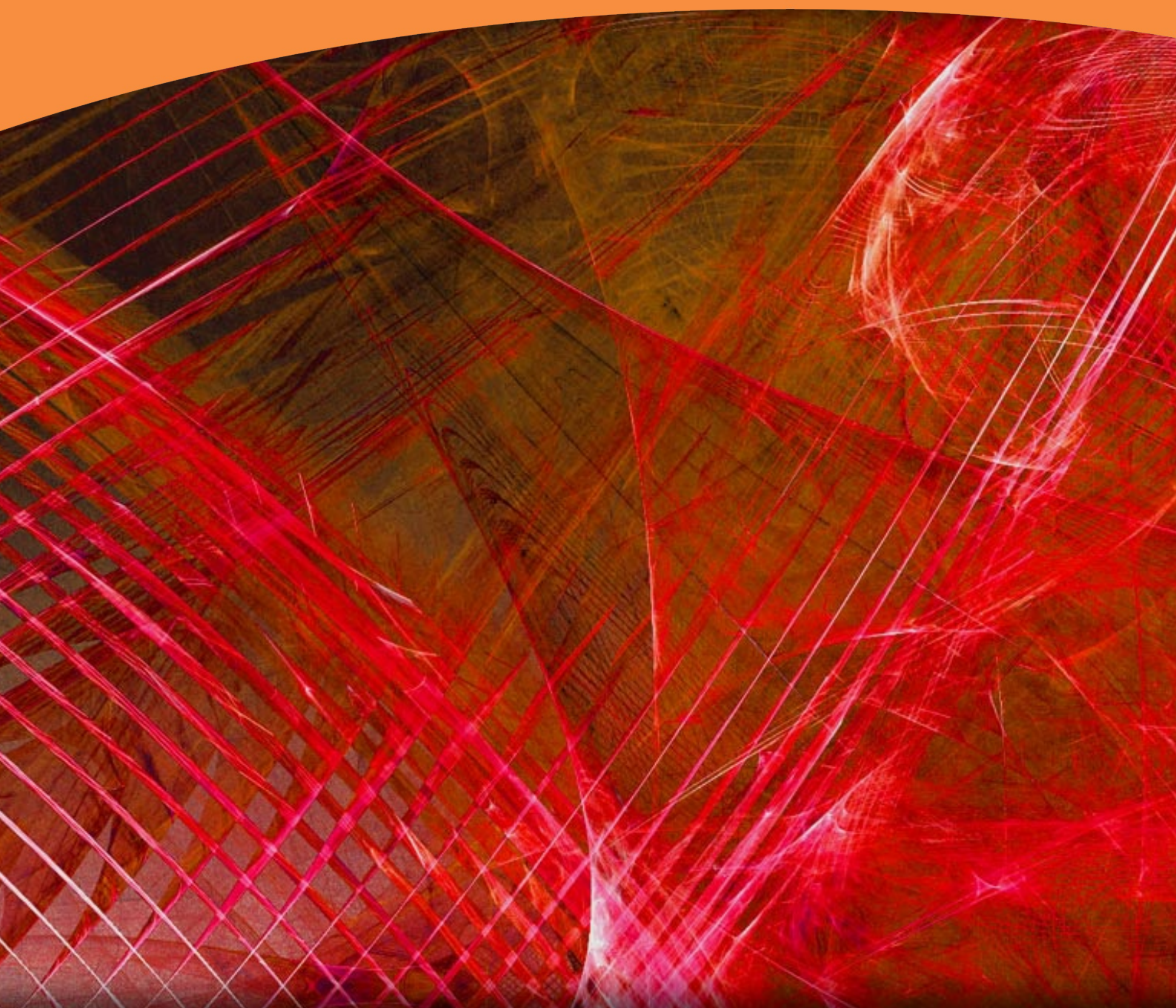# Calculus of Residua

Complex Functions Theory a-2

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Contents

Introduction

We have in *Ventus: Complex Functions Theory a-1* characterized the analytic functions by their complex differentiability and by *Cauchy-Riemann's equation*. We obtained a lot of important results by arguing on line integrals in $\mathbb{C}$. In this way we proved the *Cauchy's Integral Theorem* and *Cauchy's Integral Formula*.

In this book we shall follow an alternative approach by proving that *locally* every analytic function is described by its *Taylor series*. Historically this was the original definition of an analytic function, introduced by *Lagrange* as early as in 1797. The advantage of this approach is that it is easy to calculate on series. The disadvantage is that this approach is not global.

By combining the two aspects of analytic functions it is possible in the following to use *Cauchy-Riemann's equations*, when they are most convenient, and series when these give a better description, so we can benefit from that we have two equivalent, though different theories of the analytic functions.

Complex Functions Theory is here described in an **a** series and a **c** series. The **c** series gives a lot of supplementary and more elaborated examples to the theory given in the **a** series, although there are also some simpler examples in the **a** series. When reading a book in the **a** series the reader is therefore recommended also to read the corresponding book in the **c** series. The present **a** series is divided into four successive books, which will briefly be described below.

a-1 The book *Elementary Analytic Functions* is defining the battlefield. It introduces the analytic functions using the *Cauchy-Riemann equations*. Furthermore, the powerful results of the *Cauchy Integral Theorem* and the *Cauchy Integral Formula* are proved, and the most elementary analytic functions are defined and discussed as our building stones. The important applications of Cauchy's two results mentioned above are postponed to **a-2**.

a-2 The book *Power Series* is dealing with the correspondence between an analytic function and its complex power series. We make a digression into the theory of *Harmonic Functions*, before we continue with the *Laurent series* and the *Residue Calculus*. A handful of simple rules for computing the residues is given before we turn to the powerful applications of the residue calculus in computing certain types of *trigonometric integrals*, *improper integrals* and the sum of some not so simple series. We include a residuum formula for the computation of the *Mellin transform* of some simple functions, and finally we show that the sum of some series can also be found easily by using *Complex Functions Theory*.

a-3 The book *Stability, Riemann surfaces, and Conformal maps* is planned to be written soon. It will start with the connection between analytic functions and *Geometry*. We prove some classical criteria for stability in *Cybernetics*. Then we discuss the inverse of an analytic function and the consequence of extending this to the so-called *multi-valued functions*. Finally, we give a short review of the *conformal maps* and their importance for solving a *Dirichlet problem*.

a-4 The book *Laplace Transform* will be the next one in this series. It will focus on this transform and the related $\mathfrak{z}$-*transform*, which in some sense may be considered as a discrete Laplace transform. Both transforms are of paramount importance in some engineering sciences. This book will be supported by examples in *Ventus: Complex Functions Theory c-11*.

a-5 and a-6 Future plans. The plan is then to continue with a book on *Polynomials*. Contrary to the common thought, the theory of polynomials is far from trivial. It is important, because polynomials are always used as the first approximations. Also, the topic *Linear Difference Equations* is of interest and far from trivial. However, the latter two books are postponed for a while.

The author is well aware of that the topics above only cover the most elementary parts of *Complex Functions Theory*. The aim with this series has been hopefully to give the reader some knowledge of the mathematical technique used in the most common technical applications.

Leif Mejlbro
17th August 2010

1 Power Series

1.1 Accumulations points and limes superior

We shall later on need *limes superior* so we shall recall the definition from *Real Calculus*.

Let (c_n) be any *real sequence*. An *accumulation point* $c \in \mathbb{R}$ of (c_n) is a real number, such that for every $\varepsilon > 0$ there exists an element c_n from the sequence, such that $|c_n - c| < \varepsilon$, or formally,

$$\forall \varepsilon > 0 \, \exists n \in \mathbb{N} : |c_n - c| < \varepsilon.$$

We extend for convenience this definition to also include the following cases, where we consider $+\infty$ (or $-\infty$) as a (generalized) accumulation point of the sequence (c_n), if for every constant $C > 0$ there is an $n \in \mathbb{N}$, such that $c_n > C$ (or $c_n < -C$), i.e. formally,

$$\forall C > 0 \, \exists n \in \mathbb{N} : c_n > C \qquad \text{for } +\infty,$$

$$\forall C > 0 \, \exists n \in \mathbb{N} : c_n < -C \qquad \text{for } -\infty.$$

Example 1.1.1 The sequence

$$1, -1, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{1}{3}, -\frac{1}{3}, \ldots, n, -n, \frac{1}{n}, -\frac{1}{n}, \ldots,$$

has clearly the accumulation points $-\infty$, 0, $+\infty$. It is not hard to prove that when we include $\pm\infty$ as possible accumulation points, then every real sequence has at least one accumulation point. In the present example we have got three accumulation points. $\Diamond$

If $(c_n) \to c$ for $n \to +\infty$, then the limit c is the only accumulation point. When $c \neq \pm\infty$, we say that (c_n) *converges* towards c. When $c = +\infty$ or $-\infty$, we say that (c_n) *diverges* towards c. Notice that a divergent sequence does not necessarily diverge *towards* $+\infty$ or $-\infty$. Two simple counterexamples are $c_n = (-1)^n$ (a bounded, though not convergent sequence with the two accumulation points ± 1) and $c_n = (-1)^n n$ (an unbounded sequence, where $+\infty$ and $-\infty$ are the two (generalized) accumulation points).

We mention without proof the converse result.

Theorem 1.1.1 *Let (c_n) be a real sequence, where (c_n) has only one accumulation point c.*

1) If $c \in \mathbb{R}$, then (c_n) converges towards c for $n \to +\infty$, i.e. $\lim_{n \to +\infty} c_n = c$.

2) If $c = +\infty$ (or $= -\infty$), then (c_n) diverges towards c, i.e. $\lim_{n \to +\infty} c_n = c$.

Example 1.1.2 Usually a real sequence has many accumulation points. A very extreme example is the following. It is well-known that all rational numbers in the interval $[0, 1]$, say, are countable, so they can in principle be written as a sequence (q_n), $q_n \in \mathbb{Q} \cap [0, 1]$. The countable set $\mathbb{Q} \cup [0, 1]$ is dense everywhere in $[0, 1]$, hence every point in $[0, 1]$ is an accumulation point! $\Diamond$

Based on the discussion above we finally introduce

Definition 1.1.1 *Let (c_n) be a* real *sequence. Then we define its* limes superior, *$\limsup_{n \to +\infty} c_n$, as the largest accumulation point c of (c_n).*
If $c \in \mathbb{R}$ is finite, then for every $\varepsilon > 0$ there are only finitely many $n \in \mathbb{N}$, for which $c_n > c + \varepsilon$, and infinitely many $n \in \mathbb{N}$, for which $c - \varepsilon < c_n < c + \varepsilon$.
If $c = +\infty$, then for every $C > 0$ there are infinitely many n, for which $c_n > C$.
If instead $c = -\infty$, then for every $C < 0$ only finitely many $c_n > C$.

Similarly, we can define *limes inferior* $\liminf_{n \to +\infty} c_n$, as the smallest accumulation point c of the real sequence (c_n), so

$$\liminf_{n \to +\infty} c_n = -\limsup_{n \to +\infty} \{-c_n\}.$$

However, we shall not need $\liminf_{n \to +\infty} c_n$ in the following.

It should be emphasized that the introduction of *limes superior* relies heavily on the usual *ordering* of $\mathbb{R}$. For complex sequences, limes superior does not make sense at all. We shall only need $\limsup$ to define the *radius of convergence* of the complex series in the following, and this only requires the $\limsup$ of a *real* sequence.

1.2 Power series

We shall typically deal with *power series* of the type

$$(1) \quad \sum_{n=0}^{+\infty} a_n \left(z - z_0 \right)^n, \qquad z \in \mathbb{C},$$

where we in general *define* $a_0 \left(z - z_0 \right)^0$ in *Complex Functions Theory* as a_0. The coefficients a_n are complex numbers, and the *expansion point* $z_0 \in \mathbb{C}$ is fixed for all terms of (1).

From the symbol (1) we define the corresponding *sequential sequence* of functions $s_n = s_n(z)$, given by

$$(2) \quad s_n = s_n(z) = \sum_{j=0}^{n} a_j \left(z - z_0 \right)^j, \qquad z \in \mathbb{C},$$

i.e. the n-th element $s_n(z)$ is the sum of the first $n + 1$ terms of (1).

Definition 1.2.1 *Consider the series (1) with its corresponding sequential sequence (2), and let $\Omega \neq \emptyset$ be an open set. We say that the series (1) converges towards the limit function $f(z)$ for $z \in \Omega$, if*

$$\lim_{n \to +\infty} s_n(z) = \lim_{n \to +\infty} \sum_{j=0}^{n} a_j \left(z - z_0 \right)^j = f(z) \qquad \text{for all } z \in \Omega.$$

The convergence of the series (1) is therefore derived from the corresponding sequential sequence (2). It must here be emphasized that the sequential sequence $s_n(z) = \sum_{j=1}^{n} a_j \left(z - z_0 \right)^j$ must *not* be confused with the sequence $\left(a_n \left(z - z_0 \right)^n \right)_{n \in \mathbb{N}_0}$, which is obtained from (1) by just deleting the sum sign. Such a misunderstanding may cause some disastrous conclusions.

We mention the well-known result that if a real series of continuous functions $\sum_{n=0}^{+\infty} f_n(x)$ has a convergent majoring series $\sum_{n=0}^{+\infty} c_n < +\infty$, i.e. all $c_n \geq 0$ are constants, and $|f_n(x)| \leq c_n$ for all relevant x, then $\sum_{n=0}^{+\infty} f_n(x)$ is *absolutely and uniformly convergent*, and its sum function is continuous.

We immediately extend this result to complex series of continuous functions, because we have

$$\left. \begin{array}{c} |\Re f_n(z)| \\[2mm] |\Im f_n(z)| \end{array} \right\} \leq |f_n(z)| \leq c_n \qquad \text{and} \qquad \sum_{n=0}^{+\infty} c_n < +\infty,$$

and we can use the argument above on the real series $\sum_{n=0}^{+\infty} \Re f_n(z)$ and $\sum_{n=0}^{+\infty} \Im f_n(z)$.

We shall now more generally turn to the *complex* power series. Given (1), i.e. $\sum_{n=0}^{+\infty} a_n \left(z - z_0 \right)^n$, and consider the *real* sequence of the absolute value of the coefficients $(|a_n|)$. We introduce the number λ by

$$(3) \quad (0 \leq) \quad \lambda := \limsup_{n \to +\infty} \sqrt[n]{|a_n|} \quad (\leq +\infty).$$

Then we have the following theorem

11

Theorem 1.2.1 *The power series $\sum_{n=0}^{+\infty} a_n \left(z - z_0 \right)^n$ is absolutely convergent for every $z \in \mathbb{C}$, for which $\lambda \left| z - z_0 \right| < 1$, and divergent for every $z \in \mathbb{C}$, for which $\lambda \left| z - z_0 \right| > 1$.*

PROOF.

1) If $\lambda \left| z - z_0 \right| < 1$, then

$$\limsup_{n \to +\infty} \sqrt[n]{\left| a_n \left(z - z_0 \right)^n \right|} < 1.$$

It follows from the definition of *limes superior* that we can find a constant $k \in [0, 1[$ and an $N \in \mathbb{N}$, such that

$$\sqrt[n]{\left| a_n \left(z - z_0 \right)^n \right|} \leq k, \qquad \text{thus} \quad \left| a_n \left(z - z_0 \right)^n \right| \leq k^n \quad \text{for all } n \geq N.$$

Since $k \in [0, 1[$, the sum $\sum_{n=N}^{+\infty} k^n$ is convergent, hence $\sum_{n=0}^{+\infty} a_n \left(z - z_n \right)^n$ is absolutely convergent for every such $z \in \mathbb{C}$, satisfying $\lambda \left| z - z_0 \right| < 1$.

2) If instead $\lambda \left| z - z_0 \right| > 1$, then

$$\limsup_{n \to +\infty} \sqrt[n]{\left| a_n \left(z - z_0 \right)^n \right|} > 1,$$

so $\left| a_n \left(z - z_0 \right)^n \right| > 1$ for infinitely many $n \in \mathbb{N}$, and the *necessary condition*, $\left| a_n \left(z - z_0 \right)^n \right| \to 0$, $n \to +\infty$, for the convergence of (1) is *not* fulfilled. $\square$

It follows from Theorem 1.2.1 that if $0 < \lambda < +\infty$, then the power series $\sum_{n=0}^{+\infty} a_n \left(z - z_0 \right)^n$ is absolutely convergent in the open disc $B\left(z_0, \dfrac{1}{\lambda} \right)$, and it is divergent in the (open) complementary set $\mathbb{C} \setminus B\left[z_0, \dfrac{1}{\lambda} \right]$ of the closed disc $B\left[z_0, \dfrac{1}{\lambda} \right]$. It will be shown below in Example 1.2.1 that by the primitive test of Theorem 1.2.1 alone nothing can be said about the convergence/divergence of the power series on the circle $\left| z - z_0 \right| = \dfrac{1}{\lambda}$, which separates the open domain of convergence from the open domain of divergence.

For completeness, if $\lambda = 0$, then $\lambda \left| z - z_0 \right| = 0 < 1$ for all $z \in \mathbb{C}$, so the power series is convergent in all of $\mathbb{C}$, and if $\lambda = +\infty$, then $\lambda \left| z - z_0 \right| < 1$ is only satisfies at the point z_0, which is not an open set.

The investigation above leads us to define the *radius of convergence* of the power series as the number

$$(4) \quad \varrho := \frac{1}{\lambda} = \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{\left| a_n \right|}}, \qquad 0 \leq \varrho \leq +\infty.$$

If $\varrho > 0$, we call the open disc $B\left(z_0, \varrho \right) = \left\{ z \in \mathbb{C} \mid \left| z - z_0 \right| < \varrho \right\}$ the *disc of convergence*. For convenience we say that $B\left(z_0, +\infty \right) = \mathbb{C}$ is "a disc of radius $+\infty$".

Example 1.2.1 On the *circle of convergence* $|z - z_0| = \varrho$ we do not get further information from Theorem 1.2.1. We mention with only sketches of proofs the following four (not exhausting) possibilities of convergence/divergence, where we for comparison in all four cases have chosen $z_0 = 0$ and $\varrho = 1$.

1) The series $\sum_{n=1}^{+\infty} \dfrac{1}{n^2} z^n$ is *absolutely convergent* for $|z| = 1$.

2) The series $\sum_{n=1}^{+\infty} z^n$ is *divergent* for $|z| = 1$.

3) The series $\sum_{n=1}^{+\infty} \dfrac{1}{n} z^n$ is *divergent* for $z = 1$, and it is *conditionally convergent* (i.e. the convergence depends on the order of the terms) for $|z| = 1$ and $z \neq 1$.

4) For every $a \in \mathbb{R}$ we let $[a] \in \mathbb{Z}$ denote the *integer part* of a, i.e. the largest integer $n \in \mathbb{Z}$, for which $n \leq a$. The power series

$$\sum_{n=1}^{+\infty} \frac{1}{n} (-1)^{[\sqrt{n}]} z^n$$

is *conditionally convergent* everywhere on the circle of convergence $|z| = 1$.

The former two examples are easily proved. In the latter two one has to apply *Dirichlet's criterion*, known from real calculus. This is straightforward in 3), but difficult in 4). $\lozenge$

If eventually all $a_n \neq 0$ (e.g. for $n \geq N$), then it is sometimes easier to apply the following result instead of Theorem 1.2.1.

Theorem 1.2.2 *Given a power series $\sum_{n=0}^{+\infty} a_n \left(z - z_0\right)^n$, where $a_n \neq 0$ for all $n \geq N$. If the quotient sequence*

(5) $\left(\left| \dfrac{a_n}{a_{n+1}} \right| \right)_{n \geq N}$

is convergent, *then it has the radius of convergence*

(6) $\varrho = \lim\limits_{n \to +\infty} \left| \dfrac{a_n}{a_{n+1}} \right|.$

Remark 1.2.1 Notice that (5) may be defined without being convergent, and yet the series may have $\varrho > 0$ (which then must be found e.g. by using Theorem 1.2.1 instead). One such example is

$$\sum_{n=0}^{+\infty} \{2 + (-1)^n\}^n \, z^n, \qquad \text{for } \varrho = \frac{1}{3},$$

where

$$\left| \frac{a_{2n-1}}{a_{2n}} \right| = \frac{1}{3^{2n}} \to 0 \qquad \text{and} \qquad \left| \frac{a_{2n}}{a_{2n+1}} \right| = 3^{2n} \to +\infty$$

for $n \to +\infty$. $\lozenge$

PROOF. We consider the real series $\sum_{n=0}^{+\infty} |a_n| \cdot |z - z_0|^n$. Let $z \neq z_0$, and write

$$b_n = |a_n| \cdot |z - z_0| \qquad \text{and} \qquad \lambda' = \frac{1}{\varrho'} = \lim\limits_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

where we shall prove that $\varrho' = \varrho$, or, equivalently, $\lambda' = \lambda$. We get

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{|a_{n+1}| \cdot |z - z_0|^{n+1}}{|a_n| \cdot |z - z_0|^n} = \left| \frac{a_{n+1}}{a_n} \right| \cdot |z - z_0| \to \lambda' \, |z - z_0| \qquad \text{for } n \to +\infty.$$

If $\lambda' \, |z - z_0| < 1$, then choose k, such that $\lambda' \, |z - z_0| < k < 1$. Due to the convergence there is an $N \in \mathbb{N}$, such that

$$\frac{b_{n+1}}{b_n} \leq k \qquad \text{for all } n \geq n \geq N,$$

from which we conclude that

$$0 < b_{N+p} \leq k \cdot b_{N+p-1} \leq \cdots \leq k^p \cdot b_N \qquad \text{for all } p \in \mathbb{N},$$

and since $0 \leq k < 1$, the series

$$\sum_{n=0}^{+\infty} b_n = \sum_{n=0}^{+\infty} |a_n| \cdot |z - z_0|^n$$

is convergent in this case.

If instead $\lambda' |z - z_0| > 1$, then choose k, such that

$$\lambda' |z - z_0| > k > 1.$$

There is an $N \in \mathbb{N}$, such that

$$\frac{b_{n+1}}{b_n} \geq k \qquad \text{for all } n \geq N,$$

hence

$$b_{N+p} \geq k \cdot b_{N+p-1} \geq \cdots \geq k^p \cdot b_N \to \infty \qquad \text{for } p \to +\infty,$$

and the series $\sum_{n=0}^{+\infty} b_n$ is clearly divergent in this case.

The uniquely determined number λ' satisfies the same condition as λ in Theorem 1.2.1, hence $\lambda' = \lambda$, and thus $\varrho' = \varrho$. $\square$

Example 1.2.2 IMPORTANT! The simplest example of a power series, which is not a polynomial, is the *geometric series*

$$\sum_{n=0}^{+\infty} z^n.$$

In this case, all $a_n = 1$, so $\lambda = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} = 1$, and $\varrho = \dfrac{1}{\lambda} = 1$ and $z_0 = 0$. Thus, the *geometric series* is absolutely convergent, if $|z| < 1$ and divergent for $|z| \geq 1$, because then $|z|^n \geq 1$ for all $n \in \mathbb{N}$ and the necessary condition of convergence is not fulfilled in this case.

The *geometric series* is important, because it in some sense is the prototype of all power series of finite radius of convergence. We shall therefore find its sum function in the open disc $|z| < 1$.

More precisely, we claim that the sum function is

$$f(z) = \frac{1}{1 - z} \qquad \text{for } |z| < 1.$$

In fact, by the usual algorithm of division we obtain

$$f(z) = \frac{1}{1 - z} = 1 + z + z^2 + \cdots + z^n + \frac{z^{n+1}}{1 - z} \qquad \text{for } |z| < 1.$$

The corresponding sequential sequence is given by

$$s_n(z) = 1 + z + z^2 + \cdots + z^n,$$

and we see that

$$(7) \quad |f(z) - s_n(z)| = \left| f(z) - \left\{ 1 + z + z^2 + \cdots + z^n \right\} \right| = \left| \frac{z^{n+1}}{1 - z} \right| \leq \frac{|z|^{n+1}}{1 - |z|}.$$

Now, $\dfrac{|z|^{n+1}}{1-|z|} \to 0$ for $n \to +\infty$, when $|z| < 1$ is kept fixed. It therefore follows from (7) that

(8) $\quad \dfrac{1}{1-z} = \sum\limits_{n=0}^{+\infty} z^n, \qquad$ pointwise for $|z| < 1$.

Let $K \subset B(0,1)$ be any compact set of the open unit disc. There is an $r \in [0, 1[$, such that also $K \subseteq B[0, r]$. Then we conclude from (7) for all $z \in K$ that

$$\left| f(z) - s_n(z) \right| = \left| f(z) - \left\{ 1 + z + z^2 + \cdots + z^n \right\} \right| \leq \frac{r^{n+1}}{1-r} \to 0 \text{ for } n \to +\infty,$$

so the convergence is *uniform* over every compact subset K of $B(0,1)$. Then $f'(z)$ can be found by Theorem 3.4.2 in *Ventus: Complex Functions Theory a-1* by termwise differentiation, i.e.

$$\frac{1}{1-z} = \sum\limits_{n=0}^{+\infty} z_n, \qquad \frac{1}{(1-z)^2} = \sum\limits_{n=1}^{+\infty} n\, z^{n-1} = \sum\limits_{n=0}^{+\infty} (n+1)z^n,$$

$$\frac{2}{(1-z)^3} = \sum\limits_{n=2}^{+\infty} n(n-1)z^{n-2} = \sum\limits_{n=0}^{+\infty} (n+2)(n+1)z^n,$$

etc. for $|z| < 1$, so the coefficients of $\dfrac{k!}{(1-z)^{k+1}}$ are polynomials of degree k in n. This implies that if a series is given by polynomial coefficients

$\quad p_k(n) = a_k n^k + \cdots + a_1 n + a_0, \qquad$ for all $n \in \mathbb{N}_0$,

then the sum function of $\sum_{n=0}^{+\infty} p_k(n)z^n$ in $B(0,1)$ is a linear combination of

$$\frac{1}{1-z}, \qquad \frac{1}{(1-z)^2}, \qquad \frac{2}{(1-z)^3}, \qquad \cdots, \qquad \frac{k!}{(1-z)^{k+1}}. \qquad \Diamond$$

Theorem 1.2.3 *Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ be two power series with the same expansion point $z_0 = 0$, and assume that they are both absolutely convergent for $|z| < r$. Then the power series of their sum is given by*

(9) $\quad (f+g)(z) = f(z) + g(z) = \sum\limits_{n=0}^{+\infty} (a_n + b_n)\, z^n \qquad$ *at least for $|z| < r$.*

In some cases, (9) may be convergent in an even larger disc.

The easy proof is left to the reader. That the sum may be convergent in a larger disc can be seen from the following example, where we choose

$$f(z) = \sum\limits_{n=0}^{+\infty} z^n \qquad \text{and} \qquad g(z) = -\sum\limits_{n=0}^{+\infty} z^n \qquad \text{both convergent only for } |z| < 1.$$

Clearly,

$$(f + g)(z) = f(z) + g(z) = \sum_{n=0}^{+\infty} 0 \cdot z^n = 0 \qquad \text{for all } z \in \mathbb{C}.$$

Note that we have only *proved* that $f + g \equiv 0$ in the disc $|z| < 1$, but the strong property of being analytic implies that 0 is the unique analytic continuation to the largest possible set $\mathbb{C}$. This shows that if we only argue on series and the situation is not as clear cut as the above, then we could get into some situations, where *Cauchy-Riemann's equations* would be better to apply.

The following theorem is difficult to apply in practice, and the unexperienced reader should avoid to use it. We shall, however, later on need a part of the proof, and it is furthermore quite naturally to show a theorem on multiplication, once we have obtained Theorem 1.2.3. Therefore, the reader should check the *proof* and is at the same time *warned* against using Theorem 1.2.4 in practice. Such applications are only for very skilled persons.

Theorem 1.2.4 *Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ be as in Theorem 1.2.3, i.e. $z_0 = 0$, and they are both absolutely convergent for $z < r$. Then their product $f \cdot g$ has also a power series, and this is given by* Cauchy multiplication,

$$(10) \quad (f \cdot g)(z) = f(z) \cdot g(z) = \sum_{n=0}^{+\infty} c_n z_n,$$

where the coefficients c_n are given by the discrete convolution *of the sequences (a_n) and (b_n), which is defined by*

$$(11) \quad c_n := \sum_{k=0}^{n} a_k b_{n-k}, \qquad for\ n \in \mathbb{N}_0.$$

PROOF. It is given that

$$\limsup_{n \to +\infty} \sqrt[n]{|a_n|} = \frac{1}{\varrho_1} \leq \frac{1}{r} \qquad \text{and} \qquad \limsup_{n \to +\infty} \sqrt[n]{|b_n|} = \frac{1}{\varrho_2} \leq \frac{1}{r},$$

so $\varrho_1, \varrho_2 \geq r$. Choose any $0 < s < r$. There exists a constant $C = C_s$ only depending on s, such that

$$|a_n| \leq \frac{C}{s^n} \qquad \text{and} \qquad |b_n| \leq \frac{C}{s^n}.$$

We shall first estimate (11),

$$|c_n| \leq \sum_{k=0}^{n} |a_k|\,|b_{n-k}| \leq \sum_{k=0}^{n} \frac{C}{s^k} \cdot \frac{C}{s^{n-k}} = \frac{C^2}{s^n} \sum_{k=0}^{n} 1 = \frac{(n+1)C^2}{s^n},$$

so

$$\sqrt[n]{|c_n|} \leq \frac{1}{s} \sqrt[n]{n+1} \cdot \sqrt[n]{C^2},$$

where $\lim_{n \to +\infty} \sqrt[n]{n+1} \cdot \sqrt[n]{C^2} = 1$, so we conclude that

$$\limsup_{n \to +\infty} \sqrt[n]{|c_n|} \leq \frac{1}{s}.$$

This holds for all $s < r$, so we also have

$$\limsup_{n \to +\infty} \sqrt[n]{|c_n|} \leq \frac{1}{r},$$

and the series of the right hand side of (10) is indeed absolutely convergent for $|z| < r$, hence

$$(12) \quad \sum_{n=0}^{+\infty} \sum_{k=0}^{n} |a_k| \cdot |b_{n-k}| \cdot |z|^n$$

is convergent for $|z| < r$.

18

We note *formally*, if we collect the terms according to their power that

$$
\begin{aligned}
f(z) \cdot g(z) &= \sum_{j=1}^{+\infty} a_j z^j \cdot \sum_{m=0}^{+\infty} b_m z^m = \sum_{j=0}^{+\infty} \sum_{m=0}^{+\infty} a_j b_m z^{j+m} \\
&= \sum_{n=0}^{+\infty} \left\{ \sum_{k=0}^{n} a_k b_{n-k} \right\} z^n = \sum_{n=0}^{+\infty} c_n z^n,
\end{aligned}
$$

and (12) shows that this formal series is absolutely convergent for $|z| < r$.

We shall now prove that the c_n given by (11) in reality gives the right series expansion of the product, and not just formally.

We put

$$
f_N(z) = a_0 + a_1 z + \cdots + a_N z^N \qquad \text{and} \qquad g_N(z) = b_0 + b_1 z + \cdots + b_N z^N.
$$

Let $|z| < r$. Then clearly,

$$
f(z) = \lim_{N \to +\infty} f_N(z) \qquad \text{and} \qquad g(z) = \lim_{N \to +\infty} g_N(z).
$$

We have

$$
(13) \quad |(fg)_N(z) - f_N(z) g_N(z)| \le \sum_{n=N+1}^{+\infty} \sum_{k=0}^{n} |a_k| \cdot |b_{n-k}| \cdot |z|^n,
$$

because all terms of degree $\le N$ have disappeared on the left hand side, and no term from $(fg)_N(z)$ enters the right hand side.

Due to (12), for fixed z, $|z| < r$, and every $\varepsilon > 0$ there is an $N_0 \in \mathbb{N}$, such that the right hand side of (13) is smaller than ε for every $N > N_0$. This shows that

$$
f(z)g(z) = \lim_{N \to +\infty} f_N(z) g_N(z) = \lim_{N \to +\infty} (f \cdot g)_N(z) = (f \cdot g)(z). \qquad \square
$$

The following important theorem contains a lot of information, much more than one would guess at a first glance.

Theorem 1.2.5 *Let* $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ *be a power series of radius of convergence* $\varrho > 0$. *Then the power series is* uniformly convergent *on every compact subset* $K \subset B(z_0, \varrho)$. *The sum function*

$$
(14) \quad f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n, \qquad \text{for } z \in B(z_0, \varrho)
$$

is analytic in $B(z_0, \varrho)$, *and the derivative* $f'(z)$ *is obtained by termwise differentiation*

$$
(15) \quad f'(z) = \sum_{n=1}^{+\infty} n a_n (z - z_0)^{n-1}, \qquad \text{for } z \in B(z_0, \varrho).
$$

The sum function is differentiable of any order $p \in \mathbb{N}$ *with e.g. its derivative of order* p *given by*

$$
(16) \quad f^{(p)}(z) = \sum_{n=p}^{+\infty} n(n-1) \cdots (n-p+1) a_n (z - z_0)^{n-p}, \qquad \text{for } z \in B(z_0, \varrho).
$$

PROOF. Assume that $K \subset B(z_0, \varrho)$ is compact, i.e. K is closed and $B(z_0, \varrho)$ is open. Hence, there is an $r \in [0, \varrho[$, such that $K \subseteq B[z_0, \varrho]$. Then

$$\limsup_{n \to +\infty} \sqrt[n]{|a_n|} = \frac{1}{\varrho} < \frac{1}{r},$$

and it follows in exactly the same way as in the proof of Theorem 1.2.4 that there is a constant $C = C_s$ corresponding to $s \in]r, \varrho[$, such that

$$|a_n| \leq \frac{C}{s^n} \qquad \text{for all } n \in \mathbb{N}.$$

If $z \in K$, then we get the estimate

$$\left| \sum_{n=0}^{+\infty} a_n (z - z_0)^n \right| \leq \sum_{n=0}^{+\infty} \frac{C}{s^n} \cdot r^n = C \sum_{n=0}^{+\infty} \left\{ \frac{r}{s} \right\}^n = C \cdot \frac{s}{s-r},$$

because $0 \leq r/s < 1$, so the latter series is convergent, and its sum is independent of the choice of $z \in K$, proving the uniform convergence.

It follows from Corollary 3.4.3 in *Ventus: Complex Functions Theory* that (14) represents an analytic function of derivative (15). Finally, (16) is obtained by p successive termwise differentiations. $\square$

A very simple application of Theorem 1.2.5 with an unexpectedly large effect is to put $z = z_0$ into (16), in which case we only get a contribution from the term $n = p$. Thus,

$$(17) \quad f^{(p)}(z_0) = p!\, a_p, \qquad \text{i.e.} \qquad a_p = \frac{1}{p!} f^{(p)}(z_0).$$

Then by insertion of (17) into (14) we get

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(z_0) \cdot (z - z_0)^n, \qquad \text{for } z \in B(z_0, \varrho),$$

and we have proved

Corollary 1.2.1 *Let $f(z)$ be the sum function of a power series $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ of radius of convergence $\varrho > 0$. Then $f(z)$ is given by its* Taylor series, *expanded from the centre z_0, in $B(z_0, \varrho)$, i.e.*

$$(18) \quad f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(z_0) \cdot (z - z_0)^n, \qquad \text{for } z \in B(z_0, \varrho).$$

It follows immediately from Corollary 1.2.1 that we have

Theorem 1.2.6 The Identity Theorem. *Assume that the two power series*

$$\sum_{n=0}^{+\infty} a_n (z - z_0) \qquad and \qquad \sum_{n=0}^{+\infty} A_n (z - z_0)^n,$$

expanded from the same point z_0 have positive radii of convergence and share the same sum function $f(z)$ in their common domain. Then the two series are identical, i.e. $a_n = A_n$ for all $n \in \mathbb{N}$.

20

PROOF. This follows directly from (17), because

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = A_n. \qquad \square$$

Then we turn to the indefinite integrals.

Theorem 1.2.7 *Let $f(z)$ be the sum function of a power series $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$, expanded from z_0 and of radius of convergence $\varrho > 0$. The indefinite integral $F(z)$ of $f(z)$, for which also $F(z_0) = 0$, is in the disc $B(z_0, \varrho)$ given by the termwise integrated series*

$$(19) \quad F(z) = \sum_{n=0}^{+\infty} \frac{1}{n+1} a_n (z - z_0)^{n+1}.$$

PROOF. It follows from the definition (4) that

$$\limsup_{n \to +\infty} \sqrt[n]{\left| \frac{a_n}{n+1} \right|} = \limsup_{n \to +\infty} \left\{ \frac{1}{\sqrt[n]{n+1}} \cdot \sqrt[n]{|a_n|} \right\} = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} = \frac{1}{\varrho},$$

so the series of $F(z)$ and $f(z)$ have the same radius of convergence $\varrho > 0$. They are both expanded from the same point z_0, and it follows from Theorem 1.2.5 that $F'(z) = f(z)$, and the claim is proved. $\square$

Referring to Theorem 3.3.5 of *Ventus: Complex Functions Theory a-1* we conclude that every indefinite integral $f(z)$ in $B(z_0, \varrho)$ has the structure $F(z) + c$ for some uniquely determined constant $c \in \mathbb{C}$. $\square$

Example 1.2.3 Using Theorem 1.2.7 on the *geometric series* $\sum_{n=0}^{+\infty} z^n$ for $|z| < 1$ it follows that the indefinite integral $F(z)$ of $\dfrac{1}{1-z}$ in this disc, for which $F(0) = 0$, is given by

$$F(z) = \sum_{n=0}^{+\infty} \frac{1}{n+1} z^{n+1} = \sum_{n=1}^{+\infty} \frac{1}{n} z^n.$$

On the other hand,

$$G(z) := \mathrm{Log}\left(\frac{1}{1-z}\right) = -\mathrm{Log}(1-z) \qquad \text{for } |z| < 1,$$

is also analytic in this disc, and we have

$$G'(z) = \frac{1}{1-z} = f(z).$$

Hence, $G(z)$ is also an indefinite integral of $f(z)$, so $G(z) = F(z) + c$ for some $c \in \mathbb{C}$. Finally we see that $F(0) = G(0) = 0$, so $c = 0$, and we have proved another important result,

$$(20) \quad \mathrm{Log}\left(\frac{1}{1-z}\right) = -\mathrm{Log}(1-z) = \sum_{n=1}^{+\infty} \frac{1}{n} z^n, \qquad \text{for } |z| < 1. \quad \lozenge$$

We shall emphasize in this *Ventus: Complex Functions Theory* series that one must *always* specify the domain of convergence of a series, because otherwise one could easily jump to very wrong conclusions. It is of course legal to try to find a *formal* solution of a problem, but once a formal series solution has been found, one should immediately find the domain of validity, outside which the result is not reliable.

1.3 Expansion of an analytic function in a power series

We proved in Section 1.2 that the sum function $f(z)$ of a power series expansion from z_0 and of radius of convergence $\varrho > 0$ is analytic in $B(z_0, \varrho)$ and that $f(z)$ in $B(z_0, \varrho)$ is given by its Taylor series expanded from the center z_0 of the disc. Furthermore, Theorem 3.4.2 of *Ventus: Complex Functions Theory a-1* showed that every analytic function is infinitely often (complex) differentiable.

Remark 1.3.1 The situation is different for real functions in $C^{\infty}(\mathbb{R})$, because far from all of them can be extended to an analytic function by "just writing $z \in \mathbb{C}$ instead of $x \in \mathbb{R}$", a wrong statement which is frequently met. It is possible and even not too difficult to construct a real C^{∞} function which cannot at any point $x_0 \in \mathbb{R}$ be extended to an analytic function in any complex neighbourhood of $x_0 \in \mathbb{R}$. We shall give an example in Remark 1.3.2 where this phenomenon occurs in one point, from which this general result can be derived by some advanced, though standard mathematical procedure. $\lozenge$

We shall now show a converse result, namely that every analytic function *locally* is a sum function of a power series. Once we have proved this result, we have shown that every analytic function can be treated either by *Cauchy-Riemann's equations*, or by *local power series*.

