# Complex Functions c-1

Examples concerning Complex Numbers

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# Contents

# Introduction

This is the first book containing examples from the *Theory of Complex Functions*. All the following books will have this book as their background.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
27th May 2008

# 1 The complex numbers

**Example 1.1** *Split a complex fraction into its real and imaginary part.*

Let $a + ib \neq 0$ and $c + id$ be two complex numbers, where $a$, $b$, $c$, $d \in \mathbb{R}$. Since in general,

$$z \cdot \overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2,$$

we get by a multiplication with the complex conjugated of the denominator in both the numerator and the denominator that

$$\frac{c + id}{a + ib} = \frac{c + id}{a + ib} \cdot \frac{a - ib}{a - ib} = \frac{ac + bd}{a^2 + b^2} + i \cdot \frac{ad - bc}{a^2 + b^2},$$

and we immediately split into the real and the imaginary part.

In particular,

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i \cdot \frac{y}{x^2 + y^2} \qquad \text{for } z \neq 0.$$

**Example 1.2** *Write the following complex numbers in the form $x + iy$:*

$(a)$   $(1 + i)^2$,     $(b)$   $\dfrac{3 + 4i}{1 - 2i}$,     $(c)$   $\dfrac{1 + i}{1 - i}$.

**a.** By a small computation,

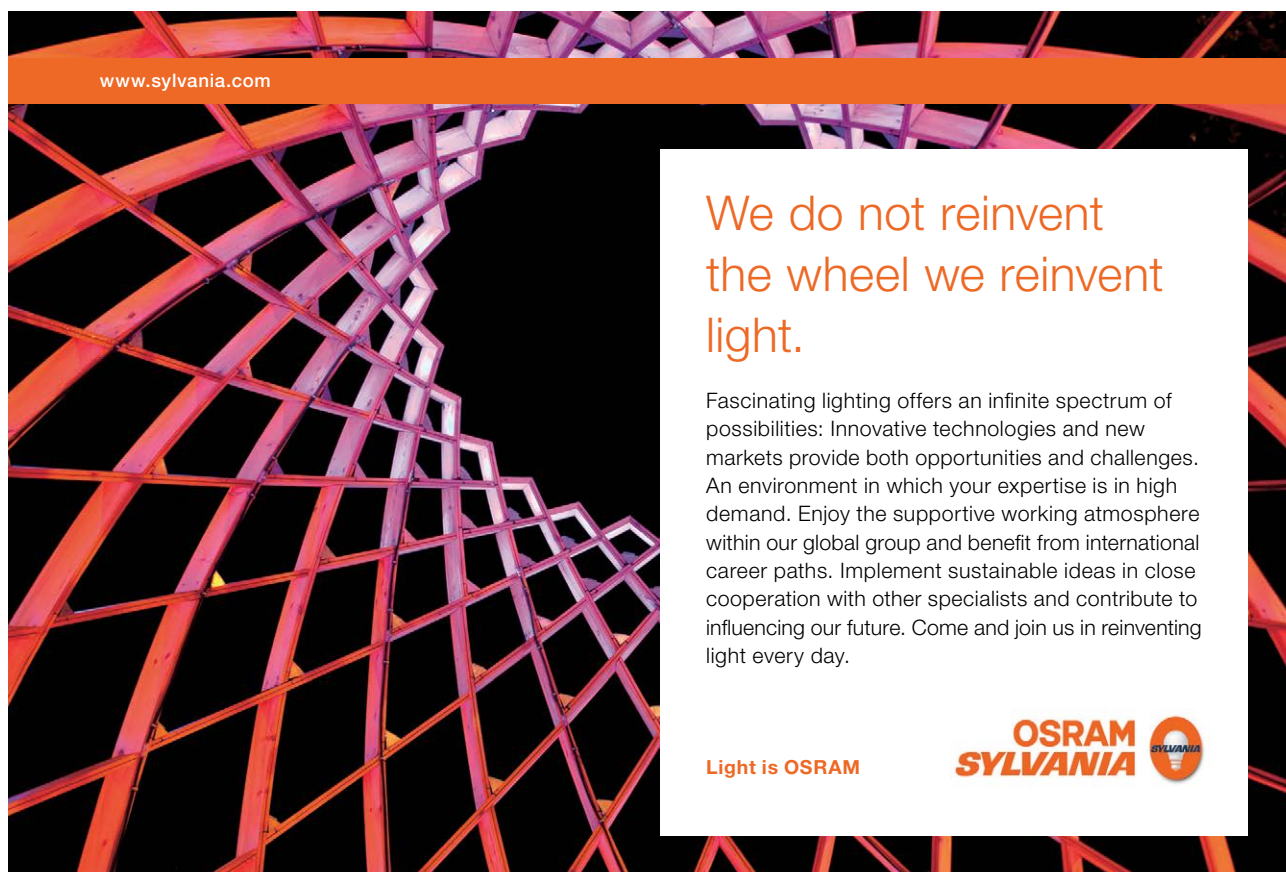$$(1 + i)^2 = 1^2 + i^2 + 2 \cdot 1 \cdot i = 1 - 1 + 2i = 2i.$$

**b.** THE STANDARD METHOD, i.e. a multiplication by the complex conjugated of the denominator in both the numerator and the denominator gives

$$\frac{3 + 4i}{1 - 2i} = \frac{3 + 4i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} = \frac{1}{5} \left\{ 3 - 8 + i(4 + 6) \right\} = \frac{1}{5} \left\{ -5 + 10i \right\} = -1 + 2i.$$

ALTERNATIVELY,

$$3 + 4i = -\{1 - 4 - 2 \cdot 2i\} = -(1 - 2i)^2 = (1 - 2i)(-1 + 2i),$$

which gives by insertion

$$\frac{3 + 4i}{1 - 2i} = \frac{(1 - 2i)(-1 + 2i)}{1 - 2i} = -1 + 2i.$$

**c.** THE STANDARD METHOD:

$$\frac{1 + i}{1 - i} = \frac{1 + i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{1}{2}(1 + i)^2 = \frac{2i}{2} = i.$$

ALTERNATIVELY, apply polar coordinates, because

$$1 + i = \sqrt{2} \exp\left(i \frac{\pi}{4}\right) \qquad \text{and} \qquad 1 - i = \sqrt{2} \exp\left(-i \frac{\pi}{4}\right),$$

hence

$$\frac{1 + i}{1 - i} = \frac{\sqrt{2} \exp\left(i \dfrac{\pi}{4}\right)}{\sqrt{2} \exp\left(-i \dfrac{\pi}{4}\right)} = \exp\left(i \frac{\pi}{2}\right) = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i.$$

**Example 1.3** *Write the following complex numbers in the form $x + iy$:*

$(a)$   $\dfrac{1}{-1 + 3i}$,     $(b)$   $(7 + \pi i)(\pi + i)$,

$(c)$   $(i + 1)(i - 2)(i + 3)$,     $(d)$   $\dfrac{2 + i}{2 - i}$.

**a.** THE STANDARD METHOD,

$$\frac{1}{-1 + 3i} = \frac{1}{-1 + 3i} \cdot \frac{-1 - 3i}{-1 - 3i} = \frac{-1 - 3i}{10} = -\frac{1}{10} - \frac{3}{10} i.$$

**b.** SIMPLE MULTIPLICATION,

$$(7 + \pi i)(\pi + i) = 7\pi - \pi + i\left(\pi^2 + 7\right)?6\pi + i\left(\pi^2 + 7\right).$$

**c.** SIMPLE MULTIPLICATIONS,

$$(i + 1)(i - 2)(i + 3) = \{-1 - 2 + i(-2 + 1)\}(3 + i)$$
$$-(3 + i)(3 + i) = -\{9 - 1 + 6i\} = -8 - 6i.$$

**d.** THE STANDARD METHOD,

$$\frac{2 + i}{2 - i} = \frac{2 + i}{2 - i} \cdot \frac{2 + i}{2 + i} = \frac{1}{5}(4 - 1 + 4i) = \frac{3}{5} + \frac{4}{5}i.$$

**Example 1.4** *Write the following complex numbers in the form $x + iy$:*

$$(a) \quad i^{26} - 3i^7 + i^6\left(1 - i^3\right) - (-i)^{18}, \qquad (b) \quad \frac{(2 + 3i)(-1 + 2i)}{2 + i} - \frac{1 - i}{1 - 2i}.$$

**a.** THE STANDARD METHOD, in which we use that $i^2 = -1$ and $i^4 = 1$, etc.,

$$i^{26} - 3i^7 + i^6\left(1 - i^3\right) - (-i)^{18} = i^2 - 3i^3 + i^2(1 + i) - i^2$$
$$= -1 + 3i - (1 + i) + 1 = -1 + 2i.$$

**b.** THE STANDARD METHOD gives

$$\frac{(2 + 3i)(-1 + 2i)}{2 + i} - \frac{1 - i}{1 - 2i} = \frac{(2 + 3i)(2 + i)i}{2 + i} - \frac{1 - i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i}$$
$$= (2 + 3i)i - \frac{1}{5}(1 + 2 + i\{-1 + 2\}) = -3 + 2i - \frac{3}{4} - \frac{1}{5}i = -\frac{18}{5} + \frac{9}{5}i.$$

**Example 1.5** *Write the following complex numbers in the form $x + iy$:*

$$(a) \quad (2 + 3i) + (5 - 2i), \qquad\qquad (b) \quad (1 - i)(2 + i),$$
$$(c) \quad \frac{1 - i}{3 + i}. \qquad\qquad\qquad\qquad (d) \quad \frac{i}{1 + i} + \frac{1 + i}{i}.$$

**a.** TRIVIAL,

$$(2 + 3i) + (5 - 2i) = 7 + i.$$

**b.** STANDARD MULTIPLICATION,

$$(1 - i)(2 + i) = 2 + 1 + i(-2 + 1) = 3 - i.$$

**c.** MULTIPLY THE NUMERATOR AND THE DENOMINATOR BY THE CONJUGATED OF THE DENOMINATOR,

$$\frac{1-i}{3+i} = \frac{1-i}{3+i} \cdot \frac{3-i}{3-i} = \frac{3-1-4i}{10} = \frac{1}{5} - \frac{2}{5}\, i.$$

**d.** MULTIPLY THE NUMERATOR AND THE DENOMINATOR BY THE CONJUGATED OF THE DENOMINATOR,

$$\frac{i}{1+i} + \frac{1+i}{i} = \frac{i}{1+i} \cdot \frac{1-i}{1-i} + 1 - i = \frac{i+1}{2} + 1 - i = \frac{3}{2} - \frac{1}{2}\, i.$$

**Example 1.6** *Prove that*

$$\left| \frac{(3-4i)(2+i)}{(2-4i)(6+8)} \right| = \frac{1}{4}.$$

We show three methods, of which the first one is recommended.

1) THE DIRECT METHOD. The simplest method is to take the absolute value separately of each factor:

$$\left| \frac{(3-4i)(2+i)}{(2-4i)(6+8)} \right| = \frac{|3-4i| \cdot |2+i|}{2|1-2i| \cdot 2|3+4i|} = \frac{1}{4} \cdot \frac{\sqrt{3^2+4^2} \cdot \sqrt{2^2+1^2}}{\sqrt{1^2+2^2} \cdot \sqrt{3^2+4^2}} = \frac{1}{4}.$$

2) ALTERNATIVELY, though less convenient we first compute the product,

$$
\begin{aligned}
\frac{(3-4i)(2+i)}{(2-4i)(6+8i)} &= \frac{6+4+i(3-8)}{12+32+i(-24+16)} = \frac{10-5i}{44-8i} = \frac{10-5i}{44-8i} \cdot \frac{44+8i}{44+8i} \\
&= \frac{440+40+i(-220+80)}{1936+64} = \frac{480-140i}{2000} = \frac{24-7i}{100},
\end{aligned}
$$

hence

$$\left| \frac{(3-4i)(2+i)}{(2-4i)(6+8i)} \right| = \left| \frac{24-7i}{100} \right| = \frac{\sqrt{24^2+7^2}}{100} = \frac{\sqrt{576+49}}{100} = \frac{\sqrt{625}}{100} = \frac{25}{100} = \frac{1}{4}.$$

3) ALTERNATIVELY we also have the following variant of **2.**,

$$\frac{(3-4i)(2+i)}{(2-4i)(6+8i)} = \frac{6+4+i(3-8)}{12+32+i(-24+16)} = \frac{10-5i}{44-8i},$$

and then we proceed in the following way,

$$\left| \frac{(3-4i)(2+i)}{(2-4i)(6+8i)} \right| = \frac{|10-5i|}{|44-8i|} = \frac{5|2-i|}{4|11-2i|} = \frac{5\sqrt{4+1}}{4\sqrt{121+4}} = \frac{5\sqrt{5}}{4\sqrt{125}} = \frac{5\sqrt{5}}{4 \cdot 5\sqrt{5}} = \frac{1}{4}.$$

**Example 1.7** *Compute* $P(1+i)$, *where*

$$P(z) = z^5 + 2i\, z^3 - z.$$

Here we suggest two solutions, of which the former is the most obvious, which that latter which is recommended is much easier.

1) THE OBVIOUS SOLUTION. Using the binomial formula we get

$$
\begin{aligned}
P(1+i) &= (1+i)^5 + 2i(1+i)^3 - (1+i) \\
&= \left\{1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5\right\} + 2i\left\{1 + 3i + 3i^2 + i^3\right\} - 1 - i \\
&= 1 - 10 + 5 + i(5 - 10 + 1) + 2i(1 - 3 + i\{3 - 1\}) - 1 - i \\
&= -4 - 4i - 4i - 4 - 1 - i = -9 - 9i.
\end{aligned}
$$

2) ALTERNATIVELY the computations become much easier, if we note that

$$(1+i)^2 = 2i.$$

Then

$$P(1+i) = (1+i)\left\{(2i)^2 + 2i \cdot 2i - 1\right\} = (-4 - 4 - 1)(1+i) = -9 - 9i.$$

**Example 1.8** *We write as usual* $z = x + iy$. *Split the following expressions into their real and imaginary part:*

(a)     $z^3$,                        (b)     $z \cdot \overline{z}$,

(c)     $\dfrac{\overline{z}}{z}$,     $z \neq 0$,          (d)     $\dfrac{z - i}{1 - \overline{z}\,i}$,     $z \neq i$.

**a.** By computing

$$z^3 = (x + iy)^3 = x^3 + 3x^2 \cdot iy + 3x \cdot (iy)^2 + (iy)^3 = x^3 - 3xy^2 + i\left(3x^2 y - y^3\right).$$

it follows immediately that

$$\operatorname{Re}\left\{z^3\right\} = x^3 - 3xy^2 \qquad \text{og} \qquad \operatorname{Im}\left\{x^3\right\} = 3x^2 y - y^3.$$

**b.** It follows from

$$z \cdot \overline{z} = |z|^2 = x^2 + y^2,$$

that

$$\operatorname{Re}\left\{z \cdot \overline{z}\right\} = x^2 + y^2 \qquad \text{og} \qquad \operatorname{Im}\left\{z \cdot \overline{z}\right\} = 0.$$

**c.** By a standard computation we first get

$$\frac{\overline{z}}{z} = \frac{x - iy}{x + iy} = \frac{(x - iy)^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{2xy}{x^2 + y^2} \cdot i.$$

Then it immediately follows that

$$\operatorname{Re}\left\{\frac{\overline{z}}{z}\right\} = \frac{x^2 - y^2}{x^2 + y^2} \qquad \text{og} \qquad \operatorname{Im}\left\{\frac{\overline{z}}{z}\right\} = -\frac{2xy}{x^2 + y^2}.$$

**d.** We get by the standard procedure (multiplication of the numerator and the denominator by the conjugated of the denominator)

$$
\begin{aligned}
\frac{z - i}{1 - \overline{z}\,i} &= \frac{x + iy - i}{1 - (x - iy)i} = \frac{x + i(y - 1)}{(-y + 1) - ix} \cdot \frac{(-y + 1) + ix}{(-y + 1) + ix} \\
&= \frac{x(-y + 1) - x(y - 1) + i\{-(y - 1)^2 + x^2\}}{x^2 + (y - 1)^2},
\end{aligned}
$$

hence

$$\operatorname{Re}\left\{\frac{z - i}{1 - \overline{z}\,i}\right\} = -\frac{2x(y - 1)}{x^2 + (y - 1)^2}, \qquad \operatorname{Im}\left\{\frac{z - i}{1 - \overline{z}\,i}\right\} = \frac{x^2 - (y - 1)^2}{x^2 + (y - 1)^2}.$$

**Example 1.9** *Express the following by means of $x$ and $y$:*

(a)    $|z-1|^2,$                                (b)   $|z|^4,$

(c)    $\left|\dfrac{z+1}{z-1}\right|, \quad z \neq 1,$         (d)        $\left|\dfrac{z+i}{1-iz}\right|, \quad z \neq -i.$

**a.** A direct computation gives

$$|z-1|^2 = |(x-1)+iy|^2 = (x-1)^2 + y^2 \quad \left(= x^2 + y^2 - 2x + 1\right).$$

**b.** Again by direct computation,

$$|z|^4 = \left(|z|^2\right)^2 = \left(x^2 + y^2\right)^2 \quad \left(= x^4 + 2x^2 y^2 + y^4\right).$$

**c.** And once more direct computation (for $z \neq 1$),

$$\left|\frac{z+1}{z-1}\right| = \left|\frac{x+1+iy}{x-1+iy}\right| = \sqrt{\frac{(x+1)^2 + y^2}{(x-1)^2 + y^2}} \quad \left(= \sqrt{\frac{x^2 + y^2 - 2x + 1}{x^2 + y^2 - 2x + 1}}\right).$$

**d.** It follows from

$$\frac{z+i}{1-iz} = \frac{i(z+i)}{i(1-iz)} = \frac{i(z+i)}{z+i} = i \qquad \text{for } z \neq -i,$$

that

$$\left|\frac{z+i}{1-iz}\right| = |i| = 1.$$

ALTERNATIVELY, put $z = x + iy$. Then we get for $z \neq -i$,

$$\left|\frac{z+i}{1-iz}\right| = \left|\frac{x+i(y+1)}{1-i(x+iy)}\right| = \left|\frac{x+i(y+1)}{(1+y)-ix}\right| = \frac{\sqrt{x^2+(y+1)^2}}{\sqrt{(1+y)^2+x^2}} = 1,$$

where the latter computation may be performed in different ways.

**Example 1.10** *Let $z \in \mathbb{C} \setminus \{0\}$. Prove that $\mathrm{Re}\left\{\dfrac{1}{z}\right\} > 0$, if and only if $\mathrm{Re}\{z\} > 0$.*

We shall give two methods of solution:

1) ANALYTICALLY: We get for $z \neq 0$,

$$\frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2},$$

hence

$$\mathrm{Re}\left\{\frac{1}{z}\right\} = \frac{z}{x^2 + y^2} > 0, \quad \text{if and only if} \quad x = \mathrm{Re}\{z\} > 0.$$

2) POLAR COORDINATES (and geometrically). If we put $z = r\,, e^{i\theta}$, then $\dfrac{1}{z} = \dfrac{1}{r}\, e^{-i\theta}$, hence

$$\operatorname{Re}\{z\} = r\cos\theta \qquad \text{and} \qquad \operatorname{Re}\left\{\frac{1}{z}\right\} = \frac{1}{r}\cos(-\theta) = \frac{1}{r}\cos\theta,$$

and it follows that $\operatorname{Re}\{z\}$ and $\operatorname{Re}\left\{\dfrac{1}{z}\right\}$ have the same sign for $z \neq 0$.

**Example 1.11** Let $\dfrac{x - iy}{x + iy} = a + ib$, where $x, y, a, b \in \mathbb{R}$, $x^2 + y^2 \neq 0$. Vis, at $a^2 + b^2 = 1$.

