Lecture Notes in Mathematics 2113

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Algebraic K-theory of Crystallographic Groups

The Three-Dimensional Splitting Case

Springer

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ISBN 978-3-319-08152-6 ISBN 978-3-319-08153-3 (eBook)
DOI 10.1007/978-3-319-08153-3
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2014946579

Mathematics Subject Classification (2014): 20H15, 19B28, 19A31, 19D35

Printed on acid-free paper

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To the memory of Almir Alves (1965–2009)

Acknowledgments

The authors would like to thank Tom Farrell for originally suggesting this project to us, and for his invaluable encouragement and support while it was being completed. We are also grateful to Jean-François Lafont for his encouragement, many helpful discussions, and for his comments on preliminary versions of this manuscript. The graphics in this monograph were kindly produced by Dennis Burke.

This work is dedicated to the memory of Almir Alves. Pedro Ontaneda has written a memorial essay describing Almir's remarkable life. It can be found (at the time of this writing) at:

www.math.binghamton.edu/dept/alves/pedro/Almir2.html

Almir grew up in poverty, attending school for the first time when he was 10 years old. He was a lifelong fighter, who flourished through perseverance, and became a Ph.D. mathematician with a gift for explaining complicated ideas in a simple way. He is greatly missed by all of us.

Pedro Ontaneda informed us that Almir, a few days before his death, told him that he knew how to prove the splitting formulas for K_0 and K_{-1}.

This project was supported in part by the NSF, under the second author's grants DMS-0805605 and DMS-1207712.

Contents

Chapter 1
Introduction

Algebraic K-theory is a branch of algebra dealing with linear algebra over a general ring R instead of a field. It associates to any ring R a sequence of abelian groups $K_n(R)$. The first two of these groups, K_0 and K_1, are easy to describe in concrete terms. For instance, a finitely generated projective R-module defines an element of $K_0(R)$, and an invertible matrix over R has a "determinant" in $K_1(R)$. The entire sequence of groups $K_n(R)$ behaves something like a homology theory of rings.

Algebraic K-theory plays an important role in many areas of mathematics, especially number theory, algebraic topology, and algebraic geometry. In particular, the class group of a number field F is essentially $K_0(R)$, where R is the ring of algebraic integers in F, and "Whitehead torsion" in topology is an element of $K_1(\mathbb{Z}\pi)$, where π is the fundamental group of the space being studied. The K-theory of group rings is computable via algebraic number theory when the group is finite and it is understood to some extent. It has been computed for many classes of torsion-free groups (see, for example, [Bas64,FH81,FH73,FJ87,Pl80] and [Wal78]). Results in the case of infinite groups with torsion include work of:

- Bass and Murthy on finitely generated groups [BM67],
- Bürgisser on arithmetic groups [B83],
- Connolly and Koźniewski [CK90], Pearson [P98] and Tsapogas [T95] on crystallographic groups,
- Farrell and Hsiang [FH70] and Farrell and Jones [FJ95] on virtually cyclic groups,
- Lück and Stamm on cocompact planar groups [LS00],
- Berkove, Farrell, Juan-Pineda, and Pearson on Bianchi groups [BFJ-PP00],
- Berkove, Juan-Pineda, and Pearson on Fuchsian groups [BJ-PP01],
- Berkove, Juan-Pineda, and Lu on mapping class groups [BJ-PL04],
- Lück on a certain finite extension of the Heisenberg group [L05],
- Lafont and Ortiz on hyperbolic three-simplex reflection groups [LO09], and
- Lafont, Magurn, and Ortiz on hyperbolic reflection groups [LMO10].

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_1

The goal of this monograph is to compute the lower algebraic K-theory of the *split three-dimensional crystallographic groups*; i.e., the groups $\Gamma \leq \mathrm{Isom}(\mathbb{R}^3)$ that fit into a split short exact sequence

$$1 \to L \to \Gamma \to H \to 1,$$

where L is a discrete cocompact additive subgroup of $\mathbb{R}^3$ (i.e., a *lattice* in $\mathbb{R}^3$), H is a finite subgroup of the orthogonal group $O(3)$, and the map $\Gamma \to H$ sends an isometry $\gamma \in \Gamma$ to its linear part. It is well known that every crystallographic group fits into a similar short exact sequence, although the sequence does not split in general. Note also that a group Γ might admit another type of splitting without being a split crystallographic group in our sense. One such example is the Klein bottle group, a two-dimensional crystallographic group that factors as $\mathbb{Z} \rtimes \mathbb{Z}$, but admits no factorization of the above form since it is torsion-free. There are 73 split three-dimensional crystallographic groups up to isomorphism, representing a third of the 219 isomorphism classes of three-dimensional crystallographic groups in all. The split crystallographic groups are also called *splitting groups*, since every n-dimensional crystallographic group embeds in a split n-dimensional crystallographic group (its splitting group) as a subgroup of finite index (see [Ra94, pp. 312–313]). Thus, the 73 split three-dimensional crystallographic groups contain the remaining three-dimensional crystallographic groups as subgroups of finite index.

Our computation of the K-groups uses the fundamental work of Farrell and Jones [FJ93], who established an isomorphism

$$Wh_n(\Gamma) \cong H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}),$$

where $E_{\mathcal{VC}}(\Gamma)$ is a model for the classifying space of Γ with isotropy in the family of virtually cyclic subgroups, and Γ is a cocompact discrete subgroup of a virtually connected Lie group. In particular, their work applies to crystallographic groups of all dimensions.

The first computations of the lower algebraic K-theory of crystallographic groups were made by Kimberly Pearson [P98], who completely handled the two-dimensional case. Alves and Ontaneda [AO06] derived the following general formula for the Whitehead group of a three-dimensional crystallographic group:

$$Wh(\Gamma) \cong \bigoplus_{G \in \mathcal{I}} Wh(G),$$

where $\mathcal{I}$ is the set of conjugacy classes of maximal infinite virtually cyclic subgroups. They also prove that the above direct sum is finite.

In this monograph, we will extend the results of [AO06] in two directions: first, we will derive similar general formulas for $K_{-1}(\mathbb{Z}\Gamma)$, $\tilde{K}_0(\mathbb{Z}\Gamma)$, and $Wh(\Gamma)$, where Γ is an arbitrary (not necessarily split) three-dimensional crystallographic group. These general results are presented in Chap. 5, which is largely self-contained. The main result is Theorem 5.1, a splitting formula for the lower algebraic K-theory of three-dimensional crystallographic groups:

$$Wh_n(\Gamma) \cong H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{\hat{\ell} \in \mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty}).$$

The indexing set $\mathcal{T}''$ consists of a selection of one line from each Γ-orbit of lines, and may be taken to be finite, since the groups in question are trivial in all but finitely many cases. In the case $n = 1$, the first summand in the above formula vanishes, and the second summand can be identified with the right side of the formula from [AO06] (above). The second goal is to make explicit calculations of the lower algebraic K-groups of the split three-dimensional crystallographic groups; that is, to describe completely the isomorphism types of $K_{-1}(\mathbb{Z}\Gamma)$, $\tilde{K}_0(\mathbb{Z}\Gamma)$, and $Wh(\Gamma)$ as abelian groups. In order to do this, it is necessary to identify the split three-dimensional crystallographic groups as explicitly as possible. We do this by specifying all possible pairings (L, H) (up to a certain type of equivalence), where L and H are as above, and the action of H (a subgroup of $O(3)$) on L is the obvious one. Our work in classifying these groups is contained in Chaps. 2–4; the results are summarized in Table 4.1. Chapters 6–9 contain parallel computations, for all 73 of the split three-dimensional crystallographic groups, of the first and second summands from the splitting formula; Chaps. 6 and 7 describe the first summand, and Chaps. 8 and 9 describe the second summand. In Chap. 10, we summarize the results of the calculations in Table 10.1, and give a pair of examples illustrating how to assemble the various pieces of the calculation from the preceding chapters.

Now let us give a more detailed, chapter-by-chapter description of this work.

Chapters 2–4 give a complete classification of the split three-dimensional crystallographic groups. The presentation is almost entirely self-contained, assumes no prior knowledge of crystallographic groups, and indeed involves little more than basic group theory and linear algebra.

Chapter 2 describes the classification of *three-dimensional point groups* (*point groups* hereafter), which are the subgroups of $O(3)$ that leave a lattice $L \leq \mathbb{R}^3$ invariant. Our treatment here is standard for the most part, and the basic elements of our classification can be found in Chap. 4 of [Sc80] (although we assume no prior familiarity with that source). One difference is that we need to find explicit groups of matrices for our computations in later chapters, so we give a more detailed classification than any that we were able to find in the literature. We begin our analysis by proving that point groups are finite, and satisfy the *crystallographic restriction* [Sc80, p. 32]: every element of a point group has order 1, 2, 3, 4, or 6. The next step is to classify the orientation-preserving point groups H. The classification heavily exploits the fact that every $h \in H$ is a rotation about an axis (which we call a *pole*, following [Sc80]). Using a counting argument (Proposition 2.1; see also [Sc80, p. 45]), one can enumerate all of the possible numbers of orbits of poles, and determine the possible orders of the elements h that act on a given pole (Proposition 2.2; also [Sc80, pp. 46–49]). It turns out that the latter numerical information completely determines the orientation-preserving point group H up to conjugacy within $O(3)$; we carefully argue a particular case of this fact in our proof of Theorem 2.1, which gives simple descriptions of the orientation-preserving point groups. There are 11 in all. It is then straightforward to classify the remaining point groups. If a point group H contains the antipodal map (-1), then it can be expressed as $\langle H^+, (-1)\rangle$, where H^+ is the orientation-preserving subgroup; thus there are also 11 point groups that contain (-1) (Theorem 2.2). The remaining point groups (which are not subgroups of $SO(3)$, but also do not contain the inversion (-1)) are all necessarily subgroups of index 2 inside of the 11 point groups containing the inversion, and may therefore be recovered as kernels of surjective homomorphisms $\phi : H \to \mathbb{Z}/2$, where H contains the inversion (-1). This scheme of classification is carried out in Sect. 2.4; there are 10 additional groups of this last type, making 32 in all. The chapter concludes with an attempt at an intuitive description of the "standard point groups", which are simply the preferred forms (up to conjugacy) of the point groups that are used throughout the rest of the monograph. (We also describe a few non-standard point groups that arise naturally in our arguments.) The standard point groups are described by their generators in Figs. 2.1 and 2.2.

Chapter 3 contains a detailed classification of the possible arithmetic equivalence classes (Definition 3.1, which is taken from [Sc80, p. 34]) of pairs (L, H), where L is a lattice and H is a point group such that $H \cdot L = L$. We will eventually show (Theorem 4.1) that an arithmetic equivalence class uniquely determines a split crystallographic group up to isomorphism, so Chap. 3 contains the heart of the classification of split crystallographic groups. Our general approach to the classification of the pairs (L, H) is as follows. We begin with a standard point group H; for the sake of illustration, let us assume that $H = S_4^+ \times (-1)$, the group of all signed permutation matrices. We can then deduce certain facts about the lattice L. For instance, H contains rotations about the coordinate axes, so an elementary argument (Lemma 3.1(1)) shows that there are lattice points on each of the coordinate axes. Moreover, since H acts transitively on the coordinate axes, the lattice points on the coordinate axes having minimal norm must all have the same norm, which we can assume is 1 up to arithmetic equivalence (which permits rescaling of the lattice L). It follows directly that $\mathbb{Z}^3 \leq L$, and that $c(1, 0, 0)$, $c(0, 1, 0)$, and $c(0, 0, 1)$ are not in L if $0 < c < 1$. (This conclusion is recorded in Proposition 3.1(3), but in somewhat different language.) From here, it is straightforward to argue that there are only three possibilities for the lattice L—see Lemma 3.2 and Corollary 3.1. It is then possible to show that all three of these possible lattices result in different arithmetic classes of pairs (L, H) (Theorem 3.2). The arguments in all of Chap. 3 follow the same pattern: for a fixed standard point group H, we attempt to "build" L using properties of H, as in the above example. We are able to show that L can always be chosen from a list of only seven lattices, which are described by generating sets in Corollaries 3.1 and 3.2. The arguments from Chap. 3, while elementary, are significantly more detailed than what can be found in [Sc80]. We arrive at a total of 73 arithmetic classes, which agrees with the count from [Sc80, p. 34].

Chapter 4 contains the crucial Theorem 4.1, which shows that the arithmetic classes of pairs (L, H) are in exact correspondence with the isomorphism classes of split crystallographic groups. In the proof, we appeal to Ratcliffe [Ra94], which is the only place where the argument of Chaps. 2–4 fails to be self-contained. The split three-dimensional crystallographic groups are classified up to isomorphism in Table 4.1. One interesting feature of our classification is that all 73 groups are contained in 7 basic maximal groups as subgroups of finite index. This fact will be heavily used in later chapters. We denote these maximal groups Γ_i ($i = 1, \ldots, 7$).

Chapter 5 contains a proof of the main splitting result, Theorem 5.1. This chapter is independent of the previous ones, and the main results apply generally, to all three-dimensional crystallographic groups, not just to the split ones. We begin by giving an explicit description of a model for $E_{\mathcal{VC}}(\Gamma)$, for any crystallographic group Γ (our construction comes from [Fa10]). Begin with a copy of $\mathbb{R}^3$, suitably cellulated to make it a Γ-CW complex. This is a model for $E_{\mathcal{FIN}}(\Gamma)$. For each $\ell \in L \subseteq \mathbb{R}^3$ that generates a maximal cyclic subgroup of L, we define a "space of lines" $\mathbb{R}^2_\ell$, consisting of the set of all lines $\hat{\ell} \subseteq \mathbb{R}^3$ having ℓ as a tangent vector. Each space $\mathbb{R}^2_\ell$ is isometric to $\mathbb{R}^2$ with respect to a suitable metric (see Sect. 5.2). The space $\mathbb{R}^3 * \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is a model for $E_{\mathcal{VC}}(\Gamma)$ (Proposition 5.2). The general form of this classifying space allows us to deduce a preliminary splitting result (Proposition 5.4):

$$
\begin{aligned}
&H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \\
&\quad \cong H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)) \\
&\quad \to E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})).
\end{aligned}
$$

Here $\Gamma(\ell)$ is the (finite index) subgroup of Γ that takes the line ℓ to a line parallel to ℓ (possibly reversing the direction), and $\mathcal{VC}_{\langle \ell \rangle}$ is the family of subgroups consisting of: (a) finite subgroups of $\Gamma(\ell)$ and (b) virtually cyclic subgroups of $\Gamma(\ell)$ that contain a translation $\tilde{\ell} \in L$ parallel to ℓ. The indexing set $\mathcal{T}$ consists of a single choice of maximal cyclic subgroup $\langle \ell \rangle \leq L$ from each H-orbit.

The next step is to compute the sum of cokernels on the right side of the formula above. We are thus led to consider the classifying space $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$; we use the model $\mathbb{R}^3 * \mathbb{R}^2_\ell$. By making a detailed analysis of the possible cell stabilizers in $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$ (Lemma 5.2 and Corollary 5.1), we are able to conclude that the great majority of these stabilizers are "negligible". Here the class of *negligible* groups is carefully chosen so that $Wh_n(G) \cong 0$ when G is negligible and $n \leq 1$, and so that the property of being negligible is closed under passage to subgroups. (See Definition 5.3 and Lemma 5.1.) As a result, we are able to argue (Propositions 5.5 and 5.6) that the only cells from the classifying space $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$ that make a contribution to the cokernels for $n \leq 1$ come from a subcomplex E that is a disjoint union

$$E = \coprod_{\hat{\ell}} E_{\mathcal{VC}}(\Gamma_{\hat{\ell}}).$$

The general splitting formula (Theorem 5.1) now follows readily from Proposition 5.4.

The remainder of the monograph uses Theorem 5.1 and the Farrell-Jones isomorphism to compute the lower algebraic K-theory of the split crystallographic groups. It now suffices to make separate computations of

$$H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \quad \text{and} \quad \bigoplus_{\hat{\ell} \in \mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty}),$$

where the latter sum is indexed over a choice from each Γ-orbit of a line $\hat{\ell} \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_{\ell}$ with non-negligible stabilizer.

Our task in Chap. 6 is to describe explicit Γ-CW structures on $\mathbb{R}^3$, where Γ ranges over the groups Γ_i ($i = 1, \ldots, 7$). This involves using Poincaré's Fundamental Polyhedron Theorem (Theorem 6.1, adapted from [Ra94, p. 711]) to produce a fundamental domain for the action of each Γ_i, which leads to the desired Γ-equivariant cellulation (Theorem 6.2). We consider each group Γ_i in great detail, producing the desired cellulation and recording the non-negligible cell stabilizers (Theorems 6.3–6.9).

In Chap. 7, we compute $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$, for all 73 split crystallographic groups Γ. The chapter begins by summarizing the isomorphism types of the groups $Wh_n(G)$, where $n \leq 1$ and G is a finite subgroup of a crystallographic group (see Table 7.1). Most of the groups $Wh_n(G)$ were known before (the reference [LO09] collects the previously known results), but for three groups ($\mathbb{Z}/4 \times \mathbb{Z}/2$, $\mathbb{Z}/6 \times \mathbb{Z}/2$, and $A_4 \times \mathbb{Z}/2$) we make original calculations of $Wh_n(G)$ ($n \leq 0$). These calculations are contained in Sects. 7.1.1, 7.1.2, and 7.1.3, respectively. Once we have understood the above groups $Wh_n(G)$ ($n \leq 1$), we are ready to use a spectral sequence due to Quinn [Qu82] to compute the groups $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$. The main additional ingredient that we will need is information about the cell stabilizers in a model for $E_{\mathcal{FIN}}(\Gamma)$, where Γ is any of the 73 split crystallographic groups. It is here that we use the fact that each split crystallographic group Γ is a finite index subgroup of some Γ_i, where $i \in \{1, \ldots, 7\}$. The model for $E_{\mathcal{FIN}}(\Gamma_i)$ (which was explicitly described in Chap. 6) is also naturally a model for $E_{\mathcal{FIN}}(\Gamma)$; it is necessary only to recompute the cell stabilizer information for the action of the smaller group. This is done using a simple procedure (Procedure 7.1) and the resulting cell stabilizer information is recorded in Tables 7.2, 7.3, 7.4, 7.5, and 7.6. From here, it is usually straightforward to compute $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$: in almost all cases, the vertices make the only contribution to the calculation, so the group in question is a direct sum of K-groups of the vertex stabilizers. (A more precise statement is given in Lemma 7.1.) We can therefore obtain a computation simply by referring to the relevant tables. There are five more difficult cases, in which there is a non-trivial contribution from the edges. We make detailed calculations of $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$ in each of these cases; the calculations are contained in Examples 7.4–7.8. Examples 7.6 and 7.8 are especially notable, since they provide the first examples of infinite groups with torsion such that $\tilde{K}_0(\mathbb{Z}\Gamma)$ has elements of infinite order. The calculations of Chap. 7 are summarized in Table 7.8.

The final step is to compute the second summand from Theorem 5.1. This is done in Chaps. 8 and 9, which are organized like Chaps. 6 and 7. In Chap. 8, we must consider the action of each group Γ_i ($i = 1, \ldots, 7$) on the associated space of lines $\coprod_{\langle \ell \rangle} \mathbb{R}^2_{\ell}$, which (up to isometry) is a countably infinite disjoint union of planes. We are able to show that, for each Γ_i ($i = 1, \ldots, 5$), only two or three of these planes make a contribution to K-theory, and that no plane makes a contribution to K-theory when $i = 6$ or 7 (Proposition 8.3). It therefore becomes feasible to determine the required actions and their fundamental domains; this is done in Theorems 8.1–8.5, where we also compute (one possible choice of) the indexing set $\mathcal{T}''$ explicitly, for all of the groups Γ_i, $i = 1, \ldots, 5$. We describe each of the lines $\hat{\ell} \in \mathcal{T}''$ by an explicit parametrization, and compute the *strict stabilizer group* $\overline{\Gamma}_{\hat{\ell}} = \{\gamma \in \Gamma \mid \gamma_{|\hat{\ell}} = id_{\hat{\ell}}\}$, where Γ is any of the groups Γ_i, $i = 1, \ldots, 5$.

In the beginning of Chap. 9, we describe how to compute the indexing sets $\mathcal{T}''$ and the strict stabilizers of lines $\hat{\ell} \in \mathcal{T}''$ for all of the split crystallographic groups Γ. This involves a simple procedure (Procedure 9.1) that computes $\mathcal{T}''$ and the strict stabilizers of lines $\hat{\ell} \in \mathcal{T}''$ for a group Γ', provided that the latter information is already known for a larger group Γ such that $[\Gamma : \Gamma'] < \infty$. Procedure 9.1 is directly analogous to Procedure 7.1, and similarly exploits the fact that every split crystallographic group sits inside of one of the Γ_i, $i = 1, \ldots, 7$. The strict stabilizer information and $\mathcal{T}''$ is recorded in Tables 9.1, 9.2, 9.3, 9.4, and 9.5. (In particular, it follows from our calculations that $\mathcal{T}''$ is always finite, as we claimed above.) We then describe a procedure (Procedure 9.2) that computes the stabilizer of a line, provided that its strict stabilizer is given to us. Using this procedure, we compute the stabilizers of all of the lines in $\mathcal{T}''$ (for arbitrary Γ); the results are summarized in Tables 9.6, 9.7, 9.8, 9.9, and 9.10. It is then enough to determine the isomorphism types of the relevant cokernels from Theorem 5.1; these are summarized in Table 9.11. (We note also that Sect. 9.3 contains original computations of a few of the cokernels from Table 9.11.) By the end of Chap. 9, we have completely reduced the problem of computing the second summand from Theorem 5.1 to a matter of consulting the relevant tables.

Chapter 10 contains Table 10.1, a complete summary of the isomorphism types of the lower algebraic K-groups of the split crystallographic groups Γ. (A group is omitted from the table if all of its K-groups are trivial.) Chapter 10 also contains a few examples, which are intended to help the reader to assemble the calculations in this work.

It is natural to ask if our arguments can be generalized to other classes of crystallographic groups. The first class to consider is that of the remaining three-dimensional crystallographic groups—there are 146 more up to isomorphism. We expect that all of our basic arguments will still be applicable. The splitting formula (Theorem 5.1) applies to all three-dimensional crystallographic groups (as we have argued in Chap. 5). Moreover, each three-dimensional crystallographic group sits inside of one of the Γ_i ($i = 1, \ldots, 7$) as a subgroup of finite index, so some procedure for passing to subgroups of finite index should still work. It seems likely that the classification itself, and the sheer number of cases, will be the biggest obstacles to a complete calculation of the lower algebraic K-theory for all 219 three-dimensional crystallographic groups. The authors intend to complete such a calculation in later work.

The obstacles to extending our arguments to dimension four seem much more substantial. Aside from the greater difficulty of classifying four-dimensional crystallographic groups (and the large number of such groups), it seems that some basic features of our approach are likely to fail. For instance, the splitting formula in Theorem 5.1 depends on having a large number of negligible cell stabilizers, which in turn depends heavily on the low codimension of cells in $\mathbb{R}^3$ and $\mathbb{R}^2_\ell$ (see, especially, Lemma 5.2 and Corollary 5.1). It is therefore not clear whether an analogous formula can be proved in dimension four. In any event, higher-dimensional cells should make more contributions to the calculations in dimension four, which will make these calculations much more complicated.

Chapter 2
Three-Dimensional Point Groups

In this chapter, we give a complete classification of the 32 three-dimensional point groups. Our treatment is self-contained, but the principles of the classification come from [Sc80, Chap. 4]. Our goal is to describe the point groups as explicit groups of matrices.

2.1 Preliminaries

Definition 2.1. A *lattice* in $\mathbb{R}^n$ is a discrete, cocompact subgroup of the additive group $\mathbb{R}^n$.

We say that $H \leq O(n)$ is a *point group* (of dimension n) if there is a lattice $L \leq \mathbb{R}^n$ such that H leaves L invariant, i.e., $H \cdot L = L$. Here we will be interested only in the case $n = 3$, so "point group" will refer to a three-dimensional point group, and "lattice" will mean a lattice in $\mathbb{R}^3$.

Our first goal is to classify the point groups. We will prove some preliminary facts in this section, and complete the classification in Sect. 2.4.

First, we set some conventions. If ℓ is a line defined by a system $\mathcal{S}$ of equations, we sometimes denote this line $\ell(\mathcal{S})$. For instance, $\ell(y = z = 0)$ denotes the x-axis. Similarly, if a plane P is defined by the equation E, we sometimes denote this plane $P(E)$. For instance, $P(z = 0)$ denotes the xy-plane.

Lemma 2.1. *Let H be a point group.*

1. $|H| < \infty$*;*
2. *Any $h \in H$ leaves some line ℓ invariant: $h \cdot \ell = \ell$. If $h \neq 1$ is orientation-preserving, then h fixes a unique line ℓ, and acts as a rotation about this line.*
3. *(The Crystallographic Restriction) If $h \in H$, then $|h| \in \{1, 2, 3, 4, 6\}$.*
4. *Let $\ell \subseteq \mathbb{R}^3$ be a one-dimensional subspace. The group $H^+_\ell = \{h \in H \cap SO(3) \mid h_{|\ell} = id_\ell\}$ is a cyclic group of rotations about the axis ℓ, of order* 1*,* 2*,* 3*,* 4*, or* 6.

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Proof. 1. Choose an element $v \in L - \{0\}$. Since L is discrete, $H \leq O(3)$, and $H \cdot L = L$, there can be at most finitely many translates of v under the action of H. It follows that the stabilizer group H_v satisfies $[H : H_v] < \infty$.

We next choose some $v_1 \in L$ such that $\{v, v_1\}$ is a linearly independent set, and conclude by the same reasoning that $[H_v, H_{\{v,v_1\}}] < \infty$, where $H_{\{v,v_1\}}$ is the subgroup of H that fixes both v and v_1.

It follows that $H_{\{v,v_1\}}$ has finite index in H. But clearly $H_{\{v,v_1\}}$ has order at most 2 (since it acts orthogonally on $\mathbb{R}^3$ and fixes a plane), so H itself must be finite.

2. Let $h \in SO(3)$. It follows from basic linear algebra that 1 is an eigenvalue of h, so h fixes a non-trivial subspace. If h fixed a subspace of dimension greater than or equal to 2, then h would necessarily be the identity since $h \in SO(3)$. The second statement follows, since h must act in an orientation-preserving fashion on the plane perpendicular to ℓ, which means that it is a rotation in this plane.

The first statement follows easily from the second one, since any $h \in O(3) - SO(3)$ can be expressed as $h = (-1)\hat{h}$ where $(-1) \in O(3)$ is the antipodal map and $\hat{h}$ is orientation-preserving.

3. It is sufficient to prove the statement for the case in which $h \in SO(3) - \{1\}$, since any $\bar{h} \in O(3) - SO(3)$ can be expressed as a product of an orientation-preserving element and the antipodal map (-1). By (2), h fixes a line, which we can assume is the z-axis.

Let L be a lattice that is invariant under the action of H on $\mathbb{R}^3$. We claim that the xy-plane contains a non-zero vector $v \in L$. Let $v_1 \in L - \ell(x = y = 0)$. The vector $v_1 - hv_1$ is non-zero and lies in the xy-plane, proving the claim. Now suppose that v_2 is the smallest vector in $L \cap P(z = 0)$; we can assume that $||v_2|| = 1$.

We consider the set $\{v_2, hv_2, \ldots, h^{n-1}v_2\}$, where $|h| = n$. No two of the elements in this set are equal, and any two $h^i \cdot v_2$, $h^j \cdot v_2$, $(i \neq j)$ must satisfy

$$||h^i \cdot v_2 - h^j \cdot v_2|| \geq 1,$$

by the minimality of the norm of v_2. Moreover,

$$\{v_2, \ldots, h^{n-1}v_2\} \subseteq \{(x, y, 0) \mid x^2 + y^2 = 1\}.$$

Thus, we have n points arranged on the unit circle in such a way that no two are closer than 1 unit. It follows easily from this that $n \leq 6$, since the circumference of the circle is 2π.

We now rule out the case $|h| = 5$. If $|h| = 5$, then h is rotation about the z-axis through $2\pi/5$ radians. One checks that

$$||v_2 + h^2 \cdot v_2|| = 2\cos\left(\frac{2\pi}{5}\right) < 1.$$

This is a contradiction.

4. This is a straightforward observation based on (2) and (3).

2.2 Classification of Orientation-Preserving Point Groups

Now we turn to the classification of point groups, based on [Sc80]. We would like to point out that Propositions 2.1 and 2.2, and Definition 2.2, are based on [Sc80, pp. 45–48]. Our contribution here is to describe the point groups explicitly in terms of generating sets.

Definition 2.2. Let $H \leq SO(3)$ be a non-trivial point group, and let $h \in H - \{1\}$. If ℓ is a one-dimensional subspace of $\mathbb{R}^3$ that is fixed by h, then we say that ℓ is a *pole* of h. We also say that ℓ is a pole of H.

Suppose $H \leq SO(3)$ is a non-trivial point group. Lemma 2.1(2) implies that each $\gamma \in H - \{1\}$ has a unique pole. We let $\mathcal{L}$ be the set of all poles of H. This set must be finite by Lemma 2.1(1) and (2). For each $\ell \in \mathcal{L}$, choose unit vectors $v_\ell^+, v_\ell^- \in \ell$ ($v_\ell^+ \neq v_\ell^-$). Let $\mathcal{L}^+ = \{v_\ell^+ \mid \ell \in \mathcal{L}\}$ and $\mathcal{L}^- = \{v_\ell^- \mid \ell \in \mathcal{L}\}$. It is not difficult to see that H acts on the set $\mathcal{L}$ (and, thus, on the set $\mathcal{L}^+ \cup \mathcal{L}^-$). We will sometimes call the elements of $\mathcal{L}^+ \cup \mathcal{L}^-$ *pole vectors*. We let $\mathcal{T}$ denote a choice of orbit representatives of $\mathcal{L}^+ \cup \mathcal{L}^-$ under this action. Finally, we let $H_v = \{h \in H \mid hv = v\}$, and let $\mathcal{O}_H(v)$ be the orbit of v under the action of H.

Proposition 2.1. *Let $H \leq SO(3)$ be a non-trivial point group.*

$$2 - \frac{2}{|H|} = \sum_{v \in \mathcal{T}} \left(1 - \frac{1}{|H_v|}\right).$$

Proof. First, we note that

1. $|H| - 1 + |\mathcal{L}^+| = \sum_{v^+ \in \mathcal{L}^+} |H_{v^+}|$
2. $|H| - 1 + |\mathcal{L}^-| = \sum_{v^- \in \mathcal{L}^-} |H_{v^-}|$

The proofs that (1) and (2) hold are identical. Formula (1) follows directly from the observations that $H_{v_1^+} \cap H_{v_2^+} = \{1\}$ if $v_1^+ \neq v_2^+$, and that H is the union of the H_v as v ranges over $\mathcal{L}^+$ (Lemma 2.1(2)). Thus, every element of H is counted exactly once on the right side of (1), except for the identity, which is counted $|\mathcal{L}^+|$ times.

It follows from (1) and (2) that

$$2|H| - 2 = \sum_{v \in \mathcal{L}^+ \cup \mathcal{L}^-} (|H_v| - 1) \Rightarrow 2 - \frac{2}{|H|} = \sum_{v \in \mathcal{L}^+ \cup \mathcal{L}^-} \left(\frac{|H_v|}{|H|} - \frac{1}{|H|}\right)$$

$$\Rightarrow 2 - \frac{2}{|H|} = \sum_{v \in \mathcal{T}} \left(1 - \frac{|O_H(v)|}{|H|}\right)$$

$$\Rightarrow 2 - \frac{2}{|H|} = \sum_{v \in \mathcal{T}} \left(1 - \frac{1}{|H_v|}\right).$$

Proposition 2.2. *Let $1 \neq H \leq SO(3)$ be a point group. The number $|\mathcal{T}|$ of orbits under the action of H on $\mathcal{L}^+ \cup \mathcal{L}^-$ is either 2 or 3. If $|\mathcal{T}| = 2$, then H leaves a line invariant, and is therefore a cyclic group of rotations. Its order must be 2, 3, 4, or 6. If $|\mathcal{T}| = 3$, then let v_1, v_2, v_3 be orbit representatives chosen so that $|H_{v_1}| \leq |H_{v_2}| \leq |H_{v_3}|$. We let $\alpha = |H_{v_1}|$, $\beta = |H_{v_2}|$, and $\gamma = |H_{v_3}|$. The only possibilities for the triple (α, β, γ) and the order of H are as follows:*

1. $(2, 2, n)$ *($n \in \{2, 3, 4, 6\}$), and* $|H| = 2n$*;*
2. $(2, 3, 3)$*, and* $|H| = 12$*;*
3. $(2, 3, 4)$*, and* $|H| = 24$.

Proof. Suppose that $|\mathcal{T}| = 1$. Proposition 2.1 implies that

$$2 - \frac{2}{|H|} = 1 - \frac{1}{|H_v|}.$$

This has no solutions, since the left side will always be at least 1, and the right side is less than 1. Thus, $|\mathcal{T}|$ is never equal to 1.

Suppose that $|\mathcal{T}| = 2$. Let $\mathcal{T} = \{v_1, v_2\}$. By the previous Proposition

$$2 - \frac{2}{|H|} = 2 - \frac{1}{|H_{v_1}|} - \frac{1}{|H_{v_2}|} \Rightarrow \frac{|H|}{|H_{v_1}|} + \frac{|H|}{|H_{v_2}|} = 2$$
$$\Rightarrow |O(v_1)| + |O(v_2)| = 2.$$

It follows that $|O(v_1)| = |O(v_2)| = 1$, so v_1 and v_2 are both fixed by all of H. If v_1 and v_2 formed a linearly independent set, it would follow that $H = 1$ (since every element of H would then act as the identity on the plane spanned by v_1, v_2). We have ruled this out by our hypothesis. The only possibility is that v_1 and v_2 are in the same one-dimensional subspace. The rest of the picture is now clear: every element of H is a rotation about the axis determined by v_1 and v_2. The conclusion follows.

Suppose that $|\mathcal{T}| = 3$. Let $\mathcal{T} = \{v_1, v_2, v_3\}$. We suppose that v_1, v_2, v_3 are chosen as in the statement of the proposition. We apply Proposition 2.1:

$$2 - \frac{2}{|H|} = 3 - \frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma} \Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1 + \frac{2}{|H|}.$$

We are therefore led to consider solutions (α, β, γ) of the inequality:

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} > 1,$$

$C_1^+ = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$D_2^+ = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$
$C_2^+ = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$D_3^+ = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$
$C_3^+ = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$	$D_4^+ = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$
$C_4^+ = \left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$A_4^+ = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$
$C_6^+ = \left\langle \frac{1}{3} \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \right\rangle$	$S_4^+ = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$
	$D_6^+ = \left\langle \frac{1}{3} \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$

Fig. 2.1 Orientation-preserving point groups

where $1 < \alpha \leq \beta \leq \gamma$. The only such solutions are: (i) $(2, 2, n)$ (n arbitrary), (ii) $(2, 3, 3)$, (iii) $(2, 3, 4)$, and (iv) $(2, 3, 5)$. If we keep in mind that each integer in these 3-tuples is $|H_v|$, for some $v \in \mathbb{R}^3$, where $H \leq SO(3)$ is a point group, then we see that each represents the order of a cyclic subgroup of H. It follows from the crystallographic restriction that we can ignore the solution $(2, 3, 5)$ and all solutions $(2, 2, n)$ where $n \notin \{1, 2, 3, 4, 6\}$. This leads us to the solutions described in (1)–(3) in the statement of the proposition.

Suppose that $|\mathcal{T}| \geq 4$. We consider the equation from the statement of Proposition 2.1. Note that each term in the sum on the right is at least $1/2$, since $|H_v| \geq 2$. It follows that the right side is at least 2. The left side is clearly less than 2, so it is impossible that $|\mathcal{T}| \geq 4$. This completes the proof.

Theorem 2.1 (Classification of Orientation-Preserving Point Groups). *Suppose that $H \leq SO(3)$ acts on a lattice L in $\mathbb{R}^3$. The group H is conjugate within $O(3)$ to one of the groups in Fig. 2.1.*

Proof. We will show that the triple $(2, 3, 4)$ from Proposition 2.2 uniquely determines the group S_4^+ up to conjugacy within $O(3)$.

Suppose that $H \leq SO(3)$ is a point group, $\mathcal{T} = \{v_1, v_2, v_3\}$, $|H_{v_1}| = 2$, $|H_{v_2}| = 3$, and $|H_{v_3}| = 4$. Proposition 2.2 says that $|H| = 24$. It follows that $|\mathcal{O}_H(v_3)| = 6$.

By Lemma 2.1(4), the groups H_{v_i} are cyclic, for $i = 1, 2, 3$. Let $h \in H_{v_3}$ be a generator of H_{v_3}, i.e., $\langle h \rangle = H_{v_3}$, and $|h| = 4$. We consider the action of h on the set $\mathcal{O}_H(v_3)$. Certainly h fixes v_3 (by definition) and h can fix at most one other vector in $\mathcal{L}^+ \cup \mathcal{L}^-$, namely $-v_3$ (for, otherwise, h would fix a 2-element linearly independent set, and would necessarily be the identity, since $h \in SO(3)$). It follows that h acts on $\mathcal{O}_H(v_3) - \{v_3\}$ with at most one fixed point. After considering the possible cycle types in the latter action, we easily conclude that h must be a 4-cycle. Therefore h must have another fixed point in $\mathcal{O}_H(v_3) - \{v_3\}$, and this fixed point must be $-v_3$. It follows, in particular, that $-v_3 \in \mathcal{O}_H(v_3)$.

Let $O_H(v_3) = \{v_3, -v_3, v_4, hv_4, h^2v_4, h^3v_4\}$ (without loss of generality). Since $O_H(v_3)$ is H-invariant, we have

$$v_3 - v_3 + v_4 + hv_4 + h^2v_4 + h^3v_4 = (1 + h + h^2 + h^3)v_4 = 0,$$

for otherwise the sum on the left would be a non-zero vector that is held invariant by all of H, and this would force H to be cyclic, by Lemma 2.1(4). The group H is, however, not cyclic by the crystallographic restriction (Lemma 2.1(2)). Now

$$0 = v_3\cdot(1+h+h^2+h^3)v_4 = v_3\cdot v_4 + hv_3\cdot hv_4 + h^2v_3\cdot h^2v_4 + h^3v_3\cdot h^3v_4 = 4(v_3\cdot v_4).$$

It follows that v_3 is perpendicular to $v_4, hv_4, h^2v_4, h^3v_4$.

We can repeat the above argument for each element of $O_H(v_3)$ to conclude that for any $v \in O_H(v_3)$: i) $-v \in O_H(v_3)$, and ii) $v \perp \hat{v}$ for any $\hat{v} \in O_H(v_3) - \{v, -v\}$.

Now choose $h_1 \in H_{v_2}$ such that $\langle h_1 \rangle = H_{v_2}$ (and so $|h_1| = 3$). We consider the action of h_1 on $O_H(v_3)$. We claim that the cycle type of h_1 as a permutation can only be $(***)(***)$. Indeed, if any power n of h_1 fixes $\tilde{v} \in O_H(v_3)$, then the vectors v_2 and $\tilde{v}$ are linearly independent, and therefore span a plane that is fixed by h_1^n. Since h_1^n is also orientation-preserving, we must have $h_1^n = 1$. Thus the action of $\langle h_1 \rangle$ on $O_H(v_3)$ is free, from which the claim easily follows.

We consider the set $B = \{v_3, h_1v_3, h_1^2v_3\}$. It is easy to check that $B \cap -B = \emptyset$, since the assumption $B \cap -B \neq \emptyset$ quickly forces h_1 to have the wrong cycle type. It follows that B is an orthonormal basis for $\mathbb{R}^3$. The element h_1 has the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the ordered basis $[v_3, h_1v_3, h_1^2v_3]$.

We return to the element h, which permutes $B \cup -B = O_H(v_3)$. Since $|h| = 4$, and $hv_3 = v_3$, we must have

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mp 1 \\ 0 & \pm 1 & 0 \end{pmatrix}.$$

It is now clear that the change of basis matrix sending $\mathbf{x}$ to v_3, $\mathbf{y}$ to h_1v_3, and $\mathbf{z}$ to $h_1^2v_3$ is orthogonal and conjugates $\langle h, h_1 \rangle$ to S_4^+ (as described in Fig. 2.1). Since $|H| = 24$, the equality $H = \langle h, h_1 \rangle$ is forced. Moreover, it is not difficult to argue that the matrices in the statement of the theorem generate a group isomorphic to S_4. One possible approach is to examine the action of the group on the diagonals of the cube $[-1, 1]^3$.

We have now argued that the 3-tuple $(2, 3, 4)$ from Proposition 2.2 determines the group S_4^+ uniquely up to conjugacy. One can argue that the cardinality of $\mathcal{T}$, the order of H, and the 3-tuple (α, β, γ) (if applicable) always determine the group H in the remaining cases as well. (There are eleven cases in all, including the one above and the case of the trivial group.) We omit the details, but note that the triple $(2, 2, n)$ (for $n = 2, 3, 4, 6$) determines D_n^+, and $(2, 3, 3)$ determines A_4^+, while the various cyclic groups of rotations from the statement of Proposition 2.2 are accounted for by the groups C_n^+, for $n = 1, 2, 3, 4, 6$.

2.3 Classification of Point Groups with Central Inversion

Suppose $H \leq O(3)$ is a point group, and H contains the central inversion (-1) (i.e., the antipodal map). Let H^+ be the orientation-preserving subgroup of H. We note that $[H : H^+] = 2$, and $H = \langle H^+, (-1)\rangle$. The following observation has an obvious proof:

Proposition 2.3. *Let H_1, H_2 be point groups containing (-1). The groups H_1, H_2 are conjugate in $O(3)$ if and only if H_1^+ and H_2^+ are conjugate in $O(3)$.* □

Theorem 2.2. *Let H be a point group containing the central inversion (-1). The group H is conjugate to one of the eleven groups $\langle H^+, (-1)\rangle$, where H^+ is one of the orientation-preserving point groups from Theorem 2.1.*

Proof. This follows easily from Proposition 2.3 and Theorem 2.1.

2.4 Classification of the Remaining Point Groups and Summary

Let $H \leq O(3)$ be a point group such that: (a) $H \not\leq SO(3)$, and (b) $(-1) \notin H$. Consider the group $\hat{H} = \langle H, (-1)\rangle$. This group contains the central inversion (-1), so it is conjugate to one of the eleven from the previous section. (Note that L is closed under additive inverses, so the group $\langle H, (-1)\rangle$ is still a point group.) We note that $[\hat{H} : H] = 2$, so $H \trianglelefteq \hat{H}$, and therefore it is the kernel of some surjective homomorphism $\phi : \langle \hat{H}^+, (-1)\rangle \to \mathbb{Z}/2\mathbb{Z}$, where $\hat{H}^+$ denotes the orientation-preserving subgroup of $\hat{H}$. Thus, to find all possibilities for H up to conjugacy, we can examine the kernels of all such homomorphisms ϕ as $\langle \hat{H}^+, (-1)\rangle$ ranges over all eleven possibilities. We need only consider homomorphisms $\phi : \langle \hat{H}^+, (-1)\rangle \to \mathbb{Z}/2\mathbb{Z}$ such that: (a) $\phi((-1)) = 1$, and (b) $\phi(\hat{H}^+) \not\leq \ker \phi$.

Lemma 2.2. *Let G be a group generated by the set S, and let (-1) be a central element of order two in G. Let $\phi : G \to \mathbb{Z}/2\mathbb{Z}$ be a homomorphism satisfying $\phi((-1)) = 1$. The kernel of ϕ is generated by*

$$\{s \in S \mid \phi(s) = 0\} \cup \{(-1)s \in S \mid \phi(s) = 1\}.$$

Proof. Let g be in the kernel of ϕ. Since S is a generating set for G, we must have $g = s_1 \ldots s_k$, for appropriate $s_i \in S \cup S^{-1}$. Let $S_0 = \{s \in S \cup S^{-1} \mid \phi(s) = 0\}$ and $S_1 = \{s \in S \cup S^{-1} \mid \phi(s) = 1\}$. Since g is in the kernel of ϕ, it must be that

$C_2' = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	$D_4' = \left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$
$C_4' = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	$D_4'' = \left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$
$C_6' = \left\langle \frac{1}{3} \begin{pmatrix} -2 & -2 & 1 \\ 1 & -2 & -2 \\ -2 & 1 & -2 \end{pmatrix} \right\rangle$	$D_6' = \left\langle \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$
$D_2' = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$D_6'' = \left\langle \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$
$D_3' = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$S_4' = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$

Fig. 2.2 The remaining point groups

elements of S_1 occur an even number of times in the string $s_1 \ldots s_k$. We can replace each such element s_i with $(-1)s_i$ without changing the product of the string.

Theorem 2.3. *Let $H \leq O(3) - SO(3)$ be a point group such that $(-1) \notin H$. The group H is conjugate within $O(3)$ to one of the groups listed in Fig. 2.2.*

Proof. Most of the proof is a straightforward application of Lemma 2.2. A few cases are worth some additional remarks.

Consider the case in which $[\langle D_4^+, (-1)\rangle : H] = 2$. We use the equality

$$D_4^+ = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle = \langle A, B \rangle,$$

where A and B (respectively) are the matrices in the term between the equal signs. Note that A is the 180° rotation about the line ℓ defined by the equations $x = y$ and $z = 0$, and B is the 180° rotation about the x-axis. It is easy to see that these are generators, as claimed. There are three homomorphisms ϕ_1, ϕ_2, ϕ_3 to consider (all of which send (-1) to 1): (i) $\phi_1(A) = 1$, $\phi_1(B) = 1$; (ii) $\phi_2(A) = 1$, $\phi_2(B) = 0$; (iii) $\phi_3(A) = 0$, $\phi_3(B) = 1$. We note that the kernels of ϕ_2 and ϕ_3 are conjugate in $O(3)$, since there is an element $\lambda \in O(3)$ conjugating A to B and B to A. The first two homomorphisms have the kernels D_4'' and D_4', respectively, by Lemma 2.2. We note that $C_4^+ \leq D_4''$ and $D_2^+ \leq D_4'$, and this distinguishes the groups up to conjugacy within $O(3)$. Thus, if H is as in Theorem 2.3, and $\langle H, (-1)\rangle$ is conjugate to $\langle D_4^+, (-1)\rangle$, then H is conjugate either to D_4' or to D_4''.

The case in which $[\langle D_6^+, (-1)\rangle : H] = 2$ is analogous to the previous one. We use the generating elements

$$A = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and proceed as before.

For the case in which $[\langle S_4^+, (-1)\rangle : H] = 2$ we use a generating set T for S_4^+ different from the one appearing in Theorem 2.1; here

$$T = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

We note that the first matrix has order 3, and must therefore be sent to the identity in any homomorphism $\phi : \langle S_4^+, (-1)\rangle \to \mathbb{Z}/2\mathbb{Z}$. It follows that there is just one homomorphism such that $\phi(-1) = 1$ and $S_4^+ \not\leq \ker\phi$; the kernel of this homomorphism is S_4' by Lemma 2.2.

Definition 2.3. We say that a point group $H \leq O(3)$ is *standard* if it is one of the 32 described in Theorems 2.1–2.3.

2.5 Descriptions of Selected Point Groups

In this section, we will attempt to give simple descriptions of the 32 standard point groups. Our goal is to help the reader develop a working knowledge of these groups, which will be essential in subsequent chapters. We will also introduce certain non-standard point groups that will arise naturally later.

2.5.1 The Orientation-Preserving Standard Point Groups

There are 11 of these in all, as listed in Fig. 2.1.

- Five are cyclic: C_i^+ ($i = 1, 2, 3, 4, 6$). If $i \in \{1, 2, 4\}$, then C_i^+ is generated by a rotation of order i about the z-axis. If $i \in \{3, 6\}$, then C_i^+ is generated by a rotation of order i about the line $x = y = z$.
- The dihedral group D_2^+ consists of all of the 180° rotations about the coordinate axes, and the identity. We can also describe D_2^+ algebraically: it is the group of 3×3 diagonal matrices with an even number -1s down the diagonal, where all other entries on the diagonal are 1.
- The dihedral group D_4^+ is generated by 180° rotations about the lines $\ell(y = z = 0)$ and $\ell(x = y; z = 0)$. (Note that the group elements in question are not the generators listed in Fig. 2.1.) The xy-plane is invariant, and D_4^+ acts as the group of symmetries of the square $[-1, 1]^2$ in that plane. We can also describe D_4^+ algebraically: D_4^+ consists of all matrices having determinant 1 and the form SP, where S is a *sign matrix* (i.e., a matrix whose off-diagonal entries are 0, and whose diagonal entries are ± 1) and P is either the identity matrix or the permutation matrix that interchanges the first two columns.
- The dihedral group D_3^+ is generated by 180° rotations about the lines $\ell(x + y = 0; z = 0)$ and $\ell(x = 0; y + z = 0)$. The plane $P(x + y + z = 0)$ is invariant;

the elements of order 3 are rotations through 120° about the axis $\ell(x = y = z)$. Algebraically, D_3^+ is the group of matrices having the form SP, where P is any permutation matrix and S is either the identity matrix (if $\det(P) = 1$) or the antipodal map (if $\det(P) = -1$).

- The dihedral group D_6^+ is generated by 180° rotations about the lines $\ell(x + y + z = 0; 2x + y = 0)$ and $\ell(x + y = 0; z = 0)$. It leaves the plane $P(x + y + z = 0)$ invariant. The additive group $\langle(1, -1, 0), (0, -1, 1)\rangle$ is a lattice in $P(x + y + z = 0)$. The six lattice points of smallest norm describe a regular hexagon. The group D_6^+ acts as the group of symmetries of this hexagon.
- The group A_4^+ consists of all matrices of the form SP, where P is a permutation matrix that permutes the coordinate axes cyclically and S is a signed matrix with an even number of -1s on the diagonal.
- The group S_4^+ is the group of signed permutation matrices with determinant equal to 1.

2.5.2 The Standard Point Groups with Inversion

The 11 standard point groups H that contain the inversion (-1) all have the form $\langle H^+, (-1)\rangle$, where H^+ is one of the 11 orientation-preserving standard point groups. Thus, the groups H have descriptions similar to the ones that were given in Sect. 2.5.1; we briefly give details for the $H = \langle H^+, (-1)\rangle$ when H^+ is neither cyclic nor D_6^+.

- If $H^+ = D_2^+$, then H is the group of sign matrices.
- If $H^+ = D_4^+$, then H is the set of all matrices expressible in the form SP, where S is an arbitrary sign matrix and P is either the identity or the permutation matrix which interchanges the first two coordinates.
- If $H^+ = D_3^+$, then H is the set of all matrices of the form SP, where S is either the identity or (-1), and P is an arbitrary permutation matrix.
- If $H^+ = A_4^+$, then H is the set of all matrices of the form SP, where S is an arbitrary sign matrix, and P is a cyclic permutation matrix.
- If $H^+ = S_4^+$, then H is the full group of signed permutation matrices.

2.5.3 The Remaining Standard Point Groups

We now briefly describe the remaining point groups (as listed in Theorem 2.3).

- The group C_2' is generated by reflection across the xy-plane. The group C_4' is generated by a rotation about the z-axis of 90°, followed by reflection in the xy-plane.

The group C_6' is generated by a rotation through 120° about the line $x = y = z$, followed by reflection across the plane $P(x + y + z = 0)$ (the complementary subspace to $\ell(x = y = z)$).

- The group D_2' consists of the sign matrices having a 1 in the lower right corner.
- The group D_3' is simply the group of permutation matrices.
- The group D_4' (like D_4^+) leaves the xy-plane invariant, and acts as the group of symmetries of the square $[-1, 1]^2$ in that plane. However, the elements that behave like reflections in the coordinate axes of $P(z = 0)$ (when we consider the restriction of the action to $P(z = 0)$) are actually rotations in the ambient $\mathbb{R}^3$. (In other words, $D_2^+ \leq D_4'$.) The elements that behave like reflections in the lines $\ell(x = y; z = 0)$ and $\ell(x = -y; z = 0)$ are reflections in the planes $P(x = y)$ and $P(x = -y)$, respectively.

  We can also give a simple algebraic description of D_4': it is the group of all matrices having the form SP, where S is a sign matrix with an even number of negative (i.e., -1) entries, and P is either the identity matrix or the permutation matrix that interchanges the first two coordinates.