Theorem 1.3.1 *Let $f : \Omega \to \mathbb{C}$ be an analytic function in an open domain Ω, and let $z_0 \in \Omega$ be any fixed point. The* Taylor series *of $f(z)$ expanded from z_0 is convergent in (at least) the largest open disc $B(z_0, \varrho) \subset \Omega$ of centre z_0. In $B(z_0, \varrho)$ the sum function of the Taylor series is $f(z)$, thus*

$$(21) \quad f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(z_0) \cdot (z - z_0)^n \qquad \text{for all } z \in B(z_0, \varrho).$$

PROOF. It follows from *Cauchy's inequalities*, cf. *Ventus: Complex Functions Theory a-1*, Theorem 3.4.5, that

$$\left| f^{(n)}(z_0) \right| \le \frac{M_r \cdot n!}{r^n} \qquad \text{for every } n \in \mathbb{N}_0 \text{ and } r \in \,]0, \varrho[,$$

where

$$M_r = \max\{|f(z)| \mid |z - z_0| = r\}.$$

If $M_r = 0$, then the Taylor series is the zero series, which of course is convergent.

If $M_r > 0$, then the radius of convergence of the Taylor series is at least

$$\lim_{n \to +\infty} \left\{ \frac{n!}{\dfrac{M_r n!}{r^n}} \right\}^{\frac{1}{n}} = r \cdot \lim_{n \to +\infty} \frac{1}{\sqrt[n]{M_r}} = r.$$

This holds for every $r \in \,]0, \varrho[$, so we conclude that the Taylor series of $f(z)$ has at least ϱ as radius of convergence.

Choose any $r \in \,]0, \varrho[$ and any point $z \in B(z_0, r)$. Then $|z - z_0| < r$, so if ζ lies on the circle $|\zeta - z_0| = r$, then we have the estimate

$$\left| \sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \right| \le \frac{1}{r} \sum_{n=0}^{+\infty} \left\{ \frac{|z - z_0|}{r} \right\}^n = \frac{1}{r - |z - z_0|},$$

from which follows that the series $\sum_{n=0}^{+\infty} \dfrac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$ is uniformly convergent in ζ for $|\zeta - z_0| = r$.

Using (95) of Theorem 3.4.2 of *Ventus: Complex Functions Theory a-1* we get

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \qquad \text{for } n \in \mathbb{N}_0.$$

Insert the Taylor series, then interchange summation and integration followed by a reduction and then finally apply *Cauchy's integral formula* to get

$$\sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(z_0) \cdot (z - z_0)^n = \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \cdot (z - z_0)^n$$

$$= \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} f(\zeta) \sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \, d\zeta = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} f(\zeta) \cdot \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{\zeta - z}} \, d\zeta$$

$$= \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} d\zeta = f(z).$$

Finally, notice that to every $z \in B(z_0, \varrho)$ we can choose $r < \varrho$, such that $z \in B(z_0, r)$, where the computation above is valid, and the theorem is proved. $\square$

Some basic power series.

It follows from Theorem 1.3.1 that all known *real Taylor series* are immediately extended to *complex Taylor series*, because the Taylor series only depends on its sequence of coefficients, $\left\{ \dfrac{1}{n!} f^{(n)}(z_0) \right\}$, derived by differentiation. We therefore get the complex Taylor functions of the following well-known functions. The reader is highly recommended to learn all these by heart, as the appear over and over again in the following, as well as in applications outside these books.

1) $\qquad \exp z = \sum_{n=0}^{+\infty} \dfrac{1}{n!} z^n, \qquad\qquad\qquad z \in \mathbb{C},$

2) $\qquad \cos z = \sum_{n=0}^{+\infty} \dfrac{(-1)^n}{(2n)!} z^{2n}, \qquad\qquad z \in \mathbb{C},$

3) $\qquad \sin z = \sum_{n=0}^{+\infty} \dfrac{(-1)^n}{(2n+1)!} z^{2n+1}, \qquad\quad z \in \mathbb{C},$

4) $\qquad \cosh z = \sum_{n=0}^{+\infty} \dfrac{1}{(2n)!} z^{2n}, \qquad\qquad\quad z \in \mathbb{C},$

5) $\qquad \sinh z = \sum_{n=0}^{+\infty} \dfrac{1}{(2n+1)!} z^{2n+1}, \qquad\quad z \in \mathbb{C},$

6) $\qquad \mathrm{Log}(1+z) = \sum_{n=0}^{+\infty} \dfrac{(-1)^n}{n+1} z^{n+1}, \qquad |z| < 1,$

7) $\qquad (1+z)^\alpha := \sum_{n=0}^{+\infty} \binom{\alpha}{n} z^n, \qquad\qquad |z| < 1, \qquad \alpha \in \mathbb{C},$

8) $\qquad \dfrac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad\qquad\qquad\quad |z| < 1.$

Formula 7) is strictly speaking the *definition* of what later is called the *principal value* of the (usually) multiply defined function $(1+z)^\alpha$. If $\alpha = n \in \mathbb{N}_0$, then $(1+z)^n$ is of course a polynomial instead, so it is uniquely defined for $z \in \mathbb{C}$, and not multiply defined in this exceptional case.

The Taylor series 1)–5) are found by means of Theorem 1.3.1, because it is in all these cases easy to find $f^{(n)}(0)$.

Formula 6) is obtained from (20) by writing $-z$ instead of z and then change sign. We notice that $\text{Log}(1+z)$ itself is defined in the open domain $\Omega = \mathbb{C}\backslash]-\infty,-1]$, so the largest open disc contained in Ω of centre 0 is $B(0,1)$.
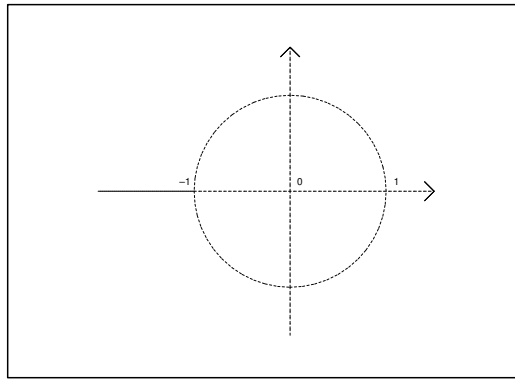

Figure 1: The domain of the Taylor series of $\text{Log}(1+z)$ expanded from $z_0 = 0$ is the open unit disc.

Formula 7) is here considered as a definition of the *principal value* of $(1+z)^\alpha$, where we define the general *binomial coefficients* by

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \qquad \alpha \in \mathbb{C}, \quad n \in \mathbb{N}_0,$$

with n factors in both the numerator and the denominator. Notice that if $\alpha = n \in \mathbb{N}_0$, then the series of $(1+z)^n$ is a polynomial of degree n, and the domain is all of $\mathbb{C}$.

Remark 1.3.2 Again the situation is different in the real case, $C^\infty(\mathbb{R})$. It is not hard to construct a real $\varphi \in C^\infty(\mathbb{R})$ and a corresponding point $x_0 \in \mathbb{R}$, such that the (real) Taylor series of $\varphi(x)$ is convergent everywhere in $\mathbb{R}$, and such that

$$\varphi(x) \neq \sum_{n=0}^{+\infty} \varphi^{(n)}(x_0) \cdot (x - x_0)^n \qquad \text{for every } x \in \mathbb{R} \setminus \{x_0\}.$$

One simple example of such a function is

$$\varphi(x) = \begin{cases} \exp\left(-\dfrac{1}{|x|}\right) & \text{for } x \in \mathbb{R} \setminus \{0\}, \\ \\ 0 & \text{for } x = 0. \end{cases}$$

Clearly, $\varphi(x)$ is C^∞ outside $x = 0$, and for $x = 0$ we use the definition of a converging sequence of difference quotients and one of the rules of magnitudes of functions (exponentials dominate polynomials) to prove that

$$\frac{\varphi(x) - \varphi(0)}{x - 0} = \frac{1}{x} \cdot \exp\left(-\frac{1}{|x|}\right) \to 0 \qquad \text{for } x \to 0,$$

25

and then by induction also for higher derivatives to get $\varphi^{(n)}(0) = 0$ for all $n \in \mathbb{N}_0$. In all cases we get $\varphi^{(n)}(0) = 0$. Thus, the Taylor series is the zero series, and it is obvious that $\varphi(x) > 0$ for $x \neq 0$. $\Diamond$

Theorem 1.3.1 shows that *if* a function $f(z)$ is analytic in the disc $B(z_0, r)$, then the function $f(z)$ is in this set alone given by the sequence $\left\{ \dfrac{1}{n!} f^{(n)}(z_0) \right\}_{n \in \mathbb{N}_0}$. This sequence is sometimes called the *germ* of the analytic function $f(z)$ in $B(z_0, r)$ expanded from z_0.

Remark 1.3.3 From an application point of view it is strange that an analytic map $f(z)$ is uniquely determined in a whole disc $B(z_0, r)$, if we just know its germ $\left\{ \dfrac{1}{n!} f^{(n)}(z_0) \right\}_{n \in \mathbb{N}_0}$ at the centre z_0. This is again a warning to the reader that the analytic functions may be easy to handle in practice, but they have there limitations, and they cannot provide us with a universal model of the real world. In particular, it is annoying that they can never directly describe causality. $\Diamond$

Remark 1.3.4 The real function $f(x) = \dfrac{1}{1 + x^2}$, $x \in \mathbb{R}$, is of class $C^\infty(\mathbb{R})$. However, its Taylor series,

$$f(x) = \frac{1}{1 + x^2} = \sum_{n=0}^{+\infty} \left(-x^2\right)^n = \sum_{n=0}^{+\infty} (-1)^n x^{2n},$$

is only convergent for $|x| < 1$. This looks like an enigma, as long as we only consider $x \in \mathbb{R}$.
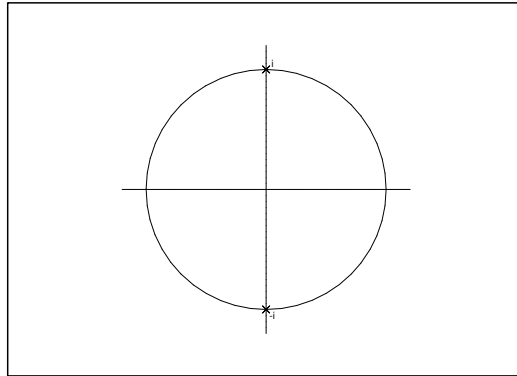
Figure 2: The largest open disc of centre 0 not containing the two singularities $\pm i$.

In the complex plane we see that $f(x)$ is a real rational function, so it is uniquely extended to the analytic function

$$f(z) = \frac{1}{1 + z^2}, \qquad \text{for } z \in \mathbb{C} \setminus \{\pm i\} = \Omega.$$

The largest open disc contained in $\Omega \subset \mathbb{C}$ of centre 0 is $B(0, 1)$, so in the *complex plane* we see why the radius of convergence is only 1, when $z_0 = 0$. The complex singularities $\pm i$ have therefore a profound influence on the convergence of a real Taylor series. This strange phenomenon has puzzled many students, who had no knowledge of *Complex Functions Theory.* $\Diamond$

We finally prove

Theorem 1.3.2 Weierstraß's Double Series Theorem. *Let* $\{g_n(z)\}_{n \in \mathbb{N}_0}$ *be a sequence of functions which are all analytic in the same disc* $B(0, \varrho)$. *Assume that the series*

$$f(z) = \sum_{n=0}^{+\infty} g_n(z)$$

is uniformly convergent in every smaller closed disc $B[0, r]$, $r < \varrho$. *Then we obtain the power series of* $f(z)$ *by first expanding all the* $g_n(z)$ *and then collect all terms of the same power of* z.

27

PROOF. It follows already from Corollary 3.4.3 of *Ventus: Complex Functions Theory a-1* that $f(z)$ is analytic in $B(0, \varrho)$ and that

$$(22) \quad f^{(p)}(0) = \sum_{n=0}^{+\infty} g_n^{(p)}(0).$$

Hence, for $z \in B(0, \varrho)$,

$$f(z) = \sum_{p=0}^{+\infty} \frac{1}{p!} f^{(p)}(0) \, z^p = \sum_{p=0}^{+\infty} \left\{ \sum_{n=0}^{+\infty} \frac{1}{p!} g_n^{(p)}(0) \right\} z^p \qquad \square$$

Example 1.3.1 Clearly

$$\cos z = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} \, z^{2n}, \qquad \text{for } z \in \mathbb{C},$$

is continuous, so there is a $\varrho > 0$, such that

$$|1 - \cos z| < 1 \qquad \text{for } z \in B(0, \varrho).$$

If we put

$$g_n(z) := (1 - \cos z)^n,$$

then we can in principle find the power series of $g_n(z)$ by *Cauchy multiplication*. Choosing $r < \varrho$ we get in $B(0, r)$ that

$$\frac{1}{\cos z} = \frac{1}{1 - (1 - \cos z)} = \sum_{n=0}^{+\infty} (1 - \cos z)^n = \sum_{n=0}^{+\infty} g_n(z),$$

so using (22) we can in principle find the *Taylor coefficients* of $\dfrac{1}{\cos z}$, i.e. its germ. We therefore obtain a power series expansion of $\dfrac{1}{\cos z}$ in the disc $B(0, r)$.

We shall not go into details with the sketch above, because the computations are fairly big and extremely tedious. We shall, however, point out the following unexpected result: Since $\dfrac{1}{\cos z}$ is analytic in the set $\mathbb{C} \setminus \left\{ \dfrac{\pi}{2} + p\pi \ \Big| \ p \in \mathbb{Z} \right\}$, it follows from Theorem 1.3.1 that the Taylor series is convergent in the disc $B\left(0, \dfrac{\pi}{2}\right)$. This is far from trivial, because

$$\left| \cos\left(\frac{\pi}{2} i\right) - 1 \right| > \frac{3}{2} > 1, \qquad \text{for all } z \in B(0, \varrho).$$

It follows by the continuity that the chosen ϱ above, which was used to compute the Taylor series, must satisfy the inequality $\varrho < \dfrac{\pi}{2}$, and yet the final domain of the Taylor series constructed on $B(0, \varrho)$ is the larger set $B\left(0, \dfrac{\pi}{2}\right)$. $\diamond$

1.4   Linear differential equations

One of the big issues of the theory of power series is the solution method of finding an analytic solution of a linear differential equation with polynomial coefficients.

Existence and Uniqueness Theorems.

We start with the following theorem of existence and uniqueness of the solution.

Theorem 1.4.1 *Let Ω be an open subset of $\mathbb{C}$, and let $a_0(z)$, ..., $a_n(z)$ and $g(z)$ be analytic functions in Ω. Let $z_0 \in \Omega$ be a point for which $a_0(z_0) \neq 0$.*
For any given complex numbers c_0, c_1, ..., c_{n-1}, there exists one and only one function $f(z)$, which is analytic in a neighbourhood ω of z_0, such that

(23) $a_0(z) \dfrac{d^n f}{dz^n} + \cdots + a_{n-1}(z) \dfrac{df}{dz} + a_n(z)f(z) = g(z) \qquad$ *for all $z \in \omega$,*

and such that

(24) $f(z_0) = c_0, \quad f'(z_0) = c_,, \quad \ldots, \quad f^{(n-1)}(z_0) = c_{n-1}.$

One may choose ω as the largest open disc $B(z_0, \varrho) \subseteq \Omega$, which does not contain any zero of $a_0(z)$.

SKETCH OF PROOF. The equation is only considered in $B(z_0, \varrho)$, where $a_0(z) \neq 0$. To ease matters we norm the differential equation (23), which means that we divide it by $a_0(z)$, so that the coefficient of the highest order term is $a_0(z) = 1$.

First *assume* that $f(z)$ is indeed a solution in $B(z_0, \varrho)$. Let $0 < r < \varrho$. By *Cauchy's inequalities* (cf. Theorem 3.4.5 of *Ventus: Complex Functions Theory a-1*) there exists a constant M_r, such that for all $j = 1$, ..., n, and all $k \in \mathbb{N}_0$,

(25) $\left| a_j^{(k)}(z_0) \right| \leq \dfrac{M_r k!}{r^k} \qquad$ and $\qquad \left| g^{(k)}(z_0) \right| \leq \dfrac{M_r k!}{r^k}.$

Using that $a_0(z) = 1$ we get by a rearrangement of (23) that

$$f^{(n)}(z) = g(z) - \sum_{j=1}^{n-1} a_{n-j}(z)\, f^{(j)}(z),$$

thus by k differentiations,

(26) $f^{(n+k)}(z) = g^{(k)}(z) - \displaystyle\sum_{j=1}^{n-1} \sum_{q=0}^{k} \binom{k}{q} a_{n-j}^{(k-q)}(z)\, f^{(j+q)}(z).$

Using that

$$f(z_0) = c_0, \quad f'(z_0) = c_1, \quad \ldots, \quad f^{(n-1)}(z_0) = c_{n-1},$$

are given, we find $f^{(p)}(z_0)$ for all $p \in \mathbb{N}_0$.

It follows that *if* an analytic solution exists, then it must be unique, because its Taylor coefficients are uniquely determined.

It is seen by insertion of the formal Taylor series into (23) that if the series is convergent, then it must be a solution.

The difficult part is to prove the existence. The idea is again to apply *Cauchy's inequalities*. We assume that

$$(27) \qquad \left| f^{(q)}(z_0) \right| \le C_q \cdot \frac{q!}{r^q} \qquad \text{for } q = 0,\, 1,\, \ldots,\, n + p - 1,$$

where the C_q are given constants. Then proceed in the following way (the complicated proof is left to the interested reader). Put (25) and (27) into (26) to find constants C_{n+p} as small as possible, such that

$$(28) \qquad \left| f^{(n+p)}(z_0) \right| \le C_{n+p} \cdot \frac{(n+p)!}{r^{n+p}}.$$

The sequence (C_q) is clearly increasing. What is more important, however, is that it is bounded from above. (The difficult proof is left to the interested reader.) Hence, there exists a function $C(r)$, depending on $r < \varrho$, such that

$$\left| f^{(n)}(z_0) \right| \le C(r) \cdot \frac{n!}{r^n} \qquad \text{for all } n.$$

Then it follows from *Cauchy's inequalities* that the Taylor series in convergent in $B(z_0, r)$.

This procedure can be performed for every $r < \varrho$, so we conclude that the Taylor series is convergent in at least the disc $B(z_0, \varrho)$, and the existence follows. $\square$

Remark 1.4.1 It should be mentioned that the Taylor series in some cases (though not in all) can be extended over (some of) the zeros of the coefficient $a_0(z)$, in which case the radius of convergence of the Taylor series becomes larger than ϱ. $\Diamond$

Corollary 1.4.1 *Every linear and homogenous differential equation (23) with $g(z) \equiv 0$ and of analytic coefficients has in a neighbourhood ω of every point $z_0 \in \Omega$ for which $a_0(z_0) \ne 0$, precisely n linearly independent solutions.*

PROOF. This follows immediately from the fact that every solution is uniquely determined by the n constants c_0, c_1, …, c_{n-1} of (24). $\square$

Practical procedures for solving a linear differential equations of analytical coefficients.

In this subsection we shall in some examples demonstrate three standard procedures os solving a linear differential equation of analytical coefficients. These are

1) Inspection

2) Calculation of the germ $\left\{ f^{(n)}(z_0) \right\}_{n \in \mathbb{N}_0}$

3) Method of power series.

Of these, *inspection* is the most difficult one. However, when it succeeds, it is also the most elegant method. It requires some skill in manipulation.

The method of calculation of the *germ* may not always be applicable, but when it succeeds, it is usually straightforward.

The *method of power series* is the most commonly used method, because it is easy to understand, and for the novice it is felt as *the* procedure. Shortly described, one inserts a *formal* power series $f(z) = \sum_{n=0}^{+\infty} b_n (z - z_0)^n$ into (23), where the constants b_n are the unknowns. Collecting the terms of the same power of the result we obtain a *recursion formula* (or a *difference equation*) in the b_n, which then is solved.

Finally – and this is very important – the constructed series is still *formal*, so we *must* always finish the task by computing the radius of convergence. In fact, the formal series may in some cases be divergent for every z with the exception of the expansion point z_0 itself, because $\varrho = 0$.

Referring to Theorem 1.4.1 we see that $\varrho = 0$ is possible, when $a_0(z_0) = 0$, in which case we cannot conclude anything about the existence of a convergent power series solution. One may insert a formal series and obtain a result, but then we *must* check if it has a positive radius of convergence.

Remark 1.4.2 We mention without proof that the zeros of $a_0(z)$ determine all the possible radii of convergence, so one can by knowing these make a qualified guess of ϱ, which is either one of the numbers $|z_j|$, where the z_j are all the zeros of $a_0(z)$, or $+\infty$. Note also, that the number of zeroes could be infinite, e.g. for $a_0(z) = \sin z$, $\cos z$, $\sinh z$ or $\cosh z$. $\Diamond$

Example 1.4.1 We shall demonstrate the three methods on the simple equation

(29) $f'(z) - f(z) = 0$.

First we see that according to Corollary 1.4.1 there is, apart from a constant factor, just one solution.

Then use Remark 1.4.2. Since $a_0(z) \equiv 1$ does not have any zeros in $\mathbb{C}$, we may expect that the radius of convergence is $\varrho = +\infty$, which by Remark 1.4.2 is the only possibility.

First method, inspection. By checking our arsenal of known common analytic functions we immediately see that

$$\frac{d}{dz} e^z = e^z,$$

so $f(z) = e^z$ satisfies the differential equation (29). We therefore conclude by the beginning of this example that the complete solution is given by

(30) $f(z) = c \cdot e^z, \qquad c \in \mathbb{C}.$

Variant of the first method. We may instead multiply (29) by a so-called *integrating factor*. By this method we shall use the well-known rules of calculations,

(31) $f'(z) \cdot g(z) + f(z) \cdot g'(z) = \dfrac{d}{dz}(f \cdot g) \qquad$ and $\qquad f'(z) \cdot g(z) - f(z) \cdot g'(z) = \{g(z)\}^2 \dfrac{d}{dz}\left(\dfrac{f}{g}\right),$

in the apparently "unusual direction". In other words, one shall search for structures in the equation of sums of products of the type

$$f'(z)g(z) + f(z)g'(z) \qquad \text{or} \qquad f'(z)g(z) - f(z)g'(z),$$

and then apply (31).

In the present case, the integrating factor is $e^{-z} \neq 0$ for all $z \in \mathbb{C}$, hence (29) is equivalent to

(32) $0 = e^{-z} f'(z) - e^{-z} f(z) = e^{-z} \dfrac{df}{dz} + \dfrac{de^{-z}}{dz} \cdot f(z) = \dfrac{d}{dz}\left\{e^{-z} f(z)\right\}.$

Then we immediately get by indefinite integration of (32) that

$$e^{-z} f(z) = c, \qquad \text{i.e.} \qquad f(z) = c \cdot e^{-z} \qquad \text{for } c \in \mathbb{C},$$

and we have again found (30).

This variant has the advantage that it is often possible by some small computations to solve the original equation, and there will usually be no problem of finding the domain of the solution, because series do not occur in this method. Its disadvantage is that it is not always possible to find an integrating factor by inspection, and furthermore, when this can be done, it requires some experience and skill.

Second method. Determination of the germ at $z_0 = 0$. the equation is of first order, so it suffices to assume that $f(0) = c \in \mathbb{C}$. It follows by induction from equation (29) that

$$f^{(n)}(z) = f^{(n-1)}(z) \qquad \text{for all } n \in \mathbb{N},$$

and then by recursion that

$$f^{(n)}(0) = f(0) = c \qquad \text{for all } n \in \mathbb{N}.$$

The formal power series is then

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(0) \, z^n = c \sum_{n=0}^{+\infty} \frac{1}{n!} \, z^n = c \cdot e^z, \qquad c \in \mathbb{C},$$

where we recognize the power series of the *exponential*.

This recognition also implies that since e^z is defined in all of $\mathbb{C}$, the radius of convergence must be $\varrho = +\infty$.

This also follows from (6), because if $c \neq 0$, then

$$\varrho = \lim_{n \to +\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to +\infty} \left| \frac{c}{n!} \cdot \frac{(n+1)!}{c} \right| = \lim_{n \to +\infty} (n+1) = +\infty.$$

Notice also that the official definition (4) gives

$$\varrho = \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{|a_n|}} = \frac{\lim_{n \to +\infty} \sqrt[n]{n!}}{\lim_{n \to +\infty} \sqrt[n]{|c|}} = \lim_{n \to +\infty} \sqrt[n]{n!}.$$

In order to compute $\sqrt[n]{n!}$ we need *Stirling's formula*

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e} \right)^n \qquad \left[\text{meaning that } \frac{n!}{\sqrt{2\pi n} \left(\dfrac{n}{e} \right)^n} \to 1 \text{ for } n \to +\infty \right],$$

or better, the estimate

$$\sqrt{2\pi n} \cdot \left(\frac{n}{e} \right)^n \exp\left(\frac{1}{2n+1} \right) < n! < \sqrt{2\pi n} \cdot \left(\frac{n}{e} \right)^n \exp\left(\frac{1}{12n} \right) \quad \text{for all } n \in \mathbb{N},$$

Then in the present case,

$$\varrho = \lim_{n \to +\infty} \sqrt[n]{n!} = \lim_{n \to +\infty} \left\{ \sqrt[2n]{2\pi n} \cdot \frac{n}{e} \right\} = +\infty.$$

The advantage of this method is that if the initial conditions are given, then it sometimes is easy to find the germ $\left\{\dfrac{1}{n!}\, f^{(n)}(z_0)\right\}$ directly. The disadvantage is that the recursion formula in other cases may be extremely complicated and unsolvable in practice.

Third method. The power series method. We assume that a solution of (29) is given by a power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{for } |z| < \varrho,$$

where *the task is not finished, before we also have found ϱ and checked that $\varrho > 0$*. First note that by the change of index $n \curvearrowright n+1$, followed by a corresponding change of the lower bound $n = 1$ to $n = 0$ (always check, if the first terms in the two series are equal)

$$f'(z) = \sum_{n=1}^{+\infty} n\, a_n\, z^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1}\, z^n.$$

Then we get by insertion into (29) that

$$
\begin{aligned}
0 &= f'(z) - f(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1} - \sum_{n=0}^{+\infty} a_n z^n \\
&= \sum_{n=0}^{+\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} \left\{ (n+1) a_{n+1} - a_n \right\} z^n.
\end{aligned}
$$

(Check that the lower bounds of the sums are identical, before we add them; if not, add or remove some terms to get the same lower bounds.) The unique power series expansion of 0 is then written in two ways,

$$
\sum_{n=0}^{+\infty} \left\{ (n+1) a_{n+1} - a_n \right\} z^n = 0 = \sum_{n=0}^{+\infty} 0 \cdot z^n.
$$

By the *identity theorem* the coefficients of the two series are equal, so by identification we get the following *recursion formula*

(33) $(n+1) a_{n+1} - a_n = 0$ for all $n \in \mathbb{N}_0$ (i.e. in the common range of summation).

The easiest way to solve (33) is to multiply it by $n! \neq 0$ and put $b_n = n! a_n$, because then we first get by a rearrangement, $(n+1)! a_{n+1} = n! a_n$, and then by a very simple recursion,

$$
b_{n+1} = b_n \ (= n! a_n) \ = b_{n-1} = \cdots = b_0 = 0! a_0 = c,
$$

from which we get $a_n = \dfrac{c}{n!}$ as previously.

Another method is the following. Assume that $a_o = c$. Put $n = 0$ into (33) to get

$$
(0+1) a_{0+1} = a_0 = c, \qquad \text{i.e.} \qquad a_1 = \frac{c}{1}.
$$

for $n = 1$ we get

$$
(1+1) a_{1+1} = 2 a_2 = a_1 = \frac{c}{1}, \qquad \text{i.e.} \qquad a_2 = \frac{c}{2!}.
$$

For $n = 2$ we get

$$
(2+1) a_{2+1} = 3 a_3 = a_2 = \frac{c}{2!}, \qquad \text{i.e.} \qquad a_3 = \frac{c}{3!}.
$$

Based on these three results we assume that

(34) $a_n = \dfrac{c}{n!}$ for some $n \in \mathbb{N}_0$,

and then we shall prove that (34) also holds for the successor, i.e. when n is replaced by $n+1$, because then (34) by induction holds for all $n \in \mathbb{N}_0$. Clearly, we have just proved that (34) is true for $n = 0$, 1 and 2, so it holds indeed for some $n \in \mathbb{N}_0$. However, it follows from the recursion formula (33) that

$$
(n+1) a_{n+1} = a_n = \frac{c}{n!}, \qquad \text{i.e.} \qquad a_{n+1} = \frac{c}{(n+1)!},
$$

which is precisely (34) with n replaced by $n+1$, and (34) follows by induction for all $n \in \mathbb{N}_0$.

The formal power series solution is then given by

$$f(z) = c \sum_{n=0}^{+\infty} \frac{1}{n!} \, z^n = c \cdot e^z, \qquad \text{for all } z \in \mathbb{C},$$

where we have once again recognized the power series of the exponential. If we do not see this, we must instead apply one of the methods from the second method above to find $\varrho = +\infty$. $\Diamond$

The simple Example 1.4.1 above had only constant coefficients. We shall in general only consider linear differential equations of *polynomial* coefficients, in which case it is always possible to find a linear recursion formula by the method of power series, although this recursion formula still may be difficult to solve. We shall by the following two examples also demonstrate the impact of the zeros of $a_0(z)$ on the radius of convergence.

Example 1.4.2 Solve the differential equation

(35) $(1 - z)f'(z) = f(z)$,

using $z_0 = 0$ as point of expansion.

First method. Inspection. We get by a rearrangement of (35),

$$0 = (1 - z)f'(z) - 1 \cdot f(z) = \frac{d}{dz}\{(1 - z)f(z)\},$$

hence by an indefinite integration, $(1 - z)f(z) = c$, for a constant $c \in \mathbb{C}$, and thus

(36) $f(z) = \dfrac{c}{1 - z} \qquad$ for $z \in \mathbb{C} \setminus \{1\}$.

Only the zero solution (for $c = 0$) can be extended to all of $\mathbb{C}$.

Second method, determination of the germ $\dfrac{1}{n!} \, f^{(n)}(0)$. The expansion point is $z_0 = 0$, where $a_0(0) = 1 \neq 0$. The only zero of $a_0(z) = 1 - z$ is $z = 1$, so the power series solution is at least convergent in the open disc $B(0, 1)$. When we differentiate (35) and then rearrange the result, we get

$$(1 - z)f''(z) = 2f'(z).$$

A comparison with (35) suggest that the general structure is possibly

(37) $(1 - z)f^{(n)}(z) = nf^{(n-1)}(z) \qquad$ for $n \in \mathbb{N}$.

This is at least true for $n = 1$ and for $n = 2$. When (37) is differentiated, we get

$$(1 - z)f^{(n+1)}(z) - f^{(n)}(z) = nf^{(n)}(z),$$

hence

$$(1 - z)f^{(n+1)}(z) = (n + 1)f^{(n)}(z),$$

which is (37) with n replaced by $n + 1$, and (37) follows in general by induction.

Putting $z = 0$ into (37) it follows by recursion that

$$f^{(n)}(0) = n f^{(n-1)}(0) = n(n-1)f^{(n-2)}(0) = \cdots = n! f(0),$$

and the Taylor series is given by

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(0) z^n = f(0) \sum_{n=0}^{+\infty} \frac{n!}{n!} z^n = f(0) \sum_{n=0}^{+\infty} z^n.$$

The radius of convergence is of course $\varrho = 1$, and the sum function of the *geometric series* $\sum_{n=0}^{+\infty} z^n$ is $\dfrac{1}{1-z}$, cf. also Section 1.3. The complete solution is

$$f(z) = \begin{cases} \dfrac{c}{1-z} & \text{for } z \in \mathbb{C} \setminus \{1\} \quad \text{and} \quad c \neq 0, \\[3mm] 0 & \text{for } z \in \mathbb{C} \qquad \text{and} \quad c = 0. \end{cases}$$

Third method. The method of power series. Assume that the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{of radius of convergence } \varrho > 0,$$

is a solution of (35). Then we get for $|z| < \varrho$,

$$\begin{aligned} 0 &= (1-z)f'(z) - f(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1} - \sum_{n=1}^{+\infty} n a_n z^n - \sum_{n=0}^{+\infty} a_n z^n \\ &= \sum_{n=0}^{+\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{+\infty} (n+1) a_n z^n = \sum_{n=0}^{+\infty} (n+1) \{a_{n+1} - a_n\} z^n. \end{aligned}$$

Hence, the zero function can be written in two ways as a convergent power series for $|z| < \varrho$ for some $\varrho > 0$,

$$\sum_{n=0}^{+\infty} (n+1) \{a_{n+1} - a_n\} z^n = 0 = \sum_{n=0}^{+\infty} 0 \cdot z^n.$$

It follows from the *identity theorem* that corresponding coefficients are equal, hence we get the following recursion formula for $n \in \mathbb{N}_0$,

$$(n+1)\{a_{n+1} - a_n\} = 0, \qquad \text{i.e.} \qquad a_{n+1} = a_n, \qquad \text{because } n+1 \neq 0.$$

Thus by recursion,

$$a_{n+1} = a_n = a_{n-1} = \cdots = a_0 \qquad \text{for all } n \in \mathbb{N}_0,$$

and the series is given by

$$f(z) = a_0 \sum_{n=0}^{+\infty} z^n.$$

For $a_0 = 0$ we get the zero series, which is convergent for all $z \in \mathbb{C}$. If $a_0 \neq 0$, then the radius of convergence can be found by either (25,

$$\varrho = \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{|a_0|}} = 1,$$

or by (26),

$$\varrho = \lim_{n \to +\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to +\infty} \left| \frac{a_0}{a_0} \right| = \lim_{n \to +\infty} 1 = 1.$$

Alternatively, the result

$$f(z) = \frac{c}{1-z}, \qquad z \in \mathbb{C} \setminus \{1\} \text{ for } c \in \mathbb{C} \setminus \{0\} \text{ constant,}$$

follows from Example 1.2.2. ◊

We still need to give an example, in which none of the standard procedures above is applicable with success. This is given by the following.

Example 1.4.3 Apply the three standard procedures to the inhomogeneous equation

(38) $z^2 f'(z) - f(z) = -z,$

with the expansion point $z_0 = 0$.

The coefficients are polynomial. However, $a_0(z_0) = a_0(0) = 0$, so nothing can be concluded from Theorem 1.4.1.

First method. Inspection. When $z \neq 0$ we multiply (38) by the *integrating factor* $\dfrac{1}{z^2} \exp\left(\dfrac{1}{z}\right) \neq 0$. Then we get, reading the equation above from the right to the left,

$$-\frac{1}{z} \exp\left(\frac{1}{z}\right) = \exp\left(\frac{1}{z}\right) \cdot f'(z) - \frac{1}{z^2} \exp\left(\frac{1}{z}\right) \cdot f(z) = \frac{d}{dz}\left\{\exp\left(\frac{1}{z}\right) \cdot f(z)\right\}.$$

Clearly, the problem would be solved, if we could find an indefinite integral of $-\dfrac{1}{z} \exp\left(\dfrac{1}{z}\right)$ in a neighbourhood of the expansion point $z_0 = 0$, but this is not possible with the available methods known so far in this book.

Second method. Determination of the germ $\left\{\dfrac{1}{n!} f^{(n)}(0)\right\}$. Putting $z = 0$ into (38) we get $f(0) = 0$. Then by successive differentiations of (38),

$$z^2 f''(z) + (2z - 1)f(z) = -1, \qquad\qquad f'(0) = 1,$$

$$z^2 f^{(3)}(z) + (4z - 1)f''(z) + 2f'(z) = 0, \qquad f''(0) = 2,$$

$$z^2 f^{(4)}(z) + (6z - 1)f^{(3)}(z) + 6f''(z) = 0, \qquad f^{(3)}(0) = 12.$$

It is left to the reader to prove by induction that in general,

$$z^2 f^{(n+1)}(z) + (2nz - 1)f^{(n)}(z) + n(n - 1)f^{(n-1)}(z) = 0 \qquad \text{for all } n \geq 1.$$

Hence, for $z = 0$,

$$f^{(n)}(0) = n(n - 1)f^{(n-1)}(0), \qquad \text{for } n \geq 2.$$

We divide this equation by $n!(n - 1)!$, and then we get by a simple recursion,

$$\frac{f^{(n)}(0)}{n!(n - 1)!} = \frac{f^{(n-1)}(0)}{(n - 1)!(n - 2)!} = \cdots = \frac{f^{(2)}(0)}{2!1!} = \frac{2}{2} = 1,$$

so the Taylor coefficients are

$$\frac{1}{n!} f^{(n)}(0) = (n - 1)! \qquad \text{for } n \in \mathbb{N}.$$

Then the *formal* Taylor series becomes

$$\sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(0)z^n = \sum_{n=1}^{+\infty} (n - 1)! z^n.$$

Its radius of convergence is computed by (6),

$$\varrho = \lim_{n \to +\infty} \frac{(n-1)!}{n!} = \lim_{n \to +\infty} \frac{1}{n} = 0,$$

so the Taylor series is only convergent for $z = 0$, and this *formal solution* is useless in the applications.

Third method. The power series method. Assume that (38) has the convergent power series solution

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{for } |z| < \varrho.$$

It follows by insertion into (38) that

$$
\begin{aligned}
z^2 f'(z) - f(z) &= z^2 \sum_{n=1}^{+\infty} n a_n z^{n-1} - \sum_{n=0}^{+\infty} a_n z^n = \sum_{n01}^{+\infty} n a_n z^{n+1} - \sum_{n=0}^{+\infty} a_n z^n \\
&= \sum_{n=2}^{+\infty} (n-1) a_{n-1} z^n - \sum_{n=0}^{+\infty} a_n z^n = -a_0 + \sum_{n=1}^{+\infty} \{(n-1) a_{n-1} - a_n\} z^n.
\end{aligned}
$$

This expression is equal to $-z$, if $-a_0 = 0$ and $0 \cdot a_0 - a_1 = -1$, i.e. $a_1 = 1$, and in general,

$$a_n = (n-1) a_{n-1}, \qquad \text{for } n \geq 2.$$

Then we get by recursion (the details are left to the reader),

$$a_n = (n.1)! a_1 = (n-1)!,$$

so the *formal* series solution is

$$\sum_{n=1}^{+\infty} (n-1)! z^n, \qquad \text{unfortunately with } \varrho = 0.$$

Thus, this series is divergent, whenever $z \neq 0$, and the method is not applicable. $\Diamond$

Example 1.4.3 shows that if $a_0(z_0) = 0$, then Theorem 1.4.1 does not apply, and the problem may not be solvable by any of the three suggested standard procedures. We shall later on in connection with *Laurent series* prove that we may in some cases be able to solve such a linear differential equation, even if $a_0(z_0) = 0$, while in other cases this is not possible, because the equation in reality should be solved on a so-called *Riemann surface* cf. *Ventus: Complex Functions Theory a-3*.

1.5 Zeros of analytical functions

We have already in Section 1.4 seen that the zeros of an analytic function may have some influence on the behaviour of the function. We shall in this section see by using some abstract topological results shown in *Ventus: Complex Functions Theory a-1* that the zeros of an *analytic* function also have other unexpected consequences, which are stronger than possibly similar results for real $C^\infty(\mathbb{R})$ functions.

