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Rémi Sentis

Mathematical Models and Methods for Plasma Physics, Volume 1

Fluid Models

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Fluid Models

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To Ghislaine

Foreword

It is well known that in plasma physics, many different models are used: fluid or kinetic models (where the ion and electron populations are characterized by a distribution function in the velocity space); but also hybrid models where kinetic equations are coupled with fluid systems. This book is restricted to some fluid models. It will be followed by another book dedicated to kinetic models. A first motivation for undertaking this work is to understand and justify (from a mathematical point of view) the derivations of some classical-specific models from more general ones. Even if these derivations are generally assumed to be well known by physicists, it seems to me that it is worth emphasizing the underlying assumptions and describing some apparently simple approximations that are more complex than they look.[1] So, I have tried to give justifications of such approximations by using asymptotic analysis whenever possible.

Another main motivation is the following: precise statements of the systems of partial differential equations with the appropriated boundary conditions are necessary to perform efficient numerical simulations mainly in multidimensional cases. Of course, this work does not aim to deal mainly with the related numerical methods, which is an entire subject unto itself; but some enlightenment is presented in relation to numerical methods for some classical problems.

Since it was necessary to make a choice for this book, I have focused on models involving the quasi-neutrality approximation, on problems related to laser propagation in a plasma, and on coupling plasma waves and electromagnetic waves.

Despite the heaviness of the notation, I have tried to keep it more or less in the physics literature. So, I hope that this work will be helpful for young physicists who are often overwhelmed by so many different models, and who are not sure of the exact underlying assumptions (perhaps also for vocabulary purposes). Of course,

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[1]For example, as F. F. Chen notices about the notion of quasi-neutrality in the textbook *Introduction to Plasma Physics'* first published in 1974 (Academic Press, New York), "it is usually possible to assume the electric charge is zero and the divergence of the electrostatic field is not zero at the same time. It is a fundamental trait of plasmas, one which is difficult for novices to understand."

I hope also that applied mathematicians will find in this work a stimulating introduction to the "marvellous world" of plasma physics and an incitement to address actual plasma physics models (and not only theoretical plasma models). They can find a few open problems (referred in the index at the end of the book) related to these actual models, which are very rich from a mathematical point of view.

The book is organized as follows.

In a short first chapter, after a historical account, we display heuristically some classical characteristic quantities that always appear in plasma physics.

In Chap. 2, we emphasize some basic introductory features which for the most part are used in the other chapters: the principles of the massless-electron approximation which is presupposed if one wants to perform the quasi-neutrality approximation (Sect. 2.1), then the justification of this quasi-neutrality approximation (Sect. 2.2). This is made by a mathematically rigorous asymptotic analysis (using a small parameter related to the Debye length). In the last section, we emphasize the approximations made to derive the two-temperature Euler system and the magneto-hydrodynamics (more precisely the so-called electron-magneto-hydrodynamics or *electron-MHD* model). These quasi-neutral models may be found, of course, in standard physics textbooks, but it seemed useful to focus on their derivation and to gather some properties related to these models, especially the formulation of these systems in the Lagrange framework, the spectral properties of their hyperbolic part, and the boundary conditions. Indeed, the results presented here were scattered across many different research papers and were not easily accessible by physicists up to now.

In Chap. 3, our concern is with models related to laser wave propagation in a plasma, and also in the interaction between laser and plasma. The heart of most models related to laser–plasma interaction is the paraxial approximation for the laser wave, which is based on time and space envelope modelling, and is related to the classical WKB (Wentzel–Kramers–Brillouin) expansion. So in Sect. 3.1, we explain firstly how the time envelope technique works; secondly, the WKB expansion is explained in our framework in order to state classical results on geometric optics; and thirdly, the WKB expansion is used one order further for the paraxial approximation.

Section 3.2 is devoted to a problem related to laser–plasma interaction, when the coupling of the laser waves with the ion acoustic waves is accounted for; this is the well-known Brillouin instability (the coupling with the Langmuir waves is here neglected). The novelty is that we focus on a crucial mathematical structure of the so-called *three-wave coupling system* which is in evidence even in a one-dimensional framework. In this system, for each wave we must address a transport equation involving a coupling term related to the product of the fields corresponding to the two other waves. It is worth noticing that this kind of three-wave coupling system was introduced 40 years ago by Kadomstev for modelling plasma turbulence. It is very well-known in the physics literature, but very few mathematical studies have been published up to now in this area.

In Chap. 4, we address firstly, for the sake of completeness, the classical problem of Langmuir waves and, secondly, the coupling of these electron waves with an ion population, which leads to the Zakharov equations. Even if these equations are well known by some mathematical teams, it seemed important to recall precisely the link to the physics and the approximations that are made, as well as to recall the main mathematical properties of the different models related to these Zakharov equations.

Chapter 5 is concerned with laser–plasma interaction involving electron waves. First, we focus on the modelling of Raman instabilities where the paraxial model for the laser beam is coupled with a model for the Langmuir waves. We give some enlightenment on this kind of model which is trickier than the Brillouin instability model, since we need to deal with a space-and-time envelope of the fields corresponding to the Langmuir waves. It is worth noticing that the modelling and the numerical simulations of the Raman instabilities are still fields of intense research. In the second part of the chapter, we briefly give some comments on specific models for dealing with interaction between an ultra-intense laser pulse and a plasma; so we address the system that consists of the Maxwell equations coupled with a fluid system for the electrons; moreover, we show how an envelope description may be useful in some cases.

In the final chapter, we consider two kinds of modelling. In the first section, we consider plasmas with two ion species, besides the electron population: after some closures, it leads to various models all based on the two-temperature Euler equations in the quasi-neutral framework; our aim here is to specify the approximations made to derive these models describing the averaged ion fluid. In the same way the role played by the different assumptions is clarified. The second section is devoted to models in a different framework: the weakly ionized plasmas. Besides the main flow of neutral particles, there are ions and electrons that undergo collisions with the neutral particles. We deal also here with the quasi-neutral approximation, and we justify the so-called ambipolar diffusion approximation by an asymptotic analysis analogue to the one of Chap. 2.

Paris, France Rémi Sentis

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Chapter 1
Introduction: Some Plasma Characteristic Quantities

Abstract The first chapter is devoted to a heuristic presentation of some basic concepts in plasma physics and the definition of some plasma characteristic quantities.

Keywords Debye length • Plasma frequency • Alfven speed • Magnetic pressure • Magneto-hydrodynamics • Electron magneto-hydrodynamics • Ohm's law

1.1 Historical Account

A plasma consists of an ion population and an electron population (with possibly neutral particles) whose evolutions are coupled through collision processes and long-range electromagnetic forces. The term *plasma* was first used in physics by Langmuir and Tonks in 1929 when they observed in an electric discharge tube that there were periodic variations of the electron density, although it was known much earlier that ions and electrons exhibit collective behavior in the *fourth state of matter* due to long-range Coulomb forces. A decade after these historic observations, plasma physics developed in many other directions including magneto-hydrodynamics (MHD), wave propagation in plasmas, and confinement of plasmas using magnetic fields.

Indeed, in 1938–1939, in experiments producing electrical discharges in gas, Merril and Webb demonstrated the part played by the oscillations of an electron beam in a plasma [85], and Loeb published his book *Fundamental Processes of Electrical Discharges* [80]. Moreover, Ramo analyzed the propagation of an electric charge wave in a one-dimensional device. In 1941, the work of Bailey's team on electron mobility was published [66]. After the work of the Russian physicist Klarfeld, the phenomena of electron plasma waves, also called *Langmuir waves*, became progressively clearer.

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Astrophysical observations gave rise to fundamental developments for magnetized plasmas; indeed, after his first observations in 1942, Alfven stated that the dynamics of ionized gases, which are a conductor, may by treated in a unified theory with classical fluid dynamics. This theory was explained in 1950 in the classical book *Cosmical Electrodynamics* [1]. Alfven noted in his introduction that "in cosmic physics the positive space charge in a volume is always approximately equal to [the] negative space charge"; moreover, he pointed out clearly the so-called *plasma approximation* by claiming that "the presence of an electric field is essential" even with a negligible electric charge. In his description of so-called *magneto-hydrodynamics*—the most popular model for magnetized plasmas—the modelling was based on a classical single-fluid hydrodynamic system where the magnetic pressure tensor was added to the matter pressure.

After Alfven's pioneering works, at the beginning of the 1950s, MHD was used by Herfolson, Thompson, Lunquist [83], Chandrasekhar [26] and other physicists. A book by Cowling [34] was devoted to this subject; moreover, it is worth noting the works of Delcroix and co-workers [40, 41] in the field of weakly ionized plasmas. Their studies focused on laboratory plasmas, but also on the explanation of atmospheric phenomena related to the Earth's magnetic field (e.g., the magnetic reconnecting processes by which magnetic field lines suddenly change their topology, boreal auroras, etc.. . .). Since that period, MHD models have also been used for modelling astrophysics problems related to solar flares, sunspots, star formation, etc.. . .

On the other hand, since 1949, various theories of electromagnetic wave propagation through nonuniform magnetized plasmas have been developed, e.g., see Ginzburg [58,59]. In 1958, Van Allen discovered the radiation belts surrounding the Earth, using data transmitted from the satellite Explorer. This was the beginning of exploring the Earth's magnetosphere via satellite, and it opened up a new field of plasma physics: it was discovered that a layer of partially ionized gas in the upper atmosphere reflects radio waves but may be also responsible for deficiencies in radio communications.

In all this early literature related to MHD, the current-carrying fluid was assumed to be neutral. But, in 1947, Bohm and Aller observed the importance of the *Debye length*, which is defined as a length over which the electrostatic field of a single charge is shielded by the response of the surrounding charges (see the last section of this Chap.). They noted that although the electron charge and the ion charge are locally almost equal, it was important to make a distinction between these two populations particularly when the Debye length is not small with respect to the characteristic length of the plasma, mainly near electrodes [19]. This fact was also pointed out by Loeb [81] and in the second edition of Alfven's book in 1963.

This was of particular importance in the devices designed by some pioneer Russian scientists such as Artsimovich, Safranov, and Yurchenko from 1960, where the plasma was confined in a vessel by external magnetic fields—the popular *tokamak*,[1] cf. [5, 6, 102]. Some years later, one of the main topics was the study of plasma instabilities, which have been extensively studied, for example, by Kadomstev [72] (indeed, when there is a small perturbation of the electromagnetic fields, the response of the two components is the opposite and thus generates an electrically charged plasma wave). Moreover, electron thermal conduction was systematically studied by Spitzer, cf. [111].

In another direction, research on thermonuclear fusion has been carried out since 1950 independently by both the United States and the Soviet Union; this has led to the publication of a number of immensely important and influential papers. The best-known books are those of Landau-Lifschitz [76] and Zel'dovich-Raizer [117]. Since it was necessary to perform numerical simulations to predict the behavior of very hot plasma, theoretical plasma physics emerged as a mathematically rigorous discipline in these years.

Finally, the development of high-powered lasers after 1960 opened up a new field of plasma physics: indeed, when such a laser beam strikes a solid target, the matter vaporizes and becomes ionized. A lot of studies related to laser–plasma interaction have been performed following the pioneering publication of Basov and Krokhin [9].

Nowadays, plasma physics is a very wide-ranging domain with many different applications, including astrophysics, electrical discharges, hot magnetized plasmas for magnetic confinement fusion, laser-heated plasmas for inertial confinement fusion (ICF), generation of high-energy particle beams by laser-plasma interaction, and spatial plasmas.

1.2 Notations

We denote by ∇ the gradient and by Δ the Laplace operator with respect to the spatial variable. For two vector fields $\mathbf{Q}$, $\mathbf{Q}'$, let $\mathbf{Q}\times\mathbf{Q}'$ be the vector product and $\mathbf{QQ}'$ be the tensor product, that is, $(\mathbf{QQ}')_{ij} = (\mathbf{Q})_i(\mathbf{Q}')_j$. Denote by $\nabla.\mathbf{Q}$ the divergence of $\mathbf{Q}$. The tensors are denoted by a special font such as $\mathbb{T}$ (or sometimes like $\underset{\leftrightarrow}{\sigma}$). Denote by $\nabla.\mathbb{T}$ the divergence of the tensor $\mathbb{T}$; thus, $\nabla.(\mathbf{QQ}')$ is the vector with components that are $(\nabla.(\mathbf{QQ}'))_i = \nabla.(\mathbf{QQ}'_i)$ and by $\mathbb{T} : \mathbb{T}'$ the contracted product of two tensors. The identity tensor is denoted by $\mathbb{I}$.

Moreover, we define the tensor $\nabla\mathbf{Q}$ by $(\nabla\mathbf{Q})_{ij} = \frac{\partial}{\partial x_i}(\mathbf{Q})_j$.

---

[1] The principle of tokamak experiments is to confine a dilute plasma in a torus-shaped vessel, applying strong poloïdal and toroïdal magnetic fields. The plasma must be heated by radio-frequency waves and the ions are accelerated by the magnetic field.

The complex conjugate is denoted by $\bar{\bullet}$. For a quantity f, instead of $f + \overline{f}$ we write $f + c.c.$ ($c.c.$ for complex conjugate).

Finally, if $\mathcal{O} = \mathbf{R}^3$ or if it is an open set, we denote by $L^2(\mathcal{O})$ the set of functions whose square is integrable, it is a Hilbert space endowed with the norm $\|u\|_{L^2} = [\int_{\mathcal{O}} |u(x)|^2 dx]^{1/2}$; moreover, $L^1(\mathcal{O})$ is the set of functions whose modulus is integrable it is a Banach space endowed with the norm $\|u\|_{L^1}$; and $L^\infty(\mathcal{O})$ is the space of bounded functions on $\mathcal{O}$ endowed with the norm $\|u\|_\infty = \sup_{x\in\mathcal{O}} |u(x)|$. We denote by $H^1(\mathcal{O})$ the space of functions u in $L^2(\mathcal{O})$ such that ∇u is also in $L^2(\mathcal{O})$ endowed with the natural norm $[\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}]^{1/2}$.

We denote by $C(0, t; L^p(\mathcal{O}))$ the space of functions depending on time and space variables that are continuous from $[0, t]$ into $L^p(\mathcal{O})$.

For physical quantities, the following general notations will be used.

q_e, m_e are the charge and the mass, respectively, of the electrons ($q_e > 0$),

m_0 is the mass of ions (when there is only one ion species),

Z is the ionization level of ions (assumed to be a positive constant), so the ion charge is Zq_e,

N_0, N_e are the ion and electron density, respectively (expressed as numbers by volume unit),

$\mathbf{U}$, $\mathbf{U}_e$ are the ion and electron velocities, respectively,

P_0, P_e denote the ion and electron pressures, respectively,

T_0, T_e denote the ion and electron temperatures, respectively (the Boltzmann constant is included in the expression of temperatures, so T_e, T_0 are expressed in energy units),

$\rho = m_0 N_0$ is the specific ion density of the plasma (called also the specific plasma density),

$\mathbf{J}$ is the total electric current, $\mathbf{J} = q_e(ZN_0\mathbf{U} - N_e\mathbf{U}_e)$,

$\mathcal{E}_0$, $\mathcal{E}_e$ are the ion and electron internal energies, respectively (expressed in energy units by volume unit),

$\varepsilon_0 = \mathcal{E}_0/\rho$, $\varepsilon_e = \mathcal{E}_e/\rho$ are the ion and electron internal specific energies, respectively (expressed in energy units by mass unit),

c is the speed of light,

ε^0, μ^0 are the vacuum dielectric and permeability constants, respectively ($\mu^0\varepsilon^0c^2 = 1$),

$\mathbf{E}$, $\mathbf{B}$ are the electric and the magnetic fields, respectively.[2]

We denote also by N_{ref} a characteristic value of the electron density and by T_{ref} a characteristic value of the electron temperature.

[2]We use always the rationalized MKSA units (except for the temperature, 1 K being equal to 1.38×10^{-23} J); so the electric and magnetic field are expressed in Volt/m and in Tesla and the Lorentz force reads as $q_e(\mathbf{E} + \mathbf{U} \times \mathbf{B})$.

We define also
$\lambda_D = (\varepsilon^0 T_{\text{ref}}/(q_e^2 N_{\text{ref}}))^{1/2}$, the Debye length,
$\omega_p = (q_e^2 N_{\text{ref}}/(\varepsilon^0 m_e))^{1/2}$, the electron plasma frequency,
$v_{T,e} = \sqrt{T_e/m_e}$ and $v_{T,0} = \sqrt{T_0/m_0}$, the electron and ion thermal velocity,
c_s, the ion sound speed of a plasma,
$v_{Al} = \frac{1}{m_0 N_{\text{ref}} \mu^0} |\mathbf{B}|^2$, the Alfven speed of a magnetized plasma,
$\mathbb{P}_B = \frac{1}{\mu^0} \left(\frac{\mathbb{I}}{2} |\mathbf{B}|^2 - \mathbf{BB} \right)$, the magnetic pressure tensor,
ν_{e0}, the Coulomb collision coefficient of between electrons and ions,
$\nu_{e0} N_0/m_e = \tau_e^{-1}$, the Coulomb collision frequency.

Note to the Reader. The proof of the mathematical results are given when they are not too complicated and I hope that they may be understood by the physicists who are not in the habit of the details of mathematical analysis. These proofs are postponed to the end of each subsection. Moreover, some simple technical calculations are written in the form [Indeed... and ... □].

Some useful formulas of tensor analysis and some usual results of functional analysis are recalled in the appendix.

1.3 Heuristics for Introducing Some Plasma Characteristic Quantities

In this book, we limit ourselves to a physical framework where electron and ion populations may be described with fluid quantities, which corresponds to the cases where the collisional mean free paths of the particles are small if compared to a characteristic length of observation. Therefore, for both populations, for electrons and ions, the distribution functions with respect to the microscopic velocity are assumed to be close to Maxwellian functions, that is, functions that are proportional to the Gauss function, $\exp -(m_e \frac{|\mathbf{v}|^2}{2T_e})$ (where $\mathbf{v}$ denotes the microscopic velocity). So, in this work, the electron population is at local thermodynamic equilibrium with a temperature T_e. Besides T_e, it is fully characterized by its density N_e, its macroscopic velocity $\mathbf{U}_e$, and its pressure P_e which depends on N_e and T_e through an *equation of state:* this equation of state links the pressure and the internal energy

$$\mathcal{E}_e = \frac{3}{2} N_e T_e$$

and is of the perfect gas law type, precisely:

$$P_e = N_e T_e = \frac{2}{3} \mathcal{E}_e.$$

(let us note that for some models, physicists introduce other equations of state). In this framework, the electron momentum balance reads in the following form

$$m_e[\partial_t(N_e\mathbf{U}_e) + \nabla.(N_e\mathbf{U}_e\mathbf{U}_e)] + \nabla P_e = \text{electromagnetic forces.}$$

For the ion population, we assume in the same way that the population is at thermodynamic equilibrium, and its distribution function is close to a Maxwellian function at temperature T_0; in the simple case without internal degree of freedom, it is proportional to the Gauss function, $\exp -(m_0\frac{|\mathbf{v}|^2}{2T_0})$; so, besides T_0, the ion population is fully characterized by its density N_0, its macroscopic velocity $\mathbf{U}$, and its pressure P_0; this pressure is linked to the internal energy through an equation of state of perfect gas law type

$$P_0 = N_0T_0, \qquad \mathcal{E}_0 = \frac{N_0T_0}{2/3}.$$

Notice that sometimes it is assumed that the ion molecules own internal degrees of freedom; then the distribution function at thermodynamic equilibrium is a modified Gauss function and one may take an equation of state of the following type: $P_0 = N_0T_0$ and $\mathcal{E}_0 = \frac{N_0T_0}{\gamma_0-1}$ (where constant γ_0 is such that $1 < \gamma_0 \leq 3$).

In this work, we focus on the case where the interactions with neutral particles are negligible (except in the last chapter where the neutral flow is prevailing). For the sake of presentation, we assume also that the ionization level is a constant Z; but, of course, for realistic plasmas with heavy ions, a crucial difficulty is to evaluate accurately this ionization level which depends strongly on the temperature and the density.

Before gathering some typical characteristic quantities classically used in plasma physics, we give by a heuristic way some classical features related to *quasi-neutrality approximation* also called *plasma approximation.* It presupposes another approximation, the so-called *massless-electron approximation* which is valid only at a large enough time scale (see Chap. 2, Sect. 2.1); it claims that the electron inertia may be neglected (indeed the electron mass is much smaller than the ion mass). So the electron momentum balance reduces to an equilibrium between the pressure force and the electromagnetic forces ($q_eN_e\mathbf{E}+q_eN_e\mathbf{U}_e\times\mathbf{B}$) and it reads in the form (using the definition of the current)

$$\nabla(N_eT_e) + q_eN_e\mathbf{E} = \mathbf{Q}, \quad \text{with } \mathbf{Q} = (\mathbf{J} - q_eZN_0\mathbf{U})\times\mathbf{B}. \tag{1.1}$$

We have not accounted here for the plasma resistivity (i.e., the collisions with the ions are neglected). Then plugging this relation (called generalized Ohm's law) into the fundamental *Gauss relation* $\varepsilon^0\nabla.\mathbf{E} = q_e(ZN_0 - N_e)$, we get the classical Poisson equation

$$-\lambda_D^2\frac{1}{T_{\text{ref}}}\nabla.\left(\frac{1}{N_e}\nabla(N_eT_e)\right) + \lambda_D^2\frac{1}{T_{\text{ref}}}\nabla.(\mathbf{Q}\frac{1}{N_e}) = \frac{ZN_0}{N_{\text{ref}}} - \frac{N_e}{N_{\text{ref}}}.$$

Here, it appears the so-called *Debye length* λ_D defined by

$$\lambda_D^2 = \frac{\varepsilon^0 T_{\text{ref}}}{q_e^2 N_{\text{ref}}}.$$

An important characterization of a plasma is the number of electrons in a *Debye sphere* (the radius of which is equal to the Debye length λ_D)

$$\lambda_D^3 N_{\text{ref}} = N_{\text{ref}}^{-1/2}(\varepsilon^0 T_{\text{ref}})^{3/2} q_e^{-3}$$

which is a nondimensional number, called a *plasma parameter*. In the sequel, we consider only phenomena where this plasma parameter is large. They are called weakly coupled plasmas. In such a plasma, a charged particle located at the centrum of a Debye sphere does not modify the equilibrium at the exterior of the sphere. The Debye length is like a screening length.

When the Debye length λ_D is small with respect to the characteristic length of the variation of the ion density, one sees heuristically that the solution N_e of this equation satisfies

$$N_e \simeq Z N_0$$

(see the asymptotic analysis below in Chap. 2, Sect. 2.2); it is the *quasi-neutrality approximation* (as it is explained by Zel'dovich [117]: since the electric field is "strong enough" in a hot plasma, "the electrons are rigidly coupled with the ions through the electric forces").

Now, denote by $v_{\text{th,i}} = (T_{\text{ref}}/m_0)^{1/2}$, the *ion thermal velocity* which is also the characteristic value of the ion velocity. In the same way, denote by $v_{\text{th,e}} = (T_{\text{ref}}/m_e)^{1/2}$ the *electron thermal velocity* (notice that in some physics textbooks, the thermal velocity is often defined as $(3T_{\text{ref}}/m_e)^{1/2}$ which is the quadratic mean velocity). We assume also that the electron temperature T_e is smaller than 10^8 K, so $v_{\text{th,e}}$ is smaller than 0.13 times the light velocity c and $v_{\text{th,i}}$ is smaller than 0.003 times c; thus a nonrelativistic framework is sufficient.

Let us now assume that the quasi-neutrality approximation holds: $N_e \simeq ZN_0$. By a classical way, the ion momentum balance equation accounting for the electromagnetic forces reads as

$$m_0 \left[\frac{\partial}{\partial t}(N_0 \mathbf{U}) + \nabla.(\mathbf{UU}N_0) \right] + \nabla P_0 = q_e Z N_0 \mathbf{E} + q_e Z N_0 \mathbf{U} \times \mathbf{B}$$

So combining with relation (1.1), it reads also

$$m_0 \left[\frac{\partial}{\partial t}(N_0 \mathbf{U}) + \nabla.(\mathbf{UU}N_0) \right] + \nabla P_0 + \nabla(Z N_0 T_e) = \mathbf{J} \times \mathbf{B}. \qquad (1.2)$$

On the left-hand side, we notice the gradient of electron pressure added to ion pressure. Now, we assume that the classical relation curl $\mathbf{B} = \mu^0 \mathbf{J}$ holds and that the magnetic field satisfies $\nabla.\mathbf{B} = 0$ and (which comes from a simplification of the Maxwell–Ampère equation). Applying the vectorial identity

$$\mathbf{A} \times \text{curl}\,(\mathbf{A}) = \nabla.\left(\frac{\mathbb{I}}{2}|\mathbf{A}|^2 - \mathbf{A}\mathbf{A}\right) + (\nabla.\mathbf{A})\mathbf{A} \tag{1.3}$$

(see the appendix at the end of the book) to the magnetic field, we see that the right-hand side of (1.2) reads as

$$\mathbf{J} \times \mathbf{B} = -\nabla.\mathbb{P}_B, \qquad \mathbb{P}_B = \frac{1}{\mu^0}\left(\frac{\mathbb{I}}{2}|\mathbf{B}|^2 - \mathbf{B}\mathbf{B}\right).$$

The tensor $\mathbb{P}_B$ is called the magnetic pressure tensor (see Chap. 2, Sect. 2.3 for a justification).

Thus, gathering (1.2) and the continuity equation $\frac{\partial}{\partial t} N_0 + \nabla.(\mathbf{U}N_0) = 0$, leads to the hydrodynamic system

$$\begin{aligned} N_0(\frac{\partial}{\partial t} + \mathbf{U}.\nabla)N_0^{-1} - \nabla.\mathbf{U} &= 0 \\ m_0 N_0(\frac{\partial}{\partial t} + \mathbf{U}.\nabla)\mathbf{U} + \nabla(P_0 + ZN_0T_e) + \frac{1}{\mu^0}\nabla.\left(\frac{\mathbb{I}}{2}|\mathbf{B}|^2 - \mathbf{B}\mathbf{B}\right) &= 0 \end{aligned}$$

We may address two limit cases, the first one corresponding to a magnetic pressure negligible with respect to the material pressure and the second one corresponding to a material pressure negligible with respect to the a magnetic pressure.

First, if the magnetic pressure is negligible, we check that there exists a characteristic sound speed (called *ion sound speed*) which is equal to $((P_0 + ZN_0T_e)/m_0N_0)^{1/2} = ((T_0 + ZT_e)/m_0)^{1/2}$ up to a multiplicative constant (in the order of 1); thus, this ion sound speed is in the order of the ion velocity $v_{\text{th,i}}$.

Now, if the material pressure is negligible with respect to the magnetic one, we may check that a characteristic wave velocity appears, the so-called Alfven speed v_{Al}. Indeed, consider the linearization of the previous momentum equation in the case of a homogeneous plasma with zero velocity

$$m_0 N_{\text{ref}} \frac{\partial}{\partial t}\mathbf{U} + \frac{1}{\mu^0}\nabla.\left(\frac{\mathbb{I}}{2}|\mathbf{B}|^2 - \mathbf{B}\mathbf{B}\right) = 0,$$

And the characteristic speed $v_{\rm Al}$ is defined by

$$v_{\rm Al}^2 = \frac{1}{m_0 N_{\rm ref} \mu^0} |\mathbf{B}|^2.$$

As a matter of fact, in one-dimensional geometry, it is a speed of a magnetic perturbation wave corresponding to the component of the magnetic field which is orthogonal to the direction of the wave propagation; see Chap. 2, Sect. 2.4 for a mathematical justification.

Using the characteristic values of the electron velocity and of the Debye length λ_D, we may define a time characteristic value for electron evolution as ω_p^{-1} where ω_p is given by

$$\omega_p = v_{\rm th,e}/\lambda_D = \sqrt{N_{\rm ref} q_e^2/(\varepsilon^0 m_e)}$$

This quantity is usually called the *electron plasma frequency*.

Another characteristic of a plasma is the Coulomb collision coefficient ν_{e0} which is related to the long-range Coulomb interaction between ions and electrons; it is proportional to the collision frequency τ_e^{-1} of electrons against ions

$$m_e \tau_e^{-1} = \nu_{e0} N_0,$$

and this frequency is expressed as (see, e.g., [38])

$$\tau_e^{-1} \approx \frac{4}{3} \frac{Z q_e^4}{(\varepsilon^0 m_e)^{1/2}} \frac{N_{\rm ref}}{T_e^{3/2}} \log \Lambda = \frac{4Z}{3} \log \Lambda \frac{N_{\rm ref}^{1/2} q_e}{(\varepsilon^0 m_e)^{1/2}} \cdot \frac{1}{\lambda_D^3 N_{\rm ref}}.$$

Here $\log\Lambda$. which is called the Coulomb logarithm, is in the order of some units; roughly speaking, it is proportional to the logarithm of the plasma parameter (it is a characteristic quantity of the plasma that depends on the density, the averaged electron and ion temperatures, and the ion species).

Notice that $\tau_e^{-1} \ll \omega_p$, since the plasma parameter $\lambda_D^3 N_{\rm ref}$ is large.

Let us give here some orders of magnitude for these quantities.

First, recall that the ratio between proton and electron mass is equal to 1837.

According to the type of plasmas, the electron and ion temperatures and the electron density may range over several orders of magnitude; but in this work, we are only concerned with plasmas that are *nondegenerated*, i.e., the electron temperature T_e is larger than the Fermi temperature defined by $T_{\rm Fermi} = h^2(3\pi^2 N_e)^{2/3}/(2m_e)$ [here h is Planck constant]. For a plasma with an electron density of $10^{27}\ {\rm m}^{-3}$, the Fermi temperature is equal to 3000 K; so for this density of plasma one only has to consider hot plasmas.

For instance, in the cavity of an ICF target, the ion and electron temperature are in the order of 10^6 or 10^7 K[3] and the density N_e of 10^{27} m^{-3}. At a typical value of the temperature of 10^6 K, the ion and electron thermal velocity is equal to 10^5 and 42×10^5 m/s. Then, in this case the Debye length is about 2.3×10^{-9} m and the plasma frequency about 1.8×10^{15} s^{-1}. Now, in the capsule of a ICF target the electron density may rise up to 10^{30} m^{-3} and the electron temperature is larger than 10^7 K.

In the sun corona, the temperature is in the order of 10^6 K and the density is in the order of 10^{14} m^{-3}.

In the ionosphere, they are in the order of 10^3 K and 10^{12} m^{-3}, respectively. Then we have also $T_e \geq T_{\text{Fermi}}$. The Debye is in the order of 2.3×10^{-3} m and the plasma frequency is in the order of 6×10^7 s^{-1}; moreover, the Alfvén speed is equal to 0.07 m/s for a magnetic field equal to 1 Gauss (10^5 T).

[3] Since the energy to move an electron against a potential of one *Volt* is equal to 1.60×10^{-19} J, the eV (electron-Volt) is also used as unit of temperature; it corresponds to 11,600 K.

Chapter 2
Quasi-Neutrality and Magneto-Hydrodynamics

Abstract In this chapter, we justify firstly the massless-electron approximation from the general ion–electron electrodynamic model. Secondly, we present the quasi-neutrality approximation, which is the heart of most of the fluid models presented in this book; this approximation is rigorously proved by an asymptotic analysis where a small parameter related to the Debye length goes to zero. We then present the two-temperature Euler system which is the basic model for quasi-neutral plasmas; in this framework we deal also with thermal conduction and radiative coupling. Lastly, we introduce the well-known model called electron magneto-hydrodynamics (MHD) which is the fundamental model for all magnetized plasmas. We give some details about the related boundary conditions.

Some crucial mathematical properties related to the "ideal part" of the previous models are displayed at the end of this chapter.

Keywords Debye length • Massless-electron approximation • Quasi-neutrality approximation • Poisson equation • Two-temperature Euler system • Electron magneto-hydrodynamics • Ideal magneto-hydrodynamics • Boundary conditions for MHD

2.1 Massless-Electron Approximation

Before dealing with the massless-electron approximation, we first state the general electrodynamic model related to the *ion–electron Euler system* coupled with *Maxwell equations* (cf. the first subsection below). From this model, we may address either modelling based on high-frequency electron waves, called *Langmuir waves,* which develop at the electron time scale ω_p^{-1} (in this framework the ions are assumed to be either at rest or mobile; see Chap. 4 for this topic) or modelling related to the evolution of the ion population at a time scale much larger than ω_p^{-1}.

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In the second framework, which corresponds to the topic of the second subsection and the other sections of this chapter, we address an observation time scale T_{obs} that is in the order of $L_{\mathrm{plasma}}/v_{\mathrm{th,i}}$ or $L_{\mathrm{plasma}}/v_{\mathrm{Al}}$ (where L_{plasma} is a characteristic length of variations of the plasma density) and the picture is the following

$$T_{\mathrm{obs}} \sim \frac{L_{\mathrm{plasma}}}{v_{\mathrm{th,i}}}, \frac{L_{\mathrm{plasma}}}{v_{\mathrm{Al}}} \gg \omega_p^{-1} = \frac{\lambda_D}{v_{\mathrm{th,e}}}$$

Since one has always $\lambda_D \leq L_{\mathrm{plasma}}$, one sees that $\frac{L_{\mathrm{plasma}}}{v_{\mathrm{th,i}}} \gg \frac{\lambda_D}{v_{\mathrm{th,e}}}$, so the previous ordering is very general and corresponds to the case where electron inertia is neglected with respect to the ion inertia. Thus, we can make the *massless-electron approximation*. It consists of assuming that, in the ion–electron Euler–Maxwell system, the electron mass m_e is negligible compared to ion mass m_0; so the electron momentum balance equation reduces to the so-called *generalized Ohm's law* which links the electric field to the electron density (we stress that there are various ways to state such a Ohm's law depending on the physical effects to be accounted for). Moreover, if there is no external electromagnetic source, no phenomena travel at the speed of light and a simplified version of the Maxwell equations corresponding to an infinite speed of light may be used. So we are led to the so-called *ion Euler–Poisson system* [system ($\mathcal{M}$) below] where the electric field reduces to an electrostatic one; it is valid even if the Debye length λ_D is not very small with respect to the characteristic length.

Now, in the first subsection, we recall the general electrodynamic model. It is a classical one (cf. [38, 112]) although it is almost never used for numerical simulations because the order of magnitude of the characteristic times of the subsystems are very different. It is worth focusing on it because it gives the conservation balance for the mass, momentum, and energy of the two populations, and it enables us to derive numerous fluid models with formal asymptotics.

2.1.1 The Ion–Electron Electrodynamic Model

Let us state the system corresponding to the classical conservation laws for the two populations of ions and electrons. First of all, the continuity equations for both species are

$$\frac{\partial N_0}{\partial t} + \nabla.(N_0 \mathbf{U}) = 0, \tag{2.1}$$

$$\frac{\partial N_e}{\partial t} + \nabla.(N_e \mathbf{U}_e) = 0. \tag{2.2}$$

Denote by ν_{e0} the Coulomb collision coefficient, it is related to collision frequency between electrons and ions which is equal to $\nu_{e0}N_0/m_e$. Then, the ion momentum balance equation reads as

$$m_0\frac{\partial}{\partial t}(N_0\mathbf{U})+m_0\nabla.(N_0\mathbf{U}\mathbf{U})+\nabla P_0 = q_e Z(N_0\mathbf{E}+N_0\mathbf{U}\times\mathbf{B})-\nu_{e0}N_e N_0(\mathbf{U}-\mathbf{U}_e). \tag{2.3}$$

On the right-hand side, the first term corresponds to the Lorentz force and the second one to the Coulomb collisions between the two species; at this level it reduces to a simple friction force proportional to the relative velocity $(\mathbf{U}-\mathbf{U}_e)$ (but we stress that there are different expressions of this friction force).

Moreover, the ion internal energy equation reads classically

$$\frac{\partial}{\partial t}\mathcal{E}_0 + \nabla.(\mathbf{U}\mathcal{E}_0) + P_0\nabla.\mathbf{U} = \Omega_{0e}, \tag{2.4}$$

where the term Ω_{0e} is related to the energy exchange between the ion and electron populations due to the Coulomb collisions. The relationship between this term and the corresponding term Ω_{e0} for the electrons will be given below [see relation (2.16)].

In the same way as above, the electron momentum balance equation reads as

$$m_e\frac{\partial}{\partial t}(N_e\mathbf{U}_e)+m_e\nabla.(N_e\mathbf{U}_e\mathbf{U}_e)+\nabla P_e = -q_e(N_e\mathbf{E}+N_e\mathbf{U}_e\times\mathbf{B})+\nu_{e0}N_e N_0(\mathbf{U}-\mathbf{U}_e), \tag{2.5}$$

and the electron internal energy equation

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathbf{U}_e\mathcal{E}_e) + P_e\nabla.\mathbf{U}_e + \nabla.\mathbf{q}_{\text{th,e}} = \Omega_{e0}. \tag{2.6}$$

Here $\mathbf{q}_{\text{th,e}}$ denotes the heat flux for the electron energy; its simplest expression is the so-called Spitzer flux which is proportional to $\nabla T_e^{7/2}$; the details about Spitzer flux are given below in Sect. 2.3.1, see also [111].

Of course, the previous energy equations may be stated in term of ion and electron total energy, i.e.,

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{E}_0 + m_0 N_0\frac{1}{2}|\mathbf{U}|^2) + \nabla.(\mathbf{U}(\mathcal{E}_0 + m_0 N_0\frac{1}{2}|\mathbf{U}|^2)) + \nabla.(\mathbf{U}P_0)\\ = \Omega_{0e} + q_e N_0\mathbf{U}.\mathbf{E} - \nu_{e0}N_e N_0\mathbf{U}.(\mathbf{U}-\mathbf{U}_e),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{E}_e+m_e N_e\frac{1}{2}|\mathbf{U}_e|^2)+\nabla.(\mathbf{U}_e(\mathcal{E}_e+m_e N_e\frac{1}{2}|\mathbf{U}_e|^2))+\nabla.(\mathbf{U}_e P_e)+\nabla\mathbf{q}_{\text{th,e}}\\ = \Omega_{e0} - q_e N_e\mathbf{U}_e.\mathbf{E} + \nu_{e0}N_e N_0\mathbf{U}_e.(\mathbf{U}-\mathbf{U}_e).\end{aligned} \tag{2.8}$$

We now deal with the Coupling with the Electromagnetic Fields.

First, using the electric current $\mathbf{J} = q_e Z N_0 \mathbf{U} - q_e N_e \mathbf{U}_e$, the electromagnetic fields $\mathbf{E}, \mathbf{B}$ satisfy the full Maxwell equations (respectively Maxwell–Ampère and Maxwell–Faraday relations)

$$\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} - \text{curl}\,\mathbf{B} + \mu^0 \mathbf{J} = 0, \tag{2.9}$$

$$\frac{\partial}{\partial t}\mathbf{B} + \text{curl}\,\mathbf{E} = 0. \tag{2.10}$$

It is also necessary to account for the electric Gauss relation

$$\varepsilon^0 \nabla.\mathbf{E} = q_e(Z N_0 - N_e). \tag{2.11}$$

and the magnetic Gauss relation which reads as $\nabla.\mathbf{B} = 0$.

According to the continuity equations (2.1) and (2.2), we see that the electric current satisfies the so-called *consistency relation* for electric charge

$$q_e \frac{\partial}{\partial t}(Z N_0 - N_e) + \nabla.\mathbf{J} = 0. \tag{2.12}$$

This consistency relation implies that relation (2.11) holds always if it holds at initial time (indeed, we have $\frac{\partial}{\partial t}\nabla.\mathbf{E} + \frac{1}{\varepsilon^0}\nabla.\mathbf{J} = 0$).

We now give the classical conversation relations related to this electrodynamic model.

(a) Momentum Balance Relation.

Adding (2.3) and (2.5), we get

$$\begin{aligned} &\frac{\partial}{\partial t}(m_e N_e \mathbf{U}_e + m_0 N_0 \mathbf{U}) + \nabla.\,(m_e N_e \mathbf{U}_e \mathbf{U}_e + m_0 N_0 \mathbf{U}\mathbf{U}) + \nabla(P_0 + P_e) \\ &\quad = \varepsilon^0 \mathbf{E}(\nabla.\mathbf{E}) + \mathbf{J} \times \mathbf{B} \end{aligned} \tag{2.13}$$

On the other hand, according to the Maxwell equations, one checks that the electromagnetic momentum $\mathbf{E} \times \mathbf{B}$ (equal to the Poynting vector, up a multiplicative constant) satisfies

$$\frac{1}{c^2 \mu^0}\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \frac{1}{\mu^0}(\text{curl}\,\mathbf{B}) \times \mathbf{B} - \varepsilon^0(\text{curl}\,\mathbf{E}) \times \mathbf{E} + \mathbf{J} \times \mathbf{B} = 0.$$

Now, using the tensor $\mathbb{P}_B = \frac{1}{\mu^0}(\frac{1}{2}\mathbb{I}|\mathbf{B}|^2 - \mathbf{B}\mathbf{B})$ and identity (1.3), recall that

$$-\frac{1}{\mu^0}\text{curl}\,\mathbf{B} \times \mathbf{B} = \nabla.\mathbb{P}_B$$

so introducing the tensor $\mathbb{S} = \mathbb{P}_B + \varepsilon^0 \left(\frac{\mathbb{I}}{2}|\mathbf{E}|^2 - \mathbf{E}\mathbf{E}\right)$, we see that

$$\frac{1}{c^2\mu^0}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \nabla.\mathbb{S} + q_e\mathbf{E}(ZN_0 - N_e) + \mathbf{J}\times\mathbf{B} = 0,$$

therefore, we get a classical result of the conversation of global momentum

$$\frac{\partial}{\partial t}\left[\frac{1}{c^2\mu^0}(\mathbf{E}\times\mathbf{B}) + m_e N_e \mathbf{U}_e + m_0 N_0 \mathbf{U}_0\right]$$
$$+ \nabla.\,(\mathbb{S} + m_e N_e \mathbf{U}_e\mathbf{U}_e + m_0 N_0 \mathbf{U}\mathbf{U}) + \nabla(P_0 + P_e) = 0.$$

(b) Energy Balance Relation.

Using the classical vector identity $\nabla.(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.\mathrm{curl}\,\mathbf{A} - \mathbf{A}.\mathrm{curl}\,\mathbf{B}$, we get for the electromagnetic energy,

$$\frac{\partial}{\partial t}\mathcal{E}_{\mathrm{electro}} + \nabla.(\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}) + \mathbf{J}.\mathbf{E} = 0, \quad \text{with } \mathcal{E}_{\mathrm{electro}} = \varepsilon^0\frac{|\mathbf{E}|^2}{2} + \frac{1}{\mu^0}\frac{|\mathbf{B}|^2}{2} \tag{2.14}$$

where $\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}$ is the Poynting vector. For the plasma energy balance, we first add (2.7) and (2.8), then on the right-hand side, we get the term

$$\mathbf{J}.\mathbf{E} + \Omega_{e0} + \Omega_{0e} - \nu_{e0}N_e N_0|\mathbf{U} - \mathbf{U}_e|^2$$

So we arrive at the following global energy balance relation

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + m_0 N_0\frac{1}{2}|\mathbf{U}|^2\right)$$
$$+ \left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}_e\cdot)\right)\times\left(\mathcal{E}_e + m_e N_e\frac{1}{2}|\mathbf{U}_e|^2\right) + \frac{\partial\mathcal{E}_{\mathrm{electro}}}{\partial t} \tag{2.15}$$
$$+\nabla.\left(\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}_e) + \nabla.\mathbf{q}_{\mathrm{th,e}} = \Omega_{e0} + \Omega_{0e} - \nu_{e0}N_e N_0|\mathbf{U} - \mathbf{U}_e|^2.$$

According to the energy conservation principle, the right-hand side must be zero; thus we must have

$$\Omega_{e0} = -\Omega_{0e} + \nu_{e0}N_e N_0|\mathbf{U} - \mathbf{U}_e|^2 \tag{2.16}$$

Moreover, the classical form of Ω_{0e} is the following

$$\Omega_{0e} = \omega_{0e}m_0 N_0(T_e - T_0) \tag{2.17}$$

where ω_{0e} is a positive quantity with an inverse ${\omega_{0e}}^{-1}$ known as the characteristic time of temperature relaxation. Notice that expression (2.17) is obtained by taking the two first moments of the Vlasov–Fokker–Planck equations for the ion and electron population evolution (see the companion book on "Kinetic Models", [118]), where ω_{0e} depends on the electron temperature and the electron density, e.g., it may be assumed to be proportional to

$$N_e Z^2 (\log \Lambda)/T_e^{3/2}$$

2.1.2 *The Ion Euler System with Massless-Electron Approximation*

Consider the evolution of a plasma with an observation time T_{obs} large enough compared with the inverse of the plasma frequency (but with a Debye length λ_D not necessarily very small with respect to L_{plasma}).

$$T_{\text{obs}} \sim \frac{L_{\text{plasma}}}{v_{\text{th,i}}} \gg \omega_p^{-1}, T_{\text{obs}} \gg \frac{L_{\text{plasma}}}{c},$$
$$L_{\text{plasma}} \sim \lambda_D \quad \text{or} \quad L_{\text{plasma}} \gg \lambda_D.$$

This situation is very frequent. For instance in the ionosphere problems, recall some orders of magnitude: the Debye is in the order of 2.3×10^{-3} m and the inverse of the plasma frequency is in the order of 1.5×10^{-8} s, and for electrical discharge concerning small Earth-orbiting satellites the characteristic time is larger than 10^{-5} s and the characteristic length is larger than 10^{-2} m.

In the same way, for cavity plasmas in Inertial Confinement Fusion, the Debye length is about 2.3×10^{-9} m and the inverse of plasma frequency about 6×10^{-16} s, but the characteristic time is larger than 10^{-12} s and the characteristic length is larger than 10^{-5} m ; indeed the variation of laser intensity is in the order of 10^{-11} s and the size of the target is in the order of a few millimeters.

So, in this framework electron inertia may be neglected with respect to the ion inertia, and the characteristic speed of the phenomena is much smaller than the electron thermal speed (and, of course, the speed of light).

Let us first stress that we can assume that the speed of light c is infinite in the previous general electrodynamic model by assuming that the displacement current $\frac{1}{c^2\mu^0}\frac{\partial \mathbf{E}}{\partial t}$ in (2.9) is negligible with respect to the electric current $\mathbf{J}$. In this framework, the Maxwell equations and Gauss relations reduce to

$$\text{curl}\,\mathbf{B} = \mu^0 \mathbf{J}, \tag{2.18}$$

and

$$\begin{cases} \text{(i)} & \text{curl}\,\mathbf{E} = -\partial_t \mathbf{B}, \\ \text{(ii)} & \nabla.\mathbf{B} = 0, \\ \text{(iii)} & \varepsilon^0 \nabla.\mathbf{E} = q_e(ZN_0 - N_e). \end{cases} \tag{2.19}$$

Here there is no formal asymptotic analysis with a small parameter and we do not take care of the relation $\varepsilon^0\mu^0c^2 = 1$, in this approximation ε^0 and μ^0 are solid values (and we simply set $c^{-1} \simeq 0$). It is worth noticing that once the densities N_0, N_e are known one can find $\mathbf{B}$ and $\mathbf{E}$ satisfying (2.18) and (2.19) only if the electric current satisfies

$$\nabla.\mathbf{J} = 0.$$

Thus, the approximation "infinite speed of light" is not appropriated with any general model for the electrons: it is made when the electron velocity is evaluated thanks to the electric current $\mathbf{J}$, in particular in the framework of the massless-electron approximation.

With the approximation of "infinite speed of light", from (2.18), the electromagnetic momentum balance reduces to

$$\nabla.\mathbb{P}_B + \mathbf{J}\times\mathbf{B} = 0,$$

[using identity (1.3)].

Now, according to the above vector identity for $\nabla.(\mathbf{E}\times\mathbf{B})$, we see that the magnetic energy balance reads as

$$\frac{1}{2\mu^0}\frac{\partial}{\partial t}|\mathbf{B}|^2 + \frac{1}{\mu^0}\nabla.(\mathbf{E}\times\mathbf{B}) + \mathbf{J}.\mathbf{E} = 0. \tag{2.20}$$

Thus, for the global energy (the sum of ion energy, electron energy, and magnetic energy), we get

$$\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_0+m_0N_0\frac{1}{2}|\mathbf{U}|^2)+\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}_e\cdot)\right)(\mathcal{E}_e)+\nabla.(P_0\mathbf{U}+P_e\mathbf{U}_e)+\nabla.\mathbf{q}_{\text{th,e}}$$
$$= -\mathbf{J}.\mathbf{E} = -\frac{1}{2\mu^0}\frac{\partial}{\partial t}|\mathbf{B}|^2 - \nabla.\left(\frac{1}{\mu^0}\mathbf{E}\times\mathbf{B}\right).$$

This conservative balance is the same kind as the one displayed in (2.14) but $\mathcal{E}_{\text{electro}}$ is replaced by the magnetic energy $\frac{1}{2\mu^0}|\mathbf{B}|^2$.

In the framework of the massless-electron approximation, which is the framework of the rest of the chapter, the electron density and velocity are not characterized by the mass and momentum conservation laws. The picture is the following.

On the one hand, the electron density is evaluated thanks to the Poisson equation which is obtained by inserting an expression of the generalized Ohm's law (see (2.21) below) into the electric Gauss relation. On the other hand, we claim that the electric current $\mathbf{J}$ is given by (2.18) and the electron velocity defined by $N_e\mathbf{U}_e = ZN_0\mathbf{U}-q_e^{-1}\mathbf{J}$. Moreover, the magnetic field is the solution of the evolution equation obtained by inserting the expression of $\mathbf{E}$ given by (2.21) into the Maxwell–Faraday equation (2.19(i)).

More precisely, the massless-electron approximation corresponds to the case where the electron mass m_e is assumed to be very small with respect to the ion mass.

Then a formal asymptotics (corresponding to a small parameter $m_e/m_0 \to 0$) leads to the following model. Since the electron and ion density are in the same order of magnitude and the electron velocity $\mathbf{U}_e$ is in the same order of magnitude as the ion one $\mathbf{U}$, we check that the electron momentum is negligible with respect to the ion momentum and that the electron kinetic energy $m_e N_e \frac{1}{2}|\mathbf{U}_e|^2$ is negligible with respect to the ion kinetic energy. For a rigorous result, see the analysis in the one-dimensional framework in [63] or [71] and also [2].

Therefore, from the electron momentum balance equation (2.5), we have firstly the relation

$$\begin{aligned}\nabla P_e + q_e N_e \mathbf{E} + q_e Z N_0 \mathbf{U} \times \mathbf{B} - \mathbf{J} \times \mathbf{B} &= \nu_{e0} N_e N_0 (\mathbf{U} - \mathbf{U}_e) \\ &= \nu_{e0} N_0 (\mathbf{U}(N_e - Z N_0) + \mathbf{J} q_e^{-1}).\end{aligned}$$

But it is usual in the ion momentum equation (2.3) and in the previous equation to replace the term $\nu_{e0} N_0 (\mathbf{U} - \mathbf{U}_e)$ with a closure of the form $q_e \chi \mu^0 \mathbf{J}$ where χ is a positive function depending on the ion density and temperature. Then we get the relation

$$\nabla P_e + q_e N_e \mathbf{E} + q_e Z N_0 \mathbf{U} \times \mathbf{B} - \mathbf{J} \times \mathbf{B} - q_e N_e \chi \mu^0 \mathbf{J} = 0, \tag{2.21}$$

which is called the *generalized Ohm's law* (see e.g., [108] or [38]). Coefficient $\chi\mu^0$ is called the specific electric resistivity of the plasma.

Secondly, combining relation (2.21) with (2.3), we get a new relation for momentum balance

$$\begin{aligned}m_0 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + m_0 \nabla.\,(N_0 \mathbf{U}\mathbf{U}) + \nabla(P_0 + P_e) &= \mathbf{J} \times \mathbf{B} + \varepsilon^0 \mathbf{E}(\nabla.\mathbf{E}) \\ &= -\nabla.\mathbb{P}_B + \varepsilon^0 \mathbf{E}(\nabla.\mathbf{E}). \end{aligned}\tag{2.22}$$

Thirdly, according to the previous remarks, the electron global energy $\left(\mathcal{E}_e + m_e N_e \frac{1}{2} |\mathbf{U}_e|^2\right)$ reduces to its internal energy $\mathcal{E}_e$. Thus, using (2.16), (2.8) reads now as follows

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e) + \nabla.(P_e \mathbf{U}) - \nabla.\left(\frac{5}{2} T_e \frac{\mathbf{J}}{q_e}\right) \\ &\quad + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} - q_e N_e \mathbf{U}_e.\mathbf{E} + q_e \mu^0 N_e \chi \mathbf{J}.\mathbf{U}.\end{aligned}$$

or equivalently

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e) + \nabla.(P_e \mathbf{U}) - \nabla.\left(\frac{5}{2} T_e \frac{\mathbf{J}}{q_e}\right) + \nabla.\mathbf{q}_{\text{th,e}} \\ &= -\Omega_{0e} + \mathbf{U}_e.\nabla P_e + q_e \mu^0 N_e \chi \mathbf{J}.(\mathbf{U} - \mathbf{U}_e)\end{aligned}\tag{2.23}$$

(indeed, according to (2.21), we have $q_e N_e \mathbf{U}_e.\mathbf{E} + \mathbf{U}_e.\nabla P_e - q_e \mu^0 N_e \chi \mathbf{J}.\mathbf{U}_e = 0$). Note that the term $\frac{5}{2} T_e$ in (2.23) stands for $(\mathcal{E}_e + P_e)/N_e$.

The ion internal energy $\mathcal{E}_0$ is given by (2.4).

The key point is to define the electron density N_e; this is done by gathering the electric Gauss relation and Ohm's law (2.21); see below.

Now, using the evaluation of $\mathbf{E}$ given by the generalized Ohm's law (2.21), the Maxwell–Faraday equation reads now as an evolution equation for the field

$$\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}\,(\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B}) = \mathrm{curl}\left(\frac{1}{q_e N_e}\nabla P_e - \frac{1}{\mu^0 q_e N_e}\mathrm{curl}\,\mathbf{B}\times\mathbf{B}\right) - \mathrm{curl}\,(\chi \mathrm{curl}\,\mathbf{B}) \tag{2.24}$$

It is called the diffusion magnetic equation. On the right-hand side, the last term is the usual resistive diffusion operator; moreover, the quadratic term, $\mathrm{curl}(\frac{1}{N_e}\mathrm{curl}\,\mathbf{B}\times\mathbf{B})$ is called the Hall's effect term (it is taken into account only if the electron density is small enough and if the magnetic field is strong enough).

Lastly, we have the Gauss relation

$$\nabla.\mathbf{B} = 0;$$

of course, if this relation holds at initial time, it holds at any time.

(*a*) *The Ion Euler–Poisson Model Without Resistivity.*

For the sake of this presentation, we state first the Euler–Poisson model by neglecting the resistive term in Ohm's law, i.e., $\chi = 0$; afterwards we will reintroduce the resistivity. So, in this framework the Ohm's law reduces to

$$q_e(\mathbf{E} + \frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B}) = -\frac{1}{N_e}\nabla(N_e T_e) + \frac{1}{N_e}\mathbf{J}\times\mathbf{B}, \tag{2.25}$$

and, according to the electric Gauss relation, we get

$$-\frac{\varepsilon^0}{q_e}\nabla.(\frac{1}{N_e}\nabla(N_e T_e)) - \frac{\varepsilon^0}{q_e}\nabla.(q_e\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B} - \frac{1}{N_e}\mathbf{J}\times\mathbf{B}) = q_e(ZN_0 - N_e).$$

Then, we are led to the following equation for defining the electron density N_e

$$-\frac{\lambda_D^2}{T_{\mathrm{ref}}}\left[\nabla.(T_e\nabla(\log N_e)) + \Delta T_e\right] - \frac{\lambda_D^2}{T_{\mathrm{ref}}}\nabla.\left(\frac{1}{N_e}(q_e ZN_0\mathbf{U}\times\mathbf{B} - \mathbf{J}\times\mathbf{B})\right) = \frac{ZN_0}{N_{\mathrm{ref}}} - \frac{N_e}{N_{\mathrm{ref}}}. \tag{2.26}$$

It is a nonlinear Poisson equation and it is crucial in plasma modelling when the quasi-neutrality assumption is not valid. Of course, this nonlinear elliptic equation needs to be supplemented by boundary conditions, for instance, Neumann conditions if the electrostatic field may be set to zero on the boundaries. But in simulations where one has to account for probes or electrodes, one must use Dirichlet conditions or more implicit boundary conditions (e.g., related to the electric current); this kind of problem is related to the so-called plasma sheath theory (see, e.g., [3, 109]).

Denoting $N_e/N_{\text{ref}} = e^{\Phi}$, the nonlinear Poisson equation reads also as

$$-\frac{\lambda_D^2}{T_{\text{ref}}}\nabla.(T_e\nabla\Phi)+e^{\Phi}-\frac{\lambda_D^2}{T_{\text{ref}}}\nabla.((q_e Z N_0 \mathbf{U}\times\mathbf{B}-\mathbf{J}\times\mathbf{B})\frac{e^{-\Phi}}{N_{\text{ref}}}) = \frac{ZN_0}{N_{\text{ref}}}+\frac{\lambda_D^2}{T_{\text{ref}}}\Delta T_e \tag{2.27}$$

Note that relation $N_e = N_{\text{ref}}e^{\Phi}$, where Φ is an electric potential divided by a temperature, is called the Maxwell–Boltzmann relation; it is often used in physics literature when the magnetic effects are not important (then (2.25) reduces $q_e\mathbf{E} = -T_e\nabla\Phi - \nabla T_e$). Notice that if the reference density N_{ref} is modified, we must also modify the potential Φ by adding a constant; as a matter of fact, in physical applications, these constants are fixed by the boundary conditions (see, e.g., [65]).

Summary. Assuming that $\mathbf{B}$ solves (2.24), the model consists of the following Euler–Poisson system. Recall that $\mathcal{E}_e = \frac{3}{2}N_eT_e$, $P_e = T_eN_e$ and $\mathbf{J} = \frac{1}{\mu^0}\text{curl}\,\mathbf{B}$.

$$(\mathcal{M})\quad\boxed{\begin{array}{ll}
\text{(i)} & \frac{\partial}{\partial t}N_0+\nabla.(N_0\mathbf{U})=0,\\
\text{(ii)} & m_o\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(N_0\mathbf{U})+\nabla(P_0+P_e)+\nabla.\mathbb{P}_B=q_e\mathbf{E}(ZN_0-N_e),\\
\text{(iii)} & \left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_0)+P_0\nabla.\mathbf{U}=\Omega_{0e},\\
\text{(iv)} & \left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e)+P_e\nabla.\mathbf{U}-\nabla.\left(\frac{5T_e}{2q_e}\mathbf{J}\right)+\nabla.\mathbf{q}_{\text{th,e}}\\
& \quad =-\Omega_{0e}+\frac{\nabla P_e}{N_e}.\left(\mathbf{U}(ZN_0-N_e)-\frac{\mathbf{J}}{q_e}\right),\\
\text{(v)} & -\frac{\lambda_D^2}{T_{\text{ref}}}\left[\nabla.(T_e\nabla(\log N_e))+\Delta T_e\right]-\frac{\lambda_D^2}{T_{\text{ref}}}\nabla.\left(\frac{1}{N_e}(q_eZN_0\mathbf{U}\times\mathbf{B}-\mathbf{J}\times\mathbf{B})\right)\\
& \quad =\frac{ZN_0-N_e}{N_{\text{ref}}}.
\end{array}}$$

with

$$\mathbf{E}+\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B} = -\frac{1}{q_e}\frac{\nabla P_e}{N_e}+\frac{1}{q_e}\frac{1}{N_e}\mathbf{J}\times\mathbf{B}. \tag{2.28}$$

Of course, this system needs to be supplemented with boundary conditions; for the nonlinear Poisson equation, the simplest one is the Neumann condition.

Equation (iv) reads also as (2.23). We can state once more an energy balance accounting only for ion and electron energy:

$$\left(\frac{\partial}{\partial t}+\nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_e+\mathcal{E}_0+\frac{m_0}{2}N_0\,|\mathbf{U}|^2\right)+\nabla.\left((P_e+P_0)\mathbf{U}+\frac{5T_e}{2q_e}\mathbf{J}\right)+\nabla.\mathbf{q}_{\text{th,e}}=\mathbf{J}.\mathbf{E}. \tag{2.29}$$

[indeed, if we multiply by $(ZN_0 - N_e)\mathbf{U}$ the relation defining the field $\mathbf{E}$, we get

$$-q_e(ZN_0 - N_e)\mathbf{U}.\mathbf{E} = -(ZN_0 - N_e)\mathbf{U}.\frac{\nabla P_e}{N_e} + (\frac{ZN_0}{N_e} - 1)\mathbf{U}.\mathbf{J} \times \mathbf{B} = \mathbf{J}.\mathbf{E}$$

moreover $ZN_0\frac{1}{N_e}\mathbf{U}.(\mathbf{J}\times\mathbf{B}) - \mathbf{J}.\nabla P_e\frac{1}{q_e N_e} = \mathbf{J}.\mathbf{E}$, so we have the previous balance by using equation (ii) multiplied by $\mathbf{U}$ and remembering that $\nabla.\mathbb{P}_B = -\mathbf{J}\times\mathbf{B}$. $\square$]

In the previous relation, term $\mathbf{J}.\mathbf{E}$ is the one which appears in the magnetic energy equation [see (2.20)].

Remark 1. To my knowledge, it is an open problem to make a rigorous asymptotic analysis leading to a massless-electron model in the multidimensional framework (i.e., to let the ratio of electron mass to ion mass m_e/m_0 converge to zero in the ion–electron Euler system); see [63] for a proof in a monodimensional case. $\square$

Notice that the nonlinear Poisson equation (v) may be replaced by a simpler one if necessary and in equation (iv), term $(ZN_0 - N_e)\mathbf{U}$ may be neglected with respect to $\mathbf{J}/q_e$.

This system looks quite complicated. As a matter of fact, it is an open problem to show its well-posedness from a mathematical point of view. Nevertheless, we have a result that is a clue in this direction: it shows that equation (v) above is well-posed and that its solution N_e is positive.

If we denote

$$\mathbf{Q} = \frac{1}{N_{\text{ref}}}(\mathbf{J}\times\mathbf{B} - q_e Z N_0 \mathbf{U}\times\mathbf{B}), \qquad g = \frac{1}{T_{\text{ref}}}\Delta T_e,$$

(2.26) reads as

$$-\lambda_D^2\nabla.(\frac{T_e}{T_{\text{ref}}}\nabla\log N_e) + \frac{N_e}{N_{\text{ref}}} + \lambda_D^2\frac{1}{T_{\text{ref}}}\nabla.(\mathbf{Q}\frac{N_{\text{ref}}}{N_e}) = \frac{ZN_0}{N_{\text{ref}}} + \lambda_D^2 g. \quad (2.30)$$

Let $\mathcal{O}$ be a bounded set with a smooth boundary and denote by $\mathbf{n}$ the outwards normal to the boundary $\partial\mathcal{O}$ of $\mathcal{O}$. Equation (2.30) needs to be supplemented with a boundary condition on $\partial\mathcal{O}$. For the sake of simplicity, we can take one of the following conditions on $\partial\mathcal{O}$

$$\text{(i)}\ \frac{\partial}{\partial\mathbf{n}}(N_e T_e) = 0, \qquad \text{or} \quad \text{(ii)}\ \frac{\partial}{\partial\mathbf{n}}N_e = 0. \quad (2.31)$$

The first condition corresponds to the case where the normal gradient of the electron pressure is zero, i.e., the electric field is tangential to the boundary (if there is no magnetic effect); the second condition is a simplification of the first one.

Assume that $N_0, g, \mathbf{Q}$ and $\nabla.\mathbf{Q}$ belong to $L^\infty(\mathcal{O})$, that N_0 is strictly positive, and that T_e is a strictly positive bounded function. So we have:

Proposition 1. *Assume that* $\inf_x(\frac{Z}{N_{ref}}N_0 + \lambda_D^2 g) > 0$. *For (2.30) supplemented with one of the conditions (2.31) on* $\partial\mathcal{O}$, *there is a unique solution* N_e *in the cone* *of functions of $H^1(\mathcal{O})$ which are strictly positive and bounded, if $\lambda_D^2 \, \|\nabla.\mathbf{Q}\|_{L^\infty}$ is small enough. Moreover, we have*

$$\frac{1}{N_{ref}} N_e(x) \geq \frac{1}{2} \inf_x \left(\frac{Z N_0}{N_{ref}} + \lambda_D^2 g \right)$$

Of course, we have also $T_e \nabla \log(N_e T_e) \in L^2(\mathcal{O})$.

We may also address this problem in the case where the spatial domain is the whole space $\mathbf{R}^3$ (or $\mathbf{R}^2$) and the result is the same, provided that N_0 is in $L^2(\mathcal{O})$.

It is worth noting that (2.30) is also sometimes called the Poincaré equation (cf. [96]), at least in its simple form $-\lambda_D^2 \Delta(\log N_e) + N_e = N_0$, and it arises in many physical areas.

(b) A Simplified Model without Magnetic Effects

In the case where the magnetic effects are neglected, we may address a simplified model based on the barotropic approximation for the ion pressure: it is given by a closure with respect to N_0, for instance, P_0 is defined by $\mathcal{P}_0(N) = \mu_p N^\gamma$ (where μ_p and $\gamma \geq 1$). In this model, there is only one energy equation (the coupling term Ω_{0e} disappears) and the four unknowns N_0, $\mathbf{U}$, T_e and N_e satisfy the system.

$$\frac{\partial}{\partial t} N_0 + \nabla.(N_0 \mathbf{U}) = 0, \tag{2.32}$$

$$m_0 \left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot) \right) (N_0 \mathbf{U}) + \nabla \mathcal{P}_0(N_0) = -Z N_0 \frac{1}{N_e} \nabla(N_e T_e), \tag{2.33}$$

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot) \right) \left(\frac{3}{2} N_e T_e \right) + N_e T_e \nabla.\mathbf{U} = (Z N_0 - N_e) \frac{1}{N_e} \mathbf{U}.\nabla(N_e T_e), \tag{2.34}$$

$$-\frac{\lambda_D^2}{T_{\mathrm{ref}}} \nabla.(\frac{1}{N_e} \nabla(N_e T_e)) = \frac{Z N_0 - N_e}{N_{\mathrm{ref}}}. \tag{2.35}$$

In the momentum equation, the right-hand side may also be stated in a conservative form, which reads as

$$-\nabla(N_e T_e) + \varepsilon^0 (\nabla.(\mathbf{E}\mathbf{E}) - \frac{1}{2} \nabla.|\mathbf{E}|^2)$$

with $q_e \mathbf{E} = \dfrac{1}{N_e} \nabla(N_e T_e)$.

As usual, we need to supplement this system with boundary conditions on $\partial\mathcal{O}$. For the three first equations, it is sufficient to state two conditions (for this topic, see the last section of this chapter); so, for the sake of simplicity, we can state

$$\frac{\partial}{\partial \mathbf{n}} T_e = 0, \qquad \mathbf{n}.\mathbf{U} = 0.$$

For elliptic equation (2.35), we consider condition (2.31 (ii)) (which is now identical to (2.31 (i))

We also have to supplement this system with initial conditions, that is to say $N_0(0) = N_0^{\text{ini}}$, $\mathbf{U}(0) = \mathbf{U}^{\text{ini}}$, $T_e(0) = T_e^{\text{ini}}$, where these initial values satisfy the boundary conditions and N_0^{ini}, T_e^{ini} are strictly positive.

Then, we may define the ion internal energy by $\mathcal{E}_0(N) = P_0(N)/(\gamma - 1)$ if $\gamma \neq 1$ and $\mathcal{E}_0(N) = \mu_p(N \log(N) + 1)$ otherwise, and we have the following result.

Proposition 2. *For system (2.32)–(2.35), supplemented with the previous boundary conditions, we have the energy balance relation*

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\frac{3}{2}N_eT_e + \mathcal{E}_0(N_0) + \frac{m_0}{2}N_0|\mathbf{U}|^2\right) + \nabla.(\mathbf{U}(P_e + \mathcal{P}_0(N_0))) = 0 \tag{2.36}$$

Moreover, as soon as the temperature T_e remains strictly positive and bounded, the electron density N_e is a positive bounded function such that $\nabla \log(N_eT_e) \in L^2(O)$.

Using the boundary conditions, property (2.36) implies that for all times t, we get

$$\int_{\mathcal{O}} \left(\frac{3}{2}N_eT_e + \mathcal{E}_0(N_0) + \frac{m_0}{2}N_0|\mathbf{U}|^2\right) dx = C_0$$

that is to say there is a good balance for the global energy which is $\frac{3}{2}N_eT_e + \mathcal{E}_0(N_0) + \frac{m_0}{2}N_0|\mathbf{U}|^2$.

(*c*) *The Ion Euler–Poisson Model with Resistivity.*

In order to account for the resistive term in the generalized Ohm's law (2.21), we need to modify the previous system. Due to (2.21), the following expression of $\mathbf{E}$

$$\mathbf{E} + \frac{ZN_0}{N_e}\mathbf{U} \times \mathbf{B} = -\frac{1}{q_e}\frac{\nabla P_e}{N_e} + \frac{1}{q_e}\frac{1}{N_e}\mathbf{J} \times \mathbf{B} + \chi\mu^0\mathbf{J}$$

may be plugged in Gauss relation, so we get the modified nonlinear Poisson

$$-\varepsilon^0\nabla.(\frac{1}{N_e}\nabla(N_eT_e)) - \varepsilon^0\nabla.(q_e\frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B} - \frac{1}{N_e}\mathbf{J} \times \mathbf{B}) + \varepsilon^0\nabla.(\mu^0 q_e\chi\mathbf{J}) = q_e^2(ZN_0 - N_e).$$

Or with the same notations as above

$$-\lambda_D^2\nabla.(\frac{T_e}{T_{\text{ref}}}\nabla \log N_e) + \frac{N_e}{N_{\text{ref}}} + \lambda_D^2\frac{1}{T_{\text{ref}}}\nabla.(\mathbf{Q}\frac{N_{\text{ref}}}{N_e}) = \frac{ZN_0}{N_{\text{ref}}} + \lambda_D^2(g - \mu^0 q_e\mathbf{J}.\nabla\chi).$$

From a mathematical point of view this equation is the same as in the case $\chi = 0$ (only the right-hand side is changed).

The model for the evolution of N_0, $\mathbf{U}$, $\mathcal{E}_0$ is the same as in system ($\mathcal{M}$); the only difference is that there is a resistive term in the evolution equation (v) for $\mathcal{E}_e$

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)(\mathcal{E}_e) + P_e\nabla.\mathbf{U} - \nabla.\left(\frac{5T_e}{2q_e}\mathbf{J}\right) + \nabla.\mathbf{q}_{\text{th,e}} \tag{2.37}$$

$$= -\Omega_{0e} + \frac{\nabla P_e}{N_e}.\left(\mathbf{U}(ZN_0 - N_e) - \frac{\mathbf{J}}{q_e}\right) + q_e\mu^0 N_e\chi\mathbf{J}.(\mathbf{U} - \mathbf{U}_e).$$

We can easily check that the global energy balance (2.29) is still true. It is worth noticing the supplementary term $N_e \chi \mathbf{J}.(\mathbf{U} - \mathbf{U}_e)$, which is called the Joule effect term. Indeed, we will see below that in the quasi-neutral case, $\mathbf{J}$ and $N_e(\mathbf{U} - \mathbf{U}_e)$ will be equal up to the constant q_e; then this term equal to $\chi|\mathbf{J}|^2$ will be positive. It corresponds to the heating of the plasma occurring due to the electric current and the energy exchange due to the friction between electrons and ions.

2.2 Quasi-Neutrality Approximation

As above, we make the massless-electron approximation, but we assume moreover that the Debye length λ_D is very small if compared to the characteristic length L_{plasma}. In this framework corresponding to the so-called *plasma approximation,* the picture is now

$$\lambda_D \ll L_{\text{plasma}}, \qquad T_{\text{obs}} \sim \frac{L_{\text{plasma}}}{v_{\text{th,i}}} \gg \omega_p^{-1}, \qquad T_{\text{obs}} \gg \frac{L_{\text{plasma}}}{c}$$

Then the electron density is close to the ion density $N_e \simeq ZN_0$ (this is the quasi-neutrality approximation), but as claimed previously the electric field cannot be set to zero.

Our aim is to explain how this quasi-neutrality may be justified from a mathematical point of view. This analysis will be performed from the ion Euler–Poisson system in two different physical frameworks

- The first case corresponds to the case where the magnetic phenomena are negligible and a simple Ohm's law suffices; see (2.38)
- The second case corresponds to the general case, where the magnetic field needs to be accounted for and Ohm's law reads as (2.25)

In both cases, starting from the Gauss relation

$$\lambda_D^2 \frac{q_e}{T_{\text{ref}}} \nabla.\mathbf{E} = \frac{1}{N_{\text{ref}}}(ZN_0 - N_e)$$

where the Debye length comes into sight, we derive a nonlinear Poisson equation of the form (2.26).

2.2.1 *Asymptotic Analysis in the Nonmagnetized Case*

We introduce a dimensionless spatial variable related to L_{Plasma} and we set $\lambda = \lambda_D / L_{\text{Plasma}}$ which is small with respect to 1.

Here, we assume that instead of (2.25) we simply have

$$q_e \mathbf{E} = -\frac{1}{N_e} \nabla(N_e T_e) \tag{2.38}$$

then the potential $\log(N_e/N_{\text{ref}})$ is solution to a nonlinear Poisson equation. Set

$$\Phi_\lambda = \log(N_e/N_{\text{ref}})$$

in order to make clear the dependency with respect to the parameter λ, so Poisson equation [which is a simplified form of (2.26)] reads as

$$-\lambda^2\nabla.\left(\frac{T_e}{T_{\text{ref}}}\nabla\Phi_\lambda\right) = \frac{ZN_0}{N_{\text{ref}}} - e^{\Phi_\lambda} + \lambda^2 g \tag{2.39}$$

recalling that $g = \frac{1}{T_{\text{ref}}}\Delta T_e$. In the same way, in the sequel, denote the electric field by $\mathbf{E}_\lambda$ instead of $\mathbf{E}$.

The asymptotic analysis of this equation when the small parameter tends to 0 was addressed in [23] about 20 years ago in the case of a constant temperature. A wide range of literature is devoted to this subject; see, e.g., [110, 115] and the references therein. Notice also that it is interesting to address such an asymptotic analysis with a vanishing Debye length in order to design numerical schemes for Euler–Poisson models in the case where quasi-neutral and non–quasi-neutral regions coexist in the same simulation domain; see, e.g., [35, 36, 39].

Here for the sake of completeness we state a generalization of the initial result of [23] in the case of a nonconstant temperature. We first assume that the spatial domain $\mathcal{O}$ is a smooth bounded open set; of course, (2.39) needs to be supplemented with a boundary condition, and as previously we set

$$\frac{\partial\Phi_\lambda}{\partial\mathbf{n}} = 0; \tag{2.40}$$

This corresponds to the case of an insulating material. It also possible to address a boundary condition of the type Φ_λ equal to given data (see below). Now, to state a rigorous mathematical result, we need to make some technical assumptions. First, we assume

$$\inf_x T_e > 0,\ \inf_x N_0 > 0,\ T_e \in L^\infty(\mathcal{O}),\ \ N_0 \in H^1(\mathcal{O}) \cap L^\infty(\mathcal{O}),\ \ \Delta T_e \in L^\infty(\mathcal{O}). \tag{2.41}$$

Moreover, for the sake of simplicity, we impose

$$\frac{\partial}{\partial\mathbf{n}}N_0 = 0, \qquad \text{on } \partial\mathcal{O}.$$

Proposition 3. *Assume that assumptions (2.41) hold. When λ goes to 0, the unique solution Φ_λ of nonlinear equation (2.39) with the boundary condition (2.40) satisfies*[1]

$$N_{ref}\exp(\Phi_\lambda) \to ZN_0 \qquad \textit{in } L^2(\mathcal{O}),$$
$$\nabla\Phi_\lambda \to \nabla\log(ZN_0) \qquad \textit{in } L^2(\mathcal{O}) \textit{ weakly.}$$

[1] A sequence u_n converges weakly to u, if $\int u_n v \to \int uv$ for any v in L^2.

The proof is given below. Notice that there are also results with weaker assumptions on the regularity of N_0 (see the proof). This proposition shows first that the quasi-neutrality holds

$$N_e \simeq ZN_0, \qquad P_e \simeq ZN_0T_e.$$

Moreover, this implies that the electric field $\mathbf{E}_\lambda$ satisfies

$$q_e\mathbf{E}_\lambda = -T_e\nabla\Phi_\lambda - \nabla T_e \to -\frac{1}{N_0}\nabla(N_0T_e) \qquad \text{in } L^2(\mathcal{O}) \text{ weakly,}$$

that is to say, it may be approximated in the following way

$$q_e\mathbf{E}_\lambda \simeq -\frac{1}{N_0}\nabla(N_0T_e). \tag{2.42}$$

This last relation for the electric field is a very usual one. In the case where the electron temperature is almost constant, this relation reads $q_e\mathbf{E}_\lambda \simeq -T_e\nabla(\log N_0)$ as it appears in the beginning of the handbook [27].

Other boundary conditions of Dirichlet type may be addressed; for instance. for the modelling of a plasma surrounded by a conducting material. As a matter of fact, if a given electric potential may be applied on two different parts of the boundary that correspond to conducting material, we can make the following modelling: assume that Γ_1 and Γ_2 are two smooth parts of the boundary $\partial\mathcal{O}$ that are disconnected, such that the potential Φ is given by a constant: B_1 on Γ_1 and B_2 on Γ_2.

Then we have the following result.

Proposition 4. *With the same assumptions as above, there exists a unique solution Φ_λ in $H^1(\mathcal{O})$ to (2.39) with the boundary conditions*

$$\Phi_\lambda = B_1 \quad \text{on } \Gamma_1; \quad \Phi_\lambda = B_2 \quad \text{on } \Gamma_2; \quad \frac{\partial}{\partial\mathbf{n}}\Phi_\lambda = 0 \quad \text{on } \partial\mathcal{O}\setminus(\Gamma_1\cup\Gamma_2) \tag{2.43}$$

Moreover, when λ goes to 0,we have

$$N_{ref}e^{\Phi_\lambda} \to ZN_0 \qquad \text{in } L^2(\mathcal{O}), \tag{2.44}$$

$$\nabla\Phi_\lambda \to \nabla\log(ZN_0) \qquad \text{in } L^2(\mathcal{O}) \text{ weakly.} \tag{2.45}$$

Generally, there exists boundary layers on the parts Γ_1 and Γ_2, since $N_{\text{ref}}\exp(B_q)$ is not equal to ZN_0.

See also [3,64] for other boundary condition problems related to the electrostatic sheath phenomena. We stress that it may lead to technical difficulties related to boundary layers where the quasi-neutrality fails.

Of course, we may address the case where the spatial domain $\mathcal{O}$ is the full space $\mathcal{O} = \mathbf{R}^3$ (or $\mathbf{R}^2$) and the asymptotic result may be proven in the same way.

Remark 2. Instead of (2.39), we may state the Poisson equation in the form

$$-\lambda^2\nabla.\left(\frac{T_e}{T_{\mathrm{ref}}}\nabla(\log P_e)\right) = \frac{ZN_0}{N_{\mathrm{ref}}} - \frac{P_e}{N_{\mathrm{ref}}T_e}, \qquad \frac{\partial P_e}{\partial \mathbf{n}}|_{\partial\mathcal{O}} = 0,$$

which is interesting if one prescribes a boundary condition on $\partial\mathcal{O}$ of the type $\frac{\partial P_e}{\partial \mathbf{n}} = 0$. Then, denoting $P_{e,\lambda}$ instead of P_e, Proposition 3 reads as follows: when λ goes to 0, the solution $P_{e,\lambda}$ of this equation satisfies

$$P_{e,\lambda} \to ZN_0T_e \qquad \text{in } L^2(\mathcal{O}) \text{ strongly;}$$
$$\nabla(\log P_{e,\lambda}) \to \nabla(\log N_0T_e) \qquad \text{in } L^2(\mathcal{O}) \text{ weakly.}$$ □

Remark 3. In the quasi-neutral approximation, determination of the electric field is a local problem in the interior of the domain, as it appears in (2.42); evaluation of $\mathbf{E}$ is performed with the local fluid variables only. But near the boundary $\partial\mathcal{O}$,this evaluation is more complex due to the boundary layers. □

2.2.2 *Asymptotic Analysis in the Magnetized Case*

We now address the more realistic case corresponding to a magnetized plasma (called also *current-carrying plasma*). We assume that the evolution of the field $\mathbf{B}$ is known (recall that the electron velocity is given by $\mathbf{U}_e = \frac{1}{N_e}N_0\mathbf{U} - \frac{1}{N_eq_e}\mathbf{J}$).

We consider the generalized Ohm's law (2.25), i.e.,

$$q_e(\mathbf{E} + \frac{ZN_0}{N_e}\mathbf{U}\times\mathbf{B}) = -\frac{1}{N_e}\nabla(N_eT_e) + \frac{1}{N_e}\mathbf{J}\times\mathbf{B} \tag{2.46}$$

(if we want to account for the resistivity, it suffices to add to the right-hand side a term of the type $\chi\mu^0\mathbf{J}$). The electron density $N_e = N_{\mathrm{ref}}e^{\Phi_\lambda}$ is given by the solution of the nonlinear Poisson equation (2.27), which reads as

$$-\lambda^2\nabla.(\frac{T_e}{T_{\mathrm{ref}}}\nabla\Phi_\lambda) + e^{\Phi_\lambda} + \lambda^2\frac{1}{T_{\mathrm{ref}}}(\nabla.\mathbf{Q}e^{-\Phi_\lambda}) = \frac{ZN_0}{N_{\mathrm{ref}}} + \lambda^2 g \tag{2.47}$$

Heuristically, if λ vanishes, we see that e^{Φ_λ} is very close to the function $\frac{ZN_0}{N_{\mathrm{ref}}}$, i.e., the quasi-neutrality holds, and we may use the following approximations

$$N_e = N_{\mathrm{ref}}e^{\Phi_\lambda} \simeq ZN_0, \qquad P_e \simeq ZN_0T_e, \tag{2.48}$$

$$\frac{1}{N_e}\nabla N_e \simeq \frac{1}{N_0}\nabla N_0 \tag{2.49}$$

So, according to (2.46), we get

$$q_e(\mathbf{E}+\mathbf{U}\times\mathbf{B})\simeq -\frac{1}{N_0}\nabla(N_0T_e)+\frac{\mathbf{J}\times\mathbf{B}}{ZN_0}$$

Let us justify these approximations by an asymptotic analysis. So, we address equation (2.47) as above in a bounded open set $\mathcal{O}$, with boundary conditions (2.40)

Proposition 5. *Assume that (2.41) holds and that* $g, \mathbf{Q}$ *and* $\nabla.\mathbf{Q}$ *are bounded in* L^∞*. Then, when* λ *goes to* 0*, the unique solution* Φ_λ *to (2.47) with boundary condition (2.40) satisfies*

$$N_{ref}e^{\Phi_\lambda}\to ZN_0 \qquad \text{in } L^2(\mathcal{O}) \tag{2.50}$$

$$\nabla\Phi_\lambda\to\nabla\log N_0, \qquad \text{in } L^2(\mathcal{O}) \text{ weakly} \tag{2.51}$$

Thus, denoting the electric field by $\mathbf{E}_\lambda$ instead of $\mathbf{E}$, it satisfies

$$q_e\mathbf{E}_\lambda = -T_e\nabla\Phi_\lambda-\nabla T_e+\mathbf{Q}e^{-\Phi_\lambda}\to -\frac{1}{N_0}\nabla(N_0T_e)+\mathbf{Q}\frac{N_{\mathrm{ref}}}{ZN_0} \quad \text{in } L^2(\mathcal{O}) \text{ weakly}$$

To justify the statement of quasi-neutral models, we must now address terms where the electric field appears in the general massless-electron model above. In particular, we need to deal with the term $\varepsilon^0\mathbf{E}_\lambda(\nabla.\mathbf{E}_\lambda)$ in the momentum equation (2.22). With the previous notations, we have $q_e\mathbf{E}_\lambda = -T_e\nabla\Phi_\lambda-\nabla T_e+\mathbf{Q}e^{-\Phi_\lambda}$ (recall that $\mathbf{Q}=0$ in the nonmagnetized case) and

$$\frac{\varepsilon^0}{N_{\mathrm{ref}}}\mathbf{E}_\lambda(\nabla.\mathbf{E}_\lambda) = \left(T_e\nabla\Phi_\lambda+\nabla T_e-\mathbf{Q}e^{-\Phi_\lambda}\right)\frac{\lambda^2}{T_{\mathrm{ref}}}\left(\nabla.(\frac{T_e}{T_{\mathrm{ref}}}\nabla\Phi_\lambda)-\nabla.(\mathbf{Q}e^{-\Phi_\lambda})+g\right)$$

We now claim

Proposition 6. *Assume that the same assumptions as the one of Proposition 5 hold, then we have if* $\lambda\to 0$*,*

$$\frac{\varepsilon^0}{N_{\mathrm{ref}}}\mathbf{E}_\lambda(\nabla.\mathbf{E}_\lambda)\to 0 \qquad \text{in } L^1(\mathcal{O}).$$

[**Indeed**, according to Proposition 5, we get

$$\lambda^2(\nabla.(\mathbf{Q}e^{-\Phi_\lambda})-\nabla.(\frac{T_e}{T_{\mathrm{ref}}}\nabla\Phi_\lambda)) = \frac{ZN_0}{N_{\mathrm{ref}}}-e^{\Phi_\lambda}+\lambda^2 g\to 0 \qquad \text{in } L^2(\mathcal{O});$$

moreover, we know that $\nabla\Phi_\lambda$ is bounded in $L^2(\mathcal{O})$. Therefore, $\lambda^2\nabla\Phi_\lambda(\nabla.(\mathbf{Q}e^{-\Phi_\lambda}) -\nabla.(\frac{T_e}{T_{\mathrm{ref}}}\nabla\Phi_\lambda))\to 0$ in L^1.□]

Summary (About quasi-neutrality). The previous proposition means that either the plasma is resistive or not, and the term $\varepsilon^0\mathbf{E}(\nabla.\mathbf{E})$ may be neglected; then (2.22) becomes simply

$$m_0 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + m_0 \nabla.(N_0 \mathbf{U}\mathbf{U}) + \nabla P_0 + \nabla P_e = \mathbf{J} \times \mathbf{B}. \qquad (2.52)$$
$$= -\nabla.\mathbb{P}_B$$

Thus, the basic principles of quasi-neutral approximation may be summarized as follows.

