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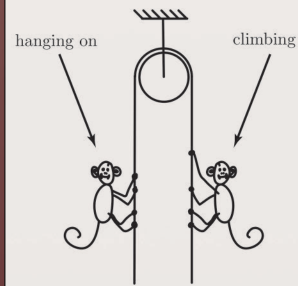
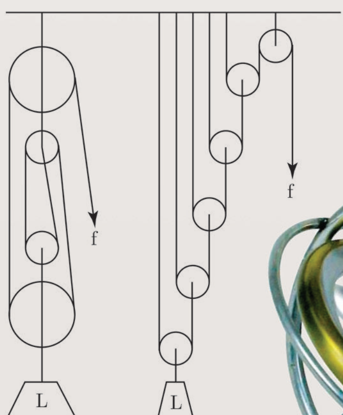
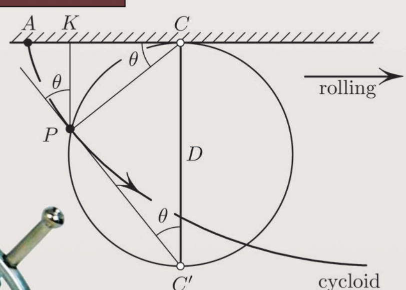
Classical Mechanics with Calculus of Variations and Optimal Control

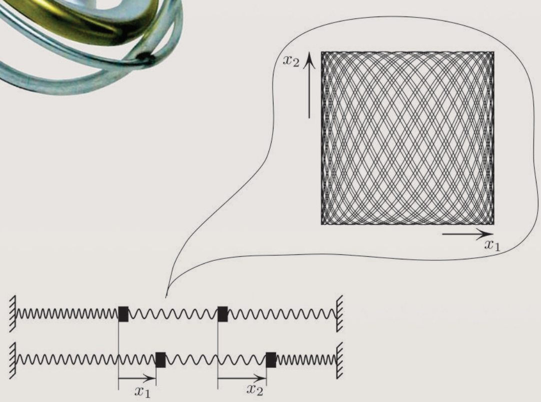
An Intuitive Introduction

Mark Levi









American Mathematical Society
Mathematics Advanced Study Semesters

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Series Foreword: MASS and REU at Penn State University

This book is part of a collection published jointly by the American Mathematical Society and the MASS (Mathematics Advanced Study Semesters) program as a part of the Student Mathematical Library series. The books in the collection are based on lecture notes for advanced undergraduate topics courses taught at the MASS and/or Penn State summer REU (Research Experiences for Undergraduates). Each book presents a self-contained exposition of a non-standard mathematical topic, often related to current research areas, accessible to undergraduate students familiar with an equivalent of two years of standard college mathematics and suitable as a text for an upper division undergraduate course.

Started in 1996, MASS is a semester-long program for advanced undergraduate students from across the USA. The program's curriculum amounts to sixteen credit hours. It includes three core courses from the general areas of algebra/number theory, geometry/topology and analysis/dynamical systems, custom designed every year; an interdisciplinary seminar; and a special colloquium. In addition, every participant completes three research projects, one for each core course. The participants are fully immersed into mathematics, and this, as well as intensive interaction among the students, usually leads to a dramatic increase in their mathematical enthusiasm and achievement. The program is unique for its kind in the United States.

The summer mathematical REU program is formally independent of MASS, but there is a significant interaction between the two: about half of the REU participants stay for the MASS semester in the fall. This makes it possible to offer research projects that require more than seven weeks (the length of the REU program) for completion. The summer program includes the MASS Fest, a two to three day conference at the end of the REU at which the participants present their research and that also serves as a MASS alumni reunion. A nonstandard feature of the Penn State REU is that, along with research projects, the participants are taught one or two intense topics courses.

Detailed information about the MASS and REU programs at Penn State can be found on the website `www.math.psu.edu/mass`.

Contents

Preface

Classical mechanics was developed by many of the greatest minds of the past, including Archimedes, Galileo, Newton, Huygens, the Bernoullis,[1] Lagrange, Gauss, Jacobi, Hamilton, and Poincaré, among others.

Their efforts made classical mechanics a multifaceted gem which is beautiful whichever way one turns it.

From the practical side, a remarkable feature of mechanics is its ability to predict things by pure thought, starting with almost nothing. For example, knowing only the gravitational acceleration $g \approx 10m/s^2$ near the Earth's surface, and Newton's laws, we can predict the frequency of oscillations of a pendulum, or the speed of precession of a spinning top, or the period of an orbiting satellite knowing its distance to the Earth's center, etc.

On the theoretical side, classical mechanics interlaces with almost every branch of mathematics: Euclidean geometry, differential equations, dynamical systems, differential geometry, topology, algebra, number theory.

[1] There was considerable rivalry between Johann and Jacob Bernoulli; ironically, nowadays some people confuse the two brothers, or even think they were the same person.

From yet another direction, mechanics actually explains and even suggests some theorems of geometry and calculus (see [**14**] for more details and references).

In such an old and fundamental subject it is difficult to say something that is simultaneously original (O), interesting (I) and correct (C). When pressed, I tried to satisfy (I) and (C) at a minimum. On occasion, (O) may have been satisfied, at least in a pedagogical sense.

Intended audience. This text is suitable for courses from junior to beginning/intermediate graduate, on the topics in the title. Each section corresponds roughly to one lecture, and it is not unreasonable to expect to cover 2 to 4 chapters in a semester. This means that the book contains enough material for more than one course. Many of the 180 problems (most with solutions or hints) can be used in lectures; in fact, much of the pleasure in the subject is derived from problems. I also hope that some ideas or problems may seem new not only to students but to specialists as well — at least in a pedagogical sense.

Physical and mathematical background. I assume that the reader has been exposed to beginning courses in physics, in vector calculus, and in basic linear algebra. Nevertheless, I recall some of the key concepts and facts as the need arises.

Some highlights of this book. 1. *Ideas come before formulas.* As an example, some texts *define* kinetic energy as $mv^2/2$, never explaining what it *really* is (it is the work required to bring the mass to speed v, see page 10 for details), and why, for instance, not $mv^2/3$, or mv^3? The reason is hidden from the reader, who deserves to be told *why*, and not only *how*. I tried to avoid such abuse throughout.

Chapter 8 is especially devoted to explaining the motivation behind Hamiltonian mechanics so that nothing is pulled out of thin air.

2. Having kept in mind Einstein's quote: "If you can't explain it to a six year old, you don't understand it yourself", I followed, as best I could, his advice: "Everything must be made as simple as possible. But not simpler."

3. I describe an *equivalence* between the dynamics of particles and the statics of springs (page 45). This equivalence is fruitful in a few ways: (i) it yields the Euler–Lagrange equation and Liouville's theorem with almost no work, at least in special cases; (ii) it gives a surprisingly obvious physical interpretation to these abstract mathematical statements, and (iii) it shows *why* Liouville's theorem holds, giving the bare essence of the reason for the result, not hidden by many steps.

4. Chapter 8 explains some fundamental ideas of Hamiltonian mechanics from one basic principle. Just as with the kinetic energy example above, many of the concepts in that field are often "pulled out of thin air".[2] Instead, I tried to show how one basic starting point leads to all of the above concepts automatically.

5. The table on page 280 shows a revealing analogy between Hamiltonian dynamics and statics of springs. In my mind, this analogy could be taken as an unspoken guiding principle of Hamiltonian formalism. Hamiltonian mechanics turns out to be a kind of "spring theory".

6. Some miscellaneous items: (i) a new heuristic way to minimize integrals $\int F(y)ds$ (page 188), (ii) some fun problems, such as one on finding the center of curvature using a bike, or on the hydrostatic nature of the tension in hanging cables, or on an interesting property of a mobile, etc., (iii) a remarkably short proof, due to Lagrange, of the ellipticity of planetary motions.

7. I included a geometrical discussion of Pontryagin's Maximum Principle of optimal control. This topic really belongs to Hamiltonian mechanics/ray optics and so is a natural fit for this book. It is remarkable that the Maximum Principle is, loosely speaking, a version of Huygens's principle. Optimal control is a standard topic in engineering courses but is rarely taught to mathematicians and physicists; I tried to bring this neglected child of Hamiltonian mechanics, often abused in engineering literature, back home. Many engineering

[2] These concepts, which the reader is not expected to know at this stage, include the Legendre transform, Hamilton–Jacobi equations, Liouville's theorem, Poincaré integral invariants, Noether's theorem.

texts effectively hide the geometry of Pontryagin's principle; I tried to bring this simple geometry to the surface.[3]

Analogies in general.[4] The use of analogies goes back to Archimedes who interpreted geometrical objects mechanically and then used mechanical intuition to discover new geometrical theorems. Later examples include the so-called Kirchhoff's kinetic analogy — actually, the *equivalence* — between the dynamics of a free rigid body on the one hand and statics of elastic rods on the other, Poincaré's analogy of phase flow of a Hamiltonian system to a fluid flow (see Arnold [**1**] for more details), Riemann's interpretation of analytic functions in terms of fluid flow, and much, much more. Aubry's discovery, simultaneously with Mather, of the Aubry–Mather theory (1982) — one of the key advances of dynamical systems in the twentieth century — was driven by a mechanical analogy. Analogy unifies our understanding by showing that seemingly unrelated things are, in certain aspects, the same.

Analogies in this book. In that vein, the dynamics of a single particle is analogous (in fact, equivalent) to the statics of a Hookean spring (see the table on page 45). For a general mechanical system (not just a single particle), there is also an analogy different from the one just mentioned (see the table on page 280). Each of these analogies opens a new view — they certainly did for me when I realized them. Some things that are not obvious and abstract for one side of the analogy are obvious and palpable for the other, as the table just mentioned illustrates.

Outline of the book. The book consists of four parts: (I) Dynamics (Chapters 1–3), (II) Variational Principles (Chapters 4–6), (III) Optimal control (Chapter 7) and (IV) Foundations of Hamiltonian mechanics (Chapter 8).

[3] The texts [**15**] and [**2**] give a very nice and a much more extensive presentation of these topics than this book.

[4] The subtitle of a remarkable recent book "Surfaces and Essences: Analogy as the Fuel and Fire of Thinking" by Hofstadter and Sander represents accurately, I think, the pivotal role of analogy in all thinking, and in particular, in classical mechanics.

Originally, I was planning Chapter 8 as a small book, but the manuscript grew, thanks in part to Sergei Gelfand's encouragement and despite my attempts to keep it short. This main chapter ended up being the last one in the book, which may seem an injustice. Here is a slightly more detailed outline of the book.

Chapters 1–3: Dynamics. Chapter 1 deals with one-dimensional motion, introducing many key ideas of mechanics with a minimum of technicalities. Chapter 2 discusses several degrees of freedom, including Kepler's problem and vibrations. Discussion of free rigid body motion (such as a tumbling asteroid, or a football, Chapter 3) completes the first part of the book.

Chapters 4–7: Calculus of Variations, Optimal Control. This topic is presented in four parts: First, variational principles of mechanics (Chapter 4); second, some classical problems from calculus of variations (Chapter 5); third, the Jacobi criterion for the minimum — a kind of a second derivative test for a minimum (Chapter 6); and fourth, Pontryagin's Maximum Principle of optimal control (Chapter 7). This principle turns out to be essentially a restatement of Huygens's principle.

Chapter 8: Hamiltonian approach motivated. This chapter contains a heuristic view of Hamiltonian and Lagrangian mechanics. Ever since having learned classical mechanics in school, I had a nagging suspicion that many texts on mechanics leave something very fundamental unspoken. Why, for example, does one introduce the Hamiltonian, or the Legendre transform, or the momentum, or the symplectic 1-form $p\,dx$?[5] The best explanation I saw was that the end — the beauty of the resulting theory — justifies the unmotivated means. It turns out, however, that *if we start the presentation of Hamiltonian mechanics with the right question, then all the concepts mentioned above fall into our laps automatically and unavoidably;* most of the theory begins to look simple and natural. This presentation is given in the ten short sections of Chapter 8. Section 11 of this chapter gives a static analog of the preceding theory; with

[5]All of these concepts are introduced in the first chapter in a conventional way, and then again in Chapter 8 in a way that shows the inevitability of these concepts.

this analogy we realize that many of the seemingly abstract concepts have an elementary physically palpable explanation. It is remarkable that, in hindsight, much of the Hamiltonian theory would have been obvious to Archimedes, if only it were restated in terms of a certain static mechanical model. Finally, the last section of Chapter 8 gives a quick description of the analogy of mechanics to optics. This traditional analogy is explained beautifully in Gelfand and Fomin's Calculus of Variations [**8**] (Appendix 1); see also Arnold's Mechanics [**1**].

Classical mechanics as a branch of mathematics. Classical mechanics deals with idealized objects, such as "point masses", "rigid bodies", "rods". These objects are imaginary approximations of actual physical things. For example, in studying planetary motion, we may treat the Earth as a point. In this respect classical mechanics is a sister of geometry, which also deals with idealized objects; to the geometer a "point" is not an ink dot on paper but an imaginary dot of zero size. When applying classical mechanics in practice one must of course remember that, for instance, the "rigid body" of classical mechanics is not exactly rigid in reality.[6] A "point mass" is a good approximation unless we look at it too closely. Classical mechanics serves very well except at microscopic scales (quantum mechanics) or at macroscopic ones (relativity). We assume, for example, that the position of a point mass can be precisely defined, that inertial frames exist, and that the time is well defined throughout the reference frame. These statements are strictly true only in the idealized world of classical mechanics.

The mystery of variational principles. (The reader who has not seen variational principles should skip this paragraph, or else see page 167 first.) Newton's laws (N) are equivalent to variational principles (V), as proven in many texts on classical mechanics. Logically, there is nothing to complain about: (N) is taken as axioms (suggested by

[6]An interesting mistake along these lines was made with the first Explorer satellite. The orbiting satellite, shaped as an elongated cylinder, was given an original spin around its long axis; unexpectedly, it developed wobbles and after several hours ended up tumbling end-over-end, contrary to what the theory of rigid body dynamics predicts (Chapter 4). It was soon realized that the satellite was not exactly rigid (it had flexible antennas), so that the theory did not quite apply.

experiments and by physical intuition, itself the result of experiments we do as part of physical activity), (V) is proven; the picture seems complete. But there is a mystery about (V): how can a mindless projectile "know" to take the path of least action?[7] After all, in order to choose the path of least action, doesn't one have to compare the action of this path to the actions of other paths? That a brainless projectile "knows" about nearby paths and chooses the "best" seems so surprising that the logical proof (N) $\Rightarrow$ (V) looks a bit legalistic and insufficient. One senses that something else must be going on. Yet no book on classical mechanics that I know of mentions this question.[8] But it is precisely this question that stimulated Feynman in his remarkable discovery of path integrals ([**7**], [**6**]). Variational principles become much less mysterious through the idea of phase cancellation, as explained briefly in the next paragraph.

Classical mechanics leading to quantum mechanics. In relation to the previous item, it is remarkable that the Maupertuis' principle (discovered by Leibnitz around 1707), or Hamilton's principle ($\delta \int L\,dt = 0$), can in retrospect be seen as a loud hint at the wave nature of particles, available long before the first shoots of quantum mechanics appeared. In optics, Fermat's principle of least time ($\delta \int dt = 0$) is explained as follows: the waves travel between two points A and B along all possible paths; but the waves traveling along paths close to the path of least time arrive in sync, with nearly equal phases, and hence add up; by contrast, the waves traveling along noncritical paths arrive at B with disparate phases and thus cancel with their neighbors, contributing almost nothing at B. In other words, the light is not intelligent but rather omnipresent; some people mistook phase cancellation for intelligence. Fermat's principle ($\delta \int dt = 0$) is thus due to the wave nature of light, suggesting that Maupertuis' principle, or Hamilton's principle, is analogously due to wave nature of particles. The alternative would be to ascribe this principle to the supernatural [**6**]. In light of this (with apologies for the pun) it may seem striking that so much time passed from the formulation of Hamilton's principle ($\delta \int L\,dt = 0$) to the discovery of

[7] Loosely speaking, action is the integral of the difference between the kinetic and the potential energies. A precise definition is given by (1.22), on page 19.

[8] Perhaps due to the long tradition because the subject is so, uh, classical.

the wave nature of particles. Feynman's derivation of the Schrödinger equation using classical mechanics is given on page 286. The remarkable feature of this derivation is that it uses almost nothing besides classical mechanics and the analogy with electromagnetic optics to arrive at the Schrödinger equation.

Acknowledgements. Selected topics of this book were presented in a course in the MASS (Mathematics Advanced Studies Semester) at Penn State in the fall of 2011. I am very grateful to Sergei Gelfand for encouraging me to convert these early notes into a text and for his kind, sober and wise suggestions. My sincere thanks go to the anonymous referees; their few compliments made me feel better and their numerous criticisms made the manuscript better. I gratefully acknowledge partial support by NSF grant DMS-0605878.

Chapter 1

One Degree of Freedom

1. The setup

In this chapter we consider the simplest class of mechanical systems: a point mass confined to a straight line or a curve; Figure 1 gives three examples. Such systems are referred to as the systems with *one degree of freedom*[1].

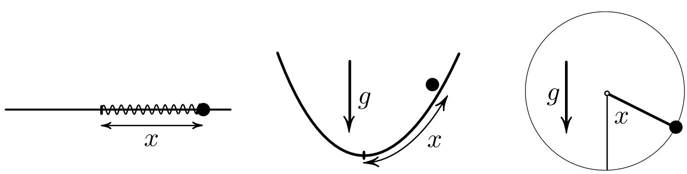

Figure 1. One-degree-of-freedom systems.

One-degree-of-freedom systems are "building blocks" for a larger class of more complex mechanical systems — the so-called completely integrable systems. Such systems (Kepler's problem is an example) reduce to a collection of decoupled one-degree-of-freedom systems. Many fundamental ideas and concepts in mechanics can be illustrated already on one-degree-of-freedom systems, as we do in this chapter.

[1]More generally, the degree of freedom of a mechanical system is the number of quantities which define the position of all the particles in the system. Other examples of one-degree-of-freedom systems include a rigid body spinning on a given axis (with the angle playing the role of coordinate), or a wheel rolling on a straight line without sliding. Higher degree of freedom systems are discussed in the next chapter. Examples of those include a particle in the plane (two coordinates define its position), a double pendulum (two angles define the positions of the masses), or a particle constrained to move on a surface in space.

2. Equations of motion

In this section I state Newton's second law, and then derive the equations of motion for each of the examples in Figure 1. An alternative way to derive the equations of motion, discovered by Lagrange and used more commonly, is described in Section 7.

Newton's second law for a point mass. Consider a particle of mass m subject to *net force* $\mathbf{F}$. By net force, also called the *resultant* force, one means the vector sum of *all* forces acting on the particle.[2] Newton's law states that the vector acceleration $\mathbf{a}$ of the particle is caused by the net force $\mathbf{F}$ and is proportional to that force:

$$m\mathbf{a} = \mathbf{F}, \quad \text{or} \quad \mathbf{a} = \frac{1}{m}\mathbf{F}, \tag{1.1}$$

where the coefficient of proportionality m is referred to as the (inertial) *mass*. In other words, the particle accelerates in the direction of the net force and with intensity proportional to the force and inverse proportional to the mass.

Exercise 1.1. A particle moves in space under the influence of a force $\mathbf{F}$. Must the particle's velocity $\mathbf{v}$ point in the same direction as $\mathbf{F}$?

Answer. No. For a flying projectile, for instance, $\mathbf{F}$ and $\mathbf{v}$ are not aligned since one points straight down while the other is tangent to the trajectory.

Motion on the line under a frictionless force. We consider a particle constrained to a straight line, which we take to be the x-axis. We assume that the particle located at x is subject to a force $F(x)$ acting in the direction of the x-axis. Note that this force is assumed to be independent of the velocity $\dot{x}$; the friction is thus excluded. Since the particle's acceleration is $a = \frac{d^2}{dt^2}x = \ddot{x}$, we can rewrite Newton's law as

$$\boxed{m\ddot{x} = F(x)} \tag{1.2}$$

Here $x = x(t)$ is a function of time, so that Newton's law becomes an ordinary differential equation for the unknown function $x(t)$. We now consider several important examples.

[2] One of the most common mistakes in mechanics is forgetting to include all of the forces in Newton's law.

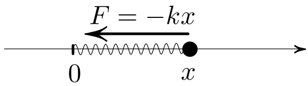

Figure 2. The harmonic oscillator. The picture shows a zero length spring, i.e., 0 is the equilibrium.

1. The free fall. The simplest example $F = \text{const.}$ was solved by Galileo in his study of free fall. Actually, Galileo's main discovery in this area was not the solution of the differential equation $\ddot{x} = -g$, but rather his realization that m does not enter this equation.

2. The harmonic oscillator. The next simplest example, where $F = -kx$ ($k = \text{const.} > 0$), is referred to as the *harmonic oscillator*, Figure 2. The minus sign indicates that the force is *restoring*. We can think of $F = -kx$ as the tension of a *linear zero length spring*, i.e., of a spring whose relaxed length is zero or, if one prefers, as the deflection from the relaxed length of a Hookean spring with nonzero relaxed length.[3] Equation (1.2) with $F = -kx$ becomes

$$\ddot{x} + \omega^2 x = 0, \quad \text{where} \quad \omega^2 = \frac{k}{m}. \tag{1.3}$$

The general solution of this ordinary differential equation is of the form $x(t) = A\cos(\omega t - \varphi)$, where A (the amplitude) and φ (the phase) are arbitrary constants which can be determined from the initial data.

3. A bead on a wire is an entire class of examples leading to (1.2). Figure 3(A) shows a bead sliding without friction on a rigid wire in the vertical plane under the influence of gravity. To write the equation of motion, let us use the arc length parameter s, the distance along the curve from some chosen point to the bead. Projecting Newton's law (1.1) onto the tangent to the wire we get the scalar equation

$$ma = F, \tag{1.4}$$

where $a = \ddot{s}$ is the tangential acceleration of the bead, and where F is the sum of projections of all the forces acting on the bead upon the tangent to the wire. As the figure illustrates, of the two forces acting on the bead the reaction $\mathbf{N}$ contributes zero (since there is no friction). The only contributing force is the projection of gravity:

[3] Some springs come pre-stressed in such a way that $F = -kx$ holds reasonably well as long as the spring is actually stretched.

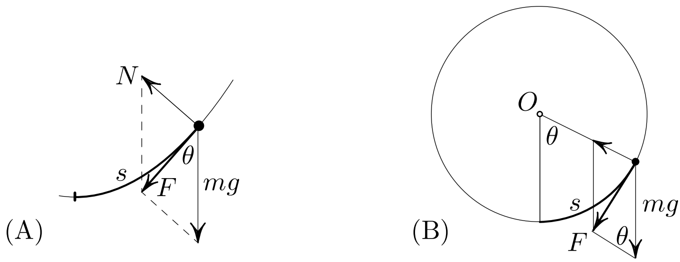

Figure 3. The bead on a wire; the circular wire corresponds to the pendulum.

$-mg \sin\theta$, where θ is the angle between the tangent and the vertical. Note that $\theta = \theta(s)$ is a function of s determined by the shape of the wire. Summarizing, (1.4) becomes

$$\boxed{\ddot{s} = -g \sin\theta(s)} \tag{1.5}$$

This equation is of the same form as (1.2), except that s is a coordinate along a curve. We now consider two shapes of the track: the circle and the cycloid.

4. The pendulum. The circular wire in Figure 3(B) need not be a physical wire: the bead can be constrained to the circle by a weightless rod hinged at O, giving us a pendulum. We have $s = L\theta$ if both s and the angle θ are measured from the same point on the circle, and where L is the length of the rod. Substituting this into (1.5) we get the differential equation for the angle θ (more convenient to use than s) of the pendulum:

$$\ddot{\theta} = -\frac{g}{R} \sin\theta. \tag{1.6}$$

5. Huygens's pendulum. The usual pendulum, used as a clock, has one shortcoming: its period depends on the amplitude. Huygens discovered how to fix this problem — one of the nicest discoveries in the history of calculus. First, Huygens showed that the cycloid (see Figure 4) is that special curve for which *the period of the bead's oscillations is independent of the amplitude* (see Problem 1.18 for a hint to a short proof). Second, Huygens showed how to make a mass travel on a cycloid in a practical way, Figure 5: Consider the piece ACB of the cycloid from Figure 4, where A and B are the lowest

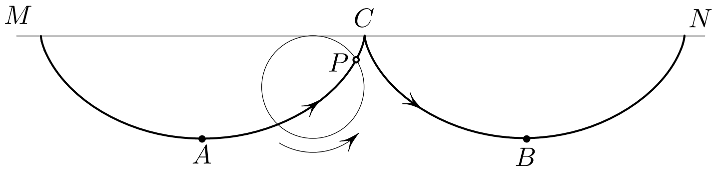

Figure 4. A circular wheel rolls on the line MN without sliding. The cycloid is a curve described by a point P on the rolling wheel. In our discussion MN is horizontal and the wheel is below MN.

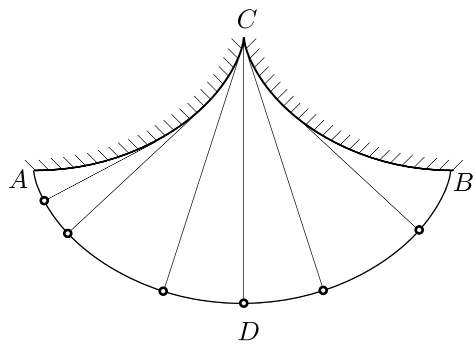

Figure 5. Hugyens's pendulum: CA and CB are arcs of a cycloid; the string attached at C has the same length as these arcs. Then the free end of the string will trace a congruent cycloid, and, moreover, the period of the resulting pendulum will be in dependent of the amplitude.

points, and treat this piece as an obstacle impenetrable by a string attached at C. The length of the string is chosen to be the same as that of the semi-arcs CA and CB. If we hang a weight at the free end of the string and let it swing, part of the string will hug the obstacle arcs and part will be straight; we obtained a pendulum of variable length. Huygens's second discovery was that *the path of the weight is a congruent cycloid(!)*

High quality grandfather clocks have special suspension mechanisms based on Huygens's discovery.

Remark 1.1. Note that the string in Figure 5 is normal to ADB and tangent to CA or CB. In other words, *the arc ACB is the envelope*

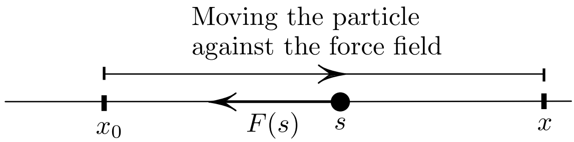

Figure 6. Potential energy for one-degree-of-freedom systems.

of the family of lines normal to ADB. Such an envelope of the family of normal lines to a planar curve is called the *evolute* of this curve. Huygens therefore showed that *the evolute of a cycloid* (ADB) *is a congruent cycloid* (ACB).[4]

3. Potential energy

The setting. Figure 6 shows a point mass on the line in a force field $F(x)$: the particle whose position is x is subject to the force $F(x)$. The mass-spring system is a prime example; in that case, $F(x) = -kx$, where x is the position of the particle relative to the equilibrium.

Intuitively, potential energy of the point mass at x is the work that I must do *against* the force $F(x)$ to bring the particle to the location x from some given reference location x_0. That is, I must apply force $-F(s)$ at s, in order to balance $F(s)$, thus dragging the particle from x_0 to x. This suggests the formal definition.

Definition. Potential energy $U(x)$ at x of a point mass in the force field $F(x)$ is defined as

$$\boxed{U(x) = -\int_{x_0}^{x} F(s)\,ds} \tag{1.7}$$

Remark 1.2. One might wonder whether the above motivation of the definition is imprecise: when I apply force $-F(s)$ to the mass, the net force on the mass becomes $-F(s) + F(s) = 0$, so why would it move at all? To answer this question, the mass will move, by inertia, *if* given an arbitrarily small initial speed, since the net force $= 0$ everywhere. So technically, to move the mass from x_0 to x I must spend work $U(x)$ plus an arbitrarily small quantity. The formal

[4]See Problems 1.19 and 1.20 on page 55.

definition (1.7) avoids this point, which is a plus, but does not seem motivated on its own, which is a minus.

Remark 1.3. If F points from x_0 to x, then the work done against F is negative. For example, in moving a weight from the tabletop to the floor I do negative work; that is, the work is done for me.

Potential energy and force. Differentiating both sides of the definition (1.7) yields, by the Fundamental Theorem of Calculus[5]:

$$\boxed{F(x) = -U'(x)} \tag{1.8}$$

Thus the force can be recovered from the potential energy. Note that large force is characterized by steep changes of energy.

Many potential energies. The potential energy depends on the choice of the reference point x_0. Choosing a different x_0 amounts to changing U by an additive constant. This constant does not affect the force $F(x)$ since the derivative in (1.8) kills the constant.

A geometrical interpretation of (1.8). Figure 7 shows that the force acts "down the slope" of the graph of $U(x)$. Imagine gravity $g = 1$ pointing down in Figure 7. Then the tangential component of this gravity upon a bead sliding on the graph of U is $F_{\text{tang}} = -\sin\tan^{-1} U'(x) = -U'(x) + o(U'(x))$. For small $U'(x)$ the force $-U'(x)$ is close to the tangential component of gravity upon a bead on the wire. For large slopes $U'(x)$ the approximation fails, although it still does give the correct sign of the force.[6]

Potential energy is defined up to an arbitrary constant. Indeed, with a different choice of the reference location $\tilde{x}_0 \neq x_0$, the corresponding potential energy $\tilde{U}(x) = -\int_{\tilde{x}_0}^{x} F(s)\,ds$ differs from $U(x)$ by

$$U(x) - \tilde{U}(x) = \int_{\tilde{x}_0}^{x_0} F(s)\,ds = \text{const.},$$

a quantity independent of x. In fact, any of the anti-derivatives $U(x) = -\int F(x)$ (defined up to a constant) is a potential energy.

[5] A reminder: $\frac{d}{dx}\int_a^x F(s)\,ds = F(x)$. Differentiation undoes the integration.
[6] See Problem 1.3 on page 51

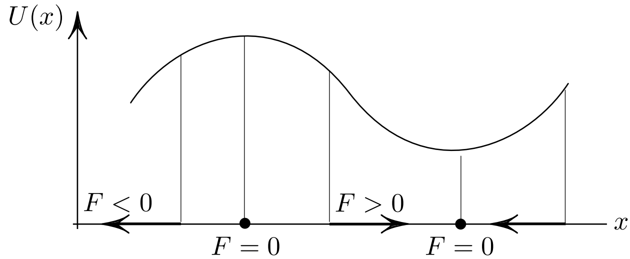

Figure 7. Force equals the negative slope of the potential energy graph.

Particle constrained to a curve. Our definition of the potential energy for a point mass on the straight line extends to a point mass constrained to a curve (Figure 1); definition (1.7) still applies, where x now stands for the arc length measured along the curve, and where $F(x)$ stands for the force in the tangential direction to the curve.

Examples of potential energy.

1. A Hookean spring. $U(x) = \int_0^x (-F(s))\, ds = \int_0^x (-(-ks))\, ds = \frac{1}{2}kx^2$, where x is the amount by which the spring was stretched from its relaxed length. Here is an elementary derivation of the result avoiding integrals: when pulling the spring from $x = 0$ to $x \neq 0$, I must apply the average force $\frac{0+kx}{2} = kx/2$; the work equals this force times the distance x, reproducing the above result.

2. Potential energy in a constant gravitational field. The particle is subject to the gravitational force $F(z) = -mg$ in the direction of the negative z-axis. The work required to move the particle from z_0 to z against F is $U(z) = (mg) \cdot (z - z_0) = mgh$, where h is the height of z above z_0.

3. Potential energy in the gravitational field of a star. Let us place the origin at the center of the star, Figure 8. The star's gravitational pull on a point mass is $F(x) = -\frac{k}{x^2}$ (k is a constant whose value is not important here). We treat the star as a point mass; otherwise we must take $x > R$, the radius of the star. Let us choose $x_0 = \infty$ for the reference "point". We have

$$U(x) = -\int_{\infty}^{x} \left(-\frac{k}{x^2}\right) dx = -\int_{x}^{\infty} \frac{k}{x^2}\, dx = -\frac{k}{x}. \tag{1.9}$$

This agrees with intuition: the work we do when moving the mass from infinity towards the star is negative, meaning that the gravitational force is doing the work for us. Moving "downhill" requires negative amount of work.

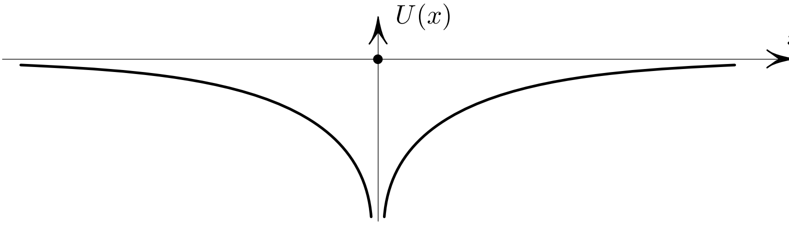

Figure 8. Gravitational potential energy $U = -k/x$. Gravitational force $F(x) = -U'(x) = -k/x^2$.

4. For a bead on a curve as in Figure 3, the force $F(s) = -mg\sin\theta$, where $\theta = \theta(s)$ is the angle between the tangent and the horizontal. Potential energy is $U(s) = -\int_{s_0}^{s} F(\sigma)\, d\sigma$, giving

$$U(s) = \int_{s_0}^{s} mg \underbrace{\sin\theta d\sigma}_{dh} = mg \int_{y_0}^{y} dh = mg(y - y_0). \tag{1.10}$$

We see that the potential energy depends only on the height and not on the horizontal position.