Theorem 1.5.1 *Assume that $f : \Omega \to \mathbb{C}$ is analytic and not the zero function. For every given point $z \in \Omega$ there exists an $n \in \mathbb{N}_0$, such that $f^{(n)}(z) \neq 0$.*

We here adopt the convenient notation, $f^{(0)}(z) := f(z)$ (no derivatives of $f(z)$).

PROOF. Given an open domain Ω in $\mathbb{C}$, and an analytic function $f : \Omega \to \mathbb{C}$. We define a subset $E \subseteq \Omega$ by

(39) $E := \left\{ z \in \Omega \mid f^{(n)}(z) = 0 \text{ for all } n \in \mathbb{N}_0 \right\}.$

This means that E is the set of points $z \in \Omega$, for which both $f(z) = 0$ and all its derivatives $f^{(n)} = 0$.

If $z_0 \in E$, then clearly the germ of the Taylor series is just the zero sequence. According to Theorem 1.3.1, the Taylor series has its sum function $f(z)$ in the largest open disc $B(z_0, \varrho) \subseteq \Omega$, hence $f(z) = 0$ for every $z \in B(z_0, \varrho)$, so by the definition (39) of the set E we have proved that $B(z_0, \varrho) \subseteq E$, proving that E is an open subset of Ω, cf. Definition 2.1.1 in *Ventus: Complex Functions Theory a-1*.

On the other hand, the complementary set

$$\Omega \setminus E = \bigcup_{n=0}^{+\infty} \left\{ z \in \Omega \mid f^{(n)}(z) \neq 0 \right\} = \bigcup_{n=0}^{+\infty} \left\{ f^{(n)} \right\}^{\circ -1} (\mathbb{C} \setminus \{0\})$$

is also open, because $\mathbb{C} \setminus \{0\}$ is open and every derivative $f^{(n)}$ is continuous, cf. Definition 2.1.2 in *Ventus: Complex Functions Theory a-1*, so each set $\left\{ f^{(n)} \right\}^{\circ -1} (\mathbb{C} \setminus \{0\})$ is open, and every union of open sets is again open.

Then the open domain $\Omega = E \cup \{\Omega \setminus E\}$ is written as a disjoint union of two open sets. Since every domain by definition is connected, it follows from Corollary 2.1.1 in *Ventus: Complex Functions Theory a-1* that either $E = \Omega$ or $E = \emptyset$.

If $E = \Omega$, then f is identically zero, which was excluded in the assumptions. Hence $E = \emptyset$, so for every given $z \in \Omega$ there exists an $n \in \mathbb{N}_0$, such that $f^{(n)}(z) \neq 0$, and the theorem is proved. $\square$

Assume that $f(z)$ is analytic and not identically zero. Let $z_0 \in \Omega$ be a zero, i.e. $f(z_0) = 0$. It follows from Theorem 1.5.1 that there is at least one $n \in \mathbb{N}$, such that $f^{(n)}(z_0) \neq 0$.

Definition 1.5.1 *Let $z_0 \in \Omega$ be a zero of the analytic function $f : \Omega \to \mathbb{C}$, where $f \neq 0$. We say that the zero z_0 has the* order, *or* multiplicity $n \in \mathbb{N}$, *if n is the smallest integer for which $f^{(n)}(z_0) \neq 0$.*

In order to motivate this definition we consider the power series expansion of f with the zero z_0 of order n as expansion point. We get in a neighbourhood of z_0 that

$$
\begin{aligned}
f(z) &= \frac{1}{n!} f^{(n)}(z_0) \cdot (z - z_0)^n + \frac{1}{(n+1)!} f^{(n+1)}(z_0) \cdot (z - z_0)^{n+1} + \cdots \\
&= (z - z_0)^n \cdot \left\{ \frac{1}{n!} f^{(n)}(z_0) + \frac{1}{(n+1)!} f^{(n+1)}(z_0) \cdot (z - z_0) + \cdots \right\} = (z - z_0)^n \cdot g(z),
\end{aligned}
$$

where $g(z)$ is analytic in the same neighbourhood of z_0 as $f(z)$, and where furthermore, $g(z_0) \neq 0$. This means that $z = z_0$ is precisely n times a zero of f, explaining the notation. Furthermore, notice

41

that since $g(z)$ is continuous, we can find a (possibly smaller) open neighbourhood ω of z_0, such that $g(z) \neq 0$ for all $z \in \omega$. Then clearly also $f(z) \neq 0$ for every $z \in \omega \setminus \{z_0\}$, and we have proved

Theorem 1.5.2 *Let* $f : \Omega \to \mathbb{C}$, $f \neq 0$, *be analytic. Every zero* z_0 *of* f *is an* isolated point, *which means that there is an open neighbourhood of* z_0, *such that* z_0 *is the only zero in* ω.

Theorem 1.5.2 immediately implies the following stronger version of Theorem 1.2.6, *The Identity Theorem.*

Theorem 1.5.3 The Identity Theorem. *Let* $f : \Omega \to \mathbb{C}$ *and* $g : \Omega \to \mathbb{C}$ *be analytic functions in the same domain* Ω. *If the set* $\{z \in \Omega \mid f(z) = g(z)\}$ *has an* accumulation point *lying* <u>in the set</u> Ω, *then* f *and* g *are identical.*

Remark 1.5.1 The underlined assumption that the accumulation point lies in the set Ω is very important. We shall in Example 1.5.1 show that what without this assumption, we cannot conclude that the two functions are equal. $\diamond$

PROOF. First recall that $z_0 \in \Omega$ is an accumulation point of a set $A \subseteq \Omega$, if for every $r > 0$,

$$\{z \in A \mid 0 < |z - z_0| < r\} \neq \emptyset.$$

Notice that the inequality $0 < |z - z_0|$ excludes z_0 from this set.

Assume that $z_0 \in \Omega$ is an accumulation point of the set

$$E := \{z \in \Omega \mid f(z) = g(z)\}.$$

Choosing $r = \dfrac{1}{n}$, $n \in \mathbb{N}$, and then $z_n \in E$, such that $f(z_n) = g(z_n)$, we define a sequence (z_n), for which $z_n \to z_0$ for $n \to +\infty$. Now, $f - g$ is continuous, and $f(z_n) - g(z_n) = 0$ for all $n \in \mathbb{N}$. Since $z_0 \in \Omega$, it follows from the continuity that also $f(z_0) - g(z_0) = 0$, so $z_0 \in E$.

It follows from the definition of the sequence (z_n) that z_0 is *not* an isolated zero of the analytic function $f - g$, so we conclude from Theorem 1.5.2 that $f - g$ is the zero function, and thus $g = g$ by a rearrangement. $\square$

Example 1.5.1 Consider the zero function 0 and the function $\sin\dfrac{1}{z}$ for $z \in \Omega = \mathbb{C} \setminus \{0\}$. Clearly, the two functions are different from each other. On the other hand, both functions are zero on the set

$$E := \left\{ \frac{1}{n\pi} \ \middle| \ n \in \mathbb{Z} \setminus \{0\} \right\},$$

which has $z_0 = 0 \notin \Omega$ as an accumulation point, because $\dfrac{1}{n\pi} \to 0$ for $n \to +\infty$, and also for $n \to -\infty$. $\diamond$

Another unexpected consequence is the following theorem.

Theorem 1.5.4 *Let $\varphi : I \to \mathbb{C}$ be a function defined either on a real interval I, or on a differential curve I in $\mathbb{C}$. If Ω is an open domain in $\mathbb{C}$, which contains I, then there is at most one analytic function $f : \Omega \to \mathbb{C}$, such that*

$$f(x) = \varphi(x) \qquad for \qquad x \in I, \ an\ interval$$

$$f(z) = \varphi(z) \qquad for \qquad z \in I, \ a\ piecewise\ C^1\ curve\ in\ \mathbb{C}.$$

PROOF. In both cases all points of I are trivially accumulation points of I, so the theorem follows immediately from Theorem 1.5.3. $\square$

The reader should be surprised that the values of an analytic function on a one-dimensional curve uniquely determines $f : \Omega \to \mathbb{C}$ in its domain. This means that if we change the analytic function f on e.g. a real interval to another *analytic* function g, then we change without any time delay f

to g over all of Ω. This implies that analytic functions are not suited directly to describe *causality*, because e.g. the impact of a sudden force applied to a system will evolve in time through the system and not immediately. Fortunately, it is possible to describe indirectly the causality (approximately) by analytic functions. However, this is not the right place to go further into this discussion.

One should also note that Theorem 1.5.4 states that there is *at most one* analytic function $f : \Omega \to \mathbb{C}$, the restriction of which to an interval is a given function $\varphi : I \to \mathbb{C}$. It is not hard to construct a function in $C^\infty(\mathbb{R})$ which cannot be extended to any analytic function, not even locally!

1.6 Simple Fourier series

It follows from the definition $e^{i\Theta} = \cos\Theta + i\sin\Theta$ for $\Theta \in \mathbb{R}$ that $e^{i\Theta}$ has the period 2π, so $e^{in\Theta}$ has for fixed $n \in \mathbb{N}$ the period $\dfrac{2\pi}{n}$ in Θ.

The classical *Fourier Series Theory* in real calculus states that every piecewise $C^1([0, 2\pi])$ function can be represented in the complex form

$$(40)\quad \varphi(\Theta) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\Theta},$$

where

$$c_n := \frac{1}{2\pi} \int_0^{2\pi} \varphi(\Theta)\, e^{-in\Theta}\, \mathrm{d}\Theta \qquad \text{for } n \in \mathbb{Z}.$$

Remark 1.6.1 The symbol $\sim$ in (40) indicates that $\varphi(\Theta)$ is equal to its Fourier series in the sense of L^2, i.e.

$$\lim_{N \to +\infty} \int_0^{2\pi} \left| \varphi(\Theta) - \sum_{n=-N}^{N} c_n e^{in\Theta} \right|^2 \mathrm{d}\Theta = 0,$$

which can also be interpreted as *convergence in energy*. It can be proved that if $\int_0^{2\pi} |\varphi|^2 \mathrm{d}\Theta < +\infty$, or just $\int_0^{2\pi} |\varphi(\Theta)|^p \mathrm{d}\Theta < +\infty$ for some $p > 1$, then (40) holds with pointwise equality sign for almost every $\Theta \in [0, 2\pi]$. (For $p = 1$ this statement is wrong as proved by Kolmogorov in the early 1920s.) The proof of the statement above is extremely difficult, and the result is of limited value, because one usually cannot specify for which Θ (40) holds with equality sign, even if we know that it holds "for almost every Θ. $\Diamond$

Consider an analytic function $f : \Omega \to \mathbb{C}$, where Ω is an open neighbourhood of 0. Then we can construct the convergent Taylor series

$$(41)\quad f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{for } |z| < \varrho.$$

44

Using polar coordinates $z = re^{i\Theta}$ for fixed $r \in \,]0, \varrho[$, we obtain the Fourier series of a function φ_r, given by

$$(42) \quad \varphi_r(\Theta) := f\left(re^{i\Theta}\right) = \sum_{n=0}^{+\infty} \{a_n r^n\} e^{in\Theta},$$

so when we compare with (40) we see that we have the *Fourier coefficients*

$$c_n = a_n r^n \quad \text{for } n \in \mathbb{N}_0 \qquad \text{and} \qquad c_n = 0 \quad \text{for } n \in \mathbb{Z}_-.$$

In (42) the sum is only over $n \in \mathbb{N}_0$, while we in (40) use all $n \in \mathbb{Z}$ in the sum. This suggests that at least concerning Fourier series, it would be quite natural also to allow power series (41), where negative exponents occur. Such series are indeed very useful in the applications. However, they cannot be defined everywhere in a disc $B(0, \varrho)$, because they are at least divergent for $z = 0$, where $\lim_{z \to 0} z^{-n} = \infty$.

We shall later in Chapter 3 study such power series of negative exponents. Such series are called *Laurent series*. The motivation for their introduction is here given by the fact that we are missing some very natural terms in the Fourier series (42), but it will turn up that there is far more in these Laurent series than one would expect at a first glance. Their main applications are in the so-called *residue calculus*, which is a powerful device to compute many definite integrals and infinite sums, including some which cannot be computed by methods from the real calculus. The Laurent series with no positive exponent are furthermore used in the theory of the $\mathfrak{z}$ *transform,*, which is a discrete form of the *Laplace transform*.

Example 1.6.1 Let us play a little with this connection between *Complex Functions Theory* and *Fourier series*. We know that

$$e^z = e^x \cos y + i\, e^x \sin y, \qquad z = x + iy \in \mathbb{C}.$$

Using polar coordinates,

$$z = r\, e^{i\Theta}, \qquad \text{i.e.} \qquad x = r \cdot \cos\Theta \quad \text{and} \quad y = r \cdot \sin\Theta,$$

we get by insertion,

$$e^z = e^{r(\cos\Theta + i\sin\Theta)} = e^{r\cos\Theta}\{\cos(r\sin\Theta) + i\sin(r\sin\Theta)\},$$

and

$$e^z = \sum_{n=0}^{+\infty} \frac{1}{n!} z^n = \sum_{n=0}^{+\infty} \frac{1}{n!} r^n e^{in\Theta} = \sum_{n=0}^{+\infty} \frac{r^n}{n!} \cos n\Theta + i \sum_{n=1}^{+\infty} \frac{r^n}{n!} \sin n\Theta.$$

By identifying the real and the imaginary parts,

$$e^{r\cos\Theta} \cos(r\sin\Theta) = \sum_{n=0}^{+\infty} \frac{r^n}{n!} \cos n\Theta, \qquad e^{r\cos\Theta} \sin(r\sin\Theta) = \sum_{n=1}^{+\infty} \frac{r^n}{n!} \sin n\Theta.$$

Notice that the summation starts at $n = 1$ in the latter sum, because $\sin 0 \cdot \Theta = 0$ for $n = 0$, no matter Θ. $\Diamond$

Example 1.6.1 is only a demonstration of what we already can obtain. The following results are more important, so for this reason they are not relegated to an example.

This time we use that

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n = \sum_{n=0}^{+\infty} r^n e^{in\Theta} = \sum_{n=0}^{+\infty} r^n \cos n\Theta + i \sum_{n=1}^{+\infty} r^n \sin n\Theta \qquad \text{for } r = |z| < 1,$$

and that also

$$(43) \quad \frac{1}{1-z} = \frac{1-x}{(1-x)^2 + y^2} + i \frac{y}{(1-x)^2 + y^2}.$$

Using polar coordinates the dominator becomes

$$(1-x)^2 + y^2 = 1 - 2x + x^2 + y^2 = 1 + r^2 - 2r\cos\Theta,$$

so

$$\frac{1}{1-z} = \frac{1 - r\cos\Theta}{1 + r^2 - 2r\cos\Theta} + i\,\frac{r\sin\Theta}{1 + r^2 - 2r\cos\Theta} = \sum_{n=0}^{+\infty} r^n\cos n\Theta + i\sum_{n=1}^{+\infty} r^n\sin n\Theta, \qquad 0 \le r < 1.$$

When we identify the real and the imaginary parts we obtain the following convergent Fourier series,

$$(44) \quad \begin{cases} \dfrac{1 - r\cos\Theta}{1 + r^2 - 2r\cos\Theta} &=& \sum_{n=0}^{+\infty} r^n\cos n\Theta, \\[4mm] \dfrac{r\sin\Theta}{1 + r^2 - 2r\cos\Theta} &=& \sum_{n=1}^{+\infty} r^n\sin n\Theta, \end{cases} \qquad \text{for } 0 \le r < 1 \text{ and } \Theta \in \mathbb{R}.$$

We shall later also need the Fourier expansion of

$$\frac{1+z}{1-z} = \frac{2}{1-z} - 1 \qquad \text{for } |z| < 1.$$

It follows immediately from (44) that

$$\frac{1+z}{1-z} = 2\,\frac{1 - r\cos\Theta}{1 + r^2 - 2r\cos\Theta} - 1 + i\,\frac{2r\cos\Theta}{1 + r^2 - 2r\cos\Theta}$$

$$(45) \qquad = \frac{1 - r^2}{1 + r^2 - 2r\cos\Theta} + i\,\frac{2r\sin\Theta}{1 + r^2 - 2r\cos\Theta}$$

$$= 1 + 2\sum_{n=1}^{+\infty} r^n\cos n\Theta + 2i\sum_{n=1}^{+\infty} r^n\sin n\Theta \qquad \text{for } 0 \le r < 1 \text{ and } \Theta \in \mathbb{R}.$$

We shall finally prove a special case of *Parseval's equation.* This result will be applied later.

Theorem 1.6.1 Parseval's equation. *Let*

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad and \qquad g(z) = \sum_{n=0}^{+\infty} b z^n$$

be two analytic functions defined in $B(0, \varrho)$. *Then for every* $r \in [0, \varrho[$,

$$(46) \quad \frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\Theta}\right)\overline{g\left(re^{i\Theta}\right)}\,d\Theta = \sum_{n=0}^{+\infty} a_n\,\overline{b}_n\,r^{2n}.$$

PROOF. Since both series are absolutely convergent for $r = |z| < \varrho$, we may apply termwise multiplication, from which we get

$$(47) \quad f\left(re^{i\Theta}\right)\overline{g\left(re^{i\Theta}\right)} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} a_n\overline{b}_m r^{n+m}e^{i(n-m)\Theta},$$

which for fixed r is absolutely and uniformly convergent, because $\sum_m \sum_n \left| a_n \overline{b}_m \right| r^{n+m}$ is a converging majoring series. Hence, we may integrate (47) termwise. Then notice that

$$\int_0^{2\pi} e^{i(n-m)\Theta}\, \mathrm{d}\Theta = 0, \qquad \text{if } n \neq m,$$

so the only relevant terms of (47) by this integration, are the terms given by $n = m$, and we get trivially,

$$\frac{1}{2\pi} \int_0^{2\pi} f\left(re^{i\Theta}\right) \overline{g\left(re^{i\Theta}\right)}\, \mathrm{d}\Theta = \frac{1}{2\pi} \sum_{n=0}^{+\infty} a_n \overline{b}_n r^{2n} \int_0^{2\pi} 1\, \mathrm{d}\Theta = \sum_{n=0}^{+\infty} a_n \overline{b}_n r^{2n}. \qquad \square$$

Corollary 1.6.1 *Let* $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ *for* $|z| < \varrho$. *Then*

$$(48) \quad \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\Theta}\right) \right|^2\, d\Theta = \sum_{n=0}^{+\infty} |a_n|^2\, r^{2n}.$$

PROOF. Just put $g = f$ and $b_n = a_n$ into (46). $\square$

1.7   The maximum principle

Another strange property of a non-constant analytic function $f : \Omega \to \mathbb{C}$ is that the continuous function $|f(z)|$ can never have a local maximum at an interior point $z_0 \in \Omega$. We have more precisely

Theorem 1.7.1 The maximum principle *Let* $f : \Omega \to \mathbb{C}$ *be analytic in an open domain* Ω. *If* $|f(z)|$ *has a local maximum at a point* $z_0 \in \Omega$, *then* $f(z)$ *is constant in* Ω.

PROOF.   Assume that $|f(z)|$ has a local maximum at an interior point $z_0 \in \Omega$. Then we have a convergent series expansion

$$(49) \quad f(z) = \sum_{n=0}^{+\infty} a_n \left(z - z_0\right)^n \qquad \text{for } |z - z_0| < \varrho.$$

Choose $r \in]0, \varrho[$ and put $z = z_0 + re^{i\Theta}$. It follows from *Parseval's equation* (48) that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(z_0 + re^{i\Theta}\right) \right|^2\, \mathrm{d}\Theta = \sum_{n=0}^{+\infty} |a_n|^2\, r^{2n}.$$

By assumption, $|f(z_0)|$ is a local maximum, so due to the continuity we can choose $r \in]0, \varrho[$, such that $|f(z)| \leq |f(z_0)|$ in the closed disc $B[z_0, r]$, and we get the estimate

$$\sum_{n=0}^{+\infty} |a_n|^2\, r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(z_0 + re^{i\Theta}\right) \right|^2\, \mathrm{d}\Theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|^2\, \mathrm{d}\Theta = |f(z_0)|^2 = |a_0|^2\,,$$

hence,

$$\sum_{n=0}^{+\infty} |a_n|^2 \, r^{2n} \leq |a_o|^2 \,.$$

Since we have chosen $r > 0$, this is only possible, when all $a_n = 0$ for $n \in \mathbb{N}$, and we get by insertion into (49) that $f(z) \equiv a_0$ in Ω. $\square$

There is of course a corresponding *minimum principle*. This is, however, slightly more complicated, because every zero of $f(z)$ for natural reasons is a minimum point of the function $|f(z)|$. The simplest way to formulate it is

Theorem 1.7.2 The minimum principle. *Assume that the analytic function $f : \Omega \to \mathbb{C}$ is not constant in the open domain Ω. If $|f(z)|$ has a local minimum at an interior point $z_0 \in \Omega$, then $f(z_0) = 0$, i.e. z_0 must necessarily be a zero of $f(z)$.*

PROOF. Assume that $f(z_0) \neq 0$. Then there is an open subdomain $\omega \subseteq \Omega$, such that $f(z) \neq 0$ for all $z \in \omega$, where also $z_0 \in \omega$. The function $g(z) = \dfrac{1}{f(z)}$ must then be analytic in ω, so when $|f(z)| > 0$ has a local minimum at $z_0 \in \omega$, then $|g(z)|$ has a local maximum at the same point $z_0 \in \omega$. By the maximum principle this is not possible, so we conclude that $f(z_0) = 0$. $\square$

Theorem 1.7.3 *Let $f : \Omega \to \mathbb{C}$ be analytic in a bounded open domain, and assume that $f : \overline{\Omega} \to \mathbb{C}$ is continuous on the closure $\overline{\Omega}$ of Ω. Then $|f(z)|$ has its maximum lying on the boundary.*

PROOF. It follows from the assumptions that $\overline{\Omega}$ is compact. The function $|f| : \overline{\Omega} \to \mathbb{C}$ is continuous, so it has a maximum on $\overline{\Omega}$. (Main theorem for continuous real functions.) By the maximum principle, the maximum cannot be attained at an interior point, unless the function is constant, so the maximum is attained at a boundary point, no matter if $|f|$ is constant or not. $\square$

The conclusion of Theorem 1.7.3 is not true for unbounded closed domains. A simple counterexample is

$$f(z) = e^z \qquad \text{for } z \in \Omega := \{z \in \mathbb{C} \mid \Re z > 0\}.$$

Clearly, $|f(z)| = e^z$ is unbounded in Ω, so no maximum exists. On the boundary, however, i.e. on the imaginary axis, we have $|f(iy)| = |e^{iy}| = 1$, which is bounded.

The following is a partial inverse.

Theorem 1.7.4 Phragmèn-Lindelöf's theorem. *Assume that the function $f(z)$ is analytic in the half plane $\Re z > 0$, and assume that $f(z)$ is bounded and continuous on the boundary (the imaginary axis), i.e. $|f(iy)| \leq M$ for some constant $M > 0$ and all real $y \in \mathbb{R}$.*
Furthermore, assume that there exist real constants $a < 1$ and $K > 0$, such that

(50) $\quad |f(z)| < K \cdot \exp(r^a), \qquad$ *for all $z = r\,e^{i\Theta}$ for which $\Re z \geq 0$.*

Then $|f(z)| \leq M$ everywhere in the half plane $\Re z \geq 0$.

PROOF. Choose any $b \in \,]a, 1[$ and any $\varepsilon > 0$. When we consider the function $g(z) = f(z) \exp\left(-\varepsilon\, z^b\right)$, we get the estimate

$$|g(z)| \;=\; |f(z)| \exp\left(-\varepsilon\, r^b \cos(b\Theta)\right) \leq |f(z)| \exp\left(-\varepsilon\, r^b \cos\left(\frac{b\pi}{2}\right)\right)$$

$$(51) \qquad \leq \quad K \cdot \exp\left(r^a - \varepsilon \cos\left(\frac{b\pi}{2}\right) \cdot r^b\right).$$

Let z_0 be any point in the half plane and choose R, such that $|z_0| < R$, and such that $|g(z)| \leq M$ on the semicircle $|z| = R$, $\Re z \geq 0$.
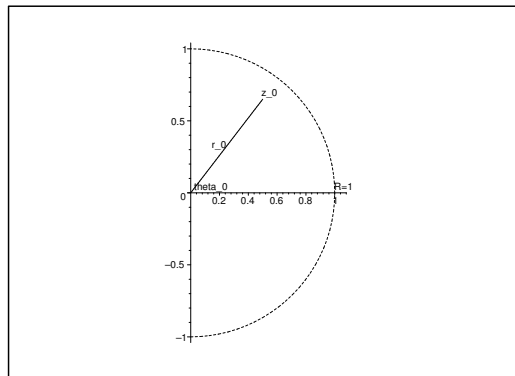

Figure 3: Phragmèn-Lindelöf's theorem.

This is possible, due to the choices of the constants ε and b, and due to the estimate (51).

The closed half disc is compact, so it follows from Theorem 1.7.3 that $|g(z_0)| \leq M$, hence

$$|f(z_0)| = |g(z_0)| \cdot \left|\exp\left(\varepsilon z_0^b\right)\right| \leq M \cdot \exp\left(\varepsilon \cdot r_0 \cos\left(b\Theta_0\right)\right).$$

This holds for every $\varepsilon > 0$, hence also by the limit process $\varepsilon \to 0+$, by which we get $|f(z_0)| \leq M$. Since z_0 was chosen arbitrarily in the half plane, the theorem is proved. $\square$

Notice that (50) is not fulfilled for any $a < 1$ for the function $f(z) = e^z$, so the theorem cannot be extended to $a = 1$.

There exists, however, another version of Phragmèn-Lindelöf's theorem, in which we allow $a \geq 1$, though the conclusion of course becomes weaker.

Theorem 1.7.5 Weak Phragmèn-Lindelöf's theorem. *Let $f(z)$ be analytic in a vertical strip $x_1 \leq \Re z \leq x_2$ (which means more precisely that $f(z)$ is analytic in some open domain containing this strip). We assume that $|f(z)| \leq 1$ on the boundary of this strip.*
If there exist constants $a > 0$ and $K > 0$, such that

$$(52) \quad |f(z)| \leq K \cdot \exp\left(|z|^a\right), \qquad \text{when } x_1 \leq \Re z \leq x_2,$$

then $|f(z)| \leq 1$ everywhere in the vertical strip.

PROOF. Let $\lambda > a$. Since $x = \Re z$ is bounded in the strip, there exists an y_0, such that

$$(53) \quad |f(z)| \leq \exp\left(|y|^\lambda\right), \qquad \text{when } |y| \geq y_0 \text{ and } x \in [x_1, x_2].$$

Choose $p \in \mathbb{N}$, such that $m = 2 + 4p > \lambda$.
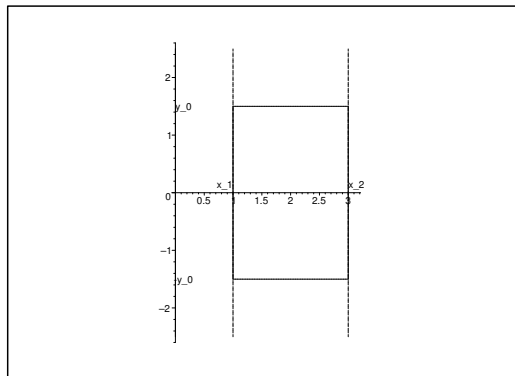

Figure 4: Proof of variant of Phragmèn-Lindelöf's theorem.

If $z = r\, e^{i\Theta}$ in the strip is large in the sense that $|\Im z| = |y| \geq y_0$, then

$$z^m = r^m \left(\cos m\Theta + i \sin m\Theta\right),$$

51

where $m\Theta$ lies close to π. More precisely, there is a $q \in \mathbb{Z}$, such that $|m\Theta - (\pi + 2q\pi)|$ is sufficiently small.

Choose any $\varepsilon > 0$ and consider the function

(54) $\ g_\varepsilon(z) = f(z) \exp\left(\varepsilon z^m\right).$

If z lies in the strip $x_1 \leq \Re z \leq x_2$, then we get the estimate

$$|g_\varepsilon(z)| \leq \exp\left(|y|^\lambda\right) \exp\left(\varepsilon\, r^m \cos m\Theta\right).$$

This shows that the function $g_\varepsilon(z)$ is bounded for large $y_0 = y_0(\varepsilon)$, and $|g_\varepsilon(z)| \leq 1$, when $|y| = y_0$ in the strip. The latter follows from the choice of m, by which we get $\cos m\Theta \leq -\alpha < 0$.

Since $|g_\varepsilon(z)| \leq \exp\left(\varepsilon x_2^m\right)$ on the edges of the rectangle, we conclude that $|g_\varepsilon(z)| \leq \exp\left(\varepsilon x_2^m\right)$ for all z inside the rectangle. Finally, since we already have proved that $|g_\varepsilon(z)| \leq 1$, when z in the strip satisfies $|\Im z| \geq y_0$, we conclude that

$$|g_\varepsilon(z)| \leq \exp\left(\varepsilon x_2^m\right) \qquad \text{for all } \varepsilon > 0 \text{ and } x_1 \leq \Re z \leq x_2.$$

Hence, it follows from (54) for every fixed z in the strip that

$$|f(z)| \leq \exp\left(\varepsilon x_2^m\right) \cdot \exp\left(\varepsilon |z|^m\right) \qquad \text{for all } \varepsilon > 0.$$

By taking the limit $\varepsilon \to 0+$ for every fixed z in the strip it follows that $|f(z)| \leq 1$. $\square$

The following result is often useful in the most unexpected situations.

Theorem 1.7.6 Schwarz's lemma. *Let $f : B(a, R) \to \mathbb{C}$ be analytic, where $f(a) = 0$ and $|f(z)| \leq M$ for every $z \in B(a, R)$. Then*

(55) $\ |f(z)| \leq \dfrac{M}{R}\,|z - a| \qquad \text{for all } z \in B(a, R).$

If we have equality in (55) at some point $z \in B(a, R)$, $z \neq a$, then $f(z)$ has the structure

(56) $\ f(z) = e^{i\Theta} \cdot \dfrac{M}{R} \cdot (z - a), \qquad \text{for all } z \in B(a, R).$

PROOF. It follows from the assumption $f(a) = 0$ that

$$f(z) = \sum_{n=1}^{+\infty} a_n (z - a)^n = (z - a) \sum_{n=0}^{+\infty} a_{n+1}(z - a)^n \qquad \text{for } z \in B(a, R).$$

If we put $g(z) = \sum_{n=0}^{+\infty} a_{n+1}(z - a)^n$, then

$$g(z) = \begin{cases} \dfrac{f(z)}{z - a} & \text{for } z \in B(a, R),\ z \neq a, \\[2mm] a_1 & \text{for } z = a, \end{cases}$$

and $g : B(a, R) \to \mathbb{C}$ is analytic.

Let $r \in]0, R[$ and $|z - a| = r$. Then we get the estimate

$$|g(z)| \leq \frac{M}{r},$$

hence by the maximum principle,

$$|g(z)| \leq \frac{M}{r} \qquad \text{for all } z \in B(a, r).$$

Since $r < R$ can be chosen arbitrarily close to R, we conclude by taking the limit $r \to R-$ that

$$|g(z)| \leq \frac{M}{R} \qquad \text{for all } z \in B(a, R),$$

hence

$$|f(z)| \leq \frac{M}{R} \cdot |z - a|.$$

Then assume that we have equality at a point $z_0 \in B(a, R)$, $z_0 \neq a$. Then $|g(z)|$ has its maximum at the point z_0 in the interior of $B(a, R)$, so $g(z)$ is constant by the maximum principle, hence

$$g(z) = e^{i\Theta} \cdot \frac{M}{R}, \quad \text{and thus} \quad f(z) = e^{i\Theta} \cdot \frac{M}{R} \cdot (z - a). \qquad \square$$
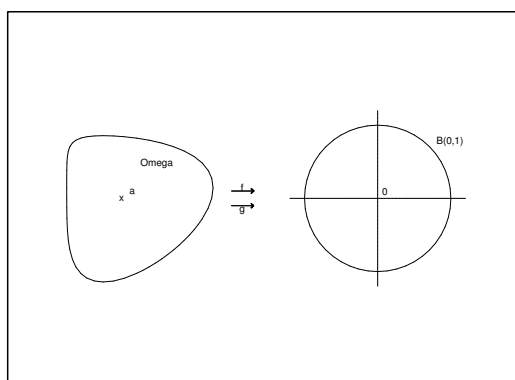
Figure 5: Proof of Corollary 1.7.1.

One nice application of *Schwarz's lemma* is the following

Corollary 1.7.1 *Let Ω be an open domain with $a \in \Omega$. If there exists an analytic map f, which maps Ω bijectively onto $B(0, 1)$, such that $f(a) = 0$, then every bijective analytic map $g : \Omega \to B(0, 1)$, for which $g(a) = 0$, is given by*

(57) $g(z) = e^{i\Theta} f(z) \qquad$ *for some $\Theta \in [0, 2\pi[$.*

Thus, these maps are uniquely determined apart from a rotation of the disc $B(0, 1)$.

53

PROOF. Clearly, every map of the form (57) satisfies the given conditions.

Let $g : \Omega \to B(0,1)$ be any bijective analytic function for which $g(a) = 0$.

The composite map $g \circ f^{-1} : B(0,1) \to B(0,1)$ is bijective, and $g \circ f^{-1}(0) = g(a) = 0$. Using *Schwarz's lemma* it follows from $\left| g \circ f^{-1}(z) \right| \leq |z|$.

Now, the inverse of $g \circ f^{-1}$, i.e. $f \circ g^{-1}$, fulfils precisely the same properties, so we also have $\left| f \circ g^{-1}(w) \right| \leq |w|$.

Then, by putting $w = g \circ f^{-1}(z)$,

$$|z| = \left| f \circ g^{-1}(w) \right| \leq |w| = \left| g \circ f^{-1}(z) \right| \leq |z|,$$

so we must have equality. Then by *Schwarz's lemma*,

$$e^{i\Theta} \cdot \left(f \circ g^{-1} \right)(w) = w.$$

Finally, putting $w = g(\zeta)$, we get $g(\zeta) = e^{i\Theta} f(\zeta)$, and the corollary is proved. $\square$

2 Harmonic Functions

2.1 Harmonic functions

The *harmonic functions* are solutions of the *Laplace equation,*

(58) $\Delta u := \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,$

which has many applications in the plane i the applied sciences. We shall see that the harmonic functions are also closely connected with the analytic functions, so many results on harmonic functions can easily be derived from the theory of analytic functions.

Let $f : \Omega \to \mathbb{C}$ be an analytic function in the open domain Ω. Then we have proved in *Ventus: Complex Functions Theory a-1* that f is of class $C^\infty(\Omega)$. If we split f into its real and imaginary parts, $f = u + i\,v$, then we get by Cauchy-Riemann's equations

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y},$$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\,\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - i\,\frac{\partial^2 u}{\partial y \partial x} = \cdots,$$

etc., from which we conclude that $\Re f = u(x,y)$ and $\Im f = v(x,y)$ are both of class C^∞ in the *real* variables $(x, y) \in \Omega$. In particular, we may interchange the order of differentiation. Since $f(z)$ is analytic we can apply Cauchy-Riemann's equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

hence by differentiation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y},$$

so by adding these two expressions we get the *Laplace equation* (58).

Similarly, we prove that $\Delta v = 0$.

We introduce formally,

Definition 2.1.1 *Let $\Omega \subseteq \mathbb{R}^2$ be an open domain. A real function $u \in C^2(\Omega)$ is called* harmonic *in Ω, if $\Delta u = 0$ in Ω.*

It follows from the above that

Theorem 2.1.1 *If $f(z) = u + iv$ is analytic in an open domain $\Omega \subseteq \mathbb{C}$, then its real and imaginary parts are both harmonic functions in $\Omega \subseteq \mathbb{R}^2$. Here the planar domain Ω is considered both as a subset of $\mathbb{C}$ and of $\mathbb{R}^2$, depending on the context.*

The harmonic functions are of course important in every two dimensional *potential theory.* We list some elementary harmonic functions in Table 1, page 66.

Definition 2.1.2 *Assume that u and v are harmonic in the open domain Ω. If the pair (u, v) satisfies Cauchy-Riemann's equations, then we say that v* (the second coordinate) *is the* harmonic conjugated *of u* (the first coordinate), *or that the pair (u, v) is an* harmonic conjugated pair.

We notice that if (u, v) is a harmonic conjugated pair, then $(-v, u)$ is also a harmonic conjugated pair, while (v, u) is only harmonic conjugated, when both u and v are constant functions. Thus, harmonic conjugation is *not* a symmetric relation.

It follows immediately from *Cauchy-Riemann's equations* that

Theorem 2.1.2 *If (u, v) is an harmonic conjugated pair in Ω, then $f = u + iv$ is an analytic function in Ω.*

Assume that $u(x, y)$ is harmonic in an open and *simply connected* domain Ω, i.e.

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \qquad \text{for } (x, y) \in \Omega.$$

It follows that the differential form

$$-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy$$

is closed, so it is exact, Ω being simply connected.

Consider a fixed $z_0 \in \Omega$. It follows from the above that the function

$$(59) \quad v(x, y) := \int_{z_0}^{z} \left\{ -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right\}, \qquad z = x + iy \in \Omega,$$

is uniquely defined in Ω.

Since u is of class $C^2(\Omega)$, it follows that $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are of class $C^1(\Omega)$, so we conclude that $v \in C^2(\Omega)$. In particular,

$$(60) \quad dv = \frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dx,$$

hence *Cauchy-Riemann's equations* follow by identification. Since $u, v \in C^2(\Omega)$, we conclude that $f(z) = u + iv$ is analytic, and we have proved

Theorem 2.1.3 *If $u(x, y)$ is harmonic in a* simply connected *open domain Ω, then there exists an analytic function $f : \Omega \to \mathbb{C}$, such that $u(x, y) = \Re f(z)$ in Ω. In particular, u is of class $C^\infty(\Omega)$, and (u, v) is a harmonic conjugated pair, when the function $v(x, y)$ is given by (59).*

Example 2.1.1 If Ω is an open domain, which is not simply connected, and $u(x, y)$ is harmonic in Ω, then there does not necessarily exist an analytic function f defined in all of Ω, such that $\Re f(z) = u(x, y)$. A very important counterexample is

$$u(x, y) = \frac{1}{2} \ln\left(x^2 + y^2\right) = \ln|z| = \ln r,$$

which is harmonic in the (not simply connected) domain $\mathbb{C} \setminus \{0\}$. By considering the analytic function $\text{Log } z$ in the sliced and connected domain $\mathbb{C} \setminus \{\mathbb{R}_- \cup \{0\}\}$ we see that

$$\Re \text{ Log } z = \frac{1}{2} \ln \left(x^2 + y^2 \right) \qquad \text{for } z \in \mathbb{C} \setminus \{\mathbb{R}_- \cup \{0\}\},$$

where $\text{Log } z$ cannot be extended analytically to all of $\mathbb{C} \setminus \{0\}$. $\Diamond$

Assume that u, v_1 and v_2 are harmonic functions in the same domain Ω, and that both (u, v_1) and (u, v_2) are harmonic conjugated pairs. Then $f_1(z) := u + iv_1$ and $f_2(z) := u + iv_2$ are both analytic in Ω, so

$$f(z) = f_1(z) - f_2(z) = i \{v_1 - v_2\}$$

is analytic in Ω. Then by *Cauchy-Riemann's equations*,

$$\frac{\partial}{\partial x} \{v_1 - v_2\} = \frac{\partial}{\partial y} \{v_1 - v_2\} = 0,$$

hence $v_1 = v_2 + C$ for some constant $C \in \mathbb{R}$, and we have proved

Corollary 2.1.1 *If $u(x,y)$ is harmonic in a* simply connected *open domain Ω, and $z_0 \in \Omega$ is a fixed point, then all harmonic conjugated functions of u are given by*

$$(61) \quad v(x,y) := \int_{z_0}^{z} \left\{ -\frac{\partial u}{\partial y} \, dx + \frac{\partial u}{\partial x} \, dy \right\} + C, \qquad C \in \mathbb{R} \text{ arbitrary.}$$

If $v(x,y)$ is harmonic in a simply connected domain Ω, it follows similarly that all harmonic functions u, for which (u,v) is a harmonic conjugated pair, are given by

$$(62) \quad u(x,y) := \int_{z_0}^{z} \left\{ \frac{\partial v}{\partial y} \, dx - \frac{\partial v}{\partial x} \, dy \right\} + C, \qquad C \in \mathbb{R} \text{ arbitrary,}$$

(notice the change of sign compared with (61)), and $f(z) = u + iv$ is analytic in Ω.