  We also note that $C_4' \leq D_4'$.
- The group D_4'' (like D_4^+ and D_4') leaves the xy-plane invariant, and acts as the group of symmetries of the square $[-1, 1]^2$. Each element of D_4'' that acts as a reflection in the (restricted) action of D_4'' on the xy-plane is also a reflection of the ambient $\mathbb{R}^3$ across a plane.

  The algebraic description of D_4'' is also easy: it is the group of all signed permutation matrices having a 1 in the lower right corner.

  It is clear that $D_2' \leq D_4''$ and $C_4^+ \leq D_4''$.
- The group D_6' can be generated in the following way:

$$D_6' \cong D_3' \times \left\langle \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} \right\rangle.$$

  The latter matrix is reflection across the plane $P(x + y + z = 0)$. In fact, we can factor $\mathbb{R}^3$ orthogonally as $P(x + y + z = 0) \times \ell(x = y = z)$. With respect to this factorization, D_3' acts on the first factor (leaving it invariant), and acts trivially on the second factor; the above reflection acts trivially on the first factor and as inversion on the second factor. We note also that $C_6' \leq D_6'$.
- The group D_6'' is analogous to D_4''. In the orthogonal factorization $\mathbb{R}^3 = P(x + y + z = 0) \times \ell(x = y = z)$, D_6'' acts as the full group of symmetries of a regular hexagon in the first factor, and trivially in the second. We note also that $C_6^+ \leq D_6''$.
- The group S_4' consists of all matrices of the form SP, where P is an arbitrary permutation matrix and S is a sign matrix with an even number of negative entries. In particular, we have the inclusion $D_4' \leq S_4'$.

2.5.4 Some Non-standard Point Groups

The property of being a point group is (clearly) inherited under passage to subgroups, but the property of being a *standard* point group is not. The arguments of subsequent chapters will frequently involve passing to subgroups, which will mean that we cannot always consider only the standard point groups. Here we describe a few of the non-standard point groups that will arise in practice.

- First, let

$$\hat{D}'_4 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

  This group is conjugate within $O(3)$ to the standard point group D'_4, and its action on $\mathbb{R}^3$ is similar to that of D'_4 in most respects: the group $\hat{D}'_4$ leaves the xy-plane invariant, and acts as the group of symmetries of the square $[-1, 1]^2$ in that plane. The elements that behave like reflections in the coordinate axes of $P(z = 0)$ (when we consider the restriction of the action to $P(z = 0)$) are also reflections in the ambient $\mathbb{R}^3$ (i.e., they are the reflections in the xz- and yz-planes, respectively). The elements that behave like reflections in the lines $\ell(x = y; z = 0)$ and $\ell(x = -y; z = 0)$ (the diagonals of the square) are actually rotations through 180° about those lines.
- The group

$$\hat{D}'_6 = \left\langle \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

  is conjugate within $O(3)$ to D'_6. We can factor $\hat{D}'_6$ as follows:

$$\hat{D}'_6 \cong D_3^+ \times \left\langle \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} \right\rangle.$$

  Each part of the factorization leaves the subspaces $P(x + y + z = 0)$ and $\ell(x = y = z)$ invariant. (As noted in the description of D'_6, the latter matrix is reflection in the plane $P(x + y + z = 0)$, and so acts trivially on the first factor.) We note that $C'_6 \leq \hat{D}'_6$.
- Finally, we describe some non-standard variations of point groups that are isomorphic to D_2 or D_4. Each standard point group of this type (with the exception of D_2^+) has a distinguished coordinate axis, namely the z-axis, which is invariant under the action of the group. For instance, each of the standard point groups D_4^+, D'_4, and D''_4 leaves the z-axis invariant. We will denote the non-standard point group with a different distinguished axis by a subscript of 1 or 2, where 1 indicates that the x-axis is distinguished, and 2 indicates that the y-axis is distinguished. For instance, $D_{4_1}^+$ denotes the group of all matrices having the form SP, where $\det(SP) = 1$, S is a sign matrix, and P is either the identity or

the permutation matrix that interchanges the y- and z-coordinates. The group $D_{4_2}^{+}$ has the same description, except that the matrix P may be either the identity or the permutation matrix interchanging the x- and z-coordinates. The groups D_{4_i}' and D_{4_i}'' for $i = 1, 2$ have analogous descriptions.

Similarly, for $i = 1, 2$, we let D_{2_i}' denote the group of sign matrices having a 1 in the ith position on the diagonal.

Chapter 3
Arithmetic Classification of Pairs (L, H)

Let L be a lattice in $\mathbb{R}^3$, and let $H \leq O(3)$ be a point group such that $H \cdot L = L$. In this chapter, we classify pairs (L, H) up to arithmetic equivalence (defined below). The equivalence classes of pairs (L, H) are in one-to-one correspondence with isomorphism classes of split crystallographic groups (see Chap. 4).

3.1 Definition of Arithmetic Equivalence and a Lemma

Now we introduce one of the central tools in the classification of split three-dimensional crystallographic groups. The definition below is due to Schwarzenberger [Sc80, p. 34].

Definition 3.1. Let $H \leq O(3)$ be a point group, and let L be a lattice in $\mathbb{R}^3$ satisfying $H \cdot L = L$. We say that two pairs (L', H'), (L, H) are *arithmetically equivalent*, and write $(L', H') \sim (L, H)$, if there is an invertible linear transformation $\lambda \in GL_3(\mathbb{R})$ such that:

1. $\lambda L' = L$, and
2. $\lambda H' \lambda^{-1} = H$.

The following lemma will be used heavily in our classification of pairs (L, H) up to arithmetic equivalence.

Lemma 3.1. *Let H be a point group acting on the lattice L.*

1. *If $h \in H \cap SO(3)$, $h \neq 1$, and ℓ is the (unique) line fixed by h, then ℓ contains a non-zero element of L.*
2. *If $h \in H \cap SO(3)$, $h \neq 1$, and ℓ is the unique line fixed by h, then $P = \{v \in \mathbb{R}^3 \mid v \perp \ell\}$ contains a non-zero element of L.*
3. *If $h \in H$ is reflection in the plane P, then $L \cap P$ is free abelian of rank two.*

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_3

Proof. 1. Let $v \in L$ be such that $v \not\perp \ell$ and $v \notin \ell$. Thus, we can write $v = v_1 + v_2$, where $v_1 \perp \ell$, $v_2 \in \ell$, and neither v_1 nor v_2 is 0. Assume that $|h| = n$; we can write

$$(1 + h + h^2 + \ldots + h^{n-1})v = nv_2 + (1 + h + \ldots + h^{n-1})v_1,$$

since v_2 is fixed by h. We note that nv_2 and $(1 + h + \ldots + h^{n-1})v_1$ are perpendicular, essentially by orthogonality of h, and that $(1 + h + h^2 + \ldots + h^{n-1})v_1$ is h-invariant. It follows that $(1 + h + h^2 + \ldots + h^{n-1})v_1 \in \ell$, and therefore can only be 0 (since it is perpendicular to $v_2 \in \ell - \{0\}$). Now clearly $(1+h+h^2+\ldots+h^{n-1})v = nv_2 \in \ell - \{0\}$ and $(1+h+h^2+\ldots+h^{n-1})v \in L$.

2. Let $x \in L - \ell$. It follows that $x - hx \neq 0$. Now we show that $x - hx \in P$. Let $v \in \ell - \{0\}$.

$$v \cdot x = hv \cdot hx = v \cdot hx,$$

so $v \cdot (x - hx) = 0$, and $x - hx \in P$.

3. Our assumptions imply that we can choose an ordered basis for $\mathbb{R}^3$ in such a way that h is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

over that basis. It is then clear that the transformation $1+h$ has a one-dimensional null space.

Let v_1, v_2, v_3 be a linearly independent subset of L. It follows that the vectors $(1 + h)v_1$, $(1 + h)v_2$, and $(1 + h)v_3$ span a two-dimensional subspace of $\mathbb{R}^3$. It follows, in particular, that the group $G = \langle (1 + h)v_1, (1 + h)v_2, (1 + h)v_3 \rangle$ has rank 2. (The rank can be no larger, since G is a discrete subgroup of a real subspace of dimension 2; the rank can be no smaller, since the generators span a real vector subspace of dimension 2.) Since h fixes each element of the generating set, $G \subseteq P$.

Finally, we note that $G \leq P \cap L$, so the latter group must have rank at least 2. It cannot have rank more than 2 since it is a discrete additive subgroup of a two-dimensional real vector space.

3.2 Full Sublattices in Pairs (L, H), Where H Contains (-1)

In this section, we take our first steps toward classifying the pairs (L, H) up to arithmetic equivalence (i.e., classifying the split crystallographic groups—see Theorem 4.1). Our strategy is to build the lattice L around the point group H. The lattice L (for any choice of H) will be made to contain one of two specific lattices in a certain way (related to the definition of "fullness" below). Our technical results here will be important in the subsequent classification of pairs (L, H).

Definition 3.2. We let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The lattices $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle$ and $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ are, respectively, the *cubic lattice* L_C and the *prismatic lattice* $L_{\mathcal{P}}$.

Definition 3.3. If L is any lattice, then a subgroup $\hat{L} \leq L$ is *full* in L if $\hat{L}$ is the maximal subgroup of L that is contained in the span of $\hat{L}$ as a vector space.

Proposition 3.1. *Let H be a standard point group acting on the lattice L; suppose $(-1) \in H$. We let H^+ denote the orientation-preserving subgroup of H.*

1. *If $H^+ = C_1^+$, then $(L, H) \sim (L_C, H)$.*
2. *If $H^+ = C_2^+, C_4^+$, or D_4^+, then $(L, H) \sim (L', H)$, where $L_C \leq L'$ and each of the subgroups $\langle \mathbf{x}, \mathbf{y} \rangle$, $\langle \mathbf{z} \rangle$ is full in L'.*
3. *If $H^+ = D_2^+, A_4^+$, or S_4^+, then $(L, H) \sim (L', H)$ where $L_C \leq L'$ and each of the subgroups $\langle \mathbf{x} \rangle$, $\langle \mathbf{y} \rangle$, $\langle \mathbf{z} \rangle$ is full in L'.*
4. *If $H^+ = C_3^+, C_6^+$, or D_6^+, then $(L, H) \sim (L', H)$, where $L_{\mathcal{P}} \leq L'$ and each of the subgroups $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$, $\langle \mathbf{v}_1 \rangle$ is full in L'.*
5. *If $H^+ = D_3^+$, then $(L, H) \sim (L', H)$ where $L_{\mathcal{P}} \leq L'$ and each of the subgroups $\langle \mathbf{v}_1 \rangle$, $\langle \mathbf{v}_2 \rangle$, $\langle \mathbf{v}_3 \rangle$ is full in L'.*

Proof. 1. This is easy. If $H^+ = C_1^+ = 1$, then $H = \langle (-1) \rangle$. There is $\lambda \in GL_3(\mathbb{R})$ such that $\lambda L = L_C$. It is obvious that $\lambda(-1)\lambda^{-1} = (-1)$, so $(L, H) \sim (\lambda L, \lambda H \lambda^{-1}) = (L_C, H)$.

2. Suppose $H^+ = C_2^+$. The group $\langle C_2^+, (-1) \rangle$ contains the reflection across the xy-plane, so $L \cap P(z = 0)$ is free abelian of rank two, by Lemma 3.1(3). It follows that there is some

$$\lambda = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in GL_3(\mathbb{R})$$

such that $\lambda(L \cap P(z = 0)) = L_C \cap P(z = 0) = \langle \mathbf{x}, \mathbf{y} \rangle$. Any such λ commutes with $\langle C_2^+, (-1) \rangle$. It follows that (L, H) is arithmetically equivalent to a pair $(\hat{L}, H)$ such that $\langle \mathbf{x}, \mathbf{y} \rangle$ is a full subgroup of $\hat{L}$. By Lemma 3.1(1), there is some $\alpha \mathbf{z} \in \hat{L}$, where $\alpha \neq 0$ and $|\alpha|$ is minimal. We multiply $\hat{L}$ by

$$\lambda' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}.$$

The matrix λ' commutes with H and $\lambda' \hat{L} = L'$ has the desired properties.

Suppose that $H^+ = C_4^+$. By Lemma 3.1(2), there is some non-zero $v \in L \cap P(z = 0)$, since the generator of C_4^+ acts as a rotation about the z-axis. We choose v to have the minimal norm of all non-zero vectors in $L \cap P(z = 0)$. After multiplying by a scalar matrix, we can assume that $||v|| = 1$. After applying a suitable rotation λ (which, as in the case of C_2^+, is a block matrix with a 2 by 2 block in the upper left corner), we can assume $v = \mathbf{x}$, i.e., $\mathbf{x} \in \lambda L = \hat{L}$, $\mathbf{x}$ is the non-zero vector of minimal norm in $\hat{L} \cap P(z = 0)$, and $\lambda H \lambda^{-1} = H$ (since any two rotations in the xy-plane commute). We conclude that $\mathbf{y} \in \hat{L}$ since H^+ contains a rotation through 90° about the z-axis. It easily follows from the minimality of the norm of $\mathbf{x}$ in $\hat{L} \cap P(z = 0)$ that $\langle \mathbf{x}, \mathbf{y} \rangle$ is a full subgroup of $\hat{L}$. We can then continue as before (that is, rescale along the z-axis while leaving the xy-plane alone) to get a new lattice L' having the additional property that $\langle \mathbf{z} \rangle$ is a full subgroup of L', and $(L, H) \sim (L', H)$.

Suppose that $H^+ = D_4^+$. The group D_4^+ contains 180° rotations about the axes $\ell(y = z = 0)$, $\ell(x = z = 0)$, $\ell(z = 0 = x - y)$, and $\ell(z = 0 = x + y)$. We claim that a smallest non-zero vector in $L \cap P(z = 0)$ lies on one of these axes. (A non-zero vector in $L \cap P(z = 0)$ exists by Lemma 3.1(2).) We can assume first that $\mathbf{x}$ is the smallest non-zero vector in $L \cap \ell(y = z = 0)$ (after multiplying by a suitable scalar matrix if necessary). (Such a non-zero vector in $L \cap \ell(y = z = 0)$ exists, by Lemma 3.1(1).) It follows that $\mathbf{y}$ is likewise the smallest non-zero vector in $L \cap \ell(x = z = 0)$, since there is a rotation in D_4^+ taking the x-axis to the y-axis. Now let $v = \alpha\mathbf{x} + \beta\mathbf{y}$ be the smallest non-zero vector in $L \cap P(z = 0)$. Apply the 180° rotation about the x-axis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha \\ -\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 2\alpha \\ 0 \\ 0 \end{pmatrix} \in L.$$

It follows that $2\alpha \in \mathbb{Z}$, since $\langle \mathbf{x} \rangle$ is a full subgroup of L. By similar reasoning (applying the rotation about the y-axis), $2\beta \in \mathbb{Z}$. This means that either: (i) $\mathbf{x}$ is a smallest non-zero vector in $L \cap P(z = 0)$, or (ii) $\frac{1}{2}(\mathbf{x} + \mathbf{y})$ is a smallest non-zero vector in $L \cap P(z = 0)$. This proves the claim, since both of these lattice points lie on axes of rotation in D_4^+.

Thus a smallest vector in $L \cap P(z = 0)$ is either $\mathbf{x}$ or we can assume that this smallest vector in $L \cap P(z = 0)$ is $\mathbf{x}$ after multiplying by a scalar and rotating 45° about the z-axis. (The latter rotation normalizes H.) It now follows directly that $(L, H) \sim (\hat{L}, H)$, where $\langle \mathbf{x}, \mathbf{y} \rangle$ is a full subgroup of $\hat{L}$. One then easily produces L' in which $\mathbf{z}$ is full as well (as in previous cases), and $(\hat{L}, H) \sim (L', H)$.

3. Suppose $H^+ = D_2^+$. Since D_2^+ contains rotations about each of the coordinate axes, Lemma 3.1(1) implies that there are vectors $\alpha\mathbf{x}$, $\beta\mathbf{y}$, and $\gamma\mathbf{z}$ $(\alpha, \beta, \gamma \neq 0)$ such that each generates a full subgroup. We can scale each vector independently to arrive at the desired L', and the suggested matrix commutes with H, so $(L, H) \sim (L', H)$.

   If $H^+ = A_4^+$ or S_4^+, then $D_2^+ \leq H^+$ and there are again vectors $\alpha\mathbf{x}$, $\beta\mathbf{y}$, and $\gamma\mathbf{z}$ $(\alpha, \beta, \gamma > 0)$, each generating a full subgroup of L. This time $\alpha = \beta = \gamma$ since H^+ permutes the coordinate axes. After multiplying by α^{-1}, we get the desired L'.
4. Suppose $H^+ = C_3^+$ or C_6^+. Let v be a smallest non-zero vector in $L \cap P(x + y + z = 0)$. Such a vector exists by Lemma 3.1(2), since $P(x + y + z = 0)$ is

perpendicular to the axis of rotation $\ell(x = y = z)$. After applying an appropriate rotation about $\ell(x = y = z)$ and multiplying in the plane $P(x + y + z = 0)$ by an appropriate scalar, we can assume that $v = \mathbf{x} - \mathbf{y} \in L$. (All of the suggested matrices commute with H.) It follows easily that $-\mathbf{y} + \mathbf{z} \in L$, as well, since H contains a rotation through 120° about the line $\ell(x = y = z)$. Since $\mathbf{x} - \mathbf{y}$ has the minimal possible norm of all non-zero vectors in $L \cap P(x + y + z = 0)$, $\langle \mathbf{x}-\mathbf{y}, -\mathbf{y}+\mathbf{z} \rangle$ must be full. (Here the lattice points $\langle \mathbf{x}-\mathbf{y}, -\mathbf{y}+\mathbf{z} \rangle$ describe a grid in the plane $P(x+y+z = 0)$ made up of equilateral triangles. If $\langle \mathbf{x}-\mathbf{y}, -\mathbf{y}+\mathbf{z} \rangle \subsetneq L \cap P(x + y + z = 0)$, then we could choose a point $v' \in L \cap P(x + y + z = 0)$ outside of the grid, and then the difference between v' and the nearest member of $\langle \mathbf{x} - \mathbf{y}, -\mathbf{y} + \mathbf{z} \rangle$ would violate the minimality of $||\mathbf{x} - \mathbf{y}||$.)

Since $L \cap \ell(x = y = z) \neq 0$ by Lemma 3.1(1), there is $\alpha \neq 0$ such that $\langle \alpha(\mathbf{x} + \mathbf{y} + \mathbf{z}) \rangle$ is a full subgroup of L'. We scale this vector to obtain the desired conclusion. This scaling can be done while leaving the perpendicular plane $P(x + y + z = 0)$ fixed.

Now we suppose that $H^+ = D_6^+$. We claim that a non-zero vector of minimal norm in $L \cap P(x + y + z = 0)$ must lie on one of the axes $\ell \subseteq P(x + y + z = 0)$ of rotation for D_6^+. (There are 6 in all, and each makes a 30° angle with the axes closest to it.) One of the axes is the line $\ell(x + y = 0 = z)$, and it follows from Lemma 3.1(1) that we can assume that $\mathbf{x} - \mathbf{y} \in L$ after scaling (if necessary), and that $\mathbf{x} - \mathbf{y}$ generates a full subgroup of L. It then follows quickly that $\langle -\mathbf{y} + \mathbf{z} \rangle$ is also a full subgroup of L, since there is an element of D_6^+ that carries the subspace $\langle \mathbf{x} - \mathbf{y} \rangle$ to $\langle -\mathbf{y} + \mathbf{z} \rangle$.

Let v be a smallest non-zero vector in $L \cap P(x + y + z = 0)$. Suppose $v = \alpha\mathbf{x} + \beta\mathbf{y} + (-\alpha - \beta)\mathbf{z}$. We can assume that $\alpha, \beta \geq 0$. (Indeed, either two or more of the numbers α, β, $-\alpha - \beta$ are nonnegative, or two or more are nonpositive. If two or more are nonnegative, then we can apply a suitable cyclic permutation matrix to arrange that the first two entries are nonnegative, yielding the desired result. It two or more entries of v are nonpositive, then we apply a suitable cyclic permutation matrix to arrange that the first two entries are nonpositive, and then we apply the antipodal map. All of the matrices in question lie in $\langle D_6^+, (-1) \rangle$.)

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix} = \begin{pmatrix} -\beta \\ -\alpha \\ \alpha+\beta \end{pmatrix}.$$

Adding the two column vectors in the above expression, we arrive at $(\alpha - \beta)(\mathbf{x} - \mathbf{y})$, which is in L since each of the above column vectors is in L. It follows that $\alpha - \beta \in \mathbb{Z}$, since $\langle \mathbf{x} - \mathbf{y} \rangle$ is a full subgroup of L.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ \alpha+\beta \\ -\beta \end{pmatrix}.$$

Adding the two column vectors in the above expression, we arrive at $(-\alpha - 2\beta)(-\mathbf{y} + \mathbf{z})$, which is in L since each of the above column vectors is in L. It follows that $-\alpha-2\beta \in \mathbb{Z}$, since $\langle -\mathbf{y}+\mathbf{z} \rangle$ is a full subgroup of L. This now implies that $3\alpha, 3\beta \in \mathbb{Z}$. Let's suppose that $\alpha = m/3$ and $\beta = n/3$, where m and n are non-negative integers. The minimality of the norm of v in $L \cap P(x+y+z=0)$ implies that

$$m^2 + mn + n^2 \leq 9,$$

and m and n are congruent modulo 3 by the condition $\alpha - \beta \in \mathbb{Z}$. It is routine to check that either: (i) m and n are both divisible by three (and so α, β are integers, one 0 and the other 1), or (ii) $m = n = 1$. In the first case, it is clear that v lies on one of the axes $\ell(y+z=0=x)$, or $\ell(x+z=0=y)$, and these are axes for rotations in D_6^+. In the second case, we have $v = \alpha(\mathbf{x}+\mathbf{y}-2\mathbf{z})$. Since v makes an angle of 30° with the vector $\mathbf{x}-\mathbf{z}$ (which lies on an axis of rotation), it follows that v itself lies on an axis of rotation. This proves the claim.

We summarize. We can assume that a smallest non-zero vector in $L \cap P(x+y+z=0)$ is $\mathbf{x}-\mathbf{y}$ after: (i) scaling in the plane $P(x+y+z=0)$, if a smallest vector in $L \cap P(x+y+z=0)$ lies on $\ell(x+y=0; z=0)$ (or on one of its orbits under the action of D_6^+), or (ii) rotating by 30° about the axis $\ell(x=y=z)$ otherwise.

The remainder of the argument is easy, and follows the lines of the cases $H = C_3^+$ and $H = C_6^+$.

5. By Lemma 3.1(1), there is some $v \in L \cap \ell(x+y=0=z)$, where $v \neq 0$. After multiplying by a suitable scalar matrix, we can assume that $\langle \mathbf{x}-\mathbf{y} \rangle$ is a full subgroup of L. Since there are elements in D_3^+ that move the subspace $\langle \mathbf{x}-\mathbf{y} \rangle$ to the subspace $\langle -\mathbf{y}+\mathbf{z} \rangle$, it follows that $\langle -\mathbf{y}+\mathbf{z} \rangle$ is also full in L. Thus, $\langle \mathbf{v}_i \rangle$ is full in L, for $i = 2, 3$. We can rescale along the line $\ell(x=y=z)$ while holding the plane $P(x+y+z=0)$ fixed; the suggested transformation λ commutes with D_3^+ for any scaling factor. There is some non-zero $v \in \ell(x=y=z) \cap L$. If we assume that v is chosen so that $||v||$ is minimal, then scaling along $\ell(x=y=z)$ by a factor of $1/||v||$ yields the desired lattice L'.

3.3 Description of Possible Lattices L

In this section, we will show that each pair (L, H) is equivalent to a pair (L', H), where L' is one of seven lattices.

Lemma 3.2. *Suppose that H is a standard point group which stabilizes the lattice L, suppose $L_C \leq L$, and each of $\langle \mathbf{x} \rangle$, $\langle \mathbf{y} \rangle$, $\langle \mathbf{z} \rangle$ is a full subgroup of L.*

1. *If H contains the 180° rotation through the x-, y-, or z-axis, then, for any $v \in L$, $v = \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z}$, we have that 2α, 2β, or $2\gamma \in \mathbb{Z}$ (respectively).*
2. *If H contains the reflection across the plane $P(z=0)$, $\langle \mathbf{x}, \mathbf{y} \rangle$ is a full subgroup of L, and $v = \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} \in L$, then $2\alpha, 2\beta, 2\gamma \in \mathbb{Z}$.*

Proof. 1. Let $v \in L$ have the form indicated in the lemma. We suppose that H contains the rotation λ through 180° about the x-axis. We have

$$\lambda \cdot v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \\ -\gamma \end{pmatrix} \in L.$$

We conclude that $2\alpha\mathbf{x} \in L$, so $2\alpha \in \mathbb{Z}$ by the fullness of the subgroup $\langle \mathbf{x} \rangle$.

2. Let $v \in L$ once again have the form indicated in the lemma. We get

$$\lambda \cdot v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix} \in L.$$

It follows directly that both $2\alpha\mathbf{x} + 2\beta\mathbf{y}$ and $2\gamma\mathbf{z}$ are in L. But now it follows that $2\alpha, 2\beta \in \mathbb{Z}$ and $2\gamma \in \mathbb{Z}$, by the fullness of the subgroups $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\langle \mathbf{z} \rangle$, respectively.

Corollary 3.1. *Suppose that the standard point group H contains the involution (-1), and suppose that the orientation-preserving subgroup H^+ of H is one of $C_1^+, C_2^+, D_2^+, C_4^+, D_4^+, A_4^+$, or S_4^+. Suppose that L is a lattice such that $H \cdot L = L$. The pair (L, H) is arithmetically equivalent to (L', H), where L' is equal to one of the following lattices (or the image of one of these under a permutation of coordinate axes):*

$$\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle ; \left\langle \mathbf{x}, \mathbf{y}, \frac{\mathbf{x}+\mathbf{y}+\mathbf{z}}{2} \right\rangle ; \left\langle \mathbf{x}, \mathbf{y}, \frac{\mathbf{x}+\mathbf{z}}{2} \right\rangle ; \left\langle \frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{x}+\mathbf{z}}{2}, \frac{\mathbf{y}+\mathbf{z}}{2} \right\rangle .$$

Moreover, in case $H^+ = C_2^+$, C_4^+, or D_4^+, we can arrange that L' contains $\langle \mathbf{x}, \mathbf{y} \rangle$ as a full subgroup. In particular, for $H^+ = C_2^+$, C_4^+, or D_4^+, L' is not the last of these four lattices.

Proof. If $H^+ = C_1^+$, then $(L, H) \sim (L_C, H)$ by Proposition 3.1(1).

We now show that any pair (L, H) with $H^+ \neq 1$ is equivalent to (L', H), where $2L' \leq L_C$. If $H^+ = C_2^+$, C_4^+, or D_4^+, then, by Proposition 3.1(2), $(L, H) \sim (L', H)$, where $L_C \leq L'$ and $\langle \mathbf{x}, \mathbf{y} \rangle$, $\langle \mathbf{z} \rangle \subseteq L'$ are full subgroups. Since, in each case, H contains the reflection across the plane $P(z = 0)$, we get the desired conclusion from Lemma 3.2(2). If H^+ is any of the remaining groups, then $(L, H) \sim (L', H)$, where $L_C \leq L'$ and each of $\langle \mathbf{x} \rangle$, $\langle \mathbf{y} \rangle$, and $\langle \mathbf{z} \rangle$ are full subgroups of L', by Proposition 3.1(3). In these cases, H (indeed, H^+) contains the rotations through 180° about each of the coordinate axes, and the first part of Lemma 3.2 directly implies the desired conclusion.

Note that $L' \cap [0, 1]^3$ generates L' (since $L_C \leq L'$). The first part of the corollary (and fullness of the subgroups $\langle \mathbf{x} \rangle$, $\langle \mathbf{y} \rangle$, and $\langle \mathbf{z} \rangle$) implies $L' \cap [0, 1]^3$ consists of the corners of the cube $[0, 1]^3$, and possibly some subcollection of seven other points: the center of the cube $\frac{1}{2}(\mathbf{x} + \mathbf{y} + \mathbf{z})$, and the centers of the two-dimensional faces.

Now we run through the cases. Of course, if $L' \cap [0, 1]^3$ consists only of the set of the corners of $[0, 1]^3$, then $L' = \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle$. If $L' \cap [0, 1]^3$ contains the center of the cube, then no other points can be in $L' \cap [0, 1]^3$ (other than the corners) without violating the fullness of one of the subgroups $\langle \mathbf{x} \rangle$, $\langle \mathbf{y} \rangle$, or $\langle \mathbf{z} \rangle$. This case thus yields the second lattice mentioned in the conclusion. The only cases left to consider are those in which any additional lattice points occur in the middle of two-dimensional faces of $[0, 1]^3$. It is obvious that these lattice points must appear in pairs (on opposing pairs of faces), and it is an elementary exercise to show that it is impossible for exactly 4 of these centers to be lattice points. It follows that either 2 opposing center points are in L', or all 6 are in L'. These yield (respectively) the last two lattices mentioned in the conclusion.

The final statement is a consequence of the fact that the L' constructed in this proof has $\langle \mathbf{x}, \mathbf{y} \rangle$ as a full subgroup in the specified cases.

We note finally that all of the lattices in the corollary are invariant under the permutation of coordinate axes, with the exception of the second-to-last.

Lemma 3.3. *Suppose that H is a standard point group which stabilizes the lattice L, suppose $L_{\mathcal{P}} \leq L$, and each of $\langle \mathbf{v}_1 \rangle$, $\langle \mathbf{v}_2 \rangle$, $\langle \mathbf{v}_3 \rangle$ is a full subgroup of L.*

1. *If H contains the $180°$ rotation through the vector space spanned by $\mathbf{v}_2$, and a rotation through $120°$ about the line $\ell(x = y = z)$, then, for any $\mathbf{v} \in L$, $\mathbf{v} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3$, we have 3β, $3\gamma \in \mathbb{Z}$ and $\beta - \gamma \in \mathbb{Z}$.*
2. *If H contains a rotation through $120°$ about the line $\ell(x = y = z)$ and $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ is a full subgroup of L, then, for any $\mathbf{v} \in L$, $\mathbf{v} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3$, we have 3β, $3\gamma \in \mathbb{Z}$ and $\beta - \gamma \in \mathbb{Z}$.*
3. *If H contains a rotation through $120°$ about the line $\ell(x = y = z)$ then, for any $\mathbf{v} \in L$, $\mathbf{v} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3$, we have $3\alpha \in \mathbb{Z}$.*
4. *If H contains both the reflection across the plane $P(x+y+z = 0)$ and a rotation through $120°$ about the line $\ell(x = y = z)$, and $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ is a full subgroup of L, then $L = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$.*

Proof. 1. Let $\mathbf{v}$ be as in the statement of the lemma. We first apply the rotation R_1 through $180°$ about $\ell(x + y = 0; z = 0)$:

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha+\beta \\ \alpha-\beta-\gamma \\ \alpha+\gamma \end{pmatrix} = \begin{pmatrix} -\alpha+\beta+\gamma \\ -\alpha-\beta \\ -\alpha-\gamma \end{pmatrix} \in L.$$

It follows that $\mathbf{v} + R_1 \mathbf{v} = (2\beta + \gamma)\mathbf{v}_2 \in L$. Since $\langle \mathbf{v}_2 \rangle$ is a full subgroup of L, it follows that $2\beta + \gamma \in \mathbb{Z}$. Now apply the rotation R_2 through $120°$ about $\ell(x = y = z)$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha+\beta \\ \alpha-\beta-\gamma \\ \alpha+\gamma \end{pmatrix} = \begin{pmatrix} \alpha-\beta-\gamma \\ \alpha+\gamma \\ \alpha+\beta \end{pmatrix} \in L.$$

It follows that

$$\mathbf{v} - R_2 \mathbf{v} = (2\beta + \gamma)\, \mathbf{v}_2 + (\gamma - \beta)\, \mathbf{v}_3 \in L.$$

This means that $(\gamma-\beta)\mathbf{v}_3 \in L$, since $(2\beta+\gamma)\mathbf{v}_2 \in L$ by the previous calculation. Since $\langle \mathbf{v}_3 \rangle$ is full in L, it must be that $\gamma - \beta \in \mathbb{Z}$. The desired conclusions follow easily.

2. This is easier than the proof of (1). We need consider only the last displayed equation, and the desired conclusions follow from the fullness of $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ in L.
3. We note that the rotation R_2 (from (1)) simply permutes the coordinates of any vector

$$\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = (\alpha + \beta)\mathbf{x} + (\alpha - \beta - \gamma)\mathbf{y} + (\alpha + \gamma)\mathbf{z}$$

cyclically. It is not difficult to see that

$$\left(1 + R_2 + R_2^2\right) \cdot \mathbf{v} = 3\alpha(\mathbf{x} + \mathbf{y} + \mathbf{z}) = 3\alpha\mathbf{v}_1.$$

Since $\langle \mathbf{v}_1 \rangle$ is full in L, we have $3\alpha \in \mathbb{Z}$.
4. Let R_3 denote the reflection in question. Let $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \in L$ be arbitrary. It follows that

$$(1 - R_3)\mathbf{v} = (\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3) - (-\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3) = 2\alpha\mathbf{v}_1 \in L.$$

The fullness of $\langle \mathbf{v}_1 \rangle$ in L implies that $2\alpha \in \mathbb{Z}$. Since $2\alpha \in \mathbb{Z}$ and $3\alpha \in \mathbb{Z}$ (by (3)), $\alpha \in \mathbb{Z}$. It follows that $\beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \in L$. Fullness of $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ in L implies that $\beta, \gamma \in \mathbb{Z}$.

Corollary 3.2. *Suppose that H contains the involution (-1), and the subgroup H^+ of H is one of C_3^+, D_3^+, C_6^+, or D_6^+. Let L be a lattice such that $H \cdot L = L$. The pair (L, H) is equivalent to a pair (L', H) where L' is one of the following:*

$$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle, \left\langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \right\rangle, \left\langle \mathbf{v}_1, \frac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_3 \right\rangle.$$

Indeed, we can assume L' is the first lattice if $H^+ = C_6^+$ or D_6^+, or that it is one of the first two lattices if $H^+ = C_3^+$.

Proof. Suppose first that $H^+ = C_6^+$ or D_6^+. We can apply Proposition 3.1(4), and conclude that $(L, H) \sim (L', H)$, where $L_{\mathcal{P}} \leq L'$ and each of the subgroups $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$, $\langle \mathbf{v}_1 \rangle$ is full in L'. Since H contains both the reflection across the plane $P(x+y+z=0)$ and the rotation through 120° about $\ell(x = y = z)$, Lemma 3.3(4) shows that $L' = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$.

Next suppose $H^+ = C_3^+$. We can again apply Proposition 3.1(4), and conclude that $(L, H) \sim (L', H)$, where $L_{\mathcal{P}} \leq L'$ and each of the subgroups $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$, $\langle \mathbf{v}_1 \rangle$ is full in L'. One possibility is that $L' = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$; we suppose otherwise. Let us consider a typical $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \in L'$. By Lemma 3.3((2) and (3)) and Proposition 3.1(4), we have that $3\alpha, 3\beta, 3\gamma, \gamma - \beta \in \mathbb{Z}$. Since $L' \neq \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$, we can find $\mathbf{v}$ so that not all of α, β, γ are integers. Indeed, fullness of $\langle \mathbf{v}_1 \rangle$ and

$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ in L' (and the inclusions $3\alpha, 3\beta, 3\gamma, \gamma - \beta \in \mathbb{Z}$) imply that none of α, β, γ are integers if one is not. It is now routine to show that one of the vectors

$$\frac{1}{3}(\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3) \quad \text{or} \quad \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$$

is in L'. After applying the 180° rotation about $\ell(x = y = z)$, we can assume that $\frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \in L'$, since the rotation in question preserves H.

We claim that $L' = \langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \rangle$. Suppose $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \in L'$. We can conclude, as before, that $3\alpha, 3\beta, 3\gamma, \gamma - \beta \in \mathbb{Z}$.

$$\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = (\alpha - \beta)\mathbf{v}_1 + (3\beta)\frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + (\gamma - \beta)\mathbf{v}_3.$$

Since $\mathbf{v}$ and every other term of the right-most sum is in L', so is $(\alpha - \beta)\mathbf{v}_1$. It follows that $\alpha - \beta \in \mathbb{Z}$. The equation above displays $\mathbf{v}$ as an integral combination of elements in $\langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \rangle$. This completes the proof in the case of $H^+ = C_3^+$.

Suppose that $H^+ = D_3^+$. We apply Proposition 3.1(5): $(L, H) \sim (L', H)$, where $L_{\mathcal{P}} \leq L'$ and each of the subgroups $\langle \mathbf{v}_1 \rangle$, $\langle \mathbf{v}_2 \rangle$ and $\langle \mathbf{v}_3 \rangle$ is full in L'. The lattice L' could be $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ or $\langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \rangle$. Indeed, an argument essentially identical to the one for the case $H^+ = C_3^+$ shows that these are the only possibilities if $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ is full in L'. Suppose that $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ is not full in L'. Let $\beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \in L'$, where, by Lemma 3.3(1), $3\beta, 3\gamma, \gamma - \beta \in \mathbb{Z}$. We assume that one of β, γ (equivalently, both of β, γ) are not integers. It follows quickly that $\frac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3) \in L'$.

We claim that $L' = \langle \mathbf{v}_1, \frac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_3 \rangle$. Let $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \in L'$. By (1) and (3) of Lemma 3.3, we have $3\alpha, 3\beta, 3\gamma, \gamma - \beta \in \mathbb{Z}$.

$$\mathbf{v} = \alpha\mathbf{v}_1 + (3\beta)\left(\frac{1}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3\right) + (\gamma - \beta)\mathbf{v}_3 \in L'.$$

It follows that $\alpha\mathbf{v}_1 \in L'$, which implies that $\alpha \in \mathbb{Z}$, by fullness of $\langle \mathbf{v}_1 \rangle$ in L'. This completes the proof.

3.4 Classification of Pairs (L, H), Where $(-1) \in H$

We now sort the pairs (L, H) up to arithmetic equivalence, assuming that $(-1) \in H$. The following theorem lists 24 such pairs as distinct possibilities; the theorem leaves open the possibility that some of these pairs will be equivalent. We will see in Theorem 3.2 that indeed all of them are different.

Theorem 3.1. *Let $L \leq \mathbb{R}^3$ be a lattice, and let $H \leq O(3)$ be a standard point group acting on L, such that $(-1) \in H$. The pair (L, H) is equivalent to one of the 24 on the following list.*

1. *If $H^+ = A_4^+$, or S_4^+, then $(L, H) \sim (L', H)$, where*

$$L' = L_C, \left\langle \frac{1}{2}(\mathbf{x}+\mathbf{y}+\mathbf{z}), \mathbf{y}, \mathbf{z} \right\rangle, \text{ or } \left\langle \frac{1}{2}(\mathbf{x}+\mathbf{y}), \frac{1}{2}(\mathbf{x}+\mathbf{z}), \frac{1}{2}(\mathbf{y}+\mathbf{z}) \right\rangle.$$

   (There are six possibilities in all.)
2. *If $H^+ = D_2^+$, then $(L, H) \sim (L', H)$, where L' is any of the lattices mentioned in Corollary 3.1. (There are four possibilities.)*
3. *If $H^+ = C_2^+$, C_4^+, or D_4^+, then $(L, H) \sim (L', H)$, where*

$$L' = \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \text{ or } \left\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x}+\mathbf{y}+\mathbf{z}) \right\rangle.$$

   (There are six possibilities.)
4. *If $H^+ = C_1^+$, then $(L, H) \sim (L', H)$, where $L' = L_C$. (There is only one possibility.)*
5. *If $H^+ = C_6^+$, or D_6^+, then $(L, H) \sim (L', H)$, where $L' = L_{\mathcal{P}}$. (There are two possibilities.)*
6. *If $H^+ = C_3^+$, then $(L, H) \sim (L', H)$, where*

$$L' = L_{\mathcal{P}} \text{ or } \left\langle \frac{1}{3}(\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \right\rangle.$$

   (There are two possibilities.)
7. *If $H^+ = D_3^+$, then $(L, H) \sim (L', H)$, where L' can be any of the lattices listed in Corollary 3.2. (There are three possibilities.)*

Proof. We note first that all of the pairs (L', H) mentioned as possibilities above truly occur (i.e., $H \cdot L' = L'$).

1. Let $H^+ = A_4^+$ or S_4^+. Corollary 3.1 gives us four possibilities. We rule out the third possibility, $\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x}+\mathbf{z}) \rangle$ (and its images under permutations of the coordinate axes), since it fails to be invariant under the action of H.
2. Let $H^+ = D_2^+$. In this case, all four lattices from Corollary 3.1 are possible. We note that H is normalized by any permutation of the coordinate axes, so the lattices

$$\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x}+\mathbf{z}) \rangle, \langle \mathbf{x}, \frac{1}{2}(\mathbf{x}+\mathbf{y}), \mathbf{z} \rangle, \text{ and } \langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{y}+\mathbf{z}) \rangle$$

   all lead to arithmetically equivalent pairs.
3. Let $H^+ = C_4^+$ or D_4^+. Corollary 3.1 gives us five possibilities. The third lattice mentioned in Corollary 3.1 is really a collection of three distinct lattices. We can

dispose with these three lattices, either because: (i) they are not invariant under the action of H (and so cannot occur as part of the pair (L, H)), or (ii) they do not contain $\langle \mathbf{x}, \mathbf{y} \rangle$ as a full subgroup (which we can arrange by Corollary 3.1). It follows that the lattices in the statement of the theorem are the only possibilities up to arithmetic equivalence.

Now suppose $H^+ = C_2^+$. We conclude, exactly as in the previous paragraph, that there are five possibilities for L', by Corollary 3.1. We can rule out

$$\left\langle \mathbf{x}, \mathbf{z}, \frac{\mathbf{x}+\mathbf{y}}{2} \right\rangle$$

since we can assume that $\langle \mathbf{x}, \mathbf{y} \rangle$ is full in L. Now we note that the lattices

$$\left\langle \mathbf{x}, \mathbf{y}, \frac{\mathbf{x}+\mathbf{z}}{2} \right\rangle, \left\langle \mathbf{x}, \mathbf{y}, \frac{\mathbf{y}+\mathbf{z}}{2} \right\rangle$$

are both equivalent to the second lattice from Corollary 3.1, by the matrices

$$\lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

4. Is clear.
5. Follows immediately from Corollary 3.2, as do (6) and (7).

The following lemma uses only basic group theory and linear algebra. The proof is left to the reader.

Lemma 3.4. *Let $\lambda \in GL_3(\mathbb{R})$ normalize the point group H.*

1. *If $\ell \subseteq \mathbb{R}^3$ is a pole such that the stabilizer group $H_\ell^+ = \{h \in H^+ \mid h_{|\ell} = id_\ell\}$ has order n, then $\lambda\ell$ is also a pole of H^+, and $H^+_{\lambda\cdot\ell}$ has order n.*
2. *If two vectors $v_1, v_2 \in \mathcal{L}^+ \cup \mathcal{L}^-$ are in the same orbit under the action of H, then $||\lambda v_1|| = ||\lambda v_2||$, and λv_1, λv_2 are in the same orbit under the action of H.*
3. *If $V \subseteq \mathbb{R}^3$ is an H-invariant vector subspace, then λV is also an H-invariant vector subspace.* □

Theorem 3.2. *No two of the* 24 *arithmetic classes of pairs* (L, H) *from Theorem 3.1 are the same.*

Proof. We first note that if two pairs (L_1, H_1), (L_2, H_2) are arithmetically equivalent, then H_1 and H_2 are isomorphic groups (in fact, conjugate in $GL_3(\mathbb{R})$). It follows that if two pairs from the list in Theorem 3.1 are the same, then their point groups must be the same (since different point groups of the given type have different isomorphism types). Thus, we can assume that $H_1 = H_2$ in the following arguments.

1. Suppose $H^+ = A_4^+$ or S_4^+. Let (L_1, H), (L_2, H) be distinct pairs, where L_1 and L_2 are as described in Theorem 3.1(1). Suppose $(L_1, H) \sim (L_2, H)$. This means that there is λ such that $\lambda L_1 = L_2$ and $\lambda H \lambda^{-1} = H$. By Lemma 3.4(1), λ must permute the coordinate axes. Since $\langle \mathbf{x} \rangle$, $\langle \mathbf{y} \rangle$, $\langle \mathbf{z} \rangle$ are full subgroups of both L_1 and L_2, it must be that λ is a signed permutation matrix. Such a matrix fixes each of the lattices from Theorem 3.1(1), which is a contradiction.
2. Follows the exact pattern of (1).
3. Suppose $H^+ = C_2^+, C_4^+$, or D_4^+. Let (L_1, H), (L_2, H) be distinct pairs, where L_1 and L_2 are chosen from the possibilities in Theorem 3.1(3). We can assume, without loss of generality, that $L_1 = \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle$ and $L_2 = \langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x} + \mathbf{y} + \mathbf{z}) \rangle$. Suppose $(L_1, H) \sim (L_2, H)$; let $\lambda \in GL_3(\mathbb{R})$ satisfy $\lambda L_1 = L_2$ and $\lambda H \lambda^{-1} = H$. We claim that λ has the form

$$\lambda = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

   Moreover, if we assume that λ has the latter form, then λ must have integral entries by the fullness of $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\langle \mathbf{z} \rangle$ in both L_1 and L_2. It will then follow that λL_1 is a sublattice of L_C, a contradiction.

   We turn to a proof of the claim. First, assume that $H^+ = C_2^+$ or C_4^+. Since λ normalizes H, it must be that λ actually commutes with the generator of C_2^+, which is the unique element of H having positive determinant and order 2. It now follows from a straightforward calculation that λ has the required block form. If $H^+ = D_4^+$, then we appeal to parts (3) and (1) (respectively) of Lemma 3.4: since the xy-plane is the unique two-dimensional H-invariant subspace, it must be preserved by λ; since the z-axis is the unique one-dimensional subspace to be an axis of rotation for an element of order 4 in H^+, it must be preserved. It follows directly that λ has the required form in this case as well. This proves the claim.
4. Is trivial.
5. Is also trivial.
6. Suppose $(L_{\mathcal{P}}, \langle C_3^+, (-1) \rangle) \sim (\langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \rangle, \langle C_3^+, (-1) \rangle)$. Suppose that $\lambda \in GL_3(\mathbb{R})$ satisfies $\lambda L_{\mathcal{P}} = \langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \rangle$ and $\lambda H \lambda^{-1} = H$. We claim that λ has the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

   as a matrix over the ordered basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Indeed, if this is the case, then all of the entries must be integers by the fullness of $\langle \mathbf{v}_1 \rangle$ and $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ in both lattices. It will then follow that $\lambda L_{\mathcal{P}} \leq L_C$, a contradiction.

   We prove the claim. Note that $\ell(x = y = z)$ is the unique one-dimensional H-invariant subspace, and so must be invariant under λ by Lemma 3.4(3). Similarly, the plane $P(x + y + z = 0)$ is the unique two-dimensional H-invariant subspace, so it is also invariant under λ. The claim now follows directly, since $\{\mathbf{v}_2, \mathbf{v}_3\}$ spans $P(x + y + z = 0)$ and $\{\mathbf{v}_1\}$ spans $\ell(x = y = z)$.

7. Let $H = \langle D_3^+, (-1)\rangle$, $L_1 = L_{\mathcal{P}}$, $L_2 = \langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3\rangle$, and $L_3 = \langle \mathbf{v}_1, \frac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_3\rangle$. We can conclude that $(L_1, H) \not\sim (L_2, H)$ exactly as in (6). If $(L_2, H) \sim (L_3, H)$, where $\lambda L_2 = L_3$ and $\lambda H \lambda^{-1} = H$, then we conclude as in (6) that λ has the same block form (as a matrix over the same ordered basis), although, in the current case, we can conclude only that the upper left entry is ± 1, by fullness of $\langle \mathbf{v}_1\rangle$ in both L_2 and L_3. This leads to a contradiction, since $\lambda(\frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)) \notin L_3$.

   Finally, suppose $(L_1, H) \sim (L_3, H)$. Let $\lambda \in GL_3(\mathbb{R})$ satisfy $\lambda L_1 = L_3$ and $\lambda H \lambda^{-1} = H$. By Lemma 3.4(1), the vector subspaces $\langle \mathbf{v}_2\rangle$, $\langle \mathbf{v}_3\rangle$, and $\langle \mathbf{v}_2 - \mathbf{v}_3\rangle$ must be permuted by λ. Since H acts transitively on these lines, by Lemma 3.4(2), $||\lambda \mathbf{v}_2|| = ||\lambda \mathbf{v}_3|| = \alpha\sqrt{2}$, say. By Lemma 3.4(1), the vector subspace spanned by $\mathbf{v}_1$ must be λ-invariant as well, since it is the unique one-dimensional vector subspace that is the axis for an element of order 3 in H^+. It follows that λ has the block form from (6) over the ordered basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. In addition, we know that the upper left entry is an integer, by fullness of $\langle \mathbf{v}_1\rangle$ in L_1 and L_3. The final two column vectors of λ must be linearly independent elements of the set

$$\{\pm\alpha\mathbf{v}_2, \pm\alpha\mathbf{v}_3, \pm\alpha(\mathbf{v}_2 - \mathbf{v}_3)\}.$$

   It follows that at least one has the form $\pm\alpha\mathbf{v}_i$ (for $i \in \{2, 3\}$). Since $\langle \mathbf{v}_2\rangle$ and $\langle \mathbf{v}_3\rangle$ are full subgroups of both L_1 and L_3, it follows that α must be an integer. Thus λ must have integral entries. It follows that $\lambda L_1 \subseteq L_C$, a contradiction.

3.5 The Classification of the Remaining Pairs (L, H)

In this section, we conclude the classification of pairs (L, H) up to arithmetic equivalence. Our approach is to reduce the problem of classifying the remaining pairs to the (previously solved) problem of classifying pairs in which the group H contains the inversion.

Theorem 3.3. *Let $L \leq \mathbb{R}^3$ be a lattice, and let $H \leq O(3)$ be a standard point group acting on L; suppose $(-1) \notin H$. The pair (L, H) is equivalent to exactly one on the following list:*

1. *If $H \leq SO(3)$, then the classification of pairs (L, H) is exactly the same as that for the group $\langle H, (-1)\rangle$, as described in Theorem 3.1. (This case accounts for 24 different possibilities.)*
2. *If $H = C_2'$, C_4', C_6', D_3', D_4'', D_6'', or S_4', then any pair (L, H) is equivalent to one of the (L', H), where L' is one of the lattices listed in Theorem 3.1 for $\langle H, (-1)\rangle$. Moreover, any two of the resulting pairs are distinct. (There are a total of 14 possibilities.)*
3. *Suppose $H = D_2'$, D_4', or D_6'.*

a. *If* $H = D_4'$*, then* $(L, H) \sim (L', H')$ *where* $L' = L_C$ *or* $\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x}+\mathbf{y}+\mathbf{z})\rangle$ *and* $H' = D_4'$ *or* $\hat{D}_4'$*.*
b. *If* $H = D_6'$*, then* $(L, H) \sim (L', H')$ *where* $L' = L_{\mathcal{P}}$ *and* $H' = D_6'$ *or* $\hat{D}_6'$*.*
c. *If* $H = D_2'$*, then* $(L, H) \sim (L', H)$*, where* L' *is any of the lattices mentioned in Corollary 3.1, or* $(L, H) \sim (L', D_{2_2}')$*, where*

$$L' = \left\langle \mathbf{x}, \mathbf{y}, \frac{\mathbf{x}+\mathbf{z}}{2} \right\rangle.$$

(There are 11 *possibilities.)*

Proof. We note first that it is impossible for a pair (L, H) to be counted twice in the different cases (1), (2), and (3), since the point groups in question are distinguished either by isomorphism type or by their orientation-preserving subgroups. (Note, in particular, that the groups D_n' and D_n'' $(n = 4, 6)$ cannot be conjugate even in $GL_3(\mathbb{R})$ by the descriptions in Sect. 2.5.3.) It is therefore enough to consider each of the cases (1), (2), and (3) individually.