5. The pendulum is a special case of the preceding item. According to (1.10), the potential energy is mgy, where y is the height of the mass above a reference point. We have $y - y_0 = L(1-\cos\theta) = L(1-\sin\frac{s}{L})$, where y_0 is the y-coordinate of the lower equilibrium. Using (1.10) we get

$$U(s) = mgL(1-\cos\theta) = mgL(1-\cos\frac{s}{L}).$$

4. Kinetic energy

Some texts *define* kinetic energy of a point mass m by the formula:

$$K = \frac{mv^2}{2},$$

where m is the mass and v is the speed. This definition is simple, but it feels unmotivated, and in the end it serves to hide the idea.

Instead, let's define the kinetic energy as the *work required to bring the mass from rest to speed* v, and then prove the formula. We will then see *why* the formula is like it is; for example, why is v squared, and where does $1/2$ come from? As an extra benefit, we will understand *why* the total energy is conserved, with no further calculations.

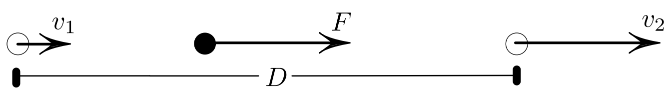

Figure 9. Kinetic energy for one-degree-of-freedom systems.

Proof of $K = mv^2/2$. The work done by the force F is

$$\int F\,dx = \int_0^T F(t)v(t)dt = \int_0^T ma\,vdt = m\int_0^T \frac{d}{dt}\left(\frac{v^2}{2}\right)dt. \tag{1.11}$$

Since $v(0) = 0$ and $v(T) = v$, the fundamental theorem of calculus shows that the integral is $mv^2/2$, as claimed. Note that the work does not depend on how the acceleration varies with time. This is a piece of good luck, since otherwise kinetic energy as we defined it would have depended on the way the particle was accelerated, and thus would have been a meaningless concept. ♢

A calculus-free explanation of $K = mv^2/2$. First, let us accelerate our mass with a constant force F, from speed 0 to v. Let D be the distance the mass travels during its speed-up, and let T be the time of travel. The work done is

$$F \cdot D = ma \cdot D = m\frac{v}{\cancel{T}} \cdot v_{\text{average}}\cancel{T} = mv \cdot \frac{v}{2} = \frac{mv^2}{2},$$

as claimed. Note that the size of F "washed out": a larger F would have meant a smaller D to gain the same speed v (as any driver knows); the product FD would have remained the same. We now see where 1/2 in $mv^2/2$ came from (the averaging of the speed to find D), and why v is squared (both F and D depend on v linearly). To see without calculus why a time-varying force F produces the same result $mv^2/2$, let's break up the travel time into many short intervals, so that during each interval we can treat F as nearly constant. If the speed changes from v_k to v_{k+1} during the kth interval, the work done during this interval is F is $\frac{mv_{k+1}^2}{2} - \frac{mv_k^2}{2}$, as a calculation similar to the one above shows. Thus the work done during acceleration from rest to speed v is the telescoping sum

$$\left(\frac{mv_1^2}{2} - \frac{0^2}{2}\right) + \left(\frac{mv_2^2}{2} - \frac{mv_1^2}{2}\right) + \cdots + \left(\frac{mv^2}{2} - \frac{mv_{n-1}^2}{2}\right) = \frac{mv^2}{2}.$$

♢

Conservation of energy. Replacing $t = 0$ and $t = T$ in (1.11) by any two times t_1 and t_2 results in

$$\int_{x_1}^{x_2} F(x)\, dx = \frac{mv_2^2}{2} - \frac{mv_1^2}{2},$$

where $x_1,\ v_1$ and $x_2,\ v_2$ are the corresponding positions and velocities of the particle. Substituting $F(x) = -U'(x)$ (see (1.8)) we get

$$U(x_1) - U(x_2) = \frac{mv_2^2}{2} - \frac{mv_1^2}{2};$$

potential energy lost equals kinetic energy gained. Or, we can rewrite this as

$$\frac{mv_1^2}{2} + U(x_1) = \frac{mv_2^2}{2} + U(x_2):$$

the total energy of a particle moving in a force field $F(x)$ does not change with time. To summarize, the above physically meaningful definition of K also yields energy conservation as a byproduct. In the next section we give a more streamlined presentation of conservation of energy.

5. Conservation of total energy

Here is an alternative short statement and proof of the law of conservation of energy.

In the same setting as before, we are considering a particle confined to a straight line or a curve and subject to the tangential force $F(x)$, where x denotes the arc length position of the particle.

According to Newton's second law we have

$$m\ddot{x} = F(x). \tag{1.12}$$

It is important to note that the force is assumed to depend on the position x only, and not on the velocity $v = \dot{x}$. This excludes from discussion frictional forces which do depend on the velocity; for such velocity-dependent forces the following theorem fails.

Theorem. *The total energy is constant for any motion* $x = x(t)$ *governed by* (1.12):

$$\boxed{K + U = \frac{m\dot{x}^2}{2} + U(x) = \text{const.}} \tag{1.13}$$

We are not claiming, of course, that different motions have the same energy — only that each motion individually has its own energy which does not change in time.

Proof. It suffices to show that $\frac{d}{dt}(K+U) = 0$, where K and U are evaluated along any solution of (1.12). Differentiating, we get:

$$\frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + U(x)\right) = m\ddot{x}\dot{x} + U'(x)\dot{x} \overset{(1.8)}{=} (m\ddot{x} - F(x))\dot{x} \overset{(1.12)}{=} 0. \tag{1.14}$$

◇

Remark 1.4. It is instructive *to read* (1.14) *backwards*. That is, we could have multiplied both sides of Newton's law $m\ddot{x} - F(x) = 0$ by $\dot{x}$ and then noticed that the left-hand side is just $\frac{d}{dt}(K+U)$. Incidentally, the idea of multiplying by $\dot{x}$ is suggested by the fact that $F\dot{x} = Fv$ is the power expended by the force F applied to a moving particle.[7]

[7]Indeed, $Fv = F\,dx/dt$; since $F dx$ is the work, $F\,dx/dt$ is the work per unit time, i.e., the power.

Mechanics and ODEs. Newton's law (1.12) is an ordinary differential equation (for the unknown function $x = x(t)$); in that sense mechanics is a branch of the theory of ordinary differential equations. A solution is completely specified by a pair $x(0), \dot{x}(0)$ of initial data, according to the theorem on uniqueness and existence for ODEs, provided F is "nice" (continuous differentiability, for instance, qualifies as nice, see [**3**] for details). In the idealized world of classical mechanics the future is completely determined by the present, provided the forces are nice functions of the position.[8]

That mechanics is governed by second order differential equations is an experimental fact. One could imagine a world in which particles' motions were governed by, say, a third order differential equation, or even by an integral equation which involves memory of the past. For instance, we could imagine that a particle held perfectly still and released will, in the absence of external forces, start moving spontaneously in memory to its earlier history, a bit like a raw egg spinning on the table, when stopped momentarily and then released, would start moving again. In fact, Newton's law is only an approximation of fuzzy quantum mechanical objects, and so the "second order" is only an approximate fact and not as intrinsic as might seem, and is probably devoid of deep philosophical meaning.[9]

Examples of energy conservation. In particular examples, conservation of energy (1.13) looks as follows.

1. For a free falling projectile governed by $m\ddot{z} = -mg$:

$$\frac{m\dot{z}^2}{2} + mgz = E.$$

[8] A surprising counterexample to this determinism is a bead on the wire whose graph is $y = -x^{4/3}$. The bead can stay at the equilibrium at $x = 0$ for all time. But the bead can also leave this equilibrium at any time without violating Newton's second law. The same holds for the similar example of a particle in the repelling potential $U = -x^{4/3}$; see Problem 2.33 on page 136.

[9] This descent from the philosophical to technical — or ascent, depending on one's scale of values — happens in science quite often. Fermat's principle is an example. According to Fermat, light rays follow paths of least time. In the early days some saw in this divine intervention. Later, however, a much more prosaic explanation was found: Fermat's principle is the result of phase cancellation or addition. In this case philosophy turned out to be a sophisticated way of admitting, perhaps unknowingly, that one doesn't quite know what's going on.

2. For a mass-spring system governed by $m\ddot{x} = -kx$:

$$\frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = E.$$

3. For the pendulum governed by $m\ddot{s} = -mgL\sin(s/L)$:

$$\frac{m\dot{s}^2}{2} + mgL(1 - \cos(s/L)) = E,$$

or, in terms of the angle $\theta = s/L$:

$$\frac{\dot{\theta}^2}{2} + \frac{g}{L}(1 - \cos\theta) = \text{const.}$$

4. For a particle of unit mass in a two-well potential $U = -x^2 + x^4$ ($\ddot{x} = -U'(x)$):

$$\frac{\dot{x}^2}{2} - x^2 + x^4 = E.$$

6. The phase plane

We continue to discuss Newton's law,

$$m\ddot{x} = F(x),$$

for the motion of a particle on the line. We now describe a fundamental and beautiful way to think of this system geometrically.

Introducing the phase plane. Complete information on the particle's future consists of *two* pieces of data $(x, \dot{x})$ at some t. It is natural to visualize $(x, \dot{x})$ as a point in the plane, thus treating the velocity as the second dimension. The point $(x, \dot{x})$ is referred to as the *phase point*, since it contains full information about the "state" or the "phase" of the system. Thus every point in the plane represents a particle at x with velocity $\dot{x} = y$, and vice versa.

We could call the phase plane the "odometer–speedometer plane", since x is the odometer reading of the particle, and $y = \dot{x}$ is its speedometer reading; see Figure 10.

Seeing the acceleration. As any driver knows, acceleration of the car ahead is much harder to see than its velocity. In the phase plane, however, we can see the acceleration geometrically – namely, as the y-component of phase velocity in the phase plane, Figure 11.

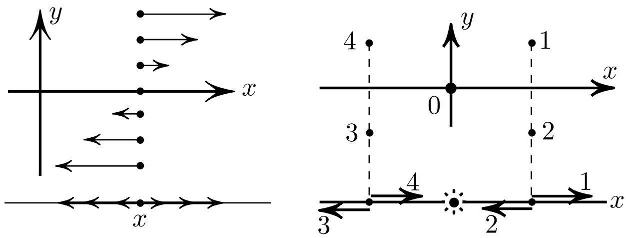

Figure 10. Left: One point on the x-axis can have many velocities as represented by points on the vertical line through x. Right: different velocities for the same position are seen as different y-coordinates in the phase plane.

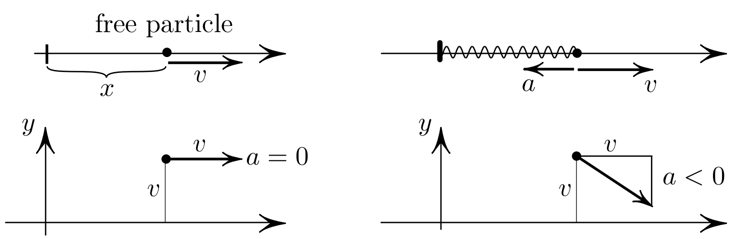

Figure 11. Acceleration seen geometrically, as the vertical component of the velocity vector in the phase plane: $a = \ddot{x} = \dot{y}$. A free particle is shown on the left; its acceleration is zero, and so its velocity vector is horizontal; a particle with negative acceleration is shown on the right.

Phase velocity field. Any mechanical system $m\ddot{x} = F(x)$ is described by a vector field in the plane, as explained in this paragraph. Calling the velocity $\dot{x} = y$, we have $\dot{y} = \ddot{x} = F(x)$ according to Newton's law. Summarizing, we have

$$\begin{cases} \dot{x} = y, \\ \dot{y} = F(x)/m \end{cases} \tag{1.15}$$

or, in vector form,

$$\dot{\mathbf{z}} = \mathbf{V}(\mathbf{z}), \quad \mathbf{z} = (x, y), \quad \mathbf{V} = (y, F(x)/m). \tag{1.16}$$

This is simply a restatement of Newton's law; we traded one second order equation $m\ddot{x} = F(x)$ for two first order equations. But the advantage of the new vector form (1.15) is in this geometrical interpretation: to every point (x, y) in the plane, our system (1.15) assigns the velocity $(\dot{x}, \dot{y})$. Better said, the system (1.15) defines an imaginary fluid flow in the plane, where the velocity of fluid at the point $\mathbf{z} = (x, y)$ is $\mathbf{V}(\mathbf{z}) = (y, m^{-1}F(x))$. Solving the equation amounts to finding the position of any fluid particle at any time.[10]

Interpreting Newton's law $m\ddot{x} = F(x)$ as a fluid flow (1.16) opens a new vista. For example, it becomes easy to view several motions of a mechanical system simultaneously, and to see how they fit together — something that is near-impossible to do by a direct physical observation. Furthermore, we can analyze geometrical features of the flow: is it incompressible? does it stretch in some directions more than in others?, and so on. These features of the "fluid" can then be translated back to mechanics of the particle to give remarkable insights which direct physical intuition does not give. Our goal now is to show how to draw the pattern of the fluid flow given by (1.15). The pattern of flow lines in the phase plane is called the *phase portrait.*

How to draw phase portraits. The key to constructing phase portraits for (1.15) is to use the conservation of energy,

$$\frac{y^2}{2} + U(x) = E = \text{const. for all } \ t, \tag{1.17}$$

where $y = \dot{x}$ and where $x = x(t)$ is any solution of $\ddot{x} = F(x)$. The constant E here depends on the choice of initial data. Geometrically, (1.17) states that the phase trajectories are level curves of the function $y^2/2 + U(x)$, and the question reduces to understanding the pattern of level curves of the energy function. Here are three examples.

1. The harmonic oscillator $\ddot{x} = -x$ corresponds to the system[11]

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases}$$

The trajectories are circles centered at the origin. Indeed, $U = x^2/2$, and the conservation of energy (1.17) gives $\dot{y}^2/2 + \dot{x}^2/2 = c$. The

[10] The solution exists and is unique.

[11] This is the dimensionless version of $m\ddot{x} = -kx$. The rescaling can be achieved by choosing the new time $\tau = at$ in such a way that the period in new units becomes 2π, or $2\pi = a\, 2\pi\sqrt{m/k}$, giving $a = \sqrt{k/m}$; see also Problem 1.23 on page 57.

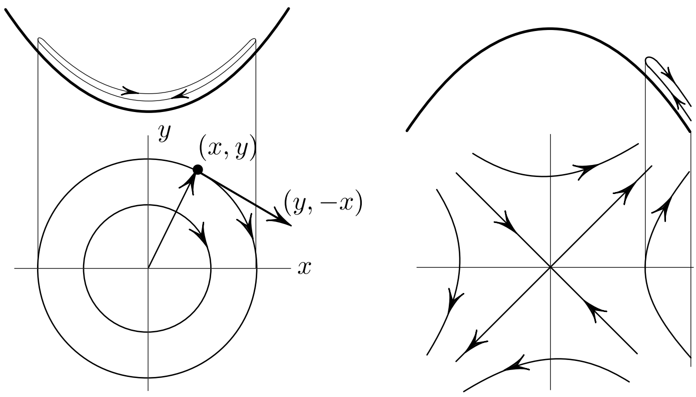

Figure 12. An harmonic oscillator (linear restoring force) and a linear repelling force.

motion on the circles is clockwise, since $\dot{x} = y > 0$ in the upper half-plane. Incidentally, to see that the motion is circular requires no calculation: the velocity vector $(y, -x)$ is perpendicular to the position vector of (x, y) (since their slopes are negative reciprocal).

2. Linear repelling force: $\ddot{x} = x$. The equivalent system is

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x. \end{cases}$$

The trajectories are hyperbolas of the form $y^2 - x^2 = c$, Figure 12, as follows from (1.17) or by direct computation:

$$\frac{d}{dt}(y^2 - x^2) = 2(y\dot{y} - x\dot{x}) = 2(yx - xy) = 0.$$

3. The pendulum. The equation of the pendulum in a rescaled form is $\ddot{x} + \sin x = 0$; here x is the angle between the pendulum and the downward vertical (see Problem 1.23 on page 57 for the details of rescaling). Figure 13 shows the phase portrait; all possible motions can be seen at a glance. Closed orbits correspond to oscillatory motions; unbounded orbits correspond to the pendulum rotating. The regions of these two motions are separated by the so-called heteroclinic orbits which asymptotically approach the upside-down equilibrium as $t \to \infty$ and also as $t \to -\infty$.

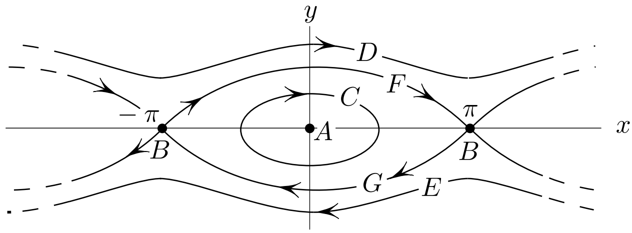

Figure 13. Phase portrait of the pendulum: A – the hanging equilibrium; B – the upside-down equilibrium (represented by $(\pi, 0)$ and its translates by 2π); C – an oscillatory motion; D – clockwise tumbling motion; F and G – motions approaching the unstable equilibrium as $t \to \pm\infty$ (the so-called *heteroclinic motions*).

7. Lagrangian equations of motion

Lagrange's equations were discovered roughly 100 years after Newton's laws. Although Lagrange's equations are equivalent to Newton's laws, they occupy a more central position in mechanics, as Figure 17 on page 287 illustrates. Lagrange's equations also lead to quantum mechanics (as explained on the same page). Since this chapter deals with particles in one dimension, I will formulate Lagrange's equation for this case, postponing the almost identical statement for higher degrees of freedom to Chapter 2.

Lagrange's equation. Consider, as before, a particle moving along the x-axis subject to the force $F(x)$ acting in the direction of the line.

Consider the *difference* of the particle's kinetic and potential energies:

$$L(x, \dot{x}) = \frac{m\dot{x}^2}{2} - U(x); \tag{1.18}$$

this function of $x, \dot{x}$ is called the *Lagrangian*.[12] Let us first treat $x, \dot{x}$ as two independent variables (and not (yet) as functions of t), so at this stage $\dot{x}$ is an independent variable, and not the time-derivative

[12]I know of no *dynamical* interpretation of this difference. However, this difference does have a direct *static* meaning, at the price of reinterpreting t and x. This is described in Section 19.

of some function $x(t)$. Differentiating L first by $\dot{x}$ and then by x, we discover:

$$L_{\dot{x}} = m\dot{x} \quad \text{and} \quad L_x = -U'(x), \tag{1.19}$$

where the subscripts denote partial derivatives: $L_x = \frac{\partial L}{\partial x}$ and $L_{\dot{x}} = \frac{\partial L}{\partial \dot{x}}$. In other words, both the momentum $m\dot{x}$ and the force $-U'(x)$ are partial derivatives of a single function L. Hence Newton's equation $m\ddot{x} = -U'(x)$ can be rewritten as

$$\frac{d}{dt} L_{\dot{x}} = L_x. \tag{1.20}$$

It should be emphasized that $L_{\dot{x}}$, L_x denote partial derivatives when x, $\dot{x}$ are treated as independent variables; however, when taking the derivative $\frac{d}{dt}$ in (1.20), we treat x as the function of time. Equation (1.20) is called the *Euler–Lagrange equation.* The recipe for generating equation of motion from the Lagrangian L applies verbatim in cases much more general than the one just considered.

8. The variational meaning of the Euler–Lagrange equation

Euler–Lagrange equation (1.20) has a remarkable hidden meaning. Loosely speaking, any solution of this equation, i.e., any physical motion, corresponds to the "shortest" path in the (t, x)-plane, in a certain sense which we now make precise.

Functionals and critical functions. Let us fix two points $A_0(t_0, x_0)$ and $A_1(t_1, x_1)$ in the (t, x)-plane. Take any differentiable function $x = x(t)$ whose graph connects connects A_0 and A_1:

$$x(t_0) = x_0, \quad x(t_1) = x_1, \tag{1.21}$$

(see Figure 14), and define the "length" of the graph, called the *action*, as the integral of the difference of kinetic and potential energies:

$$\mathcal{S}[x] = \int_{t_0}^{t_1} \left(\frac{m\dot{x}^2}{2} - U(x) \right) dt = \int_{t_0}^{t_1} L(x, \dot{x})\, dt. \tag{1.22}$$

This integral assigns a real number $\mathcal{S}[x]$ to any given (continuously differentiable) function $x = x(t)$ with fixed ends as in (1.21). Square

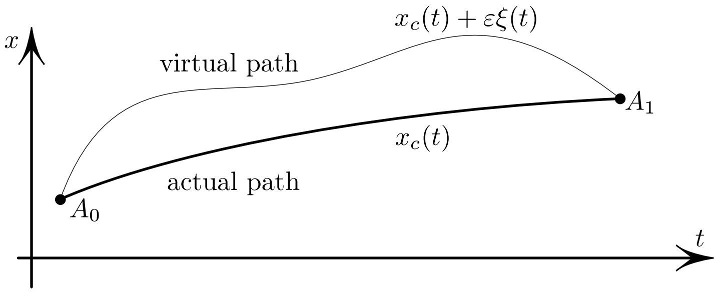

Figure 14. The graph of the actual motion $x = q(t)$ is the "shortest" curve between points $A_0(t_0, x_0)$ and $A_1(t_1, x_1)$ in the sense of the "distance" $\int_{t_0}^{t_1} L\,dt$.

brackets in $\mathcal{S}[x]$ remind us that x is a function. Such scalar-valued functions of a function are called *functionals*.

The minimum, or more generally, a critical function of the functional $\mathcal{S}$ is defined as follows. A function $x_c : [t_0, t_1] \mapsto \mathbb{R}$ is said to be a *critical function* of the functional (1.21)–(1.22) if

$$\frac{d}{d\varepsilon}\mathcal{S}[x_c + \varepsilon\xi]\bigg|_{\varepsilon=0} = 0 \tag{1.23}$$

for any smooth function ξ with $\xi(t_0) = \xi(t_1) = 0$; see Figure 14. Smoothness of ξ must be assumed for the derivative in (1.22) to exist. According to the definition (1.23), the functional changes with zero speed under any deformation of the critical curve.

Variational meaning of dynamical equations. Now we have the most remarkable and fundamental fact in mechanics, which is a culmination of several discoveries and known as *Hamilton's principle*:

The function $x = x(t)$ represents an actual motion, i.e., obeys (1.20)
if and only if
$x(t)$ is a critical function of the action integral (1.22).

Superficially, this is just a mathematical theorem (restated and proven as Theorem 1.1 on page 22). However, more is going on: the fact that actual motions are the "shortest" curves in space-time (t, x) reflects the fact that classical mechanics is the limiting case of quantum mechanics (see pages xviii and 286 for further discussion).

Remark. If $t_1 - t_0$ is sufficiently small, then the actual motion minimizes the action (1.22). This is explained in Chapter 6.

Analogy between functionals and real functions of several variables. To get more intuition on functionals and their critical functions just defined, note that our functional $\mathcal{S}[x]$ is an analog of a real function of many variables. Indeed, imagine discretizing $x(t)$, i.e., replacing it by a sampling of its values at n points in $[t_0, t_1]$. The integral $\mathcal{S}$ would then be replaced by a function of these values — it would become, in other words, a usual function of n variables. Recall the calculus definition of the critical point of such a function (with the goal of finding a familiar analog of (1.23)). A point $\mathbf{x}_c \in \mathbb{R}^n$ is said to be a *critical point* of a function $f(\mathbf{x})$ of n variables $\mathbf{x} = (x_1, \ldots, x_n)$ if the directional derivative at $\mathbf{x}_c$ is zero in every direction, i.e.,

$$\frac{d}{d\varepsilon} f(\mathbf{x}_c + \varepsilon \boldsymbol{\xi})\bigg|_{\varepsilon=0} = 0 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n.$$

In other words, f changes with zero instantaneous speed at the critical point $\mathbf{x}_c$. This is precisely the analog of the definition (1.23) of the stationary function. The left-hand side in (1.23) is simply a directional derivative of $\mathcal{S}$ in the function space.

9. Euler–Lagrange equations — general theory

In the preceding section we defined a critical function of a functional

$$\mathcal{S}[x] = \int_{t_0}^{t_1} L(x, \dot{x})\, dt, \quad x(t_0) = x_0, \ x(t_1) = x_1, \tag{1.24}$$

with prescribed endpoints (t_0, x_0), (t_1, x_1). The definition did not rely on the special form $L = K - U$ and we abandon this assumption here; we will only need L to have continuous partial derivatives with respect to its arguments $x, \dot{x}$ up to order two (with x, $\dot{x}$ treated as independent variables).

In this section we derive the "first derivative test" for $\mathcal{S}[x]$, namely, a necessary condition for $x_c(t)$ to be a critical function.

Euler and Lagrange independently found the famous answer described in the following theorem.

Theorem 1.1 (The Euler–Lagrange equation)**.** *Assume that* $L(x, \dot{x})$ *has two continuous derivatives in its variables (at this stage* $\dot{x}$ *is treated as an independent variable, and not* dx/dt*.) If* $x = x(t)$ *is a critical function (we drop the subscript c from now on) of the functional* (1.24)*, and if* x *has two continuous derivatives, then* x *satisfies the differential equation*

$$\boxed{\frac{d}{dt} L_{\dot{x}} - L_x = 0} \tag{1.25}$$

where

$$L_x = \frac{\partial}{\partial x} L(x, \dot{x}), \quad L_{\dot{x}} = \frac{\partial}{\partial v} L(x, v)|_{v=\dot{x}} \,.$$

Note that x *and* $\dot{x}$ *are treated as independent variables when taking these partial derivatives; however,* $\frac{d}{dt}$ *in* (1.25) *treats both* x *and* $\dot{x}$ *as functions of* t*.*

Example. For the harmonic oscillator, $L = m\dot{x}^2/2 - kx^2/2$ we have $L_{\dot{x}} = m\dot{x}, \;\; L_x = -kx$; substituting these into (1.25), we get $m\ddot{x} + kx = 0$, as expected.

Proof. Let $x = x(t)$ be a critical function of $\mathcal{S}$. By the definition, x satisfies

$$\frac{d}{d\varepsilon} \int_{t_0}^{t_1} L(x + \varepsilon\xi, \dot{x} + \varepsilon\dot{\xi})_{|\varepsilon=0} = 0$$

for all differentiable ξ vanishing at $t_0, \; t_1$. Differentiation by ε can be applied to the integrand. Applying the chain rule and then setting $\varepsilon = 0$ we get

$$\int_{t_0}^{t_1} (L_x \xi + L_{\dot{x}} \dot{\xi}) dt = 0, \tag{1.26}$$

where $L = L(x, \dot{x})$. Note that in applying the chain rule we had to treat x and $\dot{x}$ as independent variables in taking the partials $L_x, \; L_{\dot{x}}$. Let us now integrate by parts the second term in (1.26). Using $\xi(t_0) = \xi(t_1) = 0$ to get rid of the boundary term we get

$$\int_{t_0}^{t_1} \left(L_x - \frac{d}{dt} L_{\dot{x}} \right) \xi \; dt = 0. \tag{1.27}$$

Note that $\frac{d}{dt}$ must treat x and $\dot{x}$ as functions of t.

Now since the function ξ is arbitrary (apart from the assumptions mentioned before), we expect the expression $f(t) = L_x - \frac{d}{dt}L_{\dot{x}}$ to be identically zero (as desired). Indeed, assume for a moment the contrary: $f(\bar{t}) \neq 0$, say, $f(\bar{t}) > 0$ for some $\bar{t} \in [t_0, t_1]$. Since f is continuous by our assumptions, we have $f > 0$ on a whole interval I containing $\bar{t}$. Let us then choose $\xi > 0$ on I and $\xi = 0$ elsewhere. But then $\int_{t_0}^{t_1} f(t)\xi(t)\,dt > 0$, in contradiction with (1.27). This completes the proof. ◇

Remark. As stated in the footnote to the theorem, it actually suffices to assume that the critical function $x(t)$ has just one continuous derivative; the existence of the second derivative then follows. The idea of the proof is very nice: instead of integrating by parts the second term in (1.26), integrate the first! This gives

$$\int_{t_0}^{t_1} \left(-\int_{t_0}^{t} L_x d\tau + L_{\dot{x}} \right) \dot{\xi}\, dt = 0;$$

using arbitrariness of ξ, it is easy to show that

$$-\int_{t_0}^{t} L_x d\tau + L_{\dot{x}} = \text{const.}$$

Since the first term is continuously differentiable (C^1), so is $L_{\dot{x}}(x, \dot{x})$. But this implies that $\dot{x}$ itself is continuously differentiable (I omit details which involve using the implicit function theorem). In other words, x is twice continuously differentiable, as claimed.

10. Noether's theorem/Energy conservation

The following is actually a special case of Noether's theorem — the general case is described on pages 267 and 270 when we consider higher degrees of freedom.

Theorem 1.2. *For any solution x of the Euler–Lagrange equation* (1.25) *we have*

$$\dot{x}L_{\dot{x}} - L = \text{const.} \tag{1.28}$$

Proof goes by differentiation: Using the chain rule and the Euler–Lagrange equation, we get

$$\frac{d}{dt}(\dot{x}L_{\dot{x}} - L) = \cancel{\ddot{x}L_{\dot{x}}} + \dot{x}\frac{d}{dt}L_{\dot{x}} - L_x\dot{x} - \cancel{L_{\dot{x}}\ddot{x}} = \dot{x}\underbrace{\left(\frac{d}{dt}L_{\dot{x}} - L_x\right)}_{=0 \text{ by } (1.25)} = 0.$$

$\diamondsuit$

Although this proof is short, it does not explain "what is going on." A more illuminating proof, which shows what is happening geometrically, and works for the higher-degree-of-freedom case, is given on page 270.

Example. For the special case: $L = \frac{m\dot{x}^2}{2} - U(x)$, Noether's theorem recovers the conservation of energy:

$$\dot{x}L_{\dot{x}} - L = \frac{m\dot{x}^2}{2} + U(x) = \text{const.},$$

as a direct substitution of this L into (1.28) shows. Note that $\dot{x}L_{\dot{x}} - L$ turned out to be the total energy.

11. Hamiltonian equations of motion

We already saw one remarkable way to reformulate Newton's law $m\ddot{x} = -U'(x)$, as the Euler–Lagrange equation. There is yet another reformulation which combines a beautiful symmetry with a yet additional insight. At this point I describe only *how* to transform Euler–Lagrange's equation into a Hamiltonian system, leaving out the motivation (which can be found in Chapter 8).

The momentum and the Hamiltonian. Let us define the momentum

$$m\dot{x} = p \tag{1.29}$$

and express the total energy $H = m\dot{x}^2/2 + U$ in terms of p by substituting $\dot{x} = p/m$:

$$H(x,p) = \frac{p^2}{2m} + U(x). \tag{1.30}$$

The energy thus expressed in terms of position and momentum is called the *Hamiltonian* of the system. It is instructive to take partial derivatives of H with x, p treated as two independent variables:

$$H_x(x,p) = U'(x) = -F(x), \quad H_p(x,p) = \frac{p}{m}.$$

Using the first equation, Newton's law $m\ddot{x} = F(x)$ becomes $\dot{p} = -H_x$; using the second equation, we can rewrite the definition of p as $\dot{x} = p/m = H_p$. Summarizing, we have

$$\left\{ \begin{aligned} \dot{x} &= H_p(x,p), \\ \dot{p} &= -H_x(x,p). \end{aligned} \right. \tag{1.31}$$

This system is equivalent to Newton's equation. This elegant system looks even more elegant in vector form,

$$\dot{\mathbf{z}} = J\,\nabla H(\mathbf{z}), \tag{1.32}$$

where

$$\mathbf{z} = \begin{pmatrix} x \\ p \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \nabla H = \begin{pmatrix} H_x \\ H_p \end{pmatrix}.$$

Note that J rotates vectors by $\pi/2$ clockwise, and we arrive at a remarkable connection between Hamiltonian and gradient systems: any Hamiltonian vector field (1.31)–(1.32) is obtained from the gradient vector field $\dot{\mathbf{z}} = \nabla H(\mathbf{z})$ by the $\pi/2$ rotation clockwise, Figure 15.

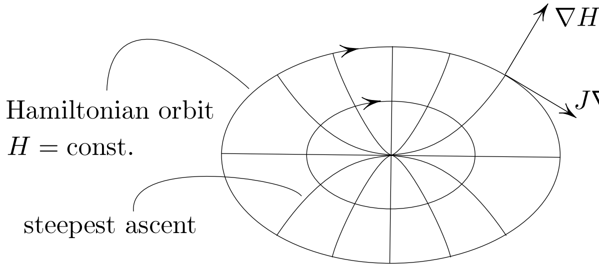

Figure 15. Hamiltonian vector field is orthogonal to the gradient vector field ∇H.

Trajectories are level curves of H. Indeed, H is constant along each solution of (1.31):

$$\frac{d}{dt}H(\mathbf{z}(t)) = H_x\dot{x} + H_p\dot{p} = H_x H_p + H_p(-H_x) = 0.$$

This can be seen geometrically from (1.32): $\dot{\mathbf{z}} \perp \nabla H$, and thus $\dot{\mathbf{z}}$ is tangent to a level curve of H; this means that $\mathbf{z}(t)$ stays on the level curve.

Time-dependent Hamiltonians. Consider the motion of a particle in a potential which depends on time:

$$\ddot{x} = -U_x(x,t); \tag{1.33}$$

examples include a pendulum whose pivot undergoes vertical oscillations:

$$\ddot{\theta} = -\frac{1}{L}(g + a(t))\sin\theta,$$

where a is the acceleration of the pivot, or a mass hanging on a spring whose end is oscillating in the vertical direction, etc.

