\left(\frac{\bar{r}\sqrt{x}}{4\kappa^2 u}\right) \mathrm{d}x \, \mathrm{d}u. \tag{5.54}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{4\kappa^2} \int_0^1 \exp\left(-\frac{\bar{r}^2 + x}{4\kappa^2}\right) I_0\left(\frac{\bar{r}\sqrt{x}}{2\kappa^2}\right) dx. \tag{5.55}$$

Wave equation ($\alpha = 2$)

a) $0 < \kappa < 1$

$$\overline{T} = \begin{cases} 1, & 0 \le \bar{r} < 1 - \kappa, \\ 1 - \Lambda_0\left(\arcsin\sqrt{\dfrac{2\bar{r}}{1 + \bar{r} + \kappa}}, k\right) + \dfrac{\bar{r} - \kappa}{\pi\sqrt{\bar{r}}} \mathbf{K}(k), & 1 - \kappa < \bar{r} < 1 + \kappa, \\ 0, & 1 + \kappa < \bar{r} < \infty. \end{cases} \tag{5.56}$$

b) $\kappa = 1$

$$\overline{T} = \begin{cases} 1 - \Lambda_0\left(\arcsin\sqrt{\dfrac{2\bar{r}}{2 + \bar{r}}}, k\right) + \dfrac{\bar{r} - 1}{\pi\sqrt{\bar{r}}} \mathbf{K}(k), & 0 \le \bar{r} < 2, \\ 0, & 2 < \bar{r} < \infty. \end{cases} \tag{5.57}$$

c) $1 < \kappa < \infty$

$$\bar{T} = \begin{cases} 1 - \Lambda_0\left(\arcsin\sqrt{\dfrac{\kappa+\bar{r}-1}{\kappa+\bar{r}+1}}, \dfrac{1}{k}\right) \\ \qquad\qquad - \dfrac{1}{\pi k\sqrt{\bar{r}}}\,\mathbf{K}\left(\dfrac{1}{k}\right), & 0 \le \bar{r} < \kappa - 1, \\ 1 - \Lambda_0\left(\arcsin\sqrt{\dfrac{2\bar{r}}{\kappa+\bar{r}+1}}, k\right) \\ \qquad\qquad + \dfrac{\bar{r}-\kappa}{\pi\sqrt{\bar{r}}}\,\mathbf{K}(k), & \kappa - 1 < \bar{r} < 1 + \kappa, \\ 0, & 1 + \kappa < \bar{r} < \infty. \end{cases} \tag{5.58}$$

where $\Lambda_0(\varphi, k)$ is the Heuman Lambda function,

$$\Lambda_0(\varphi, k) = \frac{2}{\pi}\left[\mathbf{E}(k)\, F(\varphi, k') + \mathbf{K}(k)\, E(\varphi, k') - \mathbf{K}(k)\, F(\varphi, k')\right], \tag{5.59}$$

$F(\varphi, k)$ and $E(\varphi, k)$ are elliptic integrals of the first and second kind, $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are complete elliptic integrals of the first and second kind, respectively, k and k' are the same as in (5.18).

The solution (5.51) is shown in Figs. 5.12–5.14.

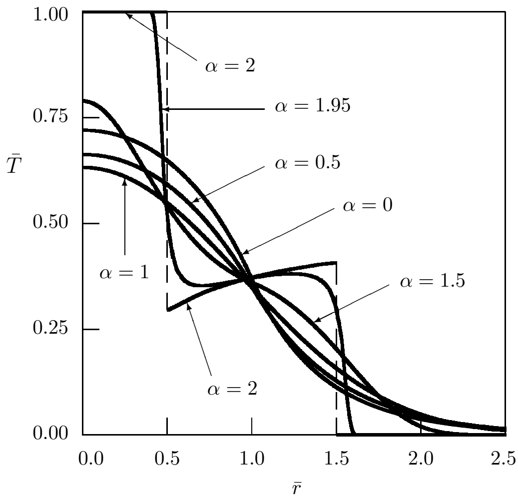

Figure 5.12: Evolution of the unit-box signal in a plane (the first Cauchy problem; $\kappa = 0.5$) [162]

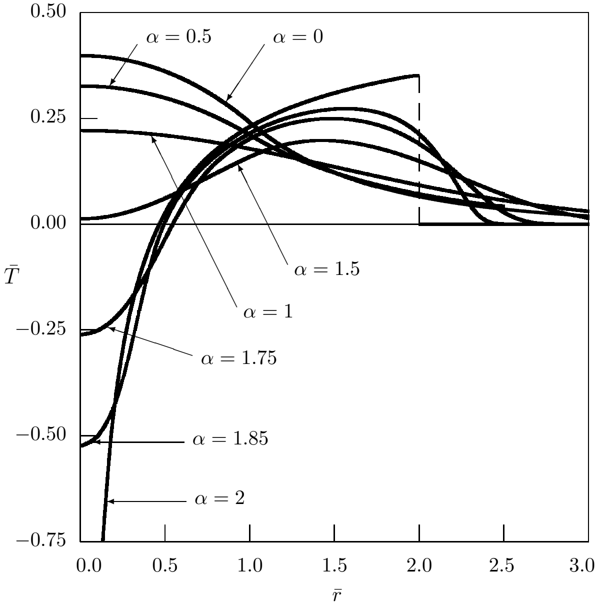

Figure 5.13: Evolution of the unit-box signal in a plane (the first Cauchy problem; $\kappa = 1$) [162]

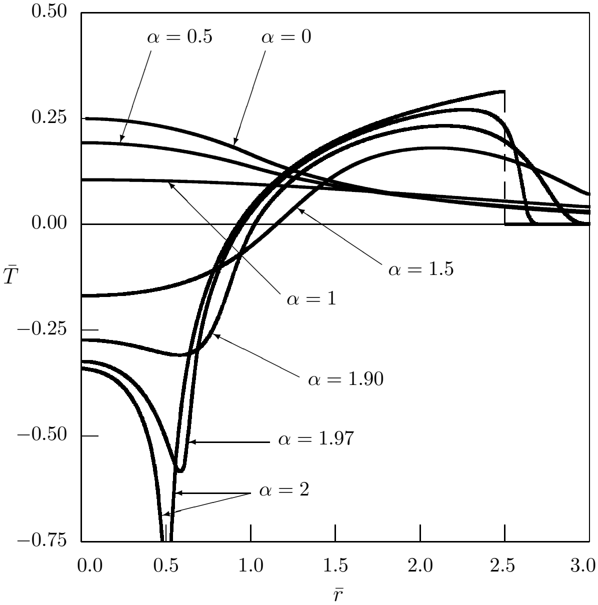

Figure 5.14: Evolution of the unit-box signal in a plane (the first Cauchy problem; $\kappa = 1.5$) [162]

5.2.2 Second Cauchy problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{5.60}$$

$$t = 0: \quad T = 0, \qquad\qquad 1 < \alpha \le 2, \tag{5.61}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = \begin{cases} w_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \qquad 1 < \alpha \le 2. \tag{5.62}$$

The solution [182]:

$$T = w_0 R t \int_0^\infty E_{\alpha,2}\left(-a\xi^2 t^\alpha\right) J_1(R\xi)\, J_0(r\xi)\, \mathrm{d}\xi. \tag{5.63}$$

Dependence of solution (5.63) on distance is shown in Figs. 5.15–5.17 for various values of κ and α ($\overline{T} = T/(w_0 t)$, $\bar{r} = r/R$).

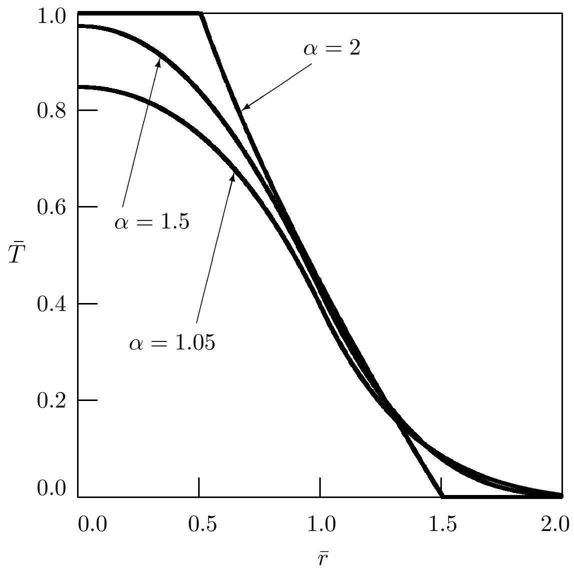

Figure 5.15: Evolution of the unit-box signal in a plane (the second Cauchy problem; $\kappa = 0.5$) [182]

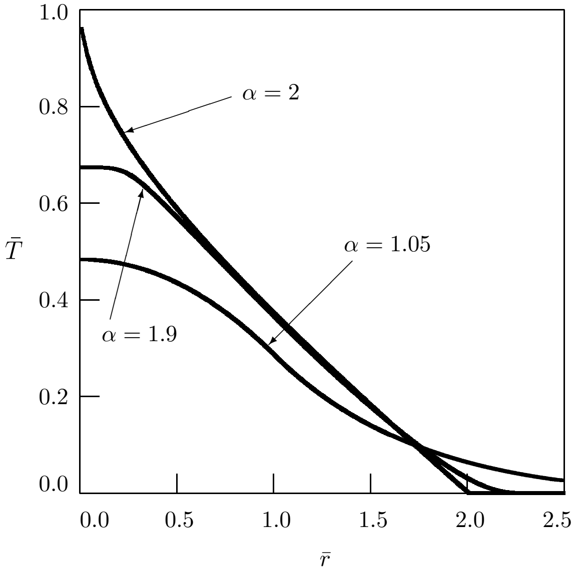

Figure 5.16: Evolution of the unit-box signal in a plane (the second Cauchy problem; $\kappa = 1$)

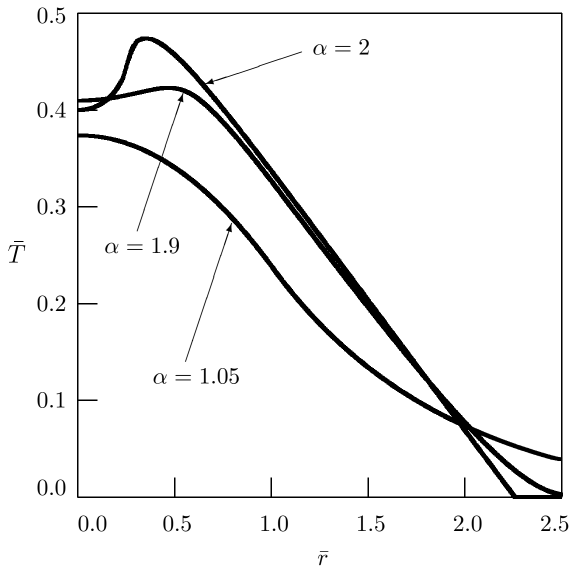

Figure 5.17: Evolution of the unit-box signal in a plane (the second Cauchy problem; $\kappa = 1.25$) [182]

5.2.3 Source problem

We consider the diffusion-wave equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \delta(t)\begin{cases} q_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \tag{5.64}$$

under zero initial conditions. The solution has the following form [182]:

$$T = q_0 R t^{\alpha-1}\int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\, J_0(r\xi)\, J_1(R\xi)\,\mathrm{d}\xi. \tag{5.65}$$

Figures 5.18–5.20 show the solution (5.65) for various values of α and κ ($\overline{T} = t^{1-\alpha}T/q_0$).

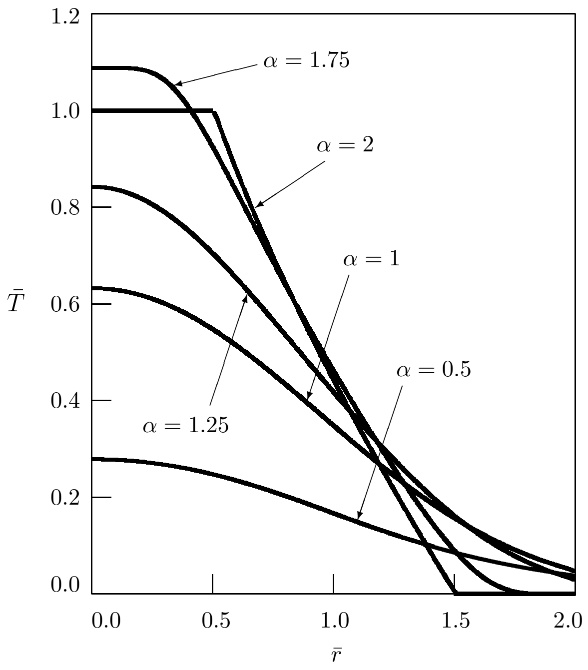

Figure 5.18: Evolution of the unit-box signal in a plane (the source problem; $\kappa = 0.5$) [182]

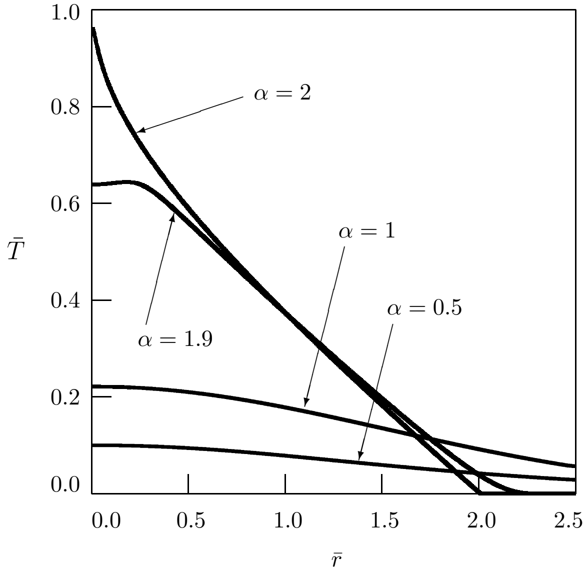

Figure 5.19: Evolution of the unit-box signal in a plane (the source problem; $\kappa = 1$)

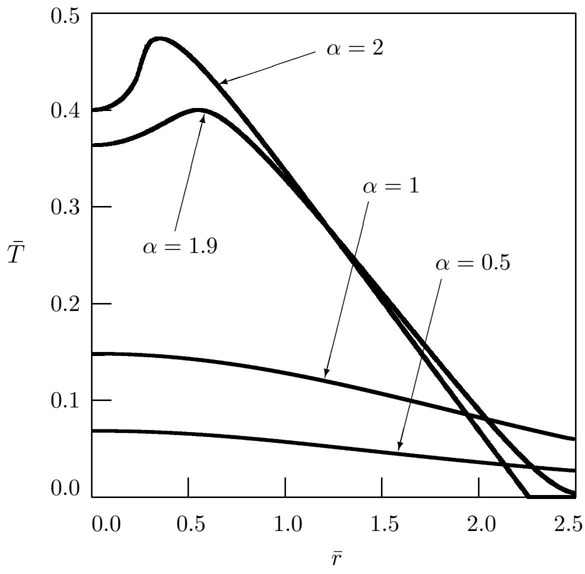

Figure 5.20: Evolution of the unit-box signal in a plane (the source problem; $\kappa = 1.25$) [182]

5.3 Domain $0 < r < R$

5.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{5.66}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{5.67}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{5.68}$$

$$r = R: \quad T = g(t). \tag{5.69}$$

The solution:

$$T(r,t) = \int_0^R f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho\,\mathrm{d}\rho + \int_0^R F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho\,\mathrm{d}\rho$$
$$+ \int_0^t \int_0^R \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{5.70}$$

The fundamental solutions under zero Dirichlet boundary condition have the form

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{2}{R^2}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix} \frac{J_0(r\xi_k)J_0(\rho\xi_k)}{J_1^2(R\xi_k)} \tag{5.71}$$

with the sum over all positive roots of the zero-order Bessel function

$$J_0(R\xi_k) = 0. \tag{5.72}$$

They are obtained using the Laplace transform with respect to time t and the finite Hankel transform (2.96) with respect to the radial coordinate r.

The fundamental solution to the Dirichlet problem under zero initial condition is expressed as

$$\mathcal{G}_g(r,t) = -\frac{aRg_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,\rho,t)}{\partial \rho}\bigg|_{\rho=R}. \tag{5.73}$$

For numerical calculations the following nondimensional quantities are introduced:

$$\bar{r} = \frac{r}{R}, \qquad \bar{\rho} = \frac{\rho}{R}, \qquad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}, \qquad \overline{\mathcal{G}}_f = \frac{R^2}{p_0}\,\mathcal{G}_f,$$

$$\overline{\mathcal{G}}_F = \frac{R^2}{w_0 t}\,\mathcal{G}_F, \qquad \overline{\mathcal{G}}_\Phi = \frac{R^2}{q_0 t^{\alpha-1}}\,\mathcal{G}_\Phi, \qquad \overline{\mathcal{G}}_g = \frac{R^2}{a g_0 t^{\alpha-1}}\,\mathcal{G}_g. \tag{5.74}$$

Dependence of the fundamental solution $\overline{\mathcal{G}}_f$ on the nondimensional distance r is shown in Figs. 5.21–5.22. Dependence of the fundamental solution $\overline{\mathcal{G}}_F$ on distance is presented in Figs. 5.23–5.25. Dependence of the fundamental solution $\overline{\mathcal{G}}_\Phi$ on distance is depicted in Figs. 5.26–5.28. The fundamental solution to the Dirichlet problem $\overline{\mathcal{G}}_g$ is shown in Fig. 5.29.

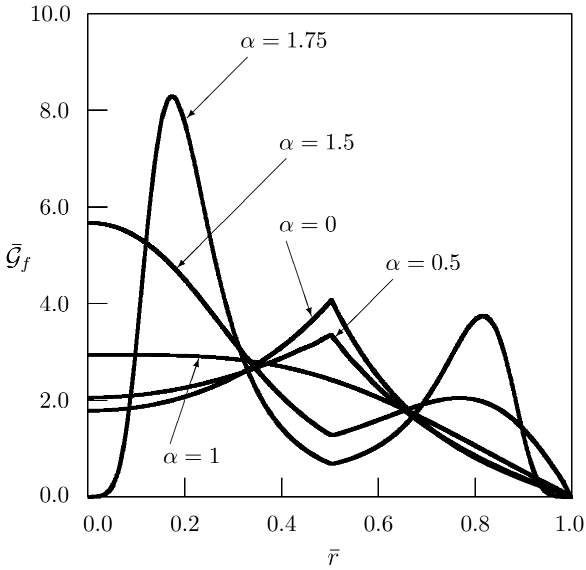

Figure 5.21: The fundamental solution to the first Cauchy problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$)

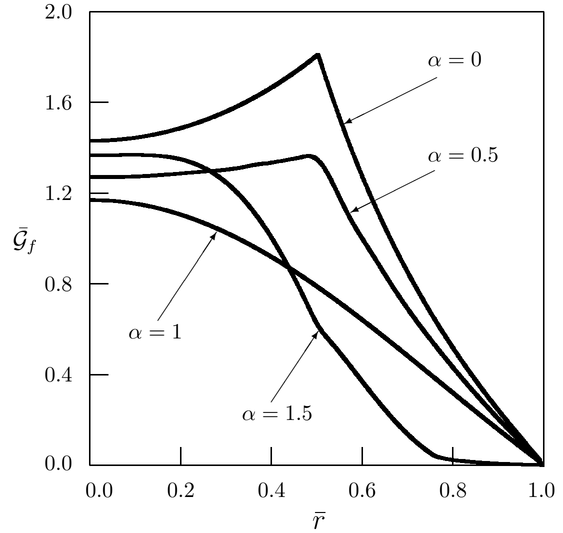

Figure 5.22: The fundamental solution to the first Cauchy problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$)

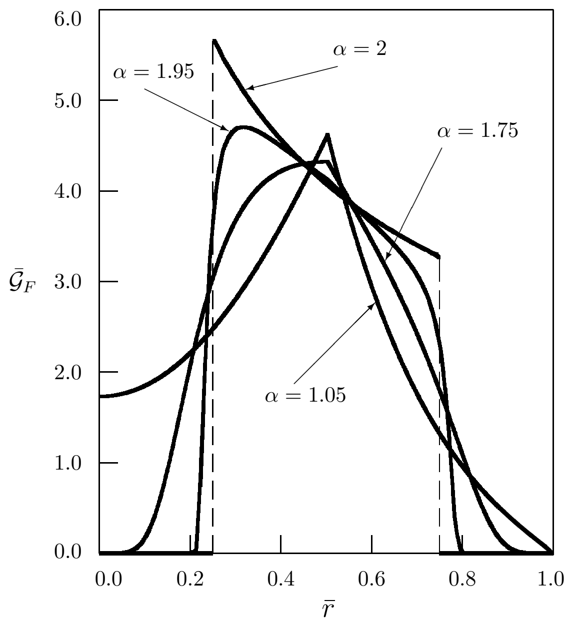

Figure 5.23: The fundamental solution to the second Cauchy problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$)

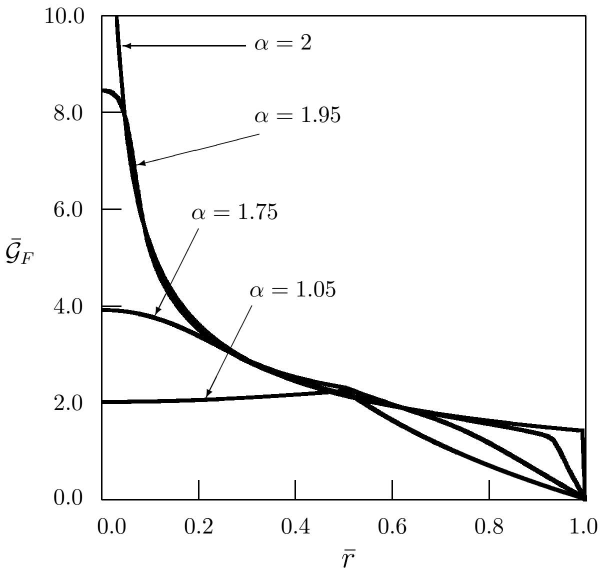

Figure 5.24: The fundamental solution to the second Cauchy problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$)

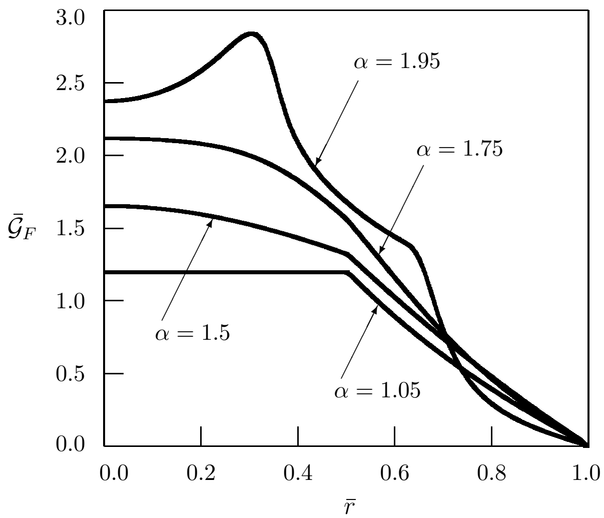

Figure 5.25: The fundamental solution to the second Cauchy problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.75$)

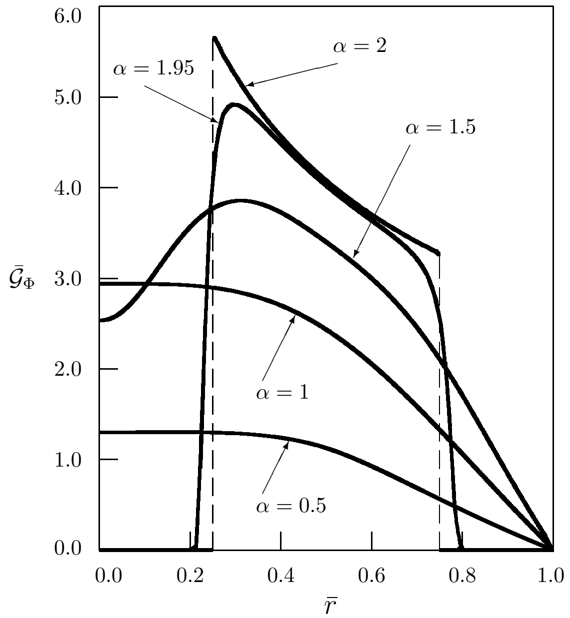

Figure 5.26: The fundamental solution to the source problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$)

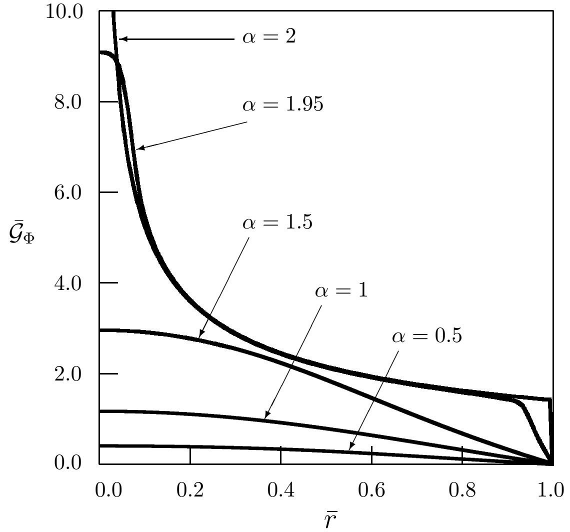

Figure 5.27: The fundamental solution to the source problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$)

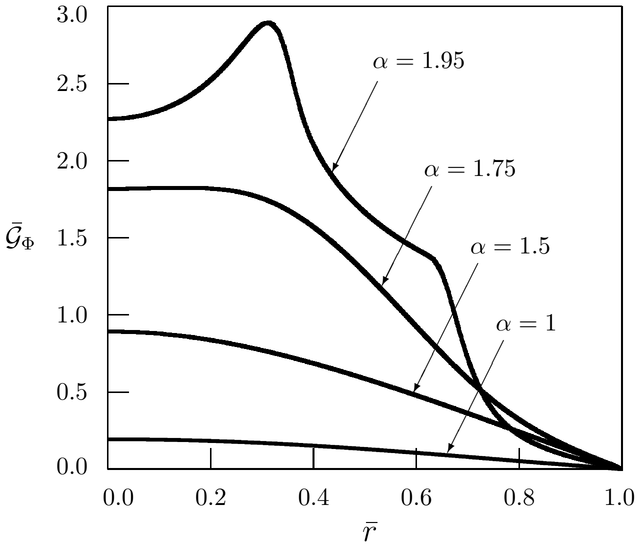

Figure 5.28: The fundamental solution to the source problem in a cylinder under zero Dirichlet boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.75$)

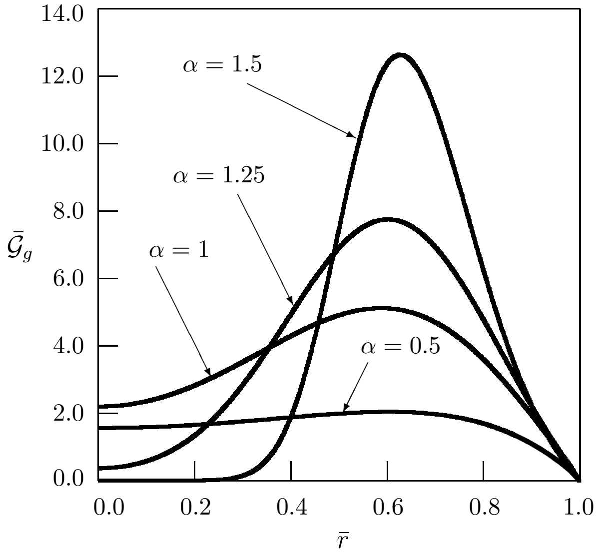

Figure 5.29: The fundamental solution to the Dirichlet problem in a cylinder; $\kappa = 0.25$

Constant source strength. Here we consider the fractional diffusion-wave equation with constant source term $Q_0 = \text{const}$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + Q_0 \tag{5.75}$$

under zero initial conditions

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{5.76}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{5.77}$$

and zero Dirichlet boundary condition

$$r = R: \quad T = 0, \tag{5.78}$$

having the solution [149]

$$T = \frac{Q_0}{4a}\left(R^2 - r^2\right) - \frac{2Q_0}{aR}\sum_{k=1}^{\infty} E_\alpha\left(-a\xi_k^2 t^\alpha\right)\frac{J_0(r\xi_k)}{\xi_k^3\, J_1(R\xi_k)}. \tag{5.79}$$

Helmholtz equation ($\alpha \to 0$)

$$T = \frac{Q_0}{4a}\left(R^2 - r^2\right) - \frac{2Q_0}{aR}\sum_{k=1}^{\infty}\frac{1}{1 + a\xi_k^2}\frac{J_0(r\xi_k)}{\xi_k^3\, J_1(R\xi_k)}. \tag{5.80}$$

Classical diffusion equation ($\alpha = 1$)

$$T = \frac{Q_0}{4a}\left(R^2 - r^2\right) - \frac{2Q_0}{aR}\sum_{k=1}^{\infty}\exp\left(-a\xi_k^2 t\right)\frac{J_0(r\xi_k)}{\xi_k^3\, J_1(R\xi_k)}. \tag{5.81}$$

The solution (5.81) is presented in [26].

Wave equation ($\alpha = 2$)

$$T = \frac{Q_0}{4a}\left(R^2 - r^2\right) - \frac{2Q_0}{aR}\sum_{k=1}^{\infty}\cos\left(\sqrt{a}\xi_k t\right)\frac{J_0(r\xi_k)}{\xi_k^3\, J_1(R\xi_k)}. \tag{5.82}$$

The results of numerical calculations are shown in Figs. 5.30 and 5.31 with $\overline{T} = aT/(Q_0 R^2)$.

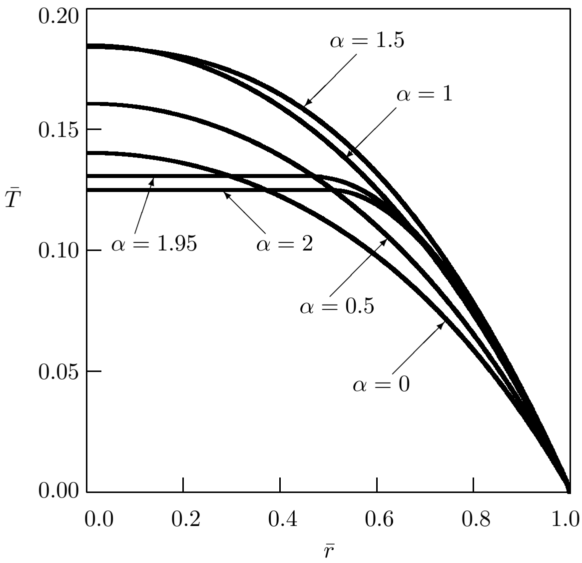

Figure 5.30: Dependence of temperature in a cylinder on distance (the constant source strength; $\kappa = 0.5$ [149]

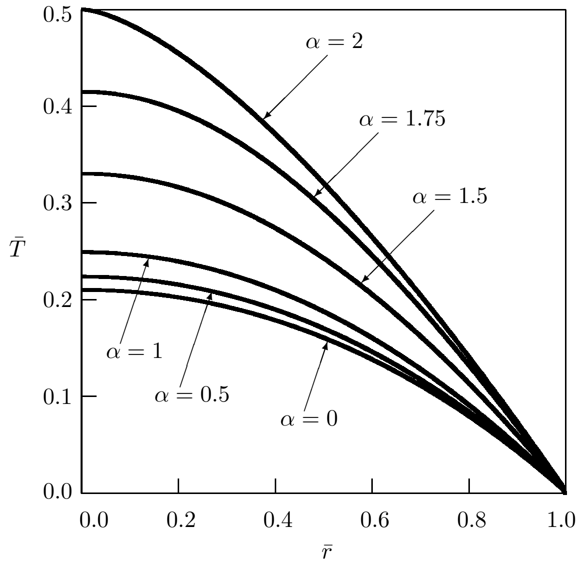

Figure 5.31: Dependence of temperature in a cylinder on distance (the constant source strength; $\kappa = 1$ [149]

Dirichlet problem with constant boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{5.83}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{5.84}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{5.85}$$

$$r = R: \quad T = T_0. \tag{5.86}$$

The solution has the form:

$$T = T_0\left[1 - 2\sum_{k=1}^{\infty} E_\alpha\left(-a\xi_k^2 t^\alpha\right)\frac{J_0(r\xi_k)}{R\xi_k\, J_1(R\xi_k)}\right]. \tag{5.87}$$

The solution (5.87) was obtained by Narahari Achar and Hanneken [126], but their numerical analysis of this solution and conclusions from such an analysis need improvement (see [149]). The results of numerical calculations according to (5.87) are presented in Figs. 5.32–5.34 for typical values of the parameter κ with ($\overline{T} = T/T_0$).

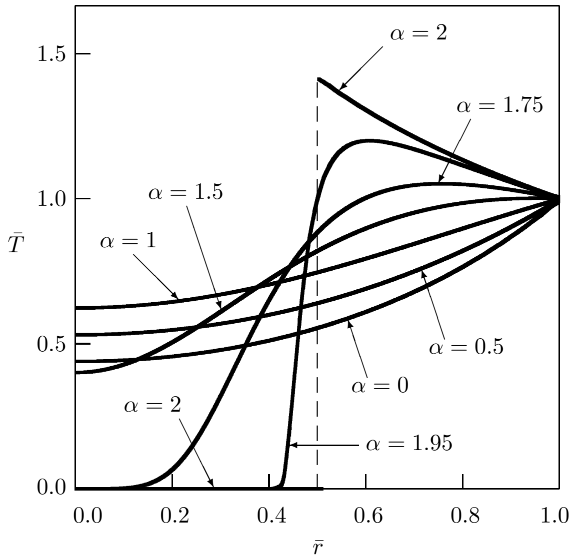

Figure 5.32: Dependence of temperature in a cylinder on distance (the constant boundary condition; $\kappa = 0.5$ [149]

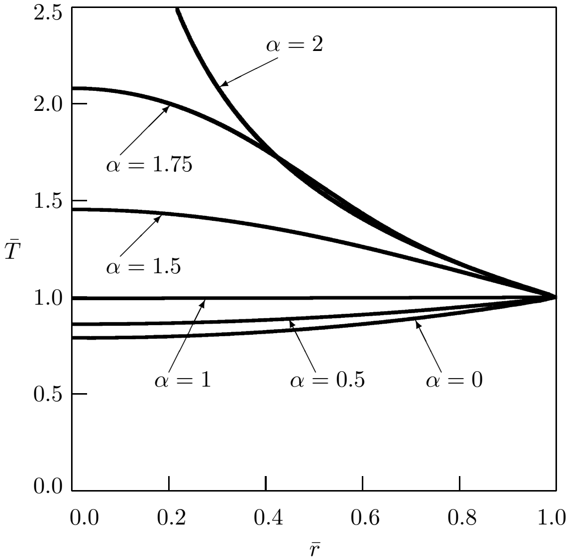

Figure 5.33: Dependence of temperature in a cylinder on distance (the constant boundary condition; $\kappa = 1$ [149]

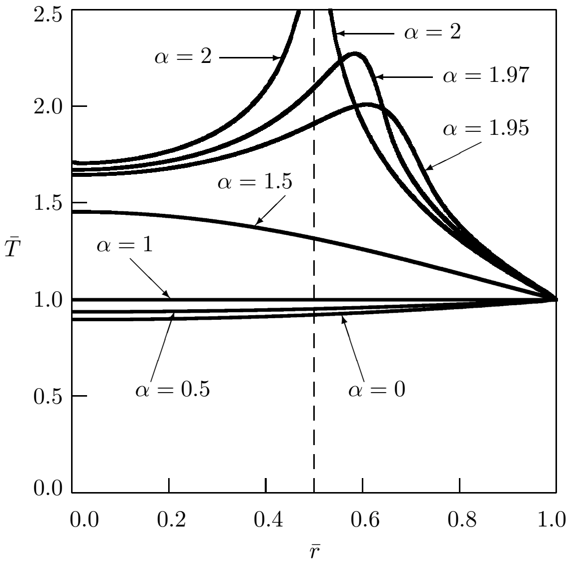

Figure 5.34: Dependence of temperature in a cylinder on distance (the constant boundary condition; $\kappa = 1.5$ [149]

5.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{5.88}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{5.89}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{5.90}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(t). \tag{5.91}$$

The solution:

$$T(r,t) = \int_0^R f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho\,\mathrm{d}\rho + \int_0^R F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho\,\mathrm{d}\rho$$

$$+ \int_0^t \int_0^R \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{5.92}$$

The fundamental solutions under zero Neumann boundary condition have the form

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{2}{R^2}\begin{pmatrix} p_0 \\ w_0 t \\ q_0 t^{\alpha-1}/\Gamma(\alpha) \end{pmatrix}$$

$$+ \frac{2}{R^2}\sum_{k=1}^{\infty}\begin{pmatrix} p_0\,E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix}\frac{J_0(r\xi_k)J_0(\rho\xi_k)}{J_0^2(R\xi_k)}, \tag{5.93}$$

with sum over all positive roots of the first-order Bessel function

$$J_1(R\xi_k) = 0. \tag{5.94}$$

The solutions were obtained using the Laplace transform with respect to time and the finite Hankel transform (2.100) with respect to the radial coordinate.

Dependence of the fundamental solution $\overline{\mathcal{G}}_f$ on nondimensional distance r is shown in Figs. 5.35–5.36. Dependence of the fundamental solution $\overline{\mathcal{G}}_F$ on distance is presented in Figs. 5.37–5.39. Dependence of the fundamental solution $\overline{\mathcal{G}}_\Phi$ on distance is depicted in Figs. 5.40–5.42. The nondimensional quantities are the same as in (5.74). For $\kappa = 0.25$ the fundamental solutions under zero Dirichlet and Neumann boundary conditions behave very similarly (the solutions do not "feel" the boundary condition), but for $\kappa = 0.5$ and $\kappa = 0.75$ there appears significant difference.

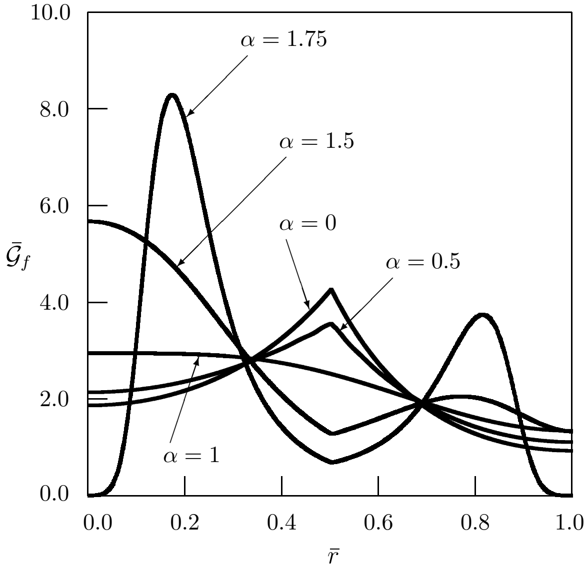

Figure 5.35: The fundamental solution to the first Cauchy problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$)

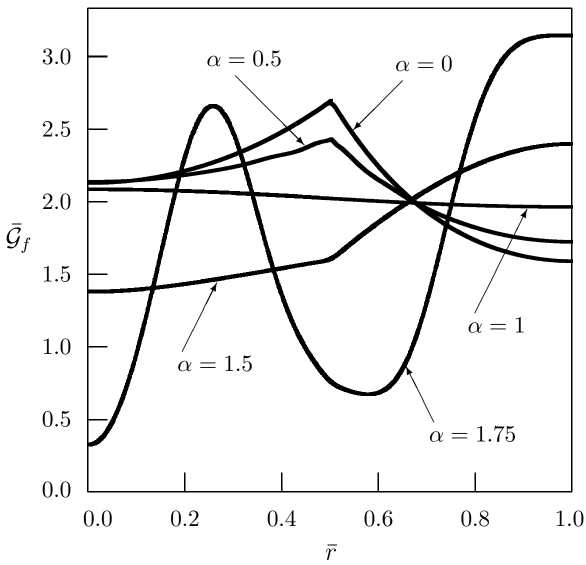

Figure 5.36: The fundamental solution to the first Cauchy problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$)

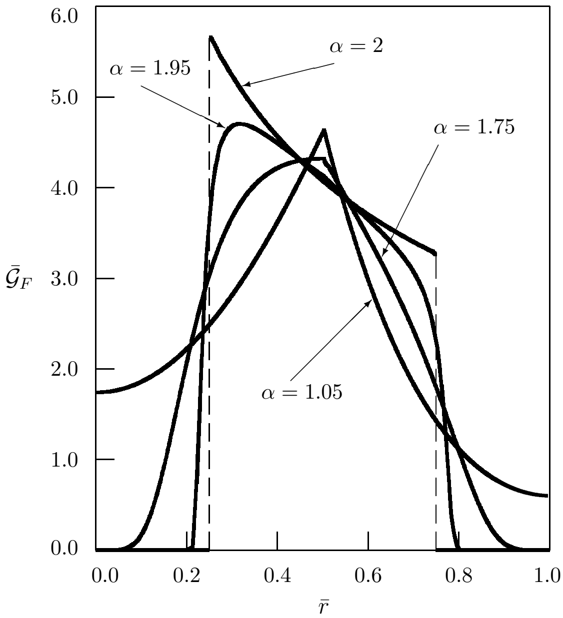

Figure 5.37: The fundamental solution to the second Cauchy problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$)

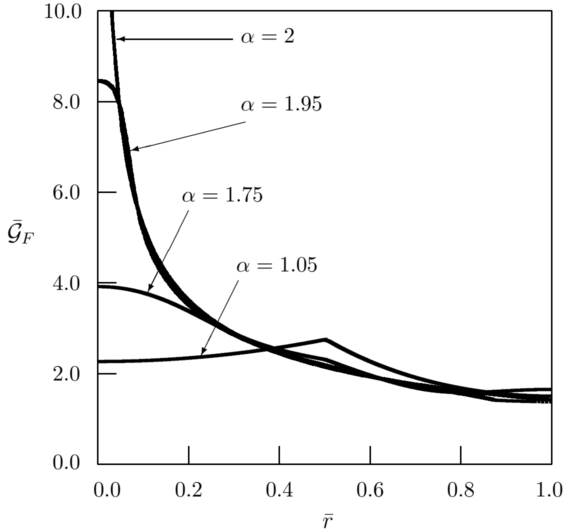

Figure 5.38: The fundamental solution to the second Cauchy problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$)

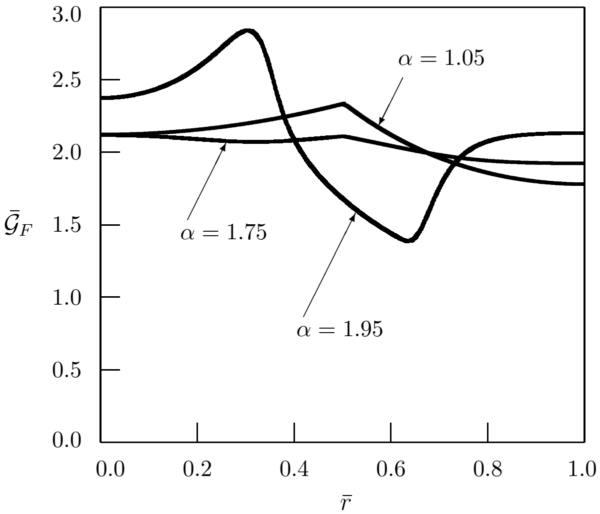

Figure 5.39: The fundamental solution to the second Cauchy problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.75$)

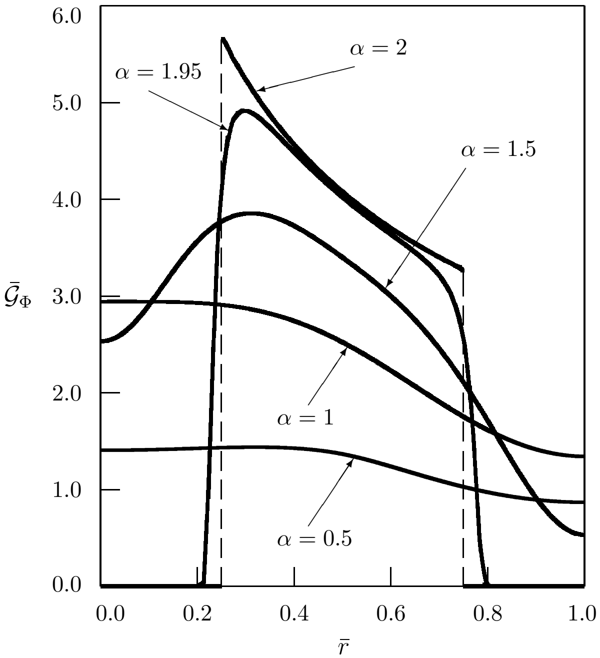

Figure 5.40: The fundamental solution to the source problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$)

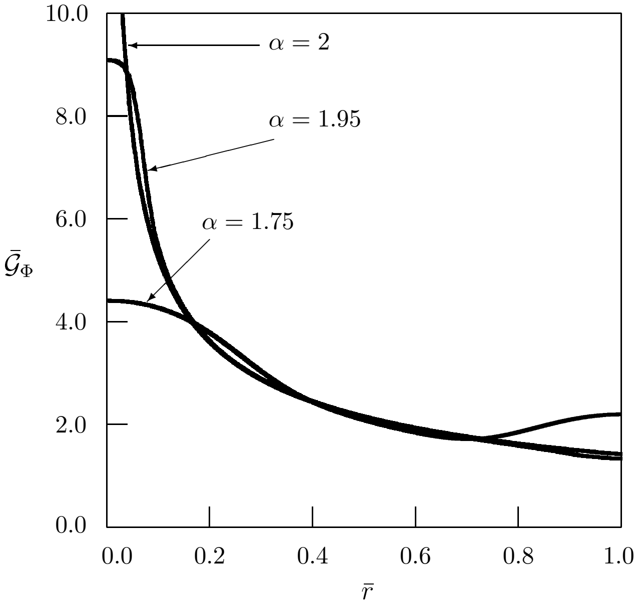

Figure 5.41: The fundamental solution to the source problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$)

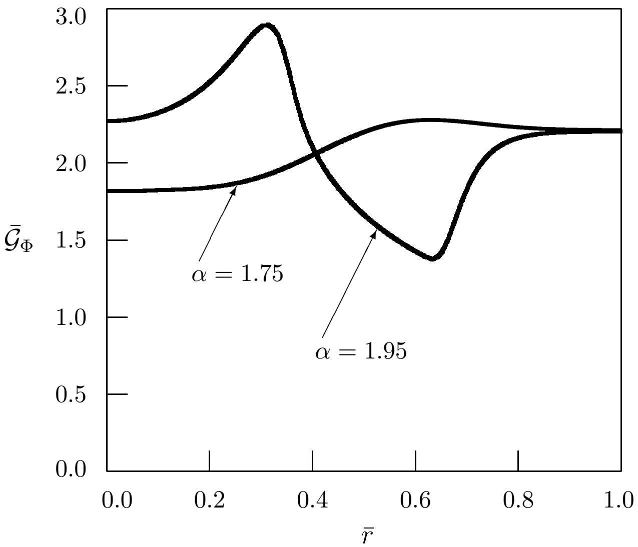

Figure 5.42: The fundamental solution to the source problem in a cylinder under zero Neumann boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.75$)

Fundamental solution to the mathematical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_m}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_m}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_m}{\partial r}\right), \tag{5.95}$$

$$t = 0: \quad \mathcal{G}_m = 0, \qquad 0 < \alpha \le 2, \tag{5.96}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_m}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{5.97}$$

$$r = R: \quad \frac{\partial \mathcal{G}_m}{\partial r} = g_0\, \delta(t). \tag{5.98}$$

The solution reads:

$$\mathcal{G}_m(r,t) = \frac{2ag_0 t^{\alpha-1}}{R}\left[\frac{1}{\Gamma(\alpha)} + \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_k^2 t^\alpha\right)\frac{J_0(r\xi_k)}{J_0(R\xi_k)}\right]. \tag{5.99}$$

The solution (5.99) is shown in Figs. 5.43 and 5.44, where $\overline{\mathcal{G}}_m = Rt^{1-\alpha}\mathcal{G}_m/(ag_0)$.

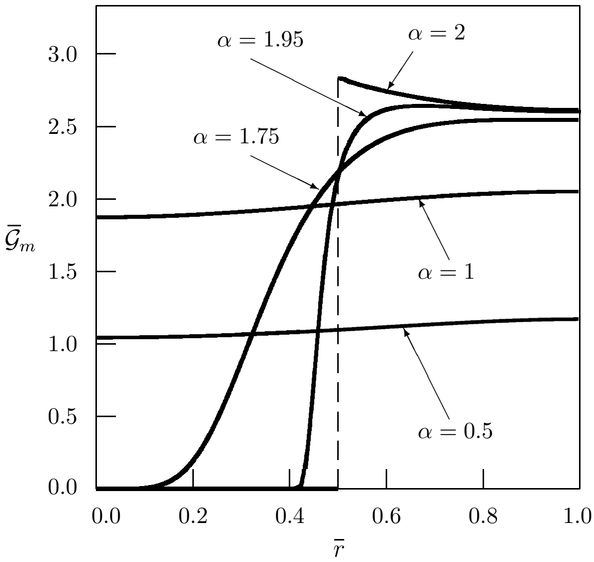

Figure 5.43: The fundamental solution to the mathematical Neumann problem for a cylinder; $\kappa = 0.5$

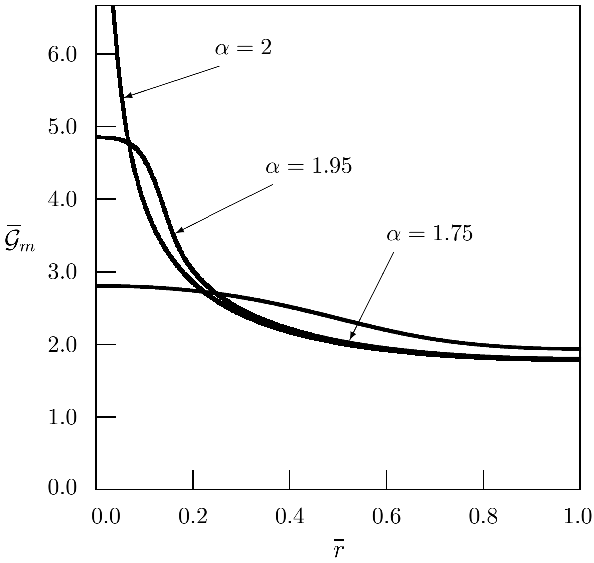

Figure 5.44: The fundamental solution to the mathematical Neumann problem for a cylinder; $\kappa = 1$

Constant boundary value of the normal derivative. In the case when a constant boundary value of the normal derivative is considered,

$$r = R: \quad \frac{\partial T}{\partial r} = g_0, \tag{5.100}$$

the solution has the following form [174]:

$$T = \frac{2ag_0 t^\alpha}{R\Gamma(1+\alpha)} + \frac{g_0}{R}\left[\frac{r^2}{2} - \frac{R^2}{4} - 2\sum_{k=1}^{\infty} E_\alpha\left(-a\xi_k^2 t^\alpha\right)\frac{J_0(r\xi_k)}{\xi_k^2\, J_0(R\xi_k)}\right]. \tag{5.101}$$

The particular case of (5.101) corresponding to the classical diffusion equation ($\alpha = 1$) coincides with the corresponding solution presented in [26].

The results of numerical calculations are presented in Fig. 5.45 and Fig. 5.46 with $\overline{T} = T/(g_0 R)$.

Fundamental solution to the physical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_p}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_p}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_p}{\partial r}\right), \tag{5.102}$$

$$t = 0: \quad \mathcal{G}_p = 0, \qquad 0 < \alpha \le 2, \tag{5.103}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_p}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{5.104}$$

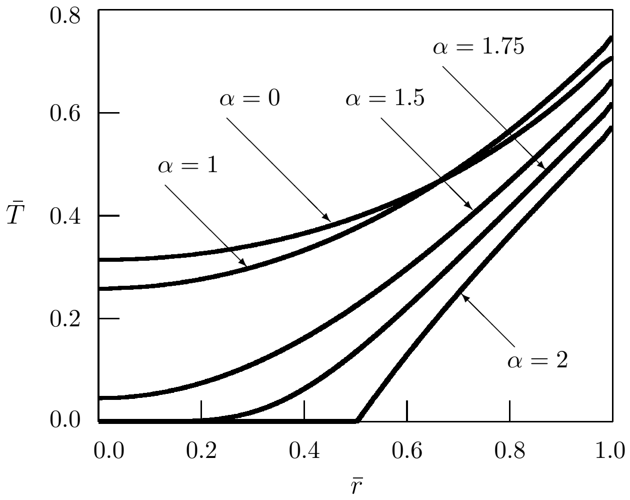

Figure 5.45: Dependence of temperature in a cylinder on distance (the constant normal derivative of temperature at the boundary; $\kappa = 0.5$)

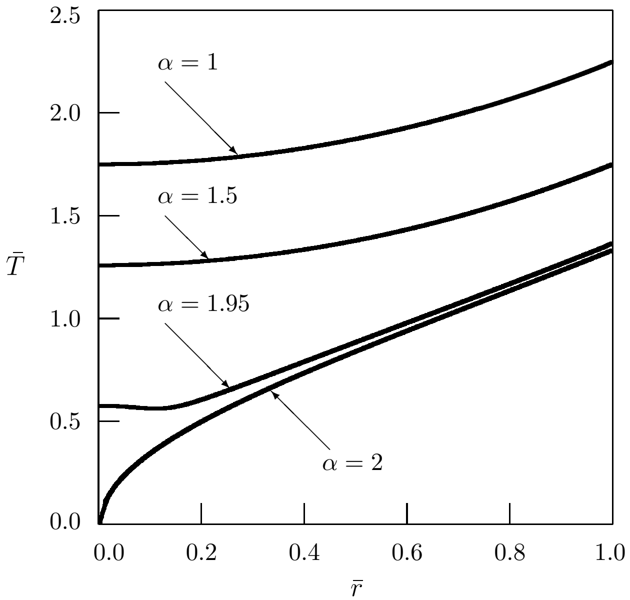

Figure 5.46: Dependence of temperature in a cylinder on distance (the constant normal derivative of temperature at the boundary; $\kappa = 1$)

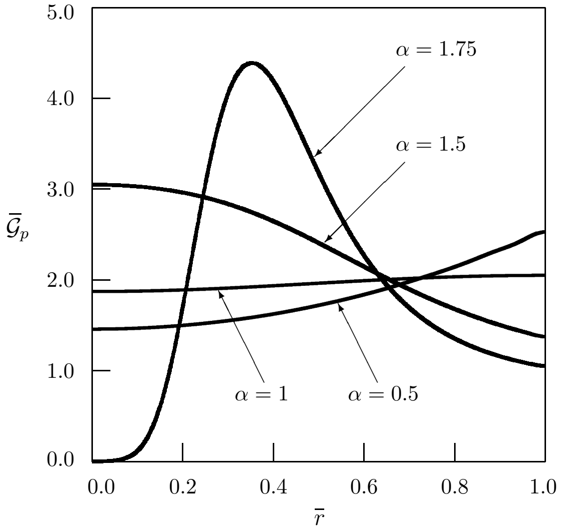

Figure 5.47: The fundamental solution to the physical Neumann problem for a cylinder; $\kappa = 0.5$

$$r = R: \quad D_{RL}^{1-\alpha} \frac{\partial \mathcal{G}_p}{\partial r} = g_0\, \delta(t), \qquad 0 < \alpha \le 1, \tag{5.105}$$

$$r = R: \quad I^{\alpha-1} \frac{\partial \mathcal{G}_p}{\partial r} = g_0\, \delta(t), \qquad 1 < \alpha \le 2. \tag{5.106}$$

The solution

$$\mathcal{G}_p(r,t) = \frac{2ag_0}{R} \left[1 + \sum_{k=1}^{\infty} E_\alpha \left(-a\xi_k^2 t^\alpha \right) \frac{J_0(r\xi_k)}{J_0(R\xi_k)} \right] \tag{5.107}$$

is shown in Figs. 5.47 and 5.48 with $\overline{\mathcal{G}}_p = R\mathcal{G}_p/(ag_0)$.

Constant heat flux at the boundary

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) \tag{5.108}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{5.109}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{5.110}$$

$$r = R: \quad D_{RL}^{1-\alpha} \frac{\partial T}{\partial r} = g_0, \quad 0 < \alpha \le 1, \tag{5.111}$$

$$r = R: \quad I^{\alpha-1} \frac{\partial T}{\partial r} = g_0, \quad 1 < \alpha \le 2. \tag{5.112}$$

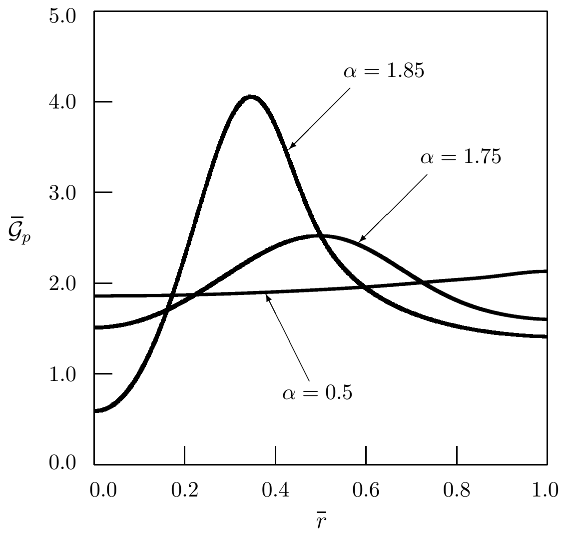

Figure 5.48: The fundamental solution to the physical Neumann problem for a cylinder; $\kappa = 1$

The solution [174]:

$$T = \frac{2ag_0 t}{R}\left[1 + \sum_{k=1}^{\infty} E_{\alpha,2}\left(-a\xi_k^2 t^\alpha\right)\frac{J_0(r\xi_k)}{J_0(R\xi_k)}\right]. \tag{5.113}$$

The results of numerical calculations of the solution (5.113) are presented in Fig. 5.49 and Fig. 5.50 with $\overline{T} = t^{\alpha-1}T/(g_0 R)$.

5.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{5.114}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{5.115}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{5.116}$$

$$r = R: \quad HT + \frac{\partial T}{\partial t} = g(t). \tag{5.117}$$

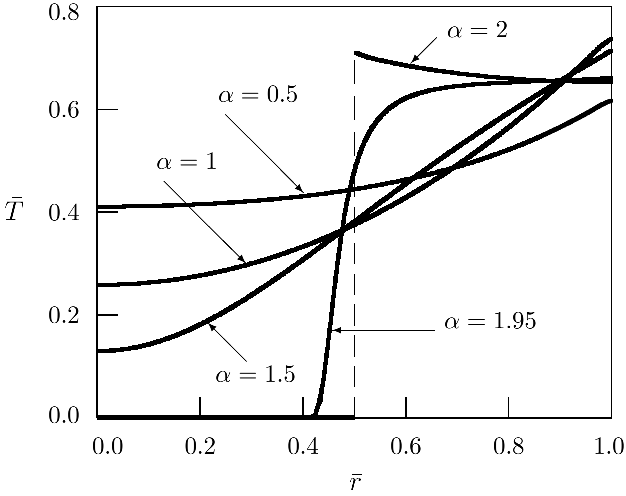

Figure 5.49: Dependence of temperature in a cylinder on distance (the constant heat flux at the boundary; $\kappa = 0.5$)

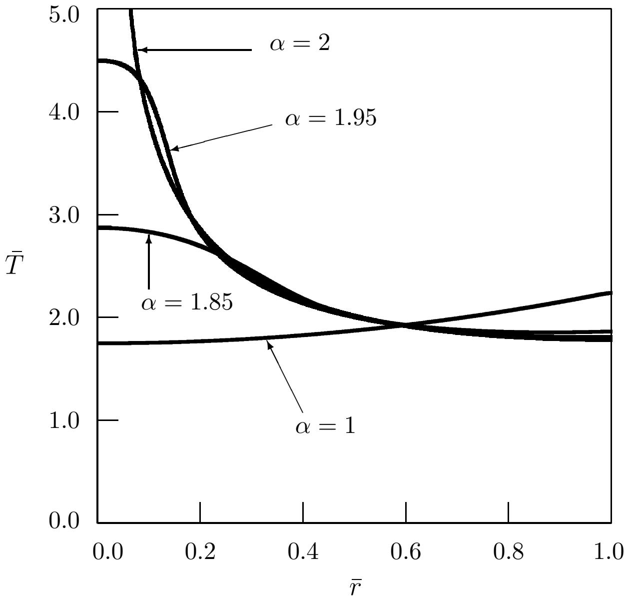

Figure 5.50: Dependence of temperature in a cylinder on distance (the constant heat flux at the boundary; $\kappa = 1$

The solution:

$$T(r,t) = \int_0^R f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho\,\mathrm{d}\rho + \int_0^R F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho\,\mathrm{d}\rho$$
$$+ \int_0^t \int_0^R \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{5.118}$$

The fundamental solutions [186]

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{2}{R^2}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix}$$
$$\times \frac{\xi_k^2}{\xi_k^2 + H^2}\,\frac{J_0(r\xi_k)J_0(\rho\xi_k)}{J_0^2(R\xi_k)} \tag{5.119}$$

with sum over all positive roots of the transcendental equation

$$\xi_k J_1(R\xi_k) = H J_0(R\xi_k) \tag{5.120}$$

are obtained using the Laplace transform with respect to time t and the finite Hankel transform (2.104) with respect to the radial coordinate r.

The fundamental solution to the mathematical Robin problem under zero initial condition is expressed as

$$\mathcal{G}_g(r,t) = \frac{aRg_0}{q_0}\,\mathcal{G}_\Phi(r,\rho,t)\Big|_{\rho=R}. \tag{5.121}$$

Dependence of the fundamental solution $\mathcal{G}_f$ on nondimensional distance r is shown in Figs. 5.51–5.52 ($\overline{H} = RH$, $\overline{\mathcal{G}}_f = R^2\mathcal{G}_f/p_0$). The fundamental solution $\mathcal{G}_F$ is depicted in Figs. 5.53–5.55 with $\overline{\mathcal{G}}_F = R^2\mathcal{G}_F/(w_0 t)$. The fundamental solution $\mathcal{G}_\Phi$ is presented in Figs. 5.56–5.58 for various values of α, κ and $\bar{H}$, where $\overline{\mathcal{G}}_\Phi = R^2 t^{1-\alpha}\mathcal{G}_\Phi/q_0$. The fundamental solution to the mathematical Robin boundary value problem under zero initial conditions $\mathcal{G}_g(r,t)$ is shown in Figs. 5.59 and 5.60 with $\overline{\mathcal{G}}_g = R\mathcal{G}_g T^{1-\alpha}/(ag_0)$. The fundamental solutions under Robin boundary conditions for $\kappa = 0.25$ do not “feel” the boundary condition, but for $\kappa = 0.5$ and $\kappa = 0.75$ there appears significant difference between solutions under Dirichlet, Neumann and Robin boundary conditions.

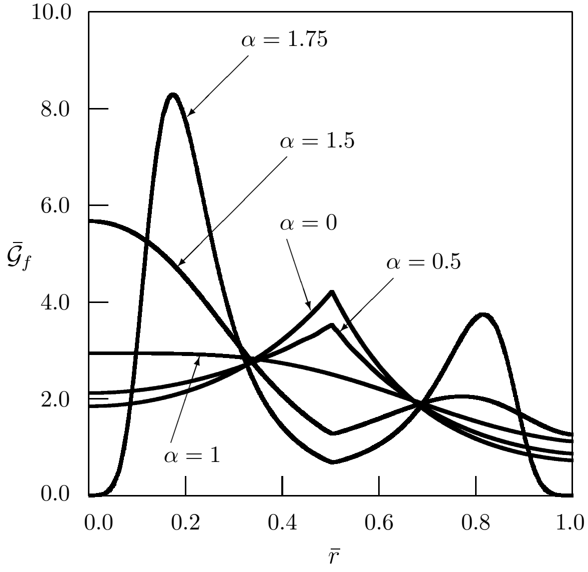

Figure 5.51: The fundamental solution to the first Cauchy problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$, $\bar{H} = 1$) [186]

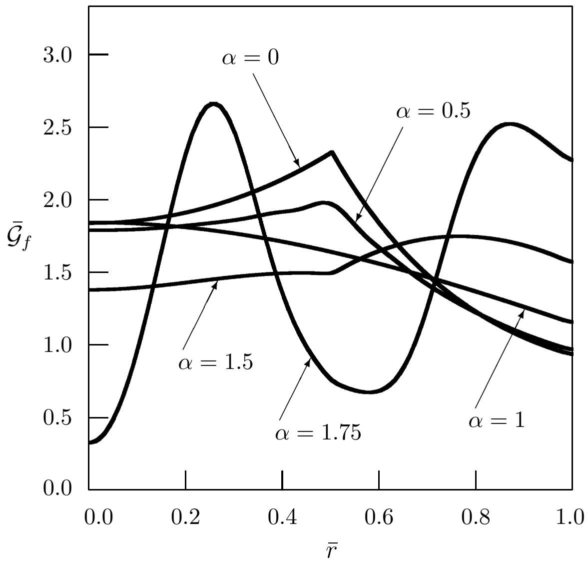

Figure 5.52: The fundamental solution to the first Cauchy problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$, $\bar{H} = 1$) [186]

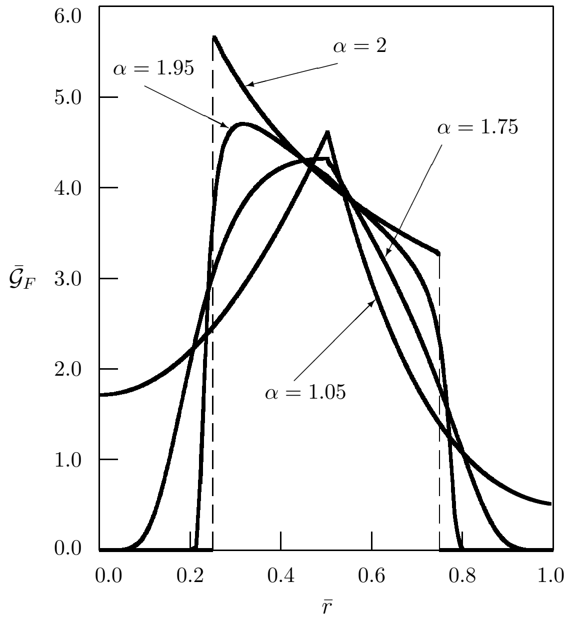

Figure 5.53: The fundamental solution to the second Cauchy problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$, $\bar{H} = 1$) [186]

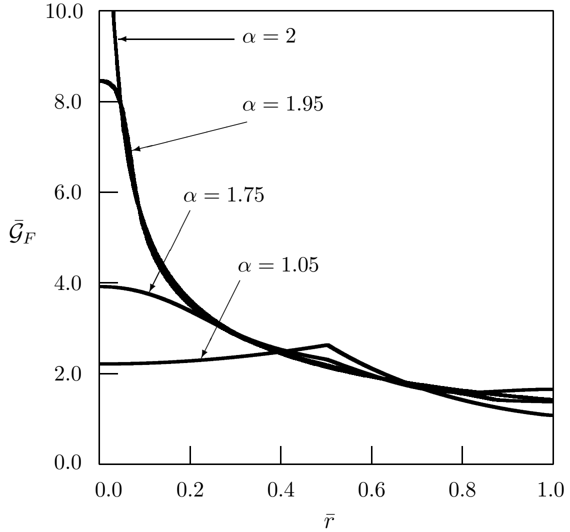

Figure 5.54: The fundamental solution to the second Cauchy problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$, $\bar{H} = 1$) [186]

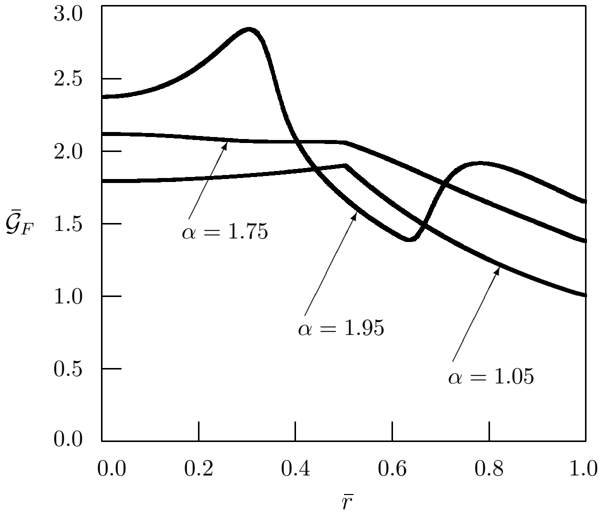

Figure 5.55: The fundamental solution to the second Cauchy problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.75$, $\bar{H} = 1$) [186]

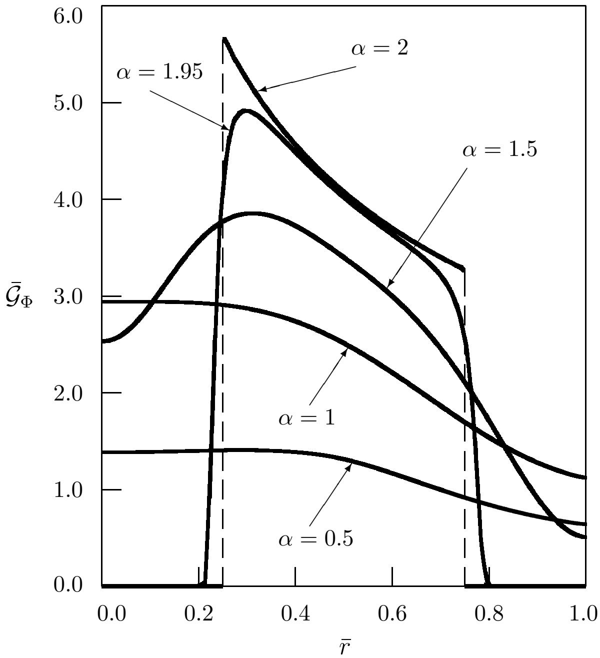

Figure 5.56: The fundamental solution to the source problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.25$, $\bar{H} = 1$) [186]

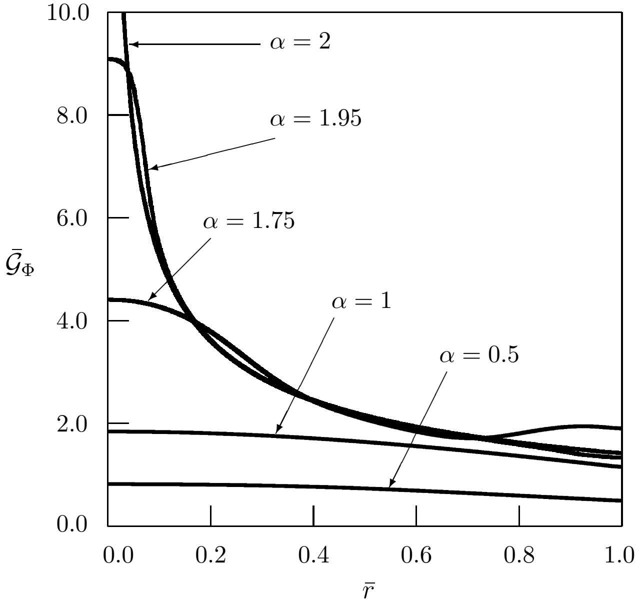

Figure 5.57: The fundamental solution to the source problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.5$, $\bar{H} = 1$) [186]

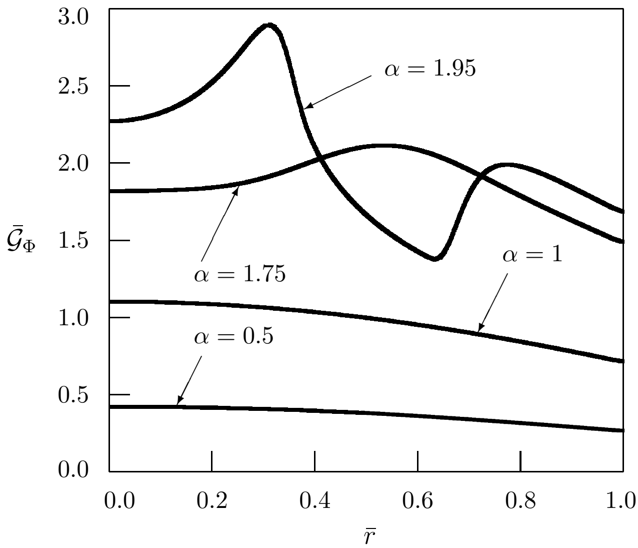

Figure 5.58: The fundamental solution to the source problem in a cylinder under zero Robin boundary condition ($\bar{\rho} = 0.5$, $\kappa = 0.75$, $\bar{H} = 1$) [186]

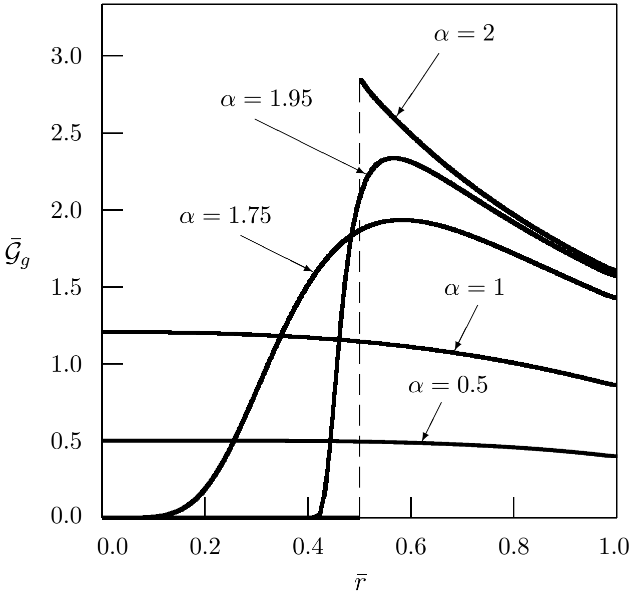

Figure 5.59: The fundamental solution to the Robin problem for a cylinder under zero initial conditions ($\kappa = 0.5$, $\bar{H} = 1$) [186]

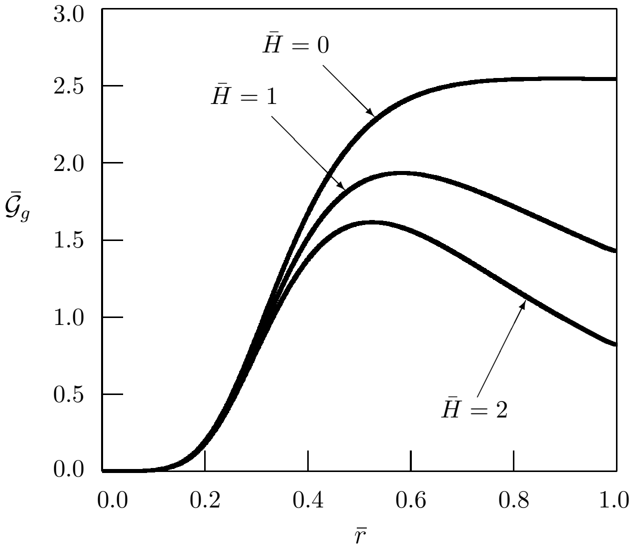

Figure 5.60: The fundamental solution to the Robin problem for a cylinder under zero initial conditions ($\alpha = 1.75$, $\kappa = 1$) [186]

5.4 Domain $R < r < \infty$

5.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{5.122}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{5.123}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{5.124}$$

$$r = R: \quad T = g(t). \tag{5.125}$$

The zero condition at infinity is also assumed:

$$\lim_{r\to\infty} T(r,t) = 0. \tag{5.126}$$

The solution:

$$T(r,t) = \int_R^\infty f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho\,\mathrm{d}\rho + \int_R^\infty F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho\,\mathrm{d}\rho$$
$$+ \int_0^t\int_R^\infty \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{5.127}$$

The fundamental solutions under zero Dirichlet boundary condition,

$$\begin{pmatrix}\mathcal{G}_f(r,\rho,t)\\ \mathcal{G}_F(r,\rho,t)\\ \mathcal{G}_\Phi(r,\rho,t)\end{pmatrix} = \int_0^\infty \begin{pmatrix} p_0\,E_\alpha(-a\xi^2t^\alpha)\\ w_0t\,E_{\alpha,2}(-a\xi^2t^\alpha)\\ q_0t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2t^\alpha)\end{pmatrix}\frac{J_0(r\xi)\,Y_0(R\xi)-Y_0(r\xi)\,J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)}$$
$$\times\left[J_0(\rho\xi)\,Y_0(R\xi)-Y_0(\rho\xi)\,J_0(R\xi)\right]\xi\,\mathrm{d}\xi, \tag{5.128}$$

are obtained using the Laplace transform with respect to time t and the Weber transform (2.108), (2.117) with respect to the radial coordinate r.

Dependence of the fundamental solution $\bar{\mathcal{G}}_f = R^2\mathcal{G}_f/p_0$ on nondimensional distance $\bar r = r/R$ with $\bar\rho = \rho/R$ and $\kappa = \sqrt{a}t^\alpha/R$ is shown in Fig. 5.61. The fundamental solution $\bar{\mathcal{G}}_F = R^2\mathcal{G}_F/(w_0t)$ is presented in Figs. 5.62 and 5.63. The fundamental solution to the source problem under zero Dirichlet boundary condition $\bar{\mathcal{G}}_\Phi = R^2\mathcal{G}_\Phi/(q_0t^{\alpha-1})$ is depicted in Figs. 5.64 and 5.65.