- *One prescribes* $N_e = ZN_0$ and $P_e = ZN_0 T_e$.
- *For ion modelling, one has a classical fluid dynamic system, e.g., (2.52) for the momentum equation.*
- *The electrostatic field* $\mathbf{E}$ *is given by Ohm's law, which may read in the form:*

$$q_e Z N_0 (\mathbf{E} + \mathbf{U} \times \mathbf{B}) - \mathbf{J} \times \mathbf{B} + \nabla P_e = 0, \qquad (2.53)$$

or, accounting for the resistive effect,

$$q_e Z N_0 (\mathbf{E} + \mathbf{U} \times \mathbf{B}) - \mathbf{J} \times \mathbf{B} + \nabla P_e = q_e Z N_0 \chi \mu^0 \mathbf{J}. \qquad (2.54)$$

In these relations, the field $(\mathbf{E} + \mathbf{U} \times \mathbf{B})$ comes into sight; it is a natural quantity in all the MHD models. Moreover, it is worth noticing that the electron velocity never appears explicitly in these models (it is always evaluated thanks to $\mathbf{U}$ and $\mathbf{J}$) and that, generally, one has to account for the evolution of the electron temperature.

2.2.3 Proofs of the Propositions of Sects. 2.1 and 2.2

Proof of Proposition 1. Let us set $\lambda = \lambda_D$ which is fixed. For the sake of conciseness, we set $T_{\text{ref}} = 1$ and $T_e = T$. Let us address equation (2.30) where N_e/N_{ref} is denoted by u ; moreover $f = \frac{ZN_0}{N_{\text{ref}}} + \lambda^2 g$ and $r = -\nabla.\mathbf{Q}$ are assumed to be bounded. It reads

$$-\lambda^2 \nabla. \left(\frac{T}{u} \nabla u \right) + \lambda^2 \frac{1}{u^2} \mathbf{Q}.\nabla u + u = f + \lambda^2 \frac{r}{u}. \qquad (2.55)$$

It is supplemented with Neumann boundary condition $\frac{\partial}{\partial \mathbf{n}} u = 0$; with the other boundary condition $\frac{\partial}{\partial \mathbf{n}}(Tu) = 0$, the proof is similar. There exist two constants α and f_∞ independent of λ such that

$$0 < 2\alpha = \inf_x f, \qquad f(x) \le f_\infty$$

Denote also $T_{\text{m}} = \inf T$, $q_\infty = \sup_{k=1,3} \|\mathbf{Q}_k\|_\infty$ and $r_\infty = \sup \|r\|_\infty$.

We will prove the existence and the uniqueness of a solution in the convex subset $K = \{v \in L^2(\mathcal{O}) / \alpha \le \inf v \le \beta\}$ of the space $L^2(\mathcal{O})$, where β is defined below.

Existence of a Solution (Based on the Fixed-Point Schauder Theorem[2]). For all functions v in K, let us define the operator S from K into $L^2(\mathcal{O})$ defined by $S(v) = u$, where u is the unique solution in $H^1(\mathcal{O})$ to the linear equation

$$-\lambda^2 \nabla . \left(\frac{T}{v}\nabla u\right) + \lambda^2 \frac{1}{v^2}\mathbf{Q}.\nabla u + u = f + \lambda^2 \frac{r}{v} \tag{2.56}$$

with boundary condition $\frac{\partial}{\partial \mathbf{n}} u = 0$. Since $|\frac{r}{v}| \leq \frac{r_\infty}{\alpha}$, according to the maximum principle,[3] we have

$$2\alpha - \lambda^2 \frac{r_\infty}{\alpha} \leq u \leq f_\infty + \lambda^2 \frac{r_\infty}{\alpha}.$$

We assume now that λ is smaller than $\lambda_0 = \alpha/\sqrt{r_\infty}$; therefore, we get

$$\alpha \leq u \leq \beta$$

where $\beta = f_\infty + \alpha$ and $u = S(v) \in K$.

To show that S is compact and continuous on $L^2(\mathcal{O})$, let us consider a sequence v_n in K that is bounded in $L^2(\mathcal{O})$ and we set $S(v_n) = u_n$. So we have

$$\lambda^2 \left\langle \frac{T}{v_n}\nabla u_n, \nabla u_n \right\rangle + \lambda^2 \left\langle \frac{1}{v_n^2}\mathbf{Q}.\nabla u_n, u_n \right\rangle + \|u_n\|^2 = \lambda^2 \left\langle r\frac{1}{v_n}, u_n \right\rangle,$$

(in the sequel $\|\cdot\|$ means $\|\cdot\|_{L^2(\mathcal{O})}$). This implies

$$\lambda^2 \frac{T_\mathrm{m}}{\beta}\|\nabla u_n\|^2 + \|u_n\|^2 \leq \lambda^2 (C\|u_n\| + \frac{q_\infty}{\alpha^2}\|\nabla u_n\|\|u_n\|) \leq \lambda^2 \left(\frac{T_\mathrm{m}}{2\beta}\|\nabla u_n\|^2 + \frac{\beta q_\infty^2}{2T_\mathrm{m}\alpha^4}\|u_n\|^2 + C\|u_n\|\right).$$

Therefore, the sequence ∇u_n is bounded in $L^2(\mathcal{O})$ and u_n is in a compact set of $L^2(\mathcal{O})$.

Assume now that the sequence converges to v_* in $L^2(\mathcal{O})$. We have seen that the sequence u_n is bounded in $H^1(\mathcal{O})$; thus a subsequence, still denoted u_n, converges to a function u_* in $L^2(\mathcal{O})$ and ∇u_n converges weakly to ∇u_*. Moreover, for each test function ψ we have

$$\lambda^2 \left\langle \nabla u_n, \frac{T}{v_n}\nabla \psi \right\rangle + \lambda^2 \left\langle \mathbf{Q}.\nabla u_n, \frac{1}{v_n^2}\psi \right\rangle + \langle u_n, \psi\rangle = \lambda^2 \left\langle r\frac{1}{v_n}, \psi \right\rangle$$

[2]Let S be a mapping from a convex subset of a Banach space into itself, if S is continuous and compact with respect to the Banach topology, then S has a fixed point.

[3]See result 1 in the Appendix.

Now since the mappings $v \mapsto \frac{1}{v}$ and $v \mapsto \frac{1}{v^2}$ are Lipschitz on K, we know that $\frac{T}{v_n}\nabla\psi \to \frac{T}{v_*}\nabla\psi$ and $\frac{1}{v_n^2}\psi \to \frac{1}{v_*^2}\psi$ in L^2 strongly; thus we can pass to the limit in the previous relation and we get

$$\lambda^2 \left\langle \nabla u_*, \frac{T}{v_*}\nabla\psi \right\rangle + \lambda^2 \left\langle \mathbf{Q}.\nabla u_*, \frac{1}{v_*^2}\psi \right\rangle + \langle u_*, \psi \rangle = \lambda^2 \left\langle r\frac{1}{v_*}, \psi \right\rangle$$

Therefore, u_* is a solution to (2.56) with $v = v_*$. If another subsequence of u_n converges to a function u^* in $L^2(\mathcal{O})$, this function u^* is also a solution to (2.56) with $v = v_*$.

Since this solution is unique, we have $u^* = u_*$ and the entire sequence u_n converges to u_* in $L^2(\mathcal{O})$

$$S(v_n) \to u_* = S(v_*) \qquad \text{in } L^2(\mathcal{O})$$

The operator S is continuous from K into K endowed with the norm of $L^2(\mathcal{O})$. According to the Schauder theorem, there exists a fixed point $u = S(u)$; it belongs to $H^1(\mathcal{O})$ and is a solution of (2.55).

Uniqueness of the Solution. Assume that there exist two solutions u and U of (2.55) belonging to $H^1(\mathcal{O})$ which are strictly positive and bounded: for some α and β we have

$$\alpha \le u \le \beta, \qquad \alpha \le U \le \beta,$$

and they satisfy

$$-\lambda^2\nabla.\,(T\nabla(\log u - \log U)) + \lambda^2\frac{1}{u^2}\mathbf{Q}.\nabla u - \lambda^2\frac{1}{U^2}\mathbf{Q}.\nabla U + u - U = \lambda^2\left(\frac{r}{u} - \frac{r}{U}\right).$$

Then multiplying by $\log(u/U)$, we get

$$\lambda^2\left\langle \nabla\log\frac{u}{U}, T\nabla\log\frac{u}{U}\right\rangle - \lambda^2\left\langle \frac{1}{u} - \frac{1}{U}, \mathbf{Q}.\nabla(\log\frac{u}{U})\right\rangle + \left\langle u - U, \log\frac{u}{U}\right\rangle = \lambda^2\left\langle \frac{r}{u} - \frac{r}{U}, \log\frac{u}{U}\right\rangle. \tag{2.57}$$

If we had have $\mathbf{Q} = 0$, $r = 0$, the uniqueness would come from relation $(u - U)\log\frac{u}{U} \ge 0$. In the general case, denoting $z = (u - U)/U$, we see that $|z| \le \beta/\alpha$ and notice that

$$\left\langle \frac{1}{U(1+z)} - \frac{1}{U}, \mathbf{Q}.\nabla\log(1+z)\right\rangle \le C_0\|\nabla\log(1+z)\|\,\|z\|.$$

So, (2.57) implies

$$\lambda^2 T_{\mathrm{m}}\|\nabla\log(1+z)\|^2 + \langle Uz, \log(1+z)\rangle \le \lambda^2 C_0\|\nabla\log(1+z)\|\,\|z\| + \frac{r_\infty}{\alpha^2}\beta\lambda^2\,\langle z, \log(1+z)\rangle\,.$$

Then we have

$$\alpha \langle z, \log(1+z)\rangle \leq \frac{\lambda^2}{4T_{\rm m}} C_0^2 \|z\|^2 + \frac{r_\infty}{\alpha^2}\beta\lambda^2 \langle z, \log(1+z)\rangle\,.$$

Thus, for $\lambda^2 \leq \alpha^3/(2\beta r_\infty)$, we get

$$\langle z, \log(1+z)\rangle \leq \frac{\lambda^2}{2T_{\rm m}} C_0^2 \|z\|^2.$$

and $z = 0$ if λ satisfies also $\lambda^2 C_0^2 \leq 2T_{\rm m}\frac{\alpha}{\beta}\log(1+\frac{\beta}{\alpha})$. □

Proof of Proposition 2. According to (2.32) we have $N_0^{-1}(\partial_t + \mathbf{U}.\nabla)(N_0) = -\nabla.\mathbf{U}$, so using (2.33) we get in a classical way

$$N_0(\partial_t + \mathbf{U}.\nabla)(N_0^{-1}) = \nabla.\mathbf{U},$$

$$m_0 N_0(\partial_t + \mathbf{U}.\nabla)(\mathbf{U}) = -\nabla\mathcal{P}_0(N_0) - ZN_0\frac{1}{N_e}\nabla(N_e T_e).$$

Since $(N_0^{-1}\mathcal{E}_0(N_0))' = N_0^{-2}\mathcal{P}_0(N_0)$, we get

$$\begin{aligned} N_0(\partial_t + \mathbf{U}.\nabla)(m_0|\mathbf{U}|^2 + N_0^{-1}\mathcal{E}_0(N_0)) &= N_0 N_0^{-2}\mathcal{P}_0(N_0)(\partial_t + \mathbf{U}.\nabla)(N_0) - \mathbf{U}.\nabla\mathcal{P}_0(N_0) - ZN_0\frac{1}{N_e}\mathbf{U}.\nabla(N_e T_e),\\ &= -\mathcal{P}_0(N_0)\nabla.\mathbf{U} - \mathbf{U}.\nabla\mathcal{P}_0(N_0) - ZN_0\frac{1}{N_e}\mathbf{U}.\nabla(N_e T_e) \end{aligned}$$

Moreover, using the identity $N_0(\partial_t + \mathbf{U}.\nabla)w = \partial_t(N_0 w) + \nabla(N_0 w\mathbf{U})$, if we add this relation and (2.34), we get

$$\begin{aligned} &(\partial_t + \nabla(\mathbf{U}.))\,(\mathcal{E}_e) + (\partial_t + \nabla(\mathbf{U}.))\left(m_0 N_0 \frac{1}{2}|\mathbf{U}|^2 + \mathcal{E}_0(N_0)\right) + N_e T_e \nabla.\mathbf{U} + \nabla(\mathcal{P}_0(N_0)\mathbf{U})\\ &= -ZN_0\frac{1}{N_e}\mathbf{U}.\nabla(N_e T_e) + (ZN_0 - N_e)\frac{1}{N_e}\nabla(N_e T_e).\mathbf{U} = -\mathbf{U}.\nabla(N_e T_e). \end{aligned}$$

Then, the result follows. □

Proof of Proposition 3. For λ small enough, the function $f_\lambda = ZN_0/N_{\rm ref} + \lambda^2 g$ satisfies $f_\lambda \geq 2\alpha$ for α strictly positive (independent of λ); it is bounded in L^2 uniformly with respect to λ. According to Proposition 1, there exists a unique solution Φ_λ in $H^1(\mathcal{O})$ to the equation

$$-\lambda^2\nabla.(T\nabla\Phi_\lambda) + e^{\Phi_\lambda} = f_\lambda. \tag{2.58}$$

and we have $\exp\Phi_\lambda \geq \alpha$.

We now multiply (2.58) by Φ_λ and integrate with respect to the space variable

$$\lambda^2\langle T\nabla\Phi_\lambda, \nabla\Phi_\lambda\rangle + \left\langle e^{\Phi_\lambda} - 1, \Phi_\lambda\right\rangle = \langle f_\lambda - 1, \Phi_\lambda\rangle\,.$$

Using the identity $(e^{\psi} - e^{\varphi})(\psi - \varphi) \geq (\psi - \varphi)^2 \min(e^{\psi}, e^{\varphi})$ for each ψ, φ (i.e., the convexity of the exponential function), we get

$$\lambda^2 T_{\mathrm{m}} \|\nabla \Phi_\lambda\|^2 + \alpha \|\Phi_\lambda\|^2 \leq \frac{\alpha}{2} \|\Phi_\lambda\|^2 + \frac{1}{2\alpha} \| f_\lambda - 1\|^2, \tag{2.59}$$

so there exists a constant C_0 (independent of λ) such that $\|\Phi_\lambda\| \leq C_0$. Using this bound, we get the estimate

$$\lambda^2 T_{\mathrm{m}} \|\nabla \Phi_\lambda\|^2 \leq C_0 \| f_\lambda - 1\|.$$

Thus, it exists C_1 such that

$$\lambda \|\nabla \Phi_\lambda\| \leq C_1. \tag{2.60}$$

We now denote $F = \log(ZN_0/N_{\mathrm{ref}})$, so multiplying (2.58) by $\Phi_\lambda - F$, we get

$$\lambda^2 \langle T \nabla \Phi_\lambda, \nabla (\Phi_\lambda - F) \rangle + \left\langle \Phi_\lambda - F, e^{\Phi_\lambda} - ZN_0/N_{\mathrm{ref}} \right\rangle = \lambda^2 \langle \Phi_\lambda - F, g \rangle .$$

Since $\langle T \nabla \Phi_\lambda, \nabla (\Phi_\lambda - F) \rangle \geq \langle T \nabla F, \nabla (\Phi_\lambda - F) \rangle$, according to the above mentioned property of the exponential function, we see that it exists α_0 independent of λ such that

$$\begin{aligned} \alpha_0 \|\Phi_\lambda - F\|^2 &\leq \left\langle \Phi_\lambda - F, e^{\Phi_\lambda} - ZN_0/N_{\mathrm{ref}} \right\rangle \\ &\leq \lambda^2 T_\infty \|\nabla \Phi_\lambda\| \|\nabla F\| + \lambda^2 \|\Phi_\lambda\| \|g\| - \lambda^2 \langle F, g \rangle \end{aligned}$$

Due to (2.60) and the hypothesis of the proposition, we have when λ goes to 0

$$\Phi_\lambda - F = \Phi_\lambda - \log(ZN_0/N_{\mathrm{ref}}) \to 0 \quad \text{in } L^2,$$

$$e^{\Phi_\lambda} - ZN_0/N_{\mathrm{ref}} \to 0 \quad \text{in } L^2. \tag{2.61}$$

Since $\nabla \Phi_\lambda$ is bounded in $L^2(\mathcal{O})$, according to result 4 in the Appendix, we get $\nabla \Phi_\lambda \rightharpoonup \nabla F$ in $L^2(\mathcal{O})$ weakly; so (2.45) holds. □

Remark 4. If the smoothness assumption on N_0 is false, i.e., ∇N_0 is not in $L^2(\mathcal{O})$, then using the bound of Φ_λ in $L^2(\mathcal{O})$, we may show that $\Phi_\lambda \to \log(ZN_0)$ in $L^2(\mathcal{O})$ weakly, but one can only prove that $\nabla \Phi_\lambda \to \nabla \log(ZN_0)$ in distribution meaning.

□

Proof of Proposition 4. We will use the following lemmas, the proof of which is given below.

Lemma 1. *Let Θ_λ be the solution of*

$$-\lambda^2 \Delta\Theta_\lambda + \Theta_\lambda = 0, \tag{2.62}$$

supplemented with the boundary condition: $\Theta_\lambda|_{\Gamma_1} = 1$, $\Theta_\lambda|_{\Gamma_2} = 0$, $\frac{\partial}{\partial \mathbf{n}}\Theta_\lambda|_{\partial\mathcal{O}\setminus\Gamma_1 \cup\Gamma_2} = 0$. *Then, we have*

$$\text{(i)} \quad \int_{\mathcal{O}} \Theta_\lambda \le C_* \lambda, \qquad \text{(ii)} \quad \|\nabla\Theta_\lambda\| \le C_0 \frac{1}{\lambda}, \qquad \text{(iii)} \quad \int_{\partial\mathcal{O}} \left|\frac{\partial\Theta_\lambda}{\partial\mathbf{n}}\right| \le C_1 \frac{1}{\lambda}.$$

Lemma 2. *With the assumptions of the proposition, the solution Φ_λ of (2.39) and (2.43) is such that there exists C_2 with*

$$\int_{\Gamma_1\cup\Gamma_2} \left|\frac{\partial\Phi_\lambda}{\partial\mathbf{n}}\right| \le C_2 \frac{1}{\lambda}.$$

As for the Neumann boundary condition, for all λ, there exists a unique solution Φ_λ to the Poisson equation

$$-\lambda^2 \nabla.(T\nabla\Phi_\lambda) + e^{\Phi_\lambda} = f_\lambda$$

supplemented with (2.43). Moreover, for λ small enough, there exist two constants α, β (independent of λ) such that

$$\log\alpha \le \Phi_\lambda \le \log\beta.$$

Multiplying the Poisson equation by Φ_λ, we get as above

$$\lambda^2 \left\langle T, |\nabla\Phi_\lambda|^2 \right\rangle + \left\langle e^{\Phi_\lambda} - 1, \Phi_\lambda \right\rangle = \langle f_\lambda - 1, \Phi_\lambda\rangle + \lambda^2 \int_{\Gamma_1\cup\Gamma_2} \Phi_\lambda T \frac{\partial\Phi_\lambda}{\partial\mathbf{n}}.$$

Since $\left\langle e^{\Phi_\lambda} - 1, \Phi_\lambda \right\rangle \ge \alpha\Phi_\lambda^2$, according to Lemma 2, one sees that

$$\lambda^2 T_{\min} \|\nabla\Phi_\lambda\|^2 + \frac{\alpha}{2}\|\Phi_\lambda\|^2 \le C_3 + \lambda C_2 C_4,$$

therefore we get

$$\|\nabla\Phi_\lambda\| \le C_5/\lambda.$$

On the other hand, with $F = \log(ZN_0/N_{\text{ref}})$, multiplying the Poisson equation by $\Phi_\lambda - F$, we get

$$\lambda^2 \langle T\nabla(\Phi_\lambda - F), \nabla\Phi_\lambda\rangle + \left\langle \Phi_\lambda - F, e^{\Phi_\lambda} - \frac{ZN_0}{N_{\text{ref}}} \right\rangle =$$

$$\lambda^2 \sum_{q=1,2} \int_{\Gamma_q} (B_q - F) T \frac{\partial\Phi_\lambda}{\partial\mathbf{n}} + \lambda^2 \langle \Phi_\lambda - F, g\rangle$$

Thus, using the inequality $\langle T\nabla(\Phi_\lambda - F), \nabla\Phi_\lambda\rangle \geq \langle T\nabla(\Phi_\lambda - F), \nabla F\rangle$, we get according to Lemma 2:

$$\alpha\|\Phi_\lambda - F\|^2 \leq \lambda^2 T_\infty \|\nabla F\| \|\nabla\Phi_\lambda\| + \lambda C_2(\sup(B_1, B_2) + \|F\|_\infty) T_\infty + C_6\lambda^2.$$

So we see that $\|\Phi_\lambda - F\|^2 = O(\lambda)$ and the remaining part of the proposition follows. □

Proof of the Lemma 1. Denote by $\delta(x)$ the distance from x to the boundary Γ_1. Let now θ_λ be the solution of $-\lambda^2\Delta\theta_\lambda + \theta_\lambda = 0$, supplemented with the boundary condition $\theta_\lambda|_{\partial\mathcal{O}} = 1$. We have the usual bound (cf. Lemma 2 of [17])

$$0 \leq \theta_\lambda(x) \leq e^{-\delta(x)/4\lambda}, \qquad \text{for } \delta(x) \geq 4\lambda/3$$

Then, for each local map $\mathcal{O}_q$, one knows that there exists a local coordinate system x_1, x_2, x_3 such that the boundary is of the form $x_3 = f(x_1, x_2)$ and a constant r such that

$$\delta(x) \geq r(x_3 - f(x_1, x_2)).$$

so integrating over x, we get

$$\int_{\mathcal{O}\cap\mathcal{O}_q} \theta_\lambda(x)dx \leq \int_{\mathcal{O}\cap\mathcal{O}_q} e^{-\delta(x)/4\lambda}dx \leq \int\int_{x\in\mathcal{O}_q}\left(\int_0^{+\infty} e^{-r(x_3 - f(x_1,x_2))/4\lambda}dx_3\right)dx_1dx_2 \leq C\lambda.$$

Then $\int_{\mathcal{O}} \theta_\lambda(x)dx \leq C_*\lambda$. Now, since $0 \leq \Theta_\lambda \leq \theta_\lambda$, we get point (i).

Moreover, Θ_λ is the minimum function in $H^1(\mathcal{O})$ for the functional $\Theta \longmapsto \lambda^2\|\nabla\Theta\|^2 + \|\Theta\|^2$ with the constraints $\Theta_\lambda|_{\Gamma_1} = 1$, $\Theta_\lambda|_{\Gamma_2} = 0$. Thus, there exists C_0 such that $\lambda^2\|\nabla\Theta_\lambda\|^2 \leq C_0^2$ (it suffices to compare with a smooth function ζ such that $\zeta|_{\Gamma_1} = 1$, $\zeta|_{\Gamma_2} = 0$). Now consider a positive smooth test function ξ (independent of λ) such that $\xi|_{\Gamma_1} = 1$, $\xi|_{\Gamma_2} = 0$ and multiply (2.62) by ξ and integrate over the domain $\mathcal{O}$, we have

$$0 \leq \lambda^2\int_{\Gamma_1}\frac{\partial}{\partial\mathbf{n}}\Theta_\lambda = \lambda^2\int_{\mathcal{O}}\nabla\xi\nabla\Theta_\lambda dx + \int_{\mathcal{O}}\xi\Theta_\lambda dx \leq \lambda C_0\|\nabla\xi\| + \lambda C_*.$$

In the same way, multiplying (2.62) by η such that $\eta|_{\Gamma_1} = 0$, $\eta|_{\Gamma_2} = 1$ and integrating over the domain $\mathcal{O}$, we get

$$0 \leq -\lambda^2\int_{\Gamma_2}\frac{\partial}{\partial\mathbf{n}}\Theta_\lambda = -\lambda^2\int_{\mathcal{O}}\nabla\eta\nabla\Theta_\lambda dx - \int_{\mathcal{O}}\eta\Theta_\lambda dx \leq \lambda C_0\|\nabla\eta\| + \lambda C_*,$$

then the last point follows. □

Proof of Lemma 2. Since Φ_λ is bounded by $\log\beta$, we first notice that

$$1 + \Phi_\lambda \leq \exp(\Phi_\lambda) \leq \Phi_\lambda + \gamma.$$

with $\gamma = \max(\beta, 1 - \log\alpha)$

Let $a^1 = \inf_\lambda \inf_x (f_\lambda)$, and $a^2 = \sup_\lambda \sup_x (f_\lambda)$. Now, let ϕ^1 and ϕ^2 be the solutions of the linear problems

$$-\lambda^2 \nabla. \left(T \nabla \phi^1_\lambda\right) + \phi^1_\lambda = a^1 - \gamma$$
$$-\lambda^2 \nabla. \left(T \nabla \phi^2_\lambda\right) + \phi^2_\lambda = a^2 - 1$$

with the same boundary conditions $\phi^q_\lambda\big|_{\Gamma_1} = B_1$, $\phi^q_\lambda\big|_{\Gamma_2} = B_2$. So, according to the maximum and minimum principle, we check that

$$\phi^1_\lambda \le \Phi_\lambda \le \phi^2_\lambda.$$

Therefore, we get on each part Γ_1 and Γ_2 of the boundary

$$\left|\frac{\partial}{\partial \mathbf{n}} \Phi_\lambda\right| \le \max(\left|\frac{\partial}{\partial \mathbf{n}} \phi^1_\lambda\right|, \left|\frac{\partial}{\partial \mathbf{n}} \phi^2_\lambda\right|)$$

So if we prove that

$$\int_{\Gamma_1} \left|\frac{\partial}{\partial \mathbf{n}} \phi^q_\lambda\right| = O(\frac{1}{\lambda}), \qquad \text{for } q = 1, 2 \tag{2.63}$$

and the same bounds for Γ_2, then the lemma follows.

According to linearity properties, to show (2.63), it suffices to prove this relation for the following toy problem. So consider Ψ solution to

$$-\lambda^2 \nabla. (T \nabla \Psi) + \Psi = 0, \qquad \Psi|_{\Gamma_1} = 1, \qquad \Psi|_{\Gamma_2} = 0, \qquad \left.\frac{\partial}{\partial \mathbf{n}} \Psi\right|_{\partial \mathcal{O} \setminus \Gamma_2 \cup \Gamma_1} = 0.$$

This solution is positive and bounded by 1. So, multiplying it by Θ_λ, we get

$$\int_{\Gamma_1} T \frac{\partial \Psi}{\partial \mathbf{n}} = \int_{\mathcal{O}} T \nabla \Psi . \nabla \Theta_\lambda + \frac{1}{\lambda^2} \langle \Psi, \Theta_\lambda \rangle .$$

and multiplying (2.62) by $T\Psi$,

$$\int_{\Gamma_1} T \Psi \frac{\partial \Theta_\lambda}{\partial \mathbf{n}} = \int_{\mathcal{O}} T \nabla \Psi . \nabla \Theta_\lambda + \frac{1}{\lambda^2} \langle T\Psi, \Theta_\lambda \rangle .$$

Therefore, we have

$$0 \le T_{\min} \int_{\Gamma_1} \frac{\partial \Psi}{\partial \mathbf{n}} \le T_\infty \int_{\Gamma_1} \left|\frac{\partial \Theta_\lambda}{\partial \mathbf{n}}\right| + \frac{1}{\lambda^2} \langle (1 - T)\Psi, \Theta_\lambda \rangle + C.$$

Then, according to Lemma 1, we see that the right-hand side is a $O(\lambda^{-1})$ and that (2.63) holds for Γ_1. For Γ_2, it suffices to integrate the equation satisfied by Ψ over the domain $\mathcal{O}$ and to use Lemma 1. □

Proof of Proposition 5. Recall that $f_\lambda = ZN_0/N_{\text{ref}} + \lambda^2 g$ is uniformly bounded, then the equation reads as

$$-\lambda^2\nabla.\,(T\nabla\Phi_\lambda) + \lambda^2\nabla.(e^{-\Phi_\lambda}\mathbf{Q}) + e^{\Phi_\lambda} = f_\lambda$$

According to Proposition 1, we know that $\frac{1}{2}(\inf_x f_\lambda(x)) = \alpha \leq \exp\Phi_\lambda \leq \beta$ with α, β independent from λ.Then, multiplying this equation by Φ_λ, we get

$$\lambda^2\langle T\nabla\Phi_\lambda, \nabla\Phi_\lambda\rangle + \left\langle \Phi_\lambda, e^{\Phi_\lambda} - 1\right\rangle = \langle \Phi_\lambda, f_\lambda - 1\rangle + \lambda^2\left\langle e^{-\Phi_\lambda}\mathbf{Q}, \nabla\Phi_\lambda\right\rangle$$
$$\lambda^2 T_{\text{m}}\|\nabla\Phi_\lambda\|^2 \leq C(1+\lambda^2) + \lambda^2 C\|\nabla\Phi_\lambda\|\|e^{-\Phi_\lambda}\|.$$

Then by the same argument as above, there exists a constant C_* independent from λ such that

$$\lambda\|\nabla\Phi_\lambda\| \leq C_*$$

Moreover, multiplying by $\Phi_\lambda - F$, where $F = \log(f_\lambda)$, we get

$$\lambda^2\langle T\nabla\Phi_\lambda, \nabla(\Phi_\lambda - F)\rangle + \left\langle \Phi_\lambda - F, e^{\Phi_\lambda} - f_\lambda\right\rangle = \lambda^2\left\langle \Phi_\lambda - F, \nabla.(e^{-\Phi_\lambda}\mathbf{Q})\right\rangle.$$

Since $\langle T\nabla\Phi_\lambda, \nabla(\Phi_\lambda - F)\rangle \geq \langle T\nabla F, \nabla(\Phi_\lambda - F)\rangle$, we obtain

$$\alpha\|\Phi_\lambda - F\|^2 \leq \left\langle \Phi_\lambda - g, e^{\Phi_\lambda} - f\right\rangle \leq \lambda^2 T_\infty\|\nabla\Phi_\lambda\|\|\nabla F\| + \lambda^2(\|\Phi_\lambda\| + \|F\|)(C + CC_*).$$

thus $\|\Phi_\lambda - F\| \to 0$. Then, the result follows. □

2.3 Two-Temperature Euler Models and Magneto-Hydrodynamics

We now perform the quasi-neutrality approximation that is justified above. With this approximation, different quasi-neutral models may be stated; these models depend on the physics to be accounted for. We address in the first subsection plasmas without magnetic effect, which leads to the usual two-temperature Euler system that we address in different physical situations.

In the second subsection, we describe the popular magneto-hydrodynamic (MHD) system, which is relevant if we must account for magnetic effects; we address this in the two-temperature framework, which enables us to emphasize the energy exchange among ion population, electron population, and magnetic energy. To get the one-temperature MHD system, it suffices to mingle the ion and electron temperature, as explained below.

It is worth noticing that the two-temperature Euler system with radiative coupling and thermal conduction is the basis of most of the computational codes for solar astrophysics and for the simulation of Inertial Confinement Fusion experiments, while the electron MHD system is the basis of all models of reduced MHD (see below) which are used in practice for the fluid simulations of the evolution of tokamak plasmas.

2.3.1 The Two-Temperature Euler System

Here, we do not account for the time evolution of the magnetic field. Firstly, we address the case where no electric current is accounted for and we state a simple system including the evolution of the ion temperature and of the electron temperature without radiative phenomenon. Secondly, we give some enlightenment on a simple thermal conduction model and on a radiative diffusion model. Lastly, we deal with the case of current-carrying plasmas.

As was explained above, since the quasi-neutrality approximation is valid, the general prescription is that $N_e = ZN_0$, $P_e = ZN_0T_e$ and the Ohm's law reduces to

$$q_e\mathbf{E} = -(ZN_0)^{-1}\nabla P_e = -{N_0}^{-1}\nabla(N_0T_e)$$

Now, the ion momentum equation reads

$$m_0\frac{\partial}{\partial t}(N_0\mathbf{U}) + m_0\nabla.(N_0\mathbf{U}\mathbf{U}) + \nabla P_0 + \nabla P_e = 0. \tag{2.64}$$

Recall that Ω_{0e} is proportional to $(T_e - T_0)$ (see formula (2.68) below) and the ion and the electron energy balance equations (2.4) and (2.37) simply read as

$$\frac{\partial}{\partial t}\mathcal{E}_0 + \nabla.(\mathcal{E}_0\mathbf{U}) + P_0\nabla.\mathbf{U} = \Omega_{0e}, \tag{2.65}$$

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e\mathbf{U}) + P_e\nabla.\mathbf{U} + \nabla\mathbf{q}_{\mathrm{th,e}} = -\Omega_{0e}. \tag{2.66}$$

Then, for the ion total energy balance, we get the relation

$$\left(\frac{\partial}{\partial t} + \nabla.\,(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + m_0N_0\frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U}) + \mathbf{U}.\nabla P_e = \Omega_{0e},$$

and for the global energy balance relation

$$\left(\frac{\partial}{\partial t} + \nabla.\,(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + \mathcal{E}_e + m_0N_0\frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}) + \nabla.\mathbf{q}_{\mathrm{th,e}} = 0. \tag{2.67}$$

It is classical to use the density $\rho = m_0 N_0$ and to introduce the specific internal energies

$$\varepsilon_0 = \frac{\mathcal{E}_0}{m_0 N_0}, \qquad \varepsilon_e = \frac{\mathcal{E}_e}{m_0 N_0} = \frac{3}{2}\frac{Z}{m_0} T_e.$$

Recall that the relation between pressure and internal energies is given by

$$P_e = \frac{2}{3}\mathcal{E}_e = \frac{2}{3}\rho\varepsilon_e = Z N_0 T_e, \qquad P_0 = (\gamma - 1)\mathcal{E}_0 = (\gamma - 1)\rho\varepsilon_0 = N_0 T_0.$$

(i.e., an equation of state of perfect gas law type). Then besides the continuity equation (2.1)

$$\frac{\partial}{\partial t} N_0 + \nabla.(N_0 \mathbf{U}) = 0,$$

the system that consists of (2.64)–(2.66) is closed. It is formally equivalent to the system (2.64), (2.66), and (2.67).

Now, introduce the Lagrangian derivative

$$\frac{D}{Dt}\bullet = \frac{\partial}{\partial t}\bullet + \mathbf{U}.\nabla(\bullet).$$

According to the continuity equation, we check that

$$\rho\frac{D}{Dt}\bullet = \frac{\partial}{\partial t}(\rho\bullet) + \nabla.(\mathbf{U}\rho\bullet).$$

Summary. Using the Lagrangian derivatives, the system may be stated as follows

$$(\mathcal{E}2\mathcal{T}) \quad \boxed{\begin{array}{ll} \text{(i)} & \rho\frac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0, \\[2ex] \text{(ii)} & \rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) = 0, \\[2ex] \text{(iii)} & \rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e}, \\[2ex] \text{(iv)} & \rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} \end{array}}$$

Of course, equation (iii) may be replaced by the following equation for the total energy

$$\rho\frac{D}{Dt}\left(\varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} = 0$$

that is to say

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U}\cdot)\right)\left(\mathcal{E}_0 + \mathcal{E}_e + \rho\frac{1}{2}|\mathbf{U}|^2\right) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} = 0$$

It is worthwhile noticing that although the roles of ions and electrons seem to be similar in (2.65) and (2.66), it is not the case. Indeed, we know that when shocks occur, some information is missing if we state only the system (2.64)–(2.66). In its reduced form without the thermal conduction term $\mathbf{q}_{\text{th,e}}$, it has been analyzed recently from a mathematical point of view; it is related to the definition of the non-conservative products like $P_e\nabla.\mathbf{U}$ when $\mathbf{U}$ is discontinuous. It may be shown that one has to add some information on the entropy deposition, see [37] for example. Zel'dovich and Raizer noted a long time ago in [117] that this entropy deposition needs to be made on ion population only. This fact has been justified also in [32] with an asymptotic analysis (in the case where the Debye length goes to zero). In [31], it has been proved in one dimension that the solution of the ion Euler system coupled with the nonlinear Poisson equation converge to the solution of (2.1) and (2.64) in the case of constant temperatures.

Notice also that without accounting for the thermal conduction $\nabla.\mathbf{q}_{\text{th,e}}$, this system may enter into the framework of the well-posed Lagrangian system described in [44]. Its numerical treatment is a whole subject not covered in this book, but it is related to the hyperbolic property of this system; this property is analyzed in the last section of this chapter.

2.3.1.1 Accounting for the Thermal Conduction

More than 50 years ago, Spitzer [111] stated a usual formula for the electron thermal conduction in hot plasmas. In its simplest term, it reads as

$$\mathbf{q}_{\text{th,e}} = -\frac{\kappa}{m_0}\nabla T_e^{7/2}$$

where κ is a positive coefficient that is roughly speaking proportional to the inverse of the Coulomb logarithm $\log\Lambda$ (recall that $\log\Lambda$ is in the order of some units and may be considered as a very smooth function of the space variable).

Notice that the accurate modelling of the electron thermal conduction is a very active area; besides [111], see pioneering works such as [54, 79] and more recent works [94, 104] and the references therein (as a matter of fact, in hot plasmas the heat flux is not simply proportional to the gradient of the temperature, so it is called a nonlocal heat flux).

To achieve the statement of the evolution system ($\mathcal{E}2\mathcal{T}$), notice that the coupling term Ω_{0e} between the ion and electron temperatures reads generally as

$$\Omega_{0e} = \frac{\rho}{m_0}\zeta(T_e)(T_e - T_0) \tag{2.68}$$

where the inverse of the relaxation time ζ between the two temperature is given by

$$\zeta(T_e) = \rho\zeta_C / T_e^{3/2}$$

here the coefficient ζ_C is proportional to $Z^2(\log\Lambda)$, see [38, 111] for more details.

From $(\mathcal{E}2\mathcal{T})$, let us focus on the subsystem corresponding to the two-temperature evolution equations with thermal conduction and two-temperature coupling; it reads

$$\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \frac{\rho}{m_0}\zeta(T_e)(T_e - T_0),$$
$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} - \nabla.(\frac{\kappa}{m_0}\nabla T_e^{7/2}) = \frac{\rho}{m_0}\zeta(T_e)(T_0 - T_e).$$

Here, we can consider a "hydrodynamics part" corresponding to the pressure terms and a "thermal part" corresponding to the conduction and the two-temperature coupling. This thermal part reads as follows

$$\rho\frac{3Z}{2}\frac{\partial}{\partial t}T_e - \nabla.(\kappa\nabla T_e^{7/2}) = \rho\zeta(T_e)(T_0 - T_e) \tag{2.69}$$

$$\rho\frac{3}{2}\frac{\partial}{\partial t}T_0 = \rho\zeta(T_e)(T_e - T_0), \tag{2.70}$$

where the density $\rho = \rho(x)$ is frozen and is strictly positive, κ is a strictly positive function depending on x, and ζ is a strictly positive function of x, T_e. We now consider this system on a bounded open set $\mathcal{O}$ with smooth boundary and state a result for its well-posedness. Of course, it needs to be supplemented with boundary conditions for T_e and initial conditions $T_e^{\text{ini}}, T_0^{\text{ini}}$. For the sake of simplicity, we consider the Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}T_e = 0$$

and we assume that two constants α and β exist and are such that

$$0 < \alpha \leq T_e^{\text{ini}} \leq \beta, \qquad \alpha \leq T_0^{\text{ini}} \leq \beta. \tag{2.71}$$

Now, we need to make technical assumptions. The function ζ satisfies the following properties: two constants K, κ_0 exist and are such that

$$\kappa(x) \geq \kappa_0 > 0; \ |\zeta(x, T_e) - \zeta(x, T_e')| \leq K|T_e - T_e'| \text{ for all } x, T_e, T_e' \text{ s.t. } \alpha \leq T_e, T_e' \leq \beta \tag{2.72}$$

Then we have the following result (the proof of which is given at the end of the section).

Proposition 7. *Assume that (2.72) holds and the initial conditions satisfy (2.71) and belong to* $H^1(\mathcal{O})$ *and* $L^2(\mathcal{O})$*. Then for system (2.69), (2.70) and for any final time* t_f*, there is a unique solution* $(T_e(t), T_0(t))$ *belonging to the spaces* $L^2(0, t_f; H^1(\mathcal{O})) \cap C(0, t_f, L^2(\mathcal{O}))$ *and* $C(0, t_f; L^2(\mathcal{O}))$ *and such that*

$$\alpha \leq T_e(t) \leq \beta, \qquad \alpha \leq T_0(t) \leq \beta \tag{2.73}$$

Of course, according to (2.73), if the density ρ is bounded, the function $\zeta(x, T_e)$ which is proportional to $\rho(x) T_e^{-3/2}$ satisfies the Lipschitz condition (2.72).

Notice that if assumption (2.72) is not true and if the initial conditions $(T_e^{\text{ini}}, T_0^{\text{ini}})$ are zero in a subdomain there are some technical difficulties to prove the uniqueness of the solution (up to my knowledge, it is an open problem).

2.3.1.2 Accounting for Radiative Coupling

In the case of hot plasmas, such as in stellar plasmas or in Inertial Confinement Fusion plasmas, radiative coupling has also to be taken into account. So, we describe here a simple model to address radiative phenomena: the frequency-dependent radiative diffusion. To account for these phenomena, we must introduce the radiative energy density $\mathcal{E}_\nu(t, x)$ corresponding to the energy of the photon population with frequency ν, at time t and position x; the frequency variable ν belongs to the half-line $\mathbf{R}^+$ (a photon of frequency ν has a energy equal to $h\nu$ where h is the Planck constant). Notice that $\mathcal{E}_\nu$ is evaluated in the fluid's reference frame.

Recall that each hot material emits spontaneously and continuously photonic radiation over a wide spectrum of frequency; if it is at local thermodynamic equilibrium, the emitted radiation is described by Planck's function which is a universal function depending only on the electron temperature T_e and frequency variable ν and which reads

$$B_\nu(T_e) = \frac{8\pi}{c^3} h\nu^3 (\exp(\frac{h\nu}{T_e}) - 1)^{-1}$$

up to a multiplicative factor $c/(4\pi)$. It is called black body emission. The integral of $B_\nu(T_e)$ over all the frequency is equal to $a_\mathrm{r} T_e^4$ (here a_r is the universal radiation constant).

Denote by $\sigma_\nu = \sigma_\nu(T_e)$ the frequency-dependent absorption coefficient (also called opacity); it depends on the temperature and on the material characteristic. For instance, for fully ionized plasma, we may use the so-called Kramer formula for the opacity

$$\sigma_\nu = C_0 \frac{1}{\nu^3} \frac{1}{{T_e}^{1/2}} (\exp(\frac{h\nu}{T_e}) - 1)$$

where the coefficient C_0 is a function of the atomic number of the material and its density. Moreover, denote by σ_{th} the so-called Thomson scattering coefficient given by $\sigma_{\text{Th}} = \sigma_{\text{cons}} N_e$ (where σ_{cons} is a universal constant).

Now, besides evolution equations for $\rho, \mathbf{U}, \varepsilon_0, \varepsilon_e$, we have to state the evolution equation for the radiative energy density $\mathcal{E}_\nu$; it reads

$$\frac{\partial}{\partial t}\mathcal{E}_\nu + \nabla.(\mathbf{U}\mathcal{E}_\nu) + \frac{1}{3}(\mathcal{E}_\nu - \frac{\partial(\nu\mathcal{E}_\nu)}{\partial \nu})\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3(\sigma_\nu + \sigma_{\text{Th}})}\nabla\mathcal{E}_\nu\right) = c\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu) + G_\nu(T_e, \mathcal{E}_\nu).$$

or in the Lagrangian framework

$$\rho\frac{D}{Dt}\left(\frac{\mathcal{E}_\nu}{\rho}\right) + \frac{1}{3}(\mathcal{E}_\nu - \frac{\partial(\nu\mathcal{E}_\nu)}{\partial \nu})\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3(\sigma_\nu + \sigma_{\text{Th}})}\nabla\mathcal{E}_\nu\right) = c\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu) + G_\nu(T_e, \mathcal{E}_\nu).$$

The term $\frac{\partial(\nu\mathcal{E}_\nu)}{\partial\nu}$ is related to the so-called Doppler effect, i.e., the frequency shift due to the expansion or compression of the matter. Moreover, the operator $G_\nu(T_e, \cdot)$ corresponds to the so-called Compton effect; it is defined by

$$G_\nu(T_e, \mathcal{E}_\nu) = \sigma_{\text{C}\nu}(4T_e\mathcal{E}_\nu - h\nu\mathcal{E}_\nu(1 + \frac{\mathcal{E}_\nu}{h\nu^3c^{-3}})) + \frac{\partial}{\partial\nu}\left(\sigma_{\text{C}\nu}h\nu^2\mathcal{E}_\nu(1 + \frac{\mathcal{E}_\nu}{h\nu^3c^{-3}}) + \sigma_{\text{C}\nu}T_e\nu^6\frac{\partial}{\partial\nu}(\frac{\mathcal{E}_\nu}{\nu^4})\right)$$

($\sigma_{\text{C}\nu}$ is a coefficient close to the Thomson coefficient σ_{Th}); the Compton effect is negligible when the plasma is not very hot. One notices in these equations the radiative diffusion term $\nabla.\left(\frac{c}{3(\sigma_\nu+\sigma_{\text{th}})}\nabla\mathcal{E}_\nu\right)$ and the term $\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu)$ related to the emission/absorption phenomena between radiation and matter.

Notice that the quantity

$$\mathcal{E}_r = \int_0^{+\infty} \mathcal{E}_\nu d\nu,$$

corresponds to the total radiative energy (evaluated in the fluid's reference frame) and we may define the radiative pressure by $\frac{1}{3}\mathcal{E}_r$.

Now, the radiative phenomena are coupled with the plasma model in the following way. In system $(\mathcal{E}2\mathcal{T})$, equation (iv) must account for the exchange term between radiation and matter; it reads now as

$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + c\int_0^{+\infty}[\sigma_\nu\mathcal{E}_\nu - \sigma_\nu B_\nu(T_e)]d\nu - \int_0^{+\infty} G_\nu(T_e, \mathcal{E}_\nu)d\nu$$

Moreover, equation (ii) is modified by the following way

$$\rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e + \frac{1}{3}\mathcal{E}_r) = 0.$$

Then, the energy balance equation reads

$$\rho\frac{D}{Dt}(\varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2 + \frac{\mathcal{E}_r}{\rho}) + \nabla.((P_0 + P_e + \frac{1}{3}\mathcal{E}_r)\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} - \nabla.\int_0^{+\infty}\frac{c\nabla\mathcal{E}_\nu}{3(\sigma_\nu + \sigma_{\text{th}})}d\nu = 0. \tag{2.74}$$

For this modelling and the mathematical analysis, see, e.g., [87, 97] and more recently [78] and the references therein. Without accounting for the hydrodynamics part, it has been proved that with appropriated assumptions, the reduced system

$$\frac{\partial}{\partial t}\mathcal{E}_\nu - \nabla.\left(\frac{c}{3(\sigma_\nu + \sigma_{\rm th})}\nabla\mathcal{E}_\nu\right) = c\sigma_\nu(B_\nu(T_e) - \mathcal{E}_\nu),$$
$$\rho\frac{\partial}{\partial t}\varepsilon_e = c\int_0^{+\infty}\sigma_\nu\mathcal{E}_\nu d\nu - c\int_0^{+\infty}\sigma_\nu B_\nu(T_e)d\nu.$$

is well-posed; see [8]. But the full system is quite difficult to handle.

There is also a simpler radiative model called the gray-diffusion approximation which may be derived from the previous one by making some elementary closures. We emphasize it now (disregarding the Compton effect).

Instead of the frequency-dependent energy density $\mathcal{E}_\nu$, we need to consider only one evolution equation for the gray energy density $\mathcal{E}_r$. Then, we define the Planck averaged opacity $\sigma_P(T_e)$ as

$$\sigma_P(T_e) = \int\sigma_\nu B_\nu(T_e)d\nu.\left(\int B_\nu(T_e)d\nu\right)^{-1}$$

and the term $\int\sigma_\nu(B_\nu(T_e)-\mathcal{E}_\nu)d\nu$ for the emission/absorption phenomena is simply replaced by $\sigma_P(a_{\rm r}T_e^4 - \mathcal{E}_r)$. The evolution equation for $\mathcal{E}_r$ reads as

$$\frac{\partial}{\partial t}\mathcal{E}_r + \nabla.(\mathbf{U}\mathcal{E}_r) + \frac{1}{3}\mathcal{E}_r\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = c\sigma_P(a_{\rm r}T_e^4 - \mathcal{E}_r) \tag{2.75}$$

or in the Lagrangian framework

$$\rho\frac{D}{Dt}\left(\frac{\mathcal{E}_r}{\rho}\right) + \frac{1}{3}\mathcal{E}_r\nabla.\mathbf{U} - \nabla.\left(\frac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = c\sigma_P(a_{\rm r}T_e^4 - \mathcal{E}_r),$$

where the Rosseland averaged opacity $\sigma_R(T_e)$ defined as

$$\frac{1}{\sigma_R(T_e)} = \int_0^{+\infty}\frac{1}{\sigma_\nu + \sigma_{\rm th}}\frac{\partial B_\nu}{\partial T_e}(T_e)d\nu.\left(\int_0^{+\infty}\frac{\partial B_\nu}{\partial T_e}(T_e)d\nu\right)^{-1}$$

See, e.g., [77, 97] or [21] for such a model.

If we disregard the thermal diffusive term, the previous system reads as

$$
(\mathcal{E}2\mathcal{TR}) \quad \boxed{\begin{aligned}
&\rho\frac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0,\\
&\rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e + \tfrac{1}{3}\mathcal{E}_r) = 0,\\
&\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \tfrac{\rho}{m_0}\zeta(T_e)(T_e - T_0),\\
&\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} = \tfrac{\rho}{m_0}\zeta(T_e)(T_0 - T_e) + c\sigma_P(\mathcal{E}_r - a_{\mathrm{r}}T_e^4),\\
&\rho\frac{D}{Dt}\left(\frac{\mathcal{E}_r}{\rho}\right) + \tfrac{1}{3}\mathcal{E}_r\nabla.\mathbf{U} - \nabla.\left(\tfrac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = c\sigma_P(a_{\mathrm{r}}T_e^4 - \mathcal{E}_r).
\end{aligned}}
$$

Of course, there is a energy balance equation that reads as

$$
\rho\frac{D}{Dt}\left(\varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2 + \frac{\mathcal{E}_r}{\rho}\right) + \nabla.\left((P_0 + P_e + \frac{1}{3}\mathcal{E}_r)\mathbf{U}\right) - \nabla.\left(\frac{c}{3\sigma_R}\nabla\mathcal{E}_r\right) = 0.
$$

We may notice that in model $(\mathcal{E}2\mathcal{TR})$ as for system $(\mathcal{E}2\mathcal{T})$, there are difficulties when shocks occur which are related to the definition of the nonconservative products such as $P_e\nabla.\mathbf{U}$ or $\frac{1}{3}\mathcal{E}_r\nabla.\mathbf{U}$. As a matter of fact, we need to add some information on the entropy deposition: the two quantities $\mathcal{E}_r$ and $\mathcal{E}_e$ play a similar part, and from a physical point of view one needs to claim that the entropy deposition is made on the ion population only.

Remark 5. The usual Rosseland approximation corresponds to the fact that in system $(\mathcal{E}2\mathcal{TR})$ we set $\mathcal{E}_r = a_{\mathrm{r}}T_e^4$ and we gather together the two last equations and replace them by

$$
\rho\frac{D}{Dt}(\varepsilon_e + \frac{a_{\mathrm{r}}}{\rho}T_e^4) + (P_e + \frac{1}{3}a_{\mathrm{r}}T_e^4)\nabla.\mathbf{U} - \nabla.\left(\frac{c a_{\mathrm{r}}}{3\sigma_R}\nabla T_e^4\right) = \frac{\rho}{m_0}\zeta(T_e)(T_0 - T_e),
$$

For mathematical results related to the well-posedness of this system and to a justification of the Rosseland approximation, see, e.g., [8]. For some other models and numerical methods, see [15]. For details of the modelling of radiative hydrodynamics using with transport equations, see [25]. □

2.3.1.3 Accounting for Electric Current

We deal now with current-carrying plasmas: the magnetic field **B** is either an external field or is given by an evolution equation (see the following subsection where the electron magneto-hydrodynamics model is complete). For the sake of legibility, we disregard the radiative coupling phenomena.

So, we address a two-temperature quasi-neutral model with given fields $\mathbf{B}$ and $\mathbf{J}$ (with $\nabla.\mathbf{B} = 0$) and we take a generalized Ohm's law given by (2.54); that is to say

$$\mathbf{E} + \mathbf{U} \times \mathbf{B} - \frac{1}{q_e N_e}\mathbf{J} \times \mathbf{B} + \frac{1}{q_e N_e}\nabla P_e = \chi\mu^0\mathbf{J}.$$

Recall that the ion momentum equation (2.22) reads

$$m_0\frac{\partial}{\partial t}(N_0\mathbf{U}) + m_0\nabla.(N_0\mathbf{U}\mathbf{U}) + \nabla P_0 + \nabla P_e = \mathbf{J} \times \mathbf{B} \tag{2.76}$$

or equivalently

$$m_0\frac{\partial}{\partial t}(N_0\mathbf{U}) + m_0\nabla.(N_0\mathbf{U}\mathbf{U}) + \nabla P_0 = q_e N_e(\mathbf{E} + \mathbf{U} \times \mathbf{B}) - q_e N_e \mu^0 \chi\mathbf{J}. \tag{2.77}$$

Let us go back now to the ion and electron energy equations (2.4) and (2.37). Thanks to the quasi-neutrality, since $\mathbf{U} - \mathbf{U}_e = \mathbf{J}(q_e N_e)^{-1}$, the ion energy equation reads

$$\frac{\partial}{\partial t}\mathcal{E}_0 + \nabla.(\mathcal{E}_0\mathbf{U}) + P_0\nabla.\mathbf{U} = \Omega_{0e}, \tag{2.78}$$

and the electron equation may read in one of the two equivalent forms

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e\mathbf{U}) + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \nabla.\left(\frac{5}{2}T_e\frac{\mathbf{J}}{q_e}\right) - \frac{1}{N_e q_e}\mathbf{J}.\nabla P_e + \chi\mu^0|\mathbf{J}|^2. \tag{2.79}$$

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e\mathbf{U}) + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \nabla.\left(\frac{5}{2}T_e\frac{\mathbf{J}}{q_e}\right) + \mathbf{J}.(\mathbf{E} + \mathbf{U} \times \mathbf{B}).$$

One moves from one form to the other by using the following relation (obtained thanks to the Ohm's law)

$$\mathbf{J}.(\mathbf{E} + \mathbf{U} \times \mathbf{B}) = -\frac{1}{N_e q_e}\mathbf{J}.\nabla P_e + \chi\mu^0|\mathbf{J}|^2.$$

Using the specific internal energies, the previous system reads as

$$(\mathcal{E}2\mathcal{T}\mathbf{J})\quad\begin{cases}
\text{(i)} & \rho\dfrac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0, \\
\text{(ii)} & \rho\dfrac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) = \mathbf{J}\times\mathbf{B}, \\
\text{(iii)} & \rho\dfrac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e}, \\
\text{(iv)} & \rho\dfrac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \nabla.\left(\dfrac{5T_e}{2q_e}\mathbf{J}\right) \\
& \quad -\dfrac{1}{q_e N_e}\mathbf{J}.\nabla P_e + \chi\mu^0|\mathbf{J}|^2.
\end{cases}$$

The term $\chi\mu^0|\mathbf{J}|^2$ corresponds to the so-called Joule effect or Ohmic heating.

Of course, multiplying (2.76) by $\mathbf{U}$ and combining with (2.54), we get as above the global energy balance

$$\frac{\partial}{\partial t}(\mathcal{E}_e + \mathcal{E}_0 + m_0 N_0 \frac{1}{2}|\mathbf{U}|^2) + \nabla.(\mathcal{E}_e\mathbf{U}_e + E_0\mathbf{U}) + \nabla.(P_0\mathbf{U} + P_e\mathbf{U}_e) + \nabla\mathbf{q}_{\text{th,e}} = \mathbf{J}.\mathbf{E}$$

(the term $\mathbf{J}.\mathbf{E}$ being related to the work of electromagnetic forces).

Remark 6. The two terms $\nabla.\left(\frac{5}{2q_e}T_e\mathbf{J}\right)$ and $\frac{1}{ZN_0q_e}\mathbf{J}.\nabla P_e$ are not always taken into account in the literature, but they appear, e.g., in [21, 38].

The first term may be considered as a part of the thermal conduction flux. But the second one is the counterpart of the term $\mathbf{J}.\mathbf{E}$ and it is necessary to have the right energy balance (recall that $-\frac{1}{ZN_0}\nabla P_e$ is the main term in the expression that defines $\mathbf{E}$). □

2.3.2 Electron Magneto-Hydrodynamics

In the framework of the quasi-neutral approximation, we now account for the full electron magneto-hydrodynamics , i.e., the magnetic field equation is coupled with the previous system: the field $\mathbf{B}$ obeys an evolution equation coming from the Maxwell–Faraday equation and the generalized Ohm's law.

Beside the first paragraph where the conductivity σ is a scalar, we address in a second paragraph the case where the conductivity is a tensor (which is generally used in the cases where the magnetic field is very strong).

2.3.2.1 Case with Scalar Conductivity

Here the electric resistivity[4] χ is of Spitzer type: i.e., it is the inverse of the electric conductivity σ (up to the constant μ^0); we have with the notations introduced above.

$$(\chi\mu^0)^{-1} = \sigma = Zq_e^2/\nu_{e0}$$

so it is also proportional to the Coulomb collision frequency $\tau_e = m_e/(\nu_{e0}N_{\rm ref})$.

Notice that the electric conductivity is strongly related to the conductivity function (4.11) given below by $\underline{\sigma}(\omega) = (ZN_0q_e^2)(\nu_{e0}N_0 - im_e\omega)^{-1}$ since the Debye length is negligible; as a matter of fact, we get

$$\mathrm{Re}(\underline{\sigma}(\omega)^{-1}) = \sigma^{-1} = \chi\mu^0. \tag{2.80}$$

When we take the "curl" of relation (2.54), we get

$$\mathrm{curl}(\mathbf{E} + \mathbf{U}\times\mathbf{B}) = -\mathrm{curl}(\frac{1}{q_eN_e}\nabla P_e) + \mathrm{curl}(\chi\mu_0\mathbf{J}) + \mathrm{curl}(\frac{1}{q_eN_e}\mathbf{J}\times\mathbf{B}).$$

Therefore, the Maxwell–Faraday equation leads to

$$\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}(\mathbf{U}\times\mathbf{B}) + \mathrm{curl}\,(\chi\mathrm{curl}\,\mathbf{B}) = \mathrm{curl}\left(\frac{\nabla P_e}{q_eN_e}\right) - \mathrm{curl}\,\left(\frac{1}{q_eN_e\mu^0}\mathrm{curl}\,\mathbf{B}\times\mathbf{B}\right),$$

with the constraint that $\mathbf{B}$ has to satisfy the magnetic Gauss relation

$$\nabla.\mathbf{B} = 0. \tag{2.81}$$

Of course, if $\mathbf{B}$ is initially divergence-free, it remains divergence-free always. According to this constraint and identity (A.1), we may write the left-hand side of this evolution equation in other forms

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}(\mathbf{U}\times\mathbf{B}) &= \frac{D}{Dt}\mathbf{B} + \mathbf{B}(\nabla.\mathbf{U}) - (\mathbf{B}.\nabla)\mathbf{U} \\ &= \rho\frac{D}{Dt}(\frac{\mathbf{B}}{\rho}) - (\mathbf{B}.\nabla)\mathbf{U}.\end{aligned}$$

To get the last line, relation $\nabla.\mathbf{U} = N_0D_tN_0^{-1} = \rho D_t\rho^{-1}$ has been used.

On momentum equation [$\mathcal{E}2\mathcal{T}\mathbf{J}$(ii)], since the current is curl $\mathbf{B}/\mu^0$, we display the magnetic pressure tensor $\nabla.\mathbb{P}_B$. Then, we get the following system for the evolution of $\rho, \mathbf{U}, \varepsilon_0, \varepsilon_e, \mathbf{B}$

[4] As a matter of fact, the electric resistivity is defined as $\mu^0\chi$; it is also denoted by η in some physics textbooks.

$$(\mathcal{MHD})\quad\left\{\begin{array}{ll}
\text{(i)} & \rho\dfrac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0,\\[2ex]
\text{(ii)} & \rho\dfrac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) + \nabla.\mathbb{P}_B = 0,\\[2ex]
\text{(iii)} & \rho\dfrac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e},\\[2ex]
\text{(iv)} & \rho\dfrac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \nabla.\left(\dfrac{5T_e}{2q_e}\dfrac{\text{curl}\mathbf{B}}{\mu^0}\right)\\
& \quad -\dfrac{\nabla P_e}{q_e N_e}.\dfrac{\text{curl}\mathbf{B}}{\mu^0} + \dfrac{\chi}{\mu^0}\,|\text{curl}\mathbf{B}|^2\\[2ex]
\text{(v)} & \rho\dfrac{D}{Dt}\left(\dfrac{\mathbf{B}}{\rho}\right) - (\mathbf{B}.\nabla)\mathbf{U} + \text{curl}\,(\chi\text{curl}\mathbf{B}) = \text{curl}\left(\dfrac{\nabla P_e}{q_e N_e}\right)\\
& \quad -\text{curl}\left(\dfrac{1}{q_e N_e \mu^0}\text{curl}\mathbf{B}\times\mathbf{B}\right),
\end{array}\right.$$

with the constraint (2.81).

Of course, we may state this system in an Euler framework, it suffices to replace equation (i) by

$$\frac{\partial}{\partial t}\rho + \nabla.(\rho\mathbf{U}) = 0$$

and in the other equations to replace $\rho\dfrac{D}{Dt}X$ by $\frac{\partial}{\partial t}(\rho X) + \nabla.(\rho X\mathbf{U})$. Then the last equation reads as

$$\frac{\partial}{\partial t}\mathbf{B}+\nabla.(\mathbf{U}\mathbf{B})-(\mathbf{B}.\nabla)\mathbf{U}+\text{curl}\,(\chi\text{curl}\mathbf{B}) = \text{curl}\left(\frac{\nabla P_e}{q_e N_e}\right)-\text{curl}\left(\frac{1}{q_e N_e\mu^0}\text{curl}\mathbf{B}\times\mathbf{B}\right)$$

In equation (v), the quadratic term (curl $\mathbf{B}\times\mathbf{B}$), which is related to the Hall effect, may be neglected if the magnetic field is not very strong. Note that there is no contribution of the Hall effect term in the energy balance (indeed, we have always $\mathbf{B}.\text{curl}(N_e^{-1}\text{curl}\mathbf{B}\times\mathbf{B}) = 0$).

The term $\nabla.(\dfrac{5T_e}{2q_e}\dfrac{\text{curl}\mathbf{B}}{\mu^0})$ is called the thermoelectric one; it is zero if the electron temperature is constant.

It is worth noticing that in equation (v) the term with $(\nabla P_e)/N_e$ corresponds to the so-called *self-generated* magnetic field and reads also as $\text{curl}(N_e^{-1}\nabla P_e) = \nabla P_e\times\nabla(N_e^{-1})$; it is an external source that is not zero if the gradient of the plasma density and the gradient of the electron temperature are not parallel.

It has its counterpart at the right-hand side of equation (iv). Therefore, it is possible to neglect these terms with $(\nabla P_e)/N_e$ both in equation (iv) and (v).

Energy Balance

Let us recall that the magnetic energy is equal to $\frac{1}{2\mu^0}|\mathbf{B}|^2$. Recall that $\nabla\mathbf{U}$ denotes the tensor $(\nabla\mathbf{U})_{i,j} = \partial_{x_i}\mathbf{U}_j$. Let us multiply the last equation of $(\mathcal{MHD})$ by $\mathbf{B}/\mu^0$. For the first part of the magnetic energy balance relation, we get

$$\frac{1}{\mu^0}\mathbf{B}.\left(\frac{D}{Dt}\mathbf{B} + \mathbf{B}(\nabla.\mathbf{U}) - (\mathbf{B}.\nabla)\mathbf{U}\right) = \frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\mu^0}\right) + \frac{1}{\mu^0}[|\mathbf{B}|^2\,\nabla.\mathbf{U} - \mathbf{B}((\mathbf{B}.\nabla)\mathbf{U})]$$

$$= \rho\frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\rho\mu^0}\right) + \mathbb{P}_B : \nabla\mathbf{U},$$

indeed we have as usual $\rho\dfrac{D}{Dt}(\dfrac{|\mathbf{B}|^2}{2\rho}) = \frac{1}{2}\dfrac{D}{Dt}|\mathbf{B}|^2 - \frac{1}{2}|\mathbf{B}|^2(\nabla.\mathbf{U})$ and obviously $\mathbf{B}((\mathbf{B}.\nabla)\mathbf{U}) = \mathbf{BB} : \nabla\mathbf{U}$. For the second part, using identity (A.2), we get

$$\mathbf{B}.\mathrm{curl}\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e} - \frac{1}{\mu^0 q_e N_e}\mathbf{J}\times\mathbf{B}\right) =$$

$$\nabla.\left(\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e} + \frac{1}{\mu^0 q_e N_e}\mathbf{J}\times\mathbf{B}\right)\times\mathbf{B}\right) + \mu^0\mathbf{J}.\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e}\right).$$

So we may state the magnetic energy balance equation

$$\rho\frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\rho\mu^0}\right) + \mathbb{P}_B : (\nabla\mathbf{U}) + \nabla.\left(\left(\chi\mathbf{J} - \frac{1}{\mu^0}\frac{\nabla P_e}{q_e N_e} + \frac{1}{\mu^0 q_e N_e}\mathbf{J}\times\mathbf{B}\right)\times\mathbf{B}\right) + \mu^0\chi|\mathbf{J}|^2 - \mathbf{J}.\frac{\nabla P_e}{q_e N_e} = 0.$$

Introduce the global energy

$$E_{\mathrm{tot}} = \frac{1}{2\mu^0}|\mathbf{B}|^2 + \mathcal{E}_e + \mathcal{E}_0 + \frac{1}{2}m_0 N_0 |\mathbf{U}|^2.$$

Therefore, multiplying the momentum equation by $\mathbf{U}$ and using the tensor identity (A.5), we get finally

$$\rho\frac{D}{Dt}\left(\frac{E_{\mathrm{tot}}}{\rho}\right) + \nabla.((P_0 + P_e)\mathbf{U}) + \nabla.(\mathbb{P}_B.\mathbf{U}) + \nabla.\mathbf{q}_{\mathrm{th,e}}$$

$$= \nabla.\left(\frac{\chi}{\mu^0}\mathbf{B}\times\mathrm{curl}\mathbf{B} + \frac{1}{\mu^0}\mathbf{B}\times\left(-\frac{\nabla P_e}{q_e N_e} + \frac{1}{q_e N_e}\mathbf{J}\times\mathbf{B}\right)\right).$$

Since the right-hand side is a divergence of a vector, we see that this energy balance relation is a conservative one.

Remark 7. The usual simplified resistive electron-MHD system is obtained when one disregards the Hall effect terms and the self-generated magnetic field (this is justified in the cases where the magnetic field is not very strong) and the thermo-electric term (justified if the electron temperature is quite constant).

More precisely, it corresponds withdrawing the right hand side of equation [$\mathcal{MHD}$(v)] and keeping only the Joule effect term (and the two-temperature coupling) in the electron energy equation; so [$\mathcal{MHD}$(iv), (v)] are replaced by

$$\rho \frac{D}{Dt}\varepsilon_e + P_e \nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \frac{\chi}{\mu^0}\,|\text{curl}\mathbf{B}|^2 \tag{2.82}$$

$$\frac{\partial}{\partial t}\mathbf{B} - \text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\chi\text{curl}\mathbf{B}) = 0. \tag{2.83}$$

Of course, the above global energy balance relation reads now as

$$\rho\frac{D}{Dt}\left(\frac{E_{\text{tot}}}{\rho}\right) + \nabla.((P_0+P_e)\mathbf{U}) + \nabla.(\mathbb{P}_B.\mathbf{U}) + \nabla.\mathbf{q}_{\text{th,e}} = \nabla.\left(\frac{\chi}{\mu^0}\mathbf{B}\times\text{curl}\mathbf{B}\right). \quad \square$$

2.3.2.2 Case with a Tensor Conductivity

We now address the case of the strongly magnetized plasma and we want to determine a link between the electric current **J** and the fields **E** and **B**. Therefore, we define the unit vector $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ and the electron Larmor frequency

$$\omega_{ce} = \frac{q_e}{m_e}|\mathbf{B}|$$

and we begin with the Ohm's law (2.54) which we write in the following form

$$\mathbf{J}\times\mathbf{b}\frac{m_e\omega_{ce}}{\nu_{e0}N_0} + \mathbf{J} - \frac{q_e^2 Z}{\nu_{e0}}(\mathbf{E}+\mathbf{U}\times\mathbf{B}) = \frac{q_e}{\nu_{e0}N_0}\nabla P_e. \tag{2.84}$$

Thanks to this expression, we now perform a closure by dropping the r.h.s. term related to the thermo-electric effect. By a classic way, $\mathbf{E}_\parallel = \mathbf{b}(\mathbf{E}.\mathbf{b})$ denotes the parallel component and $\mathbf{E}_\perp = \mathbf{b}\times(\mathbf{E}\times\mathbf{b})$ the perpendicular component of the electrostatic field. Since we have $\mathbf{b}\times(\mathbf{W}\times\mathbf{b})=\mathbf{W}$, for all vectors **W** orthogonal to **b**, it may be checked that (2.84) leads to (χ_s is the scalar resistivity defined as above by $\nu_{e0}q_e^{-2}\frac{1}{Z\mu^0}$)

$$\mathbf{J} = \frac{1}{\mu^0\chi_s}\mathbf{E}_\parallel + \frac{1}{\mu^0\chi_s}\frac{1}{1+\frac{\omega_{ce}^2}{\nu^2}}(\mathbf{E}_\perp+\mathbf{U}\times\mathbf{B}) + \frac{1}{\mu^0\chi_s}\frac{1}{\frac{\nu}{\omega_{ce}}+\frac{\omega_{ce}}{\nu}}\mathbf{b}\times(\mathbf{E}_\perp+\mathbf{U}\times\mathbf{B})$$

This expresses the desired closure

$$\mathbf{J} = \underset{\leftrightarrow}{\sigma}(\mathbf{E}+\mathbf{U}\times\mathbf{B}) \tag{2.85}$$

where the tensor $\underset{\leftrightarrow}{\sigma}$ may read in the system of coordinates defined by $\mathbf{b}, \mathbf{b}\times(\hat{\mathbf{E}}\times\mathbf{b}), (\mathbf{b}\times\hat{\mathbf{E}})$ (where $\hat{\mathbf{E}} = \mathbf{E}\frac{1}{|\mathbf{E}|}$) as follows

$$\mu^0 \underset{\leftrightarrow}{\sigma} = \begin{bmatrix} \chi_s^{-1} & 0 & 0 \\ 0 & \chi_s^{-1}(1+\frac{\omega_{ce}^2}{\nu^2})^{-1} & 0 \\ 0 & 0 & \chi_s^{-1}(\frac{\nu}{\omega_{ce}}+\frac{\omega_{ce}}{\nu})^{-1} \end{bmatrix}.$$

Moreover, the resistivity tensor $\underset{\leftrightarrow}{\chi}$ is given by $\underset{\leftrightarrow}{\chi}\,\mu^0 = \underset{\leftrightarrow}{\sigma}^{-1}$ and reads as

$$\underset{\leftrightarrow}{\chi} = \begin{bmatrix} \chi_s & 0 & 0 \\ 0 & \chi_s(1+\frac{\omega_{ce}^2}{\nu^2}) & 0 \\ 0 & 0 & \chi_s(\frac{\nu}{\omega_{ce}}+\frac{\omega_{ce}}{\nu}) \end{bmatrix}.$$

We can now proceed as in the previous paragraph. Since

$$\text{curl}\,\mathbf{E} = -\text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\mathbf{E}+\mathbf{U}\times\mathbf{B}) = -\text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\underset{\leftrightarrow}{\sigma}^{-1}\mathbf{J}),$$

the Maxwell–Faraday equation leads to the evolution equation of the magnetic field.

$$\frac{\partial}{\partial t}\mathbf{B} - \text{curl}(\mathbf{U}\times\mathbf{B}) + \text{curl}\,(\underset{\leftrightarrow}{\chi}\,\text{curl}\mathbf{B}) = 0.$$

As above, this equation may read also as

$$\rho\frac{D}{Dt}(\frac{\mathbf{B}}{\rho}) - (\mathbf{B}.\nabla)\mathbf{U} + \text{curl}\left(\underset{\leftrightarrow}{\chi}\,\text{curl}\mathbf{B}\right) = 0 \tag{2.86}$$

We may state the MHD system in this framework

$$(\mathcal{MHD}_T)\quad \begin{cases} \text{(i)} & \rho\frac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0, \\ \text{(ii)} & \rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0+P_e) + \nabla.\mathbb{P}_B = 0, \\ \text{(iii)} & \rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e}, \\ \text{(iv)} & \rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \frac{1}{\mu^0}\text{curl}\mathbf{B}.\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B}, \\ \text{(v)} & \rho\frac{D}{Dt}(\frac{\mathbf{B}}{\rho}) - (\mathbf{B}.\nabla)\mathbf{U} + \text{curl}\left(\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B}\right) = 0. \end{cases}$$

Let us now focus on the Joule effect term

$$s_{\text{Joule}} = \frac{1}{\mu^0}\text{curl}\mathbf{B}.\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B} = \mathbf{J}.(\mathbf{E} + \mathbf{U}\times\mathbf{B})$$

In the above-defined system of coordinates $\{\mathbf{b}, \mathbf{b}\times(\hat{\mathbf{E}}\times\mathbf{b}), (\mathbf{b}\times\hat{\mathbf{E}})\}$ we may compute this term. Since $\mathbf{J} = \underset{\leftrightarrow}{\sigma}.(\mathbf{E}+\mathbf{U}\times\mathbf{B})$, we get $(\mathbf{E}+\mathbf{U}\times\mathbf{B}) = \mu^0\chi_s\mathbf{J}_\parallel + \mu^0\chi_s(1+\frac{\omega_{ce}^2}{\nu^2})\mathbf{J}_\perp$ where $\mathbf{J}_\parallel = (\mathbf{J}.\mathbf{b}).\mathbf{b}$. and $\mathbf{J}_\perp = \mathbf{J} - \mathbf{J}_\parallel$; then we see that

$$s_{\text{Joule}} = \mu^0\chi_s|\mathbf{J}_\parallel|^2 + \mu^0\chi_s(1+\frac{\omega_{ce}^2}{\nu^2})|\mathbf{J}_\perp|^2.$$

We notice that the term with $(\frac{\nu}{\omega_{ce}} + \frac{\omega_{ce}}{\nu})$ in the resistivity tensor does not appear in this formula (this term corresponds to the component of $\underset{\leftrightarrow}{\chi}.\mathbf{J}$ which is orthogonal to $\mathbf{E}$).

Remark 8 (Energy Balance). We now state the magnetic energy balance. Since, according to (A.2), we have

$$\mathbf{B}.\text{curl}\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}\right) = \nabla.\left(\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}\right)\times\mathbf{B}\right) + \mu^0\mathbf{J}.\underset{\leftrightarrow}{\chi}\,\mathbf{J}$$

we get

$$\rho\frac{D}{Dt}\left(\frac{|\mathbf{B}|^2}{2\rho\mu^0}\right) + \mathbb{P}_B : \nabla\mathbf{U} + \nabla.\left(\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}\right)\times\mathbf{B}\right) + \mu^0\mathbf{J}.\underset{\leftrightarrow}{\chi}.\mathbf{J} = 0.$$

Thus, for the global energy balance relation, we get as above

$$\rho\frac{D}{Dt}\left(\frac{E_{\text{tot}}}{\rho}\right) + \nabla.((P_0+P_e)\mathbf{U}) + \nabla.(\mathbb{P}_B.\mathbf{U}) = \nabla.\left(\mathbf{B}\times\left(\frac{1}{\mu^0}\underset{\leftrightarrow}{\chi}.\text{curl}\mathbf{B}\right)\right). \quad \square$$

Remark 9. As a matter of fact it is also possible to add in the r.h.s. of $[\mathcal{MHD}_T(\text{v})]$ the term $\text{curl}\left(\frac{\nabla P_e}{q_e N_e}\right)$, related to the self-generated magnetic field. □

Remark 10. As in the case of scalar conductivity, the conductivity tensor $\underset{\leftrightarrow}{\sigma}$ is closely related to the conductivity tensor function $\underset{\leftrightarrow}{\sigma}(\omega)$ proposed in Sect. 4.1.1 of Chap. 4 which links the envelope fields $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{E}}$ by a relation of the type $\tilde{\mathbf{J}} = \underset{\leftrightarrow}{\sigma}(\omega)\tilde{\mathbf{E}}$ (where the envelope fields are such that $\mathbf{E} = \tilde{\mathbf{E}}e^{i\mathbf{k}.\mathbf{x}-i\omega t}$, $\mathbf{J} = \tilde{\mathbf{J}}e^{i\mathbf{k}.\mathbf{x}-i\omega t}$). □

2.3.2.3 Boundary Conditions. Axi-Symmetric Geometry Case

(a) Boundary Conditions in the 3D Framework

We analyze here some different boundary conditions for the MHD models, but we disregard the two energy equations (indeed, for the sake of simplicity, one may assume that the ion and electron temperatures are somehow constant near the boundary). Roughly speaking, there are two kinds of material boundaries for such models: either the boundary corresponds to an insulation material or a conducting material. Of course, the wall material may also own magnetic properties; then we need to deal with impedance conditions on this boundary (but this is more tricky).

Since we assume that the temperatures are given, $c_s^2 \nabla \rho$ may replace $\nabla(P_e + P_0)$ with an appropriated sound velocity c_s, so we may address the following simplified problem near the boundary

$$\frac{\partial}{\partial t}\rho + \nabla.(\rho \mathbf{U}) = 0 \tag{2.87}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{U}) + \nabla.(\rho \mathbf{U}\mathbf{U}) + c_s^2 \nabla \rho + \nabla.\mathbb{P}_B = 0 \tag{2.88}$$

$$\frac{\partial}{\partial t}\mathbf{B} - \mathrm{curl}(\mathbf{U} \times \mathbf{B}) + \mathrm{curl}\,(\chi \mathrm{curl}\mathbf{B}) = 0 \tag{2.89}$$

with $\mathbb{P}_B = \frac{1}{\mu^0}(\frac{\mathbb{I}}{2}\,|\mathbf{B}|^2 - \mathbf{B}\mathbf{B})$.

In the sequel, we denote as usual by $\mathbf{n}$ the outwards unit normal vector to the boundary Γ of domain $\mathcal{O}$.

It is worth noticing first that curl(χcurl $\mathbf{B}$) is a diffusion-like term; then for the boundary conditions, there are two different cases: firstly, χ is strictly positive near the boundary; secondly, χ is zero near the boundary. We focus here only on the first case corresponding to a resistive model, which is assumed to be valid up to the boundary. Therefore, we do not account for the existence of a "vacuum" near the boundary. ("Vacuum" means that the ion density is zero near the boundary. This case is quite difficult because the electron population is nonzero and a corresponding electric current needs to be addressed.)

Since χ is strictly positive, (2.89) is of parabolic type and its boundary condition is specific and may be imposed independently of the two other equations.

It is also worth recalling that system (2.87) and (2.88), which reads in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho \mathbf{U} \end{pmatrix} + \nabla.\left(\mathbb{F}\begin{pmatrix} \rho \\ \rho \mathbf{U} \end{pmatrix}\right) = 0, \qquad \text{with } \mathbb{F}\begin{pmatrix} \rho \\ \rho \mathbf{U} \end{pmatrix} = \begin{pmatrix} \rho \mathbf{U} \\ \rho \mathbf{U}\mathbf{U} + c_s^2 \rho \mathbb{I} + \mathbb{P}_B, \end{pmatrix}$$

is hyperbolic[5] and the eigenvalues of the Jacobian matrix are equal to $\mathbf{n}.\mathbf{U} \pm c_s$; indeed, $\mathbb{P}_B$ does not depend on ρ and $\mathbf{U}$, thus it does not come into the Jacobian matrix (notice that the computation of the eigenvalues for the full MHD system is made in Sect. 2.4 below). One knows (see, e.g., [60]) that if the flow near the boundary is supersonic with an outgoing velocity, i.e., $\mathbf{n}.\mathbf{U} > c_s$, then no boundary condition needs to be imposed; moreover, if it is subsonic, i.e., $|\mathbf{n}.\mathbf{U}| \leq c_s$, one must impose a scalar equation for the boundary condition.

Generally, the plasma flow is subsonic and we must handle one boundary condition related to the normal ion velocity $\mathbf{n}.\mathbf{U}$. This point is quite sensitive, since there are sheath effects according to the presence of a electric current at the boundary; moreover, the quasi-neutrality does not hold in the neighborhood of the conductor in a width of about a few tens of typical Debye lengths which is called the Langmuir sheath (and is related to the Child–Langmuir problem when an electric potential is imposed). There is a wide range of literature related to this problem; see, e.g., [28] for a review.

Nevertheless, in the case where there is no external electric circuit and the surface of the material is not insulated, there is a crude approximation known as the Bohm criterion, which claims that the ion velocity near the boundary must satisfy

$$\mathbf{n}.\mathbf{U} = c_s.$$

On the contrary, if the surface is perfectly insulated, we can assume that $\mathbf{n}.\mathbf{U} = 0$.

Let us address now the boundary condition problem for (2.89). But firstly, let us go back to the classical calculus for the magnetic energy. If we multiply (2.89) by $\mathbf{B}$ and integrate over the spatial domain, using the vector identity (A.5) and setting $S = -\int_{\mathcal{O}} \chi |\text{curl}\mathbf{B}|^2 dx + \int_{\mathcal{O}} (\mathbf{U} \times \mathbf{B}.\text{curl}\mathbf{B}) dx$ the usual inner term, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{O}} |\mathbf{B}|^2 dx &= \int_{\mathcal{O}} \mathbf{B}.\text{curl}(\mathbf{U} \times \mathbf{B} - \chi \text{curl}\mathbf{B})\, dx = S + \int_{\Gamma} \mathbf{n}.(\mathbf{U} \times \mathbf{B} - \chi \text{curl}\mathbf{B}) \times \mathbf{B} d\Gamma(x) \\ &= S + \int_{\Gamma} ((\mathbf{U} \times \mathbf{B} - \chi \text{curl}\mathbf{B}) \times \mathbf{n})[(\mathbf{B} \times \mathbf{n}) \times \mathbf{n}] d\Gamma(x)\end{aligned}$$

[using the usual formula $\mathbf{n}.(\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \times \mathbf{n}).((\mathbf{B} \times \mathbf{n}) \times \mathbf{n})$]. This shows that the normal component of the Pointing vector $(\mathbf{U} \times \mathbf{B} - \chi \text{curl}\mathbf{B}) \times \mathbf{B}$ on the boundary is related to the tangential components of the two vectors $\mathbf{E} = -\mathbf{U} \times \mathbf{B} + \chi \text{curl}\mathbf{B}$ and $\mathbf{B}$.

We now address two kinds of boundary material.

1. *If it is a purely conductive material*, since the electric potential of the conductor is constant, we must have $\mathbf{E} \times \mathbf{n} = 0$, which implies the following condition.

[5]For a one-dimensional space variable, a system of the form $\partial_t \mathbf{Y} + \frac{\partial}{\partial x}(\mathbf{F}(\mathbf{Y})) = 0$ is called hyperbolic if all the eigenvalues of the Jacobian matrix $\partial \mathbf{F}/\partial \mathbf{Y}$ are real and there exists a complete set of eigenvectors. For a three-dimensional space variable, a system $\partial_t \mathbf{Y} + \Sigma_j \frac{\partial}{\partial x_j}(\mathbb{F}_j(\mathbf{Y})) = 0$ is called hyperbolic if one has the analogous property for the Jacobian matrix $\frac{\partial}{\partial \mathbf{Y}}(\omega_1 \mathbb{F}_1 + \omega_2 \mathbb{F}_2 + \omega_3 \mathbb{F}_3)$ for all coefficients $\omega_1, \omega_2, \omega_3$.

$$(\mathbf{U} \times \mathbf{B} - \chi \mathrm{curl}\mathbf{B}) \times \mathbf{n} = 0.$$

2. *If it is not a purely conductive material*, one needs to impose a boundary condition that links the tangential components of the electric field **E** and the magnetic one **B**; so with a positive coefficient α depending on the material, we may impose the so-called impedance boundary condition

$$\alpha(\mathbf{U} \times \mathbf{B} - \chi \mathrm{curl}\mathbf{B}) \times \mathbf{n} + (\mathbf{B} \times \mathbf{n}) \times \mathbf{n} = 0.$$

This type of condition is usual in electromagnetism. Notice that the case $\alpha = 0$ corresponds to a pure insulated material.