1. Let H be an orientation-preserving standard point group, and let L be a lattice satisfying $H \cdot L = L$. There is some pair $(L', \langle H, (-1)\rangle)$ from the statement of Theorem 3.1 such that $(L, \langle H, (-1)\rangle) \sim (L', \langle H, (-1)\rangle)$; that is, we can find $\lambda \in GL_3(\mathbb{R})$ such that $\lambda\langle H, (-1)\rangle\lambda^{-1} = \langle H, (-1)\rangle$, and $\lambda L = L'$. We note that $\lambda H \lambda^{-1} = H$ since the latter groups are the orientation-preserving subgroups of $\lambda\langle H, (-1)\rangle\lambda^{-1}$ and $\langle H, (-1)\rangle$, respectively. It follows that $(L, H) \sim (L', H)$. Thus, any pair (L, H) corresponds to one of the 24 listed in Theorem 3.1.
   We now need to show that there are no repetitions on the given list of 24 pairs (L, H). Suppose that $(L_1, H_1) \sim (L_2, H_2)$, where each of H_1, H_2 is an orientation-preserving standard point group, and L_1, L_2 are lattices chosen from the statement of Theorem 3.1. We first note that H_1 and H_2 must be isomorphic by the definition of arithmetic equivalence, and therefore equal since no two groups from the list in Fig. 2.1 are isomorphic.
   Thus, we assume that $(L_1, H) \sim (L_2, H)$, where L_1, L_2, and $H = H_1 = H_2$ are all still as above. Let $\lambda \in GL_3(\mathbb{R})$ be such that $\lambda L_1 = L_2$ and $\lambda H \lambda^{-1} = H$. This λ shows that $(L_1, \langle H, (-1)\rangle) \sim (L_2, \langle H, (-1)\rangle)$. It follows that $L_1 = L_2$, by Theorem 3.1, completing the proof.
2. Let H be one of the point groups from (2), and let L be a lattice satisfying $H \cdot L = L$. There is some pair $(L', \langle H, (-1)\rangle)$ from the statement of Theorem 3.1 such that $(L, \langle H, (-1)\rangle) \sim (L', \langle H, (-1)\rangle)$; that is, we can find $\lambda \in GL_3(\mathbb{R})$ such that $\lambda\langle H, (-1)\rangle\lambda^{-1} = \langle H, (-1)\rangle$, and $\lambda L = L'$.
   The group H is unique in the following sense. If $K \leq \langle H, (-1)\rangle$ satisfies: (i) $[\langle H, (-1)\rangle : K] = 2$; (ii) K does not contain the inversion; (iii) $K^+ \cong H^+$ (where H^+ and K^+ denote the orientation-preserving subgroups), and (iv) $K \cong H$, then $K = H$. (This can be proved by enumerating the homomorphisms $\phi : \langle H, (-1)\rangle \to \mathbb{Z}/2\mathbb{Z}$ such that $\phi(-1) = 1$ and $\phi(H) = \mathbb{Z}/2\mathbb{Z}$. The group K must occur as the kernel of some such ϕ, and the given conditions force

$K = H$. Note that we have already seen this method of argument in the proof of Theorem 2.3.)

Now note that $\lambda H \lambda^{-1}$ is a subgroup of $\langle H, (-1)\rangle$ satisfying (i)–(iv). It follows that $\lambda H \lambda^{-1} = H$. We have now shown that $(L, H) \sim (L', H)$, where L' is one of the lattices that is paired with $\langle H, (-1)\rangle$ in Theorem 3.1.

We should next show that there are no repetitions in our list, but the proof of the latter fact follows the pattern from the final paragraph of the proof of (1).

3. Suppose that $H = D_4'$. We consider the pair (L, H). Add the element (-1) to the point group to get $(L, \langle H, (-1)\rangle)$. By the arithmetic classification of pairs with central inversion (Theorem 3.1(3)), we know that $(L, \langle H, (-1)\rangle) \sim (L', \langle H, (-1)\rangle)$, where

$$L' = \langle \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle \quad \text{or} \quad \left\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x} + \mathbf{y} + \mathbf{z})\right\rangle.$$

Let us suppose that $\lambda L = L'$ and $\lambda\langle H, (-1)\rangle\lambda^{-1} = \langle H, (-1)\rangle$. There are two possibilities for $\lambda H \lambda^{-1}$: D_4' and $\hat{D}_4'$. (One again sees this by enumerating the homomorphisms from $\langle D_4^+, (-1)\rangle$ to $\mathbb{Z}/2\mathbb{Z}$.) This leads to four possibilities for (L', H'), where $\lambda H \lambda^{-1} = H'$; we shall see that all are different.

Completely analogous reasoning shows that if $H = D_6'$ then there are two possibilities: $(L_{\mathcal{P}}, D_6')$ and $(L_{\mathcal{P}}, \hat{D}_6')$.

Suppose $H = D_2'$. Consider the pair (L, H). We add the element (-1) to the point group to get the pair $(L, \langle H, (-1)\rangle)$. By Theorem 3.1(2), $(L, \langle H, (-1)\rangle) \sim (L', \langle H, (-1)\rangle)$, where L' is any of the lattices listed in Corollary 3.1. Let $\lambda \in GL_3(\mathbb{R})$ satisfy: (i) $\lambda L = L'$, and (ii) $\lambda\langle H, (-1)\rangle\lambda^{-1} = \langle H, (-1)\rangle$. It is not difficult to check that $\lambda H \lambda^{-1} = H'$ is one of the following groups:

$$\langle R_{xy}, R_{yz}\rangle, \quad \langle R_{xy}, R_{xz}\rangle, \quad \langle R_{yz}, R_{xz}\rangle$$

where R_{xy} (for instance) is the reflection across the xy-plane. We note that $\langle R_{yz}, R_{xz}\rangle = D_2'$. Thus, to summarize: we've shown that $(L, D_2') \sim (L', H')$, where L' is one of the standard lattices from Corollary 3.1 and H' is one of the groups above. All three of the latter groups are clearly conjugate to D_2', by an element $\hat{\lambda} \in GL_3(\mathbb{R})$ that simply permutes the coordinate axes (i.e., $\hat{\lambda} H' \hat{\lambda}^{-1} = D_2'$). We conclude that $(L, D_2') \sim (\hat{\lambda} L', D_2')$, where $\hat{\lambda}$ is a permutation matrix and L' is as above.

If $L' = \langle \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle$, $\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x} + \mathbf{y} + \mathbf{z})\rangle$, or $\langle \frac{1}{2}(\mathbf{x} + \mathbf{y}), \frac{1}{2}(\mathbf{x} + \mathbf{z}), \frac{1}{2}(\mathbf{y} + \mathbf{z})\rangle$, then $\hat{\lambda} L' = L'$. It follows that these three possibilities give rise to three arithmetic classes of the form (L', D_2') (and all three are different, as we'll see). If $L' = \langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x} + \mathbf{z})\rangle$, then L' is not necessarily invariant under $\hat{\lambda}$ and there are two essentially different pairs of the form $(\hat{\lambda} L', D_2')$:

$$\left\langle \mathbf{x}, \mathbf{z}, \frac{1}{2}(\mathbf{x} + \mathbf{y})\right\rangle \quad \text{and} \quad \left\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x} + \mathbf{z})\right\rangle.$$

(The case in which $L' = \langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{y} + \mathbf{z})\rangle$ is identical with that in which $L' = \langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x} + \mathbf{z})\rangle$ up to arithmetic equivalence, since the transposition that flips the x- and y-coordinates normalizes D_2'.)

Now we need to show that all 11 of the above pairs are arithmetically distinct. As always, it is enough to consider the subcases (a), (b), and (c) separately.

Consider first the case in which $H = D_2'$. There are five such pairs; the only two that might be equal are $(\langle \mathbf{x}, \mathbf{z}, \frac{1}{2}(\mathbf{x}+\mathbf{y})\rangle, D_2')$ and $(\langle \mathbf{x}, \mathbf{y}, \frac{1}{2}(\mathbf{x}+\mathbf{z})\rangle, D_2')$. (Any other choice of pairs is distinct by Theorem 3.1(2): here we consider the usual reduction to point groups containing inversion.) If λ normalizes D_2' and sends one lattice to the other, then Lemma 3.4(1) and fullness of the subgroups $\langle \mathbf{x}\rangle$, $\langle \mathbf{y}\rangle$, $\langle \mathbf{z}\rangle$ in both lattices imply that λ factors as AB, where A is a diagonal matrix with 1s and -1s on the diagonal, and B is a permutation matrix which leaves the vector subspace $\langle \mathbf{z}\rangle$ invariant. No such matrix can send the first lattice to the second one. It follows that all five pairs with point group D_2' are distinct. We apply a transposition $\bar{\lambda} \in GL_3(\mathbb{R})$ of the y- and z-coordinates to the first of these pairs, $(\langle \mathbf{x}, \mathbf{z}, \frac{1}{2}(\mathbf{x} + \mathbf{y})\rangle, D_2')$, to arrive at the pair (L', D_{2_2}') from the statement of the theorem.

Now we consider the case in which $H = D_4'$. The possible arithmetic classes are represented by four pairs (L_i, D_4'), $(L_i, \hat{D}_4')$, where L_i $(i \in \{1, 2\})$ is one of two lattices. We first note that two such pairs (L', H'), (L'', H') will represent different classes if $L' \neq L''$ by Theorem 3.1(3).

Thus, suppose $\hat{L}$ is one of the two possible lattices from (3). Suppose $(\hat{L}, D_4') \sim (\hat{L}, \hat{D}_4')$; suppose $\lambda \in GL_3(\mathbb{R})$ satisfies $\lambda\hat{L} = \hat{L}$ and $\lambda D_4'\lambda^{-1} = \hat{D}_4'$. The condition $\lambda D_4'\lambda^{-1} = \hat{D}_4'$ implies that λ leaves the xy-plane invariant (here we can apply Lemma 3.4(3) with $H = \langle D_4^+, (-1)\rangle$). Now λ must send the poles of D_4' to those of $\hat{D}_4'$. It follows that λ sends the groups $\langle \mathbf{x}\rangle$, $\langle \mathbf{y}\rangle$ to $\langle \mathbf{x} + \mathbf{y}\rangle$, $\langle \mathbf{x} - \mathbf{y}\rangle$ (not necessarily in that order), and all groups in question are full in $\hat{L}$. It follows that λ restricts to a similarity on $P(z = 0)$. This leads to a contradiction, in the following way. In the pair $(\hat{L}, D_4')$, the smallest non-zero lattice point in $P(z = 0)$ lies on a pole, but in the pair $(\hat{L}, \hat{D}_4')$ the smallest non-zero lattice point in $P(z = 0)$ does not (instead it lies on a plane of reflection for $\hat{D}_4'$, either $P(x = 0)$ or $P(y = 0)$). The fact that λ maps $P(z = 0)$ to itself by a similarity implies that a smallest lattice point in $\hat{L} \cap P(z = 0)$ must be sent to another such. This is the contradiction.

The proof for the case $H = D_6'$ is similar.

Chapter 4
The Split Three-Dimensional Crystallographic Groups

Definition 4.1. An *n-dimensional crystallographic group* Γ is a discrete, cocompact subgroup of the group of isometries of Euclidean n-space. Each $\gamma \in \Gamma$ can be written in the form $v_\gamma + A_\gamma$, where $v_\gamma \in R^n$ is a translation and $A_\gamma \in O(n)$. There is a natural map $\pi : \Gamma \to O(n)$ sending $v_\gamma + A_\gamma$ to A_γ, and this map is easily seen to be a homomorphism. We get a short exact sequence as follows:

$$L \rightarrowtail \Gamma \twoheadrightarrow H,$$

where $H = \pi(\Gamma) \leq O(n)$ and L is the kernel. (By [Ra94, Theorem 7.4.2], L is a lattice in $\mathbb{R}^n$, and so necessarily isomorphic to $\mathbb{Z}^n$.) We note that H acts naturally on L, which makes H a point group in the sense of Definition 2.1. We say that H is the *point group* of Γ. The group Γ is a *split n-dimensional crystallographic group* if the above sequence splits, i.e., if there is a homomorphism $s : H \to \Gamma$ such that $\pi s = id_H$.

From now on, all of our crystallographic groups will be three-dimensional.

Definition 4.2. Suppose that L is a lattice in $\mathbb{R}^3$ and $H \leq O(3)$ satisfies $H \cdot L = L$. We let $\Gamma(L, H)$ denote the group $\langle L, H \rangle$.

Remark 4.1. It is straightforward to verify that every $\Gamma(L, H)$ is a split crystallographic group.

Theorem 4.1. *Any split crystallographic group $\hat{\Gamma}$ is isomorphic to $\Gamma(L, H)$, for some lattice $L \leq \mathbb{R}^3$ and $H \leq O(3)$ satisfying $H \cdot L = L$. The groups $\Gamma(L, H)$ and $\Gamma(L', H')$ are isomorphic if and only if the pairs (L, H) and (L', H') are arithmetically equivalent.*

Proof. We prove the first statement. Let $\hat{\Gamma}$ be a split crystallographic group, $\hat{L}$ denote the lattice of $\hat{\Gamma}$, and $\hat{H}$ denote the point group of $\hat{\Gamma}$. Since $\hat{\Gamma}$ is split, it follows that there is a finite subgroup J of $\hat{\Gamma}$ such that $\pi : \hat{\Gamma} \to \hat{H}$ satisfies $\pi(J) = \hat{H}$. It is routine to check that $\pi_{|J} : J \to \hat{H}$ must also be injective. Since J

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_4

is a finite group of isometries of $\mathbb{R}^3$, it must be that the entire group J fixes a point $v \in \mathbb{R}^3$. We consider the isometry $T_v \in \mathrm{Isom}(\mathbb{R}^3)$, which is simply translation by the vector v. It follows that $T_v^{-1} J T_v$ fixes the origin, so the map $\pi : T_v^{-1} J T_v \to \hat{H}$ is the identity. We can therefore write

$$1 \to \hat{L} \to T_v^{-1} \hat{\Gamma} T_v \to \hat{H} \to 1,$$

where $\hat{H} \le T_v^{-1} \hat{\Gamma} T_v$. It follows directly that $\hat{\Gamma} \cong T_v^{-1} \hat{\Gamma} T_v = \langle \hat{L}, \hat{H} \rangle$, proving the first statement.

Now we prove the second statement. Assume that $\Gamma(L, H)$ and $\Gamma(L', H')$ are isomorphic. Ratcliffe [Ra94, Theorem 7.4.4] says that there is an affine bijection α of $\mathbb{R}^3$ such that $\alpha \Gamma(L, H) \alpha^{-1} = \Gamma(L', H')$. We write $\alpha = v_\alpha + A_\alpha$, where $v_\alpha \in \mathbb{R}^3$ and $A_\alpha \in GL_3(\mathbb{R})$. We note:

$$\begin{aligned} \alpha L \alpha^{-1} &= A_\alpha \cdot L; \\ \alpha H \alpha^{-1} &= \left(v_\alpha - A_\alpha H A_\alpha^{-1}(v_\alpha)\right) + A_\alpha H A_\alpha^{-1}. \end{aligned}$$

Now $\alpha \Gamma(L, H) \alpha^{-1}$ and $\Gamma(L', H')$ must have the same kernel and image under the canonical projection $\pi : \mathrm{Isom}(\mathbb{R}^3) \to O(3)$, so $A_\alpha \cdot L = L'$ and $A_\alpha H A_\alpha^{-1} = H'$. (These last two equations are between kernels and images, respectively.) It follows that (L, H) and (L', H') are arithmetically equivalent.

If two pairs (L, H) and (L', H') are arithmetically equivalent, then there is $\lambda \in GL_3(\mathbb{R})$ such that $\lambda L = L'$ and $\lambda H \lambda^{-1} = H'$. It follows easily that $\lambda \Gamma(L, H) \lambda^{-1} = \Gamma(L', H')$, so $\Gamma(L, H)$ and $\Gamma(L', H')$ are isomorphic.

Theorem 4.2 (List of Split Three-Dimensional Crystallographic Groups). *Let* **x**, **y**, *and* **z** *denote the standard coordinate vectors, and let*

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

A complete list of the split three-dimensional crystallographic groups $\langle L, H \rangle$ (up to isomorphism) appears in Table 4.1.

Proof. Table 4.1 lists all pairings of lattices and point groups from Theorems 3.1 and 3.3. We have already shown that no two of the pairs from Theorem 3.1 determine the same arithmetic equivalence class (Theorem 3.2). Also, no two of the pairs from Theorem 3.3 determine the same arithmetic class. If the pair (L_1, H_1) is chosen from the pairs listed in Theorem 3.1, and (L_2, H_2) is chosen from the pairs listed in Theorem 3.3, then $(L_1, H_1) \not\sim (L_2, H_2)$, since H_1 contains the inversion (-1) and H_2 does not, and containing the inversion will be preserved by arithmetic equivalence. Thus, all 73 pairs in Table 4.1 represent distinct equivalence classes.

It is clear from Theorems 3.1 and 3.3 that any pair (L, H) is equivalent to one on the list, since H must be conjugate to one of the standard point groups. It follows that there are exactly 73 classes of such pairs.

Table 4.1 The split three-dimensional crystallographic groups

L	H					
	$S_4^+ \times (-1)$	S_4^+	S_4'	$A_4^+ \times (-1)$	A_4^+	D_4''
$\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle$	$D_4^+ \times (-1)$	D_4^+	C_2'	$D_2^+ \times (-1)$	D_2^+	C_4'
	$C_4^+ \times (-1)$	C_4^+	D_2'	$C_2^+ \times (-1)$	C_2^+	D_4'
	$C_1^+ \times (-1)$	C_1^+	$\hat{D}_4'$			
	$S_4^+ \times (-1)$	S_4^+	S_4'	$A_4^+ \times (-1)$	A_4^+	D_4''
$\langle \frac{1}{2}(\mathbf{x} + \mathbf{y} + \mathbf{z}), \mathbf{y}, \mathbf{z} \rangle$	$D_4^+ \times (-1)$	D_4^+	C_2'	$D_2^+ \times (-1)$	D_2^+	C_4'
	$C_4^+ \times (-1)$	C_4^+	D_2'	$C_2^+ \times (-1)$	C_2^+	D_4'
	$\hat{D}_4'$					
	$S_4^+ \times (-1)$	S_4^+	S_4'	$A_4^+ \times (-1)$	A_4^+	D_2'
$\frac{1}{2}\langle (\mathbf{x} + \mathbf{y}), (\mathbf{x} + \mathbf{z}), (\mathbf{y} + \mathbf{z}) \rangle$	$D_2^+ \times (-1)$	D_2^+				
$\langle \frac{1}{2}(\mathbf{x} + \mathbf{z}), \mathbf{y}, \mathbf{z} \rangle$	$D_2^+ \times (-1)$	D_2^+	D_2'	D_{2_2}'		
	$D_6^+ \times (-1)$	D_6^+	C_6'	$C_6^+ \times (-1)$	D_6'	C_6^+
$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$	$D_3^+ \times (-1)$	$\hat{D}_6'$	C_3^+	$C_3^+ \times (-1)$	D_3'	D_3^+
	D_6''					
$\langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \rangle$	$D_3^+ \times (-1)$	D_3^+	D_3'	$C_3^+ \times (-1)$	C_3^+	
$\langle \mathbf{v}_1, \frac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_3 \rangle$	$D_3^+ \times (-1)$	D_3^+	D_3'			

The theorem now follows from Theorem 4.1.

Remark 4.2. Let $\hat{\Gamma}$ be a point group. For the sake of brevity, we will sometimes let $\hat{\Gamma}_i$ denote the split crystallographic group $\langle L_i, \hat{\Gamma} \rangle$, where L_i denotes the ith lattice (in the order that they are listed in Table 4.1). Thus, $(D_2^+)_1$ denotes the split crystallographic group generated by the point group D_2^+ and the standard cubical lattice.

We will let Γ_i denote the ith maximal split crystallographic group; i.e., the pairing of the ith lattice with the largest point group from Table 4.1. Thus, for instance, Γ_1 denotes the group $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \rtimes (S_4^+ \times (-1))$.

Chapter 5
A Splitting Formula for Lower Algebraic K-Theory

Let Γ be a three-dimensional crystallographic group with lattice L and point group H. (We do not assume that Γ is a split crystallographic group.) In this chapter, we describe a simple construction of $E_{\mathcal{VC}}(\Gamma)$ and derive a splitting formula for the lower algebraic K-theory of any three-dimensional crystallographic group.

5.1 A Construction of $E_{\mathcal{FIN}}(\Gamma)$ for Crystallographic Groups

We will need to have a specific model of $E_{\mathcal{FIN}}(\Gamma)$ for our crystallographic groups Γ.

Proposition 5.1. *If Γ is a three-dimensional crystallographic group, then there is an equivariant cell structure on $\mathbb{R}^3$ making it a model for $E_{\mathcal{FIN}}(\Gamma)$.*

Proof. For every crystallographic group Γ, there is a crystallographic group Γ' of the same dimension, called the *splitting group* of Γ ([Ra94, pp. 312–313]), and an embedding $\phi : \Gamma \to \Gamma'$. The group Γ' is a split crystallographic group in our sense, by Lemma 7 on page 313 of [Ra94]. It is therefore sufficient to prove the proposition for every split three-dimensional crystallographic group. Table 4.1 shows that all of the split crystallographic groups are subgroups of seven maximal ones (consider the pairing of the maximal point group with each of the seven lattices). We will show in Chap. 6 (without circularity) that each of these maximal groups has the required model. The proposition now follows easily.

5.2 A Construction of $E_{\mathcal{VC}}(\Gamma)$ for Crystallographic Groups

Let Γ be a three-dimensional crystallographic group. We begin with a copy of $E_{\mathcal{FIN}}(\Gamma)$, which we can identify with a suitably cellulated copy of $\mathbb{R}^3$ by Proposition 5.1. For each $\ell \in L$ such that ℓ generates a maximal cyclic subgroup of

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_5

L, we define

$$\mathbb{R}^2_\ell = \{\hat{\ell} \subseteq \mathbb{R}^3 \mid \hat{\ell} \text{ is a line parallel to } \langle \ell \rangle\},$$

where $\langle \ell \rangle$ denotes the one-dimensional vector subspace spanned by ℓ. Consider $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, where the disjoint union is over all maximal cyclic subgroups $\langle \ell \rangle$ of L. We define a metric on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ as follows. If $\ell_1, \ell_2 \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, we set $d(\ell_1, \ell_2) = \infty$ if ℓ_1 and ℓ_2 are not parallel, and $d(\ell_1, \ell_2) = K$ if ℓ_1 is parallel to ℓ_2 and $K = \min\{d_{\mathbb{R}^3}(x, y) \mid x \in \ell_1, y \in \ell_2\}$. One readily checks that d is a metric on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, and that each $\mathbb{R}^2_\ell$ is isometric to $\mathbb{R}^2$. We will therefore freely refer to the $\mathbb{R}^2_\ell$ as "planes" in what follows. Moreover, Γ acts by isometries on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$.

Next we would like to introduce an equivariant cell structure on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. Choose a plane $\mathbb{R}^2_\ell$.

Definition 5.1. Let $\pi : \Gamma \to H$ be the usual projection into the point group. We let $H_{\langle \ell \rangle} = \{h \in H \mid h \cdot \langle \ell \rangle = \langle \ell \rangle\}$ and $\Gamma(\ell) = \pi^{-1}(H_{\langle \ell \rangle})$.

It is straightforward to check that $\Gamma(\ell)$ acts on $\mathbb{R}^2_\ell$. Since $\langle \ell \rangle$ is a maximal cyclic subgroup of L, we can choose a basis $\{\ell_1, \ell_2, \ell_3\}$ of L, with $\ell_3 = \ell$. Each of the ℓ_i can be written $\ell_i = \alpha_i \ell + \hat{\ell}_i$, where $\alpha_i \in \mathbb{R}$ and $\hat{\ell}_i$ is perpendicular to ℓ. Since ℓ_1, ℓ_2, and ℓ_3 are linearly independent over $\mathbb{R}$, the same must be true of $\hat{\ell}_1$ and $\hat{\ell}_2$. The translation ℓ acts trivially on $\mathbb{R}^2_\ell$, so the action of L on $\mathbb{R}^2_\ell$ is the same as the action of $\langle \hat{\ell}_1, \hat{\ell}_2 \rangle$. In particular, the action of L has discrete orbits, from which it follows readily that the action of $\Gamma(\ell)$ on $\mathbb{R}^2_\ell$ has discrete orbits. We can therefore find a $\Gamma(\ell)$-equivariant cell structure on $\mathbb{R}^2_\ell$ making it a $\Gamma(\ell)$-CW complex.

Now we choose a (finite) left transversal $T \subseteq H$ of $\Gamma(\ell)$ in Γ. For each $t \in T$, we cellulate $\mathbb{R}^2_{t \cdot \ell}$ using the equality $\mathbb{R}^2_\ell = t \cdot \mathbb{R}^2_\ell$ (that is, for each cell $\sigma \subseteq \mathbb{R}^2_\ell$, we let $t \cdot \sigma$ be a cell in the cellulation of $\mathbb{R}^2_{t \cdot \ell}$). The result is an equivariant cellulation of all of $\Gamma \cdot \mathbb{R}^2_\ell$, which is a disjoint union of finitely many planes. We can continue in the same way, choosing a new plane $\mathbb{R}^2_{\ell'}$ and applying the same procedure, until we have cellulated all of $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. The space $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is a Γ-CW complex with respect to the resulting cellulation.

Proposition 5.2. *Let $Y = E_{\mathcal{FIN}}(\Gamma)$ and $Z = \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. The space $X = Y * Z$ is a model for $E_{\mathcal{VC}}(\Gamma)$.*

Proof. Since Y and Z are Γ-CW complexes, the join X inherits a natural Γ-CW complex structure. For $G \leq \Gamma$ and $W \in \{X, Y, Z\}$, we let $\mathrm{Fix}_W(G) = \{w \in W \mid g \cdot w = w \text{ for all } g \in G\}$. We note that $\mathrm{Fix}_W(G)$ is a subcomplex of W, and $\mathrm{Fix}_X(G) = \mathrm{Fix}_Y(G) * \mathrm{Fix}_Z(G)$, for all $G \leq \Gamma$.

Let $G \in \mathcal{VC}(\Gamma)$. There are two cases. Assume first that G is finite. In this case, $\mathrm{Fix}_X(G) = \mathrm{Fix}_Y(G) * \mathrm{Fix}_Z(G)$, where $\mathrm{Fix}_Y(G)$ is contractible by our assumptions. It follows that $\mathrm{Fix}_X(G)$ is contractible.

If G is infinite and virtually cyclic, then there is a cyclic subgroup $\langle g \rangle \leq L$ having finite index in G such that $\langle g \rangle \trianglelefteq G$. This group $\langle g \rangle$ is contained in a maximal cyclic subgroup $\langle \ell \rangle$ of L. The kernel of the action of G on $\mathbb{R}^2_\ell$ therefore contains $\langle g\rangle$. It follows that the fixed set of the action of G on $\mathbb{R}^2_\ell$ is the same as the fixed set of the action of $G/\langle g\rangle$ on $\mathbb{R}^2_\ell$. The latter group is a finite group acting by isometries, so the fixed set is contractible. In particular, $\mathrm{Fix}_X(G)\cap\mathbb{R}^2_\ell$ is contractible. Now we claim that $\mathrm{Fix}_X(G) = \mathrm{Fix}_X(G)\cap\mathbb{R}^2_\ell$ (i.e., that $\mathrm{Fix}_X(G)\subseteq\mathbb{R}^2_\ell$). Indeed, it is enough to check that $\mathrm{Fix}_X(G)\cap\mathbb{R}^2_{\ell'} = \emptyset$ for $\langle\ell'\rangle \neq \langle\ell\rangle$ and $\mathrm{Fix}_X(G)\cap Y = \emptyset$. The latter equality follows directly from the definition of Y. If $\langle\ell'\rangle \neq \langle\ell\rangle$, then g acts as $\hat{g}$ on $\mathbb{R}^2_{\ell'}$, where $\hat{g}$ is the component of g perpendicular to ℓ'. The claim follows directly.

Now suppose that $G \notin \mathcal{VC}(\Gamma)$. It follows that $rk(G\cap L)\geq 2$. One easily sees that $G\cap L$ cannot have any global fixed point in any $\mathbb{R}^2_\ell$ and $\mathrm{Fix}_Y(G\cap L)=\emptyset$ by definition. It follows that $\mathrm{Fix}_X(G)=\emptyset$, as required.

5.3 A Splitting Formula for the Lower Algebraic K-Theory

In the next chapters, we will use the following theorem to compute the lower algebraic K-theory of the integral group ring of all 73 split three-dimensional crystallographic groups. Our goal in this section is to provide a proof.

Theorem 5.1. *Let Γ be a three-dimensional crystallographic group. For $n\leq 1$, we have a splitting*

$$\begin{aligned} &H_n^\Gamma(E_{\mathcal{VC}}(\Gamma);\mathbb{KZ}^{-\infty}) \\ &\quad\cong H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma);\mathbb{KZ}^{-\infty}) \oplus \bigoplus_{\hat{\ell}\in\mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}})\to *;\ \mathbb{KZ}^{-\infty}). \end{aligned}$$

The indexing set $\mathcal{T}''$ consists of a selection of one vertex $v\in\coprod_{\langle\ell\rangle}\mathbb{R}^2_\ell$ from each Γ-orbit of such non-negligible vertices.

We refer the reader to Definition 5.3 for a definition of negligible. We will eventually see that $\mathcal{T}''$ is finite. Note that Theorem 5.1 used the following definition.

Definition 5.2. Let G be a group acting on a set X. If $A\subseteq X$, we let $G_A=\{g\in G \mid g\cdot A = A\}$.

For each maximal cyclic subgroup $\langle\ell\rangle\leq L$, we set

$$\mathcal{VC}_{\langle\ell\rangle} = \mathcal{VC}\cap\{G\leq\Gamma(\ell) \mid |G\cap\langle\ell\rangle| = \infty \text{ if } |G|=\infty\}.$$

In words, $\mathcal{VC}_{\langle\ell\rangle}$ is the collection consisting of finite subgroups of $\Gamma(\ell)$ and the infinite virtually cyclic subgroups of $\Gamma(\ell)$ that contain some non-zero multiple of the translation ℓ. It is easy to check that $\mathcal{VC}_{\langle\ell\rangle}$ is a family of subgroups in $\Gamma(\ell)$.

The point group H acts on the set of maximal cyclic subgroups in L by the rule $h\cdot\langle\ell\rangle = \langle h(\ell)\rangle$. We choose a single maximal cyclic subgroup from each orbit and call the resulting collection $\mathcal{T}$.

Proposition 5.3. *Assume that Γ is given the discrete topology. We continue to write $Y = E_{\mathcal{FIN}}(\Gamma)$ and $Z = \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ when convenient.*

1. *$\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is homeomorphic to $\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell$ by a homeomorphism that is compatible with the Γ-action. (Here the action on the latter space is given by the rule $\gamma \cdot (\gamma', \ell') = (\gamma\gamma', \ell')$.) Each $\mathbb{R}^2_\ell$ is a model for $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$.*
2. *$\coprod_{\langle \ell \rangle} Y \times \mathbb{R}^2_\ell$ is homeomorphic to $\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} (Y \times \mathbb{R}^2_\ell)$ by a homeomorphism that is compatible with the Γ-action. Each $Y \times \mathbb{R}^2_\ell$ is a model for $E_{\mathcal{FIN}}(\Gamma(\ell))$.*

The space $\coprod_{\langle \ell \rangle} Y \times \mathbb{R}^2_\ell$ can be identified with $Y \times Z \times \{1/2\} \subseteq Y * Z$ and $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ can be identified with the bottom of the join $Y * Z$.

Proof. We prove (1), the proof of (2) being similar. Consider the Γ-space $\Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell$, where Γ acts by left multiplication on the first coordinate and trivially on the second coordinate. We let $\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell$ be the usual Borel construction (so that Γ acts only on the first coordinate). We regard $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ as a Γ-space with respect to its usual action.

Define maps $\pi_1 : \Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell \to \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell$ and $\pi_2 : \Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell \to \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ by the rules $\pi_1(\gamma, x) = (\gamma, x)$ and $\pi_2(\gamma, x) = \gamma \cdot x$. Both of these are quotient maps and commute with the Γ-action.

We claim that π_1 is constant on point inverses of π_2 and π_2 is constant on point inverses of π_1. It will then follow from a well-known principle (see [Mu00, Theorem 22.2]) that there is a Γ-homeomorphism $f : \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell \to \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ such that $f \circ \pi_1 = \pi_2$.

$$
\begin{array}{ccc}
\Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell & & \\
\downarrow \pi_1 & \searrow^{\pi_2} & \\
\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell & \xrightarrow{f} & \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell
\end{array}
$$

Let $\gamma \in \Gamma$ and $x \in \mathbb{R}^2_\ell$ for some $\langle \ell \rangle \in \mathcal{T}$. One easily checks that

$$\pi_1^{-1}(\gamma, x) = \{(\gamma\tilde{\gamma}, \tilde{\gamma}^{-1} \cdot x) \mid \tilde{\gamma} \in \Gamma(\ell)\}.$$

It follows directly that $\pi_2(\pi_1^{-1}(\gamma, x)) = \{\gamma \cdot x\}$ (a singleton), as required.

If $x \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, then

$$\pi_2^{-1}(x) = \{(\hat{\gamma}, \hat{x}) \mid \hat{\gamma} \cdot \hat{x} = x\}.$$

Now we suppose that (γ_1, x_1) and (γ_2, x_2) are in $\pi_2^{-1}(x)$. It follows that $\gamma_1 \cdot x_1 = \gamma_2 \cdot x_2$, so $\gamma_2^{-1}\gamma_1 \cdot x_1 = x_2$. Since x_1 and x_2 are in the same Γ-orbit, it must be that both are in $\mathbb{R}^2_\ell$, for some $\langle \ell \rangle \in \mathcal{T}$. It then follows that $\gamma_2^{-1}\gamma_1 \in \Gamma(\ell)$. Now we apply π_1:

$$\begin{aligned}\pi_1(\gamma_1, x_1) &= (\gamma_1, x_1)\\ &= (\gamma_2(\gamma_2^{-1}\gamma_1), x_1)\\ &\sim (\gamma_2, \gamma_2^{-1}\gamma_1 \cdot x_1)\\ &= (\gamma_2, x_2)\\ &= \pi_1(\gamma_2, x_2)\end{aligned}$$

It follows that $\pi_1(\pi_2^{-1}(x))$ is a singleton, as required.

We have now demonstrated the existence of f. The remaining statements are straightforward to check.

Proposition 5.4. *Let Γ be a three-dimensional crystallographic group. For all $n \in \mathbb{Z}$, we have a splitting*

$$\begin{aligned}&H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty})\\ &\qquad\cong H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))\\ &\qquad\to E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})),\end{aligned}$$

where $H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)) \to E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})$ denotes the cokernel of the relative assembly map

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \longrightarrow H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}).$$

The proof of this proposition resembles others that have appeared in [J-PL06] and [LO09].

Proof. Let us work with the explicit model X for $E_{\mathcal{VC}}(\Gamma)$ constructed in Propositions 5.2 and 5.3. Since X is obtained as a join, there exists an obvious map $\rho : X \to [0, 1]$, which further has the property that every point pre-image is Γ-invariant. In particular, corresponding to the splitting of $[0, 1]$ into $[0, 2/3) \cup (1/3, 1]$, we get a Γ-invariant splitting of X. If we let $A = \rho^{-1}[0, 2/3)$, $B = \rho^{-1}(1/3, 1]$, then from the Mayer–Vietoris sequence in equivariant homology (and omitting coefficients in order to simplify our notation), we have that:

$$\ldots \to H_n^{\Gamma}(A \cap B) \to H_n^{\Gamma}(A) \oplus H_n^{\Gamma}(B) \to H_n^{\Gamma}(X) \to \ldots$$

Next, observe that we have obvious Γ-equivariant homotopy equivalences:

- $A = \rho^{-1}[0, 2/3) \simeq \rho^{-1}(0) = E_{\mathcal{FIN}}(\Gamma)$
- $B = \rho^{-1}(1/3, 1] \simeq \rho^{-1}(1) = \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$
- $A \cap B = \rho^{-1}(1/3, 2/3) \simeq \rho^{-1}(1/2) = \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{FIN}}(\Gamma(\ell))$

Now, using the induction structure and the fact that our equivariant generalized homology theory turns disjoint unions into direct sums, we can evaluate the terms in the Mayer–Vietoris sequence as follows:

$$\ldots \to \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \to H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma)) \oplus \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$$
$$\to H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma)) \to \ldots$$

Next, we study the relative assembly map

$$\Phi_{\langle \ell \rangle} : H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \to H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)).$$

We claim $\Phi_{\langle \ell \rangle}$ is split injective. This can be seen as follows. Consider the following commutative diagram:

$$\begin{array}{ccc} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) & \xrightarrow{\Phi_{\langle \ell \rangle}} & H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)) \\ & \searrow^{\beta} & \downarrow \alpha \\ & & H_n^{\Gamma(\ell)}(E_{\mathcal{VC}}(\Gamma(\ell)) \end{array}$$

where α and β are the relative assembly maps induced by the inclusions $\mathcal{VC}_{\langle \ell \rangle} \subset \mathcal{VC}$ and $\mathcal{FIN} \subset \mathcal{VC}$. Recall that Bartels [Bar03] has established that for *any* group G, the relative assembly map:

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbb{KZ}^{-\infty}) \to H_n^G(E_{\mathcal{VC}}(G); \mathbb{KZ}^{-\infty})$$

is split injective for all n. Using this result from Bartels, it follows that

$$\beta : H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \longrightarrow H_n^{\Gamma(\ell)}(E_{\mathcal{VC}}(\Gamma(\ell))$$

is split injective. Therefore $\Phi_{\langle \ell \rangle}$ is also split injective.

Now, for each integer n, the above portion of the Mayer–Vietoris long exact sequence breaks off as a short exact sequence (since the initial term injects). Since the map from the $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma)) \to H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma))$ is also split injective (from the Bartels result), we obtain an identification of the cokernel of the latter map with the cokernel of the map

$$\bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \longrightarrow \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))) \tag{5.1}$$

Next, since the inclusion map

$$\coprod_{\langle\ell\rangle\in\mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{FIN}}(\Gamma(\ell)) \longrightarrow \coprod_{\langle\ell\rangle\in\mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{VC}_{\langle\ell\rangle}}(\Gamma(\ell))$$

is the disjoint union of cellular $\Gamma(\ell)$-maps (for all $\langle\ell\rangle \in \mathcal{T}$), we see that the maps given in (1) split as a direct sum (over $\langle\ell\rangle \in \mathcal{T}$) of the relative assembly maps $H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \to H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle\ell\rangle}}(\Gamma(\ell)))$. This immediately yields a corresponding splitting of the cokernel, completing the proof of the proposition.

The next step is to analyze the summands

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)) \to E_{\mathcal{VC}_{\langle\ell\rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})$$

from Proposition 5.4.

We fix a maximal cyclic subgroup $\langle\ell\rangle \leq L$ for the remainder of this chapter. We note that the space $\mathbb{R}^3 * \mathbb{R}^2_\ell$ is a model for $E_{\mathcal{VC}_{\langle\ell\rangle}}(\Gamma(\ell))$, where both factors are given $\Gamma(\ell)$-equivariant cell structures and the action of $\Gamma(\ell)$ is the usual one.

Next we will need to describe the class of negligible groups.

Remark 5.1. If $G \in \mathcal{VC}$, then G has one of three possible forms:

1. G is finite, or
2. G is *infinite virtually cyclic of type I*; that is, G admits a surjective homomorphism onto $\mathbb{Z}$ with finite kernel. Such a group will necessarily have the form $G \cong F \rtimes \mathbb{Z}$, where F is the kernel of the surjection onto $\mathbb{Z}$, or
3. G is *infinite virtually cyclic of type II*; that is, G admits a surjective homomorphism onto D_∞ with finite kernel. In this case, $G \cong A *_F B$, where A, B, and F are finite groups, and F has index two in both A and B.

Definition 5.3. A group $G \in \mathcal{VC}$ is *negligible* if:

1. for each finite subgroup $H \leq G$, H is isomorphic to a subgroup of S_4 (the symmetric group on four symbols), and
2. if $G \in \mathcal{VC}_\infty$, then the finite group F from Remark 5.1 has square-free order.

(Thus, a finite group G is negligible if it is isomorphic to a subgroup of S_4. An infinite virtually cyclic group of type I is negligible if F is of square-free order and isomorphic to a subgroup of S_4. An infinite virtually cyclic group of type II is negligible if the factors A and B are isomorphic to subgroups of S_4, and F has square-free order.)

We will also say that a cell σ is negligible if its stabilizer group is negligible.

Remark 5.2. This is the first of two different definitions of "negligible" that we will use. We will need a different definition in Chaps. 8 and 9.

Definition 5.3 allows us to describe classes of cells that make no contribution to K-theory (see Lemma 5.1 below), which will let us ignore them in our work.

Lemma 5.1. *Let G be a negligible group. The groups $Wh_q(G)$ are trivial for $q \leq 1$, and the same is true for all subgroups of G.*

Proof. We first note that subgroups of negligible groups are negligible.

If G is finite and negligible, then G is isomorphic to a subgroup of S_4; i.e., $G \cong \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, D_2, D_3, D_4, A_4, or S_4. The lemma then follows from Table 7.1 and the accompanying discussion. (See also [LO09].)

Now we assume that G is negligible and infinite virtually cyclic of type I. Therefore, $G \cong F \rtimes_\alpha \mathbb{Z}$, where F is a subgroup of S_4 with square-free order. By results of Farrell and Hsiang [FH68] and Farrell and Jones [FJ95],

$$Wh_q(F \rtimes_\alpha \mathbb{Z}) \cong C \oplus NK_q(\mathbb{Z}F, \alpha) \oplus NK_q(\mathbb{Z}F, \alpha^{-1}),$$

where C is a suitable quotient of the group $Wh_{q-1}(F) \oplus Wh_q(F)$ and $q \leq 1$. Since F is finite and negligible, C is trivial. Therefore,

$$Wh_q(F \rtimes_\alpha \mathbb{Z}) \cong 2NK_q(\mathbb{Z}F, \alpha),$$

since Farrell and Hsiang also show that $NK_q(\mathbb{Z}F, \alpha) \cong NK_q(\mathbb{Z}F, \alpha^{-1})$. Since F has square-free order, $NK_q(\mathbb{Z}F, \alpha)$ is trivial for $q \leq 1$ by results of [Ha87] and [J-PR09]. (The case in which $\alpha = \mathrm{id}$ was established by Harmon [Ha87], and the general case is due to [J-PR09].) This proves the lemma in the case that G is negligible and infinite virtually cyclic of type I.

Finally, we assume that G is negligible and infinite virtually cyclic of type II. Therefore, we can write $G \cong G_1 *_F G_2$, where F has square-free order and index two in both factors, and both G_1 and G_2 are isomorphic to subgroups of S_4. By results of [Wal78] (see also [CP02]),

$$Wh_q(G) \cong X \oplus NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F]),$$

for all $q \leq 1$, where X is a suitable quotient of $Wh_q(G_1) \oplus Wh_q(G_2)$. Since G_1 and G_2 are negligible, both factors in the latter direct sum are trivial, so X is trivial. It follows that

$$Wh_q(G) \cong NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F]).$$

Let $F \rtimes_\alpha \mathbb{Z}$ be the canonical index two subgroup of G. Since $NK_q(\mathbb{Z}F, \alpha)$ is trivial for $q \leq 1$ by the previous case, it follows that $NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F])$ is also trivial for $q \leq 1$, by results of Lafont and Ortiz [LO08] (see also [DQR11] and [DKR11]). It follows that $Wh_q(G)$ is trivial for $q \leq 1$ in this case as well.

Lemma 5.2. *Let $G \leq \Gamma(\ell)$.*

1. *If there is a line $\hat{\ell}$ and a point $p \notin \hat{\ell}$ such that G fixes p and leaves $\hat{\ell}$ invariant, then G is negligible.*

2. *If G leaves a line $\hat{\ell}$ invariant and $\hat{\ell} \cap c \neq \emptyset$ for some open 2-cell $c \subseteq \mathbb{R}^3$, then G is negligible.*
3. *If G fixes two points $p_1, p_2 \in \mathbb{R}^3$ and the line $\overleftrightarrow{p_1 p_2}$ is not parallel to the line ℓ, then G is negligible.*
4. *If G leaves a strip $\hat{\ell} \times [0, K]$ invariant and acts trivially on the second factor, then G is negligible.*

Proof. 1. Let $\hat{p}$ be the point on $\hat{\ell}$ that is closest to p. We let v_1 be the vector originating at $\hat{p}$ and terminating at p. Let v_2 be a tangent vector to $\hat{\ell}$ at $\hat{p}$, and let v_3 be a vector that is perpendicular to v_1 and v_2. We note that the vectors v_1, v_2, and v_3 are pairwise orthogonal.

The point $\hat{p}$ must be fixed by G, and so G must act by orthogonal matrices with respect to the ordered basis $[v_1, v_2, v_3]$. By our assumptions, G fixes v_1. The inclusion $G \cdot v_2 \subseteq \{v_2, -v_2\}$ holds, since $\hat{\ell}$ is G-invariant. It follows from orthogonality that $G \cdot v_3 \subseteq \{v_3, -v_3\}$ as well. We conclude that G is isomorphic to a subgroup of $(\mathbb{Z}/2)^2$, and therefore negligible.

2. Suppose that G leaves $\hat{\ell}$ invariant, and $\hat{\ell} \cap c \neq \emptyset$ for some open 2-cell $c \subseteq \mathbb{R}^3$. We consider the restriction homomorphism $r : G \to \mathrm{Isom}(\hat{\ell})$. The kernel of this map is a subgroup of the stabilizer group of c. It follows that $|\ker r| = 1$ or 2. Thus, G maps into $\mathbb{Z}$ or D_∞ with 1 or $\mathbb{Z}/2$ as kernel, so G is negligible.
3. Let p_1, p_2 be fixed by G. We consider the line $\hat{\ell}$ through p_2 that is parallel to ℓ. Since $\overleftrightarrow{p_1 p_2}$ is not parallel to ℓ, $p_1 \notin \hat{\ell}$. Since $G \subseteq \Gamma(\ell)$, G leaves $\hat{\ell}$ invariant. As a result, G is negligible by (1).
4. If G leaves a strip $\hat{\ell} \times [0, K]$ invariant and acts trivially on the second coordinate, then G acts on each line $\hat{\ell} \times \{k\} \subseteq \hat{\ell} \times [0, K]$. At least one of these lines must meet an open 2-cell in $\mathbb{R}^3$, so G is negligible by (2).

Corollary 5.1. *Let $\hat{\ell} \in \mathbb{R}^2_\ell$ have a non-negligible stabilizer group. The point $\hat{\ell} \in \mathbb{R}^2_\ell$ must be a vertex, and the line $\hat{\ell} \subseteq \mathbb{R}^3$ occurs as a cellulated subcomplex in $E_{\mathcal{FIN}}(\Gamma(\ell))$.*

Proof. If $\hat{\ell}$ is not a vertex of $\mathbb{R}^2_\ell$, then the stabilizer group of $\hat{\ell}$ leaves a strip invariant in $\mathbb{R}^3$ and acts trivially on the bounded factor, so the stabilizer group of $\hat{\ell}$ is negligible by Lemma 5.2(4).

The second statement follows from the fact that $\hat{\ell} \subseteq (\mathbb{R}^3)^1$, by Lemma 5.2(2).

Definition 5.4. For each vertex $\hat{\ell} \in \mathbb{R}^2_\ell$ with non-negligible stabilizer, set

$$F(\hat{\ell}) = \hat{\ell} \subseteq \mathbb{R}^3.$$

Note that $\hat{\ell}$ is a cellulated line in $\mathbb{R}^3$ by Corollary 5.1, so $F(\hat{\ell})$ is a subcomplex of our model for $E_{\mathcal{FIN}}(\Gamma(\ell))$.

Under the same assumptions on $\hat{\ell}$, we also set

$$E(\hat{\ell}) = F(\hat{\ell}) * \hat{\ell}.$$

We note that $E(\hat{\ell})$ is a subcomplex of $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)) = \mathbb{R}^3 * \mathbb{R}^2_\ell$.

Proposition 5.5. *The subcomplexes*

$$F = \coprod_{\hat{\ell}} F(\hat{\ell}), \quad \text{and} \quad E = \coprod_{\hat{\ell}} E(\hat{\ell})$$

of $E_{\mathcal{FIN}}(\Gamma(\ell))$ and $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$ (respectively) are $\Gamma(\ell)$-equivariant. (The disjoint unions are indexed over all lines $\hat{\ell} \in \mathbb{R}^2_\ell$ with non-negligible stabilizers.)

These subcomplexes also contain the only non-negligible cells in $E_{\mathcal{FIN}}(\Gamma(\ell))$ and $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$ (respectively). In particular, the natural inclusions induce isomorphisms

$$H_n^{\Gamma(\ell)}(F; \mathbb{KZ}^{-\infty}) \cong H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}),$$
$$H_n^{\Gamma(\ell)}(E; \mathbb{KZ}^{-\infty}) \cong H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}).$$

Proof. The statement that the given subcomplexes are $\Gamma(\ell)$-equivariant follows from the $\Gamma(\ell)$-equivariance of the indexing sets.

Now we would like to show that the given subcomplexes contain the only non-negligible cells. It is good enough to do this for the subcomplex $E = \coprod_{\hat{\ell}} E(\hat{\ell})$, since

$$F = \coprod_{\hat{\ell}} F(\hat{\ell}) = E_{\mathcal{FIN}}(\Gamma(\ell)) \cap \coprod_{\hat{\ell}} E(\hat{\ell}).$$

We will consider cells in the top of the join $\mathbb{R}^3 * \mathbb{R}^2_\ell$ (i.e., in $\mathbb{R}^3$), then cells in $\mathbb{R}^2_\ell$, and finally the cells that can be described as joins of cells from $\mathbb{R}^3$ and $\mathbb{R}^2_\ell$.

We first describe the collection of all cells $c \subseteq \mathbb{R}^3$ with non-negligible stabilizers. It is clear that all such cells are 0- or 1-cells, since the stabilizer of a two-dimensional cell $\sigma \subseteq \mathbb{R}^3$ has order at most 2, and the stabilizer of a three-dimensional cell is necessarily trivial.

Suppose c is a 0-cell in $\mathbb{R}^3$ and c has non-negligible stabilizer. We consider the line $\hat{\ell}$ that is parallel to ℓ and passes through c. Thus $\hat{\ell} \in \mathbb{R}^2_\ell$ has a non-negligible stabilizer (since the stabilizer of c is contained in the stabilizer of $\hat{\ell}$), so it must be a vertex. It now follows that $c \in E(\hat{\ell})$.

Now suppose that $c \subseteq \mathbb{R}^3$ is an open 1-cell with non-negligible stabilizer. We choose two points $p_1, p_2 \in c$. The stabilizer group of c fixes both p_1 and p_2. Since the latter group is non-negligible, it must be that $\overleftrightarrow{p_1 p_2} = \hat{\ell}$ is parallel to ℓ by Lemma 5.2(3), so $\hat{\ell} \in \mathbb{R}^2_\ell$. The stabilizer group of $\hat{\ell}$ is non-negligible since it contains the stabilizer group of c. Thus, $\hat{\ell}$ is a vertex in $\mathbb{R}^2_\ell$, and $c \subseteq E(\hat{\ell})$. This concludes our analysis of cells in $\mathbb{R}^3$; we have shown that all non-negligible cells in $\mathbb{R}^3$ are contained in E.

We next consider the cells of $\mathbb{R}^2_\ell$. Corollary 5.1 shows that each open cell $c \subseteq \mathbb{R}^2_\ell$ of dimension greater than 0 has negligible stabilizer. Thus, if $c \subseteq \mathbb{R}^2_\ell$ has non-negligible stabilizer, then c is a vertex, so $c \in E(c)$.

Finally, we consider the cells $c = c_1 * c_2$ having non-negligible stabilizer, where $c_1 \subseteq \mathbb{R}^3$, $c_2 \subseteq \mathbb{R}^2_\ell$ are open cells. Since $G_c = G_{c_1} \cap G_{c_2}$, both c_1 and c_2 have non-negligible stabilizer groups. It follows from Corollary 5.1 that c_2 is a vertex (in $\mathbb{R}^2_\ell$) and a line in $\mathbb{R}^3$. Since G_c is non-negligible, we must have $c_1 \subseteq c_2$ by Lemma 5.2(1). Thus, $c \subseteq E(c_2)$. We have now shown that all of the non-negligible cells are in E.

The final statement now follows from Lemma 5.1. Indeed, consider the inclusion of F into $E_{\mathcal{FIN}}(\Gamma(\ell))$. The only cells to have non-zero K-groups are contained in the image (by the above argument and Lemma 5.1), proving that the inclusion induces an isomorphism. The other case is similar.