We have two main variants:

1) The simplest variant is the following,

$$a^2 + b^2 = |a + ib|^2 = \frac{|x - iy|^2}{|x + iy|^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1, \qquad z \neq 0.$$

2) ALTERNATIVELY, we get for $z \neq 0$ that

$$\begin{aligned}
a^2 + b^2 &= \left|\frac{x - iy}{x + iy}\right|^2 = \left|\frac{x - iy}{x + iy} \cdot \frac{x - iy}{x - iy}\right|^2 = \left|\frac{(x - iy)^2}{x^2 + y^2}\right|^2 = \frac{\left|x^2 - y^2 - 2ixy\right|^2}{\left(x^2 + y^2\right)^2} \\
&= \frac{\left(x^2 - y^2\right)^2 + 4x^2y^2}{x^4 + y^4 + 2x^2y^2} = \frac{x^4 + y^4 - 2x^2y^2 + 4x^2y^2}{x^4 + y^4 + 2x^2y^2} = \frac{x^4 + y^4 + 2x^2y^2}{x^4 + y^4 + 2x^2y^2} = 1.
\end{aligned}$$

**Example 1.12** Let $a$ and $b$ be complex numbers, for which $\overline{b}\,a + \overline{a} \neq 0$ for every $z \in \mathbb{C}$, where $|z| = 1$. Prove that, if $|z| = 1$, then

$$\left|\frac{az + b}{\overline{b}\,z + \overline{a}}\right| = 1.$$

The condition

$$\overline{b}\,z + \overline{a} \neq 0 \qquad \text{for every } z \in \mathbb{C}, \text{ for which } |z| = 1,$$

only means that $|b| \neq |a|$.
If $|z| = 1$, then

$$|z|^2 = z \cdot \overline{z} = 1,$$

hence

$$z = \frac{1}{\overline{z}} \qquad \text{and} \qquad |\overline{z}| = 1,$$

and thus

$$\left|\frac{az+b}{\overline{b}\,z+\overline{a}}\right| = \left|\frac{az+b}{\overline{b}\cdot\dfrac{1}{\overline{z}}+\overline{a}}\right| = \frac{|az+b|}{\left|\dfrac{\overline{a}\,\overline{z}+\overline{b}}{\overline{z}}\right|} = |\overline{z}|\cdot\frac{|az+b|}{|az+b|} = 1.$$

ALTERNATIVELY, put $z = e^{i\theta}$, thus $|z| = 1$. Then

$$\left|\frac{az+b}{\overline{b}\,a+\overline{a}}\right| = \left|\frac{a\,e^{i\theta}+b}{\overline{b}\,e^{i\theta}+\overline{a}}\right| = \left|\frac{a\,e^{i\theta}+b}{\left(\overline{a}\,e^{-i\theta}+\overline{b}\right)e^{i\theta}}\right| = \frac{|a\,e^{i\theta}+b|}{\left|a\,e^{i\theta}+b\right|\cdot|e^{i\theta}|} = 1.$$

**Remark 1.1** This result shows that if $|a| \neq |b|$, then the function

$$f(z) = \frac{az+b}{\overline{b}\,z+\overline{a}}$$
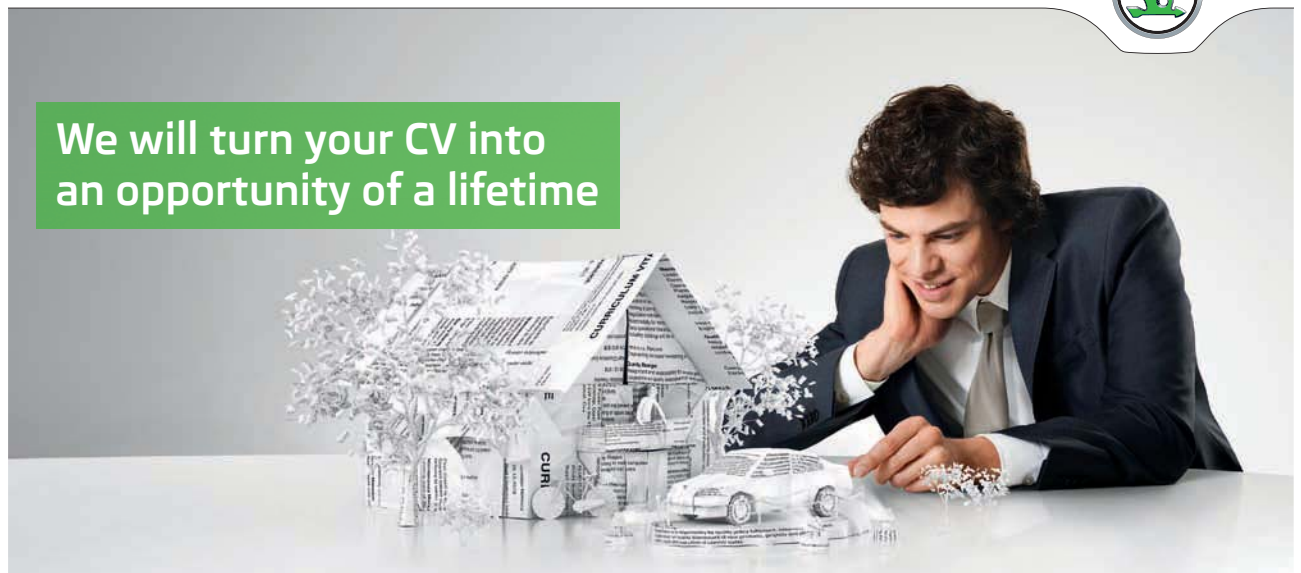
maps the unit circle into itself. $\Diamond$

**Example 1.13** *Find the maximum of $\left|z^2 + 1\right|$ on the unit disc*

$$\{z \in \mathbb{C} \mid |z| \leq 1\}.$$

We shall give two solution of which the former is in line with the *Theory of Complex Functions.* In the latter method we shall only apply real methods, which in general cannot be recommended here.

1) THE SIMPLE SOLUTION. Since

$$\left|z^2 + 1\right| \leq |z|^2 + 1 \leq 2 \qquad \text{for } |z| \leq 1,$$

the maximum must be $\leq 2$.
On the other hand we obtain the value 2 at the points $z = \pm 1$ in the closed unit disc, and we conclude that the maximum is indeed 2.

2) ALTERNATIVELY, apply the known real methods. Put

$$
\begin{aligned}
\varphi(z) &= \left|z^2 + 1\right|^2 = \left|x^2 - y^2 + 1 + 2ixy\right|^2 = \left(x^2 - y^2 + 1\right)^2 + 4x^2y^2 \\
&= x^4 + y^4 + 1 - 2x^2y^2 + 2x^2 - 2y^2 + 4x^2y^2 \\
&= x^4 + y^4 + 2x^2y^2 + 2x^2 - 2y^2 + 1.
\end{aligned}
$$

Then we find the maximum of $\left|z^2 + 1\right|$ by first finding the maximum of $\varphi(z) = \left|z^2 + 1\right|^2$.
The possible stationary points are the solutions of the following system of two equations,

$$
\begin{aligned}
\frac{\partial \varphi}{\partial x} &= 4x^3 + 4xy^2 + 4x = 4x\left(x^2 + y^2 + 1\right) = 0, \\[2mm]
\frac{\partial \varphi}{\partial y} &= 4y^3 + 4x^2y - 4y = 4y\left(x^2 + y^2 - 1\right) = 0.
\end{aligned}
$$

It follows from the former equation that $x = 0$, so by insertion into the latter equation we get the possibilities $y = 0$ and $y^2 - 1 = 0$, thus $y = -1,\ 0,\ 1$.
Hence the stationary points are

$$(x, y) = (0, -1) \sim -i, \qquad (x, y) = (0, 0) \sim 0, \qquad (x, y) = (0, 1) \sim i,$$

corresponding to

$$\varphi(-i) = 0, \qquad \varphi(0) = 1, \qquad \varphi(i) = 0.$$

We shall still go through the values on the BOUNDARY, thus we put $x^2 + y^2 = 1$. First we reduce

$$
\begin{aligned}
\varphi(z) &= x^4 + y^4 + 2x^2y^2 + 2x^2 - 2y^2 + 1 \\
&= \left(x^2 + y^2\right)^2 + 2\left(x^2 + y^2\right) + 1 - 4y^2 \\
&= 1 + 2 + 1 - 4y^2 = 4\left(1 - y^2\right), \qquad \text{for } y \in [-1, 1].
\end{aligned}
$$

Clearly, the maximum on the boundary is obtained for $y = 0$, corresponding to $z = x = \pm 1$. Then

$$\varphi(\pm 1) = 4(1 - 0) = 4,$$

and we conclude that the maximum of $\left|z^2 + 1\right|$ on the closed unit disc is $\sqrt{4} = 2$.

**Example 1.14** *It is well-known that the function $f(z) = \overline{z}$ describes a reflection with respect to the x-axis. Find a corresponding function $g(z)$, which describes a reflection with respect to the y-axis.*

Since

$$g(z) = g(x + iy) = -x + iy = -(x - iy) = -\overline{z},$$

the map is given by

$$g(z) = -\overline{z}.$$

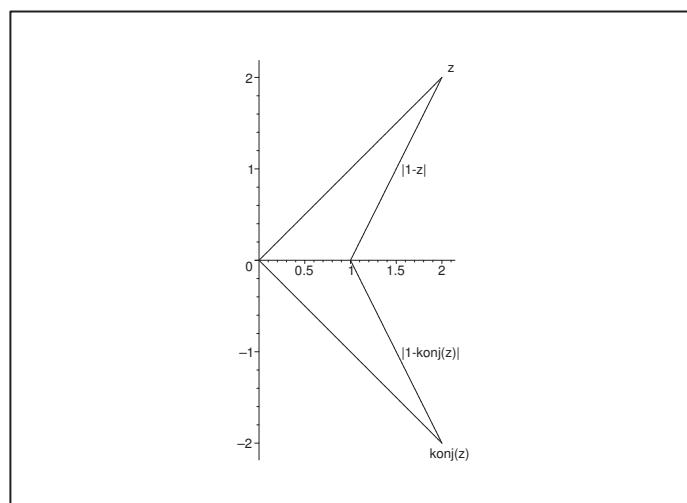**Example 1.15** *Prove that $|1 - z| = |1 - \overline{z}|$, and give a geometric interpretation of the result.*

If we put $z = x + iy$, then

$$|1 - z| = |1 - x - iy| = \sqrt{(1 - x)^2 + y^2},$$

and

$$|1 - \overline{z}| = |1 - x + iy| = \sqrt{(1 - x)^2 + y^2},$$

hence $|1 - z| = |1 - \overline{z}|$.



Since $|1 - z| = |z - 1|$, we can also write the equation in the form

$$|z - 1| = |\overline{z} - 1|.$$

The interpretation of this equation is that the distances from 1 to $z$ and to $\overline{z}$, resp. are the same.

**Example 1.16** *Sketch the set of points in $\mathbb{C}$, for which*

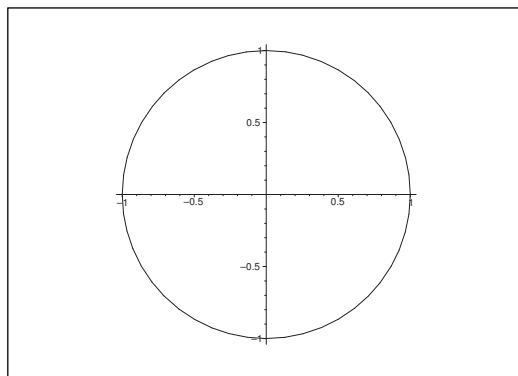(a) $|z| = 1$,    (b) $|z| < 1$,    (c) $|z - 1| = 1$,    (d) $|z - 1| \geq 1$.



Figure 1: (a) The point set described by $|z| = 1$ is the unit circle.
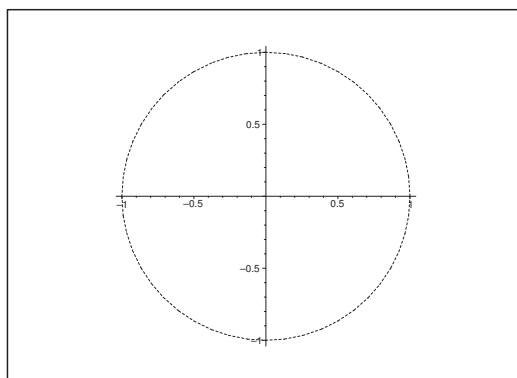
Figure 2: (b) The point set described by $|z| < 1$ is the open unit disc.
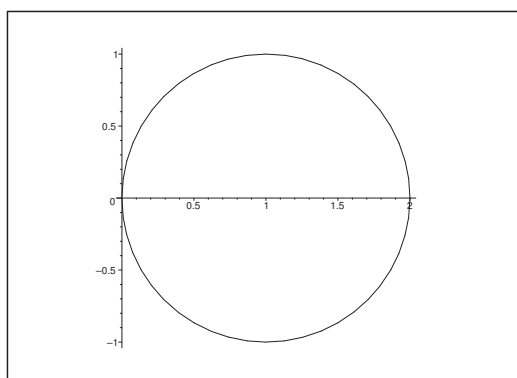


Figure 3: (c) The point set described by $|z-1| = 1$ is the circle with centre at $1 \sim (1,0)$ and radius 1.

**Example 1.17** *Sketch the set of points in $\mathbb{C}$, for which*

    $(a)$ $\operatorname{Arg} z = \dfrac{\pi}{4},$     $(b)$ $\operatorname{Re} z = 1,$     $(c)$ $\operatorname{Im} z = -1,$     $(d)$ $\operatorname{Re}(z-1) = |z|.$

**(a)** The point set is the open half line from 0 (0 excluded) in the first quadrant of the angle $\dfrac{\pi}{4}$ with the $x$-axis.

**(b)** The point set is the straight line through the point 1 on the $x$-axis which is parallel with the $y$-axis.

**(c)** The point set is the straight line through the point $-1 \sim -i$ on the $y$-axis and parallel with the $x$-axis.

**(d)** The solution set is empty. In fact,

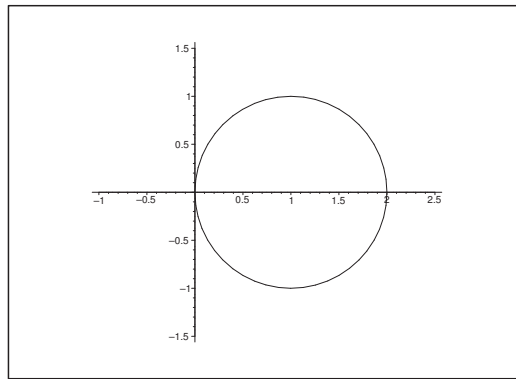    $0 \leq |z| = \operatorname{Re}(z-1) = x - 1$

Figure 4: (c) The point set described by $|z - 1| \geq 1$ is the closed complementary set of the disc with centre at $1 \sim (1, 0)$ and radius 1.
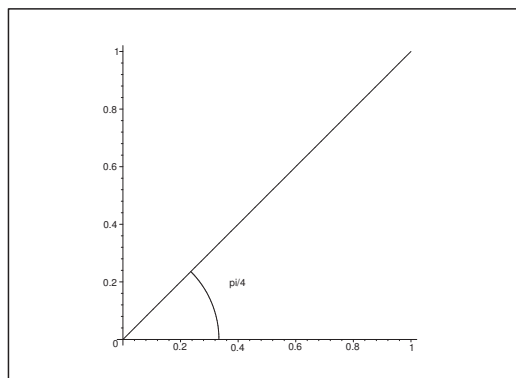


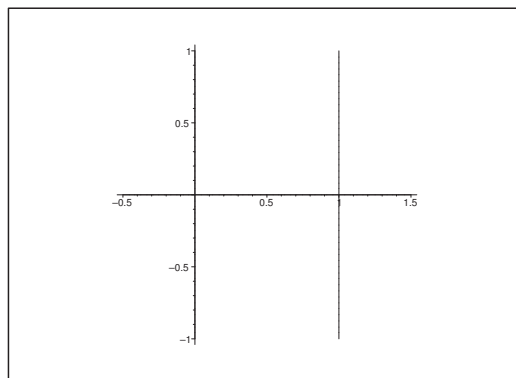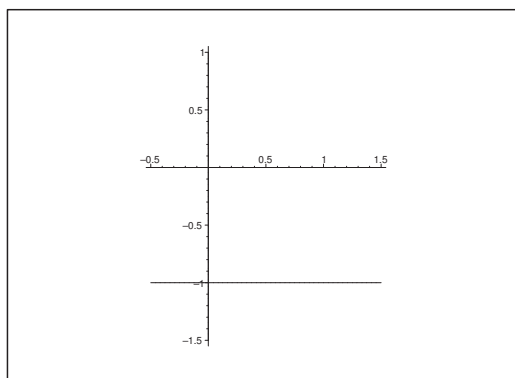Figure 5: (a) The point set described by $\operatorname{Arg} z = \dfrac{\pi}{4}$.



Figure 6: (b) The point set described by $\operatorname{Re} z = 1$.

Figure 7: (c) The point set described by $\operatorname{Im} z = -1$.

implies that $x \geq 1$, and

$$x - 1 = |z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |x| = x,$$

which again implies that $-1 \geq 0$, and that is not possible.

**Example 1.18** *Sketch the set of points in $\mathbb{C}$, for which*

(a) $Re\, z > 0$,   (b) $a < Im\, z < b$,   (c) $Re\, \dfrac{1}{z} = \dfrac{1}{R}$,   (d) $\alpha \leq \operatorname{Arg} z \leq \beta$,

*where $a$, $b$, $\alpha$, $\beta$, $R$ are real constants, satisfying*

$a < b$,       $-\pi < \alpha < \beta \leq \pi$,       $R \neq 0$.
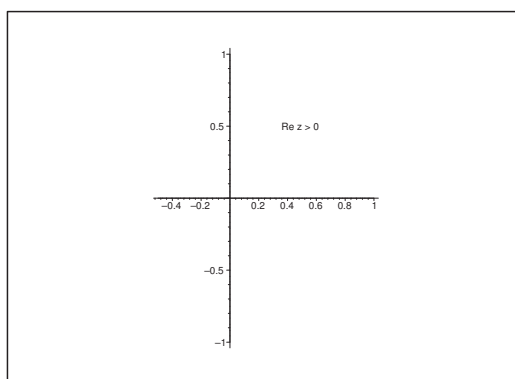


Figure 8: (a) The point set described by $\operatorname{Re} z > 0$.

**(a)** The point set described by $\operatorname{Re} z > 0$ is the open right hand half plane.
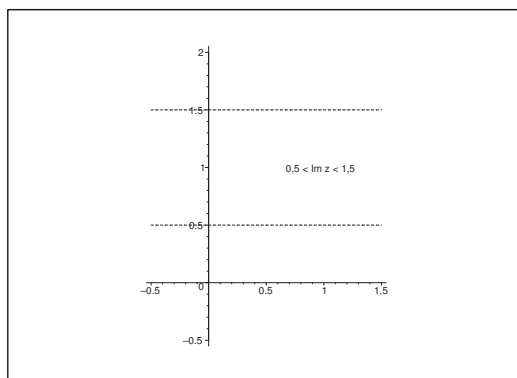
Figure 9: (b) The point set described by $a < \operatorname{Im} z < b$, where $a = 0.5$ and $b = 1.5$.
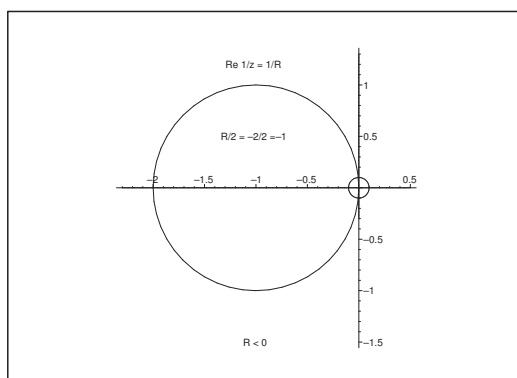


Figure 10: (c) The point set described by $\operatorname{Re} \dfrac{1}{z} = \dfrac{1}{R}$ for $R = -2$.

**(b)** The point set described by $a < \operatorname{Im} z < b$ is the open parallel strip between the lines $y = a$ and $y = b$.

**(c)** The condition $\operatorname{Re} \dfrac{1}{z} = \dfrac{1}{R}$, $R \neq 0$, is in real coordinates given by

$$\frac{x}{x^2 + y^2} = \frac{1}{R}, \qquad (x, y) \neq (0, 0),$$

thus

$$x^2 - Rx + \left(\frac{R}{2}\right)^2 + y^2 = \left|\frac{R}{2}\right|^2,$$

which we write in the standard form,

$$\left(x - \frac{R}{2}\right)^2 + y^2 = \left|\frac{R}{2}\right|^2, \qquad (x, y) \neq (0, 0).$$

This standard form shows that the point set described by $\operatorname{Re}\dfrac{1}{z} = \dfrac{1}{R}$, $R \neq 0$, is the the circle of centrum at $\left(\dfrac{R}{2}, 0\right)$ and radius $\left|\dfrac{R}{2}\right|$ with the exception of point $(0,0)$. It follows that the solution set lies in the left hand half plane, when $R < 0$, and in the right hand half plane, when $R > 0$. We have chosen $R = -2 < 0$ on the figure.
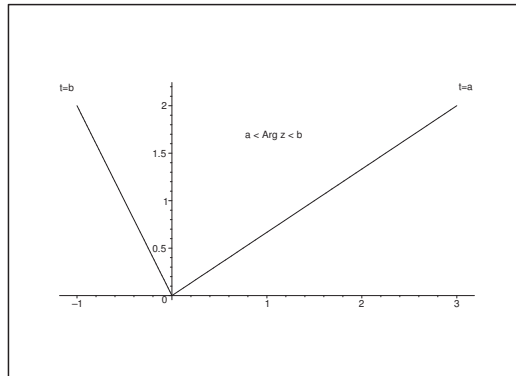


Figure 11: (d) The point set described by $a \leq \operatorname{Arg} z \leq b$ is the angular domain between the open half lines given in polar coordinates by $t = a$ and $t = b$. The half lines belong to the set with the exception of the point $(0, 0)$.

**(d)** The point set described by $\alpha \leq \operatorname{Arg} z \leq \beta$ is the angular domain between the half lines $t = \alpha$ and $t = \beta$. It is not closed, because the point $(0,0)$ does not belong to the set, neither is is open because the open half lines belong to the set.

**Example 1.19** *Assume that $c \in \mathbb{R}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Prove that*

$$\alpha z + \overline{\alpha}\,\overline{z} + c = 0$$

*is the equation of a straight line in the $(x, y)$-plane.*

It follows from $2 \operatorname{Re} w = w + \overline{w}$

$$w = \alpha z = (a + ib)(x + iy),$$

that

$$0 = \alpha z + \overline{\alpha}\,\overline{z} + c = 2 \operatorname{Re}\{(a + ib)(x + iy)\} = 2a\,x - 2b\,y + c.$$

Since $\alpha = a + ib \neq 0$, this is the equation of a straight line.

ALTERNATIVE it follows by a direct computation that

$$
\begin{aligned}
0 &= \alpha z + \overline{\alpha}\,\overline{z} + c = (a + ib)(x + iy) + (a - ib)(x - iy) + c \\
&= ax - by + i(bx + ay) + ax - by - i(bx + ay) + c = 2ax - 2by + c.
\end{aligned}
$$

**Example 1.20** *Assume that $c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ satisfy the condition $|\alpha|^2 \geq c$.*
*Sketch the set of $z \in \mathbb{C}$, for which*

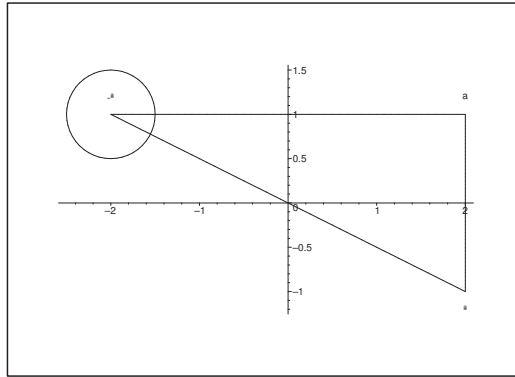$$z\,\overline{z} + \alpha\,z + \overline{\alpha}\,\overline{z} + c = 0.$$



Figure 12: The geometric solution using convenient reflections.