Remark 11 (Transparent Boundary Conditions). Generally the simulation domain needs to be truncated and it is necessary to pay attention to the treatment of the artificial boundaries. For the subsystem (2.87) and (2.88), if the flow is hypersonic with an outgoing velocity, one must not impose any boundary condition; but if the flow is subsonic, one needs to impose a transparent boundary condition. One may address this problem by using a perfectly matched layer technique as in [11] (see, e.g., [89]); one may also use a technique related to the Riemann invariants of the system ($\rho\mathbf{n}.\mathbf{U} \pm \rho c_s$), see, e.g., [60].

In all cases, one needs to deal also with boundary conditions for the resistive part ($\mathrm{curl}(\chi \mathrm{curl}\mathbf{B})$) and the analysis made above needs to be adapted for dealing with this problem.

Notice that in the case where $\chi = 0$ near the boundary, the full system (2.87)–(2.89) is hyperbolic, and the statement of the boundary conditions is different and depends on the flow characteristics—more precisely on the signs of the eigenvalues of this hyperbolic system (see Sect. 2.4). □

(b) The Two-Dimensional Axi-Symmetric Geometry

For many applications, e.g., for tokamak simulations or for Z-pinch simulations, one needs to deal with the previous system in this geometry. Let us give some notations: (r, z) denotes the coordinates (r is the distance to axis of axi-symmetric geometry), θ denotes the angular coordinate in the direction of rotation, and $\mathbf{e}_\theta$ denotes the unit vector at point (r, z, θ) in the direction of rotation. All the fields and functions are functions of (r, z) only. A vector field **A** in 3D may be decomposed into a toroidal component $A_\theta \mathbf{e}_\theta$ (where A_θ is a scalar function) and a two-dimensional poloidal component $\mathbf{A}_\Lambda = (A_r, A_z)$ which has no component according to $\mathbf{e}_\theta$; i.e., to say $\mathbf{A} = A_\theta \mathbf{e}_\theta + \mathbf{A}_\Lambda$. Recall that the divergence of the two-dimensional vector field is defined by

$$\nabla.\mathbf{A}_\Lambda = \frac{1}{r}\partial_r(rA_r) + \partial_z A_z$$

and for the curl we get

$$[\mathrm{curl}\mathbf{A}]_\theta = \nabla \times \mathbf{A}_\Lambda = \frac{\partial}{\partial z}A_r - \frac{\partial}{\partial r}A_z,$$

$$[\mathrm{curl}(A_\theta \mathbf{e}_\theta)]_r = -\frac{\partial}{\partial z}A_\theta, \qquad [\mathrm{curl}(A_\theta \mathbf{e}_\theta)]_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

Then the magnetic field $\mathbf{B}$ must be decomposed into its toroidal component B_θ and its poloidal component $\mathbf{B}_\Lambda = (B_r, B_z)$. The constraint $\nabla.\mathbf{B} = 0$ (in $\mathbf{R}^3$) reads now as

$$\nabla.\mathbf{B}_\Lambda = 0$$

Using this relation, the divergence of the magnetic tensor becomes

$$\begin{aligned}
\mu^0[\nabla.\mathbb{P}_B]_r &= \frac{B_\theta}{r}\frac{\partial}{\partial r}(rB_\theta) - B_z\frac{\partial}{\partial z}B_r + \frac{1}{2}\frac{\partial}{\partial r}B_z^2 \\
&= \frac{1}{2}\frac{\partial}{\partial r}\left(|\mathbf{B}_\Lambda|^2 + B_\theta^2\right) + \frac{1}{r}B_\theta^2 - \frac{\partial}{\partial z}(B_r B_z) - \frac{1}{r}\frac{\partial}{\partial r}(rB_r^2) \\
\mu^0[\nabla.\mathbb{P}_B]_z &= \frac{1}{2}\frac{\partial}{\partial z}B_\theta^2 - B_r\frac{\partial}{\partial r}B_z + \frac{1}{2}\frac{\partial}{\partial z}B_r^2 \\
&= \frac{1}{2}\frac{\partial}{\partial z}\left(|\mathbf{B}_\Lambda|^2 + B_\theta^2\right) - \frac{1}{r}\frac{\partial}{\partial r}(rB_r B_z) - \frac{\partial}{\partial z}B_z^2.
\end{aligned}$$

The system $(\mathcal{MHD}_T)$ may now read as follows (with the notation $\tilde{\nabla}.\bullet = \frac{\partial}{\partial r}\bullet_r + \frac{\partial}{\partial z}\bullet_z$)

$$\rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) + \nabla.\mathbb{P}_B = 0,$$

$$\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e},$$

$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + s_{\text{Joule}},$$

$$\rho\frac{D}{Dt}(\frac{B_\theta}{\rho}) - B_\theta\frac{U_r}{r} - \tilde{\nabla}.\left(\underset{\leftrightarrow}{\chi}.\frac{1}{r}\nabla(rB_\theta)\right) = 0,$$

$$\rho\frac{D}{Dt}(\frac{\mathbf{B}_\Lambda}{\rho}) - (\mathbf{B}_\Lambda.\nabla)\mathbf{U} + \mathrm{curl}\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}_\theta\right) = 0, \qquad \mathbf{J}_\theta = \frac{\mathbf{e}_\theta}{\mu^0}(\mathrm{curl}\mathbf{B}_\Lambda)_\theta$$

where

$$s_{\text{Joule}} = \frac{1}{\mu^0}\mathbf{J}_\theta.\underset{\leftrightarrow}{\chi}.\mathbf{J}_\theta + \frac{1}{\mu^0}\frac{1}{r^2}\nabla(rB_\theta).\underset{\leftrightarrow}{\chi}.\nabla(rB_\theta)$$

In a Euler framework, the two last equations read as

$$\frac{\partial B_\theta}{\partial t} + \nabla.(\mathbf{U}B_\theta) - B_\theta\frac{U_r}{r} - \tilde{\nabla}.\left(\underset{\leftrightarrow}{\chi}.\frac{1}{r}\nabla(rB_\theta)\right) = 0,$$

$$\frac{\partial \mathbf{B}_\Lambda}{\partial t} + \nabla.(\mathbf{U}\mathbf{B}_\Lambda) - (\mathbf{B}_\Lambda.\nabla)\mathbf{U} + \mathrm{curl}\left(\underset{\leftrightarrow}{\chi}.\mathbf{J}_\theta\right) = 0.$$

It is worth noting that, in this model, the two components of the magnetic field are somehow no longer coupled, the Joule effect term being the sum of the contribution of the two components.

Notice that a lot of work has been made in order to state so-called reduced models of MHD in the axi-symmetric case, see, e.g., [45] and the references therein.

(c) Boundary Conditions in the Two-Dimensional Axi-Symmetric Framework

The boundary conditions for the magnetic field state for an insulated material is as follows

$$\mathbf{B}_\theta = 0;$$

moreover, there is no boundary condition for $\mathbf{B}_\Lambda$.

For the case of a conductive material, the conditions for the two components of $\mathbf{B}$ read as

$$\mathbf{n}\times(\mathbf{U}\times\mathbf{B}_\Lambda) - \chi\mathbf{n}\times\mathrm{curl}\mathbf{B}_\Lambda = 0, \qquad c_s\mathbf{B}_\theta + \chi[\mathbf{n}_z\frac{\partial}{\partial z}\mathbf{B}_\theta + \mathbf{n}_r\frac{1}{r}\frac{\partial}{\partial r}(r\mathbf{B}_\theta)] = 0.$$

indeed $(\mathbf{U}\times\mathbf{B}_\Lambda)$ and $\mathrm{curl}\mathbf{B}_\Lambda$ are parallel to $\mathbf{e}_\theta$. Therefore, we check that the boundary conditions for B_θ and for $\mathbf{B}_\Lambda$ are no longer coupled.

=============

Proof of Proposition 7. Let us set θ, T instead of T_e, T_0. Set also $m = 7/2$ and drop the coefficient $\frac{3}{2}$(it is possible with a change of time scaling). Then system (2.69) and (2.70) reads as

$$\rho Z\frac{\partial}{\partial t}\theta - \nabla.(\kappa m\theta^{m-1}\nabla\theta) = \rho\zeta(\theta)(T-\theta)$$

$$\rho\frac{\partial}{\partial t}T = \rho\zeta(\theta)(\theta - T)$$

Assume first that there exists a solution (θ, T) belonging to $L^2(0, t_f, H^1(\mathcal{O})) \cap C(0, t_f, L^2(\mathcal{O}))$, and $C(0, t_f, L^2(\mathcal{O}))$. The first key point is to prove the maximum principle (2.73).

For all functions Y, let us define the so-called sign $-$ function $s^-(Y)$ by

$$s^-(Y) = 0 \qquad \text{if } Y \geq 0, \qquad s^-(Y) = 1 \qquad \text{if } Y < 0.$$

then we have $s^-(Y)Y \le 0$ for all Y. Notice that $s^-(Y)Y = 0$ is equivalent to $Y \ge 0$.

Let us denote

$$\tilde{\theta} = \theta - \alpha, \qquad \tilde{T} = T - \alpha$$

and set

$$I(t) \equiv Z \int_{\mathcal{O}} \rho s^-(\tilde{\theta}(t,x))\tilde{\theta}(t,x)dx + \int_{\mathcal{O}} \rho s^-(\tilde{T}(t,x))\tilde{T}(t,x)dx$$

We have, of course, $I(0) = 0$, so it suffices now to prove that

$$\partial_t I(t) \ge 0, \tag{2.90}$$

indeed, this implies $I(t) = 0$ and

$$\tilde{\theta} = \theta - \alpha \ge 0, \qquad \tilde{T} = T - \alpha \ge 0.$$

As a matter of fact, to be rigorous, we must introduce a regularized differentiable function $s^\varepsilon(Y)$ with compact support that is decreasing and that converges toward $s^-(Y)$. Then

$$I(t) \equiv \lim_{\varepsilon\to 0}\left(Z \int_{\mathcal{O}} \rho s^\varepsilon(\tilde{\theta}(t,x))\tilde{\theta}(t,x)dx + \int_{\mathcal{O}} \rho s^\varepsilon(\tilde{T}(t,x))\tilde{T}(t,x)dx\right)$$

Since $\frac{\partial s^\varepsilon}{\partial Y} \to \delta_{Y=0}$ (where δ is the Dirac distribution), we have $\frac{\partial}{\partial t}(Ys^\varepsilon(Y)) = s^\varepsilon(Y)\frac{\partial Y}{\partial t} + Y\frac{\partial s^\varepsilon}{\partial Y}\frac{\partial Y}{\partial t} \to s^-(Y)\frac{\partial Y}{\partial t}$, in the distribution meaning. Thus, we have

$$\lim_{\varepsilon\to 0} \rho Z \frac{\partial}{\partial t}(\tilde{\theta}s^\varepsilon(\tilde{\theta})) = \lim_{\varepsilon\to 0} s^\varepsilon(\tilde{\theta})\nabla.(\kappa m\theta^{m-1}\nabla\tilde{\theta}) + s^-(\tilde{\theta})\rho\zeta(\theta)(\tilde{T} - \tilde{\theta})$$

$$\lim_{\varepsilon\to 0} \rho \frac{\partial}{\partial t}(\tilde{T}s^\varepsilon(\tilde{T})) = s^-(\tilde{T})\rho\zeta(\theta)(\tilde{\theta} - \tilde{T})$$

and we get

$$\partial_t I(t) = \int_{\mathcal{O}} \rho\zeta(\theta)(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T}))dx - \lim_{\varepsilon\to 0}\int_{\mathcal{O}} \kappa m\theta^{m-1}\nabla\tilde{\theta}.\nabla s^\varepsilon(\tilde{\theta})dx$$

For the first integral, to determine the sign of the integrand there are four cases according to the values of $s^-(\tilde{\theta})$ and $s^-(\tilde{T})$. If one has $s^-(\tilde{\theta}) = 1, s^-(\tilde{T}) = 0$, then we have $\tilde{\theta} < 0$ and $\tilde{T} \ge 0$ and $(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T})) \ge 0$; in the case $s^-(\tilde{\theta}) = 0, s^-(\tilde{T}) = 1$, then we have $\tilde{T} < 0$ and $\tilde{\theta} \ge 0$ and $(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T})) \ge 0$. Therefore,

$$\int \rho\zeta(\theta)(\tilde{T} - \tilde{\theta})(s^-(\tilde{\theta}) - s^-(\tilde{T}))dx \ge 0.$$

Let us focus on the second integral. We see that $\nabla\tilde{\theta}.\nabla s^{\varepsilon}(\tilde{\theta}) = (\partial_Y s^{\varepsilon})(\tilde{\theta})|\nabla\tilde{\theta}|^2 \leq 0$ for all function $\tilde{\theta}$ in $H^1(\mathcal{O})$. And passing to the limit, we get

$$\lim_{\varepsilon\to 0}\int \kappa m\theta^{m-1}\nabla\tilde{\theta}.\nabla s^{\varepsilon}(\tilde{\theta})dx \leq 0.$$

Therefore, (2.90) follows and $\theta(t) \geq \alpha$, $T(t) \geq \alpha$ for all t.

By the same technique, we can prove that for all t

$$\theta(t) \leq \beta, \qquad T(t) \leq \beta. \tag{2.91}$$

Now let us show an a priori estimate. Assume that $\theta(t), T(t)$ is a solution of (2.69) and (2.70) such that θ and T belong to $L^2(0, t_f, H^1(\mathcal{O}))$ and $L^2(0, t_f, L^2(\mathcal{O}))$. Multiplying (2.69) by θ and (2.70) by T and integrating over the domain $\mathcal{O}$,we get

$$\frac{Z}{2}\frac{\partial}{\partial t}\int \rho\theta^2 dx + \int \kappa m\theta^{m-1}|\nabla\theta|^2 dx = \int \rho\zeta(\theta)(T-\theta)\theta dx,$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int \rho T^2 dx = -\int \rho\zeta(\theta)(T-\theta)T dx.$$

Thus, since we have $\theta \geq \alpha$, we obtain

$$\frac{\partial}{\partial t}\left[\frac{Z}{2}\int \rho\theta^2 dx + \frac{1}{2}\int \rho T^2 dx\right] + \kappa_0 m\alpha^{m-1}\int |\nabla\theta|^2 dx \leq 0$$

then for all time intervals $[0, t_f]$ we have

$$\begin{aligned} C_0\left\|\theta(t_f)\right\|^2_{L^2(\mathcal{O})} + C_0\left\|T(t_f)\right\|^2_{L^2(\mathcal{O})} + C\int_0^{t_f} \|\nabla\theta\|^2_{L^2(\mathcal{O})}dt \\ \leq \frac{Z}{2}\int \rho\theta^{\text{ini}2}dx + \frac{1}{2}\int \rho T^{\text{ini}2}dx \end{aligned} \tag{2.92}$$

Existence of a Solution. For system (2.69) and (2.70), there exists a weak solution (θ, T) belonging to $L^2(0, t_f; H^1(\mathcal{O})) \times L^2(0, t_f; L^2(\mathcal{O}))$, if (θ, T) satisfy

$$\frac{Z}{2}\frac{\partial}{\partial t}\int \rho\theta\phi dx + \int \kappa\nabla(\theta^m).\nabla\phi dx = \int \rho\zeta(\theta)(T-\theta)\phi dx$$

$$\frac{\partial}{\partial t}\int \rho T\psi dx = \int -\rho\zeta(\theta)(T-\theta)\psi dx$$

for all ψ, ϕ in $L^2(0, t_f; H^1(\mathcal{O})) \times L^2(0, t_f; L^2(\mathcal{O}))$.

We use a usual Galerkin method by approximating the original system by a system in a finite dimension.

More precisely, consider a family of finite-dimensional subspaces $(H^1_{(p)}(\mathcal{O}), L^2_{(p)}(\mathcal{O}))$ of $(H^1(\mathcal{O}), L^2(\mathcal{O}))$ such that $\cup_p(H^1_{(p)}(\mathcal{O}), L^2_{(p)}(\mathcal{O}))$ is dense in $(H^1(\mathcal{O}), L^2(\mathcal{O}))$. First, we can check that there exists a sequence of solutions

$(\theta_p(t), T_p(t))$ that are continuous from $[0, t_f]$ into $(H^1_{(p)}(\mathcal{O}) \times L^2_{(p)}(\mathcal{O}))$, which satisfy

$$\frac{Z}{2}\frac{\partial}{\partial t}\int \rho\theta_p\phi dx + \int m\kappa\theta_p^{m-1}\nabla\theta_p.\nabla\phi dx = \int \rho\zeta(\theta_p)(T_p - \theta_p)\phi dx \quad (2.93)$$

$$\frac{\partial}{\partial t}\int \rho T_p\psi dx = \int -\rho\zeta(\theta_p)(T_p - \theta_p)\psi dx \quad (2.94)$$

for all (ϕ, ψ) belonging to $(H^1_{(p)}(\mathcal{O}) \times L^2_{(p)}(\mathcal{O}))$. This is a system of ordinary differential equations.

Moreover, it is easy to check that these functions $(\theta_p(t), T_p(t))$ satisfy the bounds (2.91) and the analogous bounds of (2.92) since $\nabla((\theta_p)^m) = m\theta_p^{m-1}\nabla\theta_p$. So, we see that there exists a constant C independent from p such that

$$\int_0^{t_f} \|{\theta_p}^m\|^2_{H^1(\mathcal{O})} dt \le C, \qquad \|\theta_p(t)\|_{L^\infty(\mathcal{O})} \le \beta, \qquad \|T_p(t)\|_{L^\infty(\mathcal{O})} \le \beta,$$

for all $t \in [0, t_f]$. Thus, there exists a subsequence still denoted by θ_p, T_p and two functions θ in $L^2(0, t_f; H^1(\mathcal{O})) \cap L^\infty(0, t_f; L^\infty(\mathcal{O})$ and T in $L^\infty(0, t_f; L^\infty(\mathcal{O})$ such that

$$\theta_p \to \theta \quad \text{and} \quad {\theta_p}^m \to \theta^m \text{ in } L^2(0, t_f; L^2(\mathcal{O})) \text{ strong}$$
$$\theta_p \to \theta \text{ in } L^\infty(0, t_f; L^\infty(\mathcal{O})) \text{ weak-*}$$
$$T_p \to T \text{ in } L^\infty(0, t_f; L^\infty(\mathcal{O})) \text{ weak-*}$$

and

$$\nabla({\theta_p}^m) \to \nabla(\theta^m) \text{ in } L^2(0, t_f; L^2(\mathcal{O})) \text{ weak.}$$

Since ζ is Lipschitz continuous, we have $\zeta(\theta_p) \to \zeta(\theta)$ in $L^2(0, t_f; L^2(\mathcal{O}))$ strongly and

$$\zeta(\theta_p)(T_p - \theta_p) \to \zeta(\theta)(T - \theta) \text{ in } L^2(0, t_f; L^2(\mathcal{O})) \text{ weak.}$$

(due to the usual result 4 stated in the appendix). Therefore, we can pass to the limit in (2.93) and (2.94); thus, (θ, T) is a weak solution of the systems (2.69) and (2.70).

Now, note that the equation satisfied by θ is of the type $\partial_t\theta + A(\theta) = \xi(\theta)$ where ξ is a Lipschitz continuous function and A is a continuous operator from H^1 into the space H^{-1}. Due to the classic Aubin–Lions lemma,[6] we know that θ belongs

[6] The Aubin–Lions lemma says that if $\theta \in L^2(0, t, H^1)$ and $\partial_t\theta \in L^2(0, t, H^{-1})$, then $\theta \in C(0, t, L^2)$.(H^{-1} is the dual space of H^1).

to $C(0,t_f;L^2(\mathcal{O}))$; thus, it is a classic solution in the space $L^2(\mathcal{O})$, i.e., we have $\theta(t)-\theta(0)=\int_0^t \xi(\theta(s))ds-\int_0^t A(\theta(s))ds$ for all t.

Moreover, according to result 1 in the appendix, we know that T belongs also to $C(0,t_f;L^2(\mathcal{O}))$.

Uniqueness. Assume that θ and T belong to $L^2(0,t_f;H^1(\mathcal{O}))\cap C(0,t_f;L^2(\mathcal{O}))$ and $C(0,t_f;L^2(\mathcal{O}))$ are solutions to the system (2.69), (2.70), and that $\hat\theta=\theta+Y$ and $\hat T=T+X$ belonging to the same spaces are also solutions to this system.Then we have

$$\rho Z\frac{\partial}{\partial t}Y-\nabla.(\kappa(\nabla\hat\theta^m-\nabla\theta^m))=\rho\zeta(\theta+Y)(T+X-\theta-Y)-\rho\zeta(\theta)(T-\theta)$$

$$\rho\frac{\partial}{\partial t}X=-\rho\zeta(\theta+Y)(T+X-\theta-Y)+\rho\zeta(\theta)(T-\theta)$$

Denote now $S(.)$ the sign function ($S(Y)=1$, if $Y\geq 0$ and $S(Y)=-1$, if $Y<0$) and $S^\varepsilon(.)$ is a regularized function of $S(.)$ that increases with compact support and such that $\lim_\varepsilon S^\varepsilon(Y)=S(Y)$. Then, multiplying by $S^\varepsilon(Y)$ and integrating over $\mathcal{O}$, we get, with $\xi_\theta(Y)=\zeta(\theta+Y)-\zeta(\theta)$,

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial t}\int\rho ZYS^\varepsilon(Y)dx+\lim_\varepsilon\int\kappa\nabla((\theta+Y)^m-\theta^m).\nabla S^\varepsilon(Y)dx$$
$$=\int\rho\zeta(\hat\theta)(X-Y)S(Y)dx+\int\rho\xi_\theta(Y)(T-\theta)S(Y)dx$$

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial t}\int\rho XS^\varepsilon(X)dx=\int-\rho\zeta(\hat\theta)(X-Y)S(X)dx-\int\rho\xi_\theta(Y)(T-\theta)S(X)dx$$

Therefore, using the same kind of argument as above, we get

$$\frac{\partial}{\partial t}\int\rho[Z|Y|+|X|]dx+\lim_\varepsilon B_\varepsilon=\int\rho\zeta(\hat\theta)(X-Y)(S(Y)-S(X))dx+\int\rho\xi_\theta(Y)(T-\theta)(S(Y)-S(X))dx$$

where we have

$$B_\varepsilon=\int m\kappa((\theta+Y)^{m-1}-\theta^{m-1})\nabla Y.\nabla S^\varepsilon(Y)dx$$
$$=\int m\kappa((\theta+Y)^{m-1}-\theta^{m-1})(\partial_Y S^\varepsilon)(Y)|\nabla Y|^2dx\to 0,\qquad \text{if }\varepsilon\to 0.$$

We check that

$$\int\rho\zeta(\hat\theta)(X-Y)(S(Y)-S(X))dx\leq 0.$$

Therefore, since ξ_θ is Lipschitz continuous and $|S(Y) - S(X)| \le 2$, we have

$$\frac{\partial}{\partial t}\int \rho[Z|Y| + |X|]dx \le C\int \rho|Y|dx \le \frac{C}{Z}\int \rho[Z|Y| + |X|]dx$$

Now, according to Gronwal's lemma, we see that

$$\left(\int \rho[Z|\hat{\theta} - \theta| + |\hat{T} - T|]dx\right)(t) \le e^{Ct/Z}\int \rho[Z|\hat{\theta}^{ini} - \theta^{ini}| + |\hat{T}^{ini} - T^{ini}|]dx.$$

and uniqueness follows. □

2.4 Analysis of the Hyperbolic Part of Systems ($\mathcal{E}2\mathcal{T}$) and ($\mathcal{MHD}$)

Here we are concerned only with the "ideal part" of systems ($\mathcal{E}2\mathcal{T}$) and ($\mathcal{MHD}$); the ideal part corresponds to keeping only the terms with first-order spatial derivatives and neglecting the right-hand-side terms (see below). Our aim is to check that this ideal part is hyperbolic: we need to check that the eigenvalues of the matrix of the first-order derivatives are real. To do this, we choose the Lagrangian framework. We introduce the specific volume

$$\tau = 1/\rho,$$

which is the good unknown function for the continuity equation; indeed, we have

$$\rho\frac{D}{Dt}\tau = \nabla.\mathbf{U}.$$

For the ideal part of two models, the electron energy equation reduces to

$$\rho\frac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} = 0$$

and may be replaced by

$$\frac{D}{Dt}\varepsilon_e + P_e\frac{D}{Dt}\tau = 0.$$

But, since $\varepsilon_e = \frac{3}{2m_0}ZT_e$ and $P_e = ZN_0T_e = \frac{2}{3}\varepsilon_e/\tau$, it is equivalent to

$$\frac{D}{Dt}\log(\varepsilon_e\tau^{2/3}) = 0.$$

Then a natural quantity $\varepsilon_e \tau^{2/3} = \frac{3}{2} P_e \tau^{5/3}$ appears, called the electron entropy, and this entropy is preserved by the Lagrangian derivative (it is equal to the physical entropy up to the sign and a multiplicative constant).

Therefore, defining the specific total energy by

$$e = \varepsilon_0 + \varepsilon_e + \frac{1}{2}|\mathbf{U}|^2$$

the ideal part of system $(\mathcal{E}2\mathcal{T})$ has the following form

$$\begin{aligned} \rho \frac{D}{Dt}\tau - \nabla.\mathbf{U} &= 0, \\ \rho \frac{D}{Dt}\mathbf{U} + \nabla P_\text{p} &= 0, \\ \rho \frac{D}{Dt}e + \nabla.(\mathbf{U} P_\text{p}) &= 0, \\ \rho \frac{D}{Dt}\varepsilon_e \tau^{2/3} &= 0. \end{aligned}$$

Set $\mathbf{Y}$ as the vector (of dimension 6) of the physical state and $\mathbb{F}(\mathbf{Y})$ as the tensor, which are then characterized by

$$\mathbf{Y} = \begin{bmatrix} \tau \\ \mathbf{U} \\ e \\ \varepsilon_e \tau^{2/3} \end{bmatrix}, \qquad \text{and } \mathbb{F}(\mathbf{Y}) = \begin{bmatrix} -\mathbf{U} \\ P_\text{p}\mathbb{I} \\ \mathbf{U}P_\text{p} \\ 0 \end{bmatrix}.$$

(recall that $\mathbb{I}$ is the identity tensor), system $(\mathcal{E}2\mathcal{T})$ reads as

$$\rho \frac{D}{Dt}\left[\mathbf{Y}\right] + \nabla.\left[\mathbb{F}(\mathbf{Y})\right] = 0. \tag{2.95}$$

Now for $(\mathcal{MHD})$, the specific total energy is given by

$$e = \varepsilon_0 + \varepsilon_e + \frac{1}{2}\left|\mathbf{U}\right|^2 + \frac{1}{2\mu^0}\tau\left|\mathbf{B}\right|^2.$$

Using the identity $\nabla.(\mathbf{BU}) = \mathbf{U}(\nabla.\mathbf{B}) + (\mathbf{B}.\nabla)\mathbf{U}$ (and the fact that $\nabla.\mathbf{B} = 0$), the magnetic field equation is

$$\rho \frac{D}{Dt}(\tau \mathbf{B}) = \nabla.(\mathbf{BU}).$$

So, the ideal MHD system reads as (2.95) but the vector $\mathbf{Y}$ (of dimension 9) and $\mathbf{F}(\mathbf{Y})$ are now given by

$$\mathbf{Y} = \begin{bmatrix} \tau \\ \mathbf{U} \\ \tau\mathbf{B} \\ e \\ \varepsilon_e \tau^{2/3} \end{bmatrix}, \qquad \text{and } \mathbb{F} = \begin{bmatrix} -\mathbf{U} \\ P_{\rm p}\mathbb{I} + \mathbb{P}_B \\ -\mathbf{U}\mathbf{B} \\ P_{\rm p}\mathbf{U} + \mathbb{P}_B.\mathbf{U} \\ 0 \end{bmatrix}.$$

2.4.1 On the Galilean Transformations

We now make a regular transformation of the unknown variables $\mathbf{Z} = \mathbf{Z}_{\mathbf{u}}(\mathbf{Y})$ corresponding to a fixed translation of vector $\mathbf{u}$, (i.e., a Galilean transformation); so the density ρ is not changed. Then system (2.95) becomes $\rho \dfrac{D}{Dt}[\mathbf{Z}] + \dfrac{\partial \mathbf{Z}}{\partial \mathbf{Y}} \nabla. [\mathbb{F}(\mathbf{Y}(\mathbf{Z}))] = 0$; since the Jacobian matrix $\frac{\partial \mathbf{Z}}{\partial \mathbf{Y}}$ is constant, the new system reads

$$\rho \frac{D}{Dt}[\mathbf{Z}] + \nabla. \left[\tilde{\mathbb{F}}(\mathbf{Z})\right] = 0$$

with $\tilde{\mathbb{F}}(\mathbf{Z}) = \dfrac{\partial \mathbf{Z}}{\partial \mathbf{Y}}.\mathbb{F}(\mathbf{Y}(\mathbf{Z}))$. The system is called "invariant through a Galilean transformation," if for any fixed vector $\mathbf{u}$, after the transformation, we have $\tilde{\mathbb{F}} = \mathbb{F}$.

For the ($\mathcal{MHD}$) model, consider now a Galilean transformation related to a translation $\mathbf{u}$; since we have $\tau' = \tau$, $\varepsilon'_e = \varepsilon_e$, $\varepsilon'_0 = \varepsilon_0$, $\mathbf{U}' = \mathbf{U} - \mathbf{u}$ and $\mathbf{B}' = \mathbf{B}$, the characteristic vector $\mathbf{Z} = \mathbf{Z}_{\mathbf{u}}(\mathbf{Y})$ reads

$$\mathbf{Z} = \begin{bmatrix} \tau' \\ \mathbf{U}' \\ \tau'\mathbf{B}' \\ e' \\ \varepsilon'_e(\tau')^{2/3} \end{bmatrix} = \begin{bmatrix} \tau \\ \mathbf{U} - \mathbf{u} \\ \tau\mathbf{B} \\ e + (\frac{1}{2}|\mathbf{u}|^2 - \mathbf{U}.\mathbf{u}) \\ \varepsilon_e(\tau)^{2/3} \end{bmatrix}$$

Of course, we have $P'_{\rm p} = P_{\rm p}, \mathbb{P}'_B = \mathbb{P}_B$. Since $\mathbf{u}$ is constant, we have $\partial_t \mathbf{u} = 0$ and $\nabla.\mathbf{u} = 0$; then we see that

$$\tilde{\mathbb{F}} = \begin{bmatrix} -\mathbf{U} \\ P_{\rm p}\mathbb{I} + \mathbb{P}_B \\ -\mathbf{U}\mathbf{B} \\ P_{\rm p}\mathbf{U}' + \mathbb{P}_B.\mathbf{U}' \\ 0 \end{bmatrix} = \begin{bmatrix} -\mathbf{U}' \\ P_{\rm p}\mathbb{I} + \mathbb{P}_B \\ -\mathbf{U}'\mathbf{B}' \\ P_{\rm p}\mathbf{U}' + \mathbb{P}_B.\mathbf{U}' \\ 0 \end{bmatrix}.$$

Indeed, for the energy we have

$$\rho\frac{D}{Dt}(e-\mathbf{U}.\mathbf{u}) = \nabla.((P_{\mathrm{p}}\mathbb{I}+\mathbb{P}_B).\mathbf{U}) - \mathbf{u}.\nabla(P_{\mathrm{p}}\mathbb{I}+\mathbb{P}_B) = \nabla.((P_{\mathrm{p}}\mathbb{I}+\mathbb{P}_B).\mathbf{U}')$$

Thus, this model is invariant through this Galilean transformation.

For the model $(\mathcal{E}2\mathcal{T})$, it is the same calculus.

Moreover, for both models, it is crucial to notice that the electric field becomes $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$ and the generalized Ohm's law reads as

$$q_e Z N_0(\mathbf{E}' + \mathbf{U}' \times \mathbf{B}') + \nabla P_e = 0$$

One may also check that in these models, the flux tensor has the general form described in [43].

2.4.2 Hyperbolic Properties of Both Models

Our concern is analysis of the ideal part of both systems, so we focus on the Lagrangian framework in the one-dimensional case, i.e., all the physical quantities depend only on the x-variable. It is useful to introduce the new variable m defined by $dm = \rho dx$; the tensor $\mathbb{F}(\mathbf{Y})$ reduces to a vector $\mathbf{F}$ and the system for the ideal part reads now as

$$\frac{D}{Dt}\mathbf{Y} + \frac{\partial}{\partial m}\mathbf{F}(\mathbf{Y}) = 0. \tag{2.96}$$

(i) For Model $(\mathcal{E}2\mathcal{T})$.

Since τ, e depends only on the x-variable, the evolution of $\mathbf{U} = (U_x\, U_y, U_z)$ reduces to the one of U_x. So we have to deal with the system

$$\begin{aligned} \frac{D}{Dt}\tau - \frac{\partial}{\partial m}U_x &= 0,\\ \frac{D}{Dt}U_x + \frac{\partial}{\partial m}(P_0 + P_e) &= 0,\\ \frac{D}{Dt}e + \frac{\partial}{\partial m}(U_x(P_0 + P_e)) &= 0,\\ \frac{D}{Dt}\log(\varepsilon_e\tau^{2/3}) &= 0. \end{aligned}$$

It is a four-dimensional vector system of type (2.96) with

$$\mathbf{Y} = \begin{bmatrix} \tau \\ U_x \\ e \\ \varepsilon_e\tau^{2/3} \end{bmatrix}, \qquad \text{and } \mathbf{F}(\mathbf{Y}) = \begin{bmatrix} -U_x \\ P_{\mathrm{p}} \\ U_x P_{\mathrm{p}} \\ 0 \end{bmatrix}$$

where we set (assuming that γ_0=5/3)

$$e = \varepsilon_p + \frac{1}{2}U_x^2, \qquad \varepsilon_p = \varepsilon_0 + \varepsilon_e, \qquad P_\mathrm{p} = P_0 + P_e = \frac{2}{3}\varepsilon_p \tau^{-1}.$$

Proposition 8. *The eigenvalues of the Jacobian matrix of the flux term* $\dfrac{\partial \mathbf{F}}{\partial \mathbf{Y}}$ *are*

$$-a_o, \qquad 0, \qquad 0, \qquad a_o, \qquad \textit{where } a_o^2 = \frac{5}{3}\rho P_p$$

It can be seen that for the double eigenvalue 0, the space of the eigenvector is of dimension two, so the system is hyperbolic. We recover that when the original system (2.95) is written in a Eulerian framework, the eigenvalues of the Jacobian matrix are $-a_o/\rho, \quad 0, \quad 0, \quad a_o/\rho$ (when the plasma velocity is zero). Note that

$$\frac{a_o}{\rho} = \sqrt{\frac{5}{3}\frac{P_\mathrm{p}}{\rho}}$$

is the usual expression for the sound speed (of course, the pressure taken into account is the total pressure).

(ii) For Model ($\mathcal{MHD}$).

Recall that $\mathbf{B} = (B_x, B_y, B_z)$ and $\mathbf{U} = (U_x, U_y, U_z)$ which depends only on the x-variable. According to the constraint on the magnetic field $\nabla.\mathbf{B} = 0$, we get $\frac{\partial}{\partial x}B_x = 0$; then we have also $\frac{\partial}{\partial t}B_x = 0$; so B_x is a given constant and the system reads as

$$\begin{aligned}
\frac{D}{Dt}\tau - \frac{\partial}{\partial m}U_x &= 0,\\
\frac{D}{Dt}U_x + \frac{\partial}{\partial m}(P_\mathrm{p} + P_\mathrm{mag}) &= 0,\\
\frac{D}{Dt}U_y - \frac{B_x}{\mu^0}\frac{\partial}{\partial m}B_y &= 0,\\
\frac{D}{Dt}U_z - \frac{B_x}{\mu^0}\frac{\partial}{\partial m}B_z &= 0,\\
\frac{D}{Dt}(B_y\tau) - B_x\frac{\partial}{\partial m}U_y &= 0,\\
\frac{D}{Dt}(B_z\tau) - B_x\frac{\partial}{\partial m}U_z &= 0,\\
\frac{D}{Dt}e + \frac{\partial}{\partial m}(U_x(P_\mathrm{p} + P_\mathrm{mag})) - \frac{\partial}{\partial m}\left((U_yB_y + U_zB_z)\frac{B_x}{\mu^0}\right) &= 0,\\
\frac{D}{Dt}(\varepsilon_e\tau^{2/3}) &= 0,
\end{aligned}$$

where $e = \varepsilon_p + \frac{1}{2}|\mathbf{U}|^2 + \frac{\tau}{2\mu^0}(|B_x|^2 + |B_y|^2 + |B_z|^2)$ and P_p, ε_p defined as above, $P_{\text{mag}} = \frac{1}{2\mu^0}(B_y^2 + B_z^2 - B_x^2)$.

Of course, if $B_x = 0$, we recover model $(\mathcal{E}2\mathcal{T})$. We are now concerned with the eigenvalues of the Jacobian matrix of this system. For the sake of simplicity, assume that U_z and B_z are zero. We simplify the notations by setting $U = U_x$, $V = U_y$, $M = B_y\tau$, $\beta = B_x$. So we get

$$\varepsilon_p = e - \frac{1}{2}U^2 - \frac{1}{2}V^2 - \frac{1}{2\mu^0\tau}M^2 - \frac{\tau}{2\mu^0}\beta^2;$$

$$P_{\text{p}} + P_{\text{mag}} = \frac{2}{3}\frac{1}{\tau}\varepsilon_p + \frac{1}{2\mu^0}\left(\frac{M^2}{\tau^2} - \beta^2\right) = \frac{2}{3}\frac{1}{\tau}\left(e - \frac{U^2}{2} - \frac{V^2}{2}\right) + \frac{1}{6\mu^0\tau^2}M^2 - \frac{5}{6\mu^0}\beta^2$$

and the six-dimensional vector system reads $\frac{D}{Dt}\mathbf{Y} + \frac{\partial}{\partial m}\mathbf{F}(\mathbf{Y}) = 0$, with

$$\mathbf{Y} = \begin{bmatrix} \tau \\ U \\ V \\ M \\ e \\ \varepsilon_e\tau^{2/3} \end{bmatrix}, \qquad \text{and } \mathbf{F}(\mathbf{Y}) = \begin{bmatrix} -U \\ P_{\text{p}} + P_{\text{mag}} \\ -\frac{\beta}{\mu^0}M\tau^{-1} \\ -\beta V \\ U(P_{\text{p}} + P_{\text{mag}}) - \frac{\beta}{\mu^0}VM\tau^{-1} \\ 0 \end{bmatrix}.$$

Proposition 9. *The six eigenvalues of the Jacobian matrix* $\left.\frac{\partial\mathbf{F}}{\partial\mathbf{Y}}\right|_{\mathbf{Y}}$ *are given by*

$$-\sqrt{X_{fast}/\rho}, \qquad \text{-}\sqrt{X_{slow}/\rho}, \qquad 0, \qquad 0, \qquad \sqrt{X_{slow}/\rho}, \qquad \sqrt{X_{fast}/\rho}$$

where

$$X_{slow} = \frac{1}{2}\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right) - \frac{1}{2}\left[\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right)^2 - 4\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}\right]^{1/2}$$

$$X_{fast} = \frac{1}{2}\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right) + \frac{1}{2}\left[\left(\frac{a_0^2}{\rho} + \frac{|\mathbf{B}|^2}{\mu^0}\right)^2 - 4\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}\right]^{1/2}.$$

Therefore, the six eigenvalues λ of the original problem are the following

$$-\sqrt{\rho X_{fast}}, \qquad -\sqrt{\rho X_{slow}}, \qquad 0, \qquad 0, \qquad \sqrt{\rho X_{slow}}, \qquad \sqrt{\rho X_{fast}}.$$

Remark 12. It may be checked that if $|\mathbf{B}|$ goes to zero, the two roots X_{slow} and X_{fast} converge to 0 and a_0^2/ρ, and the eigenvalues λ converge to

$$-a_0, \qquad 0, \qquad 0, \qquad 0, \qquad 0, \qquad a_0.$$

(recall that $\pm a_0$ are exactly the eigenvalues of the model ($\mathcal{E}2\mathcal{T}$)). One may prove (see [43]) that the dimension of the eigenspace corresponding to 0 is equal to 4, thus the system is strictly hyperbolic.

Assume on the contrary that the sound speed a_0 is very small compared to the magnetic energy; then we have $\left[\left(\frac{a_0^2}{\rho}+\frac{|\mathbf{B}|^2}{\mu^0}\right)^2-4\frac{a_0^2}{\rho}\frac{\beta^2}{\mu^0}\right]^{1/2} \simeq \frac{|\mathbf{B}|^2}{\mu^0}+\frac{a_0^2}{\rho}(1-2\beta^2/|\mathbf{B}|^2)$, then we get

$$\rho X_{slow} \simeq a_0^2 \frac{\beta^2}{|\mathbf{B}|^2}$$

$$\rho X_{fast} \simeq \rho \frac{|\mathbf{B}|^2}{\mu^0} + a_0^2 (1 - \frac{\beta^2}{|\mathbf{B}|^2})$$

So, we see that the two eigenvalues λ related to X_{fast} are at first order equal to $\pm\sqrt{\rho/\mu^0}|\mathbf{B}|$ and the corresponding characteristic speeds are equal to $\pm|\mathbf{B}|/\sqrt{\rho\mu^0}$, that is to say the Alfven speed. □

2.4.3 Proofs of the Propositions of the Section

The following lemma will be useful.

Lemma 3. *If a system* $\frac{D}{Dt}[\mathbf{Y}] + \frac{\partial}{\partial m}[\mathbf{F}(\mathbf{Y})] = 0$ *is invariant through a Galilean transformation, the eigenvalues of the Jacobian matrix* $\left.\frac{\partial \mathbf{F}}{\partial \mathbf{Y}}\right|_{\mathbf{Y}}$ *are the same as the ones of the Jacobian matrix* $\left.\frac{\partial \mathbf{F}}{\partial \mathbf{Y}}\right|_{\mathbf{Y}^0}$ *when* Y^0 *is the state obtained by setting* $U = 0$.

Proof. It is clear that if we apply a transformation corresponding to a fixed translation of vector $\mathbf{u}$, the new system will have a Jacobian matrix

$$\left.\frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{Z}}\right|_{\mathbf{Z}} = \frac{\partial \mathbf{Z}}{\partial \mathbf{Y}} . \left.\frac{\partial \mathbf{F}}{\partial \mathbf{Y}}\right|_{\mathbf{Y}(\mathbf{Z})} . \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}$$

Since $\frac{\partial \mathbf{Z}}{\partial \mathbf{Y}}$ is the inverse of $\frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}$, if λ is an eigenvalue for the matrix $\left.\frac{\partial \mathbf{F}}{\partial \mathbf{Y}}\right|_{\mathbf{Y}}$, it is also an eigenvalue for the matrix $\left.\frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{Z}}\right|_{\mathbf{Z}}$. For a given point $\mathbf{x}$, we can apply this remark by choosing $\mathbf{u} = -\mathbf{U}(\mathbf{x})$, then the state $\mathbf{Z}$ corresponds to the same state as the original one, but the velocity $\mathbf{U}'$ is equal to 0 at point $\mathbf{x}$. So, according to the fact that $\tilde{\mathbf{F}} = \mathbf{F}$, the lemma follows. □

Proof of Proposition 8. The Jacobian matrix of the flux term is the matrix $\dfrac{\partial \mathbf{F}}{\partial \mathbf{Y}}$ given by

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -\frac{2}{3}\varepsilon_t \tau^{-2} & -\frac{2}{3}U_x \tau^{-1} & \frac{2}{3}\tau^{-1} & 0 \\ -\frac{2}{3}U_x \varepsilon_t \tau^{-2} & \frac{2}{3}\varepsilon_t \tau^{-1} - \frac{2}{3}U_x^2 \tau^{-1} & \frac{2}{3}U_x \tau^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the characteristic polynomial is $\lambda \det(A - \lambda I) = 0$, where A is the same as above without the last row and column, i.e.,

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -\rho P_{\mathrm{p}} & -\frac{2}{3}U_x \rho & \frac{2}{3}\rho \\ -U_x \rho P_{\mathrm{p}} & P_{\mathrm{p}} - \frac{2}{3}U_x^2 \rho & \frac{2}{3}U_x \rho \end{bmatrix}$$

According to the previous lemma, one may evaluate the eigenvalues of A after setting U_x to zero and we get

$$\det(A - \lambda I) = \lambda^3 - \frac{5}{3}\rho P_{\mathrm{p}} \lambda. \qquad \square$$

Proof of Proposition 9. The Jacobian matrix of the flux term $\left[\frac{\partial \mathbf{F}}{\partial \mathbf{Y}}\right]$ reduces (after withdrawing the last column and the last row) to the matrix given by

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -\frac{2e-|\mathbf{U}|^2}{3\tau^2} - \frac{1}{3\mu^0\tau^3} M^2 & -\frac{2}{3\tau}U & -\frac{2}{3\tau}V & \frac{1}{3\mu^0} M \frac{1}{\tau^2} & \frac{2}{3}\frac{1}{\tau} \\ \frac{\beta}{\mu^0} M \frac{1}{\tau^2} & 0 & 0 & -\frac{\beta}{\mu^0}\frac{1}{\tau} & 0 \\ 0 & 0 & -\beta & 0 & 0 \\ -U\frac{2e-|\mathbf{U}|^2}{3\tau^2} + \frac{UM^2}{3\mu^0\tau^3} + \frac{\beta VM}{\mu^0\tau^2} & P_{\mathrm{p}} + P_{\mathrm{mag}} - \frac{2}{3\tau}U^2 & -\frac{2}{3\tau}UV - \frac{\beta}{\mu^0\tau}M & \frac{1}{3\mu^0\tau^2}UM - \frac{\beta}{\mu^0\tau}V & \frac{2}{3\tau}U \end{bmatrix}$$

According to the previous lemma, one may evaluate the eigenvalues of this matrix after setting $U = V = 0$. Then, going back to the physical variables, we have to compute the characteristic polynomial $\det(A_0 - \lambda I)$ where

$$A_0 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -\rho P_{\mathrm{p}} - \rho R & 0 & 0 & \frac{1}{3\mu^0}\rho B_y & \frac{2}{3}\rho \\ \frac{\beta}{\mu^0}\rho B_y & 0 & 0 & -\frac{\beta}{\mu^0}\rho & 0 \\ 0 & 0 & -\beta & 0 & 0 \\ 0 & P_{\mathrm{p}} + P_{\mathrm{mag}} & -\frac{\beta}{\mu^0}B_y & 0 & 0 \end{bmatrix}$$

[here we have set $R = \frac{1}{3\mu^0}(2B_y^2 + \beta^2)$] and this polynomial reads as

$$\begin{vmatrix} -\lambda & -1 & 0 & 0 & 0 \\ -\rho P_\mathrm{p} - \rho R & -\lambda & 0 & \frac{1}{3\mu^0}\rho B_y & \frac{2}{3}\rho \\ \frac{\beta}{\mu^0}\rho B_y & 0 & -\lambda & -\frac{\beta}{\mu^0}\rho & 0 \\ 0 & 0 & -\beta & -\lambda & 0 \\ 0 & P_\mathrm{p} + P_\mathrm{mag} & -\frac{\beta}{\mu^0} B_y & 0 & -\lambda \end{vmatrix} = 0$$

and we get after some tedious calculus

$$-\lambda^5 + \lambda^3 \rho \left[P_\mathrm{p} + R + \frac{2}{3} P_\mathrm{p} + \frac{2}{3} P_\mathrm{mag} + \frac{\beta^2}{\mu^0} \right] + \lambda \rho^2 \left[-\frac{2}{3} \frac{\beta^2}{\mu^0} (P_\mathrm{p} + P_\mathrm{mag}) + \frac{\beta^2}{\mu^{02}} B_y^2 - \frac{\beta^2}{\mu^0} (P_\mathrm{p} + R) \right] = 0.$$

We see that 0 is, of course, a root of this equation. The other roots are obtained by solving a second-degree equation with $\lambda^2 = \rho X$, where this equation in X reads as follows (using the notation $a_0^2 = \frac{5}{3}\rho P_p$ and noticing that $\frac{2}{3} P_\mathrm{mag} + R = \frac{B_y^2}{\mu^0}$)

$$X^2 - X \left[\frac{a_0^2}{\rho} + \frac{B_y^2 + \beta^2}{\mu^0} \right] + \left[\frac{a_0^2}{\rho} \frac{\beta^2}{\mu^0} \right] = 0.$$

Its discriminant is $\Delta = \left(\frac{a_0^2}{\rho} + \frac{B_y^2 + \beta^2}{\mu^0} \right)^2 - 4 \frac{a_0^2}{\rho} \frac{\beta^2}{\mu^0}$ [which is always greater than $(\frac{a_0^2}{\rho} - \frac{\beta^2}{\mu^0})^2$]. Since $B_z = 0$, notice that $\frac{B_y^2 + \beta^2}{2\mu^0} = \frac{|\mathbf{B}|^2}{\mu^0}$ is the magnetic energy and the two roots are X_{slow} and X_{fast} given in the proposition. Then the eigenvalues of the initial Jacobian matrix are

$$-\sqrt{X_{fast}}, -\sqrt{X_{slow}}, 0, 0, \sqrt{X_{slow}}, \sqrt{X_{fast}}. \qquad \square$$

Chapter 3
Laser Propagation: Coupling with Ion Acoustic Waves

Abstract This chapter contains most of fundamental concepts for the laser–plasma interaction. We first derive the paraxial approximation for the laser propagation from the full Maxwell equations. This is done by using a time envelope model and performing the Wenzel–Kramer–Brillouin (WKB) expansion. By the way, we compare the geometrical optics approximation and the paraxial approximation. In the second part of this chapter, we focus on the modelling of the Brillouin instability which corresponds to a coupling of the laser waves and an ion acoustic wave. This leads to the so-called three-wave coupling system which was introduced 40 years ago by Kadomstev. We give some crucial mathematical properties of this system, which are new according to our knowledge and which enable a better understanding of the structure of the three-wave coupling system.

Keywords Laser-plasma interaction • Geometrical optics • Paraxial approximation • Ponderomotive force • WKB. expansion • Brillouin instability • Three-wave coupling system

In some experimental devices, such as the ones for Inertial Confinement Fusion (ICF), laser beams are designed to propagate in a plasma and to heat a specified region of this plasma. So, it is of particular importance to use accurate and well-adapted modelling for the propagation of a laser beam and its interaction with the plasma. Recall that a laser beam is a monochromatic electromagnetic planar wave with a wavelength on the order of 1 μm and different phenomena occur during the propagation. Of course, if the plasma is not very dense and the laser intensity is not very large, the beam propagates according to the laws of classical optics, accounting for the deflection (due to the variation of the index of refraction), the diffraction, and the absorption of laser energy by the plasma. When the laser intensity is large enough, there are different phenomena related to the occurrence of the so-called *ponderomotive force*, which are created by the pressure of laser light onto the plasma—e.g., the digging of the plasma that leads an autofocusing of the

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laser beam; a coupling of the laser waves with the ion plasma waves that leads to the so-called *Brillouin instability*; and, lastly a coupling of the laser waves with the electron plasma waves, which leads to the so-called *Raman instability*.

This chapter is devoted to problems related to laser propagation, laser autofusing, and the coupling with ion plasma waves. In all cases the plasma behavior is described by an ion fluid model with the quasi-neutral approximation. The coupling of the laser fields with the electron plasma waves will be addressed in Chap. 5.

In this kind of problem, some typical lengths and some typical speeds occur which range over several orders of magnitude. The laser field is assumed to be purely monochromatic, it is characterized by electromagnetic fields oscillating with a pulsation ω_0 (called *laser frequency* in the sequel). Considering the spatial variable, besides the Debye length λ_D, three other scales are relevant:

1. the typical length L_{plasma} of variation of the mean density of the plasma;
2. the typical length L_l of variation of the amplitude of the laser intensity, $2\pi L_l$ is on the order of the width of the speckles (which are hot spots of light in the laser beam); L_l is also the length of variation of the local perturbation of the plasma density;
3. the laser wavelength in the vacuum which is fixed and equal to $2\pi/k_0$, with

$$k_0 = \omega_0/c.$$

For example, in Inertial Confinement Fusion experiments with high-power intensity laser beams, the wavelength is equal to a fraction of one micron (typically 0.351μm or 0.522μm), L_l is typically on the order of one micron (then $k_0 L_l \approx 10$). For a typical plasma, the ion temperature is on the order of 10^7 Kelvin and electron temperature of 4. 10^7 Kelvin and the Debye length in the interesting region for laser interaction is smaller than 0.04 microns; L_{plasma} is larger than 100 microns.

Then, for the observation length L_{obs} of interest, the picture is the following

$$\lambda_D \ll 2\pi/k_0 \ll 2\pi L_l \ll L_{\text{obs}} \sim L_{\text{plasma}}$$

On the other hand, compared to the speed of light c, the electron thermal speed (proportional to the square root of the temperature) and the ion sound speed are very small: the electron thermal speed is on the order of $0.05c$ and the ion sound speed is on the order of $10^{-3}c$. The observation times that interest us in this chapter are on the order of

$$T_{\text{obs}} \sim \frac{L_{\text{obs}}}{v_{\text{th,i}}}$$

in the framework of laser propagation; or

$$T_{\text{obs}} \sim \frac{L_{\text{obs}}}{\sqrt{c}\sqrt{v_{\text{th,i}}}}$$

in the framework of the Brillouin instability.

The chapter is organized as follows. In the first section we deal with laser propagation and the phenomena that develop at the scale of the width of the speckles L_l (or larger): the deflection due to the variation of the optical index, the absorption, the diffraction, and the "digging" of the plasma by the ponderomotive force. One of our aims is to justify the so-called paraxial approximation. In the second subsection, we focus on the laser–plasma interaction, strictly speaking, which is related to much smaller spatial scales (on the order of the laser wavelength) and we deal with the Brillouin instabilities corresponding to an interplay between ion plasma waves and laser waves.

3.1 Laser Propagation in a Plasma

The laser propagation corresponds to the propagation of the highly oscillating electromagnetic fields with a fixed pulsation ω_0 in a medium with a non constant optical index and the coupling with the plasma hydrodynamics. So, we have to derive a model for the time envelope of these electromagnetic fields from the full Maxwell equations. In particular, we show how the electromagnetic forces may be reduced to the ponderomotive force. Within the quasi-neutral approximation, it leads to the so called *basic time envelope model* (see ($\mathcal{BTE}$) below), where the unknown functions are the time envelope of the transverse laser field, the density, and the ion velocity of the plasma. This system consists in a Schrödinger equation coupled with the ion Euler equation (e.g., it is stated in [52]).

From a numerical point of view, some realistic two-dimensional simulations based on this model are now feasible; but they require very expensive computations since at least ten cells per wavelength are necessary to get accurate results. So, the Wenzel–Kramer–Brillouin (WKB) expansion is made in order to have a simplified modelling.

We explain the derivation of this modelling below: the classical *geometrical optics* model in the second subsection, and the paraxial model in the third subsection. The justification of this last model is obtained by performing the same asymptotic expansion as the one for geometrical optics but one order further. It is done in a framework where there is an incident angle between the direction of the incoming laser wave and the gradient of the mean density of the plasma. If the incident angle is zero, one recovers the classical paraxial equation. We sketch briefly some features concerning the numerical methods for this model. Paraxial models of this type have been used for a long time in laser–plasma interaction; see, e.g., [7, 13, 50].

3.1.1 On the Time Envelope Models

Denote by N_c, the critical density defined by

$$\omega_0^2 = N_c q_e^2/(\varepsilon^0 m_e),$$

i.e., the plasma frequency corresponding to the density N_c is equal to $\omega_{0.}$. We introduce also the electron–ion collision frequency

$$\nu_{ei} = \nu_{e0} N_0 / m_e.$$

[Often, ν_{ei} is assumed to be constant in the considered spatial domain].

The starting point of the modelling is the theoretical electrodynamics model (2.1)–(2.3), (2.5) coupled with the full Maxwell equations. In this kind of problem, one generally assumes that there is no shock, then for the sake of simplicity one claims that the ion fluid is adiabatic, i.e., the ion pressure is simply proportional to the ion density to the power γ. For the electron temperature we make also a similar assumption (see below).

In the whole chapter, we assume that, at the entrance of spatial domain, the main laser wave travels in a direction defined by unit vector $\mathbf{e}_{tr}$. We have to make the following decomposition of the total electromagnetic fields. On the one hand, consider the part of these fields related to the monochromatic laser beam $\mathbf{E}^r, \mathbf{B}^r$: they are rapidly oscillating with frequency ω_0 and they are called transverse fields (for reasons given below). On the other hand, there is a slowly varying part of the electric field $\mathbf{E}^e$ called the electrostatic field. So we state

$$\mathbf{E} = \mathbf{E}^r + \mathbf{E}^e, \quad \mathbf{B} = \mathbf{B}^r. \tag{3.1}$$

In this chapter, we neglect the slowly varying part of the magnetic field.

In the same way, we decompose also the electron velocity into two components

$$\mathbf{U}_e = \mathbf{U}_e^r + \mathbf{U}_e^e,$$

the one $\mathbf{U}_e^r$, called transverse velocity, is the rapidly oscillating part (with a pulsation equal to ω_0) and the other one $\mathbf{U}_e^e$, called the electrostatic component, is the slowly varying part. Now, since the plasma is driven by the laser electromagnetic fields, we may notice that

$$|\mathbf{E}^e| \ll |\mathbf{E}^r|,$$

and in the same way,

$$|\mathbf{U}_e^e| \ll |\mathbf{U}_e^r|.$$

Since the mass ratio between the ion and electron is large, the ion velocity may be considered to be negligible with respect to the electron velocity and there is no ion contribution to the electric transverse current. So we set $\mathbf{J} = \mathbf{J}^r + \mathbf{J}^e$ with

$$\text{(i) } \mathbf{J}^r = -q_e N_e \mathbf{U}_e^r, \qquad \text{(ii) } \mathbf{J}^e = q_e Z N_0 \mathbf{U} - q_e N_e \mathbf{U}_e^e, \tag{3.2}$$

and we have also

$$|\mathbf{J}^e| \ll |\mathbf{J}^r|.$$

3.1.1.1 Decomposition of the Electromagnetic Fields

The fast oscillating electromagnetic fields satisfy the Maxwell equations

$$\text{(i)} \qquad \frac{\partial}{\partial t}\mathbf{E}^r - c^2 \text{curl}\,\mathbf{B}^r + \frac{1}{\varepsilon^0}\mathbf{J}^r = 0, \tag{3.3}$$

$$\text{(ii)} \qquad \frac{\partial}{\partial t}\mathbf{B}^r + \text{curl}\,\mathbf{E}^r = 0.$$

So, taking the time derivative of the first equation and the curl of the second one, we get the classical equation

$$\frac{\partial^2}{\partial t^2}\mathbf{E}^r + c^2 \text{curl}\,(\text{curl}\,\mathbf{E}^r) = -\frac{1}{\varepsilon^0}\frac{\partial}{\partial t}\mathbf{J}^r. \tag{3.4}$$

Remark 13. Before dealing with the evaluation of the transverse electric current $\mathbf{J}^r$, recall some classical features about a purely monochromatic planar wave. Let us assume that it propagates in a linear medium, i.e., the electromagnetic fields satisfying (3.3) are in the form

$$\mathbf{E}^r(t,\mathbf{x}) = \mathbf{E}'(\mathbf{x})\frac{1}{2}e^{-i\omega_0 t} + c.c., \qquad \mathbf{B}^r(t,\mathbf{x}) = \mathbf{B}'(\mathbf{x})\frac{1}{2}e^{-i\omega_0 t} + c.c.$$

with $\mathbf{E}'$ and $\mathbf{B}'$ time independent, and the current $\mathbf{J}^r$ is given by $\mathbf{J}^r(t,\mathbf{x}) = \mathbf{J}'(\mathbf{x})\frac{1}{2}e^{-i\omega_0 t} + c.c.$ where $\mathbf{J}' = (\sigma + i\omega_0 m)\mathbf{E}'$, with σ and m being positive constants. Then, Maxwell equations read as

$$-i(\omega_0 + i\sigma - \omega_0 m)\mathbf{E}' = c^2 \text{curl}\,\mathbf{B}', \qquad i\omega_0\mathbf{B}' = \text{curl}\,\mathbf{E}'.$$

and $\mathbf{E}'$ solves the Helmholtz equation $\omega_0^2(1 + \frac{i\sigma}{\omega_0} - m)\mathbf{E}' + c^2\Delta\mathbf{E}' = 0$ (since $\nabla.\mathbf{E}' = 0$ and curl (curl $\mathbf{E}'$) $= -\Delta\mathbf{E}'$). To enable the propagation, we must have $m < 1$ and we check that $\mathbf{E}'$ and $\mathbf{B}'$ defined by

$$\mathbf{E}'(\mathbf{x}) = \mathbf{f}e^{i\mathbf{e}_{tr}.\mathbf{x}k_0\sqrt{1-m+i\sigma/\omega_0}}, \qquad \mathbf{B}'(\mathbf{x}) = i\,\frac{\sqrt{1-m+i\sigma/\omega_0}}{c}\mathbf{e}_{tr}\times\mathbf{f}e^{i\mathbf{e}_{tr}.\mathbf{x}k_0\sqrt{1-m+i\sigma/\omega_0}},$$

are solutions of Maxwell equations with $\mathbf{f}$ a constant complex vector orthogonal to $\mathbf{e}_{tr}$.

Here $\sqrt{1-m+i\sigma/\omega_0}$ is the square root with a positive imaginary part.

So the picture is the classical one: $\mathbf{E}^r$ and $\mathbf{B}^r$ belong to the plane orthogonal to $\mathbf{e}_{tr}$, we have $\nabla.\mathbf{E}^r = \nabla.\mathbf{J}^r = 0$ and $c|\mathbf{B}^r|$ is on the order of magnitude of $|\mathbf{E}^r|$.

By using a coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ where the third component is in the direction of $\mathbf{e}_{tr}$, we have $\mathbf{f} = (f_1, f_2, 0)$ and the angle $\text{Arg}(f_2/|f_2|) - \text{Arg}(f_1/|f_1|)$ is related to the phase between the two directions of polarization of the wave. □

Let us go back to the evaluation of $\mathbf{J}^r$. Toward this aim, starting with the rapidly oscillating part of electron momentum equation (2.5), which reads as

$$\frac{\partial}{\partial t}\mathbf{J}^r+(\nabla.\mathbf{U}_e)\mathbf{J}^r+(\mathbf{U}_e.\nabla)\mathbf{J}^r-\frac{q_e}{m_e}\nabla P_e+\nu_{ei}\mathbf{J}^r = \frac{q_e^2}{m_e}N_e\mathbf{E}^r+\frac{q_e^2}{m_e}N_e\mathbf{U}_e^e\times\mathbf{B}^r-\frac{q_e}{m_e}\mathbf{J}^r\times\mathbf{B}^r, \tag{3.5}$$

we claim that this equation may be replaced by the following,

$$\frac{\partial}{\partial t}\mathbf{J}^r + \nu_{ei}\mathbf{J}^r = \frac{q_e^2}{m_e}N_e\mathbf{E}^r. \tag{3.6}$$

[**Justification** of (3.6).

According to the remark about plane waves, we may assume that $\mathbf{E}^r, \mathbf{B}^r$ and $\mathbf{J}^r, \mathbf{U}_e^r$ are in the plane orthogonal to $\mathbf{e}_{tr}$, that $|\frac{\partial}{\partial x_i}\mathbf{J}^r|$ is smaller than $k_0|\mathbf{J}^r|$, and that $c|\mathbf{B}^r|$ is on the order of magnitude of $|\mathbf{E}^r|$. Now we address the fast oscillating part of both sides of (3.5). Since $|\frac{\partial}{\partial x_i}\mathbf{J}^r|$ is smaller than $k_0|\mathbf{J}^r|$ and $|\mathbf{U}_e^e|$ is negligible with respect to c, we see that $(\mathbf{U}_e^e.\nabla)\mathbf{J}^r$ is negligible with respect to $|\frac{\partial}{\partial t}\mathbf{J}^r|$ which is on the order of $\omega_0|\mathbf{J}^r| = ck_0|\mathbf{J}^r|$; In the same way $(\nabla.\mathbf{U}_e^e)\mathbf{J}^r$ is also negligible with respect to $\frac{\partial}{\partial t}\mathbf{J}^r$. Now, since $|\mathbf{U}^e\times\mathbf{B}^r|$ is negligible with respect to $|c\mathbf{B}^r|$, it also is negligible with respect to $|\mathbf{E}^r|$. Moreover, since $\mathbf{J}^r$ and $\mathbf{B}^r$ are both oscillating at frequency ω_0, in the product $\mathbf{J}^r\times\mathbf{B}^r$ there are terms oscillating at frequency $2\omega_0$ and terms that are slowly varying with time; so their contribution to (3.5) is zero. In the same way, $(\mathbf{U}_e^r.\nabla)\mathbf{J}^r$ and $(\nabla.\mathbf{U}_e^r)\mathbf{J}^r$ do not contribute to (3.5). Then, we get

$$\frac{\partial}{\partial t}\mathbf{J}^r - \frac{q_e}{m_e}\nabla P_e + \nu_{ei}\mathbf{J}^r = \frac{q_e^2}{m_e}N_e\mathbf{E}^r.$$

Now, notice that in the electron density equation $q_e\frac{\partial}{\partial t}N_e = \nabla.\mathbf{J}^r + \nabla.\mathbf{J}^e$, the first component of the r.h.s. may be withdrawn (as a matter of fact, it is zero for a pure monochromatic planar wave); then the fast oscillating part of $\frac{\partial}{\partial t}N_e$ may also be neglected; therefore, the contribution of term $\frac{q_e}{m_e}\nabla P_e$ may be withdrawn and (3.5) reduces to (3.6). □]

Strictly speaking we need to address system (3.6) (3.4) for $\mathbf{E}^r$ and $\mathbf{J}^r$. Considering this system, we see that if

$$\nabla.\mathbf{E}^r = 0, \qquad \text{and } \nabla.\mathbf{J}^r = 0 \tag{3.7}$$

is true at the initial time, this is also true at any time. Thus, in the sequel we assume that (3.7) holds.

Using the definition of N_c, the final equation (3.6) reads now as

$$\frac{1}{\varepsilon^0}\frac{\partial}{\partial t}\mathbf{J}^r + \frac{\nu_{ei}}{\varepsilon^0}\mathbf{J}^r = \omega_0^2\frac{N_e}{N_c}\mathbf{E}^r. \tag{3.8}$$

Usually ν_{ei} is small with respect to ω_0; then we can make a last simplification of this equation by a simple perturbation argument. As a matter of fact, at order zero when ν_{ei} vanishes, we have

$$\frac{1}{\varepsilon^0}\frac{\partial \mathbf{J}^r}{\partial t} \simeq -\omega_0^2\frac{N_e}{N_c}\mathbf{E}^r;$$

moreover, since the field $\mathbf{J}^r$ is rapidly oscillating at the frequency ω_0, in a first approximation we have $-\omega_0^2\mathbf{J}^r \simeq \frac{\partial^2}{\partial t^2}\mathbf{J}^r$, so by taking the time derivative of (3.8), we get at order zero

$$\frac{1}{\varepsilon^0}\mathbf{J}^r \simeq -\frac{N_e}{N_c}\frac{\partial}{\partial t}\mathbf{E}^r$$

Therefore, at the first order with respect to ν_{ei}, (3.8) may be replaced by

$$\frac{1}{\varepsilon^0}\frac{\partial}{\partial t}\mathbf{J}^r = \omega_0^2\frac{N_e}{N_c}\mathbf{E}^r + \nu_{ei}\frac{N_e}{N_c}\frac{\partial}{\partial t}\mathbf{E}^r, \tag{3.9}$$

Using this relation and (3.7), (3.4) reads finally as a classical wave equation with diffraction and absorption:

$$\frac{\partial^2}{\partial t^2}\mathbf{E}^r - c^2\Delta\mathbf{E}^r + \omega_0^2\frac{N_e}{N_c}\mathbf{E}^r + \nu_{ei}\frac{N_e}{N_c}\frac{\partial}{\partial t}\mathbf{E}^r = 0. \tag{3.10}$$

Now, we can make the time envelope of the solution of this equation.

First, since $\nabla.\mathbf{B}^r = 0$, we introduce the potential vector $\omega_0^{-1}\mathbf{A}^r$ defined by

$$\text{curl }\mathbf{A}^r = \omega_0\mathbf{B}^r$$

with the gauge $\nabla.\mathbf{A}^r = 0$; then we have

$$\omega_0\mathbf{E}^r = -\frac{\partial}{\partial t}\mathbf{A}^r$$

Since the electromagnetic fields are rapidly oscillating with pulsation ω_0, it is convenient to introduce the time envelope and to set

$$\mathbf{A}^r(t,\mathbf{x}) = \frac{1}{2}\mathbf{A}(t,\mathbf{x})e^{-i\omega_0 t} + c.c.$$

where $\mathbf{A}$ is a slowly time-varying quantity. Of course, we have

$$\mathbf{B}^r = \frac{1}{2\omega_0}\text{curl }\mathbf{A}e^{-i\omega_0 t} + c.c. \qquad \text{and} \qquad \mathbf{E}^r = \mathbf{A}\frac{i}{2}e^{-i\omega_0 t} + c.c.. \tag{3.11}$$

Now, since $\mathbf{A}$ is slowly time varying, we can approximate $\frac{\partial^2}{\partial t^2}(e^{-i\omega_0 t}\mathbf{A})$ by the expansion

$$(-\omega_0^2\mathbf{A}+2i\omega_0\frac{\partial}{\partial t}\mathbf{A})e^{-i\omega_0 t}$$

and (3.10) reads as

$$2i\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}+\frac{1}{k_0}\Delta\mathbf{A}+k_0(1-\frac{N_e}{N_c})\mathbf{A}+i\nu_0\mathbf{A}=0, \tag{3.12}$$

with

$$\nu_0=\nu_{ei}\frac{N_e}{cN_c}$$

Notice also that due to (3.9) we have

$$\frac{1}{\varepsilon^0}\mathbf{J}^r=-(\omega_0-i\nu_{ei})\frac{N_e}{N_c}\mathbf{A}\frac{1}{2}e^{-i\omega_0 t}+c.c.. \tag{3.13}$$

See, e.g., [75] (Chaps. 3–6) and [95] (Chaps. 1 and 5) for a physical presentation of the above derivation, in particular for a justification of formulas (3.8) and (3.9) of the plasma conductivity. See also [100, 101].

Orientation

It is worth to noticing that in a plasma, electron Langmuir waves may be excited or not. That corresponds with the two different ways of dealing with the electrostatic field $\mathbf{E}^e$ and two families of models.

- *Family 1. The electron Langmuir waves are accounted for.*

In this framework, one must state wave equations for the electron density N_e and for the electrostatic field $\mathbf{E}^e$. These models are necessary if one needs to consider the Raman instabilities. They will be addressed in Chap. 4.

- *Family 2. One ignores the electron Langmuir waves.*

In this framework, one makes the electron massless approximation, so the electron density is given by the Poisson equation, as was explained in Chap. 2.

As was also noted earlier, we focus in this chapter only on the second way and neglect the coupling with the electron Langmuir waves.

3.1.1.2 Coupling Laser Waves and Ion Waves

The aim is now to derive a basic time envelope model that is a good way for the understanding the coupling of the laser waves and the acoustic ion waves. Beside the propagation, it takes into account the diffraction, the refraction, and the autofocusing of the laser beam; it is also relevant for dealing with the filamentation instability phenomena and the Brillouin instability phenomena.

Our starting point is the model that consists in the above time envelope equation for the potential vector (3.12) coupled with the Euler equations for ions and electrons (2.1)–(2.5) where the transverse components of electromagnetic fields are given by (3.11).

According to the relation (3.7), the electric Gauss relation reduces to the following for the electrostatic field $\mathbf{E}^e$:

$$\nabla.\mathbf{E}^e + \frac{1}{\varepsilon^0} q_e(ZN_0 - N_e) = 0. \tag{3.14}$$

Now, in order to obtain the low-frequency ion fluid model, we make the massless approximation and afterwards the quasi-neutrality approximation that is crucial for a useful modelling of the plasma.

- *The electrostatic field and the ponderomotive force.*

First, one checks that the slowly varying component of the electron momentum equation (2.5) reads

$$m_e \left(\frac{\partial}{\partial t}(N_e\mathbf{U}_e^e) + \nabla.(N_e\mathbf{U}_e^e\mathbf{U}_e^e) + N_e(\mathbf{U}_e^r.\nabla)\mathbf{U}_e^e + (\nabla.(N_e\mathbf{U}_e^r))\mathbf{U}_e^e\right) + \nabla P_e + q_e N_e \mathbf{E}^e =$$
$$-q_e N_e \mathbf{U}_e \times \mathbf{B}^r - m_e \nabla.(N_e \mathbf{U}_e^r \mathbf{U}_e^r) + \nu_{ei} m_e N_e (\mathbf{U} - \mathbf{U}_e^e).$$

Since we deal with the low-frequency evolution, we may take the mean value over a laser period of both sides of the equation. Denoting by the bracket $\langle . \rangle$ the time integral over a laser period $\frac{2\pi}{\omega_0}$, according to the approximation (3.13), we have $\langle N_e\mathbf{U}_e^r\rangle = 0$, $\langle \nabla.(N_e\mathbf{U}_e^r)\rangle = 0$ and also $\langle (N_e\mathbf{U}_e^r.\nabla)\mathbf{U}_e^e\rangle = 0$. According to (3.11), we have $\langle \mathbf{U}_e^e \times \mathbf{B}^r\rangle = \mathbf{U}_e^e \times \langle \mathbf{B}^r\rangle = 0$. Then, neglecting the friction term, we have

$$\frac{m_e}{N_e}\left(\frac{\partial}{\partial t}(N_e\mathbf{U}_e^e) + \nabla.(N_e\mathbf{U}_e^e\mathbf{U}_e^e)\right) + \frac{1}{N_e}\nabla P_e + q_e\mathbf{E}^e \simeq -m_e\left\langle \frac{q_e}{m_e}\mathbf{U}_e^r \times \mathbf{B}^r + \frac{1}{N_e}\nabla.(N_e\mathbf{U}_e^r\mathbf{U}_e^r)\right\rangle.$$

Due to the electron massless approximation, we neglect as usual the time derivative and assume that $m_e N_e |\mathbf{U}_e^e|^2$ is negligible with respect to $N_e T_e$, so we arrive at the following relation

$$\frac{1}{N_e}\nabla P_e + q_e\mathbf{E}^e \simeq -m_e\left\langle \frac{q_e}{m_e}\mathbf{U}_e^r \times \mathbf{B}^r + \frac{1}{N_e}\nabla.(N_e\mathbf{U}_e^r\mathbf{U}_e^r)\right\rangle \simeq -m_e\left\langle \frac{q_e}{m_e}\mathbf{U}_e^r \times \mathbf{B}^r + \nabla.(\mathbf{U}_e^r\mathbf{U}_e^r)\right\rangle, \tag{3.15}$$

We now need to evaluate a closure for the right-hand side of (3.15); this expression corresponds to the so-called *ponderomotive force* . To perform this, we use the evaluation of the transverse electron velocity given by (3.8) neglecting the absorption term $\nu_{ei}\mathbf{J}^r$ [and using (3.13)]

$$\mathbf{U}_e^r \simeq \frac{q_e}{m_e\omega_0}\mathbf{A}^r,$$

then

$$-\frac{q_e}{m_e\omega_0}\mathbf{U}_e^r \times \mathbf{B}^r - \nabla.(\mathbf{U}_e^r\mathbf{U}_e^r) \simeq -\left(\frac{q_e}{m_e\omega_0}\right)^2(\mathbf{A}^r \times \mathrm{curl}\mathbf{A}^r + \nabla.(\mathbf{A}^r\mathbf{A}^r)).$$

Thus, using $\nabla.\mathbf{A}^r = 0$, the vector identity (A.8) and the fact that $\langle|\mathbf{A}^r|^2\rangle = \frac{1}{2}|\mathbf{A}|^2$, we can see that

$$\langle\mathbf{A}^r \times \mathrm{curl}\mathbf{A}^r + \nabla.(\mathbf{A}^r\mathbf{A}^r)\rangle = \frac{1}{4}\nabla|\mathbf{A}|^2.$$

Thus, we see that (3.15) can be reduced to

$$q_e\mathbf{E}^e \simeq -\gamma_u\nabla(|\mathbf{A}|^2) - \frac{1}{N_e}\nabla P_e, \qquad \text{with } \gamma_u = \frac{q_e^2}{4m_e\omega_0^2}.$$

- *Quasi-neutrality.*

  According to (3.14), this leads to the Poisson equation

$$-\frac{\varepsilon^0}{q_e^2}\nabla.\left(\frac{1}{N_e}\nabla P_e\right) - \frac{\varepsilon^0}{q_e^2}\Delta h = ZN_0 - N_e.$$

where we have denoted by h the potential

$$h = \gamma_u|\mathbf{A}|^2$$

Using the Debye length $\lambda_D = \sqrt{\varepsilon^0 T_{\mathrm{ref}}/(N_{\mathrm{ref}}q_e^2)}$, the above equation becomes

$$-\lambda_D^2\frac{1}{T_{\mathrm{ref}}}\nabla.(T_e\nabla(\log P_e)) = \frac{1}{N_{\mathrm{ref}}}(ZN_0 - \frac{P_e}{T_e}) + \lambda_D^2\frac{1}{T_{\mathrm{ref}}}\Delta h$$

Assume that λ_D is small if compared to the characteristic length L_l. Then, as in Sect. 2.2, if the length scale is normalized by L_l, the quantity $\lambda = \lambda_D/L_l$ is a small parameter. From an heuristic point of view, we can see that P_e may be approximated by ZN_0T_e. We now bear this assertion.

Then, we write P_e^λ instead of P_e to exhibit the dependency with respect to λ and the previous equation reads as

$$-\lambda^2\frac{1}{T_{\mathrm{ref}}}\nabla.(T_e\nabla(\log P_e^\lambda)) = \frac{1}{N_{\mathrm{ref}}}(ZN_0 - \frac{P_e^\lambda}{T_e}) + \lambda^2\frac{1}{T_{\mathrm{ref}}}\Delta h \tag{3.16}$$

It is considered on a bounded smooth domain $\mathcal{O}$ and supplemented with boundary conditions, e.g.,

$$\frac{\partial}{\partial \mathbf{n}} P_e^\lambda = 0, \qquad \text{on } \partial\mathcal{O}$$

We need to make other technical assumptions:

$$\inf_x T_e > 0, \qquad \inf_x N_0 > 0, \qquad T_e, \nabla T_e \in L^\infty(\mathcal{O}), \qquad N_0 \in H^1(\mathcal{O}) \cap L^\infty(\mathcal{O}), \qquad \frac{\partial}{\partial \mathbf{n}} N_0 = 0,$$

Proposition 10. *For all λ small enough, there exists P_e^λ in $H^1(\mathcal{O})$ solution to (3.16) supplemented with the boundary condition $\frac{\partial}{\partial \mathbf{n}} P_e^\lambda = 0$ on $\partial\mathcal{O}$. Moreover, when λ goes to* 0, *one has*

$$P_e^\lambda \to Z N_0 T_e \qquad \text{in } L^2(\mathcal{O}) \text{ strongly;} \qquad \nabla(\log P_e^\lambda) \to \nabla(\log(Z N_0 T_e)) \qquad \text{in } L^2(\mathcal{O}) \text{ weakly.}$$

The proof is the same as the one made in Sect. 2.2 and is omitted. The physical meaning of these convergence results is the quasi-neutrality of the plasma and we get the following approximations

$$N_e \simeq Z N_0, \qquad P_e \simeq Z N_0 T_e,$$

$$q_e \mathbf{E}^e \simeq -\frac{1}{Z N_0} \nabla P_e - \gamma_u \nabla |\mathbf{A}|^2. \tag{3.17}$$

Now, if we use relation (3.17) in the low-frequency part of the ion momentum equation (2.3), we get

$$\frac{\partial}{\partial t}(N_0 \mathbf{U}) + \nabla.(N_0 \mathbf{U}\mathbf{U}) + \frac{1}{m_0} \nabla(P_e + P_0) = -N_0 \gamma_p \nabla |\mathbf{A}|^2, \tag{3.18}$$

$$\text{with: } \gamma_p = \frac{\gamma_u}{m_0} = \frac{Z q_e^2}{4 m_0 m_e \omega_0^2}$$

Remark 14. We have neglected in (3.15) the friction term $\frac{m_e}{m_0} \nu_{ei} (\mathbf{U} - \mathbf{U}_e^e)$. If we account for it, we can perform an asymptotic analysis analogous to the previous one and we will get the same result for the ion momentum equation. □

- *Statement of the basic time envelope model.*

From the above calculus, one checks that $|\mathbf{A}|^2$ is the time envelope of the density of the electromagnetic energy, it is called the *laser intensity*. From (3.12), one may easily check that

$$\frac{2}{c} \overline{\mathbf{A}}. \frac{\partial}{\partial t} \mathbf{A} - \overline{\mathbf{A}}. \left(i \frac{1}{k_0} \Delta \mathbf{A} + i k_0 (1 - \frac{N_e}{N_c}) \mathbf{A} \right) + \nu_0 |\mathbf{A}|^2 = 0;$$

thus, taking the real part of the integral of this relation, we get (denoting $d\Gamma$ the natural measure on the boundary $\partial\mathcal{D}$)

$$\frac{1}{c}\frac{\partial}{\partial t}\int_{\mathcal{D}}|\mathbf{A}|^2 d\mathbf{x} + \frac{1}{2k_0}\int_{\partial\mathcal{D}}\left[i\mathbf{A}.\frac{\partial}{\partial\mathbf{n}}\overline{\mathbf{A}} + c.c.\right]d\Gamma + \int_{\mathcal{D}}\nu_0|\mathbf{A}|^2 d\mathbf{x} = 0.$$

The term $\frac{1}{2}\int_{\partial\mathcal{D}}\left[i\mathbf{A}.\frac{\partial}{\partial\mathbf{n}}\overline{\mathbf{A}} + c.c.\right]d\Gamma$ corresponds to the incoming and outgoing energy through the boundary of the domain. Therefore, one checks that the density of energy lost by the laser per unit of volume is equal to

$$\nu_0|\mathbf{A}|^2.$$

According to the origin of the model, it may be checked that this energy is absorbed by the electron component of the plasma. In this kind of model, there are different ways to deal with the energy balance relations. As a matter of fact, one may state both an ion energy balance equation and an electron energy balance, but generally one assumes that the ion flow is adiabatic, i.e.,

$$P_0 = N_0^{5/3} P_{\text{ref}} N_{\text{ref}}^{-5/3},$$

and one keeps only an evolution equation for the electron energy $\varepsilon_e = \frac{3}{2}ZT_e$, where one accounts for the heating term coming from the laser energy absorption.

$$\frac{\partial}{\partial t}N_0 + \nabla.N_0\mathbf{U} = 0,$$

$$N_0\left(\frac{\partial}{\partial t} + \mathbf{U}.\nabla\right)\mathbf{U} + \frac{1}{m_0}\nabla(P_0 + P_e) = -N_0\gamma_p\nabla|\mathbf{A}|^2.$$

$$m_0 N_0 \frac{3}{2} Z\left(\frac{\partial}{\partial t} + \mathbf{U}.\nabla\right)T_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = \nu_0|\mathbf{A}|^2,$$

$$2i\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} + \frac{1}{k_0}\Delta\mathbf{A} + k_0\left(1 - \frac{ZN_0}{N_c}\right)\mathbf{A} + i\nu_0\mathbf{A} = 0.$$

In the sequel, we assume for the sake of simplicity that the electron temperature equation is somehow de-coupled from the remainder part of the system, so we do not take it into account.

Namely, according to the expression of the ion pressure, we have $\nabla P_0 = \frac{5}{3}\frac{P_{\text{ref}}}{N_{\text{ref}}^{5/3}}N_0^{2/3}\nabla N_0$. Therefore, after introducing the acoustic sound speed c_s given by

$$c_s = \sqrt{\left(\frac{5}{3}P_0 N_0^{-1} + ZT_e\right)/m_0}$$

we get

$$\nabla(P_e + P_0) = m_0 c_s^2 \nabla N_0 + Z N_0 \nabla T_e$$

It is also convenient to define the dimensionless electron density

$$N = \frac{Z}{N_c} N_0.$$

From now on, we assume that the polarization of the incoming laser wave is linear (see Chap. 5 for another kind of polarization), i.e., with the notations of Remark 13 above, we have at the entrance of the domain $\mathbf{E} = \mathbf{f} e^{i \mathbf{e}_{tr}.\mathbf{x} k_0 \sqrt{1-N}} e^{-i\omega_0 t}$ with $\mathbf{f} = f(x_1, x_2)\hat{\mathbf{f}}$ and $\hat{\mathbf{f}}$ a fixed unit vector.

Now we assume in the sequel that vector potential $\mathbf{A}$ oscillates according to this fixed unit vector $\hat{\mathbf{f}}$; so the electric field $\mathbf{E}^r$ oscillates also according to vector $\hat{\mathbf{f}}$. Moreover, we also assume that we are in the framework of the *s-polarization*: i.e., vector $\hat{\mathbf{f}}$ is orthogonal to the plane defined by the wave vector at the entrance to the spatial domain (which is here parallel to $\mathbf{e}_{tr}$) and the gradient of the mean electron density at the entrance to the spatial domain. If we assume that along the propagation of the laser beam, the wave vector corresponding to $\mathbf{A}$ and the gradient of the mean electron density remain in the same plane (i.e., if the three-dimensional aspects of the plasma behavior are not crucial), this plane remain orthogonal to $\hat{\mathbf{f}}$; thus, the vector field $\mathbf{A}(t, \mathbf{x})$ may read as $\hat{\mathbf{f}} A$ where $A(t, \mathbf{x})$ is a scalar function.