Proposition 5.6. *We have a $\Gamma(\ell)$-homeomorphism*

$$h : \coprod_{\hat{\ell}\in\mathcal{T}'} \Gamma(\ell) \times_{\Gamma(\ell)_{\hat{\ell}}} X(\hat{\ell}) \to \coprod_{\hat{\ell}} X(\hat{\ell}),$$

where $X \in \{E, F\}$ and $\mathcal{T}'$ is a selection of one vertex $\hat{\ell}$ from each $\Gamma(\ell)$-orbit of non-negligible vertices in $\mathbb{R}^2_\ell$. The domain is a $\Gamma(\ell)$-space relative to the action that is trivial on the second coordinate, and left multiplication on the first.

Moreover, each $F(\hat{\ell})$ is a model for $E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}})$, and each $E(\hat{\ell})$ is a model for $E_{\mathcal{VC}}(\Gamma_{\hat{\ell}})$.

Proof. The argument is similar to the proof of Proposition 5.3. We will prove the proposition in the case $X = E$, the other case being similar. Consider the commutative diagram:

$$\begin{array}{ccc} \Gamma(\ell) \times \coprod_{\widehat{\ell}\in\mathcal{T}'} E(\widehat{\ell}) & & \\ \downarrow \pi_1 & \searrow^{\pi_2} & \\ \coprod_{\widehat{\ell}\in\mathcal{T}'} \Gamma(\ell) \times_{\Gamma(\ell)_{\widehat{\ell}}} E(\widehat{\ell}) & \xrightarrow{\ h\ } & \coprod_{\widehat{\ell}} E(\widehat{\ell}) \end{array}$$

The space on top is a $\Gamma(\ell)$-space relative to the action $\gamma' \cdot (\gamma, x) = (\gamma'\gamma, x)$. We set $\pi_1(\gamma, x) = (\gamma, x)$ and $\pi_2(\gamma, x) = \gamma \cdot x$. As in the proof of Proposition 5.3, we will show that π_1 is constant on point inverses of π_2, and π_2 is constant on point inverses of π_1. This will establish the existence of the desired $\Gamma(\ell)$-homeomorphism h, since both π_1 and π_2 are quotient maps that commute with the $\Gamma(\ell)$-action.

Choose $(\gamma, x) \in \coprod_{\hat{\ell} \in \mathcal{T}'} \Gamma(\ell) \times_{\Gamma(\ell)_{\hat{\ell}}} E(\hat{\ell})$. We have the equality

$$\pi_1^{-1}(\gamma, x) = \{(\gamma\gamma_1, \gamma_1^{-1}x) \mid \gamma_1 \in \Gamma(\ell)_{\hat{\ell}}\}.$$

It follows directly that $\pi_2(\pi_1^{-1}(\gamma, x)) = \{\gamma \cdot x\}$, as required.

Now we choose an arbitrary $x \in \coprod_{\hat{\ell}} E(\hat{\ell})$. We have

$$\pi_2^{-1}(x) = \{(\gamma, z) \mid \gamma \cdot z = x\}.$$

We choose two elements of the latter set, (γ_1, z_1) and (γ_2, z_2). It follows directly that $\gamma_2^{-1}\gamma_1 \cdot z_1 = z_2$, so z_1 and z_2 are in the same $\Gamma(\ell)$-orbit. Given the nature of the indexing set $\mathcal{T}'$, it must be that both z_1 and z_2 are in $E(\hat{\ell})$, for some non-negligible vertex $\hat{\ell} \in \mathbb{R}^2_\ell$. It now follows that $\gamma_2^{-1}\gamma_1 \in \Gamma(\ell)_{\hat{\ell}}$. Thus,

$$\begin{aligned}
\pi_1(\gamma_1, z_1) &= (\gamma_2\gamma_2^{-1}\gamma_1, z_1) \\
&\sim (\gamma_2, \gamma_2^{-1}\gamma_1 z_1) \\
&= (\gamma_2, z_2) \\
&= \pi_1(\gamma_2, z_2)
\end{aligned}$$

It follows that π_1 is constant on point inverses of π_2, as required. The existence of the homeomorphism h follows directly.

Finally, we note that $\Gamma(\ell)_{\hat{\ell}} = \Gamma_{\hat{\ell}}$ is an infinite virtually cyclic group. Since $F(\hat{\ell})$ is simply a cellulated line, and $E(\hat{\ell})$ is the join of $F(\hat{\ell})$ with a point, both are well-known models for $E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}})$ and $E_{\mathcal{VC}}(\Gamma_{\hat{\ell}})$, respectively.

Remark 5.3. We note that $\Gamma_{\hat{\ell}}$ denotes the same subgroup of Γ no matter whether we view $\hat{\ell}$ as a vertex in $\mathbb{R}^2_\ell$ or as a line in $\mathbb{R}^3$.

Proof (Proof of Theorem 5.1). Combining Propositions 5.5, 5.6, and the fact that our equivariant generalized homology theory turns disjoint unions into direct sums, we obtain the following isomorphisms

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \cong \bigoplus_{\hat{\ell} \in \mathcal{T}'} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}), \quad \text{and}$$

$$H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle\ell\rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \cong \bigoplus_{\hat{\ell} \in \mathcal{T}'} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{VC}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}).$$

We immediately get an identification of the cokernel of the relative assembly map $H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \to H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle\ell\rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})$ with the direct sum of the cokernels of the relative assembly maps

$$H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}) \to H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{VC}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}).$$

Since $\Gamma_{\hat{\ell}} \in \mathcal{VC}$, then $E_{\mathcal{VC}}(\Gamma_{\hat{\ell}}) = \{*\}$, and the summands in Proposition 5.4 are

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)) \to E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))) \cong \bigoplus_{\hat{\ell} \in \mathcal{T}'} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *).$$

Finally, combining these observations with Proposition 5.4 completes the proof.

Chapter 6
Fundamental Domains for the Maximal Groups

The classification of split three-dimensional crystallographic groups from Theorem 4.2 shows that seven of the groups contain all of the others as subgroups. For $i = 1, \ldots, 7$, we let $\Gamma_i = \langle L_i, H_i \rangle$, where L_i is the ith lattice (in the order that the lattices are listed in Table 4.1) and H_i is the maximal point group to be paired with L_i. For instance,

$$\Gamma_4 = \left\langle \left\langle \frac{1}{2}(\mathbf{x} + \mathbf{z}), \mathbf{y}, \mathbf{z} \right\rangle, D_2^+ \times (-1) \right\rangle.$$

For the computations of K-groups in subsequent chapters, we will need to find fundamental polyhedra for the groups Γ_i. We review a case of Poincare's fundamental polyhedron theorem in Sect. 6.1, describe equivariant cell structures and cell stabilizers in Sect. 6.2, and then describe the seven fundamental domains in the remaining sections.

6.1 A Special Case of Poincare's Fundamental Polyhedron Theorem

In this section, we collect a number of results from [Ra94]. We state each result only for $\mathbb{R}^n$ and only for convex compact polyhedra, although usually the corresponding theorem (or definition) in [Ra94] is more general. All page citations in the current section are from [Ra94], unless otherwise noted.

Definition 6.1. If $P \subseteq \mathbb{R}^n$ is the set of solutions to a system of finitely many linear inequalities, and P is compact, then we say that P is a *convex compact polyhedron*. Suppose that P is m-dimensional. We let ∂P denote the topological boundary of P in the unique m-plane $\langle P \rangle$ containing P. The *interior* of P is $P - \partial P$. A *side* of a convex compact polyhedron is a non-empty maximal convex subset of ∂P. If

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_6

P has dimension $m > 0$, then each side of P is a convex compact polyhedron of dimension $m - 1$. A *ridge* of P is a side of a side of P.

(Notes: our definition of convex compact polyhedra combines Ratcliffe's definition of convex polyhedra (p. 205) with his characterization of convex compact polyhedra (Theorem 6.3.7, p. 209). The definitions of ∂P, side, and ridge occur on pages 199, 202, and 207, respectively.)

Definition 6.2. Let P be a convex compact n-dimensional polyhedron in $\mathbb{R}^n$. A *side-pairing* (cf. G-side-pairing, p. 694) for P is a set

$$\Phi = \{\phi_S \in Isom(\mathbb{R}^n) \mid S \in \mathcal{S}\}$$

indexed by the collection $\mathcal{S}$ of all sides of P such that, for each $S \in \mathcal{S}$:

1. There is a side S' of P such that $\phi_S(S') = S$.
2. If $\phi_S(S') = S$, then the isometries ϕ_S and $\phi_{S'}$ satisfy $\phi_{S'} = \phi_S^{-1}$.
3. The polyhedra P and $\phi_S(P)$ satisfy $P \cap \phi_S(P) = S$.

The pairing of side points by elements of Φ generates an equivalence relation on ∂P. The equivalence classes are called *cycles* of Φ (p. 694). We let $[x]$ denote the cycle containing x. We say that the cycle $[x]$ is a *ridge cycle* (p. 694) if some (equivalently, any) representative of $[x]$ lies in the interior of a ridge of P.

Let $[x] = \{x_1, \ldots, x_m\}$ be a finite ridge cycle of Φ. Each x_i is contained in exactly two sides of P (Theorem 6.3.6, p. 207), so x_i is paired to at most two other points of $[x]$ for each i. Therefore, we can reindex the set $\{x_1, \ldots, x_m\}$ such that

$$x_1 \simeq x_2 \ldots \simeq x_m,$$

where, for $i = 1, \ldots, m-1$, $x_i \simeq x_{i+1}$ if and only if some element $\phi(x_i) = x_{i+1}$ for some $\phi \in \Phi$. The ridge cycle $[x]$ is said to be *dihedral* (p. 695) if the sides S_1 and S_m of P are such that $x_i \in S_i$ and ϕ_{S_i} is the reflection of $\mathbb{R}^n$ in $\langle S_i \rangle$ for $i = 1, m$. (Note that if ϕ_{S_i} is a reflection in S_i, then x_i is paired with at most one other point by the relation $\simeq$, so such points x_i can only appear at the beginning or end of the sequence $x_1, \ldots, x_m$.) Otherwise, $[x]$ is said to be *cyclic* (p. 695).

If S_1 and S_2 are two sides of a convex polyhedron P, and the vectors N_1 and N_2 are the outward-pointing unit normal vectors, then the *dihedral angle* between S_1 and S_2 is

$$\theta(S_1, S_2) = \pi - \arccos(N_1 \cdot N_2).$$

The *dihedral angle sum* (p. 695) of the ridge cycle $[x]$ is

$$\theta[x] = \theta_1 + \ldots + \theta_m,$$

where θ_i is the dihedral angle between the two sides containing x_i.

The side-pairing Φ is *subproper* (p. 695) if and only if each cycle of Φ is finite, each dihedral ridge cycle of Φ has dihedral angle sum an integral submultiple of π, and each cyclic ridge cycle of Φ has dihedral angle sum an integral submultiple of 2π.

Definition 6.3. A subset R of $\mathbb{R}^n$ is a *fundamental domain* (p. 234) for a group $\Gamma \leq Isom(\mathbb{R}^n)$ if and only if

1. the set R is open in $\mathbb{R}^n$;
2. the members of $\{gR \mid g \in \Gamma\}$ are pairwise disjoint;
3. $\mathbb{R}^n = \bigcup\{g\overline{R} \mid g \in \Gamma\}$, and
4. R is connected.

We say that R is *locally finite* (p. 237) if the collection $\{g\overline{R} \mid g \in \Gamma\}$ is locally finite, i.e., if, for every $x \in \mathbb{R}^n$, there is an open ball around x meeting only finitely many members of $\{g\overline{R} \mid g \in \Gamma\}$.

A *convex compact fundamental polyhedron* (p. 247) for a discrete group $\Gamma \leq Isom(\mathbb{R}^n)$ is a convex compact polyhedron P such that the interior of P is a locally finite fundamental domain for Γ. We say that such a P is *exact* (p. 250) if and only if for each side S of P there is an element $g \in \Gamma$ such that $S = P \cap gP$. (If P is exact, then the element g is unique (Theorem 6.6.3, p. 251).)

We can now state the relevant special case of Poincaré's fundamental polyhedron theorem.

Theorem 6.1 ([Ra94, p. 711]). *If Φ is a subproper side-pairing of a convex compact polyhedron $P \subseteq \mathbb{R}^n$, then the group Γ generated by Φ is discrete, and P is an exact convex compact fundamental polyhedron for Γ.*

Proof. The general statement in [Ra94] has an additional condition in the hypothesis—namely, that the $(\mathbb{R}^n, Isom(\mathbb{R}^n))$-orbifold M obtained from P by gluing together the sides of P by Φ is complete. For convex compact Euclidean polyhedra P, this condition is automatically satisfied under the given hypothesis (Theorem 13.4.2, p. 704).

Corollary 6.1. *If P is a convex compact n-dimensional polyhedron in $\mathbb{R}^n$, and the dihedral angle between any pair of adjacent sides of P is a submultiple of π, then the group Γ generated by the reflections in all of the sides of P is discrete, and P is an exact convex compact fundamental polyhedron for Γ.*

Proof. One easily checks that the collection of reflections Φ is a side-pairing. Each cycle in this side-pairing consists of a single point $x \in P$, so the dihedral angle sum of any ridge cycle is a submultiple of π by the assumption about P. It follows that Φ is a subproper side-pairing, and one can apply Theorem 6.1.

6.2 Cell Structures and Stabilizers

6.2.1 Standard Cellulations and Equivariant Cell Structures

Definition 6.4. An *open cell of dimension n* is a space homeomorphic to $(0, 1)^n$, or, if $n = 0$, to a singleton set. If X is a Hausdorff space, then a *CW structure* on X [H, Proposition A.2] is a collection C of open cells in X such that

1. the elements of C are pairwise disjoint, and their union is X;
2. the boundary $\partial(e) = \bar{e} - e$ of an element $e \in C$ is contained in the union of elements from C of lower dimension, and
3. a subset U of X is closed if and only if $U \cap e$ is closed as a subset of e, for all $e \in C$.

Definition 6.5. Let P be a convex compact three-dimensional polyhedron in $\mathbb{R}^3$, and let Φ be a subproper side-pairing of P. If all cycles $[x]$ of Φ meet the interiors of ridges and sides of P in at most one point, then the *standard cellulation* of P is the set whose members are vertices of P, interiors of ridges of P, interiors of sides of P, and the interior of P itself.

If a cycle $[x]$ meets the interior of a ridge or a side in two points, then we call this ridge or side *bad*. (Note that, under the current hypotheses, it is impossible for a cycle $[x]$ to meet the interior of a ridge or side in more than two points.) If P has bad ridges or bad sides, then we divide each bad ridge or side exactly in half to arrive at the standard cellulation of P. The operation of subdivision is self-explanatory in the case of bad ridges. If a side S is bad, then we choose two points $x, y \in S$ such that both are in the same cycle, and both lie in the interior of S. We let ℓ denote the perpendicular bisector of $[x, y]$ in $\langle S\rangle$. The isometry $\phi_S \in \mathrm{Isom}(\mathbb{R}^3)$ maps the side S to itself, and interchanges x and y. It follows that $\phi_S(\ell) = \ell$. Now either ϕ_S fixes ℓ pointwise or ϕ_S reverses the orientation of ℓ. In the former case, we can simply subdivide the side S along ℓ. In the latter case, we must subdivide S along ℓ and also introduce a vertex at the midpoint of $\ell \cap S$.

Theorem 6.2 (Equivariant Cell Structures). *If Φ is a subproper side-pairing of a convex compact three-dimensional polyhedron $P \subseteq \mathbb{R}^3$, and $\Gamma = \langle\Phi\rangle$, then the standard cellulation C of P extends to a Γ-equivariant CW structure $\hat{C}$ on all of $\mathbb{R}^3$.*

If $g \in \Gamma$ leaves a cell $e \in \hat{C}$ invariant, then g fixes e pointwise.

Proof. Theorem 6.1 shows that P is an exact convex compact fundamental polyhedron for the action of Γ on $\mathbb{R}^n$, and that Γ is discrete. Ratcliffe [Ra94, Theorem 6.7.1] implies that

$$\mathcal{P} = \{gP \mid g \in \Gamma\}$$

is an exact tessellation of $\mathbb{R}^n$. This means [Ra94, p. 251] that $\mathcal{P}$ satisfies the following conditions:

1. if $g_1, g_2 \in \Gamma$, and $g_1 \neq g_2$, then the interiors of $g_1 P$ and $g_2 P$ are disjoint;
2. the union of the polyhedra in $\mathcal{P}$ is $\mathbb{R}^n$;
3. the collection $\mathcal{P}$ is locally finite, and
4. each side of a polyhedron in $\mathcal{P}$ is a side of exactly two polyhedra P and Q in $\mathcal{P}$. (This last condition is the definition of exactness.)

It is not difficult to see that $\Gamma \cdot C$ will be an equivariant CW complex structure on all of $\mathbb{R}^n$ if and only if the elements of $\Gamma \cdot C$ are pairwise disjoint. We therefore suppose that two cells $g_1 \cdot e_1, g_2 \cdot e_2$ have a point in common, for some $e_1, e_2 \in C$. We wish to show that $g_1 \cdot e_1 = g_2 \cdot e_2$. We can clearly assume, without loss of generality, that $g_2 = 1$ and that $\dim e_1 \leq \dim e_2$.

If e_2 is a three-dimensional open cell (i.e., the interior of P), then $g_1 \cdot e_1$ is contained in the closure of $g_1 \cdot e_2$. It follows that $g_1 \cdot e_2$ and e_2 must have a point in common, so $g_1 = 1$ by property (1) of tessellations. This implies that e_1 and e_2 have a point in common, which can only mean that $e_1 = e_2$, since C is a CW structure.

If e_2 is a two-dimensional open cell or a one-dimensional open cell contained in the interior of a bad side, then there is a unique side S of P containing e_2. We have $e_2 \subseteq \operatorname{int}(P \cup \phi_S(P))$. Since distinct translates of P can meet only in their boundaries, and $(g_1 P) \cap \operatorname{int}(S) \neq \emptyset$ by our assumption, we must have $g_1 = \phi_S$ or $g_1 = 1$, and we can assume that $g_1 = \phi_S$. The isometry ϕ_S restricts to a bijection from S' to S, where S' is a side of P, and we allow the possibility that $S' = S$. Since, by the construction of C, the map $\phi_S : S' \to S$ is a bijection mapping cells of C to cells of C, the only possibility is that $e_1 = \phi_S^{-1}(e_2)$, so $g_1 \cdot e_1 = e_2$.

Now suppose e_2 is a one-dimensional open cell contained in a unique ridge R of P. Ratcliffe [Ra94, Theorem 6.7.6] proves that if R is a ridge of a polyhedron P in an exact tessellation $\mathcal{P}$ of $\mathbb{R}^3$, then the set of all polyhedra in $\mathcal{P}$ containing R forms a cycle whose intersection is R. We avoid reproducing the definition of a cycle of polyhedra here (see [Ra94, p. 256]), but this theorem implies that R is a ridge of each of the polyhedra in the cycle, and that the interior of R lies in the interior of the union of the polyhedra in the cycle. It follows that there is some ridge R_1 of P such that $e_1 \subseteq R_1$ and $g_1 \cdot R_1 = R$.

We can conclude that $g_1 \cdot e_1 = e_2$ provided that R_1 is a bad ridge if and only if R is bad as well. Suppose that R_1 is a bad ridge. This means that there is $x \in R_1$ such that $[x]$ meets the interior of R_1 in two points, say x and x'. It follows that $g_1 \cdot [x]$ meets the interior of R in two points, $g_1 \cdot x$ and $g_1 \cdot x'$. Theorem 6.7.5 from [Ra94] says that $[x] = P \cap \Gamma x$, so $g_1 \cdot x, g_1 \cdot x' \in [x]$. It follows that R is bad. The converse is proved in the same way. It follows that $g_1 \cdot e_1 = e_2$.

Finally, there is nothing to prove if e_2 is zero-dimensional.

The final statement is an easy consequence of the fact that each cycle $[x]$ meets a given cell of C in at most one point, and the fact that $[x] = P \cap \Gamma \cdot x$.

6.2.2 Computation of Cell Stabilizers and Negligible Groups

We will soon want to compute the stabilizer groups Γ_v, where Γ is a split crystallographic group and $v \in \mathbb{R}^3$. The following lemma affords an effective procedure for computing such groups. The proof is elementary and will be omitted.

Lemma 6.1. *Assume that Γ is a split crystallographic group: $\Gamma = \langle L, H \rangle$. Let $\pi : \Gamma \to H$ be the natural projection into the point group H.*

1. *$\pi(\Gamma_v) = \{h \in H \mid v - hv \in L\}$, and*
2. *the map $\pi : \Gamma_v \to H$ is injective. Thus, we can uniquely recover Γ_v from $\pi(\Gamma_v)$.*

□

Let $\sigma \subseteq \mathbb{R}^3$ be a cell. If the stabilizer group Γ_σ and all of its subgroups K satisfy $Wh_q(K) = 0$ for $q \leq 1$, then σ makes no contribution in the calculation of $Wh_q(\Gamma)$ ($q \leq 1$). The following definition will give us a systematic way to ignore such cells, based on the isomorphism types of their stabilizer groups.

Definition 6.6. A group is *negligible* if it is isomorphic to a subgroup of S_4. We will also say that a cell is negligible if its stabilizer group is negligible in the above sense.

Remark 6.1. We note that Definition 6.6 is equivalent to Definition 5.3 for finite groups G.

Proposition 6.1. *If $\Gamma = \langle L, H \rangle$, where $H \leq S_4^+ \times (-1)$, and $v \in \mathbb{R}^3$ satisfies $2v \notin L$, then the stabilizer group Γ_v is negligible.*

Proof. Note that $v - (-1)v = 2v \notin L$, so $(-1) \notin \pi(\Gamma_v)$ by Lemma 6.1(1). The homomorphism $\pi : \Gamma_v \to H$ is injective, so Γ_v is isomorphic to a subgroup of $S_4^+ \times (-1)$ that does not contain (-1). All such groups are isomorphic to subgroups of S_4, so Γ_v is negligible.

6.3 A Fundamental Polyhedron for Γ_1

Recall that $\Gamma_1 = \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \rtimes (S_4^+ \times (-1))$. We consider the convex compact polyhedron

$$P = \left\{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq y \leq x \leq \frac{1}{2} \right\}.$$

The set P is the tetrahedron pictured in Fig. 6.1. (Note that the shape of the given tetrahedron is not intended to be accurate.)

The sides $\{S_1, S_2, S_3, S_4,\}$ are contained in the planes $y = z$, $z = 0$, $x = y$, and $x = 1/2$, respectively. The outward unit normal vectors N_1, N_2, N_3, N_4 are,

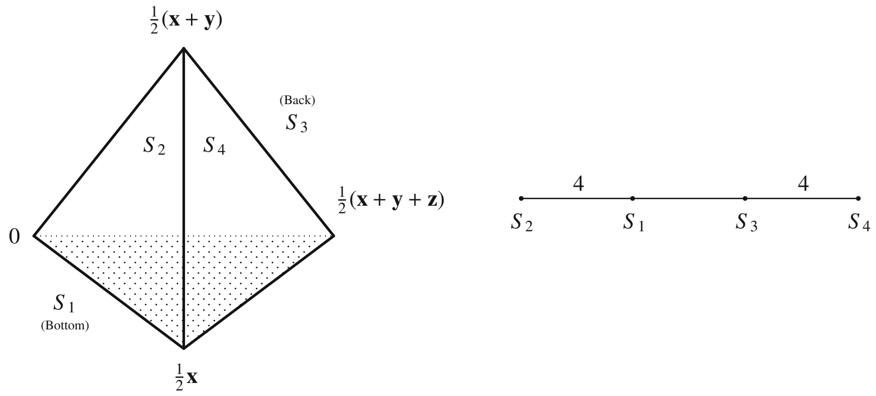


Fig. 6.1 The tetrahedron on the *left* is a fundamental domain for the action of Γ_1 on $\mathbb{R}^3$. (The precise shape of the tetrahedron is not intended to be accurate.) The group Γ_1 is a Coxeter group. Its Coxeter diagram appears on the *right*

respectively,

$$\frac{1}{\sqrt{2}}(-\mathbf{y}+\mathbf{z}), \quad -\mathbf{z}, \quad \frac{1}{\sqrt{2}}(-\mathbf{x}+\mathbf{y}), \quad \mathbf{x}.$$

The dihedral angles are easy to compute:

$$\begin{aligned} \theta(S_1,S_2) &= \pi/4, & \theta(S_1,S_3) &= \pi/3, \\ \theta(S_1,S_4) &= \pi/2, & \theta(S_2,S_3) &= \pi/2, \\ \theta(S_2,S_4) &= \pi/2, & \theta(S_3,S_4) &= \pi/4. \end{aligned}$$

Since all of these are submultiples of π, it follows from Corollary 6.1 that the group Γ_P generated by reflections in the sides of P is discrete, and that P is an exact, convex compact fundamental polyhedron for Γ_P. Furthermore,

$$\Gamma_P = \left\langle \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix} + \begin{pmatrix} -1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \right\rangle,$$

where the isometries listed between the brackets are the reflections in S_1, S_2, S_3, and S_4, respectively.

It is fairly easy to check that $\Gamma_P = \Gamma_1$, so P is an exact convex compact fundamental polyhedron for the action of Γ_1 on $\mathbb{R}^3$.

Theorem 6.3. *Let $\hat{C}$ denote the Γ_1-equivariant cell structure on $\mathbb{R}^3$ determined by the standard cellulation C of P. The quotient $\Gamma_1\backslash\mathbb{R}^3$ is P itself (see Fig. 6.1), endowed with the standard cellulation C. The vertex stabilizer groups are* *determined by the following equalities (we write Γ in place of Γ_1):*

$$\begin{aligned}\pi(\Gamma_{(0,0,0)}) &= S_4^+ \times (-1).\\ \pi(\Gamma_{(1/2,0,0)}) &= D_{4_1}^+ \times (-1).\\ \pi(\Gamma_{(1/2,1/2,0)}) &= D_4^+ \times (-1).\\ \pi(\Gamma_{(1/2,1/2,1/2)}) &= S_4^+ \times (-1).\end{aligned}$$

The stabilizer groups of all other cells are negligible.

Proof. The first statement follows from the fact that each cycle $[x]$ is a singleton, and $[x] = P \cap (\Gamma \cdot x)$.

We now describe the vertex stabilizer groups. The first equality is clear. By Lemma 6.1(1), an element $h \in S_4^+ \times (-1)$ is in $\pi(\Gamma_{(1/2,0,0)})$ if and only if $(1/2, 0, 0) - h \cdot (1/2, 0, 0) \in L$. It is clear that the latter condition is satisfied exactly when the upper left corner of the matrix h is ± 1. This proves the second equality. One proves the third equality in a similar way: Lemma 6.1(1) implies that $h \in S_4^+ \times (-1)$ is in $\pi(\Gamma_{(1/2,1/2,0)})$ if and only if the bottom right entry in h is ± 1. The fourth equality is straightforward: if $h \in S_4^+ \times (-1)$, then $h \cdot (1/2, 1/2, 1/2) = (\pm 1/2, \pm 1/2, \pm 1/2)$, where the signs may be chosen independently. It is clear then that $(1/2, 1/2, 1/2) - h \cdot (1/2, 1/2, 1/2)$ has integral entries, regardless of the choice of h.

Proposition 6.1 shows that each edge stabilizer is negligible. Indeed, the stabilizer of an edge is the same as the stabilizer of its midpoint, and each of these midpoints has at least one entry which is $1/4$. Since the edge stabilizers are negligible, the stabilizers of all higher-dimensional cells are negligible as well.

6.4 A Fundamental Polyhedron for Γ_2

We recall that

$$\Gamma_2 = \left\langle \frac{1}{2}(\mathbf{x} + \mathbf{y} + \mathbf{z}), \mathbf{y}, \mathbf{z} \right\rangle \rtimes (S_4^+ \times (-1)).$$

Throughout this section, we will write Γ in place of Γ_2. Consider the convex compact polyhedron

$$P = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq y \leq x \leq 1/2;\ x + y + z \leq 3/4\right\}.$$

It is possible to check that P is the five-sided polyhedron depicted in Fig. 6.2. The sides S_1, S_2, S_3, S_4, and S_5 are contained in the planes $y = z$, $z = 0$, $x = y$, $x = 1/2$, and $x+y+z = 3/4$, respectively. (Indeed, the sides S_i, for $i \in \{1, 2, 3, 4\}$, are contained in the same planes as the corresponding sides from Sect. 6.3.)

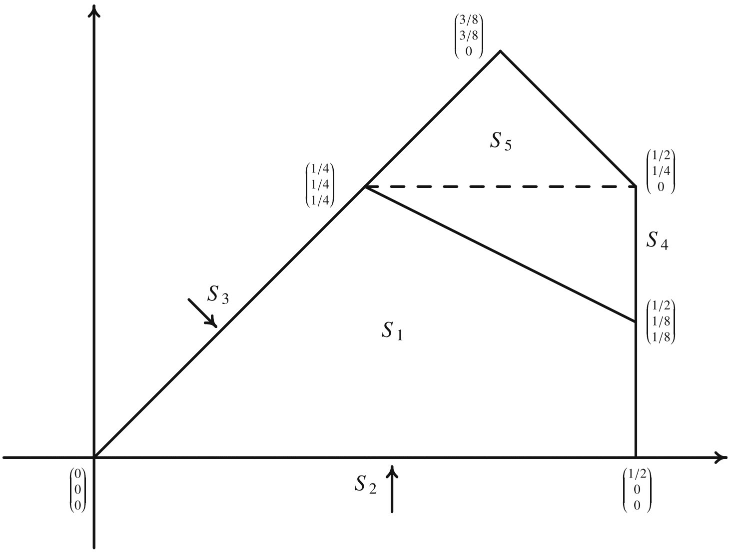


Fig. 6.2 This polyhedron (viewed from the positive z-direction) is a fundamental domain for Γ_2. The *dashed segment* represents an axis of rotation, not a division between faces. The sides S_3 and S_4 are triangular and perpendicular to the xy-plane

The outward unit normal vectors N_1, N_2, N_3, N_4, and N_5 are (respectively),

$$\frac{1}{\sqrt{2}}(-\mathbf{y}+\mathbf{z}), \quad -\mathbf{z}, \quad \frac{1}{\sqrt{2}}(-\mathbf{x}+\mathbf{y}), \quad \mathbf{x}, \quad \text{and} \quad \frac{1}{\sqrt{3}}(\mathbf{x}+\mathbf{y}+\mathbf{z}).$$

The dihedral angles are as follows:

$$\begin{aligned} \theta(S_1,S_2) &= \pi/4, & \theta(S_1,S_3) &= \pi/3, \\ \theta(S_1,S_4) &= \pi/2, & \theta(S_2,S_3) &= \pi/2, \\ \theta(S_2,S_4) &= \pi/2, & \theta(S_3,S_4) &= \pi/4, \\ \theta(S_1,S_5) &= \pi/2, & \theta(S_3,S_5) &= \pi/2, \end{aligned}$$

$$\theta(S_2,S_5) = \arccos\left(1/\sqrt{3}\right),$$

$$\theta(S_4,S_5) = \arccos\left(-1/\sqrt{3}\right)$$

We consider the collection $\Phi = \{\phi_{S_1}, \phi_{S_2}, \phi_{S_3}, \phi_{S_4}, \phi_{S_5}\}$, where ϕ_{S_i} is the reflection in the side S_i, for $i \in \{1,2,3,4\}$, and ϕ_{S_5} is the rotation 180° about the line through $(1/4,1/4,1/4)$ and $(1/2,1/4,0)$ (which is dashed in Fig. 6.2). It is rather clear that Φ is a side-pairing. We must show that Φ is a subproper side-pairing.

If x is a point in the interior of some ridge that is not a face of S_5, then it is easy to check that the dihedral angle sum of $[x]$ is a submultiple of π. There are two more kinds of ridge cycles to consider. Each has the form $\{x, \phi_{S_5}(x)\}$, where $\phi_{S_5} \in \Phi$ is the rotation about the dashed line in Fig. 6.2, and x is either on the ridge between the sides S_1 and S_5, or on the ridge between S_4 and S_5. The dihedral angle sum of $[x]$ is either

$$\theta(S_1, S_5) + \theta(S_3, S_5) \quad \text{or} \quad \theta(S_4, S_5) + \theta(S_2, S_5),$$

both of which are equal to π. It follows that Φ is a subproper side-pairing, so Theorem 6.1 applies. The polyhedron P is therefore an exact convex compact fundamental polyhedron for the action of $\langle\Phi\rangle$ on $\mathbb{R}^3$.

We need to show that $\Gamma = \langle\Phi\rangle$. We note that $\{\phi_{S_1}, \phi_{S_2}, \phi_{S_3}, \phi_{S_4}\}$ is a set that generates Γ_1 by the argument of Sect. 6.3. Furthermore, we have

$$\phi_{S_5} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

It easily follows that $\langle\Phi\rangle = \Gamma$, so P is an exact convex compact fundamental polyhedron for the action of Γ on $\mathbb{R}^3$.

Theorem 6.4. *Let $\hat{C}$ denote the Γ-equivariant cell structure on $\mathbb{R}^3$ determined by the standard cellulation C of P. The quotient $\Gamma\backslash\mathbb{R}^3$ is obtained from P by identifying the two halves of the side S_5 from Fig. 6.2. The set consisting of the vertices* $(0,0,0)$*,* $(1/2,0,0)$*, and* $(1/4,1/4,1/4)$ *maps injectively into the quotient. The stabilizers of these vertices are determined by the following equalities:*

$$\pi(\Gamma_{(0,0,0)}) = S_4^+ \times (-1).$$
$$\pi(\Gamma_{(1/2,0,0)}) = D_{4_1}^+ \times (-1).$$
$$\pi(\Gamma_{(1/4,1/4,1/4)}) = D_3^+ \times (-1).$$

All of the other stabilizers of cells in the quotient are negligible.

Proof. The first statement follows easily from a description of the cycles $[x]$, and from the fact that $[x] = P \cap (\Gamma \cdot x)$. The second statement, that the given vertices map injectively into the quotient, follows from the fact that the cycle generated by each vertex is a singleton.

We turn now to a consideration of the cell stabilizers. First, note that each of the vertices

$$(3/8, 3/8, 0), \quad (1/2, 1/4, 0), \quad (1/2, 1/8, 1/8)$$

has a negligible stabilizer group, by Proposition 6.1. This directly implies that all of the edges and faces incident with these vertices must also have negligible stabilizer groups. This leaves three vertices and two edges to consider. It is clear that the first equality in the theorem holds. The second equality holds for reasons similar to those used in establishing the second equality in Theorem 6.3. The third equality follows from Lemma 6.1(1) and the fact that $h \in S_4^+ \times (-1)$ satisfies the condition $(1/4, 1/4, 1/4) - h \cdot (1/4, 1/4, 1/4)$ if and only if $h \cdot (1/4, 1/4, 1/4) = \pm(1/4, 1/4, 1/4)$.

Finally, we note that the remaining edges are negligible by Proposition 6.1.

6.5 A Fundamental Polyhedron for Γ_3

Note that

$$\Gamma_3 = \left\langle \frac{1}{2}(\mathbf{x}+\mathbf{y}), \frac{1}{2}(\mathbf{y}+\mathbf{z}), \frac{1}{2}(\mathbf{x}+\mathbf{z}) \right\rangle \rtimes (S_4^+ \times (-1)).$$

We set $\Gamma = \Gamma_3$ in this section. Consider the convex compact polyhedron

$$P = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le y \le x,\ x + y \le 1/2\right\}.$$

A straightforward check shows that P is the tetrahedron depicted in Fig. 6.3. The sides S_1, S_2, S_3, and S_4 are contained in the planes $y = z$, $z = 0$, $x = y$, and $x + y = 1/2$, respectively. (The sides S_i for $i \in \{1, 2, 3\}$ are contained in the same planes as the corresponding sides from Sect. 6.3.)

The outward unit normal vectors N_1, N_2, N_3, and N_4 are (respectively)

$$\frac{1}{\sqrt{2}}(-\mathbf{y}+\mathbf{z}), \quad -\mathbf{z}, \quad \frac{1}{\sqrt{2}}(-\mathbf{x}+\mathbf{y}), \quad \text{and} \quad \frac{1}{\sqrt{2}}(\mathbf{x}+\mathbf{y}).$$

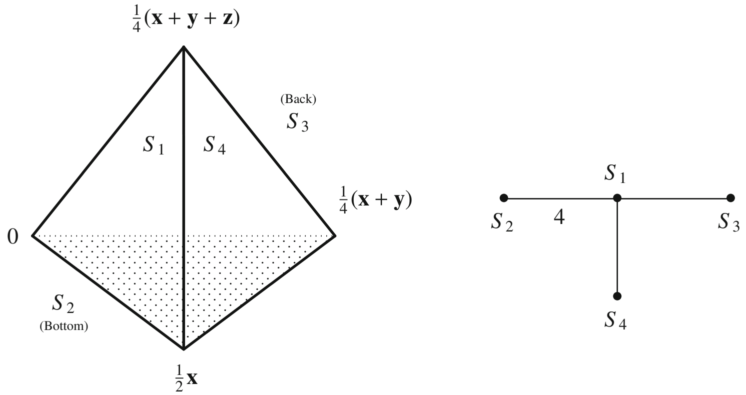


Fig. 6.3 On the *left*, we have a fundamental domain for the action of Γ_3 on $\mathbb{R}^3$. The group Γ_3 is a Coxeter group, and its Coxeter diagram appears on the *right*

A routine check shows that the dihedral angles are

$$\theta(S_1, S_2) = \pi/4, \quad \theta(S_1, S_3) = \pi/3,$$
$$\theta(S_1, S_4) = \pi/3, \quad \theta(S_2, S_3) = \pi/2,$$
$$\theta(S_2, S_4) = \pi/2, \quad \theta(S_3, S_4) = \pi/2.$$

It follows from Corollary 6.1 that Γ_P, the group generated by the reflections in the sides of the tetrahedron from Fig. 6.3, is a discrete group, and that P is an exact convex compact fundamental polyhedron for the action of Γ_P on $\mathbb{R}^3$. We have

$$\Gamma_P = \left\langle \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\\ \frac{1}{2}\\ 0 \end{pmatrix} + \begin{pmatrix} 0&-1&0\\-1&0&0\\0&0&1 \end{pmatrix} \right\rangle,$$

where the generators are reflections in the sides S_1, S_2, S_3, and S_4, respectively. It is easy to check that $\Gamma_P = \Gamma$.

It follows that P is an exact convex compact fundamental polyhedron for the action of Γ on $\mathbb{R}^3$.

Theorem 6.5. *Let $\hat{C}$ denote the Γ-equivariant cell structure on $\mathbb{R}^3$ determined by the standard cellulation C of P. The quotient $\Gamma\backslash\mathbb{R}^3$ is P itself, endowed with the standard cellulation C. The only non-negligible cell stabilizer groups are as follows:*

$$\pi(\Gamma_{(0,0,0)}) = \pi(\Gamma_{(1/2,0,0)}) = S_4^+ \times (-1), \quad \pi(\Gamma_{(1/4,1/4,0)}) \cong D_2 \times \mathbb{Z}/2.$$

Proof. The statement about the quotient follows directly from the fact that each cycle $[x]$ is a singleton.

The vertex $(1/4, 1/4, 1/4)$ has a negligible stabilizer group by Proposition 6.1. To find the vertex stabilizer of $(1/4, 1/4, 0)$ note that the group $S_4^+ \times (-1)$ translates $(1/4, 1/4, 0)$ to a total of 12 different points in $\mathbb{R}^3$. (It is easy to describe these points explicitly using the fact that $S_4^+ \times (-1)$ is the group of 3×3 signed permutation matrices.) By Lemma 6.1(1), $h \in S_4^+ \times (-1)$ is an element of $\pi(\Gamma_{(1/4,1/4,0)})$ if and only if $h \cdot (1/4, 1/4, 0) = \pm(1/4, 1/4, 0)$. It follows from this that $\pi(\Gamma_{(1/4,1/4,0)})$ has index 6 in $S_4^+ \times (-1)$, i.e., $\pi(\Gamma_{(1/4,1/4,0)})$ has order 8. It is easy to check that the following set is contained in $\pi(\Gamma_{(1/4,1/4,0)})$, and generates a group of order 8:

$$\left\{ \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&-1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&-1&0\\0&0&-1 \end{pmatrix} \right\}.$$

Therefore, $\pi(\Gamma_{(1/4,1/4,0)})$ is generated by the above matrices. One can easily show from this that $\pi(\Gamma_{(1/4,1/4,0)}) \cong D_2 \times \mathbb{Z}/2$.

A routine check using Lemma 6.1(1) shows that $\pi(\Gamma_{(0,0,0)}) = \pi(\Gamma_{(1/2,0,0)}) = S_4^+ \times (-1)$. One proves that the edge stabilizer groups are negligible by applying Proposition 6.1 to the midpoints of the edges.

6.6 A Fundamental Polyhedron for Γ_4

Note that

$$\Gamma_4 = \left\langle \frac{1}{2}(\mathbf{x}+\mathbf{z}), \mathbf{y}, \mathbf{z} \right\rangle \rtimes (D_2^+ \times (-1)).$$

We consider the convex compact polyhedron P given in Fig. 6.4. It is easy to see that

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1/2, \;\; x + z \leq 1/2\}.$$

The sides $\{S_1, S_2, S_3, S_4, S_5\}$ are contained in the planes $y = 0$, $x + z = 1/2$, $y = 1/2$, $x = 0$, and $z = 0$ respectively. The outward unit normal vectors N_1, N_2, N_3, N_4 N_5 are, respectively,

$$-\mathbf{y}, \quad \frac{1}{\sqrt{2}}(\mathbf{x}+\mathbf{z}), \quad \mathbf{y}, \quad -\mathbf{x}, \quad -\mathbf{z}.$$

The dihedral angles are as follows:

$$\begin{aligned}
&\theta(S_1, S_2) = \pi/2, \quad &\theta(S_1, S_4) = \pi/2, \quad &\theta(S_1, S_5) = \pi/2,\\
&\theta(S_2, S_3) = \pi/2, \quad &\theta(S_2, S_4) = \pi/4, \quad &\theta(S_2, S_5) = \pi/4,\\
&\theta(S_3, S_4) = \pi/2, \quad &\theta(S_3, S_5) = \pi/2, \quad &\theta(S_4, S_5) = \pi/2.
\end{aligned}$$

We consider the collection $\Phi = \{\phi_{S_1}, \phi_{S_2}, \phi_{S_3}, \phi_{S_4}, \phi_{S_5}\}$, where ϕ_{S_i} is the reflection in the side S_i, for $i \in \{1, 3, 4, 5\}$, and ϕ_{S_2} is the rotation 180° about the line through $(1/4, 0, 1/4)$ and $(1/4, 1/2, 1/4)$ (which is the line dashed in Fig. 6.4).

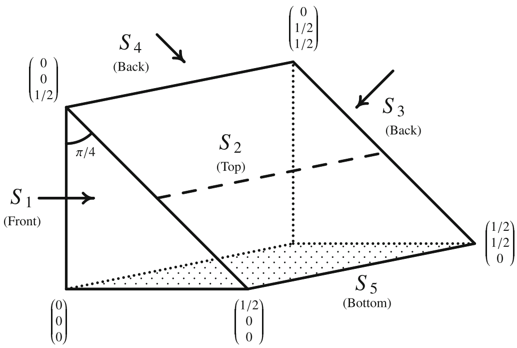


Fig. 6.4 This is a fundamental domain for the action of Γ_4 on $\mathbb{R}^3$. The *dashed line* indicates an axis of rotation

It is rather clear that Φ is a side-pairing. We must show that Φ is a subproper side-pairing.

If x is a point in the interior of some ridge that is not a face of S_2, then it is easy to check that the dihedral angle sum of $[x]$ is a submultiple of π. If x lies on the axis of rotation, then $[x]$ is again a singleton, and it follows easily that the dihedral angle sum is a submultiple of 2π. There are three more kinds of ridge cycles to consider. Each has the form $\{x, \phi_{S_2}(x)\}$, where $\phi_{S_2} \in \Phi$ is the rotation about the dashed line in Fig. 6.4, and x is either on the ridge between the sides S_1 and S_2, S_3 and S_2, or S_4 and S_2. The dihedral angle sum of $[x]$ is

$$2\theta(S_1, S_2), \quad 2\theta(S_3, S_2), \quad \text{or} \quad \theta(S_4, S_2) + \theta(S_5, S_2),$$

respectively. All of the latter sums are π or $\pi/2$. It follows that Φ is a subproper side-pairing, so Theorem 6.1 applies. The polyhedron P is therefore an exact convex compact fundamental polyhedron for the action of $\langle \Phi \rangle$ on $\mathbb{R}^3$. Furthermore, we have

$$\Gamma_P = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle,$$

where the isometries in question are ϕ_{S_i}, for $i = 1, \ldots, 5$, respectively. It is straightforward to verify that $\Gamma_P = \Gamma$.

Theorem 6.6. *Let $\hat{C}$ denote the Γ-equivariant cell structure on $\mathbb{R}^3$ determined by the standard cellulation C of P. The quotient $\Gamma \backslash \mathbb{R}^3$ is obtained from P by identifying the two halves of the side S_2 from Fig. 6.4. The set consisting of the vertices* $(0, 0, 0)$*,* $(0, 0, 1/2)$*,* $(0, 1/2, 1/2)$ *and* $(0, 1/2, 0)$ *maps injectively into the quotient. The stabilizers of these vertices are determined by the following equalities:*

$$\pi(\Gamma_{(0,0,0)}) = \pi(\Gamma_{(0,1/2,0)}) = \pi(\Gamma_{(0,0,1/2)}) = \pi(\Gamma_{(0,1/2,1/2)}) = D_2^+ \times (-1).$$

All of the other stabilizers of cells in the quotient are negligible.

Proof. First, we note that there are eight vertices in the cellulation C: the four listed in the statement of the theorem, and

$$(1/2, 0, 0), (1/2, 1/2, 0), (1/4, 0, 1/4), \text{and } (1/4, 1/2, 1/4).$$

We note that the first two of these last vertices occupy the same orbits as vertices in the statement of the theorem. The final two vertices v are negligible by Lemma 6.1(1) since $v - \phi_{S_4}(v) \notin L$, and so $|\Gamma_v| \leq 4$.

It is easy to check the equalities in the theorem using Lemma 6.1(1). The edges (and, therefore all higher-dimensional cells) have negligible stabilizers, either because they are incident with vertices having negligible stabilizers, or because of Proposition 6.1 (applied to the midpoints of the edges).

To verify the description of the quotient is straightforward.

6.7 A Fundamental Polyhedron for Γ_5

Recall that $\Gamma_5 = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle \rtimes (D_6^+ \times (-1))$. We write Γ in place of Γ_5. Consider the convex polyhedron P depicted in Fig. 6.5. One checks that

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x + y + z \leq 3/2, -2x + y + z \leq 0, x \leq z \leq y + 1\}.$$

The sides S_1, S_2, S_3, S_4, and S_5 are convex subsets of the planes $x + y + z = 3/2$, $z = y + 1$, $x + y + z = 0$, $-2x + y + z = 0$, and $x = z$, respectively. The outward unit normal vectors N_1, N_2, N_3, N_4, and N_5 are

$$\frac{1}{\sqrt{3}}\mathbf{v}_1, \quad \frac{1}{\sqrt{2}}\mathbf{v}_3, \quad \frac{-1}{\sqrt{3}}\mathbf{v}_1, \quad \frac{1}{\sqrt{6}}(-2\mathbf{v}_2 + \mathbf{v}_3), \quad \text{and} \quad \frac{1}{\sqrt{2}}(\mathbf{v}_2 - \mathbf{v}_3),$$

respectively.

It is straightforward to check that

$$\begin{aligned}
\theta(S_1, S_2) &= \pi/2, & \theta(S_1, S_4) &= \pi/2, \\
\theta(S_1, S_5) &= \pi/2, & \theta(S_2, S_3) &= \pi/2, \\
\theta(S_2, S_4) &= \pi/2, & \theta(S_2, S_5) &= \pi/3, \\
\theta(S_3, S_4) &= \pi/2, & \theta(S_3, S_5) &= \pi/2, \\
\theta(S_4, S_5) &= \pi/6.
\end{aligned}$$

Since all of these dihedral angles are submultiples of π, by Corollary 6.1 the group Γ_P generated by reflections in the sides of P is discrete, and P is an exact convex compact fundamental polyhedron for the action of Γ_P on $\mathbb{R}^3$. We have $\Gamma_P =$

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

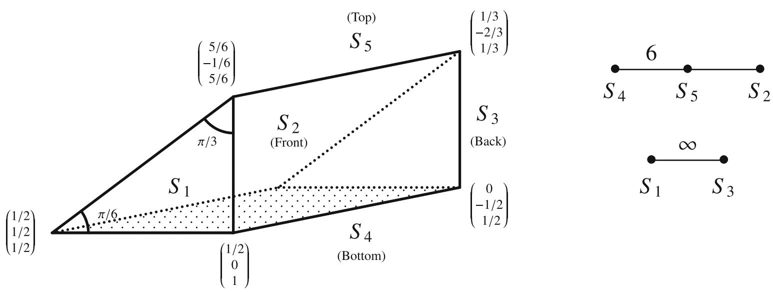


Fig. 6.5 On the *left* is a fundamental domain for the action of Γ_5 on $\mathbb{R}^3$. The unlabelled vertex at the intersection of *three dotted lines* is the origin. On the *right* is the Coxeter diagram for Γ_5

It is not difficult to check that $\Gamma_P = \Gamma$, so P is an exact convex compact fundamental polyhedron for the action of Γ on $\mathbb{R}^3$.

Theorem 6.7. *Let $\hat{C}$ denote the Γ-equivariant cell structure on $\mathbb{R}^3$ determined by the standard cellulation C of P. The quotient $\Gamma\backslash\mathbb{R}^3$ is P itself, endowed with the standard cellulation C. The non-negligible stabilizer groups are determined by the following equalities:*

$$\pi(\Gamma_{(0,0,0)}) = \pi(\Gamma_{(1/2,1/2,1/2)}) = D_6^+ \times (-1);$$
$$\pi(\Gamma_{(1/2,0,1)}) = \pi(\Gamma_{(0,-1/2,1/2)}) \cong D_2 \times \mathbb{Z}/2;$$
$$\pi(\Gamma_{(1/3,-2/3,1/3)}) = \pi(\Gamma_{(5/6,-1/6,5/6)}) = D_6';$$
$$\pi(\Gamma_{(1/4,1/4,1/4)}) = D_6''.$$

(Note that $\Gamma_{(1/4,1/4,1/4)}$ is the stabilizer group of the edge connecting $(0,0,0)$ to $(1/2,1/2,1/2)$.) The stabilizer groups of all other cells are negligible.

Proof. The statement about the quotient follows because each cycle $[x]$ is a singleton.

We now consider vertex stabilizers. The equality $\pi(\Gamma_{(0,0,0)}) = D_6^+ \times (-1)$ is trivial. The equality $\pi(\Gamma_{(1/2,1/2,1/2)}) = D_6^+ \times (-1)$ follows from Lemma 6.1(1) and the fact that $h \cdot (1/2,1/2,1/2) = \pm(1/2,1/2,1/2)$, for any $h \in D_6^+ \times (-1)$. Since $2v \notin \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$, for $v \in \{(1/3,-2/3,1/3), (5/6,-1/6,5/6)\}$, it follows from Lemma 6.1(1) that (-1) (the antipodal map) is in neither $\pi(\Gamma_{(1/3,-2/3,1/3)})$ nor $\pi(\Gamma_{(5/6,-1/6,5/6)})$. This means that each of these groups has order at most 12. It is routine to check, using Lemma 6.1(1), that D_6' is a subgroup of each of these two groups. This shows that $\pi(\Gamma_{(1/3,-2/3,1/3)}) = \pi(\Gamma_{(5/6,-1/6,5/6)}) = D_6'$. Finally, we consider $(1/2,0,1)$ and $(0,-1/2,1/2)$. Using the theory of Coxeter groups, we note that these vertices are stabilized by the groups $\langle \phi_{S_1}, \phi_{S_2}, \phi_{S_4} \rangle$, and $\langle \phi_{S_2}, \phi_{S_3}, \phi_{S_4} \rangle$, both of which are isomorphic to $D_2 \times \mathbb{Z}/2$.