It follows from

$$\begin{aligned} 0 &= z\,\overline{z} + \alpha\,z + \overline{z}\,\overline{z} + c = z\,\overline{z} + \alpha\,z + \overline{\alpha}\,\overline{z} + \alpha\,\overline{\alpha} + c - |\alpha|^2 \\ &= (z + \overline{\alpha})\,(\overline{z} + \alpha) - \left\{ |\alpha|^2 - c \right\} = |z + \overline{\alpha}|^2 - \left\{ |\alpha|^2 - c \right\}, \end{aligned}$$

and a rearrangement that

$$|z + \overline{\alpha}| = |z - (-\overline{\alpha})| = \sqrt{|\alpha|^2 - c}.$$

Then we conclude from the geometrical interpretation that the point set is a circle of centrum $-\overline{\alpha}$ (reflection of $\alpha$ with respect to the $y$-axis) and radius $\sqrt{|\alpha|^2 - c}$.
However, if $|\alpha|^2 = c$, then we only get the point $z = -\overline{\alpha}$.

ALTERNATIVELY we put $z = x + iy$ and $\alpha = a + ib$, where the assumption is that $a^2 + b^2 \geq c$. Then

$$\begin{aligned} 0 &= z\,\overline{z} + \alpha\,z + \overline{\alpha}\,\overline{z} + c = |z|^2 + 2\,\mathrm{Re}(\alpha\,z) + c = x^2 + y^2 + 2ax - 2by + c \\ &= (x + a)^2 + (y - b)^2 - \left\{ a^2 + b^2 - c \right\}, \end{aligned}$$
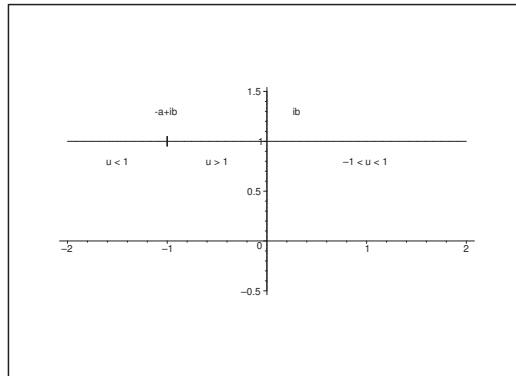
hence

$$(x + a)^2 + (y - b)^2 = a^2 + b^2 - c.$$

It follows that the point set is a circle of centrum at $(-a, b) \sim -\overline{\alpha}$ and radius $\sqrt{a^2 + b^2 + c} = \sqrt{|\alpha|^2 - c}.$

**Example 1.21** *Let $a \in \mathbb{C}$ be a constant, satisfying* Re $a > 0$.
*Find the three point sets in the z-plane, for which* $\dfrac{a - z}{\overline{a} - z}$ *is* $< 1$, $= 1$ *or* $> 1$, *respectively.*



We shall find the set of the $z$, for which

$$u = \frac{a - z}{\overline{a} + z} = \frac{-z + a}{z + \overline{a}}$$

is *real* (since otherwise the order relation does not make sense).
We get by solving the equation,

$$\overline{a}\,u + u\,z = a - z.$$

When we put $a = \alpha + i\beta$, Re $a = \alpha > 0$, then

$$z = \frac{a - \overline{a}\,u}{u + 1} = \frac{\alpha + i\beta - (\alpha - i\beta)u}{u + 1} = \alpha \cdot \frac{1 - u}{1 + u} + i\beta.$$

If $u = 1$, then $z = i\beta$.
If $u > 1$, then $x = \alpha \cdot \dfrac{1 - u}{1 + u} < 0$, hence

$$-\alpha < x < 0 \qquad \text{and} \qquad y = \beta.$$

If $u < 1$, $u \neq -1$, then we obtain the remaining parts of the line $y = \beta$, thus in a complex description, $x + i\beta$, where either

$$x < -\alpha, \quad \text{corresponding to} \quad -\infty < u < -1,$$

or

$$0 < x, \quad \text{corresponding to} \quad -1 < u < 1.$$
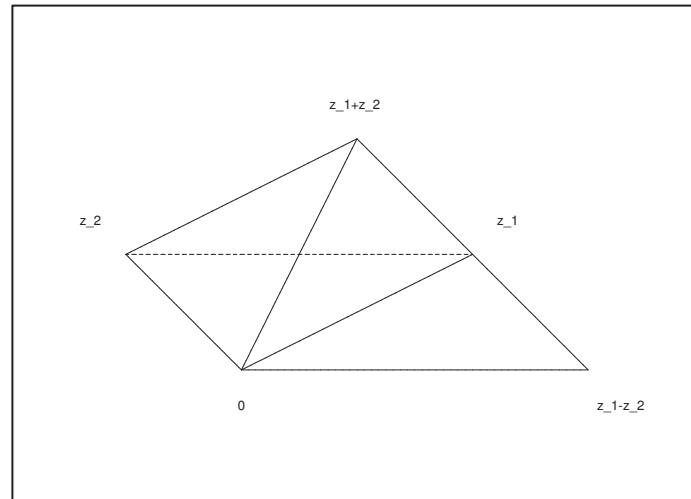
The three point sets all lie on the line $y = \beta$.

**Example 1.22** *Let $z_1$, $z_2 \in \mathbb{C}$. Prove that*

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2 \left( |z_1|^2 + |z_2|^2 \right),$$

*and give a geometrical interpretation of this result.*

If we put $z_1 = x + iy$ and $z_2 = a + ib$, then

$$
\begin{aligned}
|z_1 + z_2|^2 + |z_1 - z_2|^2 &= (x+a)^2 + (y+b)^2 + (x-a)^2 + (y-b)^2 = 2\left(x^2+y^2\right) + 2\left(y^2+b^2\right) \\
&= 2\left(x^2+y^2\right) + 2\left(a^2+b^2\right) = 2\left(|z_1|^2 + |z_2|^2\right).
\end{aligned}
$$



This means geometrically that the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.

**Example 1.23** *Let $z = x + iy$. Prove that $z^4$ is real, if and only if either $xy = 0$ or $|x| = |y|$; and that $z^4$ is imaginary, if and only if $x = \pm \left(1 \pm \sqrt{2}\right) y$.*

1) Let $z^4 = a \in \mathbb{R}$ where $a > 0$. Then

$$z = x + iy = \sqrt[4]{a} \cdot \left\{ \cos \frac{p\pi}{2} + i \sin \frac{p\pi}{2} \right\}, \qquad p = 0, 1, 2, 3,$$

thus
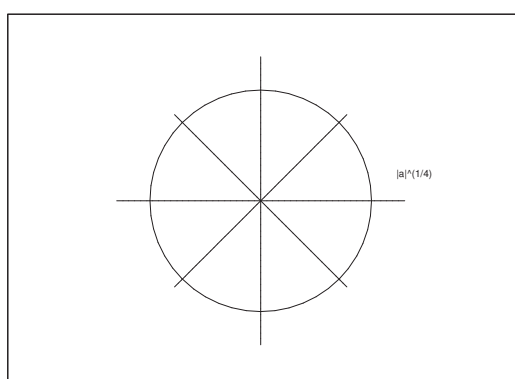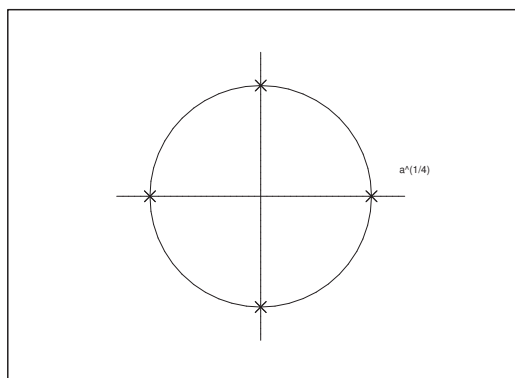
$$x = \sqrt[4]{a} \cos \frac{p\pi}{2} \qquad \text{and} \qquad y = \sqrt[4]{a} \sin \frac{p\pi}{2},$$

and we get

$$xy = \left(\sqrt[4]{a}\right)^2 \cos \frac{p\pi}{2} \cdot \sin \frac{p\pi}{2} = \frac{1}{2}\sqrt{a} \cot \sin p\pi = 0.$$

If on the other hand, $x \cdot y = 0$, then either $x = 0$ or $y = 0$, ind it follows trivially that $z^4 = (x + iy)^4 \in \mathbb{R}$.

2) If instead $a \in \mathbb{R}$ and $a < 0$, then we get

$$z = x + iy = \sqrt[4]{a} \cdot \left\{ \cos\left(\frac{\pi}{4} + \frac{p\pi}{2}\right) + i\,\sin\left(\frac{\pi}{4} + \frac{p\pi}{2}\right) \right\}.$$

It follows from

$$\left|\cos\left(\frac{\pi}{4} + p\,\frac{\pi}{2}\right)\right| = \left|\sin\left(\frac{\pi}{4} + p\,\frac{\pi}{2}\right)\right| = \frac{\sqrt{2}}{2},$$

that if $z^4 = a < 0$, then $|x| = |y|$.
If on the other hand $|x| = |y|$, then

$$z = x + iy = r \cdot \frac{\pm 1 \pm i}{\sqrt{2}},$$

and since

$$\left\{\frac{\pm 1 \pm i}{\sqrt{2}}\right\} = -1,$$

we get $z^4 = -r^4 < 0$.

3) Assume that $z^4 = i\,a$, $a \in \mathbb{R}$, where $a > 0$. Then

$$i\,a = |a| \exp\left(i\,\frac{\pi}{4}\right),$$

thus

$$z = x + iy = \sqrt[4]{a} \cdot \left\{ \cos\left(\frac{\pi}{8} + p\,\frac{\pi}{2}\right) + i\,\sin\left(\frac{\pi}{8} + p\,\frac{\pi}{2}\right) \right\}, \qquad p = 0,\,1,\,2,\,3.$$

Since $y \neq 0$, it follows that

$$
\begin{aligned}
\frac{x}{y} &= \frac{\cos\left(\dfrac{\pi}{8} + p\cdot\dfrac{\pi}{2}\right)}{\sin\left(\dfrac{\pi}{8} + p\cdot\dfrac{\pi}{2}\right)} = \frac{2\cos^2\left(\dfrac{\pi}{8} + p\cdot\dfrac{\pi}{2}\right) - 1 + 1}{2\sin\left(\dfrac{\pi}{8} + p\cdot\dfrac{\pi}{2}\right)\cos\left(\dfrac{\pi}{8} + p\cdot\dfrac{\pi}{2}\right)} = \frac{\cos\left(\dfrac{\pi}{4} + p\pi\right) + 1}{\sin\left(\dfrac{\pi}{4} + p\pi\right)} \\
&= \frac{\dfrac{1}{\sqrt{2}}\cdot(-1)^p + 1}{\dfrac{1}{\sqrt{2}}\cdot(-1)^p} = 1 \pm \sqrt{2},
\end{aligned}
$$

hence

$$x = \left(1 \pm \sqrt{2}\right) y.$$

If on the other hand, $x = \left(1 \pm \sqrt{2}\right) y$, then it follows by insertion that

$$
\begin{aligned}
z^4 &= y * 4 \left\{1 \pm \sqrt{2} + i\right\}^4 = y^4 \left\{1 + 2 - 1 \pm 2\sqrt{2} + 2i \left(1 \pm \sqrt{2}\right)\right\}^2 \\
&= y^4 \left\{2 \pm 2\sqrt{2} + 2i \left(1 \pm \sqrt{2}\right)\right\}^2 = y^4 \left(2 \pm 2\sqrt{2}\right)^2 (1+i)^2 \\
&= y^4 \left(4 + 8 \pm 8\sqrt{2}\right) \cdot 2i = 8iy^4 \left(3 \pm 2\sqrt{2}\right),
\end{aligned}
$$

i.e. precisely the required form.
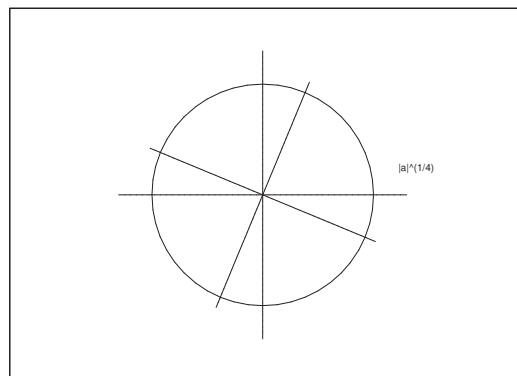


4) If $a < 0$, then

$$i\,a = |a| \exp\left(i\,\frac{3\pi}{2}\right),$$

and we get

$$z = x + iy = \sqrt[4]{|a|} \cdot \left\{\cos\left(\frac{3\pi}{8} + p\,\frac{\pi}{2}\right) + i \sin\left(\frac{3\pi}{8} + p\,\frac{\pi}{2}\right)\right\}, \quad p = 0,\,1,\,2,\,3.$$

We derive as above that

$$\frac{x}{y} = \frac{\cos\left(\frac{3\pi}{8} + p\frac{\pi}{2}\right)}{\sin\left(\frac{3\pi}{8} + p\frac{\pi}{2}\right)} = \frac{\cos\left(\frac{3\pi}{4} + p\pi\right) + 1}{\sin\left(\frac{3\pi}{4} + p\pi\right)} = \frac{-\frac{1}{\sqrt{2}}\cdot(-1)^p + 1}{\frac{1}{\sqrt{2}}\cdot(-1)^p} = -1 \mp \sqrt{2},$$

thus

$$x = -\left(1 \mp \sqrt{2}\right)y.$$

On the other hand, if $x = -\left(1 \pm \sqrt{2}\right)y$, then it follows by insertion that

$$
\begin{aligned}
z^4 &= y^4\left\{-\left(1 \pm \sqrt{2}\right) + i\right\}^4 = y^4\left\{1 + 2 - 1 \pm 2\sqrt{2} - 2i\left(1 \pm \sqrt{2}\right)\right\}^2 \\
&= y^4\left\{2 \pm 2\sqrt{2} - 2i\left(1 \pm \sqrt{2}\right)\right\}^2 = y^4\left(2 \pm 2\sqrt{2}\right)^2(1 - i)^2 \\
&= y^4\left(4 + 8 \pm 8\sqrt{2}\right)\cdot(-2i) = -8iy^4\left(3 \pm 2\sqrt{2}\right),
\end{aligned}
$$

which is precisely the required form.

**Example 1.24** *Prove that for all $z \in \mathbb{C}$,*

$$|\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}\cdot|z|.$$

If we put $z = x + iy$, this inequality is also written

$$|x| + |y| \leq \sqrt{2}\cdot\sqrt{x^2 + y^2}.$$

The left hand side is $\geq 0$, so this inequality is *equivalent* to

$$(|x| + |y|)^2 \leq 2x^2 + 2y^2,$$

hence we shall only prove this latter inequality. This follows from

$$2x^2 + 2y^2 - (|x| + |y|)^2 = 2x^2 + 2y^2 - x^2 - y^2 - 2|x|\cdot|y| = x^2 + y^2 - 2|x|\cdot|y| = (|x| - |y|)^2 \geq 0.$$

## 2 Polar form of complex numbers

**Example 2.1** *Write the following complex numbers in their polar form;*

   (a) $1 + i$,   (b) $1 + i\sqrt{3}$,   (c) $1 - i\sqrt{3}$,   (d) $-5i$.

**(a)**

$$1 + i = \sqrt{2} \exp\left(i\,\frac{\pi}{4}\right) = \sqrt{2} \exp\left(i\left\{\frac{\pi}{4} + 2p\pi\right\}\right), \qquad p \in \mathbb{Z}.$$

**(b)**

$$1 + i\sqrt{3} = 2 \exp\left(i\,\frac{\pi}{3}\right) = 2 \exp\left(i\left\{\frac{\pi}{3} + 2p\pi\right\}\right), \qquad p \in \mathbb{Z}.$$

**(c)**

$$1 - i\sqrt{3} = 2 \exp\left(-i\,\frac{\pi}{3}\right) = 2 \exp\left(i\left\{-\frac{\pi}{3} + 2p\pi\right\}\right), \qquad p \in \mathbb{Z}.$$

**(d)**

$$-5i = 5 \exp\left(-i\,\frac{\pi}{2}\right) = 5 \exp\left(i\left\{-\frac{\pi}{2} + 2p\pi\right\}\right), \qquad p \in \mathbb{Z}.$$

**Example 2.2** *Find the values of $z \in \mathbb{C}$ for which*

$$\overline{e^{i\,z}} = e^{i\,\overline{z}}.$$

If we put $z = x + iy$, then

$$\overline{e^{i\,z}} = \overline{e^{i\,x-y}} = e^{-y}e^{-i\,x}$$

and

$$e^{i\,\overline{z}} = e^{i(x-i\,y)} = e^{y}\,e^{i\,x}.$$

Hence if $\overline{e^{i\,z}} = e^{i\,\overline{z}}$, then $y = 0$ and $e^{-i\,x} = e^{i\,x}$, so

$$e^{2i\,x} = 1 = e^{2i\,p\pi},$$

and we get $x = p\pi$, i.e.

$$z = p\pi, \qquad p \in \mathbb{Z}.$$

On the other hand, if $z = p\pi$, $p \in \mathbb{Z}$, then

$$\overline{e^{i\,z}} = \overline{e^{i\,p\pi}} = (-1)^p$$

and

$$e^{i\,\overline{z}} = e^{i\,p\pi} = (-1)^p,$$

and the equation is fulfilled. Hence the complete solution is

$$z = p\pi, \qquad p \in \mathbb{Z}.$$

**Example 2.3** *Write the following complex numbers in the form of $e^z$:*

| | | | |
|---|---|---|---|
| $(a)$  $1$, | $(b)$  $-1$, | $(c)$  $i$, | $(d)$  $-i$, |
| $(e)$  $1 + i$, | $(f)$  $1 - i$, | $(g)$  $-1 + i$, | $(h)$  $-1 - i$. |

**(a)**

$$1 = e^{2i\,p\pi}, \qquad p \in \mathbb{Z}.$$

**(b)**

$$-1 = e^{i(2p+1)\pi}, \qquad p \in \mathbb{Z}.$$

**(c)**

$$i = e^{i(2p+\frac{1}{2})\pi}, \qquad p \in \mathbb{Z}.$$

**(d)**

$$-i = e^{i(2p+\frac{3}{2})\pi}, \qquad p \in \mathbb{Z}.$$

**(e)**

$$1 + i = \sqrt{2}\, e^{i(\frac{1}{4}+2p)\pi}, \qquad p \in \mathbb{Z}.$$

**(f)**

$$1 - i = \sqrt{2}\, e^{i(-\frac{1}{4}+2p)\pi}, \qquad p \in \mathbb{Z}.$$

**(g)**

$$-1 + i = \sqrt{2}\, e^{i(\frac{3}{4}+2p)\pi}, \qquad p \in \mathbb{Z}.$$

**(h)**

$$-1 - i = \sqrt{2}\, e^{i(-\frac{3}{4}+2p)\pi}, \qquad p \in \mathbb{Z}.$$

**Example 2.4** *Write the following complex numbers in the form $x + iy$:*

$$(a) \ \ \exp(i\,\pi), \qquad (b) \ \ \exp\left(\frac{2i\,\pi}{3}\right), \qquad (c) \ \ 3\exp\left(i\,\frac{\pi}{4}\right),$$

$$(d) \ \ \pi \cdot \exp\left(-\frac{\pi}{3}\right), \qquad (e) \ \ \exp\left(i\,\frac{\pi}{2}\right).$$

**(a)**

$$\exp(i\,\pi) = \cos\pi + i\,\sin\pi = -1.$$

**(b)**

$$\exp\left(\frac{2i\,\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}.$$

**(c)**

$$3 \cdot \exp\left(i\,\frac{\pi}{4}\right) = 3\,\cos\left(\frac{\pi}{4}\right) + 3i\,\sin\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + i\,\frac{3\sqrt{2}}{2}.$$

**(d)**

$$\pi \cdot \exp\left(-i\,\frac{\pi}{3}\right) = \pi\left\{\cos\left(\frac{\pi}{3}\right) - i\,\sin\left(\frac{\pi}{3}\right)\right\} = \pi \cdot \left\{\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right\} = \frac{\pi}{2} - i\,\frac{\pi\sqrt{3}}{2}.$$

**(d)**

$$\exp\left(i\,\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\,\sin\left(\frac{\pi}{2}\right) = i.$$

**Example 2.5** *Assume $e^z = e^w$. Prove that there exists a $k \in \mathbb{Z}$, such that $z = w + 2\pi\,k\,i$.*

Two complex numbers are identical, if and only if they have the same absolute value (i.e. same module) and (assuming that the modulus is $\neq 0$) if their arguments agree modulo $2\pi$.

If we put $z = x + iy$ and $w = u + iv$ into the exponential function, then

$$e^z = e^x \cdot e^{i\,y} \qquad \text{and} \qquad e^w = e^u \cdot e^{i\,v}.$$

The module is $e^x = e^u \neq 0$, hence $x = u$, and concerning the arguments we get $y \equiv v \pmod{2\pi}$, hence $y = v + 2k\,\pi$ for some $k \in \mathbb{Z}$. Finally,

$$z = x + iy = u + i(v + 2\pi\,k) = u + iv + 2\pi\,k\,i = w + 2\pi\,k\,i$$

for some $k \in \mathbb{Z}$.