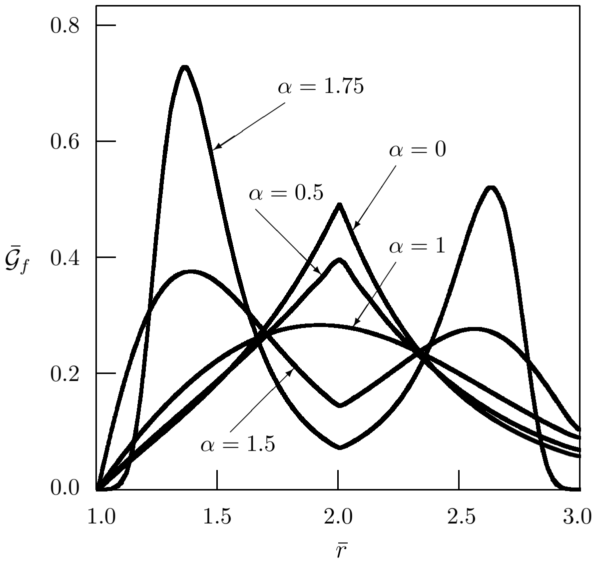

Figure 5.61: The fundamental solution to the first Cauchy problem in a body with a cylindrical hole under zero Dirichlet boundary condition ($\bar{\rho} = 2$, $\kappa = 0.5$)

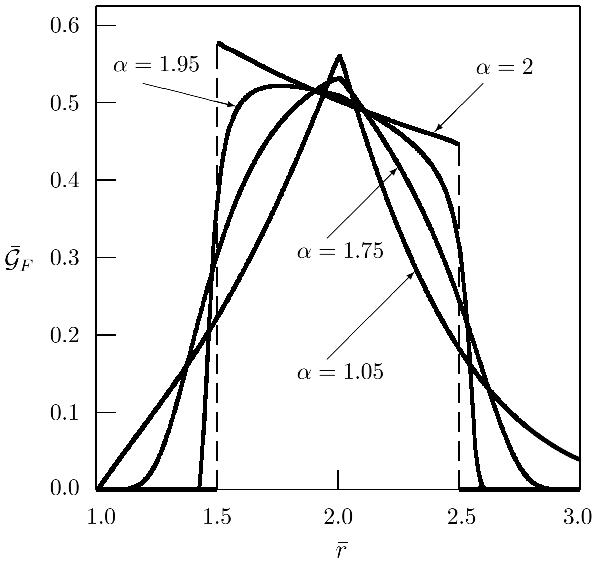

Figure 5.62: The fundamental solution to the second Cauchy problem in a body with a cylindrical hole under zero Dirichlet boundary condition ($\bar{\rho} = 2$, $\kappa = 0.5$)

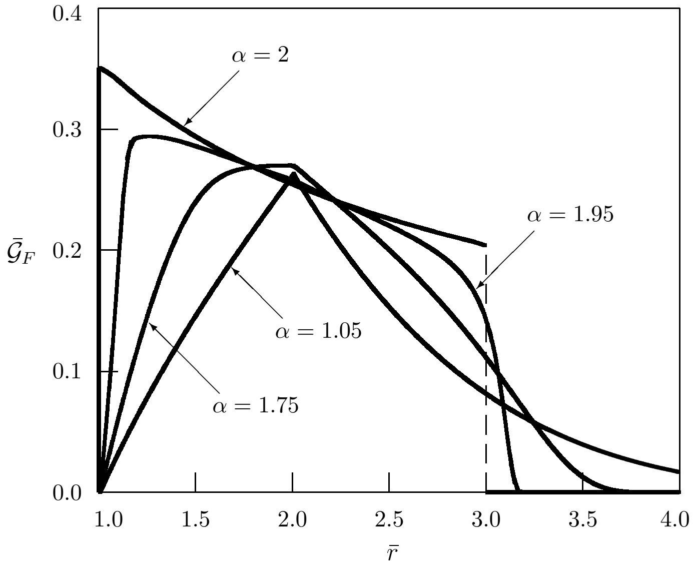

Figure 5.63: The fundamental solution to the second Cauchy problem in a body with a cylindrical hole under zero Dirichlet boundary condition ($\bar{\rho} = 2$, $\kappa = 1$)

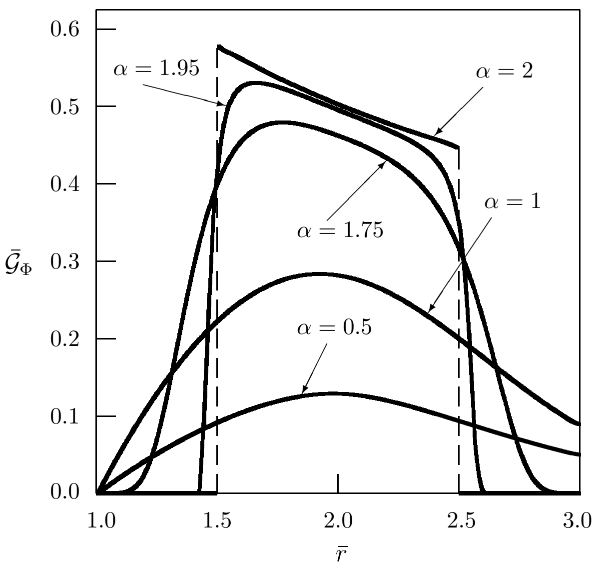

Figure 5.64: The fundamental solution to the source problem in a body with a cylindrical hole under zero Dirichlet boundary condition ($\bar{\rho} = 2$, $\kappa = 0.5$)

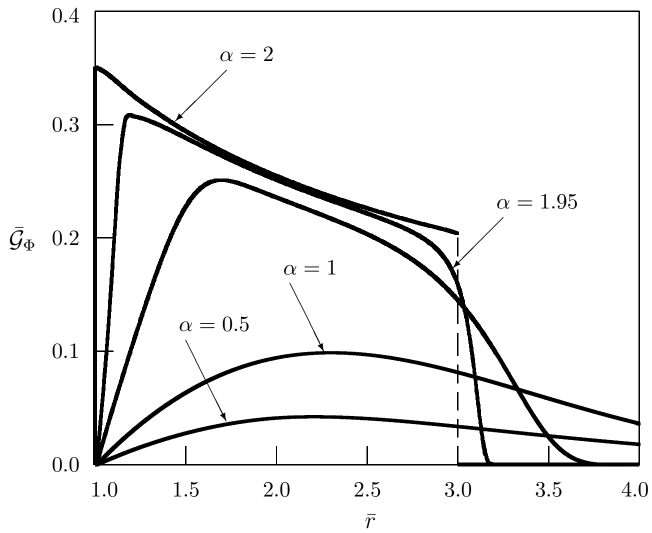

Figure 5.65: The fundamental solution to the source problem in a body with a cylindrical hole under zero Dirichlet boundary condition ($\bar{\rho} = 2$, $\kappa = 1$)

Fundamental solution to the Dirichlet problem

$$\frac{\partial^\alpha \mathcal{G}_g}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_g}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_g}{\partial r}\right), \tag{5.129}$$

$$t = 0: \quad \mathcal{G}_g = 0, \qquad 0 < \alpha \le 2, \tag{5.130}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_g}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{5.131}$$

$$r = R: \quad \mathcal{G}_g = g_0\,\delta(t). \tag{5.132}$$

The solution [157]:

$$\mathcal{G}_g(r,t) = -\frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)$$
$$\times\, \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)}\,\xi\,\mathrm{d}\xi. \tag{5.133}$$

The fundamental solution $\overline{\mathcal{G}}_g = t\mathcal{G}_g/g_0$ is shown in Fig. 5.66. The plot of solution for $\alpha = 2$ in Fig. 5.66 needs additional discussion. If we consider the axisymmetric Cauchy problem for the wave equation in a plane with initial value $T(r,0) = \delta(r-R)$, then the nondimensional solution for $0 < \kappa < 1$ has the form

$$\overline{\mathcal{G}}_f = \frac{1}{2\sqrt{1-\kappa}}\,\delta(\bar{r} - 1 + \kappa) + \frac{1}{2\sqrt{1+\kappa}}\,\delta(\bar{r} - 1 - \kappa) + (\text{a "tail"}) \tag{5.134}$$

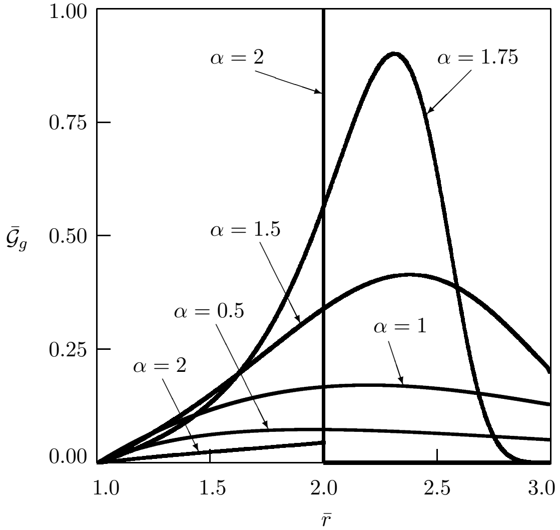

Figure 5.66: The fundamental solution to the Dirichlet problem for a body with a cylindrical hole ($\bar{\rho} = 2$, $\kappa = 1$) [157]

(see (5.16)). The first term in Eq. (5.134) presents the delta peak traveling in the direction of origin, the second term corresponds to the delta peak propagating in the direction of infinity, and the third term describes a "tail" behind the wave fronts. In the case of a cylinder with radius R ($0 \leq r \leq R$) the signaling problem for the wave equation with the Dirac delta boundary condition $T(R,t) = \delta(t)$ in the case $0 < \kappa < 1$ has a solution containing the delta peak traveling in the direction of origin and a portion of "tail" behind the wave front:

$$\overline{\mathcal{G}}_g = \frac{1}{\sqrt{1-\kappa}}\,\delta(\bar{r} - 1 + \kappa) \ + \ (\text{a "tail"}). \tag{5.135}$$

Similarly, in the case of an infinite medium with cylindrical cavity ($R \leq r < \infty$) the corresponding solution to the signaling problem contains the delta peak traveling in the direction of infinity and also a portion of "tail" behind the wave front:

$$\overline{\mathcal{G}}_g = \frac{1}{\sqrt{1+\kappa}}\,\delta(\bar{r} - 1 - \kappa) \ + \ (\text{a "tail"}). \tag{5.136}$$

It should be noted that coefficients of delta functions in (5.135) and (5.136) are twice as large as those in (5.134) (the initial delta pulse does not split in two parts). The "tails" in (5.135) and (5.136) cannot be calculated analytically as in (5.134), but can be estimated numerically.

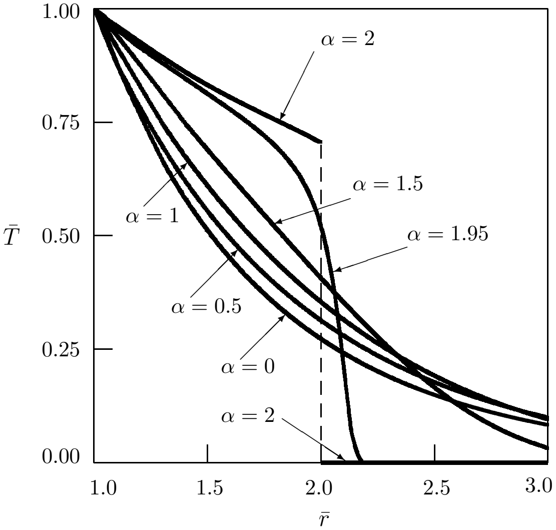

Figure 5.67: Dependence of the solution on distance (the Dirichlet problem for an infinite medium with cylindrical hole with constant boundary condition; $\kappa = 1$) [157]

Constant boundary value of temperature. In this case equations (5.129)–(5.131) are considered under the boundary condition

$$r = R: \quad T = T_0. \tag{5.137}$$

The solution has the following form [157]

$$T = T_0 + \frac{2T_0}{\pi} \int_0^\infty E_\alpha(-a\xi^2 t^\alpha) \, \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)} \, \frac{\mathrm{d}\xi}{\xi} \tag{5.138}$$

and is displayed in Fig. 5.67 with $\overline{T} = T/T_0$.

Recall that the solution to the corresponding problem for the classical heat conduction equation is well known [26, 48]:

$$T = T_0 + \frac{2T_0}{\pi} \int_0^\infty \exp\left(-a\xi^2 t\right) \, \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)} \, \frac{\mathrm{d}\xi}{\xi}. \tag{5.139}$$

5.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{5.140}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{5.141}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{5.142}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g(t), \tag{5.143}$$

$$\lim_{r \to \infty} T(r,t) = 0. \tag{5.144}$$

The solution:

$$T(r,t) = \int\limits_R^\infty f(\rho)\, \mathcal{G}_f(r,\rho,t)\, \rho\, \mathrm{d}\rho + \int\limits_R^\infty F(\rho)\, \mathcal{G}_F(r,\rho,t)\, \rho\, \mathrm{d}\rho$$

$$+ \int\limits_0^t \int\limits_R^\infty \Phi(\rho,\tau)\, \mathcal{G}_\Phi(r,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau + \int\limits_0^t g(\tau)\, \mathcal{G}_g(r,t-\tau)\, \mathrm{d}\tau. \tag{5.145}$$

The fundamental solutions under zero Neumann boundary condition have the following form:

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \int\limits_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \frac{J_0(r\xi)\, Y_1(R\xi) - Y_0(r\xi)\, J_1(R\xi)}{J_1^2(R\xi) + Y_1^2(R\xi)}$$

$$\times \left[J_0(\rho\xi)\, Y_1(R\xi) - Y_0(\rho\xi)\, J_1(R\xi) \right] \xi\, \mathrm{d}\xi \tag{5.146}$$

and are obtained using the Laplace transform with respect to time t and the Weber transform (2.108), (2.119) with respect to the radial coordinate r.

Dependence of the fundamental solution $\bar{\mathcal{G}}_f = R^2 \mathcal{G}_f / p_0$ on nondimensional distance $\bar{r} = r/R$ with $\bar{\rho} = \rho/R$ and $\kappa = \sqrt{a} t^\alpha / R$ is shown in Figs. 5.68 and 5.69. The fundamental solution $\bar{\mathcal{G}}_F = R^2 \mathcal{G}_F / (w_0 t)$ is presented in Figs. 5.70 and 5.71. The fundamental solution to the source problem under zero Neumann boundary condition $\bar{\mathcal{G}}_\Phi = R^2 \mathcal{G}_\Phi / (q_0 t^{\alpha-1})$ is depicted in Figs. 5.72 and 5.73. For $\kappa = 0.5$ the fundamental solutions under zero Dirichlet and Neumann boundary conditions are very similar (do not "feel" the boundary condition), but for $\kappa \ge 1$ the solutions under Dirichlet and Neumann boundary conditions are significantly different.

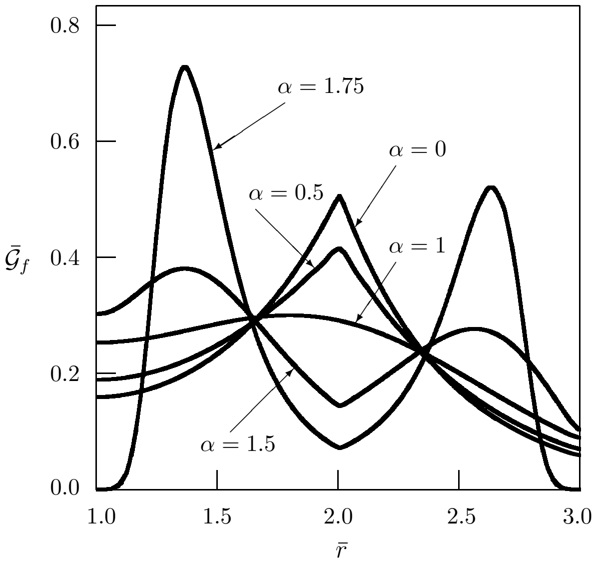

Figure 5.68: The fundamental solution to the first Cauchy problem in a body with a cylindrical hole under zero Neumann boundary condition ($\bar{\rho} = 2$, $\kappa = 0.5$)

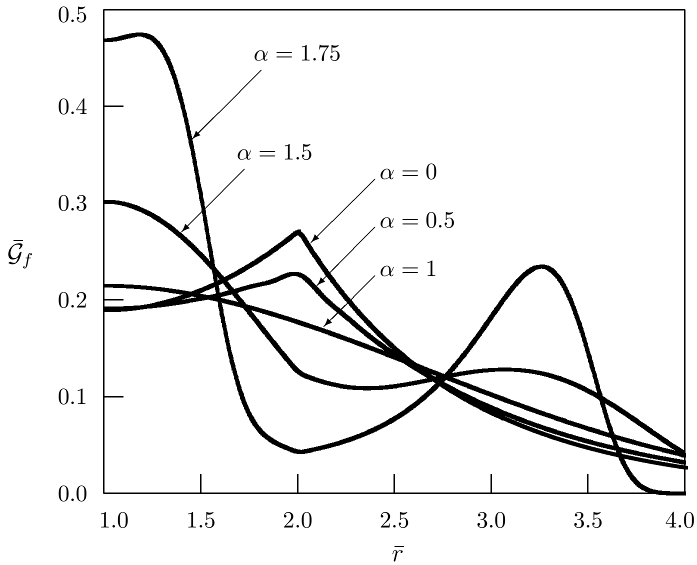

Figure 5.69: The fundamental solution to the first Cauchy problem in a body with a cylindrical hole under zero Neumann boundary condition ($\bar{\rho} = 2$, $\kappa = 1$)

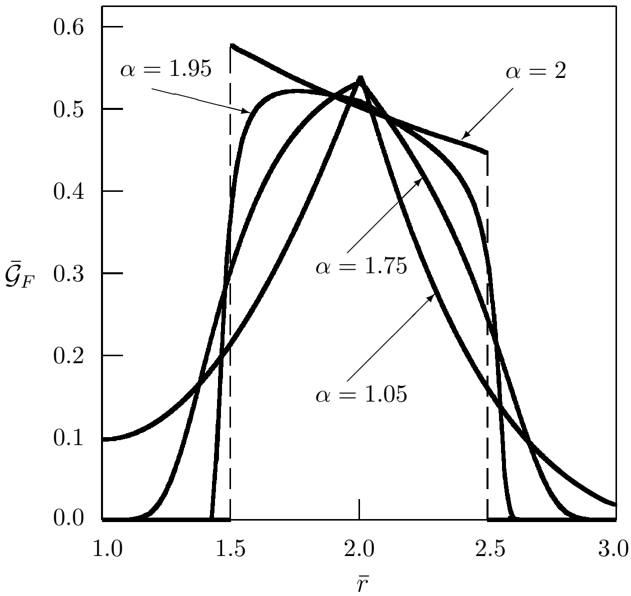

Figure 5.70: The fundamental solution to the second Cauchy problem in a body with a cylindrical hole under zero Neumann boundary condition ($\bar{\rho} = 2$, $\kappa = 0.5$)

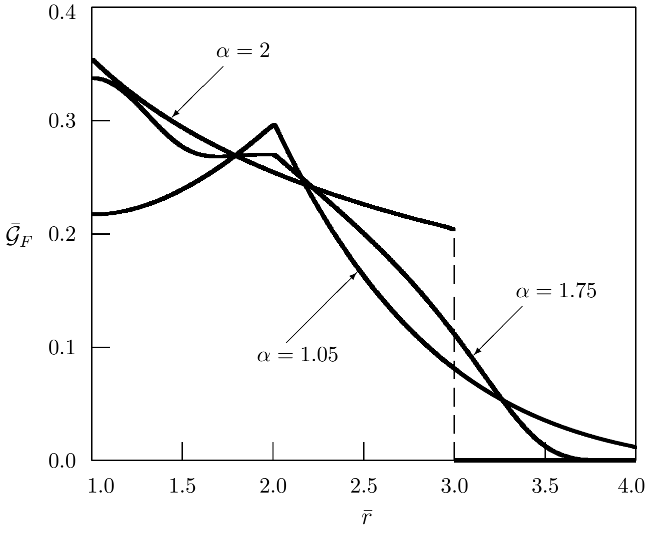

Figure 5.71: The fundamental solution to the second Cauchy problem in a body with a cylindrical hole under zero Neumann boundary condition ($\bar{\rho} = 2$, $\kappa = 1$)

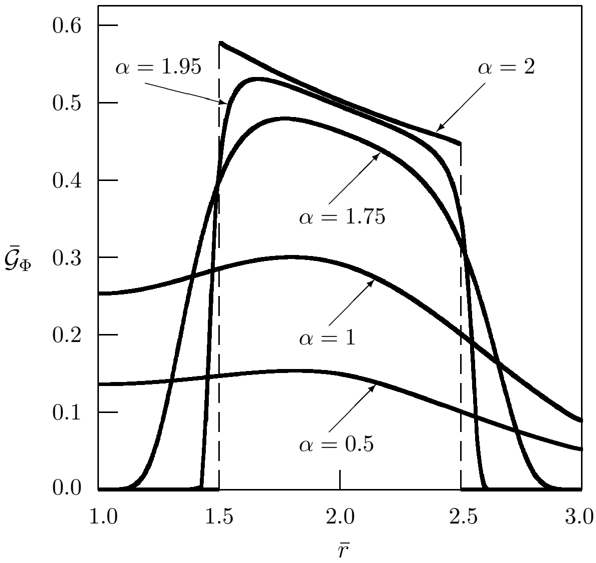

Figure 5.72: The fundamental solution to the source problem in a body with a cylindrical hole under zero Neumann boundary condition ($\bar{\rho} = 2$, $\kappa = 0.5$)

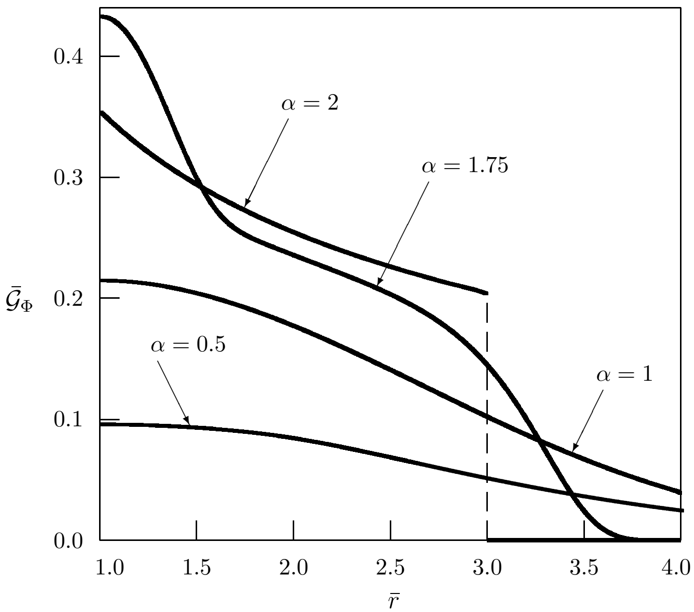

Figure 5.73: The fundamental solution to the source problem in a body with a cylindrical hole under zero Neumann boundary condition ($\bar{\rho} = 2$, $\kappa = 1$)

Fundamental solution to the mathematical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_m}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_m}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_m}{\partial r}\right), \tag{5.147}$$

$$t = 0: \quad \mathcal{G}_m = 0, \qquad 0 < \alpha \le 2, \tag{5.148}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_m}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{5.149}$$

$$r = R: \quad \frac{\partial \mathcal{G}_m}{\partial r} = -g_0\,\delta(t). \tag{5.150}$$

The solution

$$\mathcal{G}_m(r,t) = -\frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\,\frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi) + Y_1^2(R\xi)}\,\mathrm{d}\xi. \tag{5.151}$$

is depicted in Fig. 5.74 with $\overline{\mathcal{G}}_m = t\mathcal{G}_m/(Rg_0)$.

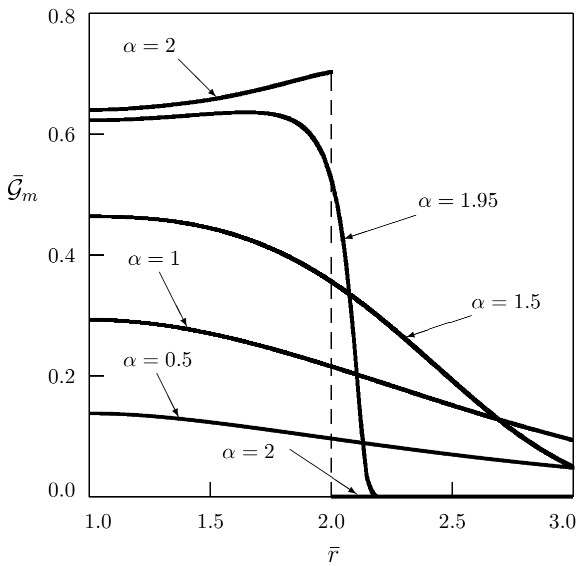

Figure 5.74: The fundamental solution to the mathematical Neumann problem for a body with a cylindrical hole ($\bar{\rho} = 2$, $\kappa = 1$) [157]

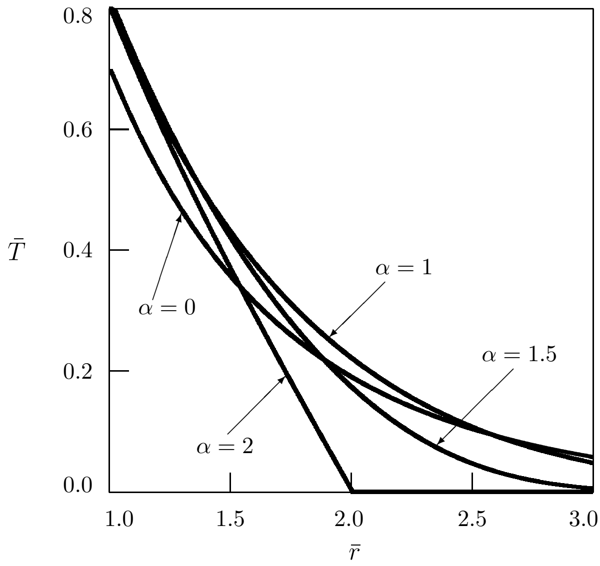

Figure 5.75: Dependence of the solution on distance (an infinite medium with a cylindrical hole and constant boundary value of normal derivative; $\kappa = 1$) [175]

Constant boundary value of normal derivative

$$r = R: \quad \frac{\partial T}{\partial r} = -g_0 = \text{const.} \tag{5.152}$$

The solution [157]

$$T = -\frac{2g_0}{\pi} \int_0^{\infty} \left[1 - E_\alpha(-a\xi^2 t^\alpha)\right] \frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi) + Y_1^2(R\xi)} \frac{\mathrm{d}\xi}{\xi^2} \tag{5.153}$$

is shown in Fig. 5.75 ($\overline{T} = T/(Rg_0)$).

Fundamental solution to the physical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_p}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_p}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_p}{\partial r}\right), \tag{5.154}$$

$$t = 0: \quad \mathcal{G}_p = 0, \qquad 0 < \alpha \le 2, \tag{5.155}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_p}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{5.156}$$

$$r = R: \quad D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial r} = -g_0\,\delta(t), \quad 0 < \alpha \le 1, \tag{5.157}$$

$$r = R: \quad I^{\alpha-1}\frac{\partial}{\partial r}\mathcal{G}_p = -g_0\,\delta(t), \quad 1 < \alpha \le 2. \tag{5.158}$$

The solution

$$\mathcal{G}_p(r,t) = -\frac{2ag_0}{\pi}\int\limits_0^\infty E_\alpha(-a\xi^2 t^\alpha)\,\frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi) + Y_1^2(R\xi)}\,\mathrm{d}\xi. \tag{5.159}$$

Constant boundary value of the heat flux

$$r = R: \quad D_{RL}^{1-\alpha}\frac{\partial T}{\partial r} = -g_0, \qquad 0 < \alpha \le 1, \tag{5.160}$$

$$r = R: \quad I^{\alpha-1}\frac{\partial T}{\partial r} = -g_0, \qquad 1 < \alpha \le 2\,. \tag{5.161}$$

The solution

$$T = -\frac{2ag_0 t}{\pi}\int\limits_0^\infty E_{\alpha,2}(-a\xi^2 t^\alpha)\,\frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi) + Y_1^2(R\xi)}\,\mathrm{d}\xi \tag{5.162}$$

is shown in Fig. 5.76 with $\overline{T} = RT/(ag_0 t)$.

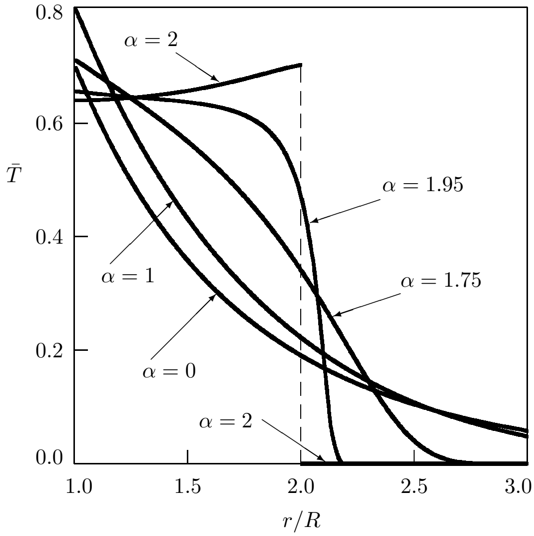

Figure 5.76: Dependence of the temperature on distance (the constant heat flux at the boundary of a body with a cylindrical hole) [175]

5.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{5.163}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{5.164}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{5.165}$$

$$r = R: \quad -\frac{\partial T}{\partial r} + HT = g(t), \tag{5.166}$$

$$\lim_{r\to\infty} T(r,t) = 0. \tag{5.167}$$

The solution:

$$T(r,t) = \int_R^\infty f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho\,\mathrm{d}\rho + \int_R^\infty F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho\,\mathrm{d}\rho$$

$$+ \int_0^t \int_R^\infty \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{5.168}$$

The fundamental solutions under zero Robin boundary condition

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}$$

$$\times \frac{Y_0(r\xi)\left[\xi J_1(R\xi) + HJ_0(R\xi)\right] - J_0(r\xi)\left[\xi Y_1(R\xi) + HY_0(R\xi)\right]}{\left[\xi J_1(R\xi) + HJ_0(R\xi)\right]^2 + \left[\xi Y_1(R\xi) + HY_0(R\xi)\right]^2}$$

$$\times \left\{ Y_0(\rho\xi)\left[\xi J_1(R\xi) + HJ_0(R\xi)\right] - J_0(\rho\xi)\left[\xi Y_1(R\xi) + HY_0(R\xi)\right] \right\} \xi\,\mathrm{d}\xi \tag{5.169}$$

are obtained using the Laplace transform with respect to time t and the Weber transform (2.108), (2.121) with respect to the radial coordinate r.

The fundamental solution to the mathematical Robin problem under zero initial conditions has the following form:

$$\mathcal{G}_g(r,t) = \frac{2ag_0 t^{\alpha-1}}{\pi} \int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)$$

$$\times \frac{Y_0(r\xi)\left[\xi J_1(R\xi) + HJ_0(R\xi)\right] - J_0(r\xi)\left[\xi Y_1(R\xi) + HY_0(R\xi)\right]}{\left[\xi J_1(R\xi) + HJ_0(R\xi)\right]^2 + \left[\xi Y_1(R\xi) + HY_0(R\xi)\right]^2}\,\xi\,\mathrm{d}\xi. \tag{5.170}$$

Chapter 6

Equations with One Space Variable in Spherical Coordinates

I feel a recipe is only a theme, which an intelligent cook can play each time with variation.

Madame Benoit

6.1 Domain $0 \le r < \infty$

6.1.1 Statement of the problem

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) + \Phi(r, t), \tag{6.1}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{6.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{6.3}$$

$$\lim_{r \to \infty} T(r, t) = 0. \tag{6.4}$$

The solution:

$$T(r, t) = \int_0^\infty f(\rho)\, \mathcal{G}_f(r, \rho, t)\, \rho^2 \, \mathrm{d}\rho + \int_0^\infty F(\rho)\, \mathcal{G}_F(r, \rho, t)\, \rho^2 \, \mathrm{d}\rho$$
$$+ \int_0^t \int_0^\infty \Phi(\rho, \tau)\, \mathcal{G}_\Phi(r, \rho, t - \tau)\, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\tau. \tag{6.5}$$

Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*,
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The fundamental solutions to the first Cauchy problem $\mathcal{G}_f(r,\rho,t)$, the second Cauchy problem $\mathcal{G}_F(r,\rho,t)$ and to the source problem $\mathcal{G}_\Phi(r,\rho,t)$ were considered in [151].

6.1.2 Fundamental solution to the first Cauchy problem

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_f}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{G}_f}{\partial r}\right), \tag{6.6}$$

$$t=0: \quad \mathcal{G}_f = p_0\,\frac{\delta(r-\rho)}{r^2}, \qquad 0<\alpha\le 2, \tag{6.7}$$

$$t=0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \qquad 1<\alpha\le 2. \tag{6.8}$$

It should be noted that the three-dimensional Dirac delta function in Cartesian coordinates $\delta(x)\,\delta(y)\,\delta(z)$ after passing to spherical coordinates takes the form $\frac{1}{4\pi r^2}\,\delta(r)$, but for the sake of simplicity we have omitted the factor 4π in the solution (6.5) as well as the factor $\frac{1}{4\pi}$ in the delta term in (6.7). The condition at infinity (6.4) is implied in all the problems in infinite domains considered in this chapter.

The Laplace transform with respect to time t and the sin-Fourier transform (2.31) with respect to the radial variable r with ξ being the transform variable give

$$\widetilde{\mathcal{G}}_f^* = \frac{\sqrt{2}\,p_0}{\sqrt{\pi}}\,\frac{\sin(\rho\xi)}{\rho\,\xi}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{6.9}$$

The inverse integral transforms result in

$$\mathcal{G}_f(r,\rho,t) = \frac{2p_0}{\pi r\rho}\int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right)\sin(r\xi)\,\sin(\rho\xi)\,d\xi. \tag{6.10}$$

It is convenient to introduce the following nondimensional quantities:

$$\bar{r} = \frac{r}{\rho}, \quad \eta = \rho\xi, \quad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{\rho}, \quad \overline{\mathcal{G}}_f = \frac{\pi\rho^3}{p_0}\,\mathcal{G}_f. \tag{6.11}$$

The factor π has been entered in the expression for $\overline{\mathcal{G}}_f$ to obtain the same scale as in [151]. Hence,

$$\overline{\mathcal{G}}_f = \frac{2}{\bar{r}}\int_0^\infty E_\alpha\left(-\kappa^2\eta^2\right)\sin(\bar{r}\eta)\,\sin(\eta)\,d\eta. \tag{6.12}$$

Let us consider several particular cases of the solution (6.12).

Helmholtz equation ($\alpha \to 0$)

$$\overline{\mathcal{G}}_f = \frac{\pi}{2\kappa\bar{r}} \left[\exp\left(-\frac{|1-\bar{r}|}{\kappa}\right) - \exp\left(-\frac{1+\bar{r}}{\kappa}\right)\right]. \tag{6.13}$$

Subdiffusion with α=1/2

$$\overline{\mathcal{G}}_f = \frac{1}{\sqrt{2}\kappa\bar{r}} \int_0^\infty \frac{1}{\sqrt{u}} \mathrm{e}^{-u^2} \left\{\exp\left[-\frac{(\bar{r}-1)^2}{8\kappa^2 u}\right] - \exp\left[-\frac{(\bar{r}+1)^2}{8\kappa^2 u}\right]\right\} \mathrm{d}u. \tag{6.14}$$

Classical diffusion equation (α=1)

$$\overline{\mathcal{G}}_f = \frac{\sqrt{\pi}}{2\kappa\bar{r}} \left\{\exp\left[-\frac{(\bar{r}-1)^2}{4\kappa^2}\right] - \exp\left[-\frac{(\bar{r}+1)^2}{4\kappa^2}\right]\right\}. \tag{6.15}$$

Wave equation (α=2)

$$\overline{\mathcal{G}}_f = \frac{\pi}{2\bar{r}} [\delta(\bar{r}-1+\kappa) + \delta(\bar{r}-1-\kappa) - \delta(\bar{r}+1-\kappa)]. \tag{6.16}$$

Figures 6.1–6.2 show dependence of the fundamental solution $\bar{\mathcal{G}}_f$ on nondimensional distance $\bar{r}$ for various values of κ and α. Usually, three distinguishing

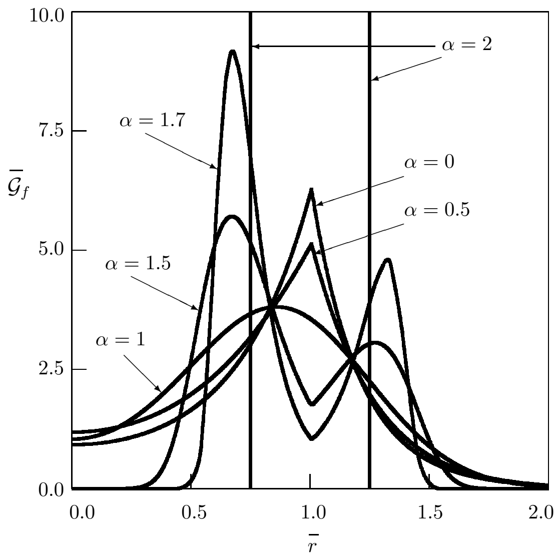

Figure 6.1: Dependence of the fundamental solution to the first Cauchy problem in a space on distance; $\kappa = 0.25$ [151]

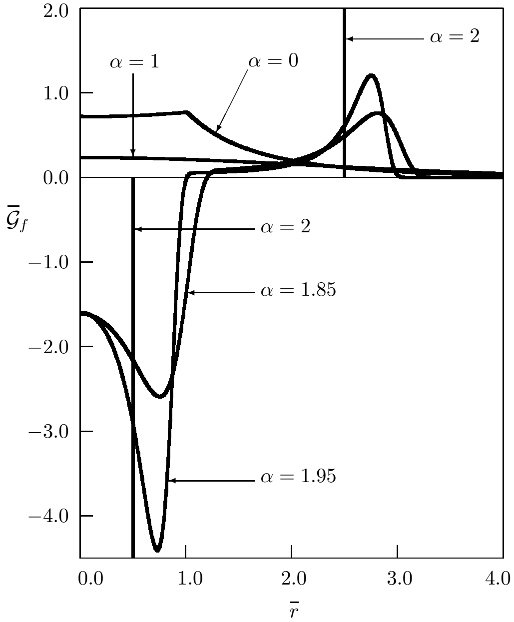

Figure 6.2: Dependence of the fundamental solution to the first Cauchy problem in a space on distance; $\kappa = 1.5$

values of the parameter κ are considered: $0 < \kappa < 1$, $\kappa = 1$ and $\kappa > 1$. For a wave equation these values correspond to three characteristic events: the wave front does not yet arrive at the origin, the wave front arrives at the origin, and the wave front reflects from the origin.

6.1.3 Fundamental solution to the second Cauchy problem

$$\frac{\partial^\alpha \mathcal{G}_F}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_F}{\partial r^2} + \frac{2}{r}\,\frac{\partial \mathcal{G}_F}{\partial r}\right), \tag{6.17}$$

$$t = 0: \quad \mathcal{G}_F = 0, \qquad 1 < \alpha \leq 2, \tag{6.18}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_F}{\partial t} = w_0\,\frac{\delta(r-\rho)}{r^2}, \qquad 1 < \alpha \leq 2. \tag{6.19}$$

The solution:

$$\mathcal{G}_F(r,\rho,t) = \frac{2w_0 t}{\pi r \rho}\int_0^\infty E_{\alpha,2}\left(-a\xi^2 t^\alpha\right)\sin(r\xi)\,\sin(\rho\xi)\,d\xi \tag{6.20}$$

with $\overline{\mathcal{G}}_F = \pi\rho^3 \mathcal{G}_F/(w_0 t)$.

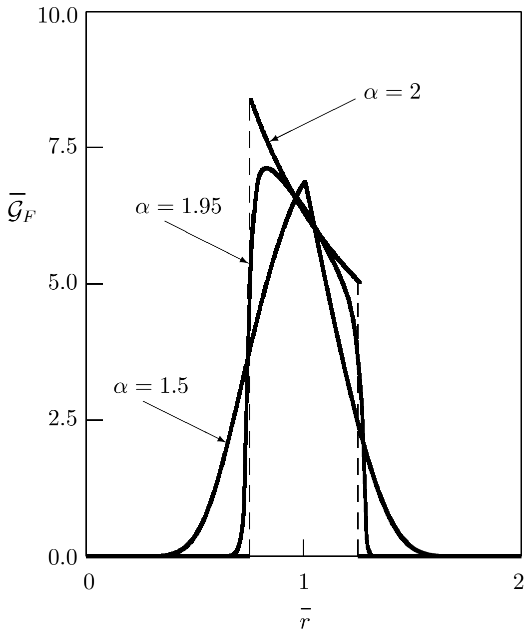

Figure 6.3: Dependence of the fundamental solution to the second Cauchy problem in a space on distance; $\kappa = 0.25$ [151]

Wave equation (α=2)

$$\overline{\mathcal{G}}_F = \begin{cases} 0, & 0 \le \bar{r} < |1-\kappa|, \\ \dfrac{\pi}{2\kappa\bar{r}}, & |1-\kappa| < \bar{r} < 1+\kappa, \\ 0, & 1+\kappa < \bar{r} < \infty. \end{cases} \tag{6.21}$$

The numerical results for the fundamental solution to the second Cauchy problem are shown in Figs. 6.3–6.5 for various values of κ and α.

6.1.4 Fundamental solution to the source problem

The fundamental solution to the source problem is obtained in a similar way and has the following form:

$$\mathcal{G}_\Phi(r,\rho,t) = \frac{2q_0 t^{\alpha-1}}{\pi r \rho} \int_0^\infty E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) \sin(r\xi)\, \sin(\rho\xi)\, d\xi \tag{6.22}$$

with $\overline{\mathcal{G}}_\Phi = \pi\rho^3 t^{1-\alpha}\mathcal{G}_\Phi/q_0$.

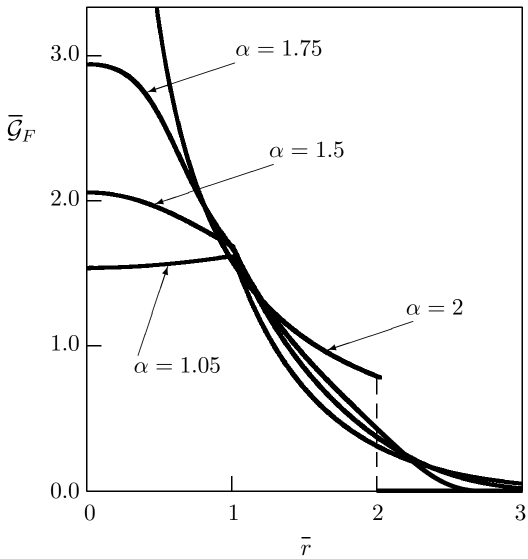

Figure 6.4: Dependence of the fundamental solution to the second Cauchy problem in a space on distance; $\kappa = 1$

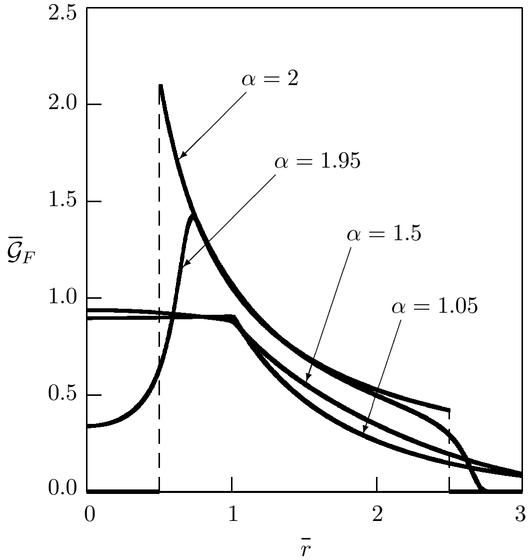

Figure 6.5: Dependence of the fundamental solution to the second Cauchy problem in a space on distance; $\kappa = 1.5$

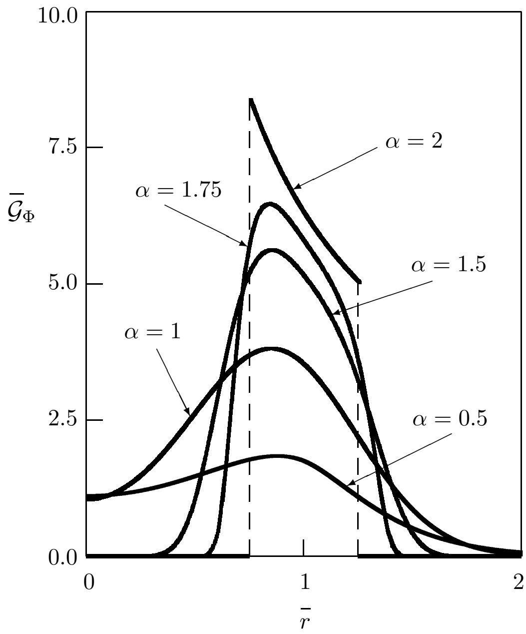

Figure 6.6: Dependence of the fundamental solution to the source problem in a space on distance; $\kappa = 0.25$ [151]

Subdiffusion with α=1/2

$$\overline{\mathcal{G}}_\Phi = \frac{1}{\sqrt{2}\kappa\bar{r}} \int_0^\infty \sqrt{u} e^{-u^2} \left\{ \exp\left[-\frac{(\bar{r}-1)^2}{8\kappa^2 u}\right] - \exp\left[-\frac{(\bar{r}+1)^2}{8\kappa^2 u}\right] \right\} du. \tag{6.23}$$

Dependence of the fundamental solution $\overline{\mathcal{G}}_\Phi$ on distance is shown in Figs. 6.6–6.8.

6.1.5 Delta-pulse at the origin

In the case of the first Cauchy problem we have

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right), \tag{6.24}$$

$$t = 0: \quad T = p_0 \frac{\delta(r)}{4\pi r^2}, \qquad 0 < \alpha \leq 2, \tag{6.25}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \leq 2. \tag{6.26}$$

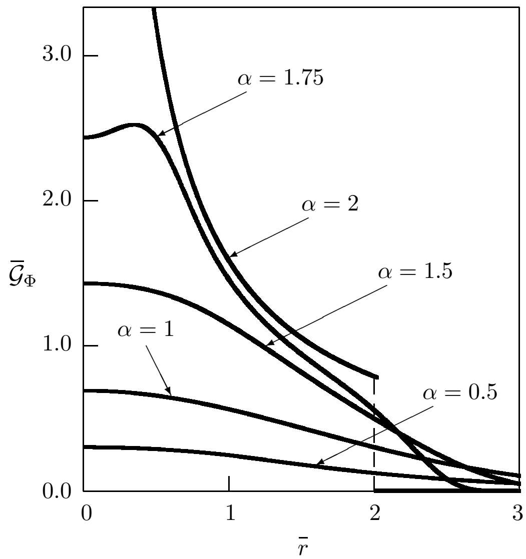

Figure 6.7: Dependence of the fundamental solution to the source problem in a space on distance; $\kappa = 1$ [151]

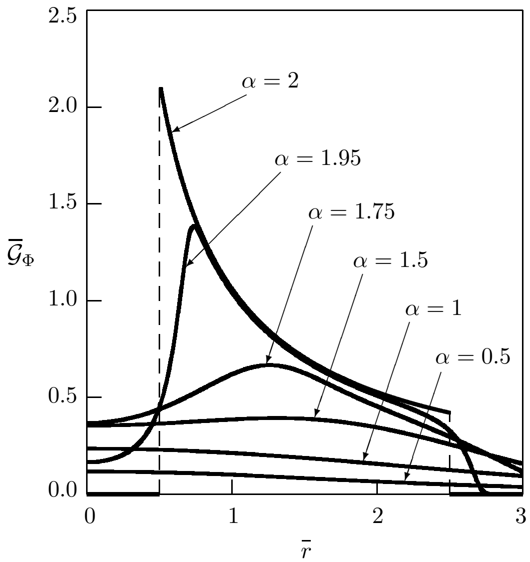

Figure 6.8: Dependence of the fundamental solution to the source problem in a space on distance; $\kappa = 1.5$ [151]

The solution [150]:

$$T = \frac{p_0}{2\pi^2 r} \int\limits_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) \sin(r\xi)\, \xi\, d\xi \tag{6.27}$$

and

$$\overline{T} = \frac{2}{\bar{r}} \int\limits_0^\infty E_\alpha\left(-\eta^2\right) \sin(\bar{r}\eta)\, \eta\, d\eta, \tag{6.28}$$

where

$$\overline{T} = \frac{4\pi^2 a^{3/2} t^{3\alpha/2}}{p_0} T, \qquad \bar{r} = \frac{r}{\sqrt{a} t^{\alpha/2}}, \qquad \eta = \sqrt{a} t^{\alpha/2} \xi. \tag{6.29}$$

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \frac{\pi}{\bar{r}}\, \mathrm{e}^{-\bar{r}}. \tag{6.30}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{T} = \frac{1}{2\sqrt{2}} \int\limits_0^\infty \exp\left(-u^2 - \frac{\bar{r}^2}{8u}\right) \frac{1}{u^{3/2}}\, \mathrm{d}u. \tag{6.31}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{\sqrt{\pi}}{2} \exp\left(-\frac{\bar{r}^2}{4}\right). \tag{6.32}$$

Now we investigate the behavior of the solution (6.27) at the origin. Due to the asymptotics (2.161) of the Mittag-Leffler function $E_\alpha(-x)$ for large values of x we can rewrite (6.27) as

$$\begin{aligned} T = {} & \frac{p_0}{2\pi^2 r} \int\limits_0^\infty \left[E_\alpha(-a\xi^2 t^\alpha) - \frac{1}{\Gamma(1-\alpha) a \xi^2 t^\alpha}\right] \sin(\xi r)\, \xi\, \mathrm{d}\xi \\ & + \frac{p_0}{2\pi^2 \Gamma(1-\alpha) a t^\alpha r} \int\limits_0^\infty \frac{\sin(\xi r)}{\xi}\, \mathrm{d}\xi. \end{aligned} \tag{6.33}$$

The first integral in (6.33) has no singularity at the origin, while the second one can be calculated analytically and yields the following singularity at the origin:

$$T \sim \frac{p_0}{4\pi \Gamma(1-\alpha) a t^\alpha} \frac{1}{r}. \tag{6.34}$$

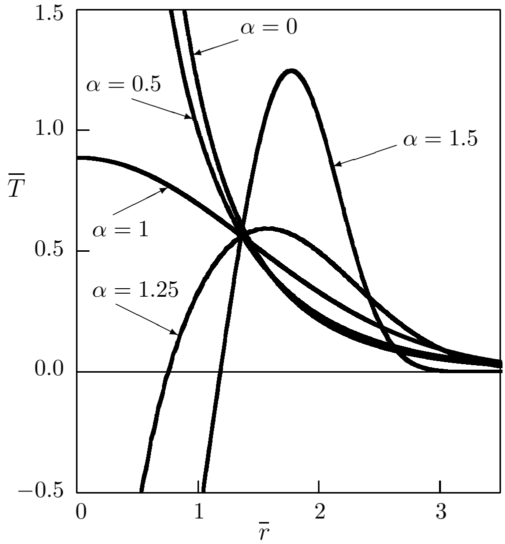

Figure 6.9: Dependence of the solution on the similarity variable $\bar{r}$ (the first Cauchy problem for a space with the delta-pulse initial condition) [150]

This result is consistent with the behavior of the solution for small r obtained in [208]. Only the solution to the classical diffusion equation has no singularity at the origin.

Dependence of the nondimensional solution $\overline{T}$ on the similarity variable $\bar{r}$ is shown in Fig. 6.9.

In the case of the second Cauchy problem with the delta-pulse initial condition we get the solution

$$T = \frac{w_0 t}{2\pi^2 r} \int_0^\infty E_{\alpha,2}\left(-a\xi^2 t^\alpha\right) \sin(r\xi)\, \xi \, d\xi. \tag{6.35}$$

The numerical results for $\overline{T} = 4\pi^2 a^{3/2} t^{3\alpha/2-1} T/w_0$ are shown in Fig. 6.10. The vertical line in Fig. 6.10 represents the Dirac delta function corresponding to the solution of the wave equation.

To investigate behavior of the solution (6.35) at the origin we recall that for large values of ξ we have the asymptotics (2.162) for the Mittag-Leffler function $E_{\alpha,2}(-x)$. As a result, the solution to the second Cauchy problem with the delta-pulse initial condition also has its singularity at the origin

$$T \sim \frac{w_0}{4\pi\Gamma(2-\alpha) a t^{\alpha-1}} \frac{1}{r}, \qquad 1 < \alpha < 2. \tag{6.36}$$

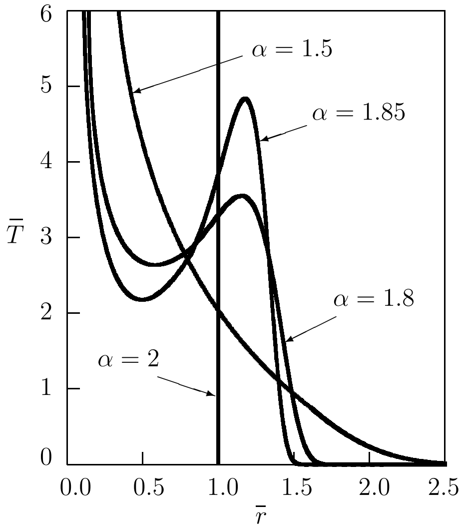

Figure 6.10: Dependence of the solution on the similarity variable $\bar{r}$ (the second Cauchy problem for a space with the delta-pulse initial condition)

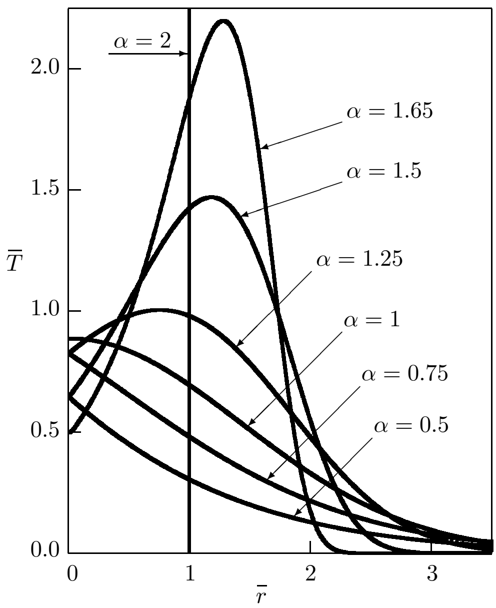

Figure 6.11: Dependence of the solution on the similarity variable $\bar{r}$ (the delta pulse source problem for a space with zero initial conditions)

The solution to the time-fractional diffusion wave equation with the source term $q_0 \frac{\delta(r)}{4\pi r^2}\,\delta(t)$ under zero initial conditions is expressed as

$$T = \frac{q_0 t^{\alpha-1}}{2\pi^2 r} \int_0^\infty E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) \sin(r\xi)\,\xi\,d\xi. \tag{6.37}$$

Due to the asymptotics of the Mittag-Leffler function $E_{\alpha,\alpha}(-x)$ (2.163) for large values of x the solution (6.37) has no singularity at the origin. The numerical results for the solution (6.37) are depicted in Fig. 6.11 for the nondimensional quantity $\overline{T} = 4\pi^2 a^{3/2} t^{\alpha/2+1} T/q_0$.

6.2 Evolution of the unit-box signal

6.2.1 First Cauchy problem

Evolution of the unit-box signal in the central symmetric case was investigated in [162].

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right), \tag{6.38}$$

$$t = 0: \quad T = \begin{cases} T_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \qquad 0 < \alpha \le 2, \tag{6.39}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{6.40}$$

The solution:

$$T = \frac{2T_0}{\pi r} \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) \left[\sin(R\xi) - R\xi\cos(R\xi)\right] \frac{\sin(r\xi)}{\xi^2}\,d\xi. \tag{6.41}$$

The nondimensional quantities used in numerical calculations are the following:

$$\overline{T} = \frac{T}{T_0}, \qquad \bar{r} = \frac{r}{R}, \qquad \kappa = \frac{\sqrt{a}\,t^{\alpha/2}}{R}. \tag{6.42}$$

Consider several particular cases of the solution (6.41).

Helmholtz equation ($\alpha \to 0$)

$$\bar{T} = \begin{cases} 1 - \dfrac{1+\kappa}{2\bar{r}}\left[\exp\left(-\dfrac{1-\bar{r}}{\kappa}\right) - \exp\left(-\dfrac{1+\bar{r}}{\kappa}\right)\right], & 0 < \bar{r} < 1, \\[2ex] \dfrac{1}{2\bar{r}}\left[(1-\kappa)\exp\left(-\dfrac{\bar{r}-1}{\kappa}\right) + (1+\kappa)\exp\left(-\dfrac{\bar{r}+1}{\kappa}\right)\right], & 1 < \bar{r} < \infty. \end{cases} \tag{6.43}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{T} = \frac{1}{\sqrt{\pi}} \int\limits_0^\infty \mathrm{e}^{-u^2} \Bigg\langle \mathrm{erf}\left(\frac{\bar{r}+1}{2\sqrt{2}\kappa\sqrt{u}}\right) - \mathrm{erf}\left(\frac{\bar{r}-1}{2\sqrt{2}\kappa\sqrt{u}}\right)$$
$$+ \frac{2\sqrt{2}\kappa\sqrt{u}}{\sqrt{\pi}\bar{r}} \left\{ \exp\left[-\frac{(\bar{r}+1)^2}{8\kappa^2 u}\right] - \exp\left[-\frac{(\bar{r}-1)^2}{8\kappa^2 u}\right] \right\} \Bigg\rangle \mathrm{d}u. \tag{6.44}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{2}\left[\mathrm{erf}\left(\frac{\bar{r}+1}{2\kappa}\right) - \mathrm{erf}\left(\frac{\bar{r}-1}{2\kappa}\right)\right]$$
$$+ \frac{\kappa}{\sqrt{\pi}\bar{r}} \left\{ \exp\left[-\frac{(\bar{r}+1)^2}{4\kappa^2}\right] - \exp\left[-\frac{(\bar{r}-1)^2}{4\kappa^2}\right] \right\}. \tag{6.45}$$

Wave equation ($\alpha = 2$)

a) $0 < \kappa < 1$

$$\overline{T} = \begin{cases} 1, & 0 \le \bar{r} < 1-\kappa, \\ \dfrac{\bar{r}-\kappa}{2\bar{r}}, & 1-\kappa < \bar{r} < 1+\kappa, \\ 0, & 1+\kappa < \bar{r} < \infty. \end{cases} \tag{6.46}$$

b) $\kappa = 1$

$$\overline{T} = \begin{cases} \dfrac{\bar{r}-1}{2\bar{r}}, & 0 < \bar{r} < 2, \\ 0, & 2 < \bar{r} < \infty. \end{cases} \tag{6.47}$$

c) $1 < \kappa < \infty$

$$\overline{T} = \begin{cases} 0, & 0 \le \bar{r} < \kappa-1, \\ \dfrac{\bar{r}-\kappa}{2\bar{r}}, & \kappa-1 < \bar{r} < \kappa+1, \\ 0, & \kappa+1 < \bar{r} < \infty. \end{cases} \tag{6.48}$$

The curves presented in Figs. 6.12–6.14 correspond to the solution (6.41) for $\kappa = 0.25$, $\kappa = 1$ and $\kappa = 1.5$, respectively.

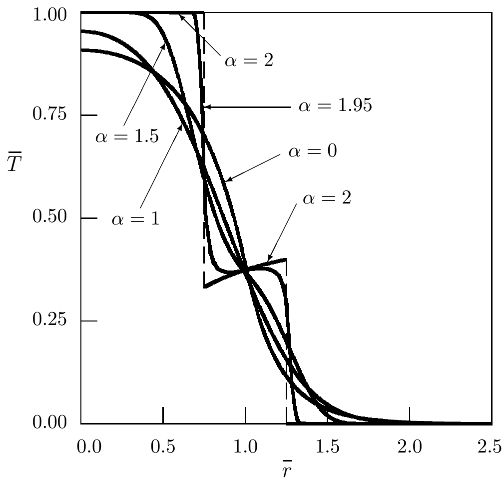

Figure 6.12: Evolution of the initial box-signal in the central symmetric case (the first Cauchy problem); $\kappa = 0.25$ [162]

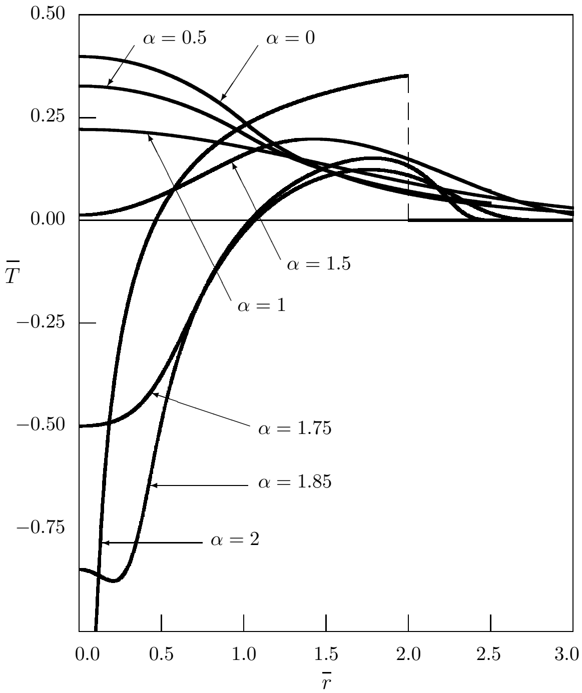

Figure 6.13: Evolution of the initial box-signal in the central symmetric case (the first Cauchy problem); $\kappa = 1$ [162]

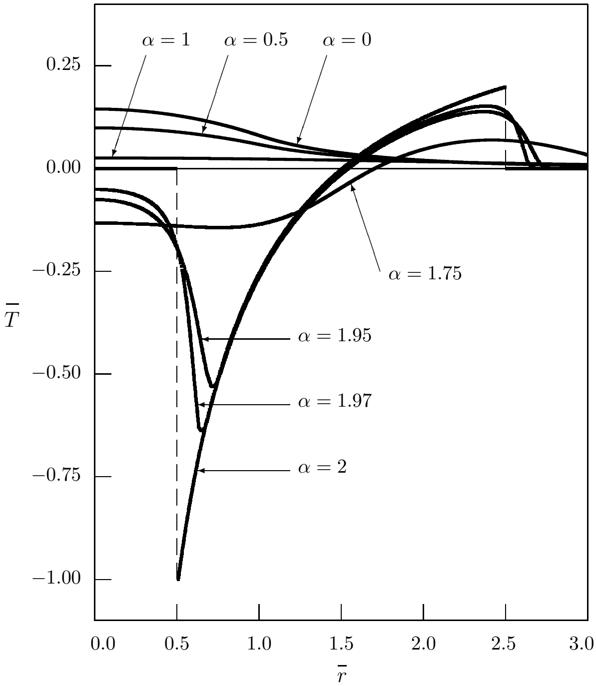

Figure 6.14: Evolution of the initial box-signal in the central symmetric case (the first Cauchy problem); $\kappa = 1.5$ [162]

6.2.2 Second Cauchy problem

Consider the central symmetric time-fractional diffusion-wave equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) \tag{6.49}$$

under initial conditions

$$t = 0: \qquad T = 0, \tag{6.50}$$

$$t = 0: \qquad \frac{\partial T}{\partial t} = \begin{cases} w_0, & 0 \le r < R, \\ 0, & R < r < \infty. \end{cases} \tag{6.51}$$

The solution has the following form [182]:

$$T = \frac{2w_0 t}{\pi r}\int_0^\infty E_{\alpha,2}(-a\xi^2 t^\alpha)\left[\sin(R\xi) - R\xi\cos(R\xi)\right]\frac{\sin(r\xi)}{\xi^2}\,\mathrm{d}\xi \tag{6.52}$$

with the special case corresponding to the standard wave equation ($\overline{T} = T/(w_0 t)$):

a) $0 < \kappa < 1$

$$\overline{T} = \begin{cases} 1, & 0 \le \bar{r} < 1 - \kappa, \\ \dfrac{1 - (\kappa - \bar{r})^2}{4\kappa\bar{r}}, & 1 - \kappa < \bar{r} < 1 + \kappa, \\ 0, & 1 + \kappa < \bar{r} < \infty. \end{cases} \tag{6.53}$$

b) $\kappa = 1$

$$\overline{T} = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{\bar{r}}{2}\right), & 0 \le \bar{r} < 2, \\ 0, & 2 < \bar{r} < \infty. \end{cases} \tag{6.54}$$

c) $\kappa > 1$

$$\overline{T} = \begin{cases} 0, & 0 \le \bar{r} < \kappa - 1, \\ \dfrac{1 - (\kappa - \bar{r})^2}{4\kappa\bar{r}}, & \kappa - 1 < \bar{r} < \kappa + 1, \\ 0, & \kappa + 1 < \bar{r} < \infty. \end{cases} \tag{6.55}$$

The curves presented in Fig. 6.15 and Fig. 6.16 correspond to the solution (6.52) for $\kappa = 0.5$ and $\kappa = 1.25$, respectively.

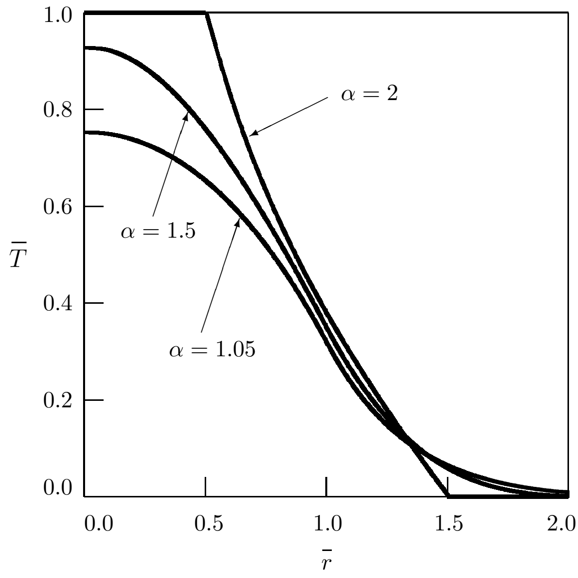

Figure 6.15: Evolution of the initial box-signal in the central symmetric case (the second Cauchy problem); $\kappa = 0.5$ [182]

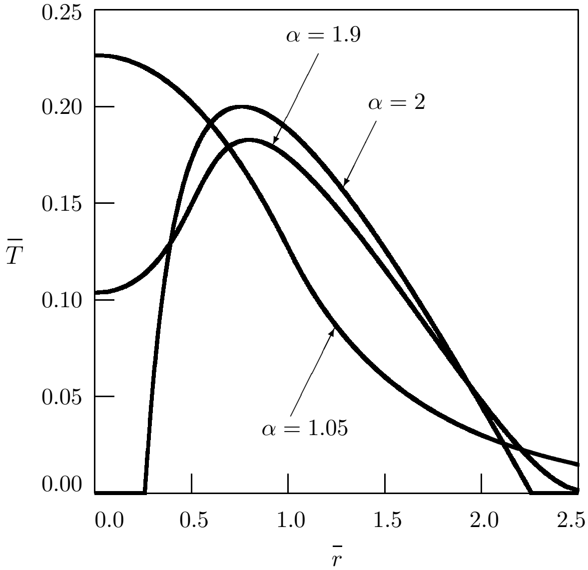

Figure 6.16: Evolution of the initial box-signal in the central symmetric case (the second Cauchy problem); $\kappa = 1.25$ [182]

6.2.3 Source problem

Consider the central symmetric time-fractional diffusion-wave equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \delta(t)\begin{cases} q_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \tag{6.56}$$

under zero initial conditions

$$t = 0: \quad T = 0, \quad 0 < \alpha \le 2, \tag{6.57}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \le 2. \tag{6.58}$$

The solution [182]:

$$T = \frac{2q_0 t^{\alpha-1}}{\pi r}\int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\left[\sin(R\xi) - R\xi\cos(R\xi)\right]\frac{\sin(r\xi)}{\xi^2}\,\mathrm{d}\xi \tag{6.59}$$

is presented in Figs. 6.17 and 6.18 with $\overline{T} = T/(q_0 t^{\alpha-1})$.

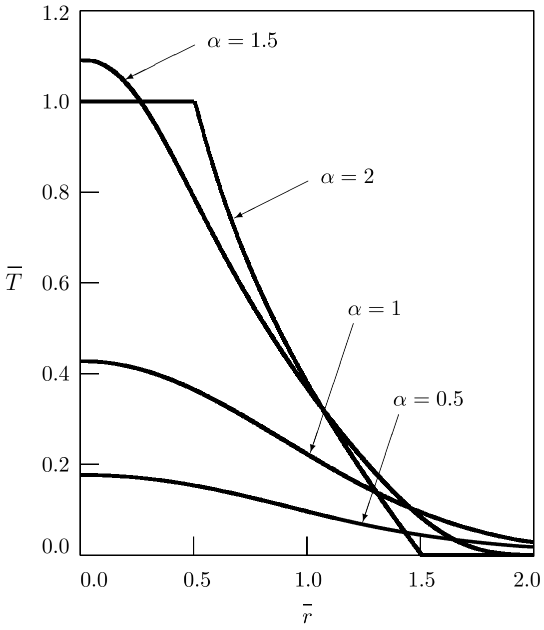

Figure 6.17: Evolution of the initial box-signal in the central symmetric case (the source problem); $\kappa = 0.5$ [182]

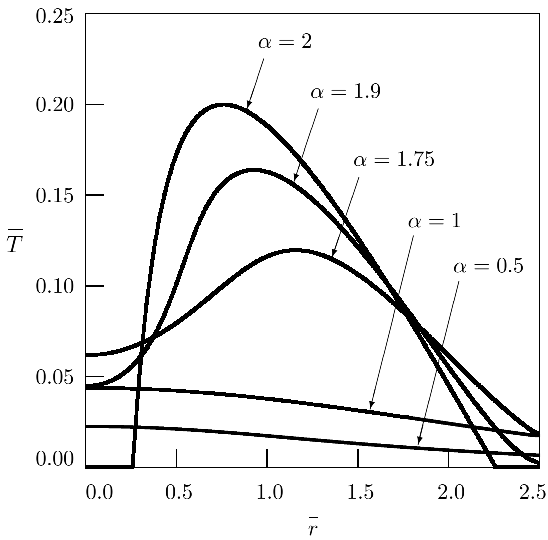

Figure 6.18: Evolution of the initial box-signal in the central symmetric case (the source problem); $\kappa = 1.25$ [182]

6.3 Domain $0 \le r < R$

6.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{6.60}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{6.61}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{6.62}$$

$$r = R: \quad T = g(t). \tag{6.63}$$

The solution:

$$T(r,t) = \int_0^R f(\rho)\, \mathcal{G}_f(r,\rho,t)\, \rho^2\, \mathrm{d}\rho + \int_0^R F(\rho)\, \mathcal{G}_F(r,\rho,t)\, \rho^2\, \mathrm{d}\rho$$

$$+ \int_0^t \int_0^R \Phi(\rho,\tau)\, \mathcal{G}_\Phi(r,\rho,t-\tau)\, \rho^2\, \mathrm{d}\rho\, \mathrm{d}\tau + \int_0^t g(\tau)\, \mathcal{G}_g(r,t-\tau)\, \mathrm{d}\tau. \tag{6.64}$$

The fundamental solutions under zero Dirichlet boundary condition

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{2}{r\rho R} \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix} \sin(r\xi_k)\, \sin(\rho\xi_k), \tag{6.65}$$

are obtained using the Laplace transform with respect to time and the finite sin-Fourier transform for a sphere (2.56) with respect to the radial coordinate. Here

$$\xi_k = \frac{k\pi}{R}. \tag{6.66}$$

The fundamental solution to the Dirichlet problem under zero initial condition is expressed as [152]

$$\mathcal{G}_g(r,t) = \frac{2ag_0 t^{\alpha-1}}{r} \sum_{k=1}^{\infty} (-1)^{k+1} \xi_k\, E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \sin(r\xi_k) \tag{6.67}$$

with the nondimensional quantities used in numerical calculations

$$\overline{\mathcal{G}}_g = \frac{t}{g_0}\, \mathcal{G}_g, \qquad \bar{r} = \frac{r}{R}, \qquad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}, \qquad \eta_k = k\pi. \tag{6.68}$$

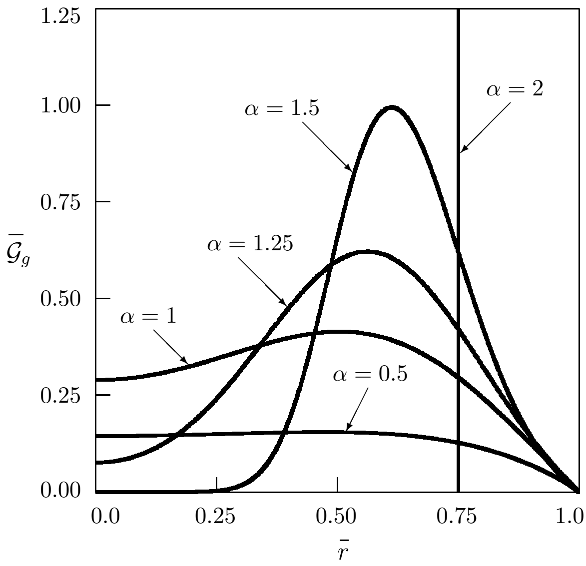

Figure 6.19: The fundamental solution to the Dirichlet problem in a sphere; $\kappa = 0.25$ [152]

Classical diffusion equation ($\alpha = 1$)

$$\overline{\mathcal{G}}_g = \frac{2\kappa^2}{\bar{r}} \sum_{k=1}^{\infty} (-1)^{k+1} \eta_k \, \exp(-\kappa^2 \eta_k^2) \sin(\bar{r}\eta_k). \tag{6.69}$$

Wave equation ($\alpha = 2$)

$$\overline{\mathcal{G}}_g = \frac{\kappa}{1-\kappa} \, \delta(\bar{r} - 1 + \kappa), \qquad 0 < \kappa < 1. \tag{6.70}$$

The results of numerical calculations for the fundamental equation to the Dirichlet problem $\overline{\mathcal{G}}_g$ (6.67) are presented in Fig. 6.19 for $\kappa = 0.25$. The vertical line at $\rho = 0.75$ corresponds to Dirac's delta solution (6.70) of the wave equation. Several problems for a sphere under Dirichlet boundary condition were solved in [152].

Delta-type instantaneous source at the origin

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) + q_0 \frac{\delta(r)}{4\pi r^2} \, \delta(t), \tag{6.71}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{6.72}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.73}$$

$$r = R: \quad T = 0. \tag{6.74}$$

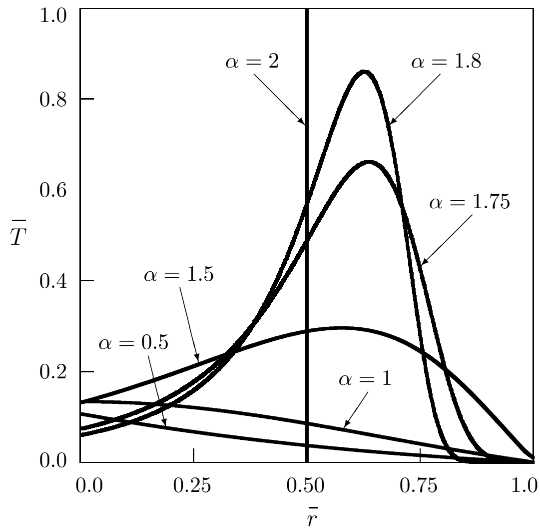

Figure 6.20: Dependence of the solution on distance (the delta-type instantaneous source at the origin; $\kappa = 0.5$) [152]

The solution [152]:

$$T = \frac{q_0 t^{\alpha-1}}{2\pi R r} \sum_{k=1}^{\infty} \xi_k \, E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \sin(\bar{r}\xi_k) \tag{6.75}$$

and $\overline{T} = R^3 t^{1-\alpha} T/q_0$.

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{2\pi\bar{r}} \sum_{k=1}^{\infty} \eta_k \, \exp(-\kappa^2\eta_k^2) \sin(\bar{r}\eta_k). \tag{6.76}$$

Wave equation ($\alpha = 2$)

$$\overline{T} = \frac{1}{4\pi\kappa^2} \delta(\rho - \kappa), \qquad 0 < \kappa < 1. \tag{6.77}$$

The curves calculated for the solution (6.75) are depicted in Fig. 6.20 for various values of α. The vertical line represents Dirac's delta solution (6.77) to the wave equation.

Constant source strength. In this case the diffusion-wave equation with the constant source strength Q_0

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + Q_0 \tag{6.78}$$

is considered under zero initial and boundary conditions. The solution has the form [152]:

$$T = \frac{Q_0}{a} \left[\frac{1}{6} \left(R^2 - r^2 \right) + \frac{2}{r} \sum_{k=1}^{\infty} (-1)^k E_\alpha(-a\xi_k t^\alpha) \frac{\sin(r\xi_k)}{\xi_k^3} \right] \tag{6.79}$$

with the nondimensional quantity $\overline{T} = aT/(Q_0 R^2)$.

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \frac{1 - \bar{r}^2}{6} + \frac{2}{\bar{r}} \sum_{k=1}^{\infty} (-1)^k \frac{1}{1 + \kappa^2 \eta_k^2} \frac{\sin(\bar{r}\eta_k)}{\eta_k^3}. \tag{6.80}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1 - \bar{r}^2}{6} + \frac{2}{\bar{r}} \sum_{k=1}^{\infty} (-1)^k \exp(-\kappa^2 \eta_k^2) \frac{\sin(\bar{r}\eta_k)}{\eta_k^3}. \tag{6.81}$$

The solution (6.81) can be found, for example, in [98].

Wave equation ($\alpha = 2$)

For $0 < \kappa < 2$ the solution to the wave equation is rearranged as

$$\overline{T} = \begin{cases} \kappa^2/2, & 0 \le \bar{r} < |1 - \kappa|, \\ \dfrac{1 - \bar{r}}{2\bar{r}} \left[\bar{r} - (1 - \kappa)^2 \right], & |1 - \kappa| < \bar{r} \le 1. \end{cases} \tag{6.82}$$

The results of numerical calculations of the solution (6.79) with typical values of the parameter κ are displayed in Figs. 6.21 and 6.22.

Constant boundary value of a function

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right), \tag{6.83}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{6.84}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.85}$$

$$r = R: \quad T = T_0. \tag{6.86}$$

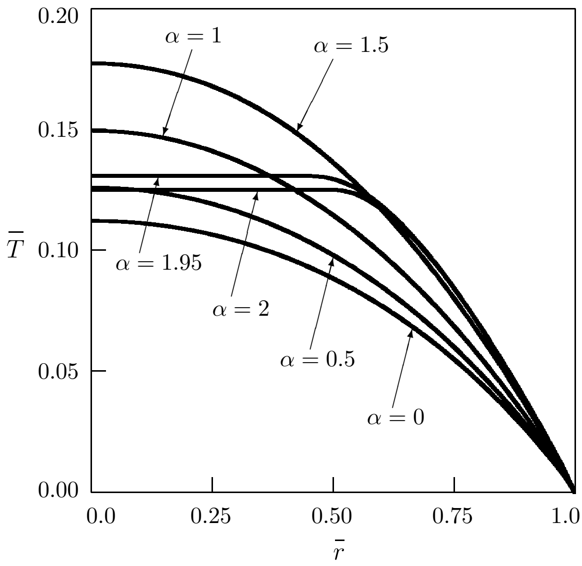

Figure 6.21: The solution to the diffusion-wave equation with constant source strength in a sphere under zero Dirichlet boundary condition; $\kappa = 0.5$ [152]

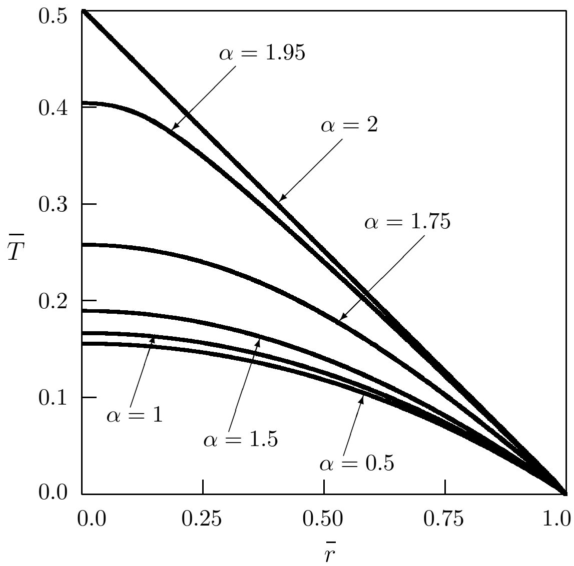

Figure 6.22: The solution to the diffusion-wave equation with constant source strength in a sphere under zero Dirichlet boundary condition; $\kappa = 1$ [152]

The solution assumes the form [152]:

$$T = T_0 + 2T_0 \sum_{k=1}^{\infty} (-1)^k E_\alpha(-a\xi^2 t^\alpha) \frac{\sin(r\xi_k)}{r\xi_k}. \tag{6.87}$$

Let us consider several particular cases.

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \frac{T}{T_0} = 1 + 2\sum_{k=1}^{\infty} (-1)^k \frac{1}{1+\kappa^2\eta_k^2} \frac{\sin(\bar{r}\eta_k)}{\bar{r}\eta_k}. \tag{6.88}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = 1 + 2\sum_{k=1}^{\infty} (-1)^k \exp(-\kappa^2\eta_k^2) \frac{\sin(\bar{r}\eta_k)}{\bar{r}\eta_k}. \tag{6.89}$$

The solution (6.89) is presented in [26].

Wave equation ($\alpha = 2$)

$$\overline{T} = \begin{cases} 0, & 0 < \bar{r} < |1-\kappa|, \\ \dfrac{1}{\bar{r}}, & |1-\kappa| < \bar{r} < 1, \end{cases} \qquad 0 < \kappa < 2. \tag{6.90}$$

Figures 6.23–6.25 show the variation of the solution with distance for typical values of the parameter κ. Figure 6.23 describes the situation typical for $0 < \kappa < 1$, Fig. 6.24 for $\kappa = 1$, and Fig. 6.25 for $1 < \kappa < 2$.

Examples of Cauchy problems. As an example of the first Cauchy problem we consider the time-fractional diffusion-wave equation with the initial conditions

$$t = 0: \quad T = T_0 \frac{R}{r}, \qquad 0 < \alpha \le 2, \tag{6.91}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.92}$$

and zero Dirichlet boundary condition.

The solution takes the form [152]

$$T = 2T_0 \sum_{k=1}^{\infty} [(-1)^{k+1} + 1] E_\alpha(-a\xi_k^2 t^\alpha) \frac{\sin(r\xi_k)}{r\xi_k} \tag{6.93}$$

with the following particular cases:

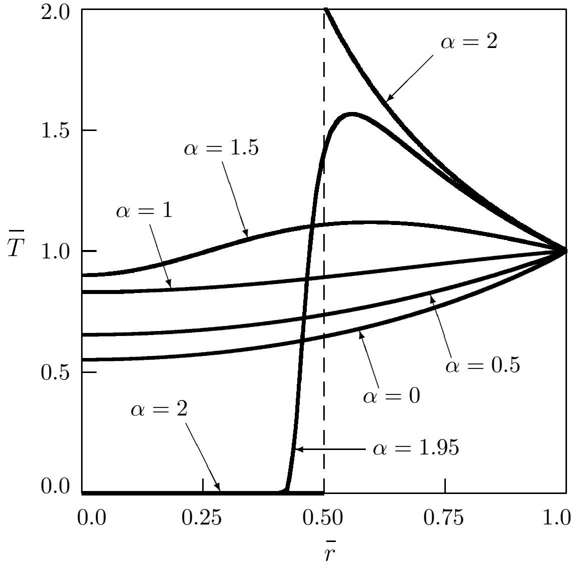

Figure 6.23: The solution to the Dirichlet problem in a sphere with constant boundary value of a function; $\kappa = 0.5$ [152]

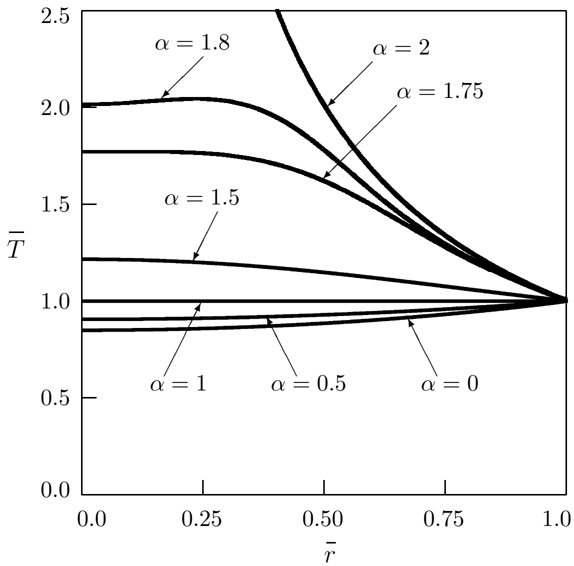

Figure 6.24: The solution to the Dirichlet problem in a sphere with constant boundary value of a function; $\kappa = 1$ [152]

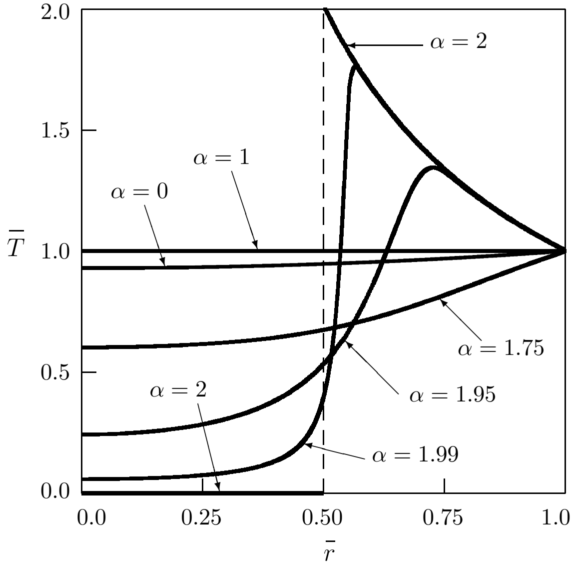

Figure 6.25: The solution to the Dirichlet problem in a sphere with constant boundary value of a function; $\kappa = 1.5$ [152]

Helmholtz equation ($\alpha \to 0$)

$$\bar{T} = \frac{T}{T_0} = 2\sum_{k=1}^{\infty}[(-1)^{k+1}+1]\frac{1}{1+\kappa^2\eta_k^2}\frac{\sin(\bar{r}\eta_k)}{\bar{r}\eta_k}. \tag{6.94}$$

Classical diffusion equation ($\alpha = 1$)

$$\bar{T} = 2\sum_{k=1}^{\infty}[(-1)^{k+1}+1]\exp(-\kappa^2\eta_k^2)\frac{\sin(\bar{r}\eta_k)}{\bar{r}\eta_k}. \tag{6.95}$$

Wave equation ($\alpha = 2$)

$$\bar{T} = \begin{cases} 0, & 0 \le \bar{r} < \kappa, \\ \dfrac{1}{r}, & \kappa \le \bar{r} < 1-\kappa, \\ 0, & 1-\kappa < \bar{r} < 1, \end{cases} \tag{6.96}$$

for $0 < \kappa < 0.5$.