Example 2.1.2 We shall show that the function $u(x,y) = x^3 - 3xy^2$ is harmonic, and find all its harmonic conjugated functions. Clearly, $u \in C^\infty \left(\mathbb{R}^2 \right)$, and

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x,$$

thus

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0,$$

and $u(x,y)$ is harmonic.

Then insert the expressions of $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ above into (61). Choosing $z_0 = 0$ we get

$$\begin{aligned} v(x,y) &= \int_{z_0}^{z} \left\{ -(6xy) \, dx + \left(3x^2 - 3y^2 \right) dy \right\} + C = \int_{(0,0)}^{(x,y)} \left\{ \left(6xy \, dx + 3x^2 \, dy \right) - 3y^2 \, dy \right\} + C \\ &= \int_{(0,0)}^{(x,y)} d \left(3x^2 y - y^3 \right) + C = 3x^2 y - y^3 + C, \qquad C \in \mathbb{R} \text{ arbitrary.} \end{aligned}$$

Here we have applied that the differential form $6xy\,dx + \left(3x^2 - 3y^2\right) dy$ is *exact*, so it can be written in the form dv. We have in particular applied the rule of the differential of products,

$$d(\varphi\psi) = \psi\,d\varphi + \varphi\,d\psi$$

in the reverse direction, i.e. from the right to the left.

Alternatively we may integrate along a simple curve composed of axiparallel segments. $\Diamond$

Remark 2.1.1 The beginner often makes the error that $\int 6xy\,dx$ and $\int \left\{3x^2 - 3y^2\right\} dy$ are computed separately, where the other variable erroneously is considered as a constant. By an addition we get the wrong result $6x^2y - y^3$, which can easily be checked by *Cauchy-Riemann's equations*, which are not fulfilled in this case. $\Diamond$

It is in some cases possible from a given harmonic function u in a simply connected domain Ω directly to find the corresponding analytic function $f(z)$, such that

$$u(x, y) = \Re f(z).$$

First, it follows from Theorem 2.1.3 that $f(z)$ exists. Then by *Cauchy-Riemann's equations*,

$$(63)\quad f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y} \qquad \left(= \frac{\partial u}{\partial x} + \frac{1}{i}\,\frac{\partial u}{\partial y}\right).$$

Since the function $u(x,y)$ is given, (63) shows that $f'(z)$ can be computed directly as an analytic function, so the right hand side of (63) must be expressed as a function in $z\,(=x+iy)$ alone. Finally, we get $f(z)$ by an indefinite integration, where the arbitrary constant is determined, such that $\Re f(z) = u(x,y)$ in Ω.

Example 2.1.3 We shall reconsider the function $u(x,y) = x^3 - 3xy^2$ from Example 2.1.2. We can now prove that $u(x,y)$ is harmonic without inserting into the Laplace equation $\Delta u = 0$. In fact, *if u were harmonic*, then there would exist an analytic function $f(z)$, such that

$$f'(z) = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - i\{-6xy\} = 3z^2.$$

Since $f'(z) = 3z^2$ has the indefinite integral $z^3 + C$, it is straightforward to check that

$$\Re z^3 = x^3 - 3xy^2 = u(x,y),$$

so u is an harmonic function.

In the present case one cannot claim that Example 2.1.3 is easier than Example 2.1.2. However, it is indeed very easy to provide examples, where

$$\frac{\partial^2 u}{\partial x^2} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2}$$

become very cumbersome to compute, while

$$\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial u}{\partial y}$$

occurring in $f'(z)$ given by (63) are easy to find, so it becomes a simple task to find $f'(z)$ as a function of z alone. $\Diamond$

2.2   The maximum principle for harmonic functions.

Given an harmonic function $u(x,y)$ in an open domain Ω, and assume that u has a local maximum (or a local minimum) at an inner point $z_0 \sim (x_0, y_0) \in \Omega$. There exists an $r > 0$, such that $B(z_0, r) \subseteq \Omega$. Since $B(z_0, r)$ is simply connected, we can find an analytic function $f(z)$, such that $\Re f(z) = u(x,y)$ locally in $B(z_0, r)$.

Since

$$\left| e^{f(z)} \right| = e^{\Re f(z)} = e(x,y) > 0,$$

we conclude that $\left| e^{f(z)} \right|$ has a maximum (a minimum) at z_0. The function $e^{f(z)}$ is analytic, hence $e^{f(z)}$ is a constant by the maximum (the minimum) principle, and $u(x,y)$ is constant in $B(z_0, r)$. Since u is continuous, it is constant in every simply connected subdomain $\omega \subseteq \Omega$, hence also in Ω itself, and it follows by contraposition that we have proved

Theorem 2.2.1 The maximum (minimum) principle for harmonic functions. *Let $u(x,y)$ be harmonic and not a constant in the open domain Ω. Then $u(x,y)$ has neither a maximum nor a minimum in Ω.*

Let $K \subseteq \Omega$ be a compact set, and let $u(x, y)$ be harmonic in Ω. Since $u(x, y)$ only has real values, it follows by the *Main Theorem for the Continuous Functions* that $u(x, y)$ has a maximum and a minimum in K. It follows from the above that these values are always obtained on the boundary of K, no matter if $u(x, y)$ is a constant or not in Ω. It therefore immediately follows that we have

Corollary 2.2.1 *Assume that Ω is an open and bounded domain. Let $u(x, y)$ be continuous on the closure $\overline{\Omega}$ and harmonic in Ω itself. Then $u(x, y)$ has its maximum and minimum on the boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$ of Ω.*

PROOF. Just notice that $\overline{\Omega}$ is compact and apply the previous argument. $\square$

In *Ventus: Complex Functions Theory a-1* we proved the *Mean Value Theorem for Analytic Functions*, i.e.

$$(64) \ \ f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + r\, e^{i\Theta}\right) \, \mathrm{d}\Theta,$$

assuming that $f(z)$ is analytic in a neighbourhood of the closed disc $B\left[z_0, r\right]$ of radius $r > 0$.

Clearly, we obtain a similar *Mean Value Theorem for Harmonic Functions* by simply taking the real part of (64). We shall more precisely state this as a theorem,

Theorem 2.2.2 Mean Value Theorem for Harmonic Functions. *Let $u(x, y)$ be harmonic in an open domain Ω containing the closed disc $B\left[z_0, r\right]$ of radius $r > 0$. Then the value $u(z_0) = u(x_0, y_0)$ at the centre of the disc is equal to the mean value of $u(x, y)$ over the circle of centre (x_0, y_0) and radius r, i.e.*

$$(65) \ \ u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + r\, e^{i\Theta}\right) \, d\Theta.$$

Finally we prove that if Ω is an open bounded domain, and $h(x, y)$ is a continuous function on the *boundary* $\partial\Omega$ of Ω, where $\partial\Omega$ is composed of continuous and piecewise differentiable curves, then the *boundary value problem*

$$(66) \ \ \begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in \Omega, \\[3mm] u(x, y) = h(x, y), & \text{for } (x, y) \in \partial\Omega, \end{cases}$$

has at most one solution.

Theorem 2.2.3 *Assume that Ω is an open and bounded domain. Let u and v be continuous functions on the closure $\overline{\Omega}$ and harmonic in Ω itself. If $u = v$ on the boundary $\partial\Omega$, then $u = v$ in all of $\overline{\Omega}$.*

PROOF. Put $\varphi = u - v$. Then φ is harmonic in Ω and $varphi = 0$ on $\partial\Omega$. It follows from Corollary 2.2.1 that since φ has its maximum and minimum (both $= 0$) on the boundary, we must have $\varphi = u - v = 0$ in all of $\overline{\Omega}$, hence $u = v$ in $\overline{\Omega}$. $\square$

Example 2.2.1 We shall here show that we cannot exclude the assumption that Ω is bounded in Theorem 2.2.3.

It follows from the obvious fact

$$\Delta(xy) = \frac{\partial^2}{\partial x^2}(xy) + \frac{\partial^2}{\partial y^2}(xy) = 0$$

that $u(x,y) = xy = \frac{1}{2}\Im z^2$ is harmonic in $\mathbb{R}^2$. Let $\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid y > 0\right\}$ denote the upper half plane. Then the boundary $\partial\Omega$ is the X-axis, and we have obviously $u(x,0) = 0$ on the X-axis.

Clearly, $u(x,y)$ has neither a maximum nor a minimum in Ω, because its range is $\mathbb{R}$.

Since $v(x,y) = 0$ is another harmonic function, for which $v(x,0) = 0$, this example illustrates that if Ω is not bounded, then the solution of the boundary value problem (66) is not unique.

We shall later in Section 2.4 explicitly solve (66) in the special case, when Ω is an open disc. $\Diamond$

2.3 The biharmonic equation

In problems from the two-dimensional elasticity theory one has to deal with the *biharmonic equation*

(67) $\Delta\Delta\Phi = \dfrac{\partial^\Phi}{\partial x^4} + 2\,\dfrac{\partial^4\Phi}{\partial x^2\partial y^2} + \dfrac{\partial^4\Phi}{\partial y^4} = 0.$

We shall in the following solve it in *simply connected domains*.

Remark 2.3.1 The conventional name "biharmonic equation" is misleading, because there are biharmonic functions which are not harmonic, while all harmonic functions trivially are biharmonic. A better name would therefore be "semiharmonic equation". It has, however, become customary to call it the biharmonic equation, so we shall stick to this notation. $\Diamond$

Theorem 2.3.1 *Let Ω be an open* simply connected *domain. Every solution of the biharmonic equation $\Delta\Delta\Phi = 0$ can be written in the form*

(68) $\Phi = \Re\left\{\overline{z}\,f(z) + g(z)\right\},$

where f and g are analytic functions in Ω.

PROOF. As mentioned above, every harmonic function is also biharmonic. Since $\Re g(z)$ is harmonic, we shall only prove that $\Re\left\{\overline{z}\,f(z)\right\}$ is biharmonic to conclude that every function Φ of the form (68) is biharmonic.

Put $f = u + iv$. Then

$$\Re\left\{\overline{z}\,f(z)\right\} = \Re\{(x - iy)(u + iv)\} = xu + yv.$$

When (68) is inserted into (67), we get

$$
\begin{aligned}
\Delta\Delta\Phi &= \Delta\left\{\frac{\partial^2}{\partial x^2}(xu+yv)+\frac{\partial^2}{\partial y^2}(xu+yv)\right\} \\
&= \Delta\left\{\frac{\partial}{\partial x}\left(u+x\frac{\partial u}{\partial x}\right)+y\frac{\partial^2 v}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial y}\left(v+y\frac{\partial v}{\partial y}\right)\right\} \\
&= \Delta\left\{2\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 v}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}+2\frac{\partial v}{\partial y}+y\frac{\partial^2 v}{\partial y^2}\right\} \\
&= 2\Delta\left\{\frac{\partial u}{\partial x}\right\}+2\Delta\left\{\frac{\partial v}{\partial y}\right\}+\Delta\{x\Delta u+y\Delta v\} \\
&= 2\frac{\partial}{\partial x}\{\Delta u\}+2\frac{\partial}{\partial y}\{\Delta v\}+\Delta(x\cdot 0+y\cdot 0)=0,
\end{aligned}
$$

and the claim is proved.

Then assume that Φ is biharmonic in Ω, i.e. $\Delta\Delta\Phi = 0$. We shall prove that Φ can be written in the form (68) for some analytic functions f and g.

It follows readily from $\Delta\Delta\Phi = \Delta(\Delta\Phi) = 0$ that $\Delta\Phi$ is harmonic. Thus there exists an analytic function h in the simply connected domain Ω, such that $\Delta\Phi = \Re h$. Using once more that Ω is simply connected we can define the indefinite integral f of the analytic function $\dfrac{1}{4}h$ in Ω, thus $f' = \dfrac{1}{4}h$. Then by a similar computation as above,

$$\Delta\left\{\Re\left(\overline{z}\,f(z)\right)\right\} = \Re h = \Delta\Phi,$$

so

$$\Delta\left\{\Phi - \Re\left(\overline{z}\,f(z)\right)\right\} = 0,$$

and we have proved that $\Phi - \overline{z}\,f(z)$ is harmonic in Ω. Thus there exists an analytic function g on Ω, such that

$$\Phi - \Re\left\{\overline{z}\,f(z)\right\} = \Re g(z),$$

and (68) is obtained by a rearrangement. $\square$

2.4   Poisson's Integral Formula

Assuming that Ω is an open bounded domain and $h(x,y)$ is a continuous function on the boundary $\partial\Omega$, we proved in Theorem 2.2.3 that the *boundary value problem*

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x,y) \in \Omega, \\[3mm] u(x,y) = h(x,y), & \text{for } (x,y) \in \partial\Omega, \end{cases}$$

has *at most* one solution.

We shall in the following prove that in the special case of $\Omega = B(0,R)$ there *exists* a solution, and we shall derive a solution formula. For practical reasons we shall in the following sometimes write $u(z)$ instead of $u(x,y)$, where as usual $z = x + iy$.

Theorem 2.4.1 Poisson's integral formula (1820). *Let Ω be an open domain, containing the closed disc $B[0,R]$ of centre $(0,0)$ and radius $R > 0$, and let $f(z)$ be analytic in Ω. Then we have for any point $z_0 = x_0 + iy_0 = re^{i\Theta} \in B(0,R)$, thus $0 < r < R$,*

$$(69) \quad u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\Theta - t)}\, u(Re^{it})\, dt,$$

and analogously,

$$v(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\Theta - t)}\, v(Re^{it})\, dt.$$

If u is harmonic in Ω, then every harmonic conjugated function $v(x,y)$ of $u(x,y)$ is given by the formula

$$(70) \quad v(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{2Rr\sin(\Theta - t)}{R^2 + r^2 - 2Rr\cos(\Theta - t)}\, u(Re^{it})\, dt + v(0).$$

Finally,

$$(71) \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z + z_0}{z - z_0}\, u(z)\, dt + i\, v(0), \qquad \text{where } z = Re^{it}.$$

PROOF. First, by *Cauchy's integral formula,*

$$(72) \quad f(z_0) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z - z_0}\, dz.$$

If $z_0 \neq 0$, then $z_1 := R^2/\overline{z}_0$ is a point outside $|z| = R$, so $f(z)/\{z - z_1\}$ is analytic in an open set containing the closed disc $B[0,R]$. Hence by *Cauchy's integral theorem,*

$$(73) \quad 0 = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z - z_1}\, dz, \qquad \text{where } z_1\overline{z}_0 = R^2.$$

Notice that $\text{Arg}\, z_0 = \text{Arg}\, z_1 = \Theta_0$.

Let $z = z(t) = Re^{it}$, $t \in [0, 2\pi]$, be a parametric description of the circle $|z| = R$. Then $\mathrm{d}z = iRe^{it}\,\mathrm{d}t = iz\,\mathrm{d}t$. Writing $f(z) = u + iv$ we get by insertion into (72),

$$(74)\quad f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{u + iv}{z - z_0}\, iz\,\mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} \frac{z}{z - z_0}\,(u + iv)\,\mathrm{d}t.$$

Apply the same substitution in (73) and conjugate the result. Using furthermore that

$$z\,\overline{z} = R^2 \qquad \text{and} \qquad \overline{z}_1 = \frac{R^2}{z_0} = \frac{z\,\overline{z}}{z_0},$$

we get

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\overline{z}}{\overline{z} - \dfrac{z\,\overline{z}}{z_0}}\,(u - iv)\,\mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} \frac{z_0}{z_0 - z}\,(u - iv)\,\mathrm{d}t,$$

thus when we multiply by -1,

$$(75)\quad 0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{z_0}{z - z_0}\,(u - iv)\,\mathrm{d}t, \qquad z = Re^{it}.$$

Formula (75) was proved, assuming that $z_0 \neq 0$, and it is trivial for $z_0 = 0$. Hence we get by adding (74) and (75),

$$
\begin{aligned}
f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{z}{z - z_0}\,(u + iv) + \frac{z_0}{z - z_0}\,(u - iv) \right\} \mathrm{d}t \\
(76) & \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{z + z_0}{z - z_0}\, u(z)\,\mathrm{d}t + \frac{i}{2\pi} \int_0^{2\pi} v(z)\,\mathrm{d}t.
\end{aligned}
$$

Now, $z = Re^{it}$ and $z_0 = re^{i\Theta}$, so be get by (45) in Section 1.6 that

$$
\begin{aligned}
\frac{z + z_0}{z - z_0} &= \frac{1 + \dfrac{r}{R}\,e^{i(\Theta - t)}}{1 - \dfrac{r}{R}\,e^{i(\Theta - t)}} = \frac{1 - \left\{\dfrac{r}{R}\right\}^2}{1 + \left\{\dfrac{r}{R}\right\}^2 - 2\dfrac{r}{R}\cos(\Theta - t)} + i\,\frac{2\dfrac{r}{R}\sin(\theta - t)}{1 + \left\{\dfrac{r}{R}\right\}^2 - 2\dfrac{r}{R}\cos(\Theta - t)} \\
(77) & \\
&= \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\Theta - t)} + i\,\frac{2Rr\sin(\Theta - t)}{R^2 + r^2 - 2Rr\cos(\Theta - t)}.
\end{aligned}
$$

Finally, it follows from the *Mean Value theorem for Harmonic Functions* that

$$\frac{1}{2\pi} \int_0^{2\pi} v(z)\,\mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} v\big(Re^{it}\big)\,\mathrm{d}t = v(0),$$

and (71) follows from (76). If (77) is put into (76), we get (69) and (70) by splitting into the real and the imaginary parts. $\square$

Example 2.4.1 Let $\Omega = B(0,1)$, and let $h(z)$ be a continuous function on the unit circle $|z| = 1$. We shall solve the *boundary value problem*,

$$
(78) \quad
\begin{cases}
\Delta u = 0, & \text{for } (x,y) \in \Omega, \\[2mm]
u(x,y) = h(x,y) & \text{for } x^2 + y^2 = 1.
\end{cases}
$$

First define a continuous function $\varphi(t)$ on $[0, 2\pi]$ by

$$
\varphi(t) = h\!\left(e^{it}\right), \quad t \in [0, 2\pi], \qquad \varphi(0) = \varphi(2\pi).
$$

Then by the classical *Theory of Fourier Series*, $\varphi(t)$ has a *Fourier series expansion*,

$$
\varphi(t) \sim \frac{1}{2}\, a_0 + \sum_{n=1}^{+\infty} \{a_n \cos nt + b_n \sin nt\},
$$

where

$$
a_n = \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \cos nt \, dt \qquad \text{and} \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \sin nt \, dt.
$$

The solution of (78) is given by (69),

$$
u\!\left(re^{i\Theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\Theta - t)} \, \varphi(t) \, dt, \qquad r \in [0, 1[.
$$

We have in Section 1.6 proved (45), i.e.

$$
(79) \quad \frac{1 - r^2}{1 + r^2 - 2r\cos(\Theta - t)} = 1 + 2 \sum_{n=1}^{+\infty} r^n \cos(n\{\Theta - t\}).
$$

When $0 \leq r < 1$ is fixed, then the series in (79) is uniformly convergent, so we can interchange summation and integration in the computation below, when (79) is inserted into the expression of $u\!\left(re^{i\Theta}\right)$,

$$
u\!\left(re^{i\Theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \, dt + \sum_{n=1}^{+\infty} r^n \cdot \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \cos(n\Theta - nt) \, dt.
$$

Here,

$$
\frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \, dt = \frac{1}{2}\, a_0,
$$

and

$$
\frac{1}{\pi} \int_0^{2\pi} \varphi(t) \cos(n\Theta - nt) \, dt = \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \cos nt \, dt \cdot \cos n\Theta + \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \sin nt \, dt \cdot \sin n\Theta = a_n \cos n\Theta + b_n \sin n\Theta,
$$

hence

$$
(80) \quad u\!\left(re^{i\Theta}\right) = \frac{1}{2}\, a_0 + \sum_{n=1}^{+\infty} r^n \{a_n \cos n\Theta + b_n \sin n\Theta\},
$$

and we have proved that the unique solution of the boundary value problem (78) is obtained from the Fourier series expansion of $\varphi(t)$ by multiplying the n-th term of the series by r^n, $r \in [0, 1]$. $\lozenge$

2.5   Electrostatic fields

A *planar electrostatic field* is a vector field in the complex plane, which at each point of its domain is given by the vector $\vec{\mathcal{E}} \sim E_x + iE_y$, corresponding to the force, which the field would exercise on a unit charge at the point.
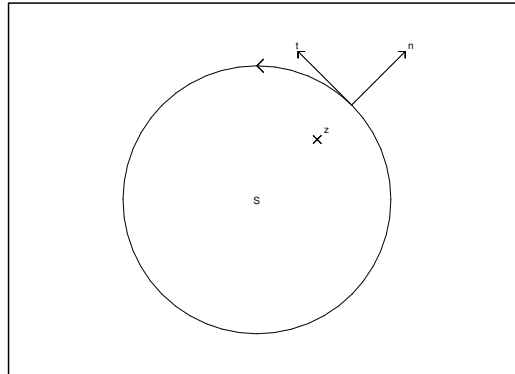

Figure 6: The tangent field and the normal field of a closed curve C in the plane.

The *flux* through a closed curve C of the field is defined by

$$N := \oint_C \vec{\mathcal{E}} \cdot \vec{n} \, \mathrm{d}s = 4\pi e,$$

(cf. Figure 6), where e is the sum of all charges inside C, and $\vec{n}$ is the outward normal vector field of the curve C, and s is the natural parameter of C defined by the curve length. At each point z,

$$(81) \ \operatorname{div} \vec{\mathcal{E}} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \lim_{C \to z} \frac{N}{S} = 4\pi\varrho,$$

where the curve C in some sense shrinks to z (choose e.g. C_r as the circle of centre z and radius r, and let $r \to 0+$), and where S is the area of the bounded domain inside C, and finally, ϱ is the *density of the charge* at the point z.

The *circulation* of $\vec{\mathcal{E}}$ along C is equal to the *work*

$$W := \oint_C \vec{\mathcal{E}} \cdot \vec{t} \, \mathrm{d}s,$$

where $\vec{t}$ denotes the unit tangent field of C. When the work is 0 along every simple closed curve C, we get by *Stokes's theorem* in two dimensions that

$$\vec{\operatorname{rot}}\, \vec{\mathcal{E}} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0.$$

This shows that the differential form $E_x \, \mathrm{d}x + E_y \, \mathrm{d}y$ is closed, thus exact in (at least) every simply connected domain. Hence, there exists a *potential* v, such that

$$E_x \, \mathrm{d}x + E_y \, \mathrm{d}y = -\mathrm{d}v,$$

(the minus sign is due to convention), so

$$\vec{\mathcal{E}} = -\frac{\partial v}{\partial x} - i\,\frac{\partial v}{\partial y} = -\vec{\mathrm{grad}}\, v.$$

If an open and simply connected domain Ω does not contain any charge, then it follows from (81) that

$$\mathrm{div}\,\vec{\mathcal{E}} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0,$$

so the differential form $-E_y\,\mathrm{d}x + E_x\,\mathrm{d}y = \mathrm{d}u$ is also exact. Then it is easy to prove the the *level curves* of the function $u(x,y)$ are the *field lines*, where the tangents define the direction of the field.

Under the given assumptions above we have constructed u and v, such that

$$\frac{\partial u}{\partial x} = -E_y, \quad \frac{\partial u}{\partial y} = E_x, \qquad \frac{\partial v}{\partial x} = -E_x, \quad \frac{\partial v}{\partial y} = -E_y.$$

It follows that u and v satisfy *Cauchy-Riemann's equations*, so they are harmonic functions in Ω, and the function

$$f(z) = u(x,y) + i\,v(x,y)$$

is analytic in Ω.

The function $f(z)$ is called the *complex potential* of the field, and the electrostatic field $\vec{\mathcal{E}}$ can be represented by

$$\vec{\mathcal{E}} = -\frac{\partial v}{\partial x} - i\,\frac{\partial u}{\partial x} = -i\,\overline{f'(z)}.$$

The field lines and the potential curves are level curves of harmonic functions. This shows that we also for (two-dimensional) electrostatic fields are interested in the solution of the boundary value problem (66).

2.6 Static temperature fields

Let $u(x,y,t)$ be a planar *temperature field* in a domain Ω. If Ω does not contain any source of heat, then it can be proved that $u(x,y,t)$ satisfies the *heat equation*

$$(82) \quad \frac{\partial u}{\partial t} = a^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\},$$

where t denotes the time, and a^2 is a positive constant. We shall not here solve (82) in general. We only note that if the temperature field does *not* depend on time (corresponding to an equilibrium state of the temperature), then $u(x,y,t) = u(x,y)$, and (82) reduces to the well-known *Laplace equation* $\Delta u = 0$, and $u(x,y)$ is an harmonic function.

$f(z)$	$u = \Re f(z)$	$v = \Im f(z)$
$i\,f(z)$	$-v = -\Im f(z)$	$u = \Re f(z)$
z	x	y
z^2	$x^2 - y^2$	$2xy$
z^3	$x^3 - 3xy^2$	$3x^2y - y^3$
$\dfrac{1}{z}$	$\dfrac{x}{x^2 + y^2}$	$-\dfrac{y}{x^2 + y^2}$
e^z	$e^x \cos y$	$e^x \sin y$
$\sin z$	$\sin x \cdot \cosh y$	$\cos x \cdot \sinh y$
$\cos z$	$\cos x \cdot \cosh y$	$-\sin x \cdot \sinh y$
$\sinh z$	$\sinh x \cdot \cos y$	$\cosh x \cdot \sin y$
$\cosh z$	$\cosh x \cdot \cos y$	$\sinh x \cdot \sin y$
$\tan z$	$\dfrac{\sin 2x}{\cos 2x + \cosh 2y}$	$\dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$
$\operatorname{Log} z$	$\ln|z| = \ln r = \dfrac{1}{2}\,\ln\left(x^2 + y^2\right)$	$\begin{cases} \operatorname{Arg} z = \Theta & \text{for } \Theta \in\,]-\pi, \pi[, \\[2mm] \operatorname{Arccot} \dfrac{x}{y} & \text{for } y > 0, \\[2mm] \operatorname{Arctan} \dfrac{y}{x} & \text{for } x > 0, \\[2mm] \operatorname{Arctan} \dfrac{y}{x} - \pi & \text{for } x < 0 \text{ and } y < 0, \\[2mm] \operatorname{Arccot} \dfrac{x}{y} - \pi & \text{for } x < 0 \text{ and } y < 0. \end{cases}$

Table 1: Some elementary analytic and harmonic functions.

3   Laurent Series and Residua

3.1   Laurent series

We previously proved in Section 1.6 that if we have given the convergent power series

(83) $f(z) = \displaystyle\sum_{n=0}^{+\infty} a_n\, z^n,$ for $|z| < R,$

then we obtain a *Fourier series* by putting $z = r\, e^{i\Theta}$, where $0 \le r < R$ and $\Theta \in \mathbb{R}$. In fact,

(84) $\varphi(\Theta) = f\left(r\, e^{i\Theta}\right) = \displaystyle\sum_{n=0}^{+\infty} a_n\, r^n\, e^{in\Theta} = \sum_{n=0}^{+\infty} c_n\, e^{in\Theta}.$

This is a special case of a Fourier series, because we usually sum from $-\infty$ to $+\infty$ in the *Theory of Fourier series*. In other words, we are missing all terms of the form $c_{-n}\, e^{-in\Theta}$, $n \in \mathbb{N}$, in (84). From

$$c_{-n}\, e^{-in\Theta} = c_n\, r^n \cdot \left(r\, e^{i\Theta}\right)^{-n} = a_{-n}\, z^{-n}$$

follows that we miss all terms of the form $a_n\, z^{-n}$ in (83). Thus, the general Fourier series force us to consider the more general functions of the form

(85) $f(z) = \displaystyle\sum_{n=-\infty}^{+\infty} a_n\, z^n,$

where we still have to discuss where this series is converging, and is representing an analytic function. Before we start on this project we sketch a useful application in the technical sciences of series of the form (85).

Let $u(t)$, $t \ge 0$, be a continuous function in time t, and assume that it is measured at the equidistant times $t = nT$, $n \in \mathbb{N}$. In this way we define a sequence (a_n) by

$$a_n = u(nT), \qquad n \in \mathbb{N}_0.$$

Such sequences are used in theoretical considerations in e.g. *Cybernetics*. One uses the so-called $\mathfrak{z}$-*transform*, which is defined by

(86) $\mathfrak{z}_T(u)(z) := \displaystyle\sum_{n=0}^{+\infty} a_n \cdot \frac{1}{z^n} = \sum_{n=0}^{+\infty} u(nT) \cdot \frac{1}{z^n},$

in a domain, where (86) is convergent. In this case all exponents are non-positive.

The $\mathfrak{z}$-transform may be considered as a discrete form of the *Laplace transform*. We shall in more details return to them in *Ventus: Complex Functions Theory a-4*. For the time being we have only used (86) as a motivation and excuse for also to consider series with negative exponents.

We shall return to mathematics.

First consider a function $f(w)$, which is analytic in w for $|w| < r$, so $f(w)$ has a convergent power series expansion

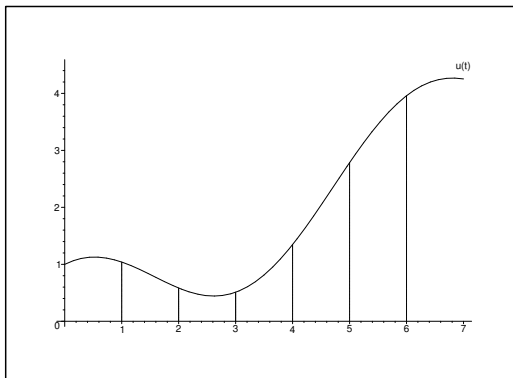(87) $f(w) = \displaystyle\sum_{n=0}^{+\infty} b_n\, w^n,$ $|w| < r.$

Figure 7: A continuous function in time, which is measured at the equidistant times $nT = n$, $n \in \mathbb{N}_0$, where we have normalized, such that $T = 1$.

Let $z_0 \in \mathbb{C}$ be fixed, and put $w = g(z) = \dfrac{1}{z - z_0}$, thus $|w| = \dfrac{1}{|z - z_0|} < r$ for $|z - z_0| > \dfrac{1}{r}$, and $g(z)$ is analytic in this domain. Hence, the composite function $(f \circ g)(z)$ is analytic for $|z - z_0| > \dfrac{1}{r}$, and when $w = \dfrac{1}{z - z_0}$ is put into (87), we get the convergent series expansion

$$(88) \quad h(z) = (f \circ g)(z) = \sum_{n=0}^{+\infty} b_n \cdot (z - z_0)^{-n} \qquad \text{for } |z - z_0| > \frac{1}{r},$$

where the series is uniformly convergent in each closed subset $A \subseteq \Omega = \mathbb{C} \setminus B\left[z_0, \dfrac{1}{r}\right]$. Note that it is not necessary here to assume that A is compact, because $g^{\circ -1}(A)$ is compact.

Since $w = 0$ corresponds to $z = \infty$, it is natural to say that $h(z)$ is *analytic at* ∞, and we put $h(\infty) = b_0$.

If $r = +\infty$, then the series (88) is convergent for all $z \neq z_0$. If $r = 0$, then the series is divergent for all $z \neq \infty$, i.e. it is only convergent at ∞ with the value b_0.

It follows from the above that a series (88) with only non-positive exponents *usually* is convergent in the complement of a closed circle, $\mathbb{C}^\star \setminus B\left[z_0, \dfrac{1}{r}\right]$, where $\mathbb{C}^\star = \mathbb{C} \cup \{\infty\}$ denotes the *extended complex plane*. If $r = +\infty$, then the domain of convergence is $C^\star \setminus \{z_0\}$, and if $r = 0$, then the series is only convergent at ∞.

Definition 3.1.1 *A Laurent series expanded from* $z_0 \in \mathbb{C}$ *is a series of the form*

$$(89) \quad \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n := \sum_{n=0}^{+\infty} a_n (z - z_0)^n + \sum_{n=1}^{+\infty} a_{-n} (z - z_0)^{-n} .$$

Its domain of convergence *is the intersection of the two domains of convergence of the two series on the right hand side of (89).*

If the "Taylor series" $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ is convergent for $|z - z_0| < R$, and the series of terms with negative exponents $\sum_{n=1}^{+\infty} a_{-n} (z - z_0)^{-n}$ is convergent for $|z - z_0| > r$, and if furthermore $r < R$, then the *domain of convergence* for the series (89) is given by the *annulus* $r < |z - z_0| < R$.
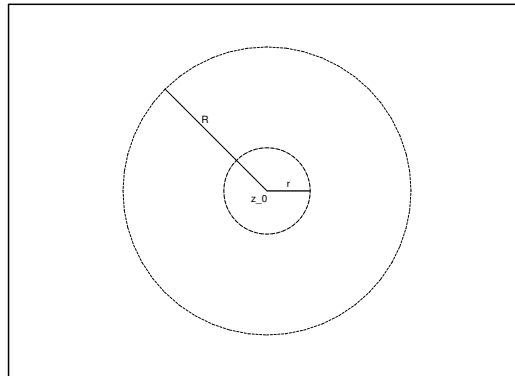

Figure 8: The domain of convergence of a Laurent series is an annulus.

Note that if $r = 0$, then the domain is $B(z_0, R) \setminus \{z_0\}$, where only the centre z_0 has been removed from the disc (a deleted neighbourhood of z_0), and if $R = +\infty$, then the domain is the complement of a closed disc, $\mathbb{C} \setminus B[z_0, r]$. If $r = 0$ and $R = +\infty$, then the domain is of course the deleted complex plane $\mathbb{C} \setminus \{z_0\}$.

If $r \geq R$, then the Laurent series is *divergent*, and (89) does not represent a(n analytic) function.

We shall prove a theorem which is analogous to Theorem 1.3.1.

Theorem 3.1.1 Laurent's theorem. *Assume that $f(z)$ is analytic in the open annulus*

$$\Omega = \left\{ z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2 \right\}, \qquad \text{where } 0 \le r_1 < r_2.$$

Then f is uniquely determined in Ω by its Laurent series

$$(90) \ \ f(z) = \sum_{n=-\infty}^{+\infty} a_n \left(z - z_0\right)^n,$$

where

$$(91) \ \ a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{\left(z - z_0\right)^{n+1}} \, dz, \qquad n \in \mathbb{Z}.$$

Here, C is any simple and closed curve in Ω which separates the two boundaries $|z - z_0| = r_1$ and $|z - z_0| = r_2$.
The series (90) is uniformly convergent in every compact subset of Ω.

PROOF. Let $C(r)$ denote the circle $|z - z_0| = r$. If $r1 < R_1 < R_2 < r_2$ and $n \in \mathbb{Z}$, then it follows from *Cauchy's integral theorem for multiply connected domains* that

$$\oint_{C(R_1)} \frac{f(z)}{\left(z - z_0\right)^{n+1}} \, \mathrm{d}z = \oint_{C(R_2)} \frac{f(z)}{\left(z - z_0\right)^{n+1}} \, \mathrm{d}z,$$

so it suffices only to consider $C = C(r)$ in (91).

For every $n \in \mathbb{Z}$ the constant a_n is uniquely determined by

$$a_n = \frac{1}{2\pi i} \oint_{C(r)} \frac{f(z)}{\left(z - z_0\right)^{n+1}} \, \mathrm{d}z, \qquad \text{where } r_1 < r < r_2.$$

We shall prove that the corresponding series $\sum_{n=-\infty}^{+\infty} a_n \left(z - z_0\right)^n$ is convergent for every $z \in \Omega$, and that its sum function is $f(z)$.

Fix $z \in \Omega$, and choose R_1 and R_2, such that

$$r_1 < R_1 < |z - z_0| < R_2 < r_2.$$

Then by *Cauchy's integral formula*,

$$(92) \ \ f(z) = \frac{1}{2\pi i} \oint_{C(R_2)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{1}{2\pi i} \oint_{C(R_1)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

If $\zeta \in C\left(R_2\right)$, then $|\zeta - z_0| = R_2$. From $|z - z_0| < R_2$ follows that

$$(93) \ \ \frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{+\infty} \left\{ \frac{z - z_0}{\zeta - z_0} \right\}^n,$$
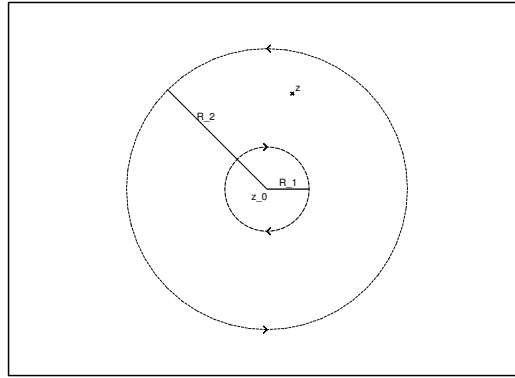
Figure 9: The paths of integration in the proof of Laurent's theorem.

where the series is *uniformly convergent* for $\zeta \in C(R_2)$. Hence, (93) can be put into the former integral on the right hand side of (92), and we are allowed to interchange summation and integration, thus

$$(94) \quad f_2(z) = \frac{1}{2\pi i} \oint_{C(R_2)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{n=0}^{+\infty} (z - z_0)^n \cdot \frac{1}{2\pi i} \oint_{C(R_2)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta = \sum_{n=0}^{+\infty} a_n (z - z_0)^n .$$

The latter integral of (92) is treated similarly. If $\zeta \in C(R_1)$, then $|\zeta - z_0| < |z - z_0|$, hence

$$(95) \quad -\frac{1}{\zeta - z} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \dfrac{\zeta - z_0}{z - z_0}} = \frac{1}{z - z_0} \sum_{n=0}^{+\infty} \left\{ \frac{\zeta - z_0}{z - z_0} \right\}^n ,$$

which is uniformly convergent for $\zeta \in C(R_1)$. Then

$$
\begin{aligned}
f_1(z) &= -\frac{1}{2\pi i} \oint_{C(R_1)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{n=0}^{+\infty} \frac{1}{(z - z_0)^{n+1}} \cdot \frac{1}{2\pi i} \oint_{C(R_1)} (\zeta - z_0)^n f(\zeta) \, d\zeta \\
(96) \\
&= \sum_{n=-1}^{-\infty} (z - z_0)^n \cdot \frac{1}{2\pi i} \oint_{C(R_1)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta = \sum_{n=-1}^{-\infty} a_n (z - z_0)^n .
\end{aligned}
$$

We get (90), when (94) and (96) are put into (92).

The series (94) and (96) are both uniformly convergent on every compact subset of Ω, so we have proved the theorem. $\square$

It is in particular easy to find the Laurent series of *rational functions*. We shall in the following give some examples which show the technique.