**Example 2.6** *Find the real and the imaginary part of $(1 + i)^{20}$.*

It follows from $(1 + i)^2 = 2\,i$ that

$$(1 + i)^{20} = \{(1 + i)\}^{10} = \{2\,i\}^{10} = 2^{10}\,i^{10} = 2^{10}\,i^2 = -1024,$$

thus

$$\operatorname{Re}\left\{(1 + i)^{20}\right\} = -1024, \qquad \text{and} \qquad \operatorname{Im}\left\{(1 + i)^{20}\right\} = 0.$$

ALTERNATIVELY we use polar coordinates. Since

$$1 + i = \sqrt{2}\,\exp\left(i\,\frac{\pi}{4}\right),$$

we get by using polar coordinates that

$$(1 + i)^{20} = \left\{\sqrt{2}\,\exp\left(i\,\frac{\pi}{4}\right)\right\}^{20} = \left(2^{\frac{1}{2}}\right)^{20} \exp\left(i\,20\,\frac{\pi}{4}\right) = 2^{10}\,e^{5i\,\pi} = 1024\,e^{i\,\pi} = -1024,$$

and it follows as before that

$$\operatorname{Re}\left\{(1 + i)^{20}\right\} = -1024, \qquad \text{and} \qquad \operatorname{Im}\left\{(1 + i)^{20}\right\} = 0.$$

**Example 2.7** *Prove for any complex number $z \neq 1$ that*

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

*Use this result for $z = e^{i\theta}$, $0 < \theta < 2\pi$, in proving that*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\frac{\theta}{2}}.$$

If $z \neq 1$, then $\dfrac{z^{n+1} - 1}{z - 1}$ is defined. Since

$$\left(z^n + z^{n-1} + \cdots + z + 1\right)(z - 1) = \left(z^{n+1} - z^n\right) + \left(z^n - z^{n-1}\right) + \cdots + \left(z^2 - z\right) + (z - 1) = z^{n+1} - 1,$$

we get precisely that

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

If $0 < \theta < 2\pi$, then $e^{i\theta} \neq 1$, hence according to the above,

$$1 + e^{i\theta} + e^{2i\theta} + \cdots e^{i(n-1)\theta} + e^{in\theta} = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}.$$

The real part is

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \operatorname{Re}\left\{ \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} \right\} = \operatorname{Re}\left\{ \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i\frac{1}{2}\theta}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \right\}$$

$$= \operatorname{Re}\left\{ \frac{\cos(n+\frac{1}{2})\theta + i\sin(n+\frac{1}{2})\theta - \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{2i\sin\frac{\theta}{2}} \right\} = \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} + \frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

**Example 2.8** *Prove for every $x \neq p\pi$, $p \in \mathbb{Z}$, that*

$$\sum_{n=1}^{+\infty} \cos^n x \cdot \sin nx = 0, \qquad \sum_{n=1}^{+\infty} \cos^n x \cdot \sin nx = \cot x.$$

Since $|\cos nx| \leq 1$ and $|\sin nx| \leq 1$, and since $|\cos x| < 1$ for $x \neq p\pi$, $p \in \mathbb{Z}$, both series are absolutely convergent.
Then we get

$$\sum_{n=1}^{+\infty} \cos^n x \cdot \cos nx + i \sum_{n=1}^{+\infty} \cos^n x \cdot \sin nx = \sum_{n=1}^{+\infty} \cos^n x \cdot \{\cos nx + i\sin nx\} = \sum_{n=1}^{+\infty} \cos^n x \cdot e^{inx}$$

$$= \sum_{n=1}^{+\infty} \left(\cos x \cdot e^{ix}\right)^n = \frac{\cos x \cdot e^{ix}}{1 - \cos x \cdot e^{ix}} \frac{\cos x}{e^{-ix} - \cos x} = \frac{\cos x}{\cos x - i\sin x - \cos x} = -\frac{1}{i}\frac{\cos x}{\sin x} = i\cot x.$$

Finally, by splitting into the real and the imaginary part,

$$\sum_{n=1}^{+\infty} \cos^n x \cdot \sin nx = 0, \qquad \sum_{n=1}^{+\infty} \cos^n x \cdot \sin nx = \cot x.$$

**Example 2.9** *Apply Moivre's formula in order to express $\cos 3\theta$ and $\sin 3\theta$ by means of $\cos\theta$ and $\sin\theta$.*

We get by Moivre's formula and the binomial formula,

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta + i\left\{3\cos^2\theta\sin\theta - \sin^3\theta\right\},$$

hence by splitting into the real and the imaginary part,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta,$$

and

$$\sin 3\theta = 3\cos^2\theta\sin\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

**Example 2.10** *Apply Moivre's formula to prove that*

$$\cos 8\theta + 28 \, \cos 4\theta + 35 = 64 \left(\cos^8 \theta + \sin^8 \theta\right).$$

**First method.** Whenever one shall apply *Moivre's formula*, one must usually do a lot of computations.

First we rewrite the left hand side to a homogenous trigonometric polynomial of degree 8, by using

$$1 = \cos^2 \theta + \sin^2 \theta.$$

We find by the binomial formula that

$$
\begin{aligned}
\cos 8\theta &= \operatorname{Re} e^{i\,8\theta} = \operatorname{Re}\left\{e^{i\,\theta}\right\}^8 = \operatorname{Re}\left(\cos\theta + i\,\sin\theta\right)^8 \\
&= \cos^8\theta - 28\,\cos^6\theta\sin^2\theta + 70\,\cos^4\theta\sin^4\theta - 28\,\cos^2\theta\sin^6\theta + \sin^8\theta, \\
\cos 4\theta &= \operatorname{Re}\left\{e^{i\,\theta}\right\}^4 = \operatorname{Re}\left(\cos\theta + i\,\sin\theta\right)^4 \\
&= \left(\cos^4\theta - 6\,\cos^2\theta\sin^2\theta + \sin^4\theta\right)\cdot\left(\cos^2\theta + \sin^2\theta\right)^2 \\
&= \left(\cos^4\theta - 6\,\cos^2\theta\sin^2\theta + \sin^4\theta\right)\cdot\left(\cos^4\theta + 2\,\cos^2\theta\sin^2 + \sin^4\theta\right) \\
&= \cos^8\theta - 4\,\cos^6\theta\sin^2\theta - 10\,\cos^4\theta\sin^4\theta - 4\,\cos^2\theta\sin^6\theta + \sin^8\theta,
\end{aligned}
$$

and

$$1 = \left(\cos^2\theta + \sin^2\theta\right)^4 = \cos^8\theta + 4\,\cos^6\theta\sin^2\theta + 6\,\cos^4\theta\sin^4\theta + 4\,\cos^2\theta\sin^6\theta + \sin^8\theta.$$

Summing up,

$$\cos 8\theta = \cos^8\theta - 28\,\cos^6\theta\sin^2\theta + 70\,\cos^4\theta\sin^4\theta - 28\,\cos^2\theta\sin^6\theta + \sin^8\theta,$$

$$28\cos 4\theta = 28\cos^8\theta - 112\cos^6\theta\sin^2\theta - 280\cos^4\theta\sin^4\theta - 112\cos^2\sin^6\theta + 28\sin^8\theta,$$

$$35 = 35\cos^8\theta + 140\cos^6\theta\sin^2\theta + 210\cos^4\theta + 140\cos^2\theta\sin^6\theta + 35\sin^8.$$

Finally, by an addition,

$$\cos 8\theta + 28\cos 4\theta + 35 = (1 + 28 + 35)\left(\cos^8\theta + \sin^8\theta\right) = 64\left(\cos^8\theta + \sin^8\theta\right).$$

**Second method.** It is here much easier to use *Euler's formulæ* and then compute from the right towards the left,

$$
\begin{aligned}
64\cos^8\theta &= \frac{64}{256}\left(e^{i\theta} + e^{-i\,\theta}\right)^8 \\
&= \frac{1}{4}\left(e^{8i\,\theta} + 8\,e^{6i\,\theta} + 28\,e^{4i\,\theta} + 56\,e^{2i\,\theta} + 70 + 56\,e^{-2i\,\theta} + 28\,e^{-4i\theta} + 8\,e^{-6i\,\theta} + e^{-8i\,\theta}\right)
\end{aligned}
$$

and analogously

$$
\begin{aligned}
64\sin^8\theta &= \frac{64}{256\cdot i^8}\left(e^{i\,\theta} - e^{-i\,\theta}\right)^8 \\
&= \frac{1}{4}\left(e^{8i\,\theta} - 8\,e^{6i\theta} + 28\,e^{4i\,\theta} - 56\,e^{2i\,\theta} + 70 - 56\,e^{-2i\,\theta} + 28\,e^{-4i\,\theta} - 8\,e^{-6i\,\theta} + e^{-8i\,\theta}\right).
\end{aligned}
$$

Finally, by an addition,

$$64\left(\cos^8\theta + \sin^8\theta\right) = \frac{1}{2}\left(e^{8i\,\theta} + 28\,e^{4i\,\theta} + 70 + 28\,e^{-4i\,\theta} + e^{-8i\,\theta}\right) = \cos 8\theta + 28\cos 4\theta + 35.$$

**Third method.** One can also apply the well-known *trigonometric relations*,

$$\cos^2 t = \frac{1 + \cos 2t}{2} \qquad \text{and} \qquad \sin^2 t = \frac{1 - \cos 2t}{2}$$

with different values ($\theta$, $2\theta$ and $4\theta$) of $t$. When we compute from the right towards the left, we get

$$64 \left(\cos^8 \theta + \sin^8 \theta\right) = 64 \left\{\left(\cos^2 \theta\right)^4 + \left(\sin^2 \theta\right)^4\right\} = \frac{2^6}{2^4} \left\{(1 + \cos 2\theta)^4 + (1 - \cos 2\theta)^4\right\}$$

$$= 4 \left\{\left(1 + \cos^2 2\theta + 2 \cos 2\theta\right)^2 + \left(1 + \cos^2 2\theta - 2 \cos 2\theta\right)^2\right\}$$

$$= 4 \cdot 2 \left\{\left(1 + \cos^2 2\theta\right)^2 + 4 \cos^2 2\theta\right\} = 8 \cdot \left\{\left(1 + \frac{1 + \cos 4\theta}{2}\right)^2 + 4 \cdot \frac{1 + \cos 4\theta}{2}\right\}$$

$$= 2 \left(3 + \cos 4\theta\right)^2 + 16 \left(1 + \cos 4\theta\right) = 2 \left(9 + 6 \cos 4\theta + \cos^2 4\theta\right) + 16 \cos 4\theta + 16$$

$$= 18 + 12 \cos 4\theta + 2 \cdot \frac{1 + \cos 8\theta}{2} + 16 \cos 4\theta + 16 = \cos 8\theta + 28 \cos 4\theta + 35.$$

**Fourth method.** The same as in the **third method**, with the only exception that we now compute from the left towards the right. It is seen that one must here use far more skill:

$$\cos 8\theta + 28 \cos 4\theta + 35 = 2 \cos^2 4\theta - 1 + 28 \cos 4\theta + 35$$

$$= 2 \left(2 \cos^2 2\theta - 1\right)^2 + 28 \left(2 \cos^2 2\theta - 1\right) + 34$$

$$= 2 \left\{2 \left(2 \cos^2 \theta - 1\right)^2 - 1\right\} + 56 \left(2 \cos^2 \theta - 1\right)^2 + 6$$

$$= 2 \left\{2 \left(4 \cos^4 \theta - 4 \cos^2 \theta + 1\right) - 1\right\}^2 + 56 \left(4 \cos^4 \theta - 4 \cos^2 \theta + 1\right) + 6$$

$$= 2 \left(8 \cos^4 \theta - 8 \cos^2 \theta + 1\right)^2 + 224 \cos^4 \theta - 224 \cos^2 \theta + 62$$

$$= 2 \left(64 \cos^8 \theta + 64 \cos^4 \theta + 1 - 128 \cos^6 \theta + 16 \cos^4 \theta - 16 \cos^2 \theta\right) + 224 \cos^4 \theta - 224 \cos^2 \theta + 62$$

$$= 64 \cos^8 \theta + 64 \cos^8 \theta - 256 \cos^6 \theta + 384 \cos^4 \theta - 256 \cos^2 \theta + 64$$

$$(1) \quad = 64 \cos^8 \theta + 64 \left(\cos^8 \theta - 4 \cos^6 \theta + 6 \cos^4 \theta - 4 \cos^2 \theta + 1\right)$$

$$= 64 \cos^8 \theta + 64 \left\{\left(1 - \sin^2 \theta\right)^4 - 4 \left(1 - \sin^2 \theta\right)^3 + 6 \left(1 - \sin^2 \theta\right)^2 - 4 \left(1 - \sin^2 \theta\right) + 1\right\}$$

$$= 64 \cos^8 \theta + 64 \left\{\sin^8 \theta - 4 \sin^6 \theta + 6 \sin^4 \theta - 4 \sin^2 \theta + 1\right.$$

$$\left. -4 \left(1 - 3 \sin^2 \theta + 3 \sin^4 \theta - \sin^6 \theta\right) + 6 \left(1 - 2 \sin^2 \theta + \sin^4 \theta - 4 + 4 \sin^2 \theta + 1\right)\right\}$$

$$= 64 \cos^8 \theta + 64 \left\{\sin^8 \theta + (-4 + 4) \sin^6 \theta + (6 - 12 + 6) \sin^4 \theta \right.$$

$$\left. +(-4 + 12 - 12 + 4) \sin^2 \theta + 1 - 4 + 6 + 1\right\}$$

$$= 64 \cos^8 \theta + 64 \sin^4 \theta.$$

**Fifth method.** This is the same as the **fourth method** until (1). If we here recognize $\left(1 - \cos^2 \theta\right)^4$, then the computations become easier in the following way,

$$\cos 8\theta + 28 \cos 4\theta + 35$$

$$= 64 \cos^8 \theta + 64 \left(\cos^8 \theta - 4 \cos^6 \theta + 6 \cos^4 \theta - 4 \cos^2 \theta + 1\right)$$

$$= 64 \cos^8 \theta + 64 \left\{1 + \frac{4}{1} \left(-\cos^2 \theta\right)^1 + \frac{4 \cdot 3}{1 \cdot 2} \left(-\cos^2 \theta\right)^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \left(-\cos^2 \theta\right)^3 + \left(-\cos^2 \theta\right)^4\right\}$$

$$= 64 \cos^8 \theta + 64 \left(1 - \cos^2 \theta\right)^4 = 64 \cos^8 \theta + 64 \left(\sin^2 \theta\right)^4 = 64 \cos^8 \theta + 64 \sin^8 \theta.$$

# 3 The binomial equation

**Example 3.1** *Solve the binomial equation*

$$z^3 = -2 - 2i.$$

Since $r = |-2 - 2i| = 2\sqrt{2}$, every root must lie on a circle of radius $\sqrt[3]{r} = \sqrt[3]{2\sqrt{2}} = \sqrt{2}$, and of centrum 0. Since $-\dfrac{3\pi}{4}$ is an argument for $-2 - 2i$, an argument for one of the three roots is given by $\theta = -\dfrac{\pi}{4}$, thus

$$z_1 = \sqrt{2}\exp\left(-i\,\frac{\pi}{4}\right) = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\,\sin\left(-\frac{\pi}{4}\right)\right\} = 1 - i.$$

The other two roots also lie on the circle $|z| = \sqrt{2}$, and since the form an equilateral triangle, we get

$$z_2 = z_1 \cdot \exp\left(\frac{2i\pi}{3}\right) = (1-i)\cdot\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2} + i\left\{\frac{\sqrt{3}}{2} + \frac{1}{2}\right\},$$

and

$$z_3 = z_1 \cdot \exp\left(\frac{4i\pi}{3}\right) = (1-i)\cdot\left(-\frac{1}{2} + -i\,\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2} - i\left\{\frac{\sqrt{3}}{2} + \frac{1}{2}\right\}.$$

**Example 3.2** *Given $a$, $b \in \mathbb{R}$. Find $x$, $y \in \mathbb{R}$ expressed by $a$ and $b$, such that*

$$(x + iy)^2 = a + ib.$$

When we compute the left hand side, we get

$$(x + iy)^2 = x^2 - y^2 + 2ixy,$$

so by a splitting into the real and the imaginary part we obtain the two equations

$$x^2 - y^2 = a \qquad \text{og} \qquad 2xy = b.$$

This *implies* that

$$a^2 + b^2 = \left(x^2 - y^2\right)^2 + 4x^2y^2 = \left(x^2 + y^2\right)^2,$$

hence

$$x^2 + y^2 = \sqrt{a^2 + b^2} \geq 0.$$

When this is compared with $x^2 - y^2 = a$, we get

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2} \quad (\geq 0), \qquad y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2} \quad (\geq 0),$$

thus

$$(2) \quad x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \quad \text{og} \quad y = \pm\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}.$$

Hence a solution is *necessarily* of the form (2). We see, however, that (2) usually gives four possibilities, and they cannot all be solutions, because we know that there are only two solutions. Hence we *must* check all our possible solutions.

The equation $x^2 - y^2 = a$ is of course always satisfied, so we turn towards $2xy = b$.
If $b = 0$, then either $x = 0$ or $y = 0$, according to (2), and the equation $2xy = b = 0$ is of course fulfilled. (In this case (2) produces actually only two solutions).
If $b \neq 0$, then a check shows that the solution is

$$x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \quad \text{og} \quad y = \pm\frac{b}{|b|}\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}},$$

where the signs are corresponding.

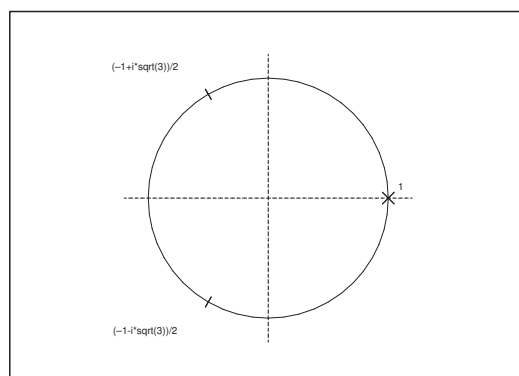**Example 3.3** *Find all the cubic roots of*

(a) 1,      (b) − 8,      (c) i.



Figure 13: (a) The cubic roots of 1.

**(a)** The three cubic roots are of course

$$1, \qquad -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \qquad -\frac{1}{2} - i\,\frac{\sqrt{3}}{2},$$

i.e.

$$e^0, \qquad \exp\left(i\,\frac{2\pi}{3}\right), \qquad \exp\left(i\,\frac{4\pi}{3}\right).$$
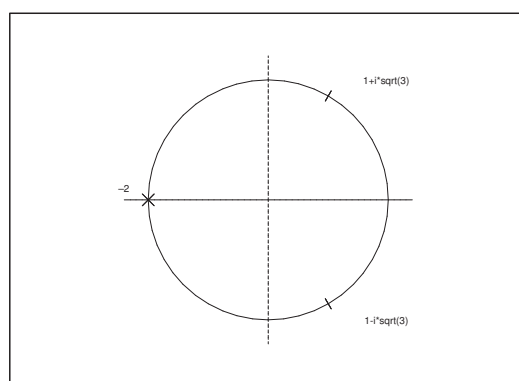


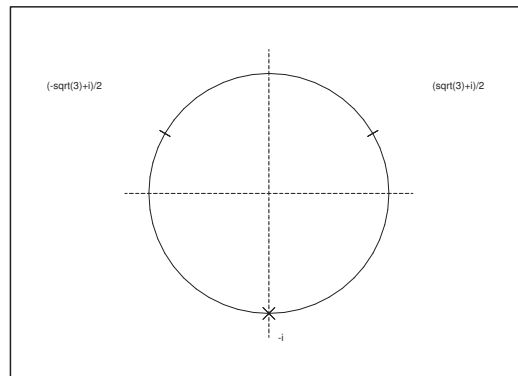Figure 14: (a) The cubic roots of −8.

**(b)** The three cubic roots of −8 are

$$-2, \qquad 1 + i\sqrt{3}, \qquad 1 - i\sqrt{3},$$

i.e.

$$2\,e^{i\pi}, \qquad 2\exp\left(i\,\frac{\pi}{3}\right), \qquad 2\exp\left(-i\,\frac{\pi}{3}\right).$$



Figure 15: (a) The cubic roots of $i$.

**(c)** The three cubic roots of $i$ are

$$-i, \qquad -\frac{\sqrt{3}}{2} + i\,\frac{1}{2}, \qquad \frac{\sqrt{3}}{2} + i\,\frac{1}{2},$$

thus

$$\exp\left(-i\,\frac{\pi}{2}\right), \qquad \exp\left(i\,\frac{5\pi}{6}\right), \qquad \exp\left(i\,\frac{\pi}{6}\right).$$

**Example 3.4** *Find all complex roots of*

$(a)$  $\sqrt{i}$,     $(b)$  $\sqrt[3]{-1+i}$,     $(c)$  $\sqrt[4]{-1}$,     $(d)$  $\sqrt[5]{-32}$.
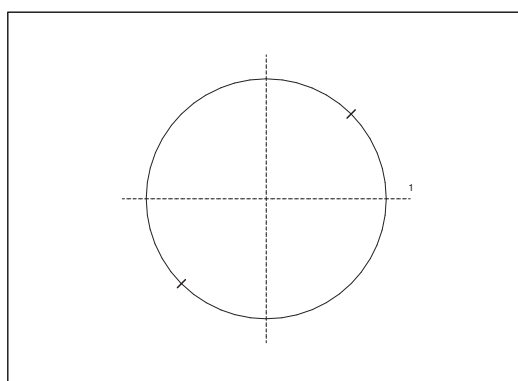


Figure 16: (a) The square roots of $i$.