Next, consider the stabilizer of the edge connecting $(0,0,0)$ to $(1/2,1/2,1/2)$, which is the same as $\Gamma_{(1/4,1/4,1/4)}$. We note that, by Lemma 6.1(1), (-1) is not in $\pi(\Gamma_{(1/4,1/4,1/4)})$, since $(1/2,1/2,1/2) \notin \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$. It is not difficult to see that $D_6'' \leq \pi(\Gamma_{(1/4,1/4,1/4)})$, and this inclusion must be an equality by order considerations.

We need to show that the remaining edges have negligible stabilizers. Once again we use the theory of Coxeter groups to simplify our work. We consider the subdiagrams of the Coxeter diagram that are determined by pairs of distinct vertices other than $\{S_1, S_3\}$ (which does not determine an edge) and $\{S_4, S_5\}$ (which is accounted for above). These subdiagrams determine subgroups that are isomorphic to D_2 or D_3, so all are negligible.

It follows easily that all remaining cells have negligible stabilizer groups.

6.8 A Fundamental Polyhedron for Γ_6

We note that

$$\Gamma_6 = \left\langle \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_2, \mathbf{v}_3 \right\rangle \rtimes (D_3^+ \times (-1)).$$

We will write Γ in place of Γ_6 throughout this section.

Let P denote the polyhedron in Fig. 6.6. The sides S_1, S_2, S_3, S_4, and S_5 are contained in the planes $x = z$, $5x + 2y + 5z = 3$, $x - y = 1$, $y = z$, and $x + y + z = 0$, respectively. The outward-pointing unit normal vectors N_1, N_2, N_3, N_4, and N_5 are, respectively,

$$\frac{1}{\sqrt{2}}(-\mathbf{x} + \mathbf{z}), \frac{1}{3\sqrt{6}}(5\mathbf{x} + 2\mathbf{y} + 5\mathbf{z}), \frac{1}{\sqrt{2}}(\mathbf{x} - \mathbf{y}), \frac{1}{\sqrt{2}}(\mathbf{y} - \mathbf{z}), \frac{-1}{\sqrt{3}}(\mathbf{x} + \mathbf{y} + \mathbf{z}).$$

The dihedral angles between faces are:

$$\theta(S_1, S_2) = \pi/2, \quad \theta(S_1, S_3) = \pi/3,$$
$$\theta(S_1, S_4) = \pi/3, \quad \theta(S_1, S_5) = \pi/2,$$
$$\theta(S_2, S_3) = \arccos(\frac{-1}{\sqrt{12}}),$$
$$\theta(S_2, S_4) = \arccos(\frac{1}{\sqrt{12}}),$$

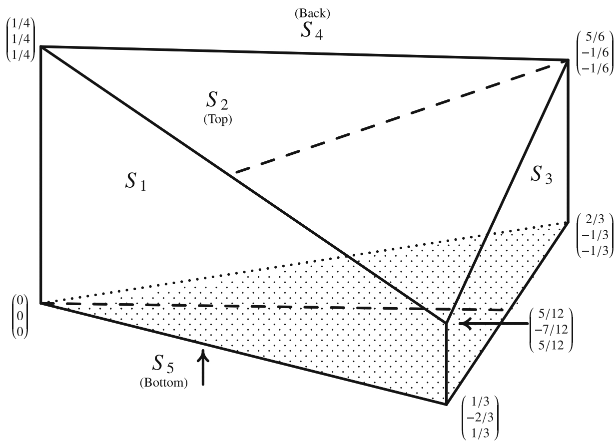


Fig. 6.6 The polyhedron pictured here is an exact convex compact fundamental polyhedron for the action of Γ_6 on $\mathbb{R}^3$. The *dashed lines* represent axes of rotation (through 180°) for certain elements of Γ_6. Note that the *base* of the figure is an equilateral triangle, but the *top* is isosceles

$$\theta(S_3, S_4) = \pi/3, \qquad \theta(S_3, S_5) = \pi/2,$$
$$\theta(S_4, S_5) = \pi/2.$$

We define $\Phi = \{\phi_{S_1}, \phi_{S_2}, \phi_{S_3}, \phi_{S_4}, \phi_{S_5}\}$ as follows. Each of ϕ_{S_1}, ϕ_{S_3}, and ϕ_{S_4} is reflection in the corresponding face of P. In particular, we note that the isometries

$$\begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix}, \begin{pmatrix} 1\\ -1\\ 0 \end{pmatrix} + \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix},$$

are ϕ_{S_1}, ϕ_{S_3}, and ϕ_{S_4}, respectively. The isometries ϕ_{S_2} and ϕ_{S_5} are rotations about the dashed lines in the faces S_2 and S_5 (respectively). The isometries

$$\begin{pmatrix} 2/3\\ -1/3\\ 2/3 \end{pmatrix} + \begin{pmatrix} 0&0&-1\\ 0&-1&0\\ -1&0&0 \end{pmatrix}, \begin{pmatrix} 0&-1&0\\ -1&0&0\\ 0&0&-1 \end{pmatrix}$$

are ϕ_{S_2} and ϕ_{S_5} (respectively).

The set Φ is a subproper side-pairing of P. We leave the checking to the reader, but note that no ridge cycle has more than two elements and the dihedral angle sum of a ridge cycle is always a submultiple of π. It follows from Theorem 6.1 that P is an exact convex compact fundamental polyhedron for the action of $\langle \Phi \rangle$ on $\mathbb{R}^3$.

Finally, we briefly argue that $\Gamma = \langle \Phi \rangle$. We notice that $\langle \phi_{S_1}, \phi_{S_4}, \phi_{S_5} \rangle = D_3^+ \times (-1)$. From this, it follows easily that $\langle \phi_{S_1}, \phi_{S_3}, \phi_{S_4}, \phi_{S_5} \rangle$ contains all of $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$, as well. It is easy to see that the group $\langle \Phi \rangle$ must contain the translation $(2/3, -1/3, 2/3)$ (since $D_3^+ \times (-1) \leq \langle \Phi \rangle$, and $\phi_{S_2} \in \langle \Phi \rangle$), so $\Gamma \leq \langle \Phi \rangle$. The reverse inclusion is clear.

It follows that P is an exact convex compact fundamental polyhedron for the action of Γ on R^3.

Theorem 6.8. *Let $\hat{C}$ denote the Γ-equivariant cell structure on $\mathbb{R}^3$ determined by the standard cellulation C of P. The quotient $\Gamma \backslash \mathbb{R}^3$ is obtained from P by identifying the two halves of each of the sides S_2 and S_5 from Fig. 6.6. The set consisting of the vertices* $(0,0,0)$ *and* $(5/6, -1/6, -1/6)$ *maps injectively into the quotient. The stabilizers of these vertices are determined by the following equalities:*

$$\pi(\Gamma_{(0,0,0)}) = \pi(\Gamma_{(5/6,-1/6,-1/6)}) = D_3^+ \times (-1).$$

All of the other stabilizers of cells in the quotient are negligible.

Proof. The statement about the quotient follows easily from a straightforward description of the cycles $[x]$, and from the fact that $[x] = P \cap (\Gamma \cdot x)$.

We turn to a description of the vertex stabilizers. Note that there are a total of 8 vertices to consider. We must describe the stabilizers of $(1/3, -1/6, 1/3)$ and $(1/2, -1/2, 0)$ since these points are endpoints of dashed lines from Fig. 6.6, and will therefore be vertices in the standard cellulation of P.

First, we note the stabilizers of the vertices $(1/4, 1/4, 1/4)$, $(5/12, -7/12, 5/12)$, $(1/3, -2/3, 1/3)$, and $(2/3, -1/3, -1/3)$ are all negligible by Proposition 6.1.

Second, we note that $\pi(\Gamma_{(1/3,-1/6,1/3)})$ and $\pi(\Gamma_{(1/2,-1/2,0)})$ do not contain the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

by Lemma 6.1(1). It follows that both of these groups have order 1, 2, or 4; this forces both of these groups to be negligible. Third, we leave the easy verification of the equalities from the statement of the theorem to the reader.

It follows easily that all of the remaining cells must have negligible stabilizers, since each such stabilizer is a subgroup of a negligible group.

6.9 A Fundamental Polyhedron for Γ_7

We note that

$$\Gamma_7 = \left\langle \mathbf{v}_1, \frac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3), \mathbf{v}_3 \right\rangle \rtimes (D_3^+ \times (-1)).$$

We write Γ in place of Γ_7. Let P denote the polyhedron in Fig. 6.7. The sides S_1, S_2, S_3, S_4, and S_5 are contained in the planes $x - 2y + z = 1$, $x + y + z = 3/2$, $x = z$, $-2x + y + z = 0$, and $x + y + z = -3/2$, respectively. The outward-pointing

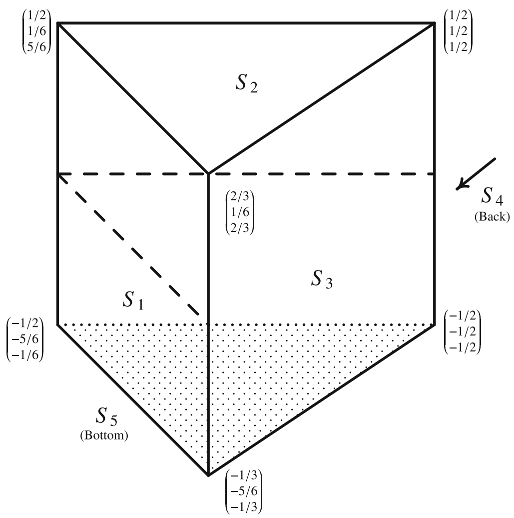


Fig. 6.7 This is a fundamental domain for the action of Γ_7 on $\mathbb{R}^3$. The *dashed line* indicate axes of rotation. Note that the *top* and *bottom* of P are triangles having angles measuring $\pi/3$, $\pi/2$, and $\pi/6$, where the right angles are in the *foreground*

unit normal vectors N_1, N_2, N_3, N_4, and N_5 are

$$\frac{1}{\sqrt{6}}(\mathbf{x} - 2\mathbf{y} + \mathbf{z}), \frac{1}{\sqrt{3}}(\mathbf{x} + \mathbf{y} + \mathbf{z}), \frac{1}{\sqrt{2}}(\mathbf{x} - \mathbf{z}), \frac{1}{\sqrt{6}}(-2\mathbf{x} + \mathbf{y} + \mathbf{z}),$$
$$\frac{-1}{\sqrt{3}}(\mathbf{x} + \mathbf{y} + \mathbf{z}),$$

respectively.

The dihedral angles between vertical sides are

$$\theta(S_1, S_4) = \pi/3, \quad \theta(S_3, S_4) = \pi/6, \quad \theta(S_1, S_3) = \pi/2,$$

and all other angles between sides have measure $\pi/2$.

We consider the set $\Phi = \{\phi_{S_1}, \phi_{S_2}, \phi_{S_3}, \phi_{S_4}, \phi_{S_5}\}$, defined respectively as follows:

$$\begin{pmatrix} 1/3 \\ -2/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

It is not difficult to check that Φ is a subproper side-pairing. We leave the checking to the reader. Note that every ridge cycle has one or two elements. All of the dihedral angle sums of the ridge cycles will be submultiplies of π, with the exception of a cycle $[x]$, where x is a point (other than the midpoint) on the ridge between S_1 and S_4. For such a cycle, the dihedral angle sum will be $2\pi/3$. This causes no problem, since the cycle $[x]$ is cyclic.

It follows from Theorem 6.1 that P is an exact convex compact fundamental polyhedron for the action of $\langle \Phi \rangle$ on $\mathbb{R}^3$. We claim that $\Gamma \subseteq \langle \Phi \rangle$, the reverse inclusion being clear. We sketch the proof. First, $D_3^+ \times (-1) = \langle \phi_{S_3}, \phi_{S_4} \rangle$. The group $\langle \phi_{S_1}, \phi_{S_3}, \phi_{S_4} \rangle$ is the group generated by $D_3^+ \times (-1)$ and $\frac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3) = \frac{1}{3}(\mathbf{x} - 2\mathbf{y} + \mathbf{z})$. It is an exercise to show that the latter group contains $\mathbf{v}_3 = -\mathbf{y} + \mathbf{z}$ as well. The equality $\Gamma = \langle \Phi \rangle$ is now clear (since $\phi_{S_2} = \mathbf{v}_1$), so that P is an exact convex compact fundamental polyhedron for the action of Γ on $\mathbb{R}^3$.

Theorem 6.9. *Let $\hat{C}$ denote the Γ-equivariant cell structure on $\mathbb{R}^3$ determined by the standard cellulation C of P. The quotient $\Gamma \backslash \mathbb{R}^3$ is obtained from P by identifying the two halves of each of the sides S_1 and S_4, and by identifying S_2 with S_5 in the obvious way (see Fig. 6.7). The set consisting of the vertices* $(0,0,0)$ *and* $(1/2, 1/2, 1/2)$ *maps injectively into the quotient. The stabilizers of these vertices are determined by the following equalities:*

$$\pi(\Gamma_{(0,0,0)}) = D_3^+ \times (-1).$$
$$\pi(\Gamma_{(1/2,1/2,1/2)}) = D_3^+ \times (-1).$$

All of the other stabilizers of cells in the quotient are negligible.

Proof. The description of the quotient follows from the same argument that we have used in the other sections.

We note that there are nine vertices in the standard cellulation of P, including the endpoints of the dashed lines from Fig. 6.7. These endpoints are:

$$(0,0,0), \quad (0,-1/3,1/3), \quad (1/6,-1/3,1/6).$$

In the quotient, there are only six vertices, however, and we may restrict our attention to the vertices from the top half of Fig. 6.7.

We note that the stabilizers of the vertices $(1/2, 1/6, 5/6)$ and $(0, -1/3, 1/3)$ are negligible by Proposition 6.1. The matrix

$$\begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$$

is in neither $\pi(\Gamma_{(2/3,1/6,2/3)})$ nor $\pi(\Gamma_{(1/6,-1/3,1/6)})$. As a result, the latter groups have index 3 (at least) in $D_3^+ \times (-1)$, so they must be negligible. It is clear that $\pi(\Gamma_{(0,0,0)}) = D_3^+ \times (-1)$, and the equality $\pi(\Gamma_{(1/2,1/2,1/2)}) = D_3^+ \times (-1)$ follows easily from Lemma 6.1(1) and the fact that $h \cdot (1/2, 1/2, 1/2) = \pm(1/2, 1/2, 1/2)$, for all $h \in D_3^+ \times (-1)$. (Note that the vertex $(-1/2, -1/2, -1/2)$ will be identified with $(1/2, 1/2, 1/2)$ in the quotient.)

Now we go on to consider edge stabilizers. Only two of these might fail to be negligible: the stabilizers of edges connecting $(0,0,0)$ to $(1/2,1/2,1/2)$ and $(-1/2,-1/2,-1/2)$, respectively. Both of the latter edges are identified in the quotient, so we need only consider the edge between $(0,0,0)$ and $(1/2,1/2,1/2)$. The stabilizer of this edge is negligible, since the stabilizer of the point $(1/4,1/4,1/4)$ is negligible by Proposition 6.1.

Chapter 7
The Homology Groups $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$

In this chapter, we compute the homology groups $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$, for all 73 split three-dimensional crystallographic groups. In order to do this, we recall that Quinn [Qu82] established the existence of a spectral sequence that converges to this homology group, with E^2-terms given by:

$$E^2_{p,q} = H_p(\Gamma \backslash E_{\mathcal{FIN}}(\Gamma)\,; \{Wh_q(\Gamma_\sigma)\}) \Longrightarrow H^{\Gamma}_{p+q}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}).$$

The chain complex that gives the homology of $\Gamma \backslash E_{\mathcal{FIN}}(\Gamma)$ with local coefficients $\{Wh_q(\Gamma_\sigma)\}$ has the form

$$0 \to \bigoplus_{\sigma^3} Wh_q(\Gamma_{\sigma^3}) \to \bigoplus_{\sigma^2} Wh_q(\Gamma_{\sigma^2}) \to \bigoplus_{\sigma^1} Wh_q(\Gamma_{\sigma^1}) \to \bigoplus_{\sigma^0} Wh_q(\Gamma_{\sigma^0}) \to 0,$$

where σ^i denotes the cells in dimension i, and the sum is over all i-dimensional cells in $\Gamma \backslash E_{\mathcal{FIN}}(\Gamma)$. The pth homology group of this complex will give us the entries for the $E^2_{p,q}$-term of the spectral sequence. Let us recall that

$$Wh_q(F) = \begin{cases} Wh(F), & q = 1 \\ \tilde{K}_0(\mathbb{Z}F), & q = 0 \\ K_q(\mathbb{Z}F), & q \leq -1. \end{cases}$$

Observe that for the groups we are interested in it is particularly easy to obtain a model for $E_{\mathcal{FIN}}(\Gamma)$: indeed, it is well known that, for a lattice in $Isom(\mathbb{R}^n)$, the Γ-space $\mathbb{R}^n$ is a model for $E_{\mathcal{FIN}}$. In our specific situation, we obtain models for $E_{\mathcal{FIN}}(\Gamma)$ having very explicit fundamental domains, namely the fundamental polyhedra given in Chap. 6.

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_7

Remark 7.1. In our models for $E_{\mathcal{FIN}}(\Gamma)$, the three-dimensional cells will have trivial stabilizers, and there will be a number of orbits of two-dimensional cells, each of which will have stabilizer 1 or $\mathbb{Z}/2$. Note that since $Wh_q(1)$ and $Wh_q(\mathbb{Z}/2)$ vanish for all $q \leq 1$, this in particular implies that *there will never be any contribution to the E^2-terms from the three-dimensional and two-dimensional cells*. In other words, $E^2_{p,q} = 0$ except possibly for $p = 0, 1$.

7.1 The Algebraic K-Theory of Cell Stabilizers in $E_{\mathcal{FIN}}(\Gamma)$

In this section, we want to find the algebraic K-theory of the finite subgroups that occur as cell stabilizers for the Γ-action on $\mathbb{R}^3$, where Γ ranges over all 73 split crystallographic groups. Recall from the discussion in Chap. 2 that these cell stabilizers are (up to conjugacy) precisely the point groups listed in Figs. 2.1 and 2.2, and in Theorem 2.2. Up to isomorphism these groups are: 1, $\mathbb{Z}/n$, D_n, $\mathbb{Z}/n \times \mathbb{Z}/2$, $D_n \times \mathbb{Z}/2$, A_4, S_4, $A_4 \times \mathbb{Z}/2$, and $S_4 \times \mathbb{Z}/2$, where $n = 2, 3, 4, 6$.

In the spectral sequence computing the homology $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$, the E^2-term is computed from the algebraic K-groups of the various cell stabilizers, so we need the algebraic K-groups for all of the finite groups appearing in the list above. For the convenience of the reader, we provide in Table 7.1 the results of the computations of these groups. Table 7.1 provides a list of all the *non-trivial* K-groups that occur amongst the finite groups we are considering.

We will now justify the results summarized in Table 7.1. It is well known that if G is a finite group, then $K_q(\mathbb{Z}G)$ is trivial for all $q \leq -2$ (see [C80a]), so we will focus only on the K-groups K_{-1}, $\tilde{K}_0$, and Wh. For all but *three* of the finite groups in our list, the lower algebraic K-theory is well known; the reference [LO09] collects references for most of the known results. Bass [Bas68, p. 695] computed $K_{-1}(\mathbb{Z}[\mathbb{Z}/6])$, Reiner and Ullom [RU74, Theorem 2.2] computed $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6])$, and Cohen [Co73] computed $Wh(\mathbb{Z}/6)$.

For the remaining groups in our list: $\mathbb{Z}/4 \times \mathbb{Z}/2$, $\mathbb{Z}/6 \times \mathbb{Z}/2$, and $A_4 \times \mathbb{Z}/2$, we detail the computations in the next few subsections.

Table 7.1 Lower algebraic K-theory of cell stabilizers in $E_{\mathcal{FIN}}$

$F \in \mathcal{FIN}$	$Wh_q \neq 0,\ q \leq -1$	$\tilde{K}_0 \neq 0$	$Wh \neq 0$
$\mathbb{Z}/6$	$K_{-1} \cong \mathbb{Z}$		
$\mathbb{Z}/4 \times \mathbb{Z}/2$		$\mathbb{Z}/2$	
$\mathbb{Z}/6 \times \mathbb{Z}/2$	$K_{-1} \cong \mathbb{Z}^3$	$(\mathbb{Z}/2)^2$	
$D_2 \times \mathbb{Z}/2$		$\mathbb{Z}/2$	
D_6	$K_{-1} \cong \mathbb{Z}$		
$D_4 \times \mathbb{Z}/2$		$\mathbb{Z}/4$	
$D_6 \times \mathbb{Z}/2$	$K_{-1} \cong \mathbb{Z}^3$	$(\mathbb{Z}/2)^2$	
$A_4 \times \mathbb{Z}/2$	$K_{-1} \cong \mathbb{Z}$	$\mathbb{Z}/2$	
$S_4 \times \mathbb{Z}/2$	$K_{-1} \cong \mathbb{Z}$	$\mathbb{Z}/4$	

7.1.1 The Lower Algebraic K-Theory of $\mathbb{Z}/4 \times \mathbb{Z}/2$

Bass [Bas68, Theorem 10.6, p. 695] proves that $K_{-1}(\mathbb{Z}G)$ vanishes if G is a finite abelian group of prime power order, so $K_{-1}(\mathbb{Z}[\mathbb{Z}/4 \times \mathbb{Z}/2]) = 0$, and Alves and Ontaneda [AO06, Sect. 5.1] show that $Wh(\mathbb{Z}/4 \times \mathbb{Z}/2) = 0$. To compute $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/4 \times \mathbb{Z}/2])$, consider the following Cartesian square

$$\begin{CD} \mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/4] @>>> \mathbb{Z}[\mathbb{Z}/4] \\ @VVV @VV{\varphi_1}V \\ \mathbb{Z}[\mathbb{Z}/4] @>>{\varphi_2}> \mathbb{F}_2[\mathbb{Z}/4] \end{CD}$$

where φ_i is reduction mod 2 for $i = 1, 2$. We let $u(\,)$ denote the group of units, and set

$$u^*(\mathbb{Z}[\mathbb{Z}/4]) = \varphi_i\{u(\mathbb{Z}[\mathbb{Z}/4])\} \subset u(\mathbb{F}_2[\mathbb{Z}/4]), \qquad i = 1, 2.$$

Let $G = \mathbb{Z}/4 \times \mathbb{Z}/2$. Since $\mathbb{Q}G \cong \mathbb{Q}[\mathbb{Z}/4] \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{Z}/2]$, and each simple component of $\mathbb{Q}[\mathbb{Z}/4]$, and of $\mathbb{Q}[\mathbb{Z}/2]$, is a full matrix algebra over $\mathbb{Q}$, then the same is true for $\mathbb{Q}G$, and consequently $\mathbb{Q}G$ satisfies the Eichler condition (i.e., no simple component of $\mathbb{Q}G$ is a totally definite quaternion algebra). By [RU74, Theorem 1.9], there is an exact sequence

$$1 \to u^*(\mathbb{Z}[\mathbb{Z}/4]) \to u(\mathbb{F}_2[\mathbb{Z}/4]) \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/4]) \to 2\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/4]) \to 0.$$

Since $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/4]) = 0$ ([RU74, Theorem 2.8]),

$$1 \to u^*(\mathbb{Z}[\mathbb{Z}/4]) \to u(\mathbb{F}_2[\mathbb{Z}/4]) \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/4]) \to 0.$$

Consequently,

$$\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/4]) \cong u(\mathbb{F}_2[\mathbb{Z}/4])/u^*(\mathbb{Z}[\mathbb{Z}/4]).$$

Next, let $\mathbb{Z}/4 = \langle \sigma \rangle$. A direct calculation shows that $u(\mathbb{F}_2[\mathbb{Z}/4]) = \langle \sigma \rangle \times \langle \sigma + \sigma^2 + \sigma^3 \rangle \cong \mathbb{Z}/4 \times \mathbb{Z}/2$. The equality $u^*(\mathbb{Z}[\mathbb{Z}/4]) = \langle \sigma \rangle$ follows from the fact that $\mathbb{Z}[\mathbb{Z}/4]$ has only trivial units (since $Wh(\mathbb{Z}/4) = 0$). Therefore, $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/4 \times \mathbb{Z}/2]) = \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/4]) = \langle \sigma + \sigma^2 + \sigma^3 \rangle \cong \mathbb{Z}/2$.

7.1.2 The Lower Algebraic K-Theory of $\mathbb{Z}/6 \times \mathbb{Z}/2$

Alves and Ontaneda [AO06, Sect. 5.1] show that $Wh(\mathbb{Z}/6 \times \mathbb{Z}/2) = 0$. To compute $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2])$, consider as before the following Cartesian square:

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/6] & \longrightarrow & \mathbb{Z}[\mathbb{Z}/6] \\ \downarrow & & \downarrow \varphi_1 \\ \mathbb{Z}[\mathbb{Z}/6] & \xrightarrow[\varphi_2]{} & \mathbb{F}_2[\mathbb{Z}/6] \end{array}$$

where φ_i is reduction mod 2 for $i = 1, 2$. Denote by $u(\,)$ the group of units and set

$$u^*(\mathbb{Z}[\mathbb{Z}/6]) = \varphi_i\{u(\mathbb{Z}[\mathbb{Z}/6])\} \subset u(\mathbb{F}_2[\mathbb{Z}/6]), \qquad i = 1, 2.$$

Let $G = \mathbb{Z}/6 \times \mathbb{Z}/2$. Since the algebra $\mathbb{Q}G$ is commutative, no simple component of $\mathbb{Q}G$ is a totally definite quaternion algebra, and therefore $\mathbb{Q}G$ satisfies the Eichler condition (see Sect. 7.1.1) and, as before, by [RU74, Theorem 1.9], there is an exact sequence

$$1 \to u^*(\mathbb{Z}[\mathbb{Z}/6]) \to u(\mathbb{F}_2[\mathbb{Z}/6]) \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/6]) \to 2\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) \to 0.$$

Reiner and Ullom [RU74, Theorem 2.2] prove that $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0$. Therefore the exact sequence above yields the exact sequence

$$1 \to u^*(\mathbb{Z}[\mathbb{Z}/6]) \to u(\mathbb{F}_2[\mathbb{Z}/6]) \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/6]) \to 0.$$

Consequently,

$$\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/6]) \cong u(\mathbb{F}_2[\mathbb{Z}/6])/u^*(\mathbb{Z}[\mathbb{Z}/6]).$$

To compute $u(\mathbb{F}_2[\mathbb{Z}/6])$, let $R = \mathbb{F}_2[\mathbb{Z}/6]$. Since R is a finite ring, the canonical group homomorphism $u(R) = GL_1(R) \to K_1(R)$ is surjective (see [Bas64, Theorem 4.2(b)]) with kernel $V(R)$ generated by the set of all $V(x, y) = (1 + xy)(1 + yx)^{-1}$, with $x, y \in R$ and $1 + xy$ invertible (see [Va70, Theorem 3.6(b)]). Since R is a finite commutative ring, $V(R) = \{1\}$, and so $K_1(R) \cong u(R)$. Now using [Ma06, Theorem 4], we have that $K_1(R) = K_1(\mathbb{F}_2[\mathbb{Z}/6]) \cong K_1(\mathbb{F}_2[\mathbb{Z}/2 \times \mathbb{Z}/3]) \cong (\mathbb{Z}/2)^c \times K_1(\mathbb{F}_2[\mathbb{Z}/3])$, where c is the number of conjugacy classes in $\mathbb{Z}/3$. A direct calculation shows that $K_1(\mathbb{F}_2[\mathbb{Z}/3]) \cong \mathbb{Z}/3$ and $c = 3$; then it follows $K_1(\mathbb{F}_2[\mathbb{Z}/6]) \cong (\mathbb{Z}/2)^3 \times \mathbb{Z}/3$. (One can also show by a direct calculation that $u(\mathbb{F}_2[\mathbb{Z}/6]) = \langle \sigma^3, 1 + \sigma + \sigma^4, \sigma^2 + \sigma^3 + \sigma^5 \rangle \times \langle \sigma^2 \rangle \cong (\mathbb{Z}/2)^3 \times \mathbb{Z}/3$, where $|\sigma| = 6$.)

Next, to compute $u(\mathbb{Z}[\mathbb{Z}/6])$, let $R = \mathbb{Z}[\mathbb{Z}/6]$. Since R is a commutative ring, $K_1(R) \cong u(R) \oplus SK_1(R)$ (see [Mi71, p. 27]). Bass et al. in [BMS67, Proposition

4.14] show that $SK_1(\mathbb{Z}H) = 1$ if H is cyclic, so it follows that $K_1(R) \cong u(R)$. For $H = \mathbb{Z}/6$, it is known that $K_1(\mathbb{Z}H) \cong \mathbb{Z}/2 \times H^{ab} \cong \mathbb{Z}/2 \times \mathbb{Z}/6$ (see [O89]), therefore $u(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}/2 \times \mathbb{Z}/6$. This implies that $u(\mathbb{Z}[\mathbb{Z}/6])$ consists of only the trivial units. Therefore $u^*(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}/6$. Consequently

$$\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/6]) \cong u(\mathbb{F}_2[\mathbb{Z}/6])/u^*(\mathbb{Z}[\mathbb{Z}/6]) \cong (\mathbb{Z}/2)^2.$$

Next, we show $K_{-1}(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/6]) \cong \mathbb{Z}^3$. Carter [C80a] proved

$$K_{-1}(\mathbb{Z}G) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s$$

where

$$r = 1 - r_{\mathbb{Q}} + \sum_{p \,|\, |G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}) \tag{7.1}$$

where p is prime and s is the number of simple components A of $\mathbb{Q}G$ with even Schur index but with A_P of odd Schur index for each prime ideal P of the center of A that divides $|G|$ (see [LMO10]).

We first recall that the group algebra $\mathbb{Q}[\mathbb{Z}/6]$ decomposes into simple components as follows:

$$\mathbb{Q}[\mathbb{Z}/6] \cong \mathbb{Q}^2 \oplus \mathbb{Q}(\zeta_6)^2.$$

Since $\mathbb{Q}[\mathbb{Z}/6 \times \mathbb{Z}/2] \cong \mathbb{Q}[\mathbb{Z}/6] \oplus \mathbb{Q}[\mathbb{Z}/6]$, we see that the Schur indices of all the simple components in the Wedderburn decomposition of $\mathbb{Q}[\mathbb{Z}/6 \times \mathbb{Z}/2]$ are equal to 1, so $s = 0$. Carter's formula (above) now tells us that $K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2])$ is torsion-free, and, from Eq. (7.1), the rank is given by

$$r = 1 - r_{\mathbb{Q}} + (r_{\mathbb{Q}_2} - r_{\mathbb{F}_2}) + (r_{\mathbb{Q}_3} - r_{\mathbb{F}_3}). \tag{7.2}$$

We now proceed to compute the various terms appearing in the above expression.

Recall that for F a field of characteristic 0, r_F just counts the number of simple components in the Wedderburn decomposition of the group algebra $F[\mathbb{Z}/6 \times \mathbb{Z}/2]$. From the discussion in the previous paragraph, we have that

$$\mathbb{Q}[\mathbb{Z}/6 \times \mathbb{Z}/2] \cong \mathbb{Q}^4 \oplus \mathbb{Q}(\zeta_6)^4.$$

yielding $r_{\mathbb{Q}} = 8$. Now by tensoring the above splitting with $\mathbb{Q}_p$ with $p = 2$ and 3, we obtain:

$$\mathbb{Q}_p[\mathbb{Z}/6 \times \mathbb{Z}/2] \cong \mathbb{Q}_p^4 \oplus (\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q})(\zeta_6)^4 \cong \mathbb{Q}_p^4 \oplus \mathbb{Q}_p(\zeta_6)^4,$$

consequently for each of the primes $p = 2, 3$, we obtain that $r_{\mathbb{Q}_2} = r_{\mathbb{Q}_3} = 8$.

Next, we consider the situation over the finite fields $\mathbb{F}_2, \mathbb{F}_3$. We first recall that the integer $r_{\mathbb{F}_p}$ counts the number of $\mathbb{F}_p$-conjugacy classes of p-regular elements in G (an element $x \in G$ is called *p-regular* if p does not divide the order of x). The $\mathbb{F}_p$-conjugacy class of an element $x \in G$ is the union of ordinary conjugacy classes of certain specific powers of x, where the powers (of x) are calculated from the Galois extension $\mathbb{F}_p(\zeta_m)$ where m is the least common multiple of the orders of p-regular elements. Note that since the fields $\mathbb{F}_p$, $\mathbb{F}_p(\zeta_m)$ are finite, and $\mathrm{Aut}(\mathbb{F}_p(\zeta_m)/\mathbb{F}_p)$ is cyclic, generated by the p-power map (since $|\mathbb{F}_p| = p$), then $\mathrm{Gal}(\mathbb{F}_p(\zeta_m)/\mathbb{F}_p) = T_m = \langle \bar{p} \rangle \leq (\mathbb{Z}/m)^\times$ (viewed as elements of $(\mathbb{Z}/m)^\times$). We refer the reader to [LMO10, Sect. 3.1] for a more complete discussion of these points.

For $p = 2$, we note that an element in $\mathbb{Z}/6 \times \mathbb{Z}/2 \cong \langle t \rangle \times \langle \sigma \rangle$ is 2-regular precisely if it has order 1 or 3. There is a single conjugacy class of elements of order one (consisting of the identity element), and the elements of order 3 form *two* conjugacy classes inside $\mathbb{Z}/6 \times \mathbb{Z}/2$; representatives for these two conjugacy classes are given by $x = (t^2, 1)$, and by $x^2 = (t^4, 1)$. Note that there will be either one or two $\mathbb{F}_2$-conjugacy classes of elements of order 3. To determine the specific powers of x, recall that the powers of x are given by considering the Galois group of the extension $\mathbb{F}_2(\zeta_3)$, viewed as elements of $(\mathbb{Z}/3)^\times$. Since the Galois group is generated by the 2-power map (i.e., by squaring), we see that the Galois group is cyclic of order 2, given by the residue classes $\{\bar{1}, \bar{2}\} \subset (\mathbb{Z}/3)^\times$. In particular, since $\bar{2}$ lies in the Galois group, we see that x and x^2 lie in the same $\mathbb{F}_2$-conjugacy class, implying that there is a *unique* $\mathbb{F}_2$-conjugacy class of elements of order 3. We conclude that there are two $\mathbb{F}_2$-conjugacy classes of 2-regular elements, giving $r_{\mathbb{F}_2} = 2$.

For $p = 3$, the 3-regular elements in $\mathbb{Z}/6 \times \mathbb{Z}/2$ have order either 1 or 2. The elements of order 2 form *three* conjugacy classes inside $\mathbb{Z}/6 \times \mathbb{Z}/2$; representatives for these three conjugacy classes are given by $(t^3, 1)$, (t^3, σ) and by $(1, \sigma)$. To determined the specific powers of these elements, recall that the powers are given by considering the Galois group of the extension $\mathrm{Gal}\big(\mathbb{F}_3(\zeta_2)/\mathbb{F}_3\big)$, viewed as elements of $(\mathbb{Z}/2)^\times$. Since the Galois group is generated by the 3-power map, we see that $\mathrm{Gal}\big(\mathbb{F}_3(\zeta_2)/\mathbb{F}_3\big) = T_2 = \langle \bar{3} \rangle = \{\bar{1}\} \subset (\mathbb{Z}/2)^\times$. We conclude that for 3-regular elements of order 2, we clearly have *three* distinct (ordinary) conjugacy classes of elements of order 2; each of these ordinary conjugacy classes is also an $\mathbb{F}_3$-conjugacy class. Also we clearly have a unique $\mathbb{F}_3$-conjugacy class of elements of order one. We conclude that overall there are four $\mathbb{F}_3$-conjugacy classes of 3-regular elements, giving $r_{\mathbb{F}_3} = 4$.

We end by substituting our calculations into the expression given in Eq. (7.2) for the rank of $K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2])$, obtaining:

$$r = 1 - 8 + (8 - 2) + (8 - 4) = 3.$$

Therefore $K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$ as claimed.

7.1.3 The Lower Algebraic K-Theory of $A_4 \times \mathbb{Z}/2$

Alves and Ontaneda in [AO06, Lemma 5.4] show that $Wh(A_4 \times \mathbb{Z}/2) = 0$.

First, we show $\tilde{K}_0(\mathbb{Z}[A_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}/2$. To see this, let H be a subgroup of a group G. For any locally free $\mathbb{Z}G$-module M its restriction to H (denoted by M_H) is a locally free $\mathbb{Z}H$-module. The mapping defined by $[M] \to [M_H]$ gives a homomorphism of $\tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}H)$.

A group H is *hyper-elementary* if H is a semidirect product $N \rtimes P$ of a cyclic normal subgroup N and a subgroup P of prime power order, where $(|N|, |P|) = 1$. Let $\mathcal{H}(G)$ consist of one representative from each conjugacy class of hyper-elementary subgroups of G. We shall need the following result presented by Reiner and Ullom in [RU74, Thm. 3.1]: for every finite group G, the map

$$\tilde{K}_0(\mathbb{Z}G) \longrightarrow \prod_{H \in \mathcal{H}(G)} \tilde{K}_0(\mathbb{Z}H) \tag{7.3}$$

is a monomorphism. Observe that all the proper subgroups of the alternating group A_4 are hyper-elementary, therefore $\mathcal{H}(A_4) = \{\mathbb{Z}/2, \mathbb{Z}/3, D_2\}$. Also, note that the hyper-elementary subgroups of $G \times \mathbb{Z}/2$ are of the form H or $H \times \mathbb{Z}/2$ for $H \in \mathcal{H}(G)$. In particular, the hyper-elementary subgroups of $A_4 \times \mathbb{Z}/2$ are all isomorphic to one of: $\mathbb{Z}/2$, $\mathbb{Z}/3$, D_2, $\mathbb{Z}/3 \times \mathbb{Z}/2$ and $D_2 \times \mathbb{Z}/2$. By the results given in Table 7.1 we have $\tilde{K}_0(\mathbb{Z}H) = 0$, for all $H \in \mathcal{H}(A_4 \times \mathbb{Z}/2)$ except for $H = D_2 \times \mathbb{Z}/2$, where $\tilde{K}_0(\mathbb{Z}H) \cong \mathbb{Z}/2$. This implies that the target of the map given in (3) is isomorphic to $\mathbb{Z}/2$, and injectivity of the map now gives us an injection $\tilde{K}_0(\mathbb{Z}[A_4 \times \mathbb{Z}/2]) \hookrightarrow \mathbb{Z}/2$. Since it is known that $\tilde{K}_0(\mathbb{Z}[A_4 \times \mathbb{Z}/2])$ is non trivial (see [EH79, Thm., p. 161]), it follows that $\tilde{K}_0(\mathbb{Z}[A_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}/2$, as claimed.

Next, we show that $K_{-1}(\mathbb{Z}[A_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}$. Here once again, we use Carter's formula, given in Eq. (7.1). We start by first recalling (see [Se77, p. 93]) that the group algebra $\mathbb{Q}A_4$ decomposes into simple components as follows:

$$\mathbb{Q}A_4 \cong \mathbb{Q} \oplus \mathbb{Q}(\zeta_3) \oplus M_3(\mathbb{Q}).$$

Since $\mathbb{Q}[A_4 \times \mathbb{Z}/2] \cong \mathbb{Q}A_4 \oplus \mathbb{Q}A_4$, we see that the Schur indices of all the simple components in the Wedderburn decomposition of $\mathbb{Q}[A_4 \times \mathbb{Z}/2]$ are equal to 1. Carter's result [C80a] now tells us that $K_{-1}(\mathbb{Z}[A_4 \times \mathbb{Z}/2])$ is torsion-free, and from Eq. (7.1), the rank is given by

$$r = 1 - r_{\mathbb{Q}} + (r_{\mathbb{Q}_2} - r_{\mathbb{F}_2}) + (r_{\mathbb{Q}_3} - r_{\mathbb{F}_3}).$$

We now proceed to compute the various terms appearing in the above expression.

For F a field of characteristic 0, r_F just counts the number of simple components in the Wedderburn decomposition of the group algebra $F[A_4 \times \mathbb{Z}/2]$. From the discussion in the above paragraph, we have that

$$\mathbb{Q}[A_4 \times \mathbb{Z}/2] \cong \mathbb{Q}^2 \oplus \mathbb{Q}(\zeta_3)^2 \oplus M_3(\mathbb{Q})^2.$$

yielding $r_{\mathbb{Q}} = 6$. Now by tensoring the above splitting with $\mathbb{Q}_p$, $p = 2$ or 3, we obtain:

$$\mathbb{Q}_p[A_4 \times \mathbb{Z}/2] \cong \mathbb{Q}_p^2 \oplus \mathbb{Q}_p(\zeta_3) \oplus M_3\big(\mathbb{Q}_p\big)^2.$$

Therefore for each of the primes $p = 2, 3$, we obtain that $r_{\mathbb{Q}_2} = r_{\mathbb{Q}_3} = 6$.

Next let us consider the situation over the finite fields $\mathbb{F}_2, \mathbb{F}_3$.

For $p = 2$, we note that elements in $A_4 \times \mathbb{Z}/2$ are 2-regular precisely if they have order 1 or 3. There is a single conjugacy class of elements of order one (the identity element). The elements of order 3 form a single conjugacy class inside $A_4 \times \mathbb{Z}/2$. We conclude that there are two $\mathbb{F}_2$-conjugacy classes of 2-regular elements, giving $r_{\mathbb{F}_2} = 2$.

For $p = 3$, the elements in $A_4 \times \mathbb{Z}/2$ which are 3-regular have order 1 or 2. Here we look at the Galois group associated to the field extension $\mathbb{F}_3(\zeta_2)$. Elements in the Galois group are generated by the third power, giving us that $\mathrm{Gal}(\mathbb{F}_3(\zeta_2)/\mathbb{F}_3) = \{\bar{1}\} \subset (\mathbb{Z}/2^{\times})$. Now we clearly have a unique $\mathbb{F}_3$-conjugacy class of elements of order one. For elements of order 2, there are *three* distinct (ordinary) conjugacy classes of elements of order two; each of these ordinary conjugacy classes is also an $\mathbb{F}_3$-conjugacy class. We conclude that overall there are four $\mathbb{F}_3$-conjugacy classes of 3-regular elements, giving $r_{\mathbb{F}_3} = 4$.

To conclude, we substitute our calculations into the expression in Eq. (7.2) for the rank of $K_{-1}(\mathbb{Z}[A_4 \times \mathbb{Z}/2])$, giving us:

$$r = 1 - 6 + (6 - 2) + (6 - 4) = 1.$$

Therefore $K_{-1}(\mathbb{Z}[A_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}$ as claimed.

7.2 The Homology of $E_{\mathcal{FIN}}(\Gamma)$

In the remainder of this chapter, we will compute the generalized homology groups $H_*^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$ for the 73 split crystallographic groups Γ.

Lemma 7.1. *Let Γ be a split three-dimensional crystallographic group. Let $\mathcal{V}_\Gamma$ consist of a selection of one non-negligible vertex from each Γ-orbit of non-negligible vertices in $E_{\mathcal{FIN}}(\Gamma)$. We have the isomorphisms:*

$$H_{-1}^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong E^2_{0,-1},$$

$$H_0^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong \bigoplus_{\sigma^0 \in \mathcal{V}_\Gamma} \tilde{K}_0(\mathbb{Z}F_{\sigma^0}) \oplus E^2_{1,-1},$$

$$H_1^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong 0.$$

Moreover, if all of the edges in $E_{\mathcal{FIN}}(\Gamma)$ have negligible stabilizer groups, then $E^2_{1,-1} = 0$ and

$$E^2_{0,-1} \cong \bigoplus_{\sigma^0 \in \mathcal{V}_\Gamma} K_{-1}(\mathbb{Z}F_{\sigma^0}).$$

Proof. The lemma follows easily from the remarks about the Quinn spectral sequence that were given in the introduction of this chapter.

We note that the possible contributions to $H_1^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$ must come from the $E^2_{0,1}$ and $E^2_{1,0}$ terms. It follows from Table 7.1 that $E^2_{0,1} \cong 0$ for all split crystallographic groups Γ. The $E^2_{1,0}$ terms come from $\tilde{K}_0(\mathbb{Z}[F_\sigma])$, where the σs are edges. The latter K-groups are always trivial, so $E^2_{1,0} \cong 0$. This proves that $H_1^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong 0$.

The isomorphism

$$H_0^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong \bigoplus_{\sigma^0 \in \mathcal{V}_\Gamma} \tilde{K}_0(\mathbb{Z}F_{\sigma^0}) \oplus E^2_{1,-1}$$

follows from the fact that one or the other of the factors

$$E^2_{0,0} \cong \bigoplus_{\sigma^0 \in \mathcal{V}_\Gamma} \tilde{K}_0(\mathbb{Z}F_{\sigma^0}) \quad \text{and} \quad E^2_{1,-1}$$

is always zero. Indeed, all edges have negligible stabilizers (so in particular $E^2_{1,-1} \cong 0$), except in the five cases that are treated in Examples 7.4–7.8. We have non-vanishing $E^2_{1,-1}$ in only two of those cases, namely Examples 7.6 and 7.8; in both of these cases, the $E^2_{0,0}$ term vanishes.

Remark 7.2. Lemma 7.1 makes the computation of $H_*^{\Gamma_i}(E_{\mathcal{FIN}}(\Gamma_i); \mathbb{KZ}^{-\infty})$ relatively straightforward, for $i \neq 5$. Indeed, Theorems 6.3, 6.4, 6.5, 6.6, 6.8, and 6.9, imply that all of the edges in $E_{\mathcal{FIN}}(\Gamma_i)$ $(i \neq 5)$ are negligible. We can let $\mathcal{V}_{\Gamma_i}$ be the set of vertices listed in the appropriate theorem (above), and then determine the isomorphism type of the group $H_*^{\Gamma_i}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$ from Lemma 7.1 and Table 7.1. We will sketch this procedure in Examples 7.1–7.3.

Procedure 7.1. *Let Γ and Γ' be split crystallographic groups, where Γ' has finite index in Γ. We describe a procedure for computing $H_*^{\Gamma'}(E_{\mathcal{FIN}}(\Gamma'); \mathbb{KZ}^{-\infty})$, assuming that the calculation of $H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$ has been done.*

1. *Select one cell from each Γ-orbit of non-negligible (relative to Γ) cells in $E_{\mathcal{FIN}}(\Gamma)$; call the resulting set C_Γ.*
2. *Let Γ' act on $E_{\mathcal{FIN}}(\Gamma)$ (which is clearly a model for $E_{\mathcal{FIN}}(\Gamma')$). Since the class of negligible groups is closed under passage to subgroups (see Definition 6.6), the only cells $\sigma \subseteq E_{\mathcal{FIN}}(\Gamma)$ such that Γ'_σ is non-negligible must be in the set $\Gamma \cdot C_\Gamma$. We can write $\Gamma = \Gamma' T$, where T is a right transversal for Γ' in Γ.*

Thus, $\Gamma \cdot C_\Gamma = \Gamma' \cdot (T \cdot C_\Gamma)$. (Note that $T \cdot C_\Gamma$ is finite when C_Γ is.) We choose a single cell from each Γ'-orbit that meets $T \cdot C_\Gamma$. Call the resulting set $\hat{C}_{\Gamma'}$. We note that if a cell $\sigma \subseteq E_{\mathcal{FIN}}(\Gamma)$ has the property that Γ'_σ is non-negligible, then σ is in $\Gamma' \cdot \hat{C}_{\Gamma'}$.

3. *It is still possible that $\hat{C}_{\Gamma'}$ contains cells σ such that Γ'_σ is negligible. We therefore recompute the cell stabilizer Γ'_σ for each $\sigma \in \hat{C}_{\Gamma'}$, removing σ from our list if Γ'_σ is negligible. The result is the desired list of cells $C_{\Gamma'}$ (which contains a single cell from each Γ'-orbit of cells σ such that Γ'_σ is non-negligible, and no cells from any other orbits).*
4. *If $C_{\Gamma'}$ consists entirely of vertices (i.e., if all of the edge stabilizers in $E_{\mathcal{FIN}}(\Gamma')$ are negligible), then we can determine $H_*^{\Gamma'}(E_{\mathcal{FIN}}(\Gamma'); \mathbb{KZ}^{-\infty})$ from Lemma 7.1 and Table 7.1. If there are non-negligible edge stabilizers, then we apply the quotient map $p : E_{\mathcal{FIN}}(\Gamma) \to \Gamma' \backslash E_{\mathcal{FIN}}(\Gamma)$ to the cells in $C_{\Gamma'}$. We note that all of the cells in $C_{\Gamma'}$ are either 0- or one-dimensional, so the image is a graph. We can then use the graph $p(C_{\Gamma'})$ to compute the $E^2_{0,-1}$ and $E^2_{1,-1}$ terms. (The remainder of the calculation is straightforward.)*

In Sect. 7.3, we will give examples to illustrate Procedure 7.1, and provide tables listing cell stabilizer information for all of the split crystallographic groups with non-negligible point groups. Table 7.8 summarizes all of the non-zero isomorphism types of the groups $H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$.

7.3 Calculations of $H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$

Let us compute some of the groups $H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$, using Procedure 7.1.

Example 7.1. Consider first the group Γ_1, which is generated by the standard cubical lattice L and the point group $S_4^+ \times (-1)$. Theorem 6.3 showed that all of the edges in $E_{\mathcal{FIN}}(\Gamma_1)$ are negligible relative to the action of Γ_1. The non-negligible vertex stabilizers are described in Table 7.2. Note that the columns of Table 7.2 are labelled by point groups, and the rows are labelled by split crystallographic groups (see Remark 4.2 for a guide to our labelling convention). If a 3-tuple $(a, b, c) \in \mathbb{R}^3$ appears in a box in the table, then the corresponding column heading is $\pi(\Gamma_{(a,b,c)})$, where Γ is the crystallographic group labelling the row, and $\pi : \Gamma \to H$ is the usual projection. Thus, for instance, the first entry in Table 7.2 tells us that $\pi(\Gamma_{(0,0,0)})$ and $\pi(\Gamma_{(1/2,1/2,1/2)})$ are both $S_4^+ \times (-1)$, where we have written Γ in place of Γ_1 (the label of that row).

We easily see from Table 7.2 that there are four vertices in C_{Γ_1}. Two of the vertices have stabilizer groups isomorphic to $S_4 \times \mathbb{Z}/2$, and two have stabilizer groups isomorphic to $D_4 \times \mathbb{Z}/2$. By Lemma 7.1 and Table 7.1, we get

$$H_{-1}^{\Gamma_1}(E_{\mathcal{FIN}}(\Gamma_1); \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^2;$$

$$H_0^{\Gamma_1}(E_{\mathcal{FIN}}(\Gamma_1); \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}/4)^4;$$

Table 7.2 Cell stabilizers in $E_{\mathcal{FIN}}(\Gamma_1)$

	$S_4^+ \times (-1)$	$D_{4_1}^+ \times (-1)$	$A_4^+ \times (-1)$	$D_4^+ \times (-1)$	$D_2^+ \times (-1)$	$C_4^+ \times (-1)$
Γ_1	$(0,0,0)$	$(\frac{1}{2},0,0)$		$(\frac{1}{2},\frac{1}{2},0)$		
	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$					
$(A_4^+ \times (-1))_1$			$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$		$(\frac{1}{2},0,0)$	
			$(0,0,0)$		$(\frac{1}{2},\frac{1}{2},0)$	
$(D_4^+ \times (-1))_1$				$(0,0,0)$	$(\frac{1}{2},0,0)$	
				$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2},\frac{1}{2})$	
				$(0,0,\frac{1}{2})$		
				$(\frac{1}{2},\frac{1}{2},0)$		
$(D_2^+ \times (-1))_1$					(8)	
$(C_4^+ \times (-1))_1$						$(0,0,0)$
						$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
						$(0,0,\frac{1}{2})$
						$(\frac{1}{2},\frac{1}{2},0)$

The "(8)" is an abbreviation for the collection of all 3-tuples (a, b, c) such that $a, b, c \in \{0, 1/2\}$

(We note that, here and in all of the other cases, $H_1^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong 0$.) These calculations are recorded in the first row of Table 7.8.