Example 3.1.1 The function $f(z) = \dfrac{1}{z-2}$ is analytic in $\mathbb{C} \setminus \{2\}$. Seen from $z_0 = 0$ the function $f(z)$ is analytic in the open disc $\Omega_1 = B(0,2)$, and in the complement $\Omega_2 = \mathbb{C} \setminus B[0,2]$ of its closure. We shall find the Laurent series of $f(z)$ in Ω_1 and in Ω_2. In both cases we apply the *geometric series* in an essential way.
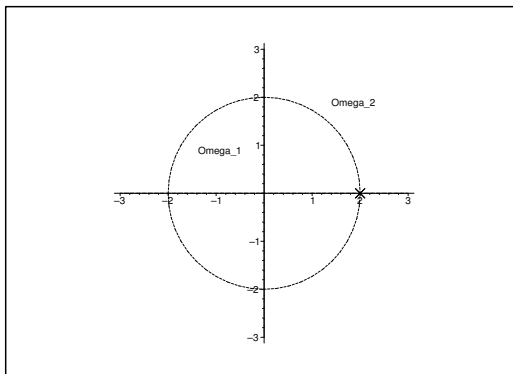

Figure 10: The two domains Ω_1 and Ω_2, in which we have a Laurent series expansion of $f(z) = \dfrac{1}{z-2}$ from $z_0 = 0$.

First consider $z \in \Omega_1 = B(0,2)$. Then $|z| < |-2| = 2$, so $\left|\dfrac{z}{2}\right| < 1$, and we get

$$(97) \quad \frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1 - \dfrac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{+\infty} \left\{\frac{z}{2}\right\}^n = -\sum_{n=0}^{+\infty} \frac{1}{2^{n+1}} z^n,$$

and $f(z)$ is in Ω_1 described by its *Taylor series* expanded from $z_0 = 0$.

Then let $z \in \Omega_2$. In this case, $|z| > |-2| = 2$, so $\left|\dfrac{2}{z}\right| < 1$, and

$$(98) \quad \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1 - \dfrac{2}{z}} = \frac{1}{z} \sum_{n=0}^{+\infty} \left\{\frac{2}{z}\right\}^n = \sum_{n=1}^{+\infty} 2^{n-1} \cdot \frac{1}{z^n},$$

corresponding to that $f(z)$ is given by a *Laurent series* in Ω_2. Since the Laurent series expansion is unique according to the theorem above in each of the domains, the problem is solved, and we have the descriptions given by (97) and (98), i.e.

$$f(z) = \begin{cases} -\sum_{n=0}^{+\infty} \dfrac{1}{2^{n+1}} \cdot z^n, & \text{for } |z| < 2, \\[2mm] \sum_{n=1}^{+\infty} 2^{n-1} \cdot \dfrac{1}{z^n}, & \text{for } |z| > 2. \end{cases}$$

The Laurent series expansion from $z_0 = 0$ does not make sense on the circle $|z| = 2$. $\Diamond$

Example 3.1.2 The function $f(z) = \dfrac{1}{(z-1)(z-2)}$ is defined and analytic in $\mathbb{C}\setminus\{1,2\}$. If we choose $z_0 = 0$, then the Laurent domains are the disc $\Omega_1 = B(0,1)$, the annulus $\Omega_2 = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$, and the complement $\Omega_3 = \mathbb{C} \setminus B[0,2]$ of a closed disc.
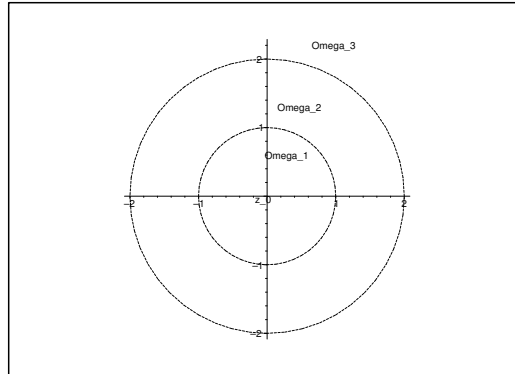

Figure 11: The three Laurent domains for $f(z) = \dfrac{1}{(z-1)(z-2)}$ and $z_0 = 0$.

First *decompose*

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Notice that we have already found the Laurent series of $\dfrac{1}{z-2}$ in Example 3.1.1.

If $z \in \Omega_1 = B(0,1)$, then $|z| < 1$, so

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2} \cdot \frac{1}{1-\dfrac{z}{2}} + \frac{1}{1-z} = \sum_{n=0}^{+\infty}\left\{1 - \frac{1}{2^{n+1}}\right\}z^n.$$

If $z \in \Omega_2$, i.e. $1 < |z| < 2$, then

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2} \cdot \frac{1}{1-\dfrac{z}{2}} - \frac{1}{z} \cdot \frac{1}{1-\dfrac{1}{z}} = -\sum_{n=0}^{+\infty}\frac{1}{2^{n+1}}z^n - \sum_{n=1}^{+\infty}\frac{1}{z^n}.$$

If $z \in \Omega_3$, then $|z| > 2$, and we get

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\dfrac{2}{z}} - \frac{1}{z} \cdot \frac{1}{1-\dfrac{1}{z}} = \sum_{n=1}^{+\infty}\left\{2^{n-1}-1\right\} \cdot \frac{1}{z^n} = \sum_{n=2}^{+\infty}\left\{2^{n-1}-1\right\} \cdot \frac{1}{z^n}. \quad \Diamond$$

Example 3.1.3 The function $f(z) = \dfrac{1}{(1-z)^2}$ is analytic in $\mathbb{C} \setminus \{1\}$. The denominator $(1-z)^2$ has the root $z = 1$ of multiplicity 2, so the direct determination of the laurent expansions from $z_0 = 0$ in $\Omega_1 = B(0,1)$ and $\Omega_2 = \mathbb{C} \setminus B[0,1]$ becomes more difficult than in the previous two examples. The trick is instead first to find the Laurent series of $g(z) = \dfrac{1}{1-z}$, where the denominator only has a simple root, and then find the Laurent series of $f(z)$ by termwise differentiation.
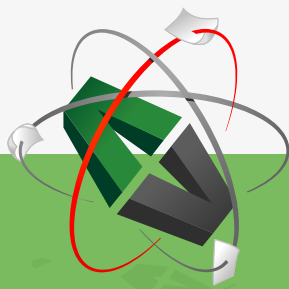
We find in $\Omega_1 = B(0,1)$,

$$g(z) = \frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1,$$

and in $\Omega_2 = \mathbb{C} \setminus B[0,1]$,

$$g(z) = \frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1 - \dfrac{1}{z}} = -\sum_{n=1}^{+\infty} \frac{1}{z^n} = -\sum_{n=1}^{+\infty} z^{-n}, \qquad |z| > 1.$$

Clearly, $f(z) = \dfrac{1}{(1-z)^2} = g'(z)$, so we get by termwise differentiation,

$$f(z) = \sum_{n=1}^{+\infty} n\, z^{n-1} = \sum_{n=0}^{+\infty} (n+1)z^n \qquad \text{for } z \in \Omega_1,$$

and

$$f(z) = \sum_{n=1}^{+\infty} n \cdot z^{-n-1} = \sum_{n=2}^{+\infty} (n-1)z^{-n} \qquad \text{for } z \in \Omega_2.$$

This method can of course be generalized to $h_n(z) = \dfrac{1}{(1-z)^{n+1}}$, in which case we apply that $h(z) = \dfrac{1}{n!}\, g^{(n)}(z)$. $\lozenge$

3.2 Fourier series II

Assume that $f(z)$ is analytic in an open annulus $r_1 < |z - z_0| < r_2$, and let $f(z)$ in this annulus be represented by its Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n\, (z - z_0)^n, \qquad r_1 < |z - z_0| < r_2.$$

Let $r_1 < r < r_2$ and $\Theta \in \mathbb{R}$, and put $z = z_0 + r\, e^{i\Theta}$. Then

(99) $\quad f\left(z_0 + r\, e^{i\Theta}\right) = \displaystyle\sum_{n=-\infty}^{+\infty} a_n\, r^n\, e^{in\Theta},$

where the series in (99) is uniformly convergent in Θ for fixed r.

We see that for given z_0 and r the series expansion (99) is the *Fourier series* of the function

$$\varphi(\Theta) := f\left(z_0 + r\, e^{i\Theta}\right).$$

This is in agreement with Laurent's theorem, because

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + r\, e^{i\Theta}\right)}{r^{n+1}\, e^{i(n+1)\Theta}} \cdot i\, r\, e^{i\Theta}\, d\Theta,$$

from which

$$a_n\, r^n = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + r\, e^{i\Theta}\right) e^{-in\Theta}\, d\Theta = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\Theta) e^{-in\Theta}\, d\Theta,$$

which is the usual formula for the n-th Fourier coefficient of $\varphi(\Theta)$.

We proved in Section 1.6 *Parseval's formula* in a special case, necessary for the proof of the *maximum principle*. We can now easily prove the general *Parseval's formula*.

Theorem 3.2.1 Parseval's formula. *Let*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n \quad and \quad g(z) = \sum_{n=-\infty}^{+\infty} b_n z^n$$

be analytic in the annulus $r_1 < |z| < r_2$. *Then for every* $r \in \ ]r_1, r_2[$,

$$(100) \quad \frac{1}{2\pi} \int_0^{2\pi} f\left(r\,e^{i\Theta}\right) \overline{g\left(r\,e^{i\Theta}\right)}\, d\Theta = \sum_{n=-\infty}^{+\infty} a_n \overline{b_n}\, r^{2n}.$$

PROOF. The only difference from the proof in Section 1.6 is that we here sum from $-\infty$ to $+\infty$ instead of from 0 to $+\infty$. $\square$

3.3   Solution of a linear differential equation by means of Laurent series

We gave in Section 1.4 a solution procedure for a linear differential equation of order n with analytic coefficients in an open domain Ω,

$$(101) \quad a_0(z)\frac{d^n f}{dz^n} + a_1(z)\frac{d^{n-1}f}{dz^{n-1}} + \cdots + a_{n-1}(z)\,f'(z) + a_n(z)\,f(z) = g(z), \qquad z \in \Omega,$$

where the coefficient $a_0(z)$ of the highest order term is not identically zero. If $a_0(z_0) \neq 0$ for $z_0 \in \Omega$, then we can apply Theorem 1.4.1. However, if $z_0 \in \Omega$ is a *singular point* for the differential equation (101), i.e. $a_0(z_0) = 0$, then we do not have a result, which guarantees that there exists an analytic solution in a (deleted) neighbourhood of z_0.

One may *assume* that (101) in the singular case has a Laurent series solution of the form

$$(102) \quad f(z) = \sum_{n=-\infty}^{+\infty} a_n\,(z - z_0)^n,$$

where the Laurent series expansion of $f^{(p)}(z)$ is *formally* found by termwise differentiation, and where z_0 clearly does not belong to the domain of convergence of $f(z)$, which is still to be found. In particular,

$$(103) \quad f'(z) = \sum_{n=-\infty}^{+\infty} n\,a_n\,(z - z_0)^{n-1}$$

in the possible domain of convergence. Since z_0 is excluded from this domain, we do not have an exceptional case in (103) for $n = 0$ as we did in Section 1.4, where we only dealt with power series. Therefore, we shall no longer be careful with the domain of summation, which always may be put equal to $\mathbb{Z}$. Hence, if we are looking for a Laurent series solution, there is no need to specify the bounds of summation, and we just write $\sum$ instead of $\sum_{-\infty}^{+\infty}$.

We use the same method as in Section 1.4 to set up a *recursion formula* for the coefficients a_n in (102). The new aspect here is that this recursion formula must be used to find both the coefficients a_n for $n > 0$, and the coefficients a_{-n} for $n > 0$. Since the recursion formula for a_{-n} of negative indices is used in the "opposite direction" of the customary one, it is highly *recommended* for the novice of this theory (it is, however, not mandatory) to put $b_n = a_{-n}$ for $n > 0$ and then find a (usual) recursion formula for the b_n.

A small and useful, though often neglected detail here is to start by finding the values of $n \in \mathbb{Z}$, for which at least one term in the recursion formula becomes zero. Such an observation will give us some information of which coefficients are zero, and which coefficients are arbitrary. Then the *formal* Laurent series is expressed by these arbitrary coefficients.

Finally, when we have found all the *formal* Laurent series (102) which satisfy the equation (101), *we* *must find its domain of convergence!!!* This point is of paramount importance and must never be forgotten, because otherwise the computations do not make sense.

This point can best be illustrated by the fact that it is not hard to construct a linear differential equation of second order (101) for which the *formal* Laurent series solution (102) contains *three* arbitrary constants, which is contradicting the theory, unless the arbitrary constants are sorted out by a discussion of the domains of convergence, which always will force at least one of the arbitrary constants to be zero in any given subdomain.

To be more precise, when we find Laurent series solutions expanded from a *singular point* z_0 of a linear differential equation of order n, then there are *at most n* linear independent Laurent series solutions, which are *convergent* in the same domain, while there may be more *formal* Laurent series solutions.

Example 3.3.1 We shall give one example, showing the method. We tried previously in Example 1.4.3 to solve the linear differential equation

(104) $z^2 f'(z) - f(z) = -z$.

We shall here try to solve it by means of Laurent series.

We first solve the corresponding *homogeneous* equation

(105) $z^2 f'(z) - f(z) = 0$.

First note that it easily follows from the rearrangement $\dfrac{f'(z)}{f(z)} = \dfrac{1}{z^2}$ for $f(z) \neq 0$ and a check that the solutions are

(106) $f(z) = c \cdot \exp\left(-\dfrac{1}{z}\right), \qquad z \neq 0$ and $c \in \mathbb{C}$ arbitrary.

We shall now *prove* (106) by using formal Laurent series instead. So assume that $f(z) = \sum a : n \, z^n$ is given by a *formal* Laurent series. Then by insertion into a reversed (105)

$$0 = z^2 f'(z) - f(z) = z^2 \sum n a_n z^{n-1} - \sum a_n z^n = \sum n a_n z^{n+1} - \sum a_n z^n$$

(107)

$$= \sum n a_n z^{n+1} - \sum a_{n+1} z^{n+1} = \sum (n a_n - a_{n+1}) z^{n+1},$$

so we get the *recursion formula*

(108) $a_{n+1} = n \, a_n \qquad$ for $n \in \mathbb{Z}$.

An obvious "zero" of (108) is $n = 0$, for which value

$$a_1 = 0 \cdot a_0 = 0,$$

no matter how a_0 is chosen. Hence, a_0 is for the time being an arbitrary constant, while $a_1 = 0$.

We shall exclude $n = 0$ in the following. This case was discussed above. The investigation is then split into the two cases, $n > 0$ and $n < 0$.

When $n > 0$ and $a_1 = 0$, it follows by recursion of (108), or by a division by $n!$, from which

$$\frac{1}{n!}\, a_{n+1} = \frac{1}{(n-1)!}\, a_n = \cdots = \frac{1}{1!}\, a_1 = 0,$$

that $a_n = 0$ for all $n \in \mathbb{N}$.

Then assume that $n < 0$ and put $m = -n > 0$ and $b_m = a_{-n}$. Then (108) becomes

$$a_{n+1} = a_{-m+1} = a_{-(m-1)} = b_{m-1} = n\, a_n = -m\, a_{-m} = -m\, b_m,$$
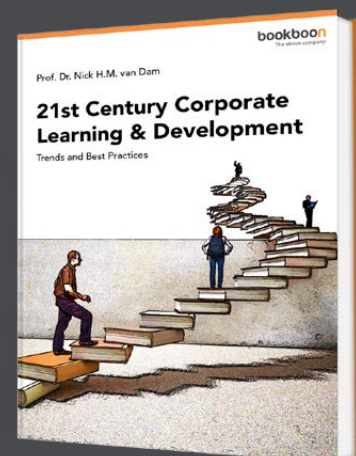
so we get the recursion formula

(109) $\quad b_{m-1} = -m\, b_m \qquad$ for $m \in \mathbb{N}$.

Notice that $m = 1$ corresponds to $-b_1 = b_0 = a_0$, where a_0 is arbitrary.

If (109) is multiplied by $(-1)^{m-1}(m-1)! \neq 0$ and we read from the right to the left, then we get by recursion

$$(-1)^m m! b_m = (-1)^{m-1}(m-1)! b_{m-1} = \cdots = (-1)^0 0! b_0 = a_0,$$

hence

$$a_{-m} = b_m = \frac{(-1)^m}{m!} a_0, \qquad m \in \mathbb{N},$$

which of course also holds for $m = 0$.

Thus we have derived the *formal* Laurent series solution of (105), which is given by

$$0 + \sum_{m=0}^{+\infty} a_{-m} z^{-m} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} a_0 \cdot \frac{1}{z^n} = a_0 \sum_{n=0}^{+\infty} \frac{1}{n!} \cdot \left\{ -\frac{1}{z} \right\}^n = a_0 \exp\left(-\frac{1}{z} \right),$$

where we have recognized the coefficients of the series expansion of the exponential function. The latter is of course convergent for all $z \in \mathbb{C}$, for which $-\dfrac{1}{z} \in \mathbb{C}$, thus for $z \in \mathbb{C} \setminus \{0\}$, and the domain of convergence is $\mathbb{C} \setminus \{0\}$. There is no need to use any other procedure of finding the radii of convergence ($r = 0$ and $R = +\infty$).

We have seen that equation (105) could be solved by guessing a Laurent series solution. However, if we try the same method on the inhomogeneous equation (104), then it follows by a modification of the computation of (107) that

$$-z = z^2 f'(z) - f(z) = \sum (n\, a_n - a_{n+1})\, z^{n+1}.$$

Here, $n = 0$ corresponds to $z^{n+1} = z$, so we get the recursion formula

$$\begin{cases} n \cdot a_n &= a_{n+1}, & \text{for } n \neq 0, \\[2mm] -a_1 &= -1, & \text{for } n = 0. \end{cases}$$

The only change from (108) is that now $a_1 = 1$. This change, however becomes disastrous, because then it follows from the first recursion formula above that

$$a_{n+1} = n\, a_n = n(n-1)a_{n-1} = \cdots = n!\, a_1 = n! \qquad \text{for } n \in \mathbb{N},$$

and trivially $a_1 = 1 = 0!$. Hence by a translation of the index, $a_n = (n-1)!$ for $n \in \mathbb{N}$. Since there is no change in the recursion formula for $n < 0$, we again get the previous solution, so in this case all *formal* Laurent series solutions are given by

$$\sum_{n=1}^{+\infty} (n-1)!\, z^n + a_0 \sum_{n=0}^{+\infty} \frac{1}{n!} (-1)^n z^{-n}.$$

The latter series is of course convergent for $z \neq 0$ with the sum function $a_0 \exp\left(-\dfrac{1}{z} \right)$, but the former series has 0 as radius of convergence, so it is only convergent for $z = 0$ (where the latter series is not convergent, unless $a_0 = 0$).

We conclude that there does not exist any *convergent* Laurent series solution of (104), *when expanded from $z_0 = 0$*. However, if we move the expansion point z_0 away from 0, then it follows from Theorem 1.4.1 that there will *always* exist even power series solutions in the open disc $B(z_0, |z_0|)$, $z_0 \neq 0$.
◇

3.4   Isolated boundary points

We have seen that we may have Laurent series which are convergent in an annulus "far away" from the expansion point z_0. It should, however, not be of any surprise that the most important case is when the domain of convergence is a *deleted disc* $D(z_0, R) = B(z_0, R) \setminus \{z_0\}$, in which case the centre z_0 is an *isolated boundary point* of the domain of convergence. In fact, it will lead us to the important *Calculus of Residua*.

Let $f : \Omega \to \mathbb{C}$ be analytic in an open domain Ω, and let $z_0 \in \mathbb{C}$ be an *isolated boundary point* of Ω. This means that $z_0 \notin \Omega$ and that we can find $R > 0$, such that the *deleted disc*

$$D(z_0, R) := B(z_0, R) \setminus \{z_0\} \subseteq \Omega.$$

In this case we can apply *Laurent's theorem* to get

$$(110) \quad f(z) = \sum_{n=-\infty}^{+\infty} a_n \, (z - z_0)^n, \qquad \text{for } z \in D(z_0, R),$$

where

$$(111) \quad a_n = \frac{1}{2\pi i} \oint_{C(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \qquad \text{for all } r \in \,]0, R[,$$

where $C(z_0, r)$ denotes the circle $|z - z_0| = r$ of centre z_0 and radius r, traversed in the positive sense of the plane.

We shall consider three cases.

3.4.1   Case I, where $a_n = 0$ for all negative n.

In this case $f(z)$ is given by an ordinary power series

$$(112) \quad f(z) = \sum_{n=0}^{+\infty} a_n \, (z - z_0)^n, \qquad z \in D(z_0, r).$$

The power series is clearly convergent and analytic in the full disc $B(z_0, r)$, so we extend $f(z)$ by (112) to all of $B(z_0, r)$, i.e. we add $f(z_0) := a_0$ to the definition.

In this case we call z_0 a *removable singularity*.

We have

Theorem 3.4.1 *If the analytic function f is* bounded *in a deleted disc $D(z_0, \varrho) = B(z_0, \varrho) \setminus \{z_0\}$, then z_0 is a* removable singularity.

PROOF. Assume that $|f(z)| \leq M$ for all $z \in D(z_0, \varrho)$. Then by an application of (111) for any $r \in \,]0, \varrho[$ and every $n \in \mathbb{Z}$,

$$|a_n| = \left| \frac{1}{2\pi i} \oint_{C(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right| \leq \frac{1}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{M}{r^n}.$$

It follows from this estimate for $n < 0$ by letting $r \to 0+$ that $a_n = 0$ for all $n < 0$. $\square$

Example 3.4.1 The function $f(z) = \dfrac{\sin z}{z}$ for $z \in \mathbb{C} \setminus \{0\}$ has a *removable singularity* at $z = 0$. In fact,

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{+\infty} \frac{(.1)^n}{(2n+1)!} z^{2n} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \qquad \text{for } z \neq 0,$$

and the series is convergent for all $z \in \mathbb{C}$, so we can put $f(0) := 1$. $\Diamond$

3.4.2   Case II, where $a_n \neq 0$ for a finite number of negative n.

In this case there is a $q \in \mathbb{N}$, such that $a_{-q} \neq 0$ and $a_n = 0$ for all $n < -q$. Then in $D(z_0, \varrho)$,

$$f(z) = \sum_{n=-q}^{+\infty} a_n (z - z_0)^n = \frac{a_{-q}}{(z - z_0)^q} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \cdots.$$

The analytic function

$$g(z) := (z - z_0)^q f(z)$$

has clearly a *removable singularity* at z_0, so it can be extended to the whole disc $B(z_0, \varrho)$ by adding the value $g(z_0) := a_{-q}$ to its definition.

In this case we say that $f(z)$ has a *pole of order q* at z_0, and we put $f(z_0) = \infty$ (the complex infinity).

We have

Theorem 3.4.2 *If $f(z) \to \infty$ for $z \to z_0$ in $D(z_0, \varrho)$, then $f(z)$ has a pole at z_0 of some* (finite) *order $q \in \mathbb{N}$.*

PROOF. This proof is not so easy as the proof of Theorem 3.4.1, because we shall show that the singularity has a *finite* order $q \in \mathbb{N}$.

From the assumption $f(z) \to \infty$ for $z \to z_0$ follows that there exists a deleted disc $D(z_0, \varrho) = B(z_0, \varrho) \setminus \{z_0\}$, such that $f(z) \neq 0$, and such that the reciprocal $h(z) = \dfrac{1}{f(z)}$ is bounded in $D(z_0, \varrho)$.

Since $h(z) \to 0$ for $z \to z_0$, the point z_0 is a removable singularity of $h(z)$, and $h(z)$ can be extended to an analytic function in the full disc $B(z_0, \varrho)$ by adding the value $h(z_0) = 0$.

Since $h(z)$ is not identically zero in $B(z_0, \varrho)$, we may apply Theorem 1.5.2 to conclude that the zero z_0 is isolated. Thus by Theorem 1.5.1 the zero z_0 has a *finite order $q \in \mathbb{N}$*, so

(113) $h(z) = (z - z_0)^q \{b_0 + b_1 (z - z_0) + \cdots\}$

in a possibly smaller open disc $B(z_0, \varrho_1)$, $\varrho_1 \in \;]0, \varrho[$. Note that $b_0 = \dfrac{1}{q!} h^{(q)}(z_0) \neq 0$, hence it follows from the continuity that ϱ_1 can be chosen, such that the latter factor $\{b_0 + b_1 (z - z_0) + \cdots\}$ in (113)

is analytic and $\neq 0$ in $B(z_0, \varrho_1)$. Then its *reciprocal* $\{b_0 + b_1 (z - z_0) + \cdots\}^{-1}$ is also *analytic* in $B(z_0, \varrho_1)$, so it even has a *Taylor series expansion*

$$\{b_0 + b_1 (z - z_0) + \cdots\}^{-1} = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad \text{for } z \in B(z_0, \varrho_1),$$

where $a_0 \neq 0$. Thus we conclude from (113) that

$$f(z) = \frac{1}{h(z)} = \frac{1}{(z - z_0)^q} \sum_{n=0}^{+\infty} a_n (z - z_0)^n = \sum_{n=-q}^{+\infty} a_{n+q} (z - z_0)^n, \qquad z \in D(z_0, \varrho_1) = B(z_0, \varrho_1) \backslash \{z_0\},$$

and we conclude that z_0 is indeed a pole of order q of $f(z)$. $\square$

Example 3.4.2 The rational function $\dfrac{z - 2}{(z^2 + 1)(z - 1)^3}$ has a pole of order 3 at $x = 1$, and *simple poles* (i.e. poles of order 1) at $z = i$ and $z = -i$. $\lozenge$

3.4.3   Case III, where $a_n \neq 0$ for infinitely many negative n.

In this case we call z_0 an *essential singularity* for $f(z)$.

We mention without proof (because the full proof is very difficult) the following theorem.

Theorem 3.4.3 Picard's theorem (1879). *If z_0 is an (isolated) essential singularity for $f(z)$, then the image $f(D(z_0, \varrho))$ is either all of $\mathbb{C}$, or $\mathbb{C} \setminus \{w_0\}$ with just one exception point w_0, where $D(z_0, \varrho)$ is* any *deleted disc contained in the domain of f.*

Picard's theorem shows that the behaviour of an analytic function f in a deleted neighbourhood of an essential singularity is very wild.

To give some understanding of *Picard's theorem* we shall below prove the following weaker result.

Theorem 3.4.4 Casorati-Weierstraß's theorem. *Let z_0 be an (isolated) essential singularity of an analytic function f. Given any deleted disc $D(z_0, \varrho) = B(z_0, \varrho) \setminus \{z_0\} \subseteq \Omega$, contained in the domain Ω of f, the image $f(D(z_0, \varrho))$ is a dense set in $\mathbb{C}$.*

PROOF. Contrariwise. Assume that we can find $\varrho > 0$, such that $D(z_0, \varrho) \subseteq \Omega$, and such that the image $f(D(z_0, \varrho))$ is *not* dense everywhere in $\mathbb{C}$. Then we can find $w_0 \in \mathbb{C}$ and $\delta > 0$, such that

$$f(D(z_0, \varrho)) \cap B\,[w_0, \delta] = \emptyset,$$

thus $|f(z) - w_0| > 0$ for every $z \in D(z_0, \varrho)$. In particular, $g(z) = \dfrac{1}{f(z) - w_0}$ is bounded and analytic in $D(z_0, \varrho)$, and z_0 is a *removable singularity* of $g(z)$.
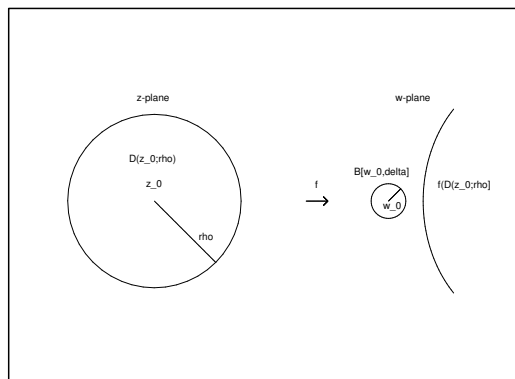

Figure 12: Proof of Casorati-Weierstraß's theorem.

There are two possibilities: If $\lim_{z \to z_0} g(z) = x \neq 0$, then it follows from the rules of computation that

$$f(z) - w_0 \to \frac{1}{c}, \quad \text{thus} \quad f(z) \to w_0 + \frac{1}{c} \qquad \text{for } z \to z_0,$$

and z_0 is a removable singularity of $f(z)$.

If instead $\lim_{z \to z_0} g(z) = 0$, then $f(z) - w_0 = \infty$, hence $f(z) \to \infty$ for $z \to z_0$, and z_0 is according to Theorem 3.4.2 a pole for $f(z)$.

In neither of the two possible cases above z_0 is an essential singularity. By contraposition of the proof we get by negating the first sentence in this proof that *for every $D(z_0, \varrho) \subseteq \Omega$ the image $f(D(z_0, \varrho))$ is dense everywhere in* $\mathbb{C}$, and the theorem is proved. $\square$

We mention two ways to show that an isolated singularity $z_0 \in \Omega$ for an analytic function $f : \Omega \to \mathbb{C}$ is an *essential singularity*.

1) If the Laurent series expansion $f(z) = \sum a_n (z - z_0)^n$ expanded from z_0 is convergent in a deleted disc $D(z_0, \varrho) = B(z_0, \varrho) \setminus \{z_0\}$, and infinitely many of the coefficients a_n for $n < 0$ are not zero, then z_0 is an *essential singularity* for $f(z)$.
Notice that we do not require that all $a_n \neq 0$ for $n < 0$, just infinitely many of the $a_{-1}, a_{-2}, \ldots$.

2) If we can find a sequence $(z_n) \subset \Omega$, such that $z_n \to z_0$, while the limit of $(f(z_n))$ does not exist for $n \to +\infty$, or *"alternatively"* (it is actually a variant of the same) if we can find two sequences $(z'_n) \to z_0$ and $(z''_n) \to z_0$ in Ω for $n \to +\infty$, such that

$$\lim_{n \to +\infty} f(z'_n) \neq \lim_{n \to +\infty} f(z''_n),$$

then z_0 is an *essential singularity* for f.
We here allow the complex infinity ∞ to be a possible limit.

Example 3.4.3 The function $f(z) = \exp\left(\dfrac{1}{z^2}\right)$, $z \in \mathbb{C} \setminus \{0\}$, has an essential singularity at $z_0 = 0$. In fact, the Laurent expansion of f is

$$\exp\left(\frac{1}{z^2}\right) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left\{\frac{1}{z^2}\right\}^n = \sum_{n=0}^{+\infty} \frac{1}{n!} \cdot \frac{1}{z^{2n}} \qquad \text{for } z \in \mathbb{C} \setminus \{0\},$$

where

$$a_n = 0 \quad \text{for } n > 0, \qquad a_{-2n+1} = 0 \quad \text{for } n > 0 \qquad \text{and} \qquad a_{-2n} = \frac{1}{n!} \quad \text{for } n > 0.$$

We see that $a_n \neq 0$ for infinitely many negative n (all even negative numbers), and $a_n = 0$ also for infinitely many negative n (all odd negative numbers).

An *alternative* proof is to choose $z'_n = \dfrac{1}{n}$ and $z''_n = \dfrac{i}{n}$. Then clearly $z'_n \to 0$ and $z''_n \to 0$ for $n \to +\infty$, while

$$f(z'_n) = \exp(n^2) \to +\infty \qquad \text{and} \qquad f(z''_n) = \exp(-n^2) \to 0 \qquad \text{for } n \to +\infty.$$

Finally, we mention that we cannot conclude that 0 is an essential singularity, if we instead of z''_n have chosen

$$\tilde{z}_n = \frac{1}{\sqrt{n^2 + i\pi}} \to 0 \qquad \text{for } n \to +\infty,$$

where the square root is any one of the two possible definitions. In this case we get

$$f(\tilde{z}_n) = \exp\left(n^2 + i\pi\right) = -\exp\left(n^2\right) \to -\infty \qquad \text{for } n \to +\infty.$$

Then

$$f(z_n') = \exp\left(n^2\right) \to +\infty \qquad \text{and} \qquad f(\tilde{z}_n) = -\exp\left(n^2\right) \to -\infty \qquad \text{for } n \to +\infty.$$

This is, however, *not* sufficient to *conclude* that $z_0 = 0$ is an essential singularity (what it is!), because the two *real* infinities, $-\infty$ and $+\infty$, in the complex plane both are identified as ∞, so we have not by this unfortunate choice excluded the possibility of a pole at 0. Therefore, we need the sequence (z_n''), where the limit of $(f(z_n''))$ is finite.

It was proved in *Ventus: Analytic Functions Theory a-1* that $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Hence,

$$f(D(0, \varrho)) \subseteq \mathbb{C} \setminus \{0\}$$

for *every* deleted disc $D(0, \varrho)$. However, according to *Picard's theorem* the image has *at most* one exception point, so we conclude that

$$f(D(0, \varrho)) = \mathbb{C} \setminus \{0\} \qquad \text{for every } \varrho > 0. \qquad \Diamond$$

3.5   Infinity ∞ as an isolated boundary point

We shall in the following sections also need to consider the case, where ∞ is an isolated boundary point. This case is just as easy as the finite case in Section 3.4. There is, however, a psychological obstacle here, because what may be obvious in the finite case if often difficult in the infinite case, and *vice versa*. We shall later benefit from the results in this section, and the gain is indeed much bigger than the effort we must use here to understand, what is going on "around ∞".

Let $f : \Omega \to \mathbb{C}$ be analytic in an open domain Ω, which has ∞ as an *isolated boundary point*. This means more precisely that there exists an $\alpha > 0$, such that $\mathbb{C} \setminus B[0, \alpha] \subset \Omega$, so every $z \in \mathbb{C}$, for which $|z| > \alpha$, belongs to Ω.
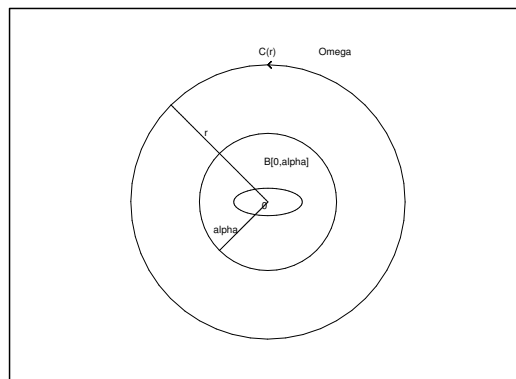

Figure 13: A deleted neighbourhood of ∞ is given by $\mathbb{C} \setminus B[0, \alpha]$.

Let $r > \alpha$, and let $C(r)$ denote the circle $\{z \in \mathbb{C} \mid |z| = r\}$. Then by *Laurent's theorem,*

$$(114) \quad f(z) = \sum_{n=-\infty}^{+\infty} a_n \, z^n \qquad \text{for } |z| > \alpha,$$

where

$$a_n = \frac{1}{2\pi i} \oint_{C(r)} \frac{f(z)}{z^{n+1}} \, \mathrm{d}z, \qquad \text{for } r > \alpha.$$

If we define an analytic function g by

$$g(w) := f\left(\frac{1}{w}\right) \qquad \text{for } \frac{1}{w} \in \Omega,$$

then we can use the discussion of Section 3.4. In fact, the domain $\left\{ w \ \middle| \ \dfrac{1}{w} \in \Omega \right\}$ of $g(w)$ contains the deleted disc

$$D\left(0, \frac{1}{\alpha}\right) := B\left(0, \frac{1}{\alpha}\right) \setminus \{0\}.$$

It follows from (114) that $g(w)$ in $D\left(0, \dfrac{1}{\alpha}\right)$ is given by the convergent Laurent series

$$g(w) = \sum_{n=-\infty}^{+\infty} a_n \left\{\frac{1}{w}\right\}^n = \sum_{n=-\infty}^{+\infty} a_{-n}\, w^n, \qquad \text{for } |w| < \frac{1}{\alpha},$$

and $z = \infty$ corresponds to $w = 0$.

We consider as in Section 3.4 three different cases:

3.5.1   Case I*, where $a_n = 0$ for all *positive* n.

In this case we get for $|z| > \alpha$,

$$f(z) = \sum_{n=-\infty}^{0} a_n\, z^n = \sum_{n=0}^{+\infty} a_{-n} \left\{\frac{1}{z}\right\}^n = \sum_{n=0}^{+\infty} a_{-n}\, w^n, \qquad \text{where } w = \frac{1}{z}.$$

When we extend $f(z)$ to $\Omega \cup \{\infty\}$ by adding the value $f(\infty) = a_0$, it follows that f is continuous in $\Omega \cup \{\infty\}$, because $z = \infty$ corresponds to $w = 0$, where $g(w) = f\left(\dfrac{1}{w}\right)$ is analytic.

In this case it is natural to say that the extended function is *analytic* in the set $\Omega \cup \{\infty\}$, and we say that ∞ is a removable singularity of $f(z)$.

When we consider the $\mathfrak{z}$-transform, we get precisely series of this type.

Example 3.5.1 It will be proved in *Ventus: Complex Functions Theory a-4* that the function $f(z) := \dfrac{z}{z-1}$ is the $\mathfrak{z}$-transform of the constant function $\varphi(t) \equiv 1$. It is of course analytic for $z \in \mathbb{C} \setminus \{1\}$.
We get for $|z| > 1$ the Laurent series expansion,

$$f(z) = \frac{z}{z-1} = \frac{1}{1 - \dfrac{1}{z}} = \sum_{n=0}^{+\infty} \frac{1}{z^n} = \sum_{n=0}^{+\infty} z^{-n}.$$

The extension is given by $f(\infty) = 1$, corresponding to

$$\lim_{z \to \infty} \frac{z}{z-1} = 1. \qquad \Diamond$$

3.5.2   Case II*, where $a_n \neq 0$ for finitely many *positive* n.

If $a_n = 0$ for all $n > q \, (> 0)$ and $a_q \neq 0$, then we get in $\mathbb{C} \setminus B[0, \alpha]$ that

$$f(z) = a_q\, z^q + \cdots + a_1\, z + a_0 + \frac{a_{-1}}{z} + \cdots,$$

and it follows that $f(z) \to \infty$ for $z \to \infty$. More precisely, $z^{-q} f(z) \to a_q$ for $z \to \infty$.

In this case we say that the analytic function $f(z)$ has a *pole of order q at ∞*, and we put $f(\infty) = \infty$.

Example 3.5.2 Every polynomial of degree n

$$P_n(z) = a_n z^n + \cdots + a_1 z + a_0, \qquad z \in \mathbb{C},$$

where $a_n \neq 0$, has a pole of order n at ∞, and we put $P_n(\infty) = \infty$. $\Diamond$

3.5.3   Case III*, where $a_n \neq 0$ for infinitely many *positive* n.

In this case we say that f has an *essential singularity at* ∞.

We clearly have an analogous result as in Section 3.4:

Theorem 3.5.1 Picard's theorem. *Let $f : \Omega \to \mathbb{C}$ be analytic with an essential singularity at ∞. For every $\mathbb{C} \setminus B[0, \alpha] \subseteq \Omega$ the image $f(\mathbb{C} \setminus B[0, \alpha])$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{w\}$ with precisely one exceptional point $w \in \mathbb{C}$.*

Example 3.5.3 <u>Important</u>. Every "nice" transcendent function like e.g.

$$\exp z, \qquad \sin z, \qquad \cos z, \qquad \sinh z, \qquad \cosh z,$$

have all an essential singularity at ∞.

It follows in particular from *Picard's theorem* that *none of them has a well-defined limit for $z \to \infty$ in* $\mathbb{C}$. They may, however, have well-defined limits on certain (one-dimensional) curves extending to ∞. For instance, $\exp z \to 0$ for $z = x \to -\infty$ along $\mathbb{R}_-$.