The curves calculated according to (6.93) are presented in Fig. 6.26.

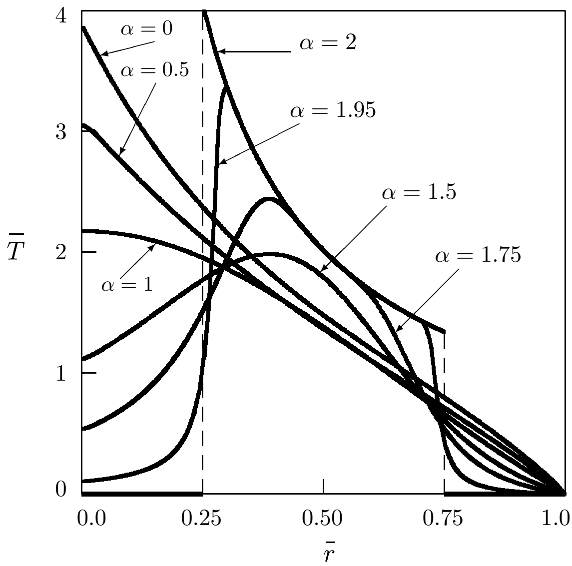

Figure 6.26: The solution (6.93) to the first Cauchy problem in a sphere under zero Dirichlet boundary condition; $\kappa = 0.25$ [152]

As an example of the second Cauchy problem we consider the diffusion-wave equation with the initial conditions

$$t = 0: \quad T = 0, \qquad 1 < \alpha \le 2, \tag{6.97}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = w_0 = \text{const}, \qquad 1 < \alpha \le 2, \tag{6.98}$$

and zero Dirichlet boundary condition.

The solution reads [152]:

$$T = 2w_0 t \sum_{k=1}^{\infty} (-1)^{k+1} E_{\alpha,2}(-a\xi^2 t^\alpha) \frac{\sin(r\xi_k)}{r\xi_k} \tag{6.99}$$

with the particular case $\alpha = 2$ describing the solution to the wave equation (with $0 < \kappa < 1$)

$$\overline{T} = \frac{T}{w_0 t} = \begin{cases} 1, & 0 \le \bar{r} < 1 - \kappa, \\ \dfrac{(1-\kappa)(1-\bar{r})}{\kappa \bar{r}}, & 1 - \kappa < \bar{r} \le 1. \end{cases} \tag{6.100}$$

The typical numerical results calculated from Eq. (6.99) are displayed in Fig. 6.27.

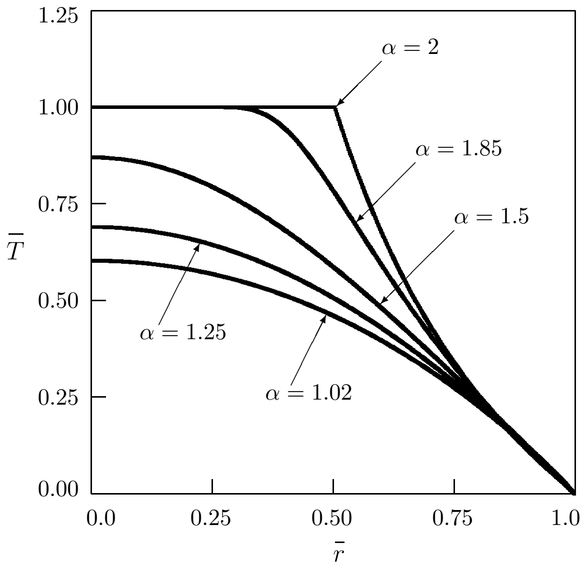

Figure 6.27: The solution to the second Cauchy problem in a sphere with constant initial value of time-derivative of a function; $\kappa = 0.5$ [152]

6.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) + \Phi(r,t), \tag{6.101}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{6.102}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{6.103}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(t). \tag{6.104}$$

The solution:

$$T(r,t) = \int_0^R f(\rho)\, \mathcal{G}_f(r,\rho,t)\, \rho^2 \,\mathrm{d}\rho + \int_0^R F(\rho)\, \mathcal{G}_F(r,\rho,t)\, \rho^2 \,\mathrm{d}\rho$$

$$+ \int_0^t \int_0^R \Phi(\rho,\tau)\, \mathcal{G}_\Phi(r,\rho,t-\tau)\, \rho^2 \,\mathrm{d}\rho \,\mathrm{d}\tau + \int_0^t g(\tau)\, \mathcal{G}_g(r,t-\tau)\, \mathrm{d}\tau. \tag{6.105}$$

The fundamental solutions under zero Neumann boundary condition have the form

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{3}{R^3} \begin{pmatrix} p_0 \\ w_0 t \\ q_0 t^{\alpha-1}/\Gamma(\alpha) \end{pmatrix}$$

$$+\frac{2}{Rr\rho} \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix} \frac{\sin(r\xi_k)\, \sin(\rho\xi_k)}{\sin^2(R\xi_k)}, \tag{6.106}$$

with sum over all positive roots of the transcendental equation

$$\tan(R\xi_k) = R\xi_k. \tag{6.107}$$

The solutions are obtained using the Laplace transform with respect to time t and the finite sin-Fourier transform for a sphere (2.60) with respect to the radial coordinate r. Several problems for a sphere under the Neumann boundary condition were considered in [173].

Fundamental solution to the mathematical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_m}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_m}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{G}_m}{\partial r}\right), \tag{6.108}$$

$$t = 0: \quad \mathcal{G}_m = 0, \qquad 0 < \alpha \le 2, \tag{6.109}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_m}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.110}$$

$$r = R: \quad \frac{\partial \mathcal{G}_m}{\partial r} = g_0\, \delta(t). \tag{6.111}$$

The solution

$$\mathcal{G}_m(r,t) = \frac{3 a g_0 t^{\alpha-1}}{R\Gamma(\alpha)} + \frac{2 a g_0 t^{\alpha-1}}{r} \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_k^2 t^\alpha\right) \frac{\sin(r\xi_k)}{\sin(R\xi_k)} \tag{6.112}$$

is shown in Figs. 6.28–6.30, where $\overline{\mathcal{G}}_m = Rt^{1-\alpha}\mathcal{G}_m/(ag_0)$.

Constant boundary value of the normal derivative. In this case the diffusion-wave equation (6.108) is solved under zero initial conditions (6.109) and (6.110) and constant boundary value of the normal derivative

$$r = R: \quad \frac{\partial T}{\partial r} = g_0 = \text{const.} \tag{6.113}$$

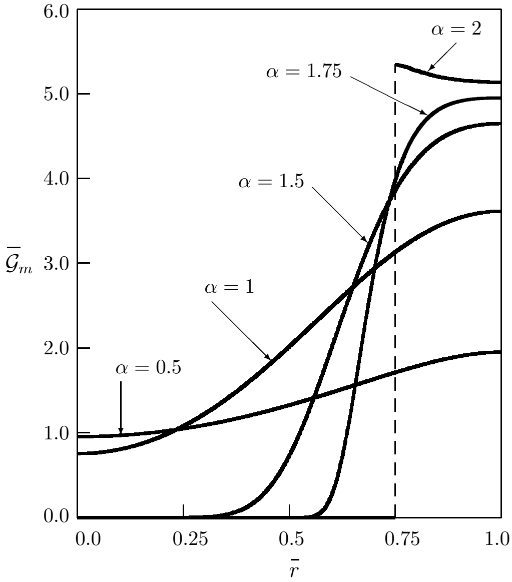

Figure 6.28: The fundamental solution to the mathematical Neumann problem for a sphere; $\kappa = 0.25$ [173]

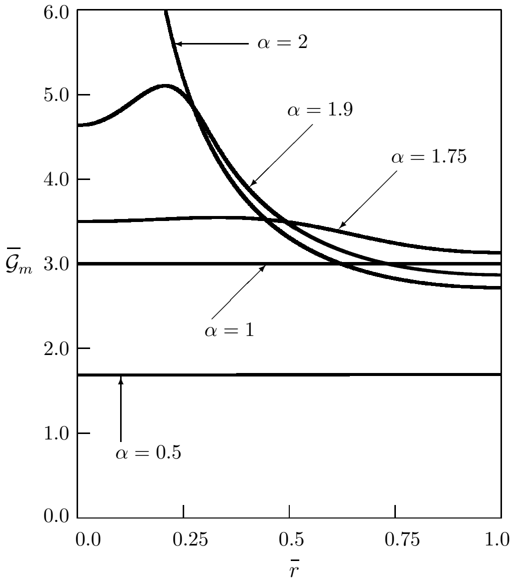

Figure 6.29: The fundamental solution to the mathematical Neumann problem for a sphere; $\kappa = 1$ [173]

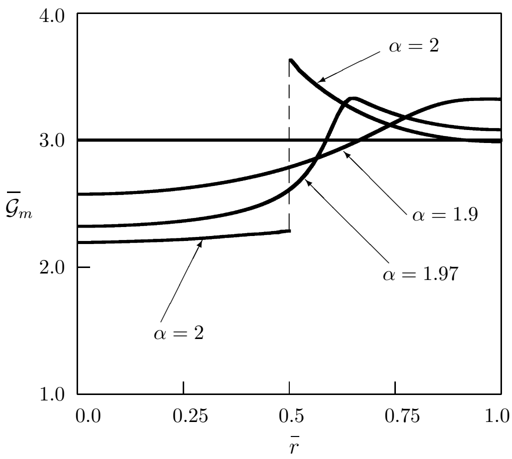

Figure 6.30: The fundamental solution to the mathematical Neumann problem for a sphere; $\kappa = 1.5$ [173]

The solution [173]:

$$T = \frac{3ag_0 t^\alpha}{R\Gamma(1+\alpha)} + \frac{2g_0}{r}\sum_{k=1}^{\infty}\frac{\sin(r\xi_k)}{\xi_k^2\sin(R\xi_k)} - \frac{2g_0}{rR^2}\sum_{k=1}^{\infty} E_\alpha\left(-a\xi_k^2 t^\alpha\right)\frac{1+R^2\xi_k^2}{\xi_k^4}\sin(r\xi_k)\sin(R\xi_k). \tag{6.114}$$

Taking into account that

$$\frac{2}{r}\sum_{k=1}^{\infty}\frac{\sin(r\xi_k)}{\xi_k^2\sin(R\xi_k)} = \frac{1}{10R}\left(5r^2 - 3R^2\right), \tag{6.115}$$

we get

$$T = \frac{3ag_0 t^\alpha}{R\Gamma(1+\alpha)} + \frac{g_0}{10R}(5r^2 - 3R^2) - \frac{2g_0}{r}\sum_{k=1}^{\infty} E_\alpha\left(-a\xi_k^2 t^\alpha\right)\frac{\sin(r\xi_k)}{\xi_k^2\sin(R\xi_k)}. \tag{6.116}$$

In the case of classical diffusion equation ($\alpha = 1$) from (6.116) we obtain [98, 144]

$$T = \frac{3ag_0 t}{R} + \frac{g_0}{10R}(5r^2 - 3R^2) - \frac{2g_0}{r}\sum_{k=1}^{\infty}\exp\left(-a\xi_k^2 t\right)\frac{\sin(r\xi_k)}{\xi_k^2\sin(R\xi_k)}. \tag{6.117}$$

The results of numerical calculations of the solution (6.116) are presented in Fig. 6.31 and Fig. 6.32 with $\overline{T} = T/(g_0 R)$.

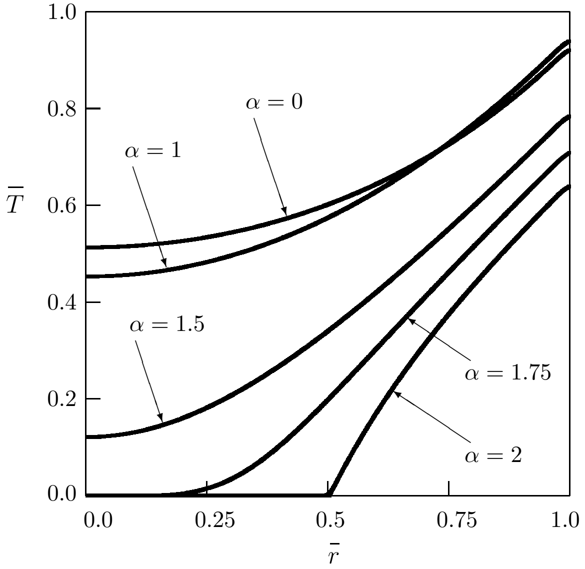

Figure 6.31: Dependence of temperature in a sphere on distance (the constant normal derivative of temperature at the boundary; $\kappa = 0.5$ [173]

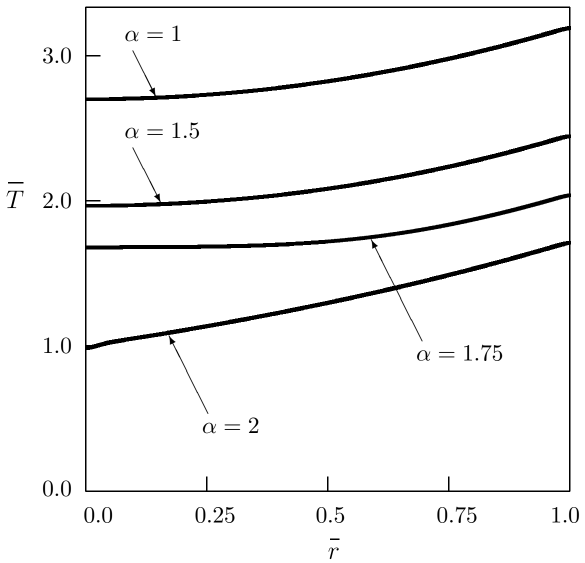

Figure 6.32: Dependence of temperature in a sphere on distance (the constant normal derivative of temperature at the boundary; $\kappa = 1$ [173]

Fundamental solution to the physical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_p}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_p}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{G}_p}{\partial r}\right), \tag{6.118}$$

$$t = 0: \quad \mathcal{G}_p = 0, \qquad 0 < \alpha \le 2, \tag{6.119}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_p}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.120}$$

$$r = R: \quad D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial r} = g_0\,\delta(t), \qquad 0 < \alpha \le 1, \tag{6.121}$$

$$r = R: \quad I^{\alpha-1}\frac{\partial \mathcal{G}_p}{\partial r} = g_0\,\delta(t), \qquad 1 < \alpha \le 2. \tag{6.122}$$

The solution:

$$\mathcal{G}_p(r,t) = \frac{3ag_0}{R} + \frac{2ag_0}{r}\sum_{k=1}^{\infty} E_\alpha\left(-a\xi_k^2 t^\alpha\right)\frac{\sin(r\xi_k)}{\sin(R\xi_k)}. \tag{6.123}$$

The solution (6.123) is shown in Fig. 6.33 with $\overline{\mathcal{G}}_p = R\mathcal{G}_p/(ag_0)$.

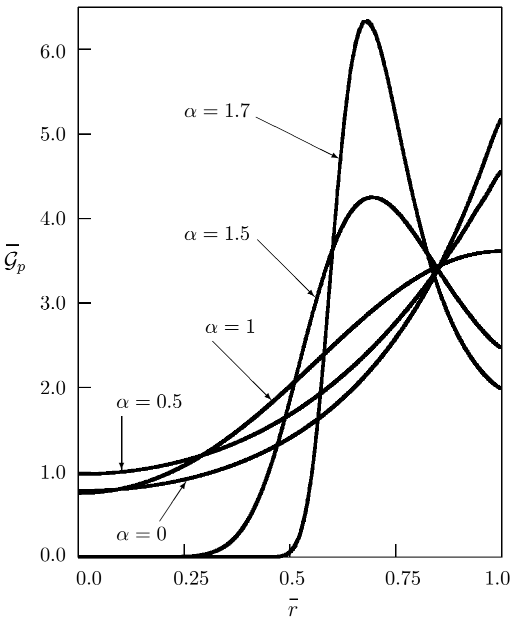

Figure 6.33: The fundamental solution to the physical Neumann problem for a sphere; $\kappa = 0.25$ [173]

Constant heat flux at the boundary

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right), \tag{6.124}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{6.125}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.126}$$

$$r = R: \quad D_{RL}^{1-\alpha}\frac{\partial T}{\partial r} = g_0, \quad 0 < \alpha \le 1, \tag{6.127}$$

$$r = R: \quad I^{\alpha-1}\frac{\partial T}{\partial r} = g_0, \quad 1 < \alpha \le 2. \tag{6.128}$$

The solution:

$$T = \frac{3ag_0 t}{R} + \frac{2ag_0 t}{r}\sum_{k=1}^{\infty} E_{\alpha,2}\left(-a\xi_k^2 t^\alpha\right)\frac{\sin(r\xi_k)}{\sin(R\xi_k)} \tag{6.129}$$

is presented in Fig. 6.34–6.36 with $\overline{T} = t^{\alpha-1}T/(g_0 R)$.

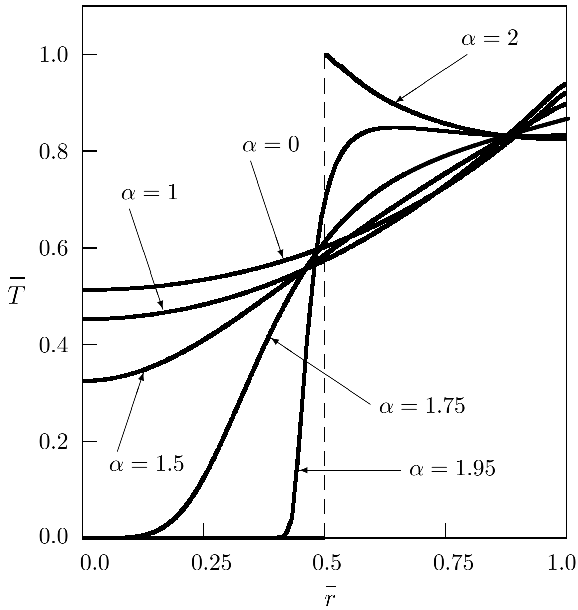

Figure 6.34: Dependence of temperature on distance (the constant heat flux at the boundary of a sphere; $\kappa = 0.5$ [173]

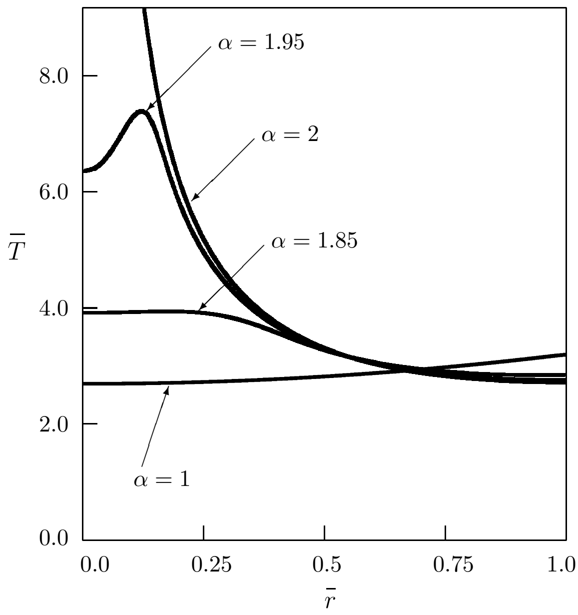

Figure 6.35: Dependence of temperature on distance (the constant heat flux at the boundary of a sphere; $\kappa = 1$ [173]

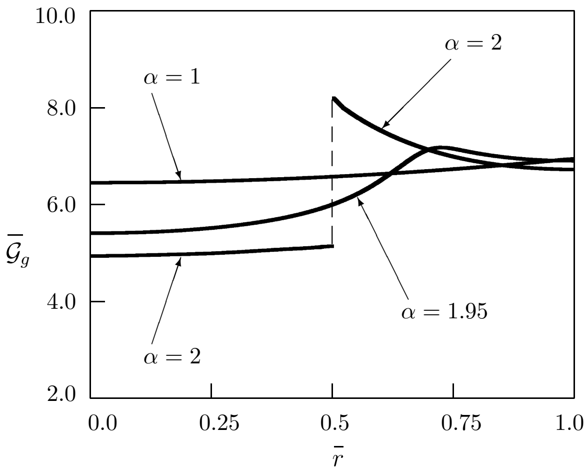

Figure 6.36: Dependence of temperature on distance (the constant heat flux at the boundary of a sphere; $\kappa = 1.5$ [173]

6.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{6.130}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{6.131}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{6.132}$$

$$r = R: \quad HT + \frac{\partial T}{\partial r} = g(t). \tag{6.133}$$

The solution:

$$T(r,t) = \int_0^R f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho^2\,\mathrm{d}\rho + \int_0^R F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho^2\,\mathrm{d}\rho$$
$$+ \int_0^t \int_0^R \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{6.134}$$

The fundamental solutions under zero mathematical Robin boundary condition have the following form

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{2}{Rr\rho}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi_k^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_k^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \end{pmatrix}$$
$$\times \frac{R\xi_k \sin(r\xi_k)\,\sin(\rho\xi_k)}{R\xi_k - \sin(R\xi_k)\,\cos(R\xi_k)}, \tag{6.135}$$

with sum over all positive roots of the transcendental equation

$$\tan(R\xi_k) = \frac{R\xi_k}{1 - RH}. \tag{6.136}$$

The solutions are obtained using the Laplace transform with respect to time and the finite sin-Fourier transform for a sphere (2.64) with respect to the radial coordinate.

Fundamental solution to the mathematical Robin problem

$$\frac{\partial^\alpha \mathcal{G}_m}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_m}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{G}_m}{\partial r}\right), \tag{6.137}$$

$$t = 0: \quad \mathcal{G}_m = 0, \qquad 0 < \alpha \le 2, \tag{6.138}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_m}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.139}$$

$$r = R: \quad H\mathcal{G}_m + \frac{\partial \mathcal{G}_m}{\partial r} = g_0\, \delta(t). \tag{6.140}$$

As usually, the boundedness condition at the origin is also assumed:

$$\lim_{r \to 0} \mathcal{G}_m(r,t) \neq \infty. \tag{6.141}$$

The solution is expressed as [189]

$$\mathcal{G}_m(r,t) = \frac{2aRg_0 t^{\alpha-1}}{r} \sum_{k=1}^{\infty} E_{\alpha,\alpha}(-a\xi_k^2 t^\alpha) \frac{\xi_k \, \sin(R\xi_k)\, \sin(r\xi_k)}{R\xi_k - \sin(R\xi_k)\cos(R\xi_k)} \tag{6.142}$$

and can be rewritten in the dimensionless form with

$$\overline{\mathcal{G}}_m = \frac{R}{ag_0 t^{\alpha-1}} \mathcal{G}_m, \quad \overline{H} = RH. \tag{6.143}$$

The results of numerical calculation of the solution (6.142) are presented in Fig. 6.37 for various values of the order of fractional derivative α.

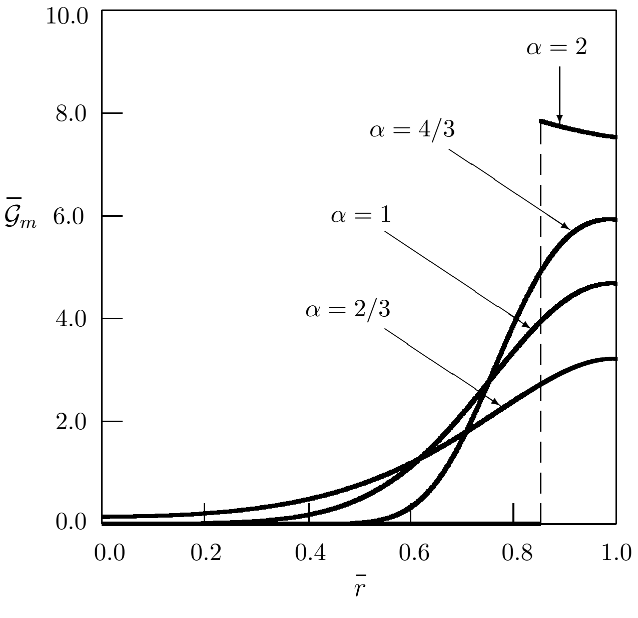

Figure 6.37: Dependence of the fundamental solution to the mathematical Robin problem for a sphere on the radial coordinate for $\overline{H} = 0.2$, $\kappa = 0.15$ [189]

Now we will investigate the approximate solution of the considered problem for small values of time t. In the case of classical heat conduction, this method was described in [98]. Applying to (6.137)–(6.140) the Laplace transform with respect to time, we obtain the following equation

$$\frac{\mathrm{d}^2\mathcal{G}_m^*}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}\mathcal{G}_m^*}{\mathrm{d}r} - \frac{s^\alpha}{a}\,\mathcal{G}_m^* = 0 \tag{6.144}$$

and the boundary condition

$$r = R: \quad H\mathcal{G}_m^* + \frac{\mathrm{d}\mathcal{G}_m^*}{\mathrm{d}r} = g_0. \tag{6.145}$$

Equation (6.144) has the solution

$$\mathcal{G}_m^*(r,s) = \frac{A}{r}\sinh\left(\sqrt{\frac{s^\alpha}{a}}r\right) + \frac{B}{r}\cosh\left(\sqrt{\frac{s^\alpha}{a}}r\right), \tag{6.146}$$

where A and B are the constants of integration. The boundedness condition at the origin (6.141) implies that $B = 0$, and from the mathematical Robin boundary condition (6.145) we get

$$A = R^2 g_0\left[(RH-1)\sinh\left(\sqrt{\frac{s^\alpha}{a}}R\right) + R\sqrt{\frac{s^\alpha}{a}}\cosh\left(\sqrt{\frac{s^\alpha}{a}}R\right)\right]^{-1}.$$

Hence, the solution (6.146) is rewritten as

$$\mathcal{G}_m^*(r,s) = \frac{R^2 g_0}{r}\left\{\exp\left[-\frac{s^{\alpha/2}(R-r)}{\sqrt{a}}\right] - \exp\left[-\frac{s^{\alpha/2}(R+r)}{\sqrt{a}}\right]\right\}$$
$$\times\left[(RH-1)\left(1-\mathrm{e}^{-2s^{\alpha/2}R/\sqrt{a}}\right) + \frac{R}{\sqrt{a}}\,s^{\alpha/2}\left(1+\mathrm{e}^{-2s^{\alpha/2}R/\sqrt{a}}\right)\right]^{-1}.$$

Based on Tauberian theorems for the Laplace transform (see, for example [34]), for small values of time t (for large values of the transform variable s) we can neglect the exponential term in comparison with 1:

$$1 \pm \mathrm{e}^{-2s^{\alpha/2}R/\sqrt{a}} \approx 1, \quad \alpha > 0,$$

and obtain

$$\mathcal{G}_m^*(r,s) \simeq \frac{R^2 g_0\sqrt{a}}{r}\left\{\exp\left[-\frac{s^{\alpha/2}(R-r)}{\sqrt{a}}\right] - \exp\left[-\frac{s^{\alpha/2}(R+r)}{\sqrt{a}}\right]\right\}$$
$$\times\,\frac{1}{\sqrt{a}(RH-1) + Rs^{\alpha/2}}. \tag{6.147}$$

Inversion of the Laplace transform taking into account the convolution theorem gives the approximate solution in terms of the Mittag-Leffler function and the Mainardi function:

$$\mathcal{G}_m(r,t) \simeq \frac{\alpha R g_0}{2r} \int\limits_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2+1}} E_{\alpha/2\,,\,\alpha/2}\left[-\frac{\sqrt{a}}{R}(RH-1)(t-\tau)^{\alpha/2}\right]$$
$$\times \left[(R-r)M\left(\frac{\alpha}{2};\frac{R-r}{\sqrt{a}\tau^{\alpha/2}}\right) - (R+r)M\left(\frac{\alpha}{2};\frac{R+r}{\sqrt{a}\tau^{\alpha/2}}\right)\right]\mathrm{d}\tau \tag{6.148}$$

or in terms of nondimensional quantities

$$\overline{\mathcal{G}}_m \simeq \frac{\alpha}{2\kappa^2\bar{r}} \int\limits_0^1 \frac{(1-u)^{\alpha/2-1}}{u^{\alpha/2+1}} E_{\alpha/2\,,\,\alpha/2}\left[-\kappa(\overline{H}-1)(1-u)^{\alpha/2}\right]$$
$$\times \left[(1-\bar{r})M\left(\frac{\alpha}{2};\frac{1-\bar{r}}{\kappa u^{\alpha/2}}\right) - (1+\bar{r})M\left(\frac{\alpha}{2};\frac{1+\bar{r}}{\kappa u^{\alpha/2}}\right)\right]\mathrm{d}u. \tag{6.149}$$

It should be emphasized that the integrand in (6.149) has no singularity at $u = 0$ due to exponential decay of the Mainardi function $M(\alpha; x)$ for $x \to \infty$ [100, 101].

Classical diffusion equation ($\alpha = 1$)

Taking into account the following formula for the inverse Laplace transform [44] (with $\gamma > 0$)

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}+\beta}\,\mathrm{e}^{-\gamma\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\gamma^2}{4t}\right) - \beta\mathrm{e}^{\beta\gamma+\beta^2 t}\mathrm{erfc}\left(\frac{\gamma}{2\sqrt{t}}+\beta\sqrt{t}\right), \tag{6.150}$$

we obtain

$$\overline{\mathcal{G}}_m \simeq \frac{1}{\bar{r}}\Bigg\{\frac{1}{\sqrt{\pi}\kappa}\exp\left[-\frac{(1-\bar{r})^2}{4\kappa^2}\right] - \frac{1}{\sqrt{\pi}\kappa}\exp\left[-\frac{(1+\bar{r})^2}{4\kappa^2}\right]$$
$$- (\overline{H}-1)\exp\left[(\overline{H}-1)(1-\bar{r}) + (\overline{H}-1)^2\kappa^2\right]\mathrm{erfc}\left[\frac{1-\bar{r}}{2\kappa}+(\overline{H}-1)\kappa\right]$$
$$+ (\overline{H}-1)\exp\left[(\overline{H}-1)(1+\bar{r}) + (\overline{H}-1)^2\kappa^2\right]$$
$$\times\ \mathrm{erfc}\left[\frac{1+\bar{r}}{2\kappa}+(\overline{H}-1)\kappa\right]\Bigg\}. \tag{6.151}$$

Wave equation ($\alpha = 2$)

$$\overline{\mathcal{G}}_m = \frac{1}{\kappa\bar{r}}\begin{cases} 0, & 0 \le \bar{r} < 1-\kappa, \\ \mathrm{e}^{(1-\bar{H})(\kappa-1+\bar{r})}, & 1-\kappa < \bar{r} \le 1. \end{cases} \tag{6.152}$$

It should be noted that for $\kappa = 0.15$ the results of numerical calculation based on (6.142) and (6.148) practically coincide.

Fundamental solution to the physical Robin problem

$$\frac{\partial^\alpha \mathcal{G}_p}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_p}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{G}_p}{\partial r},\right), \tag{6.153}$$

$$t = 0: \quad \mathcal{G}_p = 0, \qquad 0 < \alpha \le 2, \tag{6.154}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_p}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.155}$$

$$r = R: \quad H\mathcal{G}_p + D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial r} = g_0\,\delta(t), \qquad 0 < \alpha \le 1, \tag{6.156}$$

$$r = R: \quad H\mathcal{G}_p + I^{\alpha-1}\frac{\partial \mathcal{G}_p}{\partial r} = g_0\,\delta(t), \qquad 1 < \alpha \le 2, \tag{6.157}$$

$$\lim_{r\to 0} \mathcal{G}_p(r,t) \neq \infty. \tag{6.158}$$

Unfortunately, in the case under consideration the finite sin-Fourier transform (2.64) cannot be used to obtain the solution as the eventual transcendental equation for the roots ξ_k (see the transcendental equation (2.66)) would contain the Laplace transform variable s:

$$\tan(R\xi_k) = \frac{R\xi_k}{1 - s^{\alpha-1}RH}.$$

Nevertheless, the approximate solution $\mathcal{G}_p$ similar to solution under the mathematical Robin boundary condition studied above and valid for small values of time t can be obtained for several typical values of the order of fractional derivative α.

In this case, instead of the solution (6.147) we get

$$\mathcal{G}_p^*(r,s) \simeq \frac{R^2 g_0 \sqrt{a} s^{\alpha-1}}{r}\left\{\exp\left[-\frac{s^{\alpha/2}(R-r)}{\sqrt{a}}\right] - \exp\left[-\frac{s^{\alpha/2}(R+r)}{\sqrt{a}}\right]\right\}$$
$$\times\ \frac{1}{\sqrt{a}(RHs^{\alpha-1} - 1) + Rs^{\alpha/2}}. \tag{6.159}$$

Below, the following dimensionless quantities will be used:

$$\overline{\mathcal{G}}_p = \frac{R}{a g_0}\,\mathcal{G}_p, \qquad \overline{H} = R^{-1+2/\alpha} a^{1-1/\alpha} H. \tag{6.160}$$

Particular case $\alpha = 2/3$

For this value of the order of fractional derivative, the solution in the Laplace transform domain reads

$$\mathcal{G}_p^*(r,s) \simeq \frac{R\sqrt{a} g_0 s^{-2/3}}{r}\left\{\exp\left[-\frac{s^{1/3}(R-r)}{\sqrt{a}}\right] - \exp\left[-\frac{s^{1/3}(R+r)}{\sqrt{a}}\right]\right\}$$
$$\times\ \left(1 + \frac{s^{1/3} - RH}{Ra^{-1/2}s^{2/3} - s^{1/3} + RH}\right). \tag{6.161}$$

The denominator in (6.161) can be treated as a quadratic equation, which allows us to obtain the decomposition into the sum of partial fractions:

$$\frac{s^{1/3} - RH}{Ra^{-1/2}s^{2/3} - s^{1/3} + RH} = \frac{C_1}{s^{1/3} - \sigma_1} + \frac{C_2}{s^{1/3} - \sigma_2}$$

where

$$\sigma_{1,2} = \frac{\sqrt{a}}{2R}\left(1 \pm \sqrt{1 - 4\overline{\overline{H}}}\right), \qquad C_{1,2} = \pm\frac{\sqrt{a}\left(1 \pm \sqrt{1 - 4\overline{\overline{H}}}\right)^2}{4R\sqrt{1 - 4\overline{\overline{H}}}}.$$

Inversion of the Laplace transform gives the approximate fundamental solution

$$\mathcal{G}_p(r,t) \simeq \frac{g_0 R\sqrt{a}}{rt^{1/3}}\left[M\left(\frac{1}{3}; \frac{R-r}{\sqrt{a}t^{1/3}}\right) - M\left(\frac{1}{3}; \frac{R+r}{\sqrt{a}t^{1/3}}\right)\right]$$
$$+ \frac{g_0 R\sqrt{a}}{r}\int_0^t \frac{1}{(t-\tau)^{2/3}\,\tau^{1/3}}\left[M\left(\frac{1}{3}; \frac{R-r}{\sqrt{a}\tau^{1/3}}\right) - M\left(\frac{1}{3}; \frac{R+r}{\sqrt{a}\tau^{1/3}}\right)\right]$$
$$\times\left\{C_1 E_{1/3,1/3}\left[\sigma_1 (t-\tau)^{1/3}\right] + C_2 E_{1/3,1/3}\left[\sigma_2 (t-\tau)^{1/3}\right]\right\}\mathrm{d}\tau \tag{6.162}$$

or in the dimensionless form

$$\overline{\mathcal{G}}_p \simeq \frac{1}{\kappa\bar{r}}\left[M\left(\frac{1}{3}; \frac{1-\bar{r}}{\kappa}\right) - M\left(\frac{1}{3}; \frac{1+\bar{r}}{\kappa}\right)\right]$$
$$+ \frac{1}{4\bar{r}\sqrt{1 - 4\overline{\overline{H}}}}\int_0^1 \frac{1}{(1-u)^{2/3}u^{1/3}}\left[M\left(\frac{1}{3}; \frac{1-\bar{r}}{\kappa u^{1/3}}\right) - M\left(\frac{1}{3}; \frac{1+\bar{r}}{\kappa u^{1/3}}\right)\right]$$
$$\times\left\{\left(1 + \sqrt{1 - 4\overline{\overline{H}}}\right)^2 E_{1/3,1/3}\left[\frac{\kappa}{2}\left(1 + \sqrt{1 - 4\overline{\overline{H}}}\right)(1-u)^{1/3}\right]\right.$$
$$\left. - \left(1 - \sqrt{1 - 4\overline{\overline{H}}}\right)^2 E_{1/3,1/3}\left[\frac{\kappa}{2}\left(1 - \sqrt{1 - 4\overline{\overline{H}}}\right)(1-u)^{1/3}\right]\right\}\mathrm{d}u. \tag{6.163}$$

Particular case $\alpha = 4/3$

In this case the solution in the Laplace transform domain reads

$$\mathcal{G}_p^*(r,s) \simeq \frac{R\sqrt{a}g_0 s^{-1/3}}{r}\left\{\exp\left[-\frac{s^{2/3}(R-r)}{\sqrt{a}}\right] - \exp\left[-\frac{s^{2/3}(R+r)}{\sqrt{a}}\right]\right\}$$
$$\times\left(1 - \frac{RHs^{1/3} - 1}{Ra^{-1/2}s^{2/3} + RHs^{1/3} - 1}\right). \tag{6.164}$$

The denominator in (6.164) can be treated as a quadratic equation, which allows us to obtain the decomposition into the sum of partial fractions:

$$\frac{RHs^{1/3}-1}{Ra^{-1/2}s^{2/3}+RHs^{1/3}-1}=\frac{C_3}{s^{1/3}-\sigma_3}+\frac{C_4}{s^{1/3}-\sigma_4}$$

where

$$\sigma_{3,4}=-\frac{a^{1/4}}{2\sqrt{R}}\left(\overline{H}\pm\sqrt{\overline{H}^2+4}\right),\qquad C_{3,4}=\pm\frac{a^{1/4}}{4\sqrt{R}}\frac{\left(\overline{H}\pm\sqrt{\overline{H}^2+4}\right)^2}{\sqrt{\overline{H}^2+4}}.$$

Inversion of the Laplace transform gives the approximate fundamental solution

$$\begin{aligned}\mathcal{G}_p(r,t)\simeq{}&\frac{g_0R\sqrt{a}}{rt^{2/3}}\left[M\left(\frac{2}{3};\frac{R-r}{\sqrt{a}t^{2/3}}\right)-M\left(\frac{2}{3};\frac{R+r}{\sqrt{a}t^{2/3}}\right)\right]\\&-\frac{g_0R\sqrt{a}}{r}\int_0^t\frac{1}{(t-\tau)^{2/3}\,\tau^{2/3}}\left[M\left(\frac{2}{3};\frac{R-r}{\sqrt{a}\tau^{2/3}}\right)-M\left(\frac{2}{3};\frac{R+r}{\sqrt{a}\tau^{2/3}}\right)\right]\\&\times\left\{C_3E_{1/3,1/3}\left[\sigma_3(t-\tau)^{1/3}\right]+C_4\,E_{1/3,1/3}\left[\sigma_4(t-\tau)^{1/3}\right]\right\}\mathrm{d}\tau\end{aligned}\tag{6.165}$$

or in the nondimensional form

$$\begin{aligned}\overline{\mathcal{G}}_p\simeq{}&\frac{1}{\kappa\bar{r}}\left[M\left(\frac{2}{3};\frac{1-\bar{r}}{\kappa}\right)-M\left(\frac{2}{3};\frac{1+\bar{r}}{\kappa}\right)\right]\\&-\frac{1}{4\sqrt{\kappa}\bar{r}\sqrt{\overline{H}^2+4}}\int_0^1\frac{1}{(1-u)^{2/3}\,u^{2/3}}\\&\times\left[M\left(\frac{2}{3};\frac{1-\bar{r}}{\kappa u^{2/3}}\right)-M\left(\frac{2}{3};\frac{1+\bar{r}}{\kappa u^{2/3}}\right)\right]\left\{\left(\overline{H}+\sqrt{\overline{H}^2+4}\right)^2\right.\\&\times E_{1/3,1/3}\left[-\frac{\sqrt{\kappa}}{2}\left(\overline{H}+\sqrt{\overline{H}^2+4}\right)(1-u)^{1/3}\right]-\left(\overline{H}-\sqrt{\overline{H}^2+4}\right)^2\\&\times\left.E_{1/3,1/3}\left[-\frac{\sqrt{\kappa}}{2}\left(\overline{H}-\sqrt{\overline{H}^2+4}\right)(1-u)^{1/3}\right]\right\}\mathrm{d}u.\end{aligned}\tag{6.166}$$

Wave equation ($\alpha = 2$)

In the case of wave equation for temperature, taking into account that

$$\mathcal{L}^{-1}\left\{\mathrm{e}^{-\gamma s}\right\}=\delta(t-\gamma),$$

from (6.159) we get

$$\overline{\mathcal{G}}_p \simeq \frac{1}{(\overline{H}+1)\bar{r}}\,\delta(\kappa - 1 + \bar{r}) + \frac{1}{(\overline{H}+1)^2\bar{r}} \begin{cases} 0, & 0 \le \bar{r} < 1-\kappa, \\ e^{(\kappa-1+\bar{r})/(\overline{H}+1)}, & 1-\kappa < \bar{r} \le 1. \end{cases} \tag{6.167}$$

The results of numerical calculation of the approximate solutions (6.163), (6.166), and (6.167) as well as (6.151) are presented in Fig. 6.38. The vertical line in Fig. 6.38 represents the Dirac delta function.

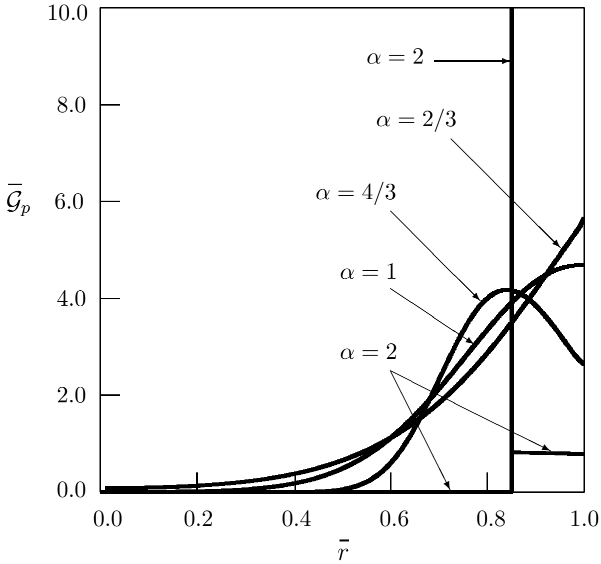

Figure 6.38: Dependence of the fundamental solution to the physical Robin problem for a sphere on radial coordinate for $\overline{H} = 0.2$, $\kappa = 0.15$ [189]

6.4 Domain $R < r < \infty$

6.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{6.168}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{6.169}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{6.170}$$

$$r = R: \quad T = g(t). \tag{6.171}$$

The zero condition at infinity is also assumed

$$\lim_{r\to\infty} T(r,t) = 0. \tag{6.172}$$

The solution:

$$T(r,t) = \int_R^\infty f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho^2\,\mathrm{d}\rho + \int_R^\infty F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho^2\,\mathrm{d}\rho$$
$$+ \int_0^t \int_R^\infty \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{6.173}$$

To obtain the fundamental solutions to the first and second Cauchy problems and to the source problem we introduce the auxiliary function $v = rT$ and the auxiliary variable $x = r - R$. Next, we use the Laplace transform with respect to time t and the sin-Fourier transform (2.25) with respect to the space coordinate x with ξ being the transform variable. Thus, we get

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{2}{\pi r\rho}\int_0^\infty \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}$$
$$\times\,\sin[(r-R)\xi]\,\sin[(\rho-R)\xi]\,\mathrm{d}\xi. \tag{6.174}$$

Similarly, the fundamental solution to the Dirichlet problem is expressed as [153]

$$\mathcal{G}_g(r,t) = \frac{2aRg_0 t^{\alpha-1}}{\pi r}\int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\,\sin[(r-R)\xi]\,\xi\,\mathrm{d}\xi. \tag{6.175}$$

The following nondimensional quantities will be used in calculations:

$$\overline{\mathcal{G}}_g = \frac{t\mathcal{G}_g}{g_0}, \qquad \bar{r} = \frac{r}{R}, \qquad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}. \tag{6.176}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{\mathcal{G}}_g = \frac{\bar{r}-1}{2\sqrt{2}\pi\kappa\bar{r}} \int_0^\infty \frac{1}{\sqrt{u}} \exp\left[-u^2 - \frac{(\bar{r}-1)^2}{8\kappa^2 u}\right] \mathrm{d}u. \tag{6.177}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{\mathcal{G}}_g = \frac{\bar{r}-1}{2\sqrt{\pi}\kappa\bar{r}} \exp\left[-\frac{(\bar{r}-1)^2}{4\kappa^2}\right]. \tag{6.178}$$

Wave equation ($\alpha = 2$)

$$\overline{\mathcal{G}}_g = \frac{\kappa}{1+\kappa}\,\delta(\bar{r}-1-\kappa). \tag{6.179}$$

Plots of the fundamental solution to the Dirichlet problem $\overline{\mathcal{G}}_g$ are shown in Fig. 6.39 for $\kappa = 1$ and various values of α.

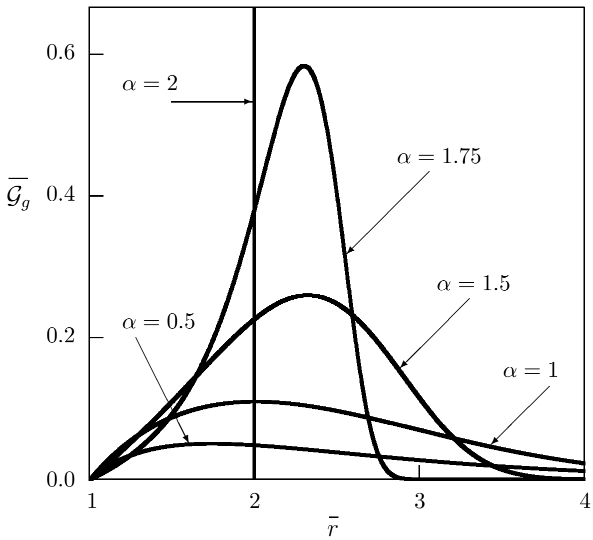

Figure 6.39: Dependence of the fundamental solution to the Dirichlet problem for a body with a spherical hole on radial coordinate; $\kappa = 1$ [153]

Constant boundary value of a function

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right), \tag{6.180}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{6.181}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{6.182}$$

$$r = R: \quad T = g_0. \tag{6.183}$$

The solution [153]:

$$T = \frac{RT_0}{r} - \frac{2RT_0}{\pi r}\int_0^\infty E_\alpha(-a\xi^2 t^\alpha)\,\frac{\sin[(r-R)\xi]}{\xi}\,\mathrm{d}\xi. \tag{6.184}$$

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \frac{T}{T_0} = \frac{1}{\bar{r}}\,\mathrm{e}^{-(\bar{r}-1)/\kappa}. \tag{6.185}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{T} = \frac{2}{\sqrt{\pi}\bar{r}}\int_0^\infty \mathrm{e}^{-u^2}\,\mathrm{erfc}\left(\frac{\bar{r}-1}{2\sqrt{2}\kappa\sqrt{u}}\right)\mathrm{d}u. \tag{6.186}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{\bar{r}}\,\mathrm{erfc}\left(\frac{\bar{r}-1}{2\kappa}\right). \tag{6.187}$$

The solution (6.187) is well known [129, 141, 215].

Wave equation ($\alpha = 2$)

$$\overline{T} = \begin{cases} \dfrac{1}{\bar{r}}, & 1 \le \bar{r} < 1+\kappa, \\ 0, & 1+\kappa < \bar{r} < \infty. \end{cases} \tag{6.188}$$

The solution (6.184) is depicted in Fig. 6.40.

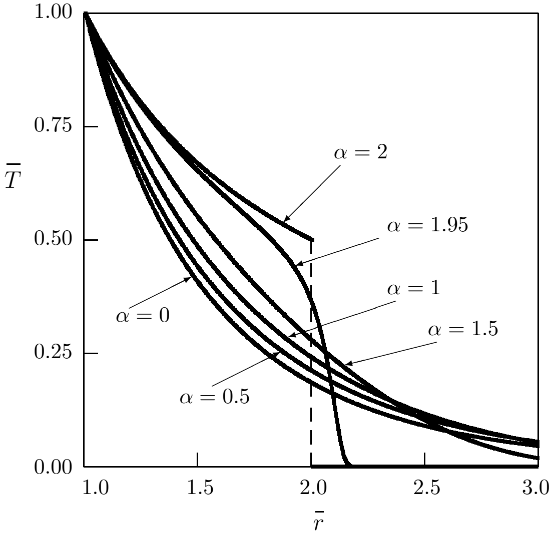

Figure 6.40: The solution to the Dirichlet problem for a body with a spherical hole (constant boundary value of temperature; $\kappa = 1$) [153]

6.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\,\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{6.189}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{6.190}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{6.191}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g(t). \tag{6.192}$$

$$\lim_{r\to\infty} T(r,t) = 0. \tag{6.193}$$

The solution:

$$T(r,t) = \int_R^\infty f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho^2\,\mathrm{d}\rho + \int_R^\infty F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho^2\,\mathrm{d}\rho$$

$$+ \int_0^t \int_R^\infty \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{6.194}$$

Consider the fundamental solution to the first Cauchy problem:

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_f}{\partial r^2} + \frac{2}{r}\,\frac{\partial \mathcal{G}_f}{\partial r}\right), \tag{6.195}$$

$$t = 0: \quad \mathcal{G}_f = p_0\,\frac{\delta(r-\rho)}{r^2}, \qquad 0 < \alpha \le 2, \tag{6.196}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{6.197}$$

$$r = R: \quad \frac{\partial \mathcal{G}_f}{\partial r} = 0. \tag{6.198}$$

Introducing the auxiliary function $v = rT$ and auxiliary spatial variable $x = r - R$ leads to the following initial-boundary-value problem for the function v with the Robin boundary condition

$$\frac{\partial^\alpha v}{\partial t^\alpha} = a\,\frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < \infty, \tag{6.199}$$

$$t = 0: \quad v = p_0\,\frac{\delta(x+R-\rho)}{\rho}, \qquad 0 < \alpha \le 2, \tag{6.200}$$

$$t = 0: \quad \frac{\partial v}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.201}$$

$$x = 0: \quad -\frac{\partial v}{\partial x} + \frac{1}{R} v = 0. \tag{6.202}$$

Using the Laplace transform with respect to time t and the sin-cos-Fourier transform (2.40) with respect to the spatial coordinate x, we obtain

$$\begin{aligned}\mathcal{G}_f(r,\rho,t) \;=\; & \frac{2p_0}{\pi r\rho}\int_0^\infty E_\alpha(-a\xi^2 t^\alpha)\,\frac{R\xi\cos[(r-R)\xi] + \sin[(r-R)\xi]}{R^2\xi^2+1} \\ & \times\,\{R\xi\cos[(\rho-R)\xi] + \sin[(\rho-R)\xi]\}\,\mathrm{d}\xi. \end{aligned} \tag{6.203}$$

Similarly, for the fundamental solutions to the second Cauchy problem and to the source problem we get

$$\begin{aligned}\begin{pmatrix}\mathcal{G}_F(r,\rho,t)\\ \mathcal{G}_\Phi(r,\rho,t)\end{pmatrix} = \frac{2}{\pi r\rho}\int_0^\infty \begin{pmatrix} w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha)\\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\end{pmatrix} \Big\{R\xi\cos[(\rho-R)\xi] \\ + \sin[(\rho-R)\xi]\Big\}\,\frac{R\xi\cos[(r-R)\xi] + \sin[(r-R)\xi]}{R^2\xi^2+1}\,\mathrm{d}\xi. \end{aligned} \tag{6.204}$$

Fundamental solution to the mathematical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_m}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_m}{\partial r^2} + \frac{2}{r}\,\frac{\partial \mathcal{G}_m}{\partial r}\right), \tag{6.205}$$

$$t = 0: \quad \mathcal{G}_m = 0, \qquad 0 < \alpha \le 2, \tag{6.206}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_m}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{6.207}$$

$$r = R: \quad -\frac{\partial \mathcal{G}_m}{\partial r} = g_0\,\delta(t). \tag{6.208}$$

The solution [153]:

$$\mathcal{G}_m(r,t) = \frac{2g_0 a t^{\alpha-1} R^2}{\pi r} \int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)\frac{R\xi\cos[(r-R)\xi] + \sin[(r-R)\xi]}{R^2\xi^2+1}\,\xi\,\mathrm{d}\xi \tag{6.209}$$

with $\overline{\mathcal{G}}_m = t\mathcal{G}_m/(g_0 R)$.

Subdiffusion with $\alpha = 1/2$

$$\begin{aligned}\overline{\mathcal{G}}_m = {} & \frac{2\kappa^2}{\sqrt{\pi}\bar{r}} \int_0^\infty u\mathrm{e}^{-u^2}\left\{\frac{1}{\sqrt{2\pi u}\kappa}\exp\left[-\frac{(\bar{r}-1)^2}{8\kappa^2 u}\right]\right. \\ & \left. - \,\mathrm{e}^{2\kappa^2 u + \bar{r} - 1}\,\mathrm{erfc}\left(\sqrt{2u}\kappa + \frac{\bar{r}-1}{2\sqrt{2u}\kappa}\right)\right\}\mathrm{d}u.\end{aligned} \tag{6.210}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{\mathcal{G}}_m = \frac{\kappa^2}{\bar{r}}\left\{\frac{1}{\sqrt{\pi}\kappa}\exp\left[-\frac{(\bar{r}-1)^2}{4\kappa^2}\right] - \mathrm{e}^{\kappa^2+\bar{r}-1}\,\mathrm{erfc}\left(\kappa + \frac{\bar{r}-1}{2\kappa}\right)\right\}. \tag{6.211}$$

Wave equation ($\alpha = 2$)

$$\overline{\mathcal{G}}_m = \begin{cases} \dfrac{\kappa}{\bar{r}}\,\mathrm{e}^{-(\kappa+1-\bar{r})}, & 1 \le \bar{r} < 1+\kappa, \\[2ex] 0, & 1+\kappa < \bar{r} < \infty, \end{cases} \tag{6.212}$$

The fundamental solution $\overline{\mathcal{G}}_m$ is presented in Fig. 6.41.

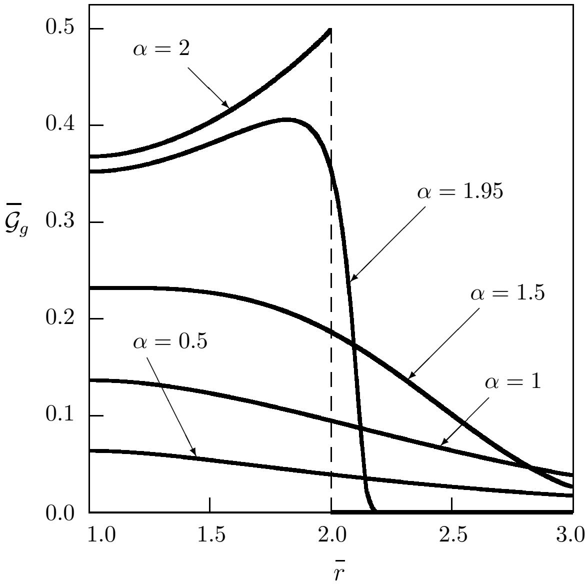

Figure 6.41: The fundamental solution to the mathematical Neumann problem in a body with a spherical hole; $\kappa = 1$) [153]

Constant boundary value of normal derivative. In this case the diffusion-wave equation is studied under zero initial conditions and constant boundary value of normal derivative:

$$r = R: \quad -\frac{\partial T}{\partial r} = g_0 = \text{const.} \tag{6.213}$$

The solution has the following form [153]:

$$T = \frac{2R^2 g_0}{\pi r} \int_0^\infty [1 - E_\alpha(-a\xi^2 t^\alpha)] \Big\{ R\xi \cos[(r-R)\xi] + \sin[(r-R)\xi] \Big\} \frac{1}{\xi(R^2\xi^2+1)} \, \mathrm{d}\xi \tag{6.214}$$

with $\overline{T} = T/(Rg_0)$.

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \frac{\kappa}{(1+\kappa)\bar{r}} \, \mathrm{e}^{-(\bar{r}-1)/\kappa}. \tag{6.215}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{T} = \frac{2}{\sqrt{\pi}\bar{r}} \int_0^\infty \mathrm{e}^{-u^2} \left[\operatorname{erfc}\left(\frac{\bar{r}-1}{2\sqrt{2u\kappa}} \right) - \mathrm{e}^{2\kappa^2 u + \bar{r} - 1} \operatorname{erfc}\left(\sqrt{2u\kappa} + \frac{\bar{r}-1}{2\sqrt{2u\kappa}} \right) \right] \mathrm{d}u. \tag{6.216}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{\bar{r}} \left[\operatorname{erfc}\left(\frac{\bar{r}-1}{2\kappa} \right) - \mathrm{e}^{\kappa^2 + \bar{r} - 1} \operatorname{erfc}\left(\kappa + \frac{\bar{r}-1}{2\kappa} \right) \right]. \tag{6.217}$$

The solution (6.217) can be found in [129].

Wave equation ($\alpha = 2$)

$$\overline{T} = \begin{cases} \dfrac{1}{\bar{r}} \left[1 - \mathrm{e}^{-(\kappa + 1 - \bar{r})} \right], & 1 \le \bar{r} < 1 + \kappa, \\ 0, & 1 + \kappa < \bar{r} < \infty. \end{cases} \tag{6.218}$$

The solution (6.214) is depicted in Fig. 6.42.

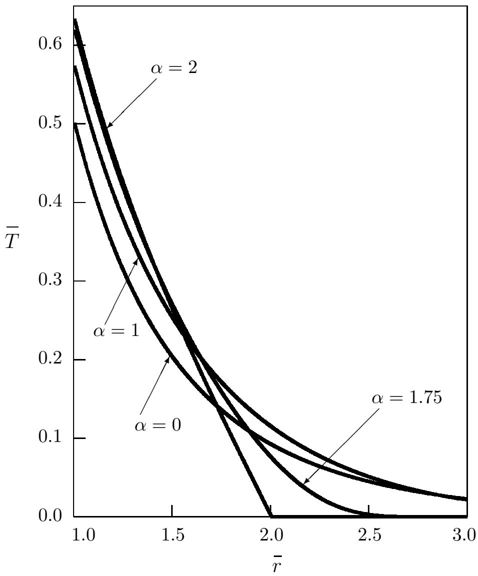

Figure 6.42: Dependence of temperature on distance (a body with a spherical hole under constant boundary value of the normal derivative of temperature); $\kappa = 1$ [153]

Fundamental solution to the physical Neumann problem

$$\frac{\partial^\alpha \mathcal{G}_p}{\partial t^\alpha} = a \left(\frac{\partial^2 \mathcal{G}_p}{\partial r^2} + \frac{2}{r} \frac{\partial \mathcal{G}_p}{\partial r} \right), \tag{6.219}$$

$$t = 0: \quad \mathcal{G}_p = 0, \qquad 0 < \alpha \le 2, \tag{6.220}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_p}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{6.221}$$

$$r = R: \quad -D_{RL}^{1-\alpha} \frac{\partial \mathcal{G}_p}{\partial r} = g_0\, \delta(t), \qquad 0 < \alpha \le 1. \tag{6.222}$$

$$r = R: \quad -I^{\alpha-1} \frac{\partial \mathcal{G}_p}{\partial r} = g_0\, \delta(t), \qquad 1 < \alpha \le 2. \tag{6.223}$$

The solution:

$$\mathcal{G}_p(r,t) = \frac{2g_0 a R^2}{\pi r} \int_0^\infty E_\alpha(-a\xi^2 t^\alpha) \frac{R\xi \cos[(r-R)\xi] + \sin[(r-R)\xi]}{R^2\xi^2 + 1}\, \xi \, \mathrm{d}\xi. \tag{6.224}$$

Constant boundary value of the heat flux

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right), \tag{6.225}$$

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{6.226}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{6.227}$$

$$r = R: \quad -D_{RL}^{1-\alpha} \frac{\partial T}{\partial r} = g_0, \qquad 0 < \alpha \le 1. \tag{6.228}$$

$$r = R: \quad -I^{\alpha-1} \frac{\partial T}{\partial r} = g_0, \qquad 1 < \alpha \le 2. \tag{6.229}$$

The solution:

$$T = \frac{2ag_0 t R^2}{\pi r} \int_0^\infty E_{\alpha,2}(-a\xi^2 t^\alpha) \Big\{ R\xi \cos[(r-R)\xi] + \sin[(r-R)\xi] \Big\} \frac{\xi}{R^2\xi^2+1}\, \mathrm{d}\xi. \tag{6.230}$$

6.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \Phi(r,t), \tag{6.231}$$

$$t = 0: \quad T = f(r), \qquad 0 < \alpha \le 2, \tag{6.232}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r), \quad 1 < \alpha \le 2, \tag{6.233}$$

$$r = R: \quad -\frac{\partial T}{\partial r} + HT = g(t). \tag{6.234}$$

$$\lim_{r\to\infty} T(r,t) = 0. \tag{6.235}$$

The solution:

$$T(r,t) = \int_R^\infty f(\rho)\,\mathcal{G}_f(r,\rho,t)\,\rho^2\,\mathrm{d}\rho + \int_R^\infty F(\rho)\,\mathcal{G}_F(r,\rho,t)\,\rho^2\,\mathrm{d}\rho$$
$$+ \int_0^t\int_R^\infty \Phi(\rho,\tau)\,\mathcal{G}_\Phi(r,\rho,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\tau + \int_0^t g(\tau)\,\mathcal{G}_g(r,t-\tau)\,\mathrm{d}\tau. \tag{6.236}$$

The fundamental solutions have the following form

$$\begin{pmatrix} \mathcal{G}_f(r,\rho,t) \\ \mathcal{G}_F(r,\rho,t) \\ \mathcal{G}_\Phi(r,\rho,t) \end{pmatrix} = \frac{2}{\pi r\rho}\int_0^\infty \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}$$
$$\times \frac{\xi\cos[(r-R)\xi] + (1/R+H)\sin[(r-R)\xi]}{\xi^2 + (1/R+H)^2}$$
$$\times \{\xi\cos[(\rho-R)\xi] + (1/R+H)\sin[(\rho-R)\xi]\}\,\mathrm{d}\xi. \tag{6.237}$$

Let us consider in more details the fundamental solutions to the mathematical $\mathcal{G}_m(r,t)$ and physical $\mathcal{G}_p(r,t)$ Robin boundary value problems [184].

Fundamental solution to the mathematical Robin problem

$$\frac{\partial^\alpha \mathcal{G}_m}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_m}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{G}_m}{\partial r}\right), \quad R < r < \infty, \tag{6.238}$$

$$t = 0: \quad \mathcal{G}_m = 0, \qquad 0 < \alpha \le 2, \tag{6.239}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_m}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.240}$$

$$r = R: \quad -\frac{\partial \mathcal{G}_m}{\partial r} + H\mathcal{G}_m = g_0\,\delta(t). \tag{6.241}$$

Usually, for the considered geometry the auxiliary function $v = r\,\mathcal{G}_m$ and the auxiliary spatial variable $x = r - R$ are used, and the problem (6.238)–(6.241) is reformulated as:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = a\frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < \infty, \tag{6.242}$$

$$t = 0: \quad v = 0, \qquad 0 < \alpha \le 2, \tag{6.243}$$

$$t = 0: \quad \frac{\partial v}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.244}$$

$$x = 0: \quad -\frac{\partial v}{\partial x} + \left(\frac{1}{R} + H\right) v = g_0 R\,\delta(t). \tag{6.245}$$

The Laplace transform with respect to time t and the sin-cos-Fourier transform with respect to the auxiliary spatial coordinate x will be used to solve the problem (6.242)–6.245). In the case under consideration, the sin-cos-Fourier transform has the following form:

$$\mathcal{F}\left\{f(x)\right\} = \widetilde{f}(\xi) = \int_0^\infty \frac{\xi\cos(x\xi) + (1/R + H)\sin(x\xi)}{\sqrt{\xi^2 + (1/R+H)^2}}\, f(x)\,\mathrm{d}x, \tag{6.246}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi)\right\} = f(x)$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\xi\cos(x\xi) + (1/R + H)\sin(x\xi)}{\sqrt{\xi^2 + (1/R+H)^2}}\, \tilde{f}(\xi)\,\mathrm{d}\xi, \tag{6.247}$$

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2 \widetilde{f}(\xi)$$

$$+ \frac{\xi}{\sqrt{\xi^2 + (1/R+H)^2}}\left[-\frac{\mathrm{d}f(x)}{\mathrm{d}x} + \left(\frac{1}{R} + H\right) f(x)\right]_{x=0}. \tag{6.248}$$

In the transform domain we obtain

$$\widetilde{v}^*(\xi, s) = \frac{aRg_0\xi}{\sqrt{\xi^2 + (1/R+H)^2}}\,\frac{1}{s^\alpha + a\xi^2}. \tag{6.249}$$

Inverting the integral transforms, we get

$$\mathcal{G}_m(r,t) = \frac{2aRg_0 t^{\alpha-1}}{\pi r} \int_0^\infty E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) \times \frac{R^2\xi^2 \cos[(r-R)\xi] + (1+RH)R\xi \sin[(r-R)\xi]}{R^2\xi^2 + (1+RH)^2} \, \mathrm{d}\xi. \tag{6.250}$$

Subdiffusion with $\alpha = 1/2$

$$\mathcal{G}_m(r,t) = \frac{2aRg_0}{\sqrt{\pi t} r} \int_0^\infty u \mathrm{e}^{-u^2} \left\{ \frac{1}{\sqrt{2\pi u a} t^{1/4}} \exp\left[-\frac{(r-R)^2}{8ua\sqrt{t}}\right] - \frac{1+RH}{R} \exp\left[\frac{1+RH}{R}(r-R) + 2\frac{(1+RH)^2}{R^2} ua\sqrt{t}\right] \times \operatorname{erfc}\left(\frac{r-R}{2\sqrt{2ua} t^{1/4}} + \frac{1+RH}{R}\sqrt{2ua} t^{1/4}\right)\right\} \mathrm{d}u. \tag{6.251}$$

Classical diffusion equation ($\alpha = 1$)

$$\mathcal{G}_m(r,t) = \frac{aRg_0}{r} \left\{ \frac{1}{\sqrt{\pi a t}} \exp\left[-\frac{(r-R)^2}{4at}\right] - \frac{1+RH}{R} \exp\left[\frac{1+RH}{R}(r-R) + \frac{(1+RH)^2}{R^2} at\right] \times \operatorname{erfc}\left(\frac{r-R}{2\sqrt{at}} + \frac{1+RH}{R}\sqrt{at}\right)\right\}. \tag{6.252}$$

Wave equation ($\alpha = 2$)

$$\mathcal{G}_m(r,t) = \begin{cases} \dfrac{\sqrt{a}Rg_0}{r} \exp\left[-\dfrac{1+RH}{R}(\sqrt{a}t - r + R)\right], & R < r < R + \sqrt{a}t, \\ 0 & R + \sqrt{a}t < r < \infty. \end{cases} \tag{6.253}$$

Dependence of nondimensional fundamental solution $\bar{\mathcal{G}}_m = t\mathcal{G}_m/(Rg_0)$ on nondimensional distance r/R is presented in Figs. 6.43 and 6.44 for $\alpha = 0.5$ and $\alpha = 1.95$, respectively, for various values of $\bar{H} = RH$. In both cases $\kappa = 1$.

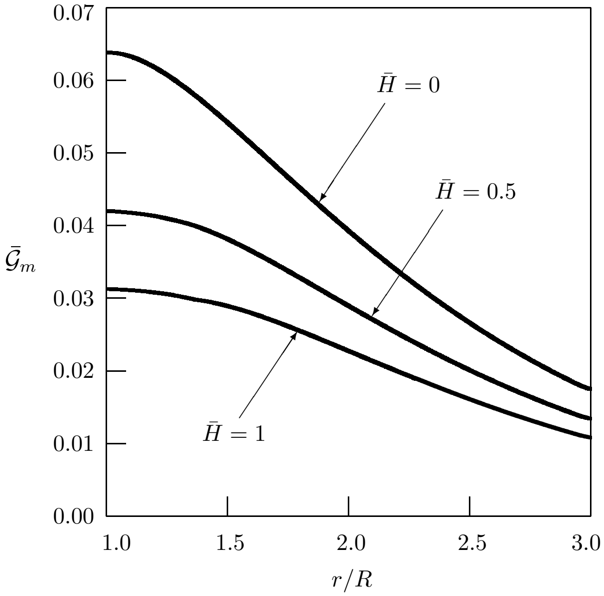

Figure 6.43: Fundamental solution to the mathematical Robin problem for a body with a spherical hole ($\alpha = 0.5$) [184]

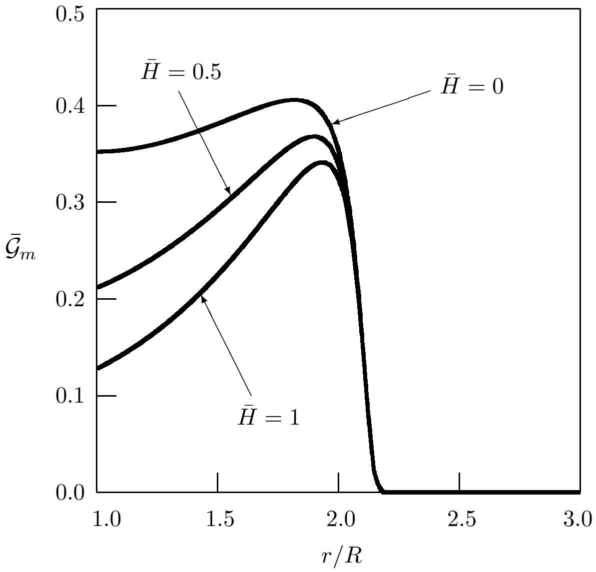

Figure 6.44: Fundamental solution to the mathematical Robin problem for a body with a spherical hole ($\alpha = 1.95$) [184]

Fundamental solution to the physical Robin problem

$$\frac{\partial^{\alpha} \mathcal{G}_p}{\partial t^{\alpha}} = a \left(\frac{\partial^2 \mathcal{G}_p}{\partial r^2} + \frac{2}{r} \frac{\partial \mathcal{G}_p}{\partial r} \right), \qquad R < r < \infty, \tag{6.254}$$

$$t = 0: \quad \mathcal{G}_p = 0, \qquad 0 < \alpha \le 2, \tag{6.255}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_p}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.256}$$

$$r = R: \quad -D_{RL}^{1-\alpha} \frac{\partial \mathcal{G}_p}{\partial r} + H \mathcal{G}_p = g_0\, \delta(t), \quad 0 < \alpha \le 1, \tag{6.257}$$

$$r = R: \quad -I^{\alpha-1} \frac{\partial \mathcal{G}_p}{\partial r} + H \mathcal{G}_p = g_0\, \delta(t), \quad 1 < \alpha \le 2. \tag{6.258}$$

In terms of the auxiliary function $v = r\, \mathcal{G}_p$ and the auxiliary spatial variable $x = r - R$, the problem (6.254)–(6.258) takes the form

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = a \frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < \infty, \tag{6.259}$$

$$t = 0: \quad v = 0, \qquad 0 < \alpha \le 2, \tag{6.260}$$

$$t = 0: \quad \frac{\partial v}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{6.261}$$

$$x = 0: \quad -D_{RL}^{1-\alpha} \frac{\partial v}{\partial x} + \left(\frac{1}{R} D_{RL}^{1-\alpha} + H \right) v = g_0 R\, \delta(t), \quad 0 < \alpha \le 1, \tag{6.262}$$

$$x = 0: \quad -I^{\alpha-1} \frac{\partial v}{\partial x} + \left(\frac{1}{R} I^{\alpha-1} + H \right) v = g_0 R\, \delta(t), \quad 1 < \alpha \le 2. \tag{6.263}$$

Applying the Laplace transform with respect to time t results in the following boundary-value problem

$$s^{\alpha} v^* = a \frac{\partial^2 v^*}{\partial x^2}, \tag{6.264}$$

$$x = 0: \quad -\frac{\partial v^*}{\partial x} + \left(\frac{1}{R} + H s^{\alpha-1} \right) v^* = g_0 R s^{\alpha-1}, \quad 0 < \alpha \le 2. \tag{6.265}$$

In this case the kernel of the sin-cos-Fourier transform with respect to the auxiliary coordinate x depends on the Laplace transform variable s:

$$\mathcal{F}\left\{f(x)\right\} = \widetilde{f}(\xi)$$

$$= \int_0^\infty \frac{\xi\cos(x\xi) + \left(1/R + Hs^{\alpha-1}\right)\sin(x\xi)}{\sqrt{\xi^2 + \left(1/R + Hs^{\alpha-1}\right)^2}}\, f(x)\,\mathrm{d}x, \tag{6.266}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi)\right\} = f(x)$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\xi\cos(x\xi) + \left(1/R + Hs^{\alpha-1}\right)\sin(x\xi)}{\sqrt{\xi^2 + \left(1/R + Hs^{\alpha-1}\right)^2}}\, \widetilde{f}(\xi)\,\mathrm{d}\xi, \tag{6.267}$$

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\widetilde{f}(\xi)$$

$$+\frac{\xi}{\sqrt{\xi^2 + \left(1/R + Hs^{\alpha-1}\right)^2}}\left[-\frac{\mathrm{d}f(x)}{\mathrm{d}x} + \left(\frac{1}{R} + Hs^{\alpha-1}\right)f(x)\right]_{x=0}. \tag{6.268}$$

In the transform domain we get

$$\widetilde{v}^*(\xi, s) = \frac{aRg_0\xi}{\sqrt{\xi^2 + \left(1/R + Hs^{\alpha-1}\right)^2}}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{6.269}$$

After inversion of the sin-cos-Fourier transform we arrive at [184]

$$\mathcal{G}_p^*(r, s) = \frac{2aRg_0}{\pi r}\int_0^\infty \frac{s^{\alpha-1}}{s^\alpha + a\xi^2}$$

$$\times\frac{R^2\xi^2\cos[(r-R)\xi] + (1 + RHs^{\alpha-1})R\xi\sin[(r-R)\xi]}{R^2\xi^2 + (1 + RHs^{\alpha-1})^2}\,\mathrm{d}\xi. \tag{6.270}$$

Inversion of the Laplace transform in (6.270) depends on the value of α. For $0 < \alpha < 1$ this equation is rewritten as

$$\mathcal{G}_p^*(r, s) = \frac{2aRg_0}{\pi r}\int_0^\infty \frac{1}{s^\alpha + a\xi^2}$$

$$\times\frac{s^{1-\alpha}R^2\xi^2\cos[(r-R)\xi] + (s^{1-\alpha} + RH)R\xi\sin[(r-R)\xi]}{(1 + R^2\xi^2)(s^{1-\alpha})^2 + 2RHs^{1-\alpha} + R^2H^2}\,\mathrm{d}\xi. \tag{6.271}$$

Next, we use the following decompositions into the sum of partial fractions:

$$\frac{1}{(1+R^2\xi^2)(s^{1-\alpha})^2+2RHs^{1-\alpha}+R^2H^2}$$
$$=\frac{i}{2HR^2\xi}\left[\frac{1}{s^{1-\alpha}+\frac{RH(1+iR\xi)}{R^2\xi^2+1}}-\frac{1}{s^{1-\alpha}+\frac{RH(1-iR\xi)}{R^2\xi^2+1}}\right], \tag{6.272}$$

$$\frac{s^{1-\alpha}}{(1+R^2\xi^2)(s^{1-\alpha})^2+2RHs^{1-\alpha}+R^2H^2}$$
$$=\frac{1}{2R\xi(R^2\xi^2+1)}\left[\frac{R\xi-i}{s^{1-\alpha}+\frac{RH(1+iR\xi)}{R^2\xi^2+1}}+\frac{R\xi+i}{s^{1-\alpha}+\frac{RH(1-iR\xi)}{R^2\xi^2+1}}\right], \tag{6.273}$$

where $i=\sqrt{-1}$.

Utilizing the convolution theorem, the solution is written as

$$\mathcal{G}_p(r,t)=\frac{aRg_0}{\pi r}\int_0^\infty\int_0^t\frac{R\xi(t-\tau)^{\alpha-1}\tau^{-\alpha}}{R^2\xi^2+1}\,E_{\alpha,\alpha}\left[-a\xi^2(t-\tau)^\alpha\right]$$
$$\times\left\{(R\xi-i)\,\mathrm{e}^{i(r-R)\xi}\,E_{1-\alpha,1-\alpha}\left[-\frac{RH(1+iR\xi)\tau^{1-\alpha}}{R^2\xi^2+1}\right]\right.$$
$$\left.+(R\xi+i)\,\mathrm{e}^{-i(r-R)\xi}\,E_{1-\alpha,1-\alpha}\left[-\frac{RH(1-iR\xi)\tau^{1-\alpha}}{R^2\xi^2+1}\right]\right\}\mathrm{d}\tau\,\mathrm{d}\xi. \tag{6.274}$$

It should be emphasized that the fundamental solution (6.274) is a real-valued function and can be written as

$$\mathcal{G}_p(r,t)=\frac{2aRg_0}{\pi r}\int_0^\infty\int_0^t\frac{R\xi(t-\tau)^{\alpha-1}\tau^{-\alpha}}{R^2\xi^2+1}\,E_{\alpha,\alpha}\left[-a\xi^2(t-\tau)^\alpha\right]$$
$$\times\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma[(m+1)(1-\alpha)]}\left(\frac{RH\tau^{1-\alpha}}{\sqrt{R^2\xi^2+1}}\right)^m$$
$$\times\sin[(r-R)\xi+(m+1)\arctan(R\xi)]\,\mathrm{d}\tau\,\mathrm{d}\xi. \tag{6.275}$$

The solution (6.274) simplifies significantly for $\alpha = 1/2$ with taking into account the representation (2.137):

$$\mathcal{G}_p(r,t) = \frac{4aRg_0}{\pi^{3/2} r} \int_0^\infty \frac{R\xi}{R^2\xi^2+1} \int_0^t \frac{1}{\sqrt{\tau(t-\tau)}}$$

$$\times \int_0^\infty u\, E_{1/2,1/2}\left(-a\xi^2\sqrt{t-\tau}\right) \exp\left(-u^2 - \frac{2RHu\sqrt{\tau}}{R^2\xi^2+1}\right)$$

$$\times \left\{ R\xi \cos\left[\left(r - R - \frac{2R^2Hu\sqrt{\tau}}{R^2\xi^2+1}\right)\xi\right]\right.$$

$$\left. + \sin\left[\left(r - R - \frac{2R^2Hu\sqrt{\tau}}{R^2\xi^2+1}\right)\xi\right]\right\} \mathrm{d}u\, \mathrm{d}\tau\, \mathrm{d}\xi. \tag{6.276}$$

Similarly, returning to (6.270) for $1 < \alpha \leq 2$, we get the required decompositions into the sum of partial fractions

$$\frac{1}{R^2\xi^2 + (1+RHs^{\alpha-1})^2}$$

$$= \frac{i}{2R\xi}\left(\frac{1}{RHs^{\alpha-1}+1+iR\xi} - \frac{1}{RHs^{\alpha-1}+1-iR\xi}\right), \tag{6.277}$$

$$\frac{1+RHs^{\alpha-1}}{R^2\xi^2 + (1+RHs^{\alpha-1})^2}$$

$$= \frac{1}{2}\left(\frac{1}{RHs^{\alpha-1}+1+iR\xi} + \frac{1}{RHs^{\alpha-1}+1-iR\xi}\right). \tag{6.278}$$

The convolution theorem allows us to invert the Laplace transform and to obtain the solution [184]:

$$\mathcal{G}_p(r,t) = \frac{aRg_0 i}{\pi H r} \int_0^\infty \int_0^t \xi \tau^{\alpha-2}\, E_\alpha\left[-a\xi^2(t-\tau)^\alpha\right]$$

$$\times \left[\mathrm{e}^{-i(r-R)\xi}\, E_{\alpha-1,\alpha-1}\left(-\frac{1+iR\xi}{RH}\tau^{\alpha-1}\right)\right.$$

$$\left. - \mathrm{e}^{i(r-R)\xi}\, E_{\alpha-1,\alpha-1}\left(-\frac{1-iR\xi}{RH}\tau^{\alpha-1}\right)\right] \mathrm{d}\tau\, \mathrm{d}\xi \tag{6.279}$$

which can be rewritten in the real-valued form as

$$\mathcal{G}_p(r,t) = \frac{2aRg_0}{\pi H r} \int_0^\infty \int_0^t \xi\, \tau^{\alpha-2}\, E_\alpha\left[-a\xi^2(t-\tau)^\alpha\right]$$
$$\times \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma[(m+1)(\alpha-1)]} \left(\frac{\tau^{1-\alpha}\sqrt{R^2\xi^2+1}}{RH}\right)^m$$
$$\times \sin[(r-R)\xi - m\arctan(R\xi)]\, \mathrm{d}\tau\, \mathrm{d}\xi. \qquad (6.280)$$

The particular case corresponding to the value $\alpha = 3/2$ is also obtained using the representation (2.137):

$$\mathcal{G}_p(r,t) = \frac{4aRg_0}{\pi^{3/2} H r} \int_0^\infty \int_0^t \int_0^\infty \frac{u\,\xi}{\sqrt{\tau}}\, E_{3/2}\left[-a\xi^2(t-\tau)^{3/2}\right]$$
$$\times \exp\left(-u^2 - \frac{2u\sqrt{\tau}}{RH}\right) \sin\left[\left(r - R + \frac{2u\sqrt{\tau}}{H}\right)\xi\right] \mathrm{d}u\, \mathrm{d}\tau\, \mathrm{d}\xi. \qquad (6.281)$$

Figure 6.45 presents the dependence of the nondimensional fundamental solution $\bar{\mathcal{G}}_p = R\mathcal{G}_p/(ag_0)$ under the physical Robin boundary condition on nondimensional distance r/R for $\alpha = 0.5$ and different values of $\overline{H} = RHt^{1-\alpha}$. In calculations we have taken $\kappa = 1$.