**(a)** The symbol $\sqrt{i}$ means the roots of the equation

$$z^2 = i = \exp\left(i\left(\frac{\pi}{2} + 2p\pi\right)\right), \qquad p \in \mathbb{Z},$$

so

$$z = \exp\left(\left(\frac{\pi}{4} + p\pi\right)\right) = \pm\exp\left(i\,\frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}\,(1+i).$$

**(b)** The symbol $\sqrt[3]{-1+i}$ is the set of roots of the equation

$$z^3 = -1 + i = \sqrt{2}\,\exp\left(i\left(\frac{3\pi}{4} + 2p\pi\right)\right), \qquad p \in \mathbb{Z},$$

so

$$z = \sqrt[6]{2}\,\exp\left(i\left(\frac{\pi}{4} + p\cdot\frac{2\pi}{3}\right)\right), \qquad p = 0,\ 1,\ 2,$$
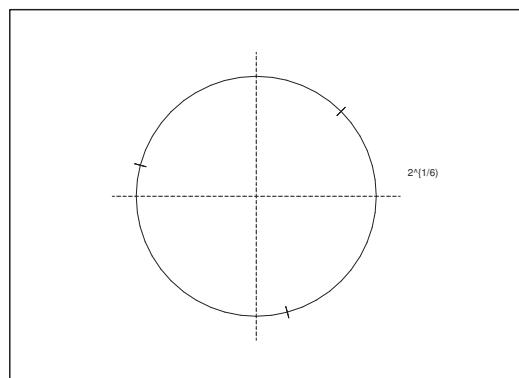
Figure 17: (a) The cubic roots of $-1 + i$.

or, more explicitly,

$$
\begin{aligned}
z_1 &= \sqrt[6]{2} \cdot \left( \frac{1}{\sqrt{2}} + i \, \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt[3]{2}} \, (1 + i), \\
z_2 &= \frac{1}{\sqrt[3]{2}} \, (1 + i) \cdot \frac{1}{2} \, (-1 + i\sqrt{3}) = \frac{1}{2 \sqrt[3]{2}} \, (-1 - \sqrt{3} + i\,\{-1 + \sqrt{3}\}), \\
z_3 &= \frac{1}{\sqrt[3]{2}} \, (1 + i) \cdot \frac{1}{2} \, (-1 - i\sqrt{3}) = \frac{1}{2 \sqrt[3]{2}} \, (-1 + \sqrt{3} - i\{1 + \sqrt{3}\}).
\end{aligned}
$$



Figure 18: (a) The quadruple roots of $-1$.

(c) The symbol $\sqrt[4]{-1}$ is the set of roots of the equation

$$
z^4 = -1 = \exp(i(\pi + 2p\pi)), \qquad p \in \mathbb{Z},
$$

so

$$
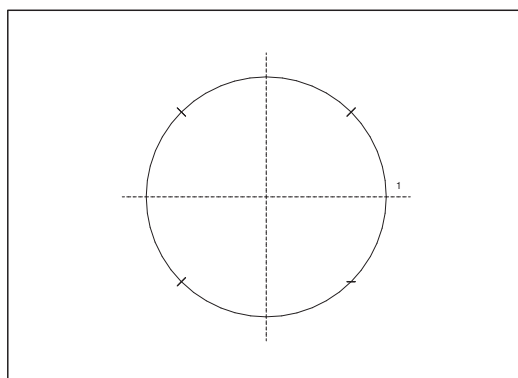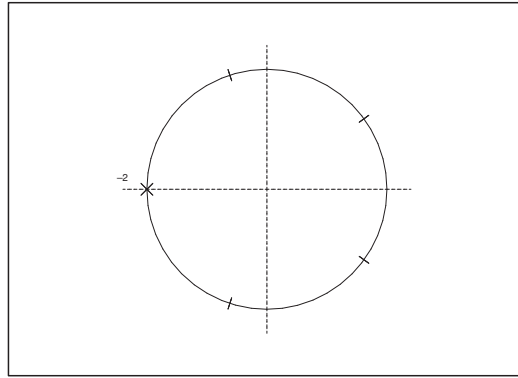z = \exp\left( i \left( \frac{\pi}{4} + p \, \frac{\pi}{2} \right) \right), \qquad p = 0,\, 1,\, 2,\, 3,
$$

Figure 19: (a) The fifth roots of $-32$.

or more explicitly,

$$z \in \left\{ \frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}} \right\}.$$

**(d)** The symbol $\sqrt[5]{-32}$ means the set of solutions of the equation

$$z^5 = -32 = 2^5\,e^{i(\pi + 2p\pi)}, \qquad p \in \mathbb{Z},$$

the solution of which are

$$z = 2\,\exp\left( i\left( \frac{\pi}{5} + p \cdot \frac{2\pi}{5} \right) \right), \qquad p = 0,\ 1,\ 2,\ 3,\ 4.$$

**Remark 3.1** Note that we for $p = 2$ obtain the trivial solution $z = -2$. The remaining roots *can* be expressed by means of square roots, if we apply the results of the folloing Example 3.5. $\Diamond$

**Example 3.5** *Prove that*

$$\cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4} \qquad and \qquad \sin\frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

HINT: *Apply that*

$$\left( \cos\frac{\pi}{5} + i\,\sin\frac{\pi}{5} \right)^5 = -1,$$

*and put $x = \cos\dfrac{\pi}{5}$.*

If we put $x = \cos\dfrac{\pi}{5} > 0$, then $\sin\dfrac{\pi}{5} = \sqrt{1 - x^2} > 0$, hence

$$\begin{aligned}
0 &= \left( \cos\frac{\pi}{5} + i\,\sin\frac{\pi}{5} \right)^5 + 1 = \left( x + i\sqrt{1 - x^2} \right)^5 + 1 \\
&= x^5 + 5i\,x^4\sqrt{1 - x^2} - 10x^3\left(1 - x^2\right) - 10i\,x^2\left(1 - x^2\right)\sqrt{1 - x^2} \\
&\qquad + 5x\left(1 - x^2\right)^2 + i\left(1 - x^2\right)^2\sqrt{1 - x^2} + 1.
\end{aligned}$$

When we split into the real and the imaginary part we get the two equations

$$(3) \quad \begin{cases} 0 = x^5 - 10x^3 \left(1 - x^2\right) + 5x \left(1 - x^2\right)^2 + 1, \\ 0 = \sqrt{1 - x^2} \cdot \left\{ 5x^4 - 10x^2 \left(1 - x^2\right) + \left(1 - x^2\right)^2 \right\}. \end{cases}$$

We have assumed that $\sqrt{1 - x^2} \neq 0$, hence (3) is reduced to

$$\begin{cases} 0 &=& x^5 + 10x^5 - 10x^3 + 5x^5 - 10x^3 + 5x + 1 = 16x^5 - 20x^3 + 5x + 1 \\ &=& (x + 1)\left(16x^4 - 16x^3 - 4x^2 + 4x + 1\right) \\ 0 &=& 5x^4 + 10x^4 - 10x^2 + x^4 - 2x^2 + 1 = 16x^4 - 12x^2 + 1, \end{cases}$$

and since $x = -1$ is not a common solution, it follows that we shall only solve the following reduced system of equations,

$$\begin{cases} 16x^4 &-& 16x^3 &-& 4x^2 &+& 4x &+& 1 &=& 0, \\ 16x^4 && && - & 12x^2 && &+& 1 &=& 0. \end{cases}$$

Then by a subtraction,

$$16x^3 - 8x^2 - 4x = 4x\left\{4x^2 - 2x - 1\right\} = 0.$$

The solutions are 0 and

$$x = \frac{2 \pm \sqrt{4 + 16}}{8} = \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4},$$

so the solution of the original equations must be one of these.

Since (3) *has* a solution $x = \cos\dfrac{\pi}{5} > 0$, and since the only positive of the possible solution is

$$x = \frac{1 + \sqrt{5}}{4},$$

we conclude that

$$\cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4},$$

hence

$$\sin\frac{\pi}{5} = \sqrt{1 - \left(\frac{1 + \sqrt{5}}{4}\right)^2} = \frac{\sqrt{16 - 1 - 6 - 2\sqrt{5}}}{4} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

ALTERNATIVELY, it follows from (3) that in particular,

$$0 = 16x^5 - 2 + x^3 + 5x + 1 = (x + 1)\left(16x^4 - 16x^3 - 4x^2 + 4x + 1\right).$$

Since

$$16x^4 - 16x^3 - 4x^2 + 4x + 1 = 16x^4 - 16x^3 - 8x^2 + 4x^2 + 4x + 1$$
$$= \left(4x^2\right)^2 - 8x^2(2x + 1) + (2x + 1)^2 = \left(4x^2 - 2x - 1\right)^2,$$

it follows that $x = \cos\dfrac{\pi}{5} \in \,]0, 1[$ fulfils the equation

$$16x^5 - 20x^3 + 5x + 1 = (x + 1)\left(4x^2 - 2x - 1\right)^2$$
$$= 4(x + 1)\left(x - \frac{2 + \sqrt{4 + 16}}{8}\right)^2\left(x - \frac{2 - \sqrt{4 + 16}}{8}\right)^2$$
$$= 4\{x - (-1)\}\left(x - \frac{1 + \sqrt{5}}{4}\right)^2\left(x - \frac{1 - \sqrt{5}}{4}\right)^2,$$

and we conclude that $x = \cos\dfrac{\pi}{5} \in \,]0, 1[$ belongs to the set

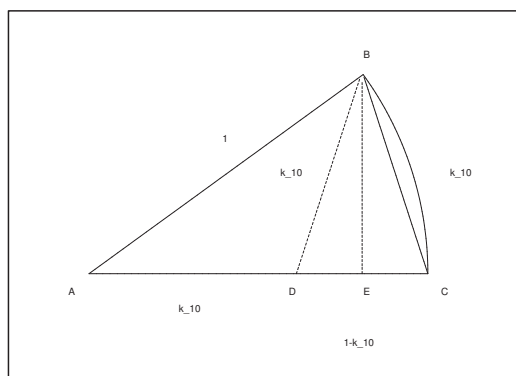$$\left\{-1, \frac{1 + \sqrt{5}}{4}, \frac{1 - \sqrt{5}}{4}\right\}.$$

Since only $\dfrac{1 + \sqrt{5}}{4}$ is positive, we conclude

$$\cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4},$$

and then

$$\sin \frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

is found as above.



ALTERNATIVELY, the example can be solved geometrically by noting that $\triangle ABC$ is the same angle as $\triangle BCD$. Then

$$\frac{|AB|}{|BC|} = \frac{|BC|}{|CD|}, \qquad \text{thus} \qquad \frac{1}{k_{10}} = \frac{k_{10}}{1 - k_{10}}.$$

We obtain the equation of second degree

$$k_{10}^2 + k_{10} - 1 = 0,$$

hence

$$k_{10} = -\frac{1}{2} \ (\overset{+}{-}) \ \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5} - 1}{2},$$

where we have exploited that $k_{10} > 0$. Finally, since $|AB| = 1$,

$$\cos \frac{\pi}{5} \ = \ |AE| = |AD| + \frac{1}{2} |DC| = k_{10} + \frac{1}{2} |DC| = k_{10} + \frac{1}{2} \ (1 - k_{10})$$

$$= \ \frac{1}{2} \ (1 + k_{10}) = \frac{1 + \sqrt{5}}{4}.$$

**Remark 3.2** The notation $k_{10}$ is due to the fact that it is the length of the cord of the regular decagon, inscribed in the unit circle. ◊

**Example 3.6** *Find all roots of the equation*

$$z^4 + i = 0.$$

We rewrite this equation as

$$z^4 = -i = \exp\left(i\left(-\frac{\pi}{2} + 2p\pi\right)\right), \qquad p \in \mathbb{Z},$$

thus

$$z = \exp\left(i\left(-\frac{\pi}{8} + p \cdot \frac{\pi}{2}\right)\right), \qquad p = 0, 1, 2, 3.$$

It follows from

$$\cos\frac{\pi}{8} = \sqrt{\frac{\cos\frac{\pi}{4} + 1}{2}} = \sqrt{\frac{1 + \frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}} = \frac{\sqrt{2 + \sqrt{2}}}{2},$$

and

$$\sin\frac{\pi}{8} = \sqrt{\frac{1 - \cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}} = \frac{\sqrt{2 - \sqrt{2}}}{2},$$

that

$$
\begin{aligned}
z_1 &= \cos\frac{\pi}{8} - i\sin\frac{\pi}{8} &&= \frac{1}{2}\left\{\sqrt{2 + \sqrt{2}} - i\sqrt{2 - \sqrt{2}}\right\}, \\[2mm]
z_2 &= i\, z_1 &&= \frac{1}{2}\left\{\sqrt{2 - \sqrt{2}} + i\sqrt{2 + \sqrt{2}}\right\}, \\[2mm]
z_3 &= -z_1 &&= \frac{1}{2}\left\{-\sqrt{2 + \sqrt{2}} + i\sqrt{2 - \sqrt{2}}\right\}, \\[2mm]
z_4 &= -i\, z_1 &&= \frac{1}{2}\left\{-\sqrt{2 - \sqrt{2}} - i\sqrt{2 + \sqrt{2}}\right\}.
\end{aligned}
$$

**Example 3.7** *Compute*

$$(-3 + 4i)^{-\frac{3}{2}}.$$

It follows by inspection that

$$-3 + 4i = 1 - 4 + 2 \cdot 2i = (1 + 2i)^2,$$

thus

$$(-3 + 4i)^{\frac{1}{2}} = \pm(1 + 2i),$$

and hence

$$
\begin{aligned}
(-3 + 4i)^{-\frac{3}{2}} &= \frac{1}{(-3 + 4i)(-3 + 4i)^{\frac{1}{2}}} = \mp\frac{1}{(-3 + 4i)(1 + 2i)} = \mp\frac{(-3 - 4i)(1 - 2i)}{25 \cdot 5} \\[2mm]
&= \pm\frac{(3 + 4i)(1 - 2i)}{125} = \pm\frac{3 + 8 - 6i + 4i}{125} = \pm\frac{11 - 2i}{125}.
\end{aligned}
$$

**Example 3.8** *Assume that $n \in \mathbb{N} \setminus \{1\}$. Prove that*

$$\sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \cdots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

HINT: *Prove that the left hand side can be written as* $\frac{1}{2^{n-1}}$ *times the product of the roots of the polynomial* $(1-z)^n - 1$, *which are different from zero.*

The equation $(1-z)^n - 1 = 0$ has the solutions

$$1 - z = \exp\left(i \frac{2p\pi}{n}\right), \qquad p = 0, 1, \ldots, n-1,$$

so when we rewrite each solution in the following way,

$$z_p = 1 - \exp\left(i \frac{2p\pi}{n}\right) = 1 - \cos \frac{2\pi}{n} - i \sin \frac{2p\pi}{n} = 1 - \left(1 - 2 \sin^2 \frac{p\pi}{n}\right) - i \cdot 2 \cos \frac{p\pi}{n} \cdot \sin \frac{p\pi}{n}$$

$$= 2 \sin \frac{p\pi}{n} \cdot \left(\sin \frac{p\pi}{n} - i \cos \frac{p\pi}{n}\right) = 2 \sin \frac{p\pi}{n} \cdot \left(\frac{1}{i} \exp\left(i \frac{p\pi}{n}\right)\right),$$

then

$$\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{i^{n-1}}{2^{n-1}}\prod_{p=1}^{n-1}\exp\left(-i\,\frac{p\pi}{n}\right)\prod_{p=1}^{n-1}z_p$$

$$= \frac{i^{n-1}}{2^{n-1}}\exp\left(-i\,\frac{\pi}{n}\cdot\frac{1}{2}(n-1)n\right)\prod_{p=1}^{n-1}z_p = \frac{1}{2^{n-1}}\exp\left((n-1)i\,\frac{\pi}{2}\right)\exp\left(-(n-1)i\,\frac{\pi}{2}\right)\prod_{p=1}^{n-1}z_p$$

$$= \frac{1}{2^{n-1}}\prod_{p=1}^{n-1}z_p.$$

Now, we also have

$$(1-z)^n - 1 = \sum_{j=0}^{n}\binom{n}{j}(-1)^j z^j - 1 = \sum_{j=1}^{n}\binom{n}{j}(-1)^j z^j = -z\prod_{p=1}^{n-1}(z_p - z)$$

$$= -\prod_{p=1}^{n-1}z_p\cdot z + \cdots + (-z)^n.$$

so by identifying the coefficients for $j=1$ we get

$$-\prod_{p=1}^{n-1}z_p = \binom{n}{1}(-1)^1 = -n,$$

hence by insertion,

$$\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

**Example 3.9** *Solve the equation $z^n = \overline{z}$ for every $n \in \mathbb{N}$.*
*Also, solve the equation, when $n \in \mathbb{Z}$.*

If $n = 1$, then the equation becomes $z = \overline{z}$. The set of complex numbers which are equal to their complex conjugated, is equal to the set of real number, hence the solution is $\mathbb{R}$.

If $n \geq 2$, then $z = 0$ is trivially a solution.
Then assume that $z \neq 0$. Put $z = r\,e^{i\theta}$, $r > 0$, which gives

$$r^n e^{in\theta} = r\,e^{-i\theta}, \qquad r > 0,$$

thus

$$r^{n-1}e^{i(n+1)\theta} = 1.$$

We conclude that $r = 1$ and $\theta = \frac{2p\pi}{n+1}$, $p = 0, 1, \ldots, n$, so the equation has the $n+2$ solutions

$$z_0 = 0 \quad\text{and}\quad z_p = \exp\left(i\,\frac{2p\pi}{n+1}\right), \quad p = 1, 2, \ldots, n+1.$$

If $n = 0$, then the equation is reduced to $\overline{z} = 1$, and it follows that the solution is $z = 1$.

Finally, let $n < 0$. If we write $n = -m$, $m \in \mathbb{N}$, then $z^{-m} = \overline{z}$. In particular, 0 is never a solution.

If $m = 1$, then $z^{-1} = \overline{z}$ is rewritten as

$$1 = \overline{z} \cdot z = |z|^2,$$

and the set of solutions is the unit circle.

When $m > 1$, we use polar coordinates $z = r\, e^{i\theta}$, so

$$1 = \overline{z}\, z^m = r^{m+1} \cdot e^{-i\theta} \cdot e^{im\theta} = r^{m+1} e^{i(m-1)\theta},$$

hence $r = 1$ and $\theta = \dfrac{2p\pi}{m-1}$. Thus we obtain in this case $m - 1 = |n| - 1 = -n - 1$ solutions,

$$z_p = \exp\left(i\,\frac{2p\pi}{m-1}\right) = \exp\left(-i\,\frac{2p\pi}{n+1}\right), \qquad p = 1, 2, \ldots, m-1 = -n-1.$$

**Example 3.10** *Prove that the function*

$$f(z) = 2x^4 - 12x^2 y^2 + 2y^4 - 3x^3 + 9xy^2 + i\left(8x^3 y - 8xy^3 - 9x^2 y + 3y^3\right),$$

*can be written as a polynomial in the complex variable $z$.*
*Then find the roots.*

Concerning polynomials, a good strategy is to identify the degrees of the pair $(x, y)$, which occur. We see that we have the degrees 4 and 3, and since

$$z^4 = (x+iy)^4 = x^4 + 4ix^3 y - 6x^2 y^2 - 4ixy^3 + y^4 = x^4 - 6x^2 y^2 + y^4 + i\left(4x^3 y - 4xy^3\right)$$

and

$$z^3 = (x+iy)^3 = x^3 + 3ix^2 y - 3xy^2 - iy^3 = x^3 - 3xy^2 + i\left(3x^2 y - y^3\right),$$

it follows that

$$
\begin{aligned}
f(z) &= 2x^4 - 12x^2 y^2 + 2y^4 - 3x^3 + 9xy^2 + i\left(8x^3 y - 8xy^3 - 9x^2 y + 3y^3\right) \\
&= 2\left(x^4 - 6x^2 y^2 + y^4 + i\left\{4x^3 y - 4xy^3\right\}\right) - 3\left(x^3 - 3xy^2 + i\left\{3x^2 y - y^3\right\}\right) \\
&= 2z^4 - 3z^3,
\end{aligned}
$$

thus

$$f(z) = 2z^4 - 3z^3 = 2z^3\left(z - \frac{3}{2}\right),$$

and the roots are $z = 0$ (of multiplicity 3) and the simple root $z = \dfrac{3}{2}$.

**Example 3.11** (ENESTRÖM'S THEOREM). *Given the real numbers $p_0$, $p_1$, ..., $p_n$, for which*

$$p_0 > p_1 > p_2 > \cdots > p_n > 0.$$

*Prove that the polynomial*

$$P(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n$$

*does not have a zero in the open unit disc $|z| < 1$.*
HINT: *Consider $(1-z)P(z)$.*

When we compute $(1-z)P(z)$ we obtain

$$
\begin{aligned}
(1-z)P(z) &= \left(p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n\right) - \left(p_0 z + p_1 z^2 + \cdots + p_{n-1} z^n + p_n z^{n+1}\right) \\
&= p_0 - (p_0 - p_1)\, z - (p_1 - p_2)\, z^2 - \cdots - (p_{n-1} - p_n)\, z^n - p_n z^{n+1}.
\end{aligned}
$$

According to we assumption we have $p_{j-1} - p_j > 0$, so when $|z| < 1$ we get the estimate

$$|(1-z)P(z)| > p_0 - (p_0 - p_1) - (p_1 - p_2) - \cdots - (p_{n-1} - p_n) - p_n = 0,$$

and we conclude that $(1-z)P(z) \neq 0$ for $|z| < 1$, so in particular that $P(z)$ does not have any zero in the open unit disc.