On the other hand, Log: $\Omega \to \mathbb{C}$, where $\Omega = \mathbb{C} \setminus \{\mathbb{R}_- \cup \{0\}\}$ does *not* have an essential singularity at ∞. The reason is that the domain Ω does not contain the complement of a disc, i.e. $\mathbb{C} \setminus B[0, \alpha]$ of centre 0, so the image "Log$(\mathbb{C} \setminus B[0, \alpha])$" is not defined. $\Diamond$

Example 3.5.4 The function $f(z) = \dfrac{1}{\sin z}$ is defined on the ∞-*connected domain*

$$\Omega = \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\}.$$

The denominator $\sin z$ has *simple zeros* for $z = p\pi$, $p \in \mathbb{Z}$, because

$$\lim_{z \to p\pi} \frac{d}{dz} \sin z = \lim_{z \to p\pi} \cos z = (-1)^p \neq 0.$$

Hence, $f(z)$ has *simple poles* at the same points.

If we put $z'_n = \dfrac{\pi}{2} + 2n\pi \to \infty$ for $n \to +\infty$, we get

$$f(z'_n) = 1.$$

If instead $z''_n = -\dfrac{\pi}{2} + 2n\pi \to \infty$ for $n \to +\infty$, then

$$f(z''_n) = -1.$$

Intuitively we would then say that ∞ is an essential singularity. *This is not true*, because Ω does not contain any complement of a disc of centre, hence ∞ is *not* an *isolated* boundary point. $\Diamond$

We may repair Example 3.5.4 by introducing the following definition.

Definition 3.5.1 *Let $f : \Omega \to \mathbb{C}$ be an analytic function in an open domain, and let*

$$z_0 \in \mathbb{C}^\star := \mathbb{C} \cup \{\infty\}.$$

If there is a sequence (z_n) of either poles *or* essential singularities *for f, such that $z_n \to z_0$ for $n \to +\infty$, then we say that z_0 is a* non-isolated essential singularity.

Example 3.5.5 It follows readily from Definition 3.5.1 that

$$\tan z, \qquad \cot z, \qquad \tanh z, \qquad \coth z,$$

all have a non-isolated essential singularity at $z_0 = \infty$, and that the function

$$\frac{1}{\sin \dfrac{1}{z}}$$

has a non-isolated essential singularity at $z_0 = 0$. $\lozenge$

From time to time we shall meet problems containing non-isolated essential singularities. This is the reason why we have given them a name in Definition 3.5.1, so we can identify them. Then in practical computations the rule of thumb is *always to avoid this type of singularity*, and instead find another method than just considering such a nasty singularity. We shall in the following only consider *isolated* singularities, i.e. *removable singularities, poles* or *essential singularities.*

3.6   Residua

Using the previous sections on *isolated singularities* we shall now introduce the important concept of a *residuum* of an analytic function at such points. The powerful applications of this theory is for practical reasons postponed to Chapter 4. In the remainder of this chapter we shall *define* the residuum and derive some easy *rules of computations* of it.

Let $f : \Omega \to \mathbb{C}$ be analytic in an open domain Ω, and let $z_0 \in \mathbb{C}$ be an isolated boundary point of Ω, i.e. a *singularity* of f. Let C be any simple closed curve in Ω surrounding z_0 and no other singularity of f. Then it follows from *Cauchy's integral theorem* that the value of the integral $\oint_C f(z) \, dz$ is the same for all such closed curves around z_0. This shows that the following definition makes sense.

Definition 3.6.1 *The* residuum *of the complex* differential form $f(z) \, dz$ *at z_0 is defined as*

(115) $res\,(f(z)\,dz; z_0) := \dfrac{1}{2\pi i} \oint_C f(z) \, dz,$

where C is any simple closed curve in Ω surrounding z_0 and no other boundary point of Ω.

In general, the correct notation $res\,(f(z)\,dz; z_0)$ is too clumsy, so we shall usually incorrectly write $res\,(f; z_0)$ instead, when there is no risk of misunderstanding. One may in the literature also find other notations, like e.g. $\mathrm{Res}[f; z_0]$ and $res_f\,(z_0)$. It should, however, be emphasized that the *residuum* is a number, which is connected with the *differential form* $f(z) \, dz$ and *not* the analytic function $f(z)$ itself. We shall later explain this point in more details.
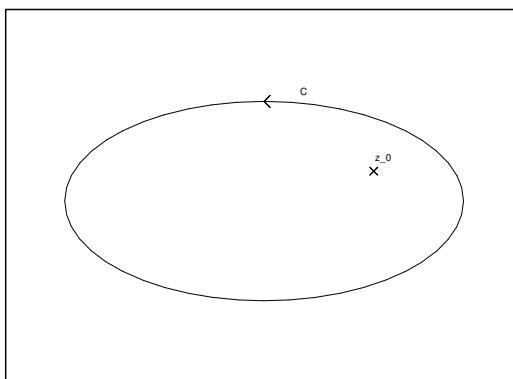
Figure 14: A simple closed curve C in Ω surrounding just one isolated singularity z_0 of $f(z)$.

Remark 3.6.1 *"Residuum",* or in some texts *"residue",* is a Latin word meaning "remaining", i.e. what is left after a part is taken away, namely the often complicated process of integrating the differential form $f(z)\,\mathrm{d}z$ along the simple closed curve C around z_0. We shall in the present text use the Latin plural "residua", but one may of course also use the English plural "residues". $\Diamond$

The first result is very simple.

Theorem 3.6.1 *Let $f(z)$ have the convergent Laurent series expansion*

(116) $f(z) = \displaystyle\sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n, \qquad for \ 0 < |z - z_0| < \varrho,$

in a deleted disc $D(z_0, \varrho) = B(z_0, \varrho) \setminus \{z_0\}$. Then

(117) $res(f; z_0) = a_{-1}.$

PROOF. The Laurent series (116) is uniformly convergent on the circle $C(z_0, r) : |z - z_0| = r$ for all fixed $r \in]0, \varrho[$. Thus, it can be integrated termwise,

$$res(f; z_0) = \frac{1}{2\pi i} \oint_{C(z_0,r)} f(z)\, dz = \sum_{n=-\infty}^{+\infty} a_n \cdot \frac{1}{2\pi i} \oint_{C(z_0,r)} (z - z_0)^n\, dz = a_{-1},$$

where we have used the result

$$\oint_{C(z_0,r)} (z - z_0)^n\, dz = \oint_{C(0,r)} z^n\, dz = \begin{cases} 0 & \text{for } n \in \mathbb{Z} \setminus \{-1\}, \\[2mm] 2\pi i & \text{for } n = -1, \end{cases}$$

proved in *Ventus: Complex Functions Theory a-1.* $\square$

Theorem 3.6.1 above is in particular applied when we shall find the residuum at an *essential singularity*. It may of course be used in general, but it is often easier to apply other methods, when the singularity is a pole.

Example 3.6.1 We shall find

$$res\left(\frac{1}{z^2(z-1)}; 0\right).$$

The Laurent series expansion from $z_0 = 0$ is in the deleted disc $0 < |z| < 1$ given by

$$\frac{1}{z^2(z-1)} = -\frac{1}{z^2} \cdot \frac{1}{1-z} = -\frac{1}{z^2} \sum_{n=0}^{+\infty} z^n = -\frac{1}{z^2} - \frac{1}{z} - \cdots . z^n - \cdots ,$$

thus $a_{-1} = -1$, and we get

$$res\left(\frac{1}{z^2(z-1)}; 0\right) = -1.$$

As a consequence – cf. (115) – we see that also

$$\oint_{|z|=\frac{1}{2}} \frac{1}{z^2(z-1)}\, dz = 2\pi i\, res\left(\frac{1}{z^2(z-1)}; 0\right) = -2\pi i.$$

Clearly, the traditional computation of the line integral in the left hand side of this equation becomes very difficult, so we have indeed derived an easier method of computation in this case. $\lozenge$

95

Example 3.6.2 It is obvious that the Laurent series expansion of $\dfrac{1}{z^2\left(z^2-1\right)}$ in a deleted disc of centre 0 only contains terms of *even exponents*, so $a_{-1}=0$, and

$$\operatorname{res}\left(\frac{1}{z^2(z^2-1)};0\right)=0,$$

in which case it is not even necessary to *find* the explicit Laurent series expansion. We just use a trivial inspection. $\diamond$

An important observation is that if z_0 is a *removable singularity*, then trivially $a_{-1}=0$, so

$$\operatorname{res}(f;z_0)=0, \qquad \text{if } z_0 \text{ is a removable singularity.}$$

This implies that if we occasionally include removable singularities in the *Calculus of Residua*, this does not matter much because such removable singularities will not contribute to the final result. This is convenient because it in general in many cases suffices to notice that z_0 is a pole of *at most* some order q.
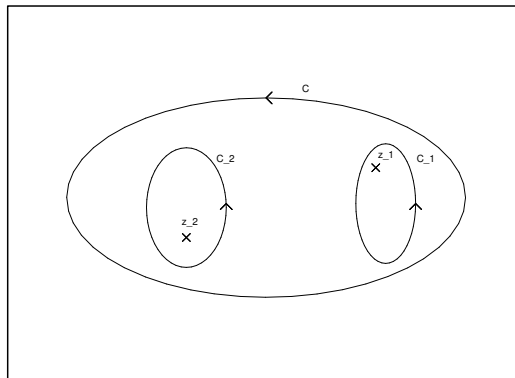

Figure 15: Cauchy's residue theorem.

Let C be a simple closed curve in the domain Ω of f, such that f is analytic inside C, with the exception of a finite number of isolated boundary points $z_1,\ \ldots,\ z_k$ of Ω, cf. Figure 15, where $k=2$.

"Inside" means here the *bounded* set surrounded by C, and the *direction* of C is in the positive sense of the complex plane, which also means that the the bounded component lies *to the left* of C seen in the direction of C.

It follows from *Cauchy's integral theorem* that

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z + \cdots + \oint_{C_k} f(z)\,\mathrm{d}z,$$

where each simple closed curve C_j (typically a small circle of centre z_j) only surrounds one singularity z_j, $j=1,\ldots,k$. Using that

$$\oint_{C_j} f(z)\,\mathrm{d}z = 2\pi i \cdot \operatorname{res}\left(f(z);z_j\right),$$

we immediately get the following *main theorem*:

Theorem 3.6.2 Cauchy's residuum theorem. *Assume that $f(z)$ is analytic in an open domain Ω, and let C be a simple closed curve in Ω oriented in the positive sense of the complex plane and with only a finite number of isolated boundary points $z_1, \ldots, z_k$ of Ω inside C (i.e. to the left of the curve), and analytic at all other points inside C. Then*

$$(118) \quad \frac{1}{2\pi i} \oint_C f(z)\, dz = res\,(f; z_1) + \cdots + res\,(f; z_k) = \sum_{j=1}^{k} res\,(f; z_j)\,.$$

The importance of this main theorem will be made clear in Chapter 4. Before this chapter we shall in the next section derive some simple rules of computation of the residuum of a function f at a pole.

3.7   Simple rules of computation of the residuum at a (finite) pole

It will in the following be convenient to consider a removable singularity as a *pole of order 0*. We shall in this section only consider residua of *finite poles* of order $q \in \mathbb{N}_0$.

If z_0 is an essential singularity of f, we either apply Theorem 3.6.1, or a technique which we shall develop in Section 3.8.

Theorem 3.7.1 *Assume that $f(z)$ has the pole z_0 of order $\leq q$ for some $q \in \mathbb{N}$. Then*

$$(119) \quad res\,(f; z_0) = \frac{1}{(q-1)!} \lim_{z \to z_0} \frac{d^{q-1}}{dz^{q-1}} \left\{ (z - z_0)^q\, f(z) \right\}.$$

PROOF. We have assumed that the order of the pole is *at most* q, hence $a_{-n} = 0$ for all $n > q$, and we have the Laurent series expansion,

$$f(z) = \frac{a_{-q}}{(z - z_0)^q} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1\,(z - z_0) + \cdots, \qquad \text{for } 0 < |z - z_0| < \varrho.$$

Notice that we do not assume that $a_{-q} \neq 0$.

Multiply this equation by $(z - z_0)^q$ to get

$$(z - z_0)^q\, f(z) = a_{-q} + \cdots + a_{-2}\,(z - z_0)^{q-2} + a_{-1}\,(z - z_0)^{q-1} + (z - z_0)^q\,\{a_0 + \cdots\}\,.$$

By $q - 1$ successive differentiations the polynomial of degree $q - 2$ disappears, so

$$\frac{d^{q-1}}{dz^{q-1}} \left\{ (z - z_0)^q\, f(z) \right\} = (q-1)!\, a_{-1} + (z - z_0)\,\{\cdots\},$$

where the dots are a shorthand for some analytic function. Then divide by $(q-1)!$ and take the limit $z \to z_0$ to get

$$\frac{1}{(q-1)!} \lim_{z \to z_0} \frac{d^{q-1}}{dz^{q-1}} \left\{ (z - z_0)^q\, f(z) \right\} = a_{-1} + 0 = a_{-1} = res\,(f; z_0)\,,$$

and the theorem is proved. $\square$

Remark 3.7.1 The proof can of course be generalized to find *all* coefficients a_{-j}, $j = 1, \ldots, 1$ (and of course also of the not so interesting coefficients a_n, $n \in \mathbb{N}_0$). By a modification of the proof above the reader easily verify that

$$(120) \quad a_{-j} = \frac{1}{(q-j)!} \lim_{z \to z_0} \frac{d^{q-j}}{dz^{q-j}} \left\{ (z - z_0)^q f(z) \right\}, \qquad j = 1, \ldots, q.$$

When $f(z)$ is a *rational function* of multiple poles, this is just *decomposition* in a new way, because a_{-j} is precisely the coefficient of $(z - z_0)^{-1}$ in the decomposition. The details are left to the reader. ◊

An important special case of Theorem 3.7.1 is the following:

Theorem 3.7.2 *Assume that z_0 is either a* simple pole *or a* removable singularity *of $f(z)$. Then*

$$(121) \quad res\,(f; z_0) = \lim_{z \to z_0} (z - z_0)\, f(z).$$

PROOF. Put $q = 1$ into (119). □

Example 3.7.1 Theorem 3.7.1 has the convenient property that one shall not find the exact order of the pole. Let us e.g. consider the function

$$f(z) = \frac{\sin^2 z}{z^5} \qquad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

Since $\dfrac{\sin z}{z}$ has a *removable singularity* at $z_0 = 0$, and

$$f(z) = \frac{\sin^2 z}{z^5} = \left\{ \frac{\sin z}{z} \right\}^2 \cdot \frac{1}{z^3},$$

we conclude that $f(z)$ has a pole of exact order 3 at $z_0 = 0$. However, if we put $q = 3$ into (119), we get

$$\operatorname{res}\left(\frac{\sin^2 z}{z^5}; 0 \right) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left\{ \frac{\sin^2 z}{z^2} \right\},$$

which clearly will give us some difficulties, if we continue this computation.

We shall not do this, for if we instead use that $z_0 = 0$ is *at most* of order $q = 5$ (> 3), then we get from (119),

$$\operatorname{res}\left(\frac{\sin^2 z}{z^5}; 0 \right) = \frac{1}{4!} \lim_{z \to 0} \frac{d^4}{dz^4} \left\{ \sin^2 z \right\} = \frac{1}{24} \lim_{z \to 0} \frac{d^3}{dz^3} \left\{ \sin 2z \right\} = \frac{1}{24} \lim_{z \to 0} 2^3 \left\{ -\cos 2z \right\} = -\frac{1}{3}.$$

In other words, by choosing a higher order than the exact one for the pole we ease the computations, at least in this particular case. $\Diamond$

Example 3.7.2 We shall find the value of the line integral along the circle $|z| = 2$,

$$\oint_{|z|=2|} \frac{e^z}{z(z-1)^2} \, \mathrm{d}z.$$

The mixture of an exponential and a rational function in the integrand will make the usual method of inserting a parametric description of the integration curve very complicated, if successful at all. Instead we notice that we have inside $|z| = 2$ two isolated singularities, namely the simple pole $z = 0$ and the double pole $z = 1$, i.e. of order 2. Then by *Cauchy's residuum theorem*,

$$\oint_{|z|=2} \frac{e^z}{z(z-1)^2} \, \mathrm{d}z = 2\pi i \left\{ \operatorname{res}(f; 0) + \operatorname{res}(f; 1) \right\}.$$

Since $z = 0$ is a *simple pole*, it follows from Theorem 3.7.2 that

$$\operatorname{res}(f; 0) = \lim_{z \to 0} z \, f(z) = \lim_{z \to 0} \frac{e^z}{(z-1)^2} = 1.$$

Since $z = 1$ is a pole of order $q = 2$, it follows from Theorem 3.7.1 that

$$\operatorname{res}(f; 1) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-1)^2 f(z) \right\} = \lim_{z \to 1} \frac{d}{dz} \left\{ \frac{e^z}{z} \right\} = \lim_{z \to 1} \frac{e^z}{z^2} (z-1) = 0.$$

Finally, we get by insertion

$$\oint_{|z|=2} \frac{e^z}{z(z-1)^2} \, \mathrm{d}z = 2\pi i \left\{ 1 + 0 \right\} = 2\pi i. \qquad \Diamond$$

99

Theorem 3.7.3 *Assume that both $A(z)$ and $B(z)$ are analytic in a neighbourhood of z_0, and assume that z_0 is a zero of* order 1 *for $B(z)$, i.e. $B(z_0) = 0$ and $B'(z_0) \neq 0$. Then*

$$(122) \quad res\left(\frac{A(z)}{B(z)}; z_0\right) = \frac{A(z_0)}{B'(z_0)}.$$

PROOF. Since $\dfrac{A(z)}{B(z)}$ has a pole of at most order 1 at z_0, it follows from Theorem 3.7.2 that

$$
\begin{aligned}
\mathrm{res}\left(\frac{A(z)}{B(z)}; z_0\right) &= \lim_{z \to z_0} (z - z_0) \cdot \frac{A(z)}{B(z)} = \lim_{z \to z_0} A(z) \cdot \frac{1}{\dfrac{B(z) - B(z_0)}{z - z_0}} \\
&= A(z_0) \cdot \frac{1}{\lim_{z \to z_0} \dfrac{B(z) - B(z_0)}{z - z_0}} = \frac{A(z_0)}{B'(z_0)}. \quad \square
\end{aligned}
$$

Example 3.7.3 We shall find the value of

$$\oint_{|z|=2} \frac{z\, e^z}{z^2 - 1}\, \mathrm{d}z.$$

Put $A(z) = z\, e^z$ and $B(z) = z^2 - 1$. Then $A(z)$ and $B(z)$ are analytic in $\mathbb{C}$, and $B(z)$ has its simple zeros at $z = \pm 1$. Let z_0 be any of the zeros ± 1. By Theorem 3.7.3,

$$\mathrm{res}(f; z_0) = \frac{A(z_0)}{B'(z_0)} = \frac{z_0 \exp(z_0)}{2z_0} = \frac{1}{2} \exp(z_0),$$

hence

$$\oint_{|z|=2} \frac{z\, e^z}{z^2 - 1}\, \mathrm{d}z = 2\pi i \left\{\mathrm{res}(f; 1) + \mathrm{res}(f; -1)\right\} = 2\pi i \cdot \frac{e^1 + e^{-1}}{2} = 2\pi i \cdot \cosh 1. \quad \Diamond$$

Example 3.7.4 Theorem 3.7.3 is in particular applied when we shall find the residua at several *simple poles*. We have e.g.

$$\oint_{|z|=2} \frac{z}{z^4 - 1}\, \mathrm{d}z = 2\pi i \left\{\mathrm{res}(f; 1) + \mathrm{res}(f; -1) + \mathrm{res}(f; i) + \mathrm{res}(f; -i)\right\},$$

where all poles , 1, -1, i and $-i$, are simple. Let z_0 be any one of these. Then $z_0^4 = 1$.

Choose $A(z) = z$ and $B(z) = z^4 - 1$, i.e. $B'(z) = 4z^3$, so it follows from Theorem 3.7.3 that

$$\mathrm{res}(f; z_0) = \frac{A(z_0)}{B'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4} \cdot \frac{z_0^2}{z_0^4} = \frac{1}{4} z_0^2,$$

hence by insertion,

$$\oint_{|z|=2} \frac{z}{z^4 - 1}\, \mathrm{d}z = \frac{2\pi i}{4} \left\{1^2 + (-1)^2 + i^2 + (-i)^2\right\} = 0. \quad \Diamond$$

In the following theorem we assume that $B(z)$ has a zero of *second order* at z_0. It is given here for completeness, but it should also be mentioned that Theorem 3.7.1 usually give smaller computations, and furthermore, it is very easy to make errors in the computations of Theorem 3.7.4 below, so the reader is warned against applying it uncritically.

Theorem 3.7.4 *Assume that $A(z)$ and $B(z)$ are analytic in a neighbourhood of z_0. Furthermore, assume that $B(z)$ has a zero of exactly* second order *at z_0. Then*

$$(123) \quad res\left(\frac{A(z)}{B(z)}; z_0\right) = \frac{6A'(z_0) B''(z_0) - 2A(z_0) B^{(3)}(z_0)}{3\left\{B''(z_0)\right\}^2}.$$

PROOF. It follows from the assumptions that

$$A(z) = a_0 + a_1(z - z_0) + \cdots, \qquad B(z) = b_2(z - z_0)^2 + b_3(z - z_0)^3 + \cdots \qquad \text{for } |z - z_0| < \varrho,$$

where $b_2 \neq 0$, and $B(z) \neq 0$ for $0 < |z - z_0| < \varrho$. We shall find the Laurent series expansion of $f(z)$ in the deleted disc $0 < |z - z_0| < \varrho$. This is given by

$$f(z) = \frac{c_{-2}}{(z - z_0)^2} + \frac{c_{-1}}{z - z_0} + \cdots = \frac{A(z)}{B(z)} = \frac{1}{(z - z_0)^2} \cdot \frac{a_0 + a_1(z - z_0) + \cdots}{b_2 + b_3(z - z_0) + \cdots}.$$

Since $res(f; z_0) = c_{-1}$, we shall only find c_{-1} of the equation $B(z)f(z) = A(z)$, i.e.

$$\left\{c_{-2} + c_{-1}(z - z_0) + \cdots\right\} \cdot \left\{b_2 + b_3(z - z_0) + \cdots\right\} = a_0 + a_1(z - z_0) + \cdots.$$

By multiplication,

$$c_{-2}b_2 + (c_{-2}b_3 + c_{-1}b_2)(z - z_0) + \cdots = a_0 + a_1(z - z_0) + \cdots.$$

It follows from the *Identity theorem* that $c_{-2} = \dfrac{a_0}{b_2}$, and

$$(124) \quad res(f; z_0) = c_{-1} = \frac{1}{b_2}\left\{a_1 - c_{-2}b_3\right\} = \frac{1}{b_2}\left\{a_1 - \frac{a_0 b_3}{b_2}\right\} = \frac{a_1 b_2 - a_0 b_3}{b_2^2}.$$

Using the notation of the theorem we get

$$A(z_0) = a_0, \quad A'(z_0) = a_1, \quad B''(z_0) = 2!b_2 = 2b_2, \quad B^{(3)}(z_0) = 3!b_3 = 6b_3.$$

We finally get by insertion into (124),

$$res(f; z_0) = \frac{A'(z_0) \cdot \frac{1}{2} B''(z_0) - A(z_0) \cdot \frac{1}{6} B^{(3)}(z_0)}{\frac{1}{4}\left\{B''(z_0)\right\}^2} = \frac{6\, A'(z_0) B''(z_0) - 2A(z_0) B^{(3)}(z_0)}{3\left\{B''(z_0)\right\}^2}. \qquad \square$$

3.8 The residuum at ∞

Assume that the analytic function $f(z)$ for $|z| > R$ is given by its convergent Laurent series expansion

$$(125) \quad f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n \qquad \text{for } |z| > R.$$

Definition 3.8.1 *Assume that $f(z)$ is analytic for $|z| > R$. Then we define the* residuum *of the differential form $f(z)\, dz$ by*

$$(126) \quad res(f(z)\, dz; \infty) := -\frac{1}{2\pi i} \oint_C f(z)\, dz,$$

where $-C$ is any simple closed curve in $z > R$ surrounding ∞.

We notice the minus sign on the right hand side of (126), and that $-C$ denotes the curve C with reversed direction, i.e. opposite the orientation of the complex plane. That $-C$ is surrounding ∞ means that ∞ lies to the left of this curve $-C$, seen in this reversed direction of C.

We have earlier proved that the value of (126) is independent of the curve C, as long as it fulfils the conditions of Definition 3.8.1.

Choose in particular $C = C(0, r)$, i.e. $|z| = r$, where $r > R$. Since we have uniform convergence of the Laurent series expansion (125) on $C(0, r)$, we can interchange summation and integration, and we get

$$\mathrm{res}(f(z)\,\mathrm{d}z; \infty) = -\frac{1}{2\pi i} \oint_{C(0,r)} f(z)\,\mathrm{d}z = -\sum_{n=-\infty}^{+\infty} \frac{1}{2\pi i} \oint_{C(0,r)}]a_n\, z^n\,\mathrm{d}z = -a_{-1},$$

so we have proved

Theorem 3.8.1 *If $f(z) = \sum_{n=-\infty}^{+\infty} a_n\, z^n$ is the convergent Laurent series expansion for $|z| > R$, then the residuum of $f(z)\,\mathrm{d}z$ at infinity is given by*

(127) $res(f(z)\,\mathrm{d}z; \infty) = -a_{-1}.$

It is easy to extend *Cauchy's residuum theorem* to also include ∞.

Theorem 3.8.2 Cauchy's residuum theorem for unbounded domains. *Assume that $f(z)$ is analytic in an open domain Ω, which contains the set $\mathbb{C} \setminus B[0, R]$ for some $R \geq 0$. Let C be any simply closed curve in Ω, such that* outside C, *i.e. to the right of C seen in its positive direction – which can also be described as the unbounded domain of $\mathbb{C}$ having C as its boundary – there are only a* finite *number of (necessarily isolated) boundary points $z_1, \ldots, z_k$ of Ω. Then*

(128) $-\dfrac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z = res(f\,\mathrm{d}z; z_1) + \cdots + res(f\,\mathrm{d}z; z_k) + res(f\,\mathrm{d}z; \infty).$
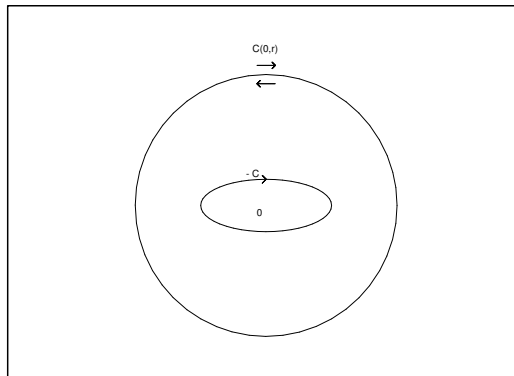

Figure 16: Cauchy's residue theorem for unbounded domains.

PROOF. Choose $r > 0$, such that $|z_j| < r$ for all $j = 1, \ldots, k$, so all the finite singularities $z_1, \ldots, z_k$ lie between C and $C(0, r)$. We get by adding and subtracting the line integral along $C(0, r)$ and applying *Cauchy's residuum theorem* for finite singularities and Definition 3.8.1, cf. Figure 16,

$$-\frac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z \;=\; -\frac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z + \frac{1}{2\pi i} \oint_{C(0,r)} f(z)\,\mathrm{d}z - \frac{1}{2\pi i} \oint_{C(0,r)} f(z)\,\mathrm{d}z$$

$$=\; \mathrm{res}(f\,\mathrm{d}z; z_1) + \cdots + \mathrm{res}(f\,\mathrm{d}z; z_k) + \mathrm{res}(f\,\mathrm{d}z; \infty). \qquad \square$$

Theorem 3.8.3 *Assume that $f : \Omega \to \mathbb{C}$ is analytic, where $\Omega = \mathbb{C} \setminus \{z_1, \ldots, z_k\}$. Then the sum of all residua, including the residuum at ∞, is equal to zero,*

(129) $\quad res(f \, \mathrm{d}z; z_1) + \cdots + res(f \, \mathrm{d}z; z_k) + res(f \, \mathrm{d}z; \infty) = 0.$

PROOF. Let C be any closed curve in Ω. Then by the two versions of *Cauchy's residuum theorem*,

$$0 = -\frac{1}{2\pi i} \oint_C f(z) \, \mathrm{d}z + \frac{1}{2\pi i} \oint_C f(z) \, \mathrm{d}z = \mathrm{res}(f \, \mathrm{d}z; z_1) + \cdots + \mathrm{res}(f \, \mathrm{d}z; z_k) + \mathrm{res}(f \, \mathrm{d}z; \infty). \quad \square$$

Theorem 3.8.4 *If $f(z)$ has a zero at ∞, then*

(130) $\quad res(f \, \mathrm{d}z; \infty) = - \lim_{z \to \infty} z \, f(z).$

In particular, $res(f \, \mathrm{d}z; \infty) = 0$, if ∞ is a zero of order ≥ 2 for f.

PROOF. By the assumption,

$$f(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots \qquad \text{for } |z| > R.$$

Then by Theorem 3.8.1,

$$\mathrm{res}(f \, \mathrm{d}z; \infty) = -a_{-1} = - \lim_{z \to \infty} z \, f(z). \quad \square$$

The residuum at ∞ is in particular applied for line integrals along simple closed curves C, for which the residuum theorem of Section 3.6 is either difficult to apply, or where its assumptions are not fulfilled at all. We shall in the following give some examples of these phenomena. Notice that we often can choose between various methods of computation. The art is then to choose the easiest one.

Example 3.8.1 We computed in Example 3.7.4 the integral

$$\oint_{|z|=2} \frac{z}{z^4 - 1} \, \mathrm{d}z$$

by applying Theorem 3.7.3. It is, however, much easier to apply Theorem 3.8.4,

$$\oint_{|z|=2} \frac{z}{z^4 - 1} \, \mathrm{d}z = - \left\{ - \oint_{|z|=2} \frac{z}{z^4 - 1} \, \mathrm{d}z \right\} = -2\pi i \cdot \mathrm{res}\left(\frac{z}{z^4 - 1} \, \mathrm{d}z; \infty \right) = 2\pi i \cdot \lim_{z \to \infty} \frac{z^2}{z^4 - 1} = 0,$$

because all (finite) singularities of the integrand lie *inside* the circle $|z| = 2$, and because the integrand has a zero of order 2 at ∞. $\Diamond$

Theorem 3.8.5 *If $f(z)$ is analytic for $|z| > R$, then*

$$(131) \quad res(f(z)\,\mathrm{d}z;\infty) = -res\left(\frac{1}{z^2}\,f\left(\frac{1}{z}\right)\mathrm{d}z;0\right).$$

PROOF. The Laurent series expansion is

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n\, z^n \qquad \text{for } |z| > R.$$

so

$$\mathrm{res}\left(\frac{1}{z^2}\,f\left(\frac{1}{z}\right)\mathrm{d}z;0\right) = \mathrm{res}\left(\sum_{n=-\infty}^{+\infty} a_n\, z^{-2-n};0\right) = a_{-1} = -\mathrm{res}(f\,\mathrm{d}z;\infty). \qquad \square$$

Remark 3.8.1 Notice that

$$-\mathrm{res}\left(\frac{1}{z^2}\,f\left(\frac{1}{z}\right)\mathrm{d}z;0\right) = \mathrm{res}\left(f\left(\frac{1}{z}\right)\mathrm{d}\left(\frac{1}{z}\right);z_0 = 0\right),$$

so (131) can be rewritten as

$$(132) \quad \mathrm{res}(f(z)\,\mathrm{d}z;\infty) = \mathrm{res}\left(f\left(\frac{1}{z}\right)\mathrm{d}\left(\frac{1}{z}\right);z_0 = 0\right),$$

which indicates why the residuum is linked to the *differential form* and not to the function $f(z)$ itself. In fact, in this way the residuum becomes invariant under transformations of z. ◊

Example 3.8.2 *Important!* The analytic function $f(z) = \dfrac{1}{z}$, $z \neq 0$, is trivially extended to ∞ by putting $f(\infty) = 0$. The Laurent series expansion of f is trivial,

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n\, z^n = \frac{1}{z} = a_{-1} \cdot \frac{1}{z},$$

thus $a_{-1} = 1$, and $\mathrm{res}(f\,\mathrm{d}z;\infty) = -1 \neq 0$ according to Theorem 3.8.1. This simple example shows that *analyticity at ∞ does <u>not</u> imply that $res(f\,dz;\infty)$ is zero!* This is in contrast to the residuum at a finite removable singularity, in which the residuum is indeed zero. Many years ago the author experienced that even trained professors in Mathematics can make errors here. Therefore this warning. ◊

Example 3.8.3 To show the power of this theory we shall show that we are now able to compute even a nasty integral as

$$\oint_{|z|=1} \frac{1}{\sin \frac{1}{z}} \, \mathrm{d}z.$$

The standard procedure of computing a line integral, using a parametric description is clearly doomed to failure. The integrand has its singularities at the zeros of $\sin \frac{1}{z}$, i.e. for $z = \frac{1}{n\pi}$ for $n \in \mathbb{Z} \setminus \{0\}$ (all simple poles lying inside $|z| = 1$), supplied with the non-isolated singularity at $z = 0$, so we cannot use the version of *Cauchy's residuum theorem* given in Section 3.6.

We then choose to apply the residuum at ∞. This is done in the following way,

$$
\begin{aligned}
\oint_{|z|=1} \frac{1}{\sin \frac{1}{z}} \, \mathrm{d}z &= -\left\{ -\oint_{|z|=1} \frac{1}{\sin \frac{1}{z}} \, \mathrm{d}z \right\} = -2\pi i \cdot \operatorname{res}\left(\frac{1}{\sin \frac{1}{z}} \, \mathrm{d}z; \infty \right) \\
&= -2\pi i \cdot \operatorname{res}\left(\frac{1}{\sin z} \, \mathrm{d}\left(\frac{1}{z} \right); 0 \right) = 2\pi i \cdot \operatorname{res}\left(\frac{1}{z^2 \sin z} \, \mathrm{d}z; 0 \right) = 2\pi i \cdot a_{-1},
\end{aligned}
$$

where a_{-1} is the coefficient of $\dfrac{1}{z}$ in the Laurent series expansion of $\dfrac{1}{z^2 \sin z}$ from $z_0 = 0$. Now, $z^2 \sin z$ clearly has a zero of order 3 at 0, so $\dfrac{1}{z^2 \sin z}$ has a pole of order 3 at 0. Using that $z^2 \sin x \neq 0$ for $0 < |z| < \pi$ we therefore have

$$\frac{1}{z^2 \sin z} = \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \cdots \qquad \text{for } 0 < |z| < \pi,$$

where the task is to find a_{-1}. If this equation is multiplied by

$$z^2 \sin z = z^2 \left\{ z - \frac{1}{6} z^3 + \cdots \right\} = z^3 \left\{ 1 - \frac{1}{6} z^2 + \cdots \right\},$$

then we get

$$\begin{aligned}
1 &= \frac{1}{z^3} \left\{ a_{-3} + a_{-2} z + a_{-1} z^2 + \cdots \right\} \cdot z^3 \left\{ 1 - \frac{z^2}{6} + \cdots \right\} \\
&= \left\{ a_{-3} + a_{-2} z + a_{-1} z^2 + \cdots \right\} \cdot \left\{ 1 - \frac{1}{6} z^2 + \cdots \right\} \\
&= a_{-3} + a_{-2} z + \left\{ a_{-1} - \frac{1}{6} a_{-3} \right\} z^2 + \cdots,
\end{aligned}$$

where the dots everywhere indicate terms of higher order. When we identify the coefficients, we get

$$a_{-3} = 1, \qquad a_{-2} = 0 \qquad \text{and} \qquad a_{-1} = \frac{1}{6} a_{-3} = \frac{1}{6}.$$

Hence by insertion,

$$\oint_{|z|=1} \frac{1}{\sin z} \, \mathrm{d}z = 2\pi i \cdot \operatorname{res}\left(\frac{1}{z^2 \sin z} \, \mathrm{d}z ; 0 \right) = 2\pi i \cdot a_{-1} = \frac{\pi i}{3}. \qquad \Diamond$$

Example 3.8.4 Since $f(z) = \sin \dfrac{1}{z}$ is analytic in $\mathbb{C} \setminus \{0\}$, it follows from Theorem 3.8.3 that

$$\operatorname{res}\left(\sin \frac{1}{z} ; \infty \right) = -\operatorname{res}\left(\sin \frac{1}{z} ; 0 \right) = -1,$$

because the Laurent series expansion of $f(z)$ from $z : 0 = 0$ is given by

$$f(z) = \sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!} \frac{1}{z^3} + \cdots, \qquad \text{for } z \in \mathbb{C} \setminus \{0\},$$

from which $a_{-1} = 1$.

Alternatively we apply Theorem 3.8.5, because

$$\frac{1}{z^2} f\left(\frac{1}{z} \right) = \frac{1}{z^2} \sin z = \frac{1}{z^2} \left\{ z - \frac{z^3}{3!} + \cdots \right\},$$

thus

$$\operatorname{res}\left(\sin \frac{1}{z} ; \infty \right) = -\operatorname{res}\left(\frac{1}{z^2} f\left(\frac{1}{z} \right) ; 0 \right) = -1. \qquad \Diamond$$

Remark 3.8.2 It must be emphasized that ∞ is *not* an isolated singularity of functions like $\dfrac{1}{\sin z}$, $\dfrac{1}{\cos z}$, $\dfrac{1}{\sinh z}$, $\dfrac{1}{\cosh z}$, $\tan x$, $\cot z$, $\tanh z$, $\coth z$, etc.. This means that the residuum at ∞ is *never* defined for these function. $\Diamond$

3.9   Summary of the Calculus of Residua

It is very important to be able to compute the residuum in Chapter 4, so we collect the basic properties in this section.

1) The *residuum* of the complex *differential form* $f(z)\,\mathrm{d}z$ at z_0 is defined as

$$\operatorname{res}(f(z)\,\mathrm{d}z; z_0) := \frac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z,$$

   where C is any simple closed curve in Ω surrounding z_0 and no other boundary point of Ω, cf. Definition 3.6.1.

2) Let $f(z)$ have the convergent Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n \left(z - z_0\right)^n, \qquad \text{for } 0 < |z - z_0| < \varrho,$$

   in a deleted disc $D(z_0, \varrho) = B(z_0, \varrho) \setminus \{z_0\}$. Then

$$\operatorname{res}(f; z_0) = a_{-1}.$$

   Cf. Theorem 3.6.1.

3) If z_0 is a *removable singularity*, then

$$\operatorname{res}(f; z_0) = 0.$$

4) Cauchy's residuum theorem. *Assume that $f(z)$ is analytic in an open domain Ω, and let C be a simple closed curve in Ω oriented in the positive sense of the complex plane and with only a finite number of isolated boundary points $z_1, \ldots, z_k$ of Ω inside C (i.e. to the left of the curve), and analytic at all other points inside C. Then*

$$\frac{1}{2\pi i} \oint_C f(z)\,dz = res(f; z_1) + \cdots + res(f; z_k) = \sum_{j=1}^{k} res(f; z_j),$$

   cf. page 93.

5) Assume that $f(z)$ has the pole z_0 of order $\leq q$ for some $q \in \mathbb{N}$. Then

$$\operatorname{res}(f; z_0) = \frac{1}{(q-1)!} \lim_{z \to z_0} \frac{d^{q-1}}{dz^{q-1}} \left\{ (z - z_0)^q\, f(z) \right\},$$

   cf. Theorem 3.7.1.