Summary. The unknowns of the basic time envelope model are the laser field A, the dimensionless electron density N, and the ion velocity $\mathbf{U}$. Neglecting the spatial gradient of the electron temperature, they satisfy

$$(\mathcal{BTE}) \quad \boxed{\begin{array}{ll} \text{(i)} & 2i \frac{1}{c}\frac{\partial A}{\partial t} + \frac{1}{k_0}\Delta A + k_0(1-N)A + i\nu_0 A = 0, \\ \text{(ii)} & \frac{\partial N}{\partial t} + \nabla.(N\mathbf{U}) = 0, \\ \text{(iii)} & \frac{\partial}{\partial t}(N\mathbf{U}) + \nabla.(N\mathbf{U}\mathbf{U}) + c_s^2 \nabla N = -N\gamma_p \nabla |A|^2. \end{array}}$$

The sound speed c_s is a smooth function of the electron temperature and of the density N. In the sequel, for the sake of simplicity, we will assume it is independent of time; sometimes we will assume that this speed is a constant.

Of course, one needs to state boundary conditions for these three equations; in particular, an inflow boundary condition needs to be prescribed for the Schrödinger equation (i); see the following subsection.

This model may be considered as the basis of all the other models of this chapter.

Remark 15. Related models. Let us now describe some models derived from the basic one. We assume here that the ion sound speed c_s is constant.

(a) By neglecting the inertial term in the momentum equation and combining with the mass conservation equation, we get a linear wave equation for the plasma response

$$\frac{\partial^2}{\partial t^2} N - c_s^2 \Delta N = \gamma_p \nabla(N \nabla |A|^2). \tag{3.19}$$

Then, we set $N = N^{\text{av}} + w$, where the average density N^{av} is a constant and w a small perturbation called an ion acoustic wave; linearizing the ponderomotive term, we get the following system

$$2i \frac{1}{c} \frac{\partial}{\partial t} A + \frac{1}{k_0} \Delta A + k_0 (1 - N^{\text{av}}) A + i \nu_0 A = k_0 w A, \tag{3.20}$$

$$\frac{\partial^2}{\partial t^2} w - c_s^2 \Delta w = \gamma_p N^{\text{av}} \Delta |A|^2. \tag{3.21}$$

(b) Another type of model is obtained by dropping the time derivative in ($\mathcal{BTE}$-i). Then we need to consider

$$\frac{1}{k_0} \Delta A + k_0 (1 - N) A + i \nu_0 A = 0, \tag{3.22}$$

$$\frac{\partial^2}{\partial t^2} N - c_s^2 \Delta N = \gamma_p \nabla(N \nabla |A|^2). \tag{3.23}$$

In some cases, when the transient phenomena have disappeared and the variation of density is very small, one may replace the last wave equation by a simpler closure (see, e.g., [98]). Using the equilibrium $c_s^2 \nabla N + \gamma_p N \nabla |A|^2 = 0$, we get

$$N = N_{\text{ref}} \exp(-\frac{\gamma_p |A|^2}{c_s^2}), \quad \text{with } N_{\text{ref}} \quad \text{constant;}$$

thus, A solves the following stationary equation of Sine–Gordon type

$$\frac{1}{k_0} \Delta A + k_0 \left(1 - N_{\text{ref}} \exp(-\frac{\gamma_p |A|^2}{c_s^2}) \right) A + i \nu_0 A = 0.$$

□

3.1.1.3 Properties of the Basic Time Envelope Model

We address here model ($\mathcal{BTE}$) and focus on the momentum balance property.

Since the low-frequency component of $\mathbf{E}^e \times \mathbf{B}^r$ is zero, we see, according to (3.1), that the low-frequency component of the magnetic momentum is

$$\mathbf{M} = i \frac{\varepsilon^0}{4\omega_0} \mathbf{A} \times \text{curl}\, \overline{\mathbf{A}} + c.c. = i \frac{\varepsilon_0}{4\omega_0} (A \nabla \overline{A} - \overline{A} \nabla A)$$

Lemma 4. *The following global momentum balance holds (where the tensor $\mathbb{S}_1$ is given below)*

$$\frac{\partial}{\partial t}(\mathbf{M} + m_0 N_0 \mathbf{U}) + \nabla.(m_0 N_0 \mathbf{U}\mathbf{U}) + \nabla(P_e + P_0) = \frac{\varepsilon^0}{4k_0^2}\nabla.\mathbb{S}_1 - \nabla(m_0\gamma_p |A|^2 N_0)$$

[**Indeed,** according to ($\mathcal{BTE}$-i), we have

$$\frac{2i}{c}\frac{\partial}{\partial t}(A\nabla\overline{A}-\overline{A}\nabla A) = \frac{1}{k_0}\left[A\nabla(\Delta\overline{A})-(\Delta\ A)\nabla\overline{A}+\overline{A}\nabla(\Delta A)-(\Delta\overline{A})\nabla A\right]$$
$$+k_0\left[A\nabla((1-N)\overline{A})+\overline{A}\nabla((1-N)A)-(1-N)(A\nabla\overline{A}+A\nabla\overline{A})\right].$$

But it is easy to check that the following identity holds

$$A\nabla(\Delta\overline{A}) - (\Delta A)\nabla\overline{A} + \overline{A}\nabla(\Delta A) - (\Delta\overline{A})\nabla A = 2\nabla.\mathbb{S}_1,$$
$$\mathbb{S}_1 = \mathbb{I}(\frac{1}{2}A(\Delta\overline{A}) + \frac{1}{2}\overline{A}(\Delta A) + |\nabla A|^2) - (\nabla A)(\nabla\overline{A}) - (\nabla\overline{A})(\nabla A). \quad (3.24)$$

Thus we get :

$$\frac{\partial}{\partial t}\mathbf{M} = \frac{\varepsilon^0}{4k_0^2}\nabla.\mathbb{S}_1 - \frac{\varepsilon^0}{4}|A|^2\nabla N.$$

Now, multiplying ($\mathcal{BTE}$-iii) by $m_0 N_c/Z$, we get the desired result.□]

In the previous relation, one sees that $m_0\gamma_p|A|^2 N_0$ may be interpreted as a laser pressure.

- *Boundary conditions and energy balance.*

Generally, one needs to consider the previous models on a bounded simulation domain $\mathcal{D}$. We must focus on the boundary conditions on the one hand for ($\mathcal{BTE}$-i) or (3.22) and on the other hand for ($\mathcal{BTE}$-ii, iii) or (3.23) on boundary $\partial\mathcal{D}$.

(a) For ($\mathcal{BTE}$-i) or (3.22), denoting by $\mathbf{e}_b$ the unit vector characterizing the propagation direction of the laser, we first need to consider the enlighted part of the boundary Γ^{in} defined by

$$\Gamma^{in} = \{\mathbf{x} \in \partial\mathcal{D}, \text{ such that } \mathbf{e}_b.\mathbf{n} < 0\}, \qquad \mathbf{n}\text{, the outwards normal vector.}$$

By assuming that the density N is a constant N^{in} on this part of the boundary Γ^{in}, we set

$$\mathbf{K}^{in} = \mathbf{e}_b(1 - N^{in})^{1/2},$$

and the incident wave is assumed to be of the form $\alpha^{in}e^{ik_0\mathbf{K}^{in}\mathbf{x}}$, knowing that $\alpha^{in} = \alpha^{in}(\mathbf{x})$ is the restriction to Γ^{in} of a smooth function. The incoming boundary condition on Γ^{in} reads as

$$(k_0^{-1}\mathbf{n}.\nabla + i\mathbf{K}^{in}.\mathbf{n})(A - \alpha^{in}e^{ik_0\mathbf{K}^{in}\mathbf{x}}) = 0. \quad (3.25)$$

On the other hand, if we set $\Gamma^{out} = \partial\mathcal{D} \setminus \Gamma^{in}$ (where $\mathbf{e}_b.\mathbf{n} \geq \mathbf{0}$), the boundary condition on Γ^{out} reads as :

$$(k_0^{-1}\mathbf{n}.\nabla - i\sqrt{1-N})A = 0. \tag{3.26}$$

Denote by $d\Gamma$ the surface measure on Γ^{in} or on Γ^{out}, we have now the following laser energy balance.

Proposition 11. *Let A, solution to (3.22), (3.25), (3.26); then we get*

$$\int_{\Gamma^{in}} \left(|\mathbf{K}^{in}.\mathbf{n}| \quad |\alpha^{in}|^2 + \frac{1}{4|\mathbf{K}^{in}.\mathbf{n}|}|k_0^{-1}\mathbf{n}.\nabla\alpha^{in}|^2 \right) d\Gamma =$$

$$\int_{\Gamma^{in}} \frac{1}{4|\mathbf{K}^{in}.\mathbf{n}|}|k_0^{-1}\frac{\partial}{\partial\mathbf{n}}A - i\mathbf{K}^{in}.\mathbf{n}A|^2 d\Gamma$$

$$+ \int_{\Gamma^{out}} \frac{1}{4|1-N|^{1/2}}|k_0^{-1}\frac{\partial}{\partial\mathbf{n}}A + i\sqrt{1-N}A|^2 d\Gamma + \int_{\mathcal{D}} \nu_0|A|^2 d\mathbf{x}.$$

[**Indeed,** one multiplies (3.22) by $\overline{A}$ and integrates by part.]

This relation may be interpreted as an energy balance equation for the laser intensity $|A|^2$; moreover, the laser energy flux through a surface with the normal $\mathbf{n}$, is equal to $\frac{1}{4|1-N|^{1/2}}|k_0^{-1}\frac{\partial}{\partial\mathbf{n}}A + i\sqrt{1-N}A|^2$. Then the energy flux incoming on Γ^{in} is equal to the flux outgoing on Γ^{in} and $\Gamma^{out,}$ respectively, plus the absorbed energy $\int \nu_0|A|^2 d\mathbf{x}$.

From a numerical point of view, since the light speed c is very large compared to the ion sound speed c_s, it is necessary to solve equation ($\mathcal{BTE}$-i) with an implicit method; as a matter of fact, the time derivative term $\frac{2i}{c}\frac{\partial}{\partial t}A$ may be considered as a perturbation of the Helmholtz equation

$$\frac{1}{k_0}\Delta A + k_0(1-N)A + i\nu_0 A = 0.$$

Then, for numerical simulations based on this model, the spatial mesh has to be fine enough (at least six cells per wavelength in each direction). Moreover, it has been noticed for a long time that the boundary treatment is crucial on Γ^{out}, at least on the part of Γ^{out} where the wave does not go away in an orthogonal direction of the boundary. This problem has been studied for a long time for the Helmholtz equation and here it is exactly the same. Different techniques may be used, but the most efficient is definitely the perfectly matched layer (PML) technique described in [11, 89]; see also [46] for a practical implementation. Let us recall briefly the principle of this technique in a finite-difference type mesh: near the boundary, one builds a layer $\mathcal{L}$ with a width that is equal to about three wavelength in which an artificial damping coefficient is plugged. Let us detail this technique in a one-dimensional framework (3.22) that reads as

$$\frac{\partial^2}{\partial x^2}A + (k_0^2(1-N) + ik_0\nu_0)A = 0.$$

Integrating on a cell with an index denoted by j and with a width equal to δx,the standard discretization yields to

$$\frac{\partial}{\partial x}A\bigg|_{j+1/2} - \frac{\partial}{\partial x}A\bigg|_{j-1/2} + \delta x \left(k_0^2(1-N)_j + ik_0\nu_{0,j}\right) A_j = 0$$

where $\frac{\partial}{\partial x}A\big|_{j+1/2} = \frac{1}{\delta x}(A_{j+1} - A_j)$; then, in the artificial layer $\mathcal{L}$,instead of this relation, one states

$$\frac{\partial}{\partial x}A\bigg|_{j+1/2} \frac{1}{1+i\sigma_{j+1/2}} - \frac{\partial}{\partial x}A\bigg|_{j-1/2} \frac{1}{1+i\sigma_{j-1/2}} + \delta x(1+i\sigma_j)\left(k_0^2(1-N)_j + ik_0\nu_{0,j}\right) A_j = 0$$

where σ_j is a real coefficient corresponding to a ad hoc function that is equal to about 10^{-2} at the interior boundary of $\mathcal{L}$ and that grows exponentially up to a value equal to about $1,000$ at the exterior boundary of $\mathcal{L}$.

(b) The problem of boundary conditions for the barotropic hydrodynamic model ($\mathcal{BTE}$-ii, iii) is classical at least in the subsonic case: one needs to give only one boundary condition. As a matter of fact, one of the transparent conditions evoked in Sect. 2.3.2.3 of the previous chapter needs to be used (and for the wave equation (3.23), it is the same kind of transparent boundary condition).

Remark 16. Numerical Simulations. Simulations based on this model have been performed, e.g., in [68] for a small spatial domain; see also [52]. But for a two-dimensional large spatial domain where we need to solve at each time step a linear system with some 10^8 degrees of freedom (corresponding to the values of A in each cell of the domain), it is a challenging work of scientific computing (see, e.g., [46]).

We present below some numerical illustrations of model ($\mathcal{BTE}$) in a two-dimensional framework for the propagation of a laser beamlet in a plasma with an initial electron density that grows linearly from 0.2 up to 0.9. Figure 3.1 shows the propagation of a beamlet with four very narrow hot spots that cross over and spread into filaments due to auto-focalization phenomena. It is worth noticing here and in the simulations corresponding to following figures that the profile of the plasma density depends on time (due to the occurrence of the ponderomotive force) so even if the entrance laser field does not depend on time, the map of the laser intensity does change with time. As a matter of fact, the general picture is always the same after a transition period, but the filaments that may be seen of Fig. 3.1 stir a little when the time increases.

Figure 3.2 (and Fig. 3.3 for a zoom) shows a map of the laser intensity without absorption for a beam that presents more than fourteen speckles at the entrance boundary, while Fig. 3.4 is the same but with a small absorption coefficient. One may notice that the main absorption phenomenon occurs in a region near a surface called *caustic surface* [which will be defined precisely in the sequel]; the regions where the speckles becomes narrow correspond to a digging of the plasma that leads to the auto-focalization of the speckles. In the above simulations, the length of the simulation domain is about 600 μm in the horizontal direction.

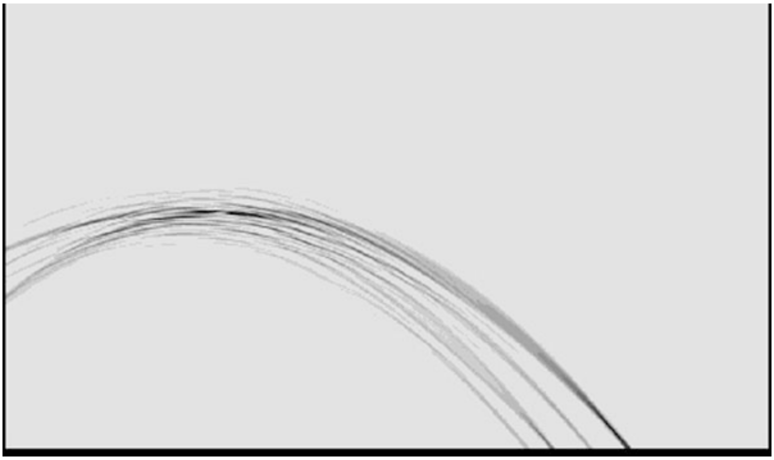

Fig. 3.1 Map of laser intensity with model (BTE) for a beamlet with four hot spots which propagates from the *bottom*

Lastly, in Fig. 3.5 we show a map of the laser intensity for a laser propagating in the same type of plasma in the case where the laser field at entrance boundary is modelized by a more sophisticated way (with a lot of speckles which overlap); this map is obtained after a long simulation time corresponding to a physical time equal to 23 ps (recall that the time for the laser to propagate through the whole plasma is equal to about 2 ps). □

Orientation

In the sequel, the basis assumption is that the laser wavelength $2\pi/k_0$ is small if compared to the length L_l. So, we introduce a small dimensionless parameter:

$$\epsilon = (k_0 L_l)^{-1}.$$

As mentioned above, for our applications, ϵ may be on the order of 0.1 or less. Since the wavelength is small, we solve approximately the basic equation ($\mathcal{BTE}$-i) by using a WKB (Wenzel–Kramer–Brillouin) asymptotic expansion technique. The classical geometrical optics approximation corresponds to the case where this asymptotic expansion is made at first order with respect to ϵ and the so-called paraxial approximation corresponds to the case where it is made at second order.

So, the geometrical optics, presented in the following subsection, is the first step of the expansion leading to the paraxial approximation, which is the focus of the next subsection. Roughly speaking, the geometrical optics approximation is a paraxial approximation without accounting for diffraction.

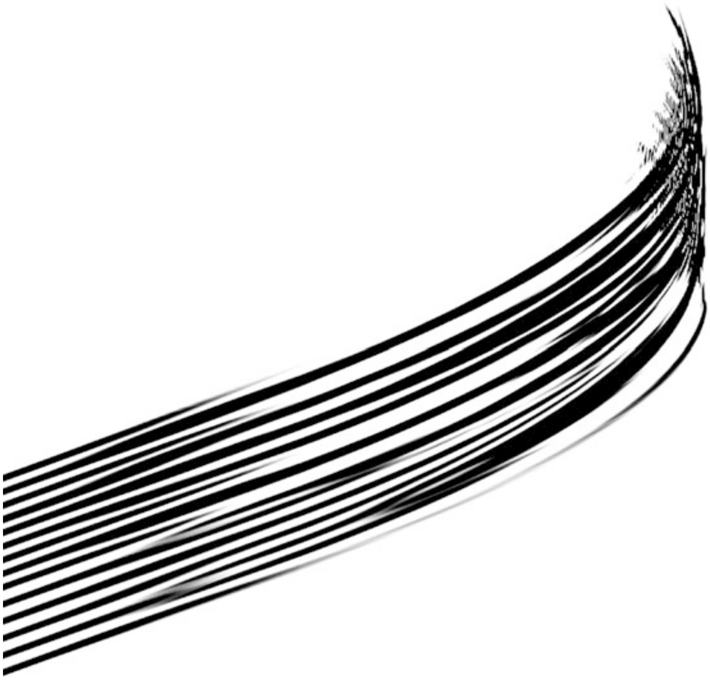

Fig. 3.2 Map of laser intensity with model (BTE) for a beamlet with fourteen hot spots that propagates from the left in a plasma with an initial density N that grows linearly from 0.2 up to 0.9 (without absorption)

3.1.2 Geometrical Optics

In most plasma experiments we have in mind, the laser wavelength is small with respect to the size of simulation domain, thus for solving Helmholtz equation (3.22) supplemented with the boundary condition (3.25), we may use the WKB expansion technique. The principle is to approximate the solution A of (3.22) by the product of a function slowly varying with the space variable (called amplitude) times a complex exponential of a phase (which varies fast with the space variable). Therefore, the amplitude is an envelope function with respect to the space variable.

This approximation at first order is called the geometrical optics. We now recall briefly its principle in our framework.

The derivation of geometrical optics model has been well known for a long time and the related ray-tracing method has been explained in the historical paper on "Geometric theory of diffraction" by Keller in 1953 (which was re-published in [74]). The geometrical optics for propagation in plasma was used already in [59]. The reader is also referred to [53] for details on the geometrical expansion ratio in the ray-tracing method.

Fig. 3.3 Detail of the previous picture near the caustic surface

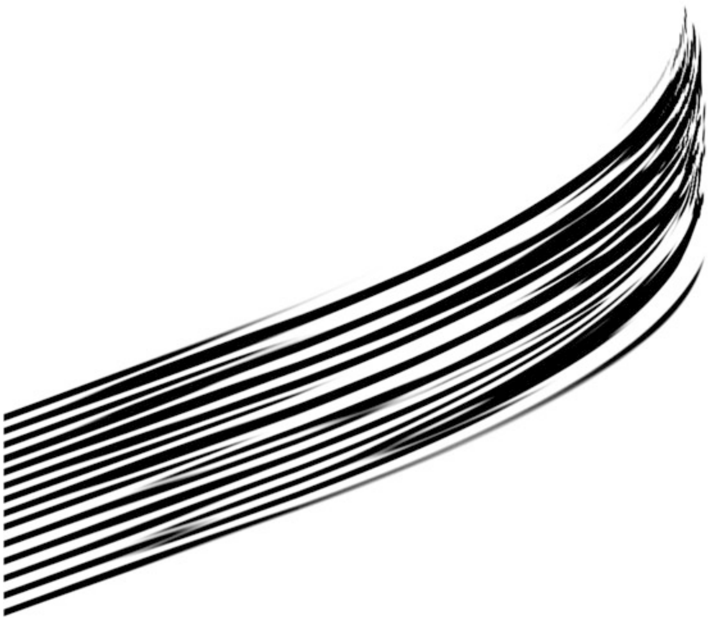

Fig. 3.4 Map of laser intensity with model (BTE) for the same beamlet in the same plasma (with a small absorption coefficient)

One knows that some difficulties arise due to the fact that a caustic phenomenon[1] may occur in the laser propagation. Here we only recall some simple features of

[1]When one performs the geometrical optics approximation, a caustic phenomenon occurs if a family of optical rays have a envelope surface (which is called the *caustic surface*).

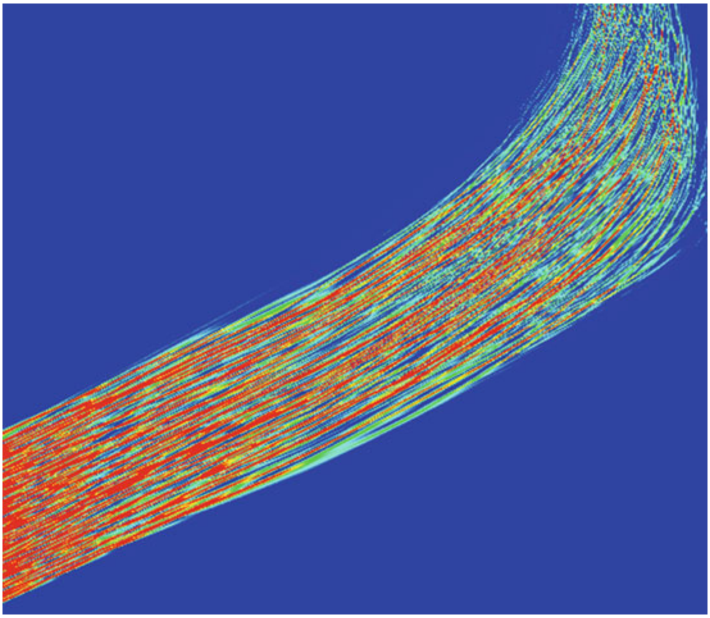

Fig. 3.5 Map of laser intensity for a sophisticated laser field at the entrance boundary, the laser propagates in the same initial plasma [simulation made using model (BTE)]

this problem and we do not worry about the laser energy transport after the caustic surface (for specific caustic problems, see, e.g., [53] and the references therein, see also [10] in a Euler framework). One assumes that the incoming field A (which is the time envelope of the physical field) does not depend on time; i.e., the time derivative may be withdrawn in this model. See the remark below for the case where one has to account for this time derivative in $(\mathcal{BTE} - i)$.

For the geometrical optics approximation, we assume that

- the plasma hydrodynamics is not coupled with the laser propagation, i.e., the dimensionless plasma density N is assumed to be a smooth function of the space variable, independent from the time variable (we write $N(\mathbf{x}) = \mathcal{N}^0(\mathbf{x})$),
- the incoming laser intensity α^{in} is also independent from the time variable (this last assumption may be removed; see the remark below).

3.1.2.1 The WKB Expansion

With the above assumptions, (3.22) may be recast in the following way

$$\epsilon L_l \Delta A + \frac{1}{\epsilon L_l}(1 - \mathcal{N}^0)A + i\nu_0 A = 0. \tag{3.27}$$

The principle of the WKB expansion is to write the solution A of this equation in the form

$$A(\mathbf{x}) \simeq (a_0(\mathbf{x}) + \epsilon a_1 + \cdots) \exp\left(i \frac{\phi(\mathbf{x})}{\epsilon L_l}\right),$$

where a_0 and ϕ are slowly space varying functions. Of course, the term $\exp\left(i \frac{\phi(\mathbf{x})}{\epsilon L_l}\right)$ is highly oscillating with respect to the space variable. The quantities ϕ and a_0 are called the phase and the amplitude of the laser field. Since

$$\exp\left(-i \frac{\phi}{\epsilon L_l}\right) \Delta \left(a_0 \exp(i \frac{\phi}{\epsilon L_l})\right) = \Delta a_0 + \frac{1}{\epsilon L_l} (2i(\nabla\phi).\nabla a_0 + i a_0 \Delta\phi) - \frac{1}{(\epsilon L_l)^2} a_0 |\nabla\phi|^2,$$

we get by plugging the previous expansion in (3.27)

$$\begin{aligned} 0 = \frac{1}{\epsilon}\frac{a_0}{L_l}\left[1 - \mathcal{N}^0 - |\nabla\phi|^2\right] + \\ + \epsilon^0 (i \nu_0 a_0 + 2i(\nabla\phi).\nabla a_0 + i a_0 \Delta\phi) \\ + \epsilon \ldots \end{aligned}$$

In order to have a nontrivial value of a_0, it is necessary for the phase ϕ to satisfy the so-called *eikonal equation*

$$|\nabla\phi|^2 = 1 - \mathcal{N}^0. \tag{3.28}$$

On the incoming part of the boundary Γ^{in}, the direction of $\nabla\phi$ needs to be parallel to the fixed vector $\mathbf{e}'_b$ then the boundary condition for the eikonal equation is (with $\mathbf{K}^{in}$ defined above)

$$\nabla\phi|_{\Gamma^{in}} = \mathbf{K}^{in}.$$

On the other hand, if we denote $\mathbf{K} = \nabla\phi$, we get the following *laser field transport equation* for the laser field a_0 (recall it is a complex function)

$$\nu_0 a_0 + 2\mathbf{K}.\nabla a_0 + a_0 \nabla.\mathbf{K} = 0. \tag{3.29}$$

On the incoming part of the boundary Γ^{in}, one has to prescribe the value of the laser field:

$$a_0|_{\Gamma^{in}} = \alpha^{in}.$$

One checks that the laser intensity $|a_0|^2$ satisfies the *energy transport equation*

$$\nu_0 |a_0|^2 + \nabla.\left(\mathbf{K}|a_0|^2\right) = 0. \tag{3.30}$$

Notice that when we integrate this relation over any small subdomain D of $\mathcal{D}$ we get an energy balance equation (recall that $d\Gamma$ is the restriction of the Lebesgue measure to a boundary of a subdomain)

$$\int_D \nu_0 |a_0|^2 d\mathbf{x} + \int_{\partial D} \mathbf{n}.\mathbf{K}|a_0|^2 d\Gamma = 0.$$

The vector $\mathbf{K}|a_0|^2 d\Gamma$ is the *laser energy flux* through the elementary surface $d\Gamma$. The quantity $\sqrt{1-\mathcal{N}^0}|a_0|^2 = |\mathbf{K}||a_0|^2$ is often called the scalar laser energy flux and it differs from the laser intensity $|a_0|^2$. One recovers here the fact that $\nu_0|a_0|^2$ is the density of absorbed laser energy. In optics, it is classical to introduce the index of refraction

$$\eta = \sqrt{1-\mathcal{N}^0}.$$

For the coupling with the dynamics of the plasma, it suffices to add in the right-hand side of the electron energy equation the absorbed laser energy, i.e.,

$$m_0 N_0 \left(\frac{\partial}{\partial t} + \mathbf{U}.\nabla \right) \varepsilon_e + P_e \nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = c\nu_0 |a_0|^2.$$

Let us stress that despite its simple form, the eikonal equation is quite complicated. As a matter of fact, it is a stationary Hamilton–Jacobi equation that reads as follows

$$H(\mathbf{x}, \nabla\phi) = 0, \qquad \text{with } H(\mathbf{x}, \mathbf{K}) = \frac{1}{2}(|\mathbf{K}|^2 - \eta^2(\mathbf{x})) \tag{3.31}$$

Ii is worth noticing that there are two cases for the solution of this eikonal equation.

The first case corresponds to a density $\mathcal{N}^0$ that is small enough; let us say η is larger than 0.8 (for an incident angle larger than $\pi/4$); then no caustic appears in the simulation domain and the solution ϕ is single-valued.

But there is another case corresponding to the case where $\mathcal{N}^0$ is larger than one in some region of the simulation domain, then the beam cannot propagate in all of the domain. As a matter of fact, there is a caustic surface $\mathcal{C}$ and the solution of the eikonal equation does not exist in the shadow region, which is "after the caustic surface $\mathcal{C}$." Moreover in the region between the illuminated surface and the caustic $\mathcal{C}$, the solution ϕ consists of two branches corresponding to a "direct path" and to a "return path" of the beam. But, in the sequel we do not worry about the "return path."

Remark 17. We assume here that one needs to account for the time derivative in $(\mathcal{BTE}-i)$; then the laser amplitude $a_0 = a_0(t,\mathbf{x})$ must satisfy

$$\frac{2}{c}\frac{\partial}{\partial t}a_0 + \nu_0 a_0 + 2\mathbf{K}.\nabla a_0 + a_0 \nabla.\mathbf{K} = 0.$$

On the incoming part of the boundary Γ^{in}, we must prescribe as above $a_0(t) = \alpha^{in}(t)$ and α^{in} may be a function depending slowly on time. Moreover, we must give the initial value of the amplitude $a_0(\mathbf{x})|_{t=0}$.

One checks that the laser intensity $|a_0|^2$ satisfies

$$\frac{1}{c}\frac{\partial}{\partial t}|a_0|^2 + \nu_0|a_0|^2 + \nabla.\left(\mathbf{K}|a_0|^2\right) = 0.$$

Notice that the laser intensity propagates with the velocity $c\mathbf{K}$ with a modulus that is $c\eta$, which is called the group velocity. □

3.1.2.2 On the Ray-Tracing Method

For dealing numerically with eikonal equation (3.31) and the energy transport equation (3.30), the most popular method is the ray-tracing method. We address this problem without accounting for the time dependence, so the incoming condition α^{in} and the profile of η are known functions. We first give some elements of the classical theory of bi-characteristics for the Hamilton–Jacobi equation (3.31) (cf. [74]); then we apply this theory to the numerical solution of system (3.31) (3.30).

The principle of the method of bi-characteristics is to define curves $(\mathbf{r}(.), \mathbf{K}(.))$ in the space $\mathbf{R}^3 \times \mathbf{R}^3$ which are parametrized by a parameter τ (belonging to $\mathbf{R}^+$) and which are solutions of the following system

$$\frac{d\mathbf{r}}{d\tau} = \frac{\partial H}{\partial \mathbf{K}}(\mathbf{r}, \mathbf{K}) = \mathbf{K}(\tau), \qquad \frac{d\mathbf{K}}{d\tau} = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{r}, \mathbf{K}) = \eta(\mathbf{r})\nabla\eta(\mathbf{r}).$$

with a starting point $\mathbf{r}^{in}$ belonging to the boundary Γ^{in} and a direction vector equal to $\mathbf{K}^{in} = \mathbf{e}_b\eta(\mathbf{r}^{in})$. We check that

$$\frac{d}{d\tau}H(\mathbf{r}(\tau), \mathbf{K}(\tau)) = \frac{\partial H}{\partial \mathbf{x}}(\mathbf{r}, \mathbf{K})\frac{\partial \mathbf{r}}{\partial \tau} + \frac{\partial H}{\partial \mathbf{K}}(\mathbf{r}, \mathbf{K})\frac{\partial \mathbf{K}}{\partial \tau} = 0;$$

then, since we have $H(\mathbf{r}^{in}, \mathbf{K}^{in}) = 0$ on the boundary, we get $H(\mathbf{r}(\tau), \mathbf{K}(\tau)) = 0$ for all τ; i.e.,

$$|\mathbf{K}(\tau)|^2 = \eta^2(\mathbf{r}(\tau))$$

Moreover, for each ray $(\mathbf{r}(.), \mathbf{K}(.))$, we introduce the solution φ of the ODE $\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial \mathbf{K}}(\mathbf{r}, \mathbf{K}).\mathbf{K}(\tau) = |\mathbf{K}(\tau)|^2$ and we define ϕ by $\phi(\mathbf{r}(\tau)) = \varphi(\tau)$, then $\frac{d}{d\tau}\phi(\mathbf{r}(\tau)) = \nabla\phi.\frac{d\mathbf{r}}{d\tau} = \nabla\phi.\frac{d\mathbf{r}}{d\tau}$ thus $\nabla\phi(\mathbf{r}(\tau)).\mathbf{K} = |\mathbf{K}(\tau)|^2$. It may check also that the rays $\mathbf{r}(.)$ are orthogonal to the surfaces $\phi = C^{\text{ant}}$; then $\nabla\phi(\mathbf{r})$ is parallel to the tangent of the rays $\mathbf{K}(.)$ and we get

$$\nabla\phi(\mathbf{r}(\tau)) = \mathbf{K}(\tau)$$

so ϕ is a local solution to the eikonal equation (3.31).

We now define a more practical parameter σ defined by $d\sigma = d\tau/|\mathbf{K}(\tau)|$; so we have $\frac{d\mathbf{r}}{d\sigma} = \frac{1}{|\mathbf{K}|}\mathbf{K}$ (which means that σ is the arc length along the ray) and $\frac{d}{d\sigma}\mathbf{K} = \frac{\eta(\mathbf{r})}{|\mathbf{K}|}\nabla\eta(\mathbf{r})$. We have also $H(\mathbf{r}(\sigma), \mathbf{K}(\sigma)) = 0$ for each ray. Therefore, the previous system reads also

$$\frac{d}{d\sigma}\mathbf{r} = \frac{1}{|\mathbf{K}|}\mathbf{K} \qquad \mathbf{r}|_{\sigma=0} = \mathbf{r}^{in} \tag{3.32}$$

$$\frac{d}{d\sigma}\mathbf{K} = \nabla\eta(\mathbf{r}), \qquad \mathbf{K}|_{\sigma=0} = \mathbf{K}^{in} \tag{3.33}$$

Address now a portion of a narrow tube D around a particular ray $\mathbf{r}^0$ between $\mathbf{r}^0(\sigma^-)$ and $\mathbf{r}^0(\sigma^+)$. That is, the normal vector $\mathbf{n}$ to ∂D at $\mathbf{r}$ is orthogonal to $\frac{d}{d\sigma}\mathbf{r}$; the sections $\Sigma(\sigma^-)$ and $\Sigma(\sigma^+)$ of the tube are normal to the tube; then upon integrating the energy balance equation (3.30) on the tube D, we get

$$\int_{\Sigma(\sigma^-)} \eta(\sigma^-)|a_0(\sigma^-, .)|^2 d\Sigma - \int_{\Sigma(\sigma^+)} \eta(\sigma^+)|a_0(\sigma^+, .)|^2 d\Sigma = \int_D \nu_0|a_0|^2 d\mathbf{x} \tag{3.34}$$

Denote $|\Sigma(\sigma)|$ as the area of the surface $\Sigma(\sigma)$. If there would be no absorption, the previous relation means that in a narrow tube, the quantity $\eta(\sigma)|a_0(\sigma)|^2|\Sigma(\sigma)|$ is constant; from a physical point of view, this means that the scalar laser energy flux multiplied by the area of the section of the elementary tube is constant.

Now introduce the Jacobian ξ of the transformation $\mathbf{r}^{in} \mapsto \mathbf{r}(\sigma)$. Upon differentiating the system (3.32) (3.33), it may be seen (see [53]) that the couple $(\xi(\sigma), \zeta(\sigma)) = (\frac{\partial\mathbf{r}}{\partial\mathbf{r}^{in}}(\sigma), \frac{\partial\mathbf{K}}{\partial\mathbf{r}^{in}}(\sigma))$ satisfies the system of ODEs (with $D^2\eta$ the Hessian matrix of η)

$$\frac{d}{d\sigma}\begin{bmatrix}\xi\\ \zeta\end{bmatrix} = \begin{bmatrix} -\frac{1}{\eta^2(\mathbf{r})}\mathbf{K}(\nabla\eta) & \frac{1}{\eta(\mathbf{r})}\mathbb{I} - \frac{1}{\eta^3(\mathbf{r})}\mathbf{K}\mathbf{K} \\ D^2\eta(\mathbf{r}) & \frac{1}{\eta^2(\mathbf{r})}(\nabla\eta)\mathbf{K}\end{bmatrix}\begin{bmatrix}\xi\\ \zeta\end{bmatrix}, \qquad \begin{bmatrix}\xi|_{\sigma=0}\\ \zeta|_{\sigma=0}\end{bmatrix} = \begin{bmatrix}\mathbb{I}\\ 0\end{bmatrix}$$

Moreover, denoting $|\xi|$ as the determinant of ξ,we see that for all σ^-, σ^+ the two sections $\Sigma(\sigma^-)$ and $\Sigma(\sigma^+)$ of a narrow tube satisfy

$$|\Sigma(\sigma^-)|/|\Sigma(\sigma^+)| = |\xi(\sigma^-)|/|\xi(\sigma^+)|$$

The determinant $|\xi|$ is called the geometrical expansion ratio (related to a ray); recall that $\eta(\mathbf{x})|a_0(\mathbf{x})|^2$ is the scalar laser energy flux. Thus, if there would be no absorption, relation (3.34) means that the scalar laser energy flux times the geometrical spreading is constant along each ray.

Our concern is now the discretization of (3.31) together with energy transport equation (3.30). First, the energy $|\alpha^{in}|^2$ may be discretized on Γ^{in}. One needs to choose a partition of the boundary Γ^{in} with small surfaces Σ^q where the centers are the discretization points $\mathbf{r}^{q,in}$. The corresponding scalar laser energy flux integrated on surface Σ^q is

$$W^{q,in} = |\Sigma^q|\eta(\mathbf{r}^{q,in})|a_0(\mathbf{r}^{q,in})|^2.$$

One defines the ray starting at $\mathbf{r}^{q,in}$ by the bi-characteristics $(\mathbf{r}^q, \mathbf{K}^q)$

$$\frac{d}{d\sigma}\mathbf{r}^q = \frac{1}{|\mathbf{K}^q|}\mathbf{K}^q, \qquad \mathbf{r}^q|_{\sigma=0} = \mathbf{r}^{q,in},$$
$$\frac{d}{d\sigma}\mathbf{K}^q = \nabla\eta(\mathbf{r}^q), \qquad \mathbf{K}^q|_{\sigma=0} = \mathbf{K}^{in}.$$

Let us also define (ξ^q, ζ^q) by the system of ODEs

$$\frac{d}{d\sigma}\begin{bmatrix}\xi^q\\ \zeta^q\end{bmatrix} = \begin{bmatrix} -\frac{1}{\eta^2(\mathbf{r}^q)}\mathbf{K}^q\nabla\eta & \frac{1}{\eta(\mathbf{r}^q)}\mathbb{I} - \frac{1}{\eta^3(\mathbf{r}^q)}\mathbf{K}^q\mathbf{K}^q \\ D^2\eta(\mathbf{r}^q) & \frac{1}{\eta^2(\mathbf{r}^q)}(\nabla\eta)\mathbf{K}^q \end{bmatrix}\begin{bmatrix}\xi^q\\ \zeta^q\end{bmatrix},$$

Let us denote $W^q(\sigma)$ as the scalar laser energy flux integrated on the narrow tube around the ray $\mathbf{r}^q(.)$, i.e.,

$$W^q(\sigma) = |\Sigma^q||\xi^q(\sigma)|\eta(\mathbf{r}^q(\sigma))|a_0(\mathbf{r}^q(\sigma))|^2.$$

Now, upon integrating (3.30) in the direction transverse to the ray, we get

$$\nu_0|a_0(\mathbf{r}^q(\sigma))|^2|\xi^q(\sigma)| + \frac{\partial}{\partial\sigma}(|\xi^q(\sigma)|\eta(\mathbf{r}^q(\sigma))|a_0(\mathbf{r}^q(\sigma))|^2) = 0,$$

therefore, we get

$$\frac{\partial}{\partial\sigma}(W^q(\sigma)) = -\frac{\nu_0(\mathbf{r}^q(\sigma))}{\eta(\mathbf{r}^q(\sigma))}W^q(\sigma), \qquad W^q|_{\sigma=0} = W^{q,in}.$$

When the ray enters in a subdomain D with a scalar energy flux $W^q(\sigma^D)$, its intensity for an abscisa $\sigma > \sigma^D$ is given by

$$W^q(\sigma) = W^q(\sigma^D)\int_{\sigma^D}^{\sigma}\exp\left(-\frac{\nu_0(\mathbf{r}^q(\sigma'))}{\eta(\mathbf{r}^q(\sigma'))}\right)d\sigma'.$$

So we get a simple method for evaluating the absorbed laser intensity in a subdomain D. Notice that to evaluate the laser intensity, it is necessary to get an evaluation of the geometrical expansion ratio $|\xi^{q,}|$ which needs to solve two supplementary ODEs for ξ^q and ζ^q.

Remark 18. It may happen that the ray reaches a caustic surface, i.e., the rays are very close to one another, and the geometrical spreading $\xi(.)$ vanishes. Since the quantity $\xi(\sigma)\eta(\sigma)|a_0(\sigma)|^2$ is finite (it would be constant if there was no absorption), one sees that the laser intensity $|a_0(\sigma)|^2$ goes to infinity. But if one handles both the geometrical spreading and the scalar laser energy flux, it is possible to use the ray-tracing method even after the caustic. This property of the geometrical optics approximation is well known and is related to the Maslov index theory; see, e.g., [53,74]. Notice that without the geometrical optics approximation, the laser intensity $|A(.)|^2$ remains finite (This fact is well known in one-dimensional geometry: indeed, consider the Airy function $a(x)$ which is a solution of $\epsilon^2 \frac{\partial}{\partial x^2} a + (1 - x/L)a = 0$ on $[0, +\infty)$, it is bounded; but making the geometrical optics approximation for $|a(x)|^2$ leads to a function that goes to infinity when $x \to L$ and that is integrable; see, e.g., [10]). □

Remark 19. Of course, the ray-tracing method may extended to the case where the boundary condition α^{in} depends on time and a finite speed of light needs to be accounted for. That is, $W^{q,in}$ depends on time. One needs to pay attention to the fact that the wave travels with the group velocity $c\eta(\mathbf{r})$ (smaller than c); then we get $d\sigma = c\eta dt$. In the case where the index of refraction is independent from the time, one may state the ray equations (3.32) (3.33) using the time variable:

$$\frac{d}{dt}\mathbf{r} = c\eta \frac{1}{|\mathbf{K}|}\mathbf{K}, \qquad \mathbf{r}|_{t=0} = \mathbf{r}^{q,in},$$

$$\frac{d}{dt}\mathbf{K} = c\eta \nabla \eta(\mathbf{r}), \qquad \mathbf{K}|_{t=0} = \mathbf{K}^{in}.$$

In the same way, we may state the differential system for the couple (ξ, ζ) using the time derivative; moreover, the scalar energy flux W^q is transported on the ray $\mathbf{r}^q$ according to the following ODE

$$\frac{\partial}{\partial t}(W^q(t)) = -c\nu_0(\mathbf{r}^q(t))W^q(t), \qquad W^q|_{t=0} = W^{q,in}.$$

□

3.1.3 The Paraxial Approximation

We now deal with the diffraction phenomenon and refraction at the small length scale. So we need to perform a more accurate approximation of equation ($\mathcal{BTE}$-i).

As before, we assume that the direction $\mathbf{e}_b$ of the incoming laser is fixed. The simulation domain $\mathcal{D}$ is assumed to be a parallelepiped and the incoming boundary is a part of hyperplane Γ^{in}. In order to study the paraxial approximation of the solution A, we need to make the following hypothesis.

- The density function is a sum of a function at the scale of L_l depending on a one-dimensional variable and a small perturbation; more precisely, there exists a unit vector $\mathbf{e}_{\text{va}}$ such that if $z = \mathbf{e}_{\text{va}}.\mathbf{x}$, the function N may be decomposed as

$$N(t, \mathbf{x}) = \mathcal{N}^0(\mathbf{x}) + \epsilon^2 G(t, \mathbf{x}) \qquad \text{and } \mathcal{N}^0(\mathbf{x}) = \mathcal{M}(\mathbf{e}_{\text{va}}.\mathbf{x}), \tag{3.35}$$

where G is a bounded function depending on the time and space variables and $\mathcal{M}(z)$ is a smooth function of z such that its derivative is small enough ($\frac{\partial}{\partial z}\mathcal{M} = O(\epsilon)$) and is equal to a constant $\mathcal{N}^{\text{in}}$ near the boundary Γ^{in}.

- The function $\mathcal{M}$ is such that there is no caustic in the simulation domain.
- The incoming laser field $\alpha^{in}(t)$ on the boundary Γ^{in} (or on one edge of the parallelepiped) is slowly time varying; more precisely, we can write

$$\alpha^{in}(t) = \tilde{\alpha}^{in}(\epsilon t)$$

with $\tilde{\alpha}^{in}$ a smooth function of the reduced time variable $T = \epsilon t$.

- The absorption length ν_0^{-1} is large enough compared to L_l, i.e., there exists ν_1 an absorption coefficient such that

$$\nu_0 = \epsilon \nu_1.$$

According to assumption (3.35), the eikonal equation reads now as (denoting $x^{\perp} = \mathbf{x} - z\mathbf{e}_{\text{va}}$)

$$|\nabla\phi(z, x^{\perp})|^2 = 1 - \mathcal{M}(z), \qquad \text{with } \nabla\phi(z, x^{\perp}) = \mathbf{e}_b\sqrt{1 - \mathcal{N}^{\text{in}}} \text{ on } \Gamma^{in}$$

and we have an explicit solution given by

$$\frac{\partial\phi}{\partial x^{\perp}} = (\mathbf{e}_b - \mathbf{e}_{\text{va}}(\mathbf{e}_{\text{va}}.\mathbf{e}_b))\sqrt{1 - \mathcal{N}^{\text{in}}}$$
$$\frac{\partial\phi}{\partial z} = \sqrt{1 - \mathcal{M}(z) - (1 - (\mathbf{e}_{\text{va}}.\mathbf{e}_b)^2)(1 - \mathcal{N}^{\text{in}})}$$

As above, we set $\mathbf{K} = \nabla\phi$; since $\frac{\partial}{\partial z}\mathcal{M} = O(\epsilon)$, we check that there exists a constant C such that

$$|\nabla.\mathbf{K}(\mathbf{x})| = C\epsilon.$$

Moreover, we make the following change of time scale $T = \epsilon t$ and define $\tilde{A}(T)$ by

$$\tilde{A}(T) = A(T/\epsilon).$$

Then, equation ($\mathcal{BTE}$-i) may be recast in the form

$$2\frac{i\epsilon}{c}\frac{\partial}{\partial T}\tilde{A} + \epsilon L_l \Delta\tilde{A} + \frac{1}{\epsilon L_l}(1 - \mathcal{N}^0)\tilde{A} - \frac{\epsilon}{L_l}G\tilde{A} + i\epsilon\nu_1\tilde{A} = 0. \tag{3.36}$$

3.1.3.1 The WKB Expansion

We now perform the WKB asymptotic expansion (with respect to ϵ) of the solution $\tilde{A}$ in a similar way as previously

$$\tilde{A}(T,.) = (a_0(T,.) + \epsilon a_1(T,.) + \epsilon^2 \dots) \exp(\frac{i\phi}{\epsilon L_l}),$$

where a_0, a_1 are now functions depending on the reduced time variable T and on the space variable.

After plugging this asymptotic expansion into (3.36), we get

$$\begin{aligned}
0 &= \epsilon^{-1}\frac{a_0}{L_l}\left[1 - \mathcal{N}^0 - |\mathbf{K}|^2\right] \\
&+ \epsilon^0 i\,(2\mathbf{K}.\nabla a_0) \\
&+ \epsilon\left[i\,(\frac{2}{c}\frac{\partial a_0}{\partial T} + 2\mathbf{K}.\nabla a_1 + \nu_1 a_0) - \frac{G}{L_l}a_0 + L_l(\Delta a_0)\right] + i\,(a_0\nabla.\mathbf{K}) \\
&+ \epsilon^2 \dots
\end{aligned}$$

According to the eikonal equation, the ϵ^{-1} terms are zero. The term of order zero needs to be also zero; thus, the transport equation for the field a_0 reduces to

$$\mathbf{K}.\nabla a_0 = 0, \qquad a_0(T) = \tilde{\alpha}^{in}(T) \qquad \text{on } \Gamma^{in}. \tag{3.37}$$

(notice that the variable T is a simple parameter). For the terms of order 1, we get

$$i\left(\frac{2}{c}\frac{\partial a_0}{\partial T} + 2\mathbf{K}.\nabla a_1 + a_0\frac{\nabla.\mathbf{K}}{\epsilon} + \nu_1 a_0\right) + L_l\Delta a_0 - \frac{G}{L_l}a_0 = 0.$$

By combining (3.37) with this last equation, we get

$$i\left(\epsilon\frac{2}{c}\frac{\partial a_0}{\partial T} + 2\mathbf{K}.\nabla(a_0 + \epsilon a_1) + a_0\nabla\mathbf{K} + \epsilon\nu_1 a_0\right) + (\epsilon L_l)\Delta a_0 - \epsilon\frac{G}{L_l}a_0 = 0.$$

Let us define the *transverse* gradient $\nabla_\perp^K$ ("transverse" means orthogonal to **K**) and the *transverse*Laplace operator by (as usual **KK** is a tensor)

$$\nabla_\perp^K \bullet = \nabla \bullet - \frac{\mathbf{K}}{|\mathbf{K}|^2}\mathbf{K}.\nabla\bullet; \qquad \Delta_\perp^K \bullet = \nabla.[(\mathbb{I} - \frac{\mathbf{KK}}{|\mathbf{K}|^2})\nabla\bullet] = \nabla.(\nabla_\perp^K\bullet).$$

According to (3.37), we can easily check that

$$\nabla a_0 = \nabla_\perp^K a_0, \qquad \Delta a_0 = \Delta_\perp^K a_0.$$

If we now set $\tilde{E} = a_0 + \epsilon a_1$, a formal calculation leads to

$$i\left(\epsilon \frac{2}{c}\frac{\partial \tilde{E}}{\partial T} + 2\mathbf{K}.\nabla \tilde{E} + \tilde{E}\nabla.\mathbf{K} + \epsilon \nu_1 \tilde{E}\right) + \epsilon L_l \Delta_\perp^K \tilde{E} - \epsilon \frac{G}{L_l}\tilde{E} = O(\epsilon^2).$$

Dropping the $O(\epsilon^2)$ and coming back to the physical variables, that is to say by defining $E(t,.) = \tilde{E}(\epsilon t,.)$, we may claim that

$$A(t,\mathbf{x}) \simeq E(t,\mathbf{x})e^{ik_0\phi}, \tag{3.38}$$

$$i\left(\frac{2}{c}\frac{\partial E}{\partial t} + 2\mathbf{K}.\nabla E + E(\nabla.\mathbf{K}) + \nu_0 E\right) + \frac{1}{k_0}(\Delta_\perp^K E) - k_0(N - \mathcal{N}^0)E = 0. \tag{3.39}$$

This equation is called a time-dependent paraxial equation. It may be noticed that, the time derivative term $\frac{1}{c}\frac{\partial E}{\partial t}$ is a perturbation term compared to the advection one $\mathbf{K}.\nabla E$; indeed the previous calculation has shown that this term is only a order ϵ term. The term $\nu_0 E$ corresponds to the laser energy absorption.

It is necessary to supplement equation (3.39) with an initial condition $E(0,.)$. Moreover, according to the incoming boundary condition (3.25), we see that the correct boundary condition for this equation on the boundary Γ^{in} is

$$(k_0^{-1}\mathbf{n}.\nabla_\perp + 2i\mathbf{K}.\mathbf{n})(E(t,.) - \alpha^{in}(t,.)) = 0. \tag{3.40}$$

Since k_0^{-1} is assumed to be small if compared to the characteristic length L_l, this condition implies that the value of E at the boundary is close to α^{in}. When $\mathbf{n}.\nabla_\perp$ is not zero, (i.e., $\mathbf{n}$ not parallel to $\mathbf{K}$), the problem may be called a tilted frame paraxial equation.

The coupling with the hydrodynamic system is made as follows

$$(\mathcal{P}ara)\quad \begin{cases} \text{(i)} & \frac{\partial}{\partial t}N + \nabla.(N\mathbf{U}) = 0, \\ \text{(ii)} & \frac{\partial}{\partial t}(N\mathbf{U}) + \nabla.(N\mathbf{U}\mathbf{U}) + c_s^2\nabla N = -N\gamma_p\nabla|E|^2, \\ \text{(iii)} & i\left(\frac{2}{c}\frac{\partial E}{\partial t} + 2\mathbf{K}.\nabla E + E(\nabla.\mathbf{K}) + \nu_0 E\right) + \frac{1}{k_0}(\Delta_\perp^K E) - k_0(N - \mathcal{N}^0)E = 0. \end{cases}$$

Of course, one must account for the boundary conditions: (3.40) for the paraxial equation (iii) and classical subsonic conditions for (i) and (ii). Instead of (i) and (ii), one may also couple equation (iii) with the wave equation (3.19).

The vector $c\mathbf{K}$ is called the group velocity.

From a practical point of view, $\mathcal{N}^0$ may be the averaged value of N in the direction orthogonal to a principal one (then the variations of N orthogonal to this principal direction are weak).

Momentum Conservation

If $\mathbf{K}$ is not constant, it is not possible to prove a rigorous momentum conservation relation as the one presented in previous sub-section. So we assume here that $\mathcal{N}^0$ is constant and thus $\mathbf{K}$ is constant. Set $\mathbf{M} = \frac{\varepsilon^0}{4\omega_0}(2k_0|E|^2\mathbf{K} + i(E\nabla\overline{E} - \overline{E}\nabla E))$ which is the low frequency component of the magnetic momentum defined for system $(\mathcal{BTE})$.

Proposition 12. *Let* $(N, \mathbf{U}, E)$ *be a solution to the system* $(\mathcal{P}ara)$*, then we get the global momentum balance relation*

$$\frac{\partial}{\partial t}(\mathbf{M} + m_0 N_0 \mathbf{U}) + \nabla.(m_0 N_0 \mathbf{U}\mathbf{U}) + \nabla(P_e + P_0) = \frac{\varepsilon^0}{4k_0^2}\nabla.\mathbb{S}_3 - \nabla(m_0\gamma_p|E|^2 N_0),$$

where $\mathbb{S}_3$ *is a tensor such that* $\mathbb{S}_3 = \mathbb{S}_1 + O(\epsilon)$ *[* $\mathbb{S}_1$ *is given by (3.24)].*

See the proof below.

Other Properties

We show here, at least in the case where the time derivative is withdrawn, that (3.40) is a good boundary condition on Γ^{in} for (3.39) and we give some enlightenments about the well-posedness of this equation. So after dropping this time derivative, (3.39) reads as a Schrödinger-type equation

$$i\,(2\mathbf{K}.\nabla E + E(\nabla.\mathbf{K}) + \nu_0 E) + \frac{1}{k_0}(\Delta_\perp^K E) - k_0(N - \mathcal{N}^0)E = 0. \tag{3.41}$$

For a given function N, this is a linear equation. Notice that in the special case where the vector $\mathbf{K}$ is parallel to the normal vector $\mathbf{n}$ to the boundary Γ^{in}, (3.41) reduces to a classical paraxial equation (it is a *Schrödinger equation* where the propagation direction plays the part of the time variable).

We give here a classical result related to laser intensity $|E|^2$.

Proposition 13. *Assume that* $(k_0^{-1}\mathbf{n}.\nabla_\perp + 2i\mathbf{n}.\mathbf{K})\alpha^{in} \in L^2(\partial\mathcal{D})$*. If* $E \in H^1(\mathcal{D})$ *is a solution to (3.41), (3.40),*

(i) then the following energy relation holds

$$\int_{\mathcal{D}} \nu_0|E|^2 d\mathbf{x} + \int_{\Gamma^{in}} |\mathbf{n}.\mathbf{K}|\,|E|^2 d\Gamma \le -Im\left(\int_{\Gamma^{in}} \overline{E}(k_0^{-1}\mathbf{n}.\nabla_\perp + 2i\mathbf{n}.\mathbf{K})\alpha^{in} d\Gamma\right). \tag{3.42}$$

(ii) Moreover, there exists a constant C depending only on $\mathbf{K}$ *such that*

$$\int_{\mathcal{D}} \nu_0|E|^2 d\mathbf{x} + \int_{\Gamma^{in}} |\mathbf{n}.\mathbf{K}|\,|E|^2 d\Gamma \le C\int_{\Gamma^{in}} |(k_0^{-1}\mathbf{n}.\nabla_\perp + 2i\mathbf{n}.\mathbf{K})\alpha^{in}|^2 d\Gamma.$$

If the absorption coefficient ν_0 is a strictly positive function, this result shows on the one hand that there exists at most one solution $E \in H^1(\mathcal{D})$ to (3.41), (3.40) for any regular function α^{in}, and on the other hand the solution is stable with respect to the boundary condition.

Remark 20. For a general domain $\mathcal{D}$, it is difficult to prove that problem (3.41), (3.40) is well-posed in the case where ν_0 is zero, but we can analyze this problem in the special case where spatial domain $\mathcal{D}$ is a half-space. For instance in two-dimensional geometry, if one assumes that $\mathcal{D} = \{x = (x_1, x_2) \in \mathbf{R}^2 \quad | x_1 > 0\}$ where the boundary Γ^{in} corresponds to $x_1 = 0$ and that $N = \mathcal{N}^0$ and $\mathbf{K} = (K_1, K_2)$ are constant, then problem (3.41), (3.40) may read as

$$i(K_1\partial_{x_1}+K_2\partial_{x_2})E+\frac{1}{2}k_0^{-1}(K_2^2(\partial^2)_{x_1x_1}-2K_1K_2(\partial^2)_{x_1x_2}+K_1^2(\partial^2)_{x_2x_2})E+i\nu_0E=0,$$

$$\frac{ik_0^{-1}}{2|\mathbf{K}|}(K_1K_2\partial_{x_2} - K_2^2\partial_{x_1})E + K_1E=K_1g, \qquad \text{on } \Gamma^{in},$$

[where $g = (1 + \frac{ik_0^{-1}}{2K_1|\mathbf{K}|}(K_1K_2\partial_{x_2} - K_2^2\partial_{x_1}))\alpha^{in}$]; it has been proved that it is well-posed even if $\nu_0 = 0$ (see [51]). □

3.1.3.2 The Classical Paraxial Equation

We now consider a laser beam that enters in the spatial domain $\mathcal{D}$ (corresponding to the plasma) without any incidence angle: the unit vector $\mathbf{e}_b$ is equal to both vector $\mathbf{e}_{\text{va}}$ and to the inwards normal vector $-\mathbf{n}$ to the boundary Γ^{in} ($\mathbf{e}_b = \mathbf{e}_{\text{va}}$), thus denote

$$z = \mathbf{e}_b.\mathbf{x}.$$

Then the solution of the eikonal equation reads now as

$$\mathbf{K}(\mathbf{x}) = \mathbf{e}_b K(\mathbf{e}_b.\mathbf{x}), \qquad K(\mathbf{e}_b.\mathbf{x}) = \sqrt{1-\mathcal{N}^0(\mathbf{x})}, \tag{3.43}$$

and (3.39), (3.40) read as

$$i\left(\frac{2}{c}\frac{\partial E}{\partial t} + 2K\frac{\partial E}{\partial z} + E\frac{\partial K}{\partial z}+\nu_0 E\right) + \frac{1}{k_0}(\Delta_\perp E) - k_0(N-\mathcal{N}^0)E = 0, \tag{3.44}$$

supplemented with an initial condition $E(t,.)|_{t=0}$ and a boundary condition

$$E(t,.) = \alpha^{in}(t,.), \qquad \text{on } \Gamma^{in}.$$

This equation is very close to the one stated at the beginning of [14] [in that reference, there is an extra term corresponding to a time derivative of the phase ϕ].

For the sake of simplicity, we assume in the sequel that the spatial domain $\mathcal{D}$ is either a half-space or a strip between the two boundaries Γ^{in} and Γ^{out} or a parallelepiped (in that case, α^{in} is nonzero only on a part of the boundary—which is included in Γ^{in}).

If $\mathcal{D}$ is the half-space $\{z \geq 0\}$, multiplying (3.44) by $\overline{E}$, integrating with respect to the transverse directions and taking the complex conjugate, we see that

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\int_{\mathbf{R}^2} |E|^2 dx^{\perp}\right) + \frac{\partial}{\partial z}\left(\int_{\mathbf{R}^2} K|E|^2 dx^{\perp}\right) + 2\int_{\mathbf{R}^2} \nu_0 |E|^2 dx^{\perp} = 0.$$

Thus, if α^{in} does not depend on t and if a stationary solution is reached, then this solution satisfies for any z positive

$$K(z)\int_{\mathbf{R}^2} |E(z,x^{\perp})|^2 dx^{\perp} + 2\int_0^z \int_{\mathbf{R}^2} \nu_0 |E(z',x^{\perp})|^2 dx^{\perp} dz' = K(0)\int_{\mathbf{R}^2} |\alpha^{in}(x^{\perp})|^2 dx^{\perp}.$$

it is an energy balance for the scalar laser energy flux which is $K(z)|E(z,x^{\perp})|^2$. Thus, if we have $\nu_0 = 0$, then the scalar laser energy flux integrated on the transverse surface is conserved when z increases (in the direction of laser propagation).

3.1.3.3 Numerics for the Classical Paraxial Equation

For the simulation of laser propagation in Inertial Confinement Fusion plasma, one often must deal numerically with the hydrodynamic system coupled with the equation of propagation (3.44) on a spatial domain $\mathcal{D}$ with sizes that are typically on the order of one millimeter (the vector $\mathbf{e}_b$ is parallel to an edge of the box).

We consider a Cartesian mesh of finite difference type. In order to have a good discretization of the speckles, the mesh size in the transverse directions needs to be on the order of a fraction of a micron; the mesh size δz in the z direction may be two or three times larger. Denote by δt the time step. For the hydrodynamic system, one must use a time step satisfying the CFL condition of the type $\max(|\mathbf{U}|, c_s)\delta t/\min(\delta x, \delta y, \delta z) < 1$. Therefore, we expect to use a numerical method that is stable with a time step such that $c_s\delta t/\delta x$ be on the order of 1; then $c\delta t/\delta z$ will be very large compared to 1.

For numerical purpose, the simulation domain is generally a parallelepiped where one edge is parallel to z and one must state boundary conditions on transverse boundaries $\Gamma_{\perp}$: it may be seen that the Neumann condition corresponds to a reflection of the light on $\Gamma_{\perp}$, which is not well-suited. When dealing with this problem, one can withdraw the time derivative and the absorption; the behavior of the solution corresponds to a solution of a Schrödinger equation $\frac{\partial E}{\partial z} - i\frac{1}{2k_0 K}\Delta_{\perp} E = 0$. From a theoretical point of view, it is tricky to have good transparent boundary conditions for this equation (these conditions involve pseudo-differential operators on $\Gamma_{\perp}$, see, e.g., [4]).

At each time step $[t^{n-1}, t^{n-1} + \delta t]$, one needs to solve successively the hydrodynamic system and the paraxial equation of propagation (3.44). For the hydrodynamics system, an explicit numerical scheme of Lagrange–Euler type may be used (e.g., in the momentum equation, the ponderomotive force $\gamma_p \nabla |E|^2$ must be taken into account by a standard centered discretization); see also [7].

Let us focus now on the treatment of the paraxial equation (3.44) in a two-dimensional geometry (for the sake of simplicity). Let y be the transverse direction and denote by $E^n_{j,m}$ the evaluation of E at time $t^n = n.\delta t$ at position $z_j = j.\delta z$ and $y_m = m.\delta y$.

If there was only a propagation phenomenon, i.e., if one had to deal with the advection operator $(\frac{1}{c}\frac{\partial}{\partial t} + K\frac{\partial}{\partial z} + \nu_0)$, we would use a time implicit scheme and we would get the values E^n_j at time t^n from step to step in the direction of propagation from the entrance side $z = 0$. Here, we perform a splitting of (3.44), with respect to the z variable. We move from $E^n_{j,\cdot}$ to $E^n_{j+1,\cdot}$ through the intermediate state $\widetilde{E^n_{j+1,\cdot}}$.

1. *The first step: advection.*

The values $E^n_{j,\cdot}$ are assumed to be known and we must solve between z_j and z_{j+1} the advection equation:

$$K\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} + \left(\nu_0 + \frac{1}{2}\frac{\partial K}{\partial z}\right)E = 0.$$

Let us denote $\mu_{j+1/2} = \frac{1}{2}(\nu_{0,j} + \nu_{0,j+1}) + (K_{j+1} - K_j)/2\delta z$. One could get $\widetilde{E^n_{j+1,\cdot}}$ by a classical upwind scheme with respect to the z variable

$$K_j\frac{\widetilde{E^n_{j+1,m}} - E^n_{j,m}}{\delta z} + \frac{1}{c}\frac{E^n_{j,m} - E^{n-1}_{j,m}}{\delta t} + \frac{\mu_{j+1/2}}{4}(E^n_{j,m} + \widetilde{E^n_{j+1,m}}) = 0. \quad (3.45)$$

This method is referred to below as the naive method. We propose now an improved method to deal with the advection equation. We do not address it directly, but the corresponding equation for the laser intensity (square of the modulus of E), i.e., we search $E = E^n$ for the solution of

$$K\frac{\partial}{\partial z}|E|^2 + \frac{1}{c\delta t}\left(|E|^2 - |E^{n-1}|^2\right) + (2\nu_0 + \frac{\partial K}{\partial z})|E|^2 = 0,$$

e.g., one may use a simple upwind scheme and set

$$|\widetilde{E^n_{j+1,m}}|^2 = \left(|E^n_{j,m}|^2(1 - \eta - \frac{\mu_{j+1/2}\delta z}{2K_j}) + \eta|E^{n-1}_{j,m}|^2\right)\left(1 + \frac{\mu_{j+1/2}\delta z}{2K_j}\right)^{-1}, \quad \text{with } \eta = \frac{1}{cK}\frac{\delta z}{\delta t}. \quad (3.46)$$

Afterwards one solves a simple advection equation for the quantity of $E^n_\cdot/|E^n_\cdot|$ which is related to the phase of $E^n_\cdot$

2. *The second step: diffraction.*

Denote now $\delta N = N - \mathcal{N}^0$. Once $\widetilde{E^n_{j+1,*}}$ is obtained, there are two ways to find $E^n_{j+1,*}$ by solving

$$K\frac{\partial E}{\partial z} = \frac{i}{2k_0}(\Delta_\perp E) - i\frac{k_0}{2}(\delta N)E.$$

In the first way, one discretizes the Laplace operator $\Delta_\perp$ by a Crank–Nicolson technique

$$K_{j+1}\frac{E^n_{j+1,m} - \widetilde{E^n_{j+1,m}}}{\delta z} = \frac{i}{2k_0}\left((\Delta_\perp \widetilde{E^n_{j+1}})_m + (\Delta_\perp E^n_{j+1})_m\right) - i\frac{k_0}{2}(\delta N)_{j+1/2}\left(\widetilde{E^n_{j+1,m}} + E^n_{j+1,m}\right), \tag{3.47}$$

where $(\Delta_\perp E)_m$ stands for the discretized form of the transverse Laplace operator [i.e., $(2E_m - E_{m+1} - E_{m-1})/\delta x_\perp^2$ in a 2D framework]. To deal with the transparent boundary conditions on $\Gamma_\perp$ from a numerical point of view, the PMLs technique [11] may be adapted to paraxial equations and appears to be very efficient.

In the second approach, one uses an analytic formula to deal with the $i\frac{k_0}{2}\delta N$ term and a spectral method for the transverse Laplace operator. More precisely, if $\mathcal{F}(E_j)$ denotes the value of the Fourier transform in the transverse direction of the function $E_j(x_\perp)$ and ξ the corresponding value of the Fourier variable, we simply set

$$\mathcal{F}(E_{j+1})(\xi) = \exp(-i\frac{\xi^2}{2Kk_0}\delta z)\mathcal{F}(\widetilde{E^n_{j+1}})(\xi),$$

$$E^n_{j+1} = \exp(i\frac{k_0}{2K}(\delta N)_{j+1/2}\delta z)\mathcal{F}^{-1}(\mathcal{F}(E_{j+1})).$$

Of course, from a numerical point of view, the fast Fourier transform (FFT) and the inverse FFT are used for $\mathcal{F}$ and $\mathcal{F}^{-1}$. To deal with nonreflective boundary conditions on $\Gamma_\perp$, it is necessary to add some ad hoc artificial absorbing coefficient in boundary layers.

Properties of the Numerical Schemes

The natural criterion for such an implicit scheme is

$$\sup\frac{\delta z}{cK_j} < \delta t.$$

We address now the problem of numerical stability. So, for j fixed, we denote by $\|E_j\|$ the l^2 norm in the transverse direction ($\|E_j\|^2 = \sum_m |E_{j,m}|^2$). For the sake of simplicity, assume that there are reflection boundary conditions (i.e., Neumann condition) on the boundary in the transverse direction: $\frac{\partial}{\partial \mathbf{n}}E = 0$.

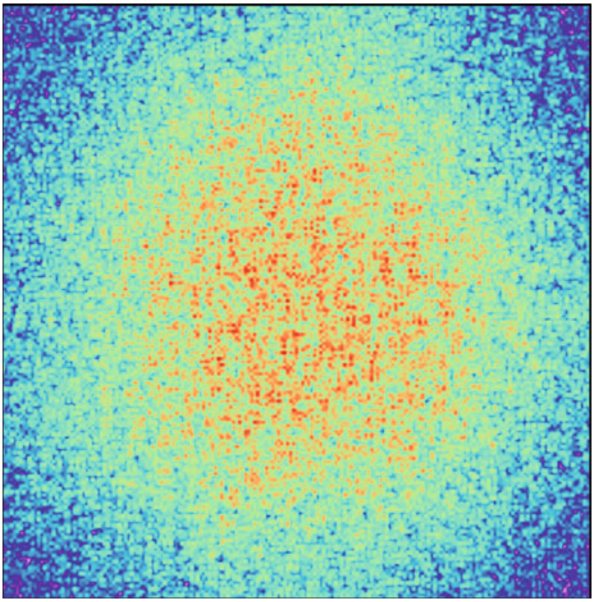

Fig. 3.6 Map of the laser intensity at the entrance boundary of the simulation domain (boundary data of the problem).

Proposition 14. *With the previous criterion on δt, the previous scheme satisfies*

$$\sup_j K_j \|E_j^n\|^2 \le \max\left(\sup_N \left(K_0\|\alpha^{in,N}\|^2\right), \sup_j K_j \|E_j^0\|^2\right) \qquad \forall n,$$

This expresses that the proposed splitting scheme is stable for the norm $\sup_j K_j \|E_j\|^2$.

Remark 21. Numerical illustrations of classical paraxial simulations. The numerical solution of the system ($\mathcal{P}ara$) based on the paraxial model with an implicit method is of course much faster than the numerical solution of system ($\mathcal{BTE}$), and it is possible to perform realistic simulations in the 3D framework. We show below some illustrations related to such a simulation with 500 million cells (with δz on the order of one wavelength and $\delta x_\perp$ on the order of one half wavelength), they have been obtained thanks to the $HERA$ code; see [7]. Figure 3.6 below shows the laser intensity map at the entrance boundary (it is the data of the problem) which shows the complexity of a realistic set of speckles in an actual laser beam. Figure 3.7 is a map of a 2D-cut of the laser intensity on a plan parallel to the propagation axis; this shows that each elementary speckle interferes with each of its neighboring speckles. Figure 3.8 shows the laser intensity at the rear side boundary; here the correlation between speckles is very different from the one observed in the entrance boundary. □

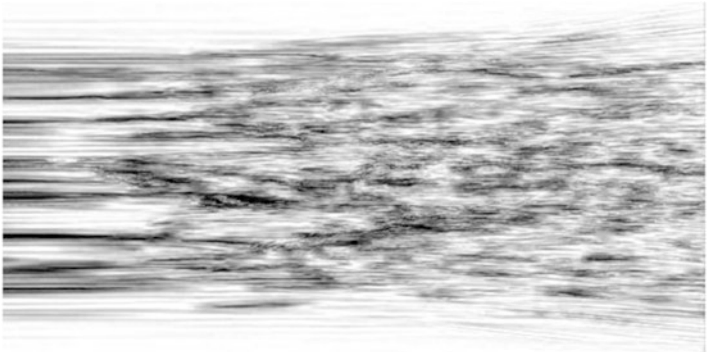

Fig. 3.7 Map of a 2D-cut of the laser intensity on a longitudinal plan parallel to the propagation axis (propagation from *left* to *right*). One notices that each elementary speckle interferes with each of its neighboring speckles when it propagates.

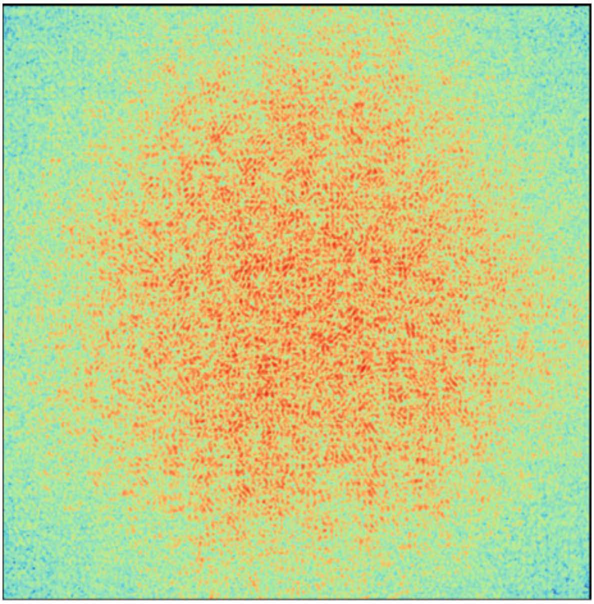

Fig. 3.8 Map of the laser intensity at the rear side boundary of the simulation domain (after some picoseconds). Compare the correlation between the speckles here and on the boundary data in Fig. 3.6

Proof of Propositions of Sect. 3.1

Proof of Proposition 12. According to ($\mathcal{P}ara$-iii), we have

$$\frac{i}{c}\frac{\partial}{\partial t}(E\nabla\overline{E}-\overline{E}\nabla E) = \frac{1}{2k_0}\left[E\nabla(\Delta_\perp\overline{E})-(\Delta_\perp E)\nabla\overline{E}+\overline{E}\nabla(\Delta_\perp E)-(\Delta_\perp\overline{E})\nabla E\right]$$
$$+\frac{1}{2}k_0\left[E\nabla((\mathcal{N}^0-N)\overline{E})+\overline{E}\nabla((\mathcal{N}^0-N)E)-(\mathcal{N}^0-N)(E\nabla\overline{E}+E\nabla\overline{E})\right]$$
$$-i\mathbf{K}.\nabla(E\nabla\overline{E}-\overline{E}\nabla E).$$

The second line of the right-hand side reads simply $-k_0|E|^2\nabla N$, and with classical identities one can prove that the following relation holds,

$$E\nabla(\Delta_\perp \overline{E}) - (\Delta_\perp E)\nabla\overline{E} + \overline{E}\nabla(\Delta_\perp E) - (\Delta_\perp \overline{E})\nabla E = \nabla.\mathbb{S}_2,$$

where

$$\mathbb{S}_2 = \mathbb{I}(E(\Delta_\perp \overline{E}) + \overline{E}(\Delta_\perp E) + 2(\nabla E).(\nabla\overline{E})) - (\nabla E)(\nabla_\perp \overline{E}) - (\nabla_\perp \overline{E})(\nabla E) - (\nabla_\perp E)(\nabla\overline{E}) - (\nabla\overline{E})(\nabla_\perp E).$$

Thus, we get:

$$\frac{4k_0}{\varepsilon^0}\frac{\partial}{\partial t}\mathbf{M} = \frac{1}{2k_0}\nabla.\overline{\overline{S_2}} - k_0|E|^2\nabla N - i\mathbf{K}.\nabla(E\nabla\overline{E} - \overline{E}\nabla E) + \mathbf{K}[i(\overline{E}\Delta_\perp E - E\Delta_\perp \overline{E}) - 2k_0\mathbf{K}.\nabla|E|^2].$$

But since we have $\nabla\overline{E}.(\mathbb{I} - \frac{\mathbf{KK}}{|\mathbf{K}|^2})\nabla_\perp E = \nabla E.(\mathbb{I} - \frac{\mathbf{KK}}{|\mathbf{K}|^2})\nabla_\perp \overline{E}$, we check that

$$-i\mathbf{K}.\nabla(E\nabla\overline{E} - \overline{E}\nabla E) + \mathbf{K}[i(\overline{E}\Delta_\perp E - E\Delta_\perp\overline{E}) = i\nabla.\left[\overline{E}\mathbf{K}(\nabla E + \nabla_\perp E)\right] + c.c..$$

In summary, we get

$$\frac{\partial}{\partial t}\mathbf{M} = \frac{\varepsilon^0}{4k_0^2}\nabla.\mathbb{S}_3 - \frac{\varepsilon^0}{4}|E|^2\nabla N, \qquad \text{with } \mathbb{S}_3 = \frac{1}{2}\mathbb{S}_2 - 2k_0^2\mathbf{KK}|E|^2 + k_0\left[i\overline{E}\mathbf{K}(\nabla E + \nabla_\perp E) + c.c.\right].$$

Then the result follows by combining with ($\mathcal{P}ara$-ii) multiplied by m_0N_c/Z. The last assertion is due to the estimate $\mathbf{K}.\nabla E = O(\epsilon)$. □

Proof of Proposition 13. (i) By multiplying (3.41) by $\overline{E}$, integrating over $\mathcal{D}$ and taking the complex conjugate, we see that

$$2\int_{\mathcal{D}}[\nu_0|E|^2 + \nabla.(\mathbf{K}|E|^2)]d\mathbf{x} = k_0^{-1}\int_{\Gamma^{in}}[-iE(\mathbf{n}.\nabla_\perp\overline{E}) + i\overline{E}(\mathbf{n}.\nabla_\perp E)]d\Gamma,$$

but according to (3.40), we have $ik_0^{-1}\mathbf{n}.\nabla_\perp E = 2\mathbf{K}.\mathbf{n}E + (ik_0^{-1}\mathbf{n}.\nabla_\perp - 2\mathbf{n}.\mathbf{K})(\alpha^{in})$, then

$$2\int_{\mathcal{D}}\nu_0|E|^2d\mathbf{x} + \int_{\Gamma^{in}}2\mathbf{n}.\mathbf{K}|E|^2d\Gamma = \int_{\Gamma^{in}}4\mathbf{n}.\mathbf{K}|E|^2d\Gamma + \int_{\Gamma^{in}}[i\overline{E}(k_0^{-1}\mathbf{n}.\nabla_\perp + 2i\mathbf{n}.\mathbf{K})(\alpha^{in}) + cc]d\Gamma,$$

and we get the result of point (i).

(ii) According to the previous point, we have

$$\int_{\mathcal{D}}\nu_0|E|^2d\mathbf{x} + |\mathbf{n}.\mathbf{K}|\int_{\Gamma^{in}}|E|^2d\Gamma \le \left(\int_{\Gamma^{in}}|(k_0^{-1}\mathbf{n}.\nabla_\perp + 2i\mathbf{n}.\mathbf{K})\alpha^{in}|^2d\Gamma\right)^{1/2}\left(\int_{\Gamma^{in}}|E|^2d\Gamma\right)^{1/2}, \tag{3.48}$$

therefore, we see that

$$\left(\int_{\Gamma^{in}}|E|^2d\Gamma\right)^{1/2} \le \frac{1}{\inf|\mathbf{n}.\mathbf{K}|}\left(\int_{\Gamma^{in}}|(k_0^{-1}\mathbf{n}.\nabla_\perp + 2i\mathbf{n}.\mathbf{K})\alpha^{in}|^2d\Gamma\right)^{1/2},$$

and plugging this upper bound into (3.48), we get the desired result. □

Proof of the Proposition 14. If one assumes that K is constant, one can see that (3.46) yields

$$|\widetilde{E^n_{j+1,m}}|^2 \leq \max\left(|E^n_{j,m}|^2, |E^{n-1}_{j,m}|^2\right),$$

Now in the general case, one can check that

$$K_{j+1}|\widetilde{E^n_{j+1,m}}|^2 \leq K_j \max\left(|E^n_{j,m}|^2, |E^{n-1}_{j,m}|^2\right). \tag{3.49}$$

For the Crank–Nicolson technique, according to (3.47), we get:

$$\sum_m K_{j+1} \frac{E^n_{j+1,m} - \widetilde{E^n_{j+1,m}}}{\delta z} (\overline{E^n_{j+1,m}} + \overline{\widetilde{E^n_{j+1,m}}}) + c.c. =$$

$$\sum_m \left(-\frac{i}{4}(\delta N)_{j+1/2,m}(\widetilde{E^n_{j+1,m}} + E^n_{j+1,m}) + \frac{i}{2k_0}((\Delta_\perp \widetilde{E^n_{j+1}})_m + (\Delta_\perp E^n_{j+1})_m)\right) (\overline{E^n_{j+1,m}} + \overline{\widetilde{E^n_{j+1,m}}}) + c.c..$$

By applying the standard relation $\Sigma_m(2u_m - u_{m+1} - u_{m-1})\overline{u_m} = \Sigma_m(u_m - u_{m+1})(\overline{u_m} - \overline{u_{m+1}}) \in \mathbf{R}$ to $u = \widetilde{E^n_{m+1}} + E^n_{m+1}$, we check easily that the right-hand side of the above equation is zero [this is the discrete counterpart of the relation $\int_{\mathbf{R}} \overline{E(y)} \Delta_y E(y) dy \in \mathbf{R}$]. Then, we have the conservation of energy during the second step $\|E^n_{j+1}\|^2 = \|\widetilde{E^n_{j+1}}\|^2$.

Thus, (3.49) yields $\|E^n_{j+1}\|^2 \leq \max\left(\|E^n_j\|^2, \|E^{n-1}_j\|^2\right)$ and by induction the result. The proof is similar for the FFT method. See [7] for the details. □

3.2 The Brillouin Instability in Laser–Plasma Interaction

We deal here with only one type of model used in laser–plasma interaction. We consider the coupling with the ion acoustic waves in order to account for the *Brillouin instability.* It occurs by the coupling of three waves: an ion acoustic wave (which is a perturbation of ion density), the main laser wave which travels forwards (called the *pump wave*) and a backscattered laser wave which travels backwards (called the *stimulated Brillouin backscattered wave*).

So, besides the macroscopic density and the velocity of the plasma, one needs to handle the time–space envelope of the main and backscattered transverse laser fields (as in the first section) as well as the space envelope of the ion acoustic waves generated by the Brillouin instability.

This modelling of Brillouin instability has been performed for a long time, but it is quite tricky; see the articles [14, 69, 82, 99] (see also [105] for a mathematical introduction to this derivation). For a physical introduction to the phenomenon of the classical three-wave coupling model, see, e.g., [88].

We first explain how to set a first Brillouin model where the plasma flow is decomposed into the classical macroscopic hydrodynamics and a wave system satisfied by the ion acoustic wave (here only the outlines of the derivation are given).

Afterwards, in the framework of a homogeneous plasma, we explain how to obtain the three-wave coupling model, which is the most popular for modelling the Brillouin instability (it is also known as the *standard decay model*).

The main assumptions are :

- The simulation domain is a strip between an entrance boundary Γ^{in} and an outgoing boundary Γ^{out}.
- The incoming laser field $\alpha^{in}(t)$ the entrance boundary Γ^{in} on the simulation domain is slowly time varying and the main laser beam is assumed to propagate according to the unit vector $\mathbf{e}_b$ normal to Γ^{in} (we set $z = \mathbf{e}_b.\mathbf{x}$).
- The dimensionless macroscopic electron density $\mathcal{N}^0(t, \mathbf{x})$ is close to N_{ref} and is a smooth function. Here N_{ref} is strictly smaller than 1 (and in practice less than 0.5). More precisely, we will set

$$N(t, \mathbf{x}) = \mathcal{N}^0(t, \mathbf{x}) + \mathcal{N}^0(t, \mathbf{x})n(t, \mathbf{x}), \qquad |n| \ll 1.$$

where n will be a perturbation of the plasma density (called ion acoustic wave).