# 4 Equations of second degree

**Example 4.1** *Let $z_0 \in \mathbb{C} \setminus \{0\}$. Solve the equation*

$$\frac{1}{z + z_0} = \frac{1}{z} + \frac{1}{z_0}.$$

Here we of course must assume that $z \neq 0$ and $z \neq -z_0$. When we multiply by the common denominator $z_0 z (z + z_0)$, we get

$$z_0 z = (z + z_0) z_0 + z (z + z_0) = z_0 z + z_0^2 + z^2 + z z_0 \quad \left( = (z + z_0)^2 \right),$$

and we obtain the equation

$$z^2 + z_0 z + z_0^2 = 0,$$

the solutions of which are

$$z = \frac{-z_0 \pm \sqrt{z_0^2 - 4 z_0^2}}{2} = z_0 \left( -\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right).$$

**Remark 4.1** Many years ago, in the 1970s, I constructed this example, because many of my students erroneously believed that one always had

$$\frac{1}{a + b} = \frac{1}{a} + \frac{1}{b}.$$

It will be proved below that this is *never* true in $\mathbb{R}$. Furthermore, it is proved that there are complex solutions, but they do not match with the wrong "rule of calculation". ◊

**Example 4.2** *Find the roots of $z^2 + (6 + i)z + 7 - 3i$.*

When we apply the solution formula, we get

$$
\begin{aligned}
z &= \frac{-(6+i) \pm \sqrt{(6+i)^2 - 4(7-3i)}}{2} = \frac{1}{2} \left\{ -6 - i \pm \sqrt{35 + 12i - 28 + 12i} \right\} \\
&= \frac{1}{2} \left\{ -6 - i \pm \sqrt{7 + 24i} \right\} = \frac{1}{2} \left\{ -6 - i \pm \sqrt{16 - 9 + 24i} \right\} \\
&= \frac{1}{2} \left\{ -6 - i \pm \sqrt{4^2 + (3i)^2 + 2 \cdot 4 \cdot 3i} \right\} \\
&= \frac{1}{2} \left\{ -6 - i \pm \sqrt{(4 + 3i)^2} \right\} = \frac{1}{2} \left\{ -6 - i \pm (4 + 3i) \right\} \\
&= \begin{cases} \frac{1}{2} \left\{ -2 + 2i \right\} &= -1 + i, \\[2mm] \frac{1}{2} \left\{ -10 - 4i \right\} &= -5 - 2i, \end{cases}
\end{aligned}
$$

so the roots are

$$z_1 = -1 + i \qquad \text{and} \qquad z_2 = -5 - 2i.$$

A CHECK shows that the sum of the roots is

$$(-1 + i) + (-5 - 2i) = -6 - i = -(6 + i),$$

i.e. the coefficient of $z$ of the opposite sign, and the product of the roots is

$$(-1 + i)(-5 - 2i) = 5 + 2 - 5i + 2i = 7 - 3i,$$

i.e. equal to the constant term of the polynomial. We see that the check is OK. $\Diamond$

**Remark 4.2** If one does not see immediately that $7 = 4^2 - 3^2$, then we may try to solve the equation $(a + ib)^2 = 7 + 24i$. This gives us the equations

$$a^2 - b^2 = 7 \qquad \text{og} \qquad 2ab = 24,$$

hence

$$\left(a^2 + b^2\right)^2 = \left(a^2 - b^2\right)^2 + (2ab)^2 = 7^2 + 24^2 = 49 + 576 = 625 = 25^2,$$

and thus

$$a^2 + b^2 = 25,$$

so when we combine it with $a^2 - b^2 = 7$ we obtain $a^2 = 16$ and $b^2 = 9$.

**Example 4.3** *Solve the equation*

$$\frac{1}{2}\left(z + \frac{1}{2}\right) = a \qquad \text{for } a \in \mathbb{C}.$$

*Prove that the equation has precisely one solution in the open unit disc $|z| < 1$, if and only if $a$ does not belong to the real interval $[-1, 1]$.*

Here we must assume that $z \neq 0$. Then by a multiplication by $2z$ and a rearrangement we get

(4)    $z^2 - 2a\,z + 1 = 0,$

the solutions of which are

$$z = a \pm \sqrt{a^2 - 1}.$$

Denote the roots by $z_1$ and $z_2$. The structure of (4) shows that the product of the roots is 1, hence $z_1 \cdot z_2 = 1$. If therefore e.g. $|z_1| \geq 1$, then we necessarily must $|z_2| \leq 1$, and *vice versa*. This shows that we have precisely one solution in the open unit disc $|z| < 1$, if and only if the other solution lies in the open complementary set $|z| > 1$ of the closed unit disc.

The condition that this is *not* the case is equivalent to $|z_1| = |z_2| = 1$. Let $z$ be a shorthand for any of these solutions. Then $z = e^{i\theta}$, if $|z| = 1$, and it follows by insertion that

$$a = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta.$$

This shows that $a$ goes through the interval $[-1, 1]$ twice, when $\theta$ goes through the parameter interval $[0, 2\pi]$ once.

Therefore, if $a \notin [-1, 1]$, then there is precisely one solution in the open unit disc $|z| < 1$.

Finally, if $a \in [-1, 1]$, then it follows from

$$z = a \pm \sqrt{a^2 - 1} = a \pm i\sqrt{1 - a^2}, \qquad 1 - a^2 \geq 0,$$

that

$$|z|^2 = a^2 + 1 - a^2 = 1,$$

and we have proved that in this case both $z_1$ and $z_2$ lie on the unit circle.

**Remark 4.3** The function

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad z \neq 0,$$

is also called *Joukovski's function.* It was many years ago applied by Joukovski in order to describe the streamlines around the wing of an aeroplane. $\Diamond$

**Example 4.4** *Prove that $1 \pm i$ are the roots of the polynomial*

$$z^4 - 2z^3 + 3z^2 - 2z + 2.$$

*Then find all its roots.*

**First method.** It is seen by *inspection* that

$$z^4 - 2z^3 + 3z^2 - 2z + 2 = \left(z^4 - 2z^3 + 2z^2\right) + \left(z^2 - 2z + 2\right)$$
$$= \left(z^2 + 1\right)\left(z^2 - 2z + 2\right) = \left(z^2 - 1\right)\left(\{z - 1\}^2 + 1\right),$$

hence the roots are $z = \pm i$ and $z = 1 \pm i$.

**Second method.** When $1 \pm i$ are roots, then

$$(z - \{1 + i\})(z - \{1 - i\}) = (\{z - 1\} - i)(\{z - 1\} + i) = (z - 1)^2 + 1 = z^2 - 2z + 2$$

must be a divisor in the polynomial. Again we find that

$$z^4 - 2z^3 + 3z^2 - 2z + 2 = \left(z^2 + 1\right)\left(z^2 - 2z + 2\right),$$

so the roots are $z = \pm i$ and $1 \pm i$.

**Example 4.5** *Given that $1 - i$ is a root, find all the roots of the polynomial*

$$z^4 - 2z^3 + 6z^2 - 8z + 8.$$

**First method.** It is seen by *inspection* that

$$\begin{aligned}
z^4 - 2z^3 + 6z^2 - 8z + 8 &= z^4 - 2z^3 + 2z^2 + 4z^2 - 8z + 8 = \left(z^2 + 2\right)\left(z^2 - 2z + 2\right) \\
&= \left(z^2 + 2^2\right)\left(\{z - 1\}^2 + 1\right),
\end{aligned}$$

and we get immediately the roots $z = \pm 2i$ and $z = 1 \pm i$.

**Second method.** The polynomial has only *real* coefficients, so the roots are either real, or pairwise complex conjugated. Thus, since $1 - i$ is a root, also $1 + i$ must be a root, hence

$$(z - \{1 - i\})(z - \{1 + i\}) = (\{z - 1\} + i)(\{z - 1\} - i) = z^2 - 2z + 2$$

must be a. By a polynomial division we get

$$z^4 - 2z^3 + 6z^2 - 8z + 8 = \left(z^2 + 4\right)\left(z^2 - 2z + 2\right),$$

and it follows as before that the roots are $z = \pm 2i$ and $z = 1 \pm i$.

**Remark 4.4** It is worth mentioning that a division by

$$z - \{1 - i\} = z - 1 + i$$

will give some very unpleasant calculations and that such a division only reduces the problem to a messy equation of third degree. Therefore, one should always exploit the the roots are either real or pairwise conjugated, if the polynomial has real coefficients. ◊

**Example 4.6** *Solve the equation* $\left(z^4 - 16\right)\left(z^3 + 1\right) = 0$.

It is seen by *inspection* that the four roots of $z^4 - 16$ are $2$, $2i$, $-2$, $-2i$, and that the three roots of $z^3 + 1$ are $-1$, $\dfrac{1}{2} + i\,\dfrac{\sqrt{3}}{2}$, $\dfrac{1}{2} - i\,\dfrac{\sqrt{3}}{2}$. Hence, the seven roots are

$$2, \quad 2i, \quad -2, \quad -2i, \quad -1, \quad \frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \quad \frac{1}{2} - i\,\frac{\sqrt{3}}{2}.$$

**Remark 4.5** The example is of course trivial, when it is solved in this way. However, it is a warning against starting by multiplying the two factors, in which case one shall get the equation of seventh degree:

$$z^7 + z^4 - 16z^3 - 16 = 0.$$

This is not very smart, even if it in this case is possible to guess the rational roots $z = -2$, $-1$ and $2$. Then we get by the usual reduction that

$$z^7 + z^4 - 16z^3 - 16 = (z + 2)(z + 1)(z - 2)\left(z^4 - z^3 + 5z^2 - 4z + 4\right) = 0,$$

and the problem has now been "reduced" to finding the roots of the equation

$$z^4 - z^3 + 5z^2 - 4z + 4,$$

which does not look very promising.

The lesson here is that one in almost every case should keep a given *factorization* and not multiply the factors. ◊

**Example 4.7** *Solve the equations*

(a)     $z^3 + 3 = 0$,                         (b)     $z^4 + 16 = 0$,
(c)     $z^8 - 2z^4 + 1 = 0$,              (d)     $z^3 + z^2 + z + 1 = 0$.

**(a)** The binomial equation is here written as

$$z^3 = -3 = 3\,e^{i(\pi + 2p\pi)}, \qquad p \in \mathbb{Z},$$

thus

$$z = \sqrt[3]{3} \cdot \exp\left(i\left\{\frac{\pi}{3} + \frac{2p\pi}{3}\right\}\right), \qquad p = 0,\ 1,\ 2.$$

More precisely,

$$z_0 = \sqrt[3]{3}\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right), \qquad z_1 = -\sqrt[3]{3}, \qquad z_2 = \sqrt[3]{3}\left(\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right).$$

**(b)** The binomial equation is here written

$$z^4 = -16 = 2^4 e^{i(\pi + 2p\pi)}, \qquad p \in \mathbb{Z},$$

thus

$$z = 2\,\exp\left(i\left\{\frac{\pi}{4} + p \cdot \frac{\pi}{2}\right\}\right), \qquad p = 0,\ 1,\ 2,\ 3.$$

More precisely,

$$z_0 = 2\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right) = \sqrt{2}\,(1 + i), \qquad z_1 = i\,z_0 = \sqrt{2}\,(-1 + i),$$

$$z_2 = -z_0 = -\sqrt{2}\,(1 + i), \qquad z_3 = -z_1 = \sqrt{2}\,(1 - i).$$

**(c)** The equation is a disguised equation of second degree in $z^4$, so

$$0 = z^8 - 2z^4 + 1 = \left(z^4 - 1\right)^2,$$

and it follows that the *eight* roots are

$$1,\ 1,\ i,\ i,\ -1,\ -1,\ -i,\ -i,$$

because the binomial equation $z^4 - 1 = 0$ has the four roots $1,\ i,\ -1,\ -i$.

**(d)** It follows by *inspection* that

$$z^3 + z^2 + z + 1)z^2(z + 1) + (z + 1) = (z + 1)\left(z^2 + 1\right),$$

and we conclude that the roots are $-1$, $i$ and $-i$.

ALTERNATIVELY it is seen that $z^3 + z^2 + z + 1$ are the first four terms of a quotient series. Therefore, if $z \neq 1$, then

$$z^3 + z^2 + z1 = \frac{z^4 - 1}{z - 1}, \qquad z \neq 1.$$

The numerator $z^4 - 1$ has the four roots $1,\ i,\ -1,\ -i$, and when $z = 1$, it follows that $z^3 + z^2 + z + 1 = 4 \neq 0$. Therefore, we conclude that the three roots are $i$, $-1$, $-i$.

**Example 4.8** *Prove that all the roots of the equation*

$$(z+1)^5 + z^5 = 0$$

*lie on the line* $x = -\dfrac{1}{2}$.
*Then prove that this is in general the case of the roots of the equation*

$$(z+1)^n + z^n = 0,$$

*for every* $n \in \mathbb{N}$.

We shall immediately prove the *general result.* Clearly, $z = 0$ is not a solution of the equation. When we divide by $z^n$, the equation becomes

$$\left(\frac{z+1}{z}\right)^n + 1 = 0.$$

By putting $w = \dfrac{z+1}{z}$ we get the binomial equation $w^n = -1$, and in particular $|w| = 1$.
We then check what the unit circle $|w| = 1$ is mapped into by the inverse transformation

$$z = \frac{1}{w-1}.$$

If we here put $w = e^{i\theta}$, $\theta \neq 2p\pi$, $p \in \mathbb{Z}$, i.e. $w \neq 1$, then

$$
\begin{aligned}
z &= \frac{1}{e^{i\theta}-1} = \frac{e^{-i\theta}-1}{(e^{i\theta}-1)(e^{-i\theta}-1)} = \frac{\cos\theta - 1 - i\sin\theta}{1+1-(e^{i\theta}+e^{-i\theta})} = \frac{\cos\theta - 1 - i\sin\theta}{2(1-\cos\theta)} \\
&= -\frac{1}{2} - i \cdot \frac{\sin\theta}{2(1-\cos\theta)}.
\end{aligned}
$$

Therefore, every root $z = x + iy$ of the original equation must therefore have the form

$$z = -\frac{1}{2} + i \cdot \frac{\sin\theta}{2(\cos\theta - 1)} = -\frac{1}{2} + i \cdot \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\left(1 - 2\sin^2\frac{\theta}{2} - 1\right)} = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2},$$

and it follows that the real part is always $x = -\dfrac{1}{2}$ as required.

**Remark 4.6** It also follows from the above that since $w^n = -1$, then

$$\theta = \frac{(2p+1)\pi}{n}, \qquad p = 0, 1, \ldots, n-1,$$

hence by an insertion,

$$z_p = -\frac{1}{2} - \frac{i}{2}\cot\left(\frac{(2p+1)\pi}{2n}\right), \qquad p = 0, 1, \ldots, n-1. \qquad \Diamond$$

**Example 4.9** *Find all the complex solutions of the equation*

$$\left\{z^2 - 2(1+i)z - 2\right\}^2 = -8i.$$

This equation can also be written

$$w^2 = -8i, \qquad \text{where} \quad w = z^2 - 2(1+i)z - 2.$$

Let us first solve the binomial equation:

$$w = \pm 2(1-i).$$

By insertion we get two equations of second degree, which are solved in the usual way:

1) If $w = 2(1-i)$, then after a reduction

$$z^2 - 2(1+i)z - 2i = 0,$$

hence

$$z = 1 + i \pm \sqrt{2i + 2i} = 1 + i \pm \sqrt{2}(1+i).$$

2) If $w = -2(1-i)$, then after a reduction

$$z^2 - 2(1+i) + 2i - 4 = 0,$$

hence

$$z = 1 + i \pm \sqrt{2i - 2i + 4} = 1 + i \pm 2.$$

Summing up, the given equation of fourth degree has the roots

$$(1 + \sqrt{2})(1+i), \qquad (1 - \sqrt{2})(1+i), \qquad 3 + i, \qquad -1 + i.$$

# 5 Rational and multiple roots in polynomials

**Example 5.1** *Find all the roots of the polynomial*

$$p(z) = z^3 - (1 + 2i)z^2 - (1 - 2i)z + 1,$$

*given that it has a multiple root.*

Since

$$p'(z) = 3z^2 - (2 + 4i)z - (1 - 2i),$$

it follows by the usual algorithm of division that

$$p(z) = \left\{ \frac{1}{3}z - \frac{1 + 2i}{9} \right\} p'(z) + \frac{4}{9}iz + \frac{4}{9} = \left\{ 13z - \frac{1 + 2i}{9} \right\} p'(z) + \frac{4}{9}i(z - i).$$

The multiple root is a root in both $p(z)$ and in $p'(z)$, so it follows that it must be $z = i$.

Since now $z = i$ is a root of at least second order, $(z - i)^2 = z^2 - 2iz - 1$ must be a divisor in $p(z)$. We obtain by division that

$$p(z) = z^3 - (1 + 2i)z^2 - (1 - 2i)z + 1 = (z - 1)(z - i)^2,$$

and the three roots are $1$, $i$ and $i$.

**Example 5.2** *Find all the roots of the equation*

$$4z^4 - 4z^3 - 25z^2 + z + 6 = 0.$$

It follows by an *inspection* that

$$4z^4 - 4z^3 - 25z^2 + z + 6 = 4z^4 - 4z^3 - 24z^2 - z^2 + z + 6$$

$$= 4z^2 \left(z^2 - z - 6\right) - \left(z^2 - z - 6\right) = \left(4z^2 - 1\right)\left(z^2 - z - 6\right) = 4\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)(z - 3)(z + 2).$$

We conclude that the roots are

$$-2, \quad -\frac{1}{2}, \quad \frac{1}{2}, \quad 3.$$

ALTERNATIVELY, the equation has only *integers* as coefficients, hence the possible rational roots must be among

$$\pm 6, \ \pm 3, \ \pm\frac{3}{2}, \ \pm\frac{3}{4}, \ \pm 2, \ \pm 1, \pm\frac{1}{2}, \ \pm\frac{1}{4},$$

thus only 16 possibilities. By using a pocket calculator we get

| | | | |
|---|---|---|---|
| $P(6) = 3432,$ | $P(1) = -18,$ | $P(-6) = 5148,$ | $P(-1) = -12,$ |
| $P(3) = 0,$ | $P(\frac{3}{4}) = -7,734374,$ | $P(-3) = 210,$ | $P(-\frac{3}{4}) = -5,859375,$ |
| $p(2) = -60,$ | $p(\frac{1}{2}) = 0,$ | $P(-2) = 0,$ | $p(-\frac{1}{2}) = 0,$ |
| $p(\frac{3}{2}) = -42,$ | $P(\frac{1}{4}) = 4,640625,$ | $P(-\frac{3}{2}) = -18,$ | $P(-\frac{1}{4}) = 4,265625.$ |

It follows from this table that $3, -2, \dfrac{1}{2}$ and $-\dfrac{1}{2}$ are indeed roots. Since a polynomial of fourth degree has precisely 4 roots (counted by multiplicity), we have found all the roots.

**Example 5.3** *Find the possible multiple roots of the polynomial*

$$4z^5 + (8 + 16i)z^4 + (-27 + 32i)z^3 - (29 + 44i)z^2 + (44 + 12i)z - 12,$$

*and then find all roots.*

This is a difficult example, because the polynomial is of fifth degree, and because the coefficients are complex. A pocket calculator will here be quite helpful, though I did not use one while I was working on this example).

It follows after a differentiation that the task is to find all the common divisors of

$$(5) \quad \begin{cases} 4z^5 + (8 + 16i)z^4 + (-27 + 32i)z^3 + (-29 - 44i)z^2 + (44 + 12i)z - 12, \\[2mm] 20z^4 + (32 + 64i)z^3 + (-81 + 96i)z^2 + (-58 - 88i)z + (44 + 12i). \end{cases}$$

The former polynomial of (5) is multiplied by med 5, and the latter with $z$. This gives

$$\begin{cases} 20z^5 + (40 + 80i)z^4 + (-135 + 160i)z^3 + (-145 - 220i)z^2 + (220 + 60i)z - 60, \\[2mm] 20z^5 + (32 + 64i)z^4 + (-81 + 96i)z^3 + (-58 - 88i)z^2 + (44 + 12i)z, \end{cases}$$

hence by a subtraction,

$$(8+16i)z^4+(-54+64i)z^3+(-87-132i)z^2+(176+48i)z-60,$$

and we have obtained a polynomial of fourth degree which also must have the multiple root as a root. The coefficients of the highest order term is $8 + 16i = 8(1 + 2i)$, so we multiply the latter polynomial by $1 - 2i$. Then

$$40z^4+(74+172)z^3+(-351+42i)z^2+(272-304i)z-60+120i,$$

which also must have the possible multiple roots as roots.

If we subtract twice times the latter polynomial of (5), from this polynomial, we obtain the following reduce polynomial, which also has the wanted roots as some of its roots,

$$(10+44i)z^3+(-189-150i)z^2+(388-128i)z+(-148+96i).$$

Since $10 + 44i = 2(5 + 22i)$, we obtain a "nicer" expression which we multiply this polynomial by $5 - 22i$, hence

$$1018z^3+(-4245+3408i)z^2+(-876-9176i)z+(1372+3736i).$$

We have now reduced our system to

$$(6) \quad \begin{cases} 20z^4+(32+64i)z^3+(-81+96i)z^2+(-58-88i)z+(44+12i), \\ 1018z^3+(-4245+3408)z^2+(-876-9176i)z+(1372+3736i). \end{cases}$$

If we multiply the former polynomial of (6) by 509 and the latter by $10z$, then we get the following rather messy system,

$$\begin{cases} 10\,180z^4+(16\,288+32\,576i)z^3+(-41\,229+48\,864i)z^2+(-29\,522-44\,792i)z+(22\,396+6\,108i), \\ 10\,180z^4+(-42\,450+34\,080i)z^3+(-8\,760-91\,760i)z^2+(13\,720+37\,360i)z, \end{cases}$$

hence by a subtraction,

$$(58\,738-1\,504i)z^3+(-32\,469+140\,624i)z^2+(-43\,242-82\,152i)z+(22\,396+6\,108i).$$

This expression is then multiplied by 1018,

$$(59\,795\,284-1\,531\,072i)z^3+(-33\,053\,442+143\,153\,232i)z^2$$
$$+(-44\,020\,356-83\,630\,736i)z+(22\,799\,128+6\,217\,944i),$$

which should be linked with the latter expression of (6) when it is multiplied by $58\,738 - 1\,504i$, thus

$$(59\,795\,284-1\,531\,072i)z^3+(-244\,217\,178+206\,563\,584i)z^2$$
$$+(-65\,255\,192-537\,662\,384i)z+(86\,207\,480+217\,381\,680i).$$

Then by a subtraction,

$$(211\,163\,736-63\,408\,532i)z^2+(21\,234\,836+454\,031\,648i)z+(-63\,408\,352-211\,163\,736i).$$

A division by 4 reduces this polynomial to

$$52\,790\,934-15\,852\,088i)z^2+(5\,308\,709+113\,507\,912i)z+(-15\,852\,088i-52\,790\,934i).$$

Then we see that 49 is also a divisor, so we can reduce to

$$(1\,077\,366-323\,512i)z^2+(108\,341+2\,316\,488i)z+(-323\,512-1\,077\,366i).$$

Then we multiply by $1\,077\,366 + 323\,512i$, in order to get

$$1\,265\,377\,512\,100z^2+(-632\,688\,756\,050+2\,530\,755\,024\,200)z-1\,265\,377\,512\,100$$
$$= 632\,688\,756\,055\left\{2z^2 + (-1 + 4i)z - 2i\right\}.$$

The task has now been reduced to find the common roots of the system

$$(7)\quad \begin{cases} 4z^5+(8+16i)z^4+(-27+32i)z^3+(-29-44i)z^2+(44+12i)z-12 = 0, \\[2mm] 2z^2+(-1+4i)z-2i = 0, \end{cases}$$

where we use the original polynomial as our first one, because it looks 'simplest', concerning the coefficients. Notice that any of the polynomials derived above could be used instead.