Exactly as before, Newton's law (1.33) can be written as a Hamiltonian system:

$$\begin{cases} \dot{x} = H_p(x,p,t), \\ \dot{p} = -H_x(x,p,t), \end{cases} \qquad H = \frac{p^2}{2m} + U(x,t) \tag{1.34}$$

or, in vector form,

$$\dot{\mathbf{z}} = J\nabla H(\mathbf{z}, t). \tag{1.35}$$

Exercise. Does H remain constant along solutions of (1.35)?

Answer. Denoting partial derivative by subscripts, we get, differentiating H along a solution of the Hamiltonian system:

$$\frac{d}{dt}H(x,p,t) = H_x\dot{x} + H_p\dot{p} + H_t = H_x H_p + H_p(-H_x) + H_t = H_t.$$

We conclude that if H depends on t, it does not remain constant.

12. The phase flow

In this and the next section we introduce two fundamental concepts — the *flow* and the *divergence*, important in their own right, and used in Liouville's theorem stated in Section 16.

Liouville's theorem states that if a vector field has zero divergence (see page 28 for the definition), then the flow generated by the vector field is incompressible, or area-preserving: as a "blob" of initial data is carried by the flow, the area of the blob remains unchanged.[13]

We assume throughout that the vector field $\mathbf{v}(\mathbf{x})$ in $\mathbb{R}^2$ is smooth on $\mathbb{R}^2$, and that for any initial condition in $\mathbb{R}^2$ the solution of the ODE

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}) \tag{1.36}$$

is defined for all t.

The phase flow. For any point $\mathbf{x}_0 \in \mathbb{R}^2$ there is a unique solution $\mathbf{x} = \mathbf{x}(t)$ of (1.36) satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_0$; this is the statement of the existence/uniqueness theorem for ODEs, applicable by the assumptions we made on $\mathbf{v}$. The standard notation for this solution is $\mathbf{x}(t) = \varphi^t \mathbf{x}_0$. We can view φ^t as the map of $\mathbb{R}^2$ which sends $\mathbf{x}_0$ to $\varphi^t \mathbf{x}_0$. Since t is arbitrary, we have a one-parameter family of these maps, also referred to as *the time t maps* (of the phase space to itself). This family of maps is called the *flow* (associated with the ODE (1.36)); the term "flow" is suggested by thinking of $\varphi^t \mathbf{x}_0$ as the position at time t of a particle of fluid.

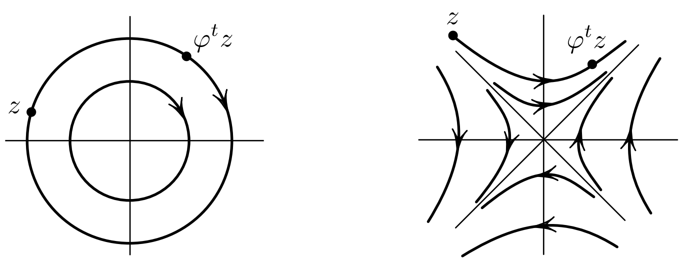

Figure 16. Flows associated with some vector fields.

[13] We are speaking of flows in $\mathbb{R}^2$; in $\mathbb{R}^3$ the area must be replaced by volume, and in $\mathbb{R}^n$ by n-volume.

Theorem 1.3. *The family of maps* $\{\varphi^t : t \in \mathbb{R}\}$ *associated with the vector field* $\mathbf{v} = \mathbf{v}(\mathbf{z})$ *satisfies the following properties:*

(1) $\varphi^t \circ \varphi^\tau = \varphi^{t+\tau}$, *for any real* t, τ.

(2) φ^0 *is an identity map:* $\varphi^0 \mathbf{x} = \mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

In other words, the family forms a group under composition.

This theorem is just a restatement of the existence and uniqueness theorem for ordinary differential equations, coupled with the fact that $\mathbf{v}$ is autonomous (i.e., does not depend on t). Without the latter assumption property (1) would fail.

13. The divergence

The concept of divergence is the second prerequisite for the formulation of Liouville's theorem. We again limit discussion to $\mathbb{R}^2$, although most ideas apply to any dimension almost verbatim.

The definition. The divergence of a vector field $\mathbf{v}$ in $\mathbb{R}^2$ at a point $\mathbf{x}$ is the outward flux per unit area through the boundary of a small region as the region shrinks to the point $\mathbf{x}$:

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = \lim_{D \to \{\mathbf{x}\}} \frac{1}{|D|} \int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds, \tag{1.37}$$

where D is a region enclosing $\mathbf{x}$ (Figure 17), $|D| = \text{area}(D)$, ∂D is the boundary of D, $\mathbf{n}$ is the unit outward normal vector and s is the arc length.[14]

The limit (1.37) exists and does not depend on the particular choice of D, provided only that $\mathbf{v}$ is a smooth function of x, y, and that D is bounded by a piecewise smooth curve without self-intersections.

[14] In dimension $n > 2$ the definition is the same, except that ds is the $(n-1)$-dimensional surface element of the n-dimensional region D.

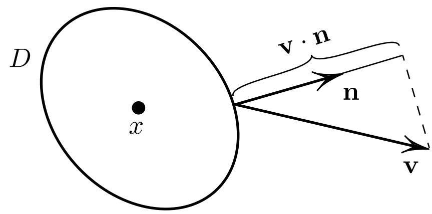

Figure 17. Definition of the divergence of a vector field.

The formula. By choosing D in the definition (1.37) to be a rectangle, $D = [x, x + dx] \times [y, y + dy]$, one arrives at the standard formula,

$$\text{div } \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}, \quad \text{where} \quad \mathbf{v} = (v_1, v_2). \tag{1.38}$$

Intuitively, the formula makes perfect sense: $\frac{\partial v_1}{\partial x}$ detects the dependence of the x-component of the velocity on x; if $\frac{\partial v_1}{\partial x}$ is large, then the horizontal velocity v_1 is greater through the right side of the box than through the left side, contributing to positive net flux out of the box. More details on the topic can be found in [**17**].

Theorem 1.4 (The Divergence Theorem)**.** *If* $\mathbf{v}$ *is a continuously differentiable vector field on a bounded domain* D *in* $\mathbb{R}^2$ *with the piecewise smooth boundary, then*

$$\int_D \text{div } \mathbf{v} \, dx = \int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds, \tag{1.39}$$

where dx *is the element of area, and* ds *is the element of arc length of the boundary* ∂D*.*

Remark 1.5. The theorem and the proof apply to the case $\mathbb{R}^n$ for any n almost verbatim.

Proof. This theorem is almost obvious if one uses the definition (1.37), rather than the computational formula (1.38). Indeed, let us divide the domain D into subdomains of small diameters $\leq \delta$, as shown in Figure 18, and denote a typical subdomain by D_i. The first step is to note that

$$\int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds = \sum_i \int_{\partial D_i} \mathbf{v} \cdot \mathbf{n} \, ds; \tag{1.40}$$

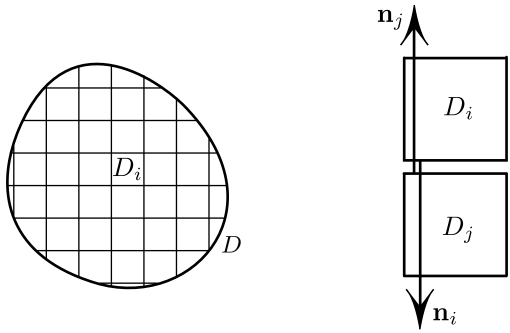

Figure 18. Proof of the divergence theorem. Right: fluxes on shared boundaries cancel.

indeed, the integrals over the shared boundaries cancel, since the outward normals on a shared boundary point in opposing directions; see Figure 18. We now apply the definition of divergence to each D_i: picking a point $\mathbf{x}_i \in D_i$ for each i, we have by (1.37) that

$$\operatorname{div} \mathbf{v}(\mathbf{x}_i) = \frac{1}{|D_i|} \int_{\partial D_i} \mathbf{v} \cdot \mathbf{n} \, ds + r_i,$$

where the remainder r_i is small if diam $(D_i) \leq \delta$ is small. Actually, the r_i are *uniformly small*: for any ε there exists δ such that $|r_i| < \varepsilon$ for *all* i, provided diam $(D_i) \leq \delta$; I omit the details of the proof, which uses continuous differentiability of $\mathbf{v}$. Rewriting the above as

$$\int_{D_i} \mathbf{v} \cdot \mathbf{n} \, dx = (\operatorname{div} \mathbf{v}(\mathbf{x}_i) - r_i) \, |D_i|, \tag{1.41}$$

and adding up, we get

$$\underbrace{\sum_i \int_{\partial D_i} \mathbf{v} \cdot \mathbf{n} \, ds}_{\int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds \text{ by } (1.40)} = \sum_i \operatorname{div} \mathbf{v}(\mathbf{x}_i) |D_i| - \underbrace{\sum_i r_i |D_i|}_{<\varepsilon |D|}. \tag{1.42}$$

The second term is the Riemann sum for the first integral in (1.39). In the limit $\varepsilon \to 0$, (1.42) becomes (1.39). This proves the divergence theorem. $\diamondsuit$

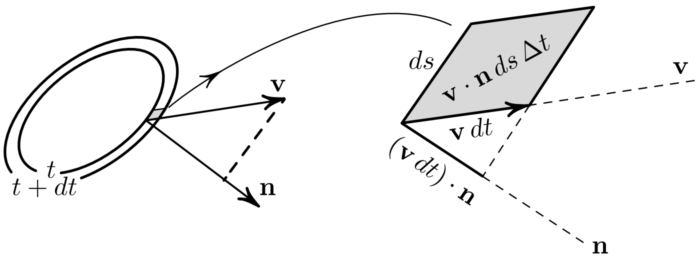

Figure 19. Computing ΔA (Steps 1–4).

14. A lemma on moving domains

The divergence in $\mathbb{R}^2$ admits a very nice interpretation: it is the exponential rate of growth of an infinitesimal area[15] carried by the vector field. In other words, the divergence is the local logarithmic rate of the area's growth. Now to make this precise, we need to express the rate of change of the volume of a moving domain; the lemma in this section does it. Incidentally, this lemma is a generalization of the fundamental theorem of calculus, as explained at the end of the section. The extension to $\mathbb{R}^n$ is verbatim and involves no new ideas; and since our goal is to learn ideas, we stick to $\mathbb{R}^2$.

Lemma 1.1. *Let $D \in \mathbb{R}^2$ be a region with a piecewise smooth boundary; let $\mathbf{v}$ be a vector field in $\mathbb{R}^2$, with the associated flow φ^t, and let $A(t) = \text{area}(\varphi^t D)$. Then*

$$A'(t) = \int_{\partial D_t} \mathbf{v}\cdot\mathbf{n}\,ds. \tag{1.43}$$

A heuristic explanation of the lemma is supplied by Figure 19: an arc ds moving with the flow for time Δt sweeps area approximately $\mathbf{v}\cdot\mathbf{n}\,ds\,\Delta t$ (shown as a shaded parallelogram), and thus the area swept by the entire boundary in time Δt is

$$\Delta A = \int_{\partial D} \mathbf{v}\cdot\mathbf{n}\,ds\,\Delta t + o(\Delta t).$$

[15]In $\mathbb{R}^n$ the n-volume should replace the area.

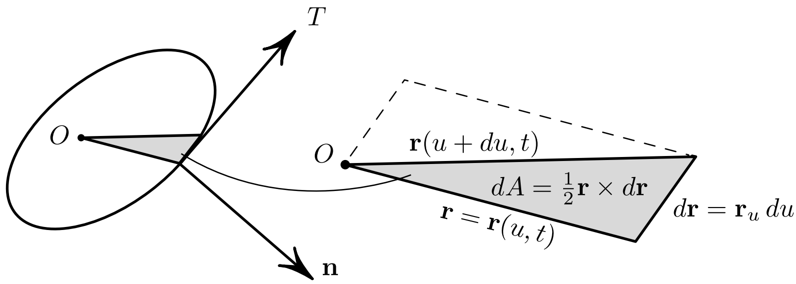

Figure 20. Explaining (1.44) geometrically.

Dividing by Δt and taking $\Delta t \to 0$ we get (1.43). Rather than massaging this heuristic argument into a rigorous proof, I give an alternative one.

Proof of the lemma in $\mathbb{R}^2$. Let us parametrize ∂D by $\mathbf{x} = \mathbf{r}(u)$, $0 \le u \le 1$, $\mathbf{r}(0) = \mathbf{r}(1)$. The curve $\partial(\varphi^t D)$ is then also parametrized by u: $\mathbf{r}(u,t) = \varphi^t \mathbf{r}(u)$. The area of $\varphi^t D$ is then given by

$$A(t) = \text{area}(\varphi^t D) = \frac{1}{2}\int_0^1 \mathbf{r}\times\mathbf{r}_u\,du \tag{1.44}$$

(see Figure 20); here $\mathbf{r}\times\mathbf{r}_u = \det(\mathbf{r}\ \mathbf{r}_u)$ is the signed area of the parallelogram generated by the pair of vectors (also called the scalar cross product). Differentiating by t and integrating the second term by parts, we get:

$$A'(t) = \frac{1}{2}\int_0^1 (\mathbf{r}_t\times\mathbf{r}_u + \mathbf{r}_t\times\mathbf{r}_{ut})\,du = \frac{1}{2}\int_0^1 (\mathbf{r}_t\times\mathbf{r}_u - \mathbf{r}_u\times\mathbf{r}_t)\,du.$$

Since $\mathbf{r}_u\times\mathbf{r}_t = -\mathbf{r}_t\times\mathbf{r}_u$, this reduces to

$$A'(t) = \int_0^1 \mathbf{r}_t\times\mathbf{r}_u\,du.$$

But $\mathbf{r}_t = \frac{d}{dt}\mathbf{r}(u,t) = \frac{d}{dt}\varphi^t\mathbf{r}(u,t) = \mathbf{v}(\mathbf{r}(u,t))$, and $\mathbf{r}_u du = \mathbf{T}\,ds$, where $\mathbf{T}$ is the unit tangent vector to $\partial\varphi^t D$. We therefore have

$$A'(t) = \int_{\partial(\varphi^t D)} \mathbf{v}\times\mathbf{T}ds = \int_{\partial(\varphi^t D)} \mathbf{v}\cdot\mathbf{n}\,ds;$$

the identity $\mathbf{v}\times\mathbf{T} = \mathbf{v}\cdot\mathbf{n}$ holds because the angles $\angle(\mathbf{v},\mathbf{T})$ and $\angle(\mathbf{v},\mathbf{n})$ are complementary. $\diamondsuit$

Proof of the lemma in $\mathbb{R}^n$. The above proof can be extended to $\mathbb{R}^n$, by replacing (1.44) with

$$V(t) = \frac{1}{n} \int \det(\mathbf{r}, \mathbf{r}_{u_1}, \ldots, \mathbf{r}_{u_{n-1}})\, du_1 \ldots du_{n-1},$$

the subscripts denoting partial derivatives.

Here is yet another dimension-independent proof of the Lemma in $\mathbb{R}^n$; the judgment on which proof is better is left to the reader. Using the change of variables $\mathbf{x} = \varphi^t \mathbf{y}$ we get:

$$A(t) = \int_{\varphi^t(D)} d\mathbf{x} = \int_D \det\, \Phi(t, \mathbf{y}) d\mathbf{y},$$

where

$$\Phi(t, \mathbf{y}) = \frac{\partial \varphi^t \mathbf{y}}{\partial \mathbf{y}} \tag{1.45}$$

is the Jacobian derivative matrix of the map φ^t. Differentiating by t, we get

$$A'(t) = \int_D \frac{d}{dt} \det \Phi\, d\mathbf{y}, \tag{1.46}$$

and our goal now is to relate the integrand to the vector field $\mathbf{v}$. This relation can only come from the definition of φ^t, according to which

$$\frac{d}{dt} \varphi^t \mathbf{y} = \mathbf{v}(\varphi^t \mathbf{y}),$$

for all $\mathbf{y} \in D$. Differentiating by the initial condition $\mathbf{y}$ and exchanging the order of differentiation we obtain

$$\frac{d}{dt} \Phi = \frac{\partial \mathbf{v}(\varphi^t \mathbf{y})}{\partial \mathbf{y}} \Phi. \tag{1.47}$$

This implies, by Abel's theorem:

$$\frac{d}{dt} \det \Phi = \operatorname{tr} \frac{\partial \mathbf{v}(\varphi^t \mathbf{y})}{\partial \mathbf{y}} \det \Phi. \tag{1.48}$$

But

$$\operatorname{tr} \frac{\partial \mathbf{v}(\varphi^t \mathbf{y})}{\partial \mathbf{y}} = \operatorname{div} \mathbf{v},$$

and (1.48) becomes

$$\frac{d}{dt} \det \Phi = \operatorname{div} \mathbf{v}(\varphi^t \mathbf{y}) \det \Phi.$$

Substituting this into (1.46) gives

$$A'(t) = \int_{D_0} \operatorname{div} \mathbf{v}(\varphi^t \mathbf{y}) \det \Phi\, d\mathbf{y}.$$

Returning to the old variables $\mathbf{x} = \varphi^t \mathbf{y}$ we arrive at the desired statement:

$$A'(t) = \int_{D_t} \operatorname{div} \mathbf{v}(\mathbf{x})\, d\mathbf{x} = \int_{\partial D_t} \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}\, ds,$$

where the last step uses the divergence theorem. ♢

Remark 1.6. The fundamental theorem of calculus (FTC) is a special case of the lemma on moving boundaries: indeed, FTC deals with the rate of change of the area $\int_a^t f(x)\, dx$ under the curve $y = f(x)$, with $a \le x \le t$, where only one piece of the boundary is moving, namely, the "right wall" $x = t$. The speed of this wall is $v = \frac{dt}{dt} = 1$, and its length $L = f(t)$. The boundary integral of the normal speed is therefore $Lv = f(t) \cdot 1 = f(t)$, so that our lemma gives

$$\frac{d}{dt} \int_a^t f(x)\, dx = f(t),$$

showing that FTC is indeed a special case.

15. Divergence as a measure of expansion

This section explains that the divergence in $\mathbb{R}^2$ is the rate of growth of an infinitesimal area per unit area (in $\mathbb{R}^n$ the same statement holds if the area is replaced by n-volume). Here is a precise statement.

Theorem 1.5. *Let $A(t) = \text{area}(\varphi^t D)$ denote the area of a domain D moving with the flow φ^t of the vector field $\mathbf{v}$. We then have:*

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = \lim_{D \to \{\mathbf{x}\}} \frac{A'(0)}{A(0)}. \tag{1.49}$$

Proof. By the lemma on moving domains

$$A'(0) = \int_{D_t} \mathbf{v} \cdot \mathbf{n}\, ds,$$

as follows from (1.43) by setting $t = 0$. Substituting this into the definition of divergence (1.37), we obtain (1.49). ♢

Equation (1.49) confirms the earlier statement that $\operatorname{div} \mathbf{v}(\mathbf{x})$ measures the instantaneous rate of change of the area of a small region surrounding $\mathbf{x}$, per unit area, as the area shrinks to $\mathbf{x}$. In other words, $\operatorname{div} \mathbf{v}(\mathbf{x})$ is the *exponential rate of growth of area* in a small neighborhood of $\mathbf{x}$, also referred to as the *logarithmic derivative* of the small area.

Incidentally, the *interest rate in continuous compounding in finance is an example of divergence* in $\mathbb{R}^1$. Indeed, the interest rate k is defined by

$$k = 100\, A'(t)/A(t),$$

where $A(t)$ is the amount of money in the account at time t, the one-dimensional version of (1.49). The vector field v in compounding is given by $v(x) = kx$, since the money grows according to $\dot{A} = kA$.

We would have used the very intuitive expression (1.49) as the definition of the divergence were it not for the fact that (1.49) requires the mention of the time and of the flow φ^t, which our definition (1.37) does not. The advantage of (1.49), however, is that it better reflects the term "divergence", since it indeed measures the expansion of area. Another advantage of (1.49) is that it suggests that if $\operatorname{div} \mathbf{v}(\mathbf{x}) = 0$ for all $\mathbf{x}$, then the flow is area-preserving. This is indeed the content of Liouville's theorem, which is discussed next.

To summarize, (1.37) and (1.49) reflect two views of the divergence: one by fixing the domain and watching the particles go by, and the other by following a moving domain. The first of these is referred to as the Eulerian approach, the second as the Lagrangian.

16. Liouville's theorem

As before, φ^t is the flow of a smooth vector field $\mathbf{v}$ in $\mathbb{R}^2$ and we assume, as always, that φ^t is well defined for all $t \in \mathbb{R}$.

Theorem 1.6 (Liouville)**.** If $\operatorname{div} \mathbf{v} = 0$, then the flow φ^t of $\mathbf{v}$ is area-preserving, that is, for any planar region D bounded by a piecewise smooth curve without self-intersections

$$A(t) = \operatorname{area}(\varphi^t D) = \text{const.} \tag{1.50}$$

Proof.

$$A'(t) \overset{(1.43))}{=} \int_{\partial(\varphi^t D)} \mathbf{v} \cdot \mathbf{n}\, ds \overset{(1.39)}{=} \int_{\varphi^t D} \operatorname{div} \mathbf{v}\, d\mathbf{x} = 0. \tag{1.51}$$

$\diamondsuit$

Remark 1.7 (On nonautonomous vector fields)**.** Liouville's theorem holds even for the time-dependent vector fields $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$. None of the statements or arguments in this section are affected by introducing the time-dependence. It suffices to assume that $\mathbf{v}$ is continuous (or even merely summable) in t, see [**3**].

Remark 1.8. To summarize its gist, Liouville's theorem amounts to little more than redefining divergence dynamically, as (1.49). In fact, this alternative definition of divergence is an infinitesimal version of Liouville's theorem. When viewed in this way, Liouville's theorem is completely transparent.

Exercise 1.2. Show that a divergence-free vector field $\mathbf{v}(\mathbf{x})$ in $\mathbb{R}^2$ is locally Hamiltonian: that is, if $\operatorname{div}\mathbf{v} = 0$ in a neighborhood of a point, then there exists a real-valued function H such that $\mathbf{v} = (H_y, -H_x)$. Such H is called a *Hamiltonian.*

Hint: Define $H(z)$ as the flux of $\mathbf{v}$ through a curve connecting z with some chosen point. Show that H does not depend on the choice of the curve (using $\operatorname{div}\mathbf{v} = 0$).

Exercise 1.3. Show that the preceding statement is only true locally, i.e., give an example of a divergence-free vector field for which a single–valued Hamiltonian function does not exist.

Answer. $\mathbf{v}(\mathbf{z}) = \frac{1}{|\mathbf{z}|^2}\mathbf{z}$ is a divergence-free flow in the punctured plane (the flow due to a point source at the origin). $H = \arg z$ satisfies $\mathbf{v} = (H_y, -H_x)$ but H is not single-valued. Any other Hamiltonian of $\mathbf{v}$ differs from H by a constant, hence there is no single-valued Hamiltonian of $\mathbf{v}$.

17. The "uncertainty principle" of classical mechanics

In this section we consider time-dependent Hamiltonian systems:

$$\begin{cases} \dot{x} = H_p(x,p,t), \\ \dot{p} = -H_x(x,p,t). \end{cases}$$

Liouville's theorem still applies, since

$$\operatorname{div}(H_p, -H_x) = H_{px} + (-H_{xp}) = 0.$$

We conclude that the flow φ^t of this system is area-preserving: for any domain $D \subset \mathbb{R}^2$ with a piecewise smooth boundary we have $\operatorname{area}(\varphi^t D) = \text{const}$. This implies that if, for some t, the image $\varphi^t D$ is squeezed in, say, the x-direction, then it must be stretched in the

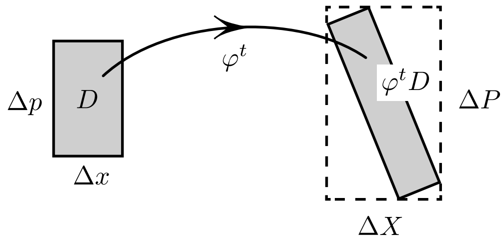

Figure 21. The uncertainty principle in classical mechanics.

p-direction. In other words, *the more we know x, the less we know p* — a kind of uncertainty principle in classical mechanics.

To get some practical conclusion, let D be a rectangle, as shown in Figure 21; as the figure illustrates, ΔX, ΔP are the horizontal and vertical widths of $\varphi^t D$ for some later time t. Since area(D) = area($\varphi^t D$), we have

$$\Delta x \Delta p \leq \Delta X \Delta P. \tag{1.52}$$

To suggest the analogy with quantum mechanics, denote area(D) = $\Delta x \Delta p = h$ (recall $\hbar$, the Planck constant); (1.52) becomes

$$\Delta X \Delta P \geq h.$$

An example with particles. The motion of particles in a potential $U(x,t)$ on the line is governed by $\ddot{x} = -U_x(x,t)$; the potential may depend on t arbitrarily. The rectangle D in Figure 21 corresponds to a "cloud" of initial data with the range Δx of positions and with the range Δp of velocities; the view in the (t,x)-plane is shown in Figure 22. Now assume that $\Delta X < \Delta x/100$; that is, assume that all the particles from the initial cloud gathered up into a narrower interval. Squeezing in the x-direction means expansion in the p-direction, according to (1.52):

$$\Delta P \geq \frac{\Delta x}{\Delta X} \Delta p > 100 \Delta p.$$

This is a remarkable conclusion: the more the particles bunch up together, the more disparate their velocities become.

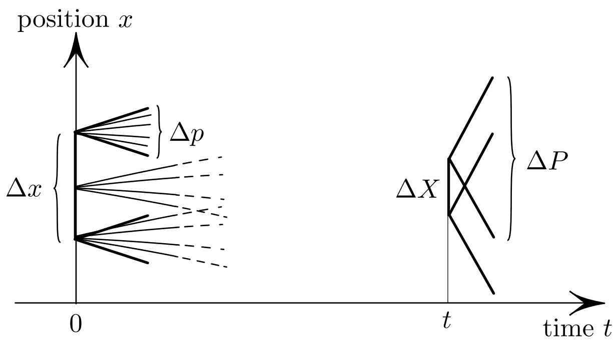

Figure 22. Classical mechanical uncertainty principle applied to trajectories or rays.

The uncertainty principle explaining how telescopes work. The discussion of the preceding paragraph has an optical interpretation. Figure 22 shows the set of solutions of $\ddot{x} + U_x(t, x) = 0$. According to the preceding paragraph, if these solutions "squeeze" through a narrow gap ΔX at some t, then ΔP (the range of their slopes at that time t) is large. Now a very similar "uncertainty principle" holds not just for the solutions of $\ddot{x} + U_x(t, x) = 0$, but also for rays passing through a binocular, or a telescope. A remarkable consequence is this: *the mere fact that the telescope converts a parallel beam of rays into a narrower parallel beam implies that the telescope magnifies distant objects.* A loose explanation is the following. Because the widths of parallel beams decrease in passing through the telescope, the angles between any two beams increase — Liouville's theorem has its optical counterpart. But this means that distant objects are magnified, since we perceive the size of an object, say the Moon, by the angles between the nearly parallel beams emitted by different parts of the object as these beams enter our eyes. Indeed, a parallel beam focuses on a dot on the retina, and the greater is the angle between the beams, the greater is the distance between two illuminated dots on the retina, and the greater is the perceived distance between the sources of the two beams.

More details on this, including an optical counterpart of Liouville's theorem, can be found in [**14**], page 129.

18. Can one hear the shape of the potential?

The famous (among mathematicians) question of Mark Kac: "can you hear the shape of the drum?" refers to the problem of recovering the shape of a vibrating membrane from the knowledge of all of the frequencies, or overtones, of its vibrational modes. A much simpler analog of this problem deals with one-dimensional oscillations of a classical particle governed by $\ddot{x} = -U'(x)$.

Question. Can one recover the shape of the potential $U(x)$ given the period $T(E)$ of oscillation of a particle as a function of its energy (or the amplitude)?[16]

The remarkable answer, due to Abel, is "yes", under mild symmetry assumptions. Let the potential U be as in Figure 23; more precisely, assume that $U(-x) = U(x)$, with $U(0) = U'(0) = 0$, and that U is monotone increasing for $x > 0$. Our goal is to produce a formula for $U(x)$ given the period as a function of energy. This goal is reached in three steps.

Step 1 is to derive the formula for the period, given the energy:

$$T(E) = 2\sqrt{2}\int_0^{x_{\max}} \frac{dx}{\sqrt{E - U(x)}}, \quad x_{\max} = U^{-1}(E) > 0. \tag{1.53}$$

To derive (1.53), we observe first that the particle oscillates back and forth between two endpoints $\pm x_{\max}$; these are the points at which the particle is instantaneously at rest, so all of its energy is potential:

$$U(x_{\max}) = U(-x_{\max}) = E.$$

The period of the oscillation is twice the time between $-x_{\max}$ and $x_{\max}$:

$$T(E) = 2\int_{-x_{\max}}^{x_{\max}} dt = 2\int_{-x_{\max}}^{x_{\max}} \frac{dx}{|\dot{x}|}$$

Now, $|\dot{x}|$ is found from

$$\frac{\dot{x}^2}{2} + U(x) = E \tag{1.54}$$

[16] Note that T is the reciprocal of the frequency; in this sense this problem is a classical mechanical analog, in one dimension, of the quantum "drum" problem of the preceding paragraph.

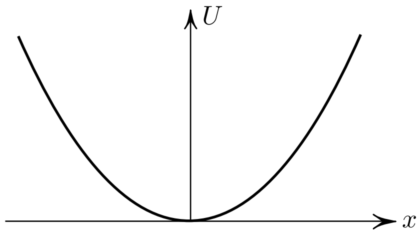

Figure 23. The unknown potential.

as $|\dot{x}| = \sqrt{2(E - U(x))}$. Substituting this into the last integral gives (1.53), after we replace $\int_{-x_{\max}}^{-x_{\max}}$ by $2\int_0^{-x_{\max}}$. We completed the first step.

Step 2: Changing the variable of integration. Our goal is to find U from (1.53); unfortunately, U enters this expression in a nonlinear way. We now eliminate this nonlinearity by choosing $U(x) = u$ as the new variable of integration. Now x becomes a function of u, namely, the inverse of the function $u = U(x)$. We have $dx = x'(u)du$, and our integral equation (1.53) becomes

$$T(E) = 2\sqrt{2}\int_0^E \frac{x'(u)}{\sqrt{E-u}}du, \tag{1.55}$$

where now the unknown is the inverse of U: $x(u) = U^{-1}(u)$.

Step 3: Solving (1.55) for the unknown function $x = x(u)$. Let $F \geq E$ be a parameter. Let us multiply both sides of (1.55) by $1/\sqrt{F-E}$ and integrate by E from 0 to F (this is done to extract $x(u)$ from the right-hand side, as will soon become clear):

$$\int_0^F \frac{T(E)}{\sqrt{F-E}}dE = 2\sqrt{2}\int_0^F\int_0^E \frac{x'(u)}{\sqrt{F-E}\sqrt{E-u}}du\,dE. \tag{1.56}$$

Let us change the order of integration on the right, with the idea of taking $x'(u)$ outside one of the integrals; see Figure 24. This integral on the right simplifies to $\pi x(F)$ (and this is the key):

$$\int_0^F x'(u)\underbrace{\left(\int_u^F \frac{1}{\sqrt{F-E}\sqrt{E-u}}dE\right)}_{=\pi,\text{ see (1.58) below}} du = \pi\int_0^F x'(v)\,du = \pi\,x(F).$$

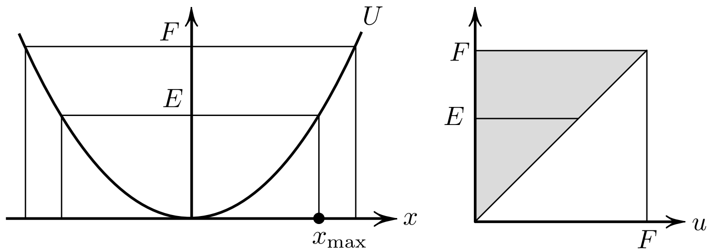

Figure 24. Solving (1.55).

Substituting this into the right-hand side of (1.56) gives

$$\boxed{x(F) = \frac{1}{2\sqrt{2}\,\pi} \int_0^F \frac{T(E)}{\sqrt{F-E}} dE} \tag{1.57}$$

We found an explicit expression $x(F) = U^{-1}(F)$ for the inverse function of U. This describes $U(x)$ completely, and solves the problem.

Proof of the identity

$$\int_u^F \frac{dE}{\sqrt{(F-E)(E-u)}} = \pi : \tag{1.58}$$

a linear substitution reduces the integral to $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} 1 - \sin^{-1}(-1) = \pi$.

Exercise 1.4. A quadratic potential $U = \frac{1}{2}x^2$ is *isochronous* in the sense that all the motions have the same period, namely 2π. Show that the solution (1.57) implies the converse: if a potential is isochronous of period 2π then it is of the form $U = \frac{1}{2}x^2$ (assuming, as we did above, that $U(0) = U'(0) = 0$ and that U is even and convex up).

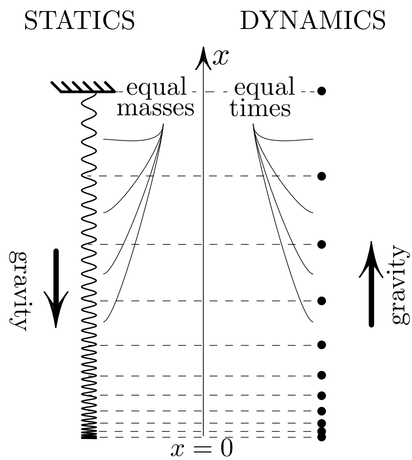

Figure 25. The statics-dynamics equivalence illustrated: hanging spring is mathematically identical to the falling projectile if we reverse the direction of gravity for the projectile.