As above, the laser wave is represented by the electromagnetic field $\mathbf{E}^r = \mathbf{E}^r(t, \mathbf{x})$ solution to

$$\frac{\partial^2}{\partial t^2}\mathbf{E}^r - c^2\Delta\mathbf{E}^r + \omega_p^2\frac{\mathcal{N}^0}{N_{\text{ref}}}(1+n)\mathbf{E}^r = 0 \tag{3.50}$$

where

$$\omega_p = \omega_0 N_{\text{ref}}^{1/2}$$

is the plasma frequency. Recall that the group velocity and the laser wave number are given by

$$c_g = c\sqrt{1 - N_{\text{ref}}}, \qquad k_p = \frac{1}{c}\sqrt{\omega_0^2 - \omega_p^2} = k_0\sqrt{1 - N_{\text{ref}}}.$$

Here the phase function ϕ satisfies the trivial eikonal equation $|\nabla\phi|^2 = k_p^2/k_0^2$; then it is given by $k_0\phi(\mathbf{x}) = \pm k_p z$. Recall that if we had not accounted for the backscattered laser wave, we would have set $\mathbf{E}^r(t, \mathbf{x}) \simeq E(t, \mathbf{x})e^{ik_p z - i\omega_0 t} + c.c.$ with E a scalar complex function solving the paraxial approximation

$$\partial_t E + c_g\partial_z E + \nu_a E - i\frac{c}{2k_0}\Delta_\perp E = i\beta_0\left(1 - \frac{\mathcal{N}^0}{N_{\text{ref}}}(1+n)\right)E.$$

$$\beta_0 = \omega_0\frac{N_{\text{ref}}}{2} = \frac{\omega_p^2}{2\omega_0}$$

But now we decompose the laser field $\mathbf{E}^r(t,\mathbf{x})$ into two components, the forward one $Ee^{ik_pz-i\omega_0 t}+c.c.$ and the backscattered one $\Phi e^{-ik_pz-i\omega_0 t}+c.c.$ (E and Φ are slowly varying functions with respect to space and time)

$$\mathbf{E}^r(t,\mathbf{x}) \simeq \left[E(t,\mathbf{x})e^{ik_pz-i\omega_0 t}+c.c\right]+\left[\Phi(t,\mathbf{x})e^{-ik_pz-i\omega_0 t}+c.c.\right],$$

As above, two waves satisfy the paraxial equations

$$\frac{\partial}{\partial t}E+c_g\partial_z E+\nu_a E-\frac{ic}{2k_0}\Delta_\perp E = -i\beta_0\frac{\mathcal{N}^0}{N_{\text{ref}}}ne^{-2ik_pz}\Phi+i\beta_0\left(1-\frac{\mathcal{N}^0}{N_{\text{ref}}}(1+n)\right)E \tag{3.51}$$

$$\frac{\partial}{\partial t}\Phi-c_g\partial_z\Phi+\nu_a\Phi-\frac{ic}{2k_0}\Delta_\perp\Phi = -i\beta_0\frac{\mathcal{N}^0}{N_{\text{ref}}}ne^{2ik_pz}E+i\beta_0\left(1-\frac{\mathcal{N}^0}{N_{\text{ref}}}(1+n)\right)\Phi \tag{3.52}$$

Notice that $\nabla|\mathbf{E}^r|^2 = \nabla\left[e^{2ik_pz}E\overline{\Phi}+e^{-2ik_pz}\Phi\overline{E}\right]+\nabla(|\Phi|^2+|E|^2)$ so upon neglecting terms like $\nabla\Phi$ with respect to $2ik_p\Phi$, we get

$$\nabla\left[e^{2ik_pz}E\overline{\Phi}\right]\simeq e^{2ik_pz}2ik_pE\overline{\Phi}.$$

Therefore, the ponderomotive effect generates a highly oscillating force with respect to z according to the frequency $2k_p$.

For the plasma response we decompose the plasma flow $N,\mathbf{U}$ into two subsystems:

- firstly, the macroscopic flow $(\mathcal{N}^0,\mathbf{U}^0)$, which is the solution of a system ($\mathcal{BTE}$-ii, iii) but with $(|E|^2+|\Phi|^2)$ replacing $|A|^2$;
- secondly, a perturbation wave that corresponds to highly oscillating functions n and $\mathbf{U}^1$,

$$N\simeq\mathcal{N}^0(1+n),\qquad \mathbf{U}\simeq\mathbf{U}^0+\mathbf{U}^1.$$

For the macroscopic flow, we get the classical system (recall that c_s is the ion acoustic sound speed and γ_p is the ponderomotive coefficient)

$$\frac{\partial}{\partial t}\mathcal{N}^0+\nabla.(\mathbf{U}^0\mathcal{N}^0)=0,$$

$$\frac{\partial}{\partial t}(\mathcal{N}^0\mathbf{U}^0)+\nabla.(\mathbf{U}^0\mathbf{U}^0\mathcal{N}^0)+c_s^2\nabla\mathcal{N}^0 = -\gamma_p\mathcal{N}^0\nabla(|E|^2+|\Phi|^2),$$

and after some calculus, we get the perturbation system for the quantities $n,\mathbf{U}^1$

$$\mathcal{N}^0(\frac{\partial}{\partial t}+\mathbf{U}^0.\nabla)n+\mathcal{N}^0\nabla.\mathbf{U}^1 = -\mathbf{U}^1.\nabla\mathcal{N}^0, \tag{3.53}$$

$$(\frac{\partial}{\partial t}+\mathbf{U}^0.\nabla)\mathbf{U}^1+c_s^2\nabla n+2\nu_L(\mathbf{U}^1-\mathbf{U}^0) = -\mathbf{U}^1.\nabla\mathbf{U}^0-\gamma_p(2ik_pE\overline{\Phi}e^{2ik_pz}+c.c.), \tag{3.54}$$

We have accounted here for a supplementary term $2\nu_L(\mathbf{U}^1 - \mathbf{U}^0)$ due to the Landau damping effect (it is stated here in its simplest form, which is a friction term). So we check that a plasma wave is generated that corresponds to a wave number

$$k_s = 2k_p.$$

It is called an ion acoustic wave. It is coupled with the main laser wave, which travels forwards, and the back-scattered laser wave, which travels backwards.

To explain how the *three-wave coupling systems* are derived, we assume in the next two subsections that

$$\mathcal{N}^0 = N_{\text{ref}} \qquad \text{and} \qquad \mathbf{U}^0 = 0$$

for the sake of simplicity. The first model comes straightforward from the previous decomposition and is called *full modified decay model* (or simply *modified decay model*). The simple way to derive the second one, the so-called *standard decay model*, is to make simplifications based on a stability analysis.

3.2.1 The Modified Decay Model in a Homogeneous Plasma

We neglect now all the transverse effects for an ion acoustic wave; then the above system for $(n, \mathbf{U}^1.\mathbf{e}_b)$ reads simply as follows if we denote $q = \mathbf{U}^1.\mathbf{e}_b c_s^{-1})$.

$$\frac{\partial}{\partial t}n + c_s\partial_z q = 0, \tag{3.55}$$

$$\frac{\partial}{\partial t}q + c_s\partial_z n + 2\nu_L q = -\gamma_p c_s^{-1}(ik_s E\overline{\Phi}e^{ik_s z} + c.c.) \tag{3.56}$$

The propagation equation for Φ reads as

$$\frac{\partial}{\partial t}\Phi - c_g\partial_z\Phi = -i\beta_0 n e^{ik_s z}E - i\beta_0 n\Phi$$

Remark 22. The momentum balance.

According to the previous system, we check easily that

$$\frac{\partial}{\partial t}(nq) + c_s\partial_z(\frac{n^2}{2} + \frac{q^2}{2}) + 2\nu_L nq = -\frac{\gamma_p k_s}{c_s}n(iE\overline{\Phi}e^{ik_s z} + c.c.)$$

$$\frac{c_s}{c_g}\frac{\partial}{\partial t}|\Phi|^2 - c_s\partial_z|\Phi|^2 = -\beta_0 n(ie^{ik_s z}E\overline{\Phi} + c.c.)$$

Thus, we see that there exists a constant $\alpha > 0$, such the following relation holds

$$\frac{\partial}{\partial t}\left[N_{\text{ref}}c_s n q - \alpha\frac{c_s}{c_g}|\Phi|^2\right] + c_s\partial_z\left[\alpha|\Phi|^2 + N_{\text{ref}}c_s^2(\frac{n^2}{2} + \frac{q^2}{2})\right] + 2N_{\text{ref}}\nu_L c_s n q = 0. \tag{3.57}$$

So, if the damping coefficient ν_L is set to zero, this is a conservation law for a perturbation of global momentum where the quantity $-\alpha\frac{c_s}{c_g}|\Phi|^2$ corresponds to the magnetic momentum of the backscattered wave. □

The source term on (3.56) is highly oscillating with respect to the space only, then the solution (n, q) is also highly oscillating with a wave number equal to k_s. We may take the space envelope M (which is not highly oscillating) of n, i.e., we set

$$n(t, z) = M(t, z)e^{ik_s z} + c.c.$$

thus, neglecting as above the terms $\partial_z\Phi, \partial_z E$ with respect to $ik_s\Phi$, $ik_s E$, the previous system is equivalent to the following equation for M

$$\frac{\partial^2}{\partial t^2}M - c_s^2(\partial_z + ik_s)^2 M + 2\nu_L\frac{\partial}{\partial t}M = -\gamma_p k_s^2 E\overline{\Phi}. \tag{3.58}$$

After neglecting the terms $\overline{M}\Phi e^{-2ik_s z}$ and $MEe^{2ik_s z}$ which are highly oscillating with respect to the space variable, the propagation equations for E and for Φ may read as

$$\frac{\partial}{\partial t}E + c_g\partial_z E = -i\beta_0 M\Phi, \tag{3.59}$$

$$\frac{\partial}{\partial t}\Phi - c_g\partial_z\Phi = -i\beta_0\overline{M}E. \tag{3.60}$$

Equations (3.58)–(3.60) make up the so-called *modified decay model* for the Brillouin instability. Of course, it must be supplemented by initial conditions $E(0, .)$, $\Phi(0, .)$, $M(0, .)$ and boundary conditions on both sides of the simulation interval for M and on the sides $z = 0$ and $z = L$ for E and Φ.

3.2.2 *The Standard Decay System in a Homogeneous Plasma*

Our aim is now to derive the so-called "standard decay model" from (3.55), (3.56), (3.59), and (3.60). In this framework we set $\nu_a = 0$ and neglect all the transverse effects. The trick is the following: one first assumes that the main laser field E is constant, then we re-introduce the evolution equation for E.

- **First step**

Assuming that E is constant, we address the linear system (3.55), (3.56), and (3.60) for (n, q, Φ) from a stability analysis point of view, then we consider it on the whole space ($z \in \mathbf{R}$). In the case when ν_L is set to zero, after introducing the space envelope m and s of the two traveling waves $(n+q)/2$ and $(n-q)/2$,

$$\frac{n+q}{2} = m(z)e^{ik_s z} + c.c., \qquad \frac{n-q}{2} = s(z)e^{ik_s z} + c.c.,$$

we get the following advection equation for m, s and Φ [in this last case, the highly oscillating term $(m+s)Ee^{2ik_s z}$ is neglected]:

$$\frac{\partial}{\partial t} m + c_s(ik_s + \partial_z)m + \nu_L(m-s) = -i\frac{\gamma_p k_s}{2c_s} E\overline{\Phi} \tag{3.61}$$

$$\frac{\partial}{\partial t} s - c_s(ik_s + \partial_z)s + \nu_L(s-m) = i\frac{\gamma_p k_s}{2c_s} E\overline{\Phi}. \tag{3.62}$$

$$\left(\frac{\partial}{\partial t} - c_g\partial_z\right)\Phi = -i\beta_0(\overline{m} + \overline{s})E \tag{3.63}$$

Now we are aimed at replacing this system with a simpler one, which consists of withdrawing the function s, i.e.,

$$\frac{\partial}{\partial t} m + c_s(ik_s + \partial_z)m + \nu_L m = -i\frac{\gamma_p k_s}{2c_s} E\overline{\Phi} \tag{3.64}$$

$$\left(\frac{\partial}{\partial t} - c_g\partial_z\right)\Phi = -i\beta_0\overline{m}E \tag{3.65}$$

From the point of view of stability analysis, we address these two systems on the whole space ($z \in \mathbf{R}$) in the case $\nu_L = 0$ (for the sake of simplicity). So we need to search m, s and Φ in the form (with $m^\sharp, s^\sharp$, and $\Phi^\sharp$ constant)

$$m(t,z) = m^\sharp e^{-i\Omega t}e^{i\zeta z}, \qquad s(t,z) = s^\sharp e^{-i\Omega t}e^{i\zeta z}, \qquad \Phi(t,z) = E\Phi^\sharp e^{i\Omega t}e^{-i\zeta z}$$

For the existence of a nontrivial solution $(m^\sharp, s^\sharp, \overline{\Phi^\sharp})$, the dispersion relation reads as follows.

$$\begin{vmatrix} \Omega - c_s(\zeta + k_s) & 0 & -\beta_0^{-1}B^2 \\ 0 & \Omega + c_s(\zeta + k_s) & \beta_0^{-1}B^2 \\ \beta_0 & \beta_0 & \Omega + c_s\zeta/\varepsilon \end{vmatrix} = 0 \tag{3.66}$$

denoting $\varepsilon = c_s/c_g$ and

$$B = |E|\sqrt{\beta_0\gamma_p k_s/2c_s} = |E|k_0\varepsilon^{-1/2}\sqrt{N_{\mathrm{ref}}\gamma_p/2}$$

For the simplified system, the dispersion relation reads as

$$\begin{vmatrix} \Omega - c_s(\zeta + k_s) & -\beta_0^{-1} B^2 \\ \beta_0 & \Omega + c_s\zeta/\varepsilon \end{vmatrix} = 0. \tag{3.67}$$

We justify first this replacement by a heuristic calculus; afterwards we give a more rigorous result.

Heuristic Justification

Dispersion relation (3.67) reads as $(\Omega - c_s(\zeta + k_s))(\Omega + c_s\zeta/\varepsilon) + B^2 = 0$. For this second-order equation in Ω, the determinant Δ_ζ is equal to $c_s^2\left(k_s + \frac{1}{\varepsilon}\zeta(1+\varepsilon)\right)^2 - 4B^2$. The growth of the instability corresponds to the case where there exists a couple (Ω_ζ, ζ) such that $\Delta_\zeta < 0$ and $\mathrm{Im}(\Omega_\zeta) > 0$ and it behaves like $\exp(t\,\mathrm{Im}(\Omega_\zeta))$. Since we have

$$\mathrm{Im}(\Omega_\zeta) = \frac{1}{2}\sqrt{-\Delta_\zeta} = \frac{1}{2}\sqrt{-c_s^2\left(k_s + \frac{1+\varepsilon}{\varepsilon}\zeta\right)^2 + 4B^2}$$

the maximum value of this quantity is obtained for

$$\zeta = \zeta_* = -k_s\frac{\varepsilon}{1+\varepsilon}.$$

The corresponding amplification coefficient is

$$\mathrm{Im}(\Omega_{\zeta_*}) = B = |E|k_0\varepsilon^{-1/2}\sqrt{N_{\mathrm{ref}}\gamma_p/2} \tag{3.68}$$

Consider now the subsystem for (s, Φ). We get the dispersion relation

$$\begin{vmatrix} \Omega + c_s(\zeta + k_s) & \beta_0^{-1} B^2 \\ \beta_0 & \Omega + c_s\zeta/\varepsilon \end{vmatrix} = 0$$

i.e., $(\Omega + c_s(k_s + \zeta))(\Omega + c_s\zeta/\varepsilon) - B^2 = 0$, so the solutions Ω^s are given by

$$\Omega^s = \frac{1}{2} \pm \frac{1}{2}\sqrt{c_s^2\left(k_s + \frac{1+\varepsilon}{\varepsilon}\zeta\right)^2 + 4B^2 - 4c_s^2(\zeta + k_s)\frac{\zeta}{\varepsilon}}$$

For $\zeta = \zeta_*$, we get

$$\Omega^s = \frac{1}{2} \pm \frac{1}{2}\sqrt{4B^2 + 4c_s^2k_s^2\frac{1}{(1+\varepsilon)^2}}.$$

Therefore, Ω^s are always real, so the s-wave does not grow with respect to the time and it is negligible with respect to the m-wave, which behaves like $\exp(tB)$. The same type of calculus may be performed in the case where ν_L is not zero. Thus, we may claim that s is negligible with respect to m when the time increases. That is, M may be replaced by m in the coupling terms in laser propagation equations.

For the density fluctuation, we get the following approximation

$$n(t,z) \simeq e^{ik_s z} m(t,z) + c.c.$$

Second Justification

A more rigorous justification is based on the following proposition; nevertheless a complete proof dealing with the four-equation system is still an open problem.

Proposition 15. *In order to have the existence of a nonreal root Ω of dispersion relation (3.66) for all values of the parameter B, it is necessary that $\zeta = \zeta_*$. For this value, there is a root Ω_{full} of dispersion relation (3.66) such that $\mathrm{Im}(\Omega_{full}) > 0$ which is close to $\mathrm{Im}(\Omega_{\zeta_*}) = B$ solution of (3.67). More precisely we have (when $B/(c_s k_s)$ is small)*

$$\mathrm{Im}(\Omega_{full})/\mathrm{Im}(\Omega_{\zeta_*}) = 1 + a_0(B/(c_s k_s))^2 + O((B/(c_s k_s))^3), \qquad \text{with } a_0 \simeq 0.51(1+\varepsilon)^2$$

This means that the equivalence between the two systems for (m, s, Φ) and (m, Φ) is justified if the quantity $B/(c_s k_s) = |E| c^{1/2} c_s^{-3/2} \sqrt{N_{\mathrm{ref}} \gamma_p / 2}$ is small with respect to 1. From a practical point of view, this constraint may be relaxed if the damping coefficient ν_L is not zero and generally it reduces to $|E| c^{1/2} c_s^{-3/2} \sqrt{N_{\mathrm{ref}} \gamma_p}$ smaller than a constant of the order of 1 (this is generally the case; see below the numerical application where $|E|$ has been replaced by α_{ref}).

Whatever system is addressed, notice that the smaller is the laser intensity $|E|^2$, the smaller is the growth of the Brillouin instability.

- **Second step**

We now address the system (3.64), (3.65), neglecting s, and we supplement it with the evolution equation for the wave E (where the highly oscillating term $\overline{m}\Phi e^{-2ik_s z}$ is neglected). So we get the so-called *standard decay* system

$$\partial_t E + c_g \partial_z E = -i\beta_0 m \Phi$$
$$\partial_t \Phi - c_g \partial_z \Phi = -i\beta_0 \overline{m} E,$$
$$\partial_t m + c_s(ik_s + \partial_z)m + \nu_L m = -i\frac{\gamma_p k_s}{2c_s} E \overline{\Phi}$$

Notice that in this system, the two characteristic speeds c_g and c_s occur, which are, respectively, on the order of the speed of light and the speed of sound. The quadratic coupling terms on the right-hand side correspond to the coupling between the three waves.

When this system is posed on the interval $[0, z_{\max}]$ it is supplemented with the following natural boundary conditions

$$E(t,0) = \alpha^{in}, \qquad \Phi(t, z_{\max}) = 0, \qquad m(t,0) = 0. \tag{3.69}$$

If the initial values of m and Φ are zero, then $\Phi(t) = m(t) = 0$ is a trivial solution. To get a nontrivial solution for the previous system, it is sufficient to have $m(0,.)$ equal to a small random noise [or to address a boundary condition on m by setting $m(t,0)$ equal to a small random noise].

3.2.3 Model with a Nonhomogeneous Plasma

We now re-introduce the space variation of the plasma density at the macroscopic scale by assuming that the macroscopic flow $(\mathcal{N}^0, \mathbf{U}^0)$ is somehow given and we set

$$\Gamma = \mathcal{N}^0 / N_{\text{ref}},$$

it is assumed to be a smooth function depending only on space and being close to one (at this scale it does not depend on time).

We now perform the calculus as above, accounting for the Landau damping term, the variation of Γ, and the macroscopic plasma velocity $U^0 = \mathbf{U}.\mathbf{e}_b$. So, without the transverse terms, the linearized system for the perturbation density reads as follows

$$\mathcal{N}^0 \left(\frac{\partial}{\partial t} n + c_s \partial_z q + U^0 \partial_z q \right) = -c_s q \partial_z^0 \mathcal{N}^0,$$

$$\frac{\partial}{\partial t} q + (c_s + U^0) \partial_z n + 2\nu_L q = -i \frac{\gamma_p k_s}{c_s} \mathcal{N}^0 E \overline{\Phi} - q \partial_z U^0.$$

using the same approximation for the ponderomotive force as before. Let us set $c_s' = c_s + U^0$. The corresponding equation for the spatial envelope m of the ion acoustic wave $\frac{n(z)+q(z)}{2}$ reads as

$$\partial_t m + c_s'(i k_s + \partial_z) m + \nu_L \Gamma m = -\frac{i \gamma_p k_s}{2 c_s} \Gamma E \overline{\Phi} - m \partial_z (U^0 + c_s \log \Gamma). \tag{3.70}$$

For the laser waves, we use the same paraxial equations (3.51), (3.52), as above:

$$\partial_t E + c_g \partial_z E + \nu_a E - i \frac{c}{2k_0} \Delta_\perp E = i \beta_0 (1 - \Gamma) E - i \beta_0 \Gamma m \Phi, \tag{3.71}$$

$$\partial_t \Phi - c_g \partial_z \Phi + \nu_a \Phi - i \frac{c}{2k_0} \Delta_\perp \Phi = i \beta_0 (1 - \Gamma) \Phi - i \beta_0 \Gamma \overline{m} E. \tag{3.72}$$

Roughly speaking, these three equations make up the full three-wave coupling system; it is the simplest one that is solved in the large three-dimensional codes *HERA* [82] or *PF3D* [14]. Of course, (3.70) may be more complicated, accounting for transverse diffusion or harmonic decomposition; see, e.g., [69, 88]. Notice that the size of the cells need to be adapted to the physical phenomena, particularly as it is shown below that the typical length of the Brillouin instability is quite small—it may be on the order of the laser wavelength. Then the longitudinal size of the cells needs to be a little smaller than this wavelength; thus, one needs to deal on a huge finite difference mesh (with hundreds of millions of cells) for handling significant simulations.

3.2.4 A Three-Wave Coupling System and Its Analysis

In the one-dimensional framework, without the zero-order terms on the right-hand sides, such as $(1-\Gamma)$, $\log\Gamma$, the system (3.71) (3.72) (3.70) reads as follows:

$$(\mathcal{TWC})\quad \boxed{\begin{array}{ll} \text{(i)} & \partial_t E + c_g \partial_z E = -i\beta_0 \Gamma m \Phi, \\ \text{(ii)} & \partial_t \Phi - c_g \partial_z \Phi = -i\beta_0 \Gamma \overline{m} E, \\ \text{(iii)} & \partial_t m + c_s'(ik_s + \partial_z) m + \nu_L \Gamma m = -\frac{i\gamma_p k_p}{2c_s} \Gamma E \overline{\Phi}. \end{array}}$$

The main mathematical difficulties of this system may be posed on a bounded interval. It is supplemented with the natural boundary conditions like (3.69) (and, of course, with initial conditions).

Notice that if the initial values $\Phi(0,.)$ and $m(0,.)$ are zero, then there is a trivial solution that is $m = \Phi = 0$ and E is the solution of $\partial_t E + c_g \partial_z E = 0$. But we shall see that this solution is unstable.

3.2.4.1 Conservation Properties

It is easy to check that if a solution (E, Φ, m) satisfies this system on an interval $[0, L]$, we get

$$\partial_t \left(\int_0^L |E|^2 dx + \int_0^L |\Phi|^2 dx \right) + c_g[|E(L)|^2 + |\Phi(0)|^2] = c_g[|E(0)|^2 + |\Phi(L)|^2],$$

This means that the time variation of the total laser energy is equal to the incoming laser intensity $c_g[|E(0)|^2 + |\Phi(L)|^2]$ minus the outgoing intensity. Moreover, we get

$$\partial_t \left(\int_0^L |m|^2 dx + Y \int_0^L |E|^2 dx \right) + 2\nu_L \Gamma \int_0^L |m|^2 dx \le c_g Y |E(0)|^2 + |m(0)|^2.$$

with

$$Y = \frac{2c_s}{\gamma_p k_p \beta^c}.$$

This is a balance relation related to the quanta $|m|^2$. Of course, we have also the following relation, which a combination of the two previous ones:

$$\partial_t \left(\int_0^L |m|^2 dx - Y \int_0^L |\Phi|^2 dx \right) + 2\nu_L \Gamma \int_0^L |m|^2 dx \le c_s' \left(|m(0)|^2 - |m(L)|^2 \right) + Y_{c_g} [|\Phi(0)|^2 + |\Phi(L)|^2].$$

This balance relation is of the same type as the one related to the global momentum in the modified decay model (3.57), and they are, of course, also true if, in a three-dimensional framework, one accounts for diffraction phenomena in this system.

3.2.4.2 Characteristic Values of the System ($\mathcal{TWC}$)

For the sake of simplicity, we do not account here for the variation of Γ, i.e., we assume $\Gamma = 1$.

For classical FCI plasma the group velocity c_g of the laser field is very close to light velocity 3×10^8 m/s. If the plasma electron temperature is on the order of some 10^7 K, the speed of sound c_s is somehow smaller than 10^5 m/s and c_s' also; thus, if we set

$$\varepsilon = \frac{c_s'}{c_g}$$

this parameter is typically on the order of 10^{-3}. Denoting α_{ref} a characteristic value of α^{in}, let us define $\hat{E}, \hat{\Phi}, \hat{M}$ as follows

$$E = \hat{E}\alpha_{\text{ref}}, \qquad \Phi = \hat{\Phi}\alpha_{\text{ref}}, \qquad m = -i\hat{m}\frac{\alpha_{\text{ref}}}{c_s'}\sqrt{2\gamma_p \frac{1 - N_{\text{ref}}}{N_{\text{ref}}}}.$$

Then if we set

$$\gamma_0 = \alpha_{\text{ref}} \frac{k_0}{c_s'} \sqrt{\frac{\gamma_p}{2} N_{\text{ref}}},$$

the previous system reads

$$\partial_t \hat{E} + \frac{c_s'}{\varepsilon} \partial_z \hat{E} = -\frac{c_s'}{\varepsilon} \gamma_0 \hat{m} \hat{\Phi},$$

$$\partial_t \hat{\Phi} - \frac{c_s'}{\varepsilon}\partial_z \hat{\Phi} = \frac{c_s'}{\varepsilon}\gamma_0 \overline{\hat{m}}\widehat{E_p},$$

$$\partial_t \hat{m} + c_s'(ik_s + \partial_z)\hat{m} + \nu_L \hat{m} = c_s'\gamma_0 \hat{E}\overline{\hat{\Phi}}.$$

Therefore, good characteristic length and time of the three-wave coupling problem are given by ${\gamma_0}^{-1}$ and $(c_s'\gamma_0)^{-1}$.

Before going on, let us give some order of magnitude of the different quantities in the application we have in mind. The typical physical values of the problem we are interested in are the following. For the plasma we take $N_{\text{ref}} = 0.15$ and with a temperature on the order of 35. 10^6 K, then $c_s \simeq 5.6\ 10^6\, cm/s$ and c_s' also. In a very hot speckle of the laser beam, we may assume that the laser intensity is about 10^{16} W/cm^2, which corresponds to α_{ref} on the order of some 10^{11} V/m. Then let's say $\alpha_{\text{ref}} \simeq 10^{11}$ V/m. With a laser wavelength equal to 0.35 μ, we have $k_0 = 1.79 \times 10^7\,\text{m}^{-1}$ and $\omega_0 = 5.3 \times 10^{15}\,\text{s}^{-1}$ then $\sqrt{\gamma_p} \simeq 3 \times 10^{-6}$ in S.I. units; thus we get

$$\gamma_0^{-1} \simeq 4 \times 10^{-7}\text{m}, \qquad (c_s\gamma_0)^{-1} \simeq 0.7 \times 10^{-12}\text{s}.$$

Notice that with these values, the growth coefficient B of the germ of the instability of (m, Φ) [see (3.68)] corresponding to $|E| = \alpha_{\text{ref}}$ and $\nu_L = 0$ is given by $B = \alpha_{\text{ref}} k_0 \varepsilon^{-1/2}\sqrt{\frac{\gamma_p}{2}N_{\text{ref}}} = \varepsilon^{-1/2}c_s\gamma_0$, so it is 70 times larger than $c_s\gamma_0$.

3.2.4.3 Dimensionless Form

To get this form of the system, define first the dimensionless time and space variables

$$t' = c_s'\gamma_0 t, \qquad x = z\gamma_0, \qquad \eta = \nu_L/\gamma_0.$$

With these units, if $\eta = 0$, the Brillouin insatiability grows like $\exp(t'\varepsilon^{-1/2})$ for the most important wave numbers, which have been exhibited in (3.68).

Then let us perform the change of notations:

$$\hat{E}(t,z) = u(t',x), \qquad \hat{\Phi}(t,z) = v(t',x), \qquad \hat{m}(t,z) = w(t',x).$$

Thus, we get the following system after writing t instead of t',

$$(\varepsilon\partial_t + \partial_x)u = -vw,$$
$$(\varepsilon\partial_t - \partial_x)v = u\overline{w},$$
$$(\partial_t + \partial_x)w + (ik_s + \eta)w = u\overline{v}.$$

A first simple property is the following one. If we set

$$U = ue^{-i\rho(t-\varepsilon x)}, \qquad V = ve^{i\rho(t+\varepsilon x)}, \qquad W = we^{2i\rho t},$$

then the previous system reads as

$$\begin{aligned} \varepsilon\partial_t U + \partial_x U &= -VW, \\ \varepsilon\partial_t V - \partial_x V &= \overline{W}U, \\ (\partial_t + \partial_x)W + (ik_s - 2i\rho + \eta)W &= U\overline{V}. \end{aligned}$$

Therefore, with a simple change of unknown functions (it suffices to set $\rho = k_s/2$), the system to be addressed may read as

$$(\varepsilon\partial_t + \partial_x)u = -vw, \tag{3.73}$$

$$(\varepsilon\partial_t - \partial_x)v = u\overline{w} \tag{3.74}$$

$$(\partial_t + \partial_x)w + \eta w = u\overline{v}, \tag{3.75}$$

This system, called the *Boyd–Kadomstev system* , is supplemented with the boundary condition

$$u(T,0) = u^{\text{in}}, \qquad v(t,L) = 0, \qquad w(t,0) = 0.$$

where $|u^{\text{in}}|$ is on the order of 1 and L is large enough compared to 1. Notice that $w(t,0)$ may also be a function of t that is small compared to 1. Of course, initial conditions need also to be prescribed, e.g.,

$$u(0,.) = u_0, \qquad v(0,.) = 0, \qquad w(0,.) = w_0$$

where u_0 and w_0 are bounded functions. In general, the L^∞-norm of u_0 is on the order of 1 and the one of w_0 is much smaller than 1. Notice that this kind of quadratic wave coupling phenomena occur in other areas; e.g., in the Rossby wave problem in classical fluid dynamics (see [47]) or for vibrating string modelling.

As for system $(\mathcal{TWC})$, the conservation properties hold and read as

$$\varepsilon\partial_t\left(\int_0^L |u|^2dx + \int_0^L |v|^2dx\right) + |u(L)|^2 + |v(0)|^2 = |u^{\text{in}}|^2, \tag{3.76}$$

$$\partial_t\left(\int_0^L |w|^2dx + \varepsilon\int_0^L |u|^2dx\right) + 2\eta\int_0^L |w|^2dx + |u(L)|^2 + |w(L)|^2 = |u^{\text{in}}|^2. \tag{3.77}$$

Remark 23. A stationary solution on the half-line.

Assume that the time derivative is cancelled in the previous problem and that it is addressed on the domain $\mathbf{R}^+$ (i.e., $L = +\infty$) with the condition $\lim_{x=\infty} v(x) = 0$, then one checks that, if ξ solves the O.D.E. $\frac{\partial^2}{\partial x^2}(\log \xi) = -|\xi|^2$ with the conditions $\xi(0) = 1, \partial_x(\log \xi)|_{x=0} = 0$, the triplet $(u^s = \xi,\ v^s = \xi,\ w^s = -\partial_x(\log \xi))$ is a stationary solution to the system (3.73)–(3.75) with $\eta = 0$. [Indeed, setting $w = -\log \xi$, the O.D.E may be written as $w'' = e^{-2w}$ with $w(0) = w'(0) = 0$. Thus, we get $(|w'|^2)' = -(e^{-2w})'$. According to the boundary conditions, that implies $|w'|^2 = 1 - e^{-2w}$ and $\xi' = -\xi\sqrt{1-\xi^2}$, so the result follows.]

But this analytic calculus has little interest since it does not provide information on the time evolution of the solution. □

To our knowledge, except the work of Novikov and Zakharov in 1984 on solitons [91] (where x ranges over the full space **R**), there was no convincing published mathematical work related to this system before [86, 106].

We first show that this system is well-posed for all $\eta \geq 0$; more precisely, we have the result [86].

Theorem 1. *Consider system (3.73)–(3.75) with initial data u_0, v_0, w_0 in $L^2_x(0, L)$ and the previous boundary conditions (u^{in} bounded); for all final time T, there exists a unique solution $(u,\ v,\ w)$ in $[C(0, T; L^2_x)]^3$.*

Moreover, if the initial data belong to L^∞_x, then u, v and w belong to $L^\infty_t(0, T; L^\infty_x)$.

The proof of this theorem is given below.

For this system, typical initial values are: u_0 on the order of 1, $v_0 = 0$, and w_0 small with respect to 1. It is a good model for the growth of the Brillouin instability; indeed for u fixed, if we consider the linear system (3.74) (3.75) on the full space **R** and if we take the Fourier transform (in x), the growth of the solution of this linear system is somehow related to the behavior of the solution $(\hat{v}(t,\xi), \hat{w}(t,\xi))$ of the ordinary differential equation (assuming that the damping coefficient η is zero)

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{v}\\ \hat{w}\end{pmatrix} = \begin{pmatrix} i\xi/\varepsilon & u/\varepsilon \\ \overline{u} & -i\xi \end{pmatrix}\begin{pmatrix}\hat{v}\\ \hat{w}\end{pmatrix},$$

and we check that for $u = u_c$ frozen, one of the eigenvalues of the 2×2 matrix has a positive real part which is given by $(|u_c|^2 - \xi^2(1+\varepsilon)^2\frac{1}{4\varepsilon})^{1/2}/\sqrt{\varepsilon}$. It is the largest for ξ vanishing; so the growth of this solution is like $\exp(|u_c|t/\sqrt{\varepsilon})$. Then we may claim that for the full system (3.73)–(3.75), there is a characteristic time that is on the order of $\sqrt{\varepsilon}$. Recall also that the characteristic value of the space variable is on the order of the laser wavelength [cf. the calculus leading to (3.68)].

When one looks at the solution at a time much larger than $\sqrt{\varepsilon}$, the problem is related to the saturation of the linear instability that occurs at the beginning of the time interval.

Therefore, it is natural to consider the problem for values of time of order 1 with respect to ε; this corresponds to addressing the problem with ε going to zero. Now, if one takes ε equal to 0 in the previous system, we may address the following non-classical system satisfied by u_*, v_*, w_*

$$\begin{aligned} \partial_x u_* &= -v_* w_*, \\ -\partial_x v_* &= u_* \overline{w_*}, \\ (\partial_t + \partial_x) w_* + \eta w_* &= u_* \overline{v_*}. \end{aligned} \tag{3.78}$$

It needs to be supplemented with only one initial condition $w|_{t=0}$ and the following boundary conditions

$$u_*(t,0) = u^{in}, \qquad v_*(t,L) = 0, \qquad w_*(t,0) = 0, \qquad \forall t.$$

In [86], it is proved that this system is well-posed in L^2_x; more precisely, there is a unique solution (u_*, v_*, w_*) in $[C(0,T; L^2_x)]^3$. It satisfies, of course, the equivalent of the previous conservation properties, i.e.,

$$|u(x)|^2 - |v(x)|^2 = |u^{\text{in}}|^2 - |v(0)|^2, \qquad \forall x,$$

$$\frac{\partial}{\partial t} \int_0^L |w|^2 dx + 2\eta \int_0^L |w|^2 dx + |u(L)|^2 + |w(L)|^2 = |u^{\text{in}}|^2.$$

Now we have the following result.

Theorem 2. *When $\varepsilon \to 0$, the solution $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ of (3.73)–(3.75) converges for almost all t in L^2_x to the solution (u_*, v_*, w_*) of the previous system (3.78).*

This result is proved in [86], and we do not give it here. Let us mention only that there is here a crucial mathematical difficulty due to the existence of an initial layer near $t = 0$, indeed u_0, v_0 is not generally equal to $u_*(0,.)$ and $v_*(0,.)$.

Remark 24. To overcome this difficulty, we need to build explicitly this initial layer. We give here some ideas how to do this.

Let us set $U(t,x) = \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}$ with a value that is in $\mathbf{C}^2$, and for all bounded functions w, we define the linear operator M_w,

$$M_w U = \begin{pmatrix} \partial_x u + wv \\ -\partial_x v - \overline{w} u \end{pmatrix}$$

Therefore, the original system is equivalent to finding w^ε and $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$ where U^ε satisfies

$$\varepsilon \partial_t U^\varepsilon + M_{w^\varepsilon} U^\varepsilon = 0 \tag{3.79}$$

with initial and boundary conditions

$$u^\varepsilon_{|t=0} = u_0, \quad v^\varepsilon_{|t=0} = 0; \quad u^\varepsilon(t,0) = u^{in}, \quad v^\varepsilon(t,L) = 0. \tag{3.80}$$

and w^ε satisfying

$$\partial_t w^\varepsilon + \partial_x w^\varepsilon = u^\varepsilon \overline{v^\varepsilon}, \quad w^\varepsilon_{|t=0} = w_0, \quad w^\varepsilon(t,0) = 0. \tag{3.81}$$

Now, for the initial layer we disregard the nonhomogeneous boundary conditions and we are led to consider the system corresponding to (3.79) after introducing the fast time variable $\tau = t/\varepsilon$. So we address the solution $\tilde{U}(\tau, .)$ of the linear system

$$\partial_\tau \tilde{U} + M_W \tilde{U} = 0 = 0, \qquad \tilde{U}|_{\tau=s} = V \tag{3.82}$$

with initial data $V = (u_0, v_0) \in [L^2_x]^2$; this system is supplemented with homogeneous boundary conditions $u(\cdot, 0) = 0, \quad v(\cdot, L) = 0$. Here W is defined by $W(\tau, x) = w_1(\varepsilon\tau, x)$ where $w_1(t, x)$ solves the simple advection equation $\partial_t w_1 + \partial_x w_1 = 0$, $w_1(0) = w_0$.

For τ larger than s, denote by $\mathcal{E}_W(\tau, s)V$ the value at time τ of the solution $\tilde{U}$ of (3.82). It is easy to check that

$$\|\mathcal{E}_W(\tau, s)V\|_{[L^2_x]^2} \leq \|V\|_{[L^2_x]^2}, \qquad \forall V \in [L^2_x]^2. \tag{3.83}$$

The linear mapping $\mathcal{E}_W(\tau, s)$ on $[L^2_x]^2$ satisfies the classical relation $\mathcal{E}_W(\tau, s')$ $\mathcal{E}_W(s', s) = \mathcal{E}_W(\tau, s)$. In particular, if W is independent of time, the mapping $\mathcal{E}_W(\tau, 0)$ is a continuous contraction semi-group on $[L^2_x]^2$. Moreover one can prove that for all bounded functions W, there are C and $\gamma > 0$ (depending on W) such that for all $\tau > s$,

$$\|\mathcal{E}_W(\tau, s)V\|_{[L^2_x]^2} \leq C e^{-\gamma(\tau - s)} \|V\|_{[L^2_x]^2}, \qquad \forall V. \tag{3.84}$$

If we consider a sequence of functions $W_n(\tau) = w^{\varepsilon_n}(\varepsilon_n \tau)$ (with $\varepsilon_n \to 0$), the inequality (3.84) is also true with C and $\gamma > 0$ independent of n, under the assumption that w^ε is a bounded family of functions such that $\|w^\varepsilon\|_{L^2(0,t_0;H^1_x)}$ and $\|\frac{\partial}{\partial t} w^\varepsilon\|_{L^2(0,t_0;L^2_x)}$ are bounded independently of ε. □

Remark 25. One crucial difficulty to deal numerically with model (3.70)–(3.72) is related to the small value of ε; as a matter of fact, the time step used by the classical upwind scheme is determined by $\delta t = \varepsilon \delta x$. But since the characteristic value of the space cell size is generally on the order of the laser wavelength; the time $\varepsilon \delta x$ is very small compared to the characteristic time of the growth of the instability which is on the order of $\sqrt{\varepsilon}\delta x$. Most of the mathematical and numerical difficulties related to this problem may be analyzed on the previous system (3.73)–(3.75) in a one-dimensional framework (the numerical treatment of the diffraction terms $\Delta_\perp$ is made by a space-marching technique). The previous remark leads one to think that a numerical method may be used where the time derivative $\varepsilon\partial_t$ is handled by a perturbation technique analogous to the one described in [7] with δt, e.g., on the order of $\sqrt{\varepsilon}\delta x$. □

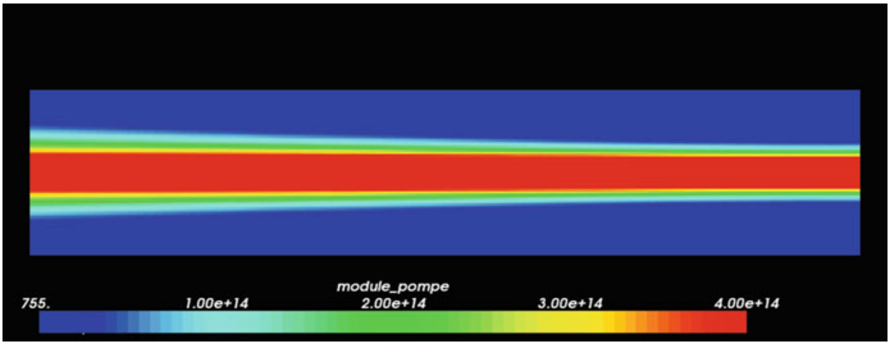

Fig. 3.9 Brillouin backscattering simulation on a toy problem in a two-dimensional framework. The wave enters the domain from the *left*. Map of the main wave intensity at 5 ps.

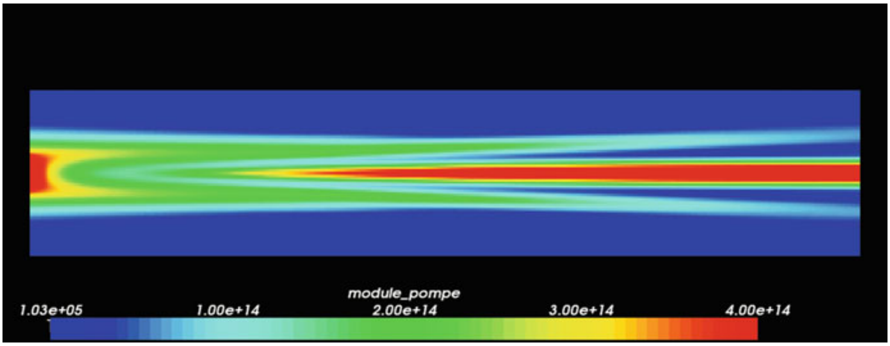

Fig. 3.10 Map of the main wave intensity at 9 ps.

Remark 26 (Numerical illustrations). We are concerned here with results obtained due to model (3.70)–(3.72) with a small value of ε (on the order of 0.002); here the initial value of v is zero and the initial value of w is very small. The simulation corresponds to a two-dimensional toy problem where one addresses only one speckle in the laser beam (with a Gaussian shape) that enters into the simulation domain from the left; so there are only about 100,000 cells (here the scheme is explicit and the time step δt is equal to $\varepsilon \delta x$). Figures 3.9, 3.10, and 3.11 show the map of the laser intensity $|E|^2$ of the main wave at different times (5 $pico$-s, 9 $pico$-s and 11 $pico$-s); moreover Fig. 3.12 shows the map of the intensity $|\Phi|^2$ of the backscattered wave for the last time. Notice that the Landau damping coefficient ν_L has been chosen to be very small (and does not correspond to a realistic plasma) in order to show a fast growth of the instability. □

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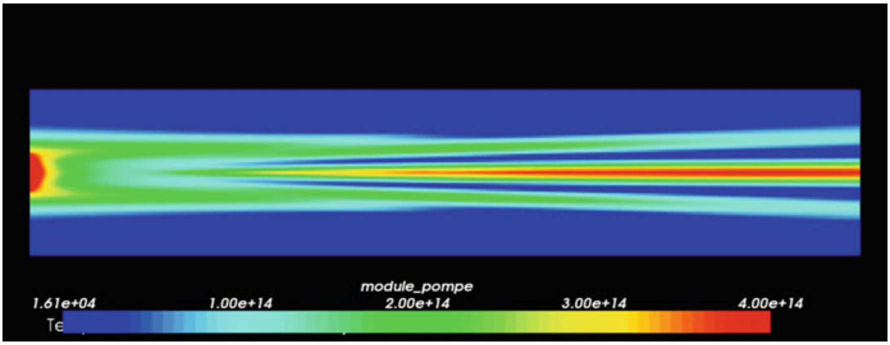

Fig. 3.11 Map of the main wave intensity at 11 ps.

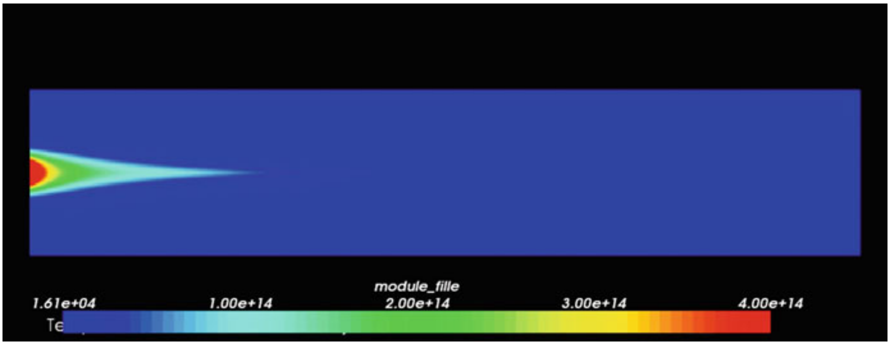

Fig. 3.12 Map of the backscattered wave intensity at 11 ps.

Proofs of Section 3.2

Proof of the Proposition 15. For these calculations we may set $c_s = 1$. The characteristic polynomial of the system (3.66) reads as follows

$$(\Omega + \frac{1}{\varepsilon}\zeta)(\Omega - (\zeta + k_s))(\Omega + (\zeta + k_s)) + B^2(\Omega + (\zeta + k_s) - \Omega + (\zeta + k_s)) = 0.$$

In order to have a nonreal root Ω even if B is small or equal to 0, it is necessary that two of three roots $(\zeta + k_s), -(\zeta + k_s), -\frac{1}{\varepsilon}\zeta$ obtained when $B = 0$, mingle together; the good situation is $\zeta + k_s = -\frac{1}{\varepsilon}\zeta$, i.e., $\zeta = \zeta_*$.

With this value, the previous polynomial reads as follows, if we set $K = \zeta_* + k_s = -\zeta_*/\varepsilon$,

$$(\Omega - K)^2(\Omega + K) + 2KB^2 = 0.$$

To find the roots of this third-order equation, we set $\Omega = K(z + 1/3)$ and $A = BK^{-1}$; it reads

$$z^3 - \frac{4}{3}z + 2(\frac{8}{27} + A^2) = 0$$

then according to Cardan's formula, we see that the determinant δ^2 is positive and we have

$$\delta = 2A(\frac{16}{27} + A^2)^{1/2} = 2A\left(\frac{4}{3\sqrt{3}} + \beta A^2\right).$$

where we denote $\beta = \frac{4}{3\sqrt{3}}A^{-2}\left(\sqrt{1+\frac{27}{16}A^2} - 1\right)$. Thus, there exists one root such that the imaginary part is positive:

$$\mathrm{Im}(z) = \frac{\sqrt{3}}{2}\left(\left[\frac{\delta}{2} + (\frac{8}{27} + A^2)\right]^{1/3} + \left[\frac{\delta}{2} - (\frac{8}{27} + A^2)\right]^{1/3}\right).$$

The two others are the complex conjugate and a real root. After some calculations, we get

$$\mathrm{Im}(z) = \frac{\sqrt{3}}{2}\left((\frac{A}{\sqrt{3}} + \frac{2}{3})^3 + (\beta - \frac{\sqrt{3}}{9})^3 A^3\right)^{1/3} + \frac{\sqrt{3}}{2}\left((\frac{A}{\sqrt{3}} - \frac{2}{3})^3 + (\beta + \frac{\sqrt{3}}{9})^3 A^3\right)^{1/3}$$

We now make an asymptotic expansion of this expression for A small. So get $\beta \simeq \beta_1 = \frac{3\sqrt{3}}{8}$

$$\begin{aligned}\mathrm{Im}(z) &= \frac{\sqrt{3}}{2}(\frac{A}{\sqrt{3}} + \frac{2}{3})\left(1 + \frac{9}{8}(\beta_1 - \frac{\sqrt{3}}{9})^3 A^3\right) + \frac{\sqrt{3}}{2}(\frac{A}{\sqrt{3}} - \frac{2}{3})\left(1 - \frac{9}{8}(\beta_1 + \frac{\sqrt{3}}{9})^3 A^3\right) + O(A^4)\\ &= A + \frac{\sqrt{3}}{3}\left[\frac{9}{8}(\beta_1 - \frac{\sqrt{3}}{9})^3 + \frac{9}{8}(\beta_1 + \frac{\sqrt{3}}{9})^3\right] A^3 + O(A^4),\end{aligned}$$

and the corresponding eigenvalues Ω_{full} for the optimal value ζ_* is such that with $a_0 = \frac{27}{8}((\frac{3}{8} - \frac{1}{9})^3 + (\frac{3}{8} + \frac{1}{9})^3)$

$$\mathrm{Im}(\Omega_{\text{full}}) = B[1 + a_0(B/K)^2 + O((B/K)^3)].$$

The result follows from the fact that $K = k_s/(1+\varepsilon)$. □

Proof of Theorem 1. Let us first give some notations. We introduce the three velocities $c_1 = \varepsilon^{-1}, c_2 = -\varepsilon^{-1}, c_3 = 1$ and $a_1 = -\varepsilon^{-1}$, $a_2 = \varepsilon^{-1}$, $a_3 = 1$.

It may be seen that the absorption term ηw in the equation for w has no importance from a mathematical point of view; thus, we can set $\eta = 0$ in the proof. Now, defining the advection operators K_i by

$$K_i = \partial_t + c_i \partial_x,$$

the original system with general initial values (u_0, v_0, w_0) reads as

$$\begin{aligned} K_1 u &= a_1 v w, \qquad & u(0,.) &= u_0, \qquad & u(t,0) &= u^{\text{in}}, \\ K_2 v &= a_2 u \overline{w}, \qquad & v(0,.) &= v_0, \qquad & v(t,L) &= 0, \\ K_3 w &= a_3 \overline{v} u, \qquad & w(0,.) &= w_0, \qquad & w(t,0) &= 0. \end{aligned} \tag{3.85}$$

The proof is based on a lemma of compensated integrability (it is a classical lemma in the L^1 framework and we extend it here in the L^2 framework). □

Lemma 5 (Compensated integrability). *There exists a constant C_A such that for all τ and for all functions u, v in $L^2(0,\tau; L^2_x)$ such that $K_1 u$ and $K_2 v$ are in $L^2(0,\tau; L^2_x)$, we have uv in $L^2(0,\tau; L^2_x)$ and we get*

$$\|uv\|^2_{L^2(0,\tau;L^2_x)} \le C_A \left[\alpha_u + \tau \, \|K_1 u\|^2_{L^2(0,\tau;L^2_x)}\right] \left[\alpha_v + \tau \, \|K_2 v\|^2_{L^2(0,\tau;L^2_x)}\right]$$

with $\alpha_u = \|u(.,0)\|^2_{L^2(0,\tau)} + \|u_0\|^2_{L^2_x}$ *and* $\alpha_v = \|v(.,L)\|^2_{L^2(0,\tau)} + \|v_0\|^2_{L^2_x}$.

The same result holds, of course, for the other products uw and vw (and for the products $u\overline{v}, \overline{w}v$).

Proof of the Lemma 5. Denote $f = K_1 u$, we have

$$u(t,x) = u_0(x - c_1 t)\mathbf{1}_{x > c_1 t} + u^{in}(\frac{c_1 t - x}{c_1})\mathbf{1}_{x < c_1 t} + \int_0^t f(t-s, x - c_1 s) ds$$

Then we get for all $t < \tau$,

$$|u(t,x)|^2 \le 2\left(|u_0(x - c_1 t)|\mathbf{1}_{x > c_1 t} + |u^{in}(\frac{c_1 t - x}{c_1})|\mathbf{1}_{x < c_1 t}\right)^2 + 2\tau F(x - c_1 t),$$

with $F(y) = \int_0^t |f(y + c_1 s, s)|^2 ds$, i.e., $|u(t,x)|^2 \le \phi_u(x - c_1 t)$ where the function ϕ_u defined on $[-c_1\tau, L]$ is given by

$$\phi_u(\sigma) = 2|u_0(\sigma)|^2 \mathbf{1}_{\sigma > 0} + 2|u^{in}(-\frac{\sigma}{c_1})|^2 \mathbf{1}_{\sigma < 0} + 2\tau F(\sigma)$$

Moreover, we see that

$$\|\phi_u\|_{L^1_x} \leq 2\alpha_u + 2\tau\,\|K_1 u\|^2_{L^2(0,\tau,L^2_x)}\,, \qquad \alpha_u = \|u_0\|^2_{L^2_x} + \|u(.,0)\|^2_{L^2_t}\,.$$

In the same way, we get $|v(t,x)|^2 \leq \phi_v(x - c_2 t)$ with ϕ_v such that for all $t \leq \tau$

$$\|\phi_v\|_{L^1_x} \leq 2\alpha_v + 2\tau\,\|K_2 v\|^2_{L^2(0,\tau,L^2_x)}\,, \qquad \alpha_v = \|v_0\|^2_{L^2_x} + \|v(.,L)\|^2_{L^2_t}\,.$$

Now with the new variables $y = (x - c_1 t)$ and $y' = (x - c_2 t)$ using the fact that $dxdt = |c_1 - c_2|^{-1} dydy'$, we get

$$\int\int |u(t,x)|^2 |v(t,x)|^2 dxdt \leq \int\int |c_1 - c_2|^{-1} \phi_u(y)\phi_v(y')dydy' \leq |c_1 - c_2|^{-1}\,\|\phi_u\|_{L^1_x}\,\|\phi_v\|_{L^1_x}$$

and the result follows. □

Proof of the First Part of the Theorem

The first part of the theorem comes from a classical continuity argument using the conservation relations (3.76), (3.77), after the following lemma is proved.

Lemma 6. *For τ small enough, there is a unique solution (u, v, w) in $[L^2(0,\tau; L^2_x)]^3$ of problem (3.85). Moreover, (u, v, w) belongs to $[C(0,\tau; L^2_x)]^3$.*

Proof of the Lemma 6. Let us set $\mathcal{U} = \{u, v, w\}$. Define the space $\mathcal{L}^{2,\tau} = (L^2(0,\tau; L^2_x))^3$ endowed by the norm $\|\mathcal{U}\|_{\mathcal{L}^{2,\tau}} = \sup_i \|\mathcal{U}_i\|_{L^2(0,\tau;L^2_x)}$. Denote $\mathcal{F}(\mathcal{U}) = \{a_1 vw, a_2 u\overline{v}, a_3 u\overline{v}\}$ and $\mathcal{K}\mathcal{U} = \{K_1\mathcal{U}_1, K_2\mathcal{U}_2, K_3\mathcal{U}_3\}$.

Existence. It is based on a fixed-point algorithm.

Let us denote by $\mathcal{U}^0 = (u^0, v^0, w^0)$ the solution of problem (3.85) without the quadratic right-hand side terms and define the sequence $\mathcal{U}^{n+1} = (u^{n+1}, v^{n+1}, w^{n+1})$ by

$$\begin{aligned} \mathcal{K}\mathcal{U}^{n+1} &= \mathcal{F}(\mathcal{U}^n) \\ \mathcal{U}^{n+1}(0,.) &= \{u_0, v_0, w_0\} \\ u^{n+1}(t,0) &= u^{in}, \qquad v^{n+1}(t,L) = 0, \qquad w^{n+1}(t,0) = 0. \end{aligned}$$

Now, denote $\mathcal{K}\mathcal{U}^n = \mathcal{G}^n$ and for fixed initial and boundary values, address the mapping

$$\mathcal{G}^n \mapsto \mathcal{G}^{n+1} = \mathcal{F}(\mathcal{U}^n) = \mathcal{F}(\mathcal{K}^{-1}(\mathcal{G}^n)).$$

and $\mathcal{G}^1 = \mathcal{F}(\mathcal{K}^{-1}(0))$. Then let us consider two elements $\mathcal{G} = \{g_1, g_2, g_3\}$ and $\hat{\mathcal{G}} = \{\widehat{g}_1, \widehat{g}_2, \widehat{g}_3\}$ and for fixed initial and boundary values, define $\mathcal{U}$ and $\hat{\mathcal{U}}$ by $\mathcal{K}\mathcal{U} = \mathcal{G}$

and $\mathcal{K}\hat{\mathcal{U}} = \hat{\mathcal{G}}$. So we may set $\mathcal{F}(\mathcal{K}^{-1}(\hat{\mathcal{G}})) - \mathcal{F}(\mathcal{K}^{-1}(\mathcal{G})) = \{a_1 z_1, a_2 z_2, a_3 z_3\}$ where we have $z_1 = \hat{v}\hat{w} - vw = \hat{v}(\hat{w} - w) + w(\hat{v} - v)$. Since $K_3(\hat{w} - w) = (\hat{g_3} - g_3)$, according to the previous lemma we know that

$$\|\hat{v}(\hat{w} - w)\|^2_{\mathcal{L}^{2,\tau}} \le C_A \left(\alpha_v + \tau \left\|\hat{\mathcal{G}}\right\|^2_{\mathcal{L}^{2,\tau}}\right) \tau \|\widehat{g_3} - g_3\|^2_{\mathcal{L}^{2,\tau}} .$$

Denoting $B^2 = \max(\|u_0\|^2_{L^2_x} + \left\|u^{in}\right\|^2_\infty, \|v_0\|^2_{L^2_x}, \|w_0\|^2_{L^2_x})$ and assuming that $\tau \le 1$, we have

$$\alpha_u \le B^2, \qquad \alpha_v \le B^2, \qquad \alpha_v \le B^2,$$

therefore, we get

$$\left\|\mathcal{F}(\mathcal{K}^{-1}(\hat{\mathcal{G}})) - \mathcal{F}(\mathcal{K}^{-1}(\mathcal{G}))\right\|^2_{\mathcal{L}^{2,\tau}} \le a_2 C_A \left(2B^2 + \tau \left\|\hat{\mathcal{G}}\right\|^2_{\mathcal{L}^{2,\tau}} + \tau \|\mathcal{G}\|^2_{\mathcal{L}^{2,\tau}}\right) \tau \left\|\mathcal{G} - \hat{\mathcal{G}}\right\|^2_{\mathcal{L}^{2,\tau}} \quad (3.86)$$

Applying this relation to the sequence $\mathcal{G}^{n+1} = \mathcal{F}(\mathcal{K}^{-1}(\mathcal{G}^n))$, we see that the sequence $q_n = \|\mathcal{G}^n\|^2_{\mathcal{L}^{2,\tau}}$ satisfies

$$q_1 = \left\|\mathcal{F}(\mathcal{K}^{-1}(0))\right\|^2_{\mathcal{L}^{2,\tau}} \le a_2 C_A B^4$$
$$q_{n+1} \le a_2 C_A (2B^2 + \tau q_n)\tau q_n \le C_2 + C_2 \tau^2 q_n^2$$

Then, one can check that if τ is such that $8C_2\tau^2 \le 1$ and $a_2 C_A B^4 2C_2\tau^2 \le 1$, the sequence q_n is decreasing and we have $\|\mathcal{G}^n\|^2_{\mathcal{L}^{2,\tau}} \le a_2 C_A B^4$ for all n. Using this bound and according to inequality (3.86), if we assume that

$$a_2 C_A B^2 \tau \le 1/3$$

then the mapping $\mathcal{G}^n \mapsto \mathcal{G}^{n+1}$ on the space $\mathcal{L}^{2,\tau}$ is strictly contracting and the entire sequence $\mathcal{G}^n$ converges to some element $\mathcal{G}$ in $\mathcal{L}^{2,\tau}$ such that

$$\mathcal{G} = \mathcal{F}(\mathcal{K}^{-1}(\mathcal{G}))$$

Thus, denoting $\mathcal{U}$ the solution of $\mathcal{K}\mathcal{U} = \mathcal{G}$ in $\mathcal{L}^{2,\tau}$, we have $\mathcal{G} = \mathcal{F}(\mathcal{U})$ and $\mathcal{K}\mathcal{U} = \mathcal{F}(\mathcal{U})$. That means that $\mathcal{U}$ is solution of (3.85).

Since the right-hand sides $\mathcal{F}(\mathcal{U})_i$ belong to $L^2(0, \tau; L^2_x)$, classical semi-group arguments imply that the solution $\mathcal{U}$ belongs to $(C(0, \tau; L^2_x))^3$.

Uniqueness. Assume that there exist two solutions $\mathcal{U}, \hat{\mathcal{U}}$; they satisfy $\mathcal{K}\mathcal{U} = \mathcal{F}(\mathcal{U})$ and $\mathcal{K}\hat{\mathcal{U}} = \mathcal{F}(\hat{\mathcal{U}})$. Then, setting $\tilde{\mathcal{U}} = \hat{\mathcal{U}} - \mathcal{U}$, we get

$$|K_i \tilde{\mathcal{U}}_i| \le |a_i| \left(|\mathcal{U}_j \tilde{\mathcal{U}}_{j'}| + |\tilde{\mathcal{U}}_j \hat{\mathcal{U}}_{j'}|\right), \qquad \text{with } (j, j') \ne i$$

According to the previous lemma, since the initial and boundary values of $\tilde{\mathcal{U}}$ are zero, we get

$$\left\|\mathcal{K}\tilde{\mathcal{U}}\right\|^2_{L^2(o,\tau,L^2_x)} \leq a_2 C_A \left[\tau \left\|\mathcal{K}\tilde{\mathcal{U}}\right\|^2_{\mathcal{L}^{2,\tau}} (B^2 + \tau \|\mathcal{K}\mathcal{U}\|^2_{\mathcal{L}^{2,\tau}}) + \tau \left\|\mathcal{K}\tilde{\mathcal{U}}\right\|^2_{\mathcal{L}^{2,\tau}} (B^2 + \tau \left\|\mathcal{K}\hat{\mathcal{U}}\right\|^2_{\mathcal{L}^{2,\tau}})\right].$$

Thus, $\tilde{\mathcal{U}} = 0$ for τ small enough. □

Proof of the Second Part of the Theorem

The proof is based on addressing the system of a two-dimensional box $[0, L]\times[0, T]$. Notice that we have the balance relation

$$\frac{\partial}{\partial t}(\varepsilon|u|^2 + |w|^2) + \frac{\partial}{\partial x}(|u|^2 + |w|^2) = 0$$

Integrating this relation on the quadrangle OP_wPP_u where the vertices are $O = (0,0)$, $P_w = (x - t, 0)$, $P = (x,t)$, $P_u = (0, t - \varepsilon x)$, and assuming that $t \geq \varepsilon x, \varepsilon(L - x)$, we get (denoting by ξ the linear abscise)

$$\int_{[P_w P]} (1-\varepsilon)|u|^2 d\xi + \int_{[P_u P]} (1-\varepsilon)|w|^2 d\xi \leq B^2.$$

In the same way, we may integrate the balance relation

$$\frac{\partial}{\partial t}(|w|^2 - \varepsilon|v|^2) + \frac{\partial}{\partial x}(|w|^2 + |v|^2) = 0$$

on the quadrangle OP_wPP_v [where $P_v = (x - \varepsilon t, L)$], we get

$$\int_{[P_w P]} (1+\varepsilon)|v|^2 d\xi + \int_{[P_v P]} \varepsilon|w|^2 d\xi \leq B^2.$$

In the same way, if $t < \varepsilon(L - x)$, we have to integrate over the surface OP_wPLP_v (where $L = (L, 0)$) and we get the same kind of bounds (with a supplementary term $\int_{[LP_v]} |w|^2 d\xi$).

In all cases, we get the following bounds for all $t \leq T$ with a constant C depending on the initial data, on ε and on T,

$$\int_0^t |u(t-s, x-s)|^2 ds \leq C$$

$$\int_0^t |v(t-s, x-s)|^2 ds \leq C$$

Then, from integrating the advection equation (3.75) along its characteristic line, we have

$$
\begin{aligned}
|w(t,x)| &\leq \int_0^t |u(t-s,x-s)\overline{v}(t-s,x-s)|ds + \|w_0\|_{L^\infty} \\
&\leq \left(\int_0^t |u(t-s,x-s)|^2 ds \int_0^t |v(t-s,x-s)|^2 ds\right)^{1/2} + \|w_0\|_{L^\infty}
\end{aligned}
$$

and w is bounded in $L^\infty_{t,x}$. In the same way, we have bounds on $\int_0^t |v(t-s,x-\varepsilon s)|^2 ds$ and $\int_0^t |w(t-s,x-\varepsilon s)|^2 ds$ and we get the results for u and for v also. □

Chapter 4
Langmuir Waves and Zakharov Equations

Abstract Here we address for the sake of completeness the modelling of the electron plasma waves, also called Langmuir waves. We recall how the coupling of these waves with the ion population leads to the system of Zakharov equations and the different approximations that are made for this derivation.

Keywords Plasma frequency • Langmuir waves • Electron plasma waves • Zakharov equations

In this chapter, we address wave phenomena with a characteristic observation time that is on the order of the electron plasma frequency ω_p^{-1}

$$T_{\text{obs}} \sim \omega_p^{-1} = \frac{\lambda_D}{v_{\text{th,e}}},$$

or on the order of

$$T_{\text{obs}} \sim \frac{\lambda_D}{v_{\text{th,i}}} = \frac{\lambda_D}{v_{\text{th,e}}}\sqrt{\frac{m_0}{m_e}},$$

but smaller than $\frac{L_{\text{plasma}}}{v_{\text{th,i}}}$. On the other hand, the Debye length λ_D is smaller than the characteristic length L_{obs} but it is not negligible. For example, we may have

$$L_{\text{obs}} \sim \lambda_D\sqrt{\frac{m_0}{m_e}}.$$

So we stress that we are not in the framework of the massless approximation (as in previous chapters). Moreover, in this chapter, we do not deal with a coupling with external electromagnetic sources. We show that in a hot plasma, electron Langmuir

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waves (also called electrostatic waves) may be generated, and they oscillate with a frequency close to the plasma frequency.

In the first section, we assume that the ions are at rest; but in the two following sections, we consider a more general situation with a coupling with ion motion, and we look at the behavior of Langmuir waves on long time intervals. So, there is interplay between the high-frequency Langmuir waves with a characteristic velocity that is on the order of the electron thermal velocity $v_{\mathrm{th,e}}$ and the classical acoustic ion waves that are characterized by the ion sound velocity which is on the order of $v_{\mathrm{th,i}}$. To obtain a practical modelling, one performs a time envelope of the high-frequency Langmuir waves; then the coupling with the acoustic ion waves leads to the well-known Zakharov equations.

4.1 Langmuir Waves Without Coupling with Ions

We are concerned with an ion population that is assumed to be at rest: $\mathbf{U} = 0$ and N_0 does not depend on time. So, we have

$$\mathbf{J} = -q_e N_e \mathbf{U}_e.$$

Let us set

$$\nu_{ei} = \nu_{e0} N_0 / m_e.$$

Moreover, for the sake of simplicity we disregard the magnetic effects, so the evolution equations for electron density and velocity read in a classical way,

$$\frac{\partial}{\partial t} N_e + \nabla.(N_e \mathbf{U}_e) = 0,$$

$$\frac{\partial}{\partial t}(N_e \mathbf{U}_e) + \nabla.(N_e \mathbf{U}_e \mathbf{U}_e) + \frac{1}{m_e} \nabla P_e = -\frac{q_e}{m_e} N_e \mathbf{E} - \nu_{ei} N_e \mathbf{U}_e.$$

It is possible to supplement this system by addressing an evolution equation for the electron temperature in the form $\left(\frac{\partial}{\partial t} + \nabla.\mathbf{U}_e \bullet\right)\left(\frac{1}{2} m_e N_e |\mathbf{U}_e|^2 + \frac{3}{2} N_e T_e\right) = \mathbf{J}.\mathbf{E} - \nabla.(P_e \mathbf{U}_e)$ (if we neglect the exchange term between ion and electron temperatures and the thermal conduction). But we make here a simplification: we do not account for the evolution of electron temperature and we assume that the electron pressure is a given function of the density $P_e = \mathcal{P}_e(N_e)$. This is the so-called barotropic model. In the simplest case, assuming that the electron population is adiabatic,[1] we set $P_e = c_e^2 N_e$, where the characteristic sound velocity c_e is given by

[1]That is, the pressure P_e is assumed to obey the law $P_e = P_{\mathrm{ref}} N_{\mathrm{ref}}^{-3} N_e^3$. Therefore, we get $\nabla P_e = 3 P_{\mathrm{ref}} N_{\mathrm{ref}}^{-3} N_e^2 \nabla N_e \simeq 3 T_{\mathrm{ref}} \nabla N_e$).

$$c_e = \sqrt{3} v_{\text{th,e}} = \sqrt{3T_{\text{ref}}/m_e}.$$

For the sake of simplicity, we also assume that the electron temperature is constant and equal to T_{ref} (sometimes in the literature, one sets the sound velocity c_e is equal to $v_{\text{th,e}}$).

In the framework of a barotropic model for electrons, one checks that there is a related energy balance, which reads as

$$\frac{\partial}{\partial t}\left(N_e(\frac{1}{2}m_e|\mathbf{U}_e|^2+\varepsilon_{ps}(N_e))\right)+\nabla.\left(N_e\mathbf{U}_e(\frac{1}{2}m_e|\mathbf{U}_e|^2+\varepsilon_{ps}(N_e))\right)+\nabla.(\mathcal{P}_e(N_e)\mathbf{U}_e)$$
$$= \mathbf{J}.\mathbf{E} - m_e \nu_{ei} N_e |\mathbf{U}_e|^2$$

where the pseudo-internal energy $\varepsilon_{ps}(.)$ is the function of N_e such that $\varepsilon'_{ps}(N_e) = N_e^{-2}\mathcal{P}_e(N_e)$.

[**Indeed**, according to the classical identity $N_e\left(\frac{\partial}{\partial t} + \mathbf{U}_e.\nabla\right)\bullet = \frac{\partial}{\partial t}(N_e\bullet) + \nabla.(\mathbf{U}_e N_e \bullet)$ we get

$$m_e N_e \left(\frac{\partial}{\partial t} + \mathbf{U}_e.\nabla\right)\mathbf{U}_e + \nabla\mathcal{P}_e(N_e) = -q_e N_e \mathbf{E} - \nu_{ei} m_e N_e \mathbf{U}_e;$$

moreover, we check that $N_e\left(\frac{\partial}{\partial t} + \mathbf{U}_e.\nabla\right)\left(\varepsilon_{ps}(N_e)\right) = -N_e^{-2}\mathcal{P}_e(N_e)\nabla.\mathbf{U}_e$. Then, combining with the previous relation multiplied by $\mathbf{U}_e$, we get the desired relation. $\square$]

In the case where $P_e = c_e^2 N_e$ the corresponding energy balance relation holds for $(c_e^2 N_e \log N_e + N_e \frac{1}{2}|\mathbf{U}_e|^2)$: the term $\mathbf{J}.\mathbf{E}$ corresponds to the electromagnetic energy; moreover, $m_e \nu_{ei} N_e |\mathbf{U}_e|^2$ corresponds to the energy that is lost by the electron wave (due to the Coulomb interactions between ions and electrons).

Since we are interested in the propagation of fluctuations, we make the following decomposition of the electron density

$$N_e = N_e^l + n_e$$

where N_e^l is slowly varying with respect to the time and n_e is highly time oscillating (with a time scale corresponding to the electron plasma frequency).

Moreover, we assume that $|n_e| \ll N_e^l$ and that N_e^l does not depend on time. So, our concern is the following system for $(n_e, \mathbf{U}_e)$

$$\frac{\partial}{\partial t} n_e + \nabla.(N_e^l \mathbf{U}_e) = 0, \tag{4.1}$$

$$N_e^l \frac{\partial}{\partial t}\mathbf{U}_e + \nabla.(N_e^l \mathbf{U}_e \mathbf{U}_e) + c_e^2 \nabla n_e = -\frac{N_{\text{ref}}}{m_e} q_e \mathbf{E} - \nu_{ei} N_e^l \mathbf{U}_e \tag{4.2}$$

There are now two ways to proceed.

- In the general case one has to account for the non-linear term $\nabla.(N_e^l \mathbf{U}_e \mathbf{U}_e)$. This type of system with a decomposition between two time scales is the basis for a derivation of the Zakharov equations; see the next section.
- In the framework we are interested in this first section, we neglect the nonlinear term $\nabla.(N_e^l \mathbf{U}_e \mathbf{U}_e)$. The aim is to introduce some elementary facts concerning this linear framework; indeed, this feature of plasma physics is crucial for a lot of models and applications (the related concepts will be useful in the next section).

In this simplified framework we get the following linear system.

$$(\mathcal{LEW}) \quad \begin{array}{|ll|} \text{(i)} & \dfrac{\partial}{\partial t} n_e + \nabla.(N_e^l \mathbf{U}_e) = 0, \\ \text{(ii)} & \dfrac{\partial}{\partial t}(N_e^l \mathbf{U}_e) + c_e^2 \nabla n_e = -\dfrac{N_{\text{ref}}}{m_e} q_e \mathbf{E} - \nu_{ei} N_e^l \mathbf{U}_e. \end{array}$$

According to the previous approximations, we set $\mathbf{J} = -q_e N_e^l \mathbf{U}_e$ and the system reads also as follows (remind that $\omega_p^2 = N_{\text{ref}} q_e^2 / (\varepsilon^0 m_e)$)

$$q_e \frac{\partial}{\partial t} n_e - \nabla.\mathbf{J} = 0 \tag{4.3}$$

$$-\frac{\partial}{\partial t}\mathbf{J} + q_e c_e^2 \nabla n_e + \omega_p^2 \varepsilon^0 \mathbf{E} = \nu_{ei} \mathbf{J} \tag{4.4}$$

Of course, it must be supplemented by an equation for the electric field. Before focusing on the analysis of this system coupled with the Gauss relation, we recall briefly in the following subsection some elementary facts concerning this system.

4.1.1 Conductivity and Dispersion Relation

Since there is no magnetic effect, the electric field is given by the Faraday equation

$$\varepsilon^0 \frac{\partial}{\partial t} \mathbf{E} = \mathbf{J}. \tag{4.5}$$

and we check that if the Gauss relation $\varepsilon^0 \nabla.\mathbf{E} = q_e(ZN_0 - N_e^l - n_e)$ is satisfied at initial time, it is always satisfied.

Assume that the solution $(n_e, \mathbf{J})$ behaves like a linear wave of the type $\exp(i\mathbf{k}.\mathbf{x} - i\omega t)$ and introduce the classical envelope functions $\widetilde{n_e}, \tilde{\mathbf{J}}$ and $\tilde{\mathbf{E}}$ by $n_e = \widetilde{n_e} e^{i\mathbf{k}.\mathbf{x} - i\omega t} + c.c.$, $\mathbf{J} = \tilde{\mathbf{J}} e^{i\mathbf{k}.\mathbf{x} - i\omega t} + c.c.$ etc. We want to set up a dispersion relation, that is to say a relation between ω and $\mathbf{k}$ such that this wave exists; we also compute the link between the fields $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{E}}$ (which is related to the plasma conductivity).

From the previous linear system, we have

$$-q_e\omega\widetilde{n_e} = \mathbf{k}.\tilde{\mathbf{J}}, \qquad (i\omega - \nu_{ei})\tilde{\mathbf{J}} + iq_e c_e^2\widetilde{n_e}\mathbf{k} + \omega_p^2\varepsilon^0\tilde{\mathbf{E}} = 0; \tag{4.6}$$

then, eliminating $\widetilde{n_e}$, we get

$$i\omega_p^2\varepsilon^0\tilde{\mathbf{E}} = (\omega + i\nu_{ei})\tilde{\mathbf{J}} - \frac{\omega_p^2}{\omega}3\lambda_D^2\mathbf{k}\mathbf{k}.\tilde{\mathbf{J}} \tag{4.7}$$

Therefore, we get the relation

$$\tilde{\mathbf{J}} = \left(\mathbb{I} - \frac{\omega_p^2 3\lambda_D^2}{\omega(\omega + i\nu_{ei})}\mathbf{k}\mathbf{k}\right)^{-1} \frac{i\omega_p^2\varepsilon^0}{(\omega + i\nu_{ei})}\tilde{\mathbf{E}}$$

$$\tilde{\mathbf{J}} = \left((\mathbb{I} - \frac{\mathbf{k}\mathbf{k}}{|\mathbf{k}|^2})\frac{1}{(\omega + i\nu_{ei})} + \frac{1}{\omega + i\nu_{ei} - \omega_p^2 3\lambda_D^2|\mathbf{k}|^2\omega^{-1}}\frac{\mathbf{k}\mathbf{k}}{|\mathbf{k}|^2}\right) i\omega_p^2\varepsilon^0\tilde{\mathbf{E}} \tag{4.8}$$

In other words,

$$\tilde{\mathbf{J}} = \underset{\leftrightarrow}{\sigma}(\omega\,, \mathbf{k})\tilde{\mathbf{E}}$$

where $\underset{\leftrightarrow}{\sigma}(\omega\,, \mathbf{k})$ is the conductivity tensor defined by the previous formula. In the reference frame where the first unit vector is $\mathbf{k}/|\mathbf{k}|$, this tensor reads as

$$\underset{\leftrightarrow}{\sigma}(\omega\,, \mathbf{k}) = i\omega_p^2\varepsilon^0 \begin{pmatrix} \frac{1}{\omega(1-\omega_p^2 3\lambda_D^2|\mathbf{k}|^2/\omega^2)+i\nu_{ei}} & 0 & 0 \\ 0 & \frac{1}{\omega+i\nu_{ei}} & 0 \\ 0 & 0 & \frac{1}{\omega+i\nu_{ei}} \end{pmatrix}$$

Moreover, the Faraday law (4.5) implies that

$$i\omega\varepsilon^0\tilde{\mathbf{E}} = \tilde{\mathbf{J}} \tag{4.9}$$

thus, for the existence of a vector $\tilde{\mathbf{E}}$, we see that the frequency ω must satisfy the relation

$$\det(\underset{\leftrightarrow}{\sigma}(\omega\,, \mathbf{k})\frac{i}{\omega\varepsilon^0} + \mathbb{I}) = 0 \tag{4.10}$$

It is the so-called dispersion relation. On the one hand, there is a solution ω of (4.10) satisfying simply

$$\omega^2 + i\nu_{ei}\omega = \omega_p^2.$$

This wave corresponds to a transverse wave (i.e., that propagates in the direction orthogonal to the electric field) $\mathbf{k}.\tilde{\mathbf{E}} = 0$ and $\widetilde{n_e} = 0$; it is a pure electromagnetic wave.

On the other hand, there is a solution ω of (4.10) that corresponds to a longitudinal wave (the electric field $\tilde{\mathbf{E}}$ is parallel to the direction of propagation $\mathbf{k}$): if we denote by $\sigma_{\|}$ the scalar conductivity

$$\sigma_{\|}(\omega, \mathbf{k}) = i\omega_p^2\varepsilon^0 \left(\omega(1 - \frac{\omega_p^2}{\omega^2}3\lambda_D^2|\mathbf{k}|^2) + i\nu_{ei}\right)^{-1}, \tag{4.11}$$

then the corresponding value of ω satisfies the dispersion relation

$$\left(\frac{\omega}{\omega_p}\right)^2 + i\nu_{ei}\frac{\omega}{\omega_p^2} = 1 + 3\lambda_D^2|\mathbf{k}|^2. \tag{4.12}$$

This wave is called the Langmuir wave; it is an electrostatic wave (since it may be exhibited also in the case where the electric field is given only by the Gauss relation).

If we define the dielectric permittivity function by $\underline{\varepsilon}(\omega, \mathbf{k}) \equiv 1 + \frac{i}{\omega\varepsilon^0}\sigma_{\|}(\omega, \mathbf{k})$ the dispersion relation (4.12) reads also $\underline{\varepsilon}(\omega, \mathbf{k}) = 0$.

Notice that the dielectric permittivity tensor of the plasma defined by $(\mathbb{I} + \underset{\leftrightarrow}{\sigma}(\omega, \mathbf{k})\frac{i}{\omega\varepsilon^0})$ is useful for modelling the propagation of radio waves in the ionosphere or the propagation of microwaves in a plasma such as magnetic confinement plasmas.

Remark 27 (Accounting for a constant magnetic field.). For the sake of completeness, we show here how the dispersion relation and formula (4.8) for the conductivity tensor should be generalized in the case where there was a magnetic field $\mathbf{B}$ (independent from space and time variable). In equation (4.6) $\tilde{\mathbf{E}}$ would be replaced by $\tilde{\mathbf{E}} - \frac{q_e}{m_e N_e^l}\tilde{\mathbf{J}} \times \mathbf{B}$ and instead of (4.7) we would have

$$i\omega_p^2\varepsilon^0\tilde{\mathbf{E}} = (\omega + i\nu_{ei})\tilde{\mathbf{J}} - \frac{\omega_p^2}{\omega}3\lambda_D^2\mathbf{k}\mathbf{k}.\tilde{\mathbf{J}} + i\frac{q_e N_{\mathrm{ref}}}{m_e N_e^l}\tilde{\mathbf{J}} \times \mathbf{B}$$

So, denoting $\omega_{ce} = \frac{q_e}{m_e}|\mathbf{B}|$ the electron cyclotronic frequency, since $\mathbf{B} = \omega_{ce}\frac{m_e}{q_e}\mathbf{b}$ with $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$, we get

$$((\omega + i\nu_{ei}) - \frac{\omega_p^2}{\omega}3\lambda_D^2\mathbf{k}\mathbf{k})\tilde{\mathbf{J}} + i\omega_{ce}\frac{N_{\mathrm{ref}}}{N_e^l}\tilde{\mathbf{J}} \times \mathbf{b} = i\omega_p^2\varepsilon^0\tilde{\mathbf{E}}.$$

This reads also $\mathbf{J} = \underset{\leftrightarrow}{\sigma}(\omega, \mathbf{k})\mathbf{E}$, but the tensor $\underset{\leftrightarrow}{\sigma}(\omega, \mathbf{k})$ expresses with values depending on the angle between $\mathbf{k}$ and $\mathbf{b}$. Since the magnetic field is constant, Faraday equation (4.5) is still valid; thus (4.9) still holds, and for the existence of a vector $\tilde{\mathbf{E}}$, the frequency ω must satisfy the dispersion relation

$$\det(\underset{\leftrightarrow}{\sigma}(\omega, \mathbf{k})\frac{i}{\omega\varepsilon^0} + \mathbb{I}) = 0$$

the solution of which is more complex than above.

Let us show how the calculus works in the case where the two vectors **k** and **b** are orthogonal. In the reference frame $(\mathbf{k}/|\mathbf{k}|, \mathbf{k}/|\mathbf{k}| \times \mathbf{b}, \mathbf{b})$, we check that we have

$$\begin{pmatrix} \omega - \frac{\alpha}{\omega} + i\nu_{ei} & -i\Omega & 0 \\ i\Omega & \omega + i\nu_{ei} & 0 \\ 0 & 0 & \omega + i\nu_{ei} \end{pmatrix} \tilde{\mathbf{J}} = i\omega_p^2 \varepsilon^0 \tilde{\mathbf{E}}$$

(with $\alpha = \omega_p^2 3\lambda_D^2 |\mathbf{k}|^2$ and $\Omega = \omega_{ce}\frac{N_{\text{ref}}}{N_e^l}$), so we get the expression

$$\underset{\leftrightarrow}{\sigma}(\omega, \mathbf{k}) = i\omega_p^2\varepsilon^0 \begin{pmatrix} (\omega + i\nu)\frac{1}{(\omega+i\nu-\frac{\alpha}{\omega})(\omega+i\nu)-\Omega^2} & i\Omega\frac{1}{(\omega+i\nu-\frac{\alpha}{\omega})(\omega+i\nu)-\Omega^2} & 0 \\ -i\Omega\frac{1}{(\omega+i\nu-\frac{\alpha}{\omega})(\omega+i\nu)-\Omega^2} & (\omega + i\nu - \frac{\alpha}{\omega})\frac{1}{(\omega+i\nu-\frac{\alpha}{\omega})(\omega+i\nu)-\Omega^2} & 0 \\ 0 & 0 & (\omega + i\nu)^{-1} \end{pmatrix}$$

There is one solution ω to the relation $\det[\underset{\leftrightarrow}{\sigma}(\omega, \mathbf{k}) - i\omega\varepsilon^0\mathbb{I}] = 0$, which corresponds to a wave $\tilde{\mathbf{E}}$ parallel to **b** given by the classical formula $\omega^2 + i\nu_{ei}\omega = \omega_p^2$. Moreover, for solutions corresponding to a wave $\tilde{\mathbf{E}}$ orthogonal to **b** (but not parallel to **k**), we need to calculate a 2×2 determinant and solve a polynomial equation with respect to the variable ω; it may be handled by a perturbation argument by imposing $\alpha = 0$ first. In that case, we get

$$(\omega + i\nu - \Omega)(\omega + i\nu + \Omega)(\omega^2 + \omega(\Omega + i\nu) + i\omega_p^2)(\omega^2 + \omega(-\Omega + i\nu) + i\omega_p^2) = 0.$$

i.e., there are solutions of the form $\omega = -i\nu \pm \Omega$ and of the form $\omega = \frac{\Omega}{2} - i\frac{\nu}{2} \pm \frac{1}{2}(\Omega^2 - 4i\omega_p^2 - 2i\Omega\nu)^{1/2}$ and $\omega = -\frac{\Omega}{2} - i\frac{\nu}{2} \pm \frac{1}{2}(\Omega^2 - 4i\omega_p^2 + 2i\Omega\nu)^{1/2}$, but, of course, the value of ω, whose real part is equal up to the sign, corresponds to the same eigenvector (with **k** changed into −**k**.) □

4.1.2 Linear Langmuir Wave Theory

Let us go back now to the linear wave model ($\mathcal{LEW}$), but for the electric field **E** we make a hypothesis that is a little weaker: we do not assume that it satisfies the Faraday equation (4.5), but we assume that it reduces to an electrostatic one (i.e., curl $\mathbf{E} = 0$) and satisfies the Gauss relation

$$q_e \nabla.\mathbf{E} = \frac{q_e^2}{\varepsilon^0}(ZN_0 - N_e^l - n_e) = \omega_p^2 \frac{m_e}{N_{\text{ref}}}(ZN_0 - N_e^l - n_e). \tag{4.13}$$

(of course, in the monodimensional framework, this is equivalent to the Faraday equation). Then taking the time derivative of equation [$\mathcal{LEW}$(i)] and the divergence of [$\mathcal{LEW}$(ii)], this leads to

$$\frac{\partial^2}{\partial t^2} n_e + \omega_p^2 n_e - c_e^2 \Delta n_e + \nu_{ei} \frac{\partial}{\partial t} n_e = \omega_p^2 (Z N_0 - N_e^l). \tag{4.14}$$

Equation (4.14) is a classical linear model for Langmuir waves, where we have exhibited a source term due to plasma heterogeneities; the term with the first derivative is a damping term.