6) Assume that z_0 is either a *simple pole* or a *removable singularity* of $f(z)$. Then

$$\operatorname{res}(f; z_0) = \lim_{z \to z_0} (z - z_0)\, f(z),$$

cf. Theorem 3.7.2

7) Assume that both $A(z)$ and $B(z)$ are analytic in a neighbourhood of z_0, and assume that z_0 is a zero of *order 1* for $B(z)$, i.e. $B(z_0) = 0$ and $B'(z_0) \neq 0$. Then

$$\operatorname{res}\left(\frac{A(z)}{B(z)}; z_0\right) = \frac{A(z_0)}{B'(z_0)},$$

cf. Theorem 3.7.3

8) (Use this result with care.) Assume that $A(z)$ and $B(z)$ are analytic in a neighbourhood of z_0. Furthermore, assume that $B(z)$ has a zero of exactly *second order* at z_0. Then

$$\operatorname{res}\left(\frac{A(z)}{B(z)}; z_0\right) = \frac{6A'(z_0)\, B''(z_0) - 2A(z_0)\, B^{(3)}(z_0)}{3\left\{B''(z_0)\right\}^2}.$$

cf. Theorem 3.7.4.

9) Assume that $f(z)$ is analytic for $|z| > R$. Then we define the *residuum* of the *differential form* $f(z)\, \mathrm{d}z$ by

$$\operatorname{res}(f(z)\, \mathrm{d}z; \infty) := -\frac{1}{2\pi i} \oint_C f(z)\, \mathrm{d}z,$$

where $-C$ is any simple closed curve in $z > R$ surrounding ∞, cf. Definition 3.8.1

10) If $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ is the convergent Laurent series expansion for $|z| > R$, then the residuum of $f(z)\, \mathrm{d}z$ at infinity is given by

$$\operatorname{res}(f(z)\, \mathrm{d}z; \infty) = -a_{-1},$$

c.f. Theorem 3.8.1.

11) Assume that $f(z)$ is analytic in an open domain Ω, which contains the set $\mathbb{C} \setminus B[0, R]$ for some $R \geq 0$. Let C be any simply closed curve in Ω, such that *to the right of C* seen in its positive direction there are only a *finite* number of (necessarily isolated) boundary points $z_1, \ldots, z_k$ of Ω. Then

$$-\frac{1}{2\pi i} \oint_C f(z)\, \mathrm{d}z = \operatorname{res}(f\, \mathrm{d}z; z_1) + \cdots + \operatorname{res}(f\, \mathrm{d}z; z_k) + \operatorname{res}(f\, \mathrm{d}z; \infty),$$

cf. Theorem 3.8.2.

12) Assume that $f : \Omega \to \mathbb{C}$ is analytic, where $\Omega = \mathbb{C} \setminus \{z_1, \ldots, z_k\}$. Then the sum of all residua, including the residuum at ∞, is equal to zero,

$$\operatorname{res}(f\, \mathrm{d}z; z_1) + \cdots + \operatorname{res}(f\, \mathrm{d}z; z_k) + \operatorname{res}(f\, \mathrm{d}z; \infty) = 0,$$

cf. Theorem 3.8.3.

13) If $f(z)$ has a *zero* at ∞, then

$$\mathrm{res}(f\,\mathrm{d}z; \infty) = -\lim_{z \to \infty} z\,f(z).$$

In particular, $\mathrm{res}(f\,\mathrm{d}z; \infty) = 0$, if ∞ is a zero of order ≥ 2 for f, cf. Theorem 3.8.4.

14) If $f(z)$ is analytic for $|z| > R$, then

$$\mathrm{res}(f(z)\,\mathrm{d}z; \infty) = -\mathrm{res}\left(\frac{1}{z^2}\,f\left(\frac{1}{z}\right)\mathrm{d}z; 0\right),$$

cf. Theorem 3.8.5.

4 Applications of the Calculus of Residua

4.1 Trigonometric integrals

We shall start this chapter by demonstrating that some trigonometric integrals are easier to compute by using the *Calculus of Residua* than the traditional method from *Real Calculus*. We first prove the following theorem:

Theorem 4.1.1 *Let $R(\xi, \eta)$ denote a function in two real variables defined in a subset of $\mathbb{R}^2$. If*

$$f(z) = R\left(\frac{z^2 - 1}{2iz}, \frac{z^2 + 1}{2z}\right)$$

is analytic in a domain $\Omega \subset \mathbb{C}$, containing the circle $|z| = 1$, then

$$(133) \quad \int_0^{2\pi} R(\sin\Theta, \cos\Theta)\, d\Theta = \oint_{|z|=1} R\left(\frac{z^2 - 1}{2iz}, \frac{z^2 + 1}{2z}\right) \frac{dz}{iz}.$$

PROOF. When we apply the parametric description $z = e^{i\Theta}$, $\Theta \in [0, 2\pi]$, of $|z| = 1$, we get $dz = i\, e^{i\Theta}\, d\Theta = i\, z\, d\Theta$. Then

$$\frac{z^2 - 1}{2iz} = \frac{1}{2i}\left\{e^{i\Theta} - e^{-i\Theta}\right\} = \sin\Theta \qquad \text{and} \qquad \frac{z^2 + 1}{2z} = \frac{1}{2}\left\{e^{i\Theta} + e^{-i\Theta}\right\} = \cos\Theta,$$

and the result follows immediately by insertion into the right hand side of (133). $\square$

Obviously, (133) should be applied from the left to the right, because then we can apply the *Calculus of Residua*. That this method is really powerful is demonstrated by the following example.

Example 4.1.1 We shall compute $\int_0^{2\pi} e^{2\cos\Theta}\, d\Theta$. The tradition substitution $t = 2\cos\Theta$ with a discussion of its intervals in which it is monotone does not look promising. Instead note that $R(\xi, \eta) = e^{2\eta}$, in which even ξ is missing, and where the function

$$R\left(\frac{z^2 - 1}{2iz}, \frac{z^2 + 1}{2z}\right) = \exp\left(2 \cdot \frac{z^2 + 1}{2z}\right) = \exp\left(z + \frac{1}{z}\right), \qquad z \neq 0,$$

is analytic in $\mathbb{C} \setminus \{0\}$. Hence, by Theorem 4.1.1,

$$\int_0^{2\pi} e^{2\cos\Theta}\, d\Theta = \oint_{|z|=1} \exp\left(z + \frac{1}{z}\right) \frac{dz}{iz} = \frac{2\pi i}{i}\, \mathrm{res}\left(\frac{1}{z}\, \exp\left(z + \frac{1}{z}\right); 0\right).$$

The unpleasant fact is of course that the only two singularities, 0 and ∞, are both *essential*. This means that the only possible method is to find a_{-1} in the Laurent series expansion. We have

$$\frac{1}{z}\, \exp\left(z + \frac{1}{z}\right) = \frac{1}{z}\, \exp z \cdot \exp\frac{1}{z} = \frac{1}{z}\sum_{m=0}^{+\infty} \frac{1}{m!}\, z^m \cdot \sum_{n=0}^{+\infty} \frac{1}{n!}\, \frac{1}{z^n}, \qquad z \in \mathbb{C} \setminus \{0\},$$

so a_{-1} is by *Cauchy multiplication* equal to the sum of the coefficients, which correspond to $m = n$ i.e. by a summation, $a_{-1} = \sum_{n=0}^{+\infty} \frac{1}{(n!)^2}$. Hence,

$$\int_0^{2\pi} e^{2\cos\Theta}\, d\Theta = 2\pi \sum_{n=0}^{+\infty} \frac{1}{(n!)^2}.$$

This series is clearly rapidly convergent. It should here be mentioned that it can be proved that

$$\sum_{n=0}^{+\infty} \frac{1}{(n!)^2} = J_0(2i),$$

where $J_0(z)$ denotes the (complex) *Bessel function* of order 0. $\Diamond$

The value of Theorem 4.1.1 lies in the fact that it can be applied when the integrand is *not* a rational function in $\cos\Theta$ and $\sin\Theta$. In many textbooks this theorem is however only formulated for such *rational functions*. In some sense this is an overkill, because the traditional method known from *Real Calculus* is often easier to apply. We shall demonstrate this in Example 4.1.2 below.

Example 4.1.2 We shall compute the integral $\int_0^{2\pi} \dfrac{\mathrm{d}\Theta}{2+\cos\Theta}$. First we get by Theorem 4.1.1,

$$\int_0^{2\pi} \frac{\mathrm{d}\Theta}{2+\cos\Theta} \;=\; \oint_{|z|=1} \frac{1}{2+\dfrac{z^2+1}{2z}}\, \frac{\mathrm{d}z}{iz} = \oint_{|z|=1} \frac{-2i}{z^2+4z+1}\,\mathrm{d}z$$

$$=\; (-2i)\cdot 2\pi i \cdot \mathrm{res}\left(\frac{1}{z^2+4z+1}; -2+\sqrt{3}\right) = 4\pi \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}} = \frac{2\pi}{\sqrt{3}},$$

where we have used that z^2+4z+1 has the two simple roots $-2\pm\sqrt{3}$, of which only $-2+\sqrt{3}$ lies inside $|z|=1$, and then used Theorem 3.7.2.

Alternatively, we get by the traditional substitution $t=\dfrac{\Theta}{2}$ and $u=\tan t$, that

$$\int_0^{2\pi} \frac{\mathrm{d}\Theta}{2+\cos\Theta} \;=\; \int_0^{2\pi} \frac{\mathrm{d}\Theta}{3\cos^2\left(\dfrac{\Theta}{2}\right)+\sin^2\left(\dfrac{\Theta}{2}\right)} = 2\cdot 2\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{3\cos^2 t+\sin^2 t}$$

$$=\; \frac{4}{3}\int_0^{+\infty} \frac{\mathrm{d}u}{1+\frac{1}{3}u^2} = \frac{4\sqrt{3}}{3}\cdot\frac{\pi}{2} = \frac{2\pi}{\sqrt{3}}. \qquad \Diamond$$

4.2 Improper integrals

Up to this point we have only considered bounded line integrals. We shall here prove that the real axis $\mathbb{R}$ in some cases – depending on the integrand – may be considered as a simple curve which is closed by adding ∞, where we identify $+\infty$ and $-\infty$ in the complex plane. Hence under some additional assumptions we shall expand *Cauchy's residuum theorem* to half planes.

Theorem 4.2.1 *Let $f : \Omega \to \mathbb{C}$ be analytic in an open domain Ω, which contains the* closed *upper half plane $\Im z \geq 0$, except for a finite number of points $z_1, \ldots, z_n$, all of which lying in the open upper half plane $\Im z_j > 0$, $j = 1, \ldots, n$. Thus,*

$$\Omega \cup \{z_1, \ldots, z_n\} \supset \{z \in \mathbb{C} \mid \Im z \geq 0\}.$$

Assume that there exist constants $R > 0$, $c > 0$ and $a > 1$, such that

(134) $\quad |f(z)| \leq \dfrac{c}{|z|} \qquad$ *for $|z| \geq R$ and $\Im z \geq 0$.*

Then the improper integral $\int_{-\infty}^{+\infty} f(x)\, dx$ along the real axis is well-defined, and its value is given by the following residuum formula,

(135) $\quad \displaystyle\int_{-\infty}^{+\infty} f(x)\, dx = 2\pi i \sum_{\Im z_j > 0} res(f; z_j) = 2\pi i \sum_{j=1}^{n} res(f; z_j).$

PROOF. First note that $\mathbb{R} + i \cdot 0 \subset \Omega$ and that f is analytic in Ω, so the restriction of f to $\mathbb{R}$ must be continuous. From $a > 1$ and (134) follows that $f(x)$ in the real has an integrable majoring function, e.g.

$$g(x) = \begin{cases} \dfrac{c}{|x|^a} & \text{for } |x| \geq R, \\[2mm] |f(x)| & \text{for } |x| < R, \end{cases}$$

and we conclude that the improper integral $\int_{-\infty}^{+\infty} f(x)\, dx$ exists, and its value is given by

$$\int_{-\infty}^{+\infty} f(x)\, dx = \lim_{r_1, r_2 \to +\infty} \int_{-r_1}^{r_2} f(x)\, dx = \lim_{r \to +\infty} \int_{-r}^{r} f(x)\, dx.$$

We then exploit that $f(z)$ is analytic in the upper half plane with the exception of only a finite number of singularities.

It follows from (134) that all singularities in the upper half plane must lie in the disc $B(0, R)$.

Let $r > R$, and let C'_r denote the circular arc of the parametric description $z = r\, e^{i\Theta}$, $\Theta \in [0, \pi]$, in the upper half plane. Let furthermore C_r be the simple closed curve which is obtained by joining C'_r and the interval $[-r, r]$ on the real axis. Then by *Cauchy's residuum theorem*,

(136) $\quad \displaystyle\int_{-r}^{r} f(x)\, dx + \int_{C'_r} f(z)\, dz = \oint_{C_r} f(z)\, dz = 2\pi i \sum_{\Im z_j > 0} res(f; z_j).$
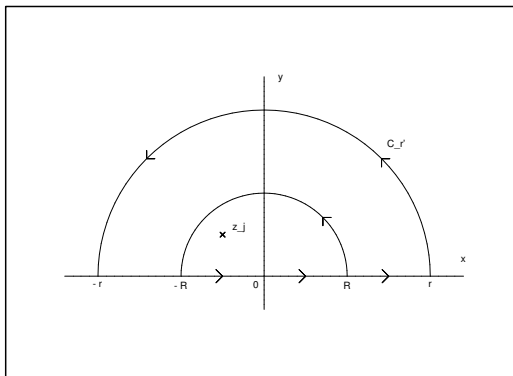
Figure 17: The path of integration in the proof of Theorem 4.2.1.

The right hand side of (136) is constant for all $r > R$. The first integral on the left hand side of (136) converges towards $\int_{-\infty}^{+\infty} f(x)\,dx$ for $r \to +\infty$. Therefore, we shall only prove that

$$\lim_{r \to +\infty} \int_{C_r'} f(z)\,dz = 0.$$

Using that $|z| = r \geq R$ for $z \in C_r'$ of length $\ell\,(C_r') = \pi r$, we get from (134) the estimate

$$\left| \int_{C_r'} f(z)\,dz \right| \leq \frac{c}{r^a}\,\ell\,(C_r') = c\,\pi \cdot r^{1-a} \to 0 \qquad \text{for } r \to +\infty,$$

because $a > 1$ by assumption. $\square$

Example 4.2.1 We choose arbitrarily

$$f(z) = \frac{1}{z^2 + 1}\,\exp\left(\frac{1}{z - i}\right), \qquad z \in \mathbb{C} \setminus \{-i, i\},$$

where it is more or less obvious that

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 1}\,\exp\left(\frac{x + i}{x^2 + 1}\right)dx = \int_{-\infty}^{+\infty} \frac{1}{x^2 + 1}\,\exp\left(\frac{x}{x^2 + 1}\right) \cdot \left\{ \cos\left(\frac{1}{x^2 + 1}\right) + i\,\sin\left(\frac{1}{x^2 + 1}\right) \right\} dx$$

cannot be computed by only using traditional methods from *Real Calculus*.

Since $\dfrac{1}{z - i} \to 0$ for $z \to \infty$, and thus $\exp\left(\dfrac{1}{z - i}\right) \to e^0 = 1$ for $z \to \infty$, it is obvious that there exists an $R > 1$, such that

$$|f(z)| \leq \frac{2}{|z|^2} \qquad \text{for } |z| \geq R,$$

and it follows from Theorem 4.2.1 that

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 1}\,\exp\left(\frac{x + i}{x^2 + 1}\right)dx = 2\pi i \cdot \text{res}\left(\frac{1}{z^2 + 1}\,\exp\left(\frac{1}{z - i}\right); i\right) = 2\pi i\,\text{res}\left(\frac{1}{w^2 + 2iw}\,\exp\left(\frac{1}{w}\right); 0\right),$$

where we have applied the linear transform $w = z - i$.

Obviously, $w_0 = 0$ is an essential singularity of $\dfrac{1}{w^2 + 2iw} \exp\left(\dfrac{1}{w}\right)$, so we shall use *Cauchy multiplication* in the Laurent series expansion from $w_0 = 0$ to find a_{-1} by collecting all coefficients of $\dfrac{1}{w}$. We get for $0 < |w| < 2$,

$$\frac{1}{w} \cdot \frac{1}{2i + w} \cdot \exp\frac{1}{w} = \frac{1}{2i} \cdot \frac{1}{w} \cdot \frac{1}{1 + \dfrac{1}{2i}} \cdot \exp\frac{1}{w} = \frac{1}{2i} \cdot \frac{1}{w} \sum_{m=0}^{+\infty} \left\{-\frac{w}{2i}\right\}^m \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{1}{w^n}.$$

Due to the factor $\dfrac{1}{w}$, the coefficient a_{-1} is given by $\dfrac{1}{2i}$ times the constant term of the product of the two series, i.e. for $m = n$. This gives

$$a_{-1} = \frac{1}{2i} \sum_{n=0}^{+\infty} \frac{1}{n!} \left\{\frac{i}{2}\right\}^n = \frac{1}{2i} \exp\left(\frac{i}{2}\right) = \frac{1}{2i} \left\{\cos\frac{1}{2} + i \sin\frac{1}{2}\right\}.$$

Thus,

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \, \exp\left(\frac{x}{x^2+1}\right) \left\{ \cos\left(\frac{1}{x^2+1}\right) + i \, \sin\left(\frac{1}{x^2+1}\right) \right\} \mathrm{d}x$$

$$= \int_{-\infty}^{+\infty} \frac{1}{x^2+1} \, \exp\left(\frac{x+i}{x^2+1}\right) \mathrm{d}x = 2\pi i \, a_{-1} = \pi \left\{ \cos\frac{1}{2} + i \, \sin\frac{1}{2} \right\}.$$

Finally we get by splitting into the real and imaginary parts,

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \, \exp\left(\frac{x}{x^2+1}\right) \cos\left(\frac{1}{x^2+1}\right) \mathrm{d}x = \pi \cdot \cos\frac{1}{2},$$

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \, \exp\left(\frac{x}{x^2+1}\right) \sin\left(\frac{1}{x^2+1}\right) \mathrm{d}x = \pi \cdot \sin\frac{1}{2}.$$

This example is of course only meant to demonstrate the power of the method, because it is most unlikely that one inn practical applications ever will need to find the exact value of these integrals.

An alternative method of computation is the following:

We adopt from the above the already derived formula

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \, \exp\left(\frac{x+i}{x^2+1}\right) \mathrm{d}x = 2\pi i \cdot \operatorname{res}\left(\frac{1}{w^2+2iw} \, \exp\left(\frac{1}{w}\right); 0\right).$$

The function

$$g(w) = \frac{1}{w^2+2iw} \, \exp\left(\frac{1}{w}\right) = \frac{1}{w(w+2i)} \, \exp\left(\frac{1}{w}\right)$$

is analytic in $\mathbb{C} \setminus \{0, -2i\}$ with only two singularities, so it follows from Theorem 3.8.3 that

$$\int_{-\infty}^{+\infty} \frac{1}{x^2} \, \exp\left(\frac{x+i}{x^2+1}\right) \mathrm{d}x = 2\pi i \cdot \operatorname{res}(g(w); 0) = 2\pi i \{-\operatorname{res}(g(w); -2i) - \operatorname{res}(g(w); \infty)\}.$$

Now $w = -2i$ is a *simple pole*, so we apply Theorem 3.7.2,

$$-\operatorname{res}(g(w); -2i) = -\lim_{z\to 2i} \frac{1}{w} \, \exp\left(\frac{1}{w}\right) = \frac{1}{2i} \, \exp\left(\frac{i}{2}\right).$$

Furthermore, $\lim_{w\to\infty} \exp\left(\dfrac{1}{w}\right) = \exp 0 = 1$, so $w = \infty$ is a zero of order 2 for

$$g(w) = \frac{1}{w^2} \cdot \frac{1}{1 + \dfrac{2i}{w}} \cdot \exp\left(\frac{1}{w}\right),$$

and we get from Theorem 3.8.4 that

$$-\operatorname{res}(g(w); \infty) = 0.$$

Summing up we get

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \, \exp\left(\frac{x+i}{x^2+1}\right) \mathrm{d}x \;=\; 2\pi i \{-\operatorname{res}(g(w); -2i) - \operatorname{res}(g(w); \infty)\}$$

$$=\; \pi \cdot \exp\left(\frac{i}{2}\right) = \pi \left\{ \cos\frac{1}{2} + i \, \sin\frac{1}{2} \right\},$$

and the results follow by taking the real and imaginary parts of this result. $\Diamond$

Example 4.2.1 showed that Theorem 4.2.1 can be successfully applied even in very complicated cases. However, the most important case is of course when $f(z)$ is a *rational function*, because the present method from *Calculus of Residua* is easier to apply than the ordinary decomposition method. We therefore explicitly formulate

Corollary 4.2.1 *Let* $f(z) = \dfrac{P(z)}{Q(z)}$ *be a quotient of two polynomials, where the denominator* $Q(x) \neq 0$
for all $x \in \mathbb{R}$.
If the degree of the denominator $Q(z)$ *is at least 2 larger than the degree of the numerator* $P(z)$, *i.e. if* $f(z)$ *has a zero of at least order 2 at* ∞, *then the improper integral* $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ *exists, and its value is given by (135), i.e.*

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\Im z_j > 0} res(f; z_j).$$

PROOF. Since degree $Q(x)-$ degree $P(z) \geq 2$, we can choose $a = 2$ and then constants $c,\, R > 0$ to fulfil the assumptions of Theorem 134, and the corollary follows. $\square$

Example 4.2.2 We shall find the value of the improper integral

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^4 + 1}.$$

The integrand $\dfrac{1}{x^4 + 1}$ is a rational function, where the denominator $x^4 + 1 \geq 1 > 0$ has degree 4, and the numerator is a constant. Hence, the assumptions of Corollary 4.2.1 are fulfilled.

The zeros of the denominator are $\exp\!\left(\dfrac{i\,p\,\pi}{4}\right)$, $p = 1,\, 3,\, 5,\, 7$, and they are all simple. Hence, the integrand has the same points as *simple poles*, of which only

$$\exp\!\left(\frac{i\pi}{4}\right) = \frac{1}{\sqrt{2}}\,(1+i) \qquad \text{and} \qquad \exp\!\left(\frac{3i\pi}{4}\right) = \frac{1}{\sqrt{2}}\,(-1+i)$$

lie in the upper half plane.

Then apply Theorem 3.7.3 with

$$A(z) = 1, \qquad B(z) = z^4 + 1 \qquad \text{and} \qquad B'(z) = 4z^3,$$

and use that $z_0^4 = -1$ for all poles, to get

$$res(f; z_0) = \frac{A(z_0)}{B'(z_0)} = \frac{1}{4z_0^3} = \frac{z_0}{4z_0^4} = -\frac{1}{4}\,z_0.$$

Finally, by Corollary 4.2.1,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1 + x^4} &= 2\pi i \left\{ \mathrm{res}\!\left(\frac{1}{1 + z^4}; \frac{1+i}{\sqrt{2}}\right) + \mathrm{res}\!\left(\frac{1}{1 + z^4}; \frac{-1+i}{\sqrt{2}}\right) \right\} \\
&= 2\pi i \left\{ -\frac{1}{4} \right\} \left\{ \frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} \right\} = -\frac{2\pi i}{4} \cdot \frac{2i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Alternatively we sketch the following clever *real decomposition*, where where we first notice that

$$x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = \left(x^2 + 1\right)^2 - \left(\sqrt{2}\,x\right)^2 = \left(x^2 + \sqrt{2}\,x + 1\right)\left(x^2 - \sqrt{2}\,x + 1\right).$$

There exist four uniquely determined constants A, B, C, $D \in \mathbb{R}$, such that

$$\frac{1}{x^4 + 1} = \frac{1}{\left(x^2 + \sqrt{2}\,x + 1\right)\left(x^2 - \sqrt{2}\,x + 1\right)} = \frac{Ax + B}{x^2 + \sqrt{2}\,x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}\,x + 1}.$$

We get by some very tedious computations,

$$A = \frac{1}{2\sqrt{2}}, \qquad B = \frac{1}{2}, \qquad C = -\frac{1}{2\sqrt{2}} \qquad \text{and} \qquad D = \frac{1}{2},$$

from which an indefinite integral of $\dfrac{1}{x^4 + 1}$ can be found, and then the value of the improper integral by taking the limits. The details are left to the reader.

A simpler *alternative* method is applying *complex decomposition*, where

$$\frac{1}{z^4 + 1} = \frac{\text{res}(f; z_1)}{z - z_1} + \frac{\text{res}(f; z_2)}{z - z_2} + \frac{\text{res}(f; z_3)}{z - z_3} + \frac{\text{res}(f; z_4)}{z - z_4},$$

and $\text{res}(f; z_j) = -\dfrac{1}{4}\,z_j$ from the above. Then pair the results to get the *real decomposition*, which then is integrated in the usual way. $\Diamond$

Obviously, Example 4.2.2 shows that *residuum formulæ* may be easier to apply than a straightforward real decomposition followed by an integration. Furthermore, Example 4.2.1 showed that we also can compute improper integrals, which could not be found by traditional real methods. However, the reader must be warned. *If not all assumptions of a residuum formula are fulfilled, then it usually give a very wrong result, even if the residuum formula itself makes sense.* We shall illustrate this by the following obvious example, but it is easy to give more subtle examples showing the same phenomena in a latent way.

Example 4.2.3 The rational function $\dfrac{z}{z^2 + 1}$ does *not* fulfil the conditions of Corollary 4.2.1, because the difference between the degrees of the denominator and the numerator is only 1. Clearly, $z = i$ is the only singularity – a simple pole – in the upper half plane, where by Theorem 3.7.3

$$\text{res}\left(\frac{z}{z^2 + 1}z^2 + 1; i\right) = \lim_{z \to i} \frac{z}{2z} = \frac{1}{2}.$$

Then a *false* application of (135) gives

$$\text{``}\int_{-\infty}^{+\infty} \frac{x}{x^2 + 1}\,\mathrm{d}x = 2\pi i \cdot \text{res}\left(\frac{z}{z^2 + 1}; i\right) = \pi i\text{''},$$

which is wrong for several reasons. The indefinite integral is $\dfrac{1}{2}\ln\left(x^2 + 1\right) \to +\infty$ for $x \to +\infty$ or $x \to -\infty$, so the improper integral does not exist. And if it did, its value ought to be real and 0 by the symmetry of the integrand, and not the complex number πi. Therefore, the reader should always be extremely careful to check *all* the assumptions of the applied theorem before using the residuum formula of this theorem. $\Diamond$

We have also the following result of improper integrals, when the integrand is a product of an analytic function and a complex exponential.

Theorem 4.2.2 *Let $f(z)$ be analytic in the open domain Ω, which contains the closed upper half plane $\Im z \geq 0$ with only exception of a finite number of singularities $z_1, \ldots, z_n$, ll lying in the open upper half plane, so $\Im z_j > 0$ for $j = 1, \ldots, n$. Assume that there exist positive constants R, a, $c > 0$, such that*

$$(137) \quad |f(z)| \leq \frac{c}{|z|^a} \qquad for \ \Im z \geq 0 \ and \ |z| \geq R.$$

For every real positive number $m > 0$ the improper integral $\int_{-\infty}^{+\infty} f(x)\, e^{imx}\, dx$ on the real line is convergent of the value

$$(138) \quad \int_{-\infty}^{+\infty} f(x)\, e^{imx}\, dx = 2\pi i \sum_{\Im z_j > 0} res\bigl(f(z)\, e^{imz}; z_j\bigr).$$

We have emphasized the important assumption that the constant m is positive, because otherwise (138) is not true. We shall, however, also deal with negative constants in Corollary 4.2.2 below.
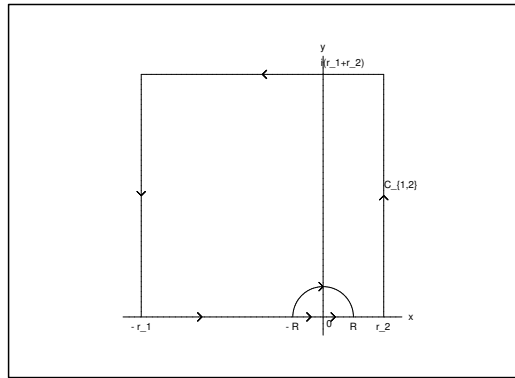
Figure 18: The path of integration in the proof of Theorem 4.2.2.

PROOF. We shall under the given assumptions prove that the limit

$$\lim_{r_1 \to +\infty} \lim_{r_2 \to +\infty} \int_{-r_1}^{r_2} f(x)\, e^{imx}\, \mathrm{d}x$$

exists and is unique, where the two limits are taken independently of each other.

Choose $r_1, r_2 > R$ and the simple closed curve $C_{1,2}$ (the boundary of a square) of Figure 18.

According to (137) all singularities in the upper half plane of the integrand $f(z)\, e^{imz}$ lie inside $C_{1,2}$, hence by *Cauchy's residuum theorem*,

$$(139) \quad \oint_{C_{1,2}} f(z)\, e^{imz}\, \mathrm{d}z = 2\pi i \sum_{j=1}^{n} \operatorname{res}\big(f(z)\, e^{imz}; z_j\big).$$

Clearly, when $r_1, r_2 > R$, then the right hand side of (139) is independent of the choices of r_1 and r_2.

On the other hand, cf. Figure 18,

$$\oint_{C_{1,2}} f(z)\, e^{imz}\, \mathrm{d}z = \int_{-r_1}^{r_2} f(x)\, e^{imx}\, \mathrm{d}x + \int_{0}^{r_1+r_2} f(r_2 + it)\, e^{im(r_2+it)} i\, \mathrm{d}t$$

$$(140) \qquad - \int_{-r_1}^{r_2} f(x + i\{r_1 + r_2\})\, e^{im(x+i\{r_1+r_2\})} \mathrm{d}x - \int_{0}^{r_1+r_2} f(-r_1 + it)\, e^{im(-r_1+it)} i\, \mathrm{d}t.$$

The positive constant $m > 0$ is fixed, so we get the following estimates for $r_1, r_2 > R$,

$$\left| \int_{0}^{r_1+r_2} f(r_2+it)\, e^{im(r_2+it)} i\, \mathrm{d}t \right| \leq \int_{0}^{r_1+r_2} |f(r_2+it)|\, e^{-mt}\, \mathrm{d}t \leq \frac{c}{r_2^a} \int_{0}^{r_1+r_2} e^{-mt}\, \mathrm{d}t$$

$$\leq \frac{c}{m\, r_2^a} \qquad \text{for } r_2 \to +\infty, \text{ because } a > 0,$$

120

and similarly for the upper horizontal line integral,

$$\left| \int_{-r_1}^{r_2} f(x + i\{r_1 + r_2\}) \, e^{im(x + i\{r_1 + r_2\})} \mathrm{d}x \right| \le \int_{r_1}^{r_2} |f(x + i\{r_1 + r_2\})| \, e^{-m(r_1 + r_2)} \mathrm{d}x$$

$$\le e^{-m(r_1 + r_2)} \int_{-r_1}^{r_2} \frac{c}{(r_1 + r_2)^a} \, \mathrm{d}x = c \cdot (r_1 + r_2)^{1-a} \cdot e^{-m(r_1 + r_2)}.$$

Since exponentials dominate polynomials, this tends towards zero if either $r_1 \to +\infty$ or $r_2 \to +\infty$.

Finally,

$$\left| \int_0^{r_1 + r_2} f(-r_1 + it) \, e^{im(-r_1 + it)} i \, \mathrm{d}t \right| \le \int_0^{r_1 + r_2} |f(-r_1 + it)| \, e^{-mt} \, \mathrm{d}t$$

$$\le \frac{c}{r_1^a} \int_0^{r_1 + r_2} e^{-mt} \, \mathrm{d}t \le \frac{c}{m \, r_1^a} \to 0 \qquad \text{for } r_1 \to +\infty.$$

When (140) is inserted into (139), we get by taking the two independent limits $r_1 \to +\infty$ and $r_2 \to +\infty$ that the improper integral $\int_{-\infty}^{+\infty} f(x) \, e^{imx} \, \mathrm{d}x$ exists with the value (138). $\square$

As mentioned above we shall also consider a negative constant m. This will, however, require that the integrand is bounded in the lower half plane, with the exception of in the neighbourhoods of the finitely many singularities.

Corollary 4.2.2 *Residuum formula for the Fourier transform.* *Let $f(z)$ be analytic in the open domain $\Omega = \mathbb{C} \setminus \{z_1, \dots, z_n\}$, where none of the singularities z_j lies on the real axis. Assume that there are positive constants $R, a, c > 0$, such that*

$$(141) \quad |f(z)| < \frac{c}{|z|^a} \qquad \text{for } |z| \ge R.$$

Then

$$(142) \quad \int_{-\infty}^{+\infty} f(x) \, e^{ixy} \, dx = \begin{cases} 2\pi i \sum_{\Im z_j > 0} res\big(f(z) \, e^{izy}; z_j\big) & \text{for } y > 0, \\[2ex] -2\pi i \sum_{\Im z_j < 0} res\big(f(z) \, e^{izy}; z_j\big) & \text{for } y < 0. \end{cases}$$

PROOF. If $y = m > 0$, then (142) follows immediately from Theorem 4.2.2.

If $y = m < 0$, then we must modify the proof of Theorem 4.2.2 by reflecting $C_{1,2}$ with respect to the x-axis and then *change the orientation* of the curve, such that the line segment on the x-axis is traversed from r_2 towards $-r_1$. In the conclusion we must reverse this direction, which causes the change of sign in (142), second line. $\square$

Example 4.2.4 A typical transition function $f(z)$ in the *Theory of Electric Circuits* is given by

$$f(z) = \frac{1}{1 + 2\pi i R C z},$$

where R denotes the resistance, and C the capacity. The corresponding *response function* is given by the *Fourier transform*

$$h(t) := \int_{-\infty}^{+\infty} f(x) \, e^{2\pi i x t} \, \mathrm{d}t = \frac{1}{2\pi i R C} \int_{-\infty}^{+\infty} \frac{1}{x - \dfrac{i}{2\pi R C}} \, e^{i 2\pi x t} \, \mathrm{d}x.$$

The only singularity of the integrand is the simple pole at $z_1 = \dfrac{i}{2\pi RC}$, and it is obvious that there are constants k, $r > \dfrac{1}{2\pi RC}$, such that $|f(z)| < \dfrac{k}{|z|}$ for $|z| > r$.

There is no singularity in the lower half plane, hence

$$h(t) = \int_{-\infty}^{+\infty} f(x)\, e^{2\pi i x t}\, \mathrm{d}x = 0 \qquad \text{for } t < 0.$$

If $t > 0$, then

$$h(t) = \int_{-\infty}^{+\infty} f(x)\, e^{2\pi i x t}\, \mathrm{d}x = \frac{2\pi i}{2\pi i RC}\, \mathrm{res}\left(\frac{e^{i 2\pi z t}}{z - \dfrac{i}{2\pi RC}} ; \frac{i}{2\pi RC} \right) = \frac{1}{RC}\, \exp\left(-\frac{t}{RC} \right). \qquad \Diamond$$

We mention in particular the case, when $f(z)$ is a rational function. The proof is trivial.

Corollary 4.2.3 *Let $P(z)$ and $Q(z)$ be polynomials. If*

1) the denominator $Q(z)$ has no zero on the real axis,

2) the denominator $Q(z)$ is of higher degree than the numerator $P(z)$,

3) the constant m is a real positive number,

then the improper integral $\int_{-\infty}^{+\infty} \dfrac{P(x)}{Q(x)} e^{imx}\, dx$ along the real axis is convergent with its value given by

$$(143) \quad \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\, e^{imx}\, dx = 2\pi i \sum_{\Im z_j > 0} res\!\left(\frac{P(z)}{Q(z)}\, e^{imz}; z_j\right).$$

We get another useful corollary, when we assume that the analytic function $f(z)$ has *real values* on the *real axis*.

Corollary 4.2.4 *Let $f(z)$ be analytic in an open domain Ω containing the* closed *upper half plane $\Im z \geq 0$ with the exception of only a finite number of points $z_1, \ldots, z_n$, none of them lying on the real axis.*
Assume that there are positive constants R, a, $c > 0$, such that (137) holds, i.e.

$$|f(z)| \leq \frac{c}{|z|^a} \qquad for\ \Im z \geq 0\ and\ |z| \geq R.$$

Finally, assume that $f(x) \in \mathbb{R}$ is real *for every $x \in \mathbb{R}$.*
Under the assumptions above the two improper integrals

$$\int_{-\infty}^{+\infty} f(x)\cos(mx)\, dx \qquad and \qquad \int_{-\infty}^{+\infty} f(x)\sin(mx)\, dx$$

exist for every positive *constant $m > 0$, and their values are given by*

$$(144) \quad \int_{-\infty}^{+\infty} f(x)\cos(mx)\, dx = \Re\left\{2\pi i \sum_{\Im z_j > 0} res\big(f(z)\, e^{imz}; z_j\big)\right\}$$

and

$$(145) \quad \int_{-\infty}^{+\infty} f(x)\sin(mx)\, dx = \Im\left\{2\pi i \sum_{\Im z_j > 0} res\big(f(z)\, e^{imz}; z_j\big)\right\}$$

PROOF. The proof is trivial. Just split (138) into its real and imaginary parts, using that $f(x)$ is *real* on the real axis. □

Remark 4.2.1 *Warning!* the value of e.g. $\int_{-\infty}^{+\infty} f(x)\cos(mx)\, \mathrm{d}x$ is *not* given by

$$(146) \quad ``2\pi i \sum_{\Im z_j > 0} res(f(z)\cos(mz)\, \mathrm{d}z; z_j),"$$

which would be natural to expect. By using (146) one makes implicitly the error that one tacitly applies *Euler's formula* $\cos mz = \dfrac{1}{2}\, e^{imz} + \dfrac{1}{2}\, e^{-imz}$ in the integral. In the latter term we have the constant $-m < 0$, violating one of the important assumptions of (138). ◊

Example 4.2.5 The simple poles of the function $\dfrac{z}{z^2+1}\,e^{iz}$ are $\pm i \notin \mathbb{R}$, and $\dfrac{z}{z^2+1} \to 0$ for $z \to \infty$, and $m = 1 > 0$, so the assumptions of Corollary 4.2.4 are satisfied. Thus,

$$\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\,e^{ix}\,\mathrm{d}x = 2\pi i \cdot \mathrm{res}\left(\frac{z}{z^2+1}\,e^{iz}; i\right) = 2\pi i \cdot \frac{i\,e^{i\cdot i}}{i+i} = \frac{\pi i}{e},$$

and we get

$$\int_{-\infty}^{+\infty} \frac{x\cos x}{x^2+1}\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+1}\,\mathrm{d}x = \frac{\pi}{e},$$

where both improper integrals are convergent. $\Diamond$

4.3   Cauchy's principal value

We considered in Section 4.2 improper integrals of the type $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$, where $f(z)$ is analytic in a domain Ω containing $\{z \in \mathbb{C} \mid \Im z \geq 0\} \setminus \{z_1, \ldots, z_n\}$, where none of the singularities $z_1, \ldots, z_n$ lie on the real axis.

We shall in this section modify the concept of integral in such a way that we may allow *simple poles* on the path of integration. We shall in the follows show that we also in this case may obtain meaningful *residuum formulæ*.

Poles of higher order, or essential singularities will not be allowed on the path of integration.