19. A dynamics-statics equivalence

A simple and yet remarkable fact is that the entire Newtonian dynamics described by $m\ddot{x} = -U'(x)$ is mathematically equivalent to *statics* of Hookean springs. Figure 25 illustrates this equivalence on a simple example. On the left, a "slinky" — a heavy Hookean spring (defined precisely in the next paragraph) — hangs motionless by one end; we choose the origin $x = 0$ to be at the free end of the spring. On the right, the picture of a falling particle is drawn *upside-down*; the mass begins to fall from $x = 0$ starting from rest. These two systems are not just analogous but are in fact mathematically *equivalent*, as explained in this section and as summarized by the dictionary on page 45.

The spring. Figure 26 shows an idealized Hookean spring, viewed as a one-dimensional object laid out on the x-axis (the spirals are only drawn to show the nonuniform stretching and not to suggest any thickness). It is natural to parametrize the particles of the spring by the mass t counted from one of the spring's ends; let $x(t)$ denote the position of the corresponding particle of the spring. At this point

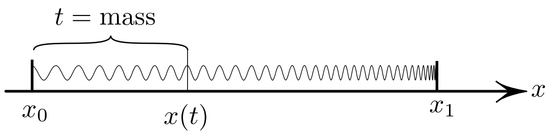

Figure 26. Position $x(t)$ of each particle on the spring is parametrized by the mass t between the particle and the attachment point.

in discussion, $x(t)$ is an arbitrary function, i.e., each particle is held forcibly in its own prescribed position. We can call x a "configuration function" of the spring. The spring is assumed to satisfy Hooke's law; furthermore, the relaxed length of the spring is assumed to be zero.[17]

Theorem 1.7. *Consider a Hookean zero length spring on the x-axis, with a configuration function $x(t)$, $0 \le t \le T$, as described in the preceding paragraph. Denote Hooke's constant of a unit mass of a spring by $k_1 = m$. Let $V(x)$ be a potential on the line (that is, $V(x)$ is the potential energy of a unit mass at x). Then the total potential energy of the spring is*

$$E_{\text{total}} = \int_0^T \left(\frac{m\dot{x}}{2} + V(x) \right) dt. \tag{1.59}$$

Proof. Consider a piece of the spring of mass dt. Hooke's constant of this piece is $k_{dt} = k_1/dt = m/dt$ (a shorter spring is stiffer, see Problem 1.1 on page 50). Therefore, the potential energy stored in stretching this piece is

$$dE_{\text{internal}} = \frac{1}{2} k_{dt} (\Delta x)^2 = \frac{m\dot{x}^2}{2} dt,$$

ignoring higher order terms. This proves (1.59) once we also observe that $V(x)\,dt$ is the potential energy of mass dt, since V was defined as the potential energy of a unit mass. $\diamondsuit$

[17]Zero length should not be a concern since this is only a thought experiment; however, a real slinky is not too far from this idealization: its relaxed length is very short compared to its "operating" length.

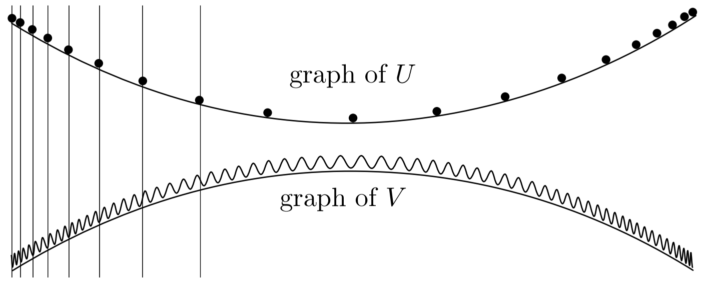

Figure 27. Equal times between positions of the particle in potential U and equal masses for the spring in potential $-U$. The equilibrium state of the spring captures the whole time history of the moving particle.

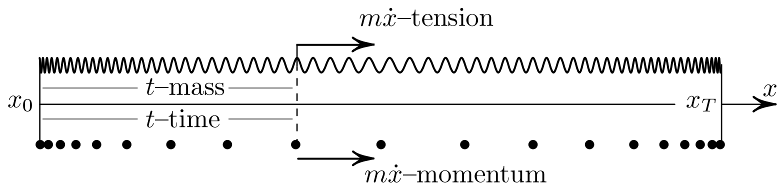

Figure 28. Equivalence between a particle in motion and a spring in equilibrium.

Now the expression (1.59) coincides with

$$\int_0^T \left(\frac{m\dot{x}^2}{2} - U(x) \right) dt,$$

the action of a particle, provided we choose $V = -U$. The two problems are therefore equivalent: the particle moving in the potential U and the spring resting in equilibrium in the potential $V = -U$. The equivalence means that the same critical function $x(t)$ describes the motion of a particle in the potential U and the static equilibrium of the spring in the potential $-U$.

Example. Figure 27 illustrates the equivalence. Note how the stretching of the spring is related to the speed of the particle. Actually, both the spring and the particle are confined to the x-axis rather than to the graphs of the potentials, as represented by the more accurate (but less intuitive) Figure 28. The following table summarizes the equivalence between the particle and the spring; a more detailed explanation is given after the table.

Dynamics of a particle	**Statics of a spring**
t – time	t – mass
m – mass	m – Hooke's constant
$x(t)$ – position of the particle at time t	$x(t)$ – position of the particle corresponding to mass parameter t
$\dot{x}$ – velocity	$\dot{x} = (dt/dx)^{-1} = (\text{density})^{-1}$
$m\dot{x}$ – momentum	$m\dot{x}$ – tension
$\frac{m\dot{x}^2}{2}$ – kinetic energy	$\frac{m\dot{x}^2}{2}\,dt$ "internal", or stretching potential energy of a mass element dt
$U(x)$ – potential energy	$V(x) = -U(x)$ potential energy of a unit mass
$K + U = \text{const.}$	$K - V = \text{const.}$
Action $\int (K - U)$	Potential energy $\int (K + V)$
$\delta \int (K - U) = 0$: $x(t)$ is a solution	$\delta \int (K + V) = 0$: $x(t)$ is an equilibrium
$\delta \int (K - U) = 0 \Leftrightarrow \ddot{x} = -U'(x)$ via Euler–Lagrange's argument	$\delta \int (K + V) = 0 \Leftrightarrow \ddot{x} = V'(x)$ by an elementary equilibrium argument (1.60)
Liouville's theorem $A(0) = A(T)$, i.e., $\int_{\varphi^T \gamma} p\,dx = \int_\gamma p\,dx$ for a closed curve γ	Work done on moving spring's ends in a cyclic fashion is zero: $-\int p_0 dx_0 + \int p_T\,dx_T = 0$

The use of this equivalence is two-fold. First, we get two physical systems analyzed for the price of one, or, putting it differently, we realize that two different physical systems are mathematically identical; and second, this equivalence gives new insights including elementary derivations of the Euler–Lagrange equation and of Liouville's theorem, as described in the next two paragraphs.

An elementary derivation of the Euler–Lagrange equation as a static equilibrium condition. Earlier we established the equivalence

$$\delta \int (K-U) = 0 \Leftrightarrow m\ddot{x} = -U'(x),$$

by showing that the right-hand side is the Euler–Lagrange equation for the functional on the left. The result is nonobvious intuitively as it relies on a formal calculation. But our analogy makes this equivalence obvious, without appealing to the Euler–Lagrange equation. Indeed, let us interpret the action $\int (K-U)\,dt$ for a particle as the potential energy $\int (K+(-U))\,dt$ of a spring in the potential $V=-U$. Then critical action for the particle motion means critical potential energy for the spring. That is, the spring is in equilibrium, and so the sum of all forces on each mass element dt is zero:

$$m\dot{x}(t+dt) + (-m\dot{x}(t)) + (-V'dt) = 0. \tag{1.60}$$

Dividing by dt and taking $dt \to 0$ gives $\ddot{x} = V'(x) = -U'(x)$. We rederived the Euler–Lagrange equation by naive means, in the special case of the Lagrangian $L = K - U$.

A quick proof of Liouville's theorem. The theorem can be restated as

$$\oint_{\gamma_0} p dx = \oint_{\gamma_T} p dx, \tag{1.61}$$

where γ_0 is a closed curve of initial data $(x, m\dot{x} = p)_{t=0}$ and γ_T is the curve formed by $(x, m\dot{x} = p)_{t=T}$ where $x = x(t)$ is a solution of $m\ddot{x} = -U'(x)$. Let us interpret the two integrals in (1.61) in terms of the spring. With the endpoints of the spring held at x_0, x_T, the external forces applied to the ends are equal to the tensions at these locations: $F_0 = -p_0 = -m\dot{x}(0)$ and $F_T = p_T = m\dot{x}(T)$, Figure 29. Let us now move the ends of the spring in a cyclic fashion (slowly, so as not to cause oscillations), bringing them back to the initial location. The net work we do in that case is zero:

$$\oint F_0 dx_0 + \oint F_T dx_T = 0,$$

and since $F_0 = -p_0$, $F_T = p_T$, we get (1.61)! This hand-waving argument takes almost no work (with apologies for two puns) and converts into a rigorous proof; see page 271.

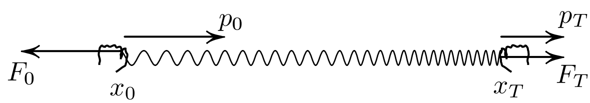

Figure 29. A hand-waving proof of Liouville's theorem: the hands execute a cyclic motion along the line, doing zero work on the spring. In addition to the two forces F_0 and F_T, there is a distributed force on the spring due to the potential V.

Remark 1.9. For any spring in equilibrium in a potential, such as in Figure 25, the difference $K - V = \text{const.}$ along the spring. This is surprising and nonobvious, but we know it is true because it is equivalent to $K + U = \text{const.}$, the conservation of energy for a particle! In particular, for the hanging spring example in Figure 25, since $V = K = 0$ at the free end, we have $K = V$: the potential energy density, i.e., the energy per unit mass due to stretching equals the potential energy density of elevation!

Exercise 1.5. Consider a Hookean zero length spring described on page 42 in equilibrium, with the points corresponding to t_0, t_1 held fixed at x_0, x_1. Let $S(x_0, x_1, t_0, t_1) = \int_{t_0}^{t_1} (K + V)\, dt$ be the potential energy of the spring; here the equilibrium function $x(t)$ is substituted into the integrand. Find the partial derivatives S_{t_0}, S_{x_0}, S_{t_1}, S_{x_1} and determine their physical interpretation.

Solution. $S_{x_1} = m\dot{x}(t_1)$ is the tension;

$$S_{t_1} = -(K - V)_{t=t_1} \tag{1.62}$$

is the difference between the internal and the external potential energy densities; similarly, $S_{x_0} = -m\dot{x}(t_0)$ and $S_{t_0} = +(K - V)_{t=t_0}$. For the proof, see Theorem 8.1 on page 260; alternatively, here is a naive physical justification. Let $x(t)$ be the equilibrium function with $x(t_0) = x_0$, $x(t_1) = x_1$, Figure 30. Let us find S_{t_1}. With the ends of the spring fixed as described, let us grab the spring at the point $A = x(t_1 - \tau)$ near the right end, Figure 30, and pull this point to the right end $x(t_1)$ and hold it there. In the process the segment of the spring corresponding to $[x_1 - \tau, x_1]$ gets collapsed to a point, while the rest of the spring stretches a little. The new potential energy of the spring is

$$P_{\text{new}} = S(x_0, x_1, t_0, t_1 - \tau) + V(x_1)\tau,$$

the second term being the potential energy of the short segment of the spring which has collapsed to a point. On the other hand, the work we did

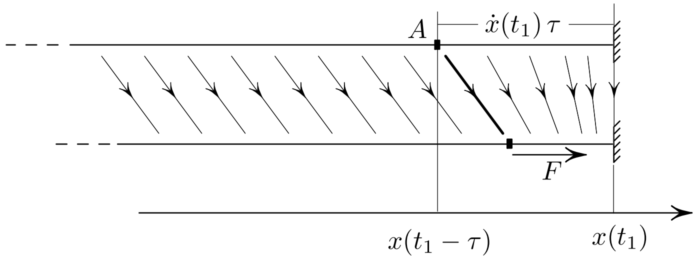

Figure 30. As we pull the point $A = x(t_1 - \tau)$ to the end $x = x_1$, the force F changes linearly with distance, from $F = 0$ to $F = m\dot{x}(t_1) + O(\tau)$.

to move the end of the spring is

$$W = F_{\text{avg}}(x(t_1) - x(t_1 - \tau)) = \frac{1}{2}m\dot{x}(t_1)^2\tau + O(\tau^2).$$

But

$$P_{\text{new}} = P_{\text{old}} + W,$$

i.e.,

$$S(x_0, x_1, t_0, t_1 - \tau) + V(x_1)\tau = S(x_0, x_1, t_0, t_1) + \frac{1}{2}m\dot{x}(t_1)^2\tau + O(\tau^2).$$

Separating the terms with S from the rest, dividing by τ and sending $\tau \to 0$ gives (1.62).[18] ♢

Remark 1.10. The same equivalence between dynamics and statics holds in higher dimension. Figure 31 shows the familiar projectile trajectory side-by-side with a hanging spring; note, however, that the gravity is in the opposite direction for the two cases. The two problems are again equivalent, with the exact same list of analogies as in the table above.

20. Chapter summary

Here are the main points of this chapter.

(1) Newton's equation: $m\ddot{x} = F(x)$.

(2) Potential energy and kinetic energy, defined.

(3) Conservation of energy: $K + U = \frac{1}{2}m\dot{x}^2 + U(x) = \text{const}$.

[18](1.62) is the Hamilton-Jacobi equation (discussed in Chapter 8).

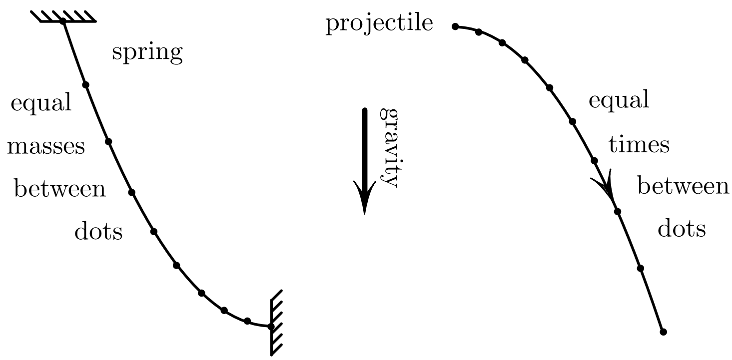

Figure 31. Equivalence between a dynamical problem and a static one in two space dimensions, illustrated on the projectile motion.

(4) The phase plane.

(5) Lagrange's equation: $\frac{d}{dt}L_{\dot{x}} - L_x = 0$, $L = K - U$.

(6) Variational origin of Lagrange's equations: $\frac{d}{dt}L_{\dot{x}} - L_x = 0 \;\Leftrightarrow\; \delta\int_{t_0}^{t_1} L\,dt = 0$.

(7) Recovering the shape of potential from the periods of oscillations.

(8) Hamilton's equations, derived from Lagrange's equation.

(9) Liouville's theorem.

(10) A classical mechanical uncertainty principle.

(11) A statics-dynamics equivalence.

(12) Two application of the statics-dynamics equivalence: (i) $\delta\int(K-U) = 0 \;\Leftrightarrow\; m\ddot{x} = -U'(x)$ and (ii) Liouville's theorem.

21. Problems

Hookean Springs.

1.1. Find Hooke's constant of the combination of two Hooke's springs (a) in parallel; (b) in series, given Hooke's constants k_1 and k_2 of the two springs.

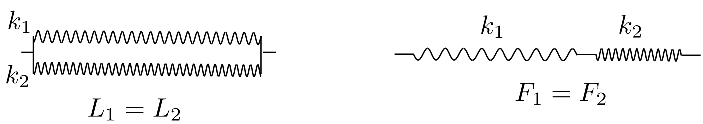

Figure 32. What is the effective Hooke's constant for springs in series and in parallel?

Solution. We show that

$$k_{\text{parallel}} = k_1 + k_2; \quad \frac{1}{k_{\text{series}}} = \frac{1}{k_1} + \frac{1}{k_2}. \tag{1.63}$$

In parallel: The key observation is

$$F = F_1 + F_2, \tag{1.64}$$

where F is the force with which the combined spring was stretched, while F_i is the force with which the ith spring is stretched. By the definition of Hooke's constant, we have $F = k_{\text{parallel}}L$, $F_1 = k_1 L$ and $F_2 = k_2 L$; note that the elongation L of both springs is the same, the second key point. Substituting this into (1.64) we obtain the first equation in (1.63).

In series: In this case, the elongation is the sum of elongations of the two springs:

$$L = L_1 + L_2, \tag{1.65}$$

Since each spring is stretched by the same force F (the second main point), we have $F = k_i L_i$, $i = 1, 2$. Thus $L_i = k_i/F$, and also $L = k_{\text{series}}/F$. Substituting these into (1.65) and cancelling F we arrive at the second formula in (1.63). $\diamondsuit$

A heuristic explanation of $\frac{1}{k_{\text{series}}} = \frac{1}{k_1} + \frac{1}{k_2}$: Hooke's constant k measures the stiffness: large k, for example, means stiff spring, since k simply measures the force needed to elongate the spring by one unit of length. The reciprocal k^{-1} therefore measures the spring's looseness. Two springs in series makes for a looser spring; the "loosenesses" in fact add, as we proved. By contrast, for the springs are connected in parallel, "stiffnesses" add.

1.2. Produce an equivalence table between the following objects from mechanics on one hand and electricity on the other. Mechanics: Hooke's constant, elongation, force ($k = F/x$). Electricity: Resistance, voltage, current ($R = V/I$) and capacitance, charge, voltage ($C = q/V$). What are the electrical analogues of the formulas (1.63)? Mere substitution of the electric analogs into proofs of (1.63) yields the formulas for resistances and for capacitances connected in parallel and in series:

$$R_{\text{parallel}}^{-1} = R_1^{-1} + R_2^{-1}, \quad R_{\text{series}} = R_1 + R_2;$$

$$C_{\text{parallel}} = C_1 + C_2, \quad C_{\text{series}}^{-1} = C_1^{-1} + C_2^{-1}.$$

A bead on a wire.

1.3. Consider the motion of a particle in a potential: $\ddot{x} = -U'(x)$. Can this equation also describe the arclength parameter of a bead sliding under gravity on an appropriately shaped wire? That is, find the curve $y = V(x)$ such that the arc length parameter s of a bead sliding on this curve under gravity ($g =$ const. pointing down the y-axis) satisfies the same equation: $\ddot{s} = -U'(s)$, and state under what conditions on U this is possible. Find V in the following two cases: (i) $U = \frac{1}{2}x^2$ and (ii) $U = -\cos x$.

Answer. The graph of V is a cycloid in case (i) and a circle in case (ii).

1.4. Figure 33 shows a bead is sliding on a wire given by $y = f(x)$ in the vertical plane subject to gravity pointing down the y-axis. Write the equation of motion for the bead in terms of its x-coordinate, using three different methods:

(1) Using Newton's law $\ddot{s} = -g\sin\theta(s)$ for the arc length $s = \int_0^x \sqrt{1 + f'(u)^2}du$.

(2) Using Lagrange's method (i.e., write the Euler–Lagrange equation).

(3) Directly from Newton's second law in $\mathbb{R}^2$. (Hint: for this method, we must consider centripetal acceleration, which has a component in the x-direction.)

1.5. A point mass is glued to the surface of a weightless cylinder which rolls without slipping on the horizontal plane. The point mass is thus traveling in a cycloid. Letting s denote the distance of the particle from the top of its trajectory, measured along the trajectory, show that $\ddot{s} = \frac{g}{D}s$: the motion is the same as for a particle with a linear repelling force. Is this equation valid for all time?

Hint. Regarding the last question, the evolution after the particle hits the floor becomes ambiguous.

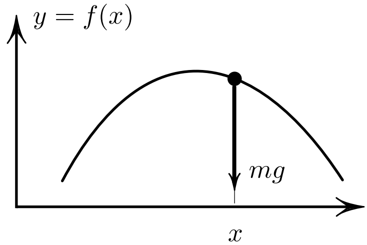

Figure 33. Deriving the equation of motion for a bead on wire with gravity.

1.6. Referring to the bead on the wire in Figure 3(A), page 4, is it true or false that the normal reaction force $N = mg\cos\theta$? That is, does N cancel the normal component of the gravity?

1.7. Consider a bead sliding frictionlessly on the curve $z = f(x)$, in the vertical (x, z)-plane with gravitational field g pointing down in the direction of the negative z-axis. Is it true that the x-coordinate satisfies $\ddot{x} = -gf'(x)$?

1.8. Consider a curve $y = f(x)$ with $f'(0) = 0$. Let s be the distance along the curve from the point with $(0, f(0))$, and let $\theta(s)$ be the angle between the x-axis and the tangent to the curve. Show that

$$\sin\theta(s) = ks + o(s),$$

where $k = f''(0)$ is the curvature of the curve at $x = 0$.[19]

Modeling, hanging chains.

1.9. A heavy homogeneous chain is hanging in equilibrium supported at two ends, Figure 34. Write the differential equation obeyed by the shape $y = f(x)$ of the chain. The chain is to be treated as a perfectly thin, perfectly flexible unstretchable curve with a uniform mass distribution along its length.

1.10. Verify the following remarkable fact: *Regardless of how the chain is suspended (Figure* 34*), its tension* $T = T_0 + \rho gh$, where T_0 is the tension at the bottom of the chain, and h is the height of the point above the bottom. Prove that the *same holds even if the chain is not freely hanging but rather is resting on a perfectly slippery surface.*

1.11. Referring to the preceding problem, can you explain why the expression $T = T_0 + \rho gh$ is so similar to the expression $p = p_0 + \rho gh$ for the

[19] Here $o(s)$ denotes a quantity which is small compared to s in the sense that $\lim_{s\to 0} o(s)/s = 0$.

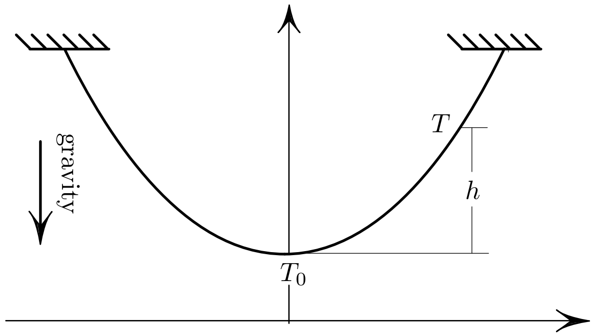

Figure 34. Tension in a hanging chain behaves like hydrostatic pressure: it varies linearly with height!

hydrostatic pressure? (The meaning of ρ in the two expression is different but similar.) Can this similarity be used to solve the preceding problem?

Solution – an outline. Imagine that the chain is a hose filled with water. The hose is perfectly flexible, weightless, unstretchable and very thin, essentially one-dimensional. The hydrostatic pressure of the water in the hose is $p = p_0 + \rho_w gh$, where ρ_w is the density of water. Now the tension of the hose caused by p is $T = pA$, where A is the cross-sectional area of the hose. Thus the tension of the water-filled hose (which in essence is the hanging chain) is $T = T_0 + \rho gh$, where $\rho = \rho_w A$, i.e., the linear density of the hose with water.

1.12. Solve Problem 1.10 by using conservation of energy instead of Newton's first law.

Solution. Focus on a segment AB of the chain; imagine taking up the length ds of the chain at B and feeding in the same length at A. The work done by taking up, i.e., pulling, is $T_B\, ds$; the work involved in feeding in, i.e., in being pulled, is $-T_A\, ds$. The net result is that the mass $dm = \rho g ds$ is lifted by height h from A to B, so that

$$T_B\, ds - T_A\, ds = \rho g\, ds\, h,$$

proving that $T_B - T_A = \rho gh$.

1.13. A thin-walled straight hose of radius r is filled with pressurized gas at pressure p. (i) Find the surface tension of the skin of the hose in the axial direction (i.e., the force required to hold closed a longitudinal slit, per unit length of the slit). (ii) Find the surface tension of the skin in the perpendicular direction.

Solution. Longitudinal tension turns out to be twice the tension across. Indeed, if we cut the hose with a plane perpendicular to the axis, the force required to hold the cut together is $\pi r^2 p$. Dividing by the circumference $2\pi r$, we get the transversal surface tension $\sigma_1 = \pi r^2 p/2\pi r = rp/2$. To find the longitudinal tension, let us cut a length L of the hose by a plane containing the axis; the force $(2r) \cdot L \cdot p$ required to hold the cut together; divided by the length $2L$ of the cut, this yields the surface tension $rp/2$, half that of the longitudinal tension. This explains why frozen pipes, and boiled sausages, always burst lengthwise.

1.14. Explain the mechanism by which wringing a towel expels water. Estimate the pressure created inside the towel, making some reasonable assumptions (and, in particular, explain what exactly could one mean by pressure in this problem). How does the pitch of a certain helix affect the squeezing efficiency? Why is it harder to squeeze out a thicker roll than a thinner one? Why is wringing much better than squeezing?

1.15. Show that the cable supporting the vertical cables in a suspension bridge is parabolic, assuming the vertical cables are very closely and equally spaced and are under equal tensions, and that the entire weight is contained in the horizontal walkway.

1.16. A long cylindrical hose with perfectly flexible walls is filled with water and is placed on the horizontal surface. Write down a differential equation that describes the shape of the (non-flat) part of the cross-section of the hose.

1.17. A chain hanging in equilibrium is shaped as the graph of a function $y = f(x)$. For any segment of this chain, the horizontal components of tensions on the two ends of the segment are in balance. Express this balance as an equality involving $f(x)$. (The resulting equation is a first integral of the second order differential equation of the catenary.)

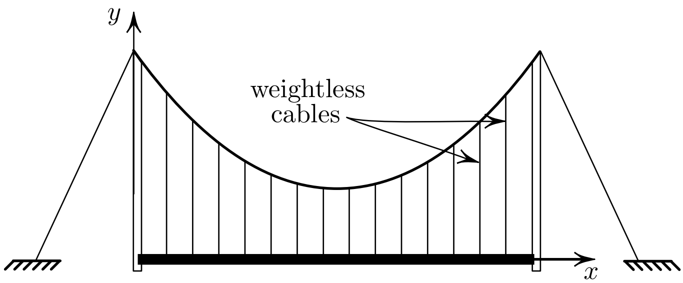

Figure 35. Suspension cable is a parabola.

Huygens's pendulum, evolutes and bike tracks.

1.18. Show that the cycloid is a tautochrone, as follows. Consider a bead sliding without friction under the influence of gravity g on a cycloid generated by a circle of diameter D, as in Figure 4, page 5. Show that $\ddot{s} = -ks$, where s is the arc length measured from the lowest point of the cycloid.

Proof (an outline). Referring to Figure 36, we show that $da = -k\,ds$, where a is the tangential acceleration of the bead. But $ds = D\sin\theta d\theta$, as the figure shows, and $a = g\cos\theta$, so that $da = -g\sin\theta d\theta$. We conclude that $da = -\frac{g}{D}ds$, implying that $a = -ks + \text{const}$. But $a = 0$ when $s = 0$, which shows that $a = -ks$, i.e., $\ddot{s} = -ks$. The period of s therefore does not depend on the amplitude, and in fact is equal to $2\pi/\sqrt{k} = 2\pi\sqrt{D/g}$.

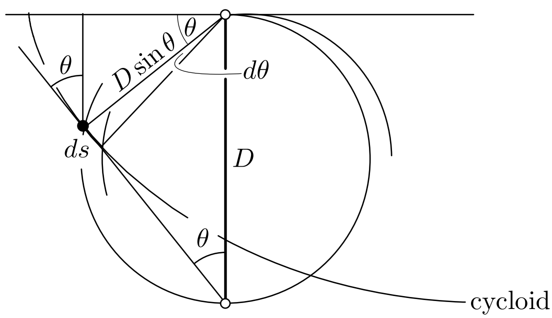

Figure 36. Brachistochrone is a tautochrone: the proof.

1.19. Consider Huygens's pendulum, or, more generally, a string fixed at point C (Figure 5, page 5), and wrapped partly around an obstacle. The free end P of the string is moved so as to keep the string taut. Prove that the velocity of P is perpendicular to the string. This would then show that the *obstacle is the evolute* (i.e., the envelope of the family of normals) of the trajectory of P.

1.20. Prove that the evolute of a curve (defined as the envelope of the family of normals to the given curve) is also the locus of the centers of curvature of the curve; see Figure 37.

Hint. Here is a kinematic argument which can be converted into a formal proof, for a small fee. If a point travels in a circle with speed v, then the radius of the circle is

$$R = v/\omega, \tag{1.66}$$

where v is the speed of P and where ω is the angular velocity of the normal to the trajectory. Now the same definition applies verbatim if the trajectory

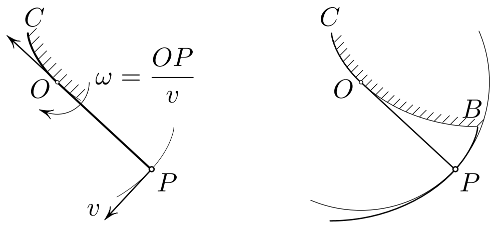

Figure 37. Proving that every point on the evolute is a center of curvature of the evolvent.

is an arbitrary curve. Now what is the angular velocity of the moving segment OP? Velocity of O aligns with OP, i.e., it has zero component normal to OP. Therefore, the *motion of O contributes nothing to the angular velocity of OP*, and we can treat O as fixed to find $\omega = v/OP$, or $OP = v/\omega$. Comparison with (1.66) shows that $R = OP$.

1.21 (Finding center of curvature using a bike)**.** Referring to Figure 38, show that *the center of curvature of the bike's rear track is at the intersection of the lines of the axles*, i.e., of the two normals to the tracks. The bike is idealized: it is a segment RF (for "rear" and "front", the points where the wheels are in contact with the ground); assume that the segment RF moves in the plane so that the length $|RF| = \text{const.}$, and that the velocity of R is aligned with RF.

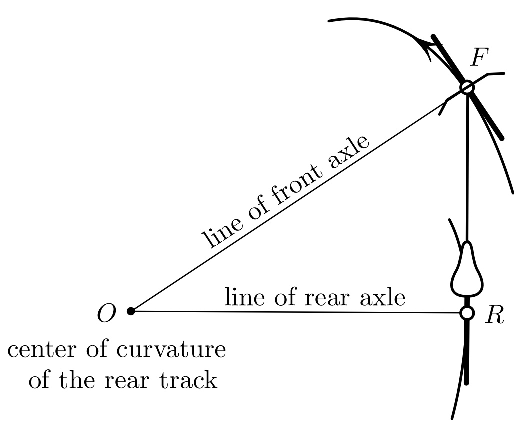

Figure 38. Proving that the point of intersection of the bike's axles is the center of curvature of the rear track.

Vibrations.

1.22. 1. Derive the equations of motion for a point mass m in Figure 39. The spring satisfies Hooke's law: the tension is $k(L - L_0)$, where L_0 is the relaxed length.

2. How many equilibria can the particle have, depending on the relationship between D and L_0?

3. Find the frequency of small vibrations near a stable equilibrium. Assume the amplitude of oscillations to be very small.

4. Show that for the zero length spring (the one for which the relaxed length $L_0 = 0$) the oscillations are harmonic.

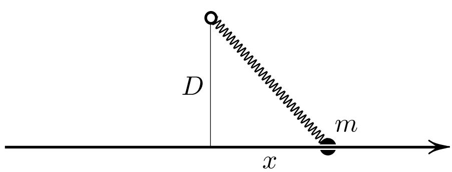

Figure 39. The mass m is constrained to the line, with no friction.

1.23. Consider the equation governing the angle θ of the pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{G}{L}\sin\theta = 0.$$

Show that the introduction of the rescaled time

$$\tau = \sqrt{\frac{g}{L}}t, \tag{1.67}$$

i.e., setting $\varphi(\tau) = \theta(t) = \theta(\sqrt{\frac{g}{L}}\tau)$ turns this equation into

$$\frac{d^2\varphi}{d\tau^2} + \sin\varphi = 0. \tag{1.68}$$

Note that $T = 2\pi\sqrt{\frac{L}{g}}$ is the period of linearized oscillations near the equilibrium, and thus our rescaling (1.67) amounts to measuring time in the more natural units of the period.

1.24. (Based on a Bond movie I saw.) Figure 40 shows idealized James Bond attached by a long rope to the top of the cliff. A villain pushes Bond off the top. Once the rope becomes taut it acts as a spring, softening what would otherwise have been a fatal jerk. Bond survives the maximal tension of the rope (see his strained position at the bottom) and resumes climbing. Show that the *maximal tension in the rope does not depend on* L_0, the

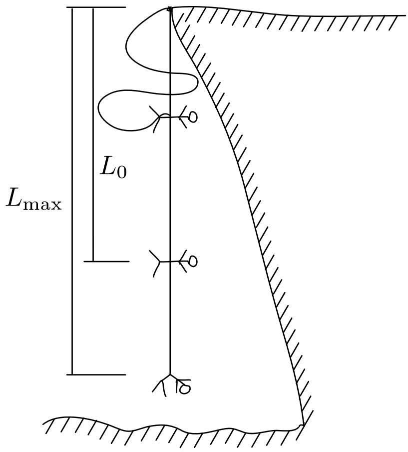

Figure 40. The maximal tension does not depend on the rope's length.

length of the unstretched rope, assuming that the rope satisfies Hooke's law: $F = k(L - L_0)$ for $L > L_0$, where $k > 0$ is a constant. Ignore the air resistance and other possible complications (such as hitting the ground too early, or the villain cutting the rope).