Then again we perform a division by polynomials, but this time we (fortunately) obtain

$$4z^5+(8+16i)z^4+(-27+32i)z^3+(-29-44i)z^2+(44+12i)z-12$$
$$= \left(2z^2+\{-1+4i\}z-2i\right)\left(2z^3+\{5+4i\}z^2+\{-3+10i\}z-6i\right).$$

According to the theory, the roots of $2z^2 + \{-1 + 4i\}z - 2i$ must be double roots of the original polynomial. Hence, we can perform another division by a polynomial, which gives us

$$4z^5+(8+16i)z^4+(-27+32i)z^3+(-29-44i)z^2+(44+12i)z-12 = \left(2z^2+\{-1+4i\}z-2i\right)^2(z+3).$$

Therefore, we shall only find the roots of

$$2z^2 + (-1 + 4i)z - 2i.$$

We get by using the usual solution formula,

$$\begin{aligned} z &= \frac{1-4i\pm\sqrt{(1-4i)^2+4\cdot 2\cdot 2i}}{4} = \frac{1-4i\pm\sqrt{-15-8i+16i}}{4} = \frac{1-4i\pm\sqrt{-15+8i}}{4} \\[2mm] &= \frac{1-4i\pm\sqrt{1+(4i)^2+2\cdot 1\cdot 4i}}{4} = \frac{1-4i\pm\sqrt{(1+4i)^2}}{4} = \frac{1-4i\pm(1+4i)}{4} = \begin{cases} \dfrac{1}{2}, \\[2mm] -2i. \end{cases} \end{aligned}$$

The factorial expansion is therefore

$$4\left(z-\frac{1}{2}\right)^2(z+2i)^2(z+3),$$

and the roots are

$$\frac{1}{2}, \quad \frac{1}{2}, \quad -2i, \quad -2i, \quad -3.$$

**Remark 5.1** Once again we see why man should always keep a factorial expansion as long as possible. One loses a lot of information by multiplying the factors. ◊

**Remark 5.2** The computations above may be performed in various ways, some of them will not give as awkward numbers as the method above. The purpose here has only been to demonstrate what may happen if one slavic follows the standard method without using one's brain. ◊

**Remark 5.3** It is also worth mentioning that there exists an ALTERNATIVE method of solution. In the actual case it will also lead to huge computations, but one advantage is that all its coefficients will remain real. The idea is that if one splits int the real and the imaginary part,

$$
\begin{aligned}
P(z) &= 4z^5 + (8+16i)z^5 + (-27+32i)z^3 + (-29-44i)z^2 + (44+12i)z - 12 \\
&= \left\{ 4z^5 + 8z^4 - 27z^3 - 29z^2 + 44z - 12 \right\} + i \left\{ 16z^4 + 32z^3 - 44z^2 + 12z \right\} \\
&= \left\{ 4z^5 + 8z^4 - 27z^3 - 29z^2 + 44z - 12 \right\} + 4iz \left\{ 4z^4 + 8z^3 - 11z^2 + 3z \right\},
\end{aligned}
$$

we conclude that the possible *real* roots necessarily must be real roots of both the real part and the imaginary part. Now, $z = 0$ is not a roots, so we shall find the common divisors of

$$
\begin{cases}
4z^5 + 8z^4 - 27z^3 - 29z^2 + 44z - 12, \\
4z^3 + 8z^2 - 11z + 3.
\end{cases}
$$

When we apply the method of division of polynomials, we get the even better result, namely that the division is successful,

$$4z^5 + 8z^4 - 27z^3 - 29z^2 + 44z - 12 = \left(4z^3 + 8z^2 - 11z + 3\right)\left(z^2 - 4\right).$$

Hence,

$$
\begin{aligned}
P(z) &= \left\{4z^5 + 8z^4 - 27z^3 - 29z^2 + 44z - 12\right\} + 4iz\left\{4z^3 + 8z^2 - 11z + 3\right\} \\
&= \left(z^2 + 4iz - 4\right)\left(4z^3 + 8z^2 - 11z + 3\right) = (z + 2i)^2\left(4z^3 + 8z^2 - 11z + 3\right).
\end{aligned}
$$

Here we can check in the latter factor, if

$$\pm 3,\ \pm\frac{3}{2},\ \pm\frac{3}{4},\ \pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{4},$$

are roots. Alternatively we may check if there are other multiple roots, i.e. we shall find the common divisors of

$$\begin{cases} 4z^3 + 8z^2 - 11z + 3, \\ 12z^2 + 16z - 11. \end{cases}$$

The former polynomial is multiplied by 3 and the latter by $z$. This gives the equivalent system

$$\begin{cases} 12z^3 + 24z^2 - 33z + 9, \\ 12z^3 + 16z^2 - 11z, \end{cases}$$

hence by a subtraction, $8z^2 - 22z + 9$, and the task has been reduced to finding a common divisor of

$$\begin{cases} 12z^2 + 16z - 11, \\ 8z^2 - 22z + 9. \end{cases}$$

First we write this system as

$$\begin{cases} 24z^2 + 32z - 22. \\ 24z^2 - 66z + 27, \end{cases}$$

Then by a subtraction, $98z - 49$, and thus $z = \dfrac{1}{2}$. Finally, by insertion,

$$4 \cdot \frac{1}{2^3} + 8 \cdot \frac{1}{2^2} - 11 \cdot \frac{1}{2} + 3 = \frac{1}{2} + 2 - \frac{11}{2} + 3 = 0,$$

and $z = \dfrac{1}{2}$ is a root of multiplicity 2.

When we divide by

$$(2z - 1)^2 = 4z^2 - 4z + 1,$$

we get

$$
\begin{aligned}
4z^3 + 8z^2 - 11z + 3 &= z\left(4z^2 - 4z + 1\right) + 12z^2 - 12z + 3 \\
&= z\left(4z^2 - 4z + 1\right) + 3\left(4z^2 - 4z + 1\right) \\
&= (z + 3)(2z - 1)^2,
\end{aligned}
$$

and we have again proved that

$$P(z) = (2z - 1)^2(z + 2i)^2(z + 3).$$

**Example 5.4** *Find all the possible multiple roots of the polynomial*

$$z^5+(9+i)z^4+(27+10i)z^3+(23+37i)z^2+(-24+60i)z-36+36i,$$

*and find all its roots.*

After we have been taught a lesson in Example 5.3 we first try to find the possible real roots. These must be the common roots of the polynomials

$$\begin{cases} z^5 + 9z^4 + 27z^3 + 23z^2 - 24z - 36 & = & P_{Re}(z) \\ \\ \qquad\quad z^4 + 10z^3 + 37z^2 + 60z + 36 & = & P_{Im}(z). \end{cases}$$

When we multiply by $z$, we get

$$z^5 + 10z^4 + 37z^3 + 60z^2 + 36z = z \cdot P_{Im}(z),$$

and we conclude that

$$P_{Re}(z) = z^5 + 9z^4 + 27z^3 + 23z^2 - 24z - 36 = (z-1)P_{Im}(z),$$

thus

$$P(z) = P_{Re}(z) + i\,P_{Im}(z) = (z - 1 + i)\left(z^4 + 10z^3 + 37z^2 + 60z + 36\right).$$

This proves that $1 - i$ is a root, and the task has been reduced to finding the multiple roots of

$$z^4 + 10z^3 + 37z^2 + 60z + 36,$$

i.e. we shall find the common divisors of

$$(8) \begin{cases} z^4 + 10z^3 + 37z^2 + 60z + 36, \\ \\ \qquad\quad 4z^3 + 30z^2 + 74z + 60. \end{cases}$$

When the former polynomial of (8) is multiplied by 4 and the latter by $z$, we obtain the following equivalent system,

$$\begin{cases} 4z^4 + 40z^3 + 148z^2 + 240z + 144, \\ \\ 4z^4 + 30z^3 + 74z^2 + 60z, \end{cases}$$

thus we get by a subtraction the polynomial

$$10z^3 + 74z^2 + 180z + 144.$$

We have now reduced (8) to the simpler and equivalent system

$$(9) \begin{cases} 10z^3 + 74z^2 + 180z + 144, \\ \\ \quad 2z^3 + 15z^2 + 37z + 30, \end{cases}$$

where the latter expression of (9) stems from the latter expression of (8) after a division by 2. When the latter expression of (9) is multiplied by 5, then

$$10z^3 + 75z^2 + 185z + 150.$$

From this we subtract the former expression of (9) in order to get

$$z^2 + 5z + 6 = (z+2)(z+3).$$

Then put $z = -2$ and $z = -3$, into $P(z) = (z - 1 + i)P_{Im}(z)$, i.e. we check the solution,

$$P(-2) = P(-3) = 0,$$

because

$$P_{Im}(-2) \;\; = \;\; 2^4 - 10 \cdot 2^3 + 37 \cdot 2^2 - 60 \cdot 2 + 36 = 16 - 80 + 148 - 120 + 36 = 0,$$

$$P_{Im}(-3) \;\; = \;\; 3^4 - 10 \cdot 3^3 + 37 \cdot 3^2 - 60 \cdot 3 + 36 = 9(9 - 30 + 37 - 20 + 4) = 0.$$

According to the theory, both $-2$ and $-3$ are *multiple* roots (of multiplicity 2), thus

$$P(z) = (z - 1 + i)(z + 2)^2(z + 3)^2,$$

and the roots are

$$1 - i, \quad -2, \quad -2, \quad -3, \quad -3.$$

ALTERNATIVELY we demonstrate in the following what happens, if we instead apply the *standard method*. It follows after a differentiation that we shall find the common divisors of

$$\begin{cases} z^5 + (9+i)z^4 + (27+10i)z^3 + (23+37i)z^2 + (-24+60i)z - 36 + 36i, \\[2mm] 5z^4 + (36+4)z^3 + (81+30i)z^2 + (46+74i)z + (-24+60i). \end{cases}$$

Multiply the former polynomial by 5 and the latter by $z$. Then

$$\begin{cases} 5z^5 + (45+5i)z^4 + (135+50i)z^3 + (115+185i)z^3 + (-120+300i)z - 180 + 180i, \\[2mm] 5z^5 + (36+4i)z^4 + (81+30i)z^3 + (46+74i)z^2 + (-24+60i)z, \end{cases}$$

hence by a subtraction

$$(9+i)z^4 + (54+20i)z^3 + (69+111i)z^2 + (-96+240i)z - 180 + 180i.$$

When we multiply by $9 - i$ we obtain the polynomial

$$82z^4 + (506+126i)z^3 + (732+930i)z^2 + (-624+2256i)z + (-1440+1800i),$$

and it follows after a division by 2 that we shall find the common divisors of

$$\begin{cases} 41z^4 + (253+3i)z^3 + (366+465i)z^2 + (-312+1128i)z + (-720+900i), \\[2mm] 5z^4 + (36+4i)z^3 + (81+30i)z^2 + (46+74i)z + (-24+60i). \end{cases}$$

Multiply the former polynomial by 5 and the latter by 41. Then we obtain the equivalent system

$$\begin{cases} 205z^4 + (1265+315i)z^3 + (1830+2325i)z^2 + (-1560+5640i)z + (-3600+4500i), \\[2mm] 205z^4 + (1476+164i)z^3 + (3321+1230i)z^2 + (1886+3034i)z + (-984+2460i). \end{cases}$$

By a subtraction,

$$(211-151i)z^3+(1491-1095i)z^2+(3446-2606i)z+(2616-2040i),$$

and we have got the system

$$\begin{cases} 5z^4+(36+4i)z^3+(81+30i)z^2+(46+74i)z+(-24+60i), \\ (211-151i)z^3+(1491-1095i)z^2+(3446-2606i)z+(2616-2040i). \end{cases}$$

The former polynomial is multiplied by $211 - 151i$, and the latter by $5z$. then

$$\begin{cases} (1055-755i)z^4+(8200-4592i)z^3+(21621-5901i)z^2+(20880+8668i)z+(3996+16284i), \\ (1055-755i)z^4+(7455-5475i)z^3+(17230-13030i)z^2+(13080-10200i)z, \end{cases}$$

and hence by a subtraction,

$$\begin{cases} (745+883i)z^3+(4391+7129i)z^2+(7800+18868i)z+(3996+16284i) \\ (211-151i)z^3+(1491-1095i)z^2+(3446-2606i)z+(2616-2040i). \end{cases}$$

We get

$$
\begin{cases}
(290\,528 + 73\,818i)z^3 + (2\,002\,980 + 841\,178i)z^2 \\
\qquad +(4\,494\,868 + 2\,803\,348i)z + (3\,302\,040 + 2\,832\,528i), \\
\\
(290\,528 + 73\,818i)z^3 + (2\,077\,680 + 500\,778i)z^2 \\
\qquad (4\,868\,368 + 1\,101\,348i)z + (3\,750\,240 + 790\,128i).
\end{cases}
$$

Then by another subtraction,

$$
(74\,700 - 340400i)z^2 + (373\,500 - 1\,702\,000i)z + (448\,200 - 2\,042\,400i).
$$

Here we can remove the common factor 100, so

$$
(747 - 3404i)z^2 + (3735 - 17020i)z + (4482 - 20424i).
$$

Now,

$$
\begin{aligned}
3735 - 17020i &= 5(747 - 3404i), \\
4482 - 20424i &= 6(747 - 3404i),
\end{aligned}
$$

so when we divide the polynomial by $747 - 3404i$, we get

$$
z^2 + 5z + 6.
$$

Then by another division,

$$
\begin{aligned}
z^5 + (9+i)z^4 + (27+10i)z^3 + (23+37i)z^2 + (-24+60i)z - 36 + 36i \\
= \left(z^2 + 5z + 6\right)\left(z^3 + (4+i)z^2 + (1+5i)z + (-6+6i)\right).
\end{aligned}
$$

Since this division was successful, the roots of $z^2 + 5z + 6$ must be double roots, so $z^2 + 5z + 6$ must again be a divisor. Then

$$
z^3 + (4+i)z^2 + (1+5i)z + (-6+6i) = \left(z^2 + 5z + 6\right)(z - 1 + i),
$$

and we finally obtain the factorial expansion

$$
P(z) = (z - 1 + i)\left(z^2 + 5z + 6\right)^2 = (z - 1 + i)(z + 2)^2(z + 3)^2.
$$

It follows that the roots are

$$
1 - i, \quad -2, \quad ,-2, \quad -3, \quad -3.
$$

# 6 Symbolic currents and voltages

**Example 6.1** *Let $\iota_1 = I_1 \sin(\omega t + \varphi_1)$ and $\iota_2 = I_2 \sin(\omega t + \varphi_2)$ be two sine currents. Find*

$$\iota = \iota_1 + \iota_2 = I \sin(\omega t + \varphi),$$

*first by using the trigonometric addition formulæ, and then by using complex currents.*

**The addition formulæ.** We get by a direct computation that

$$
\begin{aligned}
\iota &= \iota_1 + \iota_2 = I_1 \sin(\omega t + \varphi_1) + I_2 \sin(\omega t + \varphi_2) \\
&= I_1 \sin \omega t \cdot \cos \varphi_1 + I_2 \sin \omega t \cdot \cos \varphi_2 + I_1 \cos \omega t \cdot \sin \varphi_1 + I_2 \cos \omega t \cdot \varphi_2 \\
&= \{I_1 \cos \varphi_1 + I_2 \cos \varphi_2\} \sin \omega t \{I_1 \sin \varphi_1 + I_2 \sin \varphi_2\} \cos \omega t,
\end{aligned}
$$

and

$$\iota = I \sin(\omega t + \varphi) = I \cos \varphi \cdot \sin \omega t + I \sin \varphi \cdot \cos \omega t, \qquad t \in \mathbb{R}.$$

When we identify the two expressions we obtain the equations

$$
(10) \quad
\begin{cases}
I \cos \varphi = I_1 \cos \varphi_1 + I_2 \cos \varphi_2, \\
I \sin \varphi = I_1 \sin \varphi_1 + I_2 \sin \varphi_2,
\end{cases}
$$

hence

$$
\begin{aligned}
I^2 &= I^2 \cos^2 \varphi + I^2 \sin^2 \varphi = (I_1 \cos \varphi_1 + I_2 \cos \varphi_2)^2 + (I_1 \sin \varphi_1 + I_2 \sin \varphi_2)^2 \\
&= I_1^2 \cos^2 \varphi_1 + 2I_1 I_2 \cos \varphi_1 \cos \varphi_2 + I_2^2 \cos^2 \varphi_2 + I_1^2 \sin^2 \varphi_1 + 2I_1 I_2 \sin \varphi_1 \sin \varphi_2 + I_2^2 \sin^2 \varphi_2 \\
&= I_1^2 + I_2^2 + 2I_1 I_2 \cos(\varphi_1 - \varphi_2),
\end{aligned}
$$

so

$$I = \sqrt{I_1^2 + I_2^2 + 2I_1 I_2 \cos(\varphi_1 - \varphi_2)},$$

and $\varphi$ is described by the equations

$$\cos \varphi = \frac{I_1}{I} \cos \varphi_1 + \frac{I_2}{I} \cos \varphi_2, \qquad \sin \varphi = \frac{I_1}{I} \sin \varphi_1 + \frac{I_2}{I} \sin \varphi_2.$$

**The complex current.** If we instead use the complex current, we get

$$\tilde{I} = I\,e^{i\varphi} = I_1 e^{i\varphi_1} + I_2 e^{i\varphi_2} = I_1 \cos \varphi_1 + I_2 \cos \varphi_2 + i\{I_1 \sin \varphi_1 + I_2 \sin \varphi_2\},$$

and we conclude again (10).
The remaining part of the example is then treated as in the first variant.

# 7　Geometrical point sets

**Example 7.1** *Find the plan point sets which are defined by the conditions*
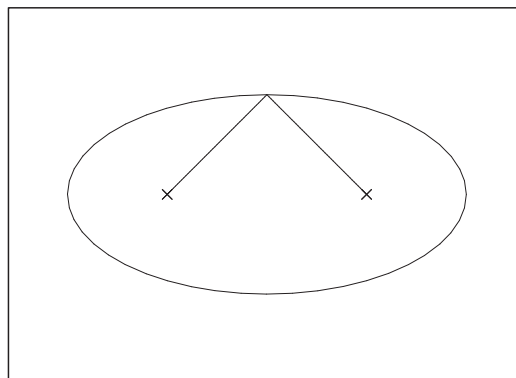
$$(a)\ \ |z - a| + |z - b| = k, \qquad (b)|z - a| + |z - b| \le k,$$

*where $a$, $b \in \mathbb{C}$ and $k \in \mathbb{R}_+$, and where $k > |a - b|$.*

**(a)** 1) The *geometric condition* is:

- Find all point $z$, for which the sum of the distances from $z$ to $a$ and from $z$ to $b$ is a constant $k$.

This is the definition of an ellipse with the focal points $a$ and $b$. If, however, $a = b$, then we get a circle instead of centrum $a$ and radius $\dfrac{k}{2}$.



2) *Analytically* this is proved in the following way:
Put

$$z = x + iy, \qquad a = a_1 + ia_2 \quad \text{and} \quad b = b_1 + ib_2.$$

Then the equation becomes

$$\sqrt{(x - a_1)^2 + (y - a_2)^2} = k - \sqrt{(x - b_1)^2 + (y - b_2)^2} \qquad (\ge 0).$$

By squaring,

$$\left(x^2 - 2a_1 x + a_1^2\right) + \left(y_2 - 2a_2 y + a_2^2\right)$$
$$= k^2 + \left(x^2 - 2b_1 x + b_1^2\right) + \left(y_2 - 2b_2 y + b_2^2\right) - 2k\sqrt{(x - b_1)^2 + (y - b_2)^2}.$$

This equation is rewritten as

$$2k\sqrt{(x - b_1)^2 + (y - b_2)^2} = 2\,(a_1 - b_1)\,x + 2\,(a_2 - b_2)\,y + \left\{k^2 + b_1^2 + b_2^2 - a_1^2 - a_2^2\right\}.$$

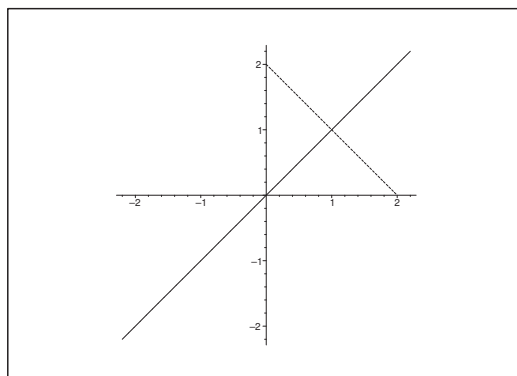By another squaring we get qualitatively the equation of an ellipse,

$$A^2 \,(x_1 - \alpha)^2 + B^2 \,(y_1 - \beta)^2 = C^2,$$

where $x_1$ and $y_1$ are linear expressions in $x$ and $y$, and where $x_1$ and $y_1$ are linearly independent.

**(b)** In this case we only add all the interior points of the ellipse.

**Example 7.2** *Find the set of all points in $\mathbb{C}$, for which*

$\quad (a)\ \ |z - 2| = |z - 2i|, \qquad (b)\ \ |z - 2| = 2|z - 2i|.$

**(a)** Geometrically, $\{z \in \mathbb{C} \mid |z - 2| = |z - 2i|\}$ is the set of all points $z$, which have the same distance from 2 and $2i$, thus a bisector.