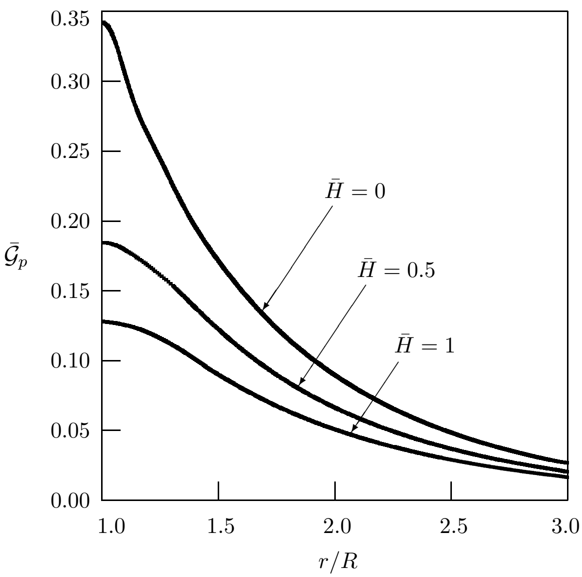

Figure 6.45: Fundamental solution to the physical Robin boundary value problem in a body with a spherical hole ($\alpha = 0.5$) [184]

6.4.4 Infinite medium with a spherical inclusion

The general statement of the problems reads as follows: to solve the time-fractional diffusion-wave equations in a spherical inclusion and in a matrix

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1 \left(\frac{\partial^2 T_1}{\partial r^2} + \frac{2}{r}\frac{\partial T_1}{\partial r} \right), \qquad 0 < r < R, \tag{6.282}$$

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2 \left(\frac{\partial^2 T_2}{\partial r^2} + \frac{2}{r}\frac{\partial T_2}{\partial r} \right), \qquad R < r < \infty, \tag{6.283}$$

under the initial conditions

$$t = 0: \quad T_1 = f_1(r), \quad 0 < r < R, \quad 0 < \alpha \le 2, \tag{6.284}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = F_1(r), \quad 0 < r < R, \quad 1 < \alpha \le 2, \tag{6.285}$$

$$t = 0: \quad T_2 = f_2(r), \quad R < r < \infty, \quad 0 < \beta \le 2 \tag{6.286}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = F_2(r), \quad R < r < \infty, \quad 1 < \beta \le 2, \tag{6.287}$$

and the boundary conditions of perfect thermal contact

$$r = R: \quad T_1(r,t) = T_2(r,t), \tag{6.288}$$

$$r = R: \quad k_1 D_{RL}^{1-\alpha} \frac{\partial T_1(r,t)}{\partial r} = k_2 D_{RL}^{1-\beta} \frac{\partial T_2(r,t)}{\partial r}, \quad 0 < \alpha \le 2, \quad 0 < \beta \le 2. \tag{6.289}$$

In the condition (6.289), $D_{RL}^\alpha f(t)$ in the case of the negative order α is understood as $I^{-\alpha} f(t)$.

The boundedness condition at the origin and the zero condition at infinity are also assumed:

$$\lim_{r\to 0} T_1(r,t) \ne \infty, \qquad \lim_{r\to\infty} T_2(r,t) = 0. \tag{6.290}$$

In what follows we restrict ourselves to the particular case when a sphere $0 \le r < R$ is at initial uniform temperature T_0 and the matrix $R < r < \infty$ is at initial zero temperature, i.e., [185]

$$t = 0: \quad T_1 = T_0, \quad 0 < r < R, \quad 0 < \alpha \le 2, \tag{6.291}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = 0, \quad 0 < r < R, \quad 1 < \alpha \le 2, \tag{6.292}$$

$$t = 0: \quad T_2 = 0, \quad R < r < \infty, \quad 0 < \beta \le 2, \tag{6.293}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = 0, \quad R < r < \infty, \quad 1 < \beta \le 2. \tag{6.294}$$

The Laplace transform with respect to time t applied to (6.282) and (6.283) leads to two ordinary differential equations

$$s^{\alpha} T_1^* - s^{\alpha-1} T_0 = a_1 \left(\frac{\mathrm{d}^2 T_1^*}{\mathrm{d}r^2} + \frac{2}{r} \frac{\mathrm{d}T_1^*}{\mathrm{d}r} \right), \qquad 0 < r < R, \tag{6.295}$$

$$s^{\beta} T_2^* = a_2 \left(\frac{\mathrm{d}^2 T_2^*}{\mathrm{d}r^2} + \frac{2}{r} \frac{\mathrm{d}T_2^*}{\mathrm{d}r} \right), \qquad R < r < \infty, \tag{6.296}$$

having the solutions

$$T_1^*(r,s) = \frac{T_0}{s} + \frac{A_1}{r} \cosh\left(\sqrt{\frac{s^{\alpha}}{a_1}}\, r \right) + \frac{B_1}{r} \sinh\left(\sqrt{\frac{s^{\alpha}}{a_1}}\, r \right), \quad 0 < r < R, \tag{6.297}$$

$$T_2^*(r,s) = \frac{A_2}{r} \exp\left(\sqrt{\frac{s^{\beta}}{a_2}}\, r \right) + \frac{B_2}{r} \exp\left(-\sqrt{\frac{s^{\beta}}{a_2}}\, r \right), \qquad R < r < \infty. \tag{6.298}$$

It follows from the conditions at the origin and at infinity (6.290) that

$$A_1 = 0, \qquad A_2 = 0. \tag{6.299}$$

The integration constants B_1 and B_2 are obtained from the perfect thermal contact boundary conditions (6.288) and (6.289):

$$B_1 = \frac{k_2 T_0 R \left(1 + R\sqrt{\frac{s^{\beta}}{a_2}} \right)}{s\,\Delta},$$

$$B_2 = \frac{T_0 R \exp\left(\sqrt{\frac{s^{\beta}}{a_2}} R \right)}{s} \left[1 + \frac{k_2 \left(1 + R\sqrt{\frac{s^{\beta}}{a_2}} \right) \sinh\left(\sqrt{\frac{s^{\alpha}}{a_1}} R \right)}{\Delta} \right],$$

where

$$\begin{aligned} \Delta = & \left[k_1 s^{\beta-\alpha} - k_2 \left(1 + R\sqrt{\frac{s^{\beta}}{a_2}} \right) \right] \sinh\left(\sqrt{\frac{s^{\alpha}}{a_1}} R \right) \\ & - R k_1 s^{\beta-\alpha} \sqrt{\frac{s^{\alpha}}{a_1}} \cosh\left(\sqrt{\frac{s^{\alpha}}{a_1}} R \right). \end{aligned}$$

Hence, the solution is written as

$$T_1^*(r,s) = \frac{T_0}{s} + \frac{k_2 T_0 R\left(1 + R\sqrt{\frac{s^\beta}{a_2}}\right)\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}r\right)}{rs\,\Delta}, \tag{6.300}$$

$$T_2^*(r,s) = \frac{T_0 R}{rs}\exp\left[-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right]$$
$$+\frac{k_2 T_0 R\left(1 + R\sqrt{\frac{s^\beta}{a_2}}\right)\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)\exp\left(-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right)}{rs\,\Delta}. \tag{6.301}$$

Now we will investigate the approximate solution of the considered problem for small values of time. In the case of classical heat conduction equation this method was described in [98, 140]. Based on Tauberian theorems for the Laplace transform (see, for example [34]), for small values of time t (the large values of the transform variable s) we can neglect the exponential term in comparison with 1:

$$1 \pm \exp\left(-2\sqrt{\frac{s^\alpha}{a_1}}R\right) \simeq 1, \tag{6.302}$$

thus obtaining

$$T_1^*(r,s) \simeq \frac{T_0}{s} + \frac{k_2 T_0 R\left(1 + R\sqrt{\frac{s^\beta}{a_2}}\right)}{rs\left[k_1 s^{\beta-\alpha}\left(1 - R\sqrt{\frac{s^\alpha}{a_1}}\right) - k_2\left(1 + R\sqrt{\frac{s^\beta}{a_2}}\right)\right]}$$
$$\times\left\{\exp\left[-\sqrt{\frac{s^\alpha}{a_1}}(R-r)\right] - \exp\left[-\sqrt{\frac{s^\alpha}{a_1}}(R+r)\right]\right\}, \tag{6.303}$$

$$T_2^*(r,s) \simeq \frac{T_0 R}{rs}\exp\left[-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right]$$
$$+\frac{k_2 T_0 R\left(1 + R\sqrt{\frac{s^\beta}{a_2}}\right)\exp\left[-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right]}{rs\left[k_1 s^{\beta-\alpha}\left(1 - R\sqrt{\frac{s^\alpha}{a_1}}\right) - k_2\left(1 + R\sqrt{\frac{s^\beta}{a_2}}\right)\right]}. \tag{6.304}$$

In the following particular cases $\alpha = 2/3$, $\beta = 4/3$; $\alpha = 1$, $\beta = 2$; $\alpha = 2$, $\beta = 1$ the denominator in (6.303) and (6.304) can be treated as a cubed equation and the decomposition into the sum of partial fractions can be obtained similarly to (6.272) and (6.273). Below we will consider another particular case when $\alpha = \beta$.

The solution reads:

$$T_1(r,t) \simeq T_0 - \frac{RT_0 k_2}{(k_2-k_1)r}\left[W\left(-\frac{\alpha}{2},1;-\frac{R-r}{\sqrt{a_1}t^{\alpha/2}}\right) - W\left(-\frac{\alpha}{2},1;-\frac{R+r}{\sqrt{a_1}t^{\alpha/2}}\right)\right] + \frac{CRT_0}{r}\int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}}\left[M\left(\frac{\alpha}{2};\frac{R-r}{\sqrt{a_1}\tau^{\alpha/2}}\right) - M\left(\frac{\alpha}{2};\frac{R+r}{\sqrt{a_1}\tau^{\alpha/2}}\right)\right] E_{\alpha/2,\alpha/2}\left[-b(t-\tau)^{\alpha/2}\right]\mathrm{d}\tau, \tag{6.305}$$

$$T_2(r,t) \simeq -\frac{RT_0 k_1}{(k_2-k_1)r} W\left(-\frac{\alpha}{2},1;-\frac{r-R}{\sqrt{a_2}t^{\alpha/2}}\right) + \frac{CRT_0}{r}\int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} \times M\left(\frac{\alpha}{2};\frac{r-R}{\sqrt{a_2}\tau^{\alpha/2}}\right) E_{\alpha/2,\alpha/2}\left[-b(t-\tau)^{\alpha/2}\right]\mathrm{d}\tau, \tag{6.306}$$

where

$$b = \frac{(k_2-k_1)\sqrt{a_1 a_2}}{R\left(k_1\sqrt{a_1}+k_2\sqrt{a_2}\right)}, \qquad C = \frac{k_1 k_2\left(\sqrt{a_1}+\sqrt{a_2}\right)}{(k_2-k_1)\left(k_1\sqrt{a_1}+k_2\sqrt{a_2}\right)},$$

$W(\alpha,\beta,z)$ is the Wright function (2.165), $M(\alpha;z)$ is the Mainardi function (2.175).

Chapter 7

Equations with Two Space Variables in Cartesian Coordinates

Но я даже пару ломаных юаней...
За эти иксы-игреки не дам.

Владимир Высоцкий [1]

7.1 Domain $-\infty < x < \infty, -\infty < y < \infty$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x, y, t), \tag{7.1}$$

$$t = 0: \quad T = f(x, y), \qquad 0 < \alpha \le 2, \tag{7.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x, y), \quad 1 < \alpha \le 2, \tag{7.3}$$

$$\lim_{x \to \pm\infty} T(x, y, t) = 0, \qquad \lim_{y \to \pm\infty} T(x, yt) = 0. \tag{7.4}$$

The solution:

$$T(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\rho, \sigma)\, \mathcal{G}_f(x-\rho, y-\sigma, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\rho, \sigma)\, \mathcal{G}_F(x-\rho, y-\sigma, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma$$

[1] *But I will not give even a few brass yuan*
For these X's and Y's.

Vladimir Vysotsky

Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*,
DOI 10.1007/978-3-319-17954-4_7

$$+\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x-\rho, y-\sigma, t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \tag{7.5}$$

with the fundamental solutions

$$\begin{pmatrix} \mathcal{G}_f(x,y,t) \\ \mathcal{G}_F(x,y,t) \\ \mathcal{G}_\Phi(x,y,t) \end{pmatrix} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$

$$\times\ \cos(x\xi)\ \cos(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{7.6}$$

7.2 Domain $0 < x < \infty,\ -\infty < y < \infty$

7.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.7}$$

$$t=0:\quad T = f(x,y), \qquad 0<\alpha\le 2, \tag{7.8}$$

$$t=0:\quad \frac{\partial T}{\partial t} = F(x,y), \quad 1<\alpha\le 2, \tag{7.9}$$

$$x=0:\quad T = g(y,t), \tag{7.10}$$

$$\lim_{x\to\infty} T(x,y,t) = 0, \qquad \lim_{y\to\pm\infty} T(x,y,t) = 0. \tag{7.11}$$

The solution is obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate y and the sin-Fourier transform (2.25) with respect to the space coordinate x and has the form:

$$\begin{aligned} T(x,y,t) &= \int_{-\infty}^{\infty} \int_0^{\infty} f(\rho,\sigma)\,\mathcal{G}_f(x, y-\sigma, \rho, t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &+ \int_{-\infty}^{\infty} \int_0^{\infty} F(\rho,\sigma)\,\mathcal{G}_F(x, y-\sigma, \rho, t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &+ \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x, y-\sigma, \rho, t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_0^t \int_{-\infty}^{\infty} g(\sigma,\tau)\,\mathcal{G}_g(x, y-\sigma, t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned} \tag{7.12}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,\rho,t) \\ \mathcal{G}_F(x,y,\rho,t) \\ \mathcal{G}_\Phi(x,y,\rho,t) \end{pmatrix} = \frac{1}{\pi^2} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big] \end{pmatrix}$$
$$\times\ \sin(x\xi)\sin(\rho\xi)\ \cos(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{7.13}$$

The fundamental solution to the Dirichlet problem is calculated as [154]

$$\mathcal{G}_g(x,y,t) = \frac{a g_0 t^{\alpha-1}}{\pi^2} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} E_{\alpha,\alpha}[-a(\xi^2+\eta^2)t^\alpha]$$
$$\times\ \xi\ \sin(x\xi)\ \cos(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{7.14}$$

Let us study the fundamental solution (7.14) in more detail. This equation is inconvenient for numerical treatment. To obtain a solution amenable for numerical calculations, we pass to polar coordinates in the (ξ,η)-plane and in the (x,y)-plane: $\xi = \varrho\cos\vartheta$, $\eta = \varrho\sin\vartheta$, $x = r\cos\varphi$, $y = r\sin\varphi$. Then (7.14) is rewritten as

$$\mathcal{G}_g(x,y,t) = \frac{a g_0 t^{\alpha-1}}{\pi^2} \int\limits_{0}^{\infty} \varrho^2 E_{\alpha,\alpha}(-a\varrho^2 t^\alpha)\,\mathrm{d}\varrho$$
$$\times \int\limits_{-\pi/2}^{\pi/2} \sin(x\varrho\cos\vartheta)\cos(y\varrho\sin\vartheta)\cos\vartheta\,\mathrm{d}\vartheta. \tag{7.15}$$

Substitution of $v = \sin\vartheta$, taking into account integral (A.22) from the Appendix, gives

$$\mathcal{G}_g = \frac{a g_0 t^{\alpha-1}\cos\varphi}{\pi} \int\limits_{0}^{\infty} \varrho^2 E_{\alpha,\alpha}(-a\varrho^2 t^\alpha)\, J_1(r\varrho)\,\mathrm{d}\varrho. \tag{7.16}$$

Dependence of the fundamental solution $\mathcal{G}_g$ on the radial coordinate r is shown in Figs. 7.1 and 7.2 for $\varphi = 0$, where the nondimensional quantities are introduced:

$$\overline{\mathcal{G}}_g = \frac{\sqrt{a}t^{\alpha/2+1}}{g_0}\,\mathcal{G}_g, \quad \bar{r} = \frac{r}{\sqrt{a}t^{\alpha/2}}. \tag{7.17}$$

Constant boundary value of a function in a local area. Of special interest is the initial-boundary-value problem (7.7)–(7.10) with the constant boundary value of temperature in the area $|y| < l$:

$$x = 0: \quad T = \begin{cases} T_0, & |y| < l, \\ 0, & |y| > l. \end{cases} \tag{7.18}$$

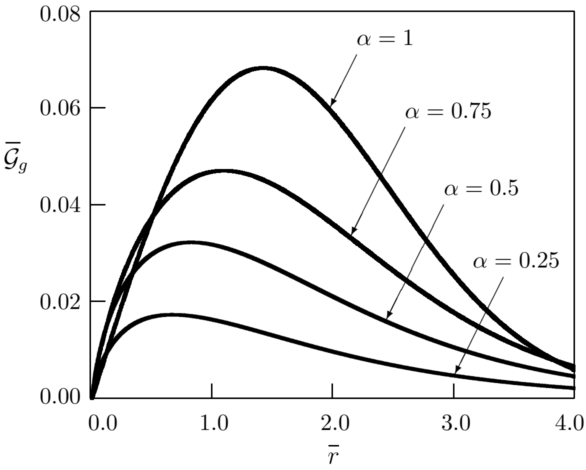

Figure 7.1: Dependence of the fundamental solution to the Dirichlet problem in a half-plane on distance for $0 < \alpha \leq 1$ [154]

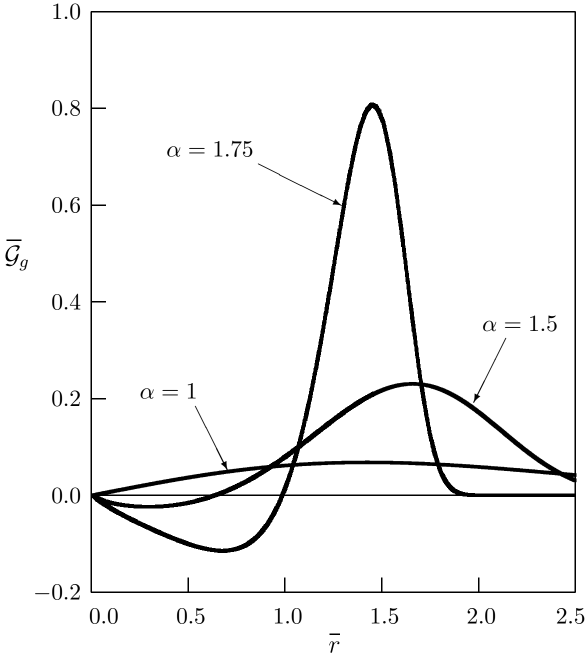

Figure 7.2: Dependence of the fundamental solution to the Dirichlet problem in a half-plane on distance for $1 \leq \alpha \leq 2$ [154]

The solution reads [154]

$$T(x,y,t) = \frac{2T_0}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(l\eta)}{\eta} \cos(y\eta)\, \mathrm{d}\eta$$
$$\times \int_0^{\infty} \frac{\xi \sin(x\xi)}{\xi^2+\eta^2} \left\{1 - E_\alpha[-a(\xi^2+\eta^2)t^\alpha]\right\} \mathrm{d}\xi. \tag{7.19}$$

Introducing polar coordinates in the (ξ,η)-plane and taking into account integrals (A.5), (A.13) and (A.24) from Appendix, we obtain

$$\overline{T} = \frac{1}{\pi}\left[\arctan\frac{1-\bar{y}}{\bar{x}} + \arctan\frac{1+\bar{y}}{\bar{x}}\right]$$
$$- \frac{\bar{x}}{\pi} \int_0^{\infty} E_\alpha(-\kappa^2\sigma^2)\, \mathrm{d}\sigma \int_{\bar{y}-1}^{\bar{y}+1} \frac{1}{\sqrt{u^2+\bar{x}^2}} J_1\left(\sigma\sqrt{u^2+\bar{x}^2}\right) \mathrm{d}u, \tag{7.20}$$

with the following nondimensional quantities

$$\overline{T} = \frac{T}{T_0}, \quad \bar{x} = \frac{x}{l}, \quad \bar{y} = \frac{y}{l}, \quad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{l}. \tag{7.21}$$

Consider several particular cases.

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \frac{\bar{x}}{\pi\kappa} \int_{\bar{y}-1}^{\bar{y}+1} \frac{1}{\sqrt{u^2+\bar{x}^2}} K_1\left(\frac{\sqrt{u^2+\bar{x}^2}}{\kappa}\right) \mathrm{d}u. \tag{7.22}$$

Subdiffusion with $\alpha = 1/2$

$$\overline{T} = \frac{2\bar{x}}{\pi^{3/2}} \int_{\bar{y}-1}^{\bar{y}+1} \frac{1}{u^2+\bar{x}^2}\, \mathrm{d}u \int_0^{\infty} \exp\left(-v^2 - \frac{\bar{x}^2+u^2}{8\kappa^2 v}\right) \mathrm{d}v. \tag{7.23}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{\bar{x}}{\pi} \int_{\bar{y}-1}^{\bar{y}+1} \frac{1}{u^2+\bar{x}^2} \exp\left(-\frac{u^2+\bar{x}^2}{4\kappa^2}\right) \mathrm{d}u. \tag{7.24}$$

Wave equation ($\alpha = 2$)

Having regard to (A.1), (A.2) and (A.44), we get a solution whose analytical form depends on κ. As $\overline{T}$ is an even function of y, we can consider only $y \geq 0$ and obtain:

a) $0 < \kappa < |1-\bar{y}|$

$$\overline{T} = \begin{cases} \dfrac{1}{2}[1+\operatorname{sign}(1-\bar{y})], & 0 < \bar{x} < \kappa, \\ 0, & \kappa < \bar{x} < \infty; \end{cases} \tag{7.25}$$

b) $|1-\bar{y}| < \kappa < 1+\bar{y}$

$$\overline{T} = \begin{cases} \dfrac{1}{2} + \dfrac{1}{\pi}\arctan\dfrac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}, & 0 < \bar{x} < \sqrt{\kappa^2-(1-\bar{y})^2}, \\ \dfrac{1}{2}[1+\operatorname{sign}(1-\bar{y})], & \sqrt{\kappa^2-(1-\bar{y})^2} < \bar{x} < \kappa, \\ 0, & \kappa < \bar{x} < \infty; \end{cases} \tag{7.26}$$

c) $1+\bar{y} < \kappa < \infty$

$$\overline{T} = \begin{cases} \frac{1}{\pi}\arctan\frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} & \\ +\frac{1}{\pi}\arctan\frac{\kappa(1+\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}}, & 0 < \bar{x} < \sqrt{\kappa^2-(1+\bar{y})^2}, \\ \frac{1}{2} + \frac{1}{\pi}\arctan\frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}, & \sqrt{\kappa^2-(1+\bar{y})^2} < \bar{x} < \sqrt{\kappa^2-(1-\bar{y})^2}, \\ \frac{1}{2}[1+\operatorname{sign}(1-\bar{y})], & \sqrt{\kappa^2-(1-\bar{y})^2} < \bar{x} < \kappa, \\ 0, & \kappa < \bar{x} < \infty. \end{cases} \tag{7.27}$$

Dependence of nondimensional solution $\overline{T}$ on the nondimensional spatial coordinate $\bar{x}$ is shown in Figs. 7.3–7.5 for various values of $\bar{y}$ and κ.

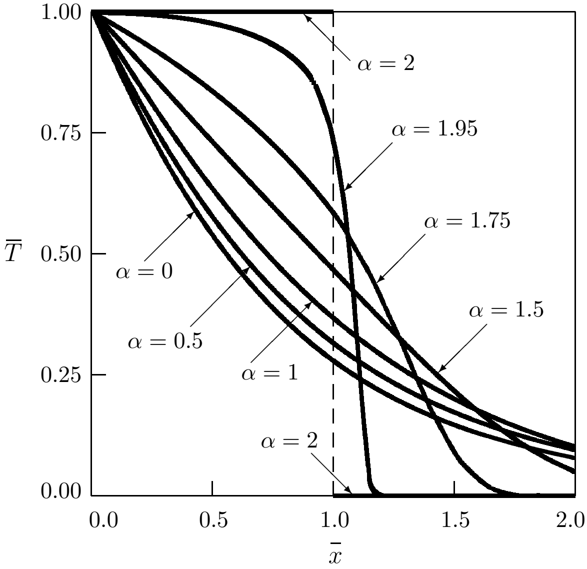

Figure 7.3: Dependence of solution on distance $\bar{x}$ for $y = 0$ and $\kappa = 1$ (a half-plane with the constant value of a function in a local area) [154]

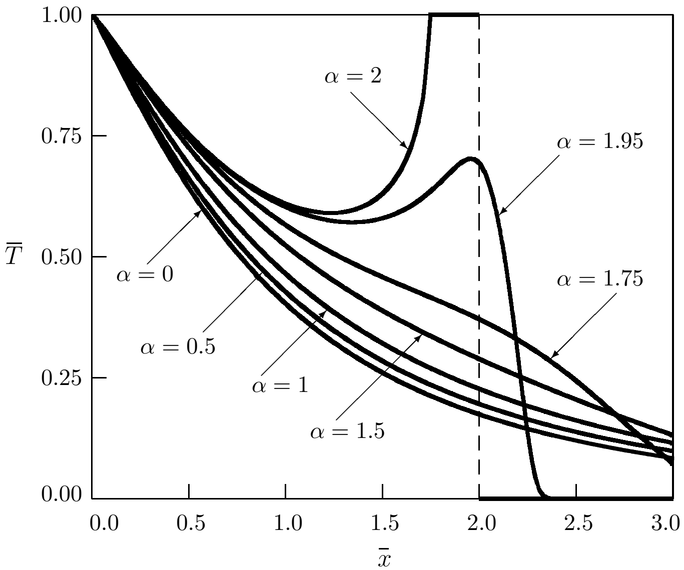

Figure 7.4: Dependence of solution on distance $\bar{x}$ for $y = 0$ and $\kappa = 2$ (a half-plane with the constant value of a function in a local area) [154]

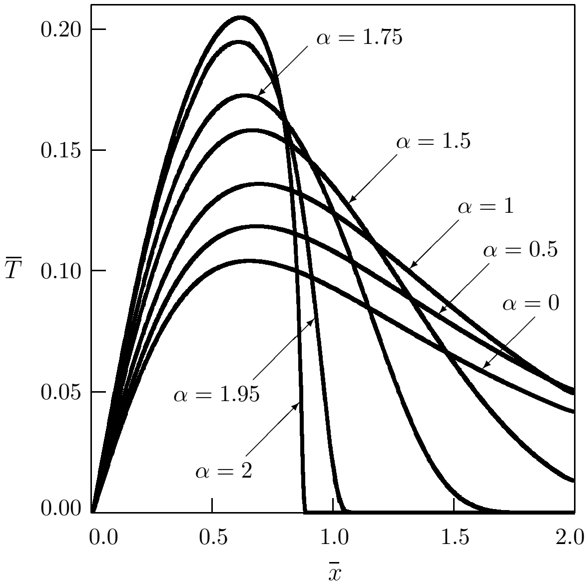

Figure 7.5: Dependence of solution on distance $\bar{x}$ for $\bar{y} = 1.5$ and $\kappa = 1$ (a half-plane with the constant value of a function in a local area) [154]

7.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x, y, t), \tag{7.28}$$

$$t = 0: \quad T = f(x, y), \qquad 0 < \alpha \le 2, \tag{7.29}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x, y), \quad 1 < \alpha \le 2, \tag{7.30}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} = g(y, t), \tag{7.31}$$

$$\lim_{x\to\infty} T(x, y, t) = 0, \qquad \lim_{y\to\pm\infty} T(x, y, t) = 0. \tag{7.32}$$

The solution is obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate y and the cos-Fourier transform (2.37) with respect to the space coordinate x and has the form:

$$T(x, y, t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} f(\rho, \sigma)\, \mathcal{G}_f(x, y - \sigma, \rho, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma$$

$$+ \int_{-\infty}^{\infty} \int_{0}^{\infty} F(\rho, \sigma)\, \mathcal{G}_F(x, y - \sigma, \rho, t))\, \mathrm{d}\rho\, \mathrm{d}\sigma$$

$$+ \int_{0}^{t} \int_{-\infty}^{\infty} \int_{0}^{\infty} \Phi(\rho, \sigma, \tau)\, \mathcal{G}_\Phi(x, y - \sigma, \rho, t - \tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau$$

$$+ \int_{0}^{t} \int_{-\infty}^{\infty} g(\sigma, \tau)\, \mathcal{G}_g(x, y - \sigma, t - \tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau \tag{7.33}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x, y, \rho, t) \\ \mathcal{G}_F(x, y, \rho, t) \\ \mathcal{G}_\Phi(x, y, \rho, , t) \end{pmatrix} = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2 + \eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2 + \eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi^2 + \eta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \cos(x\xi) \cos(\rho\xi)\ \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{7.34}$$

The fundamental solution to the mathematical Neumann problem is calculated as [154]

$$\mathcal{G}_g(x, y, t) = \frac{a g_0 t^{\alpha-1}}{\pi^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} E_{\alpha,\alpha}[-a(\xi^2 + \eta^2)t^\alpha]$$

$$\times\ \cos(x\xi)\ \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{7.35}$$

Passing to the polar coordinate, we get

$$\mathcal{G}_g(x, y, t) = \frac{a g_0 t^{\alpha-1}}{\pi^2} \int_{0}^{\infty} \varrho E_{\alpha,\alpha}(-a\varrho^2 t^\alpha)\, \mathrm{d}\varrho$$

$$\times \int_{-\pi/2}^{\pi/2} \sin(x\varrho \cos\vartheta) \cos(y\varrho \sin\vartheta)\, \mathrm{d}\vartheta \tag{7.36}$$

and after using (A.21)

$$\mathcal{G}_g = \frac{a g_0 t^{\alpha-1}}{\pi} \int_0^\infty \varrho\, E_{\alpha,\alpha}(-a\varrho^2)\, J_0(r\varrho)\, \mathrm{d}\varrho. \tag{7.37}$$

Dependence of the nondimensional fundamental solution $\overline{\mathcal{G}}_g = t\, \mathcal{G}_g / g_0$ on nondimensional radial coordinate $\bar{r}$ is shown in Figs. 7.6 and 7.7.

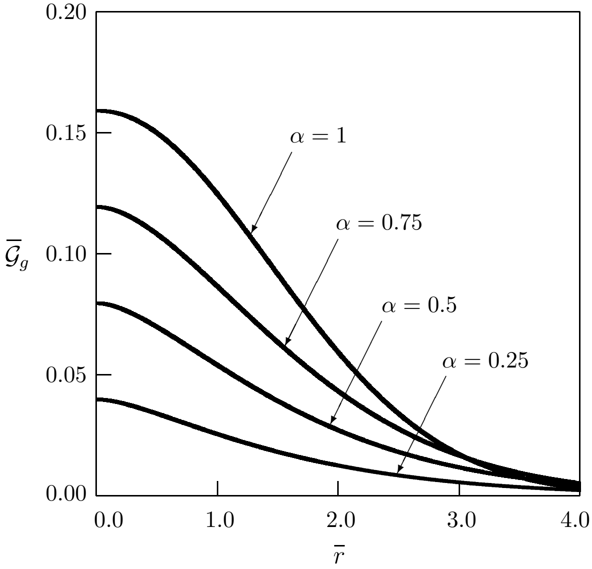

Figure 7.6: Dependence of the fundamental solution to the mathematical Neumann problem for a half-plane on distance for $0 < \alpha \leq 1$ [154]

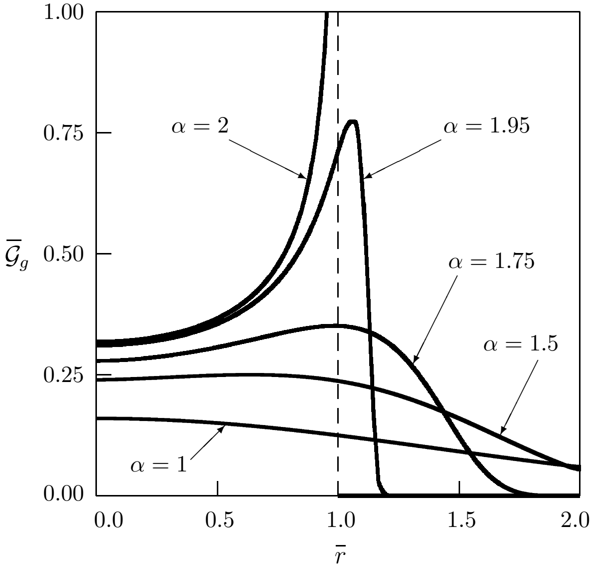

Figure 7.7: Dependence of the fundamental solution to the mathematical Neumann problem for a half-plane on distance for $1 \leq \alpha \leq 2$ [154]

The fundamental solution to the physical Neumann problem is obtained when Eqs. (7.28)–(7.30) are considered under the boundary condition

$$x = 0: \quad -D_{RL}^{1-\alpha} \frac{\partial \mathcal{G}_p}{\partial x} = g_0\, \delta(y)\, \delta(t), \qquad 0 < \alpha \le 1, \tag{7.38}$$

$$x = 0: \quad -I_{RL}^{\alpha-1} \frac{\partial \mathcal{G}_p}{\partial x} = g_0\, \delta(y)\, \delta(t), \qquad 1 < \alpha \le 2. \tag{7.39}$$

The solution has the following form:

$$\mathcal{G}_p(x, y, t) = \frac{a g_0}{\pi^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} E_\alpha[-a(\xi^2 + \eta^2) t^\alpha]\, \cos(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{7.40}$$

or

$$\mathcal{G}_p = \frac{a g_0}{\pi} \int_0^\infty \rho\, E_\alpha(-a\rho^2)\, J_0(r\rho)\, \mathrm{d}\rho. \tag{7.41}$$

Constant boundary value of the normal derivative in a local area. Consider the initial-boundary value problem (7.28)–(7.30) with the constant boundary value of normal derivative of a function in the domain $|y| < l$:

$$x = 0: \quad -\frac{\partial T}{\partial x} = \begin{cases} w_0, & |y| < l, \\ 0, & |y| > l. \end{cases} \tag{7.42}$$

The integral transforms technique leads to [154]

$$T = \frac{2 w_0}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(l\eta)}{\eta} \cos(y\eta)\, \mathrm{d}\eta$$

$$\times \int_0^\infty \frac{1}{\xi^2 + \eta^2} \left\{1 - E_\alpha[-a(\xi^2 + \eta^2) t^\alpha]\right\} \cos(x\xi)\, \mathrm{d}\xi \tag{7.43}$$

or, after passing to the polar coordinates and taking into account integrals (A.21) and (A.23) from the Appendix,

$$\overline{T} = \frac{1}{\pi} \int_0^\infty \left[1 - E_\alpha(-\kappa^2 \sigma^2)\right] \frac{1}{\sigma} \mathrm{d}\sigma \int_{\bar{y}-1}^{\bar{y}+1} J_0\left(\sigma \sqrt{u^2 + \bar{x}^2}\right) \mathrm{d}u, \tag{7.44}$$

where $\overline{T} = T/(w_0 l)$, and $\bar{x}$, $\bar{y}$ and κ are described by (7.21).

Examine several particular cases.

Helmholtz equation ($\alpha \to 0$)

$$\overline{T} = \frac{1}{\pi} \int_{\bar{y}-1}^{\bar{y}+1} K_0\left(\frac{\sqrt{u^2 + \bar{x}^2}}{\kappa}\right) \mathrm{d}u. \tag{7.45}$$

Classical diffusion equation ($\alpha = 1$)

$$\overline{T} = \frac{1}{\pi} \int\limits_0^\infty \left[1 - \exp(-\kappa^2 \sigma^2)\right] \frac{1}{\sigma} \, \mathrm{d}\sigma \int\limits_{\bar{y}-1}^{\bar{y}+1} J_0\left(\sigma \sqrt{u^2 + \bar{x}^2}\right) \mathrm{d}u. \tag{7.46}$$

Wave equation ($\alpha = 2$)

In the case of a wave equation, taking Eqs. (A.1), (A.2) and (A.45) into account, we obtain the following solution ($y \geq 0$ is considered as in (7.25)–(7.27)):

a) $0 < \kappa < |1 - \bar{y}|$

$$\overline{T} = \begin{cases} \dfrac{1}{2}(\kappa - \bar{x})[1 + \operatorname{sign}(1 - \bar{y})], & 0 < \bar{x} < \kappa, \\ 0, & \kappa < \bar{x} < \infty; \end{cases} \tag{7.47}$$

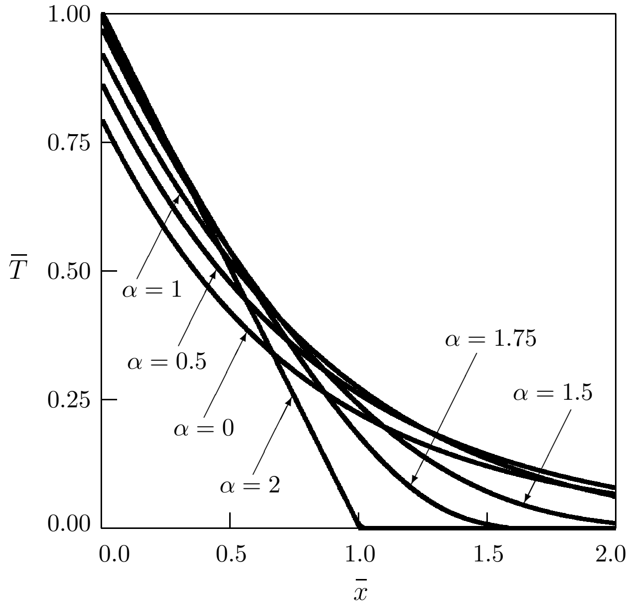

Figure 7.8: Dependence of solution on the coordinate x (the constant boundary value of the normal derivative of temperature in a local area) [154]

b) $|1-\bar{y}| < \kappa < 1+\bar{y}$

$$\overline{T} = \begin{cases} \dfrac{1}{2}(\kappa-\bar{x}) + \dfrac{\kappa}{\pi}\arcsin\dfrac{1-\bar{y}}{\sqrt{\kappa^2-\bar{x}^2}} \\ -\dfrac{\bar{x}}{\pi}\arctan\dfrac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} \\ +\dfrac{1}{2\pi}(1-\bar{y})\ln\dfrac{\kappa+\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}{\kappa-\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}, & 0<\bar{x}<\sqrt{\kappa^2-(1-\bar{y})^2}, \\ \dfrac{1}{2}(\kappa-\bar{x})[1+\operatorname{sign}(1-\bar{y})], & \sqrt{\kappa^2-(1-\bar{y})^2}<\bar{x}<\kappa, \\ 0, & \kappa<\bar{x}<\infty; \end{cases} \tag{7.48}$$

c) $1+\bar{y} < \kappa < \infty$

$$\overline{T} = \begin{cases} \frac{1}{\pi}\Big[\kappa\arcsin\frac{1-\bar{y}}{\sqrt{\kappa^2-\bar{x}^2}} + \kappa\arcsin\frac{1+\bar{y}}{\sqrt{\kappa^2-\bar{x}^2}} - \bar{x}\arctan\frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} \\ -\bar{x}\arctan\frac{\kappa(1+\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}} + \frac{1}{2}(1-\bar{y})\ln\frac{\kappa+\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}{\kappa-\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} \\ +\frac{1}{2}(1+\bar{y})\ln\frac{\kappa+\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}}{\kappa-\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}}\Big], & 0<\bar{x}<\sqrt{\kappa^2-(1+\bar{y})^2}, \\ \frac{1}{2}(\kappa-\bar{x}) + \frac{\kappa}{\pi}\arcsin\frac{1-\bar{y}}{\sqrt{\kappa^2-\bar{x}^2}} - \frac{\bar{x}}{\pi}\arctan\frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} \\ +\frac{1}{2\pi}(1-\bar{y})\ln\frac{\kappa+\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}{\kappa-\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}, \\ \qquad \sqrt{\kappa^2-(1+\bar{y})^2}<\bar{x}<\sqrt{\kappa^2-(1-\bar{y})^2}, \\ \frac{1}{2}(\kappa-\bar{x})[1+\operatorname{sign}(1-\bar{y})], & \sqrt{\kappa^2-(1-\bar{y})^2}<\bar{x}<\kappa, \\ 0, & \kappa<\bar{x}<\infty. \end{cases} \tag{7.49}$$

Dependence of nondimensional solution $\overline{T}$ on distance $\bar{x}$ is shown in Fig. 7.8 for $y=0$ and $\kappa=1$. The plots of $\overline{T}$ versus $\bar{y}$ at the boundary $x=0$ are depicted in Fig. 7.9 for $\kappa=1.5$.

Constant boundary value of the heat flux in a local area. Consider the initial-boundary value problem (7.28)–(7.30) with the constant boundary value of the

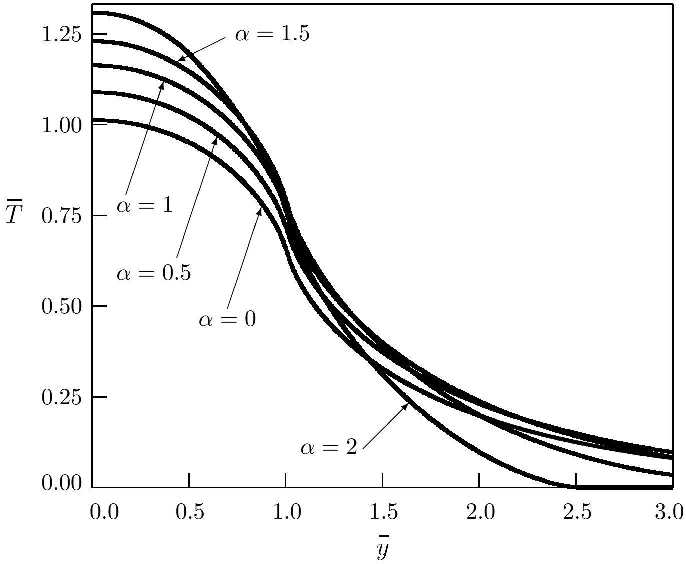

Figure 7.9: Dependence of the solution on distance $\bar{y}$ for $x = 0$ (the Neumann boundary condition for a half-plane with constant normal derivative of a function in a local area; $\kappa = 1.5$) [154]

heat flux in the domain $|y| < l$:

$$x = 0: \quad -D_{RL}^{1-\alpha} \frac{\partial T}{\partial x} = \begin{cases} w_0, & |y| < l, \\ 0, & |y| > l, \end{cases} \quad 0 < \alpha \le 1, \tag{7.50}$$

$$x = 0: \quad -I^{\alpha-1} \frac{\partial T}{\partial x} = \begin{cases} w_0, & |y| < l, \\ 0, & |y| > l, \end{cases}, \quad 1 < \alpha \le 2. \tag{7.51}$$

The integral transforms technique leads to

$$T = \frac{2aw_0 t}{\pi^2} \int\limits_{-\infty}^{\infty} \frac{\sin(l\eta)}{\eta} \cos(y\eta)\, \mathrm{d}\eta \int\limits_{0}^{\infty} E_{\alpha,2}[-a(\xi^2 + \eta^2)t^\alpha] \cos(x\xi)\, \mathrm{d}\xi \tag{7.52}$$

or, after passing to the polar coordinates [177]

$$\overline{T} = \frac{1}{\pi} \int\limits_{0}^{\infty} E_{\alpha,2}(-\kappa^2\sigma^2)\, \sigma\, \mathrm{d}\sigma \int\limits_{\bar{y}-1}^{\bar{y}+1} J_0\left(\sigma\sqrt{u^2 + \bar{x}^2}\right) \mathrm{d}u, \tag{7.53}$$

where $\overline{T} = RT/(aw_0 t)$.

Dependence of nondimensional solution $\overline{T}$ on nondimensional distance $\bar{x}$ is shown in Fig. 7.10 for $y = 0$ and $\kappa = 1$.

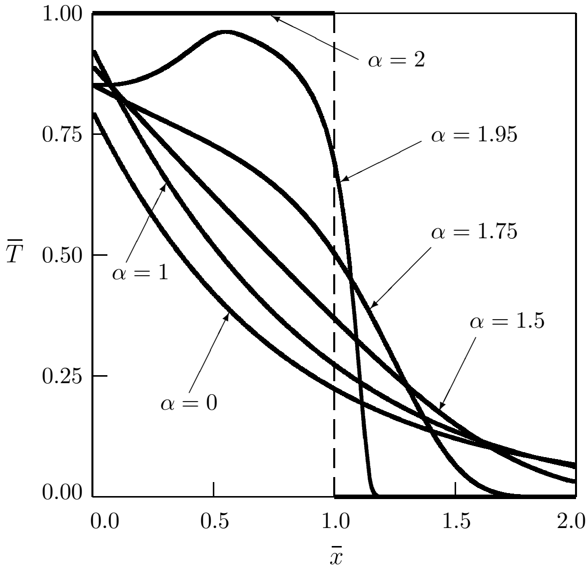

Figure 7.10: Dependence of the solution on distance $\bar{x}$ for $y = 0$ (the physical Neumann boundary condition for a half-plane with the constant heat flux in a local area; $\kappa = 1$) [177]

7.2.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi(x, y, t), \tag{7.54}$$

$$t = 0: \quad T = f(x, y), \qquad 0 < \alpha \leq 2, \tag{7.55}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x, y), \quad 1 < \alpha \leq 2, \tag{7.56}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} + HT = g(y, t), \tag{7.57}$$

$$\lim_{x \to \infty} T(x, y, t) = 0, \qquad \lim_{y \to \pm\infty} T(x, y, t) = 0. \tag{7.58}$$

The solution:

$$
\begin{aligned}
T(x,y,t) = & \int_{-\infty}^{\infty}\int_{0}^{\infty} f(\rho,\sigma)\,\mathcal{G}_f(x,y-\sigma,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\
& + \int_{-\infty}^{\infty}\int_{0}^{\infty} F(\rho,\sigma)\,\mathcal{G}_F(x,y-\sigma,\rho,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\
& + \int_{0}^{t}\int_{-\infty}^{\infty}\int_{0}^{\infty} \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,\rho,t-\tau)\ ,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
& + \int_{0}^{t}\int_{-\infty}^{\infty} g(\sigma,\tau)\,\mathcal{G}_g(x,y-\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau
\end{aligned}
\tag{7.59}
$$

where the fundamental solutions

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(x,y,\rho,t) \\ \mathcal{G}_F(x,y,\rho,t) \\ \mathcal{G}_\Phi(x,y,\rho,t) \end{pmatrix} = & \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} \begin{pmatrix} p_0\,E_\alpha\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ w_0 t\,E_{\alpha,2}\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big] \end{pmatrix} \\
\times & \frac{\xi\cos(x\xi)+H\sin(x\xi)}{\xi^2+H^2}\Big[\xi\cos(\rho\xi)+H\sin(\rho\xi)\Big]\cos(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}
\tag{7.60}
$$

are obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the coordinate y and the sin-cos-Fourier transform (2.40), (2.42) with respect to the coordinate x.

Fundamental solution to the mathematical Robin problem. The solution to the problem under the boundary condition

$$
x=0: \quad -\frac{\partial \mathcal{G}_m}{\partial x} + H\mathcal{G}_m = g_0\,\delta(y)\,\delta t
\tag{7.61}
$$

has the form [180]

$$
\begin{aligned}
\mathcal{G}_m(x,y,t) = & \frac{a g_0 t^{\alpha-1}}{\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} E_{\alpha,\alpha}[-a(\xi^2+\eta^2)t^\alpha] \\
& \times \frac{\xi^2\cos(x\xi)+H\xi\sin(x\xi)}{\xi^2+H^2}\cos(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}
\tag{7.62}
$$

and in the case of the classical diffusion equation is expressed as

$$\mathcal{G}_m(x,y,t) = \frac{a g_0}{2\sqrt{\pi a t}} \exp\left(-\frac{y^2}{4at}\right) \left[\frac{1}{\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at}\right) - H \exp\left(Hx + H^2 at\right) \operatorname{erfc}\left(\frac{x}{2\sqrt{at}} + H\sqrt{at}\right)\right]. \tag{7.63}$$

Dependence of the fundamental solution (7.62) on the space coordinates x and y is shown in Figs. 7.11–7.14 for different orders of fractional derivative and different convective heat transfer coefficients. In calculations we have introduced the following nondimensional quantities:

$$\overline{\mathcal{G}}_g = \frac{t}{g_0}\mathcal{G}_m, \quad \bar{x} = \frac{x}{\sqrt{a}t^{\alpha/2}}, \quad \bar{y} = \frac{y}{\sqrt{a}t^{\alpha/2}}, \quad \overline{H} = \sqrt{a}t^{\alpha/2}H. \tag{7.64}$$

Fundamental solution to the physical Robin problem. The initial-boundary value problem (7.54)–(7.56) is considered under the boundary condition

$$x = 0: \quad -D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial x} + H\mathcal{G}_p = g_0\,\delta(y)\,\delta(t), \quad 0 < \alpha \le 1, \tag{7.65}$$

$$x = 0: \quad -I^{\alpha-1}\frac{\partial \mathcal{G}_p}{\partial x} + H\mathcal{G}_p = g_0\,\delta(y)\,\delta(t), \quad 1 < \alpha \le 2. \tag{7.66}$$

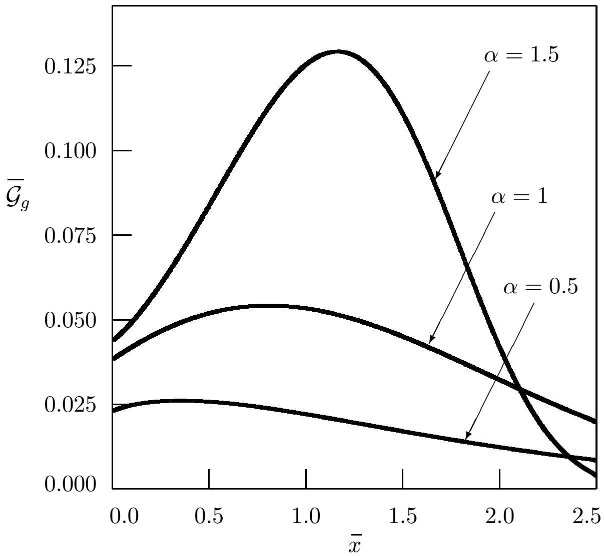

Figure 7.11: Dependence of the fundamental solution to the mathematical Robin problem for a half-plane on distance $\bar{x}$ for $y = 0$ and $\overline{H} = 1$

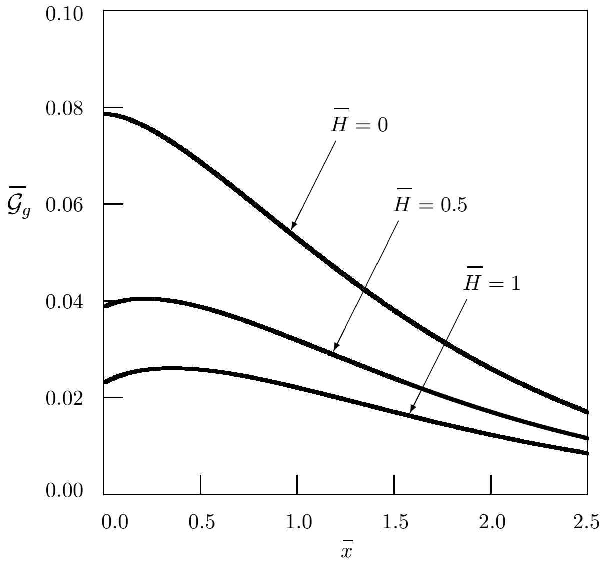

Figure 7.12: Dependence of the fundamental solution to the mathematical Robin problem for a half-plane on distance $\bar{x}$ for $y = 0$ and $\alpha = 0.5$

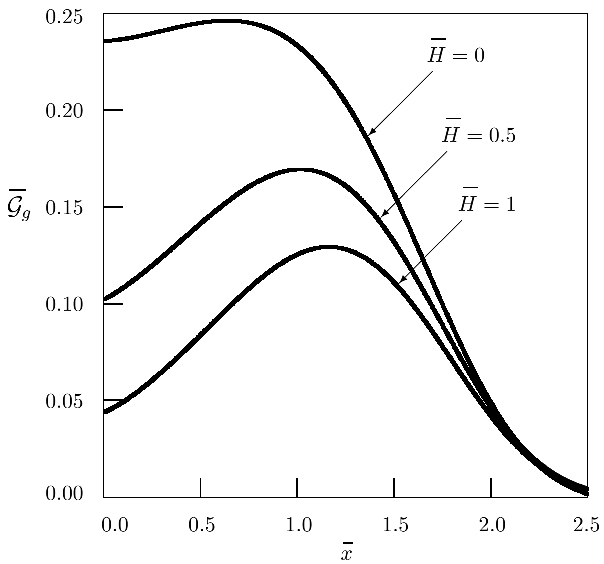

Figure 7.13: Dependence of the fundamental solution to the mathematical Robin problem for a half-plane on distance $\bar{x}$ for $y = 0$ and $\alpha = 1.5$.

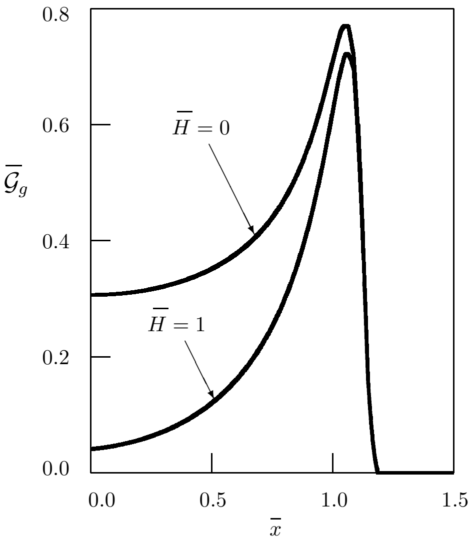

Figure 7.14: Dependence of the fundamental solution to the mathematical Robin problem for a half-plane on distance $\bar{x}$ for $y = 0$ and $\alpha = 1.95$

The Laplace transform of the boundary condition (7.65)–(7.66) gives

$$x = 0: \quad -\frac{\partial \mathcal{G}_p^*}{\partial x} + s^{\alpha-1} H \mathcal{G}_p^* = g_0\, s^{\alpha-1} \delta(y), \quad 0 < \alpha \leq 2. \tag{7.67}$$

It should be emphasized that in this case the kernel of the sin-cos-Fourier transform (2.40) with respect to the space coordinate x depends on the Laplace transform variable s and has a more complicated form than (2.42):

$$K(x, \xi, s) = \frac{\xi \cos(x\xi) + s^{\alpha-1} H \sin(x\xi)}{\sqrt{\xi^2 + (s^{\alpha-1}H)^2}}. \tag{7.68}$$

In the transform domain we obtain

$$\widetilde{\widetilde{\mathcal{G}}}_p^*(\xi, \eta, s) = \frac{a g_0}{\sqrt{2\pi}} \frac{\xi}{\sqrt{\xi^2 + (s^{\alpha-1}H)^2}} \frac{s^{\alpha-1}}{s^\alpha + a(\xi^2 + \eta^2)}. \tag{7.69}$$

The order of the inverse integral transforms is important: inversion of the Laplace transform should be carried out at the end. For $0 < \alpha \leq 1$, using the convolution theorem, we get (see [180])

$$\mathcal{G}_p(x,y,t) = \frac{ag_0}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \cos(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$- \frac{ag_0}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{H^2}{\xi^2} \cos(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$\times \int_0^t (t-\tau)^{1-2\alpha} E_{2-2\alpha,2-2\alpha}\left[-\frac{H^2}{\xi^2}(t-\tau)^{2-2\alpha}\right] E_\alpha\left[-a(\xi^2+\eta^2)\tau^\alpha\right] \mathrm{d}\tau$$

$$+ \frac{ag_0}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{H}{\xi} \sin(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$\times \int_0^t (t-\tau)^{1-2\alpha}\, E_{2-2\alpha,2-2\alpha}\left[-\frac{H^2}{\xi^2}(t-\tau)^{2-2\alpha}\right]$$

$$\times\, \tau^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)\tau^\alpha\right] \mathrm{d}\tau. \tag{7.70}$$

Similarly, for $1 < \alpha \le 2$, we obtain

$$\mathcal{G}_p(x,y,t) = \frac{ag_0 t^{\alpha-1}}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\xi}{H}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \sin(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$+ \frac{ag_0}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\xi^2}{H^2} \cos(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$\times \int_0^t (t-\tau)^{2\alpha-3} E_{2\alpha-2,2\alpha-2}\left[-\frac{\xi^2}{H^2}(t-\tau)^{2\alpha-2}\right] E_\alpha\left[-a(\xi^2+\eta^2)\tau^\alpha\right] \mathrm{d}\tau$$

$$- \frac{ag_0}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\xi^3}{H^3} \sin(x\xi)\, \cos(y\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$\times \int_0^t (t-\tau)^{2\alpha-3}\, E_{2\alpha-2,2\alpha-2}\left[-\frac{\xi^2}{H^2}(t-\tau)^{2\alpha-2}\right]$$

$$\times\, \tau^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)\tau^\alpha\right] \mathrm{d}\tau. \tag{7.71}$$

7.3 Domain $0 < x < \infty$, $0 < y < \infty$

7.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.72}$$

$$t = 0: \quad T = f(x,y), \qquad 0 < \alpha \le 2, \tag{7.73}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1 < \alpha \le 2, \tag{7.74}$$

$$x = 0: \quad T = g_1(y,t), \tag{7.75}$$

$$y = 0: \quad T = g_2(x,t), \tag{7.76}$$

$$\lim_{x\to\infty} T(x,y,t) = 0, \qquad \lim_{y\to\infty} T(x,y,t) = 0. \tag{7.77}$$

The solution:

$$\begin{aligned} T(x,y,t) = & \int_0^\infty\int_0^\infty f(\rho,\sigma)\,\mathcal{G}_f(x,y,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^\infty\int_0^\infty F(\rho,\sigma)\,\mathcal{G}_F(x,y,\rho,\sigma,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^t\int_0^\infty\int_0^\infty \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_0^\infty g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_0^\infty g_2(\rho,\tau)\,\mathcal{G}_{g2}(x,y,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\tau \end{aligned} \tag{7.78}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \begin{pmatrix} p_0\,E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$

$$\times\ \sin(x\xi)\sin(\rho\xi)\,\sin(y\eta)\,\sin(\sigma\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{7.79}$$

The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,\sigma,t) = \frac{a g_0}{q_0} \left.\frac{\partial \mathcal{G}_{\Phi}(x,y,\rho,\sigma,t)}{\partial \rho}\right|_{\rho=0}, \tag{7.80}$$

$$\mathcal{G}_{g2}(x,y,\rho,t) = \frac{a g_0}{q_0} \left.\frac{\partial \mathcal{G}_{\Phi}(x,y,\rho,\sigma,t)}{\partial \sigma}\right|_{\sigma=0}. \tag{7.81}$$

7.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.82}$$

$$t=0: \quad T = f(x,y), \qquad 0<\alpha\le 2, \tag{7.83}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1<\alpha\le 2, \tag{7.84}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(y,t), \tag{7.85}$$

$$y=0: \quad -\frac{\partial T}{\partial y} = g_2(x,t), \tag{7.86}$$

$$\lim_{x\to\infty} T(x,y,t) = 0, \qquad \lim_{y\to\infty} T(x,y,t) = 0. \tag{7.87}$$

The solution:

$$\begin{aligned} T(x,y,t) = & \int_0^\infty\int_0^\infty f(\rho,\sigma)\,\mathcal{G}_f(x,y,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^\infty\int_0^\infty F(\rho,\sigma)\,\mathcal{G}_F(x,y,\rho,\sigma,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^t\int_0^\infty\int_0^\infty \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_0^\infty g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_0^\infty g_2(\rho,\tau)\,\mathcal{G}_{g2}(x,y,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\tau \end{aligned} \tag{7.88}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$
$$\times \cos(x\xi)\,\cos(\rho\xi)\,\cos(y\eta)\,\cos(\sigma\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{7.89}$$

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,y,\sigma,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t)\bigg|_{\rho=0}, \tag{7.90}$$

$$\mathcal{G}_{m2}(x,y,\rho,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t)\bigg|_{\sigma=0}, \tag{7.91}$$

$$\mathcal{G}_{p1}(x,y,\sigma,t) = \frac{ag_0}{p_0}\,\mathcal{G}_f(x,y,\rho,\sigma,t)\bigg|_{\rho=0}, \tag{7.92}$$

$$\mathcal{G}_{p2}(x,y,\rho,t) = \frac{ag_0}{p_0}\,\mathcal{G}_f(x,y,\rho,\sigma,t)\bigg|_{\sigma=0}. \tag{7.93}$$

7.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.94}$$

$$t=0: \quad T = f(x,y), \qquad 0<\alpha\le 2, \tag{7.95}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1<\alpha\le 2, \tag{7.96}$$

$$x=0: \quad -\frac{\partial T}{\partial x} + H_1 T = g_1(y,t), \tag{7.97}$$

$$y=0: \quad -\frac{\partial T}{\partial y} + H_2 T = g_2(x,t), \tag{7.98}$$

$$\lim_{x\to\infty} T(x,y,t) = 0, \qquad \lim_{y\to\infty} T(x,y,t) = 0. \tag{7.99}$$

The solution:

$$\begin{aligned} T(x,y,t) &= \int_0^\infty\int_0^\infty f(\rho,\sigma)\,\mathcal{G}_f(x,y,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &+ \int_0^\infty\int_0^\infty F(\rho,\sigma)\,\mathcal{G}_F(x,y,\rho,\sigma,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &+ \int_0^t\int_0^\infty\int_0^\infty \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty g_2(\rho,\tau)\,\mathcal{G}_{g2}(x,y,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\tau \end{aligned} \tag{7.100}$$

with

$$\begin{aligned} &\begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \begin{pmatrix} p_0\,E_\alpha\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ w_0 t\,E_{\alpha,2}\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big] \end{pmatrix} \\ &\times \frac{[\xi\cos(x\xi)+H_1\sin(x\xi)]\,[\xi\cos(\rho\xi)+H_1\sin(\rho\xi)]}{\xi^2+H_1^2} \\ &\times \frac{[\eta\cos(y\eta)+H_2\sin(y\eta)]\,[\eta\cos(\sigma\eta)+H_2\sin(\sigma\eta)]}{\eta^2+H_2^2}\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{7.101}$$

The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(x,y,\sigma,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\rho=0}, \tag{7.102}$$

$$\mathcal{G}_{m2}(x,y,\rho,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\sigma=0}. \tag{7.103}$$

7.4 Domain $0 < x < L, -\infty < y < \infty$

7.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.104}$$

$$t = 0: \quad T = f(x,y), \qquad 0 < \alpha \le 2, \tag{7.105}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1 < \alpha \le 2, \tag{7.106}$$

$$x = 0: \quad T = g_1(y,t), \tag{7.107}$$

$$x = L: \quad T = g_2(y,t), \tag{7.108}$$

$$\lim_{y\to\pm\infty} T(x,y,t) = 0. \tag{7.109}$$

The solution:

$$\begin{aligned} T(x,y,t) = & \int_{-\infty}^{\infty}\int_0^L f(\rho,\sigma)\,\mathcal{G}_f(x,y-\sigma,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_{-\infty}^{\infty}\int_0^L F(\rho,\sigma)\,\mathcal{G}_F(x,y-\sigma,\rho,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^t\int_{-\infty}^{\infty}\int_0^L \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_{-\infty}^{\infty} g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y-\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_{-\infty}^{\infty} g_2(\sigma,\tau)\,\mathcal{G}_{g2}(x,y-\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned} \tag{7.110}$$

with

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} = \frac{1}{\pi L}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \end{pmatrix} \\ \times \sin(x\xi_k)\,\sin(\rho\xi_k)\,\cos(y\eta)\,\mathrm{d}\eta, \end{aligned} \tag{7.111}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the Dirichlet problem are calculated as

$$\mathcal{G}_{g1}(x,y,t) = \frac{a g_0}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(x,y,\rho,t)}{\partial \rho} \right|_{\rho=0}, \tag{7.112}$$

$$\mathcal{G}_{g2}(x,y,t) = -\frac{a g_0}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(x,y,\rho,t)}{\partial \rho} \right|_{\rho=L}. \tag{7.113}$$

7.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi(x,y,t), \tag{7.114}$$

$$t = 0: \quad T = f(x,y), \qquad 0 < \alpha \le 2, \tag{7.115}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1 < \alpha \le 2, \tag{7.116}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} = g_1(y,t), \tag{7.117}$$

$$x = L: \quad \frac{\partial T}{\partial x} = g_2(y,t), \tag{7.118}$$

$$\lim_{y \to \pm\infty} T(x,y,t) = 0. \tag{7.119}$$

The solution:

$$\begin{aligned} T(x,y,t) = & \int_{-\infty}^{\infty} \int_0^L f(\rho,\sigma)\, \mathcal{G}_f(x, y-\sigma, \rho, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma \\ & + \int_{-\infty}^{\infty} \int_0^L F(\rho,\sigma)\, \mathcal{G}_F(x, y-\sigma, \rho, t))\, \mathrm{d}\rho\, \mathrm{d}\sigma \\ & + \int_0^t \int_{-\infty}^{\infty} \int_0^L \Phi(\rho,\sigma,\tau)\, \mathcal{G}_\Phi(x, y-\sigma, \rho, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau \\ & + \int_0^t \int_{-\infty}^{\infty} g_1(\sigma,\tau)\, \mathcal{G}_{g1}(x, y-\sigma, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau \\ & + \int_0^t \int_{-\infty}^{\infty} g_2(\sigma,\tau)\, \mathcal{G}_{g2}(x, y-\sigma, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau \end{aligned} \tag{7.120}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,\rho,t) \\ \mathcal{G}_F(x,y,\rho,t) \\ \mathcal{G}_\Phi(x,y,\rho,t) \end{pmatrix} = \frac{1}{\pi L} \sum_{k=0}^{\infty}{}' \int_{-\infty}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \cos(x\xi_k)\ \cos(\rho\xi_k)\ \cos(y\eta)\,\mathrm{d}\eta, \tag{7.121}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,y,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,t)\bigg|_{\rho=0}, \tag{7.122}$$

$$\mathcal{G}_{m2}(x,y,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,t)\bigg|_{\rho=L}, \tag{7.123}$$

$$\mathcal{G}_{p1}(x,y,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,t)\bigg|_{\rho=0}, \tag{7.124}$$

$$\mathcal{G}_{p2}(x,y,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,t)\bigg|_{\rho=L}. \tag{7.125}$$

7.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.126}$$

$$t = 0: \quad T = f(x,y), \qquad 0 < \alpha \le 2, \tag{7.127}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1 < \alpha \le 2, \tag{7.128}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} + HT = g_1(y,t), \tag{7.129}$$

$$x = L: \quad \frac{\partial T}{\partial x} + HT = g_2(y,t), \tag{7.130}$$

$$\lim_{y\to\pm\infty} T(x,y,t) = 0. \tag{7.131}$$

The solution:

$$
\begin{aligned}
T(x,y,t) = & \int_{-\infty}^{\infty}\int_{0}^{L} f(\rho,\sigma)\,\mathcal{G}_f(x,y-\sigma,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\
& + \int_{-\infty}^{\infty}\int_{0}^{L} F(\rho,\sigma)\,\mathcal{G}_F(x,y-\sigma,\rho,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\
& + \int_{0}^{t}\int_{-\infty}^{\infty}\int_{0}^{L} \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
& + \int_{0}^{t}\int_{-\infty}^{\infty} g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
& + \int_{0}^{t}\int_{-\infty}^{\infty} g_2(\sigma,\tau)\,\mathcal{G}_{g2}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau
\end{aligned} \tag{7.132}
$$

with

$$
\begin{pmatrix} \mathcal{G}_f(x,y,\rho,t) \\ \mathcal{G}_F(x,y,\rho,t) \\ \mathcal{G}_\Phi(x,y,\rho,t) \end{pmatrix} = \frac{1}{\pi L}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \end{pmatrix}
$$

$$
\times \frac{\xi_k\cos(x\xi_k)+H\sin(x\xi_k)}{\xi_k^2+H^2+\frac{2H}{L}}\big[\xi_k\cos(\rho\xi_k)+H\sin(\rho\xi_k)\big]\cos(y\eta)\,\mathrm{d}\eta, \tag{7.133}
$$

where ξ_k are the positive roots of the transcendental equation

$$
\tan(L\xi_k) = 2H\xi_k/(\xi_k^2-H^2).
$$

The fundamental solutions to the mathematical Robin problem are calculated as

$$
\mathcal{G}_{m1}(x,y,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(x,y,\rho,t)\Big|_{\rho=0}, \tag{7.134}
$$

$$
\mathcal{G}_{m2}(x,y,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(x,y,\rho,t)\Big|_{\rho=L}. \tag{7.135}
$$

7.5 Domain $0 < x < L$, $0 < y < \infty$

7.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.136}$$

$$t = 0: \quad T = f(x,y), \qquad 0 < \alpha \le 2, \tag{7.137}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1 < \alpha \le 2, \tag{7.138}$$

$$x = 0: \quad T = g_1(y,t), \tag{7.139}$$

$$x = L: \quad T = g_2(y,t), \tag{7.140}$$

$$y = 0: \quad T = g_3(x,t). \tag{7.141}$$

The solution:

$$\begin{aligned} T(x,y,t) &= \int_0^\infty \int_0^L f(\rho,\sigma)\, \mathcal{G}_f(x,y,\rho,\sigma,t)\, \mathrm{d}\rho\, \mathrm{d}\sigma \\ &+ \int_0^\infty \int_0^L F(\rho,\sigma)\, \mathcal{G}_F(x,y,\rho,\sigma,t))\, \mathrm{d}\rho\, \mathrm{d}\sigma \\ &+ \int_0^t \int_0^\infty \int_0^L \Phi(\rho,\sigma,\tau)\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty g_1(\sigma,\tau)\, \mathcal{G}_{g1}(x,y,\sigma,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty g_2(\sigma,\tau)\, \mathcal{G}_{g2}(x,y,\sigma,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^L g_3(\rho,\tau)\, \mathcal{G}_{g3}(x,y,\rho,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\tau \end{aligned} \tag{7.142}$$

with

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} &= \frac{4}{\pi L} \sum_{k=1}^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi_k^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi_k^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi_k^2+\eta^2)t^\alpha\right] \end{pmatrix} \\ &\times \sin(x\xi_k)\sin(\rho\xi_k)\, \sin(y\eta)\, \sin(\sigma\eta)\, \mathrm{d}\eta, \end{aligned} \tag{7.143}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the Dirichlet problem are calculated as

$$\mathcal{G}_{g1}(x,y,\sigma,t) = \frac{ag_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,\rho,\sigma,t)}{\partial \rho}\bigg|_{\rho=0}, \tag{7.144}$$

$$\mathcal{G}_{g2}(x,y,\sigma,t) = -\frac{ag_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,\rho,\sigma,t)}{\partial \rho}\bigg|_{\rho=L}, \tag{7.145}$$

$$\mathcal{G}_{g3}(x,y,\rho,t) = \frac{ag_0}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,\rho,\sigma,t)}{\partial \sigma}\bigg|_{\sigma=0}. \tag{7.146}$$

7.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.147}$$

$$t=0: \quad T = f(x,y), \qquad 0<\alpha\le 2, \tag{7.148}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1<\alpha\le 2, \tag{7.149}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(y,t), \tag{7.150}$$

$$x=L: \quad \frac{\partial T}{\partial x} = g_2(y,t), \tag{7.151}$$

$$y=0: \quad -\frac{\partial T}{\partial y} = g_3(x,t). \tag{7.152}$$

The solution:

$$\begin{aligned} T(x,y,t) = & \int_0^\infty\int_0^L f(\rho,\sigma)\,\mathcal{G}_f(x,y,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^\infty\int_0^L F(\rho,\sigma)\,\mathcal{G}_F(x,y,\rho,\sigma,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^t\int_0^\infty\int_0^L \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_0^\infty g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^t\int_0^\infty g_2(\sigma,\tau)\,\mathcal{G}_{g2}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned}$$

$$+ \int_0^t \int_0^L g_3(\rho,\tau)\, \mathcal{G}_{g3}(x,y,\rho,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\tau \tag{7.153}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} = \frac{4}{\pi L} {\sum_{k=0}^{\infty}}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \cos(x\xi_k)\cos(\rho\xi_k)\, \cos(y\eta)\, \cos(\sigma\eta)\, \mathrm{d}\eta, \tag{7.154}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,y,\sigma,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t)\big|_{\rho=0}, \tag{7.155}$$

$$\mathcal{G}_{m2}(x,y,\sigma,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\rho=L}, \tag{7.156}$$

$$\mathcal{G}_{m3}(x,y,\rho,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\sigma=0}, \tag{7.157}$$

$$\mathcal{G}_{p1}(x,y,\sigma,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,\sigma,t)\Big|_{\rho=0}, \tag{7.158}$$

$$\mathcal{G}_{p2}(x,y,\sigma,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,\sigma,t)\Big|_{\rho=L}, \tag{7.159}$$

$$\mathcal{G}_{p3}(x,y,\rho,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,\sigma,t)\Big|_{\sigma=0}. \tag{7.160}$$

7.6 Domain $0 < x < L_1$, $0 < y < L_2$

7.6.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi(x,y,t), \tag{7.161}$$

$$t = 0: \quad T = f(x,y), \qquad 0 < \alpha \le 2, \tag{7.162}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1 < \alpha \le 2, \tag{7.163}$$

$$x = 0: \quad T = g_1(y,t), \tag{7.164}$$

$$x = L_1: \quad T = g_2(y,t), \tag{7.165}$$
$$y = 0: \quad T = g_3(x,t), \tag{7.166}$$
$$y = L_2: \quad T = g_4(x,t). \tag{7.167}$$

The solution:

$$\begin{aligned} T(x,y,t) &= \int_0^{L_2}\!\!\int_0^{L_1} f(\rho,\sigma)\,\mathcal{G}_f(x,y,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &+ \int_0^{L_2}\!\!\int_0^{L_1} F(\rho,\sigma)\,\mathcal{G}_F(x,y,\rho,\sigma,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ &+ \int_0^{t}\!\!\int_0^{L_2}\!\!\int_0^{L_1} \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_0^{t}\!\!\int_0^{L_2} g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_0^{t}\!\!\int_0^{L_2} g_2(\sigma,\tau)\,\mathcal{G}_{g2}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_0^{t}\!\!\int_0^{L_1} g_3(\rho,\tau)\,\mathcal{G}_{g3}(x,y,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &+ \int_0^{t}\!\!\int_0^{L_1} g_4(\rho,\tau)\,\mathcal{G}_{g4}(x,y,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\tau \end{aligned} \tag{7.168}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} = \frac{4}{L_1L_2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \begin{pmatrix} p_0\,E_\alpha\big[-a(\xi_k^2+\eta_m^2)t^\alpha\big] \\ w_0 t\,E_{\alpha,2}\big[-a(\xi_k^2+\eta_m^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta_m^2)t^\alpha\big] \end{pmatrix}$$
$$\times\ \sin(x\xi_k)\sin(\rho\xi_k)\,\sin(y\eta_m)\,\sin(\sigma\eta_m), \tag{7.169}$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$.

The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,\sigma,t) = \frac{ag_0}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,\rho,\sigma,t)}{\partial\rho}\bigg|_{\rho=0}, \tag{7.170}$$

$$\mathcal{G}_{g2}(x,y,\sigma,t) = -\frac{ag_0}{q_0}\left.\frac{\partial \mathcal{G}_\Phi(x,y,\rho,\sigma,t)}{\partial \rho}\right|_{\rho=L_1}, \tag{7.171}$$

$$\mathcal{G}_{g3}(x,y,\rho,t) = \frac{ag_0}{q_0}\left.\frac{\partial \mathcal{G}_\Phi(x,y,\rho,\sigma,t)}{\partial \sigma}\right|_{\sigma=0}, \tag{7.172}$$

$$\mathcal{G}_{g4}(x,y,\rho,t) = -\frac{ag_0}{q_0}\left.\frac{\partial \mathcal{G}_\Phi(x,y,\rho,\sigma,t)}{\partial \sigma}\right|_{\sigma=L_2}. \tag{7.173}$$

7.6.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi(x,y,t), \tag{7.174}$$

$$t=0: \quad T = f(x,y), \qquad 0 < \alpha \le 2, \tag{7.175}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y), \quad 1 < \alpha \le 2, \tag{7.176}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(y,t), \tag{7.177}$$

$$x=L_1: \quad \frac{\partial T}{\partial x} = g_2(y,t), \tag{7.178}$$

$$y=0: \quad -\frac{\partial T}{\partial y} = g_3(x,t), \tag{7.179}$$

$$y=L_2: \quad \frac{\partial T}{\partial y} = g_4(x,t). \tag{7.180}$$

The solution:

$$\begin{aligned} T(x,y,t) = & \int_0^{L_2}\int_0^{L_1} f(\rho,\sigma)\,\mathcal{G}_f(x,y,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^{L_2}\int_0^{L_1} F(\rho,\sigma)\,\mathcal{G}_F(x,y,\rho,\sigma,t))\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ & + \int_0^{t}\int_0^{L_2}\int_0^{L_1} \Phi(\rho,\sigma,\tau)\,\mathcal{G}_\Phi(x,y,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ & + \int_0^{t}\int_0^{L_2} g_1(\sigma,\tau)\,\mathcal{G}_{g1}(x,y,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned}$$

$$+ \int_0^t \int_0^{L_2} g_2(\sigma,\tau)\, \mathcal{G}_{g2}(x,y,\sigma,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau$$
$$+ \int_0^t \int_0^{L_1} g_3(\rho,\tau)\, \mathcal{G}_{g3}(x,y,\rho,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\tau$$
$$+ \int_0^t \int_0^{L_1} g_4(\rho,\tau)\, \mathcal{G}_{g4}(x,y,\rho,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\tau \qquad (7.181)$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,\rho,\sigma,t) \end{pmatrix} = \frac{4}{L_1 L_2} \sum_{k=0}^{\infty}{}' \sum_{m=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta_m^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta_m^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta_m^2)t^\alpha\big] \end{pmatrix}$$
$$\times \cos(x\xi_k)\cos(\rho\xi_k)\cos(y\eta_m)\cos(\sigma\eta_m), \qquad (7.182)$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$.

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,y,\sigma,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\rho=0}, \qquad (7.183)$$

$$\mathcal{G}_{m2}(x,y,\sigma,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\rho=L_1}, \qquad (7.184)$$

$$\mathcal{G}_{m3}(x,y,\rho,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\sigma=0}, \qquad (7.185)$$

$$\mathcal{G}_{m4}(x,y,\rho,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,\rho,\sigma,t)\Big|_{\sigma=L_2}, \qquad (7.186)$$

$$\mathcal{G}_{p1}(x,y,\sigma,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,\sigma,t)\Big|_{\rho=0}, \qquad (7.187)$$

$$\mathcal{G}_{p2}(x,y,\sigma,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,\sigma,t)\Big|_{\rho=L_1}, \qquad (7.188)$$

$$\mathcal{G}_{p3}(x,y,\rho,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,\sigma,t)\Big|_{\sigma=0}, \qquad (7.189)$$

$$\mathcal{G}_{p4}(x,y,\rho,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(x,y,\rho,\sigma,t)\Big|_{\sigma=L_2}. \qquad (7.190)$$

Chapter 8

Equations in Polar Coordinates

I'm very good at integral and differential calculus,
I know the scientific names of beings animalculous.

W.S. Gilbert

8.1 Domain $0 \leq r < \infty$, $0 \leq \varphi \leq 2\pi$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r, \varphi, t), \tag{8.1}$$

$$t = 0: \quad T = f(r, \varphi), \qquad 0 < \alpha \leq 2, \tag{8.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi), \qquad 1 < \alpha \leq 2, \tag{8.3}$$

$$\lim_{r \to \infty} T(r, \varphi, t) = 0. \tag{8.4}$$

The solution:

$$\begin{aligned} T(r, \varphi, t) = & \int_0^{2\pi}\int_0^{\infty} f(\rho, \phi)\, \mathcal{G}_f(r, \varphi, \rho, \phi, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \\ & + \int_0^{2\pi}\int_0^{\infty} F(\rho, \phi)\, \mathcal{G}_F(r, \varphi, \rho, \phi, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \\ & + \int_0^{t}\int_0^{2\pi}\int_0^{\infty} \Phi(\rho, \phi, \tau)\, \mathcal{G}_\Phi(r, \varphi, \rho, \phi, t - \tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\tau, \end{aligned} \tag{8.5}$$

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where $\mathcal{G}_f(r,\rho,\varphi,t)$ is the fundamental solution to the first Cauchy problem, $\mathcal{G}_F(r,\rho,\varphi,t)$ is the fundamental solution to the second Cauchy problem, and $\mathcal{G}_\Phi(r,\rho,\varphi,t)$ is the fundamental solution to the source problem.

Consider the fundamental solution to the first Cauchy problem.