Hint. L_0 affects both the maximal speed of fall and Hooke's constant k. Show that the two effects cancel out.

An inverse problem.

1.25. Given

$$T(E) = \int_0^{E^{1/4}} \frac{dx}{\sqrt{E - x^4}},$$

write $T'(E)$ as an integral. (Note: there is a complication when differentiating with respect to the upper limit of the integral.)

1.26. Find the derivative $T'(E)$ of the period

$$T = \int_{-x_{\max}}^{x_{\max}} \frac{dx}{\sqrt{2(E - U(x))}}, \quad x_{\max} = U^{-1}(E).$$

Assume that $U(0) = U'(0) = 0$, $U'' > 0$ and that U is even. Note: direct differentiation will not work since the integrand is infinite at the endpoints.

1.27. Show that any tautochrone symmetric with respect to a vertical axis is a cycloid. (Recall that the tautochrone is a curve in the vertical plane such that a bead sliding on this curve without friction has the period of oscillations independent of the amplitude.)

Hint. Let s be the arc length measured from the lowest point of the curve. Use Exercise 1.4 (page 41) to conclude that for a particle sliding on a tautochrone one has $\ddot{s} = -ks$ (for some $k =$ const.). Then show that the latter relation implies that the curve is a cycloid.

Hamiltonian systems.

1.28. Show that the Hamiltonian of a planar Hamiltonian system has the following interpretation: *for any two points A and B in the plane, $H(B) - H(A)$ is the flux of the Hamiltonian vector field across any curve connecting A and B.*

1.29. Let $\mathbf{v}(\mathbf{z})$ be a smooth vector field defined in the entire plane $\mathbb{R}^2$. Prove that if div $\mathbf{v} = 0$ then $\mathbf{v}$ is a Hamiltonian vector field.

Hint. Written backwards: .melborp suoiverp eht esU

1.30. Find the Hamiltonians of these systems:

$$(1.69) \qquad (1): \begin{cases} \dot{x} = \frac{x}{x^2+y^2}, \\ \dot{y} = \frac{y}{x^2+y^2}, \end{cases} \qquad (2): \begin{cases} \dot{x} = \frac{y}{x^2+y^2}, \\ \dot{y} = \frac{-x}{x^2+y^2} \end{cases}$$

One of these Hamiltonians is multiple-valued. Can you explain this using the interpretation of H as flux given by Problem 1.28?

Answer. (1) $H = \arg(x + iy)$ is multiple-valued; the flow corresponds to a point source at the origin. Geometrically, the flux through a curve connecting two points A and B picks up 2π each time the curve winds around the origin. (2) corresponds to a point vortex; $H = \ln|z|$ is single-valued.

1.31. Show that any planar vector field can be converted into a Hamiltonian vector field by adjusting the speeds, but keeping the directions at every point. In other words, show that any ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$ can be converted into a Hamiltonian system $\dot{\mathbf{x}} = \rho(\mathbf{x})\mathbf{f}(\mathbf{x})$ by an appropriate choice of $\rho : \mathbb{R}^2 \to \mathbb{R}$.

1.32 (Cows and Hamiltonian systems)**.** Consider a herd of grazing cows on a hill; we will treat this herd as a collection of particles on a surface $z = H(x, y)$. Assume that each cow avoids going up the hill — too much work — or down the hill — too hard on the front legs and too hard to graze at an angle to the horizontal (intelligence can be mistaken for laziness). In short, our cows abhor gradients. This leaves the cow with only one option: to follow a level curve on the topographic map. Assume also that the cows dislike gradients in one more way: they walk faster where the hill is steep — specifically, with speed equal to the slope of the hill at the cow's location.

(1) Show that under these assumption the (x, y)-position of each cow on the topographic map satisfies the Hamiltonian system where the Hamiltonian is the height $H(x, y)$ of the hill at (x, y). Assume that H is smooth.

(2) How does the herd's density change with time? The density is measured by the proportion of the total number of cows per unit area in the (x, y)-plane *(rather than per unit surface area of the hill).*

Answer. Each cow follows a trajectory of the Hamiltonian system $\dot{x} = H_y(x, y)$, $\dot{y} = -H_x(x, y)$, and the density remains constant along the trajectories, i.e., each cow feels equally crowded at all times: if $\rho(\mathbf{x}, t)$ is the herd's density function at time t, then $\frac{d}{dt}\rho(\mathbf{x}(t), t) = \nabla\rho \cdot \dot{x} + \rho_t = 0$.

1.33. Sketch the phase portrait for the particle in the potentials: (i) $U(x) = x - \frac{x^3}{3}$, (ii) $U(x) = x^4/4 - x^2/2$, (iii) $U(x) = -x^4/4 + x^2/2$.

Can a particle have a variable mass without exchanging matter with the outside world? The following problem gives a way to realize such a particle.

1.34. Figure 41 shows a dumbbell, free to slide along a given rigid curve. The sleeve around the curve keeps the dumbbell always perpendicular to the curve, and slides without friction. Write down the differential equation for the arclength parameter s of the sleeve. All the mass is concentrated in the two balls, each of mass $m/2$, and the length of each arm is taken to be 1 (one can always achieve this by a choice of units of length).

Solution. The velocities of the two balls are $v \pm kv$, where v is the speed of the sleeve and where $k = k(s)$ is the curvature of the track. The kinetic energy of the system is therefore

$$\frac{m/2(v + kv)^2)}{2} + \frac{m/2(v - kv)^2)}{2} = \frac{m(1 + k^2)v^2}{2}.$$

The interesting conclusion is that *the apparent mass of the sleeve is variable*: $M = m(1+k^2(s))$, i.e., it depends on the location s! Since the kinetic energy is constant (there are no external forces acting on our system):

$$\frac{m(1 + k^2)v^2}{2} = E = \text{const.},$$

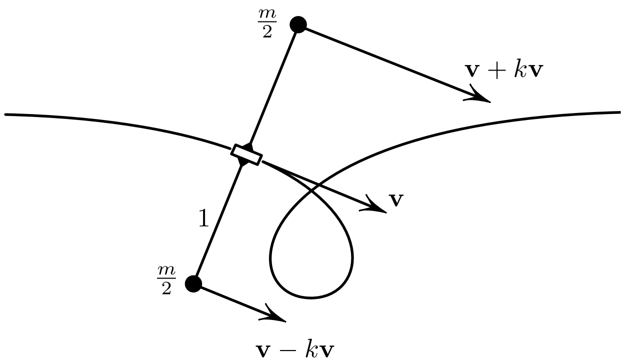

Figure 41. This dumbbell behaves as if the sleeve were a particle with variable mass! In addition, it behaves as if it were subject to a potential force, despite the absence of external forces.

we obtain $v = v(s)$ as a function of position s:

$$v = \frac{\sqrt{2E}}{\sqrt{1+k^2}} = \frac{v_0}{\sqrt{1+k^2}}, \quad \text{where} \quad v_0 = \sqrt{2E}. \tag{1.70}$$

Note that v_0 is the speed the sleeve would have on the straight section of the track. Since v changes along the track, it appears as if a tangential force were acting on the sleeve. What is the magnitude of this force? Here are the answers: differentiating v by time we obtain the acceleration

$$\ddot{s} = \dot{v} = \frac{d}{dt}\frac{v_0}{\sqrt{1+k^2}} = -v_0 \frac{kk'\dot{s}}{(1+k^2)^{3/2}} \overset{(1.70)}{=} -v_0^2 \frac{kk'}{(1+k^2)^2}; \tag{1.71}$$

by substituting $v_0 = v\sqrt{1+k^2}$ in the last term we finally get

$$a = \dot{v} = -v^2 kk'. \tag{1.72}$$

Can this apparent force reverse the direction of motion of the sleeve? No, since $v \neq 0$, according to (1.70). Interestingly, kv^2 in (1.72) is the centripetal acceleration. We conclude: the sleeve's tangential acceleration equals its centripetal acceleration times k'.

Equilibrium; stability.

1.35. Find the tensions of each segment of the cable in Figure 42, given the angles α and β.

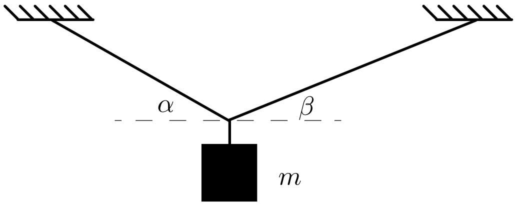

Figure 42. Find the tension of the cable.

1.36. Figure 43 shows an asymmetric dumbbell whose masses m_1 and m_2 rest on the legs of the right triangle. Find the tension of the rod and the angle α, given m_1, m_2 and θ. There is no friction; the rod is weightless.

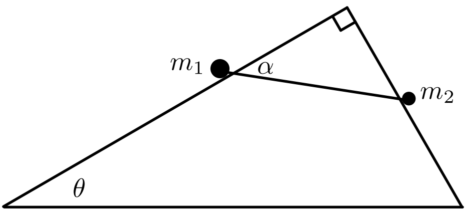

Figure 43. Towards Problem 1.36

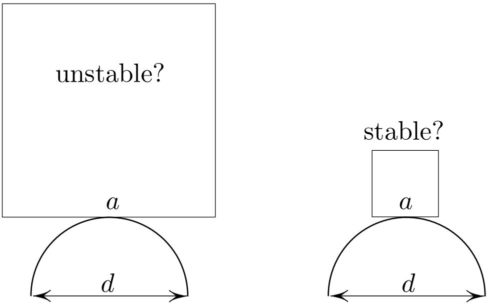

Figure 44. What size cube will be in stable balance?

1.37. Figure 44 shows a cube with side of length a resting on the top of a sphere so that the base of the cube is horizontal. There is no sliding between the cube and the sphere. Under what condition on a and d is the equilibrium stable?

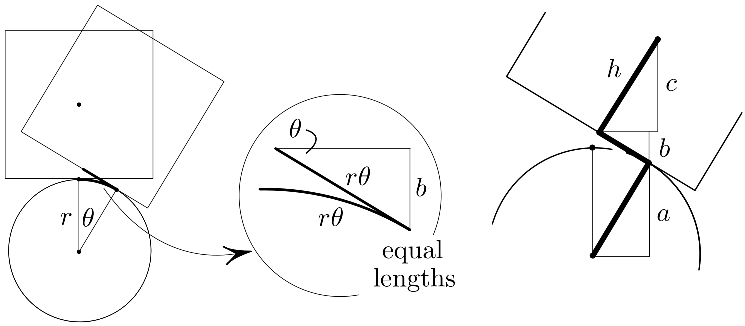

Figure 45. Towards the solution of Problem 1.37.

Solution. I present two methods: one very quick, the other general.

1. A quick method. Imagine rolling the cube past the top equilibrium as in Figure 45. Both the center of mass and the contact point move; *stability will happen if the contact moves faster than the center of mass* at the moment when the contact point is on top of the cylinder. Let ω be the angular velocity of the cube at that moment. Since the cube rotates instantaneously around the contact point, the center of mass moves with speed $\omega\frac{a}{2}$ (when the cube is on top). The contact point, on the other hand, moves with the speed $\omega\frac{d}{2}$, since the radius of the contact point rotates with the same angular velocity ω as the cube. So the contact point moves faster iff (i.e., if and only if) $\omega\frac{d}{2} > \omega\frac{a}{2}$, i.e., iff

$$d > a.$$

This is a beautiful answer: The cube is stable if and only if it does not overhang the sphere.

2. A general method. Let us roll the cube through an angle θ, Figure 45, and find the resulting potential energy $U(\theta)$ of the cube. The equilibrium corresponding to $\theta = 0$ is stable if $U(0)$ is a local minimum of U, i.e., if $U''(0) > 0$ .As the figure shows, the potential energy of the cube is, with the center of the cylinder chosen as ground level,

$$U(\theta) = mg(a+b+c) = mg(r\cos\theta + r\theta\sin\theta + h\cos\theta),$$

where $r = d/2$, $h = a/2$. Simple algebra shows that $U''(0) > 0$ iff $d > a$. $\diamondsuit$

Small vibrations.

1.38. A dumbbell balances on the cylinder in the horizontal position; see Figure 46. Find the frequency of small oscillations of the dumbbell, given the length L of the dumbbell and the radius R of the cylinder. The contact is nonsliding.

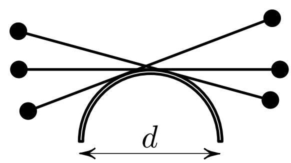

Figure 46. What is the frequency of small oscillations? Problems 1.38 and 1.39.

1.39. A dumbbell balances on a convex object, not necessarily circular, as in Figure 46, in the horizontal position. Find the frequency of small oscillations near the equilibrium, given the curvature k of the object at the topmost point and the length L of the dumbbell.

1.40. Figure 47 shows a rod with a point mass m at the end of it, attached to the cylinder; the cylinder rolls without slipping on the horizontal plane. The radius of the cylinder is R, its mass is M, and the length of the rod is L.

(1) Write the Lagrangian in terms of the angle θ.

(2) Find the period of small oscillations when $R \neq L$. Describe small oscillations in the case of $R = L$. Is this a realistic problem?

Hint. Here is a nice shortcut: the system is undergoing instantaneous rotation around C; this should allow for a quick expression of the kinetic energy.

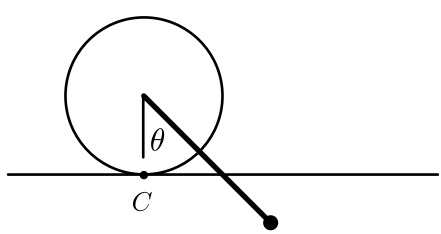

Figure 47. A rolling pendulum, Problem 1.40.

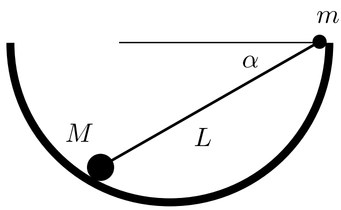

Figure 48. What is the relationship between m, M, L and α? (Problem 1.42).

1.41. Find the approximate period of small-amplitude oscillations of the pendulum (1.6), page 4.

Solution. For small θ, we have $\sin\theta = \theta + O(\theta^3)$, and we replace $\sin\theta$ with θ, obtaining an approximating equation $\ddot{\theta} + (g/R)\theta = 0$. All solutions of this equation have period $2\pi\sqrt{R/g}$.

1.42. A dumbbell in Figure 48 with masses m and M is resting in the hemispherical bowl as shown: the smaller mass is at the height of the rim, and the dumbbell forms angle α with the horizontal. (i) Find the relationship between m, M, L and α. (ii) Find the period of small vibrations near an equilibrium (assuming the bowl continues a little above the smaller mass).

Momentum, Energy.

1.43. A string of bullets strikes a box of sand resting on a frictionless horizontal plane, Figure 49. Each bullet stays buried in the sand. Find the velocity of the block after the nth impact. All the bullets have the same masses m and the same velocities v. The mass of the box is M.

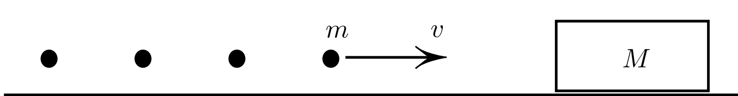

Figure 49. For Problem 1.43.

1.44. A bullet strikes a bag of sand hanging on a rope, causing the bag to swing. The bag deflects by maximal angle θ. Find the bullet's velocity, given that the masses of the bullet and of the bag are m, M, respectively, and that the length of the rope is L.

1.45. A wheeled platform of mass m is rolling without friction. Heavy rain is coming vertically down, with water constantly spilling over the edge of the platform. Does the speed of the platform change, and if so, how does it depend on time? The area of the platform is A; the intensity of the rain ρ (mass of water falling per second on a square meter of surface) is given.

Hint. The platform slows down exponentially since every second it sheds a fixed proportion of its linear momentum through the water that pours off of the platform.

1.46. A car driving in the rain gives kinetic energy to raindrops it hits. Estimate the extra power a car must expend due to this effect, given the area A of the car's frontal profile, the car's speed v, and the fact that rain is coming down at $1cm$ per hour.

1.47. A piece of bird dropping splatters against the windshield of a fast–moving car. As the result, some energy goes into heat, and some goes into the kinetic energy of the motion relative to the ground (there is also energy going into sound, etc., but let us ignore it). What is the ratio of these two energies?

Answer. The ratio equals 1 (under the simplifying assumptions stated in the problem).

1.48. A chain is lying in a pile on the ground; I am pulling one end of the chain up with a constant speed v. Would it take any force to maintain this speed if there was no gravity? In other words, does Newton's first law ("zero acceleration requires zero force") apply — why or why not? If not, what is this force? Linear density (mass per unit length) of the chain is ρ.

The preceding problem hides an interesting paradox which the next problem is asking to resolve.

1.49. Explain the following paradox. Consider the problem of the force required to pull the end of the chain lying on the floor with constant speed v, as described in Problem 1.48. Gravity is to be ignored. On the one hand, an element of mass Δm of the chain accelerates from rest to speed v in time Δt; its average acceleration is thus $a = v/\Delta t$. The average force is then $F = \Delta m a = \Delta m v/\Delta t = \rho v$, according to Newton's second law. On the other hand, the energy I am giving to the chain in time t is $mv^2/2$, where $m = \rho v t$ is the mass of the leaving the floor in time t. The power I am expending is the work I do per unit time:

$$P = \frac{mv^2/2}{t} = \rho v^3/2.$$

Thus the force I am applying is $F = P/v = \rho v^2/2$. This is half of the preceding answer. Which argument hides a mistake? What is the physical significance of the difference between the two results?

Answer. Both answers are wrong. The first answer makes a false presumption that all energy is spent on lifting; in fact, some energy ends up in the form of waves in the chain, for instance. The second answer overlooks a (surprising) possibility that the chain, when pulled up, may push against the floor. Indeed, consider, for instance, the chain made of rods linked by hinges. Consider one such rod/link lying on the ground (gravity plays no role). As we start pulling one end up, the other end will press down on the floor! The true answer to the problem lies somewhere between the given extremes, and depends on the specifics of the problem.

The following problem may seem like a fluid dynamics problem, and although technically it is, it is essentially a one-dimensional mechanics problem.

1.50. I am ejecting water from a syringe, moving the piston at a constant speed v. Water is perfectly nonviscous, and the piston is perfectly frictionless. What force, if any, must I apply to the piston? Does Newton's first law imply that this force is zero? If not, find that force, given the ratio of the diameters of the piston and of the exit hole.

The next problem shows an interesting fact: To launch a satellite to infinity takes exactly twice the amount of work of lifting it from the center of the planet to the surface, assuming that the planet has a constant density. Recall that the gravitational force inside such a homogeneous ball is a linear function of the distance to the center, Figure 50.

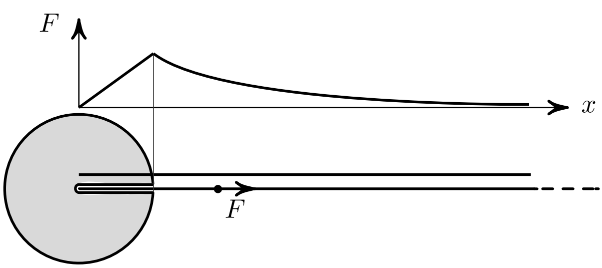

Figure 50. How much work does it take to lift a mass from the center of the asteroid to infinity? The work equals the area under the graph.

1.51. An asteroid is a perfectly round solid homogeneous sphere, Figure 50. A tunnel is drilled to the center of the asteroid, and a sample is lifted from the center first to the surface and then from the surface to infinity. What is the ratio of energies spent on the two stages? In other words, how much more (or less) work does it take to lift from the surface to infinity than from the center to the surface?

Answer. Interestingly, the second stage takes twice the work of the first. Does a similar result hold for nonconstant densities?

Miscellaneous Problems.

1.52. Consider a potential $V(x) \geq 0$ on the line $\mathbb{R}$, Figure 51, with $V(x) \to 0$ as $x \to \infty$, and with a maximum $V(x_{\max}) = H > 0$.

(1) What speeds at $x = -\infty$ would enable the particle to pass over the hill?

(2) One particle $x_1(t)$ moves freely: $x_1(t) = vt$; the other particle $x_2(t)$ moves in the potential V. Find the lag suffered by the second particle $x_2(t)$, i.e., find $\lim_{t\to\infty}(x_1(t) - x_2(t))$, given that the two have the same initial data at $t = -\infty$, i.e., that $\lim_{t\to-\infty}(vt - x_2(t)) = \lim_{t\to-\infty}(v - \dot{x}_2(t)) = 0$.

(3) Can one recover the shape of V by shooting the particle at different speeds and measuring the travel time?

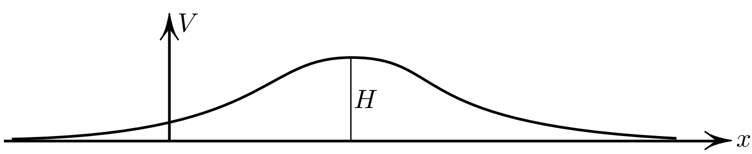

Figure 51. By how much distance does the hump delay particles? (refer to Problem 1.52).

1.53. A 4th degree polynomial $U(x) = x^4 + ax^3 + bx^2 + cx + d$ has two distinct minima. Prove that the periods of any two oscillations of a particle in this potential with the same energy are equal, Figure 52.

Hint. (*This solution assumes some knowledge of the theory of functions of complex variables.*) Consider two oscillations in the different wells with the same energy E, so that $U(x) < E$ on two intervals (x_1, x_2) and (x_3, x_4), and $U(x_i) = E$ for $i = 1, 2, 3, 4$, Figure 52. Kinetic energy $K(x) = E - U(x)$ is also a polynomial of 4th degree, with the roots at x_i. The problem amounts to proving that

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{K(x)}} = \int_{x_3}^{x_4} \frac{dx}{\sqrt{K(x)}}, \tag{1.73}$$

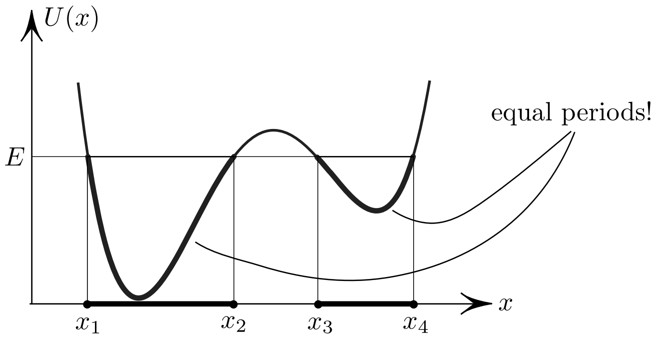

Figure 52. In a quartic potential, equality of energies implies equality of periods.

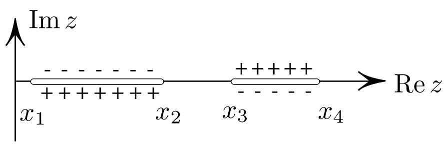

Figure 53. Signs of the chosen branch of $\sqrt{K(z)}$ on the slits' edges.

where

$$K(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

To prove (1.73), let us allow x to be complex, denoting it now by $z = x+iy$, and consider $\sqrt{K(z)}$ for complex values of z. Now K is a multiple-valued function because of the square root: Indeed, if z executes a loop around one root, say, x_1, then *arg* K changes by 2π, so that *arg* $\sqrt{K}$ changes by π, i.e. $\sqrt{K}$ changes sign. But if the loop encloses *two* roots then $\sqrt{K}$ does not change as z executes a round trip around the loop. So we cut the slits as in Figure 53 and forbid z to cross them. We thus turn $\sqrt{K(z)}$ into a single-valued function, provided we choose a particular value of the root at some fixed point and then extend to the entire plane minus the slits by continuity. In other words, $\sqrt{K(z)}$ has a single-valued branch in the plane with slits removed. Let us choose the positive sign on the upper edge of the slit $[x_3, x_4]$ as in Figure 53; the signs of our branch of $\sqrt{K}$ on the other edges of slits are determined automatically and are shown in Figure 53.

Consider now two loops A and B as in Figure 54. Loop A can be deformed into loop B by going through infinity (note that $z = \infty$ is a regular point of $\frac{1}{\sqrt{K(z)}}$). This can be seen by mapping the plane to the sphere by stereographic projection and then deforming the loops on the

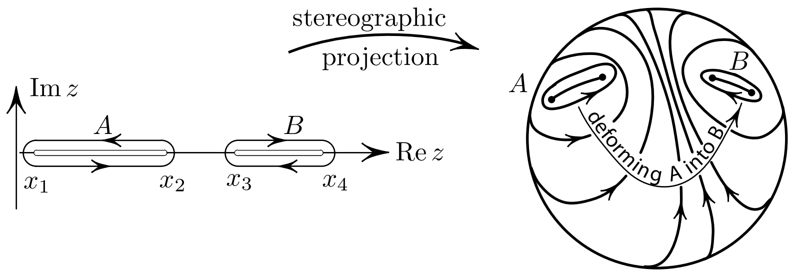

Figure 54. In a quartic potential, equal energies mean equal periods.

sphere as shown in Figure 54. The integral does not change as the contour of integration is deformed, and we conclude that

$$\int_A \frac{dx}{\sqrt{K(x)}} = \int_B \frac{dx}{\sqrt{K(x)}}.$$

But this already proves our claim (1.73), since

$$\int_A \frac{dx}{\sqrt{K(x)}} = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{K(x)}}, \quad \int_B \frac{dx}{\sqrt{K(x)}} = 2\int_{x_3}^{x_4} \frac{dx}{\sqrt{K(x)}},$$

as is clear from the sign patterns in Figure 53.

The following is a generalization of the preceding problem:

1.54. What is the analogous statement for the cubic potential with a well? What is the relationship between periods of oscillations with the same energy in a polynomial potential of degree $n > 4$?

1.55. Consider the motion of a particle of mass m in a potential $U(x)$ in the presence of linear drag force directly proportional to the particle's speed. Write down the ODE governing the motion of the particle.

Solution. The resultant force on the particle is $F = F_{\text{drag}} + F_{\text{potential}} = -k\dot{x} - U'(x)$. Newton's law then gives

$$m\ddot{x} + k\dot{x} + U'(x) = 0. \tag{1.74}$$

1.56. Prove that no periodic motions exist for the system with drag governed by (1.74) with $k \neq 0$.

Solution. *Method 1.* Assume the contrary: There exists a nonconstant periodic solution, i.e., a solution for which $(x(T), \dot{x}(T)) = (x(0), \dot{x}(0))$ with some $T > 0$. By showing that the energy $E(t) = \frac{\dot{x}^2}{2} + U(x)$ decreases: $E(T) < E(0)$, we will arrive at a contradiction. We have

$$\frac{d}{dt}E(t) = \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + U(x)\right) = \dot{x}(m\ddot{x} + U'(x)) \overset{(1.74)}{=} -k\dot{x}^2.$$

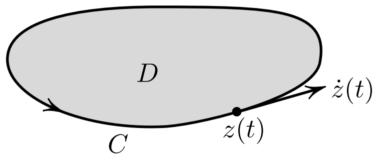

Figure 55. Area enclosed by a periodic orbit must remain constant as it flows; negative divergence implies that this area must decrease, leading to a contradiction.

Now $\dot{x} \neq 0$, with possible exception of isolated values of t, since x is a nonequilibrium solution. Therefore, $E(T) < E(0)$. $\diamondsuit$

Method 2. Assume the contrary: For some $T > 0$ we have $x(t) = x(t+T)$ for all t, Figure 55. Then the phase point $(x(t), \dot{x}(t))$ describes a closed curve C in the $(x, \dot{x})$-plane, Figure 55. Now on the one hand, the region enclosed by the trajectory is invariant under the flow, Figure 55. But on the other hand, the area of the region must decrease when carried by the flow since the divergence of the vector field is negative. Indeed, we have for the area A enclosed by C (see (1.51) on page 35)),

$$A'(t) = \int_D \operatorname{div} \mathbf{f} \, d\mathbf{x},$$

where D is the region enclosed by C. But this is a contradiction with $A = \text{const.}$, since

$$\operatorname{div} \mathbf{f} = \operatorname{div}(y, -ky - U'(x)) = -k < 0.$$

The following problem deals with a time-dependent potential.

1.57. Consider a conservative time-dependent (also called nonautonomous) system

$$\ddot{x} + U_x(x,t) = 0, \quad U_x \equiv \frac{\partial}{\partial x} U(t,x). \tag{1.75}$$

There is no frictional force here. Is the energy conserved?

1.58. The pivot of the pendulum of length L is oscillating in the vertical direction with acceleration $a(t)$. Explain why the ODE for the angle θ with the downward vertical is $L\ddot{\theta} + (g + a(t)) \sin\theta = 0$.

Does the energy conservation go hand-in-hand with the area preservation? The following problem illustrates the answer.

1.59. Consider the motion of a particle in a time-dependent potential: $\ddot{x} + U_x(t, x) = 0$ (the pendulum in the preceding problem is an example). Consider the energy $E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + U(t, x)$.

(1) Is the energy conserved during the particle's motion?

(2) Is the phase flow associated with the vector field

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -U_x(t, x). \end{cases} \tag{1.76}$$

area-preserving? In other words, does Liouville's theorem apply to time-dependent ODEs?

1.60. What are the accelerations of the mass M in Figure 56 (the pulleys are massless)? One of the answers may surprise you.

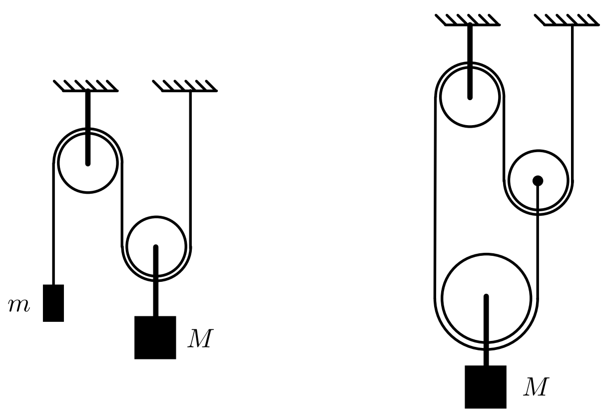

Figure 56. Some pulley problems.

1.61. 1. What force is required to hold the rope in Figure 57(A)? The shaded pulley's mass is M; the other pulleys are massless.

2. What force is required to hold the rope in Figure 57(B)?

1.62. Given that the system in Figure 57(C) is in equilibrium, what is the ratio of masses M/m?

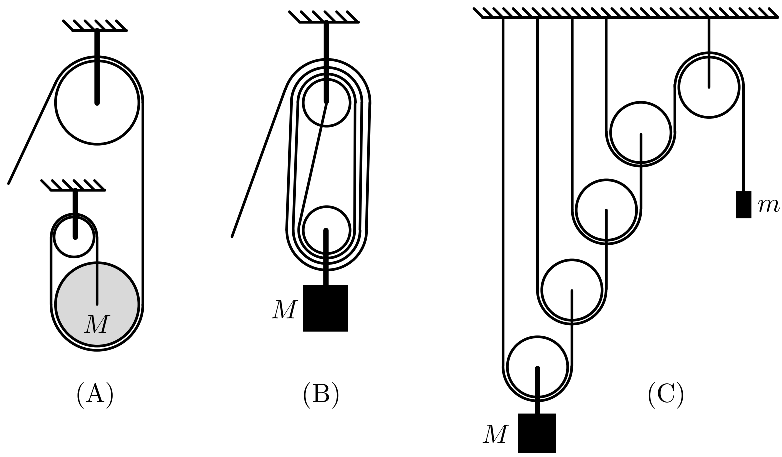

Figure 57. More pulley problems.

1.63. Figure 58 shows two identical monkeys on the rope thrown over a perfectly frictionless pulley. Initially, the two animals are at rest. Then the right one starts climbing. Describe the relative position of the two monkeys. Suddenly, one of the monkeys lets go of the rope. What happens to their relative position then?

Answer. (Written backwards: .thgieh emas eht ta niamer yehT)

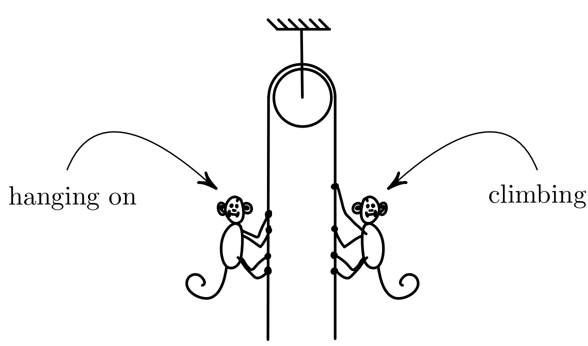

Figure 58. The right monkey is climbing; the left one is just holding on. What will happen?

The dynamics-statics equivalence.

1.64. Consider a heavy Hookean spring hanging by one end in equilibrium; see Figure 25 on page 42. Consider the energy density (per unit mass) of stretching E_s and the gravitational energy E_g per unit mass. Show that $E_s = E_g + \text{const.}$ along the spring. In particular, if the lower end of the spring is on the ground level, then $E_s = E_g$! Does this effect depend on the linearity of Hooke's law?