We first introduce

Definition 4.3.1 *Let f be analytic in an open domain Ω containing $\mathbb{R} \setminus \{x_0\}$, where the real number $x_0 \in \mathbb{R}$ is a* simple pole *of f. If the symmetric limit*

$$(147) \quad \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{+\infty} \right\} f(x)\,dx$$

exists, we say that the improper integral of $f(x)$ from $-\infty$ to $+\infty$ has its principal value *given by (147), and we write*

$$(148) \quad pv \int_{-\infty}^{+\infty} f(x)\,dx := \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{+\infty} \right\} f(x)\,dx.$$

The notation "$pv \int_{-\infty}^{+\infty} \cdots \mathrm{d}x$" indicates that there is "something wrong" with the improper integral, though it is not worse than that we obtain convergence, if we remove a small symmetric interval around the simple pole x_0 and then let this symmetric interval shrink towards x_0.

Definition 4.3.1 can clearly be extended to the case, where $f(z)$ has a finite number of simple poles on the real axis. The details are left to the reader.

The line of integration does not have always to be the real axis. Any nice curve with a finite number of simple poles may be treated in a similar way. The notation will of course be the same as the above.

Also, a function may apparently have a simple pole at $x_0 \in \mathbb{R}$, where a closer examination would show that the singularity is removable. One simple example is the function $\dfrac{\sin z}{z}$, where $z = 0$ is a pole of *at most* order 1, and we know already that it is in fact removable. We shall treat the improper integral of this important function in Example 4.3.1. For the time being we only mention that if the principal value exists, and the apparent simple poles actually are removable singularities, then "pv" can be removed,

$$\mathrm{pv} \int_{-\infty}^{+\infty} f(x) \, \mathrm{d}x = \int_{-\infty}^{+\infty} f(x) \, \mathrm{d}x.$$

In the *Real Calculus* Definition 4.3.1 can be formulated more generally. We have here restricted ourselves to analytic functions in a neighbourhood of $\mathbb{R}$ with exception of a finite number of real simple poles, because we want to find the values by using the *Calculus of Residua*.

In order to obtain these residuum formulæ we first prove the following simple lemma.

Lemma 4.3.1 *Let $f(z)$ be analytic in a deleted open disc $B(z_0, R) \setminus \{z_0\}$, and assume that the centre z_0 is a simple pole of f. Denote by*

$$C(\varepsilon) : z = z_0 + c \, e^{i\Theta}, \qquad \Theta \in [0, \pi],$$

a family of semicircles of centre z_0 and radius $\varepsilon \in \,]0, R[$. Then

(149) $\displaystyle \lim_{\varepsilon \to 0+} \int_{C(\varepsilon)} f(z) \, dz = \pi i \, res \, (f; z_0).$

PROOF. We put

$$f(z) = \frac{a}{z - z_0} + g(z),$$

where $g(z)$ is analytic in the whole disc $B(z_0, R)$, and where $a = a_{-1}$ in the Laurent series expansion, hence $a = \mathrm{res}(f; z_0)$. Then for $\varepsilon \in \left] 0, \dfrac{R}{2} \right[$,

$$\int_{C(\varepsilon)} f(z) \, \mathrm{d}z = \int_{C(\varepsilon)} \frac{a}{z - z_0} \, \mathrm{d}z + \int_{C(\varepsilon)} g(z) \, \mathrm{d}z,$$

where we have the estimate

$$\left| \int_{C(\varepsilon)} g(z) \, \mathrm{d}z \right| \leq \sup \left\{ |g(z)| \;\; \middle| \;\; z \in B\left(z_0, \frac{1}{2} R\right) \right\} \cdot \pi \varepsilon \to 0 \qquad \text{for } \varepsilon \to 0+,$$

supplied with the computation

$$\int_{C(\varepsilon)} \frac{a}{z - z_0} \, \mathrm{d}z = a \int_0^\pi \frac{1}{\varepsilon \, e^{i\Theta}} \, i \, \varepsilon \, e^{i\Theta} \, \mathrm{d}\Theta = i\pi a = \pi i \, \mathrm{res}(f; z_0),$$

and the lemma is proved. $\square$

From Lemma 4.3.1 we easily derive the following theorem, which also should be compared with Theorem 4.2.1.

125

Theorem 4.3.1 *Let $f : \Omega \to \mathbb{C}$ be analytic in an open domain Ω which contains a set of the form $\{z \in \mathbb{C} \mid \Im z \geq 0\} \setminus \{z_1, \ldots, z_n\}$.*
Assume that the (finitely many) singularities on the real axis are all simple poles, *and that there are constants $R > 0$, $c > 0$ and $a > 1$, such that*

$$(150) \quad |f(z)| \leq \frac{c}{|z|^a} \qquad \text{for } |z| \geq R \text{ and } \Im z \geq 0.$$

Then Cauchy's principal value vp $\int_{-\infty}^{+\infty} f(x)\, dx$ *is well-defined, and its value is given by*

$$(151) \quad \text{pv} \int_{-\infty}^{+\infty} f(x)\, dx = 2\pi i \sum_{\Im z_j > 0} res(f; z_j) + \pi i \sum_{\Im z_j = 0} res(f; z_j).$$

Roughly speaking, the integration line through a *simple pole* cuts its residuum into two equal halves, giving one half to the upper half plane, and the other half to the lower half plane,

$$\pi i \operatorname{res}(f; z_j) = 2\pi i \left\{ \frac{1}{2} \operatorname{res}(f; z_j) \right\}.$$
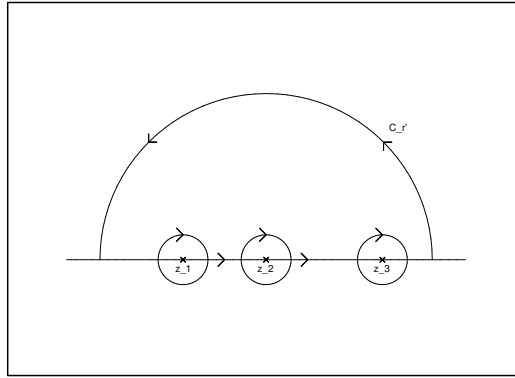
Figure 19: The path of integration in the proof of Theorem 4.3.1.

PROOF. The proof is almost the same as the proof of Theorem 4.2.1. The only modification is that we avoid all the simple poles on the real axis by small semicircles, cf. Figure 19. Using Lemma 4.3.1 and noticing that the semicircles are traversed in the negative of the complex plane we get

$$\text{pv} \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x - \pi i \sum_{\Im z_j = 0} \text{res}(f; z_j) = 2\pi i \sum_{\Im z_j > 0} \text{res}(f; z_j),$$

and (151) follows by a rearrangement. □

Remark 4.3.1 In the proof above we could of course avoid the simple poles on the real axis by small semicircles in the lower half plane. In this case we would get

$$\text{pv} \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x + \pi i \sum_{\Im z_j = 0} \text{res}(f; z_j) = 2\pi i \sum_{\Im z_j \geq 0} \text{res}(f; z_j).$$

Then note that $\Im z_j \geq 0$ and not just $\Im z_j > 0$ in the latter sum, and we obtain again (151) by using a rearrangement. ◊

Obviously, Corollary 4.2.1, Theorem 4.2.2, Corollary 4.2.2, Corollary 4.2.3 and Corollary 4.2.4 can all be extended to the principle value, if there are only a finite number of simple poles on the real axis. In each case we add half the residuum to the solution formula. It is left to the reader to formulate and prove these simple extensions.

Example 4.3.1 *Important!* We shall here compute the important improper integral

$$\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x,$$

which occurs in some applications in the technical sciences. A straight on attack on the corresponding analytic function $\dfrac{\sin z}{z}$ is doomed to failure, because $\left|\dfrac{\sin z}{z}\right|$ does not satisfy an estimate of the type (150). Furthermore, we only integrate along the positive real axis, so the problem apparently does not fit into the present theory.

127

The trick is the following: Apply *Euler's formulæ* to get

$$\int_\varepsilon^R \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2i}\int_\varepsilon^R \left\{\frac{e^{ix}}{x} - \frac{e^{-ix}}{x}\right\}\mathrm{d}x = \frac{1}{2i}\left\{\int_\varepsilon^R - \int_{-\varepsilon}^{-R}\right\}\frac{e^{ix}}{x}\,\mathrm{d}x = \frac{1}{2i}\left\{\int_\varepsilon^R + \int_{-R}^{-\varepsilon}\right\}\frac{e^{ix}}{x}\,\mathrm{d}x.$$

This implies that *if* the right hand side has a well-defined limit for $\varepsilon \to 0+$ and $R \to +\infty$, independently of each other, then the left hand side is also well-defined, and

$$\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2i}\,\mathrm{vp}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x}\,\mathrm{d}x.$$

We shall only prove that $\mathrm{vp}\int_{-\infty}^{+\infty}\dfrac{e^{ixx}}{x}\,\mathrm{d}x$ exists, i.e. we shall check the singularities of the integrand.

The only (finite) singularity of the analytic function $\dfrac{e^{iz}}{z}$ is the simple pole at $z_0 = 0$. It lies on the path of integration, so it contributes to *Cauchy's principal value* with the amount

$$\pi i \,\mathrm{res}\left(\frac{e^{iz}}{z};0\right) = \pi i \lim_{z\to 0} e^{iz} = \pi i.$$

Furthermore, $\dfrac{e^{iz}}{z} = \dfrac{1}{z}\,e^{i1z}$, where $m = 1 > 0$ and $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|^1}$ for $z \neq 0$, so $a = 1 > 0$, and it follows from Theorem 4.2.2 that the limit $R \to +\infty$ will not cause any trouble either. Hence we conclude that the improper integral $\int_0^{+\infty}\dfrac{\sin x}{x}\,\mathrm{d}x$ is convergent, and its value is

$$\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2i}\,\mathrm{pv}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x}\,\mathrm{d}x = \frac{1}{2i}\,\pi i\,\mathrm{res}\left(\frac{e^{iz}}{z};0\right) = \frac{\pi}{2}. \qquad \diamond$$

4.4   The Mellin transform

We shall in this section consider improper integrals of the form

(152) $\displaystyle\int_0^{+\infty} f(x)\,x^a\,\frac{\mathrm{d}x}{x}.$

When these integrals are considered as functions of a, we get the *Mellin transform* $\mathfrak{M}\{f\}(a)$ of the function f. This is closely related to the *two-sided Laplace transform*. In fact, if the integral (152) is absolutely convergent, then we get by the change of variable, $x = e^{-t}$,

(153) $\mathfrak{M}\{f\}(a) := \displaystyle\int_0^{+\infty} f(x)x^a\,\frac{\mathrm{d}x}{x} = \int_{-\infty}^{+\infty} f\left(e^{-t}\right)e^{-at}\,\mathrm{d}t,$

which is recognized as the *two-sided Laplace transform* of the function $g(t) = f(e^{-t})$ at the point a.

We shall not go into the applications of the Mellin transform, or the two-sided Laplace transform. The purpose of this section is only to use *Complex Functions Theory* to compute some integrals of the form (152).

In order not to make the theory too complex we have restricted ourselves to *real* $a \in \mathbb{R}$. It is possible, though far from trivial, also to allow a to be a complex number, but this would require a discussion of

the so-called *many-valued functions*, which we have postponed to *Ventus: Complex Functions Theory a-3*, and even with this discussion, the generalization becomes far from trivial and only of interest of the few.

In order to combine (153) with *Complex Functions Theory* we also must require that $f(z)$ is analytic in an open domain Ω. More precisely, $f(z)$ is analytic in all of $\mathbb{C}$ with the possible exception of only a *finite* number of singularities, none of which is lying on the positive real axis $\mathbb{R}_+$. In the literature one mostly assumes that the singularities are all poles, but except for $z = 0$, which must at most be a pole, the proof below shows that all the other singularities are allowed to be even *essential singularities*.

Then we shall fix the meaning of the factor x^a. If $a \in \mathbb{Z}$, then the definition of x^a is straightforward, and we just apply the methods of Section 4.2. We therefore assume in the following that $a \in \mathbb{R} \setminus \mathbb{Z}$.

The next problem is to *define* the analytic power function z^a for $a \in \mathbb{R} \setminus \mathbb{Z}$. first put

(154) $\mathrm{Log}_0 z := \ln|z| + i\,\mathrm{Arg}_0 z, \qquad \mathrm{Arg}_0 z \in]0, 2\pi[,$

for $z \in \Omega_1 := \mathbb{C} \setminus \{\mathbb{R}_+ \cup \{0\}\}$. Note that Log_0 is *not* the principal logarithm, Log, and Arg_0 is *not* the principal argument, Arg. They of course agree in the open upper half plane, but they are different in the open lower half plane.

Then apply (154) to *define* z^a by the "obvious" formula

(155) $z^a := \exp(a\,\mathrm{Log}_0 z) \qquad \text{for } z \in \Omega_1 = \mathbb{C} \setminus \{\mathbb{R}_+ \cup \{0\}\}.$

Then z^a is a composition of analytic functions in Ω_1, so it is also analytic in Ω_1, and a routine check shows that

$$\frac{d}{dz} z^a = \exp(a\,\mathrm{Log}_0 z) \cdot \frac{a}{z} = a\,z^{a-1} \qquad \text{for } z \in \Omega_1.$$

Furthermore, since a is *real*, it follows from (155) that

(156) $|z^a| = |\exp(a\{\ln r + i\Theta\})| = r^a \in \mathbb{R},$

a formula, which is not true, if a is complex, cf. *Ventus: Complex Functions Theory a-3*.

After these preparations we can formulate

Theorem 4.4.1 *Let f be analytic in $\Omega = \mathbb{C} \setminus \{z_1, \cdots, z_n\}$, where $z_j \notin \mathbb{R}_+$, $j = 1, \ldots, n$. Assume that there exist constants $\alpha, \beta \in \mathbb{R}$, where $\alpha < \beta$, and $C, R_0, r_0 \in \mathbb{R}_+$, such that*

(157) $|z^\alpha f(z)| \le C \qquad \text{for } |z| \le r_0, \quad z \in \Omega,$

(158) $\left|z^\beta f(z)\right| \le C \qquad \text{for } |z| \ge R_0, \quad z \in \Omega.$

Then the improper integral $\int_0^{+\infty} f(x) x^a\, \dfrac{dx}{x}$ is convergent for every $a \in]\alpha, \beta[\setminus \mathbb{Z}$, and its value is given by the residuum formula

(159) $\displaystyle \int_0^{+\infty} f(x) x^a\, \frac{dx}{x} = -\frac{\pi \exp(-\pi i a)}{\sin \pi a} \sum_{z_j \ne 0} res\bigl(f(z) z^{a-1}; z_j\bigr), \qquad \text{for } a \in]\alpha, \beta[\setminus \mathbb{Z}.$
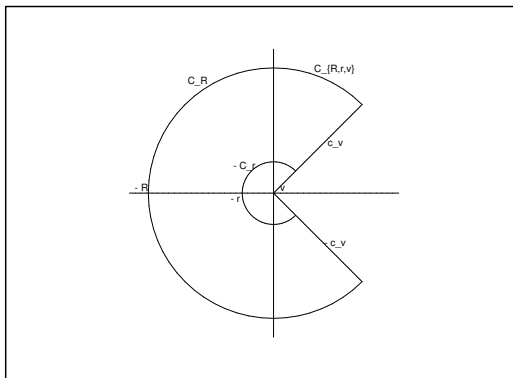
Figure 20: The path of integration in the proof of Theorem 4.4.1.

Remark 4.4.1 One usually assumes that also α, $\beta \in \mathbb{R}_+$. The proof below shows that this is not necessary. Formula (159) does not make sense for $a \in \mathbb{Z}$, because then the denominator $\sin \pi a = 0$. As mentioned earlier we may use the theory of Section 4.2 instead, or a limit process on (159). $\Diamond$

PROOF. We shall as usual find a convenient path of integration $C_{R,r,v}$ in $\Omega' = \Omega \setminus \{\mathbb{R}_+ \cup \{0\}\}$. We choose $R \geq R_0$ and $r \leq r_0$ and $v \in \left]0, \dfrac{\pi}{2}\right[$, from which we define the integration path $C_{R,r,v}$ on Figure 20 composed of two circular arcs of centre 0 and two line segments on the lines through 0 forming the angles $\pm v$ with the positive x-axis.

Assume that $C_{R,r,v}$ has been chosen, such that all singularities $\neq 0$ of $f(z)z^{a-1}$ lie inside $C_{R,r,v}$. Then we get by *Cauchy's residuum formula*

$$(160) \quad 2\pi i \sum_{z_j \neq 0} \mathrm{res}\big(f(z)z^{a-1}; z_j\big) = \oint_{C_{R,r,v}} f(z)z^{a-1}\,\mathrm{d}z = \int_{\tilde{C}_R} + \int_{-\tilde{c}_v} + \int_{-\tilde{C}_r} + \int_{\tilde{c}_v} f(z)z^{a-1}\,\mathrm{d}z,$$

where $\tilde{C}$ denotes a circular arc oriented in the positive sense of the plane seen from 0, and $\tilde{c}$ denotes a line segment oriented in the direction away from 0.

First note that

$$\int_{\tilde{c}_v} f(z)z^{a-1}\,\mathrm{d}z \;=\; \int_r^R f\big(t\,e^{iv}\big)\,e^{(a-1)(\ln t + iv)}\,e^{iv}\,\mathrm{d}t$$

$$=\; \int_r^R f\big(t\,e^{iv}\big)\,e^{(a-1)iv}\cdot e^{iv}\cdot t^{a-1}\,\mathrm{d}t \to \int_r^R f(x)z^{a-1}\,\mathrm{d}x \qquad \text{for } v \to 0+,$$

because the integrand is continuous in the closed bounded interval $[r, R]$, so we are allowed to take the limit under the sign of integration.

Since the segment $\tilde{c}_{-v}$ has the parametric description

$$z(t) = t \cdot e^{i(2\pi - v)}, \qquad t \in [r, R] \text{ and } 2\pi - v \in\,]0, 2\pi[,$$

we get

$$\mathrm{Log}_0\, z(t) = \ln t + i(2\pi - v) \qquad \text{on } \tilde{c}_{-v},$$

hence analogously,

$$\int_{-\tilde{c}_{-v}} f(z) z^{a-1}\,\mathrm{d}z = -\int_{\tilde{c}_{-v}} f(z) z^{a-1}\,\mathrm{d}z = +\int_{r}^{R} f\big(t\,e^{-iv}\big)\,t^{a-1}\,e^{(a-1)(2\pi i - iv)}\,e^{i(2\pi - v)}\,\mathrm{d}t$$

$$= \int_{r}^{R} f\big(t\,e^{-iv}\big)\,t^{a-1}\,e^{ai(2\pi - v)}\,\mathrm{d}t \to \int_{r}^{R} f(x) x^{a-1}\,e^{2\pi i a}\,\mathrm{d}x \qquad \text{for } v \to 0+\,.$$

Thus, for fixed r and R,

$$\lim_{v \to 0+} \left\{ \int_{\tilde{c}_{v}} + \int_{-\tilde{c}_{-v}} \right\} f(z) z^{a-1}\,\mathrm{d}z = \int_{r}^{R} f(x) x^{a-1} \left\{ 1 - e^{2\pi i a} \right\}\,\mathrm{d}x$$

$$= -2i\,e^{i\pi a} \cdot \frac{e^{i\pi a} - e^{-i\pi a}}{2i} \int_{r}^{R} f(x)^{a-1}\,\mathrm{d}x = -2i\,e^{i\pi a} \sin(\pi a) \int_{r}^{R} f(x) x^{a-1}\,\mathrm{d}x.$$

Next we turn to the estimates of the integrals along the two circular arcs $\tilde{C}_R$ and $\tilde{C}_r$. Here we use (156), (157) and (158) to get

$$\left| \int_{\tilde{C}_R} f(z) z^{a-1} \, \mathrm{d}z \right| = \left| \int_{\tilde{C}_R} z^\beta f(z) z^{a-\beta-1} \, \mathrm{d}z \right| \le 2\pi R \cdot C \cdot R^{a-\beta-1}$$

$$= 2\pi C \cdot R^{-(\beta-a)} \to 0 \qquad \text{for } R \to +\infty,$$

and

$$\left| \int_{\tilde{C}_r} f(z) z^{a-1} \, \mathrm{d}z \right| = \left| \int_{\tilde{C}_r} z^\alpha f(z) z^{a-\alpha-1} \, \mathrm{d}z \right| \le 2\pi r \cdot C \cdot r^{a-\alpha-1}$$

$$= 2\pi C \cdot r^{a-\alpha} \to 0 \qquad \text{for } r \to 0+,$$

where both estimates are independent of $v \in \left] 0, \dfrac{\pi}{2} \right[$.

When we apply the three limit processes $v \to 0+$, $r \to 0+$ and $R \to +\infty$, and (160), we conclude that the improper integral $\int_0^{+\infty} f(x) x^{a-1} \, \mathrm{d}x$ is well-defined and that

$$2\pi i \sum_{z_j \neq 0} \operatorname{res}\big(f(z) z^{a-1}; z_j\big) = -2i \, e^{i\pi a} \sin \pi a \int_0^{+\infty} f(x) x^{a-1} \, \mathrm{d}x,$$

and (159) follows by a rearrangement. $\square$

Example 4.4.1 The function $f(z) = \dfrac{1}{z^2+1}$, $z \in \mathbb{C} \setminus \{-i, i\}$, satisfies the estimates

$$\left| z^0 f(z) \right| \le 2 \text{ for } |z| \le r_0, \qquad \left| z^2 f(z) \right| \le 2 \text{ for } |z| \ge R_0,$$

for some constants $0 < r_0 < 1 < R_0$, and none of the simple poles πi lies on $\mathbb{R}_+$. It follows from Theorem 4.4.1 for $a \in \,]\alpha, \beta[\, = \,]0, 2[$ and $a \neq 1$ that

$$\int_0^{+\infty} \frac{z^a}{x^2+1} \cdot \frac{\mathrm{d}x}{x} = -\frac{\pi \, e^{-i\pi a}}{\sin \pi a} \left\{ \operatorname{res}\left(\frac{z^{a-1}}{z^2+1}; i \right) + \operatorname{res}\left(\frac{z^{a-1}}{z^2+1}; -i \right) \right\}$$

$$= -\frac{\pi e^{-i\pi a}}{\sin \pi a} \left\{ \frac{1}{2i} \exp\left((a-1)i\,\frac{\pi}{2} \right) - \frac{1}{2i} \exp\left((a-1)i\,\frac{3\pi}{2} \right) \right\}$$

$$= -\frac{\pi e^{-i\pi a}}{\sin \pi a} \cdot \frac{1}{2} \left\{ \frac{1}{i} \exp\left(i\,a\,\frac{\pi}{2} \right) \cdot (-i) + \frac{1}{-i} \exp\left(i\,a\,\frac{3\pi}{2} \right) \cdot i \right\}$$

$$= \frac{\pi}{\sin \pi a} \cdot \frac{1}{2} \left\{ \exp\left(-i\,a\,\frac{\pi}{2} \right) + \exp\left(i\,a\,\frac{\pi}{2} \right) \right\} = \frac{\pi \cos\left(a\,\dfrac{\pi}{2} \right)}{\sin \pi a} = \frac{\pi}{2 \sin\left(a\,\dfrac{\pi}{2} \right)}.$$

For $a = 1$ we get straightforward

$$\int_0^{+\infty} \frac{x^1}{x^2+1} \frac{\mathrm{d}x}{x} = \int_0^{+\infty} \frac{\mathrm{d}x}{x^2+1} = \frac{\pi}{2} = \lim_{a \to 1} \frac{\pi}{2 \sin\left(a\,\dfrac{\pi}{2} \right)},$$

so we have in general,

$$\int_0^{+\infty} \frac{x^{a-1}}{x^2+1}\, dx = \frac{\pi}{2\sin\left(a\dfrac{\pi}{2}\right)} \qquad \text{for } a \in\,]0,2[. \qquad \diamond$$

4.5   Residuum formulæ for sums of series

It is also possible for a large class of convergent series to compute their sum by using a residuum formula. We first prove

Theorem 4.5.1 *Let $f : \Omega \to \mathbb{C}$ be analytic in $\Omega = \mathbb{C} \setminus \{z_1, \ldots, z_k\}$, where $z_j \notin \mathbb{Z}$ for $j = 1, \ldots, k$. Assume that there exist constants R, $c > 0$ and $a > 1$, such that*

(161) $|f(z)| \le \dfrac{c}{|z|^a} \qquad for\ |z| \ge R.$

Then the series $\sum_{n=-\infty}^{+\infty} f(n)$ is convergent, and is sum is

(162) $\displaystyle \sum_{n=-\infty}^{+\infty} f(n) = -\pi \sum_{j=1}^{k} res(\cot(\pi z) \cdot f(z); z_j).$

PROOF.   Since $a > 1$, and $z_j \notin \mathbb{Z}$ for $j = 1, \ldots, k$, it follows that $\sum_{n=-\infty}^{+\infty} f(n)$ has a convergent majoring series, e.g. $\sum_{-\infty}^{+\infty} \dfrac{c_1}{|n|^a + 1}$ for some constant c_1, so it is itself convergent.

Then introduce the auxiliary function $g(z)$ by

$$g(z) := \pi \cdot \cot(\pi z) f(z).$$

Since $z_j \notin \mathbb{Z}$, $j = 1, \ldots, k$, we see that $\cot(\pi z_j)$ is well-defined for all $j = 1, \ldots, k$. Therefore, the singularities of $g(z)$ are $\{z_1, \ldots, z_k\} \cup \mathbb{Z}$, where the z_j are the singularities of f, and $n \in \mathbb{Z}$ are *simple poles* stemming from $\cot(\pi z)$.

We compute

(163) $\begin{cases} res(g; z_j) &= \pi\, res(\cot(\pi z)f(z); z_j), & j = 1, \ldots, k, \\[3mm] res(g, n) &= \pi \cdot \left[\dfrac{\cos(\pi z)}{\frac{d}{dz}\sin(\pi z)} \right]_{z=1} \cdot f(n) = f(n), & n \in \mathbb{Z}. \end{cases}$

Choose for every $N \in \mathbb{N}$ the path of integration C_N as shown on Figure 21. If $N \ge R$, then the set $\{z_1, \ldots, z_k\}$ lies inside C_N, so we get by *Cauchy's residuum theorem*,

$$\oint_{C_N} g(z)\, dz = 2\pi i \left\{ \sum_{n=-N}^{+N} f(n) + \sum_{j=1}^{k} \pi \cdot res(\cot(\pi z)f(z); z_j) \right\},$$

133
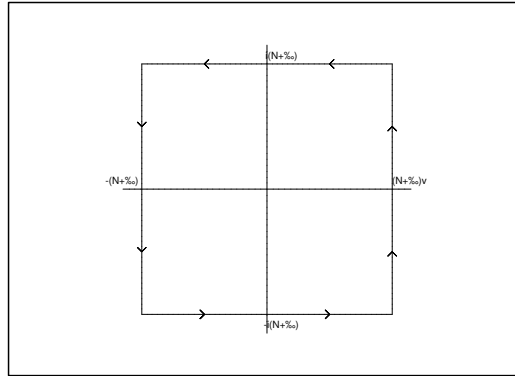
Figure 21: The path of integration C_N in the proof of Theorem 4.5.1.

and therefore by a rearrangement,

$$(164) \quad \sum_{n=-N}^{+N} f(n) = -\pi \sum_{j=1}^{k} \text{res}(f(z) \cot(\pi z); z_j) + \frac{1}{2\pi i} \oint_{C_N} g(z) \, dz.$$

The left hand side of (164) converges for $N \to +\infty$ towards the sum of the series.

Then notice that it follows from a result in *Ventus: Complex Functions Theory a-1* that

$$|\cot(\pi z)|^2 = \left| \frac{\cos(\pi z)}{\sin(\pi z)} \right|^2 = \frac{\cosh^2(\pi y) - \sin^2(\pi x)}{\cosh^2(\pi y) - \cos^2(\pi z)}.$$

Since $x = \pm \left\{ N + \frac{1}{2} \right\}$ on the vertical segments of C_N, we get on these,

$$|\cot(\pi z)|^2 \le \frac{\cosh^2(\pi y)}{\cosh^2(\pi y)} = 1,$$

so $\cot(\pi z)$ are bounded on the vertical segments.

We have $y = \pm \left\{ N + \frac{1}{2} \right\}$ on the horizontal segments of C_N, so

$$|\cot(\pi z)|^2 \le \frac{\cosh^2\left(\pi \left\{ N + \frac{1}{2} \right\} \right)}{\cosh^2\left(\pi \left\{ N + \frac{1}{2} \right\} \right) - 1} = 1 + \frac{1}{\cosh^2\left(\pi \left\{ N + \frac{1}{2} \right\} \right) - 1} \le 2,$$

because $\cosh^2\left(\pi \left\{ N + \frac{1}{2} \right\} \right) > 2$ for $N \in \mathbb{N}_0$.

It follows from the assumption (161) that $z f(z) \to 0$ for $z \to \infty$. We can therefore to every $\varepsilon > 0$ find $N_\varepsilon \in \mathbb{N}$, such that $z f(z)| < \dfrac{\varepsilon}{8}$ for all n, for which $|n| > N_\varepsilon$. Then we get the following estimate for $N > N_\varepsilon$,

$$\left| \frac{1}{2\pi i} \oint_{C_N} g(z) \, dz \right| = \frac{1}{2\pi} \left| \oint_{C_N} z f(z) \cdot \frac{\pi \cot(\pi z)}{z} \, dz \right|$$

$$\le \frac{1}{2\pi} \left\{ \frac{\varepsilon}{8} \cdot \pi \cdot 2 \cdot \frac{1}{N + \frac{1}{2}} \right\} \ell(C_N) = \frac{4(2N+1)}{8 \left(N + \frac{1}{2} \right)} \cdot \varepsilon = \varepsilon,$$

and it follows that

$$\lim_{N \to +\infty} \frac{1}{2\pi i} \oint_{C_N} g(z)\,\mathrm{d}z = 0.$$

The theorem follows by taking the limit $N \to +\infty$ in (164). $\square$

Theorem 4.5.1 is mostly applied when $f(z) = \dfrac{P(z)}{Q(z)}$ is a quotient of two polynomials with a zero of at least order 2 at ∞, i.e. degree $Q-$ degree $P \geq 2$. The remaining necessary requirement is that $Q(n) \neq 0$ for all $n \in \mathbb{Z}$.

Note also that if $z_j \notin \mathbb{Z}$ is a *simple pole* of $f(z)$, then

(165) $\operatorname{res}(\cot(\pi z)f(z); z_j) = \cot(z_j \pi) \cdot \operatorname{res}(f; z_j).$

If in particular $2z_j \in \mathbb{Z}$ is an *odd number*, then z_j becomes a *removable singularity* of $g(z) = \cot(\pi z)f(z)$, so its residuum is 0. This is in agreement with the fact that the right hand side of (165) is 0, because then $\cot(z_j \pi) = 0$. However, be aware that (165) is not true for poles of order ≥ 2.

Example 4.5.1 A simple example is given by

$$f(z) = \frac{1}{\left\{z - \frac{1}{2}\right\}^2} \qquad \text{for } z \in \mathbb{C} \setminus \left\{\frac{1}{2}\right\}.$$

Then $|f(z)| \leq \dfrac{c}{|z|^2}$ for $|z| \geq 1$ and some constant $c > 0$. Since $z_0 = \frac{1}{2} \notin \mathbb{Z}$ is the only pole, we can apply Theorem 4.5.1. Here, $z_0 = \frac{1}{2}$ is a pole of order 2 for $f(z)$, i.e. it is a pole of *at most* order 2 for the auxiliary function

$$\frac{\cot(\pi z)}{\left\{z - \frac{1}{2}\right\}^2}.$$

(The order is of course 1.) If we choose $q = 2$ in Theorem 3.7.1, we get

$$\mathrm{res}\left(\frac{\cot(\pi z)}{\left\{z - \frac{1}{2}\right\}^2}; \frac{1}{2}\right) = \frac{1}{(2-1)!} \lim_{z \to \frac{1}{2}} \frac{d}{dz} \left\{\left\{z - \frac{1}{2}\right\}^2 g(z)\right\} = \lim_{z \to \frac{1}{2}} \frac{d}{dz} \cot(\pi z)$$

$$= \lim_{z \to \frac{1}{2}} \left(-\left\{1 + \cot^2(\pi z)\right\}\pi\right) = -\pi.$$

Then the sum is given by Theorem 4.5.1,

$$\sum_{n=-\infty}^{+\infty} = -\pi \cdot \mathrm{res}\left(\frac{\cot(\pi z)}{\left\{z - \frac{1}{2}\right\}^2}; \frac{1}{2}\right) = \pi^2.$$

Using that

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\left\{n - \frac{1}{2}\right\}^2} = \sum_{n=1}^{+\infty} \frac{1}{\left\{n - \frac{1}{2}\right\}^2} + \sum_{n=0}^{+\infty} \frac{1}{\left\{n + \frac{1}{2}\right\}^2} = 2\sum_{n=0}^{+\infty} \frac{4}{(2n+1)^2} = \pi^2,$$

we have found the sum of the following important series from the *Theory of Fourier series*

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Then also

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} + \frac{1}{2^2}\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} + \frac{1}{\{2^2\}^2}\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} + \cdots$$

$$= \left\{1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right\}\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{1}{1 - \frac{1}{4}} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6},$$

which is also well-known from the *Theory of Fourier series.* $\Diamond$

Example 4.5.2 Let $f(z) = \dfrac{1}{z^2 + a^2}$, where $a > 0$ is a constant. Then $|f(z)| \leq \dfrac{c}{|z|^2}$ for $|z| \geq 2a$ for some constant $c > 0$, and $f(z)$ has only the two *simple poles* $z = \pm i a \notin \mathbb{Z}$, so we can apply (165). First compute the residua,

$$\mathrm{res}\left(\frac{1}{z^2 + a^2}; ia\right) = \frac{1}{2ia} \qquad \text{and} \qquad \mathrm{res}\left(\frac{1}{z^2 + a^2}; -ia\right) = -\frac{1}{2ia}.$$

Then it follows from Theorem 4.5.1 that

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+a^2} = -\pi \left\{ \frac{1}{2ia}\cot(i\pi a) + \frac{1}{-2ia}\cot(-i\pi a) \right\}$$

$$= -\frac{\pi}{ia}\frac{\cos(i\pi a)}{\sin(i\pi a)} = -\frac{\pi}{ia}\cdot\frac{\cosh(\pi a)}{i\cdot\sinh(\pi a)} = \frac{\pi}{a}\cdot\coth(\pi a),$$

so

$$\sum_{n=0}^{+\infty} \frac{1}{n^2+a^2} = \frac{1}{2a^2} + \frac{1}{2}\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a}\coth(\pi a).$$

We get in particular for $a = 1$ that

$$\sum_{n=0}^{+\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2}\coth\pi. \qquad \lozenge$$

Theorem 4.5.2 *Let $f : \Omega \to \mathbb{C}$ be analytic in $\mathbb{C}\setminus\{z_1,\ldots,z_k\}$, where $z_j \notin \mathbb{Z}$ for $j = 1,\ldots,k$. Assume that there exist constants R, $c > 0$ and $a > 1$, such that (161) holds, i.e.*

$$|f(z)| \le \frac{c}{|z|^a} \qquad \text{for } |z| \ge R.$$

Then the series $\sum_{n=-\infty}^{+\infty}(-1)^n f(n)$ is convergent, and its sum is given by the residuum formula

$$(166) \quad \sum_{n=-\infty}^{+\infty} (-1)^n f(n) = -\pi \sum_{j=1}^{k} res\left(\frac{f(z)}{\sin(\pi z)}; z_j \right).$$

PROOF. (Sketch) The proof is trivial modifications of the proof of Theorem 4.5.1, where we replace $\cot(\pi z)$ by $\dfrac{1}{\sin(\pi z)}$. In (163) we here get

$$res\left(\frac{\pi f(z)}{\sin(\pi z)}; n \right) = (-1)^n f(n),$$

so the left hand side of (164) is replaced by $\sum_{n=-N}^{+N}(-1)^n f(n)$. Finally, the estimate of $\dfrac{1}{|\sin(\pi z)|}$ on C_N is much easier than the estimate of $|\cot(\pi z)|$ above. $\square$

Example 4.5.3 Let $f(z) = \dfrac{1}{z^2+a^2}$, where $a > 0$ is a constant. We have already shown in Example 4.5.2 that the assumptions of Theorem 4.5.1, hence also of Theorem 4.5.2, are fulfilled. Hence,

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} = -\pi \left\{ \frac{1}{2ai}\cdot\frac{1}{\sin(i\pi a)} + \frac{-1}{2ai}\cdot\frac{1}{\sin(-i\pi a)} \right\} = -\frac{\pi}{ia}\cdot\frac{1}{\sin(i\pi a)} = \frac{\pi}{a}\cdot\frac{1}{\sinh(\pi a)}.$$

Using that

$$\frac{(-1)^{-n}}{(-n)^2+a^2} = \frac{(-1)^n}{n^2+a^2} \qquad \text{for } n \in \mathbb{Z},$$

we get

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{1}{2a^2} + \frac{1}{2} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \cdot \frac{1}{\sinh(\pi a)}.$$

Then in particular for $a = 1$,

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2 + 1} = \frac{1}{2} + \frac{\pi}{2\sinh \pi}. \qquad \Diamond$$

Finally, it should be mentioned that even if we are now able to find the exact sum of a lot more series than by *Real Calculus* alone, there are still many series that cannot be treated in this way. The simplest example is perhaps

$$\sum_{n=1}^{+\infty} \frac{1}{n^3} \approx 1.202,$$

and more generally, $\sum_{n=1}^{+\infty} \frac{1}{n^{2p+1}}$, $p \in \mathbb{N}$, for which no exact formula is known.

Index

accumulation point, 9, 42

annulus, 73

basic power series, 24

Bessel function, 112

biharmonic equation, 61

binomial coefficient, 25

boundary value problem, 60, 64, 66, 69

calculus of residua, 84

Casorati-Weierstraß's theorem, 87

Cauchy Integral formula, 7

Cauchy Integral Theorem, 7

Cauchy multiplication, 18, 28, 111, 115

Cauchy's inequalities, 23, 29, 31

Cauchy's integral formula, 24, 64, 74

Cauchy's integral theorem, 93, 96

Cauchy's integral theorem for multiply connected domains, 74

Cauchy's principal value, 124

Cauchy's residuum formula, 130

Cauchy's residuum theorem, 97, 99, 113, 120, 133

Cauchy's residuum theorem for unbounded domains, 103

Cauchy-Riemann equations, 7

Cauchy-Riemann's equations, 17, 23, 55-57, 68

causality, 44

circle of convergence, 13

circulation, 67

complex potential, 69

computation of a residuum, 97

conditional convergence, 13

convergence in L^2, 44

convergence in energy, 44

Cybernetics, 7

decomposition, 77, 98, 117, 118

deleted disc, 84

density of charge, 67

difference equation, 31

differential form, closed, 56

differential form, exact, 56, 58

direction, 96

Dirichlet problem, 7

Dirichlet's criterion, 13

disc of convergence, 12

discrete convolution, 18

domain of convergence, 72

elasticity theory, 61

electric circuits, 121

electrostatic elds, 67

essential singularity, 87, 88, 93, 95

essential singularity at ∞, 92

Euler's formulæ, 128

Euler's formula, 123

existence and uniqueness theorem, 29

expansion point, 11

exponential, 33, 36

extended complex plane, 72

field line, 68

flux, 67

Fourier coefficients, 45

Fourier series, 44, 66, 71, 79, 136

Fourier series expansion, 66

Fourier transform, 121

geometric series, 15, 22, 37, 76

Geometry, 7

germ, 26, 31, 33, 41

harmonic conjugation, 56

harmonic function, 55

harmonic functions, 7

heat equation, 69

identity theorem, 20, 35, 37, 42, 101

improper integral, 113

improper integrals, 7

inside a closed curve, 96

inspection, 32

integer part, 13

integrating factor, 32, 39

isolated boundary point at ∞, 89

isolated boundary points, 84

isolated point, 42

Lagrange, 7

Laplace transform, 7