By considering a figure we see that this bisector is

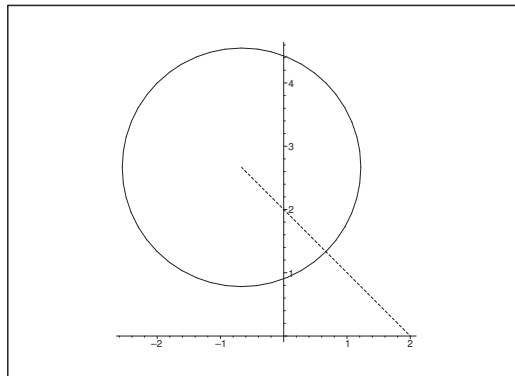$$\{z \in \mathbb{C} \mid z = t(1 + i),\, t \in \mathbb{R}\}.$$

ALTERNATIVELY the condition $|z - 2| = |z - 2i|$ is equivalent to

$$|z - 2|^2 = (x - 2)^2 + y^2 = |z - 2i|^2 = x^2 + (y - 2)^2,$$

thus

$$x^2 - 4x + 4 + y^2 = x^2 + y^2 - 4y + 4,$$

which is reduced to $y = x$.



**(b)** The equation $|z - 2| = 2|z - 2i|$ is equivalent to

$$|z - 2|^2 = (x - 2)^2 + y^2 = 4|z - 2i|^2 = 4\left\{x^2 + (y - 2)^2\right\},$$

hence

$$x^2 - 4x + 4 + y^2 = 4x^2 + 4y^2 - 16y + 16.$$

Then by a reduction,

$$3x^2 + 3y^2 + 4x - 16y + 12 = 0.$$

A division by 3 and an addition of some convenient terms give

$$x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 + y^2 - \frac{16}{3}y + \left(\frac{8}{3}\right)^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{8}{3}\right)^2 - 4 = \frac{4}{9} + \frac{64}{9} - \frac{36}{9} = \frac{32}{9},$$
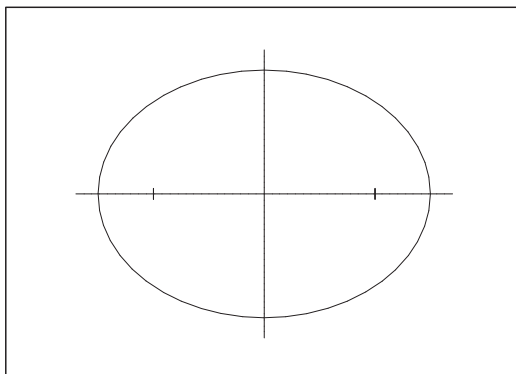
and we get

$$\left(x + \frac{2}{3}\right)^2 + \left(y - \frac{8}{3}\right)^2 = \frac{32}{9} = \left(\frac{4}{3}\sqrt{2}\right)^2.$$

The set is a circle of centrum $\left(-\dfrac{2}{3}, \dfrac{8}{3}\right)$ and of radius $\dfrac{4}{3}\sqrt{2}$.

**Example 7.3** *Find the set of points in* $\mathbb{C}$*, for which*

$(a)\ \ |z-1|+|z+1| = 3, \qquad (b)\ \ |z-1|-|z+1| = 1.$



**(a)** Here we get an ellipse, cf. Example 7.1.
If we put $z = x + iy$, then the equation is also written

$$\sqrt{(x-1)^2 + y^2} = 3 - \sqrt{(x+1)^2 + y^2} \qquad (\geq 0),$$
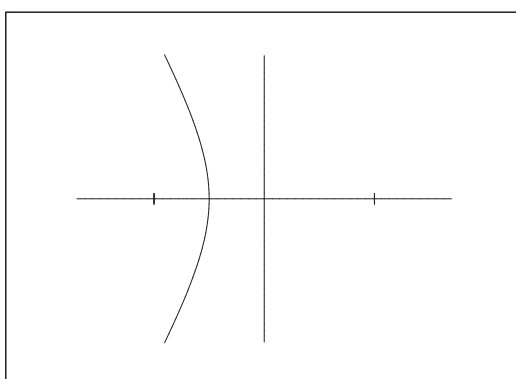
hence by a squaring,

$$x^2 - 2x + 1 + y^2 = x^2 + 2x + 1 + y^2 + 9 - 6\sqrt{(x+1)^2 + y^2}.$$

A reduction gives

$$6\sqrt{(x+1)^2 + y^2} = 9 + 4x \qquad \left(\geq 0,\ \text{dvs.}\ x \geq -\frac{9}{4}\right).$$

Then by another squaring,

$$36x^2 + 72x + 36 + 36 + 36y^2 = 81 + 16x^2 + 72x,$$

which is reduced to $20x^2 + 36y^2 = 45$, or to an equation of an ellipse of centrum $(0,0)$ and of half axes

$$\sqrt{\frac{45}{20}} = \frac{3}{2} \qquad \text{and} \qquad \sqrt{\frac{45}{36}} = \frac{\sqrt{5}}{2}.$$

**(b)** Here we get an arc of an hyperbola.

If we put $z = x + iy$, we can also write the equation

$$\sqrt{(x-1)^2 + y^2} = 1 + \sqrt{(x+1)^2 + y^2} \qquad (\geq 1),$$

hence by a squaring,

$$x^2 - 2x + 1 + y^2 = x^2 + 2x + 1 + y^2 + 1 + 2\sqrt{(x+1)^2 + y^2}.$$

This is reduced to

$$2\sqrt{(x+1)^2 + y^2} = -4x - 1 \qquad \left(\geq 0, \text{ thus } x \leq -\frac{1}{4}\right).$$

Then by another squaring,

$$4x^2 + 8x + 4 + 4y^2 = 16x^2 + 8x + 1, \qquad x \leq -\frac{1}{4},$$

which is reduced to

$$12x^2 - 4y^2 = 3, \qquad x \leq -\frac{1}{4},$$

or in its normal form,

$$\left\{\frac{x}{\frac{1}{2}}\right\}^2 - \left\{\frac{y}{\frac{\sqrt{3}}{2}}\right\}^2 = 1, \qquad x \leq -\frac{1}{4}.$$

This is the equation of a branch of an hyperbola in the left hand half plane.

**Example 7.4** *Give a geometric description of the sets*

(a)  $\left\{z \in \mathbb{C} \mid Im\left(z^2\right) > 0\right\},$      (b)  $\{z \in \mathbb{C} \mid |z - 4| > |z|\},$

(c)  $\left\{z \in \mathbb{C} \mid -\pi < Arg\, z < \frac{\pi}{2}, |z| > 2\right\}.$

**(a)** Since

$$z^2 = x^2 - y^2 + 2ixy,$$

it follows that $\text{Im}\left(z^2\right) > 0$, if and only if $xy > 0$, so the set is the union of the open first quadrant and the open third quadrant.

**(b)** *Geometrically* we shall find the set of points, the distance of which to 4 is bigger than the distance to 0. If we draw the vertical line $x = 2$, we get precisely those points for which

$$|z - 4| = |z|,$$

which is geometrical trivial.

The wanted domain is then the left hand half plane $x < 2$.
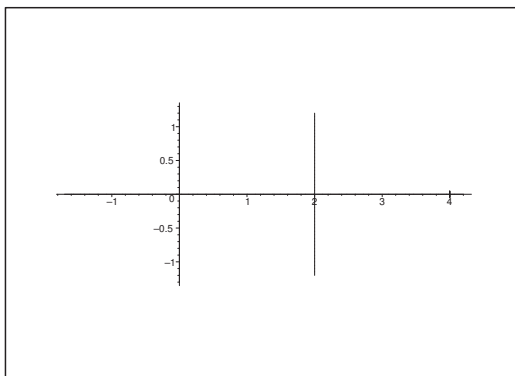
*Analytically* it follows by a squaring that $|z - 4| > |z|$ is equivalent to the inequality

$$(x - 4)^2 + y^2 > x^2 + y^2,$$

and we obtain by a reduction $x < 2$.

**(c)** The domain is the intersection of the open set $|z| > 2$ and the angular space $-\pi < \mathrm{Arg}\, z < \dfrac{\pi}{2}$,
i.e. the interior of the union of the first and third and fourth quadrant.
Thus the *complementary set* is the union of the closed second quadrant and the closed disc of centrum $(0, 0)$ and of radius $2$.

**Example 7.5** *Let $z_1$ and $z_2$ be two given points in the z-plane, and let $c \in \mathbb{R}_+$ and $k \in\, ]-\pi, \pi]$. Describe the set of points $z \in \mathbb{C}$, for which*

$(a)\quad \left| \dfrac{z - z_1}{z - z_2} \right| = c, \qquad (b)\quad Arg\, \dfrac{z - z_1}{z - z_2} = k.$

**(a)** If $c = 1$, then the equation becomes

$$|z - z_1| = |z - z_2|,$$
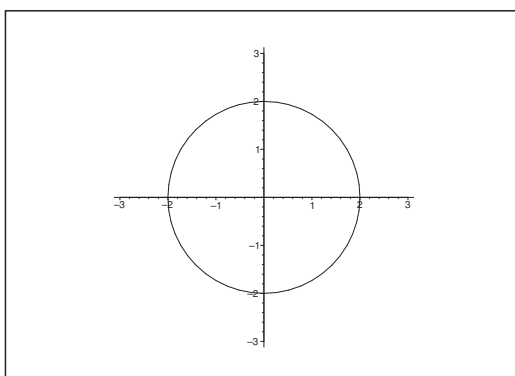
which is geometrically interpreted as the set of points $z$, which have the same distance to $z_1$ and $z_2$. This set is constructed as the line through $\dfrac{1}{2}\,(z_1 + z_2)$, and perpendicular to the vector $z_1 - z_2$.

If $c \neq 1$, then $|z - z_1| = c\,|z - z_2|$ is equivalent to

$$(x - x_1)^2 + (y - y_1)^2 = c^2\left\{(x - x_2)^2 + (y - y_2)^2\right\},$$

hence by a small computation,

$$x^2 - 2x_1 x + x_1^2 + y^2 - 2y_1 y + y_1^2 = c^2\left\{x^2 - 2x_2 x + x^2 + y^2 - 2y_2 y + y_2^2\right\},$$

and thus

$$\left(c^2 - 1\right)x^2 - 2\left(c^2 x_2 - x_1\right)x + \left(c^2 -\right)y^2 - 2\left(c^2 y_2 - y_1\right)y + c^2 x_2^2 - x_1^2 + c^2 y_2^2 - y_1^2 = 0.$$

Since $c^2 \neq 1$, we obtain a circle,

$$x^2 - 2\,\frac{c^2 x_2 - x_1}{c^2 - 1}\,x + \left(\frac{c^2 x_2 - x_1}{c^2 - 1}\right)^2 + y^2 - 2\,\frac{c^2 y_2 - y_1}{c^2 - 1}\,y + \left(\frac{c^2 y_2 - y_1}{c^2 - 1}\right)^2$$

$$= \left(\frac{c^2 x_2 - x_1}{c^2 - 1}\right)^2 + \left(\frac{c^2 y_2 - y_1}{c^2 - 1}\right)^2 + \frac{-c^2 x_2^2 + x_1 - c^2 y_2 + y_1^2}{c^2 - 1} = R^2,$$
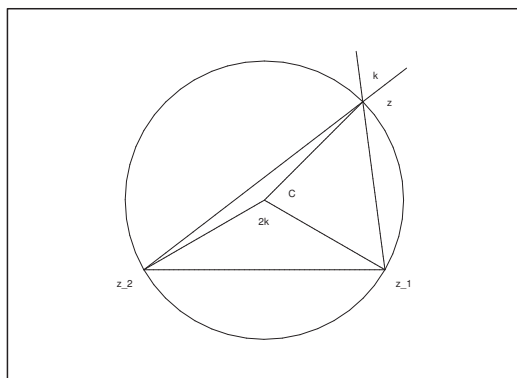
thus

$$\left\{x - \frac{c^2 x_2 - x_1}{c^2 - 1}\right\}^2 + \left\{y - \frac{c^2 y_2 - y_1}{c^2 - 1}\right\}^2 = R^2.$$
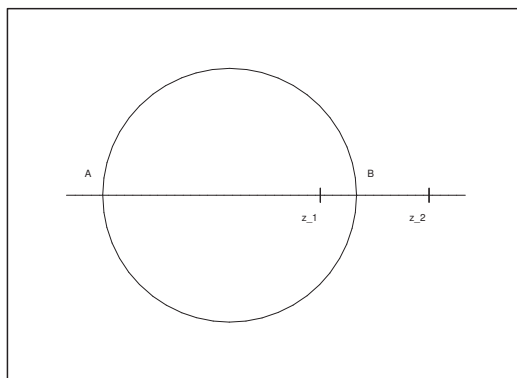


**Remark 7.1** This circle is constructed by first finding the two points $A$ and $B$ on the line $\ell$ through $z_1$ and $z_2$, such that

$$\left|\frac{z - z_1}{z - z_2}\right| = c.$$

Then $AB$ is the diameter of the circle. $\Diamond$

**(b)** The equation

$$\text{Arg}\,\frac{z - z_1}{z - z_2} = k$$

is most easy to solve geometrically when we consider the corresponding circle, which goes through the three points $z$, $z_1$ and $z_2$.

We se that $\angle z z_2 C = \angle z_2 z C = x$, because $\triangle z_2 C z$ has two radii as sides. For the same reason, $\angle z_1 z C = \angle z z_1 C = y$ and $\angle C z_2 z_1 = \angle z_1 z C = \varphi$. The sum of the angles in a triangle is always $\pi$, hence it follows from $\triangle z_1 z z_2$ that

$$2(x + y + \varphi) = \pi,$$

and analogously of $\triangle C z_1 z_2$ that

$$2\varphi + 2k = \pi,$$

hence

$$\angle z_1 z z_2 = x + y = k,$$

no matter where $z$ is lying on the circle above the the line $z_1 z_2$.

**Remark 7.2** By means of the concept of *conformal mapping*, which will be treated in a later book in this series, it is easy to prove that when $z_1$ and $z_2$ are kept fixed, then the two families of circles considered above are orthogonal. $\Diamond$

**Remark 7.3** For given $c$ and $k$ it follows that

$$\frac{z - z_1}{z - z_2} = c \cdot e^{ik} = \alpha.$$

If $\alpha \neq 1$, then

$$z = \frac{z_2 \alpha - z_1}{\alpha - 1},$$

and $z$ is uniquely determined.

Therefore, we may consider $(c, k)$ as curvilinear coordinates in $\mathbb{C} \setminus \{z_1, z_2\}$. $\Diamond$

**(b')** An ALTERNATIVE SOLUTION of the equation

$$\text{Arg}\left(\frac{z - z_1}{z - z_2}\right) = k.$$

First note that if $z \neq z_1$ and $z \neq z_2$, then

$$
\begin{aligned}
w &= u + iv = \frac{z - z_1}{z - z_2} = \frac{(z - z_1)(\overline{z} - \overline{z_2})}{|z - z_2|^2} \\
&= \frac{1}{|z - z_2|^2}\{(x - x_1) + (y - y_1)\}\{(x - x_2) - i(y - y_2)\} \\
&= \frac{1}{|z - z_2|^2}\{[(x - x_1)(x - x_2) + (y - y_2)] + i[(x - x_2)(y - y_1) - (x - x_1)(y - y_2)]\},
\end{aligned}
$$

where we do not compute the denominator $|z - z_2|^2$, because we shall only use that

$$|z - z_2|^2 > 0.$$

When we split into the real and the imaginary part, we get

$$
\begin{aligned}
u &= \frac{1}{|z-z_2|^2}\left\{x^2-(x_1+x_2)\,x+x_1x_2+y^2-(y_1+y_2)\,y+y_1y_2\right\}, \\
v &= \frac{1}{|z-z_2|^2}\left\{xy-y_1x-x_2y+x_2y_1-xy+y_2x+x_1y-x_1y_2\right\} \\
&= \frac{1}{|z-z_2|^2}\left\{(y_2-y_1)\,x-(x_2-x_1)\,y+x_2y_1-x_1y_2\right\}.
\end{aligned}
$$

A check shows that $v = 0$ describes the line through $z_1$ and $z_2$ with exception of these two points. On the other hand, we get for $v = 0$ that

$$
\mathrm{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \mathrm{Arg}(u+iv) = \mathrm{Arg}\,u = \begin{cases} 0 & \text{for } u > 0, \\[2mm] \pi & \text{for } u < 0, \end{cases}
$$

and it follows almost immediately from the expression of $u$ above (due to the squared terms) that $u > 0$, if and only if $(x, y)$ lies on one of the line segments outside $[z_1, z_2]$ (assuming that $v = 0$), and hence $u < 0$ for $(x, y) \in [z_1, z_2]$, where $[z_1, z_2]$ denotes the line segment in the plane between $z_1$ and $z_2$.
In this way we fix the curves for $k = 0$ and for $k = \pi$.

In our next case we have $v > 0$, thus $(x, y)$ lies in one of the half planes determined by the line $\ell$ through $z_1$ and $z_2$. We shall tacitly assume this in the following and not repeat ourselves.
It follows from $v > 0$ that

$$
\mathrm{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \mathrm{Arg}(u+iv) = \mathrm{Arccot}\left(\frac{u}{v}\right) \in\, ]0, \pi[,
$$

because Arccot typically is more fundamental in the *Theory of Complex Functions* than Arctan. The curves of this half plane therefore correspond to the equation

$$
\mathrm{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \mathrm{Arccot}\left(\frac{u}{v}\right) = k \in\, ]0, \pi[,
$$

or equivalently

(11) $\quad \dfrac{u(x,y)}{v(x,y)} = \cot k = c \in \mathbb{R}.$

When we multiply by $v(x, y)$ and insert the expressions of $u(x, y)$ and $v(x, y)$ we get by cancelling $|z - z_2|^2$,

$$
\begin{aligned}
x^2 - (x_1 + x_2)\,x &+ x_1x_2 + y^2 - (y_1 + y_2)\,y + y_1y_2 \\
&= c\left\{(y_2 - y_1)\,x - (x_2 - x_1)\,y + x_2y_1 - x_1y_2\right\},
\end{aligned}
$$

which is qualitatively the equation of a circle. If we put $(x, y) = (x_1, y_1)$ and $(x_2, y_2)$ into (11), it follows that both sides become 0, so the system of curves is the restriction of all circles through

$(x_1, y_1)$ and $(x_2, y_2)$ to the half plane determined by $v > 0$, i.e. determined by the line $\ell$ through $(x_1, y_1)$ and $(x_2, y_2)$.
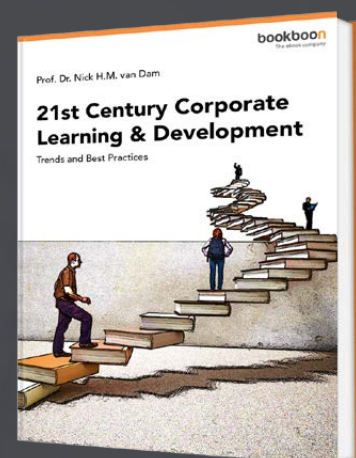
If instead $v < 0$, then we get the other half plane, and

$$\mathrm{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \mathrm{Arg}(u + iv) = \mathrm{Arccot}\left(\frac{u}{v}\right) - \pi \in \,]-\pi, 0[,$$

hence $\mathrm{Arg}\left(\dfrac{z - z_1}{z - z_2}\right) = k$ is equivalent to $\mathrm{Arccot}\left(\dfrac{u}{v}\right) = k + \pi$, i.e. to (11), because cot has period $\pi$. We again derive (11), and we find the same system of curves, only restricted to the other half plane.

**Example 7.6** *Find the domain in the z-plane, which is characterized by the condition*

$$\left| z^2 + a\,z + b \right| < R^2,$$

*where $a$, $b \in \mathbb{C}$ and $R \in \mathbb{R}_+$. Indicate the values of $R \in \mathbb{R}_+$, for which this domain is connected, and for which is it disconnected.*

Let $\alpha$ and $\beta$ be the roots of the polynomial, i.e.

$$z^2 + a\,z + b = (z - \alpha)(z - \beta).$$

If $\alpha = \beta \ \left( = -\dfrac{a}{2} \right)$, then we get an open disc of radius $R$ and centrum $-\dfrac{a}{2}$. In this case, $b = \dfrac{a^2}{4}$, and the domain is of course connected for every $R \in \mathbb{R}_+$.

If $\alpha \neq \beta$, then

$$\left| z^2 + a\,z + b \right| = |z - \alpha| \cdot |z - \beta| < R^2,$$

thus the product of the distances from $z$ to $\alpha$, and from $z$ to $\beta$, is smaller than $R^2$. (The boundary curves are also called *Cassini's rings*).

The midpoint of the line segment between $\alpha$ and $\beta$ is given by $\dfrac{\alpha + \beta}{2} = -\dfrac{a}{2}$. The set is disconnected, if this point does not belong to the point set, and it is connected, if $-\dfrac{a}{2}$ belongs to the point set. If we put $z = -\dfrac{a}{2}$, then

$$\left| z^2 + a\,z + b \right| = \left| \frac{a^2}{4} - a \cdot \frac{a}{2} + b \right| = \left| b - \frac{a^2}{4} \right|.$$

Since

$$\left| \alpha + \frac{a}{2} \right| = \left| \beta + \frac{a}{2} \right|,$$

this can also be expressed by

$$\left| z^2 + a\,z + b \right| = \left| \alpha + \frac{a}{2} \right| \cdot \left| \beta + \frac{a}{2} \right| = \left| \alpha + \frac{a}{2} \right|^2 = \left| \beta + \frac{a}{2} \right|^2 = \frac{1}{4}\,|\alpha - \beta|^2.$$

The condition of connectedness is therefore,

$$R > \sqrt{\left| b - \frac{a^2}{4} \right|}, \qquad \text{thus} \qquad |\alpha - \beta| < 2R.$$

The set is disconnected, if

$$0 < R \le \sqrt{\left| b - \frac{a^2}{4} \right|}, \qquad \text{thus} \qquad 0 < 2R \le |\alpha - \beta|.$$