In the next three problems we consider a zero length Hookean spring of mass M and Hooke's constant k. The spring is to be thought of as a negligibly thin line. Each particle of the spring is labeled by the mass t between the particle and one end of the spring. The spring is laid out along the x-axis *and is kept longitudinally deformed* so that the particle t is located at $x(t)$, where $x(\cdot)$ is a given smooth function. Each particle of the spring is thus forcibly held by some external force at a prescribed location. In addition, a force field with potential $U(x)$ is defined on the line; that is, a unit mass located at x has potential energy $U(x)$, and is therefore subject to the force $F(x) = -U'(x)$.

1.65. Given that Hooke's constant of a homogeneous linear spring is k, what is Hooke's constant of a piece of this spring, given that the mass of the piece forms a proportion $p < 1$ of the total mass of the spring?

1.66. Referring to the setting just described, do the following.

1. Write the total energy $E[x]$ of the spring in terms of the "configuration function" $x = x(t)$.

2. Write the Euler–Lagrange equation for the critical function of $E[x]$.

3. Assume that the external force holding the spring is removed, and that the spring is in equilibrium. That is, each infinitesimal element of the spring has zero net force acting on it in the direction of the x-axis. Express this equilibrium condition as a differential equation for $x(t)$, thereby rederiving the Euler–Lagrange equation from item 2 above.

4. Consider a unit mass moving on the x-axis in the force of the potential energy $-U(x)$, i.e., $F(x) = +U'(x)$. Write down the action integral $A[x] = \int(\text{kinetic} - \text{potential})\, dt$ in terms of $x(t)$, the position of a particle at time t, to show that $A[x] = E[x]$ for any function $x(t)$, and observe that the equilibrium condition of the spring is the same as Newton's second law for the particle.

1.67. A linear spring of mass M and of Hooke's constant k is hanging on its end in the gravitational field. By how much will this spring stretch under the influence of gravity? Assume the relaxed length of the spring to be zero. Note that the stretching is not uniform: the higher up the spring you go, the more it is stretched, as it carries more weight.

Chapter 2

More Degrees of Freedom

In this chapter we consider systems of n particles which interact with each other and which may also be subject to external forces. Figure 1 shows some examples: planets (viewed as point masses) moving in the gravitational field of a star; a multiple pendulum (sketched in the middle of Figure 1); point masses coupled by springs.

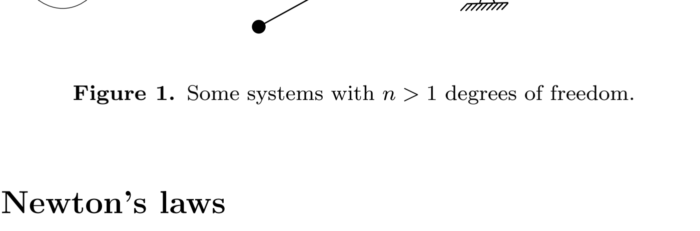

Figure 1. Some systems with $n > 1$ degrees of freedom.

1. Newton's laws

Unless stated otherwise, the reference frames used in this chapter are inertial, i.e., nonaccelerating and nonrotating. Unfortunately, this definition is meaningless since it does not specify with respect to what the acceleration and rotation are measured. One way to define an inertial frame is to say that it is the one in which Newton's laws hold. Since we have not yet stated these laws, we can instead think of frames attached to the ground or moving on a nonaccelerating train (if the Earth's rotation can be safely ignored) or as reference frames in outer space far from any gravitational bodies and not rotating relative to the stars.

The rotating frames will be discussed in connection with the Coriolis force and the centrifugal force.

Newton's second law, the foundation of classical mechanics, states that the acceleration of a particle of mass m is decided by the *resultant force* $\mathbf{F}$, i.e., by the sum of all forces acting upon the particle:

$$m\mathbf{a} = \mathbf{F}.$$

To analyze the motion of systems of several interacting particles this law does not suffice; one needs also Newton's third law:

Newton's third law: When two particles interact, the forces they exert on each other are equal in magnitude and opposite in direction; furthermore, the forces lie along the line connecting the particles.

Thanks to this law, *all the internal forces in a body cancel each other, and, moreover, produce zero torque*[1]; this suggests that without external forces the center of mass (defined in the next section) cannot accelerate. For example, the center of mass of the dumbbell with a variable-length bar in Figure 2 will not start moving spontaneously, no matter how its bar is elongated/shortened. Similarly, the dumbbell cannot start rotating, no matter how it changes its length, in the absence of external forces, thanks to the fact that the action/reaction forces act along the connecting bar. All these statements will be made precise on pages 79 and 81.

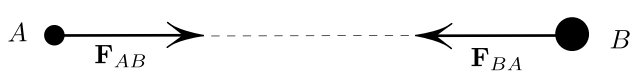

Figure 2. Newton's third law: $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$ and $\mathbf{F}_{12}$ is parallel to AB.

[1]For the definition of the torque, see page 81.

2. Center of mass

Center of mass of a collection of n masses m_i in $\mathbb{R}^3$, with position vectors $\mathbf{r}_i$, $i = 1, \ldots, n$, is defined as the weighted average of the position vectors, each vector weighted according to the mass. More precisely, we have the following.

Definition. The *center of mass* $\mathbf{r}_c$ of the system of particles is the weighted average of their positions, with weights given by the particles' masses:

$$\mathbf{r}_c \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{m_i}{m}\mathbf{r}_i = \frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{r}_i, \quad m = \sum m_i. \tag{2.1}$$

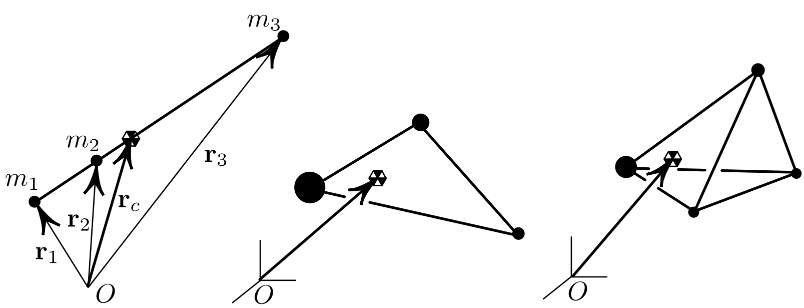

Figure 3. A collection of particles in space.

The following theorem connects this definition to the one we are used to since childhood, and also gives another interpretation of the center of mass.

Theorem 2.1. *The center of mass has the following properties.*

(1) *The center of mass of a rigid body made of finitely many particles is also the point of balance: the body suspended by its center of mass in a* uniform gravitational field *is in equilibrium in any orientation.*

(2) *The function*

$$I(\mathbf{r}) = \sum_{i=1}^{n} m_i(\mathbf{r} - \mathbf{r}_i)^2$$

achieves its minimum at the center of mass $\mathbf{r} = \mathbf{r}_c$ *of the system of particles with masses* m_i *located at* $\mathbf{r}_i$.

Proof. Part 1. (This proof uses the concept of torque introduced later, on page 81. The reader may skip this proof without dire consequences.) It suffices to show that the sum of gravitational torques around the center of mass of the body is zero, regardless of the orientation of the body. The torque of the gravitational force $\mathbf{F}_i = m_i\mathbf{g}$ relative to the center of mass $\mathbf{r}_c$ is

$$\mathbf{T}_i = (\mathbf{r}_i - \mathbf{r}_c) \times \mathbf{F}_i = m_i(\mathbf{r}_i - \mathbf{r}_c) \times \mathbf{g},$$

where $\mathbf{r}_i$ is the position vector of the particle. The sum of torques

$$\sum \mathbf{T}_i = \sum m_i(\mathbf{r}_i - \mathbf{r}_c) \times \mathbf{g} = \Big(\underbrace{\sum m_i(\mathbf{r}_i - \mathbf{r}_c)}_{=\mathbf{0} \text{ by } (2.1)}\Big)\times\mathbf{g} = \mathbf{0},$$

as claimed.

Part 2. Let us express $I(\mathbf{r})$ in terms of the displacement $\mathbf{x} = \mathbf{r} - \mathbf{r}_c$, aiming to show that that $\mathbf{x} = \mathbf{0}$ minimizes I. Substituting $\mathbf{r} = \mathbf{x}+\mathbf{r}_c$, we write

$$\begin{aligned} I(\mathbf{r}) = I(\mathbf{x} + \mathbf{r}_c) &= \sum m_i(\mathbf{x} + (\mathbf{r}_c - \mathbf{r}_i))^2 \\ &= \sum m_i(\mathbf{x}^2 + 2\mathbf{x} \cdot (\mathbf{r}_c - \mathbf{r}_i) + (\mathbf{r}_c - \mathbf{r}_i)^2). \end{aligned}$$

In the last sum the middle terms add up to zero by the definition of the center of mass (2.1), and what remains is

$$I(\mathbf{r}) = \Big(\sum m_i\Big)\mathbf{x}^2 + \sum m_i(\mathbf{r}_c - \mathbf{r}_i)^2. \tag{2.2}$$

This is minimized precisely by $\mathbf{x} = \mathbf{0}$, i.e., by $\mathbf{r} = \mathbf{r}_c$, as claimed. $\diamondsuit$

A digression: applications of the center of mass to geometry. The concept of the center of mass has some beautiful applications to geometrical problems, for instance, the following.

(1) Medians in a triangle are concurrent (i.e., all three of them intersect at one point). This theorem from geometry amounts to stating that the triangle made of three equal masses at vertices has a center of mass.

(2) Ceva's theorem: Consider three segments in a triangle ΔABC, each segment connecting a vertex with the opposite side, and dividing each side into segments of lengths a, a', b, b', c, c' (listed in the same order as the vertices). Then the three segments are concurrent if and only if $abc = a'b'c'$.

(3) Let S be a convex surface in $\mathbb{R}^3$, and let $T_{\min}$ be a tetrahedron (a triangular pyramid) of least possible volume containing S inside. Then all faces of $T_{\min}$ are tangent to S at their centroids, i.e., at the points of intersection of their medians.

Further details and references can be found in [**14**].

3. Newton's second law for multi-particle systems

Theorem 2.2. *The center of mass of any system of particles behaves as a single particle, in the sense that*

$$m\mathbf{a}_c = \mathbf{F}_{\text{ext}}, \tag{2.3}$$

where $\mathbf{a}_c = \ddot{\mathbf{r}}_c$ *is the acceleration of the center of mass,* m *is the total mass of all the particles in the system, and* $\mathbf{F}_{\text{ext}}$ *is the sum of all external forces acting on the system. Equivalently, the total linear momentum* $\mathbf{p} = \sum \mathbf{p}_i = \sum m_i \mathbf{v}_i$ *of the system satisfies*

$$\dot{\mathbf{p}} = \mathbf{F}_{\text{ext}}. \tag{2.4}$$

Proof. Let $\mathbf{F}_{ij}$ be the force the ith particle feels from the jth particle, and let $\mathbf{F}_i$ be the vector sum of all external forces on the ith particle. Newton's second law applied to the ith particle gives:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i, \quad i = 1, \ldots, n. \tag{2.5}$$

Since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ (Newton's third "action-reaction" law), the internal forces cancel upon summation of (2.5) over all i, and we get

$$\sum m_i \ddot{\mathbf{r}}_i = \underbrace{\sum \mathbf{F}_i}_{\mathbf{F}_{\text{ext}}}. \tag{2.6}$$

Dividing and multiplying by the total mass m on the left-hand side, we recognize the center of mass,

$$m \frac{d^2}{dt^2} \underbrace{\sum \frac{m_i}{m} \mathbf{r}_i}_{\mathbf{r}_c} = \mathbf{F}_{\text{ext}},$$

thus proving (2.3). Finally, (2.4) follows from (2.6) at once. ♢

4. Angular momentum, torque

To state the rotational version of Newton's second law $\mathbf{F} = m\mathbf{a} = \dot{\mathbf{p}}$, which is the goal of the next section, we must define the rotational analogues of $\mathbf{p}$ and $\mathbf{F}$. These analogs are the angular momentum and the torque.

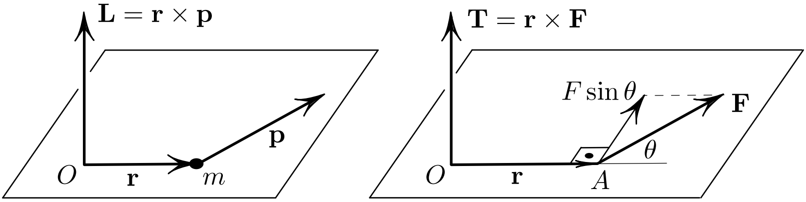

Figure 4. Definitions of the angular momentum (left) and of the torque (right).

Definition 2.1. *The angular momentum* of a point mass relative to a fixed point O in an inertial frame is the cross product

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = \mathbf{r} \times \mathbf{p}, \tag{2.7}$$

where $\mathbf{r}$ is the position vector relative to O and $\mathbf{v}$ is the velocity, Figure 4.

Note that the angular momentum depends on the choice of O. For the example of a comet orbiting the Sun, the angular momentum vector relative to the Sun's center is perpendicular to the orbit's plane (called the ecliptic), since $\mathbf{L} \perp \mathbf{r}$ and $\mathbf{L} \perp \mathbf{v}$, while $\mathbf{r}$ and $\mathbf{v}$ lie in the plane of the orbit. According to (2.7), the magnitude $L = |\mathbf{L}|$ is given by

$$L = mv_\perp r = p_\perp r,$$

where the subscript $_\perp$ denotes the component of a vector in the direction perpendicular to $\mathbf{r}$ in the plane of $\mathbf{r}$, $\mathbf{v}$.

Definition 2.2. *The angular momentum of a system of particles* is the sum of the angular momenta of all the particles in the system:

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{v}_i. \tag{2.8}$$

Definition 2.3. Consider a force $\mathbf{F}$ applied at a point A, Figure 4. The *torque* of $\mathbf{F}$ relative to the point O is the cross product

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}, \quad \text{where} \;\; \mathbf{r} = \overrightarrow{OA}. \tag{2.9}$$

The intuitive concept of the "intensity of rotation" is captured by this definition perfectly. Indeed, if we think of $\mathbf{r}$ as the handle of a wrench attached to the nut at O, then $T = rF\sin\theta$ (here $T = |\mathbf{T}|$, $r = |\mathbf{r}|$, etc.) is the product of the lever r and the perpendicular component $F\sin\theta$ is responsible for trying to rotate the wrench. Moreover, the vector $\mathbf{T}$ points along the bolt on which the nut is seated; finally, the direction of $\mathbf{T}$ along the line of the bolt tells which way the nut moves along the bolt with right-handed thread when turned according to $\mathbf{F}$.

5. Rotational version of Newton's second law; conservation of the angular momentum

Theorem 2.3. *Consider a system of ineracting particles subject to external forces. The total angular momentum* $\mathbf{L}$ *(with respect to a fixed point* O *in our inertial frame) of such a system changes according to*

$$\frac{d}{dt}\mathbf{L} = \mathbf{T}, \tag{2.10}$$

where $\mathbf{T}$ *is the sum of external torques (i.e., the torques due to forces which come from outside the system) relative to* O*:*

$$\mathbf{T} = \sum \mathbf{T}_i^{\text{ext}}.$$

Proof. Differentiating $\mathbf{L} = \sum \mathbf{L}_i$ and using $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = \mathbf{0}$ in step (A) below, we get

$$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}\sum \mathbf{r}_i \times m_i\dot{\mathbf{r}}_i \stackrel{(A)}{=} \sum \mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i = \sum \mathbf{r}_i \times \mathbf{F}_i, \tag{2.11}$$

where $\mathbf{F}_i$ is the sum of all forces, including internal ones, acting upon the ith particle. To complete the proof it remains to show that $\mathbf{F}_i$ can be replaced with $\mathbf{F}_i^{\text{ext}}$ in (2.11), i.e., that the internal torques cancel. Let us therefore separate out the external forces:

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{\text{ext}}.$$

Substituting into (2.11) we get

(2.12)
$$\frac{d}{dt}\mathbf{L} = \sum_i \mathbf{r}_i \times \left(\sum_{j\neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{\text{ext}}\right) = \sum_{j\neq i} \underbrace{\mathbf{r}_i \times \mathbf{F}_{ij}}_{\mathbf{T}_{ij}^{\text{int}}} + \sum_i \underbrace{\mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}}_{\mathbf{T}_i^{\text{ext}}}.$$

It remains to show that the sum $\sum \mathbf{T}_{ij}^{\text{int}} = \mathbf{0}$. Let us group the terms in that sum into pairs, coupling the terms containing $\mathbf{T}_{ij}$ and $\mathbf{T}_{ji}$, Figure 5; the figure suggests that each couple vanishes. Indeed,

(2.13)
$$\mathbf{F}_{ij} = -\mathbf{F}_{ji},$$

(Newton's third law) and

(2.14)
$$(\mathbf{r}_i - \mathbf{r}_j) \parallel \mathbf{F}_{ij},$$

so that

$$\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji} \overset{(2.13)}{=} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} \overset{(2.14)}{=} \mathbf{0},$$

as claimed. Only the last term remains in (2.12); the proof is complete. ♢

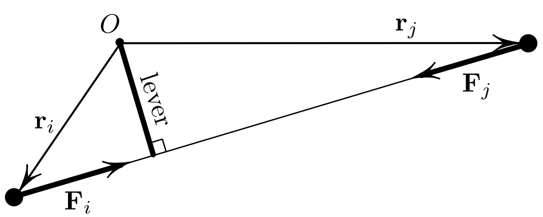

Figure 5. Forces F_{ij} and F_{ji} share the same lever and are equal in magnitude. Hence their torques cancel each other.

An immediate important corollary of (2.10) is the following.

Theorem 2.4 (Conservation of the angular momentum)**.** *If the sum of external torques upon a system of masses is zero, then the angular momentum of the system remains constant.*

Proof. $\frac{d}{dt}\mathbf{L} \overset{(2.10)}{=} \mathbf{T} = \mathbf{0}$. ♢

Some examples. 1. If we treat the solar system as a collection of point masses interacting with each other via gravitational forces, and ignore the external forces, e.g., the ones from the stars, then the total angular momentum of the system is conserved.

2. The total angular momentum of Earth + Moon is constant, if we ignore the effect of other celestial bodies. Incidentally, the Earth's axial spin is gradually slowing down over millions of years due to tidal effect of the Moon, so that the Earth is losing its angular momentum. Hence the Moon is gaining angular momentum. As the result, the Moon is increasing its distance to the Earth, since the orbital radius is an increasing function of the angular momentum.

Exercise 2.1. If a gymnast is hanging on the bar at rest, wearing perfectly slippery gloves, then the frictional torque around the bar is zero. Does this mean that the gymnast's angular momentum around the bar will remain zero regardless of how the she flexes her body?

Hint. The sum of external torques upon the gymnast need not remain zero — there is gravity.

Exercise 2.2. Is it true that the combined gravitational torque in a *constant gravitational field* upon a system of particles is the same as the torque upon a single particle of combined mass, placed at the center of mass of the system?

Answer. Yes, since the sum of torques

$$\mathbf{T} = \sum \mathbf{r}_i \times (m_i \mathbf{g}) = \left(\sum m_i \mathbf{r}_i\right) \times \mathbf{g} = \left(\sum \frac{m_i}{m} \mathbf{r}_i\right) \times m\mathbf{g} = \mathbf{r}_c \times m\mathbf{g}.$$

For nonconstant gravitational fields the result fails.

Here is another important corollary of Newton's laws for translational motion (2.3) and for rotational motion (2.10).

Theorem 2.5 (An equilibrium condition)**.** *If a rigid body — that is, a collection of point masses all joined pairwise by weightless rods — is in equilibrium, then the sum of external forces is zero and the sum of external torques upon the body is zero. Conversely, if a rigid body is initially at rest and both the sums of forces and torques vanish, then the body will remain at rest.*

Proof. For a body in equilibrium the center of mass has zero acceleration: $\mathbf{a}_c = \mathbf{0}$. Hence the sum of external forces $\mathbf{F} \overset{(2.3)}{=} m\mathbf{a}_c = \mathbf{0}$. Furthermore, for a body in equilibrium, $\mathbf{L} = \mathbf{0}$ so that the sum of external torques $\mathbf{T} \overset{(2.10)}{=} \dot{\mathbf{L}} = \mathbf{0}$. Conversely, if $\mathbf{F} = \mathbf{0}$, then $\mathbf{a}_c = \mathbf{0}$; the center of mass must be at rest since it is at rest initially; and $\dot{\mathbf{L}} = \mathbf{T} = \mathbf{0}$ implies $\mathbf{L} = \text{const.}$, and since $\mathbf{L}(0) = \mathbf{0}$ we conclude that $\mathbf{L}(t) = \mathbf{0}$ for all t, which implies that the body does not rotate.[2]

6. Circular motion: angular position, velocity, acceleration

In this and in the next two sections we consider a very special case of a particle constrained to a circle, Figure 6, so that we are really dealing with one-dimensional motions in this section. Since we just introduced the concept of torque, it is more convenient to discuss the topic here rather than in Chapter 1. The results of this section are summarized in the table on page 87.

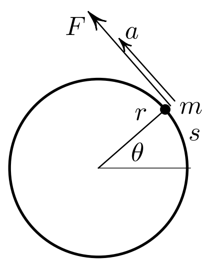

Figure 6. Introducing the moment of inertia $I = T/\alpha = mr^2$.

Angular position, velocity, accleration. Position of a particle on a circle is specified by the angle θ with a fixed direction, Figure 6. The *angular velocity* ω and the *angular acceleration* α are defined via

$$\omega = \dot{\theta}, \quad \alpha = \dot{\omega} = \ddot{\theta}. \tag{2.15}$$

[2]Strictly speaking, the last implication requires a bit more justification. This justification relies on the concepts of angular velocity $\boldsymbol{\omega}$ and of the moment of inertia I described in Chapter 3; here it is: $\mathbf{L} = I\boldsymbol{\omega} = \mathbf{0}$ implies $\boldsymbol{\omega} = \mathbf{0}$ (provided I is nongenegerate), so that the body does not rotate. Now if I is degenerate, then the body is either confined to a line or is a point, and we leave the consideration of these cases to the reader.

We have $s = \theta r$ for the length s corresponding to the angle θ, according to the definition of the radian measure of the angle, Figure 6. Differentiating this identity by t and using (2.15) gives

$$v = \omega r, \quad a = \alpha r,$$

where $a = \dot{v}$ is the tangential acceleration.

The moment of inertia is a rotational analog of mass. A precise definition comes first, followed by the reason for such a definition.

Definition 2.4. The *moment of inertia of a point mass m in the plane relative to a point O* is defined as

$$I = mr^2, \tag{2.16}$$

where r is the distance from the point mass to O. For a system of n particles in the plane, the moment of inertia relative to O is defined as the sum of moments of inertia of individual particles,

$$I = \sum_i m_i r_i^2, \tag{2.17}$$

where r_i is the distance from O to the ith particle.

To explain the definition (2.16) we note that the analog of the mass $m = \frac{F}{a}$ is the "rotational mass"

$$I = \frac{T}{\alpha}; \tag{2.18}$$

see the table on page 87. Intuitively, we expect that I does not depend on the choice of the torque T — presumably, changing the torque T also changes the angular acceleration α by the same factor, causing cancellation in (2.18). Indeed, substitute $T = rF$ into (2.18) and then use $F = ma$ and $a = \alpha r$:

$$I = \frac{T}{\alpha} = \frac{rF}{\alpha} = \frac{rma}{\alpha} = \frac{rm(\alpha r)}{\alpha} = mr^2.$$

This completes the explanation of (2.16).

7. Energy and angular momentum of rotation

The expressions $\mathbf{p} = m\mathbf{v}$ and $K = m\mathbf{v}^2/2$ have rotational counterparts which we now derive, for a planar rigid body consisting of n

masses m_i, $i = 1, \ldots, n$ forming a rigid arrangement and rotating in the plane around a point O with angular velocity ω. The more general case of spatial rotation of a rigid body is described in the next chapter.

Theorem 2.6. *Consider a planar rigid body rotating around a point O with angular velocity ω. The kinetic energy and the angular momentum relative to O are given by*

$$K = \frac{I\omega^2}{2} \tag{2.19}$$

and

$$L = I\omega, \tag{2.20}$$

where I is the moment of inertia relative to O.

Proof. An individual particle has kinetic energy

$$K_i = \frac{m_i v_i^2}{2} = \frac{m_i(\omega r_i)^2}{2} = \frac{(m_i r_i^2)\,\omega^2}{2} = \frac{I_i\,\omega^2}{2}.$$

Adding up K_i gives the result (2.19), since $I = \sum I_i$ by the definition. Similarly, the angular momentum of an individual particle is

$$L_i = m_i v_i r_i = m_i(\omega r_i) r_i = (m_i r_i^2)\,\omega = I_i\,\omega;$$

summation over i gives (2.20). ♢

Exercise 2.3. Consider a collection of particles in space. Prove that the kinetic energy of this set equals the kinetic energy of its center of mass (i.e., the kinetic energy of the particle of mass $\sum m_i$ tracking the position of C.M.) plus the sum of kinetic energies of all the particles in the reference frame of the center of mass.

Exercise 2.4. Obtain a mechanical proof of the Pythagorean Theorem using the preceding exercise, as follows. Consider two identical masses m, Figure 7, whose center of mass moves with velocity $\mathbf{a}$ and which fly away from each other with velocity $\mathbf{b}$ in the direction perpendicular to $\mathbf{a}$, Figure 7. Thus the speed c of each particle is the hypotenuse of the right triangle of velocities with legs a and b. Write the kinetic energy in two different ways: one, as $2mc^2/2$, and the other, as the energy of the center of mass plus the sum of energies of each mass relative to C.M.[3]

[3]Further details, as well as many other mechanical proofs, can be found in [**14**].

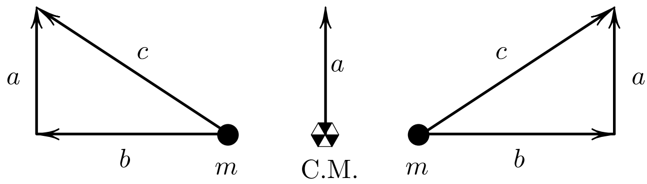

Figure 7. A kinetic energy proof of the Pythagorean theorem.

8. The rotational – translational analogy

Here is a summary of the analogy between the linear motion and the circular motion.

Linear motion	**Circular motion**
x – position	θ – angle
$v = \dot{x}$	$\omega = \dot{\theta}$ – angular velocity
$a = \dot{v} = \ddot{x}$ – acceleration	$\alpha = \dot{\omega} = \ddot{\theta}$ – angular acceleration
F – force	$T = rF$ – torque
m – mass (measure of inertia)	$I = mr^2$ – moment of inertia
$F = ma = m\dot{v}$	$T = I\alpha = I\dot{\omega}$
$p = mv$ – linear momentum	$M = I\omega$ – angular momentum
$F = \dot{p}$	$T = \dot{M}$
$K = \frac{mv^2}{2}$ – kinetic energy	$K = \frac{I\omega^2}{2}$ – kinetic energy

9. Potential force fields

We now return to a particle in $\mathbb{R}^n$, subject to force $\mathbf{F} = \mathbf{F}(\mathbf{r})$.

Potential energy. Speaking a bit loosely, the potential energy of a particle at a point $A \in \mathbb{R}^n$ is the work that I have to do to bring the particle from a reference position O to A against the force $\mathbf{F}$; formally,

$$\boxed{U(A) \stackrel{\text{def}}{=} -\int_{OA} \mathbf{F} \cdot d\mathbf{r}} \tag{2.21}$$

where OA is any curve connecting O and A. The minus sign is due to the fact that I must apply the force $-\mathbf{F}$ against the field, in order to move the particle (see the Remark 1.2 on page 6). This definition requires implicitly that the integral (2.21) be independent of the choice of path between O and A; otherwise, (2.21) is ambiguous. Potential energy is therefore defined only for special force fields, in which the integral (2.21) is path-independent.

Definition 2.5. A vector field $\mathbf{F}$ is said to be *conservative* if for any pair of points O and A, the integral (2.21) does not depend on the choice of path between O and A.

Gravitational and electrostatic fields are conservative. Had they been otherwise, we could build a perpetual motion machine. Indeed, imagine for a moment that the integral (2.21) were path-dependent: the work along one path OmA in Figure 8 is (say) less than along another path OnA. If I carry the particle along the closed path $OmAnO$, I get more energy descending AnO than I spend ascending OmA, thereby getting energy out of nothing. The conservative nature of gravitational and electrostatic fields is a physicist's way of saying that there is no free lunch.

The infinitesimal version of (2.21). Assume that $\mathbf{F}$ is a conservative field, so that the potential U is well defined by (2.21). Then we obtain

$$\boxed{\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}),} \tag{2.22}$$

recovering the force from the potential. For the proof, we compute the directional derivative of U in some direction $\mathbf{v}$:

$$D_{\mathbf{v}}U(\mathbf{r}) = \lim_{\varepsilon\to 0}\big[U(\mathbf{r}+\varepsilon\mathbf{v}) - U(\mathbf{r})\big]/\varepsilon = -\frac{1}{\varepsilon}\lim_{\varepsilon\to 0}\int_{\mathbf{r}}^{\mathbf{r}+\varepsilon\mathbf{v}} \mathbf{F}\cdot d\mathbf{x} = -\mathbf{F}(\mathbf{r})\cdot\mathbf{v}.$$

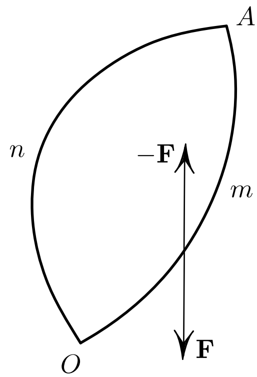

Figure 8. Definition of potential energy.

But $D_{\mathbf{v}}U(\mathbf{r}) = \nabla U \cdot \mathbf{v}$ by the definition of ∇U. We conclude that $\nabla U \cdot \mathbf{v} = -\mathbf{F}(\mathbf{r}) \cdot \mathbf{v}$. Since $\mathbf{v}$ is arbitrary, we arrive at (2.22). $\diamondsuit$

Exercise 2.5. Consider a linear force field: $\mathbf{F}(\mathbf{r}) = A\mathbf{r}$, where A is a (square) matrix. Show that $\mathbf{F}$ is conservative if and only if the matrix A is symmetric, and that the potential energy is then $U(\mathbf{r}) = \frac{1}{2}(A\mathbf{r}, \mathbf{r})$, where $(\cdot, \cdot)$ denotes the dot product.

Hint. Verify that $\oint_{[\mathbf{e}_i \mathbf{e}_j]} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}(A_{ij} - A_{ji})$, where $[\mathbf{e}_i, \mathbf{e}_j]$ is the unit square generated by the vectors $\mathbf{e}_i$, $\mathbf{e}_j$.

Exercise 2.6. Consider a unit circle centered at the origin in the plane of two unit coordinate vectors $\mathbf{e}_i$, $\mathbf{e}_j$. Show that $A_{ij} - A_{ji}$ is proportional to the torque of the force $A\mathbf{r}$ on the circle around its origin, or more precisely, to the average tangential component of the force $A\mathbf{r}$ around the circle.

Examples of conservative vector fields $\mathbf{F} = -\nabla U$.

1. Gravitational field of a point mass at the origin in $\mathbb{R}^3$:

$$\mathbf{F}(\mathbf{r}) = -\frac{\mathbf{r}}{r^3}, \quad U = -\frac{1}{r}, \tag{2.23}$$

where $r = |\mathbf{r}|$ and where we chose units so that the coefficient in front of the right-hand side in (2.23) is 1. The same scaling to achieve $k = 1$ applies to the next example.

2. Gravitational field of a point mass in $\mathbb{R}^2$, or equivalently, the electrostatic field of a wire with a uniform negative charge distribution, in the plane perpendicular to the wire:

$$\mathbf{F} = -\frac{\mathbf{r}}{r^2}, \quad U(\mathbf{r}) = \ln r. \tag{2.24}$$

3. Linear central force field:

$$\mathbf{F} = -k\mathbf{r}, \quad U(\mathbf{r}) = \frac{1}{2}k\mathbf{r}^2. \tag{2.25}$$

4. Linear force field $\mathbf{F}(\mathbf{r}) = A\mathbf{r}$ is conservative if and only if the real square matrix A is symmetric: $A^T = A$ (see Exercise 2.5 above). The corresponding potential is

$$U(\mathbf{r}) = -\frac{1}{2}(A\mathbf{r}, \mathbf{r}), \tag{2.26}$$

where $(\mathbf{u}, \mathbf{v})$ denotes the dot product of vectors $\mathbf{u}$, $\mathbf{v}$.

10. Some physical remarks

This section contains three miscellaneous observations: first, an interpretation of potential energy as hydrostatic pressure; second, an interpretation of conservativeness in terms of fluids, and third, an interpretation of the 2D curl in terms of angular accleleration. We omit some details to keep the discussion short.

Potential and the hydrostatic pressure. Imagine fluid of density $\rho = 1$ in a vessel, in a force field with potential $U(\mathbf{x})$; the field need not be constant. Let $p = p(\mathbf{r})$ be the resulting pressure in the fluid. Remarkably,

$$p(\mathbf{r}) = -U(\mathbf{r}), \tag{2.27}$$

apart from an additive constant. As an example, the pressure in a pool at depth z is $p = \rho g z = gz$ (we took $\rho = 1$), while the potential of the same gravitational field is $U = -mgz = -gz$, in agreement with (2.27). The same interpretation holds for an arbitrary conservative force field, not just the constant one.

Recall that the potential energy was defined as a certain line integral; we can therefore use the fluid and a pressure gauge as an analog computer to compute this integral via (2.27).

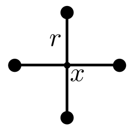

Figure 9. Curl gives the angular acceleration.