With an analogue calculus to the one above, we see that the solution n_e behaves like $n_e \simeq \widetilde{n_e} \exp(i \mathbf{k}.\mathbf{x} - i\omega t) + c.c.$ and that $(\mathbf{k}, \omega)$ must satisfy dispersion relation

$$-\omega^2 - i \nu_{ei} \omega + \omega_p^2 + 3\omega_p^2 \lambda_D^2 |\mathbf{k}|^2 = 0$$

which is exactly the dispersion relation (4.12) (this means that an electrostatic wave is a longitudinal one).

Since ν_{ei}/ω_p is always small with respect 1, this relation may be approximated by the Bohm–Gross relation

$$\omega \simeq \omega_p \sqrt{1 + 3\lambda_D^2 |\mathbf{k}|^2} - i \frac{\nu_{ei}}{2}. \tag{4.15}$$

Within this formula, we see that the wave energy $|n_e|^2$ is damped and behaves like $\exp(-\nu_{ei} t)$; indeed, if we perform an average $\langle . \rangle$ on a time interval large with respect to ω^{-1}, we get

$$\left\langle |n_e|^2 \right\rangle \simeq |\widetilde{n_e}|^2 \, 2 \left\langle e^{-i\omega t} e^{i\overline{\omega} t} \right\rangle = |\widetilde{n_e}|^2 \, 2 \left\langle e^{-2\mathrm{Re}(i\omega)t} e^{-2i\,\mathrm{Re}(\omega)t} \right\rangle = 2\, |\widetilde{n_e}|^2 \, e^{-\nu_{ei} t}$$

Moreover the frequency of the Langmuir wave, which is given by the real part of ω, is close to the plasma frequency ω_p, but is not strictly equal (this phenomenon is known as the "finite Debye length" effect). Of course, when the Debye length goes to zero, the frequency of the Langmuir wave goes to the plasma frequency ω_p.

Energy Balance

It is not feasible to state a simple energy balance relation by using only the solution n_e of (4.14). But using the original form of this system, i.e., multiplying (4.1) by $c_e^2 n_e$ and (4.2) by $N_e^l \mathbf{U}_e$, we get easily

$$\frac{1}{N_{\text{ref}}} \frac{\partial}{\partial t} \left(\frac{1}{2} c_e^2 n_e^2 + \frac{1}{2} (N_e^l)^2 |\mathbf{U}_e|^2 \right) + \frac{1}{N_{\text{ref}}} c_e^2 \nabla.(n_e N_e^l \mathbf{U}_e) = \frac{1}{m_e} \mathbf{J}.\mathbf{E} - \nu_{ei} \frac{(N_e^l)^2}{N_{\text{ref}}} |\mathbf{U}_e|^2.$$

According to the following lemma, we can see that the solution $(n_e, \mathbf{U}_e, \mathbf{E})$ of (4.1), (4.2) and (4.13) satisfies the balance relation.

$$\frac{\partial}{\partial t}\int\left[\left(\frac{m_e}{2N_{\text{ref}}}(c_e^2 n_e^2 + (N_e^l)^2|\mathbf{U}_e|^2)\right) + \frac{1}{2\varepsilon^0}|\mathbf{E}|^2\right]dx = -m_e\int \nu_{ei}\frac{(N_e^l)^2}{N_{\text{ref}}}|\mathbf{U}_e|^2 dx$$

without boring the reader about boundary conditions (of course, if the domain is bounded, we need to account for boundary terms). So, if $\nu_{ei} = 0$, we get a conservation of the global energy (the sum of the electron wave energy $\frac{m_e}{2N_{\text{ref}}}(c_e^2 n_e^2 + (N_e^l)^2|\mathbf{U}_e|^2)$ and the electrostatic energy); otherwise, this global energy decreases and the right-hand side of the above balance relation corresponds to the heating of the plasma.

Lemma 7. *Assume that the electric field* $\mathbf{E}$ *satisfies (4.13) and* $\text{curl}\mathbf{E} = 0$ *and that* n_e *satisfies (4.1) above; then we get*

$$\int_{\mathbf{R}^3}\mathbf{J}.\mathbf{E}\,dx = -\frac{1}{2\varepsilon^0}\frac{\partial}{\partial t}\int_{\mathbf{R}^3}|\mathbf{E}|^2\,dx \tag{4.16}$$

[Indeed, according to $\frac{\partial}{\partial t}n_e = \nabla.\mathbf{J}$, relation (4.13) leads to $\varepsilon^0\nabla.\mathbf{J} + \nabla.\frac{\partial}{\partial t}\mathbf{E} = 0$. Since $\mathbf{E} = -\nabla\varphi$, we get $0 = \int \varphi\nabla.\left(\varepsilon^0\mathbf{J} + \frac{\partial}{\partial t}\mathbf{E}\right)dx = \int\left(\varepsilon^0\mathbf{J} + \frac{\partial}{\partial t}\mathbf{E}\right).\mathbf{E}\,dx$. □]

4.2 Coupling of Langmuir Waves with Acoustic Waves

The aim of this section is to describe the coupling of the Langmuir waves and the ion acoustic system, which leads to the derivation of the Zakharov equations. For the sake of simplicity, we assume that there is no magnetic field and, as above, that the electric field reduces to its electrostatic part. Moreover, for the electrons, we use a barotropic modelling with $P_e = c_e^2 N_e$ where $c_e^2 = 3\mathrm{v}_{\text{th},e}^2 = 3T_{\text{ref}}/m_e$. Recall that the ion acoustic sound speed is given by

$$c_s^2 = (3ZT_{\text{ref}} + \frac{5}{3}T_0)\frac{1}{m_0} = Zc_e^2\frac{m_e}{m_0} + \frac{5}{3}T_0\frac{1}{m_0}.$$

So, the starting point of the modelling is the following system for the electron population

$$\frac{\partial}{\partial t}N_e + \nabla.(N_e\mathbf{U}_e) = 0,$$

$$\frac{\partial}{\partial t}N_e\mathbf{U}_e + \nabla.(N_e\mathbf{U}_e\mathbf{U}_e) + c_e^2\nabla N_e = -\frac{N_e}{m_e}q_e\mathbf{E} - \nu_{ei}N_e\mathbf{U}_e.$$

$$\nabla.\mathbf{E} = \frac{q_e}{\varepsilon^0}(ZN_0 - N_e)$$

which is coupled with the mass and momentum balance equations for the ion population. We assume that the electron density is a perturbation of a constant value N_{ref} ; recall that the plasma frequency ω_p is given by

$$\omega_p^2 = q_e^2 N_{\text{ref}}/\varepsilon^0 m_e$$

For the electron density, we make the same kind of decomposition as above (but we use dimensionless quantities with a reference density N_{ref}): we address a slowly time-varying part $N_e^l = \mathcal{N} N_{\text{ref}}$ and a highly oscillating part denoted by $n^h N_{\text{ref}}$ with a frequency close to ω_p and we set

$$N_e = n^h N_{\text{ref}} + \mathcal{N} N_{\text{ref}}$$

Moreover, we assume that

$$\left|n^h\right| \ll 1 \qquad \text{and} \qquad |\mathcal{N} - 1| \ll 1.$$

In the same way, we make decomposition of the electron velocity

$$\mathbf{U}_e = \mathbf{U}_e^s + \mathbf{U}_e^h$$

with $\mathbf{U}_e^h$ highly oscillating and $\mathbf{U}_e^s$ slowly time varying; the electrostatic field is also decomposed in a highly oscillating part $\mathbf{E}^h$ and a slowly time varying one $\mathbf{E}^s$

$$\mathbf{E} = \mathbf{E}^h + \mathbf{E}^s$$

Notice that we have curl $\mathbf{E}^h = 0$ and curl $\mathbf{E}^s = 0$.

(a) High-Oscillation Scale.

For the highly oscillating quantities of this model, we get

$$\frac{\partial}{\partial t} n^h + \nabla.(\mathcal{N}\mathbf{U}_e^h) = 0, \tag{4.17}$$

$$\frac{\partial}{\partial t}\mathbf{U}_e^h + c_e^2 \nabla n^h = -\frac{q_e}{m_e}\mathbf{E}^h - \nu_{ei}\mathbf{U}_e^h, \tag{4.18}$$

$$q_e \nabla.\mathbf{E}^h = -\omega_p^2 m_e n^h.$$

It is not possible to derive a linear wave equation for n^h if the multidimensional aspect needs to be taken into account. Here we will state a wave equation for the velocity $\mathbf{U}_e^h$ by taking the time derivative of (4.18) and the gradient of (4.17), so we get

$$\frac{\partial^2}{\partial t^2}\mathbf{U}_e^h - c_e^2 \nabla\nabla.(\mathcal{N}\mathbf{U}_e^h) = -\frac{1}{m_e} q_e \frac{\partial}{\partial t}\mathbf{E}^h - \nu_{ei}\frac{\partial}{\partial t}\mathbf{U}_e^h \tag{4.19}$$

Now according to the electric Gauss relation and (4.17), we have

$$q_e \nabla . \frac{\partial}{\partial t}\mathbf{E}^h = \omega_p^2 m_e \nabla.(\mathcal{N}\mathbf{U}_e^h)$$

It is worth noticing that $q_e \frac{\partial}{\partial t}\mathbf{E}^h - \omega_p^2 m_e(\mathcal{N}\mathbf{U}_e^h)$ is not zero in the multidimensional modelling. Now, gathering this relationship with (4.19), we get the following equation (using the definition of ω_p)

$$\nabla.\left[\frac{\partial^2}{\partial t^2}\mathbf{U}_e^h - c_e^2\nabla\nabla.(\mathcal{N}\mathbf{U}_e^h) + \nu_{ei}\frac{\partial}{\partial t}\mathbf{U}_e^h + \omega_p^2\mathbf{U}_e^h - \omega_p^2(1-\mathcal{N})\mathbf{U}_e^h\right] = 0 \quad (4.20)$$

Here, one proceeds classically by neglecting the multidimensional aspect (see the remark below for accounting this aspect); thus, one claims that the expression inside the [.] is zero and one denotes the velocity by a scalar U_e^h, and we get

$$\frac{\partial^2}{\partial t^2}U_e^h - c_e^2\Delta\left(\mathcal{N}U_e^h\right) = -\omega_p^2\mathcal{N}U_e^h + \nu_{ei}\frac{\partial}{\partial t}U_e^h = -\omega_p^2\mathcal{N}U_e^h. \quad (4.21)$$

Remark 28 (The multidimension aspect). If we want to account for this aspect, we must deal with another form. First, since $q_e\frac{\partial}{\partial t}\mathbf{E}^h$ is the gradient of a potential, say θ, one sees that $-\Delta\theta = \omega_p^2 m_e\nabla.(\mathcal{N}\mathbf{U}_e^h)$, thus, using the operator[2] $(-\Delta)^{-1}$, we get

$$-q_e\frac{\partial}{\partial t}\mathbf{E}^h = \omega_p^2 m_e\nabla[(-\Delta)^{-1}\nabla.(\mathcal{N}\mathbf{U}_e^h)].$$

Therefore, we get

$$\frac{\partial^2}{\partial t^2}\mathbf{U}_e^h - c_e^2\nabla\nabla.(\mathcal{N}\mathbf{U}_e^h) + \nu_{ei}\frac{\partial}{\partial t}\mathbf{U}_e^h = -\omega_p^2\nabla\left[(-\Delta)^{-1}\nabla.\left(\mathcal{N}\mathbf{U}_e^h\right)\right], \quad (4.22)$$

with the constraint that

$$\text{curl}\ \mathbf{U}_e^h = 0,$$

which is due to (4.18) and the fact that curl $\mathbf{E}^h = 0$.

[2] If one assumes that there is Neumann conditions on the boundary of the simulation domain, $-\Delta^{-1}$ defines a function up to an additive constant.

Another form is the following. According to the previous constraint on $\mathbf{U}_e^h$, setting $\chi = \nabla.\mathbf{U}_e^h$ we have $\mathbf{U}_e^h = \nabla((-\Delta)^{-1}\chi)$ and (4.19) reads as follows

$$\frac{\partial^2}{\partial t^2}\chi - c_e^2\Delta\nabla.\left(\mathcal{N}\nabla((-\Delta)^{-1}\chi)\right) + \omega_p^2\chi + \nu_{ei}\frac{\partial}{\partial t}\chi = \omega_p^2\nabla.\left((1-\mathcal{N})\nabla((-\Delta)^{-1}\chi)\right).$$

The advantage of this form is that the unknown is scalar, but the drawback is its complexity. □

Since $\mathcal{N}$ is close to 1 and is a slow time-varying function, regarding equation (4.21), one may check that the solution U_e^h (and n^h) is highly oscillating with a frequency ω that satisfies the so-called Bohm–Gross dispersion relation,

$$\omega = \omega_p\left(1 + 3|\mathbf{k}|^2\lambda_D^2\right)^{1/2} - i\frac{\nu_{ei}}{2} \tag{4.23}$$

Since $|\mathbf{k}|\lambda_D$ is small, the frequency ω is close to $\omega_p - i\frac{\nu_{ei}}{2}$; then we are led to make a WKB expansion of the function U_e^h. That is, we set

$$U_e^h(t,.) = v(t,.)e^{-i\omega_p t} + \text{c.c.} \tag{4.24}$$

where $v(t,.)$ is a slowly varying function.

Then we get

$$\frac{\partial U_e^h}{\partial t} = (\frac{\partial}{\partial t}v - i\omega_p v)e^{-i\omega_p t} + \text{c.c.}, \qquad \frac{\partial^2 U_e^h}{\partial t^2} = (\frac{\partial^2}{\partial t^2}v - 2i\omega_p\frac{\partial}{\partial t}v - \omega_p^2 v)e^{-i\omega_p t} + \text{c.c.}$$

and neglecting $\frac{\partial^2}{\partial t^2}v$ and $i\nu_{ei}\omega_p\frac{\partial}{\partial t}v$ if compared to $\omega_p^2 v$, (4.21) becomes

$$\frac{\partial}{\partial t}v - i\frac{c_e^2}{2\omega_p}\Delta(\mathcal{N}v) + \frac{\nu_{ei}}{2}v = -i\frac{\omega_p}{2}(\mathcal{N}-1)v$$

or since one has $c_e^2 = 3\text{v}_{\text{th,e}}^2 = 3\lambda_D^2\omega_p^2$

$$\frac{1}{\omega_p}\frac{\partial}{\partial t}v - i\frac{3}{2}\lambda_D^2\Delta(\mathcal{N}v) + \frac{\nu_{ei}}{2\omega_p}v = -i\frac{1}{2}(\mathcal{N}-1)v \tag{4.25}$$

If the unknown v was replaced by $\mathcal{N}v$, this equation would be a classical linear Schrödinger equation with a given potential. Notice that n^h may be expressed with the help of v and we have

$$n^h = i\omega_p^{-1}\nabla(\mathcal{N}v)e^{-i\omega_p t} + \text{c.c.}$$

(b) Slow Time Scale.

We now focus on the slow time-varying fluctuations. Let us denote by $\langle .. \rangle$ the average value of highly oscillating quantities at a low time scale. Notice that the product of two oscillating quantities $\mathbf{U}_e^h \mathbf{U}_e^h$ generates slow time varying quantities, so, from the system written at the beginning of this section, we see that

$$\frac{\partial}{\partial t}\mathcal{N} + \nabla.\mathbf{U}_e^s = 0,$$

$$\frac{\partial}{\partial t}\mathbf{U}_e^s + \left\langle \nabla.(\mathbf{U}_e^h \mathbf{U}_e^h) \right\rangle + c_e^2 \nabla \mathcal{N} = -\mathcal{N}\frac{q_e}{m_e}\mathbf{E}^s - \nu_{ei}\mathbf{U}_e^s,$$

where the electrostatic field is $\mathbf{E}^s = -q_e^{-1}\nabla\psi$ with ψ given by the Poisson equation

$$-\frac{\lambda_D^2}{T_{\text{ref}}}\Delta\psi = \frac{ZN_0}{N_{\text{ref}}} - \mathcal{N}.$$

Since one forgets the multidimensional aspect of $\mathbf{U}_e^h$, term $\nabla.(\mathbf{U}_e^h \mathbf{U}_e^h)$ is simply replaced by $\nabla(|U_e^h|^2)$ and the electron momentum equation reads as

$$m_e\left(\frac{\partial}{\partial t}\mathbf{U}_e^s + c_e^2\nabla\mathcal{N}\right) = -\mathcal{N}q_e\mathbf{E}^s - 2m_e\left\langle \nabla(|U_e^h|^2)\right\rangle - m_e\nu_{ei}\mathbf{U}_e^s \tag{4.26}$$

Thus, according to the envelope (4.24), we have

$$\left\langle \nabla(|U_e^h|^2)\right\rangle = \nabla\left\langle |e^{i\omega_p t}v + c.c.|^2\right\rangle = 2\nabla|v|^2$$

and we get

$$m_e\frac{\partial}{\partial t}\mathbf{U}_e^s + m_e c_e^2\nabla\mathcal{N} = -\mathcal{N}q_e\mathbf{E}^s - 2m_e\nabla|v|^2 - m_e\nu_{ei}\mathbf{U}_e^s.$$

Now, let address the ion wave equations with the assumption that the ion population is adiabatic. We get, with a friction term corresponding to the one of (4.26),

$$\frac{\partial}{\partial t}N_0 + \nabla.(N_0\mathbf{U}) = 0,$$

$$m_0\frac{\partial}{\partial t}(N_0\mathbf{U}) + \frac{5}{3}T_0\nabla N_0 = ZN_0 q_e\mathbf{E}^s + m_e\nu_{ei}N_{\text{ref}}\mathbf{U}_e^s$$

Thus, adding the two momentum evolution equations leads to

$$\frac{\partial}{\partial t}(m_0 N_0 \mathbf{U} + m_e N_{\text{ref}} \mathbf{U}_e^s) + \gamma_0 T_0 \nabla N_0 + m_e c_e^2 N_{\text{ref}} \nabla \mathcal{N} = (Z N_0 - N_{\text{ref}} \mathcal{N}) q_e \mathbf{E}^s - 2 N_{\text{ref}} m_e \nabla |v|^2$$
$$= N_{\text{ref}} \frac{\lambda_D^2}{T_{\text{ref}}} \Delta\psi \nabla\psi - 2 N_{\text{ref}} m_e \nabla |v|^2$$

We now make a new hypothesis: we assume that λ_D is small compared with the characteristic length of variation of the ion density N_0; this enables us to use the same philosophy as in Chap. 2 (Sect. 2.2) in order to justify the quasi-neutrality approximation. This approximation says that

$$\mathcal{N} \simeq Z N_0 / N_{\text{ref}}$$
$$\lambda_D^2 \Delta\psi \nabla\psi \simeq 0$$

Therefore, replacing $Z N_0$ by $\mathcal{N} N_{\text{ref}}$ the previous momentum equation reads as

$$\frac{\partial}{\partial t}(m_0 \frac{1}{Z} \mathcal{N} \mathbf{U} + m_e \mathbf{U}_e^s) + \frac{5}{3} \frac{1}{Z} T_0 \nabla \mathcal{N} + m_e c_e^2 \nabla \mathcal{N} = -2 m_e \nabla |v|^2$$

The last step of the derivation of the model is to neglect $m_e \mathbf{U}_e^s$ with respect to $m_0 \mathbf{U}$ in this equation.

To conclude, recalling the definition of c_s, we claim that the real scalar $\mathcal{N}$, the real vector $\mathbf{U}$, and the complex scalar v must satisfy the following system

$$\frac{\partial}{\partial t} \mathcal{N} + \nabla.(\mathcal{N} \mathbf{U}) = 0, \tag{4.27}$$

$$\frac{\partial}{\partial t}(\mathcal{N} \mathbf{U}) + c_s^2 \nabla \mathcal{N} = -2Z \frac{m_e}{m_0} \nabla |v|^2, \tag{4.28}$$

$$\frac{1}{\omega_p} \frac{\partial}{\partial t} v - i \frac{3}{2} \lambda_D^2 \Delta(\mathcal{N} v) + \frac{\nu_{ei}}{2\omega_p} v = -i \frac{1}{2} (\mathcal{N} - 1) v. \tag{4.29}$$

This is a classical wave system coupled with a Schrödinger equation satisfied by the time envelope of the Langmuir wave.

Assume that the simulation domain is the whole space $\mathbf{R}^3$. Then the solution of (4.27)–(4.29) satisfies the two following energy balance relations

$$\int \mathcal{N} \frac{\partial}{\partial t} |v|^2 d\mathbf{x} = -\nu_{ei} \int \mathcal{N} |v|^2 d\mathbf{x} \tag{4.30}$$

$$\frac{\partial}{\partial t} \int \left[\frac{1}{2} \mathcal{N}^2 + \frac{1}{2} |\mathcal{N} \mathbf{U}|^2 + 2Z \frac{m_e}{m_0} \mathcal{N} |v|^2 \right] d\mathbf{x} = -2Z \frac{m_e}{m_0} \nu_{ei} \int \mathcal{N} |v|^2 d\mathbf{x} \tag{4.31}$$

[**Indeed**, Multiplying (4.29) by $\mathcal{N}\overline{v}$ and taking the conjugate quantity, we get

$$\frac{1}{\omega_p}\left(\int \mathcal{N}\frac{\partial}{\partial t}|v|^2 + \nu_{ei}\int \mathcal{N}|v|^2\right) + \frac{3}{2}\lambda_D^2 \int [i\mathcal{N}\overline{v}\Delta(\mathcal{N}v) + \text{c.c.}] = 0$$

so relation (4.30) follows. Moreover, according to (4.28), integrating by parts the integral $\int \mathcal{N}\mathbf{U}.\nabla|v|^2$, it is classical to get

$$\frac{\partial}{\partial t}\int \left[\frac{1}{2}\mathcal{N}^2 + \frac{1}{2}|\mathcal{N}\mathbf{U}|^2\right] + 2Z\frac{m_e}{m_0}\int |v|^2\frac{\partial}{\partial t}\mathcal{N}$$
$$= -\int \left[\nabla.(\mathcal{N}^2\mathbf{U}) + 2Z\frac{m_e}{m_0}\mathcal{N}\mathbf{U}.\nabla|v|^2 + 2Z\frac{m_e}{m_0}|v|^2\nabla.(\mathcal{N}\mathbf{U})\right]$$

and the right-hand side is zero since there is no boundary. □]

Relation (4.31) shows clearly that $2Zm_e\mathcal{N}|v|^2$ is the energy density of the Langmuir wave.

4.3 The Zakharov Equations and Their Properties

We now make a change of notations in (4.27) and (4.28) by denoting $\mathbf{U}$ instead of $\mathcal{N}\mathbf{U}$. Moreover, since $\mathcal{N}$ is close to 1, one usually replaces $\Delta(\mathcal{N}v)$ by Δv. Then, we get the following system satisfied by the nondimensional density fluctuation

$$m = \mathcal{N} - 1$$

the vector $\mathbf{U}$ and the complex function v

$$(\mathcal{Z}a\mathcal{E})\quad \begin{cases} \text{(i)} & \dfrac{\partial}{\partial t}m + \nabla.\mathbf{U} = 0, \\[2ex] \text{(ii)} & \dfrac{\partial}{\partial t}\mathbf{U} + c_s^2\nabla m = -2Z\frac{m_e}{m_0}\nabla|v|^2 \\[2ex] \text{(iii)} & \dfrac{1}{\omega_p}\dfrac{\partial}{\partial t}v - i\dfrac{3}{2}\lambda_D^2\Delta v + \dfrac{\nu_{ei}}{2\omega_p}v = -i\dfrac{1}{2}mv. \end{cases}$$

This is the Zakharov system. It was introduced in 1972 [116] and is intensively used for the modelling of the plasma wave collapse and plasma turbulence. See, e.g., [90] to have a physical point of view of the system's derivation.

Of course, the two first equations of system ($\mathcal{Z}a\mathcal{E}$) may be recast with a single wave equation; i.e., one needs to address a system with two unknowns (m, v) that satisfies equation (iii) above and the following

$$\frac{\partial^2}{\partial t^2} m - c_s^2 \Delta m = 2Z \frac{m_e}{m_0} \Delta |v|^2$$

Remark 29 (On the time envelope). According to (4.21) and (4.26), with the same quasi-neutral approximation as the one above, we see that the slowly oscillating quantity $\tilde{m} = \mathcal{N} - 1$ and highly oscillating quantity U_e^h satisfy the following system

$$\frac{1}{\omega_p^2} \frac{\partial^2}{\partial t^2} U_e^h - 3\lambda_D^2 \Delta U_e^h + U_e^h(\tilde{m} - 1) + \frac{\nu_{ei}}{\omega_p^2} \frac{\partial}{\partial t} U_e^h = 0$$

$$\frac{\partial^2}{\partial t^2} \tilde{m} - c_s^2 \Delta \tilde{m} = \frac{Zm_e}{2m_0} \Delta |U_e^h|^2$$

Then it is possible to show that in a certain way the solution $(\tilde{m}, U_e^h)$ may be compared to the solution (m, v) of ($\mathcal{Z}a\mathcal{E}$) and $\tilde{m}$ is close to m and the time envelope of U_e^h is close to v (i.e., (4.24) is justified), cf. [12]. Moreover, in dimension 2 or 3, if ν_{ei} is set to 0 there is generally a blow-up of the solution and the blow-up time of the system related to (m, U_e^h) is close to the blow-up time of the Zakharov system for (m, v). □

Remark 30 (The multidimensional aspect). Due to the remark above, the multidimensional aspect may be accounted for in the following way; we need to replace v with $-\nabla((-\Delta)^{-1}\chi)$ and equation [$\mathcal{Z}\mathcal{E}$(iii)] with the equation

$$\frac{1}{\omega_p} \frac{\partial}{\partial t} \chi - i \frac{3}{2} \lambda_D^2 \Delta \chi + i \frac{\nu_{ei}}{2\omega_p} \chi = -\frac{i}{2} \nabla . \left[m \nabla((-\Delta)^{-1}\chi) \right].$$

(see, e.g., [18] for a mathematical study of this vectorial form). □

We can normalize the time variable and the space variable such that the sound speed c_s is equal to 1and that the length $\sqrt{3}\lambda_D$ is equal to 1 also.

Moreover, the amplitude of the Langmuir wave may be normalized in such a way that $2\frac{Zm_e}{m_0}$ may be replaced by 1, then the previous system reads as

$$\text{(i)}\ \frac{\partial}{\partial t} m + \nabla.\mathbf{U} = 0, \qquad \text{(ii)}\ \frac{\partial}{\partial t} \mathbf{U} + \nabla m = -\nabla |v|^2, \tag{4.32}$$

$$\frac{2i}{\omega_p} \frac{\partial}{\partial t} v + \Delta v + i\eta v = mv. \tag{4.33}$$

Of course, it must be supplemented with the initial data $m_{\text{ini}}, \mathbf{U}_{\text{ini}}, v_{\text{ini}}$.

Notice that the damping coefficient $\eta = \dfrac{\nu_{ei}}{\omega_p}$ is always small.

Let us disregard the boundary conditions and address the problem in the whole space $\mathbf{R}^d$ (with $d = 1, 2, or\ 3$). We have first two balance relations which are trivial

$$\frac{1}{\omega_p}\frac{\partial}{\partial t}\int |v(\mathbf{x})|^2 d\mathbf{x} + \eta \int |v(\mathbf{x})|^2 d\mathbf{x} = 0,$$

$$\frac{\partial}{\partial t}\int (m\mathbf{U})(\mathbf{x}) d\mathbf{x} = 0$$

In the sequel we assume that the initial conditions are such that

$$m_{\text{ini}} \in L^2(\mathbf{R}^d), \qquad \mathbf{U}_{\text{ini}} \in (L^2(\mathbf{R}^d))^d, \qquad v_{\text{ini}} \in H^1(\mathbf{R}^d) \cap L^4(\mathbf{R}^d). \tag{4.34}$$

From a mathematical point of view, the system (4.32), (4.33), may be addressed in the case where $\eta = 0$ (i.e., without a damping term in the Langmuir wave equation); indeed it suffices to set $v = \tilde{v}e^{-\eta\omega_p t/2}$ then (4.33) becomes $(\frac{2i}{\omega_p}\frac{\partial}{\partial t} + \Delta)\tilde{v} = m\tilde{v}$ and in the right-hand side of [4.32(ii)] there occurs a damping term $e^{-\eta\omega_p t}$ which does not introduce any complication in the analysis.

We can now assume $\eta = 0$. So the above relation claims that there exists a constant β such that for all t,

$$\int |v(\mathbf{x})|^2 d\mathbf{x} = \|v\|^2_{L^2} = \beta.$$

We now state an energy balance relation that makes clear the interest of space $L^4(\mathbf{R}^d)$. Indeed, if v belongs to L^4, then $|v|^2$ is in L^2; if m belongs to L^2, then also the integral $\int m|v|^2 d\mathbf{x}$ is well defined.

Proposition 16. *Assume that (4.34) holds, that (m, U, v) satisfies (4.32), (4.33), and that they are integrable (with respect to the time) in the spaces $L^2(\mathbf{R}^d), (L^2(\mathbf{R}^d))^d$, and $H^1(\mathbf{R}^d) \cap L^4(\mathbf{R}^d)$; then it exists that C is independent from time,*

$$\int \left[\frac{1}{2}m^2 + \frac{1}{2}|\mathbf{U}|^2 + m|v|^2 + |\nabla v|^2\right] d\mathbf{x} = C.$$

[**Indeed**, we may check that

$$\frac{1}{2}\frac{\partial}{\partial t}\int \left(m^2 + |\mathbf{U}|^2\right) = \int \left[|v|^2 \nabla.\mathbf{U}\right] = -\int \left[|v|^2 \frac{\partial m}{\partial t}\right]$$
$$= -\frac{\partial}{\partial t}\int \left[m|v|^2\right] + \int \left[mv\frac{\partial \overline{v}}{\partial t} + c.c.\right].$$

But we now see that

$$\int [mv\partial_t\overline{v} + c.c.] = \int [\frac{2i}{\omega_p}\partial_t v\partial_t\overline{v} + \Delta v\partial_t\overline{v} + c.c.] = -\int [\nabla v.\partial_t\nabla\overline{v} + c.c.]$$

So the result follows. □]

Numerous mathematical works has been performed on this system for thirty years, the most important are [92, 93, 103]. We give here some key simple results.

According to the previous proposition, assuming (4.34), in addition to β we can see that the following quantity is a constant (i.e., independent from time)

$$\mu = \frac{1}{2}\|m\|_{L^2}^2 + \frac{1}{2}\|\mathbf{U}\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \int \left[m|v|^2\right] d\mathbf{x}.$$

One difficulty comes from the fact that $\int \left[m|v|^2\right] d\mathbf{x}$ has no sign. We have the following result (in 1D, $\mathbf{U}$ reads as U).

Proposition 17.

(i) For a solution $(m, \mathbf{U}, v)$ of system (4.32), (4.33), in the space R^d, such that the initial data satisfy (4.34), we get (with $C_1 = 2, C_2 = 4, C_3 = 8$)

$$\frac{1}{4}\|m\|_{L^2}^2 + \frac{1}{2}\|\mathbf{U}\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \leq |\mu| + C_d\beta^{2-d/2}\|\nabla v\|_{L^2}^d. \tag{4.35}$$

(ii) Moreover, in the one-dimensional case, there exists C depending only on β and μ such that for all t, we have

$$\|m(t)\|_{L^2} \leq C, \qquad \|U(t)\|_{L^2} \leq C, \qquad \|v(t)\|_{H^1} \leq C, \qquad \|v(t)\|_{L^4} \leq C. \tag{4.36}$$

This shows clearly that the behavior of the solution of Zakharov equations (4.32) and (4.33) depends on the dimension d of the space variable: in the case $d = 1$, bound (4.35) leads to simple a priori estimates as shown below, but this is not the case for dimension $d = 2, 3$.

We now focus on the one-dimensional framework. Then, we can state the theorem first stated in [93] (the proof of the previous proposition and this theorem are given below).

Theorem 3.

(i) Assume that (4.34) holds; then there exists a weak solution (m, U, v) of system (4.32) and (4.33).

(ii) If the initial data $m_{ini}, U_{ini}, v_{ini}$ are in $H^2(\mathbf{R}) \times H^2(\mathbf{R}) \times H^3(\mathbf{R})$, there exists a unique solution (m, U, v) of system (4.32), (4.33), such that m belongs to $L^\infty(0, +\infty; H^2(\mathbf{R}))$ and v belongs to $L^\infty(0, +\infty; H^3(\mathbf{R}))$. Moreover, v is bounded in $L^\infty(0, T; L^\infty(\mathbf{R}))$ for all final times T.

For the sake of completeness, let us mention some results in the two- or three-dimensional frameworks. For initial data $m_{\text{ini}}, \mathbf{U}_{\text{ini}}, v_{\text{ini}}$, which are not zero and which belongs to $H^1 \times L^2 \times H^2$, it has been proved (see [92]) that there is a unique strong solution that is local in time and that may blow up at a finite time. It has been also proved that there exists a weak solution on $[0, T]$ for all T in the following cases:

- In 2-D, if β is small enough.
- In 3-D, if β, $\|\nabla v_{\text{ini}}\|$ and $|\mu|$ are small enough.

A very interesting open problem is to determine when one can claim that the blow-up does not occur. What are sufficient conditions on the damping coefficient η, on the initial data, and on the geometry (of course, the boundary conditions must be emphasized)?

Instabilities

One sees that if the initial value of v is zero, the solution of (4.32), (4.33), is the trivial one, i.e., the solution of $\dfrac{\partial^2}{\partial t^2}m - \Delta m = 0$. One may check that the Zakharov system is instable. More precisely if one linearizes the system in one dimension near a value $(m, U, v) = (0, 0, v_0)$, then one can prove that the solution has exponential growth.

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Proofs of Section 4.3

Proof of Proposition 17. (i) Since $\int |m|v|^2| d\mathbf{x} \leq \frac{1}{4} \|m\|^2_{L^2} + \int |v|^4 d\mathbf{x}$, it suffices to get a bound on this last term. So we check that the following classical inequality holds,

$$\int |v|^4 d\mathbf{x} \leq C_d \left(\int |v|^2 d\mathbf{x} \right)^{2-d/2} \left(\int |\nabla v|^2 d\mathbf{x} \right)^{d/2}. \tag{4.37}$$

First, in one dimension, we see that for all x we have $|v(x)|^2 \leq 2 \int |v(x')||\partial_x v(x')| dx'$, so

$$|v|^2_\infty \leq 2(\int |v(x')|^2 dx')^{1/2} (\int |\partial_x v(x')|^2 dx')^{1/2} = 2 \|v\|_{L^2} \|\partial_x v\|_{L^2} \tag{4.38}$$

then relation $\int |v|^4 dx \leq |v|^2_\infty \int |v|^2 dx$ implies (4.37).

Secondly, in two dimsionsions ($d = 2$), we have in the same way

$$|v(x_1,x_2)|^2 \le 2\int |v(x_1',x_2)||\partial_1 v(x_1',x_2)|dx_1'$$
$$\le \lambda(x_2) = 2\left(\int |v(x_1',x_2)|^2 dx_1'\right)^{1/2}\left(\int |\partial_1 v(x_1',x_2)|^2 dx_1'\right)^{1/2}$$

and an analogous bound with $\lambda^0(x_1)=2\left(\int |v(x_1,x_2')|^2 dx_2'\right)^{1/2}\left(\int |\partial_2 v(x_1,x_2')|^2 dx_2'\right)^{1/2}$. By gathering the inequalities obtained with $\int \lambda(x_2)dx_2$ and $\int \lambda^0(x_1)dx_1$, result (4.37) follows. In the three-dimensional case, the proof is analogous.

(ii) Here $d = 1$. According to the previous point, we see that

$$\frac{1}{4}\|m\|_{L^2}^2 + \frac{1}{2}\|\mathbf{U}\|_{L^2}^2 + \frac{1}{2}\|\nabla v\|_{L^2}^2 \le |\mu| + 2\beta^3$$

The bound on $\|v\|_{L^4}$ follows from (4.37).

□

Proof of Theorem 3. Here C denotes some different constants that depend only on the initial data.

(i) Let us now show first that there exists a weak solution (m, U, v) by using a Galerkin method.[3] So, we build a sequence of functions (m_q, U_q, v_q) belonging to the spaces $L^2(0,t,L_x^2) \times L^2(0,t,L_x^2) \times L^2(0,t,H_x^1)$ that solve some approximated systems. They satisfy the same bounds (4.36) as the ones stated above. So, for all time t, we have [using the bound (4.38)]

$$\left\|m_q\right\|_{L^2} \le C, \quad \left\|v_q\right\|_{L^2} \le C, \quad \left\|\partial_x v_q\right\|_{L^2} \le C, \quad \left\|v_q\right\|_{L^4} \le C,$$
$$\left\|v_q\right\|_{L^\infty} \le \sqrt{2}\left\|v_q\right\|_{L^2}^{1/2}\left\|\partial_x v_q\right\|_{L^2}^{1/2} \le C. \quad (4.39)$$

According to these bounds, we have

$$\int |m_q v_q|^2 dx \le \left\|m_q\right\|_{L^2}^2 \left\|v_q\right\|_{L^\infty}^2 \le C \quad (4.40)$$

Then, for all test functions ϕ, ψ and χ we get

$$\frac{\partial}{\partial t}\int m_q \phi\, dx = \int U_q \partial_x \phi\, dx,$$
$$\frac{\partial}{\partial t}\int U_q \psi\, dx = \int (m_q + |v_q|^2)\partial_x \psi\, dx$$
$$i\frac{\partial}{\partial t}\int v_q \chi\, dx = \int \left(\partial_x v_q \partial_x \chi - m_q v_q \partial_x \chi\right) dx$$

[3] For an introduction to Galerkin methods, see the proof of Proposition 7 above.

Now, according to bounds (4.39), (4.40), and up to the extraction of a subsequence, there exists a weak limit in L^2 of the right-hand-side terms. Since there exists also is a weak limit (m, U, v) in L^2 of a subsequence of the sequence (m_q, U_q, v_q), then up to the extraction of a subsequence we see that

$$\frac{\partial}{\partial t}\int m\phi\, dx = \int U\partial_x\phi\, dx$$
$$\frac{\partial}{\partial t}\int U\psi\, dx = \int (m+|v|^2)\partial_x\psi\, dx$$
$$i\frac{\partial}{\partial t}\int v\chi\, dx = \int (\partial_x v\partial_x\chi - mv\partial_x\chi)\, dx$$

which is the definition of a weak solution.

(ii) We now make stronger assumptions on the initial data. Here K_p denote different constants depending only on initial data and which are zero if these data are zero; moreover, C denote different constants. According to the lemma below, we check that if (m, U, v) are strong solutions on a time interval, then we have

$$\frac{\partial}{\partial t}\left[\|\partial_t m\|_{L^2}^2 + \|\partial_t U\|_{L^2}^2 + 2\|\partial_t v\|_{H^1}^2 + 2\int\left[\partial_t m\partial_t|v|^2 + m|\partial_t v|^2\right]dx\right]$$
$$= \int\left[6\partial_t m|\partial_t v|^2 + 4\mathrm{Im}(v\partial_t\overline{v})\partial_t m\right]dx$$

Thus, we get

$$\|\partial_t m\|^2 + \|\partial_t U\|^2 + 2\|\partial_t v\|_{H^1}^2 + 2\int\left[\partial_t m\partial_t|v|^2 + m|\partial_t v|^2\right]dx$$
$$= K_0 + 6\int_0^t\int\left(\partial_t m|\partial_t v|^2\right)dxdt' + 4\int_0^t\int(\mathrm{Im}(v\partial_t\overline{v})\partial_t m)\,dxdt' \quad (4.41)$$

Now, we know that $W = \partial_t v$ satisfies the following equation

$$\frac{\partial}{\partial t}W + (imW - i\Delta W)\frac{\omega_p}{2} = v(\partial_t m).$$

Since v is bounded in L^∞, we get $\frac{\partial}{\partial t}\|W\|_{L^2}^2 = \int v\overline{W}(\partial_t m)dx + c.c. \leq C\|W\|_{L^2}\|\partial_t m\|_{L^2}$, i.e., $\frac{\partial}{\partial t}\|W\|_{L^2} \leq \|\partial_t m\|_{L^2}$ and denoting $K_1 = \|\Delta v_{\text{ini}}\|_{L^2} + C\|m_{\text{ini}}\|_{L^2}$ we have the bound

$$\|W(t)\|_{L^2} \leq K_1 + C\int_0^t\|\partial_t m(s)\|_{L^2}\,ds \quad (4.42)$$

Since v is bounded, this implies for all t

$$\int_0^t \int |\overline{v}\partial_t v \partial_t m| dx dt \le K_2 \|\partial_t m\|_{L^2} + C \left[\int_0^t \|\partial_t m(s)\| \, ds \right]^2$$
$$\le K_2 \|\partial_t m\|_{L^2} + Ct \int_0^t \|\partial_t m(s)\|^2 \, ds$$

Let us focus now on the terms $\int m|\partial_t v|^2 dx$ and $\int \partial_t m |\partial_t v|^2 dx$. Due to the fact that $\|W\| . \|\partial_x W\| \le \frac{1}{2} \|W\|_{H^1}^2$ and (4.37), we get

$$2 \int m|\partial_t v|^2 dx = \|m\|_{L^2}^2 + \int |W|^4 dx \le C + 2\|W\|_{L^2}^3 \|\partial_x W\|_{L^2} \le C + \|W\|_{L^2}^2 \|W\|_{H^1}^2$$

$$2 \int \partial_t m |\partial_t v|^2 dx = \|\partial_t m\|_{L^2}^2 + \int |W|^4 dx \le \|\partial_t m\|_{L^2}^2 + \|W\|_{L^2}^2 \|W\|_{H^1}^2$$

Now, using the bound (4.42), we have

$$\int_0^t \int \partial_t m |\partial_t v|^2 dx dt \le C \int_0^t \|\partial_t m\|^2 + C \int_0^t \left(\int_0^{t'} \|\partial_t m(s)\|^2 \, ds \right) \left\|\partial_t v(t')\right\|_{H^1}^2 dt'$$

Finally, denoting

$$\zeta(t) = \|\partial_t m(t)\|_{L^2}^2 + \|\partial_t U(t)\|_{L^2}^2 + 2\|\partial_t v(t)\|_{H^1}^2$$

we have

$$\zeta(t) \le K_3 + C \int_0^t [\left\|\partial_t m(t')\right\|_{L^2}^2 + 2\left\|\partial_t v(t')\right\|_{H^1}^2] dt' + C \int_0^t \left(\int_0^{t'} \|\partial_t m(s)\|_{L^2}^2 \, ds \right) \left\|\partial_t v(t')\right\|_{H^1}^2 dt' \tag{4.43}$$

And according to Gronwall's lemma (see result 7 in the Appendix) and an iterative technique, we may see, using the assumptions on the initial data, that $\|\partial_t m(t)\|_{L^2}^2$, $\|\partial_t U(t)\|_{L^2}^2$ and $\|\partial_t v(t)\|_{H^1}^2$ are bounded.

To obtain the regularity, one has to perform a space derivative of the main equation and apply the same kind of technique as above.

To prove the uniqueness of the solutions, address two solution (m, U, v) and $(\tilde{m}, \tilde{U}, \tilde{v})$; then make the difference $\hat{m} = m - \tilde{m}, \hat{U} = U - \tilde{U}, \hat{v} = v - \tilde{v}$. Using the same kind of techniques as above, we may check for $(\hat{m}, \hat{U}, \hat{v})$ an inequality of the type (4.43) but with K_3 set to zero; therefore we get the uniqueness. □

Lemma 8. *If* (m, U, v) *are strong solutions, they satisfy the following a priori estimates*

$$2\int \left[\partial_t |v|^2 \partial_{tt} m\right] + \partial_t(\|\partial_t m\|^2 + \|\partial_t U\|^2) = 0 \quad (4.44)$$

$$\partial_t(\|\partial_t v\|^2) = \quad 2\int \mathrm{Im}\,(v\partial_t \overline{v})\,\partial_t m \quad (4.45)$$

$$-\partial_t(\|\partial_x\partial_t v\|^2) = \int m\partial_t|\partial_t v|^2 + \partial_t m\left[\partial_{tt}|v|^2 - 2|\partial_t v|^2\right] \quad (4.46)$$

Proof of the Lemma. Recall that $\int \partial_t|v|^2\partial_{tt}m = -\int(\partial_t\partial_x|v|^2)(\partial_x m + \partial_x|v|^2)$. Thus we see that

$$2\int \left[\partial_t|v|^2\partial_{tt}m\right] + \partial_t(\|\partial_t m\|^2 + \|\partial_t U\|^2)$$
$$= \int \left[-2\partial_t(\partial_x|v|^2)\partial_x m - \partial_t(\partial_x|v|^2)^2 + \partial_t(|\partial_x U|^2) + \partial_t(\partial_x m + \partial_x|v|^2)^2\right]$$
$$= \int -2(\partial_t\partial_x|v|^2)\partial_x m + 2(\partial_x U)(\partial_x\partial_t U) + \partial_t|\partial_x m|^2 + 2(\partial_t\partial_x|v|^2)\partial_x m + 2\partial_x|v|^2(\partial_t\partial_x m)$$

Since $\int \partial_x U(\partial_x\partial_t U) = \int \partial_t m\partial_x(\partial_x m + \partial_x|v|^2) = -\int(\partial_t\partial_x m)(\partial_x m + \partial_x|v|^2)$, we see that (4.44) follows.

Now recall that $2\mathrm{Im} v\partial_t\overline{v} = i\overline{v}\partial_t v - iv\partial_t\overline{v}$. Thus, we have

$$\partial_t(\|\partial_t v\|^2) - 2\int \mathrm{Im}\,(v\partial_t\overline{v})\,\partial_t m$$
$$= \int [(\partial_t\overline{v})\partial_t(i\Delta v - imv) + c.c.] - [iv\partial_t\overline{v}\partial_t m + c.c.]$$
$$= \int [(\partial_t\overline{v})\,(i\Delta\partial_t v - i\partial_t(mv) + iv\partial_t m) + c.c.] = -\int [i(\partial_t\overline{v})m\partial_t(v) + c.c.] = 0.$$

thus (4.45) follows.

Lastly, (4.46) comes from the fact that $\partial_{tt}|v|^2 - 2|\partial_t v|^2 = i\overline{v}\partial_t(\Delta v) + c.c.$ and standard calculus; see [113] for the details. □

Chapter 5
Coupling Electron Waves and Laser Waves

Abstract In this chapter, we go back to laser–plasma interaction by addressing the coupling of the laser waves with the electron plasma waves. So we derive the so-called Raman instability model. In the case of the fixed-ion assumption, it leads to a three-wave coupling system that shows the same structure as the system of Brillouin instability. In the second part of this chapter, we deal with the modelling of the interaction of an ultra-intense laser pulse and a plasma. This leads to the so-called Euler–Maxwell system. We give some mathematical properties of this system and we show how an envelope description may be useful in some cases.

Keywords Raman instability • Three-wave coupling system • Electron plasma waves • Ultra-intense leaser pulse • Euler–Maxwell system

We address in this chapter different physical phenomena where a coupling occurs between electron waves and laser electromagnetic waves. In the first section, in the framework of the laser-plasma interaction, we consider the Raman instability. There is some analogy with the Brillouin instability, although the wave velocities are not the same. Indeed, for this modelling, we made a paraxial approximation of the main laser wave and the backscattered laser wave; moreover, we addressed a time envelope of the electron Langmuir wave as the one described in the previous chapter. We must take into account three characteristic lengths: (1) the Debye length and (2) the laser wavelength—which are of the same order of magnitude—and (3) the typical value of the plasma thickness. The characteristic time of growth of the instability is much larger than the laser period.

In the two other sections, we deal with very short laser pulses of ultra-high intensity. This corresponds to a very different physics since the characteristic duration of the laser pulse is only some tens or hundreds of the laser periods. So, for the modeling, one uses Maxwell equations coupled with Euler equations for electrons—in the second section—or a closure formula—in the third section.

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5.1 Raman Instability

We assume here that the mean value of the electron density is, of course, smaller than critical density N_c (to enable the propagation of the laser wave), but it satisfies $N_{\mathrm{ref}} < N_c/4$ (for a reason that appears in the sequel). Recall that as usual

$$\omega_p = \omega_0 (N_{\mathrm{ref}}/N_c)^{1/2} \qquad \text{and} \qquad N_c = \frac{\varepsilon^0 m_e}{q_e^2}\omega_0^2.$$

The electron temperature is such that the electron thermal velocity $v_{\mathrm{th,e}}$ is smaller than the speed of light (typically 0.05 times c). For this phenomenon, the time characteristic observation time T_{obs} and the observation length L_{obs} are such that

$$\frac{\lambda_D}{v_{\mathrm{th,e}}} = \omega_p^{-1} \ll \frac{\lambda_D}{\sqrt{v_{\mathrm{th,e}} c}} \ll T_{\mathrm{obs}} \leq \frac{L_{\mathrm{obs}}}{\sqrt{v_{\mathrm{th,e}} c}}$$

$$\lambda_D \ll L_{\mathrm{obs}} \leq L_{\mathrm{plas}}$$

Notice that the typical value of T_{obs} may be approximately one picosecond, but the laser pulse may be longer than tens or hundreds of picoseconds.

We assume that the electric field is decomposed into an electrostatic field and a transverse rapidly oscillating electromagnetic field $\mathbf{E}^r$, and this last one is decomposed further into a main laser wave travelling in the direction of the unit vector $\mathbf{e}_{tr}$ and another laser called the *Raman backscattered wave* travelling in the direction $-\mathbf{e}_{tr}$. Moreover, there is a coupling between these two waves and a third wave — the electron Langmuir wave.

The electromagnetic fields are decomposed in the following way

$$\mathbf{E} = \mathbf{E}^r + \mathbf{E}^e, \qquad \mathbf{B} = \mathbf{B}^r$$

where the electrostatic component $\mathbf{E}^e(t, \mathbf{x})$ is called longitudinal and the fields $\mathbf{E}^r(t, \mathbf{x})$ and $\mathbf{B}^r(t, \mathbf{x})$ which are orthogonal to $\mathbf{e}_{tr}$, are called transverse. As in Chap. 3, the field $\mathbf{E}^r$ is the solution of the classical Maxwell equation

$$\frac{\partial^2}{\partial t^2}\mathbf{E}^r - c^2 \Delta \mathbf{E}^r + \omega_p^2 (\mathcal{N} + n) N_{\mathrm{ref}} \mathbf{E}^r = 0 \tag{5.1}$$

Since the incoming boundary condition is oscillating at frequency ω_0 so $\mathbf{E}^r$ and $\mathbf{B}^r$ are also highly oscillating at frequency close to ω_0. We assume as in Chap. 3, that the polarization of the electromagnetic wave is linear, i.e., that $\mathbf{E}^r$ may reduce to a scalar function.

Recall that the group velocity and the laser wave number are given by

$$c_g = c\sqrt{1 - N_{\mathrm{ref}}/N_{\mathrm{c}}} \qquad k_p^2 c^2 = \omega_0^2 - \omega_p^2.$$

thus $k_p = k_0 \sqrt{1 - N_{\mathrm{ref}}/N_{\mathrm{c}}}$. We set $z = \mathbf{e}_{tr}.\mathbf{x}$.

Moreover, it is assumed that the electron density is decomposed in a way analogous to the one in the previous chapter:

$$N_e(t,\mathbf{x}) = \mathcal{N}(t,\mathbf{x})N_{\text{ref}} + n(t,\mathbf{x})N_{\text{ref}}, \qquad |n| \ll 1,$$

where the mean density $\mathcal{N}$ is a slowly varying time function assumed to be close to 1, and n is a highly oscillating component (with the plasma frequency ω_p).

To account for the backscattered wave, we decompose the scalar function $\mathbf{E}^r$ into a forward field $Ee^{ik_pz-i\omega_0t}$ and the backscattered one $\Phi e^{-ik_Rz-i\omega_Rt}$, where functions E and Φ are slowly varying with respect to the space and time variables

$$\mathbf{E}^r(t,\mathbf{x}) \simeq \left[E(t,\mathbf{x})e^{ik_pz-i\omega_0t} + c.c\right] + \left[\Phi(t,\mathbf{x})e^{-ik_Rz-i\omega_Rt} + c.c.\right]. \tag{5.2}$$

As above, the two waves satisfy the paraxial propagation equations where $\beta_0 = \omega_p^2/(2\omega_0)$

$$\frac{\partial}{\partial t}E + c_g\partial_z E + \nu_a E - \frac{ic}{2k_0}\Delta_\perp E = -i\beta_0 n e^{-ik_pz+i\omega_0t-ik_Rz-i\omega_Rt}\Phi + i\beta_0(1-\mathcal{N})E \tag{5.3}$$

$$\frac{\partial}{\partial t}\Phi - c_g\partial_z\Phi + \nu_a\Phi - \frac{ic}{2k_0}\Delta_\perp\Phi = -i\beta_0 n e^{ik_pz-i\omega_0t+ik_Rz+i\omega_Rt}E + i\beta_0(1-\mathcal{N})\Phi \tag{5.4}$$

It is worth noting that the frequency ω_R and the wave number k_R of the Raman scattering are to be determined and are not equal to ω_0 and k_p (as for the Brillouin model). Note that for the propagation of the Φ-wave it is necessary to have the basic relation

$$k_R^2c^2 = \omega_R^2 - \omega_p^2. \tag{5.5}$$

Remark 31. It is also possible to account for a Raman electromagnetic wave that does not travel backwards but according to a vector $\mathbf{k}_R$. In that case, one must set

$$\mathbf{E}^r(t,\mathbf{x}) \simeq \left[E(t,\mathbf{x})e^{ik_pz-i\omega_0t} + c.c\right] + \left[\Phi(t,\mathbf{x})e^{i\mathbf{k}_R\cdot\mathbf{x}-i\omega_Rt} + c.c.\right]$$

and the previous relation reads as $|\mathbf{k}_R|^2c^2 = \omega_R^2 - \omega_p^2$. But the derivation of the model is that case is more delicate and we do not go in this direction. □

In actual physical situations, the Debye length is very small, so we are in the framework of the quasi-neutrality approximation, i.e., $\mathcal{N}N_{\text{ref}} \simeq N_0$. For the sake of explanation, we address in the following subsection an ideal case where the ion population and the mean value of the electron population are at rest, i.e., $\mathcal{N} = 1$. Afterwards, we will introduce afresh the evolution of the mean density $\mathcal{N}$ in order to get a more realistic model.

5.1.1 Model with Fixed Ions

For the electron density fluctuation, we may use the following system (coming from the general electrodynamics model (2.2) and (2.5)) where as above we denote $c_e^2 = 3\mathrm{v}_{\mathrm{th,e}} = 3T_{\mathrm{ref}}/m_e$.

$$\frac{\partial}{\partial t} n + \nabla.\mathbf{U}_e = 0$$

$$\frac{\partial}{\partial t}\mathbf{U}_e + \nabla.(\mathbf{U}_e\mathbf{U}_e) + c_e^2\nabla n + \nu_{ei}\mathbf{U}_e = -\frac{q_e}{m_e}(\mathbf{E}^e + \mathbf{E}^r + \mathbf{U}_e\times\mathbf{B}^r).$$

In the above velocity equation, note the damping term $\nu_{ei}\mathbf{U}_e$ which accounts in some sense for the well-known Landau effect (which corresponds to the fact that the distribution of electrons is not a Maxwell distribution); the coefficient ν_{ei} is called the *Landau damping coefficient*.

Since the main force is due to the electric field $\mathbf{E}^r$, the velocity $\mathbf{U}_e$ must be decomposed into a transverse component $\mathbf{U}_e^r$ that oscillates at a frequency close to ω_0 and a longitudinal component $\mathbf{U}_e^e$ that oscillates at a slower frequency. We perform an average on a laser period $2\pi/\omega_0$ using the same trick as is Chap. 3, i.e., the right-hand side becomes

$$\left\langle \frac{q_e}{m_e\omega_0}\mathbf{U}_e^r\times\mathbf{B}^r + \nabla.(\mathbf{U}_e^r\mathbf{U}_e^r)\right\rangle_{\omega_0} \simeq \left\langle \frac{1}{2}\left(\frac{q_e}{m_e\omega_0}\right)^2 \nabla|\mathbf{E}^r|^2\right\rangle_{\omega_0} = \frac{q_e^2}{4m_e^2\omega_0^2}\nabla|\mathbf{E}^r|^2 = \gamma_p\nabla|\mathbf{E}^r|^2$$

where the $\langle . \rangle_{\omega_0}$ corresponds to an averaged value over a laser period: this corresponds to the ponderomotive force.

We now address the averaged value over a laser period of the density fluctuation n, i.e.,

$$n^h = \langle n\rangle_{\omega_0} \quad \mathbf{U}_e^h = \langle \mathbf{U}_e^e\rangle_{\omega_0}$$

We will see that it is a Langmuir wave oscillating at plasma frequency ω_p. We decompose also $\mathbf{E}^e$ into a component $\mathbf{E}^h$ oscillating at frequency ω_p and a slowly time-dependent part (i.e., $\mathbf{E}^e = \mathbf{E}^h + \mathbf{E}^s$). For the sake of simplicity, we disregard the multidimensional aspects of the Langmuir wave (as in the previous chapter) and we set U_e^h instead of $\mathbf{U}_e^h$. The Langmuir waves satisfy the classical wave system supplemented by the ponderomotive force

$$\frac{\partial n^h}{\partial t} + \frac{\partial}{\partial z}U_e^h = 0,$$

$$\frac{\partial}{\partial t}U_e^h + c_e^2\frac{\partial}{\partial z}n^h + \nu_{ei}U_e^h = -\frac{q_e}{m_e}\mathbf{E}^h - \gamma_p\frac{\partial}{\partial z}|\mathbf{E}^r|^2$$

The Poisson equation reads $\varepsilon^0 \frac{\partial}{\partial z}\mathbf{E}^h = -q_e n^h$; then after time derivation we have

$$\frac{q_e}{m_e}\frac{\partial}{\partial z}\frac{\partial}{\partial t}\mathbf{E}^h = \omega_p^2 \frac{\partial}{\partial z} U_e^h$$

so without accounting for the multidimensional aspect, we get

$$\frac{q_e}{m_e}\frac{\partial}{\partial t}\mathbf{E}^h = \omega_p^2 U_e^h.$$

Consider now the ponderomotive force $\frac{\partial}{\partial z}|\mathbf{E}^r|^2$, by using the decomposition (5.2). We get

$$\frac{\partial}{\partial z}|\mathbf{E}^r|^2 = \frac{\partial}{\partial z}\left[E\overline{\Phi}e^{i(k_p+k_R)z+i(\omega_R-\omega_0)t} + c.c. + |E|^2 + |\Phi|^2\right] \simeq e^{i(\omega_R-\omega_0)t}\frac{\partial}{\partial z}(E\overline{\Phi}e^{i(k_p+k_R)z}) + c.c.,$$

indeed, we know that $|E|$ and $|\Phi|$ are slowly varying with respect to the space variable and $\frac{\partial}{\partial z}(|E|^2 + |\Phi|^2)$ are negligible with respect to $\frac{\partial}{\partial z}(E\overline{\Phi}e^{i(k_p+k_R)z})$. Using this approximation, the system for the Langmuir wave reads as

$$\frac{\partial}{\partial t}n^h + \frac{\partial}{\partial z}U_e^h = 0, \tag{5.6}$$

$$\frac{\partial}{\partial t}U_e^h + c_e^2\frac{\partial}{\partial z}n^h + \nu_{ei}U_e^h + \frac{q_e}{m_e}\mathbf{E}^h = -\gamma_p e^{i(\omega_R-\omega_0)t}\frac{\partial}{\partial z}(E\overline{\Phi}e^{i(k_p+k_R)z}) + c.c. \tag{5.7}$$

Since $\frac{\partial}{\partial z}E$, $\frac{\partial}{\partial z}\Phi$ are negligible with respect to $k_p E$ or $k_R\Phi$, we have

$$\frac{\partial}{\partial z}(E\overline{\Phi}e^{i(k_p+k_R)z}) \simeq i(k_p + k_R)e^{i(k_p+k_R)z}E\overline{\Phi};$$

moreover, $\frac{\partial}{\partial t}(E\overline{\Phi})$ is negligible with respect to $i(\omega_R - \omega_0)E\overline{\Phi}$. Thus, gathering (5.6) and (5.7) with the relation for $\mathbf{E}^h$, we get simply

$$\frac{\partial^2 U_e^h}{\partial t^2} - c_e^2\frac{\partial^2}{\partial z^2}U_e^h + \omega_p^2 U_e^h + \nu_{ei}\frac{\partial U_e^h}{\partial t} = (\omega_R - \omega_0)(k_p + k_R)\gamma_p\left[e^{i(\omega_R-\omega_0)t}e^{i(k_p+k_R)z}E\overline{\Phi} + c.c.\right]. \tag{5.8}$$

Now, address the r.h.s. of the paraxial equations (5.3) and (5.4). For instance, in the quantity $ne^{ik_pz-i\omega_0t+ik_Rz+i\omega_Rt}E$, we know that function n is oscillating at frequency ω_0 and E is slowly time varying, so the time behavior of this quantity is like $e^{+i\omega_0t-i\omega_Rt\pm i\omega_0t}$ thus it is highly time varying and it is non resonant. Thus, we may replace density n by its averaged value n^h over a laser period which is slowly varying with respect to the time and state the simplified system

$$\frac{\partial}{\partial t}E + c_g\frac{\partial}{\partial z}E - \frac{ic}{2k_0}\Delta_\perp E = -i\beta_0 n^h \Phi e^{-i(k_p+k_R)z+i(\omega_0-\omega_R)t}, \tag{5.9}$$

$$\frac{\partial}{\partial t}\Phi - c_g\frac{\partial}{\partial z}\Phi - \frac{ic}{2k_0}\Delta_\perp \Phi = -i\beta_0 n^h E e^{i(k_p+k_R)z+i(\omega_R-\omega_0)t}. \tag{5.10}$$

These two equations supplemented by evolution equations (5.6) and (5.7) for n^h, U_e^h make up a first modelling of the Raman backscattering system (of course, it is necessary to supplement this system with boundary and initial conditions).

Notice that for this system, we may check that there is a momentum balance relation where the electromagnetic momentum is related to $|E|^2 + |\Phi|^2 + E\overline{\Phi}e^{i(k_p+k_R)z+i(\omega_R-\omega_0)t} + c.c.$; this question will be addressed below.

To get a resonance condition for the previous system, we assume that the wave (U_e^h, n^h) is a travelling one, thus it behaves like

$$U_e^h = \widetilde{U_e^h}e^{iKz-i\Omega t} + c.c., \qquad n^h = \widetilde{n^h}e^{iKz-i\Omega t} + c.c.$$

with $\widetilde{U_e^h}, \widetilde{n^h}$ slowly varying with respect to z and t.

Recall that ν_{ei} is small with respect to ω_p^{-1} and the Debye length is smaller than the laser wave length (typically we have $k_0\lambda_D \leq 0.3$). Then, according to the Bohm–Gross dispersion relation (4.23), we must have

$$\Omega = \omega_p\left(1 + 3K^2\lambda_D^2\right)^{1/2} - i\frac{\nu_{ei}}{2}$$

so, using the fact that $|K| \leq 2k_p$, it leads to

$$\Omega \simeq \omega_p + \mu, \qquad \mu = \omega_p\frac{3}{2}K^2\lambda_D^2 - i\frac{\nu_{ei}}{2} \tag{5.11}$$

and we see that $|\mu|$ is smaller than ω_p. Moreover, in propagation equations (5.9), (5.10), the resonant conditions are

$$K - k_p - k_R = 0, \qquad -\Omega + \omega_0 - \omega_R = 0. \tag{5.12}$$

Here k_p, ω_0 are given and we search ω_R, k_R, Ω, K satisfying conditions (5.11) and (5.12) supplemented by (5.5).

Let us first perform the calculus in the case $\nu_{ei} = 0$; in this case, we have $\mu = \omega_p\frac{3}{2}K^2\lambda_D^2$ and for the Raman electromagnetic wave we get

$$\omega_R^2 = (\omega_0 - \omega_p - \mu)^2 \simeq \omega_0^2\left(1 - 2\frac{\omega_p}{\omega_0} + \frac{\omega_p^2}{\omega_0^2} - 2\frac{\mu}{\omega_0}(1 - \frac{\omega_p}{\omega_0}) + \frac{\mu^2}{\omega_0^2}\right)$$

Now, since $\mathrm{Re}(\mu)$ is positive and μ^2/ω_0^2 is negligible with respect to μ/ω_0, we check using (5.5) that the growth of the Raman instability is available only if $\omega_R > \omega_p$, i.e.,

$$2\omega_p/\omega_0 < 1;$$

it leads to the classical condition

$$N_{\mathrm{ref}} < 1/4$$

which has to be satisfied. Within this condition, we get

$$k_R^2 \simeq k_0^2 \left(1 - 2\frac{\omega_p}{\omega_0} - 2\frac{\mu}{\omega_0}(1 - \frac{\omega_p}{\omega_0})\right)$$

this reads

$$\left(\frac{K}{k_0} - \frac{k_p}{k_0}\right)^2 = 1 - 2\frac{\omega_p}{\omega_0} - 2\frac{\mu}{\omega_0}(1 - \frac{\omega_p}{\omega_0})$$

Then, using the previous approximation of $\mu\omega_p^{-1}$, the value of K/k_0 is the solution of the second-degree equation

$$\left(\frac{K}{k_0} - \frac{k_p}{k_0}\right)^2 + \varpi(\frac{K}{k_0})^2 = 1 - 2\frac{\omega_p}{\omega_0}, \qquad \text{with } \varpi = 3k_0^2\lambda_D^2\frac{\omega_p}{\omega_0}(1 - \frac{\omega_p}{\omega_0})$$

The two solutions of this equation are $\frac{k_p}{k_0} \pm \left[1 - 2\frac{\omega_p}{\omega_0} + \varpi(1 - 2\frac{\omega_p}{\omega_0} - \frac{k_p^2}{k_0^2})\right]^{1/2}$. Since the largest value of K corresponds to the largest value of the growth of the instability, we keep only the solution with sign +, i.e., using the expression of $k_p^2 k_0^{-2}$

$$K = k_p + k_0\sqrt{1 - 2\frac{\omega_p}{\omega_0} + \varpi(\frac{\omega_p^2}{\omega_0^2} - 2\frac{\omega_p}{\omega_0})} \simeq k_p + k_0\sqrt{1 - 2\frac{\omega_p}{\omega_0}}.$$

We now fix the values of ω_R, k_R, K, μ as above. Moreover, we set

$$U_e^h = ve^{-i\omega_p t} + \overline{v}e^{i\omega_p t}$$

according to the continuity equation, neglecting $\frac{\partial v}{\partial t}$ with respect to $i\omega_p v$, and we get

$$n^h = \frac{-i}{\omega_p}\frac{\partial v}{\partial z}e^{-i\omega_p t} + \frac{i}{\omega_p}\frac{\partial \overline{v}}{\partial z}e^{i\omega_p t}. \tag{5.13}$$

So, from equation (5.8), a calculus of the same type of the one made in the previous chapter (withdrawing the term $\frac{\partial^2 v}{\partial t^2}$) leads to the equation for the Langmuir wave

$$\frac{\partial v}{\partial t} - i\frac{3\lambda_D}{2}v_{\text{th,e}}\frac{\partial^2 v}{\partial z^2} + \frac{\nu_{ei}}{2}v = -\frac{\gamma_p}{2}\frac{\partial}{\partial z}\left(E\overline{\Phi}\theta\right)e^{-i\mu t} \tag{5.14}$$

where we have denoted

$$\theta = e^{iKz}.$$

Now, in the propagation equation (5.9) the resonant part of n^h is related to $e^{-i\omega_p t}$ and for (5.10) it is related to $e^{i\omega_p t}$. So they read as

$$\frac{\partial}{\partial t}E + c_g\frac{\partial}{\partial z}E - \frac{ic}{2k_0}\Delta_\perp E = -\frac{\beta_0}{\omega_p}\frac{\partial v}{\partial z}\Phi\overline{\theta}e^{i\mu t}, \tag{5.15}$$

$$\frac{\partial}{\partial t}\Phi - c_g\frac{\partial}{\partial z}\Phi - \frac{ic}{2k_0}\Delta_\perp \Phi = \frac{\beta_0}{\omega_p}\frac{\partial \overline{v}}{\partial z}E\theta e^{-i\mu t}. \tag{5.16}$$

The three equations (5.14)–(5.16) make up a basic ideal Raman system, where the unknown functions are v, E, and Φ.

Remark 32. It is also possible to account for the Landau damping coefficient ν_{ei} for the evaluation of ω_R, k_R, K, μ. We perform this calculus in the simple case where the $k_0\lambda_D$ term may be neglected. Then $\mu \simeq -i\nu_{ei}/2$ and we get

$$\omega_R^2 = \omega_0^2\left(1 - 2\frac{\omega_p}{\omega_0} + \frac{\omega_p^2}{\omega_0^2} - i\nu_{ei}(1 - \frac{\omega_p}{\omega_0}) - (\frac{\nu_{ei}}{2\omega_0})^2\right)$$

Then, according to this relation, condition $\omega_R > \omega_p$ reads as

$$\text{Re}(1 - 2\frac{\omega_p}{\omega_0} - i\frac{\nu_{ei}}{\omega_0}(1 - \frac{\omega_p}{\omega_0}) - (\frac{\nu_{ei}}{2\omega_0})^2) > 0.$$

or equivalently

$$2N_{\text{ref}}^{1/2} + (\frac{\nu_{ei}}{2\omega_0})^2 \leq 1$$

The standard condition for the growth of the instability ($2N_{\mathrm{ref}}^{1/2} \leq 1$) is still strengthened when $\frac{\nu_{ei}}{2\omega_0}$ is not negligible. □

Remark 33 (Momentum balance). To get a simple balance relation, we assume that $\nu_{ei} = 0$. Multiplying eq. (5.14) by $i\omega_p$ and denoting $\chi = \frac{-i}{\omega_p}\frac{\partial v}{\partial z}$, we get for the momentum fluctuation $n^h U_e^h = (v\overline{\chi} + \chi\overline{v})$ up to terms that are highly oscillating with respect to the time

$$2\omega_p \frac{\partial}{\partial t}(v\overline{\chi}+w\overline{v}) - i c_e^2 \left(\overline{\chi}\frac{\partial^2 v}{\partial z^2} - \chi\frac{\partial^2 \overline{v}}{\partial z^2}\right) = -e^{-i\mu t}\omega_p\gamma_p\overline{\chi}\frac{\partial}{\partial z}(E\overline{\Phi}\theta) + c.c.$$

$$= -e^{-i\mu t}\omega_p\gamma_p\left(iK\overline{\chi}\theta E\overline{\Phi} + \overline{\chi}\theta\frac{\partial}{\partial z}(E\overline{\Phi}) + c.c.\right)$$

Now from (5.16) we get

$$\frac{\partial}{\partial t}|\Phi|^2 - c_g\frac{\partial}{\partial z}|\Phi|^2 = -i\frac{\beta_0}{\omega_p}\overline{\chi}E\overline{\Phi}\theta e^{-i\mu t} + c.c.$$

Since $\int i\left(\overline{\chi}\frac{\partial^2 v}{\partial z^2} - \chi\frac{\partial^2 \overline{v}}{\partial z^2}\right)\omega_p^{-1} = \int(-\frac{\partial \overline{v}}{\partial z}\frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z}\frac{\partial^2 \overline{v}}{\partial z^2}) = |\frac{\partial v}{\partial z}|_{z=0}^2 - |\frac{\partial v}{\partial z}|_{z=L}^2$, we get the following relation (where a is a positive constant)

$$\frac{\partial}{\partial t}\int\left[(v\overline{\chi}+\chi\overline{v}) - a|\Phi|^2\right]dz - \frac{c_e^2}{2}\left(|\frac{\partial v}{\partial z}|_{z=0}^2 - |\frac{\partial v}{\partial z}|_{z=L}^2\right) - ac_g\left(|\Phi|_{z=0}^2 - |\Phi|_{z=L}^2\right)$$

$$= -e^{-i\mu t}\frac{\gamma_p}{2}\left(\overline{\chi}\theta\frac{\partial}{\partial z}(E\overline{\Phi}) + c.c.\right)$$

on the right-hand side. There is a term θ that is highly oscillating with respect to the space variable, so its averaged value may be neglected. Thus, this relation without the right-hand-side terms may be considered to be a good balance equation for the fluctuation of global momentum related to the Raman instability. □

Of course, in equation (5.14), it is possible to account for a transverse diffraction phenomenon by using the transverse Laplacian $\Delta_\perp$ (corresponding to the space derivative in the direction orthogonal to the main direction z).

It is crucial to point out that the system (5.14)–(5.16) is not useful from a numerical point of view since there are highly oscillating terms θ with respect to the space variable. The aim of the sequel is to simplify this system and to replace it by another one of the three-wave coupling type (analogous to the Boyd–Kadomstev system for the Brillouin instability). It is made without accounting for the diffraction terms; in a second step we will introduce afresh these diffraction terms.

It is worthwhile noting that there are different small parameters and this reduction is quite tricky, even if it has been used for a long time by physicists. We choose here to justify it by introducing a small parameter ε related to the inverse of the wave number and the inverse of the laser pulsation, and then to perform an asymptotic analysis. Note that a lot of mathematical details are still open problems, but we try here to explain the main lines of the work.

For a while we do not account for the Landau damping coefficient ν_{ei}; we will introduce it afresh at the end of the derivation. Now denote $c' = c_g/\mathrm{v}_{\mathrm{th,e}}$ the ratio between the group speed of light and the electron thermal speed (it is larger than 1; e.g., in the hot plasma we have in mind, it may be on the order of 15; but we do not make any asymptotic analysis using the parameter $1/c'$).

For the sake of conciseness, we use a length unit such that $\mathrm{v}_{\mathrm{th,e}} = 1$; the reduced space variable is now denoted by x instead of z.

Let us introduce the small parameter ε related to the ratio between the laser wavelength and the characteristic length of the plasma

$$\varepsilon = 1/(k_0 L_{\mathrm{plas}})$$

and define κ and λ by

$$K = \varepsilon^{-1}\kappa, \qquad \lambda_D = \lambda\varepsilon.$$

With the new length units, we have $\omega_p = \lambda_D^{-1} = \varepsilon^{-1}\lambda^{-1}$. Here κ^{-1} and λ are on the same order of magnitude (even if $\kappa\lambda$ is small with respect to 1). Moreover, we define β_1 by

$$\beta_0\kappa\lambda = \beta_1$$

Using the normalized space variable x, we set

$$E(z) = E'_\varepsilon(x), \qquad \Phi(z) = \Phi'_\varepsilon(x), \qquad v(z) = v'_\varepsilon(x), \qquad \theta(z) = \theta_\varepsilon(x) = e^{i\kappa x/\varepsilon.}$$

Then, with this change of variables, system (5.14)–(5.16) reads as follows without accounting for the damping coefficient ν_{ei} or the transverse diffraction operators

$$\left(\frac{\partial}{\partial t} + c'\frac{\partial}{\partial x}\right) E'_\varepsilon = -\frac{\beta_1}{\kappa}\varepsilon\frac{\partial v'_\varepsilon}{\partial x}\Phi'_\varepsilon\overline{\theta_\varepsilon}e^{i\mu t}$$

$$\left(\frac{\partial}{\partial t} - c'\frac{\partial}{\partial x}\right) \Phi'_\varepsilon = \frac{\beta_1}{\kappa}\varepsilon\frac{\partial \overline{v'_\varepsilon}}{\partial x}E'_\varepsilon\theta_\varepsilon e^{-i\mu t}$$

$$\frac{\partial v'_\varepsilon}{\partial t} - i\frac{3\lambda}{2}\varepsilon\frac{\partial^2 v'_\varepsilon}{\partial x^2} = -\frac{\gamma_p}{2}\frac{\partial}{\partial x}\left(E'_\varepsilon\overline{\Phi'_\varepsilon}\theta_\varepsilon\right)e^{-i\mu t}$$

We now define $\gamma_1 = \gamma_p \kappa/(2\varepsilon)$ (where γ_1 is of order 1) so we expect to have v'_ε of order 1 (indeed $\dfrac{\partial \theta_\varepsilon}{\partial x}$ is of order ε^{-1}) and we drop the $'$. Then the system reads

$$\left(\frac{\partial}{\partial t} + c'\frac{\partial}{\partial x}\right) E_\varepsilon = -\frac{\beta_1}{\kappa}\varepsilon\frac{\partial v_\varepsilon}{\partial x}\Phi_\varepsilon\overline{\theta_\varepsilon}e^{i\mu t}$$

$$\left(\frac{\partial}{\partial t} - c'\frac{\partial}{\partial x}\right) \Phi_\varepsilon = \frac{\beta_1}{\kappa}\varepsilon\frac{\partial \overline{v_\varepsilon}}{\partial x}E_\varepsilon\theta_\varepsilon e^{-i\mu t}$$

$$\frac{\partial v_\varepsilon}{\partial t} - i\frac{3\lambda}{2}\varepsilon\frac{\partial^2 v_\varepsilon}{\partial x^2} = -\frac{\gamma_1}{\kappa}\varepsilon\frac{\partial}{\partial x}\left(E_\varepsilon\overline{\Phi_\varepsilon}\theta_\varepsilon\right)e^{-i\mu t}$$

5.1.2 Reduction of the Model with Fixed Ions

Setting

$$W_\varepsilon = i v_\varepsilon \overline{\theta_\varepsilon} = i v_\varepsilon e^{i\kappa x/\varepsilon}, \qquad \alpha = \frac{3\lambda\kappa^2}{2}$$

the previous system may read as follows

$$\left(\frac{\partial}{\partial t} + c'\frac{\partial}{\partial x}\right) E_\varepsilon = \left(-\beta_1 W_\varepsilon \Phi_\varepsilon + i\frac{\varepsilon\beta_1}{\kappa}\frac{\partial W_\varepsilon}{\partial x}\right)e^{i\mu t}. \tag{5.17}$$

$$\left(\frac{\partial}{\partial t} - c'\frac{\partial}{\partial x}\right) \Phi_\varepsilon = \left(\beta_1 \overline{W_\varepsilon} E_\varepsilon + i\frac{\varepsilon\beta_1}{\kappa}\frac{\partial \overline{W_\varepsilon}}{\partial x}\right)e^{-i\mu t} \tag{5.18}$$

$$\frac{\partial W_\varepsilon}{\partial t} + 3\lambda\kappa\frac{\partial W_\varepsilon}{\partial x} + i\frac{\alpha}{\varepsilon}W_\varepsilon - i\frac{3\lambda}{2}\varepsilon\frac{\partial^2 W_\varepsilon}{\partial x^2} = \left(\gamma_1 E_\varepsilon\overline{\Phi_\varepsilon} - i\frac{\varepsilon\gamma_1}{\kappa}\frac{\partial}{\partial x}(E_\varepsilon\overline{\Phi_\varepsilon})\right)e^{-i\mu t} \tag{5.19}$$

this system is considered on an interval $[0, L]$ and needs to be supplemented by initial conditions and boundary ones, which are, e.g.,

$$E_\varepsilon(t, 0) = E^{in}, \qquad \Phi_\varepsilon(t, L) = 0, \qquad W_\varepsilon(t, 0) = w_\varepsilon(t, L) = 0. \tag{5.20}$$

Proposition 18. *There exists a positive constant b related to boundary value E^{in}, such that if $E_\varepsilon, \Phi_\varepsilon, W_\varepsilon$ satisfy (5.17)–(5.19), we get*

$$\frac{\partial}{\partial t}\left(\|W_\varepsilon\|_{L^2_x} + \frac{\gamma_1}{\beta_1}\|E_\varepsilon\|_{L^2_x}\right) \le b$$

$$\frac{\partial}{\partial t}\left(\|W_\varepsilon\|_{L^2_x} - \frac{\gamma_1}{\beta_1}\|\Phi_\varepsilon\|_{L^2_x}\right) \le 0$$

[**Indeed,** in a classical manner, multiplying the first equation $\overline{E_\varepsilon}$ and the third one by $\overline{W_\varepsilon}$, we get with $C = \frac{3\lambda}{2}\varepsilon$

$$\frac{\gamma_1}{\beta_1}\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)|E_\varepsilon|^2+\frac{\partial}{\partial t}|W_\varepsilon|^2+3\lambda\kappa\frac{\partial}{\partial x}|W_\varepsilon|^2=$$
$$\gamma_1\left[\left(-W_\varepsilon\Phi_\varepsilon\overline{E_\varepsilon}+i\frac{\varepsilon}{\kappa}\frac{\partial W_\varepsilon}{\partial x}\overline{E_\varepsilon}\right)-\left(-\overline{W}_\varepsilon\overline{\Phi}_\varepsilon E_\varepsilon+i\frac{\varepsilon}{\kappa}\frac{\partial\overline{W}_\varepsilon}{\partial x}E_\varepsilon\right)+c.c.\right]+C\left(i\frac{\partial^2 W_\varepsilon}{\partial x^2}\overline{W_\varepsilon}+c.c.\right)$$

Then integrating with respect to x, after integrating by parts, we get

$$\frac{\partial}{\partial t}\left(\|W_\varepsilon\|_{L^2_x}+\frac{\gamma_1}{\beta_1}\|E_\varepsilon\|_{L^2_x}\right)+\frac{\gamma_1}{\beta_1}c\left(|E_\varepsilon(L)|^2-|E^{in}|^2\right)=0.$$

The same type of calculus yields

$$-\frac{\gamma_1}{\beta_1}\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)|\Phi_\varepsilon|^2+\frac{\partial}{\partial t}|W_\varepsilon|^2+3\lambda\kappa\frac{\partial}{\partial x}|W_\varepsilon|^2=$$
$$\gamma_1\left[\left(-W_\varepsilon\Phi_\varepsilon\overline{E_\varepsilon}+i\frac{\varepsilon}{\kappa}\frac{\partial W_\varepsilon}{\partial x}\overline{E_\varepsilon}\right)-\left(-\overline{W_\varepsilon}\overline{\Phi}_\varepsilon E_\varepsilon+i\frac{\varepsilon}{\kappa}\frac{\partial\overline{W}_\varepsilon}{\partial x}E_\varepsilon\right)+c.c.\right]+C\left(i\frac{\partial^2 W_\varepsilon}{\partial x^2}\overline{W_\varepsilon}+c.c.\right)$$

and the other bound follows. □]

By neglecting $\dfrac{\partial W_\varepsilon}{\partial z}$ with respect to $\dfrac{\kappa}{\varepsilon}W_\varepsilon$ in the right-hand-side terms, we get

$$\left(\frac{\partial}{\partial t}+c'\frac{\partial}{\partial x}\right)E_\varepsilon=-\beta_1 W_\varepsilon\Phi_\varepsilon e^{i\mu t}$$
$$\left(\frac{\partial}{\partial t}-c'\frac{\partial}{\partial x}\right)\Phi_\varepsilon=\beta_1\overline{W_\varepsilon}E_\varepsilon e^{-i\mu t}$$
$$\frac{\partial W_\varepsilon}{\partial t}+3\lambda\kappa\frac{\partial W_\varepsilon}{\partial x}+i\frac{\alpha}{\varepsilon}W_\varepsilon-i\frac{3\lambda}{2}\varepsilon\frac{\partial^2 W_\varepsilon}{\partial x^2}=\gamma_1 E_\varepsilon\overline{\Phi_\varepsilon}e^{-i\mu t}$$

Define now

$$w_\varepsilon=W_\varepsilon e^{i\alpha t/\varepsilon}$$

and introduce the quantity

$$\rho=\mu-\frac{\alpha}{\varepsilon}=\frac{1}{\varepsilon\lambda}\left(\sqrt{1+3\kappa^2\lambda^2}-1-\frac{3\kappa^2\lambda^2}{2}\right)\simeq-\frac{1}{\varepsilon\lambda}\frac{9(\kappa\lambda)^4}{8}.$$

It is not exactly independent of ε, but if we assume that λ is on the order of $\varepsilon^{1/3}$ we see that ρ has a limit when ε goes to zero. It is worth noticing that term $e^{-i\rho t}$ is crucial for a good modelling of the Raman instability; it is called by the physicists a “miss-match term” (notice that if the plasma is not homogeneous, it may depend also on the space variable, and it is an open problem to account for a coefficient ρ depending on the space variable).