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_f}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathcal{G}_f}{\partial \varphi^2}\right), \tag{8.6}$$

$$t=0: \quad \mathcal{G}_f = p_0 \frac{\delta(r-\rho)}{r}\,\delta(\varphi-\phi), \qquad 0<\alpha\le 2, \tag{8.7}$$

$$t=0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \qquad 1<\alpha\le 2. \tag{8.8}$$

The Laplace transform with respect to time t gives

$$s^\alpha \mathcal{G}_f^* - s^{\alpha-1} p_0 \frac{\delta(r-\rho)}{r}\,\delta(\varphi-\phi) = a\left(\frac{\partial^2 \mathcal{G}_f^*}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_f^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathcal{G}_f^*}{\partial \varphi^2}\right). \tag{8.9}$$

Next we use the finite Fourier transform (2.72) with respect to the angular coordinate φ for 2π-periodic functions, thus obtaining

$$s^\alpha \widetilde{\mathcal{G}}_f^* - s^{\alpha-1} p_0 \frac{\delta(r-\rho)}{r}\cos\left[n(\varphi-\phi)\right] = a\left(\frac{\partial^2 \widetilde{\mathcal{G}}_f^*}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{\mathcal{G}}_f^*}{\partial r} - \frac{n^2}{r^2}\widetilde{\mathcal{G}}_f^*\right). \tag{8.10}$$

The Hankel transform (2.78) with respect to the radial variable r leads to the solution in the transform domain

$$\widehat{\widetilde{\mathcal{G}}}_f^* = p_0\, J_n(\rho\xi)\cos\left[n(\varphi-\phi)\right]\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{8.11}$$

The inverse integral transforms result in

$$\begin{aligned} \mathcal{G}_f(r,\varphi,\rho,\phi,t) &= \frac{p_0}{\pi}\sum_{n=0}^{\infty}{}' \cos\left[n(\varphi-\phi)\right] \\ &\times \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) J_n(r\xi)\, J_n(\rho\xi)\,\xi\,d\xi. \end{aligned} \tag{8.12}$$

Helmholtz equation ($\alpha \to 0$)

Evaluating integral (A.35) from Appendix, we get

$$\mathcal{G}_f = \frac{p_0}{a\pi}\sum_{n=0}^{\infty}{}'\cos\left[n(\varphi-\phi)\right]\begin{cases} I_n(r/\sqrt{a})\,K_n(\rho/\sqrt{a}), & 0\le r<\rho, \\ I_n(\rho/\sqrt{a})\,K_n(r/\sqrt{a}), & \rho<r<\infty, \end{cases} \tag{8.13}$$

where $I_n(r)$ and $K_n(r)$ are the modified Bessel functions. The sum in the right-hand side of (8.13) can be evaluated analytically taking into account that [196]

$$\sum_{n=0}^{\infty}{}' \, I_n(r)\, K_n(\rho) \cos(n\varphi) = \frac{1}{2}\, K_0 \left(\sqrt{r^2 + \rho^2 - 2r\rho\cos\varphi} \right).$$

Hence

$$\mathcal{G}_f = \frac{p_0}{2a\pi} K_0 \left(\sqrt{[r^2 + \rho^2 - 2r\rho\cos(\varphi - \phi)]/a} \right). \tag{8.14}$$

Classical diffusion equation ($\alpha = 1$)

It follows from (A.34) that

$$\mathcal{G}_f = \frac{1}{2\pi a t} \exp\left(-\frac{r^2 + \rho^2}{4at} \right) \sum_{n=0}^{\infty}{}' \cos\left[n(\varphi - \phi)\right] I_n\left(\frac{r\rho}{2at} \right). \tag{8.15}$$

Evaluating the sum in the right-hand side of (8.15) taking into account that [196]

$$\sum_{n=0}^{\infty}{}' \, I_n(r) \cos(n\varphi) = \frac{1}{2}\, \mathrm{e}^{r\,\cos\varphi},$$

we finally obtain [144]

$$\mathcal{G}_f = \frac{1}{4\pi a t} \exp\left[-\frac{r^2 + \rho^2 - 2r\rho\cos(\varphi - \phi)}{4at} \right]. \tag{8.16}$$

In a similar way we get the fundamental solutions to the second Cauchy problem and to the source problem:

$$\begin{pmatrix} \mathcal{G}_F(r, \varphi, \rho, \phi, t) \\ \mathcal{G}_\Phi(r, \varphi, \rho, \phi, t) \end{pmatrix} = \frac{1}{\pi} \sum_{n=0}^{\infty}{}' \cos\left[n(\varphi - \phi)\right]$$
$$\times \int_0^\infty \begin{pmatrix} w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} J_n(r\xi)\, J_n(\rho\xi)\, \xi \, \mathrm{d}\xi. \tag{8.17}$$

8.2 Domain $0 \le r < R,\, 0 \le \varphi \le 2\pi$

8.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} \right) + \Phi(r, \varphi, t), \tag{8.18}$$

$$t = 0: \quad T = f(r, \varphi), \qquad 0 < \alpha \le 2, \tag{8.19}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi), \qquad 1 < \alpha \le 2, \tag{8.20}$$

$$r = R: \quad T = g(\varphi, t). \tag{8.21}$$

The solution:

$$\begin{aligned} T(r, \varphi, t) = & \int_0^{2\pi} \int_0^R f(\rho, \phi)\, \mathcal{G}_f(r, \varphi, \rho, \phi, t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\phi \\ & + \int_0^{2\pi} \int_0^R F(\rho, \phi)\, \mathcal{G}_F(r, \varphi, \rho, \phi, t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\phi \\ & + \int_0^t \int_0^{2\pi} \int_0^R \Phi(\rho, \phi, \tau)\, \mathcal{G}_\Phi(r, \varphi, \rho, \phi, t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\phi \,\mathrm{d}\tau \\ & + \int_0^t \int_0^{2\pi} g(\phi, \tau)\, \mathcal{G}_g(r, \varphi, \phi, t-\tau) \,\mathrm{d}\phi \,\mathrm{d}\tau. \end{aligned} \tag{8.22}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate φ (2.72) for 2π-periodic functions and the finite Hankel transform (2.84) with respect to the radial coordinate r:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r, \varphi, \rho, \phi, t) \\ \mathcal{G}_F(r, \varphi, \rho, \phi, t) \\ \mathcal{G}_\Phi(r, \varphi, \rho, \phi, t) \end{pmatrix} = & \frac{2}{\pi R^2} {\sum_{n=0}^{\infty}}' \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \\ & \times \frac{J_n(r\xi_{nk})\, J_n(\rho\xi_{nk})}{[J_n'(R\xi_{nk})]^2} \cos[n(\varphi - \phi)], \end{aligned} \tag{8.23}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J_n(R\xi_{nk}) = 0. \tag{8.24}$$

The fundamental solution to the Dirichlet problem has the following form:

$$\begin{aligned} \mathcal{G}_g(r, \varphi, \phi, t) = & -\frac{2ag_0 t^{\alpha-1}}{\pi R} {\sum_{n=0}^{\infty}}' \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right) \\ & \times \frac{\xi_{nk}\, J_n(r\xi_{nk})}{J_n'(R\xi_{nk})} \cos\left[n(\varphi - \phi)\right]. \end{aligned} \tag{8.25}$$

8.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.26}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.27}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.28}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(\varphi,t). \tag{8.29}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) &= \int_0^{2\pi}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{2\pi}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^t\int_0^{2\pi}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau. \end{aligned} \tag{8.30}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate φ (2.72) for 2π-periodic functions and the finite Hankel transform (2.88) with respect to the radial coordinate r:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} &= \frac{1}{\pi R^2}\begin{pmatrix} p_0 \\ w_0 t \\ q_0 t^{\alpha-1}/\Gamma(\alpha) \end{pmatrix} \\ &+ \frac{2}{\pi}\sum_{n=0}^{\infty}{}'\sum_{k=1}^{\infty}\begin{pmatrix} p_0\,E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \\ &\times \frac{\xi_{nk}^2 J_n(r\xi_{nk})\,J_n(\rho\xi_{nk})}{(R^2\xi_{nk}^2 - n^2)\,[J_n(R\xi_{nk})]^2}\cos[n(\varphi-\phi)], \end{aligned} \tag{8.31}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J'_n(R\xi_{nk}) = 0. \tag{8.32}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$\mathcal{G}_m(r,\varphi,\phi,t) = \frac{aRg_0}{q_0}\mathcal{G}_\Phi(r,\rho,\varphi,t)\Big|_{\rho=R}, \tag{8.33}$$

$$\mathcal{G}_p(r,\varphi,\phi,t) = \frac{aRg_0}{p_0}\mathcal{G}_f(r,\rho,\varphi,t)\Big|_{\rho=R}. \tag{8.34}$$

8.2.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.35}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.36}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.37}$$

$$r = R: \quad \frac{\partial T}{\partial r} + HT = g(\varphi,t). \tag{8.38}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) &= \int_0^{2\pi}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{2\pi}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^t\int_0^{2\pi}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau. \end{aligned} \tag{8.39}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate φ

(2.72) for 2π-periodic functions and the finite Hankel transform (2.92) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\pi} {\sum_{n=0}^{\infty}}' \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix}$$

$$\times \frac{\xi_{nk}^2 J_n(r\xi_{nk})\, J_n(\rho\xi_{nk})}{(R^2H^2 + R^2\xi_{nk}^2 - n^2)\,[J_n(R\xi_{nk})]^2} \cos\left[n(\varphi-\phi)\right], \tag{8.40}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$\xi_{nk} J_n'(R\xi_{nk}) + H J_n(R\xi_{nk}) = 0. \tag{8.41}$$

The fundamental solution to the mathematical Robin problem under zero initial conditions is calculated as

$$\mathcal{G}_g(r,\varphi,\phi,t) = \frac{aRg_0}{q_0}\mathcal{G}_\Phi(r,\rho,\varphi,t)\bigg|_{\rho=R}. \tag{8.42}$$

8.3 Domain $R < r < \infty,\ 0 \le \varphi \le 2\pi$

8.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.43}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.44}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \quad 1 < \alpha \le 2, \tag{8.45}$$

$$r = R: \quad T = g(\varphi,t), \tag{8.46}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.47}$$

The solution:

$$
\begin{aligned}
T(r,\varphi,t) &= \int\limits_0^{2\pi}\int\limits_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int\limits_0^{2\pi}\int\limits_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int\limits_0^{t}\int\limits_0^{2\pi}\int\limits_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int\limits_0^{t}\int\limits_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau. \qquad (8.48)
\end{aligned}
$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform for 2π-periodic functions with respect to the angular coordinate φ (2.72) and the Weber transform (2.108), (2.111) with respect to the radial variable r:

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} &= \frac{1}{\pi}\sum_{n=0}^{\infty}{}' \int\limits_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2t^\alpha) \end{pmatrix} \\
&\times \frac{J_n(r\xi)\,Y_n(R\xi)-Y_n(r\xi)\,J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)}\cos\left[n(\varphi-\phi)\right] \\
&\times \Big[J_n(\rho\xi)\,Y_n(R\xi)-Y_n(\rho\xi)\,J_n(R\xi)\Big]\,\xi\,\mathrm{d}\xi. \qquad (8.49)
\end{aligned}
$$

The fundamental solution to the Dirichlet problem under zero initial conditions reads:

$$
\begin{aligned}
\mathcal{G}_g(r,\varphi,\phi,t) &= -\frac{2ag_0t^{\alpha-1}}{\pi^2}\sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2t^\alpha\right)\cos\left[n(\varphi-\phi)\right] \\
&\times \frac{J_n(r\xi)\,Y_n(R\xi)-Y_n(r\xi)\,J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)}\,\xi\,\mathrm{d}\xi. \qquad (8.50)
\end{aligned}
$$

8.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.51}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.52}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \quad 1 < \alpha \le 2, \tag{8.53}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g(\varphi,t), \tag{8.54}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.55}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) &= \int_0^{2\pi}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{2\pi}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{2\pi}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau.
\end{aligned} \tag{8.56}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform for 2π-periodic functions with respect to the angular coordinate φ (2.72) and the Weber transform (2.108), (2.113) with respect to the radial variable r:

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} &= \frac{1}{\pi}\sum_{n=0}^{\infty}{}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \\
&\times \frac{J_n(r\xi)\,Y_n'(R\xi) - Y_n(r\xi)\,J_n'(R\xi)}{[J_n'(R\xi)]^2 + [Y_n'(R\xi)]^2}\cos[n(\varphi-\phi)] \\
&\times \Big[J_n(\rho\xi)\,Y_n'(R\xi) - Y_n(\rho\xi)\,J_n'(R\xi)\Big]\,\xi\,\mathrm{d}\xi.
\end{aligned} \tag{8.57}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\mathcal{G}_m(r,\varphi,\phi,t) = \frac{2ag_0t^{\alpha-1}}{\pi^2}\sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2t^{\alpha}\right)\cos\left[n(\varphi-\phi)\right] \times \frac{J_n(r\xi)\,Y_n'(R\xi)-Y_n(r\xi)\,J_n'(R\xi)}{[J_n'(R\xi)]^2+[Y_n'(R\xi)]^2}\,\mathrm{d}\xi; \tag{8.58}$$

$$\mathcal{G}_p(r,\varphi,\phi,t) = \frac{2ag_0}{\pi^2}\sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha}\left(-a\xi^2t^{\alpha}\right)\cos\left[n(\varphi-\phi)\right] \times \frac{J_n(r\xi)\,Y_n'(R\xi)-Y_n(r\xi)\,J_n'(R\xi)}{[J_n'(R\xi)]^2+[Y_n'(R\xi)]^2}\,\mathrm{d}\xi. \tag{8.59}$$

8.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial T}{\partial\varphi^2}\right)+\Phi(r,\varphi,t), \tag{8.60}$$

$$t=0:\quad T=f(r,\varphi),\qquad 0<\alpha\le 2, \tag{8.61}$$

$$t=0:\quad \frac{\partial T}{\partial t}=F(r,\varphi),\quad 1<\alpha\le 2, \tag{8.62}$$

$$r=R:\quad -\frac{\partial T}{\partial r}+HT=g(\varphi,t), \tag{8.63}$$

$$\lim_{r\to\infty} T(r,\varphi,t)=0. \tag{8.64}$$

The solution:

$$T(r,\varphi,t) = \int_0^{2\pi}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi + \int_0^{2\pi}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi + \int_0^{t}\int_0^{2\pi}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+ \int\limits_0^t \int\limits_0^{2\pi} g(\phi, \tau)\, \mathcal{G}_g(r, \varphi, \phi, t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\tau. \tag{8.65}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform for 2π-periodic functions with respect to the angular coordinate φ (2.72) and the Weber transform (2.108), (2.115) with respect to the radial variable r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{1}{\pi} \sum_{n=0}^{\infty}{}' \int\limits_0^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \cos[n(\varphi-\phi)]$$

$$\times \frac{J_n(r\xi)[\xi Y_n'(R\xi) - HY_n(R\xi)] - Y_n(r\xi)[\xi J_n'(R\xi) - HJ_n(R\xi)]}{[\xi Y_n'(R\xi) - HY_n(R\xi)]^2 + [\xi J_n'(R\xi) - HJ_n(R\xi)]^2}$$

$$\times \Big\{ J_n(\rho\xi)[\xi Y_n'(R\xi) - HY_n(R\xi)] - Y_n(\rho\xi)[\xi J_n'(R\xi) - HJ_n(R\xi)] \Big\} \xi\, \mathrm{d}\xi. \tag{8.66}$$

The fundamental solution to the mathematical Robin problem under zero initial conditions is written as:

$$\mathcal{G}_g(r,\varphi,\phi,t) = \frac{2ag_0 t^{\alpha-1}}{\pi^2} \sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) \cos[n(\varphi-\phi)]$$

$$\times \frac{J_n(r\xi)[\xi Y_n'(R\xi) - HY_n(R\xi)] - Y_n(r\xi)[\xi J_n'(R\xi) - HJ_n(R\xi)]}{[\xi Y_n'(R\xi) - HY_n(R\xi)]^2 + [\xi J_n'(R\xi) - HJ_n(R\xi)]^2}\, \xi\, \mathrm{d}\xi. \tag{8.67}$$

8.4 Domain $0 \le r < \infty$, $0 < \varphi < \varphi_0$

8.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.68}$$

$$t=0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.69}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.70}$$

$$\varphi = 0: \quad T = g_1(r,t), \tag{8.71}$$

$$\varphi = \varphi_0: \quad T = g_2(r,t), \tag{8.72}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.73}$$

The solution:

$$
\begin{aligned}
T(r,\varphi,t) &= \int_0^{\varphi_0}\int_0^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{\varphi_0}\int_0^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{\varphi_0}\int_0^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\infty} g_1(\rho,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.74}
$$

To obtain the fundamental solutions we use the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the angular coordinate φ and the Hankel transform (2.78) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. Thus, we get [190]

$$
\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0}\sum_{n=1}^{\infty}\int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}
$$

$$
\times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\phi}{\varphi_0}\right) J_{n\pi/\varphi_0}(r\xi)\,J_{n\pi/\varphi_0}(\rho\xi)\,\xi\,\mathrm{d}\xi. \tag{8.75}
$$

The fundamental solution to the first Dirichlet problem under zero initial conditions is expressed as

$$
\mathcal{G}_{g1}(r,\varphi,\rho,t) = \frac{2ag_0 t^{\alpha-1}}{\varphi_0\rho^2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{\varphi_0}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right)
$$

$$
\times J_{n\pi/\varphi_0}(r\xi)\,J_{n\pi/\varphi_0}(\rho\xi)\,\xi\,\mathrm{d}\xi. \tag{8.76}
$$

The fundamental solution to the second Dirichlet problem under zero initial conditions is obtained from (8.76) by multiplying each term by $(-1)^{n+1}$.

8.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.77}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.78}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.79}$$

$$\varphi = 0: \quad -\frac{1}{r}\frac{\partial T}{\partial \varphi} = g_1(r,t), \tag{8.80}$$

$$\varphi = \varphi_0: \quad \frac{1}{r}\frac{\partial T}{\partial \varphi} = g_2(r,t), \tag{8.81}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.82}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) = &\int_0^{\varphi_0}\int_0^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{\varphi_0}\int_0^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{t}\int_0^{\varphi_0}\int_0^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_0^{\infty} g_1(\rho,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_0^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{8.83}$$

To obtain the fundamental solutions we use the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the angular coordinate φ and the Hankel transform (2.78) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. As a result, we get [190]

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0}\sum_{n=0}^{\infty}{}' \int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}$$

$$\times \cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\phi}{\varphi_0}\right) J_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(\rho\xi)\,\xi\,\mathrm{d}\xi. \tag{8.84}$$

The fundamental solution to the first mathematical Neumann problem under zero initial conditions has the form [190]:

$$\mathcal{G}_{m1}(r,\varphi,\rho,t) = \frac{2ag_0t^{\alpha-1}}{\varphi_0\rho}\sum_{n=0}^{\infty}{}'\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2t^{\alpha}\right)$$

$$\times\; J_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(\rho\xi)\,\xi\,\mathrm{d}\xi. \tag{8.85}$$

Classical diffusion equation ($\alpha = 1$)

Using (A.34) from the Appendix, we obtain [20, 144]

$$\mathcal{G}_{g_1}(r,\varphi,\rho,t) = \frac{g_0}{\rho\varphi_0 t}\exp\left(-\frac{r^2+\rho^2}{4at}\right)$$

$$\times\left[\frac{1}{2}\,I_0\left(\frac{r\rho}{2at}\right) + \sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\left(\frac{r\rho}{2at}\right)\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\right]. \tag{8.86}$$

Wave equation ($\alpha = 2$)

$$\mathcal{G}_{g_1}(r,\varphi,\rho,t) = \frac{2\sqrt{a}g_0}{\varphi_0\rho}\sum_{n=0}^{\infty}{}'\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\Psi(r,\rho), \tag{8.87}$$

where

a) $\sqrt{a}t < \rho$

$$\Psi(r,\rho) = \begin{cases} 0, & 0 \le r < \rho - \sqrt{a}t, \\ \dfrac{1}{2\sqrt{r\rho}}\, P_{n\pi/\varphi_0 - 1/2}\left(\dfrac{r^2+\rho^2-at^2}{2r\rho}\right), & \rho - \sqrt{a}t < r < \rho + \sqrt{a}t, \\ 0, & \rho + \sqrt{a}t < r < \infty; \end{cases}$$

b) $\sqrt{a}t = \rho$

$$\Psi(r,\rho) = \begin{cases} \dfrac{1}{2\sqrt{r\rho}}\, P_{n\pi/\varphi_0 - 1/2}\left(\dfrac{r}{2\rho}\right), & 0 < r < 2\rho, \\ 0, & 2\rho < r < \infty; \end{cases}$$

c) $\sqrt{a}t > \rho$

$$\Psi(r,\rho) = \begin{cases} -\dfrac{1}{\pi\sqrt{r\rho}} \cos\left(\dfrac{n\pi^2}{\varphi_0}\right) Q_{n\pi/\varphi_0-1/2}\left(\dfrac{at^2 - r^2 - \rho^2}{2r\rho}\right), \\ \qquad\qquad 0 \le r < \sqrt{a}t - \rho, \\ \dfrac{1}{2\sqrt{r\rho}} P_{n\pi/\varphi_0-1/2}\left(\dfrac{r^2 + \rho^2 - at^2}{2r\rho}\right), \\ \qquad\qquad \sqrt{a}t - \rho < r < \rho + \sqrt{a}t, \\ 0, \qquad\qquad \rho + \sqrt{a}t < r < \infty, \end{cases}$$

where $P_\nu(r)$ and $Q_\nu(r)$ are the Legendre functions of the first and second kind, respectively.

For the physical Neumann problem, the boundary condition at $\varphi = 0$ is formulated in terms of the normal component of the heat flux:

$$\varphi = 0: \quad -\frac{1}{r} D_{RL}^{1-\alpha} \frac{\partial G_{p_1}}{\partial \varphi} = g_0 \frac{\delta(r-\rho)}{r} \delta(t), \quad 0 < \alpha \le 1, \tag{8.88}$$

$$\varphi = 0: \quad -\frac{1}{r} I^{\alpha-1} \frac{\partial G_{p_1}}{\partial \varphi} = g_0 \frac{\delta(r-\rho)}{r} \delta(t), \quad 1 < \alpha \le 2. \tag{8.89}$$

The solution is expressed as

$$\mathcal{G}_{p1}(r,\varphi,\rho,t) = \frac{2ag_0}{\varphi_0 \rho} \sum_{n=0}^{\infty}{}' \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right)$$
$$\times J_{n\pi/\varphi_0}(r\xi) J_{n\pi/\varphi_0}(\rho\xi)\, \xi \, d\xi. \tag{8.90}$$

The fundamental solutions (8.85) and (8.90) are shown in Fig. 8.1 and Fig. 8.2, respectively, for $\varphi = 0$.

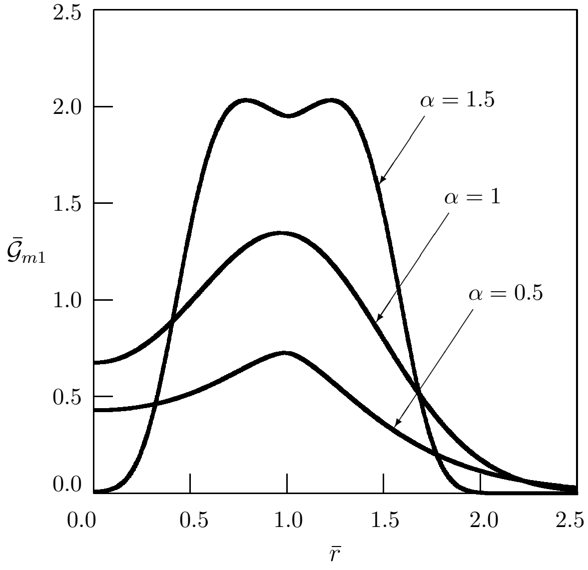

Figure 8.1: Dependence of the fundamental solution to the mathematical Neumann problem in a wedge on the radial coordinate [190]

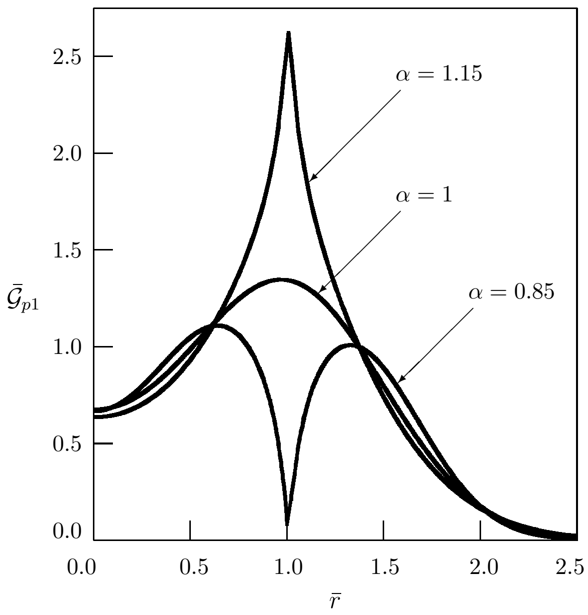

Figure 8.2: Dependence of the fundamental solution to the physical Neumann problem in a wedge on the radial coordinate [190]

8.5 Domain $0 \le r < R,\ 0 < \varphi < \varphi_0$

8.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.91}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.92}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.93}$$

$$r = R: \quad T = g_1(\varphi,t), \tag{8.94}$$

$$\varphi = 0: \quad T = g_2(r,t), \tag{8.95}$$

$$\varphi = \varphi_0: \quad T = g_3(r,t). \tag{8.96}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) = & \int_0^{\varphi_0}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
& + \int_0^{\varphi_0}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
& + \int_0^t\int_0^{\varphi_0}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
& + \int_0^t\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\
& + \int_0^t\int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
& + \int_0^t\int_0^R g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.97}$$

The problem is solved using the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the angular coordinate φ and the finite

Hankel transform (2.84) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. For the fundamental solutions we get [188]

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{4}{\varphi_0 R^2} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix}$$

$$\times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{n\pi\phi}{\varphi_0}\right) \frac{J_{n\pi/\varphi_0}(r\xi_{nk})\, J_{n\pi/\varphi_0}(\rho\xi_{nk})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nk})\right]^2}, \tag{8.98}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J_{n\pi/\varphi_0}(R\xi_{nk}) = 0. \tag{8.99}$$

For the sake of simplicity, we have used the notation ξ_{nk} for the roots (not $\xi_{n\pi/\varphi_0,k}$).

The fundamental solutions to the first and second Dirichlet problem under zero initial conditions have the following form [188]

$$\mathcal{G}_{g1}(r,\varphi,\phi,t) = -\frac{4at^{\alpha-1}}{\varphi_0 R} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$

$$\times \frac{\xi_{nk}\, J_{n\pi/\varphi_0}(r\xi_{nk})}{J'_{n\pi/\varphi_0}(R\xi_{nk})}; \tag{8.100}$$

$$\mathcal{G}_{g2}(r,\varphi,\rho,t) = \frac{4at^{\alpha-1}}{\varphi_0 R^2 \rho^2} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{n\pi}{\varphi_0} E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)$$

$$\times \frac{J_{n\pi/\varphi_0}(r\xi_{nk})\, J_{n\pi/\varphi_0}(\rho\xi_{nk})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nk})\right]^2}. \tag{8.101}$$

The fundamental solution to the third Dirichlet problem is obtained by multiplying each term in (8.101) by $(-1)^{n+1}$.

8.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.102}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.103}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.104}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g_1(\varphi,t), \tag{8.105}$$

$$\varphi = 0: \quad -\frac{1}{r}\frac{\partial T}{\partial \varphi} = g_2(r,t), \tag{8.106}$$

$$\varphi = \varphi_0: \quad \frac{1}{r}\frac{\partial T}{\partial \varphi} = g_3(r,t). \tag{8.107}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) &= \int_0^{\varphi_0}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{\varphi_0}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{\varphi_0}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{R} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{R} g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.108}$$

The problem is solved using the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the angular coordinate φ and the finite

Hankel transform (2.88) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. For the fundamental solutions we get [188]

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0 R^2} \begin{pmatrix} p_0 \\ w_0 t \\ q_0 t^{\alpha-1}/\Gamma(\alpha) \end{pmatrix}$$

$$+ \frac{4}{\varphi_0} {\sum_{n=0}^{\infty}}' \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right)$$

$$\times \frac{\xi_{nk}^2\, J_{n\pi/\varphi_0}(r\xi_{nk})\, J_{n\pi/\varphi_0}(\rho\xi_{nk})}{\left[R^2\xi_{nk}^2 - (n\pi/\varphi_0)^2\right] \left[J_{n\pi/\varphi_0}(R\xi_{nk})\right]^2}, \tag{8.109}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J'_{n\pi/\varphi_0}(R\xi_{nk}) = 0. \tag{8.110}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form

$$\mathcal{G}_{m1}(r,\varphi,\phi,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)\Big|_{\rho=R}, \tag{8.111}$$

$$\mathcal{G}_{m2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, q_0}\, \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)\Big|_{\phi=0}, \tag{8.112}$$

$$\mathcal{G}_{m3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, q_0}\, \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)\Big|_{\phi=\varphi_0}, \tag{8.113}$$

$$\mathcal{G}_{p1}(r,\varphi,\phi,t) = \frac{aRg_{01}}{p_0}\, \mathcal{G}_f(r,\varphi,\rho,\phi,t)\Big|_{\rho=R}, \tag{8.114}$$

$$\mathcal{G}_{p2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, p_0}\, \mathcal{G}_f(r,\varphi,\rho,\phi,t)\Big|_{\phi=0}, \tag{8.115}$$

$$\mathcal{G}_{p3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, p_0}\, \mathcal{G}_f(r,\varphi,\rho,\phi,t)\Big|_{\phi=\varphi_0}. \tag{8.116}$$

8.6 Domain $R \le r < \infty,\ 0 < \varphi < \varphi_0$

8.6.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.117}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.118}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.119}$$

$$r = R: \quad T = g_1(\varphi,t), \tag{8.120}$$

$$\varphi = 0: \quad T = g_2(r,t), \tag{8.121}$$

$$\varphi = \varphi_0: \quad T = g_3(r,t). \tag{8.122}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) = & \int_0^{\varphi_0}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ & + \int_0^{\varphi_0}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ & + \int_0^{t}\int_0^{\varphi_0}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ & + \int_0^{t}\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\ & + \int_0^{t}\int_R^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ & + \int_0^{t}\int_R^{\infty} g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{8.123}$$

The problem is solved using the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the angular coordinate φ and the Weber transform (2.108), (2.111) with respect to the radial variable r with

$\nu = n\pi/\varphi_0$. For the fundamental solutions to the first and second Cauchy problems and to the source problem under zero Dirichlet boundary condition we get

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0} \sum_{n=1}^{\infty} \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)$$

$$\times \sin\left(\frac{n\pi\phi}{\varphi_0}\right) \frac{J_{n\pi/\varphi_0}(r\xi)\, Y_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(R\xi)}{J^2_{n\pi/\varphi_0}(R\xi) + Y^2_{n\pi/\varphi_0}(R\xi)}$$

$$\times \left[J_{n\pi/\varphi_0}(\rho\xi)\, Y_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(\rho\xi)\, J_{n\pi/\varphi_0}(R\xi)\right] \xi\, \mathrm{d}\xi. \tag{8.124}$$

The fundamental solutions to the Dirichlet problems under zero initial conditions are calculated as

$$\mathcal{G}_{g1}(r,\varphi,\phi,t) = -\frac{4a g_{01} t^{\alpha-1}}{\pi\varphi_0} \sum_{n=1}^{\infty} \int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$

$$\times \frac{J_{n\pi/\varphi_0}(r\xi)\, Y_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(R\xi)}{J^2_{n\pi/\varphi_0}(R\xi) + Y^2_{n\pi/\varphi_0}(R\xi)}\, \xi\, \mathrm{d}\xi, \tag{8.125}$$

$$\mathcal{G}_{g2}(r,\varphi,\rho,t) = \frac{a g_{02}}{\rho^2 q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)}{\partial \phi}\right|_{\phi=0}, \tag{8.126}$$

$$\mathcal{G}_{g3}(r,\varphi,\rho,t) = -\frac{a g_{03}}{\rho^2 q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)}{\partial \phi}\right|_{\phi=\varphi_0}. \tag{8.127}$$

8.6.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.128}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.129}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.130}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g_1(\varphi,t), \tag{8.131}$$

$$\varphi = 0: \quad -\frac{1}{r}\frac{\partial T}{\partial \varphi} = g_2(r,t), \tag{8.132}$$

$$\varphi = \varphi_0: \quad \frac{1}{r}\frac{\partial T}{\partial \varphi} = g_3(r,t). \tag{8.133}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) &= \int_0^{\varphi_0}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{\varphi_0}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{\varphi_0}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_R^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_R^{\infty} g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.134}$$

The problem is solved using the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the angular coordinate φ and the Weber transform (2.108), (2.113) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. For the fundamental solutions to the first and second Cauchy problems and to the source problem under zero Neumann boundary condition we get

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} &= \frac{2}{\varphi_0}{\sum_{n=0}^{\infty}}' \int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \\
&\times \cos\left(\frac{n\pi\phi}{\varphi_0}\right) \frac{J_{n\pi/\varphi_0}(r\xi)\,Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\,J'_{n\pi/\varphi_0}(R\xi)}{[J'_{n\pi/\varphi_0}(R\xi)]^2 + [Y'_{n\pi/\varphi_0}(R\xi)]^2} \\
&\times \Big[J_{n\pi/\varphi_0}(\rho\xi)\,Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(\rho\xi)\,J'_{n\pi/\varphi_0}(R\xi)\Big]\,\xi\,\mathrm{d}\xi.
\end{aligned} \tag{8.135}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\mathcal{G}_{m1}(r,\varphi,\phi,t) = \frac{4ag_{01}t^{\alpha-1}}{\pi\varphi_0} {\sum_{n=0}^{\infty}}' \int_0^{\infty} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times \frac{J_{n\pi/\varphi_0}(r\xi)\, Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J'_{n\pi/\varphi_0}(R\xi)}{[J'_{n\pi/\varphi_0}(R\xi)]^2 + [Y'_{n\pi/\varphi_0}(R\xi)]^2}\, \mathrm{d}\xi, \tag{8.136}$$

$$\mathcal{G}_{m2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, q_0}\, \mathcal{G}_{\Phi}(r,\varphi,\rho,\phi,t)\bigg|_{\phi=0}, \tag{8.137}$$

$$\mathcal{G}_{m3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, q_0}\, \mathcal{G}_{\Phi}(r,\varphi,\rho,\phi,t)\bigg|_{\phi=\varphi_0}, \tag{8.138}$$

$$\mathcal{G}_{p1}(r,\varphi,\phi,t) = \frac{4ag_{01}}{\pi\varphi_0} {\sum_{n=0}^{\infty}}' \int_0^{\infty} E_{\alpha}(-a\xi^2 t^\alpha) \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times \frac{J_{n\pi/\varphi_0}(r\xi)\, Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J'_{n\pi/\varphi_0}(R\xi)}{[J'_{n\pi/\varphi_0}(R\xi)]^2 + [Y'_{n\pi/\varphi_0}(R\xi)]^2}\, \mathrm{d}\xi, \tag{8.139}$$

$$\mathcal{G}_{p2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, p_0}\, \mathcal{G}_{f}(r,\varphi,\rho,\phi,t)\bigg|_{\phi=0}, \tag{8.140}$$

$$\mathcal{G}_{p3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, p_0}\, \mathcal{G}_{f}(r,\varphi,\rho,\phi,t)\bigg|_{\phi=\varphi_0}. \tag{8.141}$$

Chapter 9

Axisymmetric Equations in Cylindrical Coordinates

If an idea's worth once, it's worth having twice.

Tom Stoppard

9.1 Domain $0 \le r < \infty, -\infty < z < \infty$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.1}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.3}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.4}$$

The solution:

$$\begin{aligned} T(r,z,t) =& \int_{-\infty}^{\infty}\int_0^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_{-\infty}^{\infty}\int_0^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{9.5}$$

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Using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r with ξ being the transform variable, and the exponential Fourier transform (2.20) with respect to the space coordinate z with η being the transform variable, we obtain the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$
$$\times\; J_0(r\xi)\, J_0(\rho\xi)\, \cos[(z-\zeta)\eta]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.6}$$

9.2 Domain $0 \le r < \infty,\ 0 < z < \infty$

9.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.7}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.8}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.9}$$

$$z=0: \quad T = g(r,t), \tag{9.10}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.11}$$

The solution:

$$\begin{aligned} T(r,z,t) &= \int\limits_0^\infty\int\limits_0^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int\limits_0^\infty\int\limits_0^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int\limits_0^t\int\limits_0^\infty\int\limits_0^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int\limits_0^t\int\limits_0^\infty g(\rho,\tau)\,\mathcal{G}_g(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.12}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the sin-Fourier transform (2.25) with respect to the space coordinate z:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$
$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \sin(z\eta)\, \sin(\zeta\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.13}$$

Fundamental solution to the Dirichlet problem

$$\mathcal{G}_g(r,z,\rho,t) = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right]$$
$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \sin(z\eta)\, \xi\, \eta\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.14}$$

The solution for $\rho = 0$ [160]

$$\mathcal{G}_g(r,z,0,t) = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right]$$
$$\times\ J_0(r\xi)\, \sin(z\eta)\, \xi\, \eta\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{9.15}$$

will be analyzed in detail. Passing to polar coordinates in the (η,ξ)-plane ($\eta = \sigma\cos\vartheta$, $\xi = \sigma\sin\vartheta$), (9.15) is rewritten as

$$\mathcal{G}_g = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty \sigma^3 E_{\alpha,\alpha}(-a\sigma^2 t^\alpha)\, \mathrm{d}\sigma$$
$$\times \int_0^{\pi/2} J_0(r\sigma\sin\vartheta)\, \sin(z\sigma\cos\vartheta)\, \sin\vartheta\, \cos\vartheta\, \mathrm{d}\vartheta. \tag{9.16}$$

Substitution $x = \sin\vartheta$ gives

$$\mathcal{G}_g = \frac{2ag_0 t^{\alpha-1}}{\pi}\int_0^\infty \sigma^3 E_{\alpha,\alpha}(-a\sigma^2 t^\alpha)\, \mathrm{d}\sigma \int_0^1 J_0(r\sigma x)\, \sin\left(z\sigma\sqrt{1-x^2}\right) x\, \mathrm{d}x. \tag{9.17}$$

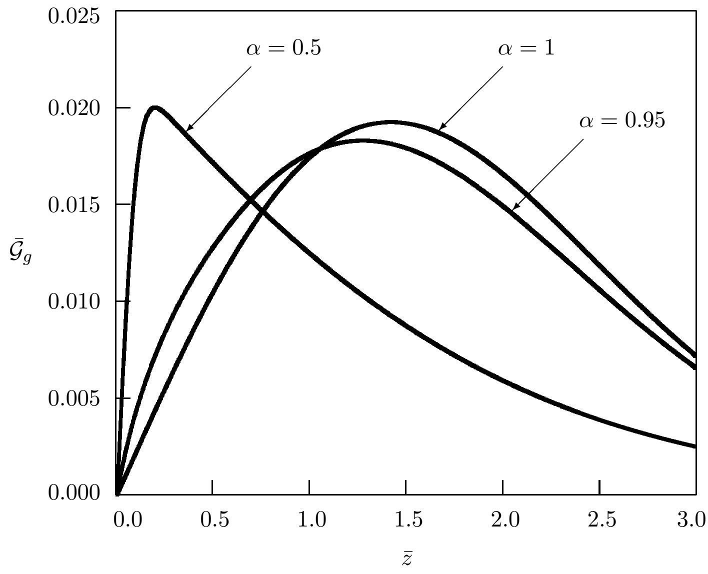

Figure 9.1: Dependence of the fundamental solution to the Dirichlet problem for a half-space on distance z for $\rho = 0$, $r = 0$, $0 \le \alpha \le 1$ [160]

With taking into account the integral (A.39) from the Appendix we obtain

$$\mathcal{G}_g = \frac{2ag_0 t^{\alpha-1} z}{\pi\left(r^2+z^2\right)} \int_0^\infty E_{\alpha,\alpha}(-a\sigma^2 t^\alpha)$$

$$\times \left[\frac{\sin\left(\sigma\sqrt{r^2+z^2}\right)}{\sqrt{r^2+z^2}} - \sigma \cos\left(\sigma\sqrt{r^2+z^2}\right) \right] \sigma \, \mathrm{d}\sigma. \tag{9.18}$$

Dependence of the nondimensional solution $\overline{\mathcal{G}}_g = at^{\alpha+1}\,\mathcal{G}_g/(2\pi g_0)$ on nondimensional spatial coordinate $\bar{z} = z\big/\left(\sqrt{a}t^{\alpha/2}\right)$ is shown in Figs. 9.1 and 9.2 for $r = 0$. As the numerical values of $\bar{\mathcal{G}}_g$ for $0 < \alpha \le 1$ and $1 < \alpha \le 2$ are widely different, the typical results for $0 < \alpha \le 1$ and $1 < \alpha \le 2$ are presented in two figures using different scales.

Constant boundary value of a function in a local area. Consider time-fractional diffusion equation (9.1) with zero source, zero initial conditions and the constant boundary value of a function in the area $0 \le r < R$:

$$z = 0: \quad T = \begin{cases} T_0, & 0 \le r < R, \\ 0, & R < r < \infty. \end{cases} \tag{9.19}$$

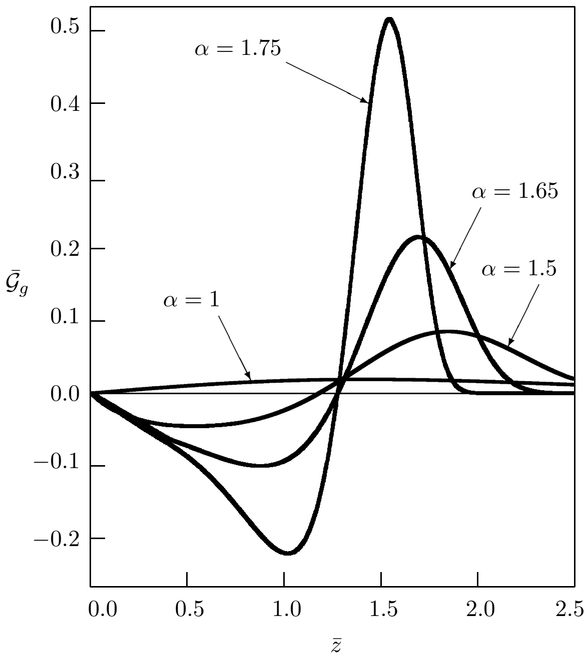

Figure 9.2: Dependence of the fundamental solution to the Dirichlet problem for a half-space on distance z for $\rho = 0$, $r = 0$, $1 \le \alpha \le 2$ [160]

The integral transforms allow us to obtain [160]

$$T = \frac{2T_0 R}{\pi} \int_0^\infty \int_0^\infty \left\{ 1 - E_\alpha[-a(\xi^2 + \eta^2)t^\alpha] \right\}$$

$$\times\ J_0(r\xi)\, J_1(R\xi)\, \frac{\eta}{\xi^2 + \eta^2}\, \sin(z\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.20}$$

Introducing polar coordinates in the (η, ξ)-plane and substituting $x = \sin\vartheta$, we arrive at

$$T = \frac{2T_0 R}{\pi} \int_0^\infty \left[1 - E_\alpha(-a\sigma^2 t^\alpha) \right] \mathrm{d}\sigma$$

$$\times \int_0^1 J_0(r\sigma x)\, J_1(R\sigma x)\, \sin(z\sigma\sqrt{1 - x^2})\, \mathrm{d}x. \tag{9.21}$$

Taking into account the integral (A.42) from Appendix leads to the expression for solution in the case $r = 0$:

$$T = T_0 \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) - \frac{2T_0}{\pi} \int\limits_0^\infty E_\alpha(-a\sigma^2 t^\alpha)$$

$$\times \left[\sin(z\sigma) - \frac{z}{\sqrt{z^2 + R^2}} \sin(\sqrt{z^2 + R^2}\sigma)\right] \frac{1}{\sigma}\, \mathrm{d}\sigma. \tag{9.22}$$

Helmholtz equation ($\alpha \to 0$)

Using (A.6), we obtain

$$T = T_0 R \int\limits_0^\infty J_0(r\xi)\, J_1(R\xi)\, \mathrm{e}^{-\, z\sqrt{\xi^2 + 1/a}} \mathrm{d}\xi. \tag{9.23}$$

This equation can be simplified in the case $r = 0$ accounting for the appropriate integral (A.38)

$$T = T_0 \left(\mathrm{e}^{-z/\sqrt{a}} - \frac{z}{\sqrt{z^2 + R^2}} \mathrm{e}^{-\sqrt{z^2+R^2}/\sqrt{a}}\right). \tag{9.24}$$

Classical diffusion equation ($\alpha = 1$)

Taking into account (A.26), the solution reads [141]

$$T = \frac{T_0 R}{2} \int\limits_0^\infty J_0(r\xi)\, J_1(R\xi) \Big[\mathrm{e}^{-z\xi} \operatorname{erfc}\left(\frac{z}{2\sqrt{at}} - \sqrt{at}\xi\right)$$

$$+ \mathrm{e}^{z\xi} \operatorname{erfc}\left(\frac{z}{2\sqrt{at}} + \sqrt{at}\xi\right)\Big] \mathrm{d}\xi. \tag{9.25}$$

In the case $r = 0$ after some algebra we get (see (A.15))

$$T = T_0 \left[\operatorname{erfc}\left(\frac{z}{2\sqrt{at}}\right) - \frac{z}{\sqrt{z^2 + R^2}} \operatorname{erfc}\left(\frac{\sqrt{z^2 + R^2}}{2\sqrt{at}}\right)\right]. \tag{9.26}$$

Wave equation ($\alpha = 2$)

For wave equation the solution has the simple form for $r = 0$:

a) $\sqrt{a}t \le R$

$$T = \begin{cases} T_0, & 0 \le z < \sqrt{a}t \\ 0, & \sqrt{a}t < z < \infty, \end{cases} \tag{9.27}$$

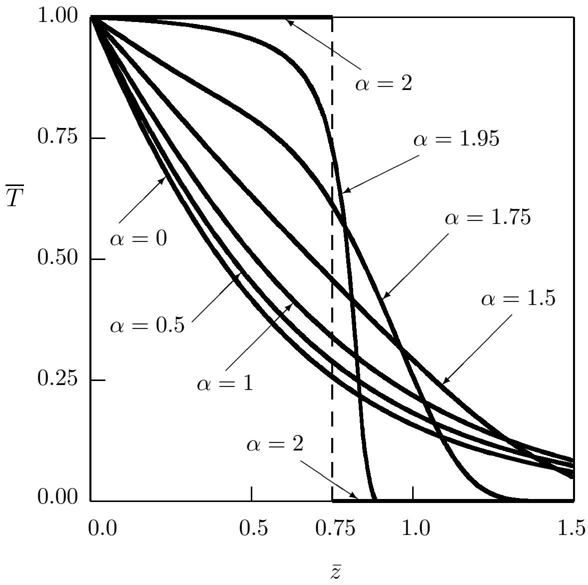

Figure 9.3: Solution to the Dirichlet problem with constant boundary value of a function in a local area of a half-space ($r = 0$, $\kappa = 0.75$) [160]

b) $\sqrt{a}t > R$

$$T = \begin{cases} T_0 \left(1 - \dfrac{z}{\sqrt{z^2 + R^2}}\right), & 0 \le z < \sqrt{at^2 - R^2}, \\ T_0, & \sqrt{at^2 - R^2} < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty. \end{cases} \tag{9.28}$$

Dependence of nondimensional solution $\overline{T} = T/T_0$ on nondimensional space coordinate $\bar{z} = z/R$ is shown in Figs. 9.3 and 9.4 for typical values $\kappa = \sqrt{a}t^{\alpha/2}/R = 0.75$ and $\kappa = 1.5$, respectively.

9.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, z, t), \tag{9.29}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.30}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.31}$$

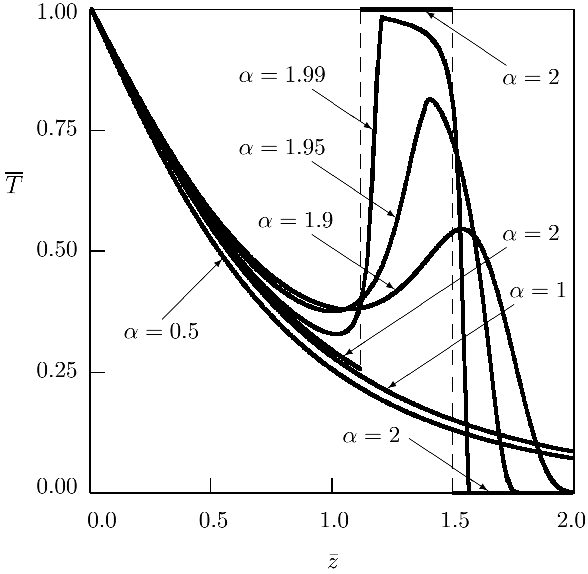

Figure 9.4: Solution to the Dirichlet problem with constant boundary value of a function in a local areal of a half-space ($r = 0$, $\kappa = 1.5$) [160]

$$z = 0: \quad -\frac{\partial T}{\partial z} = g(r, t), \tag{9.32}$$

$$\lim_{r\to\infty} T(r, z, t) = 0, \quad \lim_{z\to\infty} T(r, z, t) = 0. \tag{9.33}$$

The solution:

$$\begin{aligned} T(r, z, t) &= \int_0^\infty \int_0^\infty f(\rho, \zeta)\, \mathcal{G}_f(r, z, \rho, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^\infty \int_0^\infty F(\rho, \zeta)\, \mathcal{G}_F(r, z, \rho, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^t \int_0^\infty \int_0^\infty \Phi(\rho, \zeta, \tau)\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty g(\rho, \tau)\, \mathcal{G}_g(r, z, \rho, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau. \end{aligned} \tag{9.34}$$

The Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the cos-Fourier transform (2.37) with respect to the space coordinate z lead to the following fundamental solutions

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$
$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta)\, \cos(\zeta\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.35}$$

Fundamental solution to the mathematical Neumann problem. The diffusion-wave equation in a half-space is considered under zero initial conditions and the following boundary condition:

$$z = 0: \quad -\frac{\partial \mathcal{G}_m}{\partial z} = g_0 \frac{\delta(r-\rho)}{r}\, \delta(t). \tag{9.36}$$

The solution

$$\mathcal{G}_m(r,z,\rho,t) = \frac{2ag_0 t^{\alpha-1}}{\pi} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right]$$
$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{9.37}$$

in the case $\rho = 0$ [160]

$$\mathcal{G}_m(r,z,0,t) = \frac{2ag_0 t^{\alpha-1}}{\pi} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] J_0(r\xi)\, \cos(z\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{9.38}$$

can be simplified. Passing to polar coordinates in the (η,ξ)-plane ($\eta = \sigma\cos\vartheta$, $\xi = \sigma\sin\vartheta$)

$$\mathcal{G}_m = \frac{2ag_0 t^{\alpha-1}}{\pi} \int_0^\infty \sigma^2 E_{\alpha,\alpha}(-a\sigma^2 t^\alpha)\, \mathrm{d}\sigma$$
$$\times \int_0^{\pi/2} J_0(r\sigma\sin\vartheta)\, \cos(z\sigma\cos\vartheta)\, \sin\vartheta\, \mathrm{d}\vartheta, \tag{9.39}$$

rewriting (9.39) as

$$\mathcal{G}_m = \frac{2ag_0 t^{\alpha-1}}{\pi} \int_0^\infty \sigma^2 E_{\alpha,\alpha}(-a\sigma^2 t^\alpha)\, \mathrm{d}\sigma$$
$$\times \int_0^1 J_0(r\sigma x)\, \cos(z\sigma\sqrt{1-x^2})\, \frac{x}{\sqrt{1-x^2}}\, \mathrm{d}x, \tag{9.40}$$

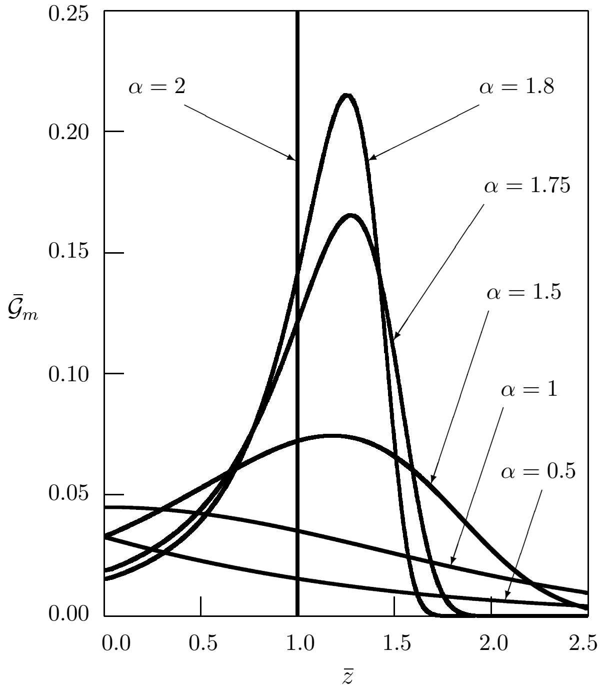

Figure 9.5: Dependence of fundamental solution to the mathematical Neumann problem for a half-space on distance z for $\rho = 0$, $r = 0$ [160]

and taking into account (A.40), we get

$$\mathcal{G}_m = \frac{2ag_0 t^{\alpha-1}}{\pi\sqrt{r^2+z^2}} \int_0^\infty \sigma E_{\alpha,\alpha}(-a\sigma^2 t^\alpha) \sin\left(\sqrt{r^2+z^2}\,\sigma\right) \mathrm{d}\sigma. \tag{9.41}$$

Dependence of nondimensional solution $\overline{\mathcal{G}}_m = \sqrt{a}t^{1+\alpha/2}\,\mathcal{G}_m/(2\pi g_0)$ on space coordinate $\bar{z} = z/(\sqrt{a}t^{\alpha/2})$ for $r = 0$ is shown in Fig. 9.5.

Constant boundary value of the normal derivative of a function in a local area. The time-fractional diffusion equation with zero source and zero initial conditions is considered under Neumann boundary condition with constant value of the normal derivative of a function in the domain $0 \le r < R$:

$$z = 0: \quad -\frac{\partial T}{\partial z} = \begin{cases} g_0, & 0 \le r < R, \\ 0, & R < r < \infty. \end{cases} \tag{9.42}$$

The integral transforms allow us to obtain

$$T = \frac{2g_0 R}{\pi} \int_0^\infty \left[1 - E_\alpha(-a\sigma^2 t^\alpha)\right] \frac{1}{\sigma} \,\mathrm{d}\sigma \times \int_0^1 J_0(r\sigma x)\, J_1(R\sigma x) \frac{\cos(z\sigma\sqrt{1-x^2})}{\sqrt{1-x^2}} \,\mathrm{d}x. \tag{9.43}$$

Equation (9.43) is simplified for $r = 0$:

$$T = g_0 \left(\sqrt{z^2+R^2} - z\right) - \frac{2g_0}{\pi} \int_0^\infty E_\alpha(-a\sigma^2 t^\alpha) \times \left[\cos(z\sigma) - \cos(\sqrt{z^2+R^2}\sigma)\right] \frac{1}{\sigma^2} \,\mathrm{d}\sigma. \tag{9.44}$$

Let us analyze several particular cases corresponding to the standard equations.

Helmholtz equation ($\alpha \to 0$)

$$T = g_0 R \int_0^\infty J_0(r\xi)\, J_1(R\xi)\, \mathrm{e}^{-\,z\sqrt{\xi^2+1/a}} \frac{1}{\sqrt{\xi^2+1/a}} \,\mathrm{d}\xi. \tag{9.45}$$

This equation is simplified for $r = 0$:

$$T = g_0 \sqrt{a} \left(\mathrm{e}^{-z/\sqrt{a}} - \mathrm{e}^{-\sqrt{z^2+R^2}/\sqrt{a}}\right). \tag{9.46}$$

Classical diffusion equation ($\alpha = 1$)

The solution for the standard heat conduction equation was obtained by Parkus [141] and has the following form:

$$T = \frac{g_0 R}{2} \int_0^\infty J_0(r\xi)\, J_1(R\xi) \Big[\mathrm{e}^{-z\xi} \operatorname{erfc}\left(\frac{z}{2\sqrt{at}} - \sqrt{at}\xi\right) - \mathrm{e}^{z\xi} \operatorname{erfc}\left(\frac{z}{2\sqrt{at}} + \sqrt{at}\xi\right)\Big] \frac{1}{\xi} \,\mathrm{d}\xi. \tag{9.47}$$

In the case $r = 0$ we have:

$$T = g_0 \left\{ \sqrt{z^2+R^2} \operatorname{erfc}\left(\frac{\sqrt{z^2+R^2}}{2\sqrt{at}}\right) - z \operatorname{erfc}\left(\frac{z}{2\sqrt{at}}\right) + 2\sqrt{\frac{at}{\pi}} \left[\exp\left(-\frac{z^2}{4at}\right) - \exp\left(-\frac{z^2+R^2}{4at}\right)\right] \right\}. \tag{9.48}$$

Wave equation ($\boldsymbol{\alpha = 2}$)

$$T = \begin{cases} g_0 R \displaystyle\int\limits_z^{\sqrt{a}t} F(x,r,z)\,\mathrm{d}x, & 0 < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty, \end{cases} \tag{9.49}$$

where

$$F(x,r,z) = \begin{cases} 1, & r + \sqrt{x^2 - z^2} < R, \\ \dfrac{1}{\pi} \arccos\left(\dfrac{x^2 + r^2 - z^2 - 1}{2r\sqrt{x^2 - z^2}}\right), & \\ & |r - \sqrt{x^2 - z^2}| < R < r + \sqrt{x^2 - z^2}, \\ 0, & R < |r - \sqrt{x^2 - z^2}|. \end{cases} \tag{9.50}$$

The solution simplifies for $r = 0$:

a) $\sqrt{a}t \le R$

$$T = \begin{cases} g_0(\sqrt{a}t - z), & 0 \le z < \sqrt{a}t \\ 0, & \sqrt{a}t < z < \infty, \end{cases} \tag{9.51}$$

b) $\sqrt{a}t > R$

$$T = \begin{cases} g_0\left(\sqrt{z^2 + R^2} - z\right), & 0 \le z < \sqrt{at^2 - R^2}, \\ g_0(\sqrt{a}t - z), & \sqrt{at^2 - R^2} < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty. \end{cases} \tag{9.52}$$

Dependence of nondimensional solution $\overline{T} = T/(Rg_0)$ on nondimensional space coordinate $\bar{z} = z/R$ is shown in Fig. 9.6 and Fig. 9.7 for $r = 0$ with $\kappa = \sqrt{a}t^{\alpha/2}/R = 0.75$ and $\kappa = 1.5$, respectively. The plot of $\overline{T}$ versus $\bar{r} = r/R$ at the boundary $z = 0$ is depicted in Fig. 9.8 for $\kappa = 0.75$.

Fundamental solution to the physical Neumann problem. In this case the time-fractional diffusion-wave equation with zero source and zero initial conditions is considered under the physical Neumann boundary condition with the given flux at the boundary:

$$\begin{aligned} z = 0: &\quad -D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial z} = g_0\,\frac{\delta(r-\rho)}{r}\,\delta(t), \quad 0 < \alpha \le 1, \\ z = 0: &\quad -I^{\alpha-1}\frac{\partial \mathcal{G}_p}{\partial z} = g_0\,\frac{\delta(r-\rho)}{r}\,\delta(t), \quad 1 < \alpha \le 2. \end{aligned} \tag{9.53}$$

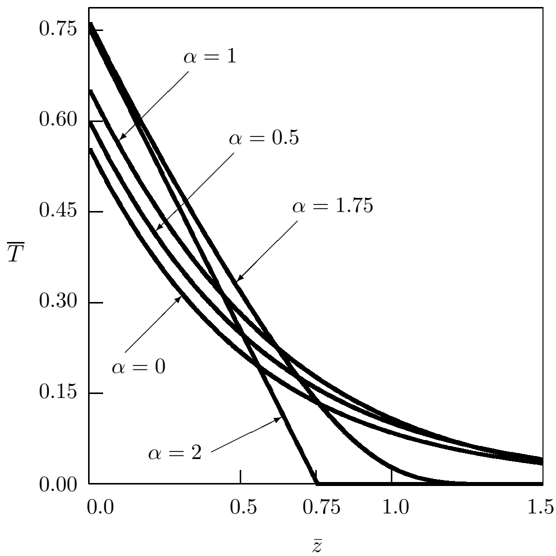

Figure 9.6: Dependence of solution on distance z for $r = 0$ (the Neumann boundary condition with constant normal derivative of a function in a local area of a half-space); $\kappa = 0.75$ [160]

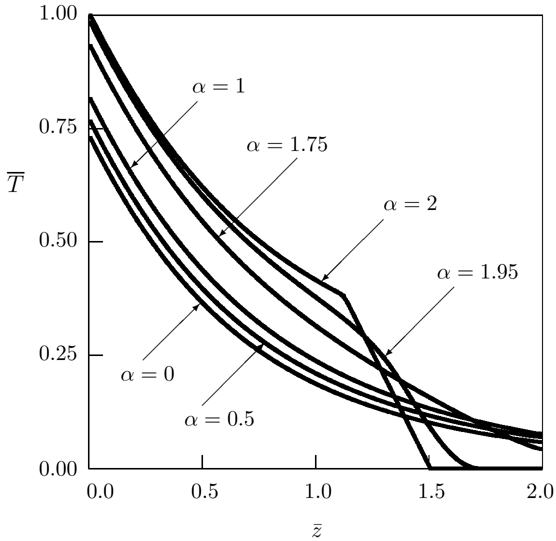

Figure 9.7: Dependence of solution on distance z for $r = 0$ (the Neumann boundary condition with constant normal derivative of a function in a local area of a half-space); $r = 0$, $\kappa = 1.5$ [160]

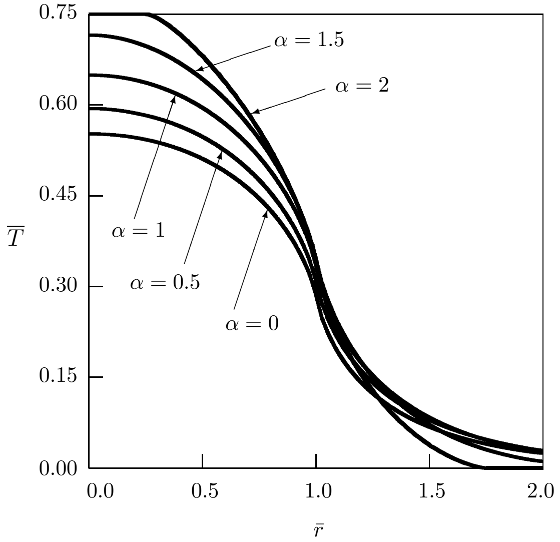

Figure 9.8: Dependence of solution on distance r for $z = 0$ (the Neumann boundary condition with constant normal derivative of a function in a local area of a half-space); $\kappa = 0.75$ [160]

The solution reads:

$$G_p(r, z, \rho, t) = \frac{2ag_0}{\pi} \int_0^\infty \int_0^\infty E_\alpha \left[-a\left(\xi^2 + \eta^2\right) t^\alpha\right] J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta)\, \xi \,\mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.54}$$

For $\rho = 0$ the solution simplifies and after passing to the polar coordinates in the (η, ξ)-plane we arrive at

$$G_p(r, z, 0, t) = \frac{2ag_0}{\pi} \int_0^\infty E_\alpha\left(-a\sigma^2 t^\alpha\right) \frac{\sin\left(\sqrt{r^2 + z^2}\,\sigma\right)}{\sqrt{r^2 + z^2}}\, \sigma \,\mathrm{d}\sigma. \tag{9.55}$$

Constant boundary value of the heat flux in a local area. Of particular interest is the problem with the constant boundary value of the heat flux in the area $0 < r < R$:

$$\begin{aligned} z = 0: \quad & -D_{RL}^{1-\alpha} \frac{\partial T}{\partial z} = \begin{cases} g_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \quad 0 < \alpha \le 1, \\ z = 0: \quad & -I^{\alpha-1} \frac{\partial T}{\partial z} = \begin{cases} g_0, & 0 \le r < R, \\ 0, & R < r < \infty, \end{cases} \quad 1 < \alpha \le 2. \end{aligned} \tag{9.56}$$

The solution has the following form:

$$T = \frac{2ag_0Rt}{\pi} \int_0^\infty \int_0^\infty E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] J_0(r\xi)\, J_1(R\xi)\, \cos(z\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.57}$$

For $r = 0$ we get

$$T = \frac{2ag_0t}{\pi} \int_0^\infty E_{\alpha,2}\left(-a\sigma^2t^\alpha\right) \left[\cos(z\sigma) - \cos\left(\sqrt{z^2+R^2}\sigma\right)\right] \mathrm{d}\sigma. \tag{9.58}$$

In particular, for the standard wave equation ($\alpha = 2$):

a) $\sqrt{a}t \le R$

$$T = \begin{cases} \sqrt{a}g_0, & 0 < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty; \end{cases} \tag{9.59}$$

b) $\sqrt{a}t > R$

$$T = \begin{cases} 0, & 0 < z < \sqrt{a}t - R, \\ \sqrt{a}g_0, & \sqrt{a}t - R < z < \sqrt{a}t, \\ 0, & \sqrt{a}t < z < \infty. \end{cases} \tag{9.60}$$

Dependence of nondimensional solution $\overline{T} = t^{\alpha-1}T/(Rg_0)$ on nondimensional spatial coordinate $\bar{z} = z/R$ is shown in Fig. 9.9 for $r = 0$, whereas the plot of $\overline{T}$ versus $\bar{r} = r/R$ at the boundary $z = 0$ is depicted in Fig. 9.10. In both cases $\kappa = 0.75$.

9.2.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.61}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.62}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.63}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + HT = g(r,t), \tag{9.64}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.65}$$

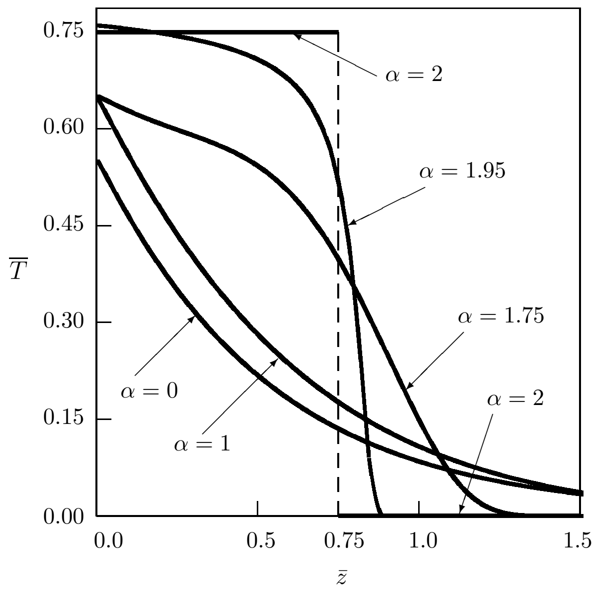

Figure 9.9: Dependence of solution on distance z for $r = 0$ (the Neumann boundary condition with the constant flux in a local area of a half-space); $\kappa = 0.75$

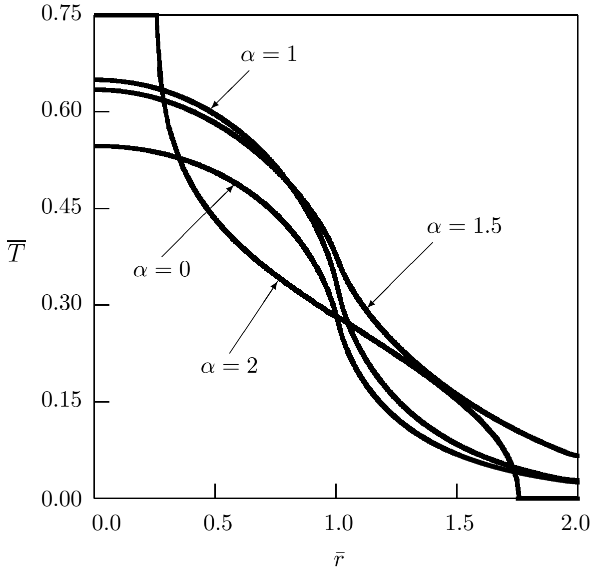

Figure 9.10: Dependence of solution on distance r for $z = 0$ (the Neumann boundary condition with the constant flux in a local area of a half-space); $\kappa = 0.75$

The solution:

$$
\begin{aligned}
T(r,z,t) = & \int_0^\infty \int_0^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_0^\infty \int_0^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_0^t \int_0^\infty \int_0^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
& + \int_0^t \int_0^\infty g(\rho,\tau)\,\mathcal{G}_g(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned}
\tag{9.66}
$$

The Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the sin-cos-Fourier transform (2.40) with respect to the space coordinate z lead to the following fundamental solutions:

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{2}{\pi} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\
& \times \frac{J_0(r\xi)\,J_0(\rho\xi)}{\eta^2+H^2}\left[\eta\cos(z\eta)+H\sin(z\eta)\right] \\
& \times \left[\eta\cos(\zeta\eta)+H\sin(\zeta\eta)\right]\xi\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}
\tag{9.67}
$$

Fundamental solution to the mathematical Robin problem. The solution was obtained in [179] and reads:

$$
\begin{aligned}
\mathcal{G}_m(r,z,\rho,t) = & \frac{2ag_0 t^{\alpha-1}}{\pi} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\
& \times \frac{J_0(r\xi)\,J_0(\rho\xi)}{\eta^2+H^2}\left[\eta\cos(z\eta)+H\sin(z\eta)\right]\xi\,\eta\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}
\tag{9.68}
$$

Consider several particular cases of the solution (9.68).

Classical diffusion equation ($\alpha = 1$)

$$\mathcal{G}_m(r,z,\rho,t) = \frac{g_0}{2t}\exp\left(-\frac{r^2+\rho^2+z^2}{4at}\right) I_0\left(\frac{r\rho}{2at}\right)\left\{\frac{1}{\sqrt{\pi a t}}\right.$$
$$\left. -H\exp\left[\left(\sqrt{at}H + \frac{z}{2\sqrt{at}}\right)^2\right]\operatorname{erfc}\left(\sqrt{at}H + \frac{z}{2\sqrt{at}}\right)\right\}. \tag{9.69}$$

Subdiffusion with $\alpha = 1/2$

$$\mathcal{G}_m(r,z,\rho,t) = \frac{g_0}{2\sqrt{\pi}t}\int_0^\infty \exp\left(-u^2 - \frac{r^2+\rho^2+z^2}{8a\sqrt{t}u}\right) I_0\left(\frac{r\rho}{4a\sqrt{t}u}\right)$$
$$\times\left\{\frac{1}{\sqrt{2\pi a u}t^{1/4}} - H\exp\left[\left(\sqrt{2au}Ht^{1/4} + \frac{z}{2\sqrt{2au}t^{1/4}}\right)^2\right]\right.$$
$$\left.\times\operatorname{erfc}\left(\sqrt{2au}Ht^{1/4} + \frac{z}{2\sqrt{2au}t^{1/4}}\right)\right\}\mathrm{d}u. \tag{9.70}$$

Wave equation ($\alpha = 2$)

Under assumption $\rho = 0$, we obtain

$$\mathcal{G}_m(r,z,0,t) = \begin{cases} \dfrac{\sqrt{a}g_0}{\sqrt{at^2-r^2}}\left[\delta\left(\sqrt{at^2-r^2}-z\right) - He^{-H\left(\sqrt{at^2-r^2}-z\right)}\right], \\ \qquad 0 < r < \sqrt{a}t, \quad 0 < z < \sqrt{at^2-r^2}, \\ 0, \qquad \sqrt{a}t < r < \infty, \quad \sqrt{at^2-r^2} < z < \infty. \end{cases} \tag{9.71}$$

Fundamental solution to the physical Robin problem. In this case the axisymmetric time-fractional diffusion-wave equation is considered under zero initial conditions and the physical Robin boundary condition

$$z = 0: \quad -D_{RL}^{1-\alpha}\frac{\partial \mathcal{G}_p}{\partial z} + H\mathcal{G}_p = g_0\,\frac{\delta(r-\rho)}{r}\,\delta(t), \quad 0 < \alpha \le 1,$$
$$z = 0: \quad -I^{\alpha-1}\frac{\partial \mathcal{G}_p}{\partial z} + H\mathcal{G}_p = g_0\,\frac{\delta(r-\rho)}{r}\,\delta(t), \quad 1 < \alpha \le 2. \tag{9.72}$$

The Laplace transform with respect to time t leads to the boundary condition

$$z = 0: \quad s^{\alpha-1}H\mathcal{G}_p^* - \frac{\partial \mathcal{G}_p^*}{\partial z} = g_0\,\frac{\delta(r-\rho)}{r}\,s^{\alpha-1}. \tag{9.73}$$

Hence, the kernel (2.42) of the sin-cos-Fourier transform (2.40) with respect to the spatial coordinate z will depend on the Laplace transform variable s,

$$K(z,\eta,s) = \frac{\eta\cos(z\eta) + s^{\alpha-1}H\sin(z\eta)}{\sqrt{\eta^2 + (s^{\alpha-1}H)^2}}, \tag{9.74}$$

and in the transform domain we obtain

$$\widehat{\widetilde{\mathcal{G}}}_p^{\,*} = g_0\, J_0(\rho\xi)\, \frac{\eta}{\sqrt{\eta^2 + (s^{\alpha-1}H)^2}}\, \frac{s^{\alpha-1}}{s^\alpha + a(\xi^2+\eta^2)}. \tag{9.75}$$

Inversion of the Laplace transform in (9.75) depends on the value of α. For $0 < \alpha \le 1$ we have [179]

$$\mathcal{G}_p(r,z,\rho,t) = \frac{2ag_0}{\pi}\int_0^\infty\int_0^\infty\int_0^t \xi\, J_0(r\xi)\, J_0(\rho\xi)\, \tau^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)\tau^\alpha\right]$$

$$\times\Bigg\{(t-\tau)^{-\alpha}\, E_{2-2\alpha,1-\alpha}\left[-\frac{H^2}{\eta^2}(t-\tau)^{2-2\alpha}\right]\cos(z\eta) + \frac{H}{\eta}\,(t-\tau)^{1-2\alpha}$$

$$+\, E_{2-2\alpha,2-2\alpha}\left[-\frac{H^2}{\eta^2}(t-\tau)^{2-2\alpha}\right]\sin(z\eta)\Bigg\}\, \mathrm{d}\tau\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{9.76}$$

whereas for $1 < \alpha \le 2$ we get

$$\mathcal{G}_p(r,z,\rho,t) = \frac{2ag_0}{\pi}\int_0^\infty\int_0^\infty\int_0^t \xi\, J_0(r\xi)\, J_0(\rho\xi)\, E_{\alpha}\left[-a\left(\xi^2+\eta^2\right)\tau^\alpha\right]$$

$$\times\Bigg\{\frac{\eta^2}{H^2}(t-\tau)^{2\alpha-3}\, E_{2\alpha-2,2\alpha-2}\left[-\frac{\eta^2}{H^2}(t-\tau)^{2\alpha-2}\right]\cos(z\eta) + \frac{\eta}{H}\,(t-\tau)^{\alpha-2}$$

$$\times\, E_{2\alpha-2,\alpha-1}\left[-\frac{H^2}{\eta^2}(t-\tau)^{2-2\alpha}\right]\sin(z\eta)\Bigg\}\, \mathrm{d}\tau\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.77}$$

9.3 Domain $0 \le r < \infty$, $0 < z < L$

9.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.78}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.79}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.80}$$

$$z=0: \quad T = g_1(r,t), \tag{9.81}$$

$$z=L: \quad T = g_2(r,t), \tag{9.82}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.83}$$

The solution:

$$\begin{aligned} T(r,z,t) =& \int_0^L\int_0^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^L\int_0^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^t\int_0^L\int_0^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty g_1(\rho,\tau)\,\mathcal{G}_{g1}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty g_2(\rho,\tau)\,\mathcal{G}_{g2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.84}$$

The problem is solved using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the finite sin-Fourier transform (2.44) with respect to the space coordinate z. For the fundamental solutions we obtain

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L}\sum_{k=1}^{\infty}\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times\ J_0(r\xi)\, J_0(\rho\xi)\, \sin(z\eta_k)\, \sin(\zeta\eta_k)\, \xi\, \mathrm{d}\xi, \tag{9.85}$$

where

$$\eta_k = \frac{k\pi}{L}. \tag{9.86}$$

The fundamental solutions to the first and second Dirichlet problems under zero initial conditions are calculated as

$$\mathcal{G}_{g1}(r,z,\rho,t) = \frac{a g_0}{q_0}\, \frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\bigg|_{\zeta=0}, \tag{9.87}$$

$$\mathcal{G}_{g2}(r,z,\rho,t) = -\frac{a g_0}{q_0}\, \frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\bigg|_{\zeta=L}. \tag{9.88}$$

9.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.89}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.90}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.91}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_1(r,t), \tag{9.92}$$

$$z = L: \quad \frac{\partial T}{\partial z} = g_2(r,t), \tag{9.93}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.94}$$

The solution:

$$
\begin{aligned}
T(r,z,t) = & \int_0^L \int_0^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\
& + \int_0^L \int_0^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\
& + \int_0^t \int_0^L \int_0^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\tau, \\
& + \int_0^t \int_0^\infty g_1(\rho,\tau)\, \mathcal{G}_{g1}(r,z,\rho,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau, \\
& + \int_0^t \int_0^\infty g_2(\rho,\tau)\, \mathcal{G}_{g2}(r,z,\rho,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau.
\end{aligned}
\tag{9.95}
$$

Using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the finite cos-Fourier transform (2.48) with respect to the space coordinate z, we get the fundamental solutions to the first and second Cauchy problems and to the source problem:

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{2}{L} {\sum_{k=0}^{\infty}}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \end{pmatrix} \\
& \times J_0(r\xi)\, J_0(\rho\xi)\, \cos(z\eta_k)\, \cos(\zeta\eta_k)\, \xi \, \mathrm{d}\xi,
\end{aligned}
\tag{9.96}
$$

where

$$
\eta_k = \frac{k\pi}{L}. \tag{9.97}
$$

The fundamental solutions to the first and second mathematical and physical Neumann problems under zero initial conditions are calculated as

$$
\mathcal{G}_{m1}(r,z,\rho,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.98}
$$

$$
\mathcal{G}_{m2}(r,z,\rho,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=L}, \tag{9.99}
$$

$$
\mathcal{G}_{p1}(r,z,\rho,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.100}
$$

$$
\mathcal{G}_{p2}(r,z,\rho,t) = \frac{a g_0}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.101}
$$

9.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.102}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.103}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.104}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + HT = g_1(r,t), \tag{9.105}$$

$$z = L: \quad \frac{\partial T}{\partial z} + HT = g_2(r,t), \tag{9.106}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.107}$$

The solution:

$$\begin{aligned} T(r,z,t) = &\int_0^L \int_0^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^L \int_0^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^t \int_0^L \int_0^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau, \\ &+ \int_0^t \int_0^\infty g_1(\rho,\tau)\,\mathcal{G}_{g1}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty g_2(\rho,\tau)\,\mathcal{G}_{g2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.108}$$

Using the Laplace transform with respect to time t, the Hankel transform of order zero (2.78) with respect to the radial coordinate r, and the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z, we get the fundamental solutions to the first and second Cauchy problems and to the source problem:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L}\sum_{k=1}^{\infty}\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_k^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)\,J_0(\rho\xi)}{\eta_k^2+H^2+\frac{2H}{L}}\left[\eta_k\cos(z\eta_k)+H\sin(z\eta_k)\right]$$

$$\times\left[\eta_k\cos(\zeta\eta_k)+H\sin(\zeta\eta_k)\right]\xi\,\mathrm{d}\xi, \tag{9.109}$$

where η_k are the positive roots of the transcendental equation

$$\tan\left(L\eta_k\right) = \frac{2H\eta_k}{\eta_k^2-H^2}. \tag{9.110}$$

The fundamental solutions to the first and second mathematical Robin problems under zero initial conditions have the following form:

$$\mathcal{G}_{m1}(r,z,\rho,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.111}$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.112}$$

9.4 Domain $0 \le r < R$, $-\infty < z < \infty$

9.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(r,z,t), \tag{9.113}$$

$$t=0: \quad T=f(r,z), \qquad 0<\alpha\le 2, \tag{9.114}$$

$$t=0: \quad \frac{\partial T}{\partial t}=F(r,z), \qquad 1<\alpha\le 2, \tag{9.115}$$

$$r=R: \quad T=g(z,t), \tag{9.116}$$

$$\lim_{z\to\pm\infty} T(r,z,t)=0. \tag{9.117}$$

The solution:

$$
\begin{aligned}
T(r,z,t) = & \int_{-\infty}^{\infty}\int_{0}^{R} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_{-\infty}^{\infty}\int_{0}^{R} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_{0}^{t}\int_{-\infty}^{\infty}\int_{0}^{R} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
& + \int_{0}^{t}\int_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \qquad (9.118)
\end{aligned}
$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.96) with respect to the radial coordinate r with ξ_n being the transform variable, and the exponential Fourier transform (2.20) with respect to the space coordinate z with η being the transform variable lead to the following fundamental solutions:

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{1}{\pi R^2}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} p_0\,E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\
& \times \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{[J_1(R\xi_k)]^2}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta, \qquad (9.119)
\end{aligned}
$$

where ξ_k are the positive roots of the transcendental equation

$$
J_0(R\xi_k) = 0. \qquad (9.120)
$$

The fundamental solution to the Dirichlet problem under zero initial conditions is expressed as

$$
\begin{aligned}
\mathcal{G}_g(r,z,\zeta,t) = & \frac{a g_0 t^{\alpha-1}}{\pi R}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\
& \times \frac{\xi_k\,J_0(r\xi_k)}{J_1(R\xi_k)}\cos[(z-\zeta)\eta]\,\mathrm{d}\eta. \qquad (9.121)
\end{aligned}
$$

Constant boundary value of temperature in a local area. In this case the time-fractional heat conduction equation in a long cylinder is considered under zero initial conditions and the following boundary condition:

$$r = R: \quad T = \begin{cases} T_0, & |z| < l, \\ 0, & |z| > l. \end{cases} \tag{9.122}$$

The solution of this problem was obtained in [191]:

$$T(r,z,t) = \frac{2T_0}{\pi R} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 + \eta^2)\, \eta\, J_1(R\xi_k)}$$

$$\times \left\{1 - E_\alpha\left[-a\left(\xi_k^2 + \eta^2\right) t^\alpha\right]\right\} \sin(l\eta)\, \cos(z\eta)\, \mathrm{d}\eta. \tag{9.123}$$

It should be emphasized that the relation [212]

$$\frac{2}{R} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 - \beta^2)\, J_1(R\xi_k)} = \frac{J_0(r\beta)}{J_0(R\beta)}$$

for $\beta = i\eta$ can be rewritten as

$$\frac{2}{R} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 + \eta^2)\, J_1(R\xi_k)} = \frac{I_0(r\eta)}{I_0(R\eta)}.$$

Hence, (9.123) takes the form

$$T(r,z,t) = \frac{T_0}{\pi} \int_{-\infty}^{\infty} \frac{I_0(r\eta)}{I_0(R\eta)}\, \frac{\sin(l\eta)\, \cos(z\eta)}{\eta}\, \mathrm{d}\eta - \frac{2T_0}{\pi R} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\xi_k\, J_0(r\xi_k)}{(\xi_k^2 + \eta^2)\, J_1(R\xi_k)}$$

$$\times\ E_\alpha\left[-a\left(\xi_k^2 + \eta^2\right) t^\alpha\right] \frac{\sin(l\eta)\, \cos(z\eta)}{\eta}\, \mathrm{d}\eta. \tag{9.124}$$

At the boundary surface $r = R$, the first integral in (9.124) satisfies the boundary condition (9.122), whereas the second one equals zero.