Conservative fields create no motion in fluids. An alternative characterization of a conservative vector field is the following. Imagine again fluid of constant density enclosed in a vessel and subject to a force field $\mathbf{F}$. *The vector field* $\mathbf{F}$ *is conservative if and only if it does not excite motion in a resting fluid.* We omit the proof.

Curl and the angular acceleration. We consider a vector field $\mathbf{F}$ in $\mathbb{R}^2$; $\mathbf{F}$ need not be conservative in this discussion. Recall that the $2D$ curl of $\mathbf{F} = \langle P, Q \rangle$ is a scalar: $\operatorname{curl} \mathbf{F} = Q_x - P_y$. Figure 9 illustrates a dynamical interpretation of the curl, as follows.[4] A rigid cross has four equal masses at the tips, as shown. The force field is acting on these masses, causing some angular acceleration α (in addition to linear acceleration $\mathbf{F}$). Interestingly, α equals the curl of $\mathbf{F}$ in the limit of a small cross:

$$\operatorname{curl} \mathbf{F} = \lim_{r \to 0} \alpha.$$

11. Conservation of energy

Consider a point mass in a conservative force field $\mathbf{F} = -\nabla U$. The particle's position vector $\mathbf{r}$ obeys Newton's second law:

$$m\ddot{\mathbf{r}} = -\nabla U(\mathbf{r}). \tag{2.28}$$

Just as in the one-degree-of-freedom case, let us define the kinetic energy of a particle as the work required to bring the particle from rest to speed $\mathbf{v}$. I claim that this definition implies

$$K = \frac{m\mathbf{v}^2}{2}, \tag{2.29}$$

[4]Interpreted kinematically, the curl is twice the average angular velocity of the imaginary fluid with velocity $\mathbf{F}$ at (x, y).

where $\mathbf{v}^2 = (\mathbf{v}, \mathbf{v})$ is the dot product, i.e., the square of the magnitude of $\mathbf{v}$. Indeed, let $\mathbf{F}(t)$ be the force causing the acceleration from zero to $\mathbf{v}$, during the time $0 \leq t \leq T$. The work done by $\mathbf{F}(t)$ is

$$K = \int_{\mathbf{r}(0)}^{\mathbf{r}(T)} \mathbf{F} \cdot d\mathbf{r} = \int_0^T m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} dt = \frac{m\dot{\mathbf{r}}^2}{2}\bigg|_0^T = \frac{m\mathbf{v}^2}{2},$$

as claimed.

Theorem 2.7. *The total energy $E = K + U$ of a particle moving according to* (2.28) *remains constant throughout the motion*

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{m\dot{\mathbf{r}}^2}{2} + U(\mathbf{r})\right) = 0. \tag{2.30}$$

Proof. Differentiating and using the chain rule on each summand in (2.30), we obtain:

$$\frac{dE}{dt} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \nabla U \cdot \mathbf{r} = (\underbrace{m\ddot{\mathbf{r}} + \nabla U}_{=0 \text{ by } (2.28)}) \cdot \mathbf{r} = 0.$$

For nonconservative fields it makes no sense to speak of potential energy, but kinetic energy is still defined. Since nonconservative fields do work on a particle executing a closed path, we expect kinetic energy of a particle to change if the particle returns to its starting point. Problem 2.13 (page 129) gives an example where the energy picked up equals the area enclosed by the particle's path.

12. Central force fields; conservation of angular momentum

Definition 2.6. A vector field which points along radial rays through the origin ("the center") and which has a constant magnitude on each sphere $r =$ const. is called *central*; see Figure 10(A). Formally, a central force field is defined by

$$\mathbf{F}(\mathbf{r}) = \lambda(r)\mathbf{r}, \tag{2.31}$$

where $\lambda : \mathbb{R}^n \to \mathbb{R}$ is any scalar function of $\mathbf{r}$.

Examples of central force fields include (2.23), (2.24) and (2.25), but not (2.26), unless $A = \lambda I$ is a multiple of an identity matrix.

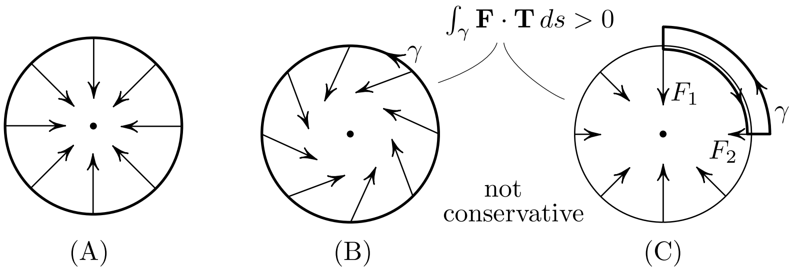

Figure 10. Force fields: (A) central, (B) nonconservative with rotational symmetry, and (C) pointing at the center but not central.

Conservativeness and rotational symmetry. Central fields have two special properties: they are conservative, *and* rotationally symmetric, meaning that $R\mathbf{F}(\mathbf{r}) = \mathbf{F}(R\mathbf{r})$ for any rotation matrix R and for any $\mathbf{r}$. The converse is also true: Any rotationally symmetric field which is conservative is a central field. A geometrical proof is explained by Figure 10(B): a field which is not central satisfies $\int_\gamma \mathbf{F}\cdot\mathbf{T}\ ds \neq 0$, and thus is not conservative. Figure 10(C) is a cautionary reminder that pointing at the center is not enough to be central: centrality also requires $|F(\mathbf{r})| = \text{const.}$ for $|\mathbf{r}| = \text{const.}$ Indeed, if this condition fails: $F_1 > F_2$, then there is a contour γ (see Figure 10(C)) for which $\oint_\gamma \mathbf{F}d\mathbf{r} > 0$, showing that $\mathbf{F}$ is nonconservative.

Conservation of the angular momentum.

Theorem 2.8. *Angular momentum of a particle moving in a central field, i.e., satisfying*

$$m\ddot{\mathbf{r}} = \lambda(r)\mathbf{r}, \quad r = |\mathbf{r}|$$

remains constant throughout the motion,

$$\mathbf{L} = m\mathbf{r}\times\dot{\mathbf{r}} = \text{const.} \tag{2.32}$$

Proof. The rotational version of Newton's law (2.10) gives

$$\frac{d}{dt}\mathbf{L} = \mathbf{T} = \mathbf{r}\times(\lambda(r)\mathbf{r}) = \mathbf{0}$$ ◇

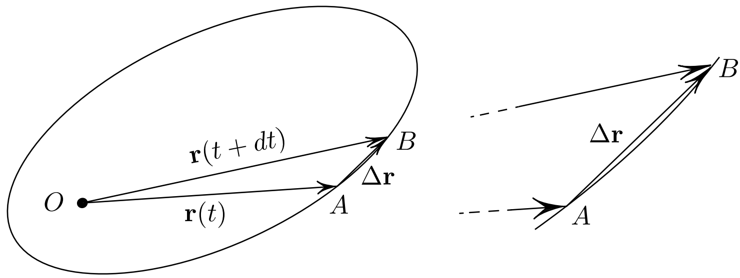

Figure 11. Kepler's law of areas is equivalent to conservation of angular momentum.

Conservation of the angular momentum has the following geometrical interpretation.

Kepler's law of equal areas: *The radius vector* $\mathbf{r}$ *of a particle moving in a central field sweeps the area at a constant rate. Equivalently,* $\mathbf{r}$ *sweeps equal areas in equal times,* Figure 11.

Proof. Kepler's law of equal areas is a consequence of $|\mathbf{L}| = \text{const.}$ To see why, let us compute an infinitesimal area ΔS of a sliver swept by $\mathbf{r}$ during the time interval $[t, t+dt]$, Figure 11. The sliver consists of the triangle OAB plus the tiny area ε of the "lens" between the chord AB and the curve (see Figure 11) so that

$$\Delta S = \frac{1}{2}|\mathbf{r}(t) \times \underbrace{(\mathbf{r}(t+dt) - \mathbf{r}(t))}_{\dot{\mathbf{r}}(t)dt+o(dt)}| + \varepsilon = \frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}|dt + o(dt) + \varepsilon;$$

here $o(dt)$ denotes a quantity small with respect to dt in the sense that $\lim_{dt\to 0}\frac{o(dt)}{dt} = 0$. Note also that $\varepsilon/dt \to 0$ as $dt \to 0$. Dividing the above by dt and taking $dt \to 0$ we then obtain

$$\frac{dS}{dt} = \frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}| \overset{(2.32)}{=} \frac{1}{2m}|\mathbf{L}| = \text{const.},$$

as claimed. ♢

13. Kepler's problem

Kepler's problem asks to describe the motion of two masses subject to mutual gravitational attraction. This section reduces the problem to a simple-looking equation (2.35) below; the reader willing to believe this reduction may wish to skip to the following section, for a remarkably short proof that Kepler's orbits are conic sections: ellipses, parabolas, or hyperbolas.

Reducing two masses to one. Consider two point masses m_1, m_2 (a planet and a star, Figure 12), subject to mutual gravitational attraction, and with no other forces. Let us choose our inertial frame with the origin at the center of mass, so that

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{0}, \quad \text{or} \quad \mathbf{r}_2 = -\frac{m_1}{m_2}\,\mathbf{r}_1, \tag{2.33}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are positions of the two masses in our frame. Such a frame is inertial since no external forces act on our system. Newton's second law for m_1 gives

$$m_1\ddot{\mathbf{r}}_1 = -Gm_1m_2\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}. \tag{2.34}$$

By (2.33), we have $\mathbf{r}_1 - \mathbf{r}_2 = \frac{m_1+m_2}{m_2}\mathbf{r}_1$. Substituting in (2.34), we get, after a brief simplification,

$$\ddot{\mathbf{r}}_1 = -k\,\frac{\mathbf{r}_1}{r_1^3}, \quad k = \frac{Gm_2^3}{(m_1+m_2)^2}.$$

It is convenient to eliminate k by setting $\mathbf{r}_1 = k^{1/3}\mathbf{r}$; the rescaled vector satisfies

$$\boxed{\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3},} \tag{2.35}$$

where $r = |\mathbf{r}|$. We thus reduced the two-body problem to one of a single particle in the central field with inverse square force.

The gist of Kepler's problem is therefore in the simple-looking equation (2.35). This equation is analyzed in the next section.

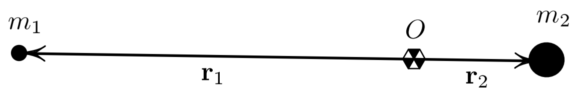

Figure 12. Reducing the two-body problem to the single particle in a central field.

14. Kepler's trajectories are conics: a short proof

The main point of this section is to give a strikingly quick and easy proof, due to Lagrange,[5] of the most famous theorem of Newton:[6]

Theorem 2.9. *Kepler's orbits, i.e., the trajectories of* (2.35), *are conics, i.e., ellipses, parabolas or hyperbolas, with the foci at the origin.*

Proof. As a preparation, let us fix any solution $\mathbf{r}(t)$ of (2.35) for the rest of this proof; assume only that the angular momentum $\mathbf{L} \neq \mathbf{0}$, since otherwise the motion is confined to a straight line through the origin and is not as interesting. The orbit of our solution lies in the plane normal to $\mathbf{L}$ and passing through the origin. Indeed, by the conservation of angular momentum (which holds since the field in (2.35) is central), we have

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{L} = \text{const.},$$

which proves that $\mathbf{r} \perp \mathbf{L}$, as claimed. Let (x, y) be Cartesian coordinates in the plane of our orbit, with the origin at the center, so that

$$\ddot{x} = -a(t)x, \quad \ddot{y} = -a(t)y, \tag{2.36}$$

where

$$a = \frac{1}{r^3}, \quad r = \sqrt{x^2(t) + y^2(t)}.$$

Since we have fixed the solution, $a = a(t)$ is a fixed function of time (rather than a function of coordinates x, y). Summarizing, both coordinates $x(t)$ and $y(t)$ satisfy the same differential equation

$$\ddot{u} = -a(t)u. \tag{2.37}$$

Now it turns out that the modified distance $w \stackrel{\text{def}}{=} r - L^2$ satisfies (2.37) as well:

$$\ddot{w} = -a(t)w; \tag{2.38}$$

[5] I am grateful to Alain Chenciner for pointing out this reference to me.

[6] To be more precise, Newton proved the converse, namely, that if the trajectories of a particle in a central force field are conics with the foci at the origin, then the force varies as the inverse square of the distance to the origin.

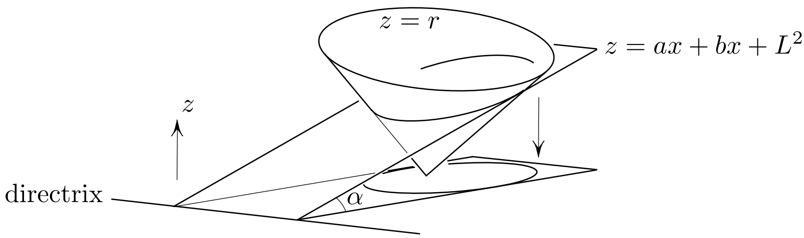

Figure 13. Showing that (2.39) describes a conic section. Eccentricity of the projected section is $e = \tan\alpha = \sqrt{a^2+b^2}$. The projection is an ellipse if $e < 1$ (as in the figure), a parabola if $e = 1$, and a hyperbola otherwise.

the proof is given shortly. But any solution of (2.37) is a linear combination of two linearly independent solutions (for the background on this, see page 99)). Now x and y are independent since $y/x \neq$ const., and thus

$$w = \alpha x + \beta y$$

for some constants α, β and for all t. Substituting $w = r - L^2$ gives

$$r = \alpha x + \beta y + L^2, \tag{2.39}$$

the equation of our trajectory! That (2.39) is a conic section is immediate from Figure 13: (2.39) is the (x, y)-projection of the intersection of the cone $z = r = \sqrt{x^2 + y^2}$ and the plane $z = \alpha x + \beta y + L^2$; and the projection of a conic section is a conic section. We now show that one of the foci of (2.39) lies at the origin, i.e., at the vertex of the cone in Figure 13. Dividing (2.39) by $e = \sqrt{\alpha^2 + \beta^2}$ results in

$$\frac{r}{e} = \frac{\alpha x + \beta y + L^2}{\sqrt{\alpha^2+\beta^2}} \stackrel{\text{def}}{=} d. \tag{2.40}$$

But d is the distance from (x, y) to the line $\alpha\overline{x} + \beta\overline{y} + L^2 = 0$ (here $(\overline{x}, \overline{y})$ denotes a point on the line), Figure 14. Now it is well known that $r/d = e$ describes a conic with the focus at the origin, with eccentricity e (see Problem 2.16 on page 130). $\diamondsuit$

Proof of (2.38). Let us place ourselves in a rotating frame with the origin at the attracting center, and with one axis passing through the

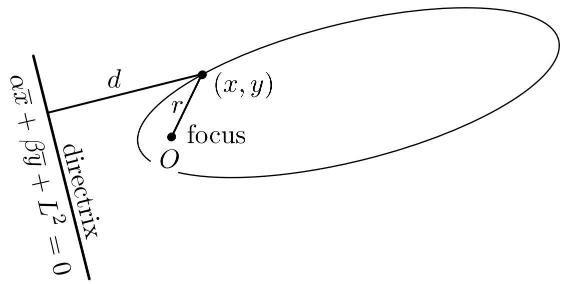

Figure 14. A conic with a focus at the origin.

planet. In the radial direction, the planet feels the force of gravitational attraction $-1/r^2$, as well as the centrifugal force $\omega^2 r$ (where ω is the instantaneous angular velocity of the radius vector):

$$\ddot{r} = -\frac{1}{r^2} + \omega^2 r. \tag{2.41}$$

But the angular momentum of the planet $L = rv_\perp = r(\omega r) = \omega r^2$, and thus

$$\omega = \frac{L}{r^2}.$$

Substituting this into (2.41) proves (2.38):

$$\underbrace{\ddot{r}}_{=\ddot{w}} = -\frac{1}{r^2} + \left(\frac{L}{r^2}\right)^2 r = -\frac{1}{r^2} + \frac{L^2}{r^3} = -\frac{1}{r^3}(\underbrace{r - L^2}_{w}).$$

$\diamondsuit$

The above proof has a gap: I should have explained why the Coriolis force does not appear (the answer: because this force has zero component in the r-direction due to the choice of the frame), and how to deal with the fact that $\omega \neq$ const.) (the fictitious force is again perpendicular to the r-direction). The reader can fill in the gap, or refer to a more formal proof given next.

An alternative proof of (2.38). This proof is more self-contained than the previous one, but is also longer and less intuitive. The magnitude of the angular momentum is given by (recall that $m = 1$ in (2.35)):

$$L = rv_\perp,$$

where $v_\perp$ is the component of velocity perpendicular to the radius-vector, so that

$$v_\perp = \frac{L}{r}. \tag{2.42}$$

The sum of kinetic and potential energies is

$$E = K + P = \frac{1}{2}(\dot{r}^2 + v_\perp^2) - \frac{1}{r} \overset{(2.42)}{=} \frac{1}{2}\left(\dot{r}^2 + \frac{L^2}{r^2}\right) - \frac{1}{r}.$$

Differentiating $E = \text{const.}$, we get

$$\left(\ddot{r} - \frac{L^2}{r^3} + \frac{1}{r^2}\right)\dot{r} = 0;$$

since $\dot{r} \neq 0$ (otherwise the orbit is a circle, which is a trivial case), we get

$$\ddot{r} - \frac{L^2}{r^3} + \frac{1}{r^2} = 0,$$

so that

$$\ddot{r} - \frac{L^2}{r^3} + \frac{1}{r^2} = 0,$$

or

$$\frac{d^2}{dt^2}(r - L^2) = -\frac{1}{r^3}(r - L^2),$$

thus proving (2.38). ♢

Remark 2.1. Figure 13 reveals a geometrical significance of the eccentricity e — namely,

$$e = \tan\alpha.$$

Note that e refers to the eccentricity of the projection (and not of the section itself).

For an even simpler geometrical interpretation of the eccentricity in the elliptical case see Problem 2.16 on page 130 and the caption of Figure 40.

Background on linear ODEs. In the above proof we used the following basic fact.

Theorem 2.10. *Consider a second order linear differential equation*

$$\ddot{w} + a(t)w = 0, \tag{2.43}$$

where $a(t)$ is a given function of time. If $x(t)$ and $y(t)$ are two linearly independent solutions[7] *of* (2.43)*, then any other solution $z(t)$ of* (2.43) *is a linear combination of these: there exist constants α and β (depending on the choice of solution z) such that*

$$z(t) = \alpha x(t) + \beta y(t) \quad \textit{for all} \quad t. \tag{2.44}$$

In other words, the space of solutions (2.43) *is a two-dimensional linear subspace in the space of twice differentiable functions on $\mathbb{R}$.*

Proof. Let $z = z(t)$ be an arbitrary solution of (2.43), and let $x(t)$ and $y(t)$ be two linearly independent solutions. For the proof, it suffices to find α and β such that

$$z(0) = \alpha x(0) + \beta y(0), \quad \dot{z}(0) = \alpha \dot{x}(0) + \beta \dot{y}(0); \tag{2.45}$$

indeed, (2.45) states that the two solutions $z(t)$ and $\alpha x(t) + \beta y(t)$ share the same initial condition, and thus (2.44) must hold by the uniqueness theorem. Now (2.45) is a linear algebraic system for the unknowns α, β; the determinant of this system is the (signed) area formed by the vectors $(x(0), \dot{x}(0))$ and $(y(0), \dot{y}(0))$. Since these vectors are not parallel by the assumption, the determinant is nonzero and thus the system has a unique solution (α, β). ♢

15. Motion in linear central fields

Having described the motion in Newtonian gravitational fields we now consider another important class of motions: those in linear central fields

$$\mathbf{F}(\mathbf{r}) = -k\mathbf{r};$$

we took the minus sign because in the most interesting examples the force is restoring, rather than repelling. Newton's equation for a point mass in such a field reads $m\ddot{\mathbf{r}} = -k\mathbf{r}$, or

$$\ddot{\mathbf{r}} = -\omega^2 \mathbf{r}, \tag{2.46}$$

where $\omega^2 = k/m$. This equation arises as a model of the spherical pendulum in a small neighborhood of its equilibrium; in that case

[7] By linear independence of two solutions we mean that the vectors $(x, \dot{x})$ and $(y, \dot{y})$ are not parallel for some t. Note that by linearity two solution vectors in the phase plane which are not parallel for some t are not parallel for all t. In particular, the two vectors are not parallel at $t = 0$; this fact will be used shortly.

$\mathbf{r} = (x, y)$ is the projection of the bob onto the horizontal (x, y)-plane, and $\omega^2 = g/L$, where L is the length of the string. Despite its simplicity, (2.46) hides a couple of surprises.

Theorem 2.11. *Trajectories of* (2.46) *are ellipses centered at the origin, Figure* 15. *Moreover, any motion under* (2.46) *is a sum of two circular motions with angular velocities* ω *and* $-\omega$*, Figure* 16(A). *Finally, when viewed in a rotating frame with angular velocity* ω*, any motion of* (2.46) *is in a circle, with constant speed, Figure* 16(B).

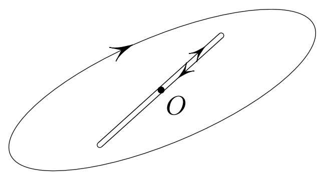

Figure 15. Trajectories of a particle in a linear central field (2.46) are ellipses centered at the origin. A collapsed ellipse is also shown.

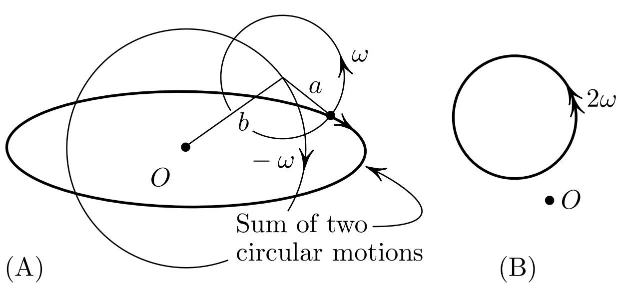

Figure 16. (A): Every motion under (2.46) is the sum of two counter-rotating circular motions. (B): In a rotating frame, the motion is circular, with constant speed, and with angular velocity 2ω. The center of the circle can be anywhere.

Proof.

1: ellipticity of the orbits. Equation (2.46) breaks up into two decoupled harmonic oscillators:

$$\ddot{x} + \omega^2 x = 0, \quad \ddot{y} + \omega^2 y = 0,$$

with solutions

$$x = a_1 \cos\omega t + b_1 \sin\omega t, \quad y = a_2 \cos\omega t + b_2 \sin\omega t. \tag{2.47}$$

One way to prove ellipticity is to rewrite this solution in vector form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix};$$

this shows that the trajectory is a linear image of the unit circle, and hence an ellipse. An alternative, less geometric approach, is to solve (2.47) for $\cos\omega t$ and $\sin\omega t$; then $\cos^2 + \sin^2 = 1$ turns into a quadratic equation in x, y. Such an equation describes a conic which in fact must be an ellipse (possibly collapsed to a segment or a point) since the curve is bounded.

2: Sum of circular motions. Using complex notation, we write $\mathbf{r} = (x, y) = x + iy$; now we can solve (2.46) in exactly the same exact way as the real counterpart $\ddot{x} = -\omega^2 x$ in a standard ODE course, by seeking solutions as exponentials. That is, we substitute the guess $\mathbf{r} = e^{\lambda t}$ into (2.46); this leads to the characteristic equation $\lambda^2 = -\omega^2$, or $\lambda = \pm i\omega$, and consequently $e^{\pm i\omega t}$ are solutions of (2.46). Thus any linear combination

$$\mathbf{r} = Ae^{i\omega t} + Be^{-i\omega t}, \tag{2.48}$$

is a solution, which proves that any motion is a sum of two circular motions. By writing $A = ae^{i\alpha}$ and $B = ae^{i\beta}$ (with real a, α, b, β), we can recast (2.48) as

$$\mathbf{r} = ae^{i(\omega t + \alpha)} + be^{-i(\omega t - \beta)}. \tag{2.49}$$

3: In the rotating frame. If the observer himself is rotating with the angular velocity $-\omega$, then the last term in (2.48) will appear constant, while the rotation $Ae^{i\omega t}$ will appear to add ω to its angular velocity. To make this more precise, consider the (X, Y)-frame rotating with angular velocity $-\omega$, i.e., clockwise, Figure 17. Any vector $\mathbf{R}$ in the rotating frame will appear to the ground observer as rotated by $-\omega t$ due to the frame's rotation. Therefore, the ground observer will see

$$\mathbf{r} = e^{-i\omega t}\mathbf{R}. \tag{2.50}$$

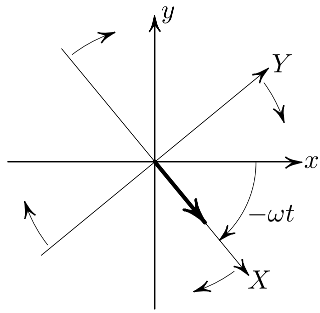

Figure 17. An inertial frame and a rotating frame.

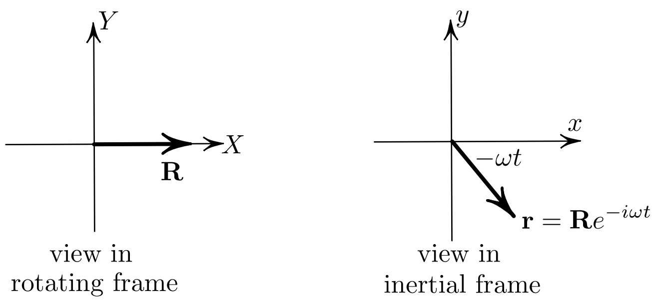

Figure 18. A particle on a rotating carousel viewed by a ground observer appears to be rotated through $-\omega t$ (the angle of the carousel's turn) as compared to the carousel observer, hence (2.50).

Substituting this into (2.48) and multiplying both sides by $e^{i\omega t}$ gives

$$\mathbf{R} = Ae^{2i\omega t} + B,$$

and proves the claim. $\diamondsuit$

Exercise 2.7. A unit point mass is subject to the a linear restoring force directed at the origin, with Hooke's constant $k = \omega^2$. Assume that the initial conditions are such that the particle oscillates along a straight line, as in Figure 15. Describe the motion of this particle as it appears to an observer sitting in a reference frame sharing the origin with the stationary frame and rotating counterclockwise with the angular velocity ω.

Answer. The apparent orbit is a circle of radius $A/2$ *passing through the origin*, where A is the amplitude of the solution in the stationary frame. The motion along this circle has constant angular velocity 2ω clockwise.

16. Linear vibrations: derivation of the equations

Linear systems of the previous section belong to a more general class:

$$M\ddot{\mathbf{x}} = -K\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{2.51}$$

where M and K are positive definite $n \times n$ matrices.[8] This section explains the reason for considering systems (2.51); their analysis is given in the next section. Note that (2.51) reminds us of the mass-spring system equation $m\ddot{x} = -kx$, and partly for that reason M is referred to as the *mass matrix*, and K as the *stiffness matrix*. Equation (2.51) describes small vibrations near an equilibrium, as explained in the next paragraph. The notation $\mathbf{x}$ in this section is used instead of $\mathbf{r}$ in the preceding sections, since in most applications $\mathbf{x}$ is not the position of one particle, but rather a collection of (generalized) coordinates of several particles. For the same reason we no longer restrict to the case $n = 2$ as we did in the preceding sections.

To explain how (2.51) arises, consider a mechanical system with the Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}(\mathcal{M}(\mathbf{q})\dot{\mathbf{q}}, \dot{\mathbf{q}}) - \mathcal{P}(\mathbf{q}), \tag{2.52}$$

where $\mathcal{M}$ is a positive definite matrix whose entries may depend on $\mathbf{q}$.[9] Let $\mathbf{q} = \mathbf{q}_0$ be an equilibrium solution of the Euler–Lagrange equation, or equivalently, that $\nabla\mathcal{P}(\mathbf{q}_0) = \mathbf{0}$.

Let $\mathbf{q}(t)$ be a near-equilibrium solution of Euler–Lagrange equation, and let $\mathbf{x}$ be its deflection from the equilibrium:

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{x}.$$

[8]Recall that a *positive definite matrix* A is, by the definition, a symmetric matrix for which the dot product $(A\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Equivalently, it is a symmetric matrix all of whose eigenvalues are positive.

[9]Kinetic energy of any mechanical system must be of such form; see Problem 2.30 on page 135 for the proof.

We now show *that the motions with both* $\mathbf{x}$, $\dot{\mathbf{x}}$ *small satisfy* (2.51), *where the matrices* M *and* K *are given by*

$$M = \mathcal{M}(\mathbf{q}_0) \quad \text{and} \quad K = \mathcal{P}''(\mathbf{q}_0), \tag{2.53}$$

where $\mathcal{P}''$ is the Hessian matrix of $\mathcal{P}$. To express the energies in terms of $\mathbf{x}$, we substitute $\mathbf{q} = \mathbf{q}_0 + \mathbf{x}$:

$$\mathcal{M}(\mathbf{q}) = \underbrace{\mathcal{M}(\mathbf{q}_0)}_{M} + O(|\mathbf{x}|), \quad P(\mathbf{q}) = P(\mathbf{q}_0) + \frac{1}{2}(\underbrace{P''(\mathbf{q}_0)}_{K}\mathbf{x}, \mathbf{x}) + O(|\mathbf{x}|^3).$$

Substituting into (2.52), we get

$$\mathcal{L} = \frac{1}{2}(M\dot{\mathbf{x}}, \dot{\mathbf{x}}) - \frac{1}{2}(K\mathbf{x}, \mathbf{x}) + \underbrace{\mathcal{P}(\mathbf{q}_0)}_{\text{constant}} + O_3,$$

where $O_3 = O(|\mathbf{x}|^3 + |\dot{\mathbf{x}}|^3)$. Now the constant term $\mathcal{P}(\mathbf{q}_0)$ does not contribute to the Euler–Lagrange equations; discarding it, as well as the cubic terms, we obtain the quadratic Lagrangian

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}(M\dot{\mathbf{x}}, \dot{\mathbf{x}}) - \frac{1}{2}(K\mathbf{x}, \mathbf{x}).$$

Now (2.51) is the Euler–Lagrange equation of this Lagrangian.[10]

Summarizing, we proved the following.

Theorem 2.12. *Linearization of the Euler–Lagrange equations with the Lagrangian* (2.52) *around an equilibrium solution* $\mathbf{q}(t) = \mathbf{q}_0$ *is given by* (2.51), *where* $M = \mathcal{M}(\mathbf{q}_0)$ *and* $K = \mathcal{P}''(\mathbf{q}_0)$ *is the Hessian matrix of* $\mathcal{U}$.

17. A nonholonomic system

In this section I make a small digression. All of the systems discussed so far involve no constraints on velocities besides those imposed by constraints on positions. For instance, a particle constrained to a curve has no constraint on the velocity besides the condition of tangency to the curve. Such systems are referred to as *holonomic*. There exist, however, systems where the constraints on the velocities are not derivable from the constraints on the coordinates. An example of such a system is a wheel rolling on the plane without sliding: indeed,

[10] This requires the identity $\nabla_{\mathbf{x}}(A\mathbf{x}, \mathbf{x}) = 2A\mathbf{x}$; see Problem 2.31 on page 135 for the proof.

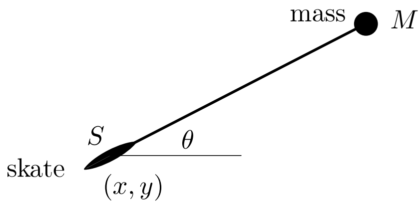

Figure 19. Chaplygin's sleigh: a skate S with a mass M attached by a weightless rod. The mass slides on ice without friction. The skate can only slide in the direction of the rod.

the no-slip condition slaves the angular velocity to the translational speed, i.e., imposes a constraint on the generalized velocity. On the other hand, there is no constraint on the position of the wheel: it can be rolled so as to touch any point of the plane with any point of the wheel, and with any orientation.[11] An even simpler example of a nonholonomic system is the so-called Chaplygin's sleigh described in this section.

The Chaplygin sleigh consists of a point mass M attached to the massless skate S by a rod, lying on a horizontal sheet of ice; Figure 19 shows the top view.[12] The skate S can only move in the direction SM, but not sideways. The mass M can slide in any direction; there is no friction. Note that there is no constraint on the position of the segment, but there is a constraint on its velocity. To express this constraint, let (x, y) be the position of S and let θ be the angle between SM and the x-axis in the plane. The configuration space of our system consists of all triples $(x, y, \theta \mod 2\pi)$. The nonskidding constraint is $(\dot{x}, \dot{y}) \parallel (\cos\theta, \sin\theta)$, i.e.,

$$-\sin\theta\, dx + \cos\theta\, dy = 0. \tag{2.54}$$

This is a constraint on the velocity $(\dot{x}, \dot{y}, \dot{\theta})$ which therefore has 2 independent variables. One could say that the Chaplygin sleigh has 2.5 degrees of freedom, since there are three independent coordinates but only two independent velocities.

[11] Proving that this is so is left as a challenge.

[12] Presumably, "sleigh" is meant to suggest a horse M attached to a sleigh S; in the present problem the "horse" simply slides in a straight line.

Geometrically, the velocity constraint (2.54) defines a tangent plane — that is, the plane of allowed infinitesimal displacements — at every point (x, y, θ) in $\mathbb{R}^3$, Figure 20. Such a field of planes is called a *distribution* of planes.