Thus, E_ε, Φ_ε and w_ε satisfy

$$\left(\frac{\partial}{\partial t} + c'\frac{\partial}{\partial x}\right) E_\varepsilon = -\beta_1 w_\varepsilon \Phi_\varepsilon e^{i\rho t} \tag{5.21}$$

$$\left(\frac{\partial}{\partial t} - c'\frac{\partial}{\partial x}\right) \Phi_\varepsilon = \beta_1 \overline{w_\varepsilon} E_\varepsilon e^{-i\rho t} \tag{5.22}$$

$$\frac{\partial w_\varepsilon}{\partial t} + 3\lambda\kappa\frac{\partial w_\varepsilon}{\partial x} - i\frac{3\lambda}{2}\varepsilon\frac{\partial^2 w_\varepsilon}{\partial x^2} = \gamma_1 E_\varepsilon \overline{\Phi_\varepsilon} e^{-i\rho t.} \tag{5.23}$$

Of course, this system needs to be supplemented with initial conditions

$$E_\varepsilon|_{t=0} = E_{\text{ini}}, \qquad \Phi_\varepsilon|_{t=0} = \Phi_{\text{ini}}, \qquad w_\varepsilon|_{t=0} = w_{\text{ini}},$$

as well as boundary conditions, e.g., (5.20) (i.e., they are of a homogeneous Dirichlet type for w_ε, but we may also have conditions of Neumann type).

The following result enables us to simplify equation (5.23). Here the final time T is fixed.

Theorem 4. *Assume that $E_{ini}, \Phi_{ini}, w_{ini}$ are bounded and belong to H^1_x. For all ε, there is a solution $(E_\varepsilon, \Phi_\varepsilon, w_\varepsilon)$ in the space $(L^\infty(0,T,L^2_x))^3$ to system (5.21)–(5.23) supplemented with the previous initial conditions and boundary conditions (5.20). Moreover, if ε goes to 0, we have*

$$E_\varepsilon \to E, \qquad \Phi_\varepsilon \to \Phi, \qquad w_\varepsilon \to w \qquad in\ L(0,T;L^2_x),$$

where (E,Φ,w) is the unique solution of the system

$$\begin{aligned} \left(\frac{\partial}{\partial t} + c'\frac{\partial}{\partial x}\right) E &= -\beta_1 w \Phi e^{-i\rho t}, \\ \left(\frac{\partial}{\partial t} - c'\frac{\partial}{\partial x}\right) \Phi &= \beta_1 \overline{w} E e^{i\rho t}, \\ \frac{\partial w}{\partial t} + 3\lambda\kappa\frac{\partial w}{\partial x} &= \gamma_1 E \overline{\Phi} e^{-i\rho t}. \end{aligned} \tag{5.24}$$

supplemented with the same initial conditions and the boundary conditions

$$E(t,0) = E^{in}, \qquad \Phi(t,L) = 0, \qquad w(t,0) = 0.$$

It is worth noticing that the proof of the existence of the solution of the system (5.21)–(5.23) is quite technical; indeed, even for a given value of E_ε, system (5.18), (5.19) is not hyperbolic. One needs to adapt techniques used in [30]. We give below only a sketch of the proof; moreover, the uniqueness of the solution of (5.21)–(5.23) seems to be an open problem.

Conclusion

Let us now go back to the physical quantities and introduce afresh the diffraction terms in the laser propagation equations and the Landau damping term. The spatial variable in the longitudinal direction is denoted afresh z, so we have $\mathbf{x} = (z, y)$ where y is a two-dimensional variable. We first check that w needs to be defined by the following relation:

$$w(\mathbf{x}) = i v \exp(iK(\frac{3}{2}K\lambda_D v_{\text{th,e}} t - z)) = i v \exp(iK(\frac{1}{2\omega_p}K c_e^2 t - z)).$$

The previous analysis is a justification for replacing the first Raman system (5.14)–(5.16) by the following one, called the simplified Raman model,

$$(\mathcal{R}am\mathcal{S}) \quad \boxed{\begin{array}{ll} \text{(i)} & \left(\frac{\partial}{\partial t} + c_g \dfrac{\partial}{\partial z}\right) E - \frac{ic}{2k_0}\Delta_\perp E = -\frac{\beta_0}{\omega_p} w\Phi e^{-i\rho t}, \\ \text{(ii)} & \left(\frac{\partial}{\partial t} - c_g \dfrac{\partial}{\partial z}\right) \Phi - \frac{ic}{2k_0}\Delta_\perp \Phi = \frac{\beta_0}{\omega_p} \overline{w} E e^{i\rho t}, \\ \text{(iii)} & \dfrac{\partial w}{\partial t} + 3\lambda_D v_{\text{th,e}} K \dfrac{\partial w}{\partial z} + \frac{\nu_{ei}}{2} w = \frac{\gamma_p}{2} K E \overline{\Phi} e^{-i\rho t}, \end{array}}$$

where the miss-match term is given by $\rho = -\frac{9}{8}(K\lambda_D)^4\omega_p$. It is called 'simplified' since there is no ion acoustic wave, that is to say $\mathcal{N} = 1$.

Of course, it must be supplemented by initial conditions and by boundary conditions; e.g., in the one-dimensional problem with $z \in [0, L]$, one may take

$$E(t)|_{z=0} = E^{in}, \qquad \Phi(t)|_{z=L} = 0, \qquad w(t)|_{z=0} = 0,$$

where E^{in} is the incoming laser field. In a multidimensional simulation one must impose transparent conditions on the boundaries corresponding to the transverse

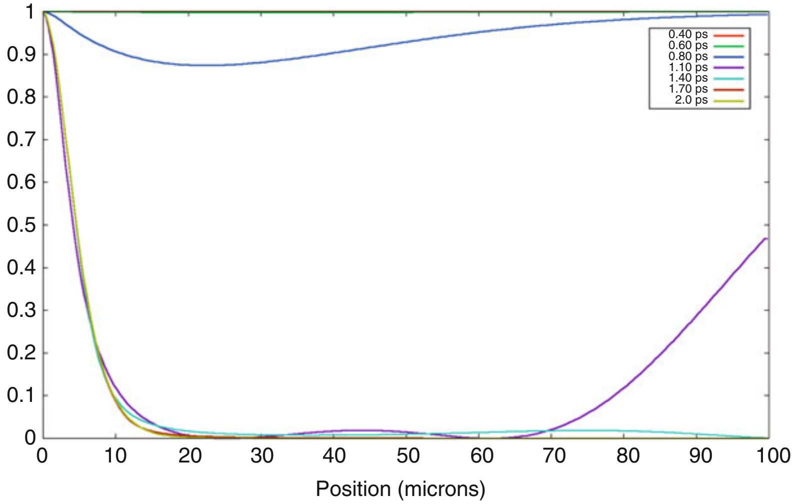

Fig. 5.1 Profile of the main laser wave intensity at times 0.4 ps, 0.6 ps, 0.8 ps, 1.1 ps, 1.4 ps, 1.7 ps, and 2.0 ps. Here, the depletion of this wave begins at 0.8 ps and is very significant at 1.1 ps

directions (to deal with the diffraction operator $\Delta_\perp$) and state the following boundary conditions in the z-direction:

$$E(t, \mathbf{x})|_{z=0} = E^{in}(y), \qquad \Phi(t, \mathbf{x})|_{z=L} = 0, \qquad w(t, \mathbf{x})|_{z=0} = 0.$$

To illustrate this modelling, we first show the results of a numerical simulation of a one-dimensional problem where one accounts for model ($\mathcal{R}am\mathcal{S}$) with $\rho = 0$ and $\nu_{ei} = 0$; $N_{\text{ref}} = 0.2\ N_c$.

In Figs. 5.1, 5.2, and 5.3 we plot the profiles of the intensity of the three waves, i.e., $|E|^2, |\Phi|^{2,}$ and $|w|^2$ versus the position z at different times. [These figures and the following ones were supplied via the courtesy of Guillaume Tran (CEA Bruyères-le-Chatel, France).]

Let us show now a numerical result of a two-dimensional simulation. Denoting z, y as the two coordinates, the incoming laser field E^{in} (at $z = 0$) is now a function of y. As a matter of fact, it is a Gauss function with a phase $\varphi(y)$; it reads as $E^{in}(y) = \exp(-|y - y_0|^2/L^2 + i\varphi(y))$ in such a way that the intensity of the laser wave propagates and focalizes at some distance (here the center of the simulation box). The map of the laser intensity $|E(z, y)|^2$ at the beginning of the simulation is plotted on Fig. 5.4.

The map of the electron wave energy $|w(t, z, y)|^2$ at different times is plotted on Fig. 5.5.

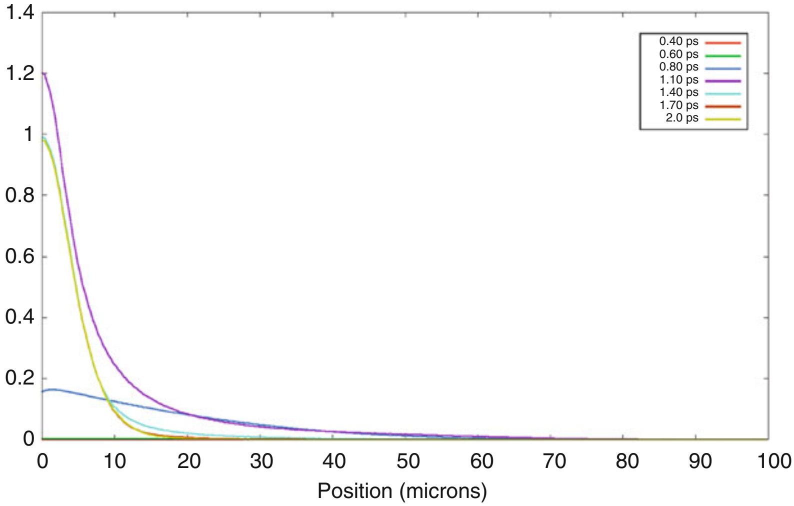

Fig. 5.2 Profile of the backscattered laser wave intensity at times 0.4 ps, 0.6 ps, 0.8 ps, 1.1 ps, 1.4 ps, 1.7 ps, and 2.0 ps

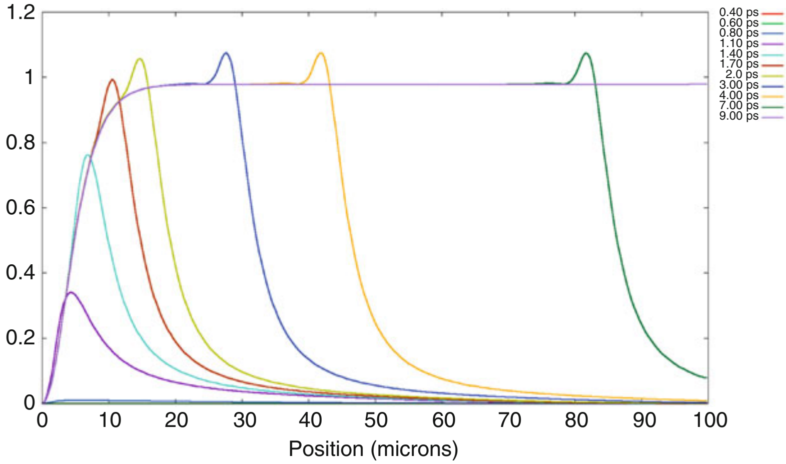

Fig. 5.3 Profile of the Langmuir wave intensity at times 0.4 ps, 0.6 ps , 0.8 ps, 1.1 ps, 1.4 ps, 1.7 ps, and 2.0 ps. The unit of intensity is normalized at its theoretical maximal value

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Sketch of the Proof of Theorem 4

We can set the constant γ_1 to 1 and drop the term $e^{i\rho t}$ (there is no difficulty related to this term from a mathematical point of view). Up to a change of notation, we may

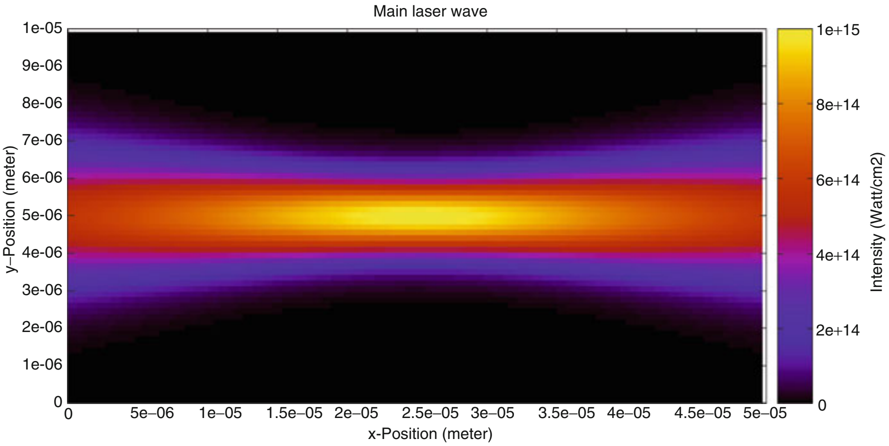

Fig. 5.4 Main laser intensity at the beginning of the simulation

write ε instead of $\frac{3\lambda}{2}\varepsilon$. Denote c instead of c' and $u_\varepsilon, v_\varepsilon$ instead of $E_\varepsilon, \Phi_\varepsilon$; moreover, u, v instead of E, Φ. Denote also by L_ε the operator $L_\varepsilon w = \varepsilon \frac{\partial^2}{\partial x^2} w_\varepsilon$. So the system to be analyzed reads as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u_\varepsilon = -\beta w_\varepsilon v_\varepsilon, \tag{5.25}$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v_\varepsilon = \beta \overline{w_\varepsilon} u_\varepsilon, \tag{5.26}$$

$$\frac{\partial w_\varepsilon}{\partial t} + 3\lambda\kappa \frac{\partial w_\varepsilon}{\partial x} - i L_\varepsilon w_\varepsilon = u_\varepsilon \overline{v_\varepsilon} \tag{5.27}$$

with the initial conditions

$$u_\varepsilon|_{t=0} = u_{\text{ini}}, \qquad v_\varepsilon|_{t=0} = v_{\text{ini}}, \qquad w_\varepsilon|_{t=0} = w_{\text{ini}},$$

and the boundary conditions

$$u_\varepsilon|_{x=0} = u^{in}, \qquad v_\varepsilon|_{x=L} = 0, \qquad w_\varepsilon|_{x=0} = 0, \qquad w_\varepsilon|_{x=L} = 0.$$

The structure of this system is the same as the Boyd–Kadomstev system up to the supplementary term $iL_\varepsilon w_\varepsilon$ in the third equation.

If there exist functions $u_\varepsilon, v_\varepsilon, w_\varepsilon$ satisfying this system,we have, according to the boundary conditions for w_ε,

$$\frac{\partial}{\partial t}\int |w_\varepsilon|^2 + 3\lambda\kappa \int \frac{\partial}{\partial x}|w_\varepsilon|^2 = \int u_\varepsilon \overline{v_\varepsilon}\overline{w_\varepsilon} + \int i\overline{w_\varepsilon} L_\varepsilon w_\varepsilon + c.c = \int u_\varepsilon \overline{v_\varepsilon}\overline{w_\varepsilon} + c.c$$

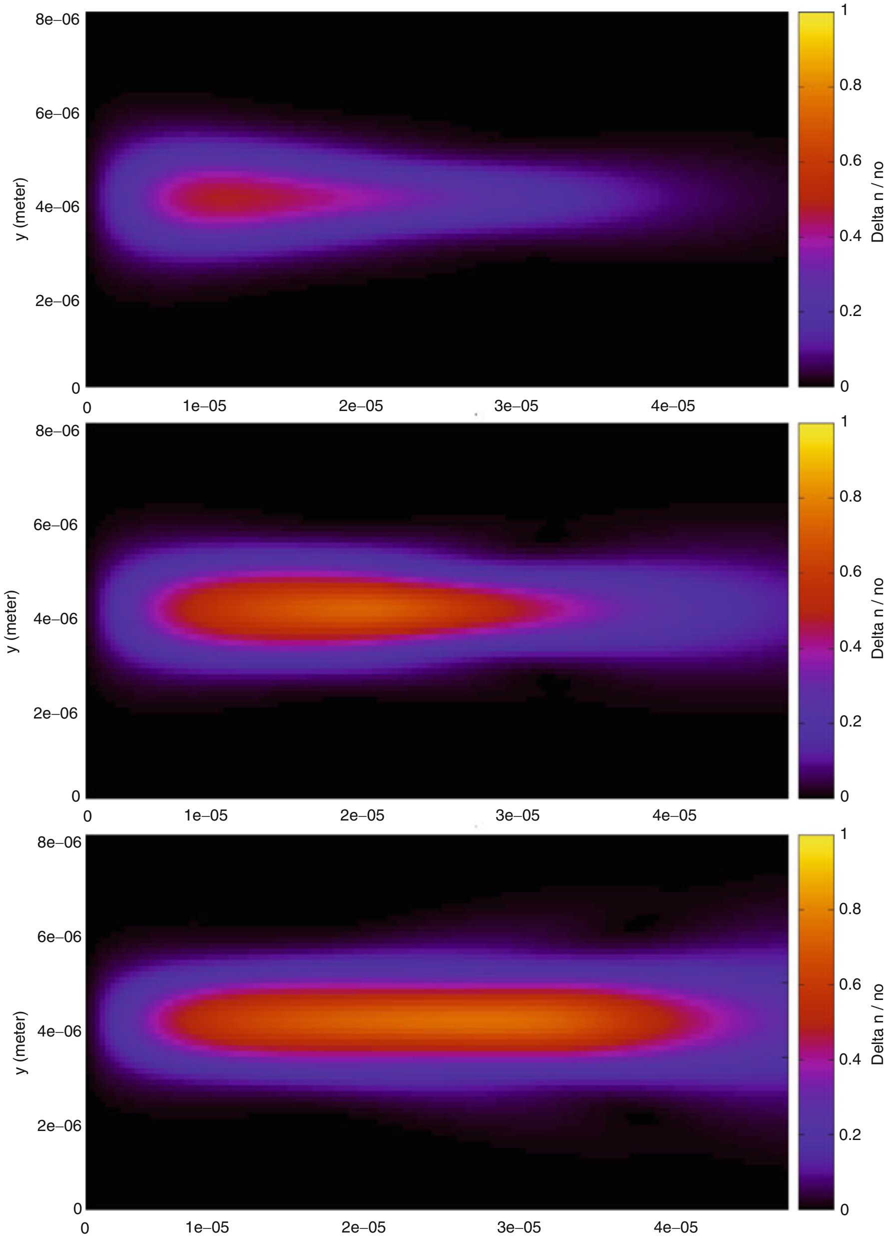

Fig. 5.5 Map of the Raman wave energy at 2 ps, 3 ps, and 4 ps, respectively

then

$$\frac{\partial}{\partial t}\left(\|u_\varepsilon\|^2_{L^2_x} + \beta\,\|w_\varepsilon\|^2_{L^2_x}\right) \le b, \tag{5.28}$$

$$\frac{\partial}{\partial t}\left(\|u_\varepsilon\|^2_{L^2_x} + \|v_\varepsilon\|^2_{L^2_x}\right) \le b, \tag{5.29}$$

$$\frac{\partial}{\partial t}\left(\|v_\varepsilon\|^2_{L^2_x} - \beta\,\|w_\varepsilon\|^2_{L^2_x}\right) \le 0. \tag{5.30}$$

Thus, there exists a constant C_0 depending on the data and independent from ε

$$\|u_\varepsilon(t)\|_{L^2_x},\ \|v_\varepsilon(t)\|_{L^2_x},\ \|w_\varepsilon(t)\|_{L^2_x} \le C_0, \qquad \text{for all } t \le T. \tag{5.31}$$

With a technique analogous to the technique used in Chap. 3 (Sect. 3.2), one can show the existence of a solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ to system (5.25)–(5.27) in the space $(L^2(0,\tau,L^2_x))^3$ for a small final time τ.

But using the bounds (5.31), a classical continuity argument shows that this solution can not blow up at a finite time. Thus, there exists a solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ in the space $(L^2(0,T,L^2_x))^3$ on the global time interval $[0,T]$.

According to the proof of the second part of Theorem 1, one knows that $(u_\varepsilon, v_\varepsilon)$ satisfy for a constant C_1 independent from ε

$$\int_0^t |v_\varepsilon(t-s,x-s)|^2 ds \le C_{1,}$$

$$\int_0^t |w_\varepsilon(t-s,x-s)|^2 ds \le C_{1.}$$

Thus, we get

$$\begin{aligned}|u_\varepsilon(t,x)| &\le \beta\int_0^t |v_\varepsilon(t-s,x-s)w_\varepsilon(t-s,x-s)|ds + \|u_{\text{ini}}\|_{L^\infty}\\ &\le C_1 + \|u_{\text{ini}}\|_{L^\infty}\end{aligned}$$

The analogous holds for v_ε, so there exists C_2 independent of ε such that

$$\|u_\varepsilon(t)\|_{L^\infty} \le C_2, \qquad \|v_\varepsilon(t)\|_{L^\infty} \le C_2$$

We now address the equation satisfied by $\partial_x u_\varepsilon$, $\partial_x v_\varepsilon$, and $\partial_x w_\varepsilon$ in order to get bounds of these functions. Using notations of the proof of Theorem 1, we have

$$K_1(\partial_x u_\varepsilon) = -\beta w_w(\partial_x v_\varepsilon) - \beta v_\varepsilon(\partial_x w_\varepsilon), \tag{5.32}$$

$$K_2(\partial_x v_\varepsilon) = \beta \overline{w_w}(\partial_x u_\varepsilon) + \beta u_\varepsilon(\partial_x \overline{w_\varepsilon}), \tag{5.33}$$

$$K_3(\partial_x w_\varepsilon) + iL_\varepsilon(\partial_x w_\varepsilon) = \overline{v_\varepsilon}(\partial_x u_\varepsilon) + u_\varepsilon(\partial_x \overline{v_\varepsilon}); \tag{5.34}$$

then we have

$$\frac{\partial}{\partial t}\left(\|\partial_x u_\varepsilon\|_{L_x^2}^2 + \|\partial_x v_\varepsilon\|_{L_x^2}^2\right) \leq b + \beta C_2 \int |(\partial_x w_\varepsilon)(\partial_x \overline{u_\varepsilon})|\, dx + \beta C_2 \int |(\partial_x \overline{w_\varepsilon})(\partial_x \overline{v_\varepsilon})|\, dx$$

Thus denoting $\chi_\varepsilon = \|\partial_x w_\varepsilon\|_{L^2}$,we see that

$$\max\left(\|\partial_x u_\varepsilon(t)\|_{L_x^2}^2, \|\partial_x v_\varepsilon(t)\|_{L_x^2}^2\right) \leq C_{\rm ini} + bt + \beta C_2 \int_0^t \chi(s) \max\left(\|\partial_x u_\varepsilon\|_{L_x^2}, \|\partial_x v_\varepsilon\|_{L_x^2}\right) ds$$

so we get

$$\sup_{s\leq t} \max\left(\|\partial_x u_\varepsilon(s)\|_{L_x^2}, \|\partial_x v_\varepsilon(s)\|_{L_x^2}\right) \leq C + C\int_0^t \chi_\varepsilon(s) ds$$

Equation (5.34) reads as $K_3(\partial_x w_\varepsilon) + iL_\varepsilon(\partial_x w_\varepsilon) = g$ with g satisfying

$$\|g(t)\|_{L^2} \leq \|v_\varepsilon\|_{L^\infty} \|\partial_x u_\varepsilon(t)\|_{L^2} + \|u_\varepsilon\|_{L^\infty} \|\partial_x v_\varepsilon(t)\|_{L^2} \leq 2C_2 \max\left(\|\partial_x u_\varepsilon(t)\|_{L_x^2}, \|\partial_x v_\varepsilon(t)\|_{L_x^2}\right)$$

Now multiplying equation (5.34) by $\partial_x w_\varepsilon$ and integrating firstly in x and secondly with respect to t, we get

$$\|\partial_x w_\varepsilon(t)\|_{L_x^2}^2 \leq \|w_{\rm ini}\|_{H_x^1}^2 + \int_0^t \|g(s)\|_{L^2} \|\partial_x w_\varepsilon(s)\|_{L^2}\, ds$$

Therefore,

$$\begin{aligned}\chi_\varepsilon(t) = \|\partial_x w_\varepsilon(t)\|_{L_x^2} &\leq C_I + \int_0^t \|g(s)\|_{L^2}\, ds \\ &\leq C_I + 2C_2 \int_0^t \max\left(\|\partial_x u_\varepsilon(s)\|_{L_x^2}, \|\partial_x v_\varepsilon(s)\|_{L_x^2}\right) ds \leq C_3 + C_4 \int_0^t \chi_\varepsilon(s) ds\end{aligned}$$

So according to Gronwall's lemma (see result 7 in the Appendix), $\chi = \|\partial_x w_\varepsilon\|_{L^2_x}$ is bounded for all $t \leq T$ by a constant independent from ε, and $\|\partial_x u_\varepsilon\|_{L^2_x}$, $\|\partial_x v_\varepsilon\|_{L^2_x}$ are also bounded by a constant independent from ε. Then, we can extract subsequences still denoted by $u_\varepsilon, v_\varepsilon, w_\varepsilon$ such that

$$u_\varepsilon \to u, \qquad u_\varepsilon \to u, \qquad u_\varepsilon \to u, \qquad \text{in } L^2(0,T;L^2_x) \text{ strongly.}$$

So, we can pass to the limit in the weak form of (5.27) which reads as

$$\frac{\partial}{\partial t}\int w_\varepsilon \zeta - 3\lambda\kappa \int w_\varepsilon \frac{\partial \zeta}{\partial x} + i\varepsilon \int \frac{\partial \zeta}{\partial x}\frac{\partial w_\varepsilon}{\partial x} = \int \zeta u_\varepsilon \overline{v_\varepsilon}$$

for any test function ζ in H^1_x and we get

$$\frac{\partial}{\partial t}\int w\zeta - 3\lambda\kappa \int w \frac{\partial \zeta}{\partial x} = \int \zeta u \overline{v}.$$

It is easy to check that u, v, w satisfy system (5.24) with appropriated boundary conditions. Since the solution of this system is unique (according to Theorem 1), the whole sequences $u_\varepsilon, v_\varepsilon, w_\varepsilon$ converge. □

5.1.3 The Raman Model with an Ion Acoustic Wave

We may now add to the previous model the time behavior of ion acoustic wave as it has been made in the previous chapter related to Zakharov equations. The derivation of this model is quite tricky and all the articles in the physics literature are not in full agreement (even in the case where multi-dimension aspects are neglected for the Langmuir waves). Nevertheless, we try here to give some enlightenments on this problem.

Remind first that the ion balance equation for mass and momentum read as

$$\frac{\partial}{\partial t} N_0 + \nabla.(N_0 \mathbf{U}) = 0,$$

$$m_0 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + \nabla P_0 = Z N_0 q_e \mathbf{E}^s + m_e \nu_{ei} N_{\mathrm{ref}} \mathbf{U}^s_e.$$

For the electron density, we perform the following decomposition the averaged value $\langle N_e \rangle_{\omega_0}$ over a laser period

$$\langle N_e \rangle_{\omega_0} = N_{\mathrm{ref}} n^h + N_{\mathrm{ref}} \mathcal{N},$$

the term n^h corresponds to the Langmuir wave.

We make a time envelope of n^h as in the previous chapter, using the slow varying time variable v that is to say $n^h = \frac{-i}{\omega_p}\frac{\partial v}{\partial z}e^{-i\omega_p t} + \frac{i}{\omega_p}\frac{\partial \overline{v}}{\partial z}e^{i\omega_p t}$. Thus, v satisfies as above equation (5.14) but with a supplementary term as for Zakharov equations:

$$\frac{\partial v}{\partial t} - i\frac{3\lambda_D}{2}\mathrm{v}_{\mathrm{th,e}}\frac{\partial^2 v}{\partial z^2} + \frac{\nu_{ei}}{2}v = -\frac{\gamma_p}{2}\frac{\partial}{\partial z}\left(E\overline{\Phi}\theta\right)e^{-i\mu t} - i\frac{\omega_p}{2}(\mathcal{N}-1)v$$

Now, in the same way as in the previous subsection, we define

$$w = i\,v\exp(iK(\frac{1}{2\omega_p}Kc_e^2 t - z))$$

and equation for v may be replaced by the following equation for w,

$$\frac{\partial w}{\partial t} + 3\lambda_D K\mathrm{v}_{\mathrm{th,e}}\frac{\partial w}{\partial z} + \frac{\nu_{ei}}{2}w = \frac{\gamma_p K}{2}E\overline{\Phi}e^{-i\rho t} - i\frac{\omega_p}{2}(\mathcal{N}-1)w$$

For the slow varying part $\mathcal{N}$, we may state, as for the Zakharov equations,

$$\frac{\partial}{\partial t}\mathcal{N} + \frac{\partial}{\partial z}\mathbf{U}_e^s = 0$$

$$\frac{\partial}{\partial t}\mathbf{U}_e^s + c_e^2\frac{\partial}{\partial z}\mathcal{N} + \nu_{ei}\mathbf{U}_e^s = -\mathcal{N}\frac{q_e}{m_e}\mathbf{E}^s - 2\frac{\partial}{\partial z}|v|^2.$$

Now, we make the quasi-neutrality approximation; then adding the momentum equations for ions and electrons, we get

$$\frac{\partial}{\partial t}(m_0 N_0\mathbf{U}+m_e N_{\mathrm{ref}}\mathbf{U}_e^s) + \frac{\partial}{\partial z}P_0 + m_e c_e^2 N_{\mathrm{ref}}\frac{\partial}{\partial z}\mathcal{N} = -m_e N_{\mathrm{ref}}2\frac{\partial}{\partial z}|v|^2.$$

Indeed, we have neglected as usual the term $(ZN_0 - N_{\mathrm{ref}}\mathcal{N})q_e\mathbf{E}^s$. Moreover, quasi-neutrality implies that $N_{\mathrm{ref}}\mathcal{N} \simeq ZN_0$ and $m_0 N_0+m_e N_{\mathrm{ref}} \simeq m_0 N_0$. This means that the ion wave satisfies

$$\frac{\partial}{\partial t}\mathcal{N} + \frac{\partial}{\partial z}(\mathcal{N}\mathbf{U}) = 0,$$

$$\frac{\partial}{\partial t}(\mathcal{N}\mathbf{U}) + c_s^2\frac{\partial}{\partial z}\mathcal{N} = -2\frac{Zm_e}{m_0}\frac{\partial}{\partial z}|v|^2.$$

Thus, denoting $\mathcal{N}\mathbf{U} = q$, instead of system ($\mathcal{R}am\mathcal{S}$), we arrive at the following system:

$$
(\mathcal{R}am)\quad
\begin{array}{ll}
\text{(i)} & \left(\frac{\partial}{\partial t}+c_g\frac{\partial}{\partial z}\right)E-\frac{ic}{2k_0}\Delta_\perp E+i\beta_0(\mathcal{N}-1)E=-\frac{\beta_0}{\omega_p}\mathcal{N}w\Phi e^{-i\rho t},\\
\text{(ii)} & \left(\frac{\partial}{\partial t}-c_g\frac{\partial}{\partial z}\right)\Phi-\frac{ic}{2k_0}\Delta_\perp\Phi+i\beta_0(\mathcal{N}-1)\Phi=\frac{\beta_0}{\omega_p}\mathcal{N}\overline{w}Ee^{i\rho t},\\
\text{(iii)} & \frac{\partial w}{\partial t}+3\lambda_D \mathrm{v}_{\mathrm{th,e}}K\frac{\partial w}{\partial z}+\frac{\nu_{ei}}{2}w+i\frac{\omega_p}{2}(\mathcal{N}-1)w=\frac{\gamma_p}{2}KE\overline{\Phi}e^{-i\rho t},\\
\text{(iv)} & \frac{\partial}{\partial t}\mathcal{N}+c_s\frac{\partial q}{\partial z}=0;\qquad \frac{\partial q}{\partial t}+c_s\frac{\partial\mathcal{N}}{\partial z}=-2\frac{Zm_e}{m_0}\frac{\partial^2}{\partial z^2}|w|^2.
\end{array}
$$

We conjecture that this system is well-posed (at least when $|w|^2$ is not too large).

Remark 34. If one wants a more realistic model, it is possible to account here for a ponderomotive force, due to coupling with the Brillouin backscattered wave, and to add on the right-hand side of the momentum equation a corresponding term; then one needs to introduce a new backscattered electromagnetic wave, let us say Ψ which solves a propagation equation

$$
\left(\frac{\partial}{\partial t}-c_g\frac{\partial}{\partial z}\right)\Psi-\frac{ic}{2k_0}\Delta_\perp\Psi+i\beta_0(1-\mathcal{N})\Psi=i\beta_0E\mathcal{N}e^{2ik_pz}
$$

and to replace system (iv) above by

$$
\frac{\partial}{\partial t}\mathcal{N}+c_s\frac{\partial q}{\partial z}=0;\qquad \frac{\partial q}{\partial t}+c_s\frac{\partial\mathcal{N}}{\partial z}=-2\frac{Zm_e}{m_0}\frac{\partial^2}{\partial z^2}|w|^2-\frac{\gamma_p}{c_s}(2ik_p\overline{\Psi}Ee^{2ik_pz}+c.c.)
$$

$\square$.

5.2 The Euler–Maxwell Model for Short Ultra-High Intensity Laser Pulses

In this section, we address briefly a special feature related to ultra-high intensity laser beam propagation in a hot plasma, where the typical intensity is larger than 10^{17} W/cm^2 for a wavelength in a vacuum equal to 1 micron (or a fraction of 1 micron). The laser pulse is also assumed to be very short: the corresponding observation time T_{obs} may be much smaller than a picosecond (recall that the time scale related to the laser period ω_0^{-1} is on the order of a femtosecond for the typical laser beam); but this observation time may range also up to some tens of picoseconds for some high-energy laser devices.

From a physical point of view, the interaction of this pulse with a preheated plasma creates a very intense electron plasma wave that perturbs the propagation of the pulse. This wake generates a non-neutral region, and a strong electrostatic field occurs that may accelerate light ions at a very high velocity. There are a lot of applications (see, e.g., [48, 57]) using such ion beams generated in this way (as in medical area, tomography, etc.).

In some applications, the plasma density may be initially larger than the critical density; then there is digging of the plasma by the laser pressure (which is in turn the same as the ponderomotive force explained above), which enables the propagation of the laser beam. So the plasma density is generally smaller than the critical density and in the same order of magnitude. Thus the plasma frequency ω_p^{-1} is on the order of magnitude of the laser frequency ω_0 (recall that $\omega_p/\omega_0 = \sqrt{N_e/N_c}$).

Then the picture is the following: for the observation time T_{obs},

$$\omega_0^{-1} \sim \frac{\lambda_D}{v_{\text{th,e}}} = \omega_p^{-1} \ll T_{\text{obs}} \sim \frac{L_{\text{obs}}}{c} \ll \frac{L_{\text{plas}}}{c}$$

moreover, the observation length L_{obs} is such that

$$\lambda_D \ll \frac{2\pi}{\omega_0} c \ll L_{\text{obs}} \ll L_{\text{plas}}$$

but the spatial domain of interest with the width L_{obs}, is often a moving frame.

As a matter of fact, the classical way to perform numerical simulations of the ultra-high intensity laser propagation is based on kinetic models of Maxwell–Vlasov type; but this kind of simulations in 3D is very expensive even on massively parallel architectures. So it may be interesting to address fluid simulations for comparison with kinetic ones and for performing simulation on long time intervals; see, e.g., [20].

This kind of model is also used for modelling other physical phenomena such as electromagnetic waves in semiconductor devices (see, e.g., [84]).

The framework of the modelling that we address now is the following: the electron population is characterized by a Maxwellian distribution and its evolution is described by the classical fluid equations. Moreover, for the sake of simplicity, the ion population is assumed to be at rest (but, of course, it would be possible to account for its evolution also by a coupling with a supplemented system) and we do not account for relativistic effects (as a matter of fact, it would be necessary to do this if the laser intensity was larger than $2.\ 10^{18}$ W/cm^2 for a laser wavelength equal to 1 μ m).

The model of interest consists of barotropic Euler equations (assuming that there is no coupling between ion and electron temperatures, and no thermal conduction) coupled with the full Maxwell equations.

$$\frac{\partial}{\partial t} N_e + \nabla.(N_e \mathbf{U}_e) = 0 \tag{5.35}$$

$$\frac{\partial}{\partial t}(N_e\mathbf{U}_e)+\nabla.(N_e\mathbf{U}_e\mathbf{U}_e)+C_P\nabla N_e^{5/3} = -N_e\frac{q_e}{m_e}(\mathbf{E}+\mathbf{U}_e\times\mathbf{B})-\nu_{ei}N_e\mathbf{U}_e. \quad (5.36)$$

$$\frac{\partial}{\partial t}\mathbf{E}-c^2\text{curl }\mathbf{B} = -\frac{1}{\varepsilon^0}\mathbf{J}, \quad (5.37)$$

$$\frac{\partial}{\partial t}\mathbf{B}+\text{curl }\mathbf{E} = 0, \quad (5.38)$$

$$\nabla.\mathbf{E} = \frac{q_e}{\varepsilon^0}(ZN_0-N_e), \qquad \text{and} \qquad \nabla.\mathbf{B} = 0. \quad (5.39)$$

with $\mathbf{J} = -q_eN_e\mathbf{U}_e$. Recall that if relations (5.39) are satisfied at the initial time, they are always satisfied.

As explained above, the hypothesis is that the pressure obeys simply the law $P_e = m_eC_PN_e^{5/3}$. Since, we have

$$\frac{1}{m_e}\nabla P_e = \frac{5P_e}{3m_eN_e}\nabla N_e$$

we check that the sound speed is equal to $c_e = (5P_e/(3m_eN_e))^{1/2}$.

There is a global energy balance, with the total energy $e_{\text{int}}(N_e)+\frac{1}{2}m_eN_e|\mathbf{U}_e|^2+\frac{\varepsilon^0}{2}|\mathbf{E}|^2+\frac{\mu_0}{2}|\mathbf{B}|^2$, where $e_{\text{int}}(.)$ corresponds to an internal energy of the plasma.

Remark 35. For the sake of completeness, let us mention that, instead of the barotropic Euler equations, it is possible to deal with the full Euler equations

$$\frac{\partial}{\partial t}N_e+\nabla.(N_e\mathbf{U}_e) = 0,$$

$$\frac{\partial}{\partial t}N_e\mathbf{U}_e+\nabla.(N_e\mathbf{U}_e\mathbf{U}_e)+\frac{1}{m_e}\nabla P_e = -N_e\frac{q_e}{m_e}(\mathbf{E}+\mathbf{U}_e\times\mathbf{B})-\nu_{ei}N_e\mathbf{U}_e.$$

$$\frac{\partial}{\partial t}\mathcal{E}_e+\nabla.(\mathcal{E}_e\mathbf{U}_e)+P_e\nabla.\mathbf{U}_e = -\nabla.\mathbf{q}_{\text{th,e}}+m_e\nu_{ei}N_e|\mathbf{U}_e|^2.$$

Then, internal energy $\mathcal{E}_e$ is proportional to the electron temperature (and there is a global energy balance for the total energy that is $\mathcal{E}_e+\frac{1}{2}m_eN_e|\mathbf{U}_e|^2+\frac{\varepsilon^0}{2}|\mathbf{E}|^2+\frac{\mu_0}{2}|\mathbf{B}|^2$). As a matter of fact, most of the properties of the barotropic Euler–Maxwell model remain true for the full Euler–Maxwell model.

Notice also that if one wants to account for relativistic effects, the left-hand side of equation (5.36) needs to be replaced by $(\frac{\partial}{\partial t}N_e\mathbf{Q}_e+\nabla.(N_e\mathbf{Q}_e\mathbf{U}_e))\frac{1}{m_e}+C_P\nabla N_e^{5/3}$ knowing that the momentum $\mathbf{Q}_e$ is related to the velocity by the relation $\mathbf{U}_e = \mathbf{Q}_e/m_e[1+\mathbf{Q}_e^2/(m_e^2c^2)]^{-1/2}$. □

5.2.1 Well-Posedness of the Model

The aim is to analyze the existence and uniqueness of the barotropic Euler–Maxwell system, and to state precisely some boundary conditions to be imposed for such a model. The time-local existence and uniqueness was first proved by Jerome [70] for the full compressible Euler–Maxwell system with regular initial value. We give here the main ideas related to this system in the barotropic framework. We do not worry about boundary conditions, which will be addressed in the following.

Let $\mathbf{W}$ denote the vector function from $\mathbf{R}^3$ in $\mathbf{R}^{10}$ defined by

$$\mathbf{W} = \begin{pmatrix} N_e \\ m_e N_e \mathbf{U}_e \\ \mathbf{E} \\ \mathbf{B} \end{pmatrix}.$$

So, using a reference frame $\mathbf{x} = (x_1, x_2, x_3)$ and denoting $\mathbf{U}_e = (U_1, U_2, U_3)$ the system reads as

$$\partial_t \mathbf{W} + \frac{\partial}{\partial x_1} \begin{pmatrix} N_e U_1 \\ m_e N_e U_1 \mathbf{U}_e + \mathbf{I}_1 P_e \\ c^2 \mathrm{S}^1 \mathbf{B} \\ \mathrm{S}^{1\mathrm{T}} \mathbf{E} \end{pmatrix} + \frac{\partial}{\partial x_2} \begin{pmatrix} N_e U_2 \\ m_e N_e U_2 \mathbf{U}_e + \mathbf{I}_2 P_e \\ c^2 \mathrm{S}^2 \mathbf{B} \\ \mathrm{S}^{2\mathrm{T}} \mathbf{E} \end{pmatrix} \tag{5.40}$$

$$+ \frac{\partial}{\partial x_3} \begin{pmatrix} N_e U_3 \\ m_e N_e U_3 \mathbf{U}_e + \mathbf{I}_3 P_e \\ c^2 \mathrm{S}^3 \mathbf{B} \\ \mathrm{S}^{3\mathrm{T}} \mathbf{E} \end{pmatrix} = \mathbf{G}_0(\mathbf{W}),$$

where we have used the following notations $\mathbf{I}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{I}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{I}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and the elementary 3×3 matrices

$$\mathrm{S}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \mathrm{S}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathrm{S}^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, the zero-order term is the following:

$$\mathbf{G}_0(\mathbf{W}) = \begin{pmatrix} 0 \\ -N_e q_e(\mathbf{E} + \mathbf{U}_e \times \mathbf{B}) - \nu_{ei} m_e N_e \mathbf{U}_e \\ \frac{1}{\varepsilon^0} q_e N_e \mathbf{U}_e \\ 0 \end{pmatrix}$$

Symmetrization

Let us first recall a classical way to symmetrize the barotropic Euler system. Instead of $(N_e, N_e\mathbf{U}_e)$, we use the unknowns $(P_e, \mathbf{U}_e)$. Since there exists C_N such that

$$m_e N_e = C_N P_e^{3/5},$$

the barotropic Euler system may be recast as

$$\frac{3}{5P_e}\frac{\partial}{\partial t}P_e + \frac{3}{5P_e}\mathbf{U}_e.\nabla P_e + \nabla.\mathbf{U}_e = 0,$$
$$C_N P_e^{3/5}\frac{\partial}{\partial t}\mathbf{U}_e + C_N P_e^{3/5}\mathbf{U}_e.\nabla\mathbf{U}_e + \nabla P_e = g,$$

denoting g as the second line of $\mathbf{G}_0(\mathbf{W})$; i.e.,

$$\mathbb{M}_0\begin{pmatrix} P_e \\ \mathbf{U}_e \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} P_e \\ \mathbf{U}_e \end{pmatrix} + \begin{pmatrix} \frac{3}{5P_e}\mathbf{U}_e.\nabla P_e & +\nabla.\mathbf{U}_e \\ \nabla P_e & +C_N P_e^{3/5}\mathbf{U}_e + \nabla P_e \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix},$$

defining the matrix $\mathbb{M}_0$ by

$$\mathbb{M}_0\begin{pmatrix} P_e \\ \mathbf{U}_e \end{pmatrix} = \begin{pmatrix} \frac{3}{5}P_e^{-1} & 0 \\ 0 & C_N P_e^{3/5}\mathbb{I} \end{pmatrix}.$$

Let us make this structure clearer by using the reference frame (x_1, x_2, x_3). In this frame, the system reads as

$$\mathbb{M}_0\begin{pmatrix} P_e \\ \mathbf{U}_e \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} P_e \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} + \mathbb{M}_1\begin{pmatrix} P_e \\ \mathbf{U}_e \end{pmatrix}\frac{\partial}{\partial x_1}\begin{pmatrix} P_e \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} + \mathbb{M}_2\begin{pmatrix} P_e \\ \mathbf{U}_e \end{pmatrix}\frac{\partial}{\partial x_2}\begin{pmatrix} P_e \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} + \mathbb{M}_3\begin{pmatrix} P_e \\ \mathbf{U}_e \end{pmatrix}\frac{\partial}{\partial x_3}\begin{pmatrix} P_e \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

where the three 4 × 4 matrices $\mathbb{M}_i(.)$ are given by

$$\mathbb{M}_1 = \begin{pmatrix} \frac{3U_1}{5P_e} & 1 & 0 & 0 \\ 1 & C_N P_e^{\frac{3}{5}} U_1 & 0 & 0 \\ 0 & 0 & C_N P_e^{\frac{3}{5}} U_1 & 0 \\ 0 & 0 & 0 & C_N P_e^{\frac{3}{5}} U_1 \end{pmatrix}, \quad \mathbb{M}_2 = \begin{pmatrix} \frac{3U_2}{5P_e} & 0 & 1 & 0 \\ 0 & C_N P_e^{\frac{3}{5}} U_2 & 0 & 0 \\ 1 & 0 & C_N P_e^{\frac{3}{5}} U_2 & 0 \\ 0 & 0 & 0 & C_N P_e^{\frac{3}{5}} U_2 \end{pmatrix}$$

$$\mathbb{M}_3 = \begin{pmatrix} \frac{3U_3}{5P_e} & 0 & 0 & 1 \\ 0 & C_N P_e^{\frac{3}{5}} U_3 & 0 & 0 \\ 0 & 0 & C_N P_e^{\frac{3}{5}} U_3 & 0 \\ 1 & 0 & 0 & C_N P_e^{\frac{3}{5}} U_3 \end{pmatrix}$$

They are symmetric. Denote now

$$\mathbf{Z} = \begin{pmatrix} P_e \\ \mathbf{U}_e \\ \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

Then, the Euler–Maxwell system reads as

$$\mathbb{A}_0 \frac{\partial}{\partial t}\mathbf{Z} + \mathbb{A}_1 . \frac{\partial}{\partial x_1}\mathbf{Z} + \mathbb{A}_2 . \frac{\partial}{\partial x_2}\mathbf{Z} + \mathbb{A}_3 . \frac{\partial}{\partial x_3}\mathbf{Z} = \mathbf{G}(\mathbf{Z}). \tag{5.41}$$

where

$$\mathbb{A}_1 = \begin{pmatrix} \mathbb{M}_1 & 0 & 0 \\ 0 & 0 & \mathrm{S}^1 \\ 0 & \mathrm{S}^{1\mathrm{T}} & 0 \end{pmatrix}, \quad \mathbb{A}_2 = \begin{pmatrix} \mathbb{M}_2 & 0 & 0 \\ 0 & 0 & \mathrm{S}^2 \\ 0 & \mathrm{S}^{2\mathrm{T}} & 0 \end{pmatrix}, \quad \mathbb{A}_3 = \begin{pmatrix} \mathbb{M}_{33} & 0 & 0 \\ 0 & 0 & \mathrm{S}^3 \\ 0 & \mathrm{S}^{3\mathrm{T}} & 0 \end{pmatrix},$$

$$\mathbb{A}_0 = \begin{pmatrix} \mathbb{M}_0 & 0 & 0 \\ 0 & \frac{1}{c^2}\mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{pmatrix}, \qquad \mathbf{G}(\mathbf{Z}) = \begin{pmatrix} 0 \\ -C_N P_e^{3/5} \frac{q_e}{m_e}(\mathbf{E} + \mathbf{U}_e \times \mathbf{B}) - \nu_{ei} C_N P_e^{\frac{3}{5}} \mathbf{U}_e \\ \mu^0 \frac{q_e}{m_e} C_N P_e^{\frac{3}{5}} \mathbf{U}_e \\ 0 \end{pmatrix}.$$

Of course, this system must be supplemented with an initial value

$$\mathbf{Z}(0) = \mathbf{Z}^{\text{ini}} = \begin{pmatrix} P_e^{\text{ini}} \\ \mathbf{U}_e^{\text{ini}} \\ \mathbf{E}^{\text{ini}} \\ \mathbf{B}^{\text{ini}} \end{pmatrix}$$

such that

$$\nabla.\mathbf{E}^{\text{ini}} = \frac{q_e}{\varepsilon^0}(ZN_0 - C_N P_e^{\text{ini}}), \qquad \text{and} \qquad \nabla.\mathbf{B}^{\text{ini}} = 0. \tag{5.42}$$

The four matrices (of order 10×10) that appear in the system (5.41) are symmetric. In the sequel, we assume that there exists a constant $P_{\text{inf}} > 0$ such that

$$P_{\text{inf}} \leq P_e^{\text{ini}} \leq P_{\text{sup}} \tag{5.43}$$

Lemma 9. *The matrices $\mathbb{M}_1\mathbb{M}_0^{-1}$ is diagonalizable with eigenvalues $(U_1, U_1, U_1 - c_e, U_1 + c_e)$ and the analogue for $\mathbb{M}_2\mathbb{M}_0^{-1}$ and for $\mathbb{M}_3\mathbb{M}_0^{-1}$*

Thus, the matrix $\mathbb{A}_1\mathbb{A}_0^{-1}$ is also diagonalizable and its eigenvalues are

$$U_1, U_1, U_1 - c_e, U_1 + c_e, -c, -c, 0, 0, c, c.$$

Notice that accounting for the constraints (5.39), it is natural that the solution to equation (5.41) belongs to a space of dimension 8 only, and as a matter of fact the two eigenvalues equal to 0 do not correspond to physical waves. In the same way, the matrices $\mathbb{A}_2\mathbb{A}_0^{-1}$ and $\mathbb{A}_3\mathbb{A}_0^{-1}$ are also diagonalizable and have analogous eigenvalues. Then, equation (5.41) is a nonlinear symmetric hyperbolic system in the Friedrichs' meaning. So, we have

Theorem 5. *Assume that there exists a constant P_∞ such that $(P_e^{ini} - P_\infty, \mathbf{U}_e^{ini}, \mathbf{E}^{ini}, \mathbf{B}^{ini})$ belongs to $[H^3(\mathbf{R}^3)]^{10}$ and that (5.42) and (5.43) hold.*

Then for a small time t_0, there exists a unique solution $\mathbf{Z}(t)$ to equation (5.41) such that $(P_e - P_\infty, \mathbf{U}_e, \mathbf{E}, \mathbf{B})$ belongs to the space $C(0, t_0, [H^3(\mathbf{R}^3)]^{10})$. Moreover, this solution is in $C(0, t_0, [L^\infty(\mathbf{R}^3)]^{10})$ and relations (5.39), (5.43) hold for all time t in $[0, t_0]$.

Remark 36. Recall that the classical Moorey's inequality (which is of the same type as Sobolev inequalities) claims that

$$\|z\|_{L^\infty(\mathbf{R}^3)} \leq C_{\text{Mo}} \|z\|_{H^2(\mathbf{R}^3)}, \qquad \forall z \in H^2(\mathbf{R}^3)$$

(with a universal constant C_{Mo}); thus, we see that if $\mathbf{Z}(t)$ is continuous from $[0, t_0]$ into $[H^3(\mathbf{R}^3)]^{10}$ it is also continuous from $[0, t_0]$ into $[L^\infty(\mathbf{R}^3)]^{10}$. □

Remark 37. Notice that if there is no electromagnetic source and if the magnetic effects are negligible, the Euler–Maxwell system becomes simply the Euler–Poisson system; that is to say

$$\mathbf{E} = -\frac{1}{q_e}\nabla\Phi, \quad -\Delta\Phi = \frac{q_e^2}{\epsilon^0}(ZN_0 - N_e)$$

which is a classical one for which we have also existence and uniqueness of a regular solution on a small time interval; see, e.g., [55]. □

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Proofs of the Sub-section

Proof of Lemma 9. We see that

$$\mathbb{M}_1\mathbb{M}_0^{-1}\binom{P_e}{\mathbf{U}_e} = \begin{pmatrix} U_1 & (m_eN_e)^{-1} & 0 & 0 \\ \frac{5}{3}P_e & U_1 & 0 & 0 \\ 0 & 0 & U_1 & 0 \\ 0 & 0 & 0 & U_1. \end{pmatrix}$$

then the eigenvaluesx are given by $U_1, U_1, U_1+\sqrt{\frac{5}{3}P_e/(m_eN_e)}, U_1-\sqrt{\frac{5}{3}P_e/(m_eN_e)}$. And by a classical way, we see that the matrix is diagonalizable. □

Sketch of the Proof of Theorem 5

We follow the proof given in [70]; the idea is to apply the general result for quasilinear symmetric hyperbolic systems based on the nonlinear operator semi-group theory of Kato in the spirit of [73]. Denote $R = \|Z^{\text{ini}}\|_{(L^\infty)^{10}}$. First, for the matrices occurring in (5.41) it is easy to check the following properties. For all $\mathbf{Z} = (P_e, \mathbf{U}_e, \mathbf{E}, \mathbf{B})$ and $\mathbf{Z}' = (P_e', \mathbf{U}_e', \mathbf{E}', \mathbf{B}')$ such that $|\mathbf{Z}|_{\mathbf{R}^{10}} \le 2R$, $|\mathbf{Z}'|_{\mathbf{R}^{10}} \le 2R$ and $P_e, P_e' \ge P_{\text{inf}}/3$, we have the following:

(i) The matrices $\mathbb{A}_i(\mathbf{Z})$ for $i = 0, 1, 2, 3$ are symmetric; moreover, the matrix $\mathbb{A}_0(\mathbf{Z})$ is definite-positive, $(\mathbb{A}_0(\mathbf{Z}))^{-1}$ is uniformly bounded.

(ii) The matrix functions $\mathbb{A}_i(\mathbf{Z})$ are bounded in the space $\mathcal{L}(\mathbf{R}^{10}, \mathbf{R}^{10})$ and depend smoothly on $\mathbf{Z}$ (in a differential way); there exists a constant C_R (depending on R and P_{inf}) such that

$$|\mathbb{A}_i(\mathbf{Z}) - \mathbb{A}_i(\mathbf{Z}')|_{\mathcal{L}(\mathbf{R}^{10},\mathbf{R}^{10})} \le C_R|\mathbf{Z} - \mathbf{Z}'|_{\mathbf{R}^{10}}$$

(iii) The vector function $\mathbf{G}(\mathbf{Z})$ depends smoothly on $\mathbf{Z}$ (in a differential way) and

$$|\mathbf{G}(\mathbf{Z}) - \mathbf{G}(\mathbf{Z}')|_{\mathbf{R}^{10}} \le C_R|\mathbf{Z} - \mathbf{Z}'|_{\mathbf{R}^{10}}, \tag{5.44}$$

Existence. First, in matrices $\mathbb{A}_i$, we replace the function P_e by $\widetilde{P_e}(P_e)$, where $\widetilde{P_e}$ is a smooth function such that

$$\begin{aligned}\widetilde{P_e}(P) &= P \qquad \text{if } P > P_{\text{inf}}/2 \\ &= P_{\text{inf}}/3 \qquad \text{if } P < P_{\text{inf}}/3\end{aligned}$$

Due to the above properties and the fact that $\widetilde{P_e}(P_e)$ is bounded from below, according to the general theory [70], there exists a solution $\mathbf{Z}(t)$ on a small time interval such that $(P_e - P_\infty, \mathbf{U}_e, \mathbf{E}, \mathbf{B})$ is continuous from $[0, t_0]$ in the space $(H^3(\mathbf{R}^3))^{10}$; i.e., we have

$$\mathbf{Z}(t) - \mathbf{Z}(0) + \int_0^t \left(\mathbb{A}_0^{-1}\mathbb{A}_1.\frac{\partial \mathbf{Z}}{\partial x_1}(t') + \mathbb{A}_0^{-1}\mathbb{A}_2.\frac{\partial \mathbf{Z}}{\partial x_2}(t') + \mathbb{A}_0^{-1}\mathbb{A}_3.\frac{\partial \mathbf{Z}}{\partial x_3}(t') \right) dt' = \int_0^t \mathbb{A}_0^{-1}\mathbf{G}(\mathbf{Z})(t')dt'.$$

Now, according to Moorey's inequality, we see that the functions $\mathbf{Z}(.)$ and $\frac{\partial}{\partial x_i}\mathbf{Z}(.)$ are continuous in $(L^\infty(\mathbf{R}^3))^{10}$, thus the previous relation yields $\|\mathbf{Z}(t) - \mathbf{Z}(0)\|_{(L^\infty)^{10}} \leq Ct$. That is to say we get for a small time t_1

$$\|\mathbf{Z}(t)\|_{(L^\infty)^{10}} \leq 2R, \qquad \inf_x \widetilde{P_e}(P_e(t,x)) \geq P_{\text{inf}}/2, \quad \forall t \leq t_1, \tag{5.45}$$

and we have $\widetilde{P_e}(P_e(t,.)) = P_e(t,.)$ for $t \leq t_1$.

Let us now focus on the *uniqueness*.

Assume that there exist two solutions $\mathbf{Z}$ and $\mathbf{Z}_1 = \mathbf{Z} + \mathbf{Y}$ that are continuous from $[0, t_0]$ in $[H^3(\mathbf{R}^3)]^{10}$; thus, they satisfy (5.45). In fact, we will prove the stability in L^2 with respect to the initial values and we want to show that on a small time interval $[0, t_1]$ with initial value $\mathbf{Z}^{\text{ini}}$ and $\mathbf{Z}_1^{\text{ini}}$, there exists a constant $C_*(t_1)$ depending on t_1 such that

$$\|\mathbf{Z}(t) - \mathbf{Z}_1(t)\|_{L^2} \leq C_*(t_1) \left\|\mathbf{Z}^{\text{ini}} - \mathbf{Z}_1^{\text{ini}}\right\|_{L^2}, \quad \forall t \leq t_1. \tag{5.46}$$

By combining the two equations satisfied by $\mathbf{Z}$ and $\mathbf{Z} + \mathbf{Y}$, denoting $\mathbb{A}_{i0} = \mathbb{A}_i(\mathbf{Z} + \mathbf{Y})$,we get

$$\begin{aligned}\mathbb{A}_{00}\frac{\partial}{\partial t}\mathbf{Y} + \mathbb{A}_{10}.\frac{\partial}{\partial x_1}\mathbf{Y} + \mathbb{A}_{20}.\frac{\partial}{\partial x_2}\mathbf{Y} + \mathbb{A}_{30}.\frac{\partial}{\partial x_3}\mathbf{Y} = \mathbf{G}(\mathbf{Z} + \mathbf{Y}) - \mathbf{G}(\mathbf{Z}) \\ -\mathbb{A}_0^D(\mathbf{Y})\frac{\partial}{\partial t}\mathbf{Z} - \mathbb{A}_1^D(\mathbf{Y}).\frac{\partial}{\partial x_1}\mathbf{Z} - \mathbb{A}_2^D(\mathbf{Y}).\frac{\partial}{\partial x_2}\mathbf{Z} - \mathbb{A}_3^D(\mathbf{Y}).\frac{\partial}{\partial x_3}\mathbf{Z},\end{aligned}$$

where $\mathbb{A}_i^D(\mathbf{Y}) = \mathbb{A}_i(\mathbf{Z} + \mathbf{Y}) - \mathbb{A}_i(\mathbf{Z})$ and $|\mathbb{A}_i^D(\mathbf{Y})| \leq C_R|\mathbf{Y}|$. Now, we multiply the right-hand side of the previous equation by $\mathbf{Y}$ and integrate over $\mathbf{R}^3$, since $2\left\langle \mathbb{A}_*\frac{\partial}{\partial t}\mathbf{Y}, \mathbf{Y}\right\rangle = \frac{\partial}{\partial t}\langle \mathbb{A}_*\mathbf{Y}, \mathbf{Y}\rangle - \left\langle (\frac{\partial}{\partial t}\mathbb{A}_*)\mathbf{Y}, \mathbf{Y}\right\rangle$ and the analogue for $\frac{\partial}{\partial x}$, we get

$$2\frac{\partial}{\partial t}\langle \mathbb{A}_{00}\mathbf{Y},\mathbf{Y}\rangle - \left\langle (\frac{\partial}{\partial t}\mathbb{A}_{00})\mathbf{Y},\mathbf{Y}\right\rangle - \left\langle (\frac{\partial}{\partial x_1}\mathbb{A}_{10})\mathbf{Y},\mathbf{Y}\right\rangle - \left\langle (\frac{\partial}{\partial x_2}\mathbb{A}_{20})\mathbf{Y},\mathbf{Y}\right\rangle - \left\langle (\frac{\partial}{\partial x_3}\mathbb{A}_{30})\mathbf{Y},\mathbf{Y}\right\rangle$$

$$\leq 2\langle \mathbf{G}(\mathbf{Z}+\mathbf{Y}) - \mathbf{G}(\mathbf{Z}),\mathbf{Y}\rangle + C\,\|\mathbf{Y}\|_{L^2}^2 \left\|\frac{\partial \mathbf{Z}}{\partial t}\right\|_\infty + C\,\|\mathbf{Y}\|_{L^2}^2 \left\|\frac{\partial \mathbf{Z}}{\partial x_i}\right\|_\infty .$$

We know that $\frac{\partial}{\partial t}\mathbf{Z}$, $\frac{\partial}{\partial x_1}\mathbf{Z}$, $\frac{\partial}{\partial x_2}\mathbf{Z}$, and $\frac{\partial}{\partial x_3}\mathbf{Z}$ are bounded in $(H^2(\mathbf{R}^3))^{10}$; thus, according to Moorey's inequality, they are bounded in $(L^\infty(\mathbf{R}^3))^{10}$. According to (5.44), we see that $\|\mathbf{G}(\mathbf{Z}+\mathbf{Y}) - \mathbf{G}(\mathbf{Z})(t)\|_{L^2}$ is bounded by $C_R\,\|\mathbf{Y}(t)\|_{L^2}$; therefore, we get for a constant C_3

$$\frac{\partial}{\partial t}\langle \mathbb{A}_{00}\mathbf{Y}(t),\mathbf{Y}(t)\rangle \leq C_3\,\|\mathbf{Y}(t)\|_{L^2}^2 .$$

Then, due to the property of $\mathbb{A}_{00}$, we see that $\|\mathbf{Y}\|_{L^2}^2 \leq C_4\langle \mathbb{A}_{00}\mathbf{Y},\mathbf{Y}\rangle$ and we get $C_4^{-1}\,\|\mathbf{Y}(t)\|_{L^2}^2 \leq \|\mathbb{A}_{00}\|_\infty\,\|\mathbf{Y}(0)\|_{L^2}^2 + C_3\int_0^t \|\mathbf{Y}(s)\|_{L^2}^2\,ds$; therefore, inequality (5.46) follows according to Gronwall's lemma (see result 7 in the Appendix). □

5.2.2 Boundary Conditions

In order to fix the ideas, we address a classical situation where the simulation domain $\mathcal{O}$ is a slab, let's say $x_3 \in [0, x_{\max}]$ or a parallelepiped $(x_1, x_2, x_3) \in \mathcal{R}\times[0, x_{\max}]$ where $\mathcal{R}$ is a rectangle, and assume that a laser beam enters in domain $\mathcal{O}$ with a normal incidence through the boundary Γ corresponding to $x_3 = 0$. Denote $\mathbf{n}$ by the outwards normal vector.

Before dealing with the electromagnetic part, let us address the fluid part of the full system. For the sake of simplicity, we assume that the flow is subsonic. Then, if we are in a one-dimensional framework, one has to give only one boundary condition on each face of the spatial domain. The simplest one is to impose the value of the density

$$N_e = N_{\text{ref}}.$$

If we are in a three-dimensional framework, then we also need to impose boundary conditions (e.g., of the type $\frac{\partial}{\partial \mathbf{n}}U_2 = \frac{\partial}{\partial \mathbf{n}}U_3 = 0$) on the faces of domain where $U_1 > 0$.

In the sequel we assume that $N_{\text{ref}} < N_c$ so $\omega_p/\omega_0 = \sqrt{N_{\text{ref}}/N_c} < 1$.

Let us address now the conditions for the electromagnetic part (5.37) and (5.38). First of all, since the eigenvalues of the one-dimensional system (related to the direction $\mathbf{n}$) are $(-c, -c, 0, 0, c, c)$, there are only two eigenvalues with the sign corresponding to an incoming wave; thus, one must impose only two scalar boundary conditions on each boundary.

Recall the features related a monochromatic plane wave in a linear medium explained in Remark 13 of Chap. 3, Sect. 3.1.1. We may build **E** and **B** solutions of (5.37) and (5.38) in a neighborhood of the boundary where N_e is equal to N_{ref} by the following way: we set $\mathbf{E}(t,\mathbf{x}) = \mathbf{E}'(\mathbf{x})e^{-i\omega_0 t} + c.c.$, $\mathbf{B}(t,\mathbf{x}) = \mathbf{B}'(\mathbf{x})e^{-i\omega_0 t} + c.c.$ and $\mathbf{J}(t,\mathbf{x}) = \mathbf{J}'(\mathbf{x})e^{-i\omega_0 t} + c.c.$ where $\mathbf{J}' = i\frac{\omega_p^2}{\omega_0}\mathbf{E}'$ (neglecting the absorption phenomenon), then $(\mathbf{E}',\mathbf{B}')$ satisfies $-i\omega_0(1-\frac{\omega_p^2}{\omega_0^2})\mathbf{E}' - c^2\text{curl }\mathbf{B}' = 0$ and $-i\omega_0\mathbf{B}' + \text{curl }\mathbf{E}' = 0$; the wave is characterized by the wave vector

$$\mathbf{k} = -\mathbf{n}\sqrt{\omega_0^2 - \omega_p^2}/c$$

and introducing a fixed vector **f** orthogonal to **k**, the electric and magnetic fields are given by

$$\mathbf{E}(t,\mathbf{x}) = e^{-i\omega_0 t}\mathbf{f}e^{i\mathbf{k}.\mathbf{x}} + c.c. \quad \text{and} \quad \omega_0\mathbf{B}(t,\mathbf{x}) = \mathbf{k}\times\mathbf{E}'(\mathbf{x})e^{-i\omega_0 t} = \mathbf{k}\times\mathbf{E}(t,\mathbf{x}). \tag{5.47}$$

Now, before dealing with the case of the incoming laser beam, let us recall a classical result for the boundary condition of the outgoing-wave type. We have the classical result.

Lemma 10. *For system (5.37) and (5.38) in a domain $\mathcal{O}$, the following boundary condition of outgoing-wave type is well-posed,*

$$\mathbf{E}\times\mathbf{n} - c(\mathbf{B}\times\mathbf{n})\times\mathbf{n} = 0 \tag{5.48}$$

and the outgoing energy density on $\partial\mathcal{O}$ is proportional to

$$|\mathbf{E}\times\mathbf{n}|^2$$

[**Indeed,** Remind the classical balance of the electromagnetic energy for any domain $\mathcal{O}$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{O}}(|\mathbf{E}|^2 + |\mathbf{B}|^2)d\mathbf{x} - \int_{\mathcal{O}}\mathbf{J}.\mathbf{E}d\mathbf{x} = -\int_{\mathcal{O}}\nabla.(\mathbf{E}\times\mathbf{B})d\mathbf{x} = \int_{\partial\mathcal{O}}(\mathbf{B}\times\mathbf{E}).\mathbf{n}d\Gamma$$

but one knows that $(\mathbf{B}\times\mathbf{E}).\mathbf{n} = -(\mathbf{E}\times\mathbf{n}).((\mathbf{B}\times\mathbf{n})\times\mathbf{n})$; then according to (5.48), we check that the last member of the previous balance relation is equal to $-\int_{\partial\mathcal{O}}\frac{1}{c}|\mathbf{E}\times\mathbf{n}|^2 d\Gamma$. □]

Of course, there are other outgoing-wave boundary conditions that are more accurate, but condition (5.48) is the simplest one. Notice that the vectors $\mathbf{E}\times\mathbf{n}$ and $(\mathbf{B}\times\mathbf{n})\times\mathbf{n}$ belong to the plane tangent to the boundary, so condition (5.48) corresponds in fact to two scalar boundary conditions.

Let us now go back to the case of an incoming laser beam that enters in Γ (i.e., the plane $x_3 = 0$). We have seen that the incoming wave needs to correspond to the wave vector $\mathbf{k} = -\mathbf{n}\sqrt{\omega_0^2 - \omega_p^2}/c$; we now assume that it is given by

$$\mathbf{E}^{\text{in}}(t, x_1, x_2, x_3) = \mathbf{f}(x_1, x_2)e^{-i\omega_0 t + i\mathbf{k}.\mathbf{x}} + c.c. = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ 0 \end{pmatrix} e^{-i\omega_0 t + i\mathbf{k}.\mathbf{x}} + c.c.$$

where the vector function $\mathbf{f}$ is slowly varying with respect to the space variable. At first order, relation (5.47) is still valid and we can use the approximation

$$c\mathbf{B}^{\text{in}}(t, x_1, x_2, x_3) \simeq \mathbf{f} \times \mathbf{n}\sqrt{1 - \frac{N_{\text{ref}}}{N_c}} e^{-i\omega_0 t + i\mathbf{k}.\mathbf{x}} + c.c. = \sqrt{1 - \frac{N_{\text{ref}}}{N_c}} \mathbf{E}^{\text{in}} \times \mathbf{n} + c.c.$$

As a matter of fact, the incoming-wave boundary condition on Γ is related to an outgoing-wave boundary condition for $\mathbf{E} - \mathbf{E}^{\text{in}}$ based on (5.48). Therefore, the boundary condition for systems (5.37) and (5.38) reads as follows (see, e.g., [49])

$$(\mathbf{E} - c(\mathbf{B} \times \mathbf{n})) \times \mathbf{n} = \left[\mathbf{E}^{\text{in}} - \sqrt{1 - \frac{N_{\text{ref}}}{N_c}} (\mathbf{E}^{\text{in}} \times \mathbf{n}) \times \mathbf{n} \right] \times \mathbf{n} = (1 + \sqrt{1 - \frac{N_{\text{ref}}}{N_c}}) \mathbf{f} \times \mathbf{n} e^{-i\omega_0 t + i\mathbf{k}.\mathbf{x}} + c.c.$$

On the other parts of the boundary of the domain $\mathcal{O}$, the boundary conditions for the electromagnetic part may be stated as (5.48) or by transparent boundary conditions, as was explained in Chap. 3.

5.3 Envelope Models for Very Short High-Intensity Laser Pulses

We are interested here in the propagation of very short intense laser pulses in weakly ionized media such as the atmosphere; it turns out that this kind of laser pulse may propagate a very long distance in weakly dense ionized gas. In this modelling, besides the classical framework of the propagation of strong electromagnetic fields which was the topic of the previous section, one must account for ionization phenomena of the medium and for the response of bounded electrons of the medium driven by the laser radiation. Here, for the sake of presentation, we deal with a model with characteristics that are representative of the realistic one, but which is deeply simplified; the aim is to give the ideas of the envelope approximation used in such modelling. So we consider the Maxwell equations with an electric current given by a simple formula like the one used in Chap. 3, Sect. 3.1.1 and we account for a supplementary term corresponding to the response of the medium which is simplified; it is a simple linear relation in Fourier space (for the realistic model, it needs to be nonlinear). See [12] for a review paper on this subject.

So, the picture is the following. The ratio of the time duration of pulse T_{pulse} and the laser period ω_p^{-1} may be only on the order of some hundreds, but also sometimes smaller and the pulse length L_{pulse} is on the order of cT_{pulse},but the observation time T_{obs} is much larger than T_{pulse} . Moreover we have

$$\omega_0^{-1} \sim \frac{\lambda_D}{\mathrm{v}_{\text{th,e}}} = \omega_p^{-1} \ll T_{\text{pulse}} \sim \frac{L_{\text{pulse}}}{c}$$

$$\lambda_D \ll \frac{2\pi}{\omega_0} c \ll L_{\text{pulse}} \ll L_{\text{obs}}.$$

Let us state now the basis system. The simulation domain $\mathcal{O}$ is, as above, either a slab $x_3 \in [0, x_{\max}]$ or a parallelepiped $(x_1, x_2, x_3) \in \mathcal{R} \times [0, x_{\max}]$ (with $\mathcal{R}$ a rectangle). The laser beam enters in $\mathcal{O}$ with a normal incidence through the boundary Γ corresponding to $x_3 = 0$. In the sequel we write z instead of x_3 and $x_\perp$ for the transverse variables (x_1, x_2).

From the theoretical point of view, the laser fields obey the Maxwell equations

$$\frac{\partial}{\partial t}\mathbf{E} - c^2 \text{curl } \mathbf{B} = -\frac{1}{\varepsilon^0}(\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t})$$

$$\frac{\partial}{\partial t}\mathbf{B} + \text{curl } \mathbf{E} = 0$$

where $\mathbf{J} = -q_e N_e \mathbf{U}_e$ and $\mathbf{P}$ denotes the polarization vector; its relation with field $\mathbf{E}$ is given below. Instead of the electron velocity equation (5.36), we state as usual

$$\frac{\partial}{\partial t}\mathbf{J} + \nu_{ei}\mathbf{J} = \frac{q_e^2}{m_e} N_e \mathbf{E} \tag{5.49}$$

and for the sake of simplicity the initial electron density N_e is assumed to be a datum. The laser pulse is centered around a central frequency ω_0, but there is broadness in the frequency domain, so we need to handle the equation in the Fourier domain. Moreover, as in Chap. 3, Sect. 3.1.1, we assume that the polarization of the electromagnetic wave is linear, i.e., fields $\mathbf{E}$, $\mathbf{J}$ and $\mathbf{P}$ may reduce to scalar functions E, J and P.

So we get

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E - \Delta E = -\frac{1}{c^2\varepsilon^0}(\frac{\partial J}{\partial t} + \frac{\partial^2 P}{\partial t^2})$$

By using the Fourier transform, we denote

$$E(t,x) = \int \hat{E}(\omega,x)e^{-i\omega t}d\omega, \quad J(t,x) = \int \hat{J}(\omega,x)e^{-i\omega t}d\omega, \quad P(t,x) = \int \hat{P}(\omega,x)e^{-i\omega t}d\omega$$

Now, the polarization vector is related to the electric field by a linear relation

$$\hat{P}(\omega) = \varepsilon^0 \chi(\omega)\hat{E}(\omega),$$

where the susceptibility $\chi(\omega)$ is a real function describing the response of the population of bounded electrons, i.e., we neglect the nonlinear effect (this is, of course, very crude, but it shows the main ideas of the time envelope technique).

Our aim is to describe very briefly an envelope approximation of this model as it is used in the [12] (for another approach of envelope approximation, see also [114]).

First, due to (5.49), we have

$$(\nu_{ei} - i\omega)\hat{J} = \frac{q_e^2}{m_e} N_e \hat{E}$$

and $\hat{E}$ satisfies for all ω the Helmholtz equation

$$\frac{\omega^2}{c^2}\hat{E} + \Delta\hat{E} = \frac{1}{c^2(1 + i\nu_{ei}\omega^{-1})}\frac{q_e^2}{m_e\varepsilon^0} N_e \hat{E} - \frac{\omega^2}{c^2}\chi(\omega)\hat{E}. \tag{5.50}$$

Recall that $q_e^2/(m_e\varepsilon^0) = \omega_0^2/N_c$ and set as usual

$$N = N_e/N_c.$$

On the boundary corresponding to Γ, we may assume that the incoming field is a given datum $E^{in}(t, x_\perp) = e^{-i\omega_0 t}A^{in}(t, x_\perp)$ where $A^{in}(., x_\perp)$ is a slowly time-varying function that is twice differentiable with respect to time. Denote its Fourier transform by $\hat{A}^{in}(., x_\perp)$; then equation (5.50) is supplemented with a boundary condition on Γ of the form

$$\left(i\frac{\partial}{\partial z} + \frac{\omega}{c}\sqrt{1-N}\right)\left(\hat{E}(\omega, .)|_{z=0} - \hat{A}^{in}(\omega - \omega_0, .)\right) = 0.$$

From now on, we withdraw the absorption term. Indeed, $\nu_{ei}\omega^{-1}$ is always very small compared to 1 and is often negligible (but all the sequel remains valid with a small constant absorption coefficient); moreover, we introduce the decomposition of the Laplace operator $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$ where $\Delta_\perp$ denotes the transverse Laplace operator. Then equation (5.50) reads as

$$\left(\omega^2(1 + \chi(\omega)) - \omega_0^2 N\right)\hat{E} + c^2\Delta_\perp\hat{E} + c^2\frac{\partial^2}{\partial z^2}\hat{E} = 0. \tag{5.51}$$

Since ω/c is very small with respect to the characteristic length of the simulation, the wave is highly oscillating with the z-variable, and by a classical way there are two components of the wave: the one travels forward and the other travels backward. In the sequel, we neglect the second one. Thus, in the same way as in Chap. 3, we can make the classical envelope approximation by setting

$$\hat{E}(\omega, z, x_\perp) = e^{ik(\omega)z}\hat{\Psi}(\omega, z, x_\perp) \tag{5.52}$$

where the function $\hat{\Psi}$ is slowly varying with respect to z.

To find the wave number $k(\omega)$, we assume that the transverse diffraction terms are less important than the propagation in the z-direction and we consider the dispersion relation obtained from (5.51), i.e.,

$$\omega^2(1+\chi(\omega)) - \omega_0^2 N - c^2 k(\omega)^2 = 0$$

so we get

$$ck(\omega) = \pm\sqrt{\omega^2(1+\chi(\omega)) - \omega_0^2 N}.$$

Of course, the wave number with sign $-$ corresponds to a backscattered wave and the wave number with sign $+$ corresponds to the main forward wave. We disregard here the backscattering phenomena, henceforth we only consider

$$ck(\omega) = \sqrt{\omega^2(1+\chi(\omega)) - \omega_0^2 N}, \tag{5.53}$$

For $\omega = \omega_0$, we get

$$k(\omega_0) = k^0 = \frac{\omega_0}{c}\sqrt{1+\chi(\omega_0) - N},$$

which is on the order of the wavelength in the vacuum ω_0/c. We check that $e^{ik(\omega)z}$ is highly oscillating with respect to z.

Let us state now the paraxial equation associated with this wave number. Since

$$\frac{\partial^2}{\partial z^2}(e^{ikz}\hat{\Psi}) = e^{ik(\omega)z}(\frac{\partial^2}{\partial z^2}\hat{\Psi} + 2ik(\omega)\frac{\partial}{\partial z}\hat{\Psi} - k(\omega)^2\hat{\Psi})$$

and $\hat{\Psi}$ is slowly varying with respect to z, the term $\frac{\partial^2}{\partial z^2}\hat{\Psi}$ is negligible with respect to $ik(\omega)\frac{\partial}{\partial z}\hat{\Psi}$ and equation (5.51) becomes

$$\left(\omega^2(1+\chi(\omega)) - \omega_0^2 N - c^2 k(\omega)^2\right)\hat{\Psi}(\omega) + c^2\Delta_\perp\hat{\Psi}(\omega) + c^2 2ik(\omega)\frac{\partial}{\partial z}\hat{\Psi}(\omega) = 0$$

but due to relation (5.53) the zero-order term vanishes and we get simply the classical paraxial equation

$$2k(\omega)\frac{\partial}{\partial z}\hat{\Psi}(\omega) = i\Delta_\perp\hat{\Psi}(\omega).$$

Now the boundary condition in $z = 0$ becomes simply $\hat{\Psi}(\omega, .) = \hat{A}^{in}(\omega - \omega_0, .)$

Of course, we can go back to the function $\hat{E}$: since $\frac{\partial}{\partial z}(\hat{\Psi}e^{ikz}) = e^{ikz}\frac{\partial}{\partial z}\hat{\Psi} + ik\hat{\Psi}e^{ikz}$; using relation (5.52), this equation reads also

$$\frac{\partial}{\partial z}\hat{E}(\omega) = ik(\omega)\hat{E}(\omega) + \frac{i}{2k(\omega)}\Delta_\perp \hat{E}(\omega)$$

supplemented with the boundary condition

$$\hat{E}(\omega, x_\perp)|_{z=0} = \hat{A}^{in}(\omega - \omega_0, x_\perp).$$

Now, taking the inverse Fourier transform, we return to the time variable

$$\begin{aligned}\frac{\partial}{\partial z}E(t) &= \int ik(\omega)\hat{E}(\omega)e^{-i\omega t}d\omega + \frac{i}{2}\int \frac{1}{k(\omega)}\Delta_\perp \hat{E}e^{-i\omega t}d\omega \\ &= ik^0E(t) + i\int (k(\omega) - k^0)\hat{E}(\omega)e^{-i\omega t}d\omega + \frac{i}{2}\int \frac{1}{k(\omega)}\Delta_\perp \hat{E}e^{-i\omega t}d\omega.\end{aligned}$$

Recall that ω is close to ω_0, so $k(\omega)$ is close to k_0 and we can approximate $\int \frac{1}{k(\omega)}\Delta_\perp \hat{E}e^{-i\omega t}d\omega$ by $\frac{1}{k^0}\int \Delta_\perp \hat{E}e^{-i\omega t}d\omega$. Moreover, we have $(k(\omega) - k^0) \simeq \frac{\partial k}{\partial \omega}\big|_0 (\omega - \omega_0) + \frac{1}{2}k^{0(2)}(\omega - \omega_0)^2$ (here $k^{0(2)}$ denotes the second derivative with respect to ω). Since $\int \hat{E}(\omega)e^{-i\omega t}i(\omega - \omega_0)d\omega = -e^{-i\omega_0 t}\frac{\partial}{\partial t}[E(t)e^{i\omega_0 t}]$, we get the following approximated equation

$$\frac{\partial}{\partial z}E(t)e^{i\omega_0 t} = ik^0E(t)e^{i\omega_0 t} - \frac{\partial k}{\partial \omega}\bigg|_0 \frac{\partial}{\partial t}[E(t)e^{i\omega_0 t}] + \frac{i}{2}k^{0(2)}\int (\omega - \omega_0)^2\hat{E}(\omega)e^{-i(\omega-\omega_0)t}d\omega + \frac{i}{2k^0}\Delta_\perp E(t)e^{i\omega_0 t}$$

So we may define the time envelope function A (which is not highly oscillating with respect to the time variable) by

$$E(t, z, x_\perp) = A(t, z, x_\perp)e^{-i\omega_0 t}$$

and we get

$$\frac{\partial}{\partial z}A(t) = ik^0A(t) - \frac{\partial k}{\partial \omega}\bigg|_0 \frac{\partial}{\partial t}A(t) + \frac{i}{2}k^{0(2)}\int (\omega - \omega_0)^2\hat{E}(\omega)e^{-i(\omega-\omega_0)t}d\omega + \frac{i}{2k^0}\Delta_\perp A(t).$$

The coefficient $\frac{\partial k}{\partial \omega}\big|_0$ is the inverse of

$$\frac{\partial \omega}{\partial k}\bigg|_0 = c_g = c(1 + \chi_0 - N)^{1/2}/(1 + \chi_0 + \omega_0\chi_0')$$

which is the group velocity of the wave (notice that in the simple case where the susceptibility of the medium is neglected, we find again the classical expression of the group velocity $c\sqrt{1 - N}$).

Let us now focus on the new term in front of $k^{0(2)}$ [the second derivative of $k(\omega)$] where the interpretation is the following. We have

$$\frac{1}{2}\int(\omega-\omega_0)^2\hat{E}(\omega)e^{-i(\omega-\omega_0)t}d\omega = \frac{1}{2}\int\omega_1^2\hat{A}(\omega_1)e^{-i\omega_1 t}d\omega_1 = -\frac{\partial^2}{\partial t^2}A(t).$$

Therefore, we get

$$\frac{\partial}{\partial z}A(t,z) = ik^0A(t,z) - \frac{1}{c_g}\frac{\partial}{\partial t}A(t,z) - ik^{0(2)}\frac{\partial^2}{\partial t^2}A(t,z) + \frac{i}{2k^0}\Delta_\perp A(t,z), \quad (5.54)$$

which is supplemented with the following condition in $z=0$,

$$A(t,.,x_\perp)|_{z=0} = A^{in}(t,x_\perp). \quad (5.55)$$

We can assume that the datum $A^{in}(.,x_\perp)$ is zero for t negative and satisfies $A^{in}(.,x_\perp)|_{t=0} = 0$ and $\frac{\partial A^{in}}{\partial t}(t,x_\perp)|_{t=0} = 0$. Now there are two ways to deal with time boundary conditions: either we may assume that equation (5.54) is posed for t ranging from $-\infty$ to $+\infty$, or we may assume this equation is posed for $t \geq 0$ and the boundary condition in $t=0$ is given by $A(.,x_\perp)|_{t=0} = 0$ and $\frac{\partial A}{\partial t}(t,x_\perp)|_{t=0} = 0$.

Remark 38 (Energy balance.).