Example 7.2. Next, we follow Procedure 7.1 with $\Gamma' = (A_4^+ \times (-1))_1$, and $\Gamma = \Gamma_1$. A right transversal T for Γ' in Γ is as follows:

$$\left\{ \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix} \right\}.$$

We can let C_Γ be the collection of vertices from the first row of Table 7.2. We note that $T \cdot C_\Gamma$ contains five vertices: the four vertices from C_Γ, and $(0, 1/2, 0)$. We now choose a cell from each Γ'-orbit meeting $T \cdot C_\Gamma$; we can simply choose $\hat{C}_{\Gamma'} = C_\Gamma$, since the new vertex $(0, 1/2, 0)$ is in the Γ'-orbit of $(1/2, 0, 0) \in C_\Gamma$. The next step is to compute the stabilizer groups Γ'_v of the vertices $v \in \hat{C}_{\Gamma'}$. This is straightforward, and amounts to computing the intersections of

$$S_4^+ \times (-1), \quad S_4^+ \times (-1), \quad D_{4_1}^+ \times (-1), \quad \text{and} \quad D_4^+ \times (-1)$$

with $A_4^+ \times (-1)$ (respectively). As a result, we see two vertices v (namely $(0, 0, 0)$ and $(1/2, 1/2, 1/2)$) such that $\pi(\Gamma'_v) = A_4^+ \times (-1)$, and two vertices such that $\pi(\Gamma'_v) = D_2^+ \times (-1)$. All of these vertices are non-negligible, so $C_{\Gamma'} = \hat{C}_{\Gamma'}$. The four vertices are recorded in the second row of Table 7.2, in the appropriate columns. It now follows from Lemma 7.1 and Table 7.1 that

$$H_{-1}^{\Gamma'}(E_{\mathcal{FIN}}(\Gamma'); \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^2;$$
$$H_0^{\Gamma'}(E_{\mathcal{FIN}}(\Gamma'); \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}/2)^4.$$

These calculations are recorded in Table 7.8.

Example 7.3. Now we follow the same procedure with $\Gamma = (A_4^+ \times (-1))_1$ and $\Gamma' = (D_2^+ \times (-1))_1$. We can let $T = C_3^+$. (Recall that C_3^+ is the group of matrices that cyclically permute the coordinates—see Fig. 2.1.) We can let C_Γ be the same collection of four vertices from the second row of Table 7.2. Applying T, we find

$$T \cdot C_\Gamma = \{(a, b, c) \in \mathbb{R}^3 \mid a, b, c \in \{0, 1/2\}\}.$$

Next, we must choose one cell from each Γ'-orbit that meets $T \cdot C_\Gamma$. In fact, all of the elements of $T \cdot C_\Gamma$ are easily seen to be in distinct Γ'-orbits. We can therefore set $\hat{C}_{\Gamma'} = T \cdot C_\Gamma$. The next step is to compute the groups $\pi(\Gamma'_v)$ for the vertices in $\hat{C}_{\Gamma'}$. The best approach here may be to use Lemma 6.1(1). We note that, for $h \in D_2^+ \times (-1)$, $(a, b, c) - h \cdot (a, b, c) = (a', b', c')$, where x' is either 0 or $2x$, for $x \in \{a, b, c\}$. It follows from Lemma 6.1(1) that $\pi(\Gamma'_v) = D_2^+ \times (-1)$ for each $v \in \hat{C}_{\Gamma'}$. This implies that each member of the latter set is non-negligible, so we can let $C_{\Gamma'} = \hat{C}_{\Gamma'}$. It now follows from Lemma 7.1 and Table 7.1 that

$$H_{-1}^{\Gamma'}(E_{\mathcal{FIN}}(\Gamma'); \mathbb{KZ}^{-\infty}) \cong 0;$$
$$H_0^{\Gamma'}(E_{\mathcal{FIN}}(\Gamma'); \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}/2)^8,$$

as recorded in Table 7.8.

Examples 7.1–7.3 illustrate the general pattern in "easy" cases—those in which the edge stabilizers are negligible. Note that, given Γ and Γ', a choice of a right transversal for Γ' in Γ is made during the application of Procedure 7.1. Of course, in a given case, several choices of transversal are possible, and these could easily give us vertices (or edges) that are different from the ones recorded in Tables 7.2, 7.3, 7.4, 7.5, 7.6, and 7.7. (In fact, we choose orbit representatives,

Table 7.3 Cell stabilizers in $E_{\mathcal{FIN}}(\Gamma_2)$

	$D_4^+ \times (-1)$	$D_3^+ \times (-1)$	$D_2^+ \times (-1)$	$C_4^+ \times (-1)$	$C_3^+ \times (-1)$
Γ_2	$(\frac{1}{2}, 0, 0)^\dagger$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$			
$(A_4^+ \times (-1))_2$			$(\frac{1}{2}, 0, 0)$		$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
$(D_4^+ \times (-1))_2$	$(0, 0, 0)$		$(\frac{1}{2}, 0, 0)$		
	$(0, 0, \frac{1}{2})$				
$(D_2^+ \times (-1))_2$			$(0, 0, 0)$		
			$(\frac{1}{2}, 0, 0)$		
			$(0, \frac{1}{2}, 0)$		
			$(0, 0, \frac{1}{2})$		
$(C_4^+ \times (-1))_2$				$(0, 0, 0)$	
				$(0, 0, \frac{1}{2})$	

Note that the dagger (†) indicates a vertex with the stabilizer group $D_{4_1} \times (-1)$. We have also omitted two entries for formatting reasons. The first row should have the origin $(0, 0, 0)$ listed with stabilizer group $S_4^+ \times (-1)$, and the second row should list the origin with the stabilizer group $A_4^+ \times (-1)$

Table 7.4 Cell stabilizers in $E_{\mathcal{FIN}}(\Gamma_3)$ and $E_{\mathcal{FIN}}(\Gamma_4)$

	$S_4^+ \times (-1)$	$A_4^+ \times (-1)$	$D_2^+ \times (-1)$	$D_2 \times (-1)$
Γ_3	$(0,0,0)$			$(\frac{1}{4}, \frac{1}{4}, 0)$
	$(\frac{1}{2}, 0, 0)$			
$(A_4^+ \times (-1))_3$		$(0,0,0)$		
		$(\frac{1}{2}, 0, 0)$		
$(D_2^+ \times (-1))_3$			$(0,0,0)$	
			$(\frac{1}{2}, 0, 0)$	
Γ_4			$(0,0,0)$	
			$(0, \frac{1}{2}, \frac{1}{2})$	
			$(0, 0, \frac{1}{2})$	
			$(0, \frac{1}{2}, 0)$	

Table 7.5 Cell stabilizers in $E_{\mathcal{FIN}}(\Gamma_6)$, and $E_{\mathcal{FIN}}(\Gamma_7)$

	$D_3^+ \times (-1)$	$C_3^+ \times (-1)$
Γ_6	$(0,0,0)$	
	$(\frac{5}{6}, \frac{-1}{6}, \frac{-1}{6})$	
$(C_3^+ \times (-1))_6$		$(0,0,0)$
		$(\frac{5}{6}, \frac{-1}{6}, \frac{-1}{6})$
Γ_7	$(0,0,0)$	
	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	

Table 7.6 Cell stabilizers in $E_{\mathcal{FIN}}(\Gamma_5)$ (part I)

	D_6'	D_6''	$\hat{D}_6'$	$C_6^+ \times (-1)$
$(D_6')_5$	$(0,0,0)$			
	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$			
	$(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$			
	$(\frac{5}{6}, -\frac{1}{6}, \frac{5}{6})$			
	$(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$			
	$(-\frac{5}{6}, \frac{1}{6}, -\frac{5}{6})$			
$(D_6'')_5$		$(0,0,0)$		
		$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		
		$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})^*$		
		$(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})^*$		
$(\hat{D}_6')_5$			$(0,0,0)$	
			$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	
$(C_6^+ \times (-1))_5$				$(0,0,0)$
				$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

An asterisk denotes the stabilizer of an edge, where the coordinates indicate the edge's midpoint. For the cell stabilizers in Γ_5 itself, we refer the reader to Theorem 6.7. Note that this table and Table 7.7 should be considered two halves of the same table (which was split in two purely for the sake of better formatting)

Table 7.7 Cell stabilizers in $E_{\mathcal{FIN}}(\Gamma_5)$ (part II)

	D_6^+	$D_3^+ \times (-1)$	C_6^+	C_6'	$C_3^+ \times (-1)$
$(\hat{D}_6')_5$				$(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$	
				$(\frac{5}{6}, \frac{-1}{6}, \frac{5}{6})$	
$(C_6^+ \times (-1))_5$			$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})^*$	$(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3})$	
				$(\frac{5}{6}, \frac{-1}{6}, \frac{5}{6})$	
$(D_6^+)_5$	$(0, 0, 0)$		$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})^*$		
	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$				
$(D_3^+ \times (-1))_5$		$(0, 0, 0)$			
		$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$			
$(C_6^+)_5$			$(0, 0, 0)$		
			$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		
			$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})^*$		
			$(\frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4})^*$		
$(C_6')_5$				$(0, 0, 0)$	
				$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	
				$(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3})$	
				$(\frac{5}{6}, \frac{-1}{6}, \frac{5}{6})$	
				$(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3})$	
				$(\frac{-5}{6}, \frac{1}{6}, \frac{-5}{6})$	
$(C_3^+ \times (-1))_5$				$(0, 0, 0)$	
				$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	

An asterisk indicates an edge stabilizer, where the coordinates are for the midpoint of the edge

too, and this could also give us different cells.) The calculation of homology is unaffected, however, since the resulting cells are necessarily in the same Γ'-orbits no matter which transversal is selected. In practice, we have always chosen the groups Γ and Γ' in such a way that $[\Gamma : \Gamma'] \leq 3$. We also favored certain transversals, such as groups of permutation matrices (as in the above examples), or the group generated by the antipodal map (when applicable).

Note also that most of the split crystallographic groups in Table 4.1 have negligible point groups, which directly implies that the homology groups in question are 0. This observation reduces the number of split crystallographic groups that must be considered.

There are only five "hard" cases. We consider these next.

Example 7.4 (The Case of Γ_5). We would like to compute the generalized homology groups $H_*^{\Gamma_5}(E_{\mathcal{FIN}}(\Gamma_5); \mathbb{KZ}^{-\infty})$. We work with the vertices and edges from the statement of Theorem 6.7. There is only one non-negligible edge, which connects $(0, 0, 0)$ to $(1/2, 1/2, 1/2)$. The only part of the calculation that is not completely straightforward is the calculation of the $E^2_{0,-1}$ and $E^2_{1,-1}$ terms; we compute these from the complex described in Fig. 7.1.

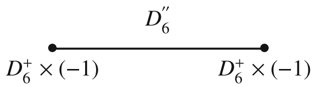


Fig. 7.1 The non-negligible edge in $\Gamma_5 \backslash E_{\mathcal{FIN}}(\Gamma_5)$. In addition to the cells pictured, there are 2 isolated vertices with stabilizer subgroup $D_2 \times \mathbb{Z}/2$ and 2 other isolated vertices with stabilizer subgroup D_6

We compute the homology of the chain complex:

$$0 \to \bigoplus_{\sigma^1 \in \mathcal{C}_{\Gamma_5}} K_{-1}(\mathbb{Z}\Gamma_{\sigma^1}) \to \bigoplus_{\sigma^0 \in \mathcal{C}_{\Gamma_5}} K_{-1}(\mathbb{Z}\Gamma_{\sigma^0}) \to 0,$$

By Fig. 7.1 and Table 7.1, the latter chain complex becomes:

$$0 \to K_{-1}(\mathbb{Z}D_6) \xrightarrow{\rho} (K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \oplus K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2])) \oplus 2K_{-1}(\mathbb{Z}D_6) \to 0.$$

The morphism ρ is determined by the map $K_{-1}(\mathbb{Z}D_6) \to K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2])$. We get $K_{-1}(\mathbb{Z}D_6) \cong \mathbb{Z}$ and $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$ by Table 7.1. We claim that the map induced by the natural inclusion $D_6 \hookrightarrow D_6 \times \mathbb{Z}/2$ is injective, and the quotient group is isomorphic to $\mathbb{Z}^2$. In order to see this, we merely note that there is a retraction from $D_6 \times \mathbb{Z}/2$ to the subgroup D_6, and hence we must have that $K_{-1}(\mathbb{Z}D_6) \cong \mathbb{Z}$ is a summand inside $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$. This immediately gives the following isomorphisms:

$$E^2_{1,-1} \cong 0;$$
$$E^2_{0,-1} \cong \mathbb{Z}^7.$$

Combining these isomorphisms with Lemma 7.1, Table 7.1, and the list of cells from Fig. 7.1, we arrive at the calculation of $H_*^{\Gamma_5}(E_{\mathcal{FIN}}(\Gamma_5); \mathbb{KZ}^{-\infty})$ recorded in Table 7.8.

Example 7.5 (The Case of $(C_6^+ \times (-1))_5$). We now consider the group $\Gamma = (C_6^+ \times (-1))_5$. We will assume that Procedure 7.1 has been followed up to (4). There are a total of 5 cells in $\mathcal{C}_\Gamma$, as recorded in Tables 7.6 and 7.7. If we apply the quotient map p to the cells $\mathcal{C}_\Gamma$, we get the complex described in Fig. 7.2.

As before, we want to compute the $E^2_{0,-1}$ and $E^2_{1,-1}$ terms. The chain complex

$$0 \to \bigoplus_{\sigma^1 \in \mathcal{C}_{\Gamma}} K_{-1}(\mathbb{Z}\Gamma_{\sigma^1}) \to \bigoplus_{\sigma^0 \in \mathcal{C}_{\Gamma}} K_{-1}(\mathbb{Z}\Gamma_{\sigma^0}) \to 0$$

becomes

$$0 \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \xrightarrow{\rho} 2K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2]) \oplus 2K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \to 0.$$

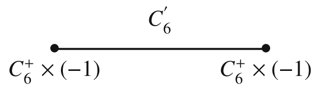


Fig. 7.2 The graph $p(C_\Gamma)$, for $\Gamma = (C_6^+ \times (-1))_5$. There are two isolated vertices (not pictured), both of which have stabilizer groups isomorphic to $\mathbb{Z}/6$

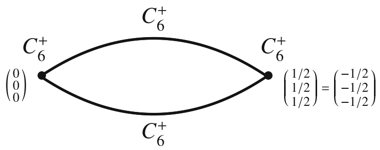


Fig. 7.3 The graph $p(C_\Gamma)$, for $\Gamma = (C_6^+)_5$. In this case, there are no isolated vertices

The argument follows the pattern from Example 7.4. Note that the morphism ρ is determined by the map $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2])$. We get $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}$ and $K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$ by Table 7.1. We claim that the map induced by the natural inclusion $\mathbb{Z}/6 \hookrightarrow \mathbb{Z}/6 \times \mathbb{Z}/2$ is injective, and the quotient group is isomorphic to $\mathbb{Z}^2$. There is a retraction from $\mathbb{Z}/6 \times \mathbb{Z}/2$ to the subgroup $\mathbb{Z}/6$, and so $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}$ is a summand inside $K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}/2]) \cong \mathbb{Z}^3$. It follows easily that

$$E^2_{1,-1} \cong 0;$$
$$E^2_{0,-1} \cong \mathbb{Z}^7.$$

This (with Lemma 7.1) directly leads to the calculation that is recorded in Table 7.8.

Example 7.6 (The Case of $(C_6^+)_5$). We set $\Gamma = (C_6^+)_5$. The quotient $p(C_\Gamma)$ appears in Fig. 7.3.

The chain complex for computing $E^2_{0,-1}$ and $E^2_{1,-1}$ is as follows:

$$0 \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \xrightarrow{\rho} K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \to 0.$$

Since $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}$, the latter complex amounts to the following:

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\rho} \mathbb{Z} \oplus \mathbb{Z} \to 0,$$

where the map ρ sends $(0,1) \mapsto (1,-1)$ and $(1,0) \mapsto (-1,1)$. It follows that $\ker(\rho) = \langle (1,1) \rangle \cong \mathbb{Z}$ and $\mathrm{im}(\rho) = \langle (1,-1) \rangle$. Therefore

$$E^2_{0,-1} \cong \mathbb{Z}, \text{ and } E^2_{1,-1} \cong \mathbb{Z}.$$

Since $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0$, this and Lemma 7.1 directly gives us the calculation that is recorded in Table 7.8.

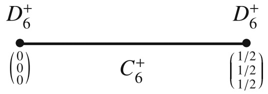


Fig. 7.4 The graph $p(C_\Gamma)$ for $\Gamma = (D_6^+)_5$. There are no isolated vertices

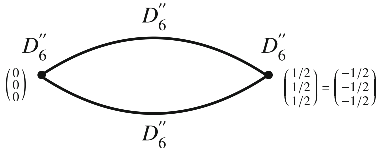


Fig. 7.5 The graph $p(C_\Gamma)$, for $\Gamma = (D_6'')_5$. There are no isolated vertices

Example 7.7 (The Case of $(D_6^+)_5$). In this case, the graph $p(C_\Gamma)$ is simply an edge—see Fig. 7.4.

The chain complex for computing $E^2_{p,-1}$ is as follows:

$$0 \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \xrightarrow{\rho} K_{-1}(\mathbb{Z}D_6) \oplus K_{-1}(\mathbb{Z}D_6) \to 0.$$

Since $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong K_{-1}(\mathbb{Z}[D_6]) \cong \mathbb{Z}$, the complex can be written

$$0 \to \mathbb{Z} \xrightarrow{\rho} \mathbb{Z} \oplus \mathbb{Z} \to 0,$$

where the map ρ sends $1 \mapsto (1,-1)$. It follows that $\ker(\rho)$ is trivial, and $im(\rho) = \langle(1,-1)\rangle \cong \mathbb{Z}$. Therefore

$$E^2_{0,-1} \cong \mathbb{Z}, \text{ and } E^2_{1,-1} \cong 0.$$

Since $\tilde{K}_0(\mathbb{Z}F) = 0$ for $F \in \{\mathbb{Z}/6, D_6\}$ (see Table 7.1), we immediately get the calculation that is recorded in Table 7.8.

Example 7.8 (The Case of $(D_6'')_5$). This case follows the exact pattern of Example 7.6. The quotient $p(C_\Gamma)$ is pictured in Fig. 7.5.

The complex for computing $E^2_{p,-1}$ is:

$$0 \to K_{-1}(\mathbb{Z}[D_6]) \oplus K_{-1}(\mathbb{Z}[D_6]) \xrightarrow{\rho} K_{-1}(\mathbb{Z}[D_6]) \oplus K_{-1}(\mathbb{Z}[D_6]) \to 0.$$

Since $K_{-1}(\mathbb{Z}D_6) \cong \mathbb{Z}$, we get

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\rho} \mathbb{Z} \oplus \mathbb{Z} \to 0,$$

Table 7.8 The homology groups $H_*^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty})$

Γ	$K_{-1} \neq 0$	$\tilde{K}_0 \neq 0$
Γ_1	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^4$
$(A_4^+ \times (-1))_1$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^4$
$(D_4^+ \times (-1))_1$		$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^4$
$(D_2^+ \times (-1))_1$		$(\mathbb{Z}/2)^8$
$(C_4^+ \times (-1))_1$		$(\mathbb{Z}/2)^4$
Γ_2	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^2$
$(A_4^+ \times (-1))_2$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^2$
$(D_4^+ \times (-1))_2$		$\mathbb{Z}/2 \oplus (\mathbb{Z}/4)^2$
$D_2^+ \times (-1))_2$		$(\mathbb{Z}/2)^4$
$(C_4^+ \times (-1))_2$		$(\mathbb{Z}/2)^2$
Γ_3	$\mathbb{Z}^2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/4)^2$
$(A_4^+ \times (-1))_3$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^2$
$(D_2^+ \times (-1))_3$		$(\mathbb{Z}/2)^2$
Γ_4		$(\mathbb{Z}/2)^4$
Γ_5	$\mathbb{Z}^7$	$(\mathbb{Z}/2)^6$
$(D_6^+)_5$	$\mathbb{Z}$	
$(C_6')_5$	$\mathbb{Z}^6$	
$(C_6^+ \times (-1))_5$	$\mathbb{Z}^7$	$(\mathbb{Z}/2)^4$
Γ	$K_{-1} \neq 0$	$\tilde{K}_0 \neq 0$
$(D_6')_5$	$\mathbb{Z}^6$	
$(C_6^+)_5$	$\mathbb{Z}$	$\mathbb{Z}$
$(D_3^+ \times (-1))_5$	$\mathbb{Z}^2$	
$(\hat{D}_6')_5$	$\mathbb{Z}^4$	
$(C_3^+ \times (-1))_5$	$\mathbb{Z}^2$	
$(D_6'')_5$	$\mathbb{Z}$	$\mathbb{Z}$
Γ_6	$\mathbb{Z}^2$	
$(C_3^+ \times (-1))_6$	$\mathbb{Z}^2$	
Γ_7	$\mathbb{Z}^2$	

where the map ρ sends $(0, 1) \mapsto (1, -1)$ and $(1, 0) \mapsto (-1, 1)$. It follows that $\ker(\rho) = \langle (1, 1) \rangle \cong \mathbb{Z}$ and $im(\rho) = \langle (1, -1) \rangle \cong \mathbb{Z}$. Therefore

$$E^2_{0,-1} \cong \mathbb{Z}, \text{ and } E^2_{1,-1} \cong \mathbb{Z}.$$

Since $\tilde{K}_0(\mathbb{Z}D_6) = 0$, these terms represent the only contributions to K-theory. See Table 7.8.

Chapter 8
Fundamental Domains for Actions on Spaces of Planes

Theorem 5.1 showed that the lower algebraic K-theory of any crystallographic group can be computed in two pieces. In Chap. 7, we completed the first half of the computation for the 73 split three-dimensional crystallographic groups. The results obtained are summarized in Table 7.8.

Next, according to Theorem 5.1, we must calculate the equivariant homology groups $H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty})$, where $\hat{\ell} \in \mathcal{T}''$. In the current chapter, we will determine which groups $H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty})$ contribute to the lower algebraic K-theory of the seven maximal crystallographic groups Γ_i, $(i = 1, \ldots, 7)$. We will describe fundamental domains for the actions of the groups Γ_i on their associated spaces of lines $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. Our arguments here are generally analogous to the ones from Chap. 6, and the organization of this chapter is similar.

8.1 Negligible Line Stabilizer Groups

As in Chap. 6, the notion of a negligible stabilizer group will be very useful. We will want to adapt the old definition (Definitions 5.3 and 6.6) to our needs in the current chapter.

The main result of this section is Proposition 8.3, which greatly simplifies the remainder of the computation of the lower algebraic K-groups of the split crystallographic groups.

Definition 8.1. Let Γ be a split crystallographic group. Let $\ell \subseteq \mathbb{R}^3$ be a line. We let $\overline{\Gamma}_\ell = \{\gamma \in \Gamma \mid \gamma_{|\ell} = \text{id}_\ell\}$. We will sometimes call this group the *strict stabilizer group of* ℓ, to distinguish it from the stabilizer group $\Gamma_\ell = \{\gamma \in \Gamma \mid \gamma \cdot \ell = \ell\}$.

If G is an infinite virtually cyclic subgroup of Γ, then $G = \Gamma_\ell$ for some line $\ell \subseteq \mathbb{R}^3$. We then let $\overline{G}$ denote $\overline{\Gamma}_\ell$.

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_8

Definition 8.2. An infinite virtually cyclic subgroup $G \leq \Gamma$ is called *negligible* if the finite subgroup F from Remark 5.1 has square-free order. We will also say that $\overline{G}$ is negligible, and that σ is negligible, if G is the stabilizer of σ.

Proposition 8.1. *If G is infinite virtually cyclic and $G \cong F \rtimes_\alpha \mathbb{Z}$ or $G \cong G_1 *_F G_2$, where F has square-free order, then $NK_q(\mathbb{Z}F, \alpha)$ or $NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F])$ (respectively) is trivial for all $q \leq 1$.*

Proof. If G has the first form, then results of [Ha87] and [J-PR09] show that $NK_q(\mathbb{Z}F, \alpha)$ is trivial for $q \leq 1$. If G has the second form, then the canonical index two subgroup of G has the form $F \rtimes_\alpha \mathbb{Z}$, so the corresponding group $NK_q(\mathbb{Z}F, \alpha)$ is trivial for $q \leq 1$, as above. Results of Lafont and Ortiz [LO08] show that when $NK_q(\mathbb{Z}F, \alpha)$ is trivial, so is $NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F])$.

Remark 8.1. If an infinite virtually cyclic group G is *negligible* in the sense of Definition 5.3, then it is also *negligible* in the sense of Definition 8.2. Note that the two notions are not the same, however. For instance, $\mathbb{Z}/6 \times D_\infty$ is *negligible* in the sense of Definition 8.2, but *non-negligible* in the sense of Definition 5.3. In fact, when we express $\mathbb{Z}/6 \times D_\infty$ as an infinite virtually cyclic group of type II, we have

$$\mathbb{Z}/6 \times D_\infty \cong (\mathbb{Z}/6 \times \mathbb{Z}/2) *_{\mathbb{Z}/6} (\mathbb{Z}/6 \times \mathbb{Z}/2),$$

so $\mathbb{Z}/6 \times D_\infty$ is negligible in the sense of Definition 8.2 (since 6 is square-free). But, since $\mathbb{Z}/6 \times \mathbb{Z}/2$ is not isomorphic to a subgroup of S_4, then it follows that $\mathbb{Z}/6 \times D_\infty$ is not negligible in the sense of Definition 5.3.

Remark 8.2. Remark 8.1 showed that the class of negligible groups in the sense of Definition 8.2 is strictly wider than the class of negligible groups in the sense of Definition 5.3. Thus, the class of non-negligible groups in the sense of Definition 5.3 is strictly wider than the class of non-negligible groups in the sense of Definition 8.2.

Recall that, in Theorem 5.1, we considered an indexing set $\mathcal{T}''$, where $\mathcal{T}''$ consisted of a choice of vertex v from each orbit of non-negligible (in the sense of Definition 5.3) vertices in $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. In practice, it will be easier to consider the smaller indexing set $\hat{\mathcal{T}}''$, where $\hat{\mathcal{T}}''$ consists of a choice of vertex v from each orbit of non-negligible (in the sense of Definition 8.2) vertices in $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. Note that the indexing set $\hat{\mathcal{T}}''$ leaves out only vertices $v \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ with stabilizers such as $\mathbb{Z}/6 \times D_\infty$, which make no contribution to Nils by Proposition 8.1. It follows that we can use the smaller indexing set $\hat{\mathcal{T}}''$ in Theorem 5.1.

In practice, we will use the indexing set $\hat{\mathcal{T}}''$, but write $\mathcal{T}''$. This means that, from now on, we will simply read Theorem 5.1 with the current definition (Definition 8.2) of negligible in mind.

Proposition 8.2. *Let $\Gamma = \langle H, L \rangle$ be a split crystallographic group. Let $\ell \subseteq \mathbb{R}^3$ be a line; let $r(\alpha) = t + \alpha v$ ($t, v \in \mathbb{R}^3$; $\alpha \in \mathbb{R}^3$) be a parametrization of ℓ. If $H_v = \{h \in H \mid h \cdot v = v\}$ has square-free order, then $\Gamma_\ell = \{\gamma \in \Gamma \mid \gamma \cdot \ell = \ell\}$ is negligible.*

Proof. Let $\gamma \in \overline{\Gamma}_\ell$; we write $\gamma = v' + h$, where h is the linear part of the isometry γ (i.e., $h = \pi(\gamma)$) and $v' \in L$. Now if $\gamma \cdot r(\alpha) = r(\alpha)$ for all $\alpha \in \mathbb{R}$, we clearly must have $h \cdot v = v$. Thus, $\pi(\gamma) \in H_v$. Since $\pi : \overline{\Gamma}_\ell \to H$ is injective by Lemma 6.1(2) and $\pi(\overline{\Gamma}_\ell) \subseteq H_v$, it follows that $\overline{\Gamma}_\ell$ has square-free order, making Γ_ℓ negligible.

Definition 8.3. If H is a point group, then a vector $v \in \mathbb{R}^3$ is a *pole vector* if there is some orientation-preserving $h \in H - \{1\}$ such that $h \cdot v = v$.

Corollary 8.1. *If the line $\ell \subseteq \mathbb{R}^3$ can be parametrized as $r(\alpha) = t + \alpha v$ where v is not a pole vector of H, then the stabilizer group Γ_ℓ is negligible.*

Proof. We note that if v is not a pole vector of H, then the only orientation-preserving $h \in H_v$ is $h = 1_H$. The orientation-preserving subgroup of H_v has index at most two in H_v, so $|H_v| \leq 2$. Thus, H_v has square-free order, and Proposition 8.2 applies.

Proposition 8.3. *Let Γ be one of the groups Γ_i, ($i = 1, \ldots, 7$). We let H denote the point group of Γ.*

1. *If $H = S_4^+ \times (-1)$, then the vertex $\hat{\ell} \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is negligible unless $\hat{\ell}$ admits a parametrization of the form $r(\alpha) = t + \alpha v$ ($t, v \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$), where $v = (1,0,0)$ or $(1,1,0)$ (or some image of the latter vectors under the action of H).*
2. *If $H = D_2^+ \times (-1)$, then the vertex $\hat{\ell} \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is negligible unless $\hat{\ell}$ admits a parametrization of the form $r(\alpha) = t + \alpha v$ ($t, v \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$), where $v = (1,0,0)$, $(0,1,0)$, or $(0,0,1)$.*
3. *If $H = D_6^+ \times (-1)$, then the vertex $\hat{\ell} \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is negligible unless $\hat{\ell}$ admits a parametrization of the form $r(\alpha) = t + \alpha v$ ($t, v \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$), where $v = (1,1,1)$, $(1,-1,0)$, or $(1,-2,1)$ (or some image of the latter vectors under the action of H).*
4. *If $H = D_3^+ \times (-1)$, then all of the vertices $\hat{\ell} \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ are negligible. In particular, the complete lower algebraic K-groups of Γ_i for $i = 6, 7$ appear in Table 7.8.*

Proof. The proofs of all four parts are similar. Proposition 2.2 showed that there are only two or three orbits of (unit) pole vectors for any point group H. Proposition 2.2 also shows that there are exactly three orbits of unit pole vectors for all four of the groups in the statement of our proposition. For each of the four groups above, we can find the three distinct orbits of unit pole vectors by inspection. For instance, if $H = D_6^+ \times (-1)$, we note that the vectors $(1,1,1)$, $(1,-1,0)$, and $(1,-2,1)$ are all pole vectors of H since the orientation-preserving isometries

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{pmatrix}$$

from H fix the given vectors (respectively). Next, it is straightforward to check that the given three vectors are in separate orbits with respect to H (even after normalization), so the normalized vectors represent the three orbits of unit pole vectors for H. The rest of (3) now follows easily from Corollary 8.1. Case (2) is easier.

The proof of (1) begins in the same way. It is straightforward to check that the vectors $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ are pole vectors, and that they represent distinct H-orbits (even after normalization). It follows directly from Corollary 8.1 that the vertex $\hat{\ell} \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is negligible unless it admits a parametrization $r(\alpha) = t + \alpha v$ with one of the latter vectors playing the role of v. We can furthermore argue that $\hat{\ell}$ is negligible when $v = (1, 1, 1)$. One checks that the orbit of $(1, 1, 1)$ contains eight vectors, which implies that $|H_v| = 6$. Thus, $\Gamma_{\hat{\ell}}$ is negligible by Proposition 8.2. Statement (1) follows.

If $H = D_3^+ \times (-1)$, then the vectors $(1, 1, 1)$, $(1, -1, 0)$, and $(0, -1, 1)$ determine the three distinct orbits of unit pole vectors. In this case, one checks that the stabilizer groups H_v have orders 6, 2, and 2 (respectively), so the desired conclusion follows from Proposition 8.2.

8.2 The Finiteness of the Indexing Set $\mathcal{T}''$

In this section, we will explicitly identify the indexing set $\mathcal{T}''$ from Theorem 5.1 for the crystallographic groups Γ_i $(i = 1, \ldots, 5)$. (Proposition 8.3 has already shown that we can take $\mathcal{T}'' = \emptyset$ when $i = 6$ or 7.) It will follow easily that $\mathcal{T}''$ is always finite for any three-dimensional crystallographic group. We also record here the strict line stabilizers for the groups Γ_i $(i = 1, \ldots, 5)$. We will need a lemma.

Lemma 8.1. *Let $\Gamma = \Gamma_i$ $(i = 1, \ldots, 5)$, and let $\ell \subseteq \mathbb{R}^3$ be a line. The group $\Gamma(\ell)$ acts isometrically on the space of lines $\mathbb{R}^2_\ell$. We let $\psi : \Gamma(\ell) \to \mathrm{Isom}(\mathbb{R}^2_\ell)$ denote the associated homomorphism, and let G denote the image of ψ.*

1. *If $\ell' \in \mathbb{R}^2_\ell$, then $\psi_{|\overline{\Gamma}_{\ell'}} : \overline{\Gamma}_{\ell'} \to G_{\ell'}$ is injective. In particular, $|\overline{\Gamma}_{\ell'}|$ divides $|G_{\ell'}|$.*
2. *Let $g \in G_{\ell'}$, and let $\gamma \in \psi^{-1}(g)$. We have $g \in \psi(\overline{\Gamma}_{\ell'})$ if and only if there is an element $k \in \ker \psi$ such that $\gamma_{|\ell'} = k_{|\ell'}$.*

Proof. The statement that $\Gamma(\ell)$ acts isometrically on $\mathbb{R}^2_\ell$ is straightforward.

1. We claim that $\psi_{|\overline{\Gamma}_{\ell'}} : \overline{\Gamma}_{\ell'} \to \mathrm{Isom}(\mathbb{R}^2_\ell)$ is injective. Suppose that $\gamma \in \ker(\psi_{|\overline{\Gamma}_{\ell'}})$. Since $\gamma \in \overline{\Gamma}_{\ell'}$, γ acts trivially on the line $\ell' \subseteq \mathbb{R}^3$, and, since $\gamma \in \ker(\psi)$, γ leaves every line parallel to ℓ' invariant as a set. Let ℓ'' be parallel to ℓ', and choose distinct points $x_1, x_2 \in \ell'$. There are points $y_1, y_2 \in \ell''$ that are the unique closest points to x_1 and x_2 (respectively) among all points on ℓ''. Since ℓ'' is invariant under γ, and x_1 and x_2 are fixed, it must be that y_1 and y_2 are fixed as well. Thus, all of ℓ'' is fixed. This shows that every line parallel to ℓ' is fixed by γ, so $\gamma = 1$. This proves the claim.

   We can therefore identify $\overline{\Gamma}_{\ell'}$ with a subgroup of $G_{\ell'}$ and apply Lagrange's Theorem.

2. We assume $g \in \psi(\overline{\Gamma}_{\ell'})$ and $\psi(\gamma) = g$. Choose $\hat{\gamma} \in \overline{\Gamma}_{\ell'}$ such that $\psi(\hat{\gamma}) = g$. We can set $k = \hat{\gamma}^{-1}\gamma$, which has the required properties. Conversely, if $\gamma_{|\ell'} = k_{|\ell'}$, then $k^{-1}\gamma \in \overline{\Gamma}_{\ell'}$, and $\psi(k^{-1}\gamma) = g$, as required.

Theorem 8.1. *For the group $\Gamma = \Gamma_1$, we can choose*

$$\mathcal{T}'' = \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ \alpha \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ 1/2 \end{pmatrix} \right\},$$

where we have expressed the vertices (i.e., lines) in $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ in parametric form.

For the vertices $v \in \mathcal{T}''$, the strict stabilizer groups $\overline{\Gamma}_v$ satisfy

$$\pi(\overline{\Gamma}_v) = D_4'', \quad D_4'', \quad D_2', \quad \langle A, B \rangle, \textit{ and } \quad \langle A, B \rangle,$$

respectively, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \textit{ and } \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note that $\langle A, B \rangle \cong D_2$.

Proof. The point group H of Γ_1 is $S_4^+ \times (-1)$, so Proposition 8.3(1) implies that the only non-negligible point stabilizers from $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ have parametrizations of the form $r(\alpha) = t + \alpha v$, where $v = (1, 0, 0)$ or $(1, 1, 0)$ (or the image of one of these vectors under the action of H).

We consider the first case. Let $r(\alpha) = t + \alpha v$, where $v = (1, 0, 0)$ (or one of its images under the action of H). It is clear that any such parametrized line $r(\alpha)$ can be moved to the plane $\mathbb{R}^2_{(0,0,\alpha)}$ by an element of Γ. Thus, we can find a complete set of Γ-orbit representatives with the given type (i.e., with tangent vector $v = (0, 0, 1)$) inside $\mathbb{R}^2_{(0,0,\alpha)}$. We consider the action of the group

$$\Gamma(0, 0, \alpha) = \{\gamma \in \Gamma \mid \gamma(0, 0, \alpha) \text{ is parallel to } (0, 0, \alpha)\}$$

on the plane $\mathbb{R}^2_{(0,0,\alpha)}$. (In words, $\Gamma(0, 0, \alpha)$ takes lines parallel to the z-axis to other lines parallel to the z-axis, possibly reversing the directions of the lines.) It is straightforward to check that $\Gamma(0, 0, \alpha) = (D_4^+ \times (-1))_1$. We can identify $\mathbb{R}^2_{(0,0,\alpha)}$ with the xy-plane $P(z = 0)$ in $\mathbb{R}^3$. With respect to this identification, the point group $D_4^+ \times (-1)$ acts by restricting its usual action on $\mathbb{R}^3$ to $P(z = 0)$, and the cubical lattice acts on $P(z = 0)$ by ignoring the third coordinate (so the translation $(1, 0, 0) \in L$ acts as $(1, 0)$ on $P(z = 0)$, and $(0, 0, 1)$ acts trivially). This action yields a homomorphism $\psi : \Gamma(0, 0, \alpha) \to \mathrm{Isom}(\mathbb{R}^2)$. The kernel of ψ is

$$\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

The image $\psi(\Gamma(0, 0, \alpha)) = G$ is described in Fig. 8.1. The translations T_1 and T_2 (pictured) generate the whole group of translations in the image. Double-tailed

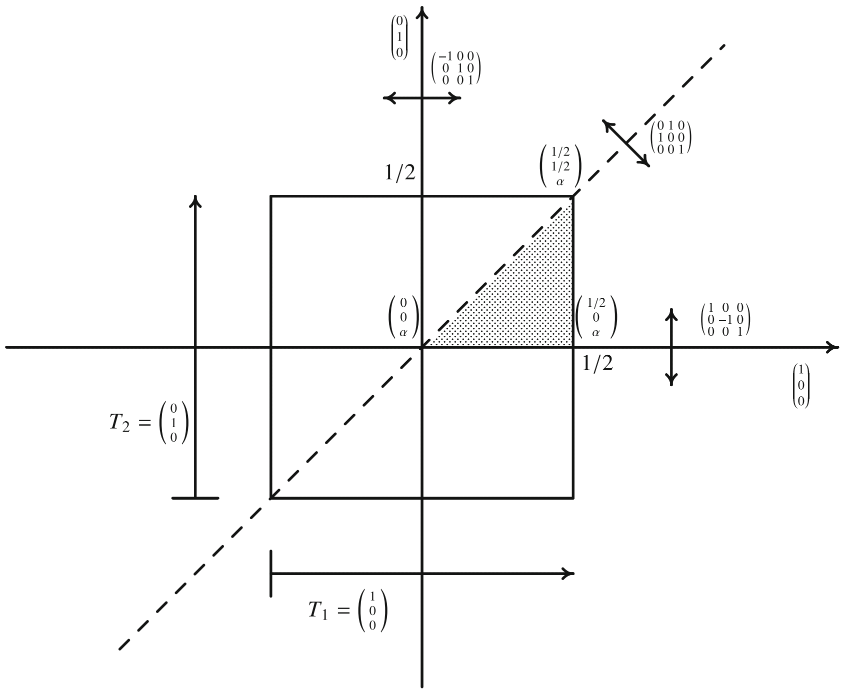


Fig. 8.1 The image of $\Gamma(0,0,\alpha)$ under the homomorphism $\psi : \Gamma(0,0,\alpha) \to \mathbb{R}^2_{(0,0,\alpha)}$. A fundamental domain for the image of ψ is *shaded*. Here $\Gamma = \Gamma_1$

arrows in the figure denote reflections, both in $\mathbb{R}^2$ and in $\mathbb{R}^3$. Using Theorem 6.1, it is straightforward to verify that the shaded triangle P is an exact convex compact fundamental polyhedron for the image of ψ. The side-pairings are all reflections (in fact, the image of ψ is a Coxeter group), so in particular all of the vertices in the triangle are in separate orbits.

Now we compute the strict stabilizers of the vertices. (Corollary 5.1 has already shown that the edges and 2-cells have negligible stabilizers.) The stabilizers $G_{(0,0,\alpha)}$ and $G_{(1/2,1/2,\alpha)}$ are easily seen to have order 8, and the stabilizer $G_{(1/2,0,\alpha)}$ has order 4. It follows from Lemma 8.1(1) that the orders of the strict stabilizers $\overline{\Gamma}_{\hat{\ell}}$ divide 8, 8, and 4 (respectively), where $\hat{\ell}$ ranges over the given lines. Consider the line $(0,0,\alpha)$. It is easy to see that the isometries

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(pictured in Fig. 8.1) are in the strict stabilizer $\overline{\Gamma}_{(0,0,\alpha)}$, and they clearly generate the group D_4''. It follows that $\overline{\Gamma}_{(0,0,\alpha)} = D_4''$. The other two strict stabilizers associated to this plane can similarly be computed by inspection of Fig. 8.1.

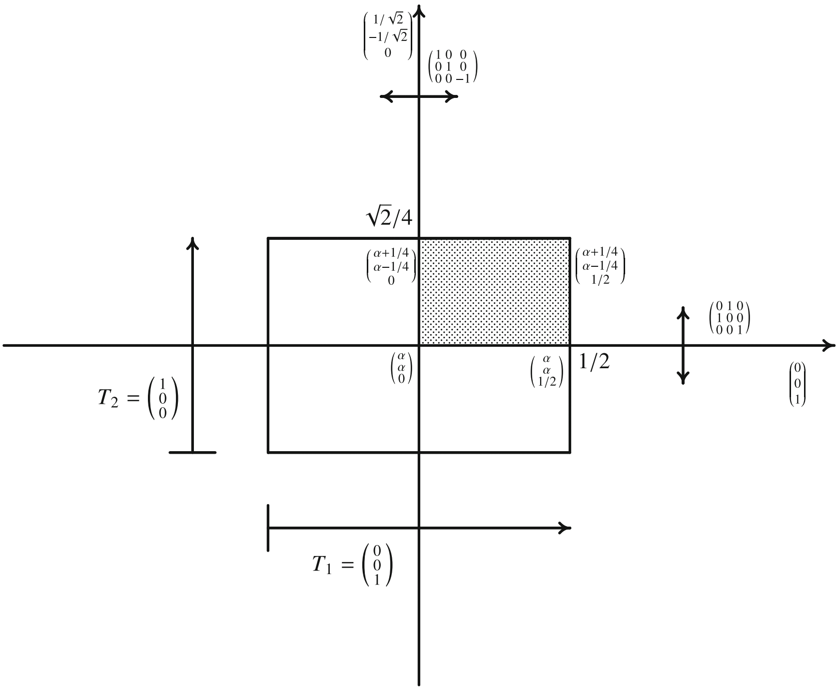


Fig. 8.2 A picture of the plane $\mathbb{R}^2_{(\alpha,\alpha,0)}$. The action of $\Gamma(\alpha,\alpha,0)$ is indicated, and a fundamental domain is *shaded*. Here $\Gamma = \Gamma_1$

Next we must consider the plane $\mathbb{R}^2_{(\alpha,\alpha,0)}$, which contains orbit representatives of all of the other relevant vertices, by Proposition 8.3. Note that in Fig. 8.2 we have identified the plane $\mathbb{R}^2_{(\alpha,\alpha,0)}$ with the plane in $\mathbb{R}^3$ through the origin and perpendicular to the vector $(1,1,0)$, namely $P(x+y=0)$. The group $\Gamma(\alpha,\alpha,0)$ is generated by the lattice $\mathbb{Z}^3 \subseteq \mathbb{R}^3$ and the point group that is generated by the reflections from Fig. 8.2 and the antipodal map. The action of $\Gamma(\alpha,\alpha,0)$ is determined as before: translations act like their projections into the plane $P(x+y=0)$, and the point group acts by restriction. We have labelled the axes with a convenient orthogonal basis for $P(x+y=0)$. The shaded rectangle is a fundamental domain; the image group G is generated by reflections in the sides of this rectangle.

We compute the strict stabilizers of the lines that form the corners of the rectangle. It is clear from Lemma 8.1(1) that the orders of these strict stabilizer groups divide 4. One checks directly that the reflections labelling the double-tailed arrows from Fig. 8.2 are both in the strict stabilizer of $(\alpha,\alpha,0)$, and they clearly generate a group of order 4. It follows that $\overline{\Gamma}_{(\alpha,\alpha,0)} = \langle A, B\rangle$. One can also check that $\pi(\overline{\Gamma}_{(\alpha,\alpha,1/2)}) = \langle A, B\rangle$ in roughly the same way: the isometry A is in $\overline{\Gamma}_{(\alpha,\alpha,1/2)}$, and the isometry $T_1 + B$ is also in $\overline{\Gamma}_{(\alpha,\alpha,1/2)}$, and these two isometries generate a group of order 4, which must therefore be all of $\overline{\Gamma}_{(\alpha,\alpha,1/2)}$.

Now we will show that the top corners of the rectangle are negligible. Consider the vertex $(\alpha + 1/4, \alpha - 1/4, 0)$. We claim that the reflection g across the top edge of the rectangle is not in the image of $\psi_| : \overline{\Gamma}_{(\alpha+1/4,\alpha-1/4,0)} \to \mathrm{Isom}(\mathbb{R}^2)$. The proof uses Lemma 8.1(2). We note that $\gamma = T_2 + A$ is an isometry in $\Gamma(\alpha, \alpha, 0)$ that leaves the given line $(\alpha + 1/4, \alpha - 1/4, 0)$ invariant. Indeed, if we write the line in the form $r(\alpha)$, then $\gamma \cdot r(\alpha) = r(\alpha + 1/2)$. Note that $g = \psi(\gamma)$. By Lemma 8.1(2), $g \in \psi(\overline{\Gamma}_{(\alpha+1/4,\alpha-1/4,0)})$ if and only if there is some $k \in \ker \psi$ such that k and γ agree on $r(\alpha)$. However, we can explicitly identify the kernel of $\psi : \Gamma(\alpha, \alpha, 0) \to \mathrm{Isom}(\mathbb{R}^2)$ as follows:

$$\ker \psi = \left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle.$$

Note that the first generator γ_1 acts by $\gamma_1 \cdot r(\alpha) = r(-\alpha)$, and the second generator γ_2 acts by $\gamma_2 \cdot r(\alpha) = r(\alpha + 1)$. It follows that there is no k in the kernel such that $k \cdot r(\alpha) = r(\alpha + 1/2)$. This proves the claim. It follows that $\psi(\overline{\Gamma}_{(\alpha+1/4,\alpha-1/4,0)})$ is proper subgroup of $G_{(\alpha+1/4,\alpha-1/4,0)}$, which has order 4. It now follows that the group $\overline{\Gamma}_{(\alpha+1/4,\alpha-1/4,0)}$ has order at most 2, and is therefore negligible. The proof that $\overline{\Gamma}_{(\alpha+1/4,\alpha-1/4,1/2)}$ is negligible follows a similar pattern.

Remark 8.3. Theorem 8.1 establishes the general pattern of all five basic cases:

- It suffices to consider two or three planes $\mathbb{R}^2_\ell$ (depending on the point group, as from Proposition 8.3); all of the required Γ-orbit representatives occur as vertices in these planes.
- Given a split crystallographic group $\Gamma = \Gamma_i$ $(i = 1, \ldots, 5)$ and one of the above planes $\mathbb{R}^2_\ell$, there is a natural homomorphism $\psi : \Gamma(\ell) \to \mathrm{Isom}(\mathbb{R}^2_\ell)$ and a natural identification of $\mathbb{R}^2_\ell$ with a two-dimensional subspace of $\mathbb{R}^3$. Specifically, choosing ℓ to be a one-dimensional subspace of $\mathbb{R}^3$ (as we may), we can identify $\mathbb{R}^2_\ell$ with the perpendicular two-dimensional subspace S. The group $\Gamma(\ell)$, which is necessarily a split crystallographic group, acts on S. An element of the point group of $\Gamma(\ell)$ acts by restriction to S, and an element of the lattice acts by its projection into S.
- In every case, we will specify generators for the image $G = \psi(\Gamma(\ell))$ in accompanying figures, and label the coordinate axes with convenient unit vectors. The group G is often, but not always, a Coxeter group. Double-tailed arrows in the figures will always denote reflections in the plane S that preserve the normal direction (as isometries of $\mathbb{R}^3$). The action of G on S will always have an exact convex compact fundamental polyhedron P. It is generally straightforward to check that the shaded region in the figures is an exact convex compact fundamental polyhedron. The proof will be omitted. Having P lets us determine orbit information: every point (line) in the given plane is in some translate of P, and the intersections of $\Gamma(\ell)$-orbits with P have the form $P \cap [x]$, where $x \in P$ and $[x]$ denotes the cycle of x (see Sect. 6.1). By Corollary 5.1, only vertices in the cellulation determined by P (see Theorem 6.2) can be non-negligible.
- We compute the strict stabilizer groups using the two parts of Lemma 8.1.

This general outline will be assumed in what follows.

Theorem 8.2. *For the group $\Gamma = \Gamma_2$, we can choose*

$$\mathcal{T}'' = \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} \right\},$$

where we have expressed the vertices (i.e., lines) in $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ in parametric form.

For the vertices $v \in \mathcal{T}''$, the strict stabilizer groups $\overline{\Gamma}_v$ satisfy

$$\pi(\overline{\Gamma}_v) = D_4'', \quad D_2', \text{ and } \quad \langle A, B \rangle,$$

respectively, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof. We will consider the same planes $\mathbb{R}^2_\ell$ as in the proof of Theorem 8.1, although they will have different actions. First, we consider the plane $\mathbb{R}^2_{(0,0,\alpha)}$, pictured in Fig. 8.3.

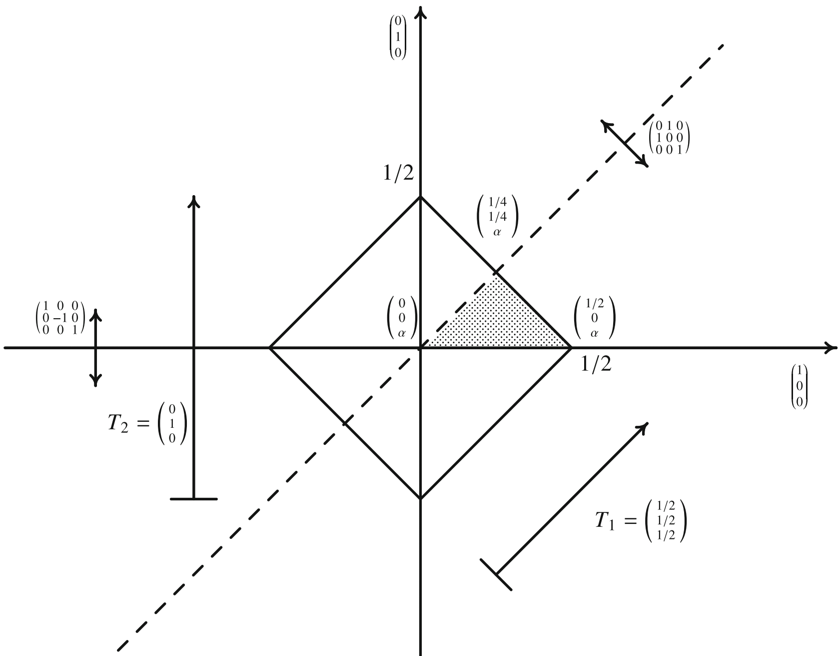


Fig. 8.3 This picture describes the action of $\Gamma(0, 0, \alpha)$ on $\mathbb{R}^2_{(0,0,\alpha)}$, where $\Gamma = \Gamma_2$. The point group of $\Gamma(0, 0, \alpha)$ is $D_4^+ \times (-1)$

It is not difficult to check that the shaded triangle is an exact convex compact fundamental polyhedron for the action of G, which is generated by reflections in the sides of the triangle. It is straightforward to check that the stabilizers $G_{(0,0,\alpha)}$, $G_{(1/2,0,\alpha)}$, and $G_{(1/4,1/4,\alpha)}$ have orders 8, 8, and 4, respectively. (This is because the images of the fundamental polyhedron cannot overlap in their interiors—see Definition 6.3.) By Lemma 8.1(1), the map $\psi_| : \overline{\Gamma}_{\ell'} \to G_{\ell'}$ is injective for each of the lines in question. It is straightforward to check, exactly as in the proof of Theorem 8.1, that $\psi_| : \overline{\Gamma}_{(0,0,\alpha)} \to G_{(0,0,\alpha)}$ is in fact an isomorphism, and that $\overline{\Gamma}_{(0,0,\alpha)} = D_4''$.