9.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.125}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.126}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.127}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(z,t), \tag{9.128}$$

$$\lim_{z \to \pm\infty} T(r,z,t) = 0. \tag{9.129}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_{-\infty}^{\infty} \int_0^R f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ & + \int_{-\infty}^{\infty} \int_0^R F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ & + \int_0^t \int_{-\infty}^{\infty} \int_0^R \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ & + \int_0^t \int_{-\infty}^{\infty} g(\zeta,\tau)\, \mathcal{G}_g(r,z,\zeta,t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau. \end{aligned} \tag{9.130}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.100) with respect to the radial coordinate r, and the exponential Fourier transform (2.20) with respect to the space coordinate z lead to the following fundamental solutions:

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) = & \frac{p_0}{\pi R^2} \int_{-\infty}^{\infty} E_\alpha\left(-a\eta^2 t^\alpha\right) \cos[(z-\zeta)\eta]\, \mathrm{d}\eta \\ & + \frac{p_0}{\pi R^2} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} E_\alpha\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \cos[(z-\zeta)\eta]\, \mathrm{d}\eta \end{aligned} \tag{9.131}$$

or, taking into account (2.186), as

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) = & \frac{p_0}{R^2\sqrt{a}t^{\alpha/2}} M\left(\frac{\alpha}{2}; \frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right) \\ & + \frac{p_0}{\pi R^2} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} E_\alpha\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \cos[(z-\zeta)\eta]\, \mathrm{d}\eta, \end{aligned} \tag{9.132}$$

where ξ_k are the positive roots of the transcendental equation

$$J_1(R\xi_k) = 0. \tag{9.133}$$

Similarly,

$$\mathcal{G}_F(r,z,\rho,\zeta,t) = \frac{w_0 t}{\pi R^2} \int\limits_{-\infty}^{\infty} E_{\alpha,2}\left(-a\eta^2 t^\alpha\right) \cos[(z-\zeta)\eta]\, \mathrm{d}\eta$$

$$+\frac{w_0 t}{\pi R^2} \sum_{k=1}^{\infty} \int\limits_{-\infty}^{\infty} E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \cos[(z-\zeta)\eta]\, \mathrm{d}\eta \tag{9.134}$$

or (see (2.187))

$$\mathcal{G}_F(r,z,\rho,\zeta,t) = \frac{w_0 t^{1-\alpha/2}}{R^2\sqrt{a}}\, W\left(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)$$

$$+\frac{w_0 t}{\pi R^2} \sum_{k=1}^{\infty} \int\limits_{-\infty}^{\infty} E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \cos[(z-\zeta)\eta]\, \mathrm{d}\eta \tag{9.135}$$

and

$$\mathcal{G}_\Phi(r,z,\rho,\zeta,t) = \frac{q_0 t^{\alpha-1}}{\pi R^2} \int\limits_{-\infty}^{\infty} E_{\alpha,\alpha}\left(-a\eta^2 t^\alpha\right) \cos[(z-\zeta)\eta]\, \mathrm{d}\eta$$

$$+\frac{q_0 t^{\alpha-1}}{\pi R^2} \sum_{k=1}^{\infty} \int\limits_{-\infty}^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \cos[(z-\zeta)\eta] \mathrm{d}\eta \tag{9.136}$$

or, taking into accout (2.188),

$$\mathcal{G}_\Phi(r,z,\rho,\zeta,t) = \frac{q_0 t^{\alpha/2-1}}{R^2\sqrt{a}}\, W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)$$

$$+\frac{q_0 t^{\alpha-1}}{\pi R^2} \sum_{k=1}^{\infty} \int\limits_{-\infty}^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2} \cos[(z-\zeta)\eta]\, \mathrm{d}\eta. \tag{9.137}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$\mathcal{G}_m(r,z,\zeta,t) = \frac{aRg_0}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.138}$$

$$\mathcal{G}_p(r,z,\zeta,t) = \frac{aRg_0}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\rho=R}. \tag{9.139}$$

9.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.140}$$

$$t=0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.141}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.142}$$

$$r=R: \quad \frac{\partial T}{\partial r} + HT = g(z,t), \tag{9.143}$$

$$\lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.144}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_{-\infty}^{\infty}\int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_{-\infty}^{\infty}\int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ & + \int_0^t\int_{-\infty}^{\infty}\int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ & + \int_0^t\int_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{9.145}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.104) with respect to the radial coordinate r, and the exponential Fourier transform (2.20) with respect to the space coordinate z lead to the following fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{1}{\pi R^2}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} p_0\,E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times\frac{\xi_k^2}{\xi_k^2+H^2}\,\frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\,\cos[(z-\zeta)\eta)]\,\mathrm{d}\eta, \tag{9.146}$$

where ξ_k are the positive roots of the transcendental equation

$$\xi_k J_1(R\xi_k) = H J_0(R\xi_k). \tag{9.147}$$

The fundamental solution to the mathematical Robin problem under zero initial conditions is calculated as

$$\mathcal{G}_m(r,z,\zeta,t) = \frac{aRg_0}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}. \tag{9.148}$$

9.5 Domain $0 \le r < R,\ 0 < z < \infty$

9.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.149}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.150}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.151}$$

$$r = R: \quad T = g_1(z,t), \tag{9.152}$$

$$z = 0: \quad T = g_2(r,t), \tag{9.153}$$

$$\lim_{z\to\infty} T(r,z,t) = 0. \tag{9.154}$$

The solution:

$$\begin{aligned} T(r,z,t) =& \int_0^\infty\int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^\infty\int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^t\int_0^\infty\int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.155}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.96) with respect to the radial coordinate r, and the sin-Fourier transform (2.25) with respect to the space coordinate z allow us to obtain the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{\pi R^2} \sum_{k=1}^{\infty} \int_0^{\infty} \begin{pmatrix} p_0\, E_\alpha \left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \\ w_0 t\, E_{\alpha,2} \left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha} \left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_1(R\xi_k)]^2} \sin(z\eta)\, \sin(\zeta\eta)\, \mathrm{d}\eta, \tag{9.156}$$

where ξ_k are the positive roots of the transcendental equation

$$J_0(R\xi_k) = 0. \tag{9.157}$$

The fundamental solutions to the Dirichlet problems under zero initial conditions are expressed as

$$\mathcal{G}_{g_1}(r,z,\zeta,t) = -\frac{aRg_{01}}{q_0} \frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \rho}\bigg|_{\rho=R}, \tag{9.158}$$

$$\mathcal{G}_{g_2}(r,z,\rho,t) = \frac{ag_{02}}{q_0} \frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\bigg|_{\zeta=0}. \tag{9.159}$$

9.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.160}$$

$$t=0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.161}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.162}$$

$$r=R: \quad \frac{\partial T}{\partial r} = g_1(z,t), \tag{9.163}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.164}$$

$$\lim_{z\to\infty} T(r,z,t) = 0. \tag{9.165}$$

The solution:

$$\begin{aligned} T(r,z,t) &= \int_0^{\infty}\int_0^{R} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^{\infty}\int_0^{R} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_0^{t}\int_0^{\infty}\int_0^{R} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_0^{\infty} g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_0^{R} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{9.166}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.100) with respect to the radial coordinate r, and the cos-Fourier transform (2.37) with respect to the space coordinate z allow us to obtain the fundamental solutions.

The fundamental solution to the first Cauchy problem under zero Neumann boundary condition is

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) &= \frac{4p_0}{\pi R^2}\int_0^{\infty} E_\alpha\left(-a\eta^2 t^\alpha\right)\,\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta \\ &+ \frac{4p_0}{\pi R^2}\sum_{k=1}^{\infty}\int_0^{\infty} E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ &\times \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos(z\eta)\cos(\zeta\eta)\,\mathrm{d}\eta \end{aligned} \tag{9.167}$$

or

$$\begin{aligned} \mathcal{G}_f(r,z,\rho,\zeta,t) &= \frac{p_0}{R^2\sqrt{a}t^{\alpha/2}}\left[M\left(\frac{\alpha}{2};\frac{z+\zeta}{\sqrt{a}t^{\alpha/2}}\right)+M\left(\frac{\alpha}{2};\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)\right] \\ &+ \frac{4p_0}{\pi R^2}\sum_{k=1}^{\infty}\int_0^{\infty} E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\ &\times \frac{J_0(r\xi_n)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos(z\eta)\cos(\zeta\eta)\mathrm{d}\eta, \end{aligned} \tag{9.168}$$

where ξ_k are the positive roots of the transcendental equation

$$J_1(R\xi_k) = 0. \tag{9.169}$$

The fundamental solution to the second Cauchy problem under zero Neumann boundary condition is

$$\begin{aligned}
\mathcal{G}_F(r,z,\rho,\zeta,t) &= \frac{4w_0 t}{\pi R^2}\int_0^\infty E_{\alpha,2}\left(-a\eta^2 t^\alpha\right)\,\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta \\
&+ \frac{4w_0 t}{\pi R^2}\sum_{k=1}^\infty \int_0^\infty E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\
&\times \frac{J_0(r\xi_k)J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\cos(z\eta)\cos(\zeta\eta)\mathrm{d}\eta
\end{aligned} \tag{9.170}$$

or

$$\begin{aligned}
\mathcal{G}_F(r,z,\rho,\zeta,t) &= \frac{w_0 t^{1-\alpha/2}}{R^2\sqrt{a}}\left[W\left(-\frac{\alpha}{2},2-\frac{\alpha}{2};-\frac{z+\zeta}{\sqrt{a}t^{\alpha/2}}\right)\right. \\
&\left.+W\left(-\frac{\alpha}{2},2-\frac{\alpha}{2};-\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)\right] + \frac{4w_0 t}{\pi R^2}\sum_{k=1}^\infty\int_0^\infty E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\
&\times \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\,\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta.
\end{aligned} \tag{9.171}$$

The fundamental solution to the source problem under zero Neumann boundary condition is

$$\begin{aligned}
\mathcal{G}_\Phi(r,z,\rho,\zeta,t) &= \frac{4q_0 t^{\alpha-1}}{\pi R^2}\int_0^\infty E_{\alpha,\alpha}\left(-a\eta^2 t^\alpha\right)\,\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta \\
&+ \frac{4q_0 t^{\alpha-1}}{\pi R^2}\sum_{k=1}^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \\
&\times \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\,\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta
\end{aligned} \tag{9.172}$$

or

$$\mathcal{G}_\Phi(r,z,\rho,\zeta,t) = \frac{q_0 t^{\alpha/2-1}}{R^2\sqrt{a}} \left[W\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{z+\zeta}{\sqrt{a}t^{\alpha/2}}\right) \right.$$

$$\left. + W\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|z-\zeta|}{\sqrt{a}t^{\alpha/2}}\right)\right] + \frac{4q_0 t^{\alpha-1}}{\pi R^2} \sum_{k=1}^{\infty} \int_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]$$

$$\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2} \cos(z\eta)\, \cos(\zeta\eta)\, \mathrm{d}\eta. \tag{9.173}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$\mathcal{G}_{m_1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.174}$$

$$\mathcal{G}_{m_2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.175}$$

$$\mathcal{G}_{p_1}(r,z,\zeta,t) = \frac{aRg_{01}}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.176}$$

$$\mathcal{G}_{p_2}(r,z,\rho,t) = \frac{ag_{02}}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\Big|_{\zeta=0}. \tag{9.177}$$

9.5.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.178}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.179}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.180}$$

$$r=R: \quad \frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.181}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.182}$$

$$\lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.183}$$

The solution:

$$\begin{aligned}
T(r,z,t) &= \int_0^\infty \int_0^R f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
&+ \int_0^\infty \int_0^R F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\
&+ \int_0^t \int_0^\infty \int_0^R \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_0^\infty g_1(\zeta,\tau)\, \mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_0^R g_2(\rho,\tau)\, \mathcal{G}_{g_2}(r,z,\rho,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau.
\end{aligned} \tag{9.184}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.104) with respect to the radial coordinate r, and the sin-cos-Fourier transform (2.40), (2.42) with respect to the space coordinate z result in the following fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{\pi R^2} \sum_{k=1}^{\infty} \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta^2\right) t^\alpha\right] \end{pmatrix}$$

$$\times \frac{\xi_k^2}{\xi_k^2+H_1^2}\, \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{\left[J_0(R\xi_k)\right]^2}\, \frac{\eta\cos(z\eta)+H_2\sin(z\eta)}{\eta^2+H_2^2}$$

$$\times \Big[\eta\cos(\zeta\eta)+H_2\sin(\zeta\eta)\Big]\, \mathrm{d}\eta. \tag{9.185}$$

The fundamental solutions to the first and second mathematical Robin problems under zero initial conditions are expressed as

$$\mathcal{G}_{m1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.186}$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}. \tag{9.187}$$

9.6 Domain $0 \le r < R$, $0 < z < L$

9.6.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, z, t), \tag{9.188}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.189}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.190}$$

$$r = R: \quad T = g_1(z, t), \tag{9.191}$$

$$z = 0: \quad T = g_2(r, t), \tag{9.192}$$

$$z = L: \quad T = g_3(r, t). \tag{9.193}$$

The solution:

$$\begin{aligned} T(r, z, t) = & \int_0^L \int_0^R f(\rho, \zeta)\, \mathcal{G}_f(r, z, \rho, \zeta, t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ & + \int_0^L \int_0^R F(\rho, \zeta)\, \mathcal{G}_F(r, z, \rho, \zeta, t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ & + \int_0^t \int_0^L \int_0^R \Phi(\rho, \zeta, \tau)\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\tau \\ & + \int_0^t \int_0^L g_1(\zeta, \tau)\, \mathcal{G}_{g_1}(r, z, \zeta, t-\tau) \,\mathrm{d}\zeta \,\mathrm{d}\tau \\ & + \int_0^t \int_0^R g_2(\rho, \tau)\, \mathcal{G}_{g_2}(r, z, \rho, t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\tau \\ & + \int_0^t \int_0^R g_3(\rho, \tau)\, \mathcal{G}_{g_3}(r, z, \rho, t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\tau. \end{aligned} \tag{9.194}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.96) with respect to the radial coordinate r with ξ_k being the transform variable, and the finite sin-Fourier transform (2.44) with respect to the space coordinate z with η_m being the transform variable allow us to obtain the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{R^2 L} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_1(R\xi_k)]^2} \sin(z\eta_m) \sin(\zeta\eta_m), \tag{9.195}$$

where ξ_k are the positive roots of the transcendental equation

$$J_0(R\xi_k) = 0 \quad \text{and} \quad \eta_m = \frac{m\pi}{L}. \tag{9.196}$$

The fundamental solutions to the Dirichlet problems under zero initial conditions are expressed as

$$\mathcal{G}_{g1}(r,z,\zeta,t) = -\frac{aRg_{01}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \rho}\right|_{\rho=R}, \tag{9.197}$$

$$\mathcal{G}_{g2}(r,z,\rho,t) = \frac{ag_{02}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\right|_{\zeta=0}, \tag{9.198}$$

$$\mathcal{G}_{g3}(r,z,\rho,t) = -\frac{ag_{03}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta}\right|_{\zeta=L}. \tag{9.199}$$

9.6.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.200}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.201}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.202}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g_1(z,t), \tag{9.203}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.204}$$

$$z = L: \quad \frac{\partial T}{\partial z} = g_3(r,t). \tag{9.205}$$

The solution:

$$
\begin{aligned}
T(r,z,t) = & \int_0^L \int_0^R f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_0^L \int_0^R F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_0^t \int_0^L \int_0^R \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
& + \int_0^t \int_0^L g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
& + \int_0^t \int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
& + \int_0^t \int_0^R g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \qquad (9.206)
\end{aligned}
$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.100) with respect to the radial coordinate r with ξ_k being the transform variable, and the finite cos-Fourier transform (2.48) with respect to the space coordinate z with η_m being the transform variable lead to the fundamental solutions:

$$
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{R^2 L} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix}
$$

$$
\times \frac{J_0(r\xi_k)\,J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2}\,\cos(z\eta_m)\,\cos(\zeta\eta_m), \qquad (9.207)
$$

where ξ_k are the nonnegative roots of the transcendental equation

$$
J_1(R\xi_k) = 0 \quad \text{and} \quad \eta_m = \frac{m\pi}{L}. \qquad (9.208)
$$

Recall that Eq. (9.208) has also the zero root $\xi_0 = 0$.

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$
\mathcal{G}_{m1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \qquad (9.209)
$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.210}$$

$$\mathcal{G}_{m3}(r,z,\rho,t) = \frac{a g_{03}}{q_0}\, \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}, \tag{9.211}$$

$$\mathcal{G}_{p1}(r,z,\zeta,t) = \frac{a R g_{01}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.212}$$

$$\mathcal{G}_{p2}(r,z,\rho,t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.213}$$

$$\mathcal{G}_{p3}(r,z,\rho,t) = \frac{a g_{03}}{p_0}\, \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.214}$$

9.6.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.215}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.216}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.217}$$

$$r = R: \quad \frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.218}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.219}$$

$$z = L: \quad \frac{\partial T}{\partial z} + H_2 T = g_3(r,t). \tag{9.220}$$

The solution:

$$\begin{aligned} T(r,z,t) = &\int_0^L \int_0^R f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^L \int_0^R F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta \\ &+ \int_0^t \int_0^L \int_0^R \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau \end{aligned}$$

$$+ \int_0^t \int_0^L g_1(\zeta, \tau)\, \mathcal{G}_{g_1}(r, z, \zeta, t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau$$
$$+ \int_0^t \int_0^R g_2(\rho, \tau)\, \mathcal{G}_{g_2}(r, z, \rho, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau$$
$$+ \int_0^t \int_0^R g_3(\rho, \tau)\, \mathcal{G}_{g_3}(r, z, \rho, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\tau. \tag{9.221}$$

The Laplace transform with respect to time t, the finite Hankel transform of order zero (2.104) with respect to the radial coordinate r with ξ_k being the transform variable, and the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z with η_m being the transform variable result in the fundamental solutions:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{4}{R^2 L} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi_k^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix}$$
$$\times \frac{\xi_k^2}{\xi_k^2+H_1^2}\, \frac{J_0(r\xi_k)\, J_0(\rho\xi_k)}{[J_0(R\xi_k)]^2}\, \frac{\eta_m \cos(z\eta_m) + H_2 \sin(z\eta_m)}{\eta_m^2 + H_2^2 + \frac{2H_2}{L}}$$
$$\times \Big[\eta_m \cos(\zeta\eta_m) + H_2 \sin(\zeta\eta_m)\Big], \tag{9.222}$$

where ξ_k and η_m are the nonnegative roots of the transcendental equations

$$\xi_k\, J_1(R\xi_k) = H_1 J_0(R\xi_k) \quad \text{and} \quad \tan(L\eta_m) = \frac{2H_2\eta_m}{\eta_m^2 - H_2^2}. \tag{9.223}$$

The fundamental solutions to the mathematical Robin problems under zero initial conditions are expressed as

$$\mathcal{G}_{m1}(r,z,\zeta,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\rho=R}, \tag{9.224}$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.225}$$

$$\mathcal{G}_{m3}(r,z,\rho,t) = \frac{ag_{03}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.226}$$

9.7 Domain $R < r < \infty$, $-\infty < z < \infty$

9.7.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.227}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.228}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.229}$$

$$r = R: \quad T = g(z,t), \tag{9.230}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.231}$$

The solution:

$$\begin{aligned} T(r,z,t) &= \int_{-\infty}^{\infty}\int_{R}^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_{-\infty}^{\infty}\int_{R}^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\ &+ \int_{0}^{t}\int_{-\infty}^{\infty}\int_{R}^{\infty} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_{0}^{t}\int_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{9.232}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.117) with respect to the radial coordinate r with ξ being the transform variable, and the exponential Fourier transform (2.20) with respect to the space coordinate z with η being the transform variable:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} \begin{pmatrix} p_0\,E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\ &\times \cos[(z-\zeta)\eta]\,\frac{J_0(r\xi)Y_0(R\xi)-Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)} \\ &\times \Big[J_0(\rho\xi)Y_0(R\xi)-Y_0(\rho\xi)J_0(R\xi)\Big]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{9.233}$$

The fundamental solution to the Dirichlet problem under zero initial conditions has the form

$$\mathcal{G}_g(r,z,\zeta,t) = -\frac{a g_0 t^{\alpha-1}}{\pi^2} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^{\alpha}\right]$$

$$\times \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)} \cos[(z-\zeta)\eta]\, \xi \, \mathrm{d}\xi \, \mathrm{d}\eta. \tag{9.234}$$

9.7.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.235}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.236}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.237}$$

$$r=R: \quad -\frac{\partial T}{\partial r} = g(z,t), \tag{9.238}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.239}$$

The solution:

$$T(r,z,t) = \int\limits_{-\infty}^{\infty}\int\limits_{R}^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$

$$+ \int\limits_{-\infty}^{\infty}\int\limits_{R}^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$

$$+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{R}^{\infty} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty} g(\zeta,\tau)\,\mathcal{G}_g(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau. \tag{9.240}$$

The Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.119) with respect to the radial coordinate r and the exponential Fourier transform (2.20) with respect to the space coordinate z result in the following fundamental solution:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \begin{pmatrix} p_0\, E_\alpha \left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2} \left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha} \left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \cos[(z-\zeta)\eta]\, \frac{J_0(r\xi)Y_1(R\xi)-Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}$$

$$\times \Big[J_0(\rho\xi)Y_1(R\xi)-Y_0(\rho\xi)J_1(R\xi)\Big]\, \xi \,\mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.241}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\begin{pmatrix} \mathcal{G}_m(r,z,\zeta,t) \\ \mathcal{G}_p(r,z,\zeta,t) \end{pmatrix} = \frac{a g_0}{\pi^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \begin{pmatrix} t^{\alpha-1}\, E_{\alpha,\alpha} \left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ E_\alpha \left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)Y_1(R\xi)-Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}\, \cos[(z-\zeta)\eta]\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.242}$$

9.7.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.243}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.244}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.245}$$

$$r=R: \quad -\frac{\partial T}{\partial r} + HT = g(z,t), \tag{9.246}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,z,t) = 0. \tag{9.247}$$

The solution:

$$T(r,z,t) = \int_{-\infty}^{\infty}\int_{R}^{\infty} f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta$$

$$+ \int_{-\infty}^{\infty}\int_{R}^{\infty} F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_R^{\infty} \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\zeta\, \mathrm{d}\tau$$

$$+\int_0^t \int_{-\infty}^{\infty} g(\zeta,\tau)\, \mathcal{G}_g(r,z,\zeta,t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\tau. \tag{9.248}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.121) with respect to the radial coordinate r and the exponential Fourier transform (2.20) with respect to the space coordinate z:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+HJ_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+HY_0(R\xi)]}{[\xi J_1(R\xi)+HJ_0(R\xi)]^2+[\xi Y_1(R\xi)+HY_0(R\xi)]^2}$$

$$\times\Big\{Y_0(r\xi)[\xi J_1(R\xi)+HJ_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+HY_0(R\xi)]\Big\}$$

$$\times \cos[(z-\zeta)\eta]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.249}$$

The fundamental solution to the mathematical Robin problem under zero initial condition is expressed as

$$\mathcal{G}_g(r,z,\zeta,t) = \frac{a g_0 t^{\alpha-1}}{\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \cos[(z-\zeta)\eta]$$

$$\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+HJ_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+HY_0(R\xi)]}{[\xi J_1(R\xi)+HJ_0(R\xi)]^2+[\xi Y_1(R\xi)+HY_0(R\xi)]^2} \xi \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.250}$$

9.8 Domain $R < r < \infty$, $0 < z < \infty$

9.8.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.251}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.252}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.253}$$

$$r = R: \quad T = g_1(z,t), \tag{9.254}$$

$$z = 0: \quad T = g_2(r,t), \tag{9.255}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.256}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_0^\infty \int_R^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ & + \int_0^\infty \int_R^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ & + \int_0^t \int_0^\infty \int_R^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\tau \\ & + \int_0^t \int_0^\infty g_1(\zeta,\tau)\, \mathcal{G}_{g_1}(r,z,\zeta,t-\tau) \,\mathrm{d}\zeta \,\mathrm{d}\tau \\ & + \int_0^t \int_R^\infty g_2(\rho,\tau)\, \mathcal{G}_{g_2}(r,z,\rho,t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\tau. \end{aligned} \tag{9.257}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.117) with respect to the radial coordinate r, and the sin-Fourier transform (2.25) with respect to the space coordinate z we obtain

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{2}{\pi} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\ & \times \sin(z\eta)\, \sin(\zeta\eta)\, \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)} \\ & \times \Big[J_0(\rho\xi)Y_0(R\xi) - Y_0(\rho\xi)J_0(R\xi)\Big]\, \xi \,\mathrm{d}\xi \,\mathrm{d}\eta. \end{aligned} \tag{9.258}$$

The fundamental solution to the first Dirichlet problem under zero initial conditions has the following form

$$\begin{aligned} \mathcal{G}_{g1}(r,z,\zeta,t) = & -\frac{4ag_{01}t^{\alpha-1}}{\pi^2} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ & \times \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi) + Y_0^2(R\xi)}\, \sin(z\eta)\, \sin(\zeta\eta)\, \xi \,\mathrm{d}\xi \,\mathrm{d}\eta, \end{aligned} \tag{9.259}$$

whereas the fundamental solution to the second Dirichlet problem is calculated as

$$\mathcal{G}_{g_2}(r,z,\rho,t) = \frac{a g_{02}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial \zeta} \right|_{\zeta=0}. \tag{9.260}$$

9.8.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.261}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.262}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.263}$$

$$r=R: \quad -\frac{\partial T}{\partial r} = g_1(z,t), \tag{9.264}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.265}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.266}$$

The solution:

$$\begin{aligned}
T(r,z,t) = & \int_0^\infty \int_R^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_0^\infty \int_R^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
& + \int_0^t \int_0^\infty \int_R^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
& + \int_0^t \int_0^\infty g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
& + \int_0^t \int_R^\infty g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{9.267}$$

The Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.119) with respect to the radial coordinate r and the cos-Fourier transform (2.37) with respect to the space coordinate z result in the following fundamental solution:

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \cos(z\eta)\, \cos(\zeta\eta)\, \frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}$$

$$\times \Big[J_0(\rho\xi)Y_1(R\xi) - Y_0(\rho\xi)J_1(R\xi)\Big]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.268}$$

The fundamental solutions to the first and second mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\begin{pmatrix} \mathcal{G}_{m1}(r,z,\zeta,t) \\ \mathcal{G}_{p1}(r,z,\zeta,t) \end{pmatrix} = \frac{4ag_{01}}{\pi^2} \int_0^\infty \int_0^\infty \begin{pmatrix} t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)}\, \cos(z\eta)\, \cos(\zeta\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{9.269}$$

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t)\bigg|_{\zeta=0}, \tag{9.270}$$

$$\mathcal{G}_{p2}(r,z,\rho,t) = \frac{ag_{02}}{p_0}\, \mathcal{G}_f(r,z,\rho,\zeta,t)\bigg|_{\zeta=0}. \tag{9.271}$$

9.8.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.272}$$

$$t = 0: \quad T = f(r,z), \qquad 0 < \alpha \le 2, \tag{9.273}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1 < \alpha \le 2, \tag{9.274}$$

$$r = R: \quad -\frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.275}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.276}$$

$$\lim_{r\to\infty} T(r,z,t) = 0, \qquad \lim_{z\to\infty} T(r,z,t) = 0. \tag{9.277}$$

The solution:

$$\begin{aligned}
T(r,z,t) &= \int\limits_0^{\infty}\int\limits_R^{\infty} f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int\limits_0^{\infty}\int\limits_R^{\infty} F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta \\
&+ \int\limits_0^{t}\int\limits_0^{\infty}\int\limits_R^{\infty} \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int\limits_0^{t}\int\limits_0^{\infty} g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int\limits_0^{t}\int\limits_R^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \qquad (9.278)
\end{aligned}$$

The Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.121) with respect to the radial coordinate r and the sin-cos-Fourier transform (2.40), (2.42) with respect to the space coordinate z give:

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} &= \frac{2}{\pi}\int\limits_0^{\infty}\int\limits_0^{\infty} \begin{pmatrix} p_0\,E_\alpha\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \end{pmatrix} \\
&\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+H_1J_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+H_1Y_0(R\xi)]}{[\xi J_1(R\xi)+H_1J_0(R\xi)]^2+[\xi Y_1(R\xi)+H_1Y_0(R\xi)]^2} \\
&\times\Big\{Y_0(r\xi)[\xi J_1(R\xi)+H_1J_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+H_1Y_0(R\xi)]\Big\} \\
&\times \frac{\eta\cos(z\eta)+H_2\sin(z\eta)}{\eta^2+H_2^2}\Big[\eta\cos(\zeta\eta)+H_2\sin(\zeta\eta)\Big]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (9.279)
\end{aligned}$$

The fundamental solution to the first mathematical Robin problem under zero initial conditions has the form

$$\begin{aligned}
\mathcal{G}_{m1}(r,z,\zeta,t) &= \frac{4ag_{01}t^{\alpha-1}}{\pi^2}\int\limits_0^{\infty}\int\limits_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \\
&\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+H_1J_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+H_1Y_0(R\xi)]}{[\xi J_1(R\xi)+H_1J_0(R\xi)]^2+[\xi Y_1(R\xi)+H_1Y_0(R\xi)]^2} \\
&\times \frac{\eta\cos(z\eta)+H_2\sin(z\eta)}{\eta^2+H_2^2}\Big[\eta\cos(\zeta\eta)+H_2\sin(\zeta\eta)\Big]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (9.280)
\end{aligned}$$

The fundamental solution to the second mathematical Robin problem is calculated as

$$\mathcal{G}_{m2}(r, z, \rho, t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t)\Big|_{\zeta=0}. \tag{9.281}$$

9.9 Domain $R < r < \infty$, $0 < z < L$

9.9.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, z, t), \tag{9.282}$$

$$t = 0: \quad T = f(r, z), \qquad 0 < \alpha \le 2, \tag{9.283}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, z), \qquad 1 < \alpha \le 2, \tag{9.284}$$

$$r = R: \quad T = g_1(z, t), \tag{9.285}$$

$$z = 0: \quad T = g_2(r, t), \tag{9.286}$$

$$z = L: \quad T = g_3(r, t), \tag{9.287}$$

$$\lim_{r\to\infty} T(r, z, t) = 0. \tag{9.288}$$

The solution:

$$\begin{aligned}
T(r, z, t) &= \int_0^L \int_R^\infty f(\rho, \zeta)\, \mathcal{G}_f(r, z, \rho, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\
&+ \int_0^L \int_R^\infty F(\rho, \zeta)\, \mathcal{G}_F(r, z, \rho, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \\
&+ \int_0^t \int_0^L \int_R^\infty \Phi(\rho, \zeta, \tau)\, \mathcal{G}_\Phi(r, z, \rho, \zeta, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\tau \\
&+ \int_0^t \int_0^L g_1(\zeta, \tau)\, \mathcal{G}_{g_1}(r, z, \zeta, t-\tau)\, \mathrm{d}\zeta \, \mathrm{d}\tau \\
&+ \int_0^t \int_R^\infty g_2(\rho, \tau)\, \mathcal{G}_{g_2}(r, z, \rho, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\tau
\end{aligned}$$

$$+\int_0^t \int_R^\infty g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \tag{9.289}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.117) with respect to the radial coordinate r, and the finite sin-Fourier transform (2.44) with respect to the space coordinate z we obtain

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L}\sum_{m=1}^{\infty}\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix}$$

$$\times \sin(z\eta_m)\,\sin(\zeta\eta_m)\,\frac{J_0(r\xi)Y_0(R\xi)-Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)}$$

$$\times\Big[J_0(\rho\xi)Y_0(R\xi)-Y_0(\rho\xi)J_0(R\xi)\Big]\,\xi\,\mathrm{d}\xi, \tag{9.290}$$

where $\eta_m = m\pi/L$.

The fundamental solution to the first Dirichlet problem under zero initial conditions has the form

$$\mathcal{G}_{g1}(r,z,\zeta,t) = -\frac{4ag_{01}t^{\alpha-1}}{\pi L}\sum_{m=1}^{\infty}\int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right]$$

$$\times\frac{J_0(r\xi)Y_0(R\xi)-Y_0(r\xi)J_0(R\xi)}{J_0^2(R\xi)+Y_0^2(R\xi)}\,\sin(z\eta_m)\,\sin(\zeta\eta_m)\,\xi\,\mathrm{d}\xi, \tag{9.291}$$

whereas the fundamental solutions to the second and third Dirichlet problems are calculated as

$$\mathcal{G}_{g2}(r,z,\rho,t) = \frac{ag_{02}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial\zeta}\bigg|_{\zeta=0}, \tag{9.292}$$

$$\mathcal{G}_{g3}(r,z,\rho,t) = -\frac{ag_{03}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,z,\rho,\zeta,t)}{\partial\zeta}\bigg|_{\zeta=L}. \tag{9.293}$$

9.9.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(r,z,t), \tag{9.294}$$

$$t=0: \quad T=f(r,z), \qquad 0<\alpha\le 2, \tag{9.295}$$

$$t=0: \quad \frac{\partial T}{\partial t}=F(r,z), \qquad 1<\alpha\le 2, \tag{9.296}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g_1(z,t), \tag{9.297}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_2(r,t), \tag{9.298}$$

$$z = L: \quad \frac{\partial T}{\partial z} = g_3(r,t), \tag{9.299}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.300}$$

The solution:

$$\begin{aligned} T(r,z,t) = & \int_0^L \int_R^\infty f(\rho,\zeta)\, \mathcal{G}_f(r,z,\rho,\zeta,t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ & + \int_0^L \int_R^\infty F(\rho,\zeta)\, \mathcal{G}_F(r,z,\rho,\zeta,t)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \\ & + \int_0^t \int_0^L \int_R^\infty \Phi(\rho,\zeta,\tau)\, \mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\tau \\ & + \int_0^t \int_0^L g_1(\zeta,\tau)\, \mathcal{G}_{g_1}(r,z,\zeta,t-\tau) \,\mathrm{d}\zeta \,\mathrm{d}\tau \\ & + \int_0^t \int_R^\infty g_2(\rho,\tau)\, \mathcal{G}_{g_2}(r,z,\rho,t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\tau \\ & + \int_0^t \int_R^\infty g_3(\rho,\tau)\, \mathcal{G}_{g_3}(r,z,\rho,t-\tau)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\tau. \end{aligned} \tag{9.301}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.119) with respect to the radial coordinate r, and the finite cos-Fourier transform (2.48) with respect to the space coordinate z, we obtain

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = & \frac{2}{L} \sum_{m=0}^{\infty}{}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix} \\ & \times \cos(z\eta_m)\, \cos(\zeta\eta_m)\, \frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi) + Y_1^2(R\xi)} \\ & \times \Big[J_0(\rho\xi)Y_1(R\xi) - Y_0(\rho\xi)J_1(R\xi)\Big]\, \xi \,\mathrm{d}\xi, \end{aligned} \tag{9.302}$$

where $\eta_m = m\pi/L$.

The fundamental solutions to the first mathematical and physical Neumann problems under zero initial conditions have the form

$$\begin{pmatrix} \mathcal{G}_{m1}(r,z,\zeta,t) \\ \mathcal{G}_{p1}(r,z,\zeta,t) \end{pmatrix} = \frac{4a g_{01}}{\pi L} \sum_{m=0}^{\infty}{}' \int_0^{\infty} \begin{pmatrix} t^{\alpha-1} E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^{\alpha}\right] \\ E_{\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^{\alpha}\right] \end{pmatrix}$$

$$\times \frac{J_0(r\xi)Y_1(R\xi)-Y_0(r\xi)J_1(R\xi)}{J_1^2(R\xi)+Y_1^2(R\xi)} \cos(z\eta_m)\cos(\zeta\eta_m)\,\mathrm{d}\xi, \tag{9.303}$$

whereas the fundamental solutions to the second and third mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m2}(r,z,\rho,t) = \frac{a g_{02}}{q_0} \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.304}$$

$$\mathcal{G}_{m3}(r,z,\rho,t) = \frac{a g_{03}}{q_0} \mathcal{G}_{\Phi}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}, \tag{9.305}$$

$$\mathcal{G}_{p2}(r,z,\rho,t) = \frac{a g_{02}}{p_0} \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=0}, \tag{9.306}$$

$$\mathcal{G}_{p3}(r,z,\rho,t) = \frac{a g_{03}}{p_0} \mathcal{G}_{f}(r,z,\rho,\zeta,t)\Big|_{\zeta=L}. \tag{9.307}$$

9.9.3 Robin boundary condition

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,z,t), \tag{9.308}$$

$$t=0: \quad T = f(r,z), \qquad 0<\alpha\le 2, \tag{9.309}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,z), \qquad 1<\alpha\le 2, \tag{9.310}$$

$$r=R: \quad -\frac{\partial T}{\partial r} + H_1 T = g_1(z,t), \tag{9.311}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.312}$$

$$z=L: \quad \frac{\partial T}{\partial z} + H_2 T = g_2(r,t), \tag{9.313}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{9.314}$$

The solution:

$$T(r,z,t) = \int_0^L \int_R^\infty f(\rho,\zeta)\,\mathcal{G}_f(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$
$$+ \int_0^L \int_R^\infty F(\rho,\zeta)\,\mathcal{G}_F(r,z,\rho,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta$$
$$+ \int_0^t \int_0^L \int_R^\infty \Phi(\rho,\zeta,\tau)\,\mathcal{G}_\Phi(r,z,\rho,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\tau$$
$$+ \int_0^t \int_0^L g_1(\zeta,\tau)\,\mathcal{G}_{g_1}(r,z,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\tau$$
$$+ \int_0^t \int_R^\infty g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau$$
$$+ \int_0^t \int_R^\infty g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,z,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \tag{9.315}$$

Using the Laplace transform with respect to time t, the Weber transform of order zero (2.108), (2.121) with respect to the radial coordinate r, and the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z we get

$$\begin{pmatrix} \mathcal{G}_f(r,z,\rho,\zeta,t) \\ \mathcal{G}_F(r,z,\rho,\zeta,t) \\ \mathcal{G}_\Phi(r,z,\rho,\zeta,t) \end{pmatrix} = \frac{2}{L}\sum_{m=1}^{\infty}\int_0^\infty \begin{pmatrix} p_0\,E_\alpha\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta_m^2\right)t^\alpha\right] \end{pmatrix}$$
$$\times \frac{Y_0(r\xi)[\xi J_1(R\xi)+H_1 J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi)+H_1 Y_0(R\xi)]}{[\xi J_1(R\xi)+H_1 J_0(R\xi)]^2 + [\xi Y_1(R\xi)+H_1 Y_0(R\xi)]^2}$$
$$\times \Big\{Y_0(r\xi)[\xi J_1(R\xi)+H_1 J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi)+H_1 Y_0(R\xi)]\Big\}$$
$$\times \frac{\eta_m\cos(z\eta_m)+H_2\sin(z\eta_m)}{\eta_m^2+H_2^2+2H_2/L}\Big[\eta_m\cos(\zeta\eta_m)+H_2\sin(\zeta\eta_m)\Big]\,\xi\,\mathrm{d}\xi, \tag{9.316}$$

where η_m are the positive roots of the transcendental equation

$$\tan(L\eta_m) = \frac{2H_2\eta_m}{\eta_m^2 - H_2^2}. \tag{9.317}$$

The fundamental solution to the first mathematical Robin problem under zero initial conditions has the form

$$\mathcal{G}_{m1}(r, z, \zeta, t) = \frac{4ag_{01}t^{\alpha-1}}{\pi L} \sum_{m=1}^{\infty} \int_0^{\infty} E_{\alpha,\alpha}\left[-a\left(\xi^2 + \eta_m^2\right)t^{\alpha}\right]$$

$$\times \frac{Y_0(r\xi)[\xi J_1(R\xi) + H_1 J_0(R\xi)] - J_0(r\xi)[\xi Y_1(R\xi) + H_1 Y_0(R\xi)]}{[\xi J_1(R\xi) + H_1 J_0(R\xi)]^2 + [\xi Y_1(R\xi) + H_1 Y_0(R\xi)]^2}$$

$$\times \frac{\eta_m \cos(z\eta_m) + H_2 \sin(z\eta_m)}{\eta_m^2 + H_2^2 + 2H_2/L} \Big[\eta_m \cos(\zeta\eta_m) + H_2 \sin(\zeta\eta_m)\Big]\, \xi \,\mathrm{d}\xi, \tag{9.318}$$

the fundamental solutions to the second and third mathematical Robin problems are calculated as

$$\mathcal{G}_{m2}(r, z, \rho, t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_{\Phi}(r, z, \rho, \zeta, t)\Big|_{\zeta=0}, \tag{9.319}$$

$$\mathcal{G}_{m3}(r, z, \rho, t) = \frac{ag_{03}}{q_0}\, \mathcal{G}_{\Phi}(r, z, \rho, \zeta, t)\Big|_{\zeta=L}. \tag{9.320}$$

Chapter 10

Equations with Three Space Variables in Cartesian Coordinates

If A is a success in life,
then A equals x plus y plus z.
Albert Einstein

10.1 Domain $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.1}$$

$$t = 0: \quad T = f(x,y,z), \qquad 0 < \alpha \le 2, \tag{10.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1 < \alpha \le 2, \tag{10.3}$$

$$\lim_{x\to\pm\infty} T(x,y,z,t) = 0, \qquad \lim_{y\to\pm\infty} T(x,y,z,t) = 0,$$
$$\lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.4}$$

The solution:

$$T(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x-\rho, y-\sigma, z-\upsilon, t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x-\rho, y-\sigma, z-\upsilon, t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x-\rho, y-\sigma, z-\upsilon, t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \tag{10.5}$$

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with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,t) \\ \mathcal{G}_F(x,y,z,t) \\ \mathcal{G}_\Phi(x,y,z,t) \end{pmatrix} = \frac{1}{8\pi^3} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \cos(x\xi)\ \cos(y\eta)\ \cos(z\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{10.6}$$

10.2 Domain $0 < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$

10.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.7}$$

$$t = 0: \quad T = f(x,y,z), \qquad 0 < \alpha \le 2, \tag{10.8}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1 < \alpha \le 2, \tag{10.9}$$

$$x = 0: \quad T = g(y,z,t), \tag{10.10}$$

$$\lim_{x\to\infty} T(x,y,z,t) = 0, \qquad \lim_{y\to\pm\infty} T(x,y,z,t) = 0,$$

$$\lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.11}$$

The solution:

$$T(x,y,z,t) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+ \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} g(\sigma,\upsilon,\tau)\,\mathcal{G}_g(x,y-\sigma,z-\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \tag{10.12}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,t) \\ \mathcal{G}_F(x,y,z,\rho,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,t) \end{pmatrix} = \frac{1}{2\pi^3} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} \begin{pmatrix} p_0 E_\alpha\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t E_{\alpha,2}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$
$$\times\ \sin(x\xi)\sin(\rho\xi)\ \cos(y\eta)\ \cos(z\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{10.13}$$

The fundamental solution to the Dirichlet problem is calculated as

$$\mathcal{G}_g(x,y,z,t) = \frac{a g_0}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,t)}{\partial \rho}\right|_{\rho=0}. \tag{10.14}$$

10.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.15}$$

$$t = 0: \quad T = f(x,y,z), \qquad 0 < \alpha \le 2, \tag{10.16}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1 < \alpha \le 2, \tag{10.17}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} = g(y,z,t), \tag{10.18}$$

$$\lim_{x\to\infty} T(x,y,z,t) = 0, \qquad \lim_{y\to\pm\infty} T(x,y,z,t) = 0,$$
$$\lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.19}$$

The solution:

$$T(x,y,z,t) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$
$$+ \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} g(\sigma,\upsilon,\tau)\,\mathcal{G}_g(x,y-\sigma,z-\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \tag{10.20}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,t) \\ \mathcal{G}_F(x,y,z,\rho,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,t) \end{pmatrix} = \frac{1}{2\pi^3} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \cos(x\xi)\cos(\rho\xi)\ \cos(y\eta)\ \cos(z\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{10.21}$$

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_m(x,y,z,t) = \frac{ag_0}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,t)\bigg|_{\rho=0}, \tag{10.22}$$

$$\mathcal{G}_p(x,y,z,t) = \frac{ag_0}{p_0}\,\mathcal{G}_f(x,y,z,\rho,t)\bigg|_{\rho=0}. \tag{10.23}$$

10.2.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.24}$$

$$t=0:\quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.25}$$

$$t=0:\quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.26}$$

$$x=0:\quad -\frac{\partial T}{\partial x} + HT = g(y,z,t), \tag{10.27}$$

$$\lim_{x\to\infty} T(x,y,z,t) = 0, \qquad \lim_{y\to\pm\infty} T(x,y,z,t) = 0,$$

$$\lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.28}$$

The solution:

$$T(x,y,z,t) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} g(\sigma,\upsilon,\tau)\,\mathcal{G}_g(x,y-\sigma,z-\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \tag{10.29}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,t) \\ \mathcal{G}_F(x,y,z,\rho,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,t) \end{pmatrix} = \frac{1}{2\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \frac{\xi\cos(x\xi)+H\sin(x\xi)}{\xi^2+H^2}\big[\xi\cos(\rho\xi)+H\sin(\rho\xi)\big]$$

$$\times\ \cos(y\eta)\,\cos(z\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{10.30}$$

The fundamental solution to the mathematical Robin problem is calculated as

$$\mathcal{G}_g(x,y,z,t) = \frac{a g_0}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,t)\Big|_{\rho=0}. \tag{10.31}$$

10.3 Domain $0 < x < \infty$, $0 < y < \infty$, $-\infty < z < \infty$

10.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.32}$$

$$t=0:\quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.33}$$

$$t=0:\quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.34}$$

$$x=0:\quad T = g_1(y,z,t), \tag{10.35}$$

$$y=0:\quad T = g_2(x,z,t), \tag{10.36}$$

$$\lim_{x\to\infty} T(x,y,z,t) = 0, \qquad \lim_{y\to\infty} T(x,y,z,t) = 0,$$

$$\lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.37}$$

The solution:

$$T(x,y,z,t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z-\upsilon,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z-\upsilon,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z-\upsilon,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z-\upsilon,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_2(\rho,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau \tag{10.38}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,t) \end{pmatrix} = \frac{2}{\pi^3}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right] \end{pmatrix}$$

$$\times\sin(x\xi)\,\sin(\rho\xi)\,\sin(y\eta)\,\sin(\sigma\eta)\,\cos(z\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{10.39}$$

The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,\sigma,t) = \frac{a g_{01}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)}{\partial\rho}\bigg|_{\rho=0}, \tag{10.40}$$

$$\mathcal{G}_{g2}(x,y,z,\rho,t) = \frac{a g_{02}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)}{\partial\sigma}\bigg|_{\sigma=0}. \tag{10.41}$$

10.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(x,y,z,t), \tag{10.42}$$

$$t=0:\quad T=f(x,y,z),\qquad 0<\alpha\le 2, \tag{10.43}$$

$$t=0:\quad \frac{\partial T}{\partial t}=F(x,y,z),\quad 1<\alpha\le 2, \tag{10.44}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} = g_1(y, z, t), \tag{10.45}$$

$$y = 0: \quad -\frac{\partial T}{\partial y} = g_2(x, z, t), \tag{10.46}$$

$$\lim_{x\to\infty} T(x, y, z, t) = 0, \qquad \lim_{y\to\infty} T(x, y, z, t) = 0,$$

$$\lim_{z\to\pm\infty} T(x, y, z, t) = 0. \tag{10.47}$$

The solution:

$$T(x, y, z, t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(\rho, \sigma, \upsilon)\, \mathcal{G}_f(x, y, z-\upsilon, \rho, \sigma, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon$$

$$+ \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} F(\rho, \sigma, \upsilon)\, \mathcal{G}_F(x, y, z-\upsilon, \rho, \sigma, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon$$

$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \Phi(\rho, \sigma, \upsilon, \tau)\, \mathcal{G}_\Phi(x, y, z-\upsilon, \rho, \sigma, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_1(\sigma, \upsilon, \tau)\, \mathcal{G}_{g1}(x, y, z-\upsilon, \sigma, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_2(\rho, \upsilon, \tau)\, \mathcal{G}_{g2}(x, y, z-\upsilon, \rho, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau \tag{10.48}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x, y, z, \rho, \sigma, t) \\ \mathcal{G}_F(x, y, z, \rho, \sigma, t) \\ \mathcal{G}_\Phi(x, y, z, \rho, \sigma, t) \end{pmatrix} = \frac{2}{\pi^3}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right] \end{pmatrix}$$

$$\times \cos(x\xi)\, \cos(\rho\xi)\, \cos(y\eta)\, \cos(\sigma\eta)\, \cos(z\zeta)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta. \tag{10.49}$$

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x, y, z, \sigma, t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, t)\bigg|_{\rho=0}, \tag{10.50}$$

$$\mathcal{G}_{m2}(x, y, z, \rho, t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, t)\bigg|_{\sigma=0}, \tag{10.51}$$

$$\mathcal{G}_{p1}(x,y,z,\sigma,t) = \frac{ag_{01}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,t)\bigg|_{\rho=0}, \tag{10.52}$$

$$\mathcal{G}_{p2}(x,y,z,\rho,t) = \frac{ag_{02}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,t)\bigg|_{\sigma=0}. \tag{10.53}$$

10.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(x,y,z,t), \tag{10.54}$$

$$t=0:\quad T=f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.55}$$

$$t=0:\quad \frac{\partial T}{\partial t}=F(x,y,z), \quad 1<\alpha\le 2, \tag{10.56}$$

$$x=0:\quad -\frac{\partial T}{\partial x}+H_1T=g_1(y,z,t), \tag{10.57}$$

$$y=0:\quad -\frac{\partial T}{\partial y}+H_2T=g_2(x,z,t), \tag{10.58}$$

$$\lim_{x\to\infty}T(x,y,z,t)=0, \qquad \lim_{y\to\infty}T(x,y,z,t)=0,$$

$$\lim_{z\to\pm\infty}T(x,y,z,t)=0. \tag{10.59}$$

The solution:

$$\begin{aligned}
T(x,y,z,t) &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z-\upsilon,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z-\upsilon,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z-\upsilon,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z-\upsilon,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_2(\rho,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau
\end{aligned} \tag{10.60}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,t) \end{pmatrix} = \frac{2}{\pi^3} \int\limits_{-\infty}^{\infty} \int\limits_0^{\infty} \int\limits_0^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \frac{\xi\cos(x\xi)+H_1\sin(x\xi)}{\xi^2+H_1^2}\big[\xi\cos(\rho\xi)+H_1\sin(\rho\xi)\big]$$

$$\times \frac{\eta\cos(y\eta)+H_2\sin(y\eta)}{\eta^2+H_2^2}\big[\eta\cos(\sigma\eta)+H_2\sin(\sigma\eta)\big]\cos(z\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{10.61}$$

The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(x,y,z,\sigma,t) = \frac{a g_{01}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)\Big|_{\rho=0}, \tag{10.62}$$

$$\mathcal{G}_{m2}(x,y,z,\rho,t) = \frac{a g_{02}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)\Big|_{\sigma=0}. \tag{10.63}$$

10.4 Domain $0 < x < \infty$, $0 < y < \infty$, $0 < z < \infty$

10.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.64}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.65}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.66}$$

$$x=0: \quad T = g_1(y,z,t), \tag{10.67}$$

$$y=0: \quad T = g_2(x,z,t), \tag{10.68}$$

$$z=0: \quad T = g_3(x,y,t), \tag{10.69}$$

$$\lim_{x\to\infty} T(x,y,z,t)=0, \qquad \lim_{y\to\infty} T(x,y,z,t)=0,$$

$$\lim_{z\to\infty} T(x,y,z,t)=0. \tag{10.70}$$

The solution:

$$\begin{aligned}
T(x,y,z,t) &= \int_0^\infty\int_0^\infty\int_0^\infty f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int_0^\infty\int_0^\infty\int_0^\infty F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int_0^t\int_0^\infty\int_0^\infty\int_0^\infty \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int_0^t\int_0^\infty\int_0^\infty g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int_0^t\int_0^\infty\int_0^\infty g_2(\rho,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z,\rho,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int_0^t\int_0^\infty\int_0^\infty g_3(\rho,\sigma,\tau)\,\mathcal{G}_{g3}(x,y,z,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau
\end{aligned} \tag{10.71}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t) \end{pmatrix} = \frac{8}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \sin(x\xi)\,\sin(\rho\xi)\,\sin(y\eta)\,\sin(\sigma\eta)\,\sin(z\zeta)\,\sin(\upsilon\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{10.72}$$

The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{01}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial \rho}\bigg|_{\rho=0}, \tag{10.73}$$

$$\mathcal{G}_{g2}(x,y,z,\rho,\upsilon,t) = \frac{a g_{02}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial \sigma}\bigg|_{\sigma=0}, \tag{10.74}$$

$$\mathcal{G}_{g3}(x,y,z,\rho,\sigma,t) = \frac{a g_{03}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial \upsilon}\bigg|_{\upsilon=0}. \tag{10.75}$$

10.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x, y, z, t), \tag{10.76}$$

$$t = 0: \quad T = f(x, y, z), \qquad 0 < \alpha \le 2, \tag{10.77}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x, y, z), \quad 1 < \alpha \le 2, \tag{10.78}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} = g_1(y, z, t), \tag{10.79}$$

$$y = 0: \quad -\frac{\partial T}{\partial y} = g_2(x, z, t), \tag{10.80}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_3(x, y, t), \tag{10.81}$$

$$\lim_{x\to\infty} T(x, y, z, t) = 0, \qquad \lim_{y\to\infty} T(x, y, z, t) = 0,$$

$$\lim_{z\to\infty} T(x, y, z, t) = 0. \tag{10.82}$$

The solution:

$$\begin{aligned} T(x, y, z, t) &= \int_0^\infty\int_0^\infty\int_0^\infty f(\rho, \sigma, \upsilon)\, \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon \\ &+ \int_0^\infty\int_0^\infty\int_0^\infty F(\rho, \sigma, \upsilon)\, \mathcal{G}_F(x, y, z, \rho, \sigma, \upsilon, t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon \\ &+ \int_0^t\int_0^\infty\int_0^\infty\int_0^\infty \Phi(\rho, \sigma, \upsilon, \tau)\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty\int_0^\infty g_1(\sigma, \upsilon, \tau)\, \mathcal{G}_{g1}(x, y, z, \sigma, \upsilon, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty\int_0^\infty g_2(\rho, \upsilon, \tau)\, \mathcal{G}_{g2}(x, y, z, \rho, \upsilon, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau \\ &+ \int_0^t\int_0^\infty\int_0^\infty g_3(\rho, \sigma, \tau)\, \mathcal{G}_{g3}(x, y, z, \rho, \sigma, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau \end{aligned} \tag{10.83}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t) \end{pmatrix} = \frac{8}{\pi^3} \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \cos(x\xi)\, \cos(\rho\xi)\, \cos(y\eta)\, \cos(\sigma\eta)\, \cos(z\zeta)\, \cos(\upsilon\zeta)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta. \tag{10.84}$$

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\bigg|_{\rho=0}, \tag{10.85}$$

$$\mathcal{G}_{m2}(x,y,z,\rho,\upsilon,t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\bigg|_{\sigma=0}, \tag{10.86}$$

$$\mathcal{G}_{m3}(x,y,z,\rho,\sigma,t) = \frac{a g_{03}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\bigg|_{\upsilon=0}, \tag{10.87}$$

$$\mathcal{G}_{p1}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{01}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\bigg|_{\rho=0}, \tag{10.88}$$

$$\mathcal{G}_{p2}(x,y,z,\rho,\upsilon,t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\bigg|_{\sigma=0}, \tag{10.89}$$

$$\mathcal{G}_{p3}(x,y,z,\rho,\sigma,t) = \frac{a g_{03}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\bigg|_{\upsilon=0}. \tag{10.90}$$

10.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.91}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.92}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.93}$$

$$x=0: \quad -\frac{\partial T}{\partial x} + H_1 T = g_1(y,z,t), \tag{10.94}$$

$$y=0: \quad -\frac{\partial T}{\partial y} + H_2 T = g_2(x,z,t), \tag{10.95}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + H_3 T = g_3(x,y,t), \tag{10.96}$$

$$\lim_{x\to\infty} T(x,y,z,t) = 0, \qquad \lim_{y\to\infty} T(x,y,z,t) = 0,$$
$$\lim_{z\to\infty} T(x,y,z,t) = 0. \tag{10.97}$$

The solution:

$$\begin{aligned}
T(x,y,z,t) &= \int_0^\infty\int_0^\infty\int_0^\infty f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\\
&+ \int_0^\infty\int_0^\infty\int_0^\infty F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\\
&+ \int_0^t\int_0^\infty\int_0^\infty\int_0^\infty \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau\\
&+ \int_0^t\int_0^\infty\int_0^\infty g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau\\
&+ \int_0^t\int_0^\infty\int_0^\infty g_2(\rho,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z,\rho,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau\\
&+ \int_0^t\int_0^\infty\int_0^\infty g_3(\rho,\sigma,\tau)\,\mathcal{G}_{g3}(x,y,z,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau
\end{aligned} \tag{10.98}$$

with

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\\ \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\end{pmatrix} &= \frac{8}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \begin{pmatrix} p_0\,E_\alpha\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right]\\ w_0 t\,E_{\alpha,2}\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right]\\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2+\zeta^2)t^\alpha\right]\end{pmatrix}\\
&\times \frac{\xi\cos(x\xi)+H_1\sin(x\xi)}{\xi^2+H_1^2}\left[\xi\cos(\rho\xi)+H_1\sin(\rho\xi)\right]\\
&\times \frac{\eta\cos(y\eta)+H_2\sin(y\eta)}{\eta^2+H_2^2}\left[\eta\cos(\sigma\eta)+H_2\sin(\sigma\eta)\right]\\
&\times \frac{\zeta\cos(z\zeta)+H_3\sin(z\zeta)}{\zeta^2+H_3^2}\left[\zeta\cos(\upsilon\zeta)+H_3\sin(\upsilon\zeta)\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.
\end{aligned} \tag{10.99}$$

The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{01}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\rho=0}, \tag{10.100}$$

$$\mathcal{G}_{m2}(x,y,z,\rho,\upsilon,t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_{\Phi}(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\sigma=0}, \tag{10.101}$$

$$\mathcal{G}_{m3}(x,y,z,\rho,\sigma,t) = \frac{ag_{03}}{q_0}\,\mathcal{G}_{\Phi}(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\upsilon=0}. \tag{10.102}$$

10.5 Domain $0 < x < L,\ -\infty < y < \infty,\ -\infty < z < \infty$

10.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.103}$$

$$t=0:\quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.104}$$

$$t=0:\quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.105}$$

$$x=0:\quad T = g_1(y,z,t), \tag{10.106}$$

$$x=L:\quad T = g_2(y,z,t), \tag{10.107}$$

$$\lim_{y\to\pm\infty} T(x,y,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.108}$$

The solution:

$$\begin{aligned}
T(x,y,z,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{L} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{L} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{L} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y-\sigma,z-\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(\sigma,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y-\sigma,z-\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau
\end{aligned} \tag{10.109}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,t) \\ \mathcal{G}_F(x,y,z,\rho,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,t) \end{pmatrix} = \frac{1}{2\pi^2 L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \sin(x\xi_k)\sin(\rho\xi_k)\ \cos(y\eta)\ \cos(z\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{10.110}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,t) = \frac{a g_{01}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,t)}{\partial \rho}\right|_{\rho=0}, \tag{10.111}$$

$$\mathcal{G}_{g2}(x,y,z,t) = -\frac{a g_{02}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,t)}{\partial \rho}\right|_{\rho=L}. \tag{10.112}$$

10.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.113}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.114}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.115}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(y,z,t), \tag{10.116}$$

$$x=L: \quad \frac{\partial T}{\partial x} = g_2(y,z,t), \tag{10.117}$$

$$\lim_{y\to\pm\infty} T(x,y,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.118}$$

The solution:

$$T(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^L \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\sigma, v, \tau)\, \mathcal{G}_{g1}(x, y-\sigma, z-v, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}\tau$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(\sigma, v, \tau)\, \mathcal{G}_{g2}(x, y-\sigma, z-v, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}\tau \tag{10.119}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,t) \\ \mathcal{G}_F(x,y,z,\rho,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,t) \end{pmatrix} = \frac{1}{2\pi^2 L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} {\sum_{k=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \cos(x\xi_k)\cos(\rho\xi_k)\ \cos(y\eta)\ \cos(z\zeta)\, \mathrm{d}\eta\, \mathrm{d}\zeta, \tag{10.120}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,y,z,t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,t)\Big|_{\rho=0}, \tag{10.121}$$

$$\mathcal{G}_{m2}(x,y,z,t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,t)\Big|_{\rho=L}, \tag{10.122}$$

$$\mathcal{G}_{p1}(x,y,z,t) = \frac{a g_{01}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,t)\Big|_{\rho=0}, \tag{10.123}$$

$$\mathcal{G}_{p2}(x,y,z,t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,t)\Big|_{\rho=L}. \tag{10.124}$$

10.5.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.125}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.126}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.127}$$

$$x=0: \quad -\frac{\partial T}{\partial x} + HT = g_1(y,z,t), \tag{10.128}$$

$$x=L: \quad \frac{\partial T}{\partial x} + HT = g_2(y,z,t), \tag{10.129}$$

$$\lim_{y\to\pm\infty} T(x,y,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.130}$$

The solution:

$$T(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{L} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{L} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y-\sigma,z-\upsilon,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{L} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y-\sigma,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$
$$+ \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y-\sigma,z-\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$
$$+ \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(\sigma,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y-\sigma,z-\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \tag{10.131}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,t) \\ \mathcal{G}_F(x,y,z,\rho,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,t) \end{pmatrix} = \frac{1}{2\pi^2 L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\,E_\alpha\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\,E_{\alpha,2}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$
$$\times\ \frac{\xi_k\cos(x\xi_k)+H\sin(x\xi_k)}{\xi_k^2+H^2+\frac{2H}{L}}\,\big[\xi_k\cos(\rho\xi_k)+H\sin(\rho\xi_k)\big]$$
$$\times\ \cos(y\eta)\,\cos(z\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{10.132}$$

where ξ_k are the positive roots of the transcendental equation $\tan(L\xi_k) = \frac{2H\xi_k}{\xi_k^2-H^2}$.

The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,t) = \frac{a g_{01}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,t)\Big|_{\rho=0}, \tag{10.133}$$

$$\mathcal{G}_{g1}(x,y,z,t) = \frac{a g_{02}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,t)\Big|_{\rho=L}. \tag{10.134}$$

10.6 Domain $0 < x < L$, $0 < y < \infty$, $-\infty < z < \infty$

10.6.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x, y, z, t), \tag{10.135}$$

$$t = 0: \quad T = f(x, y, z), \qquad 0 < \alpha \le 2, \tag{10.136}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x, y, z), \quad 1 < \alpha \le 2, \tag{10.137}$$

$$x = 0: \quad T = g_1(y, z, t), \tag{10.138}$$

$$x = L: \quad T = g_2(y, z, t), \tag{10.139}$$

$$y = 0: \quad T = g_3(x, z, t), \tag{10.140}$$

$$\lim_{y\to\infty} T(x, y, z, t) = 0, \qquad \lim_{z\to\pm\infty} T(x, y, z, t) = 0. \tag{10.141}$$

The solution:

$$T(x, y, z, t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^L f(\rho, \sigma, \upsilon)\,\mathcal{G}_f(x, y, z-\upsilon, \rho, \sigma, t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^L F(\rho, \sigma, \upsilon)\,\mathcal{G}_F(x, y, z-\upsilon, \rho, \sigma, t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^L \Phi(\rho, \sigma, \upsilon, \tau)\,\mathcal{G}_\Phi(x, y, z-\upsilon, \rho, \sigma, t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_1(\sigma, \upsilon, \tau)\,\mathcal{G}_{g1}(x, y, z-\upsilon, \sigma, t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} g_2(\sigma, \upsilon, \tau)\,\mathcal{G}_{g2}(x, y, z-\upsilon, \sigma, t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau,$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^L g_3(\rho, \upsilon, \tau)\,\mathcal{G}_{g3}(x, y, z-\upsilon, \rho, t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau \tag{10.142}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,t) \end{pmatrix} = \frac{2}{\pi^2 L} \int_{-\infty}^{\infty} \int_0^{\infty} \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \sin(x\xi_k)\, \sin(\rho\xi_k)\, \sin(y\eta)\, \sin(\sigma\eta)\, \cos(z\zeta)\, \mathrm{d}\eta\, \mathrm{d}\zeta, \tag{10.143}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,\sigma,t) = \frac{a g_{01}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)}{\partial \rho}\right|_{\rho=0}, \tag{10.144}$$

$$\mathcal{G}_{g2}(x,y,z,\sigma,t) = -\frac{a g_{02}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)}{\partial \rho}\right|_{\rho=L}, \tag{10.145}$$

$$\mathcal{G}_{g3}(x,y,z,\rho,t) = \frac{a g_{03}}{q_0} \left.\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)}{\partial \sigma}\right|_{\sigma=0}. \tag{10.146}$$

10.6.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.147}$$

$$t = 0: \quad T = f(x,y,z), \qquad 0 < \alpha \le 2, \tag{10.148}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1 < \alpha \le 2, \tag{10.149}$$

$$x = 0: \quad -\frac{\partial T}{\partial x} = g_1(y,z,t), \tag{10.150}$$

$$x = L: \quad \frac{\partial T}{\partial x} = g_2(y,z,t), \tag{10.151}$$

$$y = 0: \quad -\frac{\partial T}{\partial y} = g_3(x,z,t), \tag{10.152}$$

$$\lim_{y\to\infty} T(x,y,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.153}$$

The solution:

$$
\begin{aligned}
T(x,y,z,t) = &\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{L} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z-\upsilon,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{L} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z-\upsilon,\rho,\sigma,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon \\
&+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{L} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z-\upsilon,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z-\upsilon,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau \\
&+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} g_2(\sigma,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z-\upsilon,\sigma,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau, \\
&+ \int\limits_{0}^{t}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{L} g_3(\rho,\upsilon,\tau)\,\mathcal{G}_{g3}(x,y,z-\upsilon,\rho,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau
\end{aligned}
\tag{10.154}
$$

with

$$
\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,t) \end{pmatrix} = \frac{2}{\pi^2 L}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}{\sum_{k=0}^{\infty}}{}' &\begin{pmatrix} p_0\,E_\alpha\left[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\right] \end{pmatrix} \\
&\times\cos(x\xi_k)\,\cos(\rho\xi_k)\,\cos(y\eta)\,\cos(\sigma\eta)\,\cos(z\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta,
\end{aligned}
\tag{10.155}
$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$
\mathcal{G}_{m1}(x,y,z,\sigma,t) = \frac{a g_{01}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)\Big|_{\rho=0}, \tag{10.156}
$$

$$
\mathcal{G}_{m2}(x,y,z,\sigma,t) = \frac{a g_{02}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)\Big|_{\rho=L}, \tag{10.157}
$$

$$
\mathcal{G}_{m3}(x,y,z,\rho,t) = \frac{a g_{03}}{q_0}\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,t)\Big|_{\sigma=0}, \tag{10.158}
$$

$$\mathcal{G}_{p1}(x,y,z,\sigma,t) = \frac{ag_{01}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,t)\bigg|_{\rho=0}, \tag{10.159}$$

$$\mathcal{G}_{p2}(x,y,z,\sigma,t) = \frac{ag_{02}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,t)\bigg|_{\rho=L}, \tag{10.160}$$

$$\mathcal{G}_{p3}(x,y,z,\rho,t) = \frac{ag_{03}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,t)\bigg|_{\sigma=0}. \tag{10.161}$$

10.7 Domain $0 < x < L,\ 0 < y < \infty,\ 0 < z < \infty$

10.7.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.162}$$

$$t = 0: \quad T = f(x,y,z), \qquad 0 < \alpha \le 2, \tag{10.163}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1 < \alpha \le 2, \tag{10.164}$$

$$x = 0: \quad T = g_1(y,z,t), \tag{10.165}$$

$$x = L: \quad T = g_2(y,z,t), \tag{10.166}$$

$$y = 0: \quad T = g_3(x,z,t), \tag{10.167}$$

$$z = 0: \quad T = g_4(x,y,t), \tag{10.168}$$

$$\lim_{y\to\infty} T(x,y,z,t) = 0, \qquad \lim_{z\to\infty} T(x,y,z,t) = 0. \tag{10.169}$$

The solution:

$$T(x,y,z,t) = \int_0^\infty\int_0^\infty\int_0^L f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int_0^\infty\int_0^\infty\int_0^L F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int_0^t\int_0^\infty\int_0^\infty\int_0^L \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+ \int_0^t\int_0^\infty\int_0^\infty g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^\infty g_2(\sigma,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^L g_3(\rho,\upsilon,\tau)\,\mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^L g_4(\rho,\sigma,\tau)\,\mathcal{G}_{g4}(x,y,z,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \tag{10.170}$$

with

$$\begin{pmatrix}\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\\ \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\end{pmatrix}=\frac{8}{\pi^2 L}\int_0^\infty\int_0^\infty\sum_{k=1}^\infty\begin{pmatrix}p_0\,E_\alpha\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big]\\ w_0 t\,E_{\alpha,2}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big]\\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big]\end{pmatrix}$$

$$\times\sin(x\xi_k)\,\sin(\rho\xi_k)\,\sin(y\eta)\,\sin(\sigma\eta)\,\sin(z\zeta)\,\sin(\upsilon\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{10.171}$$

where $\xi_k = k\pi/L$.

The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t)=\frac{ag_{01}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\rho}\bigg|_{\rho=0}, \tag{10.172}$$

$$\mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t)=-\frac{ag_{02}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\rho}\bigg|_{\rho=L}, \tag{10.173}$$

$$\mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t)=\frac{ag_{03}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\sigma}\bigg|_{\sigma=0}, \tag{10.174}$$

$$\mathcal{G}_{g4}(x,y,z,\rho,\sigma,t)=\frac{ag_{04}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\upsilon}\bigg|_{\upsilon=0}. \tag{10.175}$$

10.7.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha}=a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(x,y,z,t), \tag{10.176}$$

$$t=0:\quad T=f(x,y,z),\qquad 0<\alpha\le 2, \tag{10.177}$$

$$t=0:\quad \frac{\partial T}{\partial t}=F(x,y,z),\quad 1<\alpha\le 2, \tag{10.178}$$

$$x=0:\quad -\frac{\partial T}{\partial x}=g_1(y,z,t), \tag{10.179}$$

$$x = L: \quad \frac{\partial T}{\partial x} = g_2(y, z, t), \tag{10.180}$$

$$y = 0: \quad -\frac{\partial T}{\partial y} = g_3(x, z, t), \tag{10.181}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_4(x, y, t), \tag{10.182}$$

$$\lim_{y\to\infty} T(x, y, z, t) = 0, \qquad \lim_{z\to\infty} T(x, y, z, t) = 0. \tag{10.183}$$

The solution:

$$\begin{aligned} T(x,y,z,t) = &\int_0^\infty \int_0^\infty \int_0^L f(\rho,\sigma,\upsilon)\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon \\ &+ \int_0^\infty \int_0^\infty \int_0^L F(\rho,\sigma,\upsilon)\, \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon \\ &+ \int_0^t \int_0^\infty \int_0^\infty \int_0^L \Phi(\rho,\sigma,\upsilon,\tau)\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty \int_0^\infty g_1(\sigma,\upsilon,\tau)\, \mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty \int_0^\infty g_2(\sigma,\upsilon,\tau)\, \mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty \int_0^L g_3(\rho,\upsilon,\tau)\, \mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty \int_0^L g_4(\rho,\sigma,\tau)\, \mathcal{G}_{g4}(x,y,z,\rho,\sigma,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau \end{aligned} \tag{10.184}$$

with

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t) \end{pmatrix} = \frac{8}{\pi^2 L} \int_0^\infty \int_0^\infty {\sum_{k=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_k^2+\eta^2+\zeta^2)t^\alpha\big] \end{pmatrix} \\ \times \cos(x\xi_k)\, \cos(\rho\xi_k)\, \cos(y\eta)\, \cos(\sigma\eta)\, \cos(z\zeta)\, \cos(\upsilon\zeta)\, \mathrm{d}\eta\, \mathrm{d}\zeta, \end{aligned} \tag{10.185}$$

where $\xi_k = k\pi/L$. The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\rho=0}, \tag{10.186}$$

$$\mathcal{G}_{m2}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\rho=L}, \tag{10.187}$$

$$\mathcal{G}_{m3}(x,y,z,\rho,\upsilon,t) = \frac{a g_{03}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)|_{\sigma=0}, \tag{10.188}$$

$$\mathcal{G}_{m4}(x,y,z,\rho,\sigma,t) = \frac{a g_{04}}{q_0}\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\upsilon=0}, \tag{10.189}$$

$$\mathcal{G}_{p1}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{01}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\rho=0}, \tag{10.190}$$

$$\mathcal{G}_{p2}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\rho=L}, \tag{10.191}$$

$$\mathcal{G}_{p3}(x,y,z,\rho,\upsilon,t) = \frac{a g_{03}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)|_{\sigma=0}, \tag{10.192}$$

$$\mathcal{G}_{p4}(x,y,z,\rho,\sigma,t) = \frac{a g_{04}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\upsilon=0}. \tag{10.193}$$

10.8 Domain $0 < x < L_1$, $0 < y < L_2$, $-\infty < z < \infty$

10.8.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.194}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.195}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.196}$$

$$x=0: \quad T = g_1(y,z,t), \tag{10.197}$$

$$x=L_1: \quad T = g_2(y,z,t), \tag{10.198}$$

$$y=0: \quad T = g_3(x,z,t), \tag{10.199}$$

$$y=L_2: \quad T = g_4(x,z,t), \tag{10.200}$$

$$\lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.201}$$

The solution:

$$T(x,y,z,t) = \int_{-\infty}^{\infty} \int_0^{L_2} \int_0^{L_1} f(\rho,\sigma,\upsilon)\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon$$

$$+ \int_{-\infty}^{\infty} \int_0^{L_2} \int_0^{L_1} F(\rho,\sigma,\upsilon)\, \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon$$

$$+ \int_0^t \int_{-\infty}^{\infty} \int_0^{L_2} \int_0^{L_1} \Phi(\rho,\sigma,\upsilon,\tau)\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} \int_0^{L_2} g_1(\sigma,\upsilon,\tau)\, \mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} \int_0^{L_2} g_2(\sigma,\upsilon,\tau)\, \mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} \int_0^{L_1} g_3(\rho,\upsilon,\tau)\, \mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} \int_0^{L_1} g_4(\rho,\upsilon,\tau)\, \mathcal{G}_{g4}(x,y,z,\rho,\upsilon,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau \tag{10.202}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t) \end{pmatrix}$$

$$= \frac{2}{\pi L_1 L_2} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi_k^2+\eta_m^2+\zeta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi_k^2+\eta_m^2+\zeta^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi_k^2+\eta_m^2+\zeta^2)t^\alpha\right] \end{pmatrix}$$

$$\times \sin(x\xi_k)\, \sin(\rho\xi_k)\, \sin(y\eta_m)\, \sin(\sigma\eta_m)\, \cos(z\zeta)\, \mathrm{d}\zeta, \tag{10.203}$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$. The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t) = \frac{a g_{01}}{q_0}\, \frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial \rho}\bigg|_{\rho=0}, \tag{10.204}$$

$$\mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t) = -\frac{a g_{02}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial \rho}\bigg|_{\rho=L_1}, \tag{10.205}$$

$$\mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t) = \frac{a g_{03}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial \sigma}\bigg|_{\sigma=0}, \tag{10.206}$$

$$\mathcal{G}_{g4}(x,y,z,\rho,\upsilon,t) = -\frac{a g_{04}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial \sigma}\bigg|_{\sigma=L_2}. \tag{10.207}$$

10.8.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.208}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.209}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.210}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(y,z,t), \tag{10.211}$$

$$x=L_1: \quad \frac{\partial T}{\partial x} = g_2(y,z,t), \tag{10.212}$$

$$y=0: \quad -\frac{\partial T}{\partial y} = g_3(x,z,t), \tag{10.213}$$

$$y=L_2: \quad \frac{\partial T}{\partial y} = g_4(x,z,t), \tag{10.214}$$

$$\lim_{z\to\pm\infty} T(x,y,z,t) = 0. \tag{10.215}$$

The solution:

$$T(x,y,z,t) = \int_{-\infty}^{\infty}\int_0^{L_2}\int_0^{L_1} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_{-\infty}^{\infty}\int_0^{L_2}\int_0^{L_1} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_0^t\int_{-\infty}^{\infty}\int_0^{L_2}\int_0^{L_1} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_{-\infty}^{\infty}\int_0^{L_2} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_0^{L_2} g_2(\sigma, \upsilon, \tau)\, \mathcal{G}_{g2}(x, y, z, \sigma, \upsilon, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_0^{L_1} g_3(\rho, \upsilon, \tau)\, \mathcal{G}_{g3}(x, y, z, \rho, \upsilon, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_0^{L_1} g_4(\rho, \upsilon, \tau)\, \mathcal{G}_{g4}(x, y, z, \rho, \upsilon, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau \tag{10.216}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_F(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t) \end{pmatrix}$$

$$= \frac{2}{\pi L_1 L_2} \int_0^{\infty} \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\right] \end{pmatrix}$$

$$\times \cos(x\xi_k)\, \cos(\rho\xi_k)\, \cos(y\eta_m)\, \cos(\sigma\eta_m)\, \cos(z\zeta)\, \mathrm{d}\zeta, \tag{10.217}$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$.

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=0}, \tag{10.218}$$

$$\mathcal{G}_{m2}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=L_1}, \tag{10.219}$$

$$\mathcal{G}_{m3}(x, y, z, \rho, \upsilon, t) = \frac{a g_{03}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\sigma=0}, \tag{10.220}$$

$$\mathcal{G}_{m4}(x, y, z, \rho, \upsilon, t) = \frac{a g_{04}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\sigma=L_2}, \tag{10.221}$$

$$\mathcal{G}_{p1}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{01}}{p_0}\, \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=0}, \tag{10.222}$$

$$\mathcal{G}_{p2}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=L_1}, \tag{10.223}$$

$$\mathcal{G}_{p3}(x,y,z,\rho,\upsilon,t) = \frac{a g_{03}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\sigma=0}, \tag{10.224}$$

$$\mathcal{G}_{p4}(x,y,z,\rho,\upsilon,t) = \frac{a g_{04}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\sigma=L_2}. \tag{10.225}$$

10.9 Domain $0 < x < L_1$, $0 < y < L_2$, $0 < z < \infty$

10.9.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.226}$$

$$t = 0: \quad T = f(x,y,z), \qquad 0 < \alpha \le 2, \tag{10.227}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1 < \alpha \le 2, \tag{10.228}$$

$$x = 0: \quad T = g_1(y,z,t), \tag{10.229}$$

$$x = L_1: \quad T = g_2(y,z,t), \tag{10.230}$$

$$y = 0: \quad T = g_3(x,z,t), \tag{10.231}$$

$$y = L_2: \quad T = g_4(x,z,t), \tag{10.232}$$

$$z = 0: \quad T = g_5(x,y,t), \tag{10.233}$$

$$\lim_{z\to\infty} T(x,y,z,t) = 0. \tag{10.234}$$

The solution:

$$T(x,y,z,t) = \int_0^\infty \int_0^{L_2} \int_0^{L_1} f(\rho,\sigma,\upsilon)\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\, d\rho\, d\sigma\, d\upsilon$$

$$+ \int_0^\infty \int_0^{L_2} \int_0^{L_1} F(\rho,\sigma,\upsilon)\, \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\, d\rho\, d\sigma\, d\upsilon$$

$$+ \int_0^t \int_0^\infty \int_0^{L_2} \int_0^{L_1} \Phi(\rho,\sigma,\upsilon,\tau)\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\, d\rho\, d\sigma\, d\upsilon\, d\tau$$

$$+ \int_0^t \int_0^\infty \int_0^{L_2} g_1(\sigma,\upsilon,\tau)\, \mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\, d\sigma\, d\upsilon\, d\tau$$

$$+ \int_0^t \int_0^\infty \int_0^{L_2} g_2(\sigma, \upsilon, \tau)\, \mathcal{G}_{g2}(x, y, z, \sigma, \upsilon, t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^\infty \int_0^{L_1} g_3(\rho, \upsilon, \tau)\, \mathcal{G}_{g3}(x, y, z, \rho, \upsilon, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^\infty \int_0^{L_1} g_4(\rho, \upsilon, \tau)\, \mathcal{G}_{g4}(x, y, z, \rho, \upsilon, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{L_2} \int_0^{L_1} g_5(\rho, \sigma, \tau)\, \mathcal{G}_{g5}(x, y, z, \rho, \sigma, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau \tag{10.235}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_F(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t) \end{pmatrix}$$

$$= \frac{8}{\pi L_1 L_2} \int_0^\infty \sum_{m=1}^\infty \sum_{k=1}^\infty \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\big[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \sin(x\xi_k)\, \sin(\rho\xi_k)\, \sin(y\eta_m)\, \sin(\sigma\eta_m)\, \sin(z\zeta)\, \sin(\upsilon\zeta)\, \mathrm{d}\zeta, \tag{10.236}$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$. The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{01}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)}{\partial \rho} \right|_{\rho=0}, \tag{10.237}$$

$$\mathcal{G}_{g2}(x, y, z, \sigma, \upsilon, t) = -\frac{a g_{02}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)}{\partial \rho} \right|_{\rho=L_1}, \tag{10.238}$$

$$\mathcal{G}_{g3}(x, y, z, \rho, \upsilon, t) = \frac{a g_{03}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)}{\partial \sigma} \right|_{\sigma=0}, \tag{10.239}$$

$$\mathcal{G}_{g4}(x, y, z, \rho, \upsilon, t) = -\frac{a g_{04}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)}{\partial \sigma} \right|_{\sigma=L_2}, \tag{10.240}$$

$$\mathcal{G}_{g5}(x, y, z, \rho, \sigma, t) = \frac{a g_{05}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)}{\partial \upsilon} \right|_{\upsilon=0}. \tag{10.241}$$

10.9.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.242}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.243}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.244}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(y,z,t), \tag{10.245}$$

$$x=L_1: \quad \frac{\partial T}{\partial x} = g_2(y,z,t), \tag{10.246}$$

$$y=0: \quad -\frac{\partial T}{\partial y} = g_3(x,z,t), \tag{10.247}$$

$$y=L_2: \quad \frac{\partial T}{\partial y} = g_4(x,z,t), \tag{10.248}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_5(x,y,t), \tag{10.249}$$

$$\lim_{z\to\infty} T(x,y,z,t) = 0. \tag{10.250}$$

The solution:

$$T(x,y,z,t) = \int_0^\infty\int_0^{L_2}\int_0^{L_1} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_0^\infty\int_0^{L_2}\int_0^{L_1} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+\int_0^t\int_0^\infty\int_0^{L_2}\int_0^{L_1} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^{L_2} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^{L_2} g_2(\sigma,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^{L_1} g_3(\rho,\upsilon,\tau)\,\mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t \int_0^\infty \int_0^{L_1} g_4(\rho, \upsilon, \tau)\, \mathcal{G}_{g4}(x, y, z, \rho, \upsilon, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+\int_0^t \int_0^{L_2} \int_0^{L_1} g_5(\rho, \sigma, \tau)\, \mathcal{G}_{g5}(x, y, z, \rho, \sigma, t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau, \tag{10.251}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_F(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t) \end{pmatrix}$$

$$= \frac{8}{\pi L_1 L_2} \int_0^\infty {\sum_{m=0}^{\infty}}' {\sum_{k=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_k^2 + \eta_m^2 + \zeta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \cos(x\xi_k)\, \cos(\rho\xi_k)\, \cos(y\eta_m)\, \cos(\sigma\eta_m)\, \cos(z\zeta)\, \cos(\upsilon\zeta)\, \mathrm{d}\zeta, \tag{10.252}$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$.