We assume that k^0 does not depend on $x_\perp$. For the sake of presentation assume that the simulation domain is a slab (z belongs to $[0,z_{\max}]$); multiplying equation (5.54) by $\overline{A}$ and integrating over the transverse direction, we get

$$\frac{\partial}{\partial z}\|A(t,z)\|_{L^2} = -\frac{1}{c_g}\frac{\partial}{\partial t}\|A(t,z)\|_{L^2} - ik^{0(2)}\int\frac{\partial^2}{\partial t^2}A(t,z)\overline{A}(t,z)dx_\perp - \frac{i}{2k^0}\int\nabla_\perp A(t,z)\nabla_\perp\overline{A}(t,z)dx_\perp + c.c.$$

where $\|.\|_{L^2}$ is the L^2-norm in the transverse direction; so this relation yields to the classical transport equation for the laser energy

$$c_g\frac{\partial}{\partial z}\|A(t,z)\|_{L^2} + \frac{\partial}{\partial t}\|A(t,z)\|_{L^2} = 0. \qquad \square$$

In the case where N is a constant, then k^0 is a constant also; we can go back to the classical envelope approximation by setting $A(t,z) = e^{ik^0z}\Psi(t,z)$. In the case where N depends smoothly on z, the same approximation is also possible by setting $A(t,z) = e^{i\int^z k^0(z')dz'}\Psi(t,z)$ and the previous equation reads as

$$\frac{\partial}{\partial z}\Psi(t,z) = -\frac{1}{c_g}\frac{\partial}{\partial t}\Psi(t,z) - ik^{0(2)}\frac{\partial^2}{\partial t^2}\Psi(t,z) + \frac{i}{2k^0}\Delta_\perp\Psi(t,z).$$

and the boundary conditions in $z=0$ and in $t=0$ are analogue to the previous one.

Lastly, from equation (5.54) we may perform a change of variable by setting

$$\tau = t - \frac{1}{c_g} z, \qquad \tilde{A}(\tau, z, x_\perp) = A(\tau + \frac{1}{c_g} z, z, , x_\perp),$$

i.e., the pulse is addressed in a moving frame that travels at velocity c_g. Since $\partial_z \tilde{A}(\tau, z) = \partial_z A(\tau, z) + \frac{1}{c_g} \partial_t A(\tau, z)$, and $\partial_\tau \tilde{A}(\tau, z) = \partial_t A(\tau, z)$, we get the following equation

$$\frac{\partial}{\partial z} \tilde{A}(\tau, z) = i k^0 \tilde{A}(\tau, z) - i k^{0(2)} \frac{\partial^2}{\partial \tau^2} \tilde{A}(\tau, z) + \frac{i}{2k^0} \Delta_\perp \tilde{A}(\tau, z)$$

and the boundary conditions on Γ is of the same type as above.

To conclude this section, let us notice first that in the moving frame or in the static frame the second time derivative $\frac{\partial^2}{\partial t^2} A(t, z)$ or $\frac{\partial^2}{\partial \tau^2} \tilde{A}(\tau, z)$ is only due to the nonlinearity of the dispersion relation linking k and ω. Moreover, in the moving frame the time variable is often a secondary variable since the term $k^{0(2)} \frac{\partial^2}{\partial \tau^2} A(\tau, z)$ is a corrective one.

Chapter 6
Models with Several Species

Abstract The first part of this chapter is devoted to the modelling of hot plasmas with different species of ions in the framework of the quasi-neutrality approximation. Our aim is to show how to derive models describing the averaged ion fluid and its coupling with the electron temperature equation. In the second part, we are concerned by a different framework: the weakly ionized plasmas. We show how the quasi-neutrality approximation works (in the case where the Debye length is small enough) and we justify the so-called ambipolar diffusion approximation.

Keywords Debye length • Quasi-neutrality approximation • Plasma with two ion species • Ambipolar diffusion approximation

We are now concerned with models where, besides the electron population, there are different ion species. Indeed, in realistic simulations of experiments, one needs to account for different species and, as will be explained below, the generalization of the two-temperature Euler models (and electron magnetic-hydrodynamics also) must be made carefully.

6.1 Two-Temperature Euler System for a Mixing of Two Ion Species

Here, for the sake of simplicity, we restrict ourselves to the case of two different ion species and, of course, one electron population.

In the first subsection, we state a basic model for the three populations that consist of a system of six equations that correspond to the conservation of mass, momentum, and energy for each ion species, besides an equation for the electron energy. But this full model is rarely used in two- or three-dimensional simulations, especially when one needs to take into account an elaborate electron thermal conduction

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equation; so one is led to use simpler models when one deals with an averaged ion internal energy equation. Moreover, instead of dealing with two ion velocities, one may consider a different reduced model based on closures for the relative velocity between the two ion species.

The second section is devoted to the statement of such reduced models which consist of the conservation equations of mass, average momentum, and average ion energy coupled with one for the electron energy and one for the mass concentration c.

6.1.1 The Three-Population Full Model

The two ion species are called here a and b. For each ion species, we denote the following:

m_a, m_b the ion masses, Z_a, Z_b the ionization levels, N_a, N_b the ion densities, $\mathbf{u}_a, \mathbf{u}_b$ the ion velocities, $\mathcal{E}_a$, $\mathcal{E}_b$ the internal ion energies (per unit of volume), P_a, P_b the ion pressures, T_a, T_b the ion temperatures, E_a, E_b the total ion energies, i.e. $E_q = \mathcal{E}_q + \frac{1}{2} m_q N_q |\mathbf{u}_q|^2$.

For the sake of simplicity, we assume in the sequel that the polytropic coefficient of each ion species is $\gamma_0 = \frac{5}{3}$, i.e., the ion temperatures and energies satisfy

$$\mathcal{E}_a = \frac{3}{2} N_a T_a, \qquad P_a = N_a T_a = \frac{2}{3}\mathcal{E}_a \qquad \text{and} \qquad \mathcal{E}_b = \frac{3}{2} N_b T_b,$$
$$P_b = N_b T_b = \frac{2}{3}\mathcal{E}_b. \tag{6.1}$$

As noted previously, the electron density, temperature, internal energy per unit of volume and pressure are denoted by N_e, T_e, $\mathcal{E}_e$, and P_e, respectively; we have

$$N_e = Z_a N_a + Z_b N_b, \qquad \mathcal{E}_e = \frac{3}{2} N_e T_e, \qquad P_e = N_e T_e = \frac{2}{3}\mathcal{E}_e.$$

We first state the model as it appears from first principle conservation laws (see, e.g.. [38], Sect. 1). In the sequel the electric current $\mathbf{J}$ is either zero or is given as usual by an external relation that accounts for magnetic fields (but, for the sake of conciseness, we do not write here the corresponding equation for this field). By definition, we have the relation with the electron velocity

$$\frac{\mathbf{J}}{q_e N_e} = \frac{Z_a N_a}{N_e}(\mathbf{u}_a - \mathbf{U}_e) + \frac{Z_b N_b}{N_e}(\mathbf{u}_b - \mathbf{U}_e). \tag{6.2}$$

6.1.1.1 Conservation of Ion Momentum and Coupling with the Electron Velocity

Notice first that if there would be no coupling with the electron population, the mass and momentum conservation equations would read as

$$\frac{\partial}{\partial t}N_q + \nabla.\left(N_q\mathbf{u}_q\right) = 0, \qquad [q = a, b] \tag{6.3}$$

$$m_a\frac{\partial}{\partial t}\left(N_a\mathbf{u}_a\right) + \nabla.\left(N_a\mathbf{u}_a\mathbf{u}_a + P_a\right) = \Xi_a, \tag{6.4}$$

$$m_b\frac{\partial}{\partial t}\left(N_b\mathbf{u}_b\right) + \nabla.\left(N_b\mathbf{u}_b\mathbf{u}_b + P_b\right) = \Xi_b, \tag{6.5}$$

with the following friction or drag term

$$\Xi_a = -\Xi_b = \tilde{\nu}N_aN_b(\mathbf{u}_b - \mathbf{u}_a).$$

The relaxation coefficient $\tilde{\nu}$ depends on the temperatures of the two species and on the physical characteristics of the particles (and also on the relative velocity of the two species). This coefficient is given by a closure made from the kinetic models which may be read as

$$\tilde{\nu} = m_am_b\beta\left(\frac{T_a}{m_a} + \frac{T_b}{m_b}\right)^{-3/2} \tag{6.6}$$

with a coefficient β depending on the characteristics of the two species (through a Coulomb logarithm).

Besides the interaction of the ions of both species, there is a relaxation between the ion of the one species over the electrons; these friction terms read as (where ν_{eq} is also given by a closure made in the kinetic models)

$$\nu_{eq}N_eN_q(\mathbf{U}_e - \mathbf{u}_q), \qquad q = a, b.$$

So for the momentum balance, we get

$$m_a\left(\frac{\partial}{\partial t}\left(N_a\mathbf{u}_a\right) + \nabla.\left(N_a\mathbf{u}_a\mathbf{u}_a\right)\right) + \nabla P_a = \Xi_a + \nu_{ea}N_eN_a(\mathbf{U}_e - \mathbf{u}_a) + Z_aN_aq_e\mathbf{E},$$

$$m_b\left(\frac{\partial}{\partial t}\left(N_b\mathbf{u}_b\right) + \nabla.\left(N_b\mathbf{u}_b\mathbf{u}_b\right)\right) + \nabla P_b = -\Xi_a + \nu_{eb}N_eN_b(\mathbf{U}_e - \mathbf{u}_b) + Z_bN_bq_e\mathbf{E},$$

Moreover, the generalized Ohm's law reads as

$$\nu_{ea}N_eN_a(\mathbf{U}_e - \mathbf{u}_a) + \nu_{eb}N_eN_b(\mathbf{U}_e - \mathbf{u}_b) + N_eq_e\mathbf{E} + \nabla P_e = 0. \tag{6.7}$$

Therefore, the previous system for the ion momentum reads now as

$$m_a \left(\frac{\partial}{\partial t} + \nabla . (\mathbf{u}_a \cdot) \right) (N_a \mathbf{u}_a) + \nabla P_a + \frac{Z_a N_a}{N_e} \nabla P_e$$
$$= N_a N_b \left((\tilde{\nu} + \widetilde{\nu_e})(\mathbf{u}_b - \mathbf{u}_a) - \frac{Z_a \nu_{eb} - Z_b \nu_{ea}}{q_e N_e} \mathbf{J} \right), \tag{6.8}$$

$$m_b \left(\frac{\partial}{\partial t} + \nabla . (\mathbf{u}_b \cdot) \right) (N_b \mathbf{u}_b) + \nabla P_b + \frac{Z_b N_b}{N_e} \nabla P_e$$
$$= N_a N_b \left((\tilde{\nu} + \widetilde{\nu_e})(\mathbf{u}_a - \mathbf{u}_b) + \frac{Z_a \nu_{eb} - Z_b \nu_{ea}}{q_e N_e} \mathbf{J} \right). \tag{6.9}$$

with:

$$\widetilde{\nu_e} = \frac{\nu_{ea} N_b Z_b^2 + \nu_{eb} N_a Z_a^2}{N_b Z_b + N_a Z_a}.$$

[**Indeed**, according to relation (6.2), the electron velocity satisfies $\mathbf{u}_a - \mathbf{U}_e = \frac{\mathbf{J}}{q_e N_e} + \frac{Z_b N_b}{N_e}(\mathbf{u}_a - \mathbf{u}_b)$; then the generalized Ohm's law becomes

$$-(N_a \nu_{ea} + N_b \nu_{eb}) \frac{\mathbf{J}}{q_e N_e} + \frac{1}{N_e} N_a N_b \left(\nu_{eb} Z_a - \nu_{ea} Z_b\right) (\mathbf{u}_a - \mathbf{u}_b) + q_e \mathbf{E} + \frac{\nabla P_e}{N_e} = 0, \tag{6.10}$$

So we get $q_e \mathbf{E} - \frac{\nu_{ea}}{Z_a} N_e (\mathbf{U}_e - \mathbf{u}_a) = -\frac{\nabla P_e}{N_e} + Z_b N_b (\frac{\nu_{eb}}{Z_b} - \frac{\nu_{ea}}{Z_a}) \frac{\mathbf{J}}{q_e N_e} - (Z_b N_b \frac{\nu_{ea}}{Z_a} + Z_a N_a \frac{\nu_{eb}}{Z_b} \frac{Z_b N_b}{N_e})(\mathbf{u}_a - \mathbf{u}_b)$ and the analogous for $\mathbf{U}_e - \mathbf{u}_b$, which implies (6.8) and (6.9).□]

6.1.1.2 Energy Balance and Statement of the Model

Let us first focus on the interaction between both ion internal energy. To do this, assume for a while that there would be no coupling with the electron population and no spatial dependency; then by dealing with the kinetic models (see the part related to the kinetics models), it may be checked that the variation of total energy of each species would read as

$$\frac{\partial}{\partial t} \left(\mathcal{E}_a + \frac{m_a}{2} N_a |\mathbf{u}_a|^2 \right) = \Omega^\# + \frac{m_a \mathbf{u}_a + m_b \mathbf{u}_b}{m_a + m_b} . \Xi_a = -\frac{\partial}{\partial t} \left(\mathcal{E}_b + \frac{m_b}{2} N_b |\mathbf{u}_b|^2 \right),$$

where the term $\Omega^\#$ corresponds to the ion temperature coupling due to thermal effects and is given by

$$\Omega^\# = \nu^\# N_a N_b (T_b - T_a),$$

coefficient $\nu^{\#}$ depending on the ion temperatures. In this framework, since the variation of momentum would reduce to $m_a \frac{\partial}{\partial t}(N_a \mathbf{u}_a) = \Xi_a$, $m_b \frac{\partial}{\partial t}(N_b \mathbf{u}_b) = -\Xi_a$, we would get

$$\frac{\partial}{\partial t}\mathcal{E}_a = \Omega^{\#} + \frac{m_b}{m_a + m_b}(\mathbf{u}_b - \mathbf{u}_a).\Xi_a, \qquad \frac{\partial}{\partial t}\mathcal{E}_b = -\Omega^{\#} + \frac{m_a}{m_a + m_b}(\mathbf{u}_b - \mathbf{u}_a).\Xi_a.$$

Return now to the full space-dependent model. According to the previous remark, for each ion species, the internal energy equations read simply as

$$\frac{\partial}{\partial t}\mathcal{E}_a + \nabla.(\mathcal{E}_a \mathbf{u}_a) + P_a \nabla.\mathbf{u}_a = \Omega^{\#} + \frac{m_b}{m_a + m_b}(\mathbf{u}_b - \mathbf{u}_a).\Xi_a + \Omega_{ae}, \quad (6.11)$$

$$\frac{\partial}{\partial t}\mathcal{E}_b + \nabla.(\mathcal{E}_b \mathbf{u}_b) + P_b \nabla.\mathbf{u}_b = -\Omega^{\#} + \frac{m_a}{m_a + m_b}(\mathbf{u}_b - \mathbf{u}_a).\Xi_a + \Omega_{be}. \quad (6.12)$$

We assume by a classical way that the relaxation terms between the ion and electron temperatures satisfy

$$\Omega_{ae} = \nu_a^E N_a N_e (T_e - T_a), \qquad \Omega_{be} = \nu_b^E N_b N_e (T_e - T_b),$$

where ν_a^E and ν_b^E are relaxation coefficients depending on the temperature and are given by closure from kinetics models.

Let us now state the total energy balance for each species. If one sets

$$\Omega_{ab} = \Omega^{\#} + \tilde{\nu} N_a N_b \frac{m_a \mathbf{u}_a + m_b \mathbf{u}_b}{m_a + m_b}.(\mathbf{u}_b - \mathbf{u}_a),$$

after combining with the momentum balance equations, we get for the balance equation of E_a

$$\frac{\partial}{\partial t}E_a + \nabla.((E_a + P_a)\mathbf{u}_a) + \frac{Z_a N_a}{N_e}\mathbf{u}_a \nabla P_e$$

$$= \Omega_{ab} + \widetilde{\nu_e} N_a N_b \mathbf{u}_a.(\mathbf{u}_b - \mathbf{u}_a) + \Omega_{ae} - N_a N_b (Z_a \nu_{eb} - Z_b \nu_{ea}) \frac{\mathbf{J}}{q_e N_e}.\mathbf{u}_a, \quad (6.13)$$

and the analogous for the balance equation of E_b, with $\Omega_{ba} = -\Omega_{ab}$.

Therefore, we set

$$\Omega_{0e} = \Omega_{ae} + \Omega_{be} = (\nu_a^E N_a (T_e - T_a) + \nu_b^E N_b (T_e - T_b))(Z_a N_a + Z_b N_b), \quad (6.14)$$

and for the balance equation of $E_a + E_b$, we get

$$\frac{\partial}{\partial t}(E_b + E_a) + \nabla.((E_a + P_a)\mathbf{u}_a + (E_b + P_b)\mathbf{u}_b) + (\mathbf{U}_e + \frac{\mathbf{J}}{q_e N_e}).\nabla P_e$$
$$= \Omega_{0e} + N_a N_b \left((Z_a \nu_{eb} - Z_b \nu_{ea}) \frac{\mathbf{J}}{q_e N_e}.(\mathbf{u}_b - \mathbf{u}_a) - \widetilde{\nu_e}|\mathbf{u}_a - \mathbf{u}_b|^2 \right). \qquad (6.15)$$

Now, for the electron energy equation, we get

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e \mathbf{U}_e) + P_e \nabla.\mathbf{U}_e + \nabla.\mathbf{q}_{\text{th,e}} = -\Omega_{0e} + \nu_{e,\text{av}} \frac{|\mathbf{J}|^2}{q_e^2 N_e} + \widetilde{\nu_e} N_a N_b |\mathbf{u}_a - \mathbf{u}_b|^2, \qquad (6.16)$$

where

$$\nu_{\text{e,av}} = (\nu_{ea} N_a + \nu_{eb} N_b)/(Z_a N_a + Z_b N_b). \qquad (6.17)$$

[**Indeed**, according to (6.10), the right hand side of the previous relation reads as $-\Omega_{0e} + \frac{\mathbf{J}}{q_e N_e}.\nabla P_e + \mathbf{J}.\mathbf{E} - N_a N_b \left((Z_a \nu_{eb} - Z_b \nu_{ea}) \frac{\mathbf{J}.(\mathbf{u}_a - \mathbf{u}_b)}{q_e N_e} - \widetilde{\nu_e}|\mathbf{u}_a - \mathbf{u}_b|^2\right)$. Then by adding with (6.15), we find the right balance equation for the global energy $(E_b + E_a + \mathcal{E}_e)$: only the classical source term $\mathbf{J}.\mathbf{E}$ appears by summation of the electron and ion energy balance equation.☐].

Statement of the Model

Denoting

$$\mathbf{U} = (m_a N_a \mathbf{u}_a + m_b N_b \mathbf{u}_b)/(m_a N_a + m_b N_b),$$

and using (6.2), we can express $\mathbf{U}_e$ in terms of the other velocities.

$$\mathbf{U} - \mathbf{U}_e = \frac{\mathbf{J}}{q_e N_e} + \frac{\mathbf{j}}{q_e N_e}, \qquad \text{with } \frac{\mathbf{j}}{q_e} = -\frac{m_a N_a m_b N_b}{m_a N_a + m_b N_b}\left(\frac{Z_a}{m_a} - \frac{Z_b}{m_b}\right)\left(\mathbf{u}_a - \mathbf{u}_b\right). \qquad (6.18)$$

[**Indeed**, according to (6.2), we get

$$(\mathbf{U} - \mathbf{U}_e)N_e - \frac{\mathbf{J}}{q_e} = N_e \mathbf{U} - Z_a N_a \mathbf{u}_a - Z_b N_b \mathbf{u}_b,$$
$$= \mathbf{u}_a \left(\frac{Z_a N_a + Z_b N_b}{m_a N_a + m_v N_b} - \frac{Z_a}{m_a}\right) m_a N_a + \mathbf{u}_b \left(\frac{Z_a N_a + Z_b N_b}{m_a N_a + m_v N_b} - \frac{Z_b}{m_b}\right) m_b N_b, \square].$$

The term $\mathbf{j}$ is a corrective electric current due the difference of the mobility of two components.

Using now the same trick as for (2.79), we may state the full system

$$\frac{\partial}{\partial t} N_a + \nabla.(N_a \mathbf{u}_a) = 0, \qquad \frac{\partial}{\partial t} N_b + \nabla.(N_b \mathbf{u}_b) = 0, \qquad N_e = Z_a N_a + Z_b N_b,$$

$$m_a \left(\frac{\partial}{\partial t} + \nabla.(\mathbf{u}_a .)\right)(N_a \mathbf{u}_a) + \nabla P_a + \tfrac{Z_a N_a}{N_e} \nabla P_e$$
$$= -N_a N_b \left((\tilde{\nu} + \widetilde{\nu_e})(\mathbf{u}_a - \mathbf{u}_b) + \tfrac{Z_a \nu_{eb} - Z_b \nu_{ea}}{q_e N_e} \mathbf{J}\right),$$

$$m_b \left(\frac{\partial}{\partial t} + \nabla.(\mathbf{u}_b \cdot)\right)(N_b \mathbf{u}_b) + \nabla P_b + \tfrac{Z_b N_b}{N_e} \nabla P_e$$
$$= N_a N_b \left((\tilde{\nu} + \widetilde{\nu_e})(\mathbf{u}_a - \mathbf{u}_b) + \tfrac{Z_a \nu_{eb} - Z_b \nu_{ea}}{q_e N_e} \mathbf{J}\right),$$

$$\frac{\partial}{\partial t} \mathcal{E}_a + \nabla.(\mathcal{E}_a \mathbf{u}_a) + P_a \nabla.\mathbf{u}_a = \Omega^{\#} + \tfrac{m_b}{m_a + m_b} N_a N_b \tilde{\nu} \, |\mathbf{u}_b - \mathbf{u}_a|^2 + \Omega_{ae},$$

$$\frac{\partial}{\partial t} \mathcal{E}_b + \nabla.(\mathcal{E}_b \mathbf{u}_b) + P_b \nabla.\mathbf{u}_b = -\Omega^{\#} + \tfrac{m_a}{m_a + m_b} N_a N_b \tilde{\nu} \, |\mathbf{u}_b - \mathbf{u}_a|^2 + \Omega_{be},$$

$$\left(\frac{\partial}{\partial t} + \nabla.(\mathbf{U} \cdot)\right) \mathcal{E}_e + P_e \nabla.\mathbf{U} + \nabla.\mathbf{q}_{th,e} = \nabla.\left(\tfrac{5}{2} \tfrac{T_e}{q_e} \mathbf{j}\right) - \mathbf{j}.\frac{\nabla P_e}{q_e N_e}$$
$$+ N_a N_b \widetilde{\nu_e} |\mathbf{u}_a - \mathbf{u}_b|^2 - \Omega_{0e} + F_{\mathbf{J}}$$
$$\text{with } F_{\mathbf{J}} = \nu_{\text{e,av}} \frac{|\mathbf{J}|^2}{q_e^2 N_e} + \nabla.\left(\tfrac{5}{2} \tfrac{T_e}{q_e} \mathbf{J}\right) - \mathbf{J}.\frac{\nabla P_e}{q_e N_e}.$$

Notice that the last equation is of the same type as (2.79). The true electric current **J** is replaced by the corrective current **j** and an extra term appears that is related to the friction effect between the two ion velocities. We can check that this system without the right-hand-side terms is a hyperbolic one.

6.1.2 Average-Species Models

The above system may be quite difficult to solve in two- or three-dimensional geometry, especially in the case when the friction coefficient $\tilde{\nu} N_{\text{ref}}/m_a$ or the relaxation coefficient $\nu^{\#} N_{\text{ref}}/m_a$ is large if compared to the inverse of the characteristic time of evolution of the ion population. So, it is interesting to derive some simplified models for the multispecies plasmas, especially in the cases where the relaxation coefficient is large enough (then only one ion temperature needs to be accounted for). In the next subsection, we first state the evolution equation for some averaged quantities: the ion density, the ion momentum, and the ion and electron energies. The sequel is devoted to describe several ways to derive simplified models; they differ especially by the treatment of the relative velocity.

6.1.2.1 A Model with Mass Fraction, Average Ion Velocity and Average Ion Energy

Let us now introduce the natural average physical quantities: the mass density

$$\rho = m_a N_a + m_b N_b$$

$c = m_a N_a/\rho$, the mass concentration of material a also called the mass fraction (so we have $\rho(1-c) = m_b N_b$),

$P_0 = P_a + P_b$, the global ion pressure,

$\mathcal{E}_0 = \mathcal{E}_a + \mathcal{E}_b$, the global internal ion energy,

$E_0 = E_a + E_b$, the global total ion energy,

$\mathbf{V} = \mathbf{u}_a - \mathbf{u}_b$, the relative velocity.

So, the mean ion velocity $\mathbf{U}$ satisfies

$$\mathbf{U} = c\mathbf{u}_a + (1-c)\mathbf{u}_b, \qquad \mathbf{u}_a = \mathbf{U} + (1-c)\mathbf{V}, \qquad \mathbf{u}_b = \mathbf{U} - c\mathbf{V}. \tag{6.19}$$

Define the ion specific internal energy by

$$\varepsilon_0 = \frac{\mathcal{E}_0}{\rho} = c\frac{\mathcal{E}_a}{m_a N_a} + (1-c)\frac{\mathcal{E}_b}{m_b N_b}.$$

Since the polytropic coefficients are equal for both ion species, we have a simple equation of state

$$P_0 = (\gamma_0 - 1)\mathcal{E}_0 = (\gamma_0 - 1)\rho\varepsilon_0,$$

(if the two polytropic coefficients had been different, the average equation of state would have been more complicated).

It is easy to check that, if we define the specific mixing kinetic energy by

$$K = \frac{1}{2}|\mathbf{V}|^2 c(1-c),$$

we get

$$E_0 = \mathcal{E}_0 + \frac{1}{2}\rho|\mathbf{U}|^2 + \rho K = \rho\left(\varepsilon_0 + \frac{1}{2}|\mathbf{U}|^2 + K\right).$$

We have also

$$N_e = \rho\left(c\frac{Z_a}{m_a} + (1-c)\frac{Z_b}{m_b}\right), \qquad \varepsilon_e = \frac{\mathcal{E}_e}{\rho} = \frac{3}{2}T_e\left(c\frac{Z_a}{m_a} + (1-c)\frac{Z_b}{m_b}\right).$$

Momentum Balance

After standard calculus, we get the classical relations

$$\frac{\partial}{\partial t}\rho + \nabla.(\rho\mathbf{U}) = 0, \tag{6.20}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{U}) + \nabla.(\rho\mathbf{U}\mathbf{U}) + \nabla P_0 + \nabla P_e + \nabla.(\rho\mathbf{V}\mathbf{V}c(1-c)) = 0 \tag{6.21}$$

$$\frac{\partial}{\partial t}(\rho c) + \nabla.(\rho c\mathbf{U}) + \nabla.(\rho c(1-c)\mathbf{V}) = 0, \tag{6.22}$$

The supplementary term in (6.21) corresponds to the mixing pressure. Moreover, if one defines

$$\sigma = \rho\tilde{\nu}\frac{1}{m_a m_b}, \qquad \sigma_e = \rho\widetilde{\nu_e}\frac{1}{m_a m_b} \qquad \zeta = \frac{Z_a m_a^{-1} - Z_b m_b^{-1}}{N_e}\rho,$$

one may state an equation for the relative velocity

$$\frac{\partial}{\partial t}\mathbf{V} + \nabla.(\mathbf{U}\mathbf{V}) - \nabla(\frac{|\mathbf{V}|^2}{2}(2c-1)) + \mathcal{F} + \mathcal{B} + \frac{\zeta}{\rho}\nabla P_e$$
$$= -(\sigma + \sigma_e)\mathbf{V} - (1-2c)\,(Z_a\nu_{eb} - Z_b\nu_{ea})\,\frac{\mathbf{J}}{q_e N_e}, \tag{6.23}$$

where

$$\mathcal{F} = \frac{1}{\rho c}\nabla P_a - \frac{1}{\rho(1-c)}\nabla P_b,$$

$$\mathcal{B} = -\nabla.\left(\mathbf{V}\mathbf{V}\frac{2c-1}{2}\right) + \nabla.\left(|\mathbf{V}|^2\frac{2c-1}{2}\right) + \frac{2c-1}{2}\,[(\nabla.\mathbf{V})\mathbf{V} - (\mathbf{V}.\nabla)\mathbf{V}]$$
$$+(\mathbf{V}.\nabla)\mathbf{U} - (\nabla.\mathbf{U})\mathbf{V}.$$

[**Indeed**, according to (6.8) and (6.9), we have

$$\frac{\partial}{\partial t}\mathbf{u}_a + (\mathbf{u}_a.\nabla)\mathbf{u}_a - \frac{1}{\rho c}\left(\nabla P_a + \frac{cZ_a m_a^{-1}}{cZ_a m_a^{-1} + (1-c)Z_b m_b^{-1}}\nabla P_e\right)$$
$$= -(1-c)\left((\sigma + \sigma_e)\mathbf{V} + (Z_a\nu_{eb} - Z_b\nu_{ea})\,\frac{\mathbf{J}}{q_e N_e}\right)$$

and the analogous for $\mathbf{u}_b$; thus, (6.23) follows by standard calculus.□**]**

In the one-dimensional case, $\mathcal{B}$ is equal to zero, so we may neglect it.

The Energy Balance Equations

Recall that $\mathbf{U} - \mathbf{U}_e = (\mathbf{J} - \mathbf{j})\frac{1}{q_e N_e}$ where according to (6.18), $\mathbf{j}$ is given by

$$\frac{\mathbf{j}}{q_e} = -N_e c(1-c)\zeta\mathbf{V}.$$

Then, since $\widetilde{\nu_e} N_a N_b |\mathbf{V}|^2 = 2\rho\sigma_e K$, the electron energy equation reads now as

$$\frac{\partial}{\partial t}\mathcal{E}_e + \nabla.(\mathcal{E}_e\mathbf{U}) + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{\text{th,e}} = \nabla.\left(\frac{5}{2}T_e\frac{\mathbf{j}}{q_e}\right) - \mathbf{j}.\frac{\nabla P_e}{q_e N_e} + F_{\mathbf{J}} - \Omega_{0e} + 2\rho\sigma_e K. \tag{6.24}$$

Lastly, after some tedious calculus, one checks that the ion total energy balance equation reads as

$$\frac{\partial}{\partial t}E_0 + \nabla.(E_0\mathbf{U}) + \nabla.(P_0\mathbf{U} + \rho c(1-c)\mathcal{G}\mathbf{V}) + (\mathbf{U} - \zeta c(1-c)\mathbf{V}).\nabla P_e +$$

$$+\nabla.(2\rho K\mathbf{U} - (2c-1)\rho K\mathbf{V}) = \Omega_{0e} - 2\rho\sigma_e K - \rho c(1-c)\frac{(Z_a\nu_{eb} - Z_b\nu_{ea})}{cZ_a m_b + (1-c)Z_b m_a}\frac{\mathbf{J}}{q_e}.\mathbf{V} \tag{6.25}$$

$$\text{with} \qquad \mathcal{G} = \frac{1}{m_a N_a}(P_a + \mathcal{E}_a) - \frac{1}{m_b N_b}(P_b + \mathcal{E}_b) = \left(\frac{\mathcal{E}_a}{m_a N_a} - \frac{\mathcal{E}_b}{m_b N_b}\right)\gamma_0.$$

According to (6.11) and (6.12), the evolution equation for the internal ion energy $\mathcal{E}_0$ reads as

$$\frac{\partial}{\partial t}\mathcal{E}_0 + \nabla.(\mathcal{E}_0\mathbf{U}) + P_0\nabla.\mathbf{U} = -\nabla.[\rho c(1-c)\mathcal{G}\mathbf{V}] + \rho c(1-c)\mathcal{F}\mathbf{V} + 2\sigma\rho K + \Omega_{0e}, \tag{6.26}$$

The previous balance equations may read in a natural way using the Lagrangian derivative $\frac{D}{Dt} = \frac{\partial}{\partial t}\bullet + \mathbf{U}.\nabla\bullet$.

Summary and Orientation

From the previous model, which consists of the evolution equations for ρ, $\mathbf{U}$, c, $\mathbf{V}/\rho$, $\mathcal{E}_e/\rho$, T_a and T_b, several different simplifications may be performed. The simplest one is to address an averaged ion energy equation and state an equation for the relative velocity after making a closure to determine the expression $\mathcal{F}$. For instance, one may assume that the two temperatures T_a and T_b are equal; a better assumption is related to the chemical potential of booth species (see, e.g., [61] for the kind of model in the framework of classical fluid dynamics); this corresponds to another closure for $\mathcal{G}$. So if these closures $\mathcal{F} = \mathcal{F}(c, \varepsilon_0)$ and $\mathcal{G} = \mathcal{G}(c, \varepsilon_0)$ hold, this modelling leads to the following system (using the Lagrangian derivatives) where $F_{\mathbf{J}}$ and K are given below.

$$
(\mathcal{EV})\quad
\begin{array}{c}
\rho\dfrac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0,\\[2ex]
\rho\dfrac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) + \nabla.(\rho\mathbf{V}\mathbf{V}c(1-c)) = 0,\\[2ex]
\rho\dfrac{D}{Dt}\left(\dfrac{\mathbf{V}}{\rho}\right) - \nabla\left(\dfrac{|\mathbf{V}|^2}{2}(2c-1)\right) + \mathcal{F} + \frac{\zeta}{\rho}\nabla P_e = -(\sigma+\sigma_e)\mathbf{V}\\
-(1-2c)\left(Z_a\nu_{eb} - Z_b\nu_{ea}\right)\frac{\mathbf{J}}{q_e N_e}\\[2ex]
\rho\dfrac{D}{Dt}c - \nabla.\left(\rho c(1-c)\mathbf{V}\right) = 0,\\[2ex]
\rho\dfrac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = -\nabla.[\rho c(1-c)\mathcal{G}\mathbf{V}] + \rho c(1-c)\mathcal{F}\mathbf{V} + 2\sigma\rho K + \Omega_{0e},\\[2ex]
\rho\dfrac{D}{Dt}\varepsilon_e + P_e\nabla.\mathbf{U} - \nabla.\mathbf{q}_{\text{th,e}} = -\nabla.\left(\frac{5}{2}P_e c(1-c)\zeta\mathbf{V}\right) + c(1-c)\zeta\mathbf{V}.\nabla P_e\\
-\Omega_{0e} + 2\rho\sigma_e K + F_{\mathbf{J}}.
\end{array}
$$

6.1.2.2 Simplified Models with Mass Fraction

Notice that the evolution equation for $\mathbf{V}/\rho$ is not classical and may be difficult to solve numerically, particularly in two or three dimensions. So, we now describe two ways to simplify the previous model by withdrawing the evolution equation for $\mathbf{V}$. The first method (a) is very crude; it consists of a closure for $\mathbf{V}$ only in the concentration equation. In the second method (b), we assume that $\mathbf{V}$ is given by a closure also in the other equations.

(a) Two-temperature Euler model with mass fraction.

We withdraw the terms $\mathcal{F}$ and $\mathcal{G}$ and set $K = 0$. Moreover, we assume that the two temperatures T_a and T_b are equal and are denoted by T_0. Then, according to (6.1), the mixing equation of state is the following

$$
\frac{1}{\gamma_0 - 1}\frac{1}{m_{\text{ave}}}T_0 = \varepsilon_0, \qquad P_0 = (\gamma_0 - 1)\rho\varepsilon_0 = \frac{1}{m_{\text{ave}}}\rho T_0,
$$
$$
\text{with } \frac{1}{m_{\text{ave}}} = \frac{c}{m_a} + \frac{1-c}{m_b}.
$$

Recall that for the electron equation of state, we get

$$
N_e = \rho\frac{Z_{\text{ave}}}{m_{\text{ave}}}, \qquad \varepsilon_e = \frac{3}{2}\frac{Z_{\text{ave}}}{m_{\text{ave}}}T_e, \qquad P_e = \frac{Z_{\text{ave}}}{m_{\text{ave}}}\rho T_e,
$$
$$
\text{with } \frac{Z_{\text{ave}}}{m_{\text{ave}}} = c\frac{Z_a}{m_a} + (1-c)\frac{Z_b}{m_b}.
$$

As in the framework of model ($\mathcal{E}2\mathcal{T}\mathbf{J}$) stated above, the electric current is assumed to be an external data. The system reads as

$$(\mathcal{E}2\mathcal{M}) \quad \boxed{\begin{aligned} &\rho\frac{D}{Dt}\rho^{-1} - \nabla.\mathbf{U} = 0,\\ &\rho\frac{D}{Dt}\mathbf{U} + \nabla(P_0 + P_e) = 0,\\ &\rho\frac{D}{Dt}c - \nabla.\left(\rho c(1-c)D\nabla c\right) = 0,\\ &\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = \Omega_{0e},\\ &\rho D_t\varepsilon_e + P_e\nabla.\mathbf{U} + \nabla.\mathbf{q}_{th,e} = \nabla.\left(\tfrac{5}{2}P_e c(1-c)\zeta D\nabla c\right) + F_{\mathbf{J}} - \Omega_{0e}.\end{aligned}}$$

where the diffusion coefficient D may be given by some closure (e.g., one may use the formula (6.30) given below). The resulting model is the same as ($\mathcal{E}2\mathcal{T}\mathbf{J}$), but it is supplemented by an equation for the mass fraction and there is a supplementary term $c(1-c)\zeta D\nabla c$ corresponding to the corrective current in the electron energy equation. According to (6.14), the coefficient Ω_{0e} depends on the electron temperature and is given by

$$\Omega_{0e} = \left(c\frac{\nu_a^E}{m_a} + (1-c)\frac{\nu_b^E}{m_b}\right)\frac{Z_{\text{ave}}}{m_{\text{ave}}}\rho^2(T_e - T_0). \tag{6.27}$$

We may check once again that the global energy $\rho(\varepsilon_0 + \varepsilon_e + \frac{1}{2}m_0|\mathbf{U}|^2)$ satisfies a good balance equation.

(b) Two-temperature Euler model with mass fraction and mixing energy.

We now propose another simplification of model ($\mathcal{E}\mathcal{V}$) where besides the density ρ, the mass fraction c, the averaged velocity $\mathbf{U}$, the electron and ion energy, we account for an evolution equation for the mixing kinetic energy K. The key features are the following.

- We simplify the ion internal ion equation (6.26) by withdrawing the terms with $\mathbf{V}$; then it becomes

$$\rho\frac{D}{Dt}\varepsilon_0 + P_0\nabla.\mathbf{U} = 2\sigma\rho K + \Omega_{0e}, \tag{6.28}$$

- The relative velocity is no more characterized by an evolution equation, but it is given by the relation $(\sigma + \sigma_e)\mathbf{V} + \mathcal{F} = 0$ and we build closures for $\mathcal{F}$ and $\mathcal{G}$. For instance, we may introduce a nondimensional increasing function Ψ depending only on c and satisfying $\Psi(1) = -\Psi(0) = \gamma_0$, $\Psi'(0) = \Psi'(1) = \gamma_0$ and set

$$\mathcal{F} = \frac{\nabla P_a}{m_a N_a} - \frac{\nabla P_b}{m_b N_b} \simeq \varepsilon_0 \nabla \Psi(c), \qquad \mathcal{G} = \frac{\gamma_0 \mathcal{E}_a}{m_a N_a} - \frac{\gamma_0 \mathcal{E}_b}{m_b N_b} \simeq \varepsilon_0 \Psi(c).$$

Thus, we get

$$\mathbf{V} \simeq -\frac{\varepsilon_0}{\sigma + \sigma_e} \nabla \Psi(c) = -D \nabla c \ , \qquad \text{with } D = \frac{\varepsilon_0}{\sigma + \sigma_e} \Psi'. \tag{6.29}$$

For a justification of these closures, see [107].

Using these closures, (6.22), the momentum equation and the electron energy equation read as

$$\rho \frac{D}{Dt} c - \nabla . (\rho c(1-c) D \nabla c) = 0.$$

$$\rho \frac{D}{Dt} \mathbf{U} + \nabla(P_0 + P_e + 2\rho K) = 0,$$

$$\rho \frac{D}{Dt} \varepsilon_e + P_e \nabla . \mathbf{U} - \nabla . (\rho \varepsilon_e c(1-c) \zeta D \nabla c) - P_e \nabla . (c(1-c)\zeta D \nabla c) + \nabla . \mathbf{q}_{th,e}$$
$$= F_{\mathbf{J}} - \Omega_{0e} + 2\rho \sigma_e K.$$

We must also state an evolution equation for the mixing kinetic energy K, which is obtained using (6.25) and (6.28) in such a way that the global energy balance equation is satisfied. After some tedious calculus we get

$$\begin{aligned} &\rho D_t K + 2\rho K \nabla . \mathbf{U} + \nabla . ((2c-1)\rho K D \nabla c) \\ &= c(1-c)\zeta D \nabla c . \nabla P_e + \Psi \nabla . (\rho c(1-c)\varepsilon_0 D \nabla c) \\ &+\rho(\sigma + \sigma_e) \left[c(1-c) D^2 |\nabla c|^2 - 2K \right] \end{aligned}$$

In the structure of the three energy equations, the coupling terms $2\rho\sigma K$, $2\rho\sigma_e K$, and Ω_{0e} appear as classical coupling terms with opposite signs. Of course, the global energy $\rho E_0 + \mathcal{E}_e$ satisfies a conservative balance equation. For details, see [107].

Due to formula (6.29) and expression (6.6) which leads to $\tilde{\nu} \simeq m_a m_b \beta (T_0(\frac{1}{m_a} + \frac{1}{m_b}))^{-3/2}$, we check that the diffusion coefficient $D \simeq \varepsilon_0 \Psi' m_a m_b / (\rho \tilde{\nu})$ may read in the form

$$D(c) \simeq \frac{\varepsilon_0}{\beta} \frac{\Psi'(c)}{\rho} \left(T_0 \left(\frac{1}{m_a} + \frac{1}{m_b} \right) \right)^{\frac{3}{2}}. \tag{6.30}$$

Remark 39 (On the mass fraction equation). In the above models one has to deal with a non linear equation which is our concern now

$$\rho \frac{D}{Dt} c - \nabla . (\rho c(1-c) D \nabla c) = 0.$$

If we assume for the sake of simplicity that ρD is a constant α, it reads

$$\rho \frac{D}{Dt} c - \alpha \nabla . (c(1-c)\nabla c) = 0.$$

It is clear that in one-dimension geometry, the Heaviside function ($c(x) = 0$ for $x < x_0$ and $c(x) = 1$ for $x > x_0$) is a trivial solution to this equation (therefore, if there is no mixing at initial time, then no mixing occurs).

But this solution is unstable: for an initial data which is a small perturbation of the Heaviside function, a non trivial solution exists. Indeed, with ρ constant, one may exhibit a non trivial solution which satisfies

$$\lim_{t \to 0} c(t,x) = 0 \qquad \text{for } x < x_0, \qquad \text{and } = 1 \qquad \text{for } x > x_0. \tag{6.31}$$

This solution may be built as follows. Let us define the continuous function $c_*(t,x)$ by

$$c_*(t,x) = \frac{1}{2} + \beta(x - x_0) \quad \text{for } |x - x_0| < \frac{1}{2\beta} \quad \text{and} \quad c = 0 \text{ or } 1 \qquad \text{elsewhere,}$$

where $\beta = \beta(t)$ is solution to the ordinary differential equation $\rho \frac{d}{dt}\beta = -2\alpha\beta^3$, that is to say

$$\beta(t) = \frac{1}{2}\sqrt{\frac{\rho}{\alpha t}}.$$

Since $\nabla . (c_*(1-c_*)\nabla c_*) = -2\beta^3(x - x_0)$ (for $|x - x_0| < \frac{1}{2\beta}$), one sees that this function solves the equation we are interested in; of course, it also satisfies (6.31). It may be checked that this solution is stable [if at an initial time t_0, $c(t_0, .)$ is a small perturbation of $c_*(t_0, .)$ then it remains a small perturbation afterwards]. □

6.2 Some Models for Weakly Ionized Plasmas

We address here a very different kind of plasma called *weakly ionized*. In these plasmas, there are a bath of neutral particles which drives the flow of charged particles (and is not affected by these charged particles); the density of ions and electrons is much smaller than the one of neutral particles; then only the binary collisions involving one ion or one electron and one neutral particle need to be taken into account.

To be more precise, in this chapter we consider plasmas with a single species of neutral particles and a single species of ions with an ionization level equal to 1, $Z = 1$.

Hence the macroscopic picture for the charged particles is the following: their bulk velocity is that of the neutral particles; however, their interaction with the bath of neutral particles results in particle diffusion. Beside kinetic models, there are various fluid models for such plasma flows. In the physics literature, one may found a good survey on the studies in this topic in the classical textbook [40]. Moreover, the link between various MHD models for weakly ionized plasmas is described in [16]. We focus here on modelling without any magnetic effect.

In the first subsection, we explain the link between two models: the multifluid model and the multispecies diffusion model. We show that the second model may be derived from the first one by assuming that the typical length of collisions between charged particles and neutral ones is small.

The second subsection is devoted to the approximation of the previous multispecies diffusion model by the so-called *ambipolar diffusion model*. After a heuristic presentation, we give a rigorous proof of this approximation in the framework of the electron–massless approximation using an asymptotic analysis with respect to the Debye length (with some supplementary assumptions).

6.2.1 The Multifluid Model and the Multispecies Diffusion Model

Besides the classical notations for ions and electrons, let us introduce for the neutral particles the following notations: m_n, the molecular mass, N_n, the particle density, $\mathbf{U}_n$, the velocity, and T_n, the temperature.

For notational simplicity, we introduce also the mass ratios $\beta_0^2 = m_0/m_n$, $\beta_e^2 = m_e/m_n$ (of course, β_0 is on the order of 1 and β_e is very small with respect to 1). We assume that the neutral population obeys a perfect gas law with a polytropic coefficient γ, so the pressure is given by $N_n T_n$ and the internal energy $N_n T_n/(\gamma - 1)$. The evolution of the three quantities N_n, $\mathbf{U}_n$, T_n is governed by the classical Euler system

$$
\begin{aligned}
&\frac{\partial}{\partial t} N_n + \nabla.(N_n \mathbf{U}_n) = 0\,,\\
&\frac{\partial}{\partial t} N_n \mathbf{U}_n + \nabla(N_n \mathbf{U}_n \mathbf{U}_n) + \nabla\left(N_n \frac{T_n}{m_n}\right) = 0\,,\\
&\frac{\partial}{\partial t} N_n T_n + \nabla(N_n T_n \mathbf{U}_n) + (\gamma - 1) N_n T_n \nabla.\mathbf{U}_n = 0\,.
\end{aligned}
\tag{6.32}
$$

We assume on the one hand that the ions and the neutral particles are at the same temperature (i.e., $T_0 = T_n$). On the other hand, the electron temperature is generally governed by an evolution equation that involves coupling with the other species; but we assume in the framework of our model that it is somehow given and different from the ion one.

From kinetic models, a hydrodynamic closure leads to the following Euler equations for each species; notice that the friction of the ions and electrons against neutral particles is taken into account thanks to an exchange of momentum (see, e.g., [24]). Denoting by μ_0 and μ_e, the mean collision frequency of the ions and the electrons against the neutral particles, the system reads as

$$(\mathcal{MF})\quad \boxed{\begin{aligned} &\frac{\partial}{\partial t}N_0+\nabla.(N_0\mathbf{U}) = 0,\\ &m_n\beta_0^2\left(\frac{\partial}{\partial t}(N_0\mathbf{U})+\nabla.(N_0\mathbf{U}\mathbf{U})\right) + \nabla(N_0T_n) - N_0q_e\mathbf{E} = -\beta_0\mu_0N_0(\mathbf{U}-\mathbf{U}_n),\\ &\frac{\partial}{\partial t}N_e+\nabla.\Big(N_e\mathbf{U}_e\Big) = 0,\\ &m_n\beta_e^2\left(\frac{\partial}{\partial t}(N_e\mathbf{U}_e) + \nabla.(N_e\mathbf{U}_e\mathbf{U}_e)\right) +\nabla.(N_eT_e) + N_eq_e\mathbf{E}\\ &\qquad = -\beta_e\mu_eN_e(\mathbf{U}_e-\mathbf{U}_n). \end{aligned}}$$

It is supplemented by a relation that defines the electric field $\mathbf{E}$; it is assumed to derive from a potential, so we set as usual

$$q_e\mathbf{E} = -\nabla\Phi,$$

where Φ is solution to the Poisson equation

$$-\frac{\varepsilon^0}{q_e^2}\Delta\Phi = N_0 - N_e. \tag{6.33}$$

6.2.1.1 Derivation of the Multispecies Diffusion Model

We now explain how to derive the multispecies diffusion model from model $(\mathcal{MF})$ which is an example of a system of conservation laws with a relaxation term (due to the slowing of charged particles by collisions with the neutral ones). Notice that for systems of conservation laws in the strong relaxation limit, formal expansions in the style of the Chapman–Enskog expansion have been proposed by different authors (see, e.g., [29, 33] and the references therein). We do not give the proof of convergence (it suffices to say here that the result is based on the Hilbert or Chapman–Enskog expansion methods, so the difficulties are related to the treatment of the initial conditions of the flow).

We assume that the neutral and electron temperatures T_n and T_e are in the same order of magnitude and denote by T_{ref} their characteristic value. Generally, the μ_e and μ_0 are on the same order of magnitude; they are proportional to the neutral density. We may denote their characteristic value by μ_{ref}, so the characteristic mean free path of charged particles is

$$l_C = \left(\frac{T_{\text{ref}}}{m_n}\right)^{1/2} \frac{1}{\mu_{\text{ref}}}$$

Denoting by L_{plasma} the characteristic length scale of the neutral flow (e.g., the size of a body immersed in the flow, or a typical length of the gradient of the ion density), we assume that the mean free path l_C is small if compared to L_{plasma} (it is the case when the neutral density is large enough); thus, we introduce the small parameter

$$\eta = \frac{l_C}{L_{\text{plasma}}} \ll 1\,. \tag{6.34}$$

Moreover, one sets

$$\mu_0 = \frac{\mu_0'}{\eta}, \qquad \mu_e = \frac{\mu_e'}{\eta}.$$

So, the momentum equations in multifluid model ($\mathcal{MF}$) read as

$$\beta_0^2 \frac{\partial}{\partial t}(N_0 \mathbf{U}) + \beta_0^2 \nabla.(N_0 \mathbf{U}\mathbf{U}) + \nabla\left(N_0 \frac{T_n}{m_n}\right) + N_0 \mathbf{G}_0 = -\beta_0 \frac{\mu'_0}{\eta} N_0(\mathbf{U} - \mathbf{U}_n),$$
$$\beta_e^2 \frac{\partial}{\partial t}(N_e \mathbf{U}_e) + \beta_e^2 \nabla.(N_e \mathbf{U}_e \mathbf{U}_e) + \nabla\left(N_e \frac{T_n}{m_n}\right) + N_e \mathbf{G}_e = -\beta_e \frac{\mu'_e}{\eta} N_e(\mathbf{U}_e - \mathbf{U}_n),$$

where we have

$$\mathbf{G}_0 = -\frac{q_e}{m_n}\mathbf{E}, \qquad \mathbf{G}_e = \frac{q_e}{m_n}\mathbf{E}.$$

(Of course, if there are magnetic effects, one must modify the definition of $\mathbf{G}_0$ and $\mathbf{G}_e$ by adding terms $\mathbf{U} \times \mathbf{B}$ and $\mathbf{U}_e \times \mathbf{B}$).

The Chapman–Enskog method consists of substituting the expansion $\mathbf{U}_\alpha^0 + \eta \mathbf{U}_\alpha^1 + \eta^2 \cdots$ truncated at first order in this system and insisting that this truncated expansion be consistent to the second order with the real solution. This results, of course, in a system of PDEs for the profiles N_0, N_e, and this system will be the desired approximation. Hence, consider the truncated Chapman–Enskog expansion

$$\mathbf{U} \simeq \mathbf{U}_0^0 + \eta \mathbf{U}_0^1, \qquad \mathbf{U}_e \simeq \mathbf{U}_e^0 + \eta \mathbf{U}_e^1, \tag{6.35}$$

which is substituted to $\mathbf{U}_\alpha$ in the momentum equations for both the ions and the electrons. Balancing order by order in η, one finds at order 0,

$$\mathbf{U}_0^0 = \mathbf{U}_e^0 = \mathbf{U}_n. \tag{6.36}$$

Moreover, at order one, taking account of (6.36), for $\alpha = 0$ and e, we get

$$\mathbf{U}_\alpha^1 = -\frac{\beta_\alpha}{\mu'_\alpha N_\alpha}\left(N_\alpha\left(\frac{\partial}{\partial t}\mathbf{U}_n + \mathbf{U}_n.\nabla \mathbf{U}_n\right) + \frac{1}{\beta_\alpha^2}\nabla\left(N_\alpha \frac{T_\alpha}{m_n}\right) + \frac{1}{\beta_\alpha^2} N_\alpha \mathbf{G}_\alpha\right).$$

This relation may be simplified by using the momentum conservation for the neutral particles which is

$$\frac{\partial}{\partial t}\mathbf{U}_n + \mathbf{U}_n.\nabla\mathbf{U}_n = -\frac{1}{N_n}\nabla\left(N_n\frac{T_n}{m_n}\right),$$

so we get

$$\mathbf{U}_0^1 = -\frac{1}{\mu'_0\beta_0 N_0}\left(\nabla(N_0\frac{T_n}{m_n}) + N_0\mathbf{G}_0 - \beta_0^2\frac{N_0}{N_n}\nabla(N_n\frac{T_n}{m_n})\right), \tag{6.37}$$

$$\mathbf{U}_e^1 = -\frac{1}{\mu'_e\beta_e N_e}\left(\nabla(N_e\frac{T_e}{m_n}) + N_e\mathbf{G}_e - \beta_e^2\frac{N_e}{N_n}\nabla(N_n\frac{T_n}{m_n})\right). \tag{6.38}$$

The system of equations for the approximate evolution of N_0 and N_e is then obtained by substituting $\mathbf{U}_0^0+\eta\mathbf{U}_0^1$ and $\mathbf{U}_e^0+\eta\mathbf{U}_e^1$ to $\mathbf{U}$ and $\mathbf{U}_e$, respectively, in the conservation of mass for the ions and the electrons.

With this prescription we see that the solution of the multifluid model $(\mathcal{MF})$ may be approximated by the solution of the following diffusion model which reads as

$$\frac{\partial}{\partial t}N_0 + \nabla.(N_0\mathbf{U}_n) = \nabla.\left[\frac{1}{\mu_0\beta_0}\left(\nabla(N_0\frac{T_n}{m_n}) + N_0\mathbf{G}_0 - \beta_0^2\frac{N_0}{N_n}\nabla(N_n\frac{T_n}{m_n})\right)\right],$$

$$\frac{\partial}{\partial t}N_e + \nabla.(N_e\mathbf{U}_n) = \nabla.\left[\frac{1}{\mu_e\beta_e}\left(\nabla(N_e\frac{T_e}{m_n}) + N_e\mathbf{G}_e - \beta_e^2\frac{N_e}{N_n}\nabla(N_n\frac{T_n}{m_n})\right)\right].$$

6.2.1.2 Statement of the Multispecies Diffusion Model

After withdrawing the β_e^2 term in the evolution equation for N_e, we get the multispecies diffusion model, which reads as

$$(\mathcal{DF})\quad \boxed{\begin{array}{ll} \text{(i)} & \frac{\partial}{\partial t}N_0 + \nabla.(N_0\mathbf{U}_n) = \nabla.\left[D\left(\nabla(N_0T_n) + N_0\nabla\Phi - \beta_0^2\frac{N_0}{N_n}\nabla(N_nT_n)\right)\right] \\ \text{(ii)} & \frac{\partial}{\partial t}N_e + \nabla.(N_e\mathbf{U}_n) = \nabla.\left[\frac{\chi}{\beta_e}\left(\nabla(N_eT_e) - N_e\nabla\Phi\right)\right] \\ \text{(iii)} & -\frac{\lambda_D^2}{T_{\text{ref}}}\Delta\Phi = \frac{1}{N_{\text{ref}}}(N_0 - N_e) \end{array}}$$

with

$$D = \frac{1}{\mu_0 \beta_0 m_n}, \qquad \chi = \frac{1}{\mu_e m_n}.$$

and the Debye length given by

$$\lambda_D^2 = \frac{\varepsilon^0 T_{\mathrm{ref}}}{q_e^2 N_{\mathrm{ref}}},$$

Model $(\mathcal{DF})$ can be found from the general prescription in [67]. It is also widely used in the aerodynamics literature (see, e.g., [42, 56]).

Generally, the Debye length is very small if compared to L_{plasma}, so in the sequel we assume that

$$\lambda_D \ll L_{\mathrm{plasma}}$$

It is worth noticing that in that case some difficulties may occur in the numerical solution of this coupled model; indeed, one may check that the characteristic time of the system $[\mathcal{DF}$(ii), (iii)] is on the order of the inverse of the plasma frequency $\lambda_D/\sqrt{T_{\mathrm{ref}}/m_e}$ and is very small if compared to the characteristic time of the plasma evolution $L_{\mathrm{plasma}}/\sqrt{T_{\mathrm{ref}}/m_n}$. To overcome this difficulty, which is related to the quasi-neutrality of the plasma, some authors replace the Poisson equation with a relation expressing that the electric current is zero. This current may be expressed using the above expression of $\mathbf{U}_0^1$ and $\mathbf{U}_e^1$,

$$\mathbf{J} = N_0(\mathbf{U}_n + \eta \mathbf{U}_0^1) - N_e(\mathbf{U}_n + \eta \mathbf{U}_e^1) = (N_0 - N_e)\mathbf{U}_n + \mathbf{J}_1$$

$$\text{with } \mathbf{J}_1 = \frac{\chi}{\beta_e}\left(\nabla(N_e T_e) - N_e \nabla\Phi\right) - D\left(\nabla(N_0 T_n) + N_0 \nabla\Phi - \beta_0^2 \frac{N_0}{N_n}\nabla(N_n T_n)\right).$$

But the requirement $\mathbf{J} = 0$ leads to a system that is not well posed in a two- or three-dimensional geometry, because we get one vector equation for one unknown scalar quantity Φ.

A better way to deal with this difficulty is to derive heuristically a quasi-neutral system with the requirement $\nabla.\mathbf{J} = 0$. If, at initial time, we have $(N_0 - N_e) = 0$, this requirement is equivalent to $\nabla.\mathbf{J}_1 = 0$. Indeed, according to $[\mathcal{DF}$(i), (ii)], we see that $(N_0 - N_e)$ satisfies $\frac{\partial}{\partial t}(N_0 - N_e) + \nabla.((N_0 - N_e)\mathbf{U}_n) = 0$; therefore at any time we get

$$N_0 - N_e = 0$$

Now, condition $\nabla.\mathbf{J}_1 = 0$ reads as

$$-\nabla.\left[\left(DN_0 + \frac{\chi}{\beta_e}N_e\right)\nabla\Phi\right] = \nabla.\left[D\left(\nabla(N_0 T_n) - \beta_0^2 \frac{N_0}{N_n}\nabla(N_n T_n)\right) - \frac{\chi}{\beta_e}\nabla(N_e T_e)\right].$$

Therefore, the model consists in solving the previous elliptic equation for Φ (with N_e replaced by N_0) coupled with the following diffusion equation

$$\frac{\partial}{\partial t}N_0 + \nabla.(N_0\mathbf{U}_n) = \nabla.\left[D\left(\nabla(N_0T_n) - \beta_0^2\frac{N_0}{N_n}\nabla(N_nT_n)\right)\right] + \nabla.\,(DN_0\nabla\Phi)$$

which yields the evolution of ion density N_0. This model is close to the model $(\mathcal{APD})$ which is derived more rigorously in the next subsection.

Remark 40. Approximating solutions of the multifluid model $(\mathcal{MF})$ by solutions of the multispecies diffusion model $(\mathcal{DF})$ makes sense only as long as one is dealing with smooth solutions. It is worth noticing that in the strong relaxation limit the ion velocity $\mathbf{U}$ and the electron one $\mathbf{U}_e$ are supposed to be close to the given field $\mathbf{U}_n$. Hence this theory cannot describe situations where the ion or electron densities have shocks or even high-frequency oscillations. □

6.2.2 The Ambipolar Diffusion Model

Our aim here is to justify the classical *ambipolar diffusion approximation* which is made when the Debye length is very small if compared to the characteristic length of the plasma. But, first of all, we modify the model $(\mathcal{DF})$ by making the electron–massless approximation, i.e., we let β_e vanish.

Thus, all that remains from the electron momentum conservation law is the generalized Ohm's law

$$\nabla(N_eT_e) - N_e\nabla\Phi = 0, \tag{6.39}$$

(which it is equivalent to the Maxwell–Boltzmann relation $N_e = Ce^{\Phi/T_e}$ if T_e is constant).

We assume that we are in a bounded domain; however the boundary conditions are compatible with the interior approximation so that no boundary layers are required to perform the analysis. The presence of conducting boundaries usually involves rather complicated boundary effects, which must be handled by boundary layer techniques as for fully ionized plasma, cf. Sect. 2.2; see also [3] in the complicated case of plasma erosion or electric sheaths. Notice that this derivation is also true in the whole space $\mathbf{R}^3$ but with strong assumptions related to the behavior of the ion density at infinity.

Due to the electron–massless approximation, the two first equations of model $(\mathcal{DF})$ may be replaced by

$$\text{(i)} \qquad \frac{\partial}{\partial t}N_0 + \nabla.(N_0\mathbf{U}_n) = \nabla.\,(D\,[\nabla(N_0T_n) + N_0\nabla\Phi]) - \nabla.\left(\beta_0^2 D\frac{N_0}{N_n}\nabla(N_nT_n)\right),$$

$$\text{(ii)} \qquad \nabla(N_eT_e) - N_e\nabla\Phi = 0,$$

supplemented with the same Poisson equation.

Since we may express $\nabla\Phi$ with respect to $\nabla N_e T_e$, we get the following system satisfied by N_0 and N_e.

$$\frac{\partial}{\partial t}N_0+\nabla.(N_0\mathbf{U}_n)=\nabla.\left(D\left[N_0T_e\nabla\log(N_eT_e)+\nabla(N_0T_n)\right]\right)-\nabla.\left(\beta_0^2 D\frac{N_0}{N_n}\nabla(N_nT_n)\right) \tag{6.40}$$

$$-\frac{\lambda_D^2}{T_{\text{ref}}}\nabla.(T_e\nabla\log(N_eT_e)) = \frac{1}{N_{\text{ref}}}(N_0-N_e). \tag{6.41}$$

This nonlinear equation is the same than the one studied in Sect. 2.2 of Chap. 2.

In the case where the Debye length λ_D is small compared to the macroscopic length, it is classical to make the quasi-neutral approximation. Heuristically, in the limit as λ_D vanishes, we may see that $|N_0 - N_e| \ll N_0 + N_e$ and we get an approximate common value for the ion and electron densities

$$N_e \simeq N_0\,,$$

then, replacing $\nabla\log(N_eT_e)$ by $\nabla\log(N_0T_e)$ in the electric force, one arrives at the single equation governing the dynamics of N_0:

$$(\mathcal{APD})\quad \boxed{\frac{\partial}{\partial t}N_0 - \nabla.[D\nabla((T_n+T_e)N_0)] + \nabla.(N_0\mathbf{U}_n) + \nabla.\left(\beta_0^2 D\frac{N_0}{N_n}\nabla(N_nT_n)\right) = 0.}$$

That is, we have replaced the ion pressure N_0T_n by the sum of the ion and the electron pressures at the expense of withdrawing the electric force. This kind of model may be found in [41] (in the case where $T_e = T_n$, it leads to double the ion diffusion coefficient: introducing this factor 2 is the reason why this diffusion equation is called "ambipolar"). This problem has been addressed in [62]; see also [84] for related results in a different framework—semiconductor modelling.

It is now worthwhile presenting arguments coming from asymptotic analysis in favor of this popular *ambipolar diffusion* approximation in order to clarify the limits of its validity, even if this analysis is not made on the previous system but on a simplified one where

- the drift velocity $\mathbf{U}_n + \beta_0^2\dfrac{D}{N_n}\nabla(N_nT_n)$ is not accounted for,
- the ratio between the two temperatures is a constant denoted by

$$s = \frac{T_e}{T_n},$$

- the time derivative is replaced by a time difference.

More precisely, on a time step Δt, denote by N_0^{ini} the initial value of the ion density at the beginning of the time step and by N_0 the final value of the ion density;

so the time derivative $\frac{\partial N_0}{\partial t}$ is replaced by an implicit discretization $\varpi N_0 - \varpi N_0^{\text{ini}}$, where $\varpi = 1/\Delta t$. This is justified; indeed the time evolution of the ion density is very low compared to the coupling phenomena between electron and ion (which are instantaneous in this model).

Let us now normalize the densities and temperatures by setting

$$N_0' = N_0/N_{\text{ref}}, \qquad N_e' = N_e/N_{\text{ref}}, \qquad N_n' = N_n/N_{\text{ref}}, \qquad T_e' = T_e/T_{\text{ref}},$$
$$T_n' = T_n/T_{\text{ref}}, \qquad D' = DT_{\text{ref}}$$

We use a dimensionless spatial variable related to L_{plasma} and denote $\lambda = \lambda_D/L_{\text{plasma}}$; then system (6.40) and (6.41) reads as

$$\varpi N_0' - \nabla.\left[D' N_0' T_e' \nabla \log(N_e' T_e') + D' \nabla (N_0' T_n')\right] = \varpi G/T_e,$$
$$-\lambda^2 \nabla.(T_e' \nabla \log(N_e' T_e')) = N_0' - N_e'$$

with $G = T_e N_0'^{\text{ini}}$.

We now drop the $'$ for $N_\cdot'$ and $T_\cdot'$. Instead of N_e, N_0, we introduce the new unknowns

$$p = N_e T_e, \qquad w = N_0 T_e$$

Moreover, we set $\alpha = \varpi/T_e$. All the coefficients D, α, T_e are given functions assumed to be smooth and strictly positive. This leads to the following coupled system where p and w are the new unknowns:

$$\alpha w - \nabla.\left[D w \nabla(\log p) + \frac{D}{s}\nabla w\right] = \alpha G \tag{6.42}$$

$$-\lambda^2 \nabla.(T_e \nabla(\log p)) + \frac{1}{T_e}(p - w) = 0\,. \tag{6.43}$$

The last equation, which is the classical nonlinear Poisson equation, is posed in a domain $\mathcal{O} \subset \mathbf{R}^3$ with smooth boundary $\partial\mathcal{O}$; it is supplemented by the following homogeneous Neumann conditions at the boundary $\frac{\partial}{\partial n} \log p\big|_{\partial\mathcal{O}} = 0$ (indicating that the boundary is insulating). Moreover we set $\frac{\partial N_0}{\partial n}\Big|_{\partial\mathcal{O}} = 0$ indicating that particles are reflected at the boundary. We can also assume that $\frac{\partial T_n}{\partial n}\Big|_{\partial\mathcal{O}} = 0$. Then, for system (6.42) and (6.43), we address the following boundary conditions

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\mathcal{O}} = 0, \qquad \left.\frac{\partial w}{\partial n}\right|_{\partial\mathcal{O}} = 0. \tag{6.44}$$

One might object that these boundary conditions are not extremely interesting from a physical viewpoint. However, since the ambipolar diffusion model is a consequence of quasi-neutrality, in order to establish the validity of the ambipolar diffusion up to the boundary of the domain $\mathcal{O}$ it is important that the boundary conditions exclude the possibility of source terms at the boundary that would induce a local breakdown of the quasi-neutrality. The general mathematical analysis involving boundary layers describing the departure from quasi-neutrality near the boundary is a rather difficult subject (this boundary layer analysis is related to a quasi-neutrality breakdown).

Let us now state the main result on the previous system in the limit as $\lambda \to 0$. Let us assume that the data G and the coefficients D, α and T_n verify for some positive constant C_0

$$\frac{1}{C_0} \le G(x) \le C_0\,, \qquad \inf_{\mathcal{O}} D > 0\,, \qquad \inf_{\mathcal{O}} \alpha > 0\,, \qquad T_n \in W^{1,\infty}(\mathcal{O}),$$
$$\inf_{\mathcal{O}} T_n > 0\,. \tag{6.45}$$

Proposition 19. *With the above assumptions, for all $\lambda > 0$, there exists a solution (p^λ, w^λ) of the system (6.42), (6.43), and (6.44). Moreover, there exists a function p_*, such that*

$$p^\lambda \to p_*, \qquad w^\lambda \to p_*, \qquad \textit{strongly in } L^2(\mathcal{O},)$$

when $\lambda \to 0$. The function p_* is a solution of the "ambipolar diffusion equation"

$$\alpha p_* - \nabla.\left((D\frac{1+s}{s})\nabla p_*\right) = \alpha G \tag{6.46}$$

$$\left.\frac{\partial p_*}{\partial n}\right|_{\partial\mathcal{O}} = 0\,. \tag{6.47}$$

Setting $p_* = N_*\, T_e$, this result means that

$$N_0 \simeq N_*, \qquad N_e \simeq N_*,$$

where N_* is the solution of the following limit equation

$$\varpi N_* - \nabla.\,(D\nabla(N_*(T_e + T_n))) = \varpi N_0^{\text{ini}},$$
$$\left.\frac{\partial N_*}{\partial n}\right|_{\partial\mathcal{O}} = 0\,.$$

which is exactly the time-discretized version of equation $(\mathcal{ADP})$.

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Proof of Proposition 19. For all λ fixed, we first prove the existence of a solution (p^λ, w^λ) and we then show that they satisfy some upper and lower bounds uniformly with respect to λ as $\lambda \to 0$. This the most difficult point of the proof, for which we need the assumption saying that s is constant. (The generalization of the result in the case where s is no more constant but a smooth bounded function is a open problem.)

For the sake of conciseness, we set $T = T_e$ and $\beta = 1/T_e$. □

Lemma 11. *Under the assumption (6.45), for all $\lambda > 0$, there exists a solution (p^λ, w^λ) of the system (6.42), (6.43), and (6.44) belonging to the space $H^1(O) \times H^1(O)$. Moreover, there exist positive constants C_0, C_1, C_* such that, for all $\lambda > 0$, this solution (p^λ, w^λ) satisfies the bounds*

$$C_0 \le p^\lambda(x)\,,\ \ w^\lambda(x) \le C_1\,, \qquad \textit{for all } x \in \mathcal{O}\,. \tag{6.48}$$

$$\|\nabla p^\lambda\|_{L^2} \le C_*/\lambda. \tag{6.49}$$

Proof of Lemma 11. Let us set $\kappa = D/s$.

With $w^{(0)} = G/\alpha$, define the sequences $p^{(n)}$ and $w^{(n)}$ as follows. First, $p^{(n)}$ is the solution to

$$-\lambda^2 \nabla.(T \nabla \log p^{(n)}) + \beta p^{(n)} = \beta w^{(n-1)}, \tag{6.50}$$

with the boundary conditions (6.44). Recall that the existence of a unique solution in $H^1(\mathcal{O})$ was proved in Proposition 1 of Sect. 2.2 and that it satisfies

$$\inf w^{(n-1)} \le p^{(n)} \le \sup w^{(n-1)}.$$

Secondly, $w^{(n)}$ is the unique solution in $H^1(\mathcal{O})$ to the linear equation

$$\alpha w^{(n)} - \nabla.\left[\kappa s w^{(n)} \nabla \log p^{(n)} + \kappa \nabla w^{(n)}\right] = \alpha G\,, \tag{6.51}$$

Now, we will show that $p^{(n)}$ and $w^{(n)}$ have lower and upper bounds independent of n. Thus, we introduce the so-called Slotboom variable (see [84])

$$u = p^{(n)s} w^{(n)}$$

and (6.51) becomes

$$\frac{\alpha}{p^{(n)s}} u - \nabla.\left[\kappa w^{(n)} \nabla \log u\right] = \alpha G\,.$$

We may consider this equation satisfied by u and apply the maximum and minimum principle (see result 1 in the Appendix), so we check that

$$\inf p^{(n)s} G \le u \le \sup p^{(n)s} G.$$

Thus, using the definition of u, we get

$$\inf G \leq w^{(n)} \leq \sup G \tag{6.52}$$

Therefore, the sequence $p^{(n)}$ satisfies also for all n

$$\inf G \leq p^{(n)} \leq \sup G. \tag{6.53}$$

The two sequences $w^{(n)}$ and $p^{(n)}$ are also bounded in $L^2(\mathcal{O})$). So with C_0, C_1 independent of n and λ, they satisfy (6.48).

Now, the solution of (6.50) satisfies $\lambda^2 \int (T\nabla p^{(n)} \nabla \log p^{(n)}) dx \leq \int \beta w^{(n-1)} p^{(n)} dx$; therefore, according to (6.48) we see that there exists a constant C_2 independent of n and λ such that

$$\lambda^2 \inf(T) C_1^{-1} \|\nabla \log p^{(n)}\|^2 \leq C_2. \tag{6.54}$$

Then $\left\langle \nabla w^{(n)}, w^{(n)} \kappa \nabla \log p^{(n)} \right\rangle \leq C_3 \|\nabla w^{(n)}\| \lambda^{-1}$ and the solution of (6.51) satisfies

$$(\inf \alpha) \|w^{(n)}\|^2 + (\inf \kappa) \|\nabla w^{(n)}\|^2 \leq C_4 \|w^{(n)}\| + C_5 (1 + \lambda^{-1}) \|\nabla w^{(n)}\|,$$

thus $\|\nabla w^{(n)}\| \leq C(1 + \lambda^{-1})$ and the sequences $p^{(n)}$ and $w^{(n)}$ are bounded in $H^1(\mathcal{O})$ for a fixed λ. Therefore, there are subsequences still denoted by $p^{(n)}$ and $w^{(n)}$ which converge strongly in $L^2(\mathcal{O})$ towards limits denoted by p^λ, w^λ and such that $\nabla \log p^{(n)} \rightharpoonup \nabla \log p^\lambda$ weakly in $L^2(\mathcal{O})$. So, for any test function ψ,we get

$$\left\langle \psi, \alpha w^{(n)} \right\rangle + \left\langle w^{(n)} \nabla \psi, \kappa s \nabla \log p^{(n)} \right\rangle + \left\langle \nabla \psi, \kappa \nabla w^{(n)} \right\rangle = \langle \psi, G \rangle .$$

Using the weak convergence of $\nabla \log p^{(n)}$ towards $\nabla \log p^\lambda$, we can pass to the limit in the nonlinear term $w^{(n)} \nabla \log p^{(n)}$ (see result 6 in the Appendix) and we get

$$\left\langle \psi, \alpha w^\lambda \right\rangle - \left\langle \psi, \nabla.(w^\lambda \kappa s \nabla \log p^\lambda) + \nabla.(\kappa \nabla w^\lambda) \right\rangle = \langle \psi, G \rangle$$

that means that w^λ, p^λ satisfy (6.42) and (6.43) in a weak sense.

By a classical technique, one may check that if others sub-sequences $p^{(n)}$ and $w^{(n)}$ converge strongly in $L^2(\mathcal{O})$ towards limits denoted by p_*^λ and w_*^λ, then they are necessarily equal to w^λ, p^λ. So, the entire sequences $p^{(n)}$ and $w^{(n)}$ converge strongly in $L^2(\mathcal{O})$ towards p^λ and w^λ which are classical solutions of (6.42) and (6.43).

It is also clear that they satisfy the boundary conditions (6.44).

Lastly, according to (6.54), we see that p^λ satisfies (6.49) with C_* independent from λ. □

Lemma 12 (Asymptotic behavior). *Assume that (6.45) holds. Then there exists a constant C, independent of λ, such that*

$$\|\nabla (s \log p^\lambda + \log w^\lambda)\| \leq C. \tag{6.55}$$

When $\lambda \to 0$, the solution (p^λ, w^λ) of (6.42), (6.43), and (6.44) satisfy the asymptotic estimates

$$\|p^\lambda - w^\lambda\|_{L^2} = O(\sqrt{\lambda})\,. \tag{6.56}$$

Proof of Lemma 12. Notice that (6.42) reads as

$$\alpha w^\lambda - \nabla.\left[\kappa s w^\lambda \left(\nabla(\log p^\lambda) + \nabla(\log w^\lambda)\right)\right] + \nabla.(w^\lambda \mathbf{V}) = G.$$

Multiplying it by $s\log p^\lambda + \log w^\lambda$ and integrating over the spatial domain $\mathcal{O}$, we get

$$\int \alpha w^\lambda (s\log p^\lambda + \log w^\lambda) + \int (s\nabla \log p^\lambda + \nabla \log w^\lambda)^2 \kappa w^\lambda = \int G(s\log p^\lambda + \log w^\lambda).$$

Then, according to (6.48), κw^λ is bounded from below and $\log w^\lambda$, $\log p^\lambda$ are also bounded, so C_6 exists independent of λ such that

$$\|\nabla(s\log p^\lambda + \log w^\lambda)\|^2 \le (s+1)(\log C_1)\|G - \alpha w^\lambda\|_{L^1} \inf(\kappa w^\lambda)^{-1} \le C_6^2. \tag{6.57}$$

So (6.55) holds. According to relation (6.49) we have

$$\|\nabla(\log w^\lambda)\| \le C_7/\lambda. \tag{6.58}$$

Thus, multiplying (6.43) by $\log p^\lambda - \log w^\lambda$, we have

$$\lambda^2 \int \frac{T}{s+1}[\nabla(s\log p^\lambda + \log w^\lambda) + \nabla(\log p^\lambda - \log w^\lambda)]\nabla(\log p^\lambda - \log w^\lambda)$$
$$+ \int \beta(\log p^\lambda - \log w^\lambda)\left(p^\lambda - w^\lambda\right) = 0;$$

thus we check that there exists C_8 (independent of λ) such that

$$\int \beta(\log p^\lambda - \log w^\lambda)\left(p^\lambda - w^\lambda\right) \le \lambda^2 \int \frac{T}{s+1}\nabla(s\log p^\lambda + \log w^\lambda)$$
$$\times \nabla(\log p^\lambda - \log w^\lambda) \le \lambda^2 \frac{\sup T}{s+1} C_6 \left(\frac{C_7}{\lambda} + \frac{C_7}{\lambda s} + C_6\right) \le \lambda C_8.$$

According to the identity $(\log p - \log w)\,(p-w) \ge \frac{1}{\max(p,w)}(p-w)^2$, result (6.56) follows (by using (6.48)). □

Proof of Proposition 19. According to (6.56) and (6.55), we know that there exists subsequences still denoted by p^λ, w^λ and a function χ such that we have the strong convergences in $L^2(\mathcal{O})$

$$s\log p^\lambda + \log w^\lambda \to \chi(1+s), \qquad \log p^\lambda - \log w^\lambda \to 0,$$

Thus, we get, by setting $p_* = \exp \chi$,

$$\log p^\lambda \to \log p_*, \qquad \log w^\lambda \to \log p_*, \qquad \text{in } L^2(\mathcal{O}) \text{ strong}$$

and using the Lipschitz property of the exponential,

$$p^\lambda \to p_*, \qquad w^\lambda \to p_*,$$

Moreover, it is easy to check (see point 6 of the appendix) that we have (up to the extraction of a subsequence)

$$\nabla(s \log p^\lambda + \log w^\lambda) \to \nabla((1+s)\log p_*) \qquad \text{weakly in } L^2(\mathcal{O}). \tag{6.59}$$

Now, using an arbitrary test function ξ in $H^1(\mathcal{O})$, (6.42) reads in the following form

$$\int G\,\xi dx = \int \alpha p^\lambda \xi dx + \int \kappa w^\lambda (\nabla\xi).\nabla(s \log p^\lambda + \log w^\lambda) dx.$$

So according to (6.59) and the strong convergence of w^λ and p^λ, we may let $\lambda \to 0$ and we get

$$\begin{aligned}\int G\xi dx &= \int \alpha p_* \xi dx + \int \kappa(1+s) p_* (\nabla\xi).(\nabla \log p_*) dx \\ &= \int \alpha p_* \xi dx + \int \kappa(1+s)\nabla\xi.\nabla p_* dx\end{aligned}$$

which is the weak form of the desired equation for p_*.

Assume now that there exists another subsequence (p^λ, w^λ) and another function χ^* such that $\log p^\lambda \to \chi^*$ and $\log w^\lambda \to \chi^*$. Then by the same arguments as above, we get $p^\lambda \to p^*$ and $w^\lambda \to p^*$ in L^2 strong (where $p^* = \exp \chi^*$) and p^* satisfies for all test function ξ in $H^1(\mathcal{O})$

$$\int \alpha p^* \xi dx + \int \kappa(1+s)\nabla\xi.\nabla p^* dx = \int G\xi dx.$$

Since this is a linear equation, we have $p^* = p_*$ and the entire sequences p^λ, w^λ converge. □

Appendix A

A.1 Tensor Analysis Formula

Here f is a scalar; $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors; $\mathbb{T}$ and $(\nabla\mathbf{A})$ are tensors

$$\text{curl}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla.\mathbf{B}) - (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{B}(\nabla.\mathbf{A}) - (\mathbf{B}.\nabla)\mathbf{A} \tag{A.1}$$

$$\mathbf{B}.\text{curl}(\mathbf{A}) = \nabla.(\mathbf{A}\times\mathbf{B}) + \mathbf{A}.\text{curl}(\mathbf{B}) \tag{A.2}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}, \qquad \mathbf{B}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{B}.\mathbf{C})\mathbf{B} - \mathbf{C}|\mathbf{B}|^2 \tag{A.3}$$

$$\text{curl}(\text{curl}(\mathbf{A})) = -\Delta\mathbf{A} + \nabla(\nabla.\mathbf{A}) \tag{A.4}$$

$$\nabla.(\mathbb{T}.\mathbf{A}) = \mathbb{T} : (\nabla\mathbf{A}) + \mathbf{A}.(\nabla.\mathbb{T}) \tag{A.5}$$

$$\nabla.(\mathbf{AB}) = (\nabla.\mathbf{A})\mathbf{B} + (\mathbf{A}.\nabla)\mathbf{B} \tag{A.6}$$

$$\mathbf{A}\times\text{curl}(\mathbf{B}) = (\nabla\mathbf{B}).\mathbf{A} - (\mathbf{A}.\nabla)\mathbf{B} \tag{A.7}$$

$$\mathbf{B}\times\text{curl}(\mathbf{B}) = \nabla.\left(\frac{\mathbb{I}}{2}|\mathbf{B}|^2 - \mathbf{BB}\right) + (\nabla.\mathbf{B})\mathbf{B} \tag{A.8}$$

$$\text{curl}(f\mathbf{A}) = \mathbf{A}\times(\nabla f) + f\,\text{curl}(\mathbf{A}) \tag{A.9}$$

A.2 Useful Lemmas of Functional Analysis

To help readers who are not conversant in mathematical analysis, we display here some classical results of functional analysis (the proofs may be found in any standard textbook; see, e.g., [22]).

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1. Let us address a linear elliptic equation on a domain $\mathcal{O}$ (an open set of $\mathbf{R}^d$ with smooth boundary) with bounded coefficients k, χ and $\mathbf{Q}$ (such that k and χ are strictly positive)

$$-\nabla(k\nabla u) + \mathbf{Q}.\nabla u + \chi u = \chi f$$

with boundary condition $\frac{\partial u}{\partial \mathbf{n}} = 0$ ($\mathbf{n}$ is the outwards normal) and the right-hand side f bounded; then the unique solution u satisfies

$$\min_{\mathcal{O}} f \leq u \leq \max_{\mathcal{O}} f.$$

There is an analogous result with other boundary conditions.
2. Assume that domain $\mathcal{O}$ is an open set of $\mathbf{R}^d$ with smooth boundary. Denote $t_f>0$ a final time. Assume that ξ is in $L^\infty(0, t_f; \mathcal{O})$ and f is in $\infty(0, t_f, L^2(\mathcal{O}))$. Then if ϕ^{ini} belongs to $L^2(\mathcal{O})$, there exists a unique solution to the equation

$$\frac{\partial \phi}{\partial t} + \xi\phi = f, \qquad \phi(0) = u^{ini},$$

which is in $C(0, t_f, L^2(\mathcal{O}))$.
3. If the sequences u_p and v_p are such that[1]

$$u_p \to u \text{ in } L^2(\mathcal{O}) \text{ strongly}, \qquad v_p \to v \text{ in } L^2(\mathcal{O}) \text{ weakly},$$

then $\int u_p(x)v_p(x)dx \to \int u(x)v(x)dx$.
4. If the sequences u_p and v_p are such that[2]

$$u_p \to u \text{ in } L^2(\mathcal{O}) \text{ strongly}, \qquad v_p \to v \text{ in } L^\infty(\mathcal{O}) \text{ weakly-*}$$

Then we have $u_p v_p \to uv$ in $L^2(\mathcal{O})$ weakly.
5. If the sequence u_p is such that the u_p are bounded in $H^1(\mathcal{O})$, then there exists a subsequence u_p and a function u in L^2 such that $u_p \to u$ in $L^2(\mathcal{O})$ strongly.
6. If the entire sequence u_p converges to u in $L^2(\mathcal{O})$ and the u_p are bounded in $H^1(\mathcal{O})$, then

$$\nabla u_p \rightharpoonup \nabla u, \text{ in } L^2(\mathcal{O}) \text{ weakly}.$$

[Indeed there exists a subsequence u_p and ξ in $L^2(\mathcal{O})$ such that $\nabla u_p \rightharpoonup \xi$, so for all χ in $H_0^1(\mathcal{O})$,[3] we have $\lim\langle\nabla u_p, \chi\rangle = -\lim\langle u_p, \nabla\chi\rangle = -\langle u, \nabla\chi\rangle = \langle\nabla u, \chi\rangle$. Since $H_0^1(\mathcal{O})$ is dense in $L^2(\mathcal{O})$, we have $\xi = \nabla u$].

[1] One says that $v_p \to v$ in $L^2(\mathcal{O})$ weakly when $\int v_p\varphi \to \int v\varphi$ for all ϕ in $L^2(\mathcal{O})$.
[2] One says that $v_p \to v$ in $L^\infty(\mathcal{O})$ weakly-* when $\int v_p\varphi \to \int v\varphi$ for all ϕ in $L^1(\mathcal{O})$.
[3] The subspace H^1 of functions that are zero on the boundary $\partial\mathcal{O}$.

7. Gronwall's lemma. Let y be a time-dependent function that is positive and satisfies

$$y(t) \leq a + b \int_0^t y(s)ds;$$

then we have $y(t) \leq ae^{bt}$ for all t.

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