Now we consider the strict stabilizer group $\overline{\Gamma}_{(1/2,0,\alpha)}$. It is straightforward to check that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

are both in $\overline{\Gamma}_{(1/2,0,\alpha)}$. It follows directly that $D_2' \subseteq \pi(\overline{\Gamma}_{(1/2,0,\alpha)})$. We claim that $\psi(\overline{\Gamma}_{(1/2,0,\alpha)})$ is a proper subgroup of $G_{(1/2,0,\alpha)}$, so that $D_2' = \pi(\overline{\Gamma}_{(1/2,0,\alpha)})$. We prove the claim. Consider the isometry

$$\gamma = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Note that the linear part is reflection in the line $y = -x$ in the plane from Fig. 8.3.) It is easy to check that $g = \psi(\gamma) \in G_{(1/2,0,\alpha)}$. By Lemma 8.1(2), $g \in \psi(\overline{\Gamma}_{(1/2,0,\alpha)})$ if and only if the action of γ on $(1/2, 0, \alpha)$ agrees with the action of some k from the kernel of $\psi : \Gamma(0, 0, \alpha) \to \mathrm{Isom}(\mathbb{R}^2)$. The latter kernel is

$$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

We have $\gamma \cdot r(\alpha) = r(\alpha + 1/2)$ (where $r(\alpha) = (1/2, 0, \alpha)$), however, and there is no $k \in \ker\, \psi$ that acts this way on $r(\alpha)$. This proves the claim.

One can show that $\psi(\overline{\Gamma}_{(1/4,1/4,\alpha)})$ is a proper subgroup of $G_{(1/4,1/4,\alpha)}$ in an analogous way, using the same γ as above. One then concludes that $|\overline{\Gamma}_{(1/4,1/4,\alpha)}| \leq 2$, so $\overline{\Gamma}_{(1/4,1/4,\alpha)}$ is negligible.

Next, we consider the action of $\Gamma(\alpha, \alpha, 0)$ on $\mathbb{R}^2_{(\alpha,\alpha,0)}$, as described in Fig. 8.4. The image group G acts by reflections on the sides of the shaded rectangle, which is an exact convex compact fundamental polyhedron. (In fact, G is again a Coxeter group.) It is easy to see that the four vertices forming the corners of the rectangle have stabilizers of order 4 relative to G.

The reflections in Fig. 8.4 are both easily seen to be elements of the group $\overline{\Gamma}_{(\alpha,\alpha,0)}$. Since these reflections generate a group of order 4, it follows from Lemma 8.1(1) that this is the entire strict stabilizer of $(\alpha, \alpha, 0)$ (as stated in the theorem).

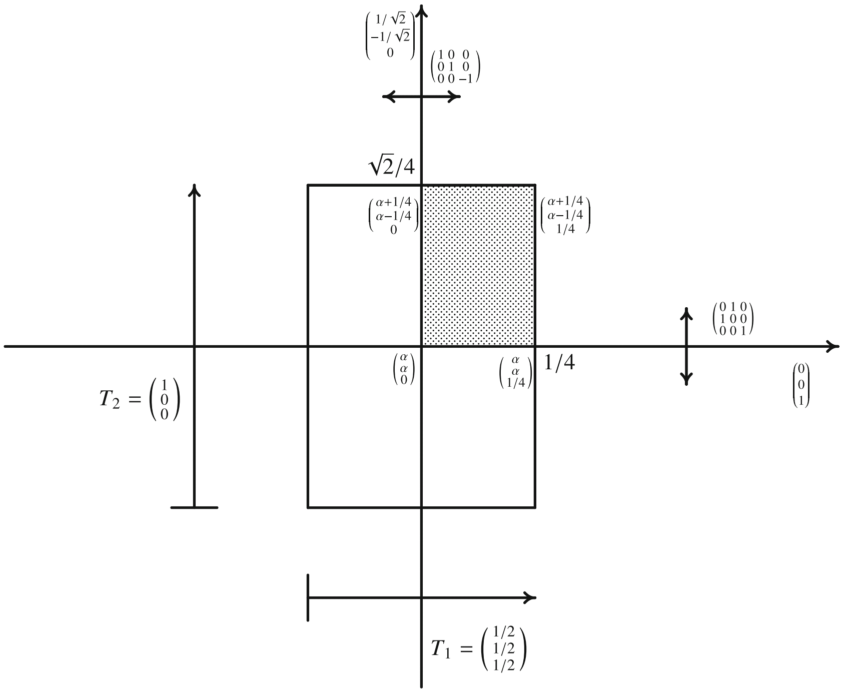


Fig. 8.4 This picture describes the action of $\Gamma(\alpha,\alpha,0)$ on $\mathbb{R}^2_{(\alpha,\alpha,0)}$, where $\Gamma = \Gamma_2$. The point group is generated by the given reflections and the antipodal map

All of the other corners have negligible stabilizers. We will prove this for the corner $(\alpha,\alpha,1/4)$, the other cases being similar. Consider the isometry $\gamma = T_1 + R_z$, where T_1 is pictured in Fig. 8.4 and R_z is multiplication by -1 in the z-coordinate; i.e., R_z labels the horizontal double-tailed arrow from Fig. 8.4. Setting $r(\alpha) = (\alpha,\alpha,1/4)$, one easily checks that $\gamma \cdot r(\alpha) = r(\alpha + 1/2)$. By Lemma 8.1(2), $g = \psi(\gamma) \in \psi(\overline{\Gamma}_{(\alpha,\alpha,1/4)})$ if and only if there is some k in the kernel

$$\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix} \right\rangle$$

such that $k \cdot r(\alpha) = r(\alpha + 1/2)$. The generators of the kernel send $r(\alpha)$ to $r(\alpha+1)$ and $r(-\alpha)$ (respectively). It follows that the required k does not exist. Thus g is in $G_{(\alpha,\alpha,1/4)}$ but not in $\psi(\overline{\Gamma}_{(\alpha,\alpha,1/4)})$, so the latter group is a proper subgroup of the former. It follows that $|\overline{\Gamma}_{(\alpha,\alpha,1/4)}| \le 2$, proving that $\overline{\Gamma}_{(\alpha,\alpha,1/4)}$ is negligible.

Theorem 8.3. *For the group $\Gamma = \Gamma_3$, we can choose*

$$\mathcal{T}'' = \left\{ \begin{pmatrix}0\\0\\\alpha\end{pmatrix}, \begin{pmatrix}1/4\\1/4\\\alpha\end{pmatrix}, \begin{pmatrix}\alpha\\\alpha\\0\end{pmatrix}, \begin{pmatrix}\alpha\\\alpha\\1/2\end{pmatrix} \right\},$$

where we have expressed the vertices (i.e., lines) in $\coprod_{\langle \ell \rangle} \mathbb{R}^2_{\ell}$ in parametric form.

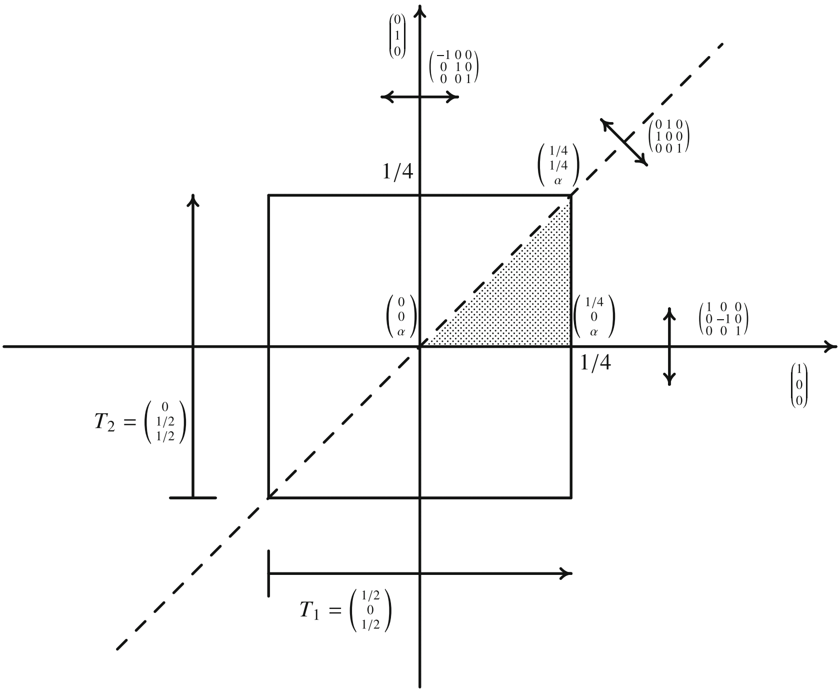


Fig. 8.5 This picture describes the action of $\Gamma(0,0,\alpha)$ on the plane $\mathbb{R}^2_{(0,0,\alpha)}$, where $\Gamma = \Gamma_3$

For the vertices $v \in \mathcal{T}''$, the strict stabilizer groups $\overline{\Gamma}_v$ satisfy

$$\pi(\overline{\Gamma}_v) = D_4'', \quad \langle A, C\rangle, \quad \langle A, B\rangle, \text{ and } \quad \langle A, B\rangle,$$

respectively, where

$$A = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \quad B = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \text{ and } \quad C = \begin{pmatrix} -1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

Proof. We first consider the action on $\mathbb{R}^2_{(0,0,\alpha)}$. The image group G is a Coxeter group, generated by reflections in the sides of the triangle pictured in Fig. 8.5. We note first that it is straightforward to verify that $\overline{\Gamma}_{(0,0,\alpha)} = D_4''$. The argument follows the same lines as in the proofs of the previous theorems.

We concentrate on the remaining two corners of the triangle. First consider the line $(1/4, 0, \alpha)$. It is clear that $G_{(1/4,0,\alpha)}$ has order 4. We claim that $\overline{\Gamma}_{(1/4,0,\alpha)}$ is negligible. This will follow from Lemma 8.1(1) and Proposition 8.1 once we show that $\psi(\overline{\Gamma}_{(1/4,0,\alpha)})$ is a proper subgroup of $G_{(1/4,0,\alpha)}$. Consider the element

$$\gamma = T_1 + \begin{pmatrix} -1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

It is easy to check that $\psi(\gamma) \in G_{(1/4,0,\alpha)}$. By Lemma 8.1(2), $g = \psi(\gamma)$ is in $\psi(\overline{\Gamma}_{(1/4,0,\alpha)})$ if and only if there is some element k of the kernel of $\psi : \Gamma(0,0,\alpha) \to \mathrm{Isom}(\mathbb{R}^2)$ that agrees with γ on $(1/4, 0, \alpha)$. We note that the kernel is

$$\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

The above generators act on $r(\alpha) = (1/4, 0, \alpha)$ by sending $r(\alpha)$ to $r(\alpha + 1)$ and $r(-\alpha)$, respectively. On the other hand, $\gamma \cdot r(\alpha) = r(\alpha + 1/2)$. It follows that $g = \psi(\gamma) \notin \psi(\overline{\Gamma}_{(1/4,0,\alpha)})$. This proves the claim.

Next, we consider the line $(1/4, 1/4, \alpha)$. Essentially the same reasoning as above, using the same γ, shows that $\psi(\overline{\Gamma}_{(1/4,1/4,\alpha)})$ is a proper subgroup of $G_{(1/4,1/4,\alpha)}$. We note that $G_{(1/4,1/4,\alpha)}$ has order 8, so that $\overline{\Gamma}_{(1/4,1/4,\alpha)}$ has order at most 4. It is straightforward to check that

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

is a group of order 4 that is a subgroup of $\overline{\Gamma}_{(1/4,1/4,\alpha)}$, so equality must hold. It easily follows that $\pi(\overline{\Gamma}_{(1/4,1/4,\alpha)}) = \langle A, C \rangle$, as in the statement of the theorem.

Next we must consider the plane $\mathbb{R}^2_{(\alpha,\alpha,0)}$. The action is described in Fig. 8.6. We note that the image group G does not act as a Coxeter group in this case: the side-pairing of the slanted side of the triangle P with itself is a rotation of 180° about the midpoint of that side. It follows that the top vertex and the right-hand vertex are in the same orbit under the action of $\Gamma_{(\alpha,\alpha,0)}$. As a result, we only need to consider the vertices $(\alpha, \alpha, 0)$ and $(\alpha, \alpha, 1/2)$. It is easy to check that $\overline{\Gamma}_{(\alpha,\alpha,0)} = \langle A, B \rangle$, so we will concentrate on the vertex $(\alpha, \alpha, 1/2)$.

We claim that $|G_{(\alpha,\alpha,1/2)}| < 8$. To prove the claim, we note that $|G_{(\alpha,\alpha,1/2)}|$ must be less than or equal to the number of translates of the fundamental domain that touch $(\alpha, \alpha, 1/2)$. There are 8 such translates, which proves that $|G_{(\alpha,\alpha,1/2)}| \leq 8$. Next, we consider the side-pairing ϕ of the slanted side of P with itself, which is rotation by 180° about the midpoint. Note that $\phi(P)$ touches $(\alpha, \alpha, 1/2)$, although $\phi \notin G_{(\alpha,\alpha,1/2)}$. There cannot be $g \in G_{(\alpha,\alpha,1/2)}$ such that $g(P) = \phi(P)$, for, by Definition 6.3, this would imply that $g = \phi$, a contradiction. Thus, the elements of $G_{(\alpha,\alpha,1/2)}$ must move P to some subset of the remaining seven translates of P that touch $(\alpha, \alpha, 1/2)$. Since different elements of $G_{(\alpha,\alpha,1/2)}$ must move P to different elements of the tessellation $\{gP \mid g \in G\}$ (again by Definition 6.3), the claim follows.

Next, we directly check that

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \leq \overline{\Gamma}_{(\alpha,\alpha,1/2)}.$$

It is easy to see that the above isometries generate a group of order 4. In view of the inequality $|G_{(\alpha,\alpha,1/2)}| < 8$ and Lemma 8.1(1), the above containment must be an equality.

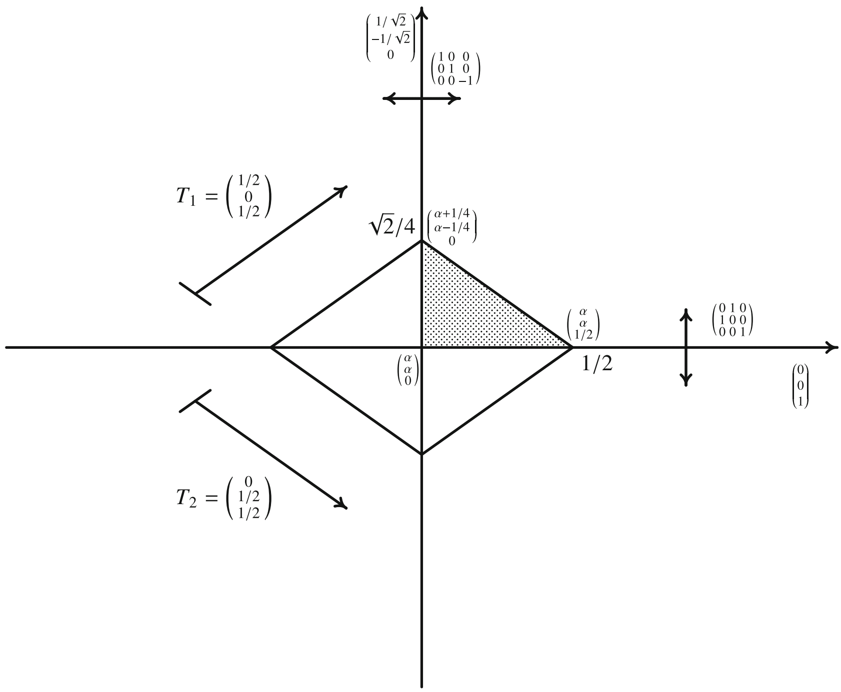


Fig. 8.6 This picture describes the action of $\Gamma(\alpha, \alpha, 0)$ on $\mathbb{R}^2_{(\alpha,\alpha,0)}$, where $\Gamma = \Gamma_3$

(Technically, Theorem 6.2 tells us that the midpoint of the slanted side in Fig. 8.6 should be a vertex v in the cellulation of the plane, so we should consider its strict stabilizer as well. However, it is clear that $|G_v| = 2$, so the strict stabilizer is necessarily negligible, by Lemma 8.1(1).)

Theorem 8.4. *For the group $\Gamma = \Gamma_4$, we can choose*

$$\mathcal{T}'' = \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha \\ 1/2 \end{pmatrix} \right\},$$

where we have expressed the vertices (i.e., lines) in $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ in parametric form (Figs. 8.7 and 8.8).

For the vertices $v \in \mathcal{T}''$, the strict stabilizer groups $\overline{\Gamma}_v$ are

$$\pi(\overline{\Gamma}_v) = D_2', \quad D_2', \quad D_{2_1}', \quad D_{2_1}', \quad D_{2_2}', \textit{ and } \quad D_{2_2}'.$$

Proof. We must consider the planes $\mathbb{R}^2_{(0,0,\alpha)}$, $\mathbb{R}^2_{(\alpha,0,0)}$, and $\mathbb{R}^2_{(0,\alpha,0)}$. These three cases pose no problems that we haven't encountered before, so we will give an explicit argument for the plane $\mathbb{R}^2_{(0,0,\alpha)}$, and leave the remaining cases to the reader.

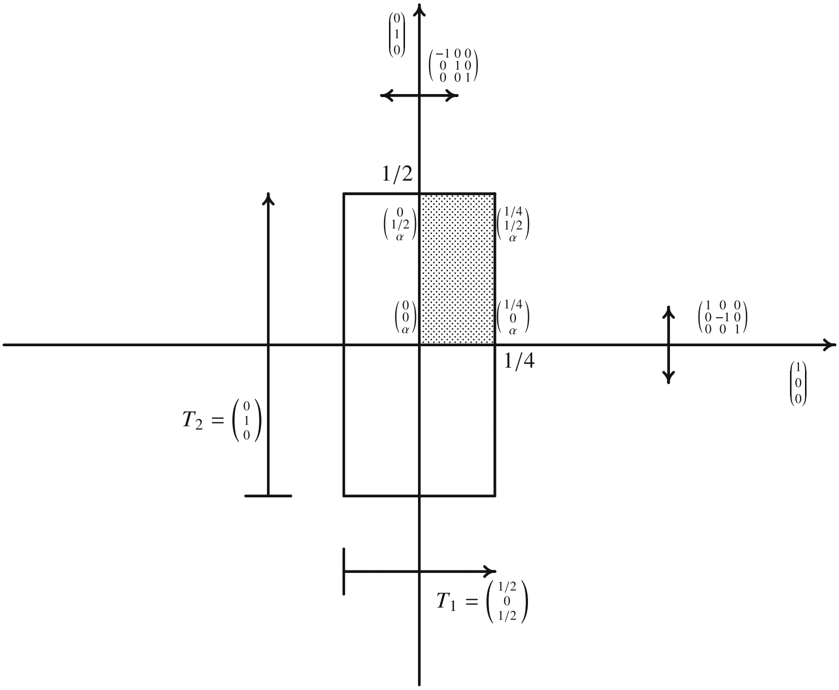


Fig. 8.7 This picture describes the action of $\Gamma(0, 0, \alpha)$ on the plane $\mathbb{R}^2_{(0,0,\alpha)}$, where $\Gamma = \Gamma_4$

We note that the image group G is generated by reflections in the sides of the shaded rectangle, and is in particular a Coxeter group. It follows that the four vertices are all in separate orbits modulo the action of $\Gamma(0, 0, \alpha)$. It is also easy to see that $|G_\ell| = 4$ for the lines ℓ in question. It is completely straightforward to verify that $D_2' \leq \overline{\Gamma}_{(0,0,\alpha)}$, so equality must hold by Lemma 8.1(1).

Next we consider the line $(0, 1/2, \alpha)$. We easily check that

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \leq \overline{\Gamma}_{(0,1/2,\alpha)}.$$

Since the isometries above will generate a group of order 4, and $|\overline{\Gamma}_{(0,1/2,\alpha)}| \leq |G_{(0,1/2,\alpha)}| = 4$ by Lemma 8.1(1), equality is forced. The equality $\pi(\overline{\Gamma}_{(0,1/2,\alpha)}) = D_2'$ follows directly.

We claim that the remaining two vertices in this plane are negligible. We prove this fact for the vertex (line) $(1/4, 0, \alpha)$, the other case being similar. Consider the isometry

$$\gamma = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

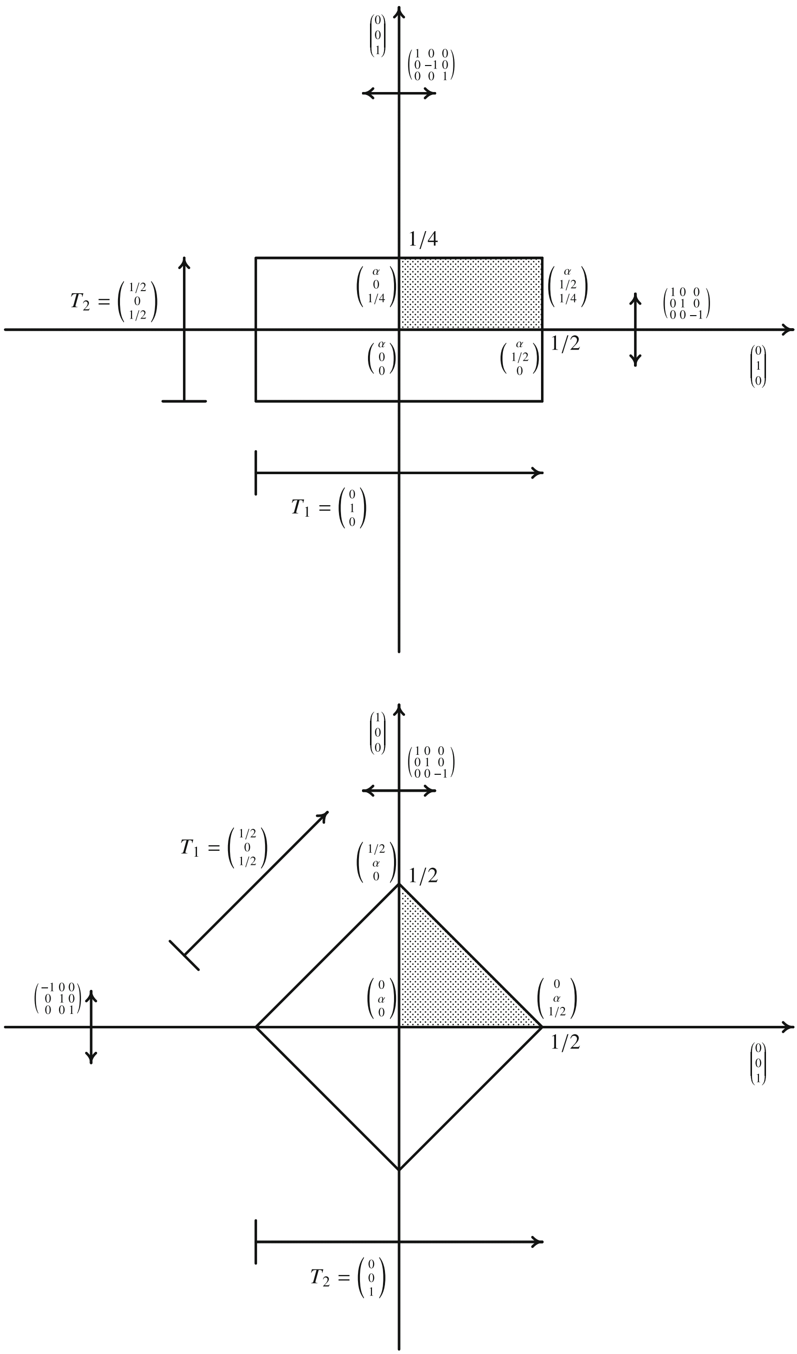


Fig. 8.8 The planes $\mathbb{R}^2_{(\alpha,0,0)}$ and $\mathbb{R}^2_{(0,\alpha,0)}$ appear on the *top* and *bottom* (respectively). The acting group is $\Gamma(\ell)$ (for appropriate ℓ) and $\Gamma = \Gamma_4$

Clearly $g = \psi(\gamma) \in G_{(1/4,0,\alpha)}$. It is sufficient to show that $g \notin \psi(\overline{\Gamma}_{(1/4,0,\alpha)})$; we will establish this by proving that there is no $k \in \ker\psi$ such that $k_{|(1/4,0,\alpha)} = \gamma_{|(1/4,0,\alpha)}$. The kernel is equal to

$$\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

These generators act on $r(\alpha) = (1/4, 0, \alpha)$ by sending $r(\alpha)$ to $r(\alpha+1)$ and $r(-\alpha)$ (respectively). The element γ acts by the rule $\gamma \cdot r(\alpha) = r(\alpha + 1/2)$. It follows that the required k does not exist, so $\overline{\Gamma}_{(1/4,0,\alpha)}$ is negligible.

Theorem 8.5. *For the group $\Gamma = \Gamma_5$, we can choose*

$$\mathcal{T}'' = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha+1/2 \\ \alpha-1/2 \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ -\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha+1/2 \\ -\alpha+1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha+1/2 \\ -2\alpha+1/2 \\ \alpha+1/2 \end{pmatrix} \right\},$$

where we have expressed the vertices (i.e., lines) in $\coprod_{\langle\ell\rangle} \mathbb{R}^2_\ell$ in parametric form.

For the vertices $v \in \mathcal{T}''$, the strict stabilizer groups $\overline{\Gamma}_v$ satisfy

$$\pi(\overline{\Gamma}_v) = D_6'', \quad \langle A, D\rangle, \quad \langle D, E\rangle, \quad \langle D, E\rangle, \quad \langle E, F\rangle, \text{ and } \quad \langle E, F\rangle,$$

respectively, where

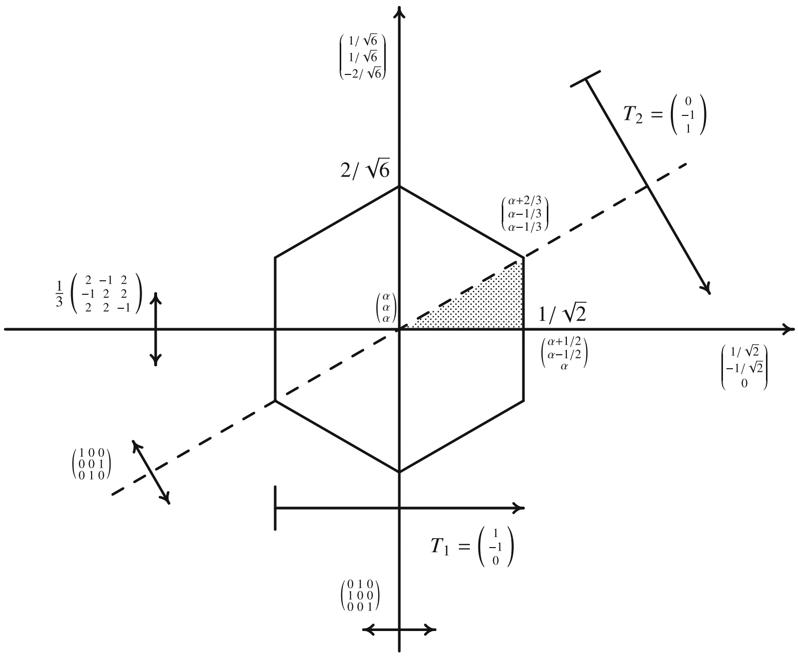
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad E = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \text{ and } \quad F = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The groups $\langle A, D\rangle$, $\langle D, E\rangle$, and $\langle E, F\rangle$ are isomorphic to D_2.

Proof. We will consider only the planes $\mathbb{R}^2_{(\alpha,\alpha,\alpha)}$ and $\mathbb{R}^2_{(\alpha,-\alpha,0)}$, leaving the plane $\mathbb{R}^2_{(\alpha,-2\alpha,\alpha)}$ (Fig. 8.10) to the reader.

We note that $\mathbb{R}^2_{(\alpha,\alpha,\alpha)}$ is pictured at the top of Fig. 8.9 with its associated action by $\Gamma(\alpha,\alpha,\alpha)$. The image group G is generated by reflections in the sides of the shaded triangle (whose interior angles are $\pi/6$, $\pi/3$, and $\pi/2$, reading clockwise from the origin). In particular, G is a Coxeter group, and all three vertices are distinct in the quotient. It also follows that the stabilizers G_ℓ of the vertices have orders 12, 6, and 4 (respectively, reading clockwise from the origin). It follows from Lemma 8.1(1) that the top corner of the fundamental domain has a negligible strict stabilizer (since the order divides six). It is also straightforward to check that $D_6'' \leq \overline{\Gamma}_{(\alpha,\alpha,\alpha)}$, and an application of Lemma 8.1(1) forces this inclusion to be an equality by counting. Next, we consider the vertex $(\alpha + 1/2, \alpha - 1/2, \alpha)$. It is straightforward to check that D and $T_1 + A$ are both in $\overline{\Gamma}_{(\alpha+1/2,\alpha-1/2,\alpha)}$, and that these isometries generate a group of order 4. It follows that $\overline{\Gamma}_{(\alpha+1/2,\alpha-1/2,\alpha)} = \langle D, T_1 + A\rangle$, so that $\pi(\overline{\Gamma}_{(\alpha+1/2,\alpha-1/2,\alpha)}) = \langle D, A\rangle$, as claimed (Fig. 8.9).

Finally, we consider the plane $\mathbb{R}^2_{(\alpha,-\alpha,0)}$, which is pictured at the bottom of Fig. 8.9. We note that the image group G is a Coxeter group, generated by reflections



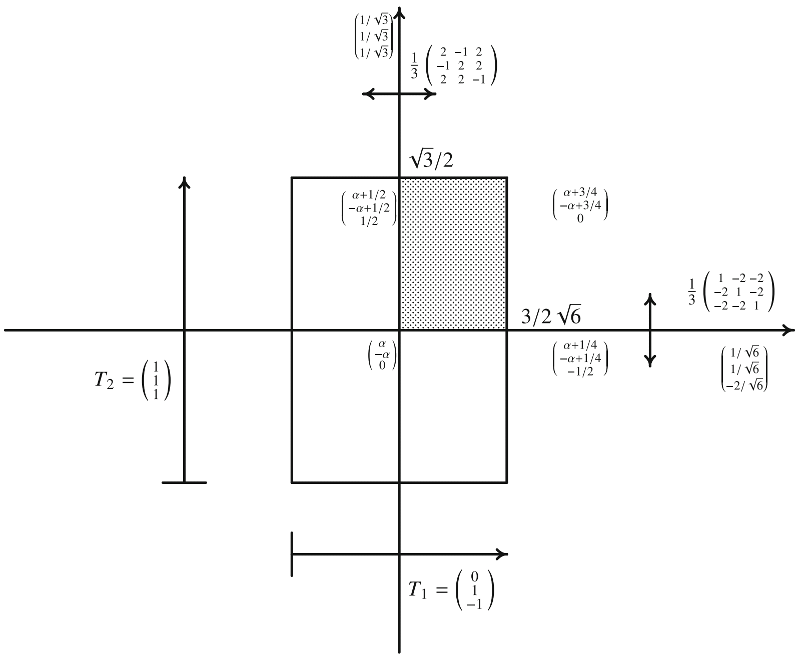


Fig. 8.9 These are the planes $\mathbb{R}^2_{(\alpha,\alpha,\alpha)}$ and $\mathbb{R}^2_{(\alpha,-\alpha,0)}$ (respectively, from the *top*), with the associated actions from $\Gamma(\ell)$, where $\Gamma = \Gamma_5$

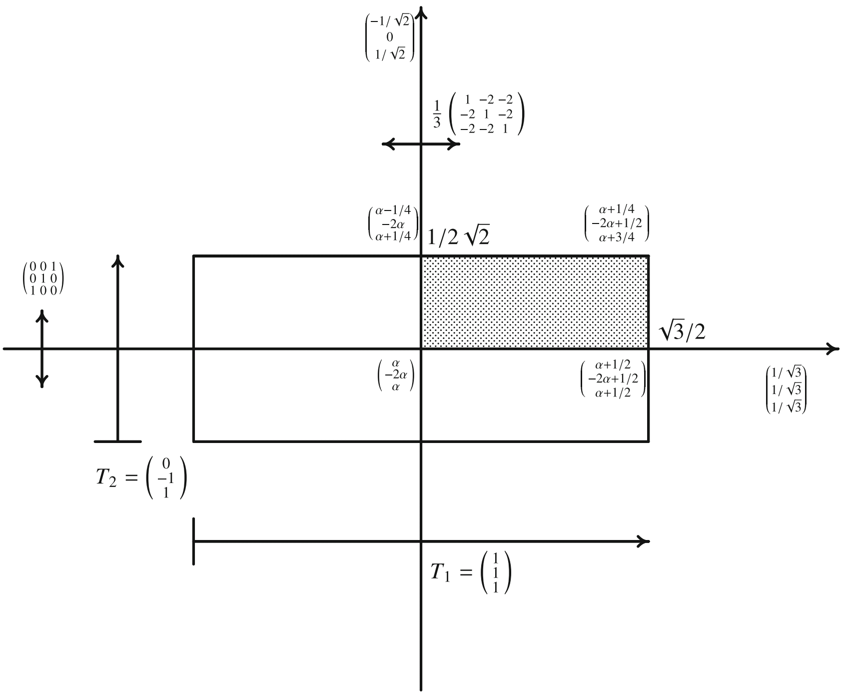


Fig. 8.10 This is the plane $\mathbb{R}^2_{(\alpha,-2\alpha,\alpha)}$ with its associated action by $\Gamma(\alpha, -2\alpha, \alpha)$, where $\Gamma = \Gamma_5$

in the sides of the shaded rectangle. It follows that all four corners are distinct in the quotient, and that the stabilizers G_ℓ have order 4, for each corner ℓ of the rectangle.

We first consider the two vertices along the vertical axis. Note first that $\langle D, E \rangle \leq \overline{\Gamma}_{(\alpha,-\alpha,0)}$, and equality is forced by Lemma 8.1(1) since the former group has order 4. Similarly, one can show that $\langle D, T_2 + E \rangle \leq \overline{\Gamma}_{(\alpha+1/2,-\alpha+1/2,1/2)}$, and equality is again forced by the same argument.

We consider the vertex $r(\alpha) = (\alpha + 1/4, -\alpha + 1/4, -1/2)$. We set $\gamma = T_1 + D$, and note that $\gamma \cdot r(\alpha) = r(\alpha - 1/2)$. We have $g = \psi(\gamma) \notin \psi(\overline{\Gamma}_{(\alpha+1/4,-\alpha+1/4,-1/2)})$ if there is no element of the kernel that acts on $r(\alpha)$ in the same way. The kernel is the group

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle;$$

these generators send $r(\alpha)$ to $r(-\alpha)$ and $r(\alpha + 1)$ (respectively). Therefore,

$$\overline{\Gamma}_{(\alpha+1/4,-\alpha+1/4,-1/2)}$$

is negligible. One argues that the remaining vertex is negligible in the same way.

Chapter 9
Cokernels of the Relative Assembly Maps for $\mathcal{VC}_\infty$

In this chapter, we will compute the contribution of the infinite virtually cyclic groups to the lower algebraic K-theory of the split three-dimensional crystallographic groups. There are three steps. First, in Sect. 9.1, we must determine the (non-negligible) strict stabilizers of lines ℓ relative to all 73 split three-dimensional crystallographic groups. This amounts to finding the indexing set $\mathcal{T}''$ from the statement of Theorem 5.1. In Chap. 8, we determined the indexing sets $\mathcal{T}''$ for the groups Γ_i ($i = 1, \ldots, 5$). We will give a procedure (Procedure 9.1) that describes, for given split crystallographic groups Γ, Γ' such that $[\Gamma : \Gamma'] < \infty$, a suitable choice of $\mathcal{T}''$ for the group Γ', assuming that a choice of $\mathcal{T}''$ for Γ is known to us. The procedure is analogous to the one that we have already used in Chap. 7 (see Procedure 7.1), when we were determining vertex stabilizers in $E_{\mathcal{FIN}}(\Gamma)$. We will also identify the strict stabilizer groups for all of the lines in question.

The next step is to rebuild the line stabilizer groups from their strict stabilizers. This is done in Sect. 9.2 using Procedure 9.2. The final step is to determine the remaining factor from Theorem 5.1:

$$\bigoplus_{\hat{\ell} \in \mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty}).$$

This will be done in Sect. 9.3. The nontrivial cokernels are summarized in Table 9.11.

9.1 Passing to Subgroups

In this section, we will list, for each split crystallographic group Γ, a complete set $\mathcal{T}''$ of orbit representatives of lines ℓ with non-negligible strict stabilizer groups. We will also describe the isomorphism types of the latter groups. In Sect. 8.2, we saw such lists for the groups Γ_i ($i = 1, \ldots, 5$). Our method of computing the strict

D.S. Farley, I.J. Ortiz, *Algebraic K-theory of Crystallographic Groups*, Lecture Notes in Mathematics 2113, DOI 10.1007/978-3-319-08153-3_9

stabilizer groups of lines for the remaining groups (Procedure 9.1 below) involves passing to finite index subgroups and generally follows the pattern of Procedure 7.1. In Sect. 9.2, we will describe how to compute the stabilizer group of a line from its strict stabilizer.

Procedure 9.1. *Let $\Gamma = \langle L, H\rangle$ and $\Gamma' = \langle L', H'\rangle$ be split crystallographic groups, where Γ' has finite index in Γ. Let $\mathcal{L}(\Gamma)$ denote a set of lines $\ell \subseteq \mathbb{R}^3$ that contains exactly one line ℓ from each Γ-orbit of lines $\hat{\ell}$ with non-negligible strict stabilizer $\overline{\Gamma}_{\hat{\ell}}$. We describe a procedure that computes an analogous set $\mathcal{L}(\Gamma')$ for Γ'.*

1. *Let T be a finite right transversal for Γ' in Γ. First choose one line from each Γ'-orbit of (unparametrized) lines in $T \cdot \mathcal{L}(\Gamma)$. Denote the resulting set $\mathcal{L}'(\Gamma')$.*
2. *For each $\ell \in \mathcal{L}'(\Gamma')$, compute the strict stabilizer $\overline{\Gamma}'_\ell$, which can be done as follows. Assume that ℓ admits the parametrization $r(\alpha) = t + \alpha v$. We have the equality $\pi(\overline{\Gamma}'_\ell) = \{h' \in H'_v \mid t - h'(t) \in L'\}$, where $\pi : \Gamma' \to H'$ is the usual projection to the point group. We recall that $\pi : \overline{\Gamma}'_\ell \to \pi(\overline{\Gamma}'_\ell)$ is an isomorphism.*
3. *We eliminate a line ℓ from the list $\mathcal{L}'(\Gamma')$ if its strict stabilizer is* negligible. *The resulting list is $\mathcal{L}(\Gamma')$.*

Next, we illustrate this procedure in a few representative examples, and summarize the results in Tables 9.1, 9.2, 9.3, 9.4, and 9.5. We note that the matrices A, B, and C from those tables are the same as the ones from Theorem 8.2, and D, E, and F are the same as the ones from Theorem 8.5.

Example 9.1. We first let $\Gamma = \Gamma_1$ and $\Gamma' = \langle L, D_4^+ \times (-1)\rangle$, where L is the standard cubical lattice. It is not difficult to see that $T = C_3^+$ is a right transversal for Γ' in Γ. Recall that a set $\mathcal{L}(\Gamma)$ was computed in Theorem 8.1; the results of that computation are summarized in the top row of Table 9.1. Following Procedure 9.1, we apply T to the five parametrized lines in the top row of Table 9.1. This results in a list of 15 lines (which are obtained by cyclically permuting the entries of the original five, by the description of C_3^+ in Theorem 2.1). Since the permutation matrix which swaps the first two coordinates is in $D_4^+ \times (-1)$, it is easy to check that we can eliminate four lines from our list, since these four are in the same Γ'-orbits as some of the other 11. One possible choice of the remaining 11 lines is as follows:

$$\begin{pmatrix} * \\ * \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ * \\ * \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ * \end{pmatrix}, \begin{pmatrix} * \\ \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha \\ 1/2 \end{pmatrix},$$

where $* \in \{0, 1/2\}$ (and must be one or the other, not both within one vector). It is now straightforward to check that no two of the above lines are in the same Γ'-orbit. We may therefore let the above collection be $\mathcal{L}'(\Gamma')$.

Next, we compute the strict stabilizers of each of the new lines. Set $H = D_4^+ \times (-1)$. For the vectors $v = \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}$, we have

$$H_v = D'_{2_1}, \quad D'_{2_2}, \quad D''_4, \quad \langle A, B\rangle, \quad \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

Table 9.1 The entries in the table are parametrized lines; if a line ℓ appears in the row and column labelled (respectively) H_1 and H_2, then $\pi(\overline{\Gamma}_\ell) = H_2$, where $\Gamma = \langle L, H_1 \rangle$ and L is the standard cubical lattice

H	D_4''	D_2'	D_{2_1}'	D_{2_2}'	C_4^+	$\langle A, B\rangle$	$\langle A, C\rangle$
$S_4^+ \times (-1)$	$(0,0,\alpha)$	$(\frac{1}{2},0,\alpha)$				$(\alpha,\alpha,0)$	
	$(\frac{1}{2},\frac{1}{2},\alpha)$					$(\alpha,\alpha,\frac{1}{2})$	
S_4^+					$(0,0,\alpha)$		
					$(\frac{1}{2},\frac{1}{2},\alpha)$		
S_4'							$(0,0,\alpha)$
							$(\frac{1}{2},\frac{1}{2},\alpha)$
$A_4^+ \times (-1)$		$(0,0,\alpha)$					
		$(\frac{1}{2},0,\alpha)$					
		$(0,\frac{1}{2},\alpha)$					
		$(\frac{1}{2},\frac{1}{2},\alpha)$					
$D_4^+ \times (-1)$	$(0,0,\alpha)$	$(\frac{1}{2},0,\alpha)$	$(\alpha,0,0)$	$(0,\alpha,\frac{1}{2})$		$(\alpha,\alpha,0)$	
	$(\frac{1}{2},\frac{1}{2},\alpha)$		$(\alpha,\frac{1}{2},0)$			$(\alpha,\alpha,\frac{1}{2})$	
			$(\alpha,\frac{1}{2},\frac{1}{2})$				
D_4^+					$(0,0,\alpha)$		
					$(\frac{1}{2},\frac{1}{2},\alpha)$		
$C_4^+ \times (-1)$					$(0,0,\alpha)$		
					$(\frac{1}{2},\frac{1}{2},\alpha)$		
C_4^+					$(0,0,\alpha)$		
					$(\frac{1}{2},\frac{1}{2},\alpha)$		
$D_2^+ \times (-1)$		$(0,0,\alpha)$	$(\alpha,0,0)$	$(0,\alpha,0)$			
		$(0,\frac{1}{2},\alpha)$	$(\alpha,\frac{1}{2},0)$	$(0,\alpha,\frac{1}{2})$			
		$(\frac{1}{2},\frac{1}{2},\alpha)$	$(\alpha,0,\frac{1}{2})$	$(\frac{1}{2},\alpha,0)$			
		$(\frac{1}{2},0,\alpha)$	$(\alpha,\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\alpha,\frac{1}{2})$			
D_2'		$(0,0,\alpha)$					
		$(\frac{1}{2},0,\alpha)$					
		$(0,\frac{1}{2},\alpha)$					
		$(\frac{1}{2},\frac{1}{2},\alpha)$					
D_4'							$(0,0,\alpha)$
							$(\frac{1}{2},\frac{1}{2},\alpha)$
$\hat{D}_4'$		$(0,0,\alpha)$					
		$(\frac{1}{2},\frac{1}{2},\alpha)$					
		$(\frac{1}{2},0,\alpha)$					
D_4''	$(0,0,\alpha)$	$(\frac{1}{2},0,\alpha)$					
	$(\frac{1}{2},\frac{1}{2},\alpha)$						

Table 9.2 The entries in the table are parametrized lines; if a line ℓ appears in the row and column labelled (respectively) H_1 and H_2, then $\pi(\overline{\Gamma}_\ell) = H_2$, where $\Gamma = \langle L, H_1\rangle$ and L is the lattice $\langle \frac{1}{2}(\mathbf{x}+\mathbf{y}+\mathbf{z}), \mathbf{y}, \mathbf{z}\rangle$

H	D_4''	D_2'	D_{2_1}'	D_{2_2}'	C_4^+	$\langle A, B\rangle$	$\langle A, C\rangle$
$S_4^+ \times (-1)$	$(0,0,\alpha)$	$(\frac{1}{2},0,\alpha)$				$(\alpha,\alpha,0)$	
S_4'							$(0,0,\alpha)$
$A_4^+ \times (-1)$		$(0,0,\alpha)$					
		$(\frac{1}{2},0,\alpha)$					
D_4''	$(0,0,\alpha)$	$(\frac{1}{2},0,\alpha)$					
S_4^+					$(0,0,\alpha)$		
$D_4^+ \times (-1)$	$(0,0,\alpha)$	$(\frac{1}{2},0,\alpha)$		$(0,\alpha,\frac{1}{2})$		$(\alpha,\alpha,0)$	
				$(0,\alpha,0)$			
D_4^+					$(0,0,\alpha)$		
$C_4^+ \times (-1)$					$(0,0,\alpha)$		
C_4^+					$(0,0,\alpha)$		
$D_2^+ \times (-1)$		$(0,0,\alpha)$	$(\alpha,0,0)$	$(0,\alpha,0)$			
		$(\frac{1}{2},0,\alpha)$	$(\alpha,\frac{1}{2},0)$	$(0,\alpha,\frac{1}{2})$			
D_4'							$(0,0,\alpha)$
$\hat{D}_4'$		$(0,0,\alpha)$					
		$(\frac{1}{2},0,\alpha)$					
D_2'		$(0,0,\alpha)$					
		$(\frac{1}{2},0,\alpha)$					

Table 9.3 The entries in the table are parametrized lines; if a line ℓ appears in the row and column labelled (respectively) H_1 and H_2, then $\pi(\overline{\Gamma}_\ell) = H_2$, where $\Gamma = \langle L, H_1\rangle$ and L is the lattice $\langle \frac{1}{2}(\mathbf{x}+\mathbf{y}), \frac{1}{2}(\mathbf{x}+\mathbf{z}), \frac{1}{2}(\mathbf{y}+\mathbf{z})\rangle$

H	D_4''	D_2'	D_{2_1}'	D_{2_2}'	C_4^+	$\langle A, B\rangle$	$\langle A, C\rangle$
$S_4^+ \times (-1)$	$(0,0,\alpha)$					$(\alpha,\alpha,0)$	$(\frac{1}{4},\frac{1}{4},\alpha)$
						$(\alpha,\alpha,\frac{1}{2})$	
S_4^+					$(0,0,\alpha)$		
S_4'							$(0,0,\alpha)$
							$(\frac{1}{4},\frac{1}{4},\alpha)$
$A_4^+ \times (-1)$		$(0,0,\alpha)$					
D_2'		$(0,0,\alpha)$					
$D_2^+ \times (-1)$		$(0,0,\alpha)$	$(\alpha,0,0)$	$(0,\alpha,0)$			

respectively. The last group has order 2, and it follows that each line of the form $(*,\alpha,\alpha)$ is negligible. We note that the strict stabilizer $\overline{\Gamma}'_{(1/2,0,\alpha)}$ has order at most 4, since $|\overline{\Gamma}'_{(1/2,0,\alpha)}|$ divides $|H_v| = 8$, and $A \notin \pi(\overline{\Gamma}'_{(1/2,0,\alpha)})$. One can check that $D_2' \leq \pi(\overline{\Gamma}'_{(1/2,0,\alpha)})$, so equality must hold. It is straightforward to check that each of the remaining parametrized lines $r(\alpha) = t + \alpha v$ satisfies $\pi(\overline{\Gamma}'_\ell) = H_v$.

Table 9.4 The entries in the table are parametrized lines; if a line ℓ appears in the row and column labelled (respectively) H_1 and H_2, then $\pi(\overline{\Gamma}_\ell) = H_2$, where $\Gamma = \langle L, H_1 \rangle$ and L is the lattice $\langle \frac{1}{2}(\mathbf{x}+\mathbf{z}), \mathbf{y}, \mathbf{z} \rangle$

H	D_2'	D_{2_1}'	D_{2_2}'
$D_2^+ \times (-1)$	$(0, 0, \alpha)$	$(\alpha, 0, 0)$	$(0, \alpha, 0)$
	$(0, \frac{1}{2}, \alpha)$	$(\alpha, \frac{1}{2}, 0)$	$(0, \alpha, \frac{1}{2})$
D_2'	$(0, 0, \alpha)$		
	$(0, \frac{1}{2}, \alpha)$		
D_{2_2}'			$(0, \alpha, 0)$
			$(0, \alpha, \frac{1}{2})$

Table 9.5 The entries in the table are parametrized lines; if a line ℓ appears in the row and column labelled (respectively) H_1 and H_2, then $\pi(\overline{\Gamma}_\ell) = H_2$, where $\Gamma = \langle L, H_1 \rangle$ and L is the lattice $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$

H	D_6''	$\langle A, D \rangle$	$\langle D, E \rangle$	$\langle E, F \rangle$
$D_6^+ \times (-1)$	(α, α, α)	$(\alpha + \frac{1}{2}, \alpha - \frac{1}{2}, \alpha)$	$(\alpha, -\alpha, 0)$	$(\alpha, -2\alpha, \alpha)$
			$(\alpha + \frac{1}{2}, -\alpha + \frac{1}{2}, \frac{1}{2})$	$(\alpha + \frac{1}{2}, -2\alpha + \frac{1}{2}, \alpha + \frac{1}{2})$
D_6''	(α, α, α)	$(\alpha + \frac{1}{2}, \alpha - \frac{1}{2}, \alpha)$		
D_6'				$(\alpha, -2\alpha, \alpha)$
				$(\alpha + \frac{1}{2}, -2\alpha + \frac{1}{2}, \alpha + \frac{1}{2})$
$\hat{D}_6'$			$(\alpha, -\alpha, 0)$	
			$(\alpha + \frac{1}{2}, -\alpha + \frac{1}{2}, \frac{1}{2})$	

It now follows that the remaining nine lines form $\mathcal{L}(\Gamma')$. The strict stabilizer groups of these lines (as computed above) are described in the relevant line of Table 9.1.

Example 9.2. Now we let $\Gamma = \langle L, D_4^+ \times (-1) \rangle$ and $\Gamma' = \langle L, D_4' \rangle$, where L is still the standard cubical lattice. We can take the group generated by the antipodal map as T. Applying the antipodal map to the nine lines of $\mathcal{L}(\Gamma)$ from Example 9.1, we get six new ones (three of the lines are invariant under the antipodal map). It is straightforward to check that these six new lines are all in Γ'-orbits of the original nine. For instance, $(-\alpha, -1/2, 0)$ (one of the new lines) is clearly in the Γ'-orbit of $(-\alpha, 1/2, 0) = (\alpha, 1/2, 0) \in \mathcal{L}(\Gamma)$. The five remaining lines are redundant by similar (easy) calculations. It follows that we can take $\mathcal{L}'(\Gamma') = \mathcal{L}(\Gamma)$.

Next, we compute the strict stabilizer $\overline{\Gamma}'_\ell$ of each $\ell \in \mathcal{L}'(\Gamma')$. Recall that each element $M \in D_4'$ can be factored as $M = SP$, where P is a permutation matrix that fixes the last coordinate and S is a diagonal matrix with an even number of -1s on the diagonal, the remaining diagonal entries being 1. We let $H = D_4'$. We need to find the stabilizer groups H_v for $v = \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x}+\mathbf{y}$; by the above description of D_4', these groups are

$$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

respectively. It follows directly that all of the lines except possibly

$$(0, 0, \alpha), \quad (1/2, 1/2, \alpha), \quad (1/2, 0, \alpha)$$

are negligible. It is straightforward to check that the first two lines ℓ satisfy $\pi(\overline{\Gamma}'_\ell) = H_{\mathbf{z}}$. For the final line ℓ, we have $|\pi(\overline{\Gamma}'_\ell)| = 2$, so the strict stabilizer of ℓ is negligible.