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=0}, \tag{10.253}$$

$$\mathcal{G}_{m2}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=L_1}, \tag{10.254}$$

$$\mathcal{G}_{m3}(x, y, z, \rho, \upsilon, t) = \frac{a g_{03}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\sigma=0}, \tag{10.255}$$

$$\mathcal{G}_{m4}(x, y, z, \rho, \upsilon, t) = \frac{a g_{04}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\sigma=L_2}, \tag{10.256}$$

$$\mathcal{G}_{m5}(x, y, z, \rho, \sigma, t) = \frac{a g_{05}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\upsilon=0}, \tag{10.257}$$

$$\mathcal{G}_{p1}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{01}}{p_0}\, \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=0}, \tag{10.258}$$

$$\mathcal{G}_{p2}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=L_1}, \tag{10.259}$$

$$\mathcal{G}_{p3}(x, y, z, \rho, \upsilon, t) = \frac{a g_{03}}{p_0}\, \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\sigma=0}, \tag{10.260}$$

$$\mathcal{G}_{p4}(x,y,z,\rho,\upsilon,t) = \frac{a g_{04}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\sigma=L_2}, \tag{10.261}$$

$$\mathcal{G}_{p5}(x,y,z,\rho,\sigma,t) = \frac{a g_{05}}{p_0}\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\upsilon=0}. \tag{10.262}$$

10.10 Domain $0 < x < L_1, 0 < y < L_2, 0 < z < L_3$

10.10.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.263}$$

$$t = 0: \quad T = f(x,y,z), \qquad 0 < \alpha \le 2, \tag{10.264}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1 < \alpha \le 2, \tag{10.265}$$

$$x = 0: \quad T = g_1(y,z,t), \tag{10.266}$$

$$x = L_1: \quad T = g_2(y,z,t), \tag{10.267}$$

$$y = 0: \quad T = g_3(x,z,t), \tag{10.268}$$

$$y = L_2: \quad T = g_4(x,z,t), \tag{10.269}$$

$$z = 0: \quad T = g_5(x,y,t), \tag{10.270}$$

$$z = L_3: \quad T = g_6(x,y,t). \tag{10.271}$$

The solution:

$$T(x,y,z,t) = \int_0^{L_3}\int_0^{L_2}\int_0^{L_1} f(\rho,\sigma,\upsilon)\, \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon$$

$$+ \int_0^{L_3}\int_0^{L_2}\int_0^{L_1} F(\rho,\sigma,\upsilon)\, \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon$$

$$+ \int_0^{t}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1} \Phi(\rho,\sigma,\upsilon,\tau)\, \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^{t}\int_0^{L_3}\int_0^{L_2} g_1(\sigma,\upsilon,\tau)\, \mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^{t}\int_0^{L_3}\int_0^{L_2} g_2(\sigma,\upsilon,\tau)\, \mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t-\tau)\, \mathrm{d}\sigma\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+\int_0^t\int_0^{L_3}\int_0^{L_1} g_3(\rho,\upsilon,\tau)\,\mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{L_3}\int_0^{L_1} g_4(\rho,\upsilon,\tau)\,\mathcal{G}_{g4}(x,y,z,\rho,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{L_2}\int_0^{L_1} g_5(\rho,\sigma,\tau)\,\mathcal{G}_{g5}(x,y,z,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{L_2}\int_0^{L_1} g_6(\rho,\sigma,\tau)\,\mathcal{G}_{g6}(x,y,z,\rho,\sigma,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\tau \tag{10.272}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t) \\ \mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t) \end{pmatrix}$$

$$= \frac{8}{L_1L_2L_3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\begin{pmatrix} p_0\,E_\alpha\left[-a(\xi_k^2+\eta_m^2+\zeta_n^2)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a(\xi_k^2+\eta_m^2+\zeta_n^2)t^\alpha\right] \\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi_k^2+\eta_m^2+\zeta_n^2)t^\alpha\right] \end{pmatrix}$$

$$\times\sin(x\xi_k)\,\sin(\rho\xi_k)\,\sin(y\eta_m)\,\sin(\sigma\eta_m)\,\sin(z\zeta_n)\,\sin(\upsilon\zeta_n), \tag{10.273}$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$, $\zeta_n = n\pi/L_3$. The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t) = \frac{ag_{01}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\rho}\bigg|_{\rho=0}, \tag{10.274}$$

$$\mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t) = -\frac{ag_{02}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\rho}\bigg|_{\rho=L_1}, \tag{10.275}$$

$$\mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t) = \frac{ag_{03}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\sigma}\bigg|_{\sigma=0}, \tag{10.276}$$

$$\mathcal{G}_{g4}(x,y,z,\rho,\upsilon,t) = -\frac{ag_{04}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\sigma}\bigg|_{\sigma=L_2}, \tag{10.277}$$

$$\mathcal{G}_{g5}(x,y,z,\rho,\sigma,t) = \frac{ag_{05}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\upsilon}\bigg|_{\upsilon=0}, \tag{10.278}$$

$$\mathcal{G}_{g6}(x,y,z,\rho,\sigma,t) = -\frac{ag_{06}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t)}{\partial\upsilon}\bigg|_{\upsilon=L_3}. \tag{10.279}$$

10.10.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(x,y,z,t), \tag{10.280}$$

$$t=0: \quad T = f(x,y,z), \qquad 0<\alpha\le 2, \tag{10.281}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(x,y,z), \quad 1<\alpha\le 2, \tag{10.282}$$

$$x=0: \quad -\frac{\partial T}{\partial x} = g_1(y,z,t), \tag{10.283}$$

$$x=L_1: \quad \frac{\partial T}{\partial x} = g_2(y,z,t), \tag{10.284}$$

$$y=0: \quad -\frac{\partial T}{\partial y} = g_3(x,z,t), \tag{10.285}$$

$$y=L_2: \quad \frac{\partial T}{\partial y} = g_4(x,z,t), \tag{10.286}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_5(x,y,t), \tag{10.287}$$

$$z=L_3: \quad \frac{\partial T}{\partial z} = g_6(x,y,t). \tag{10.288}$$

The solution:

$$T(x,y,z,t) = \int_0^{L_3}\int_0^{L_2}\int_0^{L_1} f(\rho,\sigma,\upsilon)\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int_0^{L_3}\int_0^{L_2}\int_0^{L_1} F(\rho,\sigma,\upsilon)\,\mathcal{G}_F(x,y,z,\rho,\sigma,\upsilon,t)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon$$

$$+ \int_0^{t}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1} \Phi(\rho,\sigma,\upsilon,\tau)\,\mathcal{G}_\Phi(x,y,z,\rho,\sigma,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+ \int_0^{t}\int_0^{L_3}\int_0^{L_2} g_1(\sigma,\upsilon,\tau)\,\mathcal{G}_{g1}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+ \int_0^{t}\int_0^{L_3}\int_0^{L_2} g_2(\sigma,\upsilon,\tau)\,\mathcal{G}_{g2}(x,y,z,\sigma,\upsilon,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+ \int_0^{t}\int_0^{L_3}\int_0^{L_1} g_3(\rho,\upsilon,\tau)\,\mathcal{G}_{g3}(x,y,z,\rho,\upsilon,t-\tau)\,\mathrm{d}\rho\,\mathrm{d}\upsilon\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{L_3} \int_0^{L_1} g_4(\rho, \upsilon, \tau)\, \mathcal{G}_{g4}(x, y, z, \rho, \upsilon, t - \tau)\, \mathrm{d}\rho\, \mathrm{d}\upsilon\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{L_2} \int_0^{L_1} g_5(\rho, \sigma, \tau)\, \mathcal{G}_{g5}(x, y, z, \rho, \sigma, t - \tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{L_2} \int_0^{L_1} g_6(\rho, \sigma, \tau)\, \mathcal{G}_{g6}(x, y, z, \rho, \sigma, t - \tau)\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\tau \tag{10.289}$$

with

$$\begin{pmatrix} \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_F(x, y, z, \rho, \sigma, \upsilon, t) \\ \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t) \end{pmatrix}$$

$$= \frac{8}{L_1 L_2 L_3} \sum_{n=0}^{\infty}{}' \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_k^2 + \eta_m^2 + \zeta_n^2) t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_k^2 + \eta_m^2 + \zeta_n^2) t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_k^2 + \eta_m^2 + \zeta_n^2) t^\alpha\big] \end{pmatrix}$$

$$\times \cos(x\xi_k)\, \cos(\rho\xi_k)\, \cos(y\eta_m)\, \cos(\sigma\eta_m)\, \cos(z\zeta_n)\, \cos(\upsilon\zeta_n), \tag{10.290}$$

where $\xi_k = k\pi/L_1$, $\eta_m = m\pi/L_2$, $\zeta_n = n\pi/L_3$. The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=0}, \tag{10.291}$$

$$\mathcal{G}_{m2}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=L_1}, \tag{10.292}$$

$$\mathcal{G}_{m3}(x, y, z, \rho, \upsilon, t) = \frac{a g_{03}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\sigma=0}, \tag{10.293}$$

$$\mathcal{G}_{m4}(x, y, z, \rho, \upsilon, t) = \frac{a g_{04}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\sigma=L_2}, \tag{10.294}$$

$$\mathcal{G}_{m5}(x, y, z, \rho, \sigma, t) = \frac{a g_{05}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\upsilon=0}, \tag{10.295}$$

$$\mathcal{G}_{m6}(x, y, z, \rho, \sigma, t) = \frac{a g_{06}}{q_0}\, \mathcal{G}_\Phi(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\upsilon=L_3}, \tag{10.296}$$

$$\mathcal{G}_{p1}(x, y, z, \sigma, \upsilon, t) = \frac{a g_{01}}{p_0}\, \mathcal{G}_f(x, y, z, \rho, \sigma, \upsilon, t)\Big|_{\rho=0}, \tag{10.297}$$

$$\mathcal{G}_{p2}(x,y,z,\sigma,\upsilon,t) = \frac{ag_{02}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\rho=L_1}, \tag{10.298}$$

$$\mathcal{G}_{p3}(x,y,z,\rho,\upsilon,t) = \frac{ag_{03}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\sigma=0}, \tag{10.299}$$

$$\mathcal{G}_{p4}(x,y,z,\rho,\upsilon,t) = \frac{ag_{04}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\sigma=L_2}, \tag{10.300}$$

$$\mathcal{G}_{p5}(x,y,z,\rho,\sigma,t) = \frac{ag_{05}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\upsilon=0}, \tag{10.301}$$

$$\mathcal{G}_{p6}(x,y,z,\rho,\sigma,t) = \frac{ag_{06}}{p_0}\,\mathcal{G}_f(x,y,z,\rho,\sigma,\upsilon,t)\Big|_{\upsilon=L_3}. \tag{10.302}$$

Chapter 11

Equations with Three Space Variables in Cylindrical Coordinates

It is quite a three pipe problem.
Arthur Conan Doyle
"The Adventures of Sherlock Holmes"

11.1 Domain $0 \le r < \infty$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, \varphi, z, t), \tag{11.1}$$

$$t = 0: \quad T = f(r, \varphi, z), \qquad 0 < \alpha \le 2, \tag{11.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi, z), \qquad 1 < \alpha \le 2, \tag{11.3}$$

$$\lim_{r\to\infty} T(r, \varphi, z, t) = 0, \qquad \lim_{z\to\pm\infty} T(r, \varphi, z, t) = 0. \tag{11.4}$$

The solution:

$$T(r, z, \varphi, t) = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} f(\rho, \phi, \zeta)\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$
$$+ \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} F(\rho, \phi, \zeta)\, \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau. \tag{11.5}$$

The fundamental solutions are obtained using the Laplace transform (2.1) with respect to time t, the exponential Fourier transform (2.20) with respect to the

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space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Hankel transform of order n (2.78) with respect to the radial coordinate r [168]:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sum_{n=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$
$$\times J_n(r\xi)\, J_n(\rho\xi)\, \cos[n(\varphi-\phi)]\, \cos[(z-\zeta)\eta]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{11.6}$$

Dependence of fundamental solution $\mathcal{G}_f$ on the coordinates r, φ and z is presented in Figs. 11.1–11.4, the fundamental solution $\mathcal{G}_F$ is depicted in Figs. 11.5–11.8, and Figs. 11.9–11.11 show the fundamental solution $\mathcal{G}_\Phi$. In calculations we have introduced the nondimensional quantities

$$\bar{r} = \frac{r}{\rho}, \quad \bar{z} = \frac{z}{\rho}, \quad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{\rho}, \tag{11.7}$$

$$\bar{\mathcal{G}}_f = \frac{\rho^3}{p_0}\mathcal{G}_f, \quad \bar{\mathcal{G}}_F = \frac{\rho^3}{w_0 t}\mathcal{G}_F, \quad \bar{\mathcal{G}}_\Phi = \frac{\rho^3}{q_0 t^{\alpha-1}}\mathcal{G}_\Phi. \tag{11.8}$$

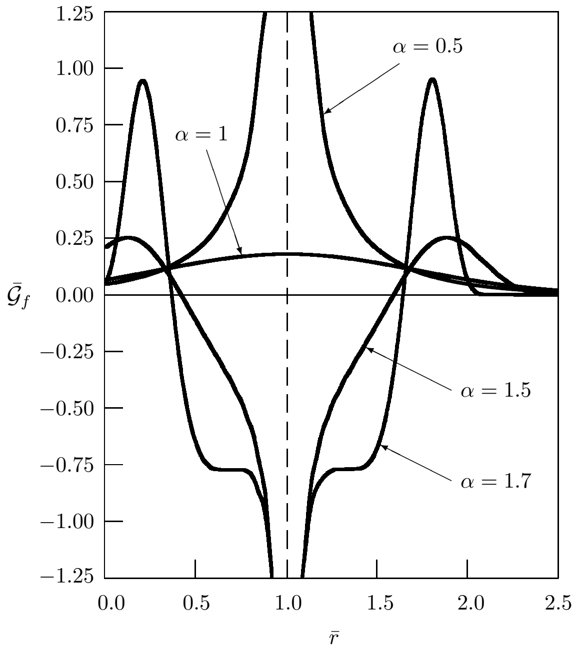

Figure 11.1: Dependence of the fundamental solution $\mathcal{G}_f$ in an infinite medium on the radial coordinate r for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $z = 0$, $\kappa = 0.5$ [168]

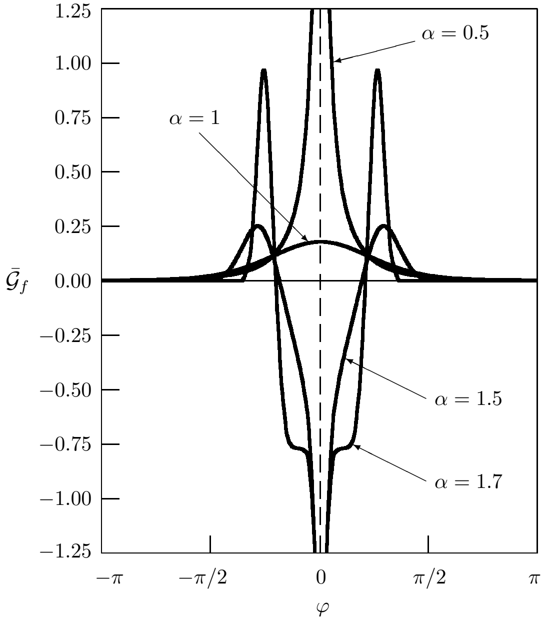

Figure 11.2: Dependence of the fundamental solution $\mathcal{G}_f$ in an infinite medium on the angular coordinate φ for $\phi = 0$, $\zeta = 0$, $r = \rho$, $z = 0$, $\kappa = 0.5$ [168]

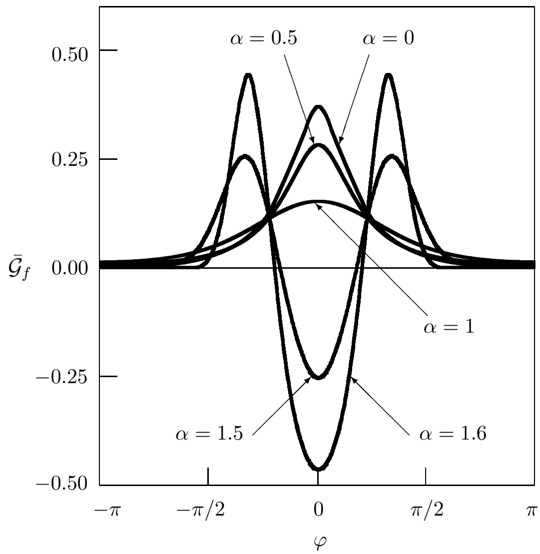

Figure 11.3: Dependence of the fundamental solution $\mathcal{G}_f$ in an infinite medium on the angular coordinate φ for $\phi = 0$, $\zeta = 0$, $r = 0.6\rho$, $z = 0$, $\kappa = 0.5$ [168]

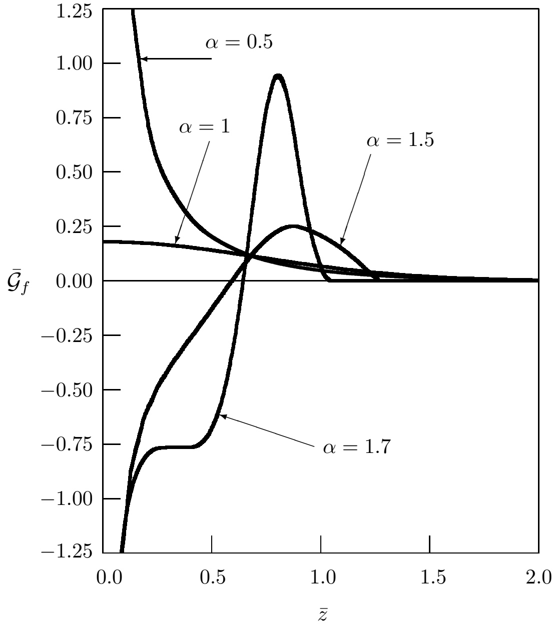

Figure 11.4: Dependence of the fundamental solution $\mathcal{G}_f$ in an infinite medium on the space coordinate z for $\phi = 0$, $\zeta = 0$, $r = \rho$, $\varphi = 0$, $\kappa = 0.5$ [168]

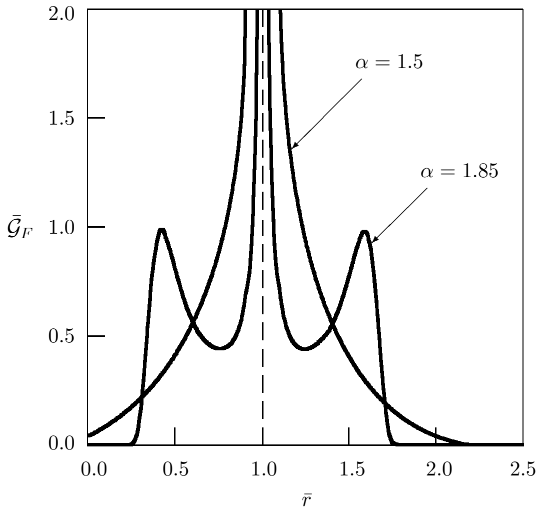

Figure 11.5: Dependence of the fundamental solution $\mathcal{G}_F$ in an infinite medium on the radial coordinate r for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $z = 0$, $\kappa = 0.5$ [168]

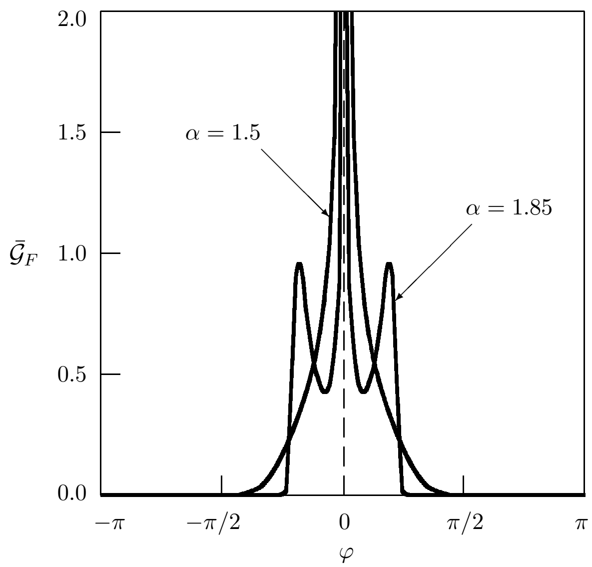

Figure 11.6: Dependence of the fundamental solution $\mathcal{G}_F$ in an infinite medium on the angular coordinate φ for $\phi = 0$, $\zeta = 0$, $r = \rho$, $z = 0$, $\kappa = 0.5$ [168]

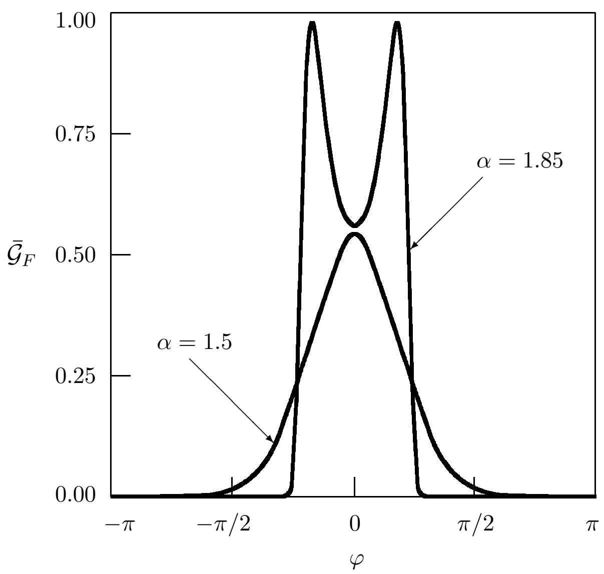

Figure 11.7: Dependence of the fundamental solution $\mathcal{G}_F$ in an infinite medium on the angular coordinate φ for $\phi = 0$, $\zeta = 0$, $r = 0.6\rho$, $z = 0$, $\kappa = 0.5$ [168]

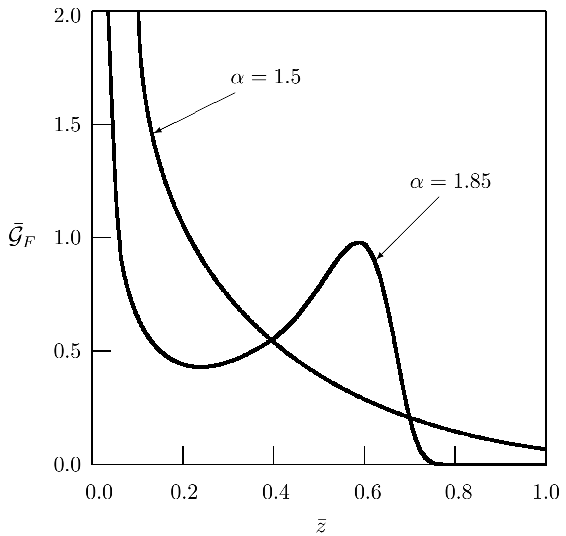

Figure 11.8: Dependence of the fundamental solution $\mathcal{G}_F$ in an infinite medium on the space coordinate z for $\phi = 0$, $\zeta = 0$, $r = \rho$, $\varphi = 0$, $\kappa = 0.5$ [168]

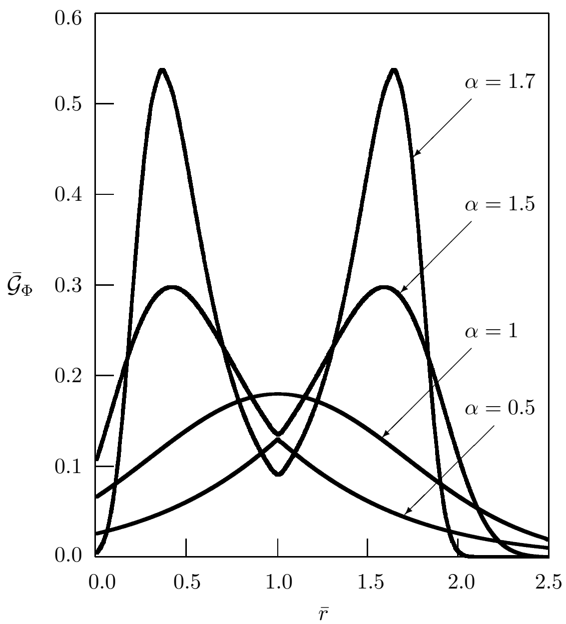

Figure 11.9: Dependence of the fundamental solution $\mathcal{G}_\Phi$ in an infinite medium on the radial coordinate r for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $z = 0$, $\kappa = 0.5$ [168]

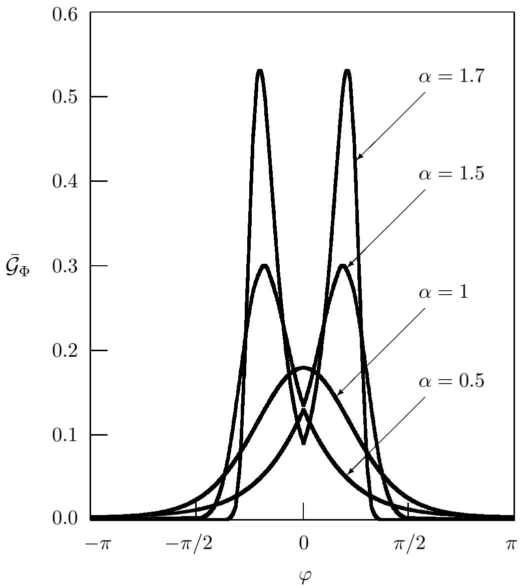

Figure 11.10: Dependence of the fundamental solution $\mathcal{G}_\Phi$ in an infinite medium on the angular coordinate φ for $\phi = 0$, $\zeta = 0$, $r = \rho$, $z = 0$, $\kappa = 0.5$ [168]

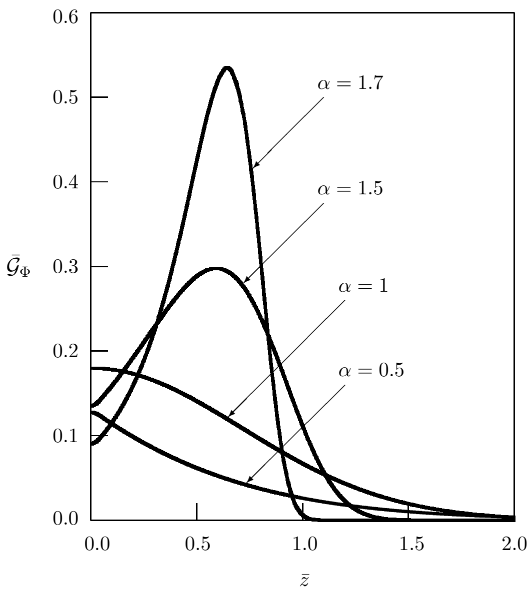

Figure 11.11: Dependence of the fundamental solution $\mathcal{G}_\Phi$ in an infinite medium on the space coordinate z for $\phi = 0$, $\zeta = 0$, $r = \rho$, $\varphi = 0$, $\kappa = 0.5$ [168]

11.2 Domain $0 \le r < \infty,\ 0 \le \varphi \le 2\pi,\ 0 < z < \infty$

11.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.9}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.10}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.11}$$

$$z = 0: \quad T = g(r,\varphi,t), \tag{11.12}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0, \qquad \lim_{z\to\infty} T(r,\varphi,z,t) = 0. \tag{11.13}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^\infty \int_0^{2\pi} \int_0^\infty f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$
$$+ \int_0^\infty \int_0^{2\pi} \int_0^\infty F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$
$$+ \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^\infty \Phi(\rho,\phi,\zeta,\tau)\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau$$
$$+ \int_0^t \int_0^{2\pi} \int_0^\infty g(\rho,\phi,\tau)\, \mathcal{G}_g(r,\varphi,z,\rho,\phi,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \tag{11.14}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the sin-Fourier transform (2.25) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Hankel transform of order n (2.78) with respect to the radial coordinate r and have the following form [171]:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{2}{\pi^2} \int_0^\infty \int_0^\infty {\sum_{n=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$
$$\times J_n(r\xi)\, J_n(\rho\xi)\, \cos[n(\varphi-\phi)]\, \sin(z\eta)\, \sin(\zeta\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{11.15}$$

The fundamental solution to the Dirichlet problem can be calculated as

$$\mathcal{G}_g(r,\varphi,z,\rho,\phi,t) = \frac{a g_0}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial \zeta} \right|_{\zeta=0}. \tag{11.16}$$

11.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2} \right) + \Phi(r,\varphi,z,t), \tag{11.17}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.18}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.19}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g(r,\varphi,t), \tag{11.20}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0, \qquad \lim_{z\to\infty} T(r,\varphi,z,t) = 0. \tag{11.21}$$

The solution:

$$\begin{aligned} T(r,z,\varphi,t) &= \int_0^\infty \int_0^{2\pi} \int_0^\infty f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\ &+ \int_0^\infty \int_0^{2\pi} \int_0^\infty F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\ &+ \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^\infty \Phi(\rho,\phi,\zeta,\tau)\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^{2\pi} \int_0^\infty g(\rho,\phi,\tau)\, \mathcal{G}_g(r,\varphi,z,\rho,\phi,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \end{aligned} \tag{11.22}$$

The Laplace transform with respect to time t, the cos-Fourier transform (2.37) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Hankel transform of order n (2.78) with respect to the radial coordinate r are used to obtain [171]:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{2}{\pi^2}\int_0^\infty\int_0^\infty \sum_{n=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$

$$\times J_n(r\xi)\, J_n(\rho\xi)\, \cos[n(\varphi-\phi)]\, \cos(z\eta)\, \cos(\zeta\eta)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{11.23}$$

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_m(r,\varphi,z,\rho,\phi,t) = \frac{ag_0}{q_0}\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\bigg|_{\zeta=0}, \tag{11.24}$$

$$\mathcal{G}_p(r,\varphi,z,\rho,\phi,t) = \frac{ag_0}{p_0}\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\bigg|_{\zeta=0}. \tag{11.25}$$

The nondimensional fundamental solution $\bar{\mathcal{G}}_m = 2\pi\rho^4\mathcal{G}_m/(g_0 a t^{\alpha-1})$ to the mathematical Neumann problem is presented in Figs. 11.12 and 11.13.

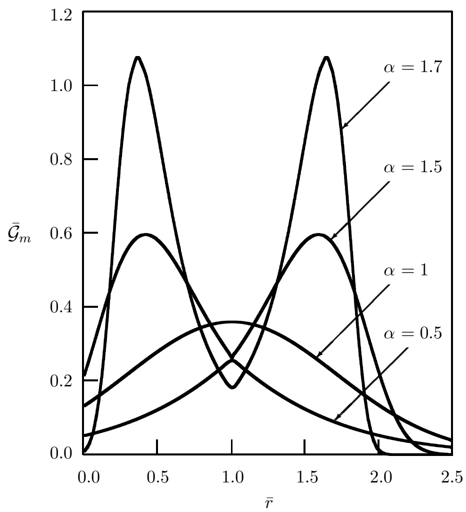

Figure 11.12: Dependence of the fundamental solution $\mathcal{G}_m$ in a half-space on the radial coordinate r for $\phi = 0$, $\varphi = 0$, $z = 0$, $\kappa = 0.5$ (the mathematical Neumann boundary condition)

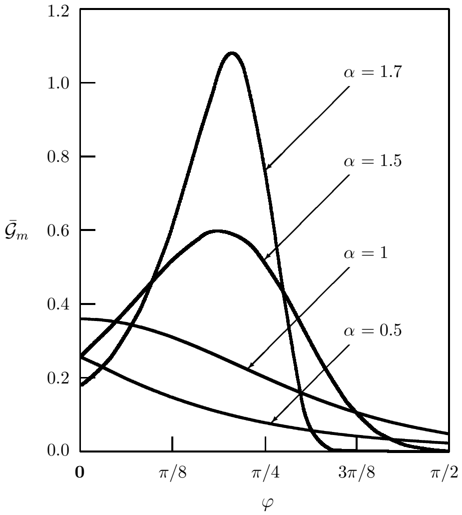

Figure 11.13: Dependence of the fundamental solution $\mathcal{G}_m$ in a half-space on the angular coordinate φ for $\phi = 0$, $r = \rho$, $z = 0$, $\kappa = 0.5$ (the mathematical Neumann boundary condition)

11.2.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, \varphi, z, t), \tag{11.26}$$

$$t = 0: \quad T = f(r, \varphi, z), \qquad 0 < \alpha \le 2, \tag{11.27}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi, z), \qquad 1 < \alpha \le 2, \tag{11.28}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + HT = g(r, \varphi, t), \tag{11.29}$$

$$\lim_{r\to\infty} T(r, \varphi, z, t) = 0, \qquad \lim_{z\to\infty} T(r, \varphi, z, t) = 0. \tag{11.30}$$

The solution:

$$T(r,z,\varphi,t)=\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty} f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty} F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int_0^{t}\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty} \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^{t}\int_0^{2\pi}\int_0^{\infty} g(\rho,\phi,\tau)\,\mathcal{G}_g(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau. \qquad (11.31)$$

The Laplace transform with respect to time t, the sin-cos-Fourier transform (2.40), (2.42) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Hankel transform of order n (2.78) with respect to the radial coordinate r result in:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{2}{\pi^2}\int_0^{\infty}\int_0^{\infty}\sum_{n=0}^{\infty}{}' \begin{pmatrix} p_0\,E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$

$$\times J_n(r\xi)\,J_n(\rho\xi)\,\cos[n(\varphi-\phi)]\,\frac{\eta\cos(z\eta)+H\sin(z\eta)}{\eta^2+H^2}$$

$$\times\left[\eta\cos(\zeta\eta)+H\sin(\zeta\eta)\right]\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (11.32)$$

The fundamental solution to the mathematical Robin problem is written as

$$\mathcal{G}_g(r,\varphi,z,\rho,\phi,t)=\frac{ag_0}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}. \qquad (11.33)$$

11.3 Domain $0 \le r < \infty$, $0 \le \varphi \le 2\pi$, $0 < z < L$

11.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.34}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.35}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.36}$$

$$z = 0: \quad T = g_1(r,\varphi,t), \tag{11.37}$$

$$z = L: \quad T = g_2(r,\varphi,t), \tag{11.38}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0. \tag{11.39}$$

The solution:

$$\begin{aligned}
T(r,z,\varphi,t) &= \int_0^L \int_0^{2\pi} \int_0^\infty f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta \\
&+ \int_0^L \int_0^{2\pi} \int_0^\infty F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta \\
&+ \int_0^t \int_0^L \int_0^{2\pi} \int_0^\infty \Phi(\rho,\phi,\zeta,\tau)\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta \, \mathrm{d}\tau \\
&+ \int_0^t \int_0^{2\pi} \int_0^\infty g_1(\rho,\phi,\tau)\, \mathcal{G}_{g1}(r,\varphi,z,\rho,\phi,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\tau, \\
&+ \int_0^t \int_0^{2\pi} \int_0^\infty g_2(\rho,\phi,\tau)\, \mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\tau.
\end{aligned} \tag{11.40}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Hankel transform of order n (2.78) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{2}{\pi L}\int\limits_0^\infty \sum_{m=1}^\infty {\sum_{n=0}^\infty}{}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta_m^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta_m^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta_m^2)t^\alpha\big] \end{pmatrix}$$

$$\times J_n(r\xi)\, J_n(\rho\xi)\, \cos[n(\varphi-\phi)]\, \sin(z\eta_m)\, \sin(\zeta\eta_m)\, \xi\, \mathrm{d}\xi, \tag{11.41}$$

where $\eta_m = m\pi/L$. The fundamental solution to the first Dirichlet problem can be calculated as

$$\mathcal{G}_{g1}(r,\varphi,z,\rho,\phi,t) = \frac{a g_{01}}{q_0}\, \frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial \zeta}\bigg|_{\zeta=0}. \tag{11.42}$$

The fundamental solution to the second Dirichlet problem $\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t)$ is obtained from the fundamental solution to the first Dirichlet problem $\mathcal{G}_{g1}(r,\varphi,z,\rho,\phi,t)$ by multiplying each term in the series by $(-1)^{m+1}$.

11.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.43}$$

$$t=0: \quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.44}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.45}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_1(r,\varphi,t), \tag{11.46}$$

$$z=L: \quad \frac{\partial T}{\partial z} = g_2(r,\varphi,t), \tag{11.47}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0. \tag{11.48}$$

The solution:

$$T(r,z,\varphi,t) = \int\limits_0^L \int\limits_0^{2\pi} \int\limits_0^\infty f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$

$$+ \int\limits_0^L \int\limits_0^{2\pi} \int\limits_0^\infty F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$

$$+ \int_0^t \int_0^L \int_0^{2\pi} \int_0^\infty \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t - \tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_0^\infty g_1(\rho, \phi, \tau)\, \mathcal{G}_{g1}(r, \varphi, z, \rho, \phi, t - \tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_0^\infty g_2(\rho, \phi, \tau)\, \mathcal{G}_{g2}(r, \varphi, z, \rho, \phi, t - \tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \tag{11.49}$$

The Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Hankel transform of order n (2.78) with respect to the radial coordinate r give:

$$\begin{pmatrix} \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t) \end{pmatrix} = \frac{2}{\pi L} \int_0^\infty {\sum_{m=0}^\infty}' {\sum_{n=0}^\infty}' \begin{pmatrix} p_0\, E_\alpha\left[- a(\xi^2 + \eta_m^2) t^\alpha \right] \\ w_0 t\, E_{\alpha,2}\left[- a(\xi^2 + \eta_m^2) t^\alpha \right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[- a(\xi^2 + \eta_m^2) t^\alpha \right] \end{pmatrix}$$

$$\times J_n(r\xi)\, J_n(\rho\xi)\, \cos[n(\varphi - \phi)]\, \cos(z\eta_m)\, \cos(\zeta\eta_m)\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{11.50}$$

where $\eta_m = m\pi/L$. The fundamental solutions to the first and second mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(r, \varphi, z, \rho, \phi, t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t)\Big|_{\zeta=0}, \tag{11.51}$$

$$\mathcal{G}_{m2}(r, \varphi, z, \rho, \phi, t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t)\Big|_{\zeta=L}, \tag{11.52}$$

$$\mathcal{G}_{p1}(r, \varphi, z, \rho, \phi, t) = \frac{a g_{01}}{p_0}\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\Big|_{\zeta=0}, \tag{11.53}$$

$$\mathcal{G}_{p2}(r, \varphi, z, \rho, \phi, t) = \frac{a g_{02}}{p_0}\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\Big|_{\zeta=L}. \tag{11.54}$$

11.3.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2} \right) + \Phi(r, \varphi, z, t), \tag{11.55}$$

$$t = 0: \quad T = f(r, \varphi, z), \qquad 0 < \alpha \le 2, \tag{11.56}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi, z), \qquad 1 < \alpha \le 2, \tag{11.57}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + HT = g_1(r, \varphi, t), \tag{11.58}$$

$$z = L: \quad \frac{\partial T}{\partial z} + HT = g_2(r, \varphi, t), \tag{11.59}$$

$$\lim_{r \to \infty} T(r, \varphi, z, t) = 0. \tag{11.60}$$

The solution:

$$\begin{aligned} T(r, z, \varphi, t) &= \int_0^L \int_0^{2\pi} \int_0^\infty f(\rho, \phi, \zeta)\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\ &+ \int_0^L \int_0^{2\pi} \int_0^\infty F(\rho, \phi, \zeta)\, \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\ &+ \int_0^t \int_0^L \int_0^{2\pi} \int_0^\infty \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^{2\pi} \int_0^\infty g_1(\rho, \phi, \tau)\, \mathcal{G}_{g1}(r, \varphi, z, \rho, \phi, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^{2\pi} \int_0^\infty g_2(\rho, \phi, \tau)\, \mathcal{G}_{g2}(r, \varphi, z, \rho, \phi, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \end{aligned} \tag{11.61}$$

The Laplace transform with respect to time t, the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Hankel transform of order n (2.78) with respect to the radial coordinate r result in

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t) \end{pmatrix} &= \frac{2}{\pi L} \int_0^\infty \sum_{m=1}^\infty {\sum_{n=0}^\infty}' \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2 + \eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2 + \eta_m^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi^2 + \eta_m^2)t^\alpha\right] \end{pmatrix} \\ &\times J_n(r\xi)\, J_n(\rho\xi)\, \cos[n(\varphi - \phi)]\, \frac{\eta_m \cos(z\eta_m) + H \sin(z\eta_m)}{\eta_m^2 + H^2 + 2H/L} \\ &\times [\eta_m \cos(\zeta\eta_m) + H \sin(\zeta\eta_m)]\, \xi\, \mathrm{d}\xi, \end{aligned} \tag{11.62}$$

where η_m are the positive roots of the transcendental equation

$$\tan(L\eta_m) = \frac{2H\eta_m}{\eta_m^2 - H^2}. \tag{11.63}$$

The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{g1}(r,\varphi,z,\rho,\phi,t) = \frac{a g_{01}}{q_0}\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.64}$$

$$\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t) = \frac{a g_{02}}{q_0}\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=L}. \tag{11.65}$$

11.4 Domain $0 \le r < R,\ 0 \le \varphi \le 2\pi,\ -\infty < z < \infty$

11.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.66}$$

$$t=0: \quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.67}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.68}$$

$$r=R: \quad T = g(\varphi,z,t), \tag{11.69}$$

$$\lim_{z\to\pm\infty} T(r,\varphi,z,t) = 0. \tag{11.70}$$

The solution:

$$\begin{aligned}
T(r,z,\varphi,t) &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta \\
&+ \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta \\
&+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\phi,\zeta,\tau)\,\mathcal{G}_g(r,\varphi,z,\phi,\zeta,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau.
\end{aligned} \tag{11.71}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform (2.84) with respect to the radial coordinate r [166]:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty}{}' \sum_{m=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_{nm}^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_{nm}^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_{nm}^2+\eta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \frac{J_n(r\xi_{nm})\, J_n(\rho\xi_{nm})}{R^2\, [J_n'(R\xi_{nm})]^2} \cos[n(\varphi-\phi)]\, \cos[(z-\zeta)\eta]\, \mathrm{d}\eta. \tag{11.72}$$

Here ξ_{nm} are the positive roots of the equation $J_n(R\xi_{nm}) = 0$. The fundamental solution to the Dirichlet problem is calculated as

$$\mathcal{G}_g(r,\varphi,z,\phi,\zeta,t) = -\frac{aRg_0}{q_0} \frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial \rho}\bigg|_{\rho=R}. \tag{11.73}$$

Figures 11.14–11.17 show dependence of the fundamental solution $\mathcal{G}_g$ on cylindrical coordinates r, φ and z. In calculations we have introduced the nondimensional quantities

$$\bar{r} = \frac{r}{R}, \quad \bar{z} = \frac{z}{R}, \quad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}, \quad \bar{\mathcal{G}}_g = \frac{R^3}{at^{\alpha-1}g_0}\mathcal{G}_g. \tag{11.74}$$

11.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.75}$$

$$t=0: \quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.76}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.77}$$

$$r=R: \quad \frac{\partial T}{\partial r} = g(\varphi,z,t), \tag{11.78}$$

$$\lim_{z\to\pm\infty} T(r,\varphi,z,t) = 0. \tag{11.79}$$

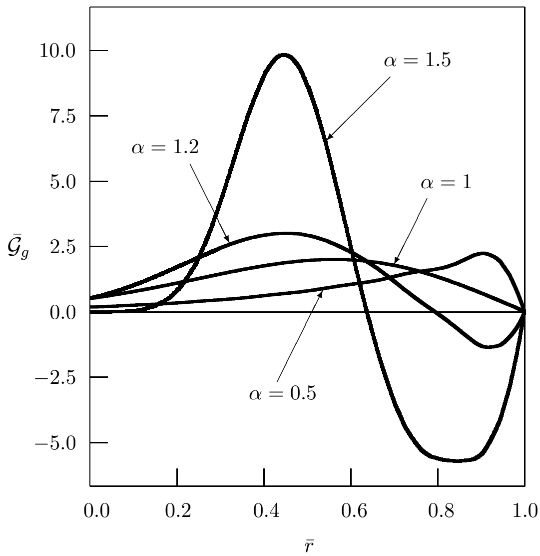

Figure 11.14: Dependence of the fundamental solution $\mathcal{G}_g(r, \varphi, z, \phi, \zeta, t)$ in an infinite cylinder on the radial coordinate r for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $z = 0$, $\kappa = 0.3$ (the Dirichlet boundary condition) [166]

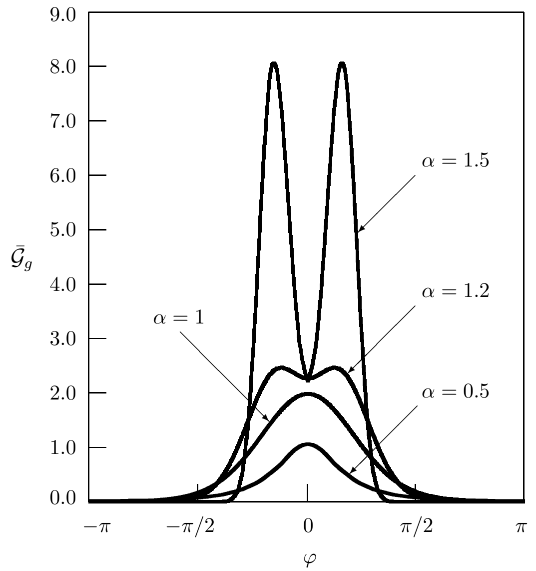

Figure 11.15: Dependence of the fundamental solution $\mathcal{G}_g(r, \varphi, z, \phi, \zeta, t)$ in an infinite cylinder on the angular coordinate φ for $\phi = 0$, $\zeta = 0$, $r = 0.6R$, $z = 0$, $\kappa = 0.3$ (the Dirichlet boundary condition) [166]

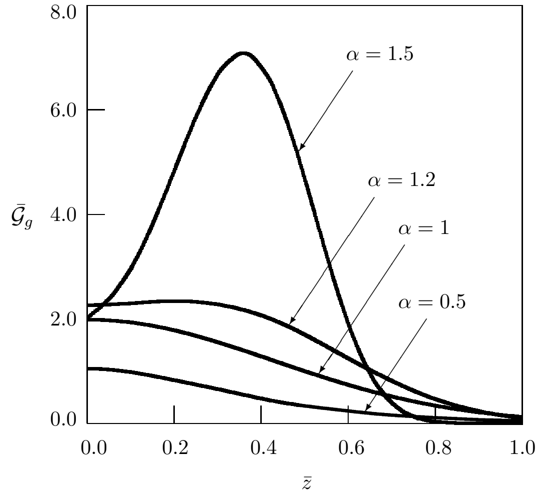

Figure 11.16: Dependence of the fundamental solution $\mathcal{G}_g(r, \varphi, z, \phi, \zeta, t)$ in an infinite cylinder on the space coordinate z for $\phi = 0$, $\zeta = 0$, $r = 0.6R$, $\varphi = 0$, $\kappa = 0.3$ (the Dirichlet boundary condition) [166]

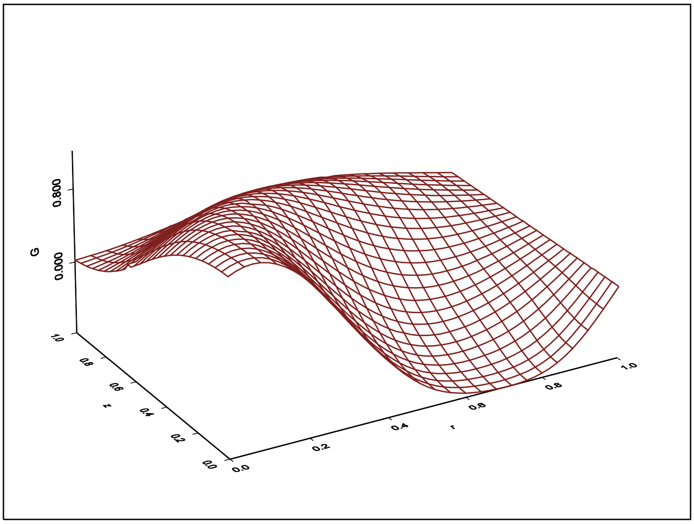

Figure 11.17: Dependence of the fundamental solution $\mathcal{G}_g(r, \varphi, z, \phi, \zeta, t)$ in an infinite cylinder on the space coordinates r and z for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $\alpha = 1.5$, $\kappa = 0.5$ (the Dirichlet boundary condition) [166]

The solution:

$$\begin{aligned} T(r,z,\varphi,t) &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta \\ &+ \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta \\ &+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\phi,\zeta,\tau)\,\mathcal{G}_g(r,\varphi,z,\phi,\zeta,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{11.80}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform (2.88) with respect to the radial coordinate r [166]:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} &= \frac{1}{2\pi^2R^2}\int_{-\infty}^{\infty} \begin{pmatrix} p_0 E_\alpha\left(-a\eta^2t^\alpha\right) \\ w_0 t E_{\alpha,2}\left(-a\eta^2t^\alpha\right) \\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left(-a\eta^2t^\alpha\right) \end{pmatrix} \\ \times \cos[(z-\zeta)\eta]\,\mathrm{d}\eta &+ \frac{1}{\pi^2}{\sum_{n=0}^{\infty}}'\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} p_0 E_\alpha\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \\ w_0 t E_{\alpha,2}\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \end{pmatrix} \\ \times \frac{\xi_{nm}^2\,J_n(\rho\xi_{nm})\,J_n(r\xi_{nm})}{[R^2\xi_{nm}^2-n^2]\,[J_n(R\xi_{nm})]^2}&\cos[n(\varphi-\phi)]\,\cos[(z-\zeta)\eta]\,\mathrm{d}\eta, \end{aligned} \tag{11.81}$$

where ξ_{nm} are the positive roots of the transcendental equation

$$J_n'(R\xi_{nm}) = 0. \tag{11.82}$$

The fundamental solutions for the mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_m(r,\varphi,z,\phi,\zeta,t) = \frac{aRg_0}{q_0}\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\bigg|_{\rho=R}, \tag{11.83}$$

$$\mathcal{G}_p(r,\varphi,z,\phi,\zeta,t) = \frac{aRg_0}{p_0}\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\bigg|_{\rho=R}. \tag{11.84}$$

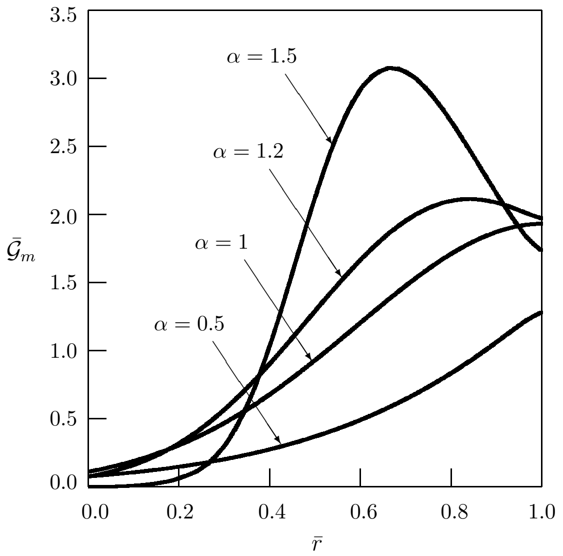

Figure 11.18: Dependence of the fundamental solution $\mathcal{G}_m(r, \varphi, z, \phi, \zeta, t)$ in an infinite cylinder on the radial coordinate r for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $z = 0$, $\kappa = 0.3$ (the mathematical Neumann problem) [166]

Dependence of the fundamental solution to the mathematical Neumann problem $\mathcal{G}_m$ on cylindrical coordinates r, φ and z is shown in Figs. 11.18–11.21, where $\bar{\mathcal{G}}_m = \frac{R^2}{at^{\alpha-1}g_0}\,\mathcal{G}_m$. The fundamental solution to the physical Neumann problem $\mathcal{G}_p$ is displayed in Fig. 11.22, where $\bar{\mathcal{G}}_p = \frac{R^2}{ag_0}\,\mathcal{G}_p$.

11.4.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, \varphi, z, t), \tag{11.85}$$

$$t = 0: \quad T = f(r, \varphi, z), \qquad 0 < \alpha \le 2, \tag{11.86}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi, z), \qquad 1 < \alpha \le 2, \tag{11.87}$$

$$r = R: \quad \frac{\partial T}{\partial r} + HT = g(\varphi, z, t), \tag{11.88}$$

$$\lim_{z \to \pm\infty} T(r, \varphi, z, t) = 0. \tag{11.89}$$

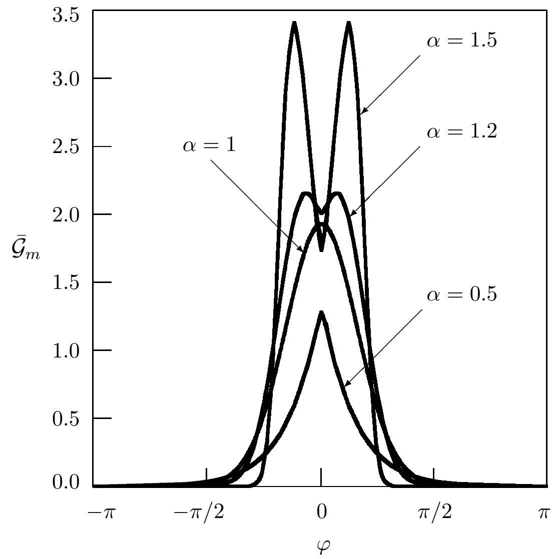

Figure 11.19: Dependence of the fundamental solution $\mathcal{G}_m(r, \varphi, z, \phi, \zeta, t)$ in an infinite cylinder on the angular coordinate φ for $\phi = 0$, $\zeta = 0$, $r = R$, $z = 0$, $\kappa = 0.3$ (the mathematical Neumann problem) [166]

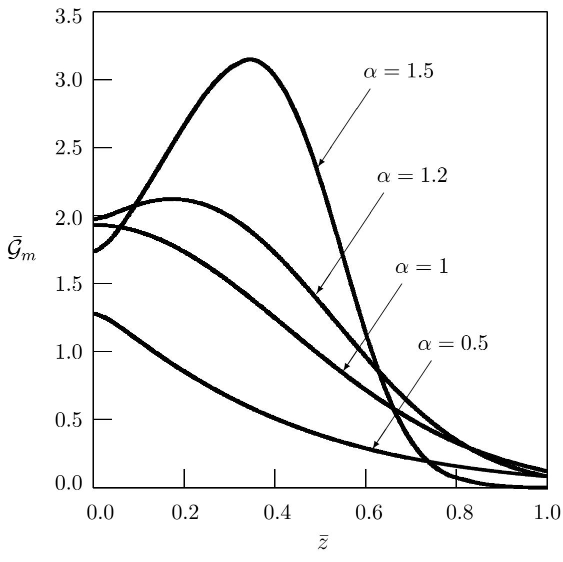

Figure 11.20: Dependence of the fundamental solution $\mathcal{G}_m(r, \varphi, z, \phi, \zeta, t)$ in an infinite cylinder on the space coordinate z for $\phi = 0$, $\zeta = 0$, $r = R$, $\varphi = 0$, $\kappa = 0.3$ (the mathematical Neumann problem) [166]

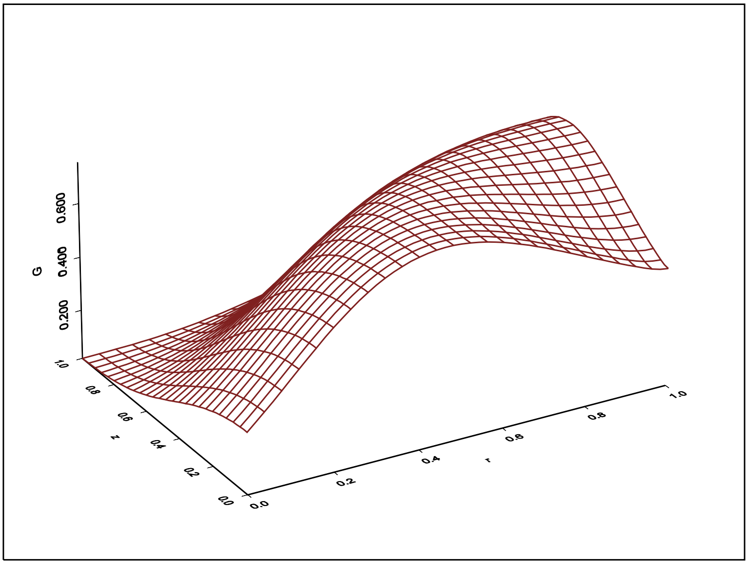

Figure 11.21: Dependence of the fundamental solution $\mathcal{G}_m(r, \varphi, z, \phi, \zeta, t)$ on the space coordinates r and z for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $\alpha = 1.5$, $\kappa = 0.5$ (the mathematical Neumann problem) [166]

The solution:

$$T(r, z, \varphi, t) = \int\limits_{-\infty}^{\infty} \int\limits_{0}^{2\pi} \int\limits_{0}^{R} f(\rho, \phi, \zeta)\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta$$

$$+ \int\limits_{-\infty}^{\infty} \int\limits_{0}^{2\pi} \int\limits_{0}^{R} F(\rho, \phi, \zeta)\, \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta$$

$$+ \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{2\pi} \int\limits_{0}^{R} \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t-\tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta \, \mathrm{d}\tau$$

$$+ \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} \int_{0}^{2\pi} g(\phi, \zeta, \tau)\, \mathcal{G}_g(r, \varphi, z, \phi, \zeta, t-\tau)\, \mathrm{d}\phi \, \mathrm{d}\zeta \, \mathrm{d}\tau. \tag{11.90}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect

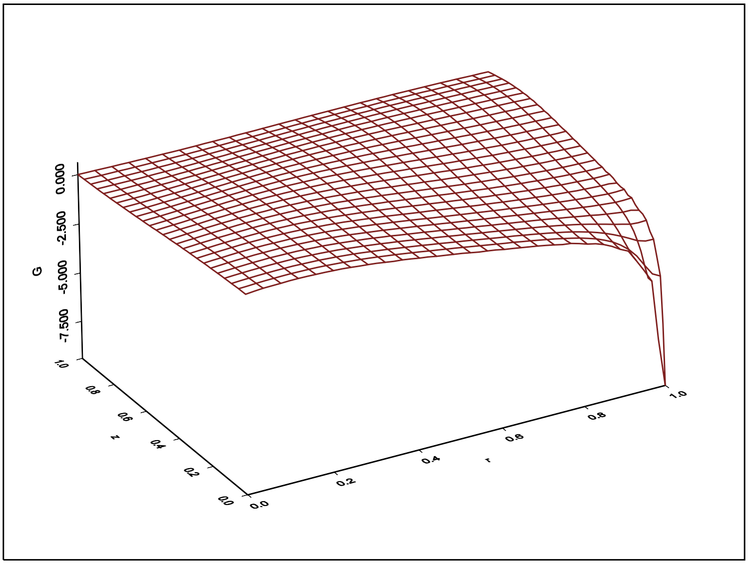

Figure 11.22: Dependence of the fundamental solution $\mathcal{G}_p(r,\varphi,z,\phi,\zeta,t)$ in an infinite cylinder on the space coordinates r and z for $\phi = 0$, $\zeta = 0$, $\varphi = 0$, $\alpha = 1.5$, $\kappa = 0.5$ (the physical Neumann problem) [166]

to the angular coordinate φ, and the finite Hankel transform (2.92) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{1}{\pi^2} \sum_{n=0}^{\infty}{}' \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \begin{pmatrix} p_0 E_\alpha\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \\ w_0 t E_{\alpha,2}\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{\xi_{nm}^2\, J_n(\rho\xi_{nm})\, J_n(r\xi_{nm})}{[R^2H^2+R^2\xi_{nm}^2-n^2]\,[J_n(R\xi_{nm})]^2} \cos[n(\varphi-\phi)]\cos[(z-\zeta)\eta]\mathrm{d}\eta. \qquad (11.91)$$

Recall that ξ_{nm} are the positive roots of the transcendental equation

$$\xi_{nm}\, J_n'(R\xi_{nm}) + HJ_n(R\xi_{nm}) = 0. \qquad (11.92)$$

For the mathematical Robin problem

$$\mathcal{G}_m(r,\varphi,z,\phi,\zeta,t) = \frac{aRg_0}{q_0}\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\bigg|_{\rho=R}. \qquad (11.93)$$

11.5 Domain $0 \le r < R,\ 0 \le \varphi \le 2\pi,\ 0 < z < \infty$

11.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.94}$$

$$t=0: \quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.95}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.96}$$

$$r=R: \quad T = g_1(\varphi,z,t), \tag{11.97}$$

$$z=0: \quad T = g_2(r,\varphi,t), \tag{11.98}$$

$$\lim_{z\to\infty} T(r,\varphi,z,t) = 0. \tag{11.99}$$

The solution:

$$\begin{aligned} T(r,z,\varphi,t) &= \int_0^\infty \int_0^{2\pi} \int_0^R f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\ &+ \int_0^\infty \int_0^{2\pi} \int_0^R F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\ &+ \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^R \Phi(\rho,\phi,\zeta,\tau)\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^\infty \int_0^{2\pi} g_1(\phi,\zeta,\tau)\, \mathcal{G}_{g1}(r,\varphi,z,\zeta,\phi,t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau \\ &+ \int_0^t \int_0^{2\pi} \int_0^R g_2(\rho,\phi,\tau)\, \mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \end{aligned} \tag{11.100}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the sin-Fourier transform (2.25) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform of order n (2.84) with respect to the radial coordinate r and have the following form

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{4}{\pi^2} \int_0^\infty \sum_{n=0}^{\infty}{}' \sum_{m=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi_{nm}^2+\eta^2)t^\alpha\right] \end{pmatrix}$$

$$\times\ \frac{J_n(r\xi_{nm})\, J_n(\rho\xi_{nm})}{R^2\left[J_n'(R\xi_{nm})\right]^2}\ \cos[n(\varphi-\phi)]\ \sin(z\eta)\ \sin(\zeta\eta)\,\mathrm{d}\eta. \tag{11.101}$$

Here $J_n(\xi_{nm}) = 0$. The fundamental solution to the first Dirichlet problem can be calculated as

$$\mathcal{G}_{g1}(r,\varphi,z,\zeta,\phi,t) = -\frac{aRg_{01}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial \rho}\bigg|_{\rho=R}, \tag{11.102}$$

the fundamental solution to the second Dirichlet problem has the following form:

$$\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial \zeta}\bigg|_{\zeta=0}. \tag{11.103}$$

11.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.104}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.105}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.106}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g_1(\varphi,z,t), \tag{11.107}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_2(r,\varphi,t), \tag{11.108}$$

$$\lim_{z\to\infty} T(r,\varphi,z,t) = 0. \tag{11.109}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^\infty \int_0^{2\pi} \int_0^R f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$
$$+ \int_0^\infty \int_0^{2\pi} \int_0^R F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$
$$+ \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^R \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\phi,\zeta,\tau)\,\mathcal{G}_{g1}(r,\varphi,z,\zeta,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_0^R g_2(\rho,\phi,\tau)\,\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau. \tag{11.110}$$

The Laplace transform with respect to time t, the cos-Fourier transform (2.37) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform of order n (2.88) with respect to the radial coordinate r lead to

$$\begin{pmatrix}\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\end{pmatrix} = \frac{2}{\pi^2R^2}\int_0^\infty \begin{pmatrix} p_0E_\alpha\big(-a\eta^2t^\alpha\big)\\ w_0tE_{\alpha,2}\big(-a\eta^2t^\alpha\big)\\ q_0t^{\alpha-1}E_{\alpha,\alpha}\big(-a\eta^2t^\alpha\big)\end{pmatrix}$$

$$\times\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta + \frac{4}{\pi^2}\sum_{n=0}^{\infty}{}'\sum_{m=1}^{\infty}\int_0^\infty \begin{pmatrix} p_0E_\alpha\big[-a(\xi_{nm}^2+\eta^2)t^\alpha\big]\\ w_0tE_{\alpha,2}\big[-a(\xi_{nm}^2+\eta^2)t^\alpha\big]\\ q_0t^{\alpha-1}E_{\alpha,\alpha}\big[-a(\xi_{nm}^2+\eta^2)t^\alpha\big]\end{pmatrix}$$

$$\times\frac{\xi_{nm}^2\,J_n(\rho\xi_{nm})\,J_n(r\xi_{nm})}{[R^2\xi_{nm}^2-n^2]\,[J_n(R\xi_{nm})]^2}\,\cos[n(\varphi-\phi)]\,\cos(z\eta)\,\cos(\zeta\eta)\,\mathrm{d}\eta, \tag{11.111}$$

where $J_n'(\xi_{nm})=0$. The fundamental solutions to the first and second mathematical and physical Neumann problems are calculated as

$$\mathcal{G}_{m1}(r,\varphi,z,\zeta,\phi,t) = \frac{aRg_{01}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\rho=R}, \tag{11.112}$$

$$\mathcal{G}_{m2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.113}$$

$$\mathcal{G}_{p1}(r,\varphi,z,\zeta,\phi,t) = \frac{aRg_{01}}{p_0}\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\rho=R}, \tag{11.114}$$

$$\mathcal{G}_{p2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{p_0}\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}. \tag{11.115}$$

11.5.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(r,\varphi,z,t), \tag{11.116}$$

$$t=0:\quad T=f(r,\varphi,z),\qquad 0<\alpha\le 2, \tag{11.117}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi, z), \qquad 1 < \alpha \le 2, \tag{11.118}$$

$$r = R: \quad \frac{\partial T}{\partial r} + H_1 T = g_1(\varphi, z, t), \tag{11.119}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r, \varphi, t), \tag{11.120}$$

$$\lim_{z \to \infty} T(r, \varphi, z, t) = 0. \tag{11.121}$$

The solution:

$$\begin{aligned} T(r, z, \varphi, t) = & \int_0^\infty \int_0^{2\pi} \int_0^R f(\rho, \phi, \zeta)\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta \\ & + \int_0^\infty \int_0^{2\pi} \int_0^R F(\rho, \phi, \zeta)\, \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta \\ & + \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^R \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t - \tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\zeta \, \mathrm{d}\tau \\ & + \int_0^t \int_0^\infty \int_0^{2\pi} g_1(\phi, \zeta, \tau)\, \mathcal{G}_{g1}(r, \varphi, z, \zeta, \phi, t - \tau)\, \mathrm{d}\phi \, \mathrm{d}\zeta \, \mathrm{d}\tau \\ & + \int_0^t \int_0^{2\pi} \int_0^R g_2(\rho, \phi, \tau)\, \mathcal{G}_{g2}(r, \varphi, z, \rho, \phi, t - \tau)\, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\tau. \end{aligned} \tag{11.122}$$

The Laplace transform with respect to time t, the sin-cos-Fourier transform (2.40) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform of order n (2.92) with respect to the radial coordinate r result in

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t) \end{pmatrix} = & \frac{4}{\pi^2} \int_0^\infty {\sum_{n=0}^\infty}' \sum_{m=1}^\infty \left(\begin{pmatrix} p_0 \, E_\alpha\left[-a(\xi_{nm}^2 + \eta^2) t^\alpha \right] \\ w_0 t \, E_{\alpha,2}\left[-a(\xi_{nm}^2 + \eta^2) t^\alpha \right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi_{nm}^2 + \eta^2) t^\alpha \right] \end{pmatrix} \right. \\ & \times \left. \frac{\xi_{nm}^2 \, J_n(\rho \xi_{nm}) \, J_n(r \xi_{nm})}{[R^2 H_1^2 + R^2 \xi_{nm}^2 - n^2] \, [J_n(R \xi_{nm})]^2} \right) \cos[n(\varphi - \phi)] \\ & \times \frac{\eta \cos(z\eta) + H_2 \sin(z\eta)}{\eta^2 + H_2^2} [\eta \cos(\zeta\eta) + H_2 \sin(\zeta\eta)] \, \mathrm{d}\eta. \end{aligned} \tag{11.123}$$

Here $\xi_{nm} J_n'(R\xi_{nm}) + H_1 J_n(R\xi_{nm}) = 0$.

The fundamental solutions to the first and second mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(r,\varphi,z,\zeta,\phi,t) = \frac{aRg_{01}}{q_0}\,\mathcal{G}_{\Phi}(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\rho=R}, \tag{11.124}$$

$$\mathcal{G}_{m2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_{\Phi}(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}. \tag{11.125}$$

11.6 Domain $0 \le r < R$, $0 \le \varphi \le 2\pi$, $0 < z < L$

11.6.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.126}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.127}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.128}$$

$$r = R: \quad T = g_1(\varphi,z,t), \tag{11.129}$$

$$z = 0: \quad T = g_2(r,\varphi,t), \tag{11.130}$$

$$z = L: \quad T = g_3(r,\varphi,t). \tag{11.131}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^L\int_0^{2\pi}\int_0^R f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int_0^L\int_0^{2\pi}\int_0^R F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int_0^t\int_0^L\int_0^{2\pi}\int_0^R \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^L\int_0^{2\pi} g_1(\phi,\zeta,\tau)\,\mathcal{G}_{g1}(r,\varphi,z,\zeta,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_0^R g_2(\rho,\phi,\tau)\,\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+\int_0^t \int_0^{2\pi} \int_0^R g_3(\rho,\phi,\tau)\, \mathcal{G}_{g3}(r,\varphi,z,\rho,\phi,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \tag{11.132}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform of order n (2.84) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix}
= \frac{4}{\pi L} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} {\sum_{n=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \end{pmatrix}$$
$$\times \frac{J_n(r\xi_{nm})\, J_n(\rho\xi_{nm})}{R^2 \left[J_n'(R\xi_{nm})\right]^2} \cos[n(\varphi-\phi)]\, \sin(z\eta_k)\, \sin(\zeta\eta_k). \tag{11.133}$$

Here ξ_{nm} are the positive roots of the equation $J_n(R\xi_{nm}) = 0$, $\eta_k = k\pi/L$. The fundamental solution to the first and second Dirichlet problems are:

$$\mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t) = -\frac{aRg_{01}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial \rho} \right|_{\rho=R}, \tag{11.134}$$

$$\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0} \left. \frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial \zeta} \right|_{\zeta=0}. \tag{11.135}$$

The fundamental solution to the third Dirichlet problem $\mathcal{G}_{g3}(r,\varphi,z,\rho,\phi,t)$ is obtained from the fundamental solution to the second Dirichlet problem $\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t)$ by multiplying each term in the series by $(-1)^{k+1}$.

11.6.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2} \right) + \Phi(r,\varphi,z,t), \tag{11.136}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.137}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.138}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g_1(\varphi,z,t), \tag{11.139}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} = g_2(r, \varphi, t), \tag{11.140}$$

$$z = L: \quad \frac{\partial T}{\partial z} = g_3(r, \varphi, t). \tag{11.141}$$

The solution:

$$T(r, z, \varphi, t) = \int_0^L \int_0^{2\pi} \int_0^R f(\rho, \phi, \zeta)\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$

$$+ \int_0^L \int_0^{2\pi} \int_0^R F(\rho, \phi, \zeta)\, \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$

$$+ \int_0^t \int_0^L \int_0^{2\pi} \int_0^R \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^L \int_0^{2\pi} g_1(\phi, \zeta, \tau)\, \mathcal{G}_{g1}(r, \varphi, z, \zeta, \phi, t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_0^R g_2(\rho, \phi, \tau)\, \mathcal{G}_{g2}(r, \varphi, z, \rho, \phi, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_0^R g_3(\rho, \phi, \tau)\, \mathcal{G}_{g3}(r, \varphi, z, \rho, \phi, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \tag{11.142}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform of order n (2.88) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t) \end{pmatrix}$$

$$= \frac{2}{\pi L R^2} \sum_{k=0}^{\infty}{}' \sum_{n=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\big(-a\eta_k^2 t^\alpha\big) \\ w_0 t\, E_{\alpha,2}\big(-a\eta_k^2 t^\alpha\big) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big(-a\eta_k^2 t^\alpha\big) \end{pmatrix} \cos(z\eta_k)\, \cos(\zeta\eta_k)$$

$$+\frac{4}{\pi L}\sum_{k=0}^{\infty}{}'\sum_{n=0}^{\infty}{}'\sum_{m=1}^{\infty}\begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \end{pmatrix}$$

$$\times\frac{\xi_{nm}^2 J_n(r\xi_{nm})\, J_n(\rho\xi_{nm})}{[R^2\xi_{nm}^2-n^2]\,[J_n(R\xi_{nm})]^2}\,\cos[n(\varphi-\phi)]\,\cos(z\eta_k)\,\cos(\zeta\eta_k). \tag{11.143}$$

Here ξ_{nm} are the positive roots of the transcendental equation $J_n'(R\xi_{nm}) = 0$ and $\eta_k = k\pi/L$.

The fundamental solutions to the mathematical Neumann problems are calculated as

$$\mathcal{G}_{m1}(r,\varphi,z,\phi,\zeta,t) = \frac{aRg_{01}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\rho=R}, \tag{11.144}$$

$$\mathcal{G}_{m2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.145}$$

$$\mathcal{G}_{m3}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{03}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=L}. \tag{11.146}$$

The corresponding results for the physical Neumann problems read:

$$\mathcal{G}_{p1}(r,\varphi,z,\phi,\zeta,t) = \frac{aRg_{01}}{p_0}\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\rho=R}, \tag{11.147}$$

$$\mathcal{G}_{p2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{p_0}\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.148}$$

$$\mathcal{G}_{p3}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{03}}{p_0}\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=L}. \tag{11.149}$$

11.6.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(r,\varphi,z,t), \tag{11.150}$$

$$t=0:\quad T=f(r,\varphi,z),\qquad 0<\alpha\le 2, \tag{11.151}$$

$$t=0:\quad \frac{\partial T}{\partial t}=F(r,\varphi,z),\qquad 1<\alpha\le 2, \tag{11.152}$$

$$r=R:\quad \frac{\partial T}{\partial r}+H_1 T=g_1(\varphi,z,t), \tag{11.153}$$

$$z=0:\quad -\frac{\partial T}{\partial z}+H_2 T=g_2(r,\varphi,t), \tag{11.154}$$

$$z=L:\quad \frac{\partial T}{\partial z}+H_2 T=g_3(r,\varphi,t). \tag{11.155}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^L \int_0^{2\pi} \int_0^R f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+ \int_0^L \int_0^{2\pi} \int_0^R F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+ \int_0^t \int_0^L \int_0^{2\pi} \int_0^R \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^L \int_0^{2\pi} g_1(\phi,\zeta,\tau)\,\mathcal{G}_{g1}(r,\varphi,z,\zeta,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_0^R g_2(\rho,\phi,\tau)\,\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_0^R g_3(\rho,\phi,\tau)\,\mathcal{G}_{g3}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau. \tag{11.156}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the finite Hankel transform of order n (2.92) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix}$$

$$= \frac{4}{\pi L} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty}{}' \sum_{m=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi_{nm}^2+\eta_k^2)t^\alpha\big] \end{pmatrix}$$

$$\times \frac{\xi_{nm}^2\, J_n(\rho\xi_{nm})\, J_n(r\xi_{nm})}{[R^2H_1^2 + R^2\xi_{nm}^2 - n^2]\,[J_n(R\xi_{nm})]^2}\,\cos[n(\varphi-\phi)]$$

$$\times \frac{\eta_k\cos(z\eta_k) + H_2\sin(z\eta_k)}{\eta_k^2 + H_2^2 + 2H_2/L}\big[\eta_k\cos(\zeta\eta_k) + H_2\sin(\zeta\eta_k)\big]. \tag{11.157}$$

Here ξ_{nm} and η_k are the positive roots of the transcendental equations

$$\xi_{nm} J_n'(R\xi_{nm}) + H_1 J_n(R\xi_{nm}) = 0 \quad \text{and} \quad \tan(L\eta_k) = 2H_2\eta_k/(\eta_k^2 - H_2^2),$$

respectively.

The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(r,\varphi,z,\phi,\zeta,t) = \frac{aRg_{01}}{q_0}\,\mathcal{G}_{\Phi}(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\rho=R}, \tag{11.158}$$

$$\mathcal{G}_{m2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0}\,\mathcal{G}_{\Phi}(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.159}$$

$$\mathcal{G}_{m3}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{03}}{q_0}\,\mathcal{G}_{\Phi}(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=L}. \tag{11.160}$$

11.7 Domain $R < r < \infty$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$

11.7.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.161}$$

$$t=0:\quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.162}$$

$$t=0:\quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.163}$$

$$r=R:\quad T = g(\varphi,z,t), \tag{11.164}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,\varphi,z,t) = 0. \tag{11.165}$$

The solution:

$$\begin{aligned} T(r,z,\varphi,t) &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta \\ &+ \int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta \\ &+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\phi,\zeta,\tau)\,\mathcal{G}_g(r,\varphi,z,\phi,\zeta,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau. \end{aligned} \tag{11.166}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.111) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} {\sum_{n=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big] \end{pmatrix}$$
$$\times \frac{J_n(r\xi)Y_n(R\xi) - Y_n(r\xi)J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)} \left[J_n(\rho\xi)Y_n(R\xi) - Y_n(\rho\xi)J_n(R\xi)\right]$$
$$\times \cos[n(\varphi-\phi)]\, \cos[(z-\zeta)\eta]\, \xi \, \mathrm{d}\xi \, \mathrm{d}\eta. \tag{11.167}$$

The fundamental solution to the Dirichlet problem is calculated as

$$\mathcal{G}_g(r,\varphi,z,\phi,\zeta,t) = -\frac{a g_0 t^{\alpha-1}}{\pi^3} \int_{-\infty}^{\infty} \int_0^{\infty} {\sum_{n=0}^{\infty}}' E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big]$$
$$\times \frac{J_n(r\xi)Y_n(R\xi) - Y_n(r\xi)J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)} \cos[n(\varphi-\phi)]\, \cos[(z-\zeta)\eta]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{11.168}$$

11.7.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.169}$$

$$t=0: \quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.170}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.171}$$

$$r=R: \quad -\frac{\partial T}{\partial r} = g(\varphi,z,t), \tag{11.172}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0, \qquad \lim_{z\to\pm\infty} T(r,\varphi,z,t) = 0. \tag{11.173}$$

The solution:

$$T(r,z,\varphi,t) = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$
$$+ \int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_0^{2\pi} \int_R^{\infty} \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_0^{2\pi} g(\phi, \zeta, \tau)\, \mathcal{G}_g(r, \varphi, z, \phi, \zeta, t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau. \tag{11.174}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.113) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t) \end{pmatrix} = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_0^{\infty} {\sum_{n=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \frac{J_n(r\xi)Y_n'(R\xi) - Y_n(r\xi)J_n'(R\xi)}{[J_n'(R\xi)]^2 + [Y_n'(R\xi)]^2} \left[J_n(\rho\xi)Y_n'(R\xi) - Y_n(\rho\xi)J_n'(R\xi)\right]$$

$$\times \cos[n(\varphi-\phi)]\, \cos[(z-\zeta)\eta]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{11.175}$$

The fundamental solutions to the mathematical and physical Neumann problems are calculated as

$$\begin{pmatrix} \mathcal{G}_m(r, \varphi, z, \phi, \zeta, t) \\ \mathcal{G}_p(r, \varphi, z, \phi, \zeta, t) \end{pmatrix} = \frac{a g_0}{\pi^3} \int_{-\infty}^{\infty} \int_0^{\infty} {\sum_{n=0}^{\infty}}' \begin{pmatrix} t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ E_\alpha\big[-a(\xi^2+\eta^2)t^\alpha\big] \end{pmatrix}$$

$$\times \frac{J_n(r\xi)Y_n'(R\xi) - Y_n(r\xi)J_n'(R\xi)}{[J_n'(R\xi)]^2 + [Y_n'(R\xi)]^2} \cos[n(\varphi-\phi)]\, \cos[(z-\zeta)\eta]\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{11.176}$$

11.7.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r, \varphi, z, t), \tag{11.177}$$

$$t = 0: \quad T = f(r, \varphi, z), \qquad 0 < \alpha \le 2, \tag{11.178}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi, z), \qquad 1 < \alpha \le 2, \tag{11.179}$$

$$r = R: \quad -\frac{\partial T}{\partial r} + HT = g(\varphi, z, t), \tag{11.180}$$

$$\lim_{r\to\infty} T(r, \varphi, z, t) = 0, \qquad \lim_{z\to\pm\infty} T(r, \varphi, z, t) = 0. \tag{11.181}$$

The solution:

$$T(r,z,\varphi,t) = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$
$$+ \int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi}\int_R^{\infty} \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$
$$+ \int_0^t\int_{-\infty}^{\infty}\int_0^{2\pi} g(\phi,\zeta,\tau)\,\mathcal{G}_g(r,\varphi,z,\phi,\zeta,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau. \tag{11.182}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the exponential Fourier transform (2.20) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.115) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sum_{n=0}^{\infty}{}' \begin{pmatrix} p_0\,E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\,E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$
$$\times\ \frac{J_n(r\xi)[\xi Y_n'(R\xi)-HY_n(R\xi)]-Y_n(r\xi)[\xi J_n'(R\xi)-HJ_n(R\xi)]}{[\xi J_n'(R\xi)-HJ_n(R\xi)]^2+[\xi Y_n'(R\xi)-HY_n(R\xi)]^2}$$
$$\times\ \{J_n(\rho\xi)[\xi Y_n'(R\xi)-HY_n(R\xi)]-Y_n(\rho\xi)[\xi J_n'(R\xi)-HJ_n(R\xi)]\}$$
$$\times\ \cos[n(\varphi-\phi)]\,\cos[(z-\zeta)\eta]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{11.183}$$

The fundamental solution to the mathematical Robin problem is written as

$$\mathcal{G}_m(r,\varphi,z,\phi,\zeta,t) = \frac{ag_0t^{\alpha-1}}{\pi^3}\int_{-\infty}^{\infty}\int_0^{\infty}\sum_{n=0}^{\infty}{}'E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right]$$
$$\times\ \frac{J_n(r\xi)[\xi Y_n'(R\xi)-HY_n(R\xi)]-Y_n(r\xi)[\xi J_n'(R\xi)-HJ_n(R\xi)]}{[\xi J_n'(R\xi)-HJ_n(R\xi)]^2+[\xi Y_n'(R\xi)-HY_n(R\xi)]^2}$$
$$\times\ \cos[n(\varphi-\phi)]\,\cos[(z-\zeta)\eta]\,\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{11.184}$$

11.8 Domain $R < r < \infty$, $0 \le \varphi \le 2\pi$, $0 < z < \infty$

11.8.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.185}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.186}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.187}$$

$$r = R: \quad T = g_1(\varphi,z,t), \tag{11.188}$$

$$z = 0: \quad T = g_2(r,\varphi,t), \tag{11.189}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0, \qquad \lim_{z\to\infty} T(r,\varphi,z,t) = 0. \tag{11.190}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^\infty \int_0^{2\pi} \int_R^\infty f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$
$$+ \int_0^\infty \int_0^{2\pi} \int_R^\infty F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta$$
$$+ \int_0^t \int_0^\infty \int_0^{2\pi} \int_R^\infty \Phi(\rho,\phi,\zeta,\tau)\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau$$
$$+ \int_0^t \int_0^\infty \int_0^{2\pi} g_1(\phi,\zeta,\tau)\, \mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau$$
$$+ \int_0^t \int_0^{2\pi} \int_R^\infty g_2(\rho,\phi,\tau)\, \mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau. \tag{11.191}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the sin-Fourier transform (2.25) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.111) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{2}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \sum_{n=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2+\eta^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big] \end{pmatrix}$$

$$\times\ \frac{J_n(r\xi)Y_n(R\xi)-Y_n(r\xi)J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)}\,[J_n(\rho\xi)Y_n(R\xi)-Y_n(\rho\xi)J_n(R\xi)]$$

$$\times\ \cos[n(\varphi-\phi)]\,\sin(z\eta)\sin(\zeta\eta)\,\xi\,d\xi\,d\eta. \tag{11.192}$$

The fundamental solutions to the first and second Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t) = -\frac{4ag_{01}t^{\alpha-1}}{\pi^3}\int\limits_0^\infty\int\limits_0^\infty \sum_{n=0}^{\infty}{}' E_{\alpha,\alpha}\big[-a(\xi^2+\eta^2)t^\alpha\big]$$

$$\times\frac{J_n(r\xi)Y_n(R\xi)-Y_n(r\xi)J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)}\cos[n(\varphi-\phi)]\,\sin(z\eta)\sin(\zeta\eta)\,\xi\,d\xi\,d\eta, \tag{11.193}$$

$$\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t) = \frac{aRg_{02}}{q_0}\,\frac{\partial \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial\zeta}\bigg|_{\zeta=0}. \tag{11.194}$$

11.8.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi(r,\varphi,z,t), \tag{11.195}$$

$$t=0:\quad T=f(r,\varphi,z),\qquad 0<\alpha\le 2, \tag{11.196}$$

$$t=0:\quad \frac{\partial T}{\partial t}=F(r,\varphi,z),\qquad 1<\alpha\le 2, \tag{11.197}$$

$$r=R:\quad -\frac{\partial T}{\partial r}=g_1(\varphi,z,t), \tag{11.198}$$

$$z=0:\quad -\frac{\partial T}{\partial z}=g_2(r,\varphi,t), \tag{11.199}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t)=0,\qquad \lim_{z\to\infty} T(r,\varphi,z,t)=0. \tag{11.200}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^\infty \int_0^{2\pi} \int_R^\infty f(\rho,\phi,\zeta)\, \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, d\rho\, d\phi\, d\zeta$$
$$+ \int_0^\infty \int_0^{2\pi} \int_R^\infty F(\rho,\phi,\zeta)\, \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\, \rho\, d\rho\, d\phi\, d\zeta$$
$$+ \int_0^t \int_0^\infty \int_0^{2\pi} \int_R^\infty \Phi(\rho,\phi,\zeta,\tau)\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\, \rho\, d\rho\, d\phi\, d\zeta\, d\tau$$
$$+ \int_0^t \int_0^\infty \int_0^{2\pi} g_1(\phi,\zeta,\tau)\, \mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t-\tau)\, d\phi\, d\zeta\, d\tau$$
$$+ \int_0^t \int_0^{2\pi} \int_R^\infty g_2(\rho,\phi,\tau)\, \mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\, \rho\, d\rho\, d\phi\, d\tau. \tag{11.201}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the cos-Fourier transform (2.37) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.113) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{2}{\pi^2} \int_0^\infty \int_0^\infty {\sum_{n=0}^{\infty}}' \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$
$$\times \frac{J_n(r\xi)Y_n'(R\xi) - Y_n(r\xi)J_n'(R\xi)}{[J_n'(R\xi)]^2 + [Y_n'(R\xi)]^2} \left[J_n(\rho\xi)Y_n'(R\xi) - Y_n(\rho\xi)J_n'(R\xi)\right]$$
$$\times \cos[n(\varphi-\phi)]\, \cos(z\eta)\cos(\zeta\eta)\, \xi\, d\xi\, d\eta. \tag{11.202}$$

The fundamental solutions to the first and second mathematical and physical Neumann problems are calculated as

$$\begin{pmatrix} \mathcal{G}_{m1}(r,\varphi,z,\phi,\zeta,t) \\ \mathcal{G}_{p1}(r,\varphi,z,\phi,\zeta,t) \end{pmatrix} = \frac{4ag_{01}}{\pi^3} \int_0^\infty \int_0^\infty {\sum_{n=0}^{\infty}}' \begin{pmatrix} t^{\alpha-1} E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$
$$\times \frac{J_n(r\xi)Y_n'(R\xi) - Y_n(r\xi)J_n'(R\xi)}{[J_n'(R\xi)]^2 + [Y_n'(R\xi)]^2}$$
$$\times \cos[n(\varphi-\phi)]\, \cos(z\eta)\cos(\zeta\eta)\, d\xi\, d\eta, \tag{11.203}$$

$$\mathcal{G}_{m2}(r,\varphi,z,\phi,\rho,t) = \frac{a g_{02}}{q_0}\,\mathcal{G}_{\Phi}(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.204}$$

$$\mathcal{G}_{p2}(r,\varphi,z,\phi,\rho,t) = \frac{a g_{02}}{p_0}\,\mathcal{G}_{f}(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}. \tag{11.205}$$

11.8.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.206}$$

$$t=0: \quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.207}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.208}$$

$$r=R: \quad -\frac{\partial T}{\partial r} + H_1 T = g_1(\varphi,z,t), \tag{11.209}$$

$$z=0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,\varphi,t), \tag{11.210}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t=0, \qquad \lim_{z\to\infty} T(r,\varphi,z,t) = 0. \tag{11.211}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^\infty\int_0^{2\pi}\int_R^\infty f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int_0^\infty\int_0^{2\pi}\int_R^\infty F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int_0^t\int_0^\infty\int_0^{2\pi}\int_R^\infty \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^\infty\int_0^{2\pi} g_1(\phi,\zeta,\tau)\,\mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_R^\infty g_2(\rho,\phi,\tau)\,\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau. \tag{11.212}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the sin-cos-Fourier transform (2.40), (2.42) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.115) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t) \\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t) \end{pmatrix} = \frac{2}{\pi^2} \int_0^\infty \int_0^\infty \sum_{n=0}^{\infty}{}' \begin{pmatrix} p_0\, E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ w_0 t\, E_{\alpha,2}\left[-a(\xi^2+\eta^2)t^\alpha\right] \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \end{pmatrix}$$

$$\times \frac{J_n(r\xi)[\xi Y_n'(R\xi) - H_1 Y_n(R\xi)] - Y_n(r\xi)[\xi J_n'(R\xi) - H_1 J_n(R\xi)]}{[\xi J_n'(R\xi) - H_1 J_n(R\xi)]^2 + [\xi Y_n'(R\xi) - H_1 Y_n(R\xi)]^2}$$

$$\times \{J_n(\rho\xi)[\xi Y_n'(R\xi) - H_1 Y_n(R\xi)] - Y_n(\rho\xi)[\xi J_n'(R\xi) - H_1 J_n(R\xi)]\}$$

$$\times \cos[n(\varphi-\phi)]\, \frac{\eta\cos(z\eta) + H_2\sin(z\eta)}{\eta^2 + H_2^2}$$

$$\times [\eta\cos(\zeta\eta) + H_2\sin(\zeta\eta)]\, \xi\, \mathrm{d}\xi\, \mathrm{d}\eta. \qquad (11.213)$$

The fundamental solutions to the first and second mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(r,\varphi,z,\phi,\zeta,t) = \frac{4ag_{01}t^{\alpha-1}}{\pi^3} \int_0^\infty \int_0^\infty \sum_{n=0}^{\infty}{}' E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right]$$

$$\times \frac{J_n(r\xi)[\xi Y_n'(R\xi) - H_1 Y_n(R\xi)] - Y_n(r\xi)[\xi J_n'(R\xi) - H_1 J_n(R\xi)]}{[\xi J_n'(R\xi) - H_1 J_n(R\xi)]^2 + [\xi Y_n'(R\xi) - H_1 Y_n(R\xi)]^2}$$

$$\times \cos[n(\varphi-\phi)]\, \frac{\eta\cos(z\eta) + H_2\sin(z\eta)}{\eta^2 + H_2^2} [\eta\cos(\zeta\eta) + H_2\sin(\zeta\eta)]\xi \mathrm{d}\xi \mathrm{d}\eta, \quad (11.214)$$

$$\mathcal{G}_{m2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0}\, \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}. \qquad (11.215)$$

11.9 Domain $R < r < \infty$, $0 \le \varphi \le 2\pi$, $0 < z < L$

11.9.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \qquad (11.216)$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \qquad (11.217)$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \qquad (11.218)$$

$$r = R: \quad T = g_1(\varphi, z, t), \tag{11.219}$$

$$z = 0: \quad T = g_2(r, \varphi, t), \tag{11.220}$$

$$z = L: \quad T = g_3(r, \varphi, t), \tag{11.221}$$

$$\lim_{r \to \infty} T(r, \varphi, z, t) = 0. \tag{11.222}$$

The solution:

$$\begin{aligned}
T(r, z, \varphi, t) &= \int_0^L \int_0^{2\pi} \int_R^\infty f(\rho, \phi, \zeta)\, \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\
&+ \int_0^L \int_0^{2\pi} \int_R^\infty F(\rho, \phi, \zeta)\, \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta \\
&+ \int_0^t \int_0^L \int_0^{2\pi} \int_R^\infty \Phi(\rho, \phi, \zeta, \tau)\, \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_0^L \int_0^{2\pi} g_1(\phi, \zeta, \tau)\, \mathcal{G}_{g1}(r, \varphi, z, \phi, \zeta, t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\zeta\, \mathrm{d}\tau \\
&+ \int_0^t \int_0^{2\pi} \int_R^\infty g_2(\rho, \phi, \tau)\, \mathcal{G}_{g2}(r, \varphi, z, \rho, \phi, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau \\
&+ \int_0^t \int_0^{2\pi} \int_R^\infty g_3(\rho, \phi, \tau)\, \mathcal{G}_{g3}(r, \varphi, z, \rho, \phi, t-\tau)\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau.
\end{aligned} \tag{11.223}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.111) with respect to the radial coordinate r:

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_F(r, \varphi, z, \rho, \phi, \zeta, t) \\ \mathcal{G}_\Phi(r, \varphi, z, \rho, \phi, \zeta, t) \end{pmatrix} &= \frac{2}{\pi L} \int_0^\infty \sum_{k=1}^\infty {\sum_{n=0}^\infty}' \begin{pmatrix} p_0\, E_\alpha\big[-a(\xi^2 + \eta_k^2)t^\alpha\big] \\ w_0 t\, E_{\alpha,2}\big[-a(\xi^2 + \eta_k^2)t^\alpha\big] \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}\big[-a(\xi^2 + \eta_k^2)t^\alpha\big] \end{pmatrix} \\
&\times \frac{J_n(r\xi)Y_n(R\xi) - Y_n(r\xi)J_n(R\xi)}{J_n^2(R\xi) + Y_n^2(R\xi)} \left[J_n(\rho\xi)Y_n(R\xi) - Y_n(\rho\xi)J_n(R\xi)\right] \\
&\times \cos[n(\varphi - \phi)]\, \sin(z\eta_k)\, \sin(\zeta\eta_k)\, \xi\, \mathrm{d}\xi,
\end{aligned} \tag{11.224}$$

where $\eta_k = k\pi/L$. The fundamental solutions to the Dirichlet problems are calculated as

$$\mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t) = -\frac{4ag_{01}t^{\alpha-1}}{\pi^2 L}\int\limits_0^\infty \sum_{k=1}^\infty \sum_{n=0}^\infty{}' E_{\alpha,\alpha}\big[-a(\xi^2+\eta_k^2)t^\alpha\big]$$

$$\times \frac{J_n(r\xi)Y_n(R\xi) - Y_n(r\xi)J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)}$$

$$\times \cos[n(\varphi-\phi)]\,\sin(z\eta_k)\sin(\zeta\eta_k)\,\xi\,\mathrm{d}\xi, \tag{11.225}$$

$$\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial\zeta}\bigg|_{\zeta=0}. \tag{11.226}$$

$$\mathcal{G}_{g3}(r,\varphi,z,\rho,\phi,t) = -\frac{ag_{03}}{q_0}\,\frac{\partial\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)}{\partial\zeta}\bigg|_{\zeta=L}. \tag{11.227}$$

11.9.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.228}$$

$$t=0: \quad T = f(r,\varphi,z), \qquad 0<\alpha\le 2, \tag{11.229}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1<\alpha\le 2, \tag{11.230}$$

$$r=R: \quad -\frac{\partial T}{\partial r} = g_1(\varphi,z,t), \tag{11.231}$$

$$z=0: \quad -\frac{\partial T}{\partial z} = g_2(r,\varphi,t), \tag{11.232}$$

$$z=L: \quad \frac{\partial T}{\partial z} = g_3(r,\varphi,t), \tag{11.233}$$

$$\lim_{r\to\infty} T(r,\varphi,z,t) = 0. \tag{11.234}$$

The solution:

$$T(r,z,\varphi,t) = \int\limits_0^L\int\limits_0^{2\pi}\int\limits_R^\infty f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int\limits_0^L\int\limits_0^{2\pi}\int\limits_R^\infty F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+\int_0^t\int_0^L\int_0^{2\pi}\int_R^\infty \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_0^L g_1(\phi,\zeta,\tau)\,\mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_R^\infty g_2(\rho,\phi,\tau)\,\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_R^\infty g_3(\rho,\phi,\tau)\,\mathcal{G}_{g3}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau. \tag{11.235}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.113) with respect to the radial coordinate r:

$$\begin{pmatrix}\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\end{pmatrix}=\frac{2}{\pi L}\int_0^\infty\sum_{k=0}^{\infty}{}'\sum_{n=0}^{\infty}{}'\begin{pmatrix}p_0\,E_\alpha\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\\ w_0 t\,E_{\alpha,2}\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\end{pmatrix}$$

$$\times\frac{J_n(r\xi)Y_n'(R\xi)-Y_n(r\xi)J_n'(R\xi)}{[J_n'(R\xi)]^2+[Y_n'(R\xi)]^2}\,[J_n(\rho\xi)Y_n'(R\xi)-Y_n(\rho\xi)J_n'(R\xi)]$$

$$\times\cos[n(\varphi-\phi)]\,\cos(z\eta_k)\cos(\zeta\eta_k)\,\xi\,\mathrm{d}\xi, \tag{11.236}$$

where $\eta_k=k\pi/L$. The fundamental solutions to the Neumann problems are calculated as

$$\begin{pmatrix}\mathcal{G}_{m1}(r,\varphi,z,\phi,\zeta,t)\\ \mathcal{G}_{p1}(r,\varphi,z,\phi,\zeta,t)\end{pmatrix}=\frac{4ag_{01}}{\pi^2 L}\int_0^\infty\sum_{k=0}^{\infty}{}'\sum_{n=0}^{\infty}{}'\begin{pmatrix}t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\\ E_\alpha\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\end{pmatrix}$$

$$\times\frac{J_n(r\xi)Y_n'(R\xi)-Y_n(r\xi)J_n'(R\xi)}{[J_n'(R\xi)]^2+[Y_n'(R\xi)]^2}$$

$$\times\cos[n(\varphi-\phi)]\,\cos(z\eta_k)\cos(\zeta\eta_k)\,\mathrm{d}\xi, \tag{11.237}$$

$$\mathcal{G}_{m2}(r,\varphi,z,\rho,\phi,t)=\frac{ag_{02}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\bigg|_{\zeta=0}, \tag{11.238}$$

$$\mathcal{G}_{m3}(r,\varphi,z,\rho,\phi,t)=\frac{ag_{03}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\bigg|_{\zeta=L}, \tag{11.239}$$

$$\mathcal{G}_{p2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{02}}{p_0}\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.240}$$

$$\mathcal{G}_{p2}(r,\varphi,z,\rho,\phi,t) = \frac{ag_{03}}{p_0}\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=L}. \tag{11.241}$$

11.9.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi(r,\varphi,z,t), \tag{11.242}$$

$$t = 0: \quad T = f(r,\varphi,z), \qquad 0 < \alpha \le 2, \tag{11.243}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi,z), \qquad 1 < \alpha \le 2, \tag{11.244}$$

$$r = R: \quad -\frac{\partial T}{\partial r} + H_1 T = g_1(\varphi,z,t), \tag{11.245}$$

$$z = 0: \quad -\frac{\partial T}{\partial z} + H_2 T = g_2(r,\varphi,t), \tag{11.246}$$

$$z = L: \quad \frac{\partial T}{\partial z} + H_2 T = g_3(r,\varphi,t), \tag{11.247}$$

$$\lim_{r\to\infty} T(r,z,t) = 0. \tag{11.248}$$

The solution:

$$T(r,z,\varphi,t) = \int_0^L \int_0^{2\pi} \int_R^\infty f(\rho,\phi,\zeta)\,\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+ \int_0^L \int_0^{2\pi} \int_R^\infty F(\rho,\phi,\zeta)\,\mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta$$

$$+ \int_0^t \int_0^L \int_0^{2\pi} \int_R^\infty \Phi(\rho,\phi,\zeta,\tau)\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\zeta\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_0^L g_1(\phi,\zeta,\tau)\,\mathcal{G}_{g1}(r,\varphi,z,\phi,\zeta,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_R^\infty g_2(\rho,\phi,\tau)\,\mathcal{G}_{g2}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_R^\infty g_3(\rho,\phi,\tau)\,\mathcal{G}_{g3}(r,\varphi,z,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau. \tag{11.249}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite sin-cos-Fourier transform (2.52) with respect to the space coordinate z, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, and the Weber transform (2.108), (2.115) with respect to the radial coordinate r:

$$\begin{pmatrix}\mathcal{G}_f(r,\varphi,z,\rho,\phi,\zeta,t)\\ \mathcal{G}_F(r,\varphi,z,\rho,\phi,\zeta,t)\\ \mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\end{pmatrix}=\frac{2}{\pi L}\int_0^\infty\sum_{k=1}^\infty\sum_{n=0}^\infty{}'\begin{pmatrix}p_0\,E_\alpha\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\\ w_0 t\,E_{\alpha,2}\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\\ q_0 t^{\alpha-1}E_{\alpha,\alpha}\left[-a(\xi^2+\eta_k^2)t^\alpha\right]\end{pmatrix}$$

$$\times\frac{J_n(r\xi)[\xi Y_n'(R\xi)-H_1Y_n(R\xi)]-Y_n(r\xi)[\xi J_n'(R\xi)-H_1J_n(R\xi)]}{[\xi J_n'(R\xi)-H_1J_n(R\xi)]^2+[\xi Y_n'(R\xi)-H_1Y_n(R\xi)]^2}$$

$$\times\{J_n(\rho\xi)[\xi Y_n'(R\xi)-H_1Y_n(R\xi)]-Y_n(\rho\xi)[\xi J_n'(R\xi)-H_1J_n(R\xi)]\}$$

$$\times\cos[n(\varphi-\phi)]\,\frac{\eta_k\cos(z\eta_k)+H_2\sin(z\eta_k)}{\eta_k^2+H_2^2+2H_2/L}$$

$$\times\left[\eta_k\cos(\zeta\eta_k)+H_2\sin(\zeta\eta_k)\right]\xi\,\mathrm{d}\xi, \tag{11.250}$$

where $\tan(L\eta_k)=2H_2\eta_k/(\eta_k^2-H_2^2)$. The fundamental solutions to the mathematical Robin problems are calculated as

$$\mathcal{G}_{m1}(r,\varphi,z,\phi,\zeta,t)=\frac{4ag_{01}t^{\alpha-1}}{\pi^2L}\int_0^\infty\sum_{k=1}^\infty\sum_{n=0}^\infty{}'E_{\alpha,\alpha}\left[-a(\xi^2+\eta_k^2)t^\alpha\right]$$

$$\times\frac{J_n(r\xi)[\xi Y_n'(R\xi)-H_1Y_n(R\xi)]-Y_n(r\xi)[\xi J_n'(R\xi)-H_1J_n(R\xi)]}{[\xi J_n'(R\xi)-H_1J_n(R\xi)]^2+[\xi Y_n'(R\xi)-H_1Y_n(R\xi)]^2}$$

$$\times\cos[n(\varphi-\phi)]\,\frac{\eta_k\cos(z\eta_k)+H_2\sin(z\eta_k)}{\eta_k^2+H_2^2+2H_2/L}$$

$$\times\left[\eta_k\cos(\zeta\eta_k)+H_2\sin(\zeta\eta_k)\right]\xi\,\mathrm{d}\xi. \tag{11.251}$$

$$\mathcal{G}_{m2}(r,\varphi,z,\rho,\phi,t)=\frac{ag_{02}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=0}, \tag{11.252}$$

$$\mathcal{G}_{m3}(r,\varphi,z,\rho,\phi,t)=\frac{ag_{03}}{q_0}\,\mathcal{G}_\Phi(r,\varphi,z,\rho,\phi,\zeta,t)\Big|_{\zeta=L}. \tag{11.253}$$

Chapter 12

Equations with Three Space Variables in Spherical Coordinates

I have answered three questions, and that is enough.

Lewis Carroll
"Alice's Adventures in Wonderland"

12.1 Domain $0 \le r < \infty,\ -1 \le \mu \le 1,\ 0 \le \varphi \le 2\pi$

Consider the time-fractional diffusion-wave equation with a source term in spherical coordinates r, θ, and φ:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial T}{\partial\theta}\right)\right.$$
$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\varphi^2}\right] + \Phi(r,\theta,\varphi,t), \tag{12.1}$$
$$0 \le r < \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi.$$

Change of variable $\mu = \cos\theta$ leads to the equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial\mu}\right]\right.$$
$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial\varphi^2}\right\} + \Phi(r,\mu,\varphi,t),$$
$$0 \le r < \infty, \quad -1 \le \mu \le 1, \quad 0 \le \varphi \le 2\pi. \tag{12.2}$$

Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*,
DOI 10.1007/978-3-319-17954-4_12

In the subsequent text we will consider immediately Eq. (12.2). For this equation the initial conditions are prescribed:

$$t = 0: \quad T = f(r, \mu, \varphi), \qquad 0 < \alpha \le 2, \tag{12.3}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \mu, \varphi), \quad 1 < \alpha \le 2. \tag{12.4}$$

As usually, the zero condition at infinity is assumed:

$$\lim_{r \to \infty} T(r, \mu, \varphi, t) = 0. \tag{12.5}$$

The solution to the initial-value problem (12.2)–(12.5) is written as:

$$\begin{aligned} T(r, \mu, \varphi, t) &= \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} f(\rho, \zeta, \phi)\, \mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t)\, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\phi \\ &+ \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} F(\rho, \zeta, \phi)\, \mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t)\, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\phi. \\ &+ \int_0^{t} \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} \Phi(\rho, \zeta, \phi, \tau)\, \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t-\tau)\, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\phi \, \mathrm{d}\tau. \end{aligned} \tag{12.6}$$

The fundamental solution $\mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t)$ is the solution to the following problem:

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a \left\{ \frac{\partial^2 \mathcal{G}_f}{\partial r^2} + \frac{2}{r} \frac{\partial \mathcal{G}_f}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial \mathcal{G}_f}{\partial \mu} \right] + \frac{1}{r^2\left(1-\mu^2\right)} \frac{\partial^2 \mathcal{G}_f}{\partial \varphi^2} \right\}, \tag{12.7}$$

$$t = 0: \quad \mathcal{G}_f = p_0 \frac{\delta(r-\rho)}{r^2}\, \delta(\mu - \zeta)\, \delta(\varphi - \phi), \quad 0 < \alpha \le 2, \tag{12.8}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{12.9}$$

Now we introduce the new looked-for function $v = \sqrt{r}\, \mathcal{G}_f$. In terms of this function, the initial value problem (12.7)–(12.9) is rewritten as

$$\frac{\partial^\alpha v}{\partial t^\alpha} = a \left\{ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{1}{4r^2} v + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial v}{\partial \mu} \right] + \frac{1}{r^2\left(1-\mu^2\right)} \frac{\partial^2 v}{\partial \varphi^2} \right\}, \tag{12.10}$$

$$t = 0: \quad v = \frac{p_0}{r^{3/2}}\, \delta(r-\rho)\, \delta(\mu - \zeta)\, \delta(\varphi - \phi), \quad 0 < \alpha \le 2, \tag{12.11}$$

$$t = 0: \quad \frac{\partial v}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{12.12}$$

Next, we use the Laplace transform (2.1) with respect to time t, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ, and the Hankel transform (2.78) of the order $n+1/2$ with respect to the radial coordinate r. It should be emphasized that the order of integral transforms is important. In the transforms domain we get

$$\widehat{\widetilde{v}}^{**}(\xi, m, n, \rho, \phi, s) = \frac{p_0}{\sqrt{\rho}}\, J_{n+1/2}(\rho\xi) P_n^m(\zeta)\cos[m(\varphi-\phi)]\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{12.13}$$

Inversion of all the integral transforms and bringing back to the fundamental solution $\mathcal{G}_f = v/\sqrt{r}$ gives [169]:

$$\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t) = \frac{p_0}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\, P_n^m(\mu)\, P_n^m(\zeta)$$
$$\times \cos[m(\varphi-\phi)] \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right)\, J_{n+1/2}(r\xi)\, J_{n+1/2}(\rho\xi)\,\xi\, d\xi, \tag{12.14}$$

where the prime near the summation symbol denotes that the term corresponding to $m=0$ in the sum should be multiplied by the factor $1/2$.

Dependence of the fundamental solution (12.14) on the radial coordinate r is presented in Fig. 12.1. The following nondimensional quantities have been

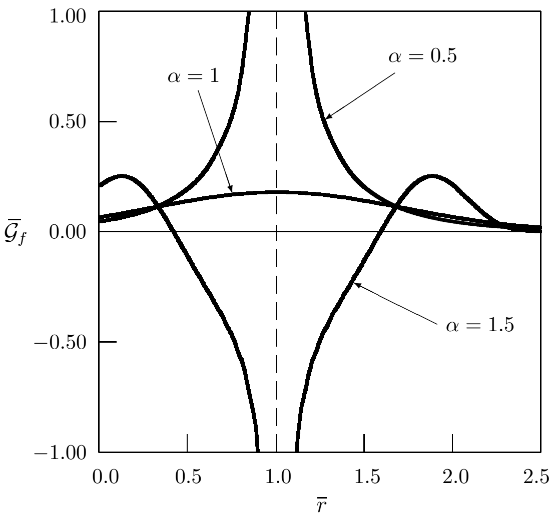

Figure 12.1: Dependence of the fundamental solution to the first Cauchy problem in an infinite medium in spherical coordinates on the radial coordinate r ($\kappa = 0.5$, $\varphi = 0$, $\phi = 0$, $\zeta = 0$, $\mu = 0$)

introduced:

$$\overline{\mathcal{G}}_f = \frac{\rho^3}{p_0}\,\mathcal{G}_f, \qquad \bar{r} = \frac{r}{\rho}, \qquad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{\rho}. \tag{12.15}$$

The fundamental solutions to the second Cauchy problem and to the source problem are obtained in a similar way and are expressed as [169]

$$\begin{pmatrix} \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)]$$

$$\times \int_0^\infty \begin{pmatrix} w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} J_{n+1/2}(r\xi)\, J_{n+1/2}(\rho\xi)\, \xi\, \mathrm{d}\xi. \tag{12.16}$$

The nondimensional fundamental solutions

$$\overline{\mathcal{G}}_F = \frac{\rho^3}{w_0 t}\,\mathcal{G}_F \quad \text{and} \quad \overline{\mathcal{G}}_\Phi = \frac{\rho^3}{q_0 t^{\alpha-1}}\,\mathcal{G}_\Phi \tag{12.17}$$

are shown in Figs. 12.2–12.4 and Figs. 12.5–12.7, respectively.

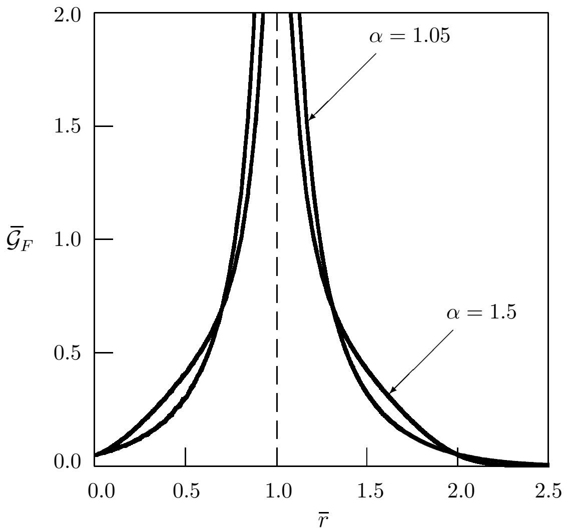

Figure 12.2: Dependence of the fundamental solution to the second Cauchy problem in an infinite medium in spherical coordinates on the radial coordinate r ($\kappa = 0.5$, $\varphi = 0$, $\phi = 0$, $\zeta = 0$, $\mu = 0$)

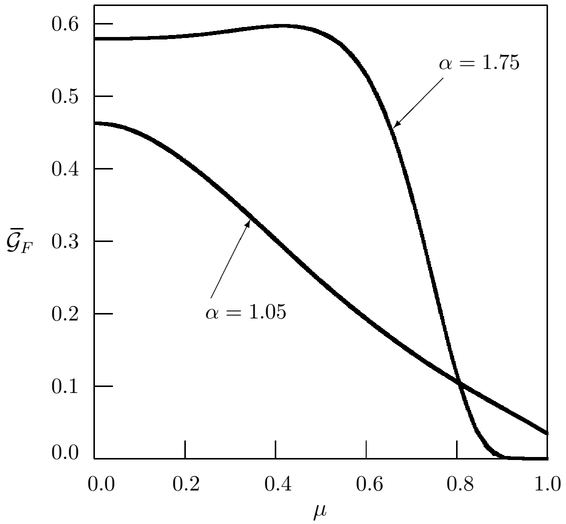

Figure 12.3: Dependence of the fundamental solution to the second Cauchy problem in an infinite medium in spherical coordinates on the coordinate μ ($\kappa = 0.5$, $\varphi = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

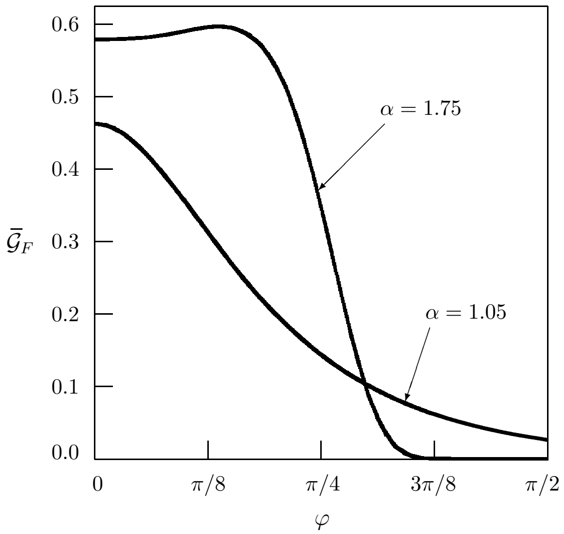

Figure 12.4: Dependence of the fundamental solution to the second Cauchy problem in an infinite medium in spherical coordinates on the angular coordinate φ ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

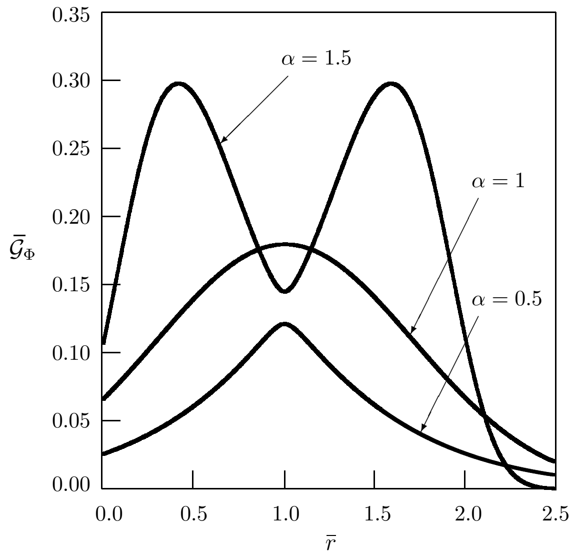

Figure 12.5: Dependence of the fundamental solution to the source problem in an infinite medium in spherical coordinates on the radial coordinate r ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

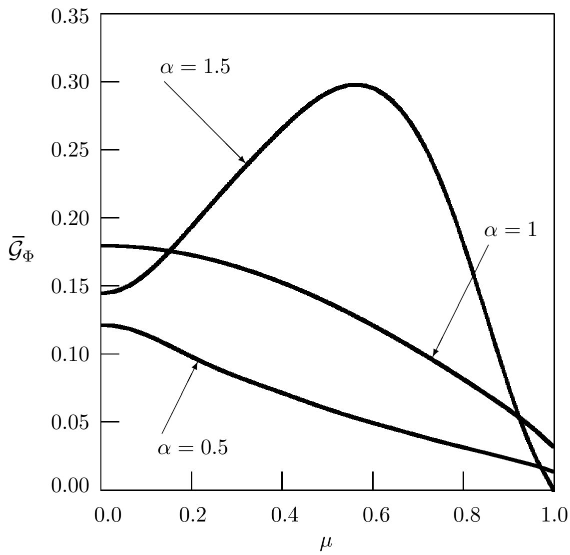

Figure 12.6: Dependence of the fundamental solution to the source problem in an infinite medium in spherical coordinates on the coordinate μ ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

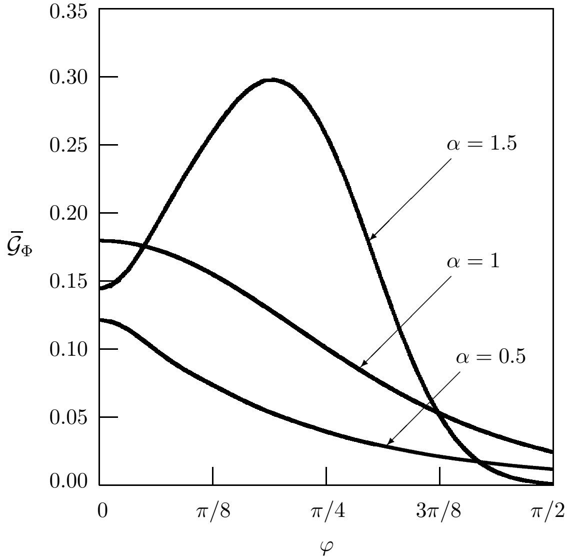

Figure 12.7: Dependence of the fundamental solution to the source problem in an infinite medium in spherical coordinates on the angular coordinate φ ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

12.2 Domain $0 \le r < R,\ -1 \le \mu \le 1,\ 0 \le \varphi \le 2\pi$

12.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\,\frac{\partial T}{\partial r} + \frac{1}{r^2}\,\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\,\frac{\partial T}{\partial \mu}\right]\right.$$
$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\,\frac{\partial^2 T}{\partial \varphi^2}\right\} + \Phi(r,\mu,\varphi,t), \tag{12.18}$$

$$t = 0: \quad T = f(r,\mu,\varphi), \qquad 0 < \alpha \le 2, \tag{12.19}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1 < \alpha \le 2, \tag{12.20}$$

$$r = R: \quad T = g(\mu,\varphi,t). \tag{12.21}$$

The solution:

$$T(r,\mu,\varphi,t) = \int_0^{2\pi}\int_{-1}^{1}\int_0^{R} f(\rho,\zeta,\phi)\,\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+\int_0^{2\pi}\int_{-1}^{1}\int_0^{R} F(\rho,\zeta,\phi)\,\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+\int_0^{t}\int_0^{2\pi}\int_{-1}^{1}\int_0^{R} \Phi(\rho,\zeta,\phi,\tau)\,\mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+\int_0^{t}\int_0^{2\pi}\int_{-1}^{1} g(\zeta,\phi,\tau)\,\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau, \qquad (12.22)$$

with the fundamental solutions obtained by the Laplace transform to with respect time, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the finite Hankel transform (2.84) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}R^2}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}{\sum_{m=0}^{n}}{}'\,(2n+1)\,\frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]$$

$$\times \begin{pmatrix} p_0\,E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \frac{J_{n+1/2}(r\xi_{nk})\,J_{n+1/2}(\rho\xi_{nk})}{\left[J'_{n+1/2}(R\xi_{nk})\right]^2}, \qquad (12.23)$$

where ξ_{nk} are the positive roots of the equation $J_{n+1/2}(R\xi_{nk})=0$.

The fundamental solution to the Dirichlet problem has the following form [176]:

$$\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t) = -\frac{ag_0 t^{\alpha-1}}{\pi\sqrt{rR}}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}{\sum_{m=0}^{n}}{}'(2n+1)\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)\,P_n^m(\zeta)$$

$$\times\cos[m(\varphi-\phi)]\,\frac{J_{n+1/2}(r\xi_{nk})}{J'_{n+1/2}(R\xi_{nk})}\,\xi_{nk}\,E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right). \qquad (12.24)$$

Two known particular cases can be obtained from Eq. (12.24).

Classical diffusion equation ($\alpha = 1$)

$$\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t) = -\frac{a g_0}{\pi\sqrt{rR}} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=0}^{n}{}' (2n+1)\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)\,P_n^m(\zeta)$$
$$\times \cos[m(\varphi-\phi)]\,\frac{J_{n+1/2}(r\xi_{nk})}{J'_{n+1/2}(R\xi_{nk})}\,\xi_{nk}\,\exp\left(-a\xi_{nk}^2 t\right). \tag{12.25}$$

This solution is presented in [20, 26].

Wave equation ($\alpha = 2$)

$$\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t) = -\frac{\sqrt{a} g_0}{\pi\sqrt{rR}} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=0}^{n}{}' (2n+1)\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)\,P_n^m(\zeta)$$
$$\times \cos[m(\varphi-\phi)]\,\frac{J_{n+1/2}(r\xi_{nk})}{J'_{n+1/2}(R\xi_{nk})}\,\sin\left(\sqrt{a}\xi_{nk} t\right). \tag{12.26}$$

Dependence of nondimensional fundamental solution $\overline{\mathcal{G}}_g$ on the coordinates $\bar{r}$, μ and φ is displayed in Figs. 12.8–12.10. Here

$$\overline{\mathcal{G}}_g = \frac{R^2}{a g_0 t^{\alpha-1}}\,\mathcal{G}_g, \qquad \bar{r} = \frac{r}{R}, \qquad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}. \tag{12.27}$$

12.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \varphi^2}\right\} + \Phi(r,\mu,\varphi,t), \tag{12.28}$$

$$t = 0: \quad T = f(r,\mu,\varphi), \qquad 0 < \alpha \le 2, \tag{12.29}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1 < \alpha \le 2, \tag{12.30}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(\mu,\varphi,t). \tag{12.31}$$

The solution:

$$T(r,\mu,\varphi,t) = \int_0^{2\pi}\int_{-1}^{1}\int_0^{R} f(\rho,\zeta,\phi)\,\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$
$$+ \int_0^{2\pi}\int_{-1}^{1}\int_0^{R} F(\rho,\zeta,\phi)\,\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

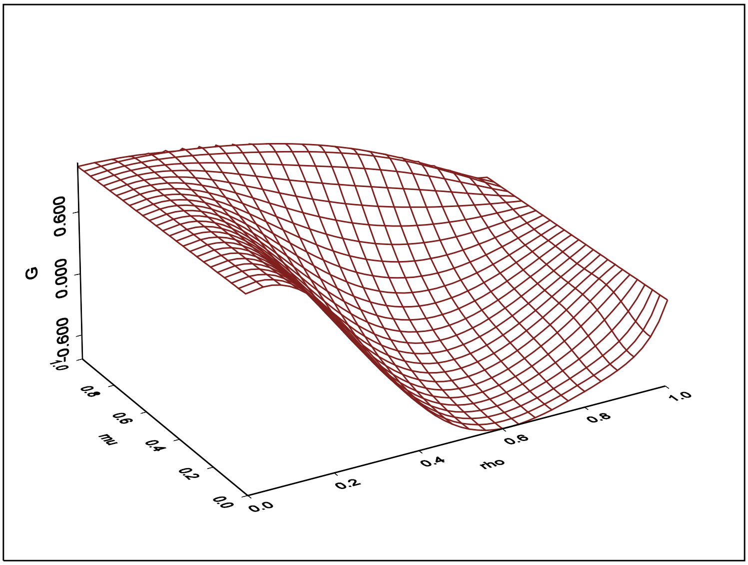

Figure 12.8: Dependence of the fundamental solution to the Dirichlet problem for a sphere $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on coordinates r and μ for $\zeta = 0$, $\phi = 0$, $\varphi = 0$, $\kappa = 0.5$ [176]

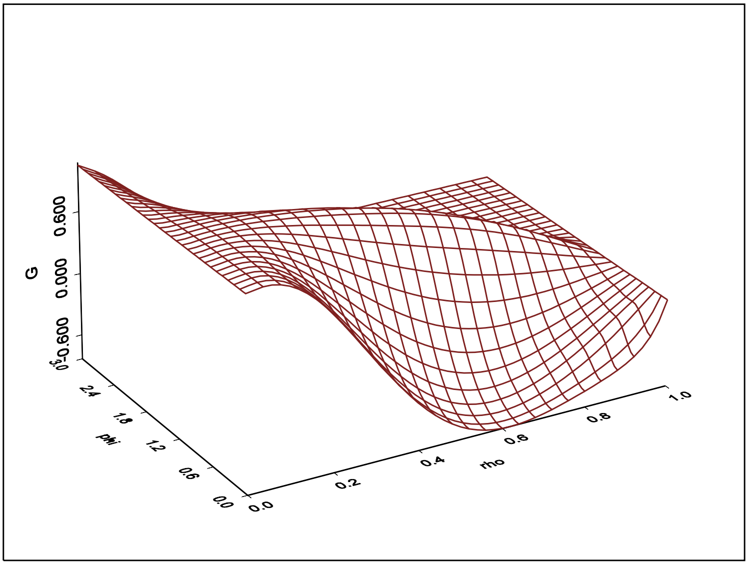

Figure 12.9: Dependence of the fundamental solution to the Dirichlet problem for a sphere $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on coordinates r and φ for $\zeta = 0$, $\phi = 0$, $\mu = 0$, $\kappa = 0.5$ [176]

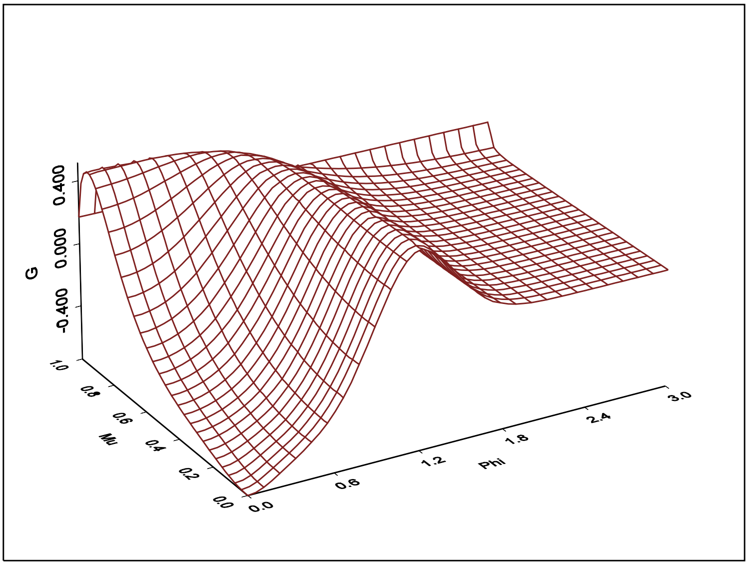

Figure 12.10: Dependence of the fundamental solution to the Dirichlet problem for a sphere $\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t)$ on coordinates μ and φ for $\zeta = 0$, $\phi = 0$, $\bar{r} = 0.75$, $\kappa = 0.5$ [176]

$$+\int_0^t\int_0^{2\pi}\int_{-1}^{1}\int_0^R \Phi(\rho,\zeta,\phi,\tau)\,\mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+\int_0^t\int_0^{2\pi}\int_{-1}^{1} g(\zeta,\phi,\tau)\,\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau. \qquad (12.32)$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform (2.72) for 2π-periodic functions with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the finite Hankel transform (2.88) of the order $n+1/2$ with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \,(2n+1)\,\frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]$$

$$\times \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \frac{\xi_{nk}\, J_{n+1/2}(r\xi_{nk})\, J_{n+1/2}(\rho\xi_{nk})}{[R^2\xi_{nk}^2 - (n+1/2)^2]\, J_{n+1/2}^2(R\xi_{nk})}, \tag{12.33}$$

where ξ_{nk} are the positive roots of the equation $J'_{n+1/2}(R\xi_k) = 0$.

The fundamental solution to the mathematical and physical Neumann problems can be calculated as

$$\mathcal{G}_m(r,\mu,\varphi,\zeta,\phi,t) = \frac{ag_0R}{q_0}\, \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t)\Big|_{\rho=R}, \tag{12.34}$$

$$\mathcal{G}_p(r,\mu,\varphi,\zeta,\phi,t) = \frac{ag_0R}{p_0}\, \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\Big|_{\rho=R}. \tag{12.35}$$

12.3 Domain $R \le r < \infty,\ -1 \le \mu \le 1,\ 0 \le \varphi \le 2\pi$

12.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right]\right.$$

$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \varphi^2}\right\} + \Phi(r,\mu,\varphi,t), \tag{12.36}$$

$$t=0: \quad T = f(r,\mu,\varphi), \qquad 0<\alpha\le 2, \tag{12.37}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1<\alpha\le 2, \tag{12.38}$$

$$r=R: \quad T = g(\mu,\varphi,t). \tag{12.39}$$

The solution:

$$T(r,\mu,\varphi,t) = \int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} f(\rho,\zeta,\phi)\,\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+ \int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} F(\rho,\zeta,\phi)\,\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+ \int_0^{t}\int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} \Phi(\rho,\zeta,\phi,\tau)\,\mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_{-1}^1 g(\zeta, \phi, \tau)\, \mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\phi\, \mathrm{d}\tau \tag{12.40}$$

with the fundamental solutions [170]

$$\begin{pmatrix} \mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)] \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}(-a\xi t^\alpha) \end{pmatrix}$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J_{n+1/2}(R\xi)}{J_{n+1/2}^2(R\xi) + Y_{n+1/2}^2(R\xi)}$$

$$\times \Big[J_{n+1/2}(\rho\xi)\, Y_{n+1/2}(R\xi) - Y_{n+1/2}(\rho\xi) J_{n+1/2}(R\xi)\Big]\, \xi\, \mathrm{d}\xi \tag{12.41}$$

obtained using the Laplace transform with respect to time t, the finite Fourier transform (2.72) for 2π-periodic functions with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the Weber transform (2.108), (2.111) of the order $n + 1/2$ with respect to the radial coordinate r.

The fundamental solution to the Dirichlet problem has the form

$$\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t) = -\frac{a g_0 t^{\alpha-1}\sqrt{R}}{\pi^2\sqrt{r}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' (2n+1) \frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)] \int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J_{n+1/2}(R\xi)}{J_{n+1/2}^2(R\xi) + Y_{n+1/2}^2(R\xi)}\, \xi\, \mathrm{d}\xi. \tag{12.42}$$

Figure 12.11 presents the dependence of the fundamental solution to the second Cauchy problem on the radial coordinate with $\bar{\mathcal{G}}_F = R^3 \mathcal{G}_F/(t w_0)$. The fundamental solution to the source problem $\bar{\mathcal{G}}_\Phi = R^3 \mathcal{G}_\Phi/(q_0 t^{\alpha-1})$ is depicted in Figs. 12.12–12.13. The fundamental solution to the Dirichlet problem is presented in Fig. 12.14 with $\bar{\mathcal{G}}_g = t\mathcal{G}_g/g_0$.

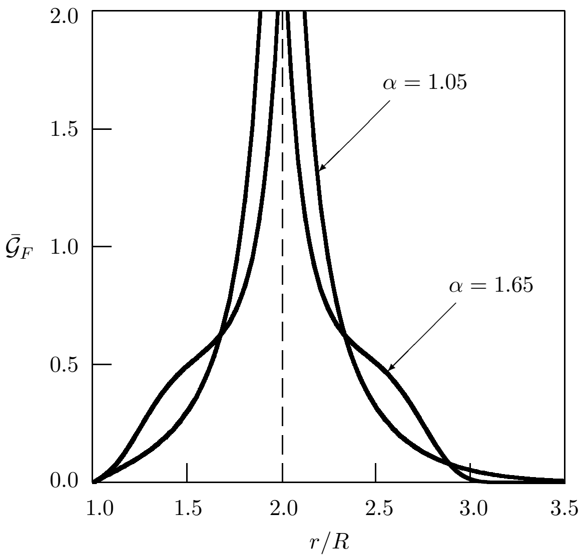

Figure 12.11: Dependence of the fundamental solution to the second Cauchy problem for a solid with a spherical hole $\mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the radial coordinate r for $\mu = 0$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

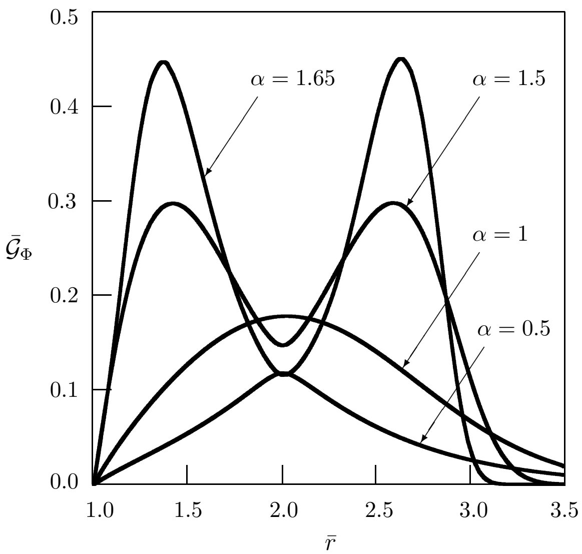

Figure 12.12: Dependence of the fundamental solution to the source problem for a solid with a spherical hole $\mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the radial coordinate r for $\mu = 0$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

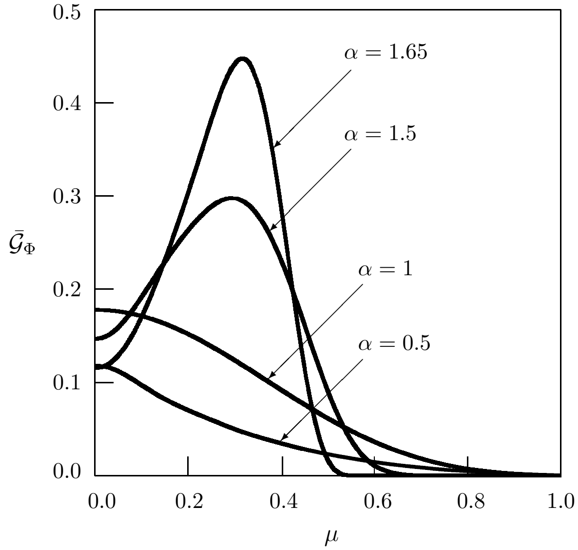

Figure 12.13: Dependence of the fundamental solution to the source problem for a solid with a spherical hole $\mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the coordinate μ for $r/R = 2$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

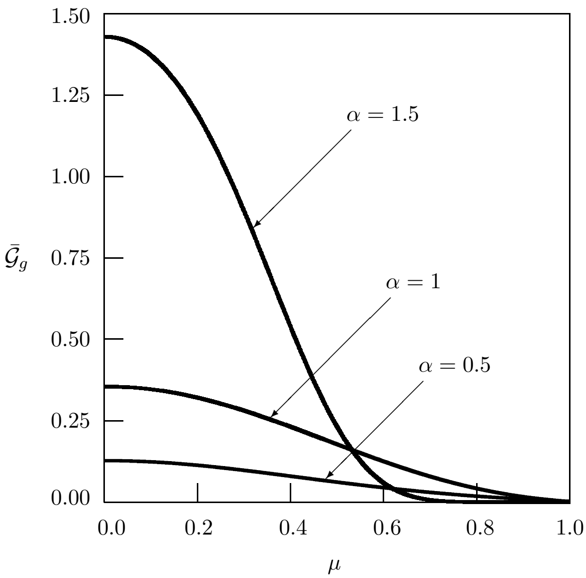

Figure 12.14: Dependence of the fundamental solution to the Dirichlet problem for a solid with a spherical hole $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on the coordinate μ for $r/R = 2$, $\varphi = 0$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

12.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right]\right.$$

$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \varphi^2}\right\} + \Phi(r,\mu,\varphi,t), \tag{12.43}$$

$$t = 0: \quad T = f(r,\mu,\varphi), \qquad 0 < \alpha \le 2, \tag{12.44}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1 < \alpha \le 2, \tag{12.45}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g(\mu,\varphi,t). \tag{12.46}$$

The solution is obtained using the Laplace transform with respect to time t, the finite Fourier transform (2.72) for 2π-periodic functions with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the Weber transform (2.108), (2.113) of the order $n+1/2$ with respect to the radial coordinate r:

$$T(r,\mu,\varphi,t) = \int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} f(\rho,\zeta,\phi)\,\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+ \int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} F(\rho,\zeta,\phi)\,\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+ \int_0^{t}\int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} \Phi(\rho,\zeta,\phi,\tau)\,\mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+ \int_0^{t}\int_0^{2\pi}\int_{-1}^{1} g(\zeta,\phi,\tau)\,\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau, \tag{12.47}$$

with the fundamental solutions

$$\begin{pmatrix} \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}}\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \, \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}$$

$$\times\, P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]\int_0^{\infty}\begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi t^\alpha) \end{pmatrix}$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y'_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J'_{n+1/2}(R\xi)}{\left[J'_{n+1/2}(R\xi)\right]^2 + \left[Y'_{n+1/2}(R\xi)\right]^2}$$

$$\times \left[J_{n+1/2}(\rho\xi)\, Y'_{n+1/2}(R\xi) - Y_{n+1/2}(\rho\xi) J'_{n+1/2}(R\xi)\right] \xi \, \mathrm{d}\xi. \tag{12.48}$$

The fundamental solutions to the mathematical and physical Neumann problems have the form

$$\begin{pmatrix} \mathcal{G}_m(r,\mu,\varphi,\zeta,\phi,t) \\ \mathcal{G}_p(r,\mu,\varphi,\zeta,\phi,t) \end{pmatrix} = \frac{a g_0 \sqrt{R}}{\pi^2 \sqrt{r}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' (2n+1) \frac{(n-m)!}{(n+m)!}$$

$$\times\ P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)] \int\limits_0^\infty \begin{pmatrix} t^{\alpha-1} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \\ E_\alpha(-a\xi t^\alpha) \end{pmatrix}$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y'_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J'_{n+1/2}(R\xi)}{\left[J'_{n+1/2}(R\xi)\right]^2 + \left[Y'_{n+1/2}(R\xi)\right]^2} \, \mathrm{d}\xi. \tag{12.49}$$

Conclusions

All tragedies are finished by a death,
All comedies are ended by a marriage.

Lord Byron

We have shown that the time-nonlocal generalization of the Fourier law with the "long-tail" power kernel results in the time-fractional diffusion-wave equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta\,T$$

with the time-fractional Caputo derivative of the order $0 < \alpha \le 2$. This equation in the case $0 < \alpha < 1$ interpolates the elliptic Helmholtz equation ($\alpha \to 0$) and the parabolic heat conduction equation ($\alpha = 1$). When $1 < \alpha < 2$, the diffusion-wave equation interpolates the standard heat conduction equation ($\alpha = 1$) and the hyperbolic wave equation ($\alpha = 2$).

The solutions to the fractional diffusion-wave equation have been obtained in different space domains. Numerical results depicted graphically are an integral part of the book. In figures, we have tried to show the transition of solutions from the Helmholtz equation through the heat conduction equation to the wave equation. Unfortunately, for reasons of space, it is impossible to exhibit the whole spectrum of numerical results. Therefore, we have restricted ourselves to the most essential and representative figures. The solutions of fractional diffusion-wave equations in the case $1 < \alpha < 2$ approximate propagating steps and humps typical for the standard wave equation in contrast to the shape of curves describing the solutions for $0 < \alpha < 1$. In particular, it is evident from figures how wave fronts arising in the case of the wave equation are approximated by the solutions of time-fractional diffusion-wave equation with α approaching the value 2.

The classical theory of heat conduction predicts that the effects of a disturbance will be felt instantaneously at distances infinitely far from its source. This limitation of the theory follows from the fact that the classical heat conduction equation is a parabolic-type equation. The hyperbolic-type transport equation allows finite wave speed for signals. The wave equation corresponding to $\alpha = 2$

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permits propagation of waves at finite speed. Nevertheless, in the case of time-fractional diffusion equation with $1 < \alpha < 2$ the propagating peaks approximating a delta function are also exhibited. The solutions for $1 < \alpha < 2$ have peaks propagating with constant speed (a feature typical for wave equation), but the strength of these peaks is decreasing with time (dissipation typical for heat conduction equation). From the analysis presented in Chapter 4, we can conclude that for $1 < \alpha < 1.36$ the solution has features more closely resembling those of the heat conduction equation. For $1.69 < \alpha < 2$ the solution resembles the solution to the wave equation and the intermediate behavior corresponds to the values $1.36 < \alpha < 1.69$. The analysis of distinguishing features of the fundamental solution to the first Cauchy problem carried out in Chapter 4 is based on introducing the similarity variable $\frac{x}{\sqrt{a}t^{\alpha/2}}$. It should be mentioned that the same method can be applied for any twice-differentiable function that depends on the similarity variable or another one in the form of the product of the power functions in x and t (see also [19, 93–97, 155, 160, 210]). The fundamental solutions for many linear partial differential equations of fractional order possess this property and can be investigated by the method presented in Chapter 4.

In this book, we have used heat conduction terminology (the generalized Fourier law, the fractional heat conduction equation). As a rule, fractional heat conduction and fractional diffusion have the same origin. Sometimes, the diffusion interpretation (the generalized Fick law, the fractional diffusion equation) may give a clearer insight into the physical aspects of the theory. At the level of individual particle motions the classical diffusion corresponds to Brownian motion which is characterized by a mean-squared displacement increasing linearly with time

$$\langle x^2 \rangle \sim at.$$

Anomalous diffusion, which is exemplified by a mean-squared displacement with the power-law time dependence

$$\langle x^2 \rangle \sim at^\alpha$$

at the level of individual particle motion, has been modeled in numerous ways. The continuous time random walk (CTRW) theory (see [114, 116, 122] and references therein) is in most common use and allows one to extend classical Brownian random walks to variable jump lengths and waiting times between successive jumps. The velocity model in a CTRW scheme [112, 236] assumes that the particle moves at constant velocity to the new site. The power-law tails make it possible to have very long waiting times, and in the subdiffusion regime ($0 < \alpha < 1$) particles on the average move slower than in ordinary diffusion which corresponds to $\alpha = 1$. In the superdiffusion regime ($1 < \alpha < 2$) particles on the average move faster than in ordinary diffusion. For dimensions higher than $D = 1$, solutions of a time-fractional diffusion-wave equation for $1 < \alpha < 2$ can be bimodal, and from a waiting time perspective correspond to the velocity model.

In this book, the integral transform technique has been used, and the fundamental solutions are expressed in terms of Mittag-Leffler functions. For particular cases of the classical heat conduction equation ($\alpha = 1$) and the wave equation ($\alpha = 2$), when the Mittag-Leffler functions are reduced to exponential and trigonometric functions, the obtained solutions coincide with those given in [26, 31, 98, 140, 144]. To obtain these particular cases of solutions the integrals presented in the Appendix have been used. It should be emphasized that infinite Fourier series appearing in solutions are well convergent. Some difficulties in calculations arise in the case of the wave equation ($\alpha = 2$). But in this case the solutions are obtained in the analytic form which allows us to test the computer routines and to decide when to truncate the infinite series. Special functions appearing in the particular cases of the solutions were computed according to routines descibed in [194]. The Mittag-Leffler functions were calculated by FORTRAN programs that implement the algorithms proposed in [52] (see equations (2.152)–(2.155)). The interested reader is also referred to the MATLAB programs [111] which also use the same algorithms.

In the finite space domains we, as a rule, have considered Dirichlet, Neumann and Robin boundary conditions. For the time-fractional diffusion-wave equation, two types of Neumann and Robin boundary conditions can be considered: the mathematical one formulated in terms of the normal derivative of the sought function and the physical one formulated in terms of the flux. These two types of boundary conditions coincide only in the case of the classical heat conduction equation ($\alpha = 1$), but for $\alpha \neq 1$ they are essentially different. It should also be mentioned that using appropriate integral transforms, the mixed boundary problems can also be considered (see, for example, [48, 144], where the mixed boundary-value problems were studied for classical partial differential equations).

In Chapter 11, for reasons of space we have not considered the wedge domains with $0 < \varphi < \varphi_0$. The solutions for such domains can be easily obtained using the corresponding results from Chapter 8.

The interested reader is also referred to the papers in which the time-fractional diffusion-wave equation was considered in Cartesian coordinates [3, 4, 25, 57, 58, 100, 101, 106, 110, 145, 146, 154, 162, 177, 181–183, 187, 220, 224]; in cylindrical coordinates [73, 126, 145, 146, 148, 149, 156, 157, 160–163, 166, 168, 171, 174, 175, 179, 182, 186, 188, 190, 197] and in spherical coordinates [89, 150–153, 162, 164, 169, 170, 173, 176, 182, 184, 185, 189, 197].

In this book, we have considered the time-fractional diffusion-wave equation with the Caputo fractional derivative. Equations with the Riemann–Liouville fractional derivative have also been studied in [69, 113, 134–139, 203, 225, 226, 229] as well as more complicated models, in particular fractional diffusion equations of distributed order (see [7, 8, 27–29, 55, 84, 108, 109, 125, 213] and references therein) and the generalized diffusion equation containing the fractional time-derivative with a weight and a scale [232].

Appendix: Integrals

Divide et impera.
Julius Caesar

Divide et integra.
Tadeusz Trajdos

The indefinite integrals are borrowed from [18, 35]:

$$\int \frac{1}{1+\varepsilon \cos x}\,\mathrm{d}x = \frac{2}{\sqrt{1-\varepsilon^2}} \arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{x}{2}\right), \quad 0<\varepsilon<1, \tag{A.1}$$

$$\int \frac{1}{p^2 - q^2\cos^2 x}\,\mathrm{d}x = \frac{1}{p^2 \sin\gamma}\ \arctan\left(\frac{\tan x}{\sin\gamma}\right), \tag{A.2}$$

where $q^2 < p^2,\ \gamma = \arccos\left(\frac{q}{p}\right)$.

To obtain the particular case of solutions for the wave equation the following integral is often used:

$$\int_0^\infty \cos(bx)\mathrm{d}x = \pi\delta(b). \tag{A.3}$$

Integrals containing elementary functions are taken from [60, 195]; integrals (A.18), (A.20), (A.23) and (A.24) have been evaluated by the author:

$$\int_0^\infty \frac{\sin(bx)}{x}\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{sign}\,b. \tag{A.4}$$

$$\int_0^\infty \frac{x}{x^2+c^2}\ \sin(bx)\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{e}^{-|bc|}\,\mathrm{sign}\,b. \tag{A.5}$$

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$$\int_0^\infty \frac{\sin(bx)}{x(x^2+c^2)}\,\mathrm{d}x = \frac{\pi}{2c^2}\left(1-\mathrm{e}^{-|bc|}\right)\operatorname{sign} b. \tag{A.6}$$

$$\int_0^\infty \frac{1}{x^2+c^2}\cos(bx)\,\mathrm{d}x = \frac{\pi}{2c}\,\mathrm{e}^{-bc}, \quad b>0, \quad c>0. \tag{A.7}$$

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,\mathrm{d}x = \frac{\pi}{2}(b-a), \quad a>0,\ b\ >0. \tag{A.8}$$

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2(x^2+c^2)}\,\mathrm{d}x = \frac{\pi}{2c^3}\left[(b-a)c+\mathrm{e}^{-bc}-\mathrm{e}^{-ac}\right], \quad a>0,\ b>0,\ c>0. \tag{A.9}$$

$$\int_0^\infty \frac{\sin(ax)-ax\cos(ax)}{x^3}\mathrm{d}x = \frac{\pi}{4}\,a^2\operatorname{sign} a. \tag{A.10}$$

$$\int_0^\infty \frac{\cos(px)\ \cos(qx)}{x^2+b^2}\,\mathrm{d}x = \begin{cases} \frac{\pi}{2b}e^{-bp}\cosh(bq), & 0<q<p, \\ \frac{\pi}{2b}e^{-bq}\cosh(bp), & 0<p<q, \end{cases} \quad b>0. \tag{A.11}$$

$$\int_0^\infty \frac{x\cos(px)\ \sin(qx)}{x^2+b^2}\,\mathrm{d}x = \begin{cases} -\frac{\pi}{2}e^{-bp}\sinh(bq), & 0<q<p, \\ \frac{\pi}{4}\,\mathrm{e}^{-2bp}, & 0<p=q, \\ \frac{\pi}{2}e^{-bq}\cosh(bp), & 0<p<q, \end{cases} \quad b>0. \tag{A.12}$$

$$\int_0^\infty \frac{\mathrm{e}^{-px}}{x}\sin(qx)\,\mathrm{d}x = \arctan\left(\frac{q}{p}\right), \qquad p>0. \tag{A.13}$$

$$\int_0^\infty x\,\mathrm{e}^{-a^2x^2}\sin(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}b}{4a^3}\exp\left(-\frac{b^2}{4a^2}\right), \quad a>0. \tag{A.14}$$

$$\int_0^\infty \frac{1}{x}\,\mathrm{e}^{-a^2x^2}\sin(bx)\,\mathrm{d}x = \frac{\pi}{2}\operatorname{erf}\left(\frac{b}{2a}\right), \quad a>0. \tag{A.15}$$

$$\int_0^\infty \mathrm{e}^{-a^2x^2} \cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{4a^2}\right), \quad a > 0. \tag{A.16}$$

$$\int_0^\infty x^2\,\mathrm{e}^{-a^2x^2} \cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{4a^3}\left(1-\frac{b^2}{2a^2}\right) \exp\left(-\frac{b^2}{4a^2}\right), \quad a > 0. \tag{A.17}$$

$$\begin{aligned}\int_0^\infty \frac{\cos(bx)-\cos(cx)}{x^2}\,\mathrm{e}^{-a^2x^2}dx &= \frac{\pi}{2}\Bigg\{c\,\mathrm{erf}\left(\frac{c}{2a}\right) - b\,\mathrm{erf}\left(\frac{b}{2a}\right) \\ &+ \frac{2a}{\sqrt{\pi}}\left[\exp\left(-\frac{c^2}{4a^2}\right) - \exp\left(-\frac{b^2}{4a^2}\right)\right]\Bigg\}, \quad a > 0.\end{aligned} \tag{A.18}$$

$$\int_0^\infty \frac{\sin\left(c\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}} \cos(bx)\,\mathrm{d}x = \begin{cases} \frac{\pi}{2} J_0\left(a\sqrt{c^2-b^2}\right), & b < c, \\ 0, & c < b. \end{cases} \tag{A.19}$$

$$\begin{aligned}&\int_0^\infty \frac{\cos(c\sqrt{x^2+a^2})}{x^2+a^2} \cos(bx)\,\mathrm{d}x \\ &= \begin{cases} \frac{\pi}{2a}\,\mathrm{e}^{-ab}, & c < b, \\ \frac{\pi}{2a}\,\mathrm{e}^{-ab} - \frac{\pi}{2}\int_b^c J_0(a\sqrt{u^2-b^2})\mathrm{d}u, & b < c. \end{cases}\end{aligned} \tag{A.20}$$

$$\int_0^1 \frac{\cos(p\sqrt{1-x^2})}{\sqrt{1-x^2}} \cos(qx)\,\mathrm{d}x = \frac{\pi}{2} J_0\left(\sqrt{p^2+q^2}\right). \tag{A.21}$$

$$\int_0^1 \sin(p\sqrt{1-x^2})\cos(qx)\,\mathrm{d}x = \frac{\pi}{2}\frac{p}{\sqrt{p^2+q^2}} J_1\left(\sqrt{p^2+q^2}\right). \tag{A.22}$$

$$\int_0^1 \frac{\cos(p\sqrt{1-x^2})}{x\sqrt{1-x^2}} \sin(qx)\,\mathrm{d}x = \frac{\pi}{2}\int_0^q J_0\left(\sqrt{p^2+u^2}\right)\mathrm{d}u, \quad q > 0. \tag{A.23}$$

$$\int_0^1 \frac{\sin(p\sqrt{1-x^2})}{x} \sin(qx)\,\mathrm{d}x = \frac{\pi}{2}\int_0^q \frac{p}{\sqrt{p^2+u^2}}\, J_1\left(\sqrt{p^2+u^2}\right) \mathrm{d}u,\ \ q>0. \quad \text{(A.24)}$$

Equations (A.25) and (A.26) are taken from [44], equation (A.27) has been obtained by the author:

$$\int_0^\infty \frac{1}{x^2+c^2}\,\mathrm{e}^{-a^2x^2}\ \cos(bx)\,\mathrm{d}x = \frac{\pi}{4c}\,\mathrm{e}^{a^2c^2}\left[\mathrm{e}^{-bc}\mathrm{erfc}\left(ac-\frac{b}{2a}\right)\right.$$
$$\left. + \mathrm{e}^{bc}\mathrm{erfc}\left(ac+\frac{b}{2a}\right)\right],\ \ a>0,\ \ b>0,\ \ c>0. \quad \text{(A.25)}$$

$$\int_0^\infty \frac{x}{x^2+c^2}\,\mathrm{e}^{-a^2x^2}\ \sin(bx)\,\mathrm{d}x = \frac{\pi}{4}\,\mathrm{e}^{a^2c^2}\left[\mathrm{e}^{-bc}\mathrm{erfc}\left(ac-\frac{b}{2a}\right)\right.$$
$$\left. - \mathrm{e}^{bc}\mathrm{erfc}\left(ac+\frac{b}{2a}\right)\right],\ \ a>0,\ \ b>0,\ \ c>0. \quad \text{(A.26)}$$

$$\int_0^\infty \frac{1}{x(x^2+c^2)}\,\mathrm{e}^{-a^2x^2}\ \sin(bx)\,\mathrm{d}x = \frac{\pi}{4c^2}\,\mathrm{e}^{a^2c^2}\left[-\,\mathrm{e}^{-bc}\mathrm{erfc}\left(ac-\frac{b}{2a}\right)\right.$$
$$\left. + \mathrm{e}^{bc}\mathrm{erfc}\left(ac+\frac{b}{2a}\right) + 2\mathrm{e}^{-a^2c^2}\,\mathrm{erf}\left(\frac{b}{2a}\right)\right],\ \ a>0,\ \ b>0,\ \ c>0. \quad \text{(A.27)}$$

Integrals containing the Bessel functions are taken from [196]; integrals (A.30), (A.36), (A.37) have been evaluated by the author:

$$\int_0^\infty \frac{x}{x^2+p^2}\,J_0(qx)\,\mathrm{d}x = K_0(pq), \quad p>0, \quad q>0. \quad \text{(A.28)}$$

$$\int_0^\infty \frac{x^2}{x^2+p^2}\,J_1(qx)\,\mathrm{d}x = pK_1(pq), \quad p>0, \quad q>0. \quad \text{(A.29)}$$

$$\int_0^\infty \frac{1}{x^2+p^2}\,J_1(qx)\,\mathrm{d}x = \frac{1}{qp^2} - \frac{1}{p}K_1(pq), \quad p>0, \quad q>0. \quad \text{(A.30)}$$

$$\int_0^\infty \mathrm{e}^{-px^2}\,J_1(qx)\,\mathrm{d}x = \frac{1}{q}\left[1-\exp\left(-\frac{q^2}{4p}\right)\right], \quad p>0, \quad q>0. \quad \text{(A.31)}$$

$$\int_0^\infty x\,\mathrm{e}^{-px^2}\,J_0(qx)\,\mathrm{d}x = \frac{1}{2p}\,\exp\left(-\frac{q^2}{4p}\right), \quad p>0, \quad q>0. \quad \text{(A.32)}$$

$$\int_0^\infty x^2 \mathrm{e}^{-px^2}\, J_1(qx)\,\mathrm{d}x = \frac{q}{4p^2}\,\exp\left(-\frac{q^2}{4p}\right), \quad p>0, \quad q>0. \tag{A.33}$$

$$\int_0^\infty e^{-ax^2} J_\nu(bx)\, J_\nu(cx)\, x\,\mathrm{d}x = \frac{1}{2a}\,\exp\left(-\frac{b^2+c^2}{4a}\right) I_\nu\left(\frac{bc}{2a}\right). \tag{A.34}$$

$$\int_0^\infty \frac{x}{x^2+a^2}\, J_0(bx)\, J_0(cx)\,\mathrm{d}x = \begin{cases} I_0(ab)\, K_0(ac), & 0<b<c, \\ I_0(ac)\, K_0(ab), & 0<c<b. \end{cases} \tag{A.35}$$

$$\int_0^\infty e^{-ax^2}\, J_0(bx)\, J_1(cx)\,\mathrm{d}x = \frac{1}{2ac}\int_0^c x\exp\left(-\frac{b^2+x^2}{4a}\right) I_0\left(\frac{bx}{2a}\right)\,\mathrm{d}x. \tag{A.36}$$

$$\int_0^\infty \frac{1}{x^2+a^2}\, J_0(bx)\, J_1(cx)\,\mathrm{d}x$$
$$= \begin{cases} \dfrac{1}{a}\, I_1(ac)\, K_0(ab), & 0<c<b, \\ \dfrac{1}{ac}\left[bI_1(ab)\, K_0(ab) + I_0(ab)\Big(bK_1(ab) - cK_1(ac)\Big)\right], & 0<b<c. \end{cases} \tag{A.37}$$

$$\int_0^\infty \exp\left(-b\sqrt{x^2+a^2}\right)\, J_1(cx)\,\mathrm{d}x$$
$$= \frac{1}{c}\left[\mathrm{e}^{-ab} - \frac{b}{\sqrt{b^2+c^2}}\,\exp\left(-a\sqrt{b^2+c^2}\right)\right]. \tag{A.38}$$

$$\int_0^1 x\sin\left(b\sqrt{1-x^2}\right) J_0(cx)\,\mathrm{d}x$$
$$= \frac{b}{b^2+c^2}\left[\frac{\sin\left(\sqrt{b^2+c^2}\right)}{\sqrt{b^2+c^2}} - \cos\left(\sqrt{b^2+c^2}\right)\right]. \tag{A.39}$$

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,\cos\left(b\sqrt{1-x^2}\right)\, J_0(cx)\,\mathrm{d}x = \frac{\sin\left(\sqrt{b^2+c^2}\right)}{\sqrt{b^2+c^2}}. \tag{A.40}$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\cos\left(b\sqrt{1-x^2}\right) J_1(cx)\,\mathrm{d}x = \frac{1}{c}\left[\cos b - \cos\left(\sqrt{b^2+c^2}\right)\right]. \tag{A.41}$$

$$\int_0^1 \sin\left(b\sqrt{1-x^2}\right) J_1(cx)\,\mathrm{d}x = \frac{1}{c}\left[\sin b - \frac{b}{\sqrt{b^2+c^2}}\sin\left(\sqrt{b^2+c^2}\right)\right]. \qquad \text{(A.42)}$$

The discontinuous Weber–Schafheitlin type integrals read [196]

$$\int_0^\infty \sin(px)\, J_0(qx)\,\mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{p^2-q^2}}, & 0<q<p, \\ 0, & 0<p<q. \end{cases} \qquad \text{(A.43)}$$

$$\int_0^\infty \cos(px)\, J_1(qx)\,\mathrm{d}x = \begin{cases} -\dfrac{q}{\sqrt{p^2-q^2}}\,\dfrac{1}{p+\sqrt{p^2-q^2}}, & 0<q<p, \\ q^{-1}, & 0<p<q. \end{cases} \qquad \text{(A.44)}$$

$$\int_0^\infty \frac{1-\cos(px)}{x} J_0(qx)\,\mathrm{d}x = \begin{cases} \ln\dfrac{p+\sqrt{p^2-q^2}}{q}, & 0<q<p, \\ 0, & 0<p<q. \end{cases} \qquad \text{(A.45)}$$

$$\int_0^\infty \frac{x}{\sqrt{x^2+p^2}} \sin\left(b\sqrt{x^2+p^2}\right) J_0(cx)\,\mathrm{d}x$$
$$= \begin{cases} \dfrac{1}{\sqrt{b^2-c^2}} \cos\left(p\sqrt{b^2-c^2}\right), & 0<c<b, \\ 0, & 0<b<c. \end{cases} \qquad \text{(A.46)}$$

The following integrals [196] appear in the case of the wave equation

$$\int_0^\infty \sin ax\, J_\nu(bx)\, J_\nu(cx)\,\mathrm{d}x$$
$$= \begin{cases} \dfrac{1}{2\sqrt{bc}} P_{\nu-1/2}\left(\dfrac{b^2+c^2-a^2}{2bc}\right), & |b-c|<a<b+c, \\ -\dfrac{\cos(\nu\pi)}{\pi\sqrt{bc}} Q_{\nu-1/2}\left(\dfrac{a^2-b^2-c^2}{2bc}\right), & b+c<a,\ 0<b<c, \\ 0, & 0<a<|b-c|, \end{cases} \qquad \text{(A.47)}$$

where $P_\nu(r)$ and $Q_\nu(r)$ are the Legendre functions of the first and second kind, respectively. In particular,

$$\int_0^\infty \sin(ax)\, J_0(bx)\, J_0(cx)\,\mathrm{d}x = \begin{cases} 0, & 0<a<|b-c|, \\ \dfrac{1}{\pi\sqrt{bc}}\mathbf{K}(k), & |b-c|<a<b+c, \\ \dfrac{1}{\pi k\sqrt{bc}}\mathbf{K}\left(\dfrac{1}{k}\right), & b+c<a<\infty. \end{cases} \qquad \text{(A.48)}$$

$$\int_0^\infty \cos(ax)\, J_1(bx)\, J_0(cx)\, \mathrm{d}\xi$$

$$= \begin{cases} 0, & 0 < a < c - b, \\ \dfrac{1}{b}\left[1 - \Lambda_0\left(\varphi_1, k\right) + \dfrac{c-a}{\pi\sqrt{bc}}\,\mathbf{K}(k)\right], & 0 < |c-b| < a < c+b, \\ \dfrac{1}{b}\left[1 - \Lambda_0\left(\varphi_2, \dfrac{1}{k}\right) - \dfrac{1}{\pi k}\sqrt{\dfrac{b}{c}}\,\mathbf{K}\left(\dfrac{1}{k}\right)\right], & 0 < b + c < a, \\ \dfrac{1}{b}, & 0 < |a-c| < a + c < b, \end{cases} \tag{A.49}$$

where $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are complete elliptic integrals of the first and second kind, respectively. Heuman's Lambda function is expressed in terms of incomplete elliptic integrals of the first and second kind $F(\varphi, k)$ and $E(\varphi, k)$ as [1]

$$\Lambda_0(\varphi, k) = \frac{2}{\pi}\left[\mathbf{E}(k)\, F(\varphi, k') + \mathbf{K}(k)\, E(\varphi, k') - \mathbf{K}(k)\, F(\varphi, k')\right].$$

Here

$$k = \frac{\sqrt{a^2 - (b-c)^2}}{2\sqrt{bc}}, \quad k' = \sqrt{1-k^2}, \quad a > 0, \quad b > 0, \quad c > 0,$$
$$\varphi_1 = \arcsin\sqrt{\frac{2c}{a+b+c}}, \quad \varphi_2 = \arcsin\sqrt{\frac{a+c-b}{a+b+c}}.$$

$$\int_0^\infty J_0(ax)\, J_0(bx)\, J_1(cx)\, \mathrm{d}x$$

$$= \begin{cases} 0, & 0 < c < |a-b|, \\ \dfrac{1}{\pi c}\arccos\left(\dfrac{a^2+b^2-c^2}{2ab}\right), & |a-b| < c < a+b, \\ \dfrac{1}{c}, & a + b < c. \end{cases} \tag{A.50}$$

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Outside of a dog, a book is man's best friend.
Inside a dog it's too dark to read.

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