This implies that $\mathcal{L}(\Gamma') = \{(0,0,\alpha), (1/2, 1/2, \alpha)\}$. The strict stabilizer groups of the latter lines are recorded in Table 9.1.

Example 9.3. Now we let $\Gamma = \langle L, D_4^+ \times (-1)\rangle$ and $\Gamma' = \langle L, \hat{D}'_4\rangle$, where L is the standard cubical lattice. We can again choose the group generated by the antipodal map as T. Exactly the same calculation as in Example 9.2 shows that we arrive at the same set of nine lines $\mathcal{L}'(\Gamma')$.

Set $H = \hat{D}'_4$. Recall that

$$\hat{D}'_4 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

One can check (by listing all of the matrices in $\hat{D}'_4$), that $D'_2 \leq \hat{D}'_4$, and that the remaining matrices can all be expressed in the form $M = SP$, where P is the permutation matrix that interchanges the first two coordinates, and S is a diagonal matrix with 1s and -1s down the diagonal, but with a -1 in the bottom corner. With this description, it is easy to check that the stabilizer groups H_v for $v = \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x}+\mathbf{y}$ satisfy

$$H_v = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle, \quad D'_2, \quad \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle,$$

respectively. It follows directly that a line $\ell \in \mathcal{L}'(\Gamma')$ has a negligible strict stabilizer group unless it has $\mathbf{z}$ as a tangent vector. Thus,

$$\mathcal{L}(\Gamma') = \{(0,0,\alpha), (1/2, 1/2, \alpha), (1/2, 0, \alpha)\}.$$

Finally, an easy check shows that $\pi(\overline{\Gamma}'_\ell) = D'_2$ for each of the latter three lines, confirming the entries in the relevant row of Table 9.1.

9.2 Reconstructing Γ_ℓ from $\overline{\Gamma}_\ell$

Let $\Gamma = \langle L, H\rangle$. Assume that ℓ is one of the parametrized lines from Tables 9.1, 9.2, 9.3, 9.4, or 9.5. We now describe a procedure for computing the stabilizer group Γ_ℓ of ℓ from its strict stabilizer $\overline{\Gamma}_\ell$ (which we computed in Sect. 9.1).

Procedure 9.2. *We may assume that ℓ has a parametrization of the form $r(\alpha) = \frac{1}{2}t + \alpha v$, where $t, v \in L$, $t \perp v$, and $\beta v \notin L$ for $\beta \in (0,1)$, since all of the parametrized lines in our tables have this form.*

1. *Find the smallest positive constant C such that there is $\gamma_T \in \Gamma$ with the property that $\gamma_T \cdot r(\alpha) = r(\alpha + C)$. (The isometry γ_T is the translation that acts on ℓ with minimal translation length.) A smallest C always exists, and we can find both C and γ_T as follows:*

   a. *Compute the set*

   $$A_{v,L} = \left\{ \frac{v_1 \cdot v}{v \cdot v} \mid v_1 \in L \right\}.$$

   This set is a finitely generated subgroup of the rational numbers, and therefore cyclic. Let $q > 0$ be a generator. Note that $1 \in A_{v,L}$, so q is the reciprocal of a positive integer.

   b. *Determine the smallest number C in the sequence q, $2q$, $3q$, ..., such that there is $h \in H$ satisfying: $h(v) = v$ and $\frac{1}{2}(t - h(t)) + Cv \in L$.*

   c. *We set*

   $$\gamma_T = \frac{1}{2}(t - h(t)) + Cv + h.$$

   We note that this entire step becomes trivial to perform if $t = 0$ or if $A_{v,L}$ is the set of integers, for then $C = 1$ is forced and we can take $\gamma_T = v$ (i.e., the translation by v). In either of these cases, we will call this step easy.

2. *Determine whether some isometry $\gamma_R \in \Gamma$ acts as a reflection on the parametrized line $r(\alpha)$; that is, determine whether there is some isometry $\gamma_R \in \Gamma$ and $D \in \mathbb{R}$ such that $\gamma_R \cdot r(\alpha) = r(D - \alpha)$. This amounts to doing the following. Fix a basis b_1, b_2, b_3 for L as a free abelian group. For each $h \in H$ such that $h(v) = -v$, determine whether the equation*

   $$c_1 b_1 + c_2 b_2 + c_3 b_3 = \frac{1}{2}(t - h(t)) + Dv$$

   has a solution for integers c_i $(i = 1, 2, 3)$ and $D \in \mathbb{R}$. If the solution exists for some $h \in H$, then we set

   $$\gamma_R = \left(\frac{1}{2}(t - h(t)) + Dv \right) + h.$$

   We note that this step becomes trivial to perform if there is no $h \in H$ such that $h(v) = -v$ (in which case there can exist no γ_R), or if there is $h \in H$ such that $h(v) = -v$ and $h(t) = \pm t$ (in which case we can set $D = 0$ and let $\gamma_R = h$ or $t + h$, respectively). Note, in particular, that the latter conditions are always satisfied if $(-1) \in H$. We call all of these cases easy.

3. *There are two possible outcomes:*

   a. *If there is no reflection γ_R as above, then $\Gamma_\ell = \langle \overline{\Gamma}_\ell, \gamma_T \rangle$, and we have the isomorphism*

   $$\Gamma_\ell \cong \pi(\overline{\Gamma}_\ell) \rtimes_\phi \mathbb{Z},$$

where the action is conjugation by $\pi(\gamma_T)$ and π is the usual projection into the point group H.

b. *If there is a reflection γ_R, then we get a free product with amalgamation:*

$$\Gamma_\ell = \langle \overline{\Gamma}_\ell, \gamma_R \rangle *_{\overline{\Gamma}_\ell} \langle \overline{\Gamma}_\ell, \gamma_T \gamma_R \rangle.$$

We can determine the abstract isomorphism type of Γ_ℓ by applying the projection π to the factors and the amalgamated subgroup.

Lemma 9.1. *Procedure 9.2 is valid, and its steps can be performed algorithmically.*

Proof. We assume that $\Gamma = \langle L, H \rangle$, $\ell \subseteq \mathbb{R}^3$, and that ℓ admits a parametrization $r(\alpha) = \frac{1}{2}t + \alpha v$, where $t, v \in L$, $t \perp v$, and $\beta v \notin L$ for $\beta \in (0, 1)$.

Consider step (1) from Procedure 9.2. The set $A_{v,L}$ is an additive subgroup of $\mathbb{R}$ because of the bilinearity of the dot product. It is a set of rational numbers since each possible lattice L is contained in $\mathbb{Q}^3$, and so each member of $A_{v,L}$ is a quotient of two rational numbers. It also follows from bilinearity that $A_{v,L}$ is generated by

$$\frac{b_1 \cdot v}{v \cdot v}, \quad \frac{b_2 \cdot v}{v \cdot v}, \quad \frac{b_3 \cdot v}{v \cdot v},$$

if $\{b_1, b_2, b_3\}$ generates L as an abelian group. It follows that $A_{v,L}$ is a finitely generated additive subgroup of $\mathbb{Q}$, and therefore cyclic. It is clear that $1 \in A_{v,L}$, so the generator is the reciprocal of some positive integer, as claimed. We note that it is straightforward to find the generator q algorithmically, given the above generating set.

It is already clear that Step 1(b) can be performed algorithmically; the procedure is guaranteed to stop since $1 \in A_{v,L}$ and the given equations can always be satisfied with $h = 1$ and $C = 1$. Step 1(c) poses no problems.

We need to show that if $\gamma \in \Gamma$ acts on ℓ by the rule $\gamma \cdot r(\alpha) = r(\alpha + C')$, then $C' \in A_{v,L}$. (It is easy to show that γ_T, as defined in Procedure 9.2, satisfies $\gamma_T \cdot r(\alpha) = r(\alpha + C)$.) Assume that $\gamma \cdot r(\alpha) = r(\alpha + C')$; we let $\gamma = \hat{t} + h$, where $\hat{t} \in L$ and $h \in H$. Note that we must have $h(v) = v$ if the line ℓ is to be acted on by a translation.

$$\begin{aligned} \gamma \cdot r(\alpha) &= (\hat{t} + h)(\frac{1}{2}t + \alpha v) \\ &= \hat{t} + \frac{1}{2}h(t) + \alpha v. \end{aligned}$$

Setting the latter expression equal to $r(\alpha + C')$, we get

$$\hat{t} + \frac{1}{2}h(t) + \alpha v = \frac{1}{2}t + C'v + \alpha v \quad \Rightarrow \quad \hat{t} = \frac{1}{2}(t - h(t)) + C'v.$$

Taking the dot product with v on both sides gives the equality $\hat{t} \cdot v = C'(v \cdot v)$, since both t and $h(t)$ are perpendicular to v. It follows that $C' \in A_{v,L}$, as claimed. Therefore, Step 1 is valid, and can be performed algorithmically.

Next we consider Step 2. It is straightforward to check that the isometry γ_R (as defined in Procedure 9.2) acts on ℓ by the rule $\gamma_R \cdot r(\alpha) = r(D - \alpha)$, and that $\gamma_R \in \Gamma$ if $\frac{1}{2}(t - h(t)) + Dv \in L$. It is now enough to solve the equation

$$c_1 b_1 + c_2 b_2 + c_3 b_3 = \frac{1}{2}(t - h(t)) + Dv,$$

where the c_i $(i = 1, 2, 3)$ are integers and $D \in \mathbb{R}$, by an algorithm (or to show, by the same algorithm, that no solution exists). This is also straightforward: we represent the right side of the equation as an $\mathbb{R}$-linear combination of the basis elements b_1, b_2, b_3, treating D as an independent variable. That is,

$$\frac{1}{2}(t - h(t)) + Dv = C_1(D)b_1 + C_2(D)b_2 + C_3(D)b_3,$$

where $C_i(D)$ is a linear function of D, for $i = 1, 2, 3$. The problem of solving the original equation amounts to the problem of solving the system of congruences $C_i(D) = 0 \bmod \mathbb{Z}$, which can clearly be done algorithmically. This shows that Step 2 is valid, and can be done algorithmically.

Finally, we note that the validity of Step 3 follows from the Bass–Serre theory of groups acting on trees [Se80].

We will now show how to apply Procedure 9.2 in a few representative cases. The results of all of the calculations are summarized in Tables 9.6, 9.7, 9.8, 9.9, and 9.10.

Example 9.4. Let us first suppose that L is the standard integral lattice, and $H \leq S_4^+ \times (-1)$. Thus, $\Gamma = \langle L, H \rangle$ is one of the subgroups $\Gamma \leq \Gamma_1$, whose strict line stabilizers are recorded in Table 9.1. Note that almost all of the parametrized lines in

Table 9.6 The structure of $\mathcal{VC}_\infty$ subgroups of $\langle\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle, H\rangle$

H	$\mathcal{VC}_\infty$
$S_4^+ \times (-1)$	$D_4 \times D_\infty$ (twice), $D_2 \times D_\infty$ (three times)
S_4^+	$D_4 *_{C_4} D_4$ (twice)
S_4'	$D_4 *_{D_2} D_4$ (twice)
$A_4^+ \times (-1)$	$D_2 \times D_\infty$ (four times)
D_4''	$D_4 \times \mathbb{Z}$ (twice), $D_2 \times \mathbb{Z}$
$D_4^+ \times (-1)$	$D_4 \times D_\infty$ (twice), $D_2 \times D_\infty$ (seven times)
D_4^+	$D_4 *_{C_4} D_4$ (twice)
$D_2^+ \times (-1)$	$D_2 \times D_\infty$ (twelve times)
$C_4^+ \times (-1)$	$C_4 \times D_\infty$ (twice)
D_2'	$D_2 \times \mathbb{Z}$ (four times)
D_4'	$D_4 *_{D_2} D_4$ (twice)
$\hat{D}_4'$	$D_4 *_{D_2} D_4$ (twice), $D_2 \times \mathbb{Z}$

Table 9.7 The structure of $\mathcal{VC}_\infty$ subgroups of $\langle\langle \frac{1}{2}(\mathbf{x}+\mathbf{y}+\mathbf{z}), \mathbf{y}, \mathbf{z}\rangle, H\rangle$

H	$\mathcal{VC}_\infty$
$S_4^+ \times (-1)$	$D_4 \times D_\infty,\ D_2 \times D_\infty, (D_2 \times \mathbb{Z}/2) *_{D_2} D_4$
S_4^+	$D_4 *_{C_4} D_4$
S_4'	$D_2 \times D_\infty$
$A_4^+ \times (-1)$	$D_2 \times D_\infty$(twice)
D_4''	$D_4 \times \mathbb{Z}$ (twice), $D_2 \times \mathbb{Z}$
$D_4^+ \times (-1)$	$D_4 \times D_\infty,\ D_2 \times D_\infty$ (three times), $(D_2 \times \mathbb{Z}/2) *_{D_2} D_4$
D_4^+	$D_4 \times \mathbb{Z},\ D_2 \rtimes_\alpha \mathbb{Z}$ (with $\alpha^2 = 1$)
$D_2^+ \times (-1)$	$D_2 \times D_\infty$ (six times)
$C_4^+ \times (-1)$	$C_4 \times D_\infty$
C_4^+	$C_4 \times \mathbb{Z}$
D_2'	$D_2 \times \mathbb{Z}$ (twice)
D_4'	$D_4 *_{D_2} D_4$
$\hat{D}_4'$	$D_4 *_{D_2} D_4$ (twice)

Table 9.8 The structure of $\mathcal{VC}_\infty$ subgroups of $\langle\langle \frac{1}{2}(\mathbf{x}+\mathbf{y}), \frac{1}{2}(\mathbf{x}+\mathbf{z}), \frac{1}{2}(\mathbf{y}+\mathbf{z})\rangle, H\rangle$

H	$\mathcal{VC}_\infty$
$S_4^+ \times (-1)$	$D_4 \times D_\infty,\ D_2 \times D_\infty$(two times), $(D_2 \times \mathbb{Z}/2) *_{D_2} D_4$
S_4^+	$D_4 *_{C_4} D_4$
S_4'	$D_4 *_{D_2} D_4$ (twice)
$A_4^+ \times (-1)$	$D_2 \times D_\infty$
D_2'	$D_2 \times \mathbb{Z}$
$D_2^+ \times (-1)$	$D_2 \times D_\infty$ (three times)

Table 9.9 The structure of $\mathcal{VC}_\infty$ subgroups of $\langle\langle \frac{1}{2}(\mathbf{x}+\mathbf{z}), \mathbf{y}, \mathbf{z}\rangle, H\rangle$

H	$\mathcal{VC}_\infty$
$D_2^+ \times (-1)$	$D_2 \times D_\infty$ (six times)
D_2'	$D_2 \times \mathbb{Z}$ (twice)
D_{2_2}'	$D_2 \times \mathbb{Z}$ (twice)

Table 9.10 The structure of $\mathcal{VC}_\infty$ subgroups of $\langle\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\rangle, H\rangle$

H	$\mathcal{VC}_\infty$
$D_6^+ \times (-1)$	$D_2 \times D_\infty$ (five times), $D_6 \times D_\infty$
D_6'	$D_2 \times \mathbb{Z}$ (twice)
$\hat{D}_6'$	$D_2 \times \mathbb{Z}$ (twice)
D_6''	$D_2 \times \mathbb{Z},\ D_6 \times \mathbb{Z}$

Table 9.1 have the property that $v \cdot v = 1$. (The four exceptions occur in the column headed $\langle A, B\rangle$.) In all such cases, $A_{v,L}$ must be $\mathbb{Z}$, since the dot product $v_1 \cdot v$ is an integer when $v_1, v \in \mathbb{Z}^3 = L$. It follows that Step 1 of Procedure 9.2 is easy in all of these cases, and we can take $\gamma_T = v$.

This leaves four more cases to be considered. The remaining lines have the form $(\alpha, \alpha, 0)$ or $(\alpha, \alpha, 1/2)$. We note that Step 1 is still easy for the line $(\alpha, \alpha, 0)$ (with respect to both crystallographic groups, since $t = 0$), so we can again take $\gamma_T = v = (1, 1, 0)$.

We consider the remaining lines. Suppose first that $H = S_4^+ \times (-1)$ and ℓ has the parametrization $r(\alpha) = (\alpha, \alpha, 1/2)$. It follows that $t = (0, 0, 1)$ and $v = (1, 1, 0)$. An easy check shows that $A_{v,L} = \frac{1}{2}\mathbb{Z}$, so that $q = 1/2$ is a generator. We must determine whether there is $h \in H$ such that $h(v) = v$ and $\frac{1}{2}(t - h(t)) + \frac{1}{2}v \in L$. There are four signed permutation matrices $h \in H$ such that $h(v) = v$:

$$\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&-1 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&-1 \end{pmatrix}.$$

All of these matrices h have the property that $\frac{1}{2}(t - h(t)) \in L$. In particular, we can have $\frac{1}{2}(t - h(t)) + \frac{1}{2}v \in L$ if and only if $\frac{1}{2}v \in L$. Since the latter inclusion is false, there is no h with the required properties.

Now we must consider the next largest element of $A_{v,L}$, namely 1. We ask for an $h \in H$ that satisfies the conditions $h(v) = v$ and $\frac{1}{2}(t - h(t)) + v \in L$. It is clear that we can set $h = 1$, and we then set $\gamma_T = v$. This is the minimal translation to act on the line $(\alpha, \alpha, 1/2)$.

The case in which $H = D_4^+ \times (-1)$ and $r(\alpha) = (\alpha, \alpha, 1/2)$ is similar, and we again have $\gamma_T = v$. It follows that the minimal translation to act on ℓ is always $\gamma_T = v$, for all of the lines in Table 9.1.

Next we consider Step 2. Recall that this step is always easy if $(-1) \in H$ (see Procedure 9.2), and we can set $\gamma_R = (-1)$ or $t + (-1)$. For instance, suppose that $H = S_4^+ \times (-1)$. Using the fact that $\pi(\gamma_T) = 1$ and $\pi(\gamma_R) = (-1)$ for each of the five parametrized lines $r(\alpha)$ in the corresponding row of Table 9.1, Step 3 of Procedure 9.2 shows that the line stabilizers are

$$(D_4 \times \mathbb{Z}/2) *_{D_4} (D_4 \times \mathbb{Z}/2) \quad \text{and} \quad (D_2 \times \mathbb{Z}/2) *_{D_2} (D_2 \times \mathbb{Z}/2),$$

where the first group appears twice, and the second appears three times. (Here we record only the isomorphism types of the stabilizer groups.) The latter groups are isomorphic to $D_4 \times D_\infty$ and $D_2 \times D_\infty$, respectively. This calculation is recorded in the first line of Table 9.6.

In fact, Step 2 is often quite easy in practice when $L = \mathbb{Z}^3$, even if $(-1) \notin H$. For instance, assume that $H = S_4^+$. We note that for

$$h = \begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&-1 \end{pmatrix},$$

the isometries $\gamma_1 = h$ and $\gamma_2 = (0, 1, 0) + h$ act on the lines $(0, 0, \alpha)$ and $(1/2, 1/2, \alpha)$ (respectively) as reflections γ_R with $D = 0$. By our results from Step 1, we therefore have $\pi(\gamma_T) = 1$ and $\pi(\gamma_R) = h$ for both of these lines. It follows from Step 3 and Table 9.1 that both line stabilizers have the form

$$\langle C_4^+, h\rangle *_{C_4^+} \langle C_4^+, h\rangle.$$

Since $\langle C_4^+, h\rangle = D_4^+$, the above amalgam reduces to the one listed in the second row of Table 9.6. (Note that this case is not "easy" in the sense described in Procedure 9.2.)

In a few cases, Step 2 is "easy" because there is no $h \in H$ that reverses the direction of v. This is the case for $H = D_2'$ and $H = D_4''$ in Table 9.1, for example.

Example 9.5. We will now run Procedure 9.2 with the group Γ_2 and the line $(1/2, 0, \alpha)$ as input (see Table 9.2). (Thus, $t = (1,0,0)$ and $v = (0,0,1)$.)

An easy calculation establishes that $A_{v,L} = \frac{1}{2}\mathbb{Z}$. We must determine whether there is $h \in H$ such that $h(v) = v$ and $\frac{1}{2}(t - h(t)) + \frac{1}{2}v \in L$. One can check that these conditions are satisfied by

$$h = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that a minimal translation γ_T satisfies $\gamma_T \cdot r(\alpha) = r(\alpha + 1/2)$, and that γ_T can be chosen so that $\pi(\gamma_T) = h$.

We turn to Step 2. This step is easy because $(-1) \in H$. We can take $\gamma_R = t + (-1)$ (and $D = 0$).

It follows that the stabilizer is isomorphic to

$$\left\langle D_2', (-1)\right\rangle *_{D_2'} \left\langle D_2', -h\right\rangle \cong (D_2 \times \mathbb{Z}/2) *_{D_2} D_4.$$

This stabilizer is recorded in the first line of Table 9.7.

Example 9.6. Consider Γ_3 and the parametrized line $(1/4, 1/4, \alpha)$ (see Table 9.3). Thus $t = (1/2, 1/2, 0)$ and $v = (0,0,1)$.

It is straightforward to check that $A_{v,L} = \frac{1}{2}\mathbb{Z}$. We therefore begin by checking for $h \in H$ that satisfy $h(v) = v$ and $\frac{1}{2}(t - h(t)) + \frac{1}{2}v \in L$. Note that D_4'' is the stabilizer group H_v. Setting

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we get a solution. It follows that there is a minimal translation γ_T such that $\pi(\gamma_T) = h$.

Step 2 is easy—we simply set $\gamma_R = t + (-1)$ (and $D = 0$).

It follows that the stabilizer group has the form

$$\langle A, C, (-1)\rangle *_{\langle A,C\rangle} \langle A, C, -h\rangle \cong (D_2 \times \mathbb{Z}/2) *_{D_2} D_4.$$

This calculation contributes the asymmetric amalgam in the first line of Table 9.8.

9.3 Cokernels of Relative Assembly Maps

In view of Theorem 5.1, we will need for our computations the cokernels of the relative assembly maps for the various maximal infinite virtually cyclic subgroups that were listed in Sect. 9.2.

Proposition 9.1 (Maximal infinite virtually cyclic subgroups). *The following list contains* all *of the* non-negligible *maximal infinite virtually cyclic groups that appear as subgroups of the 73 split three-dimensional crystallographic groups:* $C_4 \times \mathbb{Z}$, $C_4 \rtimes \mathbb{Z}$, $D_2 \times \mathbb{Z}$, $D_2 \rtimes_\alpha \mathbb{Z}$ *(with* $|\alpha| = 2$*),* $D_4 \times \mathbb{Z}$, $D_6 \times \mathbb{Z}$, $D_4 *_{C_4} D_4$, $D_4 *_{D_2} D_4$, $(D_2 \times \mathbb{Z}/2) *_{D_2} D_4$, $C_4 \times D_\infty$, *and* $D_n \times D_\infty$ *for* $n = 2, 4$, *and* 6.

We first note that for the groups $D_2 \times \mathbb{Z}$, $(D_2 \times \mathbb{Z}/2) *_{D_2} D_4$, $D_4 *_{D_2} D_4$, and $D_n \times D_\infty$ for $n = 2, 4$, the cokernels have already been computed by Lafont and the second author in [LO07, Sect. 4] and [LO09, Sects. 6.2, 6.3, 6.4]. The remaining groups in our list will be discussed in the following subsections.

Observe that by a result of Farrell and Jones [FJ95], the cokernels of the relative assembly maps $H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *)$ are automatically trivial for $n < -1$ (in fact, both the source and target groups vanish in this case). In the same paper, they establish that, for the case $n = -1$, these cokernels are finitely generated, which, by results of Farrell [F77], Ramos [Ra], and Grunewald [G07], implies that the cokernel is actually trivial. In particular, we only need to focus on the cases $n = 0$ and $n = 1$. These cokernels are precisely the elusive Bass, Farrell, and Waldhausen Nil-groups.

It follows from additional results of Farrell and Jones [FJ95] that the groups $K_{-1}(\mathbb{Z}\Gamma_{\hat{\ell}})$ are generated by the images of the groups $K_{-1}(\mathbb{Z}F)$, where F runs over finite subgroups of $\Gamma_{\hat{\ell}}$, and the maps in question are induced by inclusion.

We are able to identify all of these cokernels explicitly, with the exceptions of $NK_1(\mathbb{Z}[D_4])$ and $NK_1(\mathbb{Z}[D_6])$ (see Sects. 9.3.1 and 9.3.5). The first group is known to be an infinite torsion group of exponent 2 or 4 [We09]. We summarize the known non-trivial cokernels in Table 9.11.

9.3.1 *The Lower Algebraic K-Theory of* $C_4 \times \mathbb{Z}$, $D_4 \times \mathbb{Z}$, *and* $D_6 \times \mathbb{Z}$

The Bass–Heller–Swan decomposition yields the following isomorphism for any group F and $q \leq 1$:

$$Wh_q(F \times \mathbb{Z}) \cong Wh_{q-1}(F) \oplus Wh_q(F) \oplus NK_q(\mathbb{Z}F) \oplus NK_q(\mathbb{Z}F).$$

Also, as mentioned earlier, $K_q(\mathbb{Z}V)$ is zero for $q < -1$ (see [FJ95]).

Let us consider first the cases of $Wh_q(V)$ for $V = C_4 \times \mathbb{Z}$ and $D_4 \times \mathbb{Z}$. Since $Wh_q(C_4) \cong 0$ for $q \leq 1$ (Chap. 7, Table 7.1) and $NK_q(\mathbb{Z}C_4) \cong \bigoplus_\infty \mathbb{Z}/2$ for

Table 9.11 Cokernels of relative assembly maps for maximal infinite $V \in \mathcal{VC}_\infty$

$V \in \mathcal{VC}_\infty$	$H_0^V(E_{\mathcal{FIN}}(V) \to *) \neq 0$	$H_1^V(E_{\mathcal{FIN}}(V) \to *) \neq 0$
$C_4 \times \mathbb{Z}$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_2 \times \mathbb{Z}$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_2 \rtimes_\alpha \mathbb{Z}$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_4 \times \mathbb{Z}$	$\bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$2NK_1(\mathbb{Z}D_4)$
$D_6 \times \mathbb{Z}$	$\bigoplus_\infty \mathbb{Z}/2$	$2NK_1(\mathbb{Z}D_6)$
$C_4 \times D_\infty$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_2 \times D_\infty$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_4 *_{D_2} D_4$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_4 *_{C_4} D_4$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2 \times \mathbb{Z}/2) *_{D_2} D_4$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$D_4 \times D_\infty$	$\bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$NK_1(\mathbb{Z}D_4)$
$D_6 \times D_\infty$	$\bigoplus_\infty \mathbb{Z}/2$	$NK_1(\mathbb{Z}D_6)$

$q = 0, 1$ (see [We09]), it follows that

$$Wh_q(C_4 \times \mathbb{Z}) \cong \begin{cases} \bigoplus_\infty \mathbb{Z}/2 & q = 1 \\ \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ 0 & q \leq -1, \end{cases}$$

and the cokernels of the relative assembly maps are both isomorphic to $\bigoplus_\infty \mathbb{Z}/2$.

Since $Wh_q(D_4) \cong 0$ for $q \leq 1$, it follows that

$$Wh_q(D_4 \times \mathbb{Z}) \cong \begin{cases} 2NK_1(\mathbb{Z}D_4) & q = 1 \\ \bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4 & q = 0 \\ 0 & q \leq -1, \end{cases}$$

since the Bass Nil-group $NK_0(\mathbb{Z}D_4) \cong \bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$ was computed by Weibel in [We09]. He also showed that $NK_1(\mathbb{Z}D_4)$ is a countably infinite torsion group of exponent 2 or 4.

Since $K_{-1}(\mathbb{Z}D_6) \cong \mathbb{Z}$ and $Wh_q(D_6) \cong 0$ for $q = 0, 1$, it follows that

$$Wh_q(D_6 \times \mathbb{Z}) \cong \begin{cases} 2NK_1(\mathbb{Z}D_6) & q = 1 \\ 2NK_0(\mathbb{Z}D_6) & q = 0 \\ \mathbb{Z} & q = -1 \\ 0 & q < -1. \end{cases}$$

Proposition 9.2. $NK_0(\mathbb{Z}D_6) \cong \bigoplus_\infty \mathbb{Z}/2$.

Proof. We first show that $NK_0(\mathbb{Z}[D_6]) \cong NK_1(\mathbb{F}_2[D_3])$. Write $\mathbb{Z}[D_6] = \mathbb{Z}[D_3 \times \mathbb{Z}/2]$ as $\mathbb{Z}[D_3][\mathbb{Z}/2]$, and let $\mathbb{Z}/2 = \langle t \rangle$. Consider the following Cartesian square:

$$\begin{array}{ccc} \mathbb{Z}[D_3][\mathbb{Z}/2] & \xrightarrow{t \mapsto 1} & \mathbb{Z}[D_3] \\ {\scriptstyle t \mapsto -1}\downarrow & & \downarrow{\scriptstyle \text{mod } 2} \\ \mathbb{Z}[D_3] & \xrightarrow{\text{mod } 2} & \mathbb{F}_2[D_3] \end{array}$$

Applying the NK-functor to this Cartesian square yields the Mayer–Vietoris sequence (see [Mi71, Theorem 6.4])

$$NK_2(\mathbb{F}_2[D_3]) \to NK_1(\mathbb{Z}[D_6]) \to NK_1(\mathbb{Z}[D_3]) \oplus NK_1(\mathbb{Z}[D_3]) \to NK_1(\mathbb{F}_2[D_3])$$
$$\to NK_0(\mathbb{Z}[D_6]) \to NK_0(\mathbb{Z}[D_3]) \oplus NK_0(\mathbb{Z}[D_3]) \to \cdots .$$

Since $NK_i(\mathbb{Z}[D_3]) = 0$, for $i \leq 1$ (see [Ha87]), we obtain the desired isomorphism $NK_0(\mathbb{Z}[D_6]) \cong NK_1(\mathbb{F}_2[D_3])$, as claimed.

Next, we claim $NK_1(\mathbb{F}_2[D_3]) \cong \bigoplus_\infty \mathbb{Z}/2$. Note that the ring $\mathbb{F}_2[D_3] \cong M_2(\mathbb{F}_2) \times \mathbb{F}_2[C_2]$ (see [Ma07, Example 2]). Consider the following cartesian square

$$\begin{array}{ccc} \mathbb{F}_2[D_3] & \longrightarrow & \mathbb{F}_2[C_2] \\ \downarrow & & \downarrow \\ M_2(\mathbb{F}_2) & \longrightarrow & 0 \end{array}$$

Applying the NK-functor to this Cartesian square yields the following isomorphism

$$0 \to NK_1(\mathbb{F}_2[D_3]) \to NK_1(M_2(\mathbb{F}_2)) \oplus NK_1(\mathbb{F}_2[C_2]) \to 0.$$

Since $M_2(\mathbb{F}_2)$ is a regular ring (see [We09, Proposition 2.3]), $NK_i(M_2(\mathbb{F}_2)) \cong 0$ for all $i \in \mathbb{Z}$. It follows that $NK_1(\mathbb{F}_2[D_3]) \cong NK_1(\mathbb{F}_2[C_2]) \cong \bigoplus_\infty \mathbb{Z}/2$. For the latter isomorphism we refer the reader to [LO09].

This shows that $NK_0(\mathbb{Z}D_6) \cong NK_1(\mathbb{F}_2[D_3]) \cong \bigoplus_\infty \mathbb{Z}/2$.

We summarize our calculations as follows:

$$Wh_q(D_6 \times \mathbb{Z}) \cong \begin{cases} 2NK_1(\mathbb{Z}D_6) & q = 1 \\ \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ \mathbb{Z} & q = -1 \\ 0 & q < -1. \end{cases}$$

Remark 9.1. Consider the first Cartesian square from the above proof. The head of the associated Mayer–Vietoris sequence gives the following epimorphism, since

$NK_1(\mathbb{Z}D_3)$ vanishes:

$$NK_2(\mathbb{F}_2[D_3]) \to NK_1(\mathbb{Z}D_6) \to 0.$$

The second Cartesian square gives us an epimorphism

$$NK_2(\mathbb{F}_2[D_3]) \to NK_2(\mathbb{F}_2[\mathbb{C}_2]) \oplus NK_2(M_2(\mathbb{F}_2)) \to 0.$$

As mentioned earlier, $NK_2(M_2(\mathbb{F}_2))$ is trivial. It follows that we have an epimorphism

$$NK_2(\mathbb{F}_2[D_3]) \to NK_2(\mathbb{F}_2[\mathbb{C}_2]) \to 0.$$

This implies that $NK_2(\mathbb{F}_2[D_3])$ is non-trivial, and therefore must be an infinitely generated torsion group (see [F77]). It remains to study the first epimorphism above. We would like to show that this map is an isomorphism, but we have no proof.

9.3.2 The Lower Algebraic K-Theory of $D_2 \rtimes_\alpha \mathbb{Z}$

Let $V = D_2 \rtimes_\alpha \mathbb{Z}$, where a generator of $\mathbb{Z}$ acts by an automorphism α of order 2. Since $K_{-1}(\mathbb{Z}F) \cong 0$ for all $F \leq D_2$ (Chap. 7, Table 7.1), we have $K_{-1}(\mathbb{Z}V) \cong 0$.

Farrell and Hsiang in [FH68] show that the group $Wh_q(F \rtimes_\alpha \mathbb{Z})$ (for arbitrary α) can be expressed in the following form:

$$Wh_q(F \rtimes_\alpha \mathbb{Z}) \cong C \oplus NK_q(\mathbb{Z}F, \alpha) \oplus NK_q(\mathbb{Z}F, \alpha^{-1}),$$

where C is a suitable quotient (determined by the automorphism α) of the K-groups $Wh_{q-1}(F) \oplus Wh_q(F)$. Since $Wh_q(D_2) \cong 0$ for $q \leq 1$, it follows that $Wh_q(D_2 \rtimes_\alpha \mathbb{Z}) \cong NK_q(\mathbb{Z}D_2, \alpha) \oplus NK_q(\mathbb{Z}D_2, \alpha^{-1})$ for $q = 0, 1$. Farrell and Hsiang also show that $NK_q(\mathbb{Z}F, \alpha) \cong NK_q(\mathbb{Z}F, \alpha^{-1})$, therefore $Wh_q(D_2 \rtimes_\alpha \mathbb{Z}) \cong 2NK_q(\mathbb{Z}D_2, \alpha)$.

The group $D_2 \rtimes_\alpha \mathbb{Z}$ is the canonical index two subgroup of the group $(D_2 \times \mathbb{Z}/2) *_{D_2} D_4$, so by [DKR11] (see also [DQR11]), we know that the Waldhausen Nil-groups $NK_q(\mathbb{Z}D_2; \mathbb{Z}[(D_2 \times \mathbb{Z}/2) - D_2], \mathbb{Z}[D_4 - D_2])$ are isomorphic to the corresponding Farrell Nil-groups for the canonical index two subgroup $D_2 \rtimes_\alpha \mathbb{Z} \trianglelefteq (D_2 \times \mathbb{Z}/2) *_{D_2} D_4$. In [LO09, Sect. 6.2], Lafont and the second author showed that $NK_q(\mathbb{Z}D_2; \mathbb{Z}[(D_2 \times \mathbb{Z}/2) - D_2], \mathbb{Z}[D_4 - D_2]) \cong \bigoplus_\infty \mathbb{Z}/2$; it follows that

$$NK_q(\mathbb{Z}D_2, \alpha) \cong \bigoplus_\infty \mathbb{Z}/2, \qquad q = 0, 1.$$

Therefore, we have that

$$Wh_q(D_2 \rtimes_\alpha \mathbb{Z}) \cong \begin{cases} \bigoplus_\infty \mathbb{Z}/2 & q = 1 \\ \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ 0 & q \leq -1, \end{cases}$$

and the cokernels of the relative assembly maps are both isomorphic to $\bigoplus_\infty \mathbb{Z}/2$.

9.3.3 The Lower Algebraic K-Theory of $D_4 *_{C_4} D_4$

Since $K_{-1}(\mathbb{Z}F) \cong 0$ for all $F \leq D_4$ (Chap. 7, Table 7.1), we see that, for $V = D_4 *_{C_4} D_4$, we have that $K_{-1}(\mathbb{Z}V) \cong 0$.

For the remaining K-groups, using [CP02, Lemma 3.8], we have that $\tilde{K}_0(\mathbb{Z}V) \cong NK_0(\mathbb{Z}C_4; A_1, A_2)$, where $A_i = \mathbb{Z}[(D_4 - C_4]$ is the $\mathbb{Z}C_4$-bimodule generated by $D_4 - C_4$, for $i = 1, 2$. Similarly, we have that $Wh(V) \cong NK_1(\mathbb{Z}C_4; A_1, A_2)$, where A_1, A_2 are the bi-modules defined above.

Now by [DKR11] (see also [DQR11]), we know that the Waldhausen Nil-groups $NK_q(\mathbb{Z}C_4; A_1, A_2)$ are isomorphic to the corresponding Farrell Nil-group for the canonical index two subgroup $C_4 \times \mathbb{Z} \trianglelefteq D_4 *_{C_4} D_4$. Note that, in this case, the Farrell Nil-group is untwisted, and hence is just the Bass Nil-group $NK_q(\mathbb{Z}C_4) \cong \bigoplus_\infty \mathbb{Z}/2$ ($q = 0, 1$), so the lower algebraic K-theory of $D_4 *_{C_4} D_4$ is given by:

$$Wh_q(D_4 *_{C_4} D_4) \cong \begin{cases} \bigoplus_\infty \mathbb{Z}/2 & q = 1 \\ \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ 0 & q \leq -1. \end{cases}$$

The cokernels of the relative assembly maps are both isomorphic to $\bigoplus_\infty \mathbb{Z}/2$.

9.3.4 The Lower Algebraic K-Theory of $C_4 \times D_\infty$

First, note that $C_4 \times D_\infty \cong (C_4 \times \mathbb{Z}/2) *_{C_4} (C_4 \times \mathbb{Z}/2)$. Since $K_{-1}(\mathbb{Z}F) \cong 0$ for all $F \leq C_4 \times \mathbb{Z}/2$ (Chap. 7, Table 7.1), we see that for $V = (C_4 \times \mathbb{Z}/2) *_{C_4} (C_4 \times \mathbb{Z}/2)$, we have $K_{-1}(\mathbb{Z}V) \cong 0$.

For the remaining K-groups, we use [CP02, Lemma 3.8]. Since $\tilde{K}_0(\mathbb{Z}C_4) \cong 0$ and $\tilde{K}_0(\mathbb{Z}[C_4 \times \mathbb{Z}/2]) \cong \mathbb{Z}/2$ (Chap. 7, Table 7.1), it follows that $\tilde{K}_0(\mathbb{Z}V) \cong (\mathbb{Z}/2)^2 \oplus NK_0(\mathbb{Z}C_4; B_1, B_2)$, where $B_i = \mathbb{Z}[(C_4 \times \mathbb{Z}/2) - C_4]$ is the $\mathbb{Z}C_4$-bimodule generated by $(C_4 \times \mathbb{Z}/2) - C_4$ for $i = 1, 2$. Since $Wh(C_4) \cong Wh(C_4 \times \mathbb{Z}/2) \cong 0$ (see Chap. 7, Table 7.1), it follows that $Wh(V) \cong NK_1(\mathbb{Z}C_4; B_1, B_2)$, with B_1 and B_2 as before. The Nil-groups appearing in these computations are the Waldhausen Nil-groups.

Now by [DKR11] (see also [DQR11]), we know that the Waldhausen Nil-groups $NK_q(\mathbb{Z}C_4; B_1, B_2)$ are isomorphic to the corresponding Farrell Nil-group for the canonical index two subgroup $C_4 \times \mathbb{Z} \trianglelefteq C_4 \times D_\infty$. Note that, in this case, the Farrell Nil-group is untwisted, and hence is just the Bass Nil-group $NK_q(\mathbb{Z}C_4) \cong \bigoplus_\infty \mathbb{Z}/2$ ($q = 0, 1$). We summarize our computations as follows:

$$Wh_q(C_4 \times D_\infty) \cong \begin{cases} \bigoplus_\infty \mathbb{Z}/2 & q = 1 \\ (\mathbb{Z}/2)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ 0 & q \leq -1. \end{cases}$$

The cokernels of the relative assembly maps are both isomorphic to $\bigoplus_\infty \mathbb{Z}/2$.

9.3.5 The Lower Algebraic K-Theory of $D_6 \times D_\infty$

First, note that $D_6 \times D_\infty \cong (D_6 \times \mathbb{Z}/2) *_{D_6} (D_6 \times \mathbb{Z}/2)$. Since $K_{-1}(\mathbb{Z}D_6) \cong \mathbb{Z}$, and $K_{-1}(\mathbb{Z}D_6 \times \mathbb{Z}/2) \cong \mathbb{Z}^3$ (Chap. 7, Table 7.1), we see that for $V = (D_6 \times \mathbb{Z}/2) *_{D_6} (D_6 \times \mathbb{Z}/2)$, we have $K_{-1}(\mathbb{Z}V) \cong \mathbb{Z}^5$ (see also Example 7.4).

For the remaining K-groups, we use [CP02, Lemma 3.8]. Since $\tilde{K}_0(\mathbb{Z}D_6) \cong 0$ and $\tilde{K}_0(\mathbb{Z}[D_6 \times \mathbb{Z}/2]) \cong (\mathbb{Z}/2)^2$, we have that $\tilde{K}_0(\mathbb{Z}V) \cong (\mathbb{Z}/2)^4 \oplus NK_0(\mathbb{Z}D_6; C_1, C_2)$, where $C_i = \mathbb{Z}[(D_6 \times \mathbb{Z}/2) - D_6]$ is the $\mathbb{Z}D_6$-bimodule generated by $(D_6 \times \mathbb{Z}/2) - D_6$ for $i = 1, 2$. Since $Wh(D_6)$ and $Wh(D_6 \times \mathbb{Z}/2)$ are both trivial, it follows that $Wh(V) \cong NK_1(\mathbb{Z}D_6; C_1, C_2)$, with C_1 and C_2 as before. The Nil-groups appearing in these computations are the Waldhausen Nil-groups.

Now by [DKR11] (see also [DQR11]), we know that the Waldhausen Nil-groups $NK_q(\mathbb{Z}D_6; C_1, C_2)$ are isomorphic to the corresponding Farrell Nil-group for the canonical index two subgroup $D_6 \times \mathbb{Z} \trianglelefteq D_6 \times D_\infty$. Note that, in this case, the Farrell Nil-group is untwisted, and hence is just the Bass Nil-group $NK_q(\mathbb{Z}D_6)$. We summarize our computations as follows:

$$Wh_q(D_6 \times D_\infty) = \begin{cases} NK_1(\mathbb{Z}D_6) & q = 1 \\ (\mathbb{Z}/2)^4 \oplus \bigoplus_\infty \mathbb{Z}/2 & q = 0 \\ \mathbb{Z}^5 & q = -1 \\ 0 & q \leq -2. \end{cases}$$

Here the cokernels are simply the direct sums of two copies of the Bass Nil-groups of D_6. We do not know the isomorphism types of these groups.

Chapter 10
Summary

We can now compute the lower algebraic K-theory of the 73 split crystallographic groups. Recall that Theorem 5.1 tells us that, for all such groups Γ, we have an isomorphism

$$K_n(\mathbb{Z}\Gamma) \cong H_*^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{\hat{\ell} \in \mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty}).$$

For all 73 of our groups, we have:

- Explicitly computed in Chap. 7 the homology groups

$$H_*^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}),$$

  and summarized the results in Table 7.8.
- Described in Sect. 9.1 a suitable indexing set $\mathcal{T}''$ (as summarized in Tables 9.1, 9.2, 9.3, 9.4, and 9.5).
- Explicitly computed in Sect. 9.2 the isomorphism types of the groups associated to the indexing set $\mathcal{T}''$ (as summarized in Tables 9.6, 9.7, 9.8, 9.9, and 9.10).
- Explicitly computed in Sect. 9.3 (see Table 9.11) the cokernels

$$H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty})$$

  for all of the infinite virtually cyclic subgroups that occur as stabilizers of lines $\hat{\ell}$ from $\mathcal{T}''$, except for the groups $NK_1(\mathbb{Z}D_n)$ ($n = 4, 6$). (We note that Weibel [We09] proved that $NK_1(\mathbb{Z}D_4)$ is a countably infinite torsion group of exponent 2 or 4.)

This information makes it straightforward to apply Theorem 5.1 to a given split three-dimensional crystallographic group Γ, yielding an explicit calculation of $K_{-1}(\mathbb{Z}\Gamma)$, $\tilde{K}_0(\mathbb{Z}\Gamma)$, and $Wh(\Gamma)$. We have summarized these calculations in Table 10.1. We have entered only the non-zero terms in the table; all of the blank

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Table 10.1 Lower algebraic K-theory of split three-dimensional crystallographic groups

Γ	$K_{-1} \neq 0$	$\tilde{K}_0 \neq 0$	$Wh \neq 0$
Γ_1	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^4 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus 2NK_1(\mathbb{Z}D_4)$
$(S_4^+)_1$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(S_4')_1$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(A_4^+ \times (-1))_1$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_4'')_1$		$\bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus 2NK_1(\mathbb{Z}D_4)$
$(D_4^+ \times (-1))_1$		$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^4 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus 2NK_1(\mathbb{Z}D_4)$
$(D_4^+)_1$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2^+ \times (-1))_1$		$(\mathbb{Z}/2)^8 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(C_4^+ \times (-1))_1$		$(\mathbb{Z}/2)^4 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2')_1$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_4')_1$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(\hat{D}_4')_1$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
Γ_2	$\mathbb{Z}^2$	$(\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus NK_1(\mathbb{Z}D_4)$
$(S_4^+)_2$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(S_4')_2$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(A_4^+ \times (-1))_2$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_4'')_2$		$\bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus NK_1(\mathbb{Z}D_4)$
$(D_4^+ \times (-1))_2$		$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus NK_1(\mathbb{Z}D_4)$
$(D_4^+)_2$		$\bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus NK_1(\mathbb{Z}D_4)$
$(D_2^+ \times (-1))_2$		$(\mathbb{Z}/2)^4 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(C_4^+ \times (-1))_2$		$(\mathbb{Z}/2)^2 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(C_4^+)_2$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2')_2$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_4')_2$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(\hat{D}_4')_2$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$

Γ	$K_{-1} \neq 0$	$\tilde{K}_0 \neq 0$	$Wh \neq 0$
Γ_3	$\mathbb{Z}^2$	$(\mathbb{Z}/2) \oplus (\mathbb{Z}/4)^2 \oplus \bigoplus_\infty \mathbb{Z}/2 \oplus \bigoplus_\infty \mathbb{Z}/4$	$\bigoplus_\infty \mathbb{Z}/2 \oplus NK_1(\mathbb{Z}D_4)$
$(S_4^+)_3$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(S_4')_3$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(A_4^+ \times (-1))_3$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^3 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2')_3$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2^+ \times (-1))_3$		$(\mathbb{Z}/2)^2 \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
Γ_4		$(\mathbb{Z}/2)^4 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_2')_4$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(D_{2_2}')_4$		$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
Γ_5	$\mathbb{Z}^7$	$(\mathbb{Z}/2)^6 \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2 \oplus NK_1(\mathbb{Z}D_6)$
$(D_6^+)_5$	$\mathbb{Z}$		
$(C_6')_5$	$\mathbb{Z}^6$		
$(C_6^+ \times (-1))_5$	$\mathbb{Z}^7$	$(\mathbb{Z}/2)^4$	
$(D_6')_5$	$\mathbb{Z}^6$	$\bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2$
$(C_6^+)_5$	$\mathbb{Z}$	$\mathbb{Z}$	
$(D_3^+ \times (-1))_5$	$\mathbb{Z}^2$		
$(\hat{D}_6')_5$	$\mathbb{Z}^8$		
$(C_3^+ \times (-1))_5$	$\mathbb{Z}^2$		
$(D_6'')_5$	$\mathbb{Z}$	$\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/2$	$\bigoplus_\infty \mathbb{Z}/2 \oplus NK_1(\mathbb{Z}D_6)$
Γ_6	$\mathbb{Z}^2$		
$(C_3^+ \times (-1))_6$	$\mathbb{Z}^2$		
Γ_7	$\mathbb{Z}^2$		

squares represent entries where the corresponding group vanishes, and if Γ does not appear in the table, then $Wh_n(\mathbb{Z}\Gamma) = 0$ for all $n \leq 1$.

We conclude with a pair of examples that illustrate how the pieces of the calculation fit together.

Example 10.1. We compute the lower algebraic K-theory of $(A_4^+ \times (-1))_1 = \langle L, A_4^+ \times (-1)\rangle$, where L is the standard cubical lattice. We will write Γ in place of $(A_4^+ \times (-1))_1$ and H in place of $A_4^+ \times (-1)$. Note that

$$H_{-1}^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^2;$$
$$H_0^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}/2)^4;$$
$$H_1^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong 0,$$

by Table 7.8. (We recall that the group $Wh(G)$ is always trivial if G is a finite subgroup of a three-dimensional crystallographic group—see Table 7.1.) The next step is to compute the second summand from the formula in Theorem 5.1 (as reproduced at the beginning of this chapter). We found in Sect. 9.1 (see Table 9.1) that the parametrized lines

$$\begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ \alpha \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ \alpha \end{pmatrix}$$

represent a choice of indexing set $\mathcal{T}''$ for the latter summand. The strict stabilizers $\overline{\Gamma}_{\hat{\ell}}$ of these lines $\hat{\ell}$ (i.e., the subgroups that fix the lines pointwise) satisfy $\pi(\overline{\Gamma}_{\hat{\ell}}) = D_2'$, where $\pi : \Gamma \to H$ is the canonical projection into the point group. We can apply Procedure 9.2 to conclude that the stabilizers $\Gamma_{\hat{\ell}}$ of these lines are all isomorphic to $D_2 \times D_\infty$ (see Table 9.6). It follows that

$$\bigoplus_{\hat{\ell} \in \mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty}) \cong \left(\bigoplus_{\infty} \mathbb{Z}/2\right)^4 \cong \bigoplus_{\infty} \mathbb{Z}/2$$

for $n = 0$ and 1. (We recall that [FJ95] prove that the term in question is always trivial when $n = -1$.) See Table 9.11. It now follows directly from Theorem 5.1 that the lower algebraic K-groups of Γ are as described in Table 10.1.

Example 10.2. We compute the lower algebraic K-theory of $\Gamma_5 = \langle L, D_6^+ \times (-1)\rangle$, where L is the lattice $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\rangle$. We will write Γ in place of Γ_5 and H in place of $D_6^+ \times (-1)$. Note that

$$H_{-1}^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^7;$$
$$H_0^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}/2)^6;$$
$$H_1^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong 0,$$

by Table 7.8. Now we compute the second summand from the formula in Theorem 5.1. Theorem 8.5 showed that the parametrized lines

$$\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha+1/2 \\ \alpha-1/2 \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ -\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha+1/2 \\ -\alpha+1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha+1/2 \\ -2\alpha+1/2 \\ \alpha+1/2 \end{pmatrix}$$

represent a choice of indexing set $\mathcal{T}''$ for the latter summand (the same fact is recorded in Table 9.5). The strict stabilizers $\overline{\Gamma}_{\hat{\ell}}$ of these lines $\hat{\ell}$ satisfy

$$\pi(\overline{\Gamma}_{\hat{\ell}}) = D_6'', \ \langle A, D\rangle, \ \langle D, E\rangle, \ \langle D, E\rangle, \ \langle E, F\rangle, \ \langle E, F\rangle,$$

respectively, where $\pi : \Gamma \to H$ is the canonical projection into the point group, and the matrices A, D, E, and F are identified in Theorem 8.5. We can apply Procedure 9.2 to conclude that the stabilizers $\Gamma_{\hat{\ell}}$ of these lines are either $D_6 \times D_\infty$ or $D_2 \times D_\infty$, where the first group occurs once and the latter group occurs five times (see Table 9.10). It follows that

$$\begin{aligned} \bigoplus_{\hat{\ell} \in \mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty}) &\cong NK_n(\mathbb{Z}D_6) \oplus \left(\bigoplus_\infty \mathbb{Z}/2\right)^5 \\ &\cong NK_n(\mathbb{Z}D_6) \oplus \bigoplus_\infty \mathbb{Z}/2 \end{aligned}$$

for $n = 0$ and 1. (The given term is trivial when $n = -1$.) See Table 9.11. It now follows directly from Theorem 5.1 that the lower algebraic K-groups of Γ are as described in Table 10.1.

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