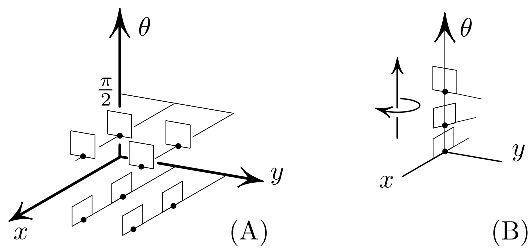

Figure 20. (A) The distribution of planes (2.54), and (B) a close-up near a vertical line.

Definition 2.7. A distribution is said to be nonintegrable if there is no surface all of whose tangent planes belong to the distribution.

Exercise 2.8. Show that the distribution of planes given by (2.54) is nonintegrable.

Solution. Assume for a moment that there exists an integral surface S in the (x, y, θ)-space, and let $P_0 = (x_0, y_0, \theta_0) \in S$. The set of all $P(x, y, \theta)$ reachable from P_0 by paths respecting the distribution is therefore two-dimensional. But one can slide the sleigh from (x_0, y_0, θ_0) to *any* (x, y, θ), as is easy to check directly; this is a contradiction.

An alternative proof of the nonintegrability is sketched in Figure 21: Assume, for a moment, that there exists an integral surface S through a; the normal to this surface is shown in the figure. But there exists a path $abcde$ tangent to the distribution which can reach a point $e \neq a$ on the normal, as the figure illustrates. Now the path $abcde$ can be made arbitrarily small. Hence the normal to S at a intersects S at another point arbitrarily close to a — a contradiction. For a more extensive treatment of the subject, including Frobenius's criterion of nonintegrability using differential forms, see, e.g., [**1**].

Exercise 2.9. Can exponential decay occur in a frictionless mechanical system for *any* initial condition?

Hint. The angle θ of the Chaplygin sleigh approaches a constant exponentially, both in the future time and in the past.

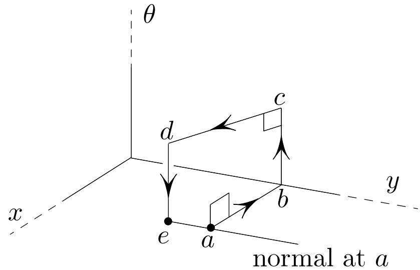

Figure 21. Proving the nonintegrability of the distribution (6.23).

18. The modal decomposition of vibrations

We now return to small vibrations $M\ddot{\mathbf{x}} = -K\mathbf{x}$ introduced in the section before last. Let us start with the special case of $M = I$:

$$\ddot{\mathbf{x}} = -K\mathbf{x}. \tag{2.55}$$

The case of general $M > 0$ reduces to this one and is discussed later. We assume throughout that K is a symmetric positive definite matrix or equivalently, that K has an orthogonal basis of eigenvectors with positive eigenvalues. A quick intuitive picture of the problem, described in the next paragraph, leads to the formal solution in the paragraph that follows.

An intuitive picture. It helps to think of (2.55) as Newton's second law for the mass $m = 1$ in the force field $\mathbf{F}(\mathbf{x}) = -K\mathbf{x}$. Such a force field is sketched in Figure 22 in the planar case, but the idea applies to any dimension with obvious changes.

The key is to focus on the special directions $\mathbf{v}$, Figure 22, where the force points directly into the origin: $K\mathbf{v} \parallel \mathbf{v}$, i.e.,

$$K\mathbf{v} = \lambda\mathbf{v} \quad \text{for some} \quad \lambda \in \mathbb{R}. \tag{2.56}$$

We thus were led to the eigenvectors ($\mathbf{v}$) and the eigenvalues λ of K. Note that (2.56) looks like Hooke's law, with λ playing the role of Hooke's constant. The familiar scalar mass-spring systems along the eigendirections are embedded in our problem! These give rise to harmonic (i.e., sinusoidal) oscillations along the eigendirections as special motions, called the *normal modes* of vibration; the general

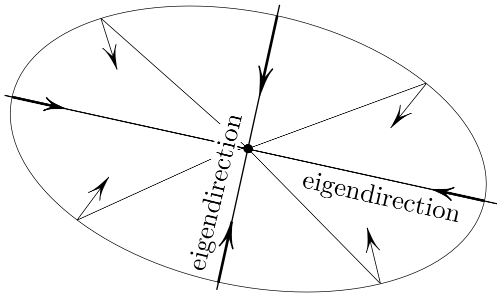

Figure 22. Equation (2.55) describes a particle in the force field $\mathbf{F} = -K\mathbf{x}$. The ellipse is an equipotential line $U\mathbf{x}) = \frac{1}{2}(A\mathbf{x}, \mathbf{x}) = \text{const.}$; the forces are normal to it.

solution will be shown to be the sum of normal modes, due to the superposition principle[13]. In the next paragraph all this is made formal.

A formal solution of (2.55). According to the preceding paragraph we first seek special solutions of the form

$$\mathbf{x}(t) = a(t)\mathbf{v}, \tag{2.57}$$

where $\mathbf{v}$ satisfies (2.56) and $a(t)$ is a scalar function to be found (we expect something sinusoidal). To find $a(t)$, substitute (2.57) into (2.55):

$$\ddot{a}(t)\mathbf{v} = -K(a(t)\mathbf{v}) \overset{(2.56)}{=} -\lambda a(t)\mathbf{v}.$$

This holds iff

$$\ddot{a} = -\lambda a,$$

i.e.,

$$a(t) = A\cos\omega t + B\sin\omega t, \quad \omega = \sqrt{\lambda},$$

where A and B are arbitrary constants. Note that ω is real, since $\lambda > 0$ thanks to the assumption $K > 0$. Summarizing,

$$\mathbf{x}(t) = (A\cos\omega t + B\sin\omega t)\mathbf{v} \tag{2.58}$$

is a solution of (2.55) if $\omega^2 = \lambda$ and $\mathbf{v}$ is an eigenvalue-eigenvector pair of K. Solutions (2.58) are called the *normal modes.*

[13]According to which any linear combination of solutions is a solution. This principle applies since our ODE (2.55) is linear.

Theorem 2.13. *Any solution of* (2.55) *with* $K > 0$ *is the sum of normal modes:*

$$\sum_{k=1}^{n} (A_k \cos \omega_k t + B_k \sin \omega_k t)\mathbf{v}_k, \tag{2.59}$$

with some constants A_k, B_k*; here* $\{\mathbf{v}_k\}_{k=1}^n$ *is an eigenvector basis of* K *and* $\omega_k^2 = \sqrt{\lambda_k}$ *is the eigenvalue corresponding to* $\mathbf{v}_k$*, Figure* 23*.*

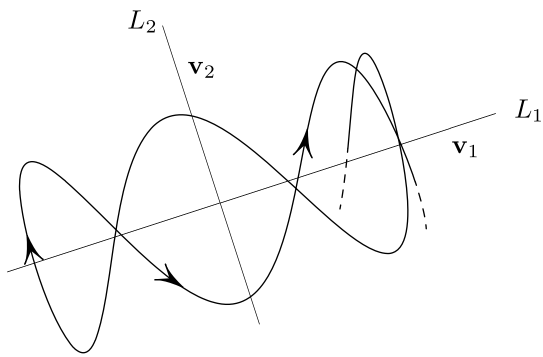

Figure 23. The planar case ($n = 2$): any trajectory of $\ddot{\mathbf{x}} = -K\mathbf{x}$ is a combination of two harmonic oscillations along L_1 and L_2.

Proof. We already showed that the summands in (2.59) are solutions, and so the sum (2.59) is also a solution by the superposition principle (which applies since our system is linear). It remains to show that no other solutions exist, i.e., that an arbitrary solution $\mathbf{x}(t)$ of (2.55) can be written as (2.59) with a proper choice of A_k, B_k. If we can choose these constants so as to match the initial data, i.e., so that

$$\begin{aligned} \mathbf{x}(0) &= \textstyle\sum_{k=1}^n A_k \mathbf{v}_k, \\ \dot{\mathbf{x}}(0) &= \textstyle\sum_{k=1}^n \omega_k B_k \mathbf{v}_k, \end{aligned} \tag{2.60}$$

then we can conclude that (2.59) holds by the uniqueness theorem for ODEs, thus completing the proof. It thus remains to show that A_k, B_k in (2.60) exist for any choice of $\mathbf{x}(0)$, $\dot{\mathbf{x}}(0)$. Now any vector is a linear combination of the vectors $\mathbf{v}_k$ since these form a basis of $\mathbb{R}^n$. In particular, $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ are linear combinations of $\mathbf{v}_k$. This completes the proof. $\diamondsuit$

Exercise 2.10. Find the coefficients A_k, B_k explicitly.

Hint. Take the dot product of both sides in (2.60) with v_k and use orthogonality of the basis. If $\mathbf{v}_k$ are unit vectors, then A_k is the projection of the initial position onto $\mathbf{v}_k$, and $\omega_k B_k$ is the projection onto $\mathbf{v}_k$ of the initial velocity.

19. Lissajous' figures and Chebyshev's polynomials

Having solved (2.55) analytically, we now describe the solutions geometrically, in the planar case: $\mathbf{x} \in \mathbb{R}^2$. The trajectories (2.59) are referred to as the *Lissajous figures.* Figure 24 shows examples for various rational frequency ratios $\rho = \omega_2/\omega_1$. Figure 25 illustrates that for an irrational ρ Lissajous' figure fills a rectangle densely, i.e., it eventually passes through an arbitrarily small neighborhood of any point in the rectangle.

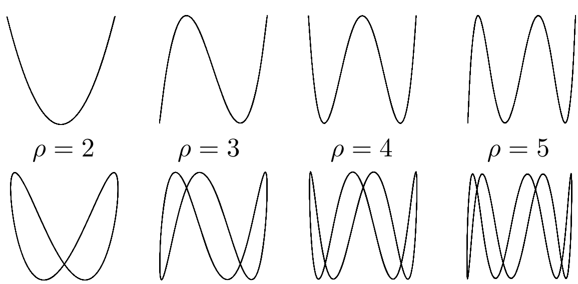

Figure 24. Lissauous figures (2.59) for different rational ratios $\rho = \omega_2/\omega_1$. The top four curves (corresponding to $B_1 = B_2 = 0$) are the graphs of Chebyshev's polynomials of degrees $2, 3, 4, 5$.

Let us choose the eigenvectors of K as the coordinate axes; the solution then is of the form

$$x = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t, \quad y = A_2 \cos \omega_2 t + B_2 \sin \omega_2 t. \tag{2.61}$$

How to turn a straight line into a Lissajous figure. Interestingly, any Lissajous figure comes from a straight line subjected to two

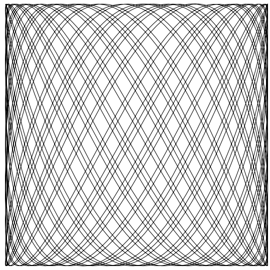

Figure 25. A Lissauous figure (2.59) for $\rho = \sqrt{3.2}$, $0 \le t \le 50$.

"roll and project" operations which I now describe. We start with an arbitrary straight line drawn on a sheet of transparency.

(1) Roll the transparency into a cylinder; the line becomes a helix.

(2) Project the cylinder onto the flat sheet parallel to the cylinder's axis; the helix projects onto a sinusoid.

(3) Roll up the flat sheet into a cylinder, rolling in the direction of the sinusoid's axis so that this axis becomes an equator of the cylinder.

(4) Project the new cylinder onto the sheet parallel to the cylinder's axis; the sinusoid's projection is a Lissajous figure!

In short, two "roll and project" operations convert a straight line into a Lissajous figure. During its metamorphosis, the straight line first becomes a helix, then a sinusoid and finally a Lissajous figure.

Remark. Compare the curves in any of the columns in Figure 24; you may notice that both curves come from a sinusoid on a cylinder by projection on the plane of the paper. Moreover, one curve changes into the other if the cylinder is turned around on its axis by an appropriate angle.

Chebyshev's polynomials. Consider a special case of Lissajous figures:

$$x = \cos t, \; y = \cos mt, \tag{2.62}$$

where m is an integer. Any solution of the equation $\ddot{\mathbf{x}} = -K\mathbf{x}$ in $\mathbb{R}^2$ with $\dot{\mathbf{x}} = 0$ reduces to this form under rescaling, provided the frequency ratio $\rho = m$ is an integer.

It turns out that (2.62) defines y as a polynomial function of x: $y = T_m(x)$, called the Chebyshev polynomial.[14] Figure 24 shows graphs of $T_m(x)$ for $m = 2, 3, 4, 5$.

Theorem 2.14. $y = \cos mt$ is a polynomial in $x = \cos t$ of degree m (with coefficients independent of t), Figure 24.

Proof. By Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ we have

$$y = \cos mt = \mathrm{Re}(e^{imt}) = \mathrm{Re}((e^{it})^m). \tag{2.63}$$

Since $t = \cos^{-1} x$, we have

$$e^{it} = \cos(\cos^{-1} x) + i\sin(\cos^{-1} x) = x + i\sqrt{1-x^2}.$$

Substituting this into (2.63) gives

$$y = \mathrm{Re}\,(x + i\sqrt{1-x^2})^m = \mathrm{Re}\sum_{k=0}^{m}\binom{m}{k} x^{m-k}(i\sqrt{1-x^2})^k.$$

But only even k contribute to the real part, and these real terms are polynomial thanks to the evenness of k. This completes the proof. $\diamondsuit$

Exercise 2.11. Find $T_m(x)$ for $m = 0, 1, 2, 3, 4$.

Answer. $T_0 = 1$, $T_1 = x$, $T_2 = 2x^2 - 1$, $T_3 = 4x^3 - 3x$, $T_4 = 8x^4 - 8x^2 + 1$. For a general recurrence relation for Chebyshev's polynomials, see Problem 2.32 on page 136.

20. Invariant 2-tori in $\mathbb{R}^4$

In studying a simple harmonic oscillator $\ddot{x} = -\omega^2 x$ it was very fruitful to consider its phase space $\mathbb{R}^2$; we saw that the phase space is foliated by (topological) circles, namely the energy ellipses $\dot{x}^2 + \omega^2 x^2 = \text{const}$. What is the corresponding picture for (2.55) with $\mathbf{x} \in \mathbb{R}^2$? Phase

[14] The French and German spelling "Tchebysheff" explains the use of T in T_m.

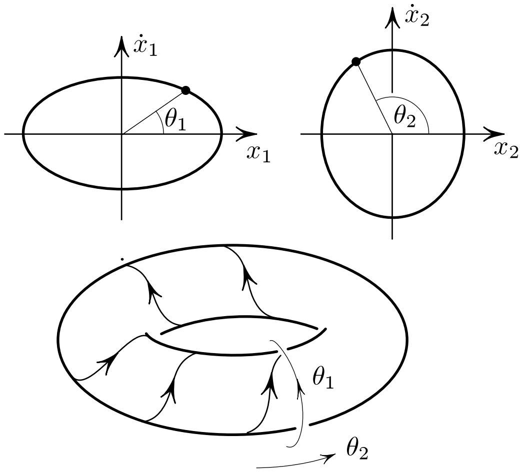

Figure 26. Two angles $\theta_1,\ \theta_2$ define a point on the 2-torus (2.66).

space for this system is $\mathbb{R}^4$, since (2.55) is equivalent to a system of first order ODEs in $\mathbb{R}^4$:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{y}, \\ \dot{\mathbf{y}} = -K\mathbf{x}, \end{cases} \quad \mathbf{x},\ \ \mathbf{y} \in \mathbb{R}^2. \tag{2.64}$$

Assume that K is diagonal; no generality is lost since the assumption amounts to choosing the coordinate axes in the $\mathbf{x}$-plane along the eigenvectors of K. The system (2.55) then decouples into two harmonic oscillators:

$$\begin{cases} \ddot{x}_1 = -\omega_1^2 x_1, \\ \ddot{x}_2 = -\omega_2^2 x_2. \end{cases} \tag{2.65}$$

For each of these, the energy is conserved:

$$y_1^2 + \omega_1^2 x_1^2 = 2E_1, \quad y_2^2 + \omega_2^2 x_2^2 = 2E_2, \tag{2.66}$$

where $y_i = \dot{x}_i$ and where $E_1,\ E_2$ are constants depending on the initial condition, Figure 26. For a fixed pair $E_1,\ E_2$, both nonzero, (2.66) defines a 2-torus in $\mathbb{R}^4 = \{(x_1, \dot{x}_1, x_2, \dot{x}_2)\}$. Indeed, referring to Figure 26, two angular variables $\theta_1,\ \theta_2$ specify the point $(x_1, \dot{x}_1, x_2, \dot{x}_2)$ in (2.66) uniquely; these angular variables can be thought of as the coordinates on the 2-torus, as Figure 26 illustrates. Now in the exceptional cases $E_1 = 0,\ E_2 > 0$ or $E_1 > 0,\ E_2 = 0$ the torus collapses to a circle, and in the totally degenerate case $E_1 = E_2 = 0$ the energy set (2.66) is an equilibrium point. We conclude: The entire space $\mathbb{R}^4$ is the union of 2-tori (corresponding to parameters $E_1 > 0,\ E_2 > 0$), two families of circles and one point. Because of the conservation of energies (2.66), the *tori are invariant under the flow of the system* (2.64).

The invariant tori in the 3-sphere. Let us now fix the total energy: $E_1 + E_2 = E > 0$, i.e., consider all solutions of (2.55) with

$$y_1^2 + \omega_1^2 x_1^2 + y_2^2 + \omega_2^2 x_2^2 = E.$$

This set is, topologically speaking, a 3-sphere $\mathbb{S}^3 \in \mathbb{R}^4$; and, as we saw, this sphere is a union of 2-tori (2.66) with $E_{1,2} > 0$ and of two circles corresponding to $(E_1, E_2) = (E, 0)$ or $(0, E)$, as illustrated in Figure 27. In this figure the sphere $\mathbb{S}^3$ is identified with $\mathbb{R}^3 \cup \{\infty\}$. In particular, the vertical line C_2 in the figure corresponds to an invariant circle. The circles C_1 and C_2 in Figure 27 correspond to the normal modes, where all the energy is concentrated in one mode.

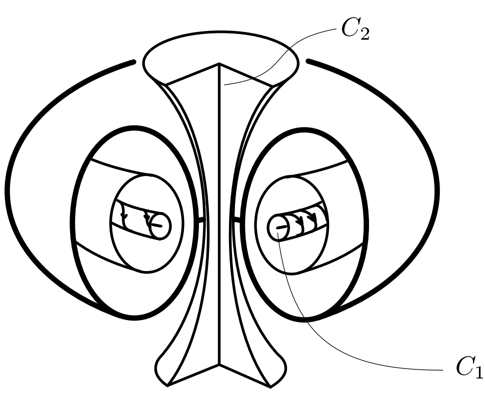

Figure 27. Invariant tori on the energy sphere $\mathbb{S}^3$ (represented in the figure as $\mathbb{R}^3$ with the point at infinity). The two circles C_1, C_2 correspond to the two normal modes.

The Hopf fibration. Continuing from the preceding paragraph, consider now the special case of equal frequencies $\omega_1 = \omega_2 = 1$ in (2.65); that is, we have the system of two identical decoupled harmonic oscillators. Let us fix the total energy $2E = 1$; all solutions with such energy lie on the sphere

$$\mathbb{S}^3 = \{x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.$$

Since all solutions are periodic, we conclude that $\mathbb{S}^3$ is a union of circles, parametrized by the time t:

$$x_1 + iy_1 = re^{it},\ x_2 + iy_2 = \sqrt{1-r^2}e^{i(t-\varphi)},\ 0 \le r \le 1, \tag{2.67}$$

where r and φ are constants. One such circle is sketched in Figure 28(a); if we keep r fixed and change φ, the circle will move around the torus, as shown in Figure 28(b); as φ changes by 2π, this circle sweeps the entire torus. We see that each torus in Figure 27 is a union of circles. The entire $\mathbb{S}^3$ is thus a union of circles; this union is called the *Hopf fibration*.

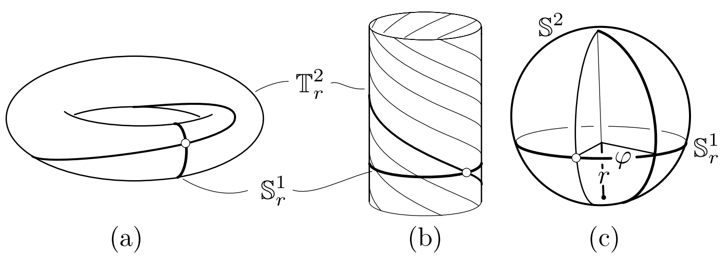

Figure 28. Each torus in Figure 27 is a union of circles. The degenerate tori — namely, the circles C_1 and C_2 from Figure 27 — map to the poles of $\mathbb{S}^2$.

Theorem 2.15. *There exists a continuous map $p : \mathbb{S}^3 \to \mathbb{S}^2$ (onto) which maps each Hopf circle* (2.67) *to a point on $\mathbb{S}^2$. Moreover, different circles map to different points.*

Nevertheless, $\mathbb{S}^3$ is *not* a direct product of $\mathbb{S}^2$ and $\mathbb{S}^1$; that is, there does not exist a continuous one-to-one correspondence between points in $\mathbb{S}^3$ on the one hand and the pairs of points from $\mathbb{S}^2$ and $\mathbb{S}^1$ on the other. There is some interesting topology behind the simple mechanical system of two harmonic oscillators.

Proof of the theorem. Any point on $\mathbb{S}^3$ can be represented by (2.67); the map

$$p := (re^{it}, \sqrt{1-r^2}e^{i(t-\varphi)}) \mapsto (R\cos\varphi, R\sin\varphi, r), \quad R = \sqrt{r(1-r)}$$

indeed satisfies the conditions of the theorem. Although this completes the proof, I would like to give the motivation for this choice of p. A typical torus from Figure 27, given by

$$z_1 = re^{i\theta_1}, \quad z_2 = \sqrt{1-r^2}e^{i\theta_2}, \quad 0 < r < 1$$

(where θ_1, θ_2 are the parameters and r defines the torus), is shown in Figure 28(a). This torus, which we denote by $\mathbb{T}^2_r$, is foliated by circles (2.67), each represented by its own value of $\varphi \bmod 2\pi$. Each Hopf circle is therefore represented by a point on the meridian of $\mathbb{T}^2_r$, as shown in Figure 28(a), (b). In short, we have a mapping from $\mathbb{T}^2_r$ to the meridian circle $\mathbb{S}^1_r$, and this map collapses Hopf circles to points. Now for each r we have such a circle $\mathbb{S}^1_r$; stacked on top of each other, with r as their height, they form a sphere (Figure 28(c)), once we note that $r = 0$ and $r = 1$ each corresponds to a point, one giving the south pole and the other giving the north pole. ♢

21. Rayleigh's quotient and a physical interpretation

The eigenvalues and eigenvectors play the key role in understanding the dynamics of $\ddot{\mathbf{x}} = -K\mathbf{x}$. In this section I give a variational description of the eigenvalues and the eigenvectors. In fact, this description can be explained physically to the mutual benefit of linear algebra and mechanics.

Theorem 2.16. *For any symmetric $n \times n$ matrix K with real entries, the critical points of the quadratic form $(K\mathbf{x}, \mathbf{x})$ restricted to the unit sphere $|\mathbf{x}| = 1$ are the eigenvectors of K, and the critical values are the corresponding eigenvalues. In particular,*

$$\lambda_{\min} = \min_{|\mathbf{x}|=1} (K\mathbf{x}, \mathbf{x}), \quad \lambda_{\max} = \max_{|\mathbf{x}|=1} (K\mathbf{x}, \mathbf{x}) \tag{2.68}$$

and

$$A\mathbf{x}_{\min} = \lambda_{\min}\mathbf{x}_{\min}, \quad A\mathbf{x}_{\max} = \lambda_{\max}\mathbf{x}_{\max}. \tag{2.69}$$

Relations (2.68) *can equivalently be written as*

$$\lambda_{\min} = \min_{\mathbf{y}\neq\mathbf{0}} \frac{(K\mathbf{y},\mathbf{y})}{(\mathbf{y},\mathbf{y})}, \quad \lambda_{\max} = \max_{\mathbf{y}\neq\mathbf{0}} \frac{(K\mathbf{y},\mathbf{y})}{(\mathbf{y},\mathbf{y})}. \tag{2.70}$$

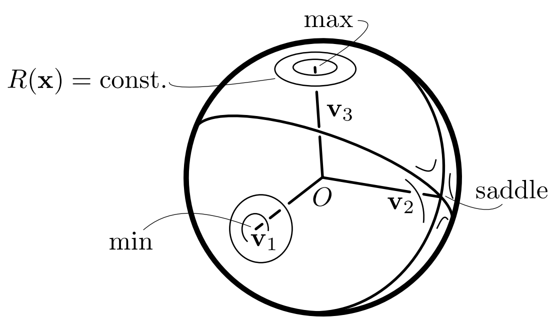

Figure 29. Level curves of Rayleigh's quotient $R = (K\mathbf{x},\mathbf{x})$, $|\mathbf{x}| = 1$. Critical values of R are the eigenvalues and critical points are the eigenvectors of K.

Proof. Recall the method of Lagrange multipliers: If $\mathbf{v}$ is a critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ subject to a constraint $g(\mathbf{x}) =$ const., then the gradients are parallel at $\mathbf{x} = \mathbf{v}$, i.e.,

$$\nabla f(\mathbf{v}) = \lambda \nabla g(\mathbf{v}) \tag{2.71}$$

for some $\lambda \in \mathbb{R}$; the converse holds as well. We conclude: $\mathbf{v}$ is a critical point of $f(\mathbf{x}) = (K\mathbf{x},\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = (\mathbf{x},\mathbf{x}) = 1$ iff for some λ

$$\nabla(K\mathbf{x},\mathbf{x}) = \lambda\nabla(\mathbf{x},\mathbf{x}) \quad \text{at} \quad \mathbf{x} = \mathbf{v},$$

i.e., $K\mathbf{v} = \lambda\mathbf{v}$. Moreover, the corresponding critical value $(K\mathbf{v},\mathbf{v}) = (\lambda\mathbf{v},\mathbf{v}) = \lambda(\mathbf{v},\mathbf{v}) = \lambda$. ♢

A physical interpretation of Theorem 2.16. Consider a point mass constrained to the unit sphere and subject to the force field $\mathbf{F}(\mathbf{x}) = -K\mathbf{x}$, Figure 30. The potential energy of this particle is

$$U(\mathbf{x}) = \frac{1}{2}(K\mathbf{x},\mathbf{x}), \tag{2.72}$$

as explained in the next paragraph; critical points of the restriction of U to $|\mathbf{x}| = 1$ are the equilibria of our point mass. At such an equilibrium the sum of forces on the particle vanishes:

$$-K\mathbf{v} + \lambda\mathbf{v} = 0;$$

here the second term is the normal reaction force of the constraint. We conclude that critical points of the restriction of U to $|\mathbf{x}| = 1$ are the eigenvectors of K, precisely as Rayleigh's criterion states.

To justify (2.72) let us drag the particle from $\mathbf{0}$ to $\mathbf{x}$ along a straight line; this takes average force $\mathbf{F}_{\text{avg}} = \frac{1}{2}(\mathbf{0} + K\mathbf{x}) = \frac{1}{2}K\mathbf{x}$. The work done is the dot product $(\mathbf{F}_{\text{avg}}, \mathbf{x}) = \frac{1}{2}(K\mathbf{x}, \mathbf{x})$.

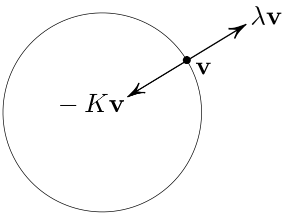

Figure 30. A physical explanation of Rayleigh's principle.

22. The Coriolis and the centrifugal forces

In this section we place ourselves in a rotating frame.[15] Newton's second law is not invariant under the change from inertial to noninertial frame. In other words, when expressing Newton's second law in the coordinates of a noninertial frame, extra terms appear in addition to $\mathbf{F}$. For a rotating frame these extra forces are the Coriolis and the centrifugal forces. To an observer sitting in a rotating frame and unaware of the frame's rotation these terms are perceived as actual forces. These fictitious forces are also referred to as inertial forces, since they are manifestations of inertia. To find the expressions for these forces we must relate the coordinates of the rotating frame to those of the inertial frame.

[15] I consider only the frames in two dimensions, rotating with a constant angular velocity, and omit the discussion of angular or linear acceleration of the frame. A more complete discussion of these topics is available in many texts, e.g., in [**9**].

Transformation to a rotating frame. Figure 31 shows a platform rotating counterclockwise with a constant angular velocity ω. A coordinate system (X, Y) drawn on the platform coincides with a stationary coordinate system (x, y) at $t = 0$. Given a point in the

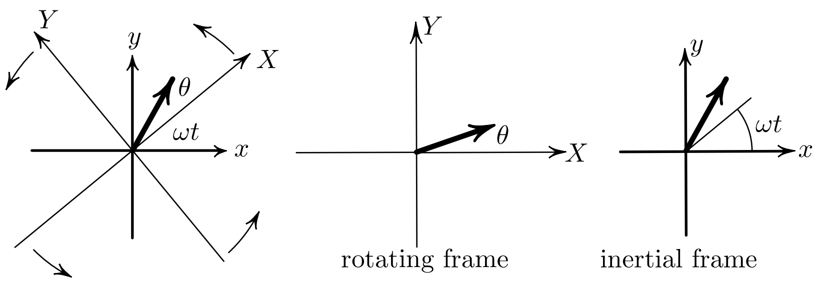

Figure 31. An explanation of (2.73).

plane, denote its coordinates in the two frames by $Z = (X, Y)$ and $z = (x, y)$. I claim that

$$z = e^{i\omega t} Z, \tag{2.73}$$

where complex notation is used: $Z = (X, Y) \equiv X + iY$, $z = x + iy$.[16] Indeed, as the last two sketches in Figure 31 explain, the inertial observer sees the particle rotated through the angle ωt, as compared to the rotating observer.

Using (2.73) we will now rewrite Newton's law

$$m\ddot{z} = \mathbf{F} \tag{2.74}$$

for a point mass subject to force $\mathbf{F}$ in the rotating frame, thus discovering the Coriolis and the centrifugal forces.

Before substituting $z = e^{i\omega t} Z$ into (2.74), we differentiate twice and collect terms:

$$\ddot{z} = e^{i\omega t}(\ddot{Z} + 2i\omega \dot{Z} - \omega^2 Z).$$

[16]Recall that there is nothing "imaginary" about the complex number $x + iy$: it is just the point (x, y) in the plane. The imaginary unit i, in particular, is just the point $(0, 1)$ on the y-axis. Recall also that multiplication of a vector by $e^{i\theta}$ rotates the vector by θ around the origin.

Now (2.74) turns (no pun intended) into

$$m\ddot{Z} = \underbrace{e^{-i\omega t}\mathbf{F}}_{\text{actual}} \underbrace{-2im\omega\dot{Z}}_{\text{Coriolis}} + \underbrace{m\omega^2 Z}_{\text{centrifugal}} . \tag{2.75}$$

Note that if $\mathbf{F}$ depends on the position, then in the last formula it must be expressed in terms of Z.

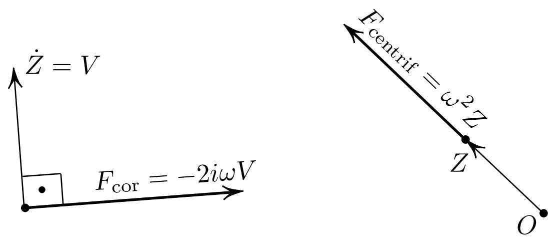

Figure 32. The Coriolis and the centrifugal forces acting on a moving particle in a counterclockwise rotating frame ($\omega > 0$).

To illustrate this result, a free particle ($\mathbf{F} = \mathbf{0}$) in a rotating frame satisfies

$$m\ddot{Z} = \underbrace{-2i\omega\dot{Z}}_{\text{Coriolis}} + \underbrace{\omega^2 Z}_{\text{centrifugal}}$$

after dividing both sides by m. An observer in this frame will have an illusion of two forces acting on the particle.

Remark 2.2. As Figure 32 illustrates, the Coriolis force *is perpendicular to the velocity of the particle*, in the same way as magnetic force on a charge is perpendicular to the charge's velocity. The Coriolis force, just as the magnetic force, does zero work since it is perpendicular to the velocity.

Remark 2.3. The centrifugal force points away from the center, Figure 32. In fact, "centrifugal" means "fleeing from the center", from Latin "effugere" (to flee), similar to "fugitive".

Exercise 2.12. Write the governing equation for the particle subject to a linear restoring force: $\ddot{z} = -\omega^2 z$ (let us take $m = 1$) in the frame rotating with angular velocity ω.

Solution. Equation (2.75) gives the answer once we note that the term $e^{-i\omega t}\mathbf{F} = e^{-i\omega t}(-\omega^2 z) = -\omega^2 Z$ cancels the centrifugal term in (2.75); the Coriolis term remains:

$$\ddot{Z} = -2i\omega \dot{Z}.$$

The motion of Z is circular, with constant angular velocity -2ω, as explained on page 102, and as is easy to see directly, since the general solution is given by

$$Z = A + Be^{-2i\omega t},$$

where A and B are arbitrary complex numbers. A is then the center of the circular orbit and $|B|$ is the radius.

23. Miscellaneous examples

This section describes three observations that I found interesting or amusing.

1. Channels through a planet. Imagine various channels drilled through a planet, as in Figure 33 (we assume that the planet is a solid ball of constant density, and that it does not rotate). Particles can slide inside each channel without friction. We place a particle in each of the channels at A, and release them. The particles will oscillate back and forth along each channel.

Suprisingly, all the periods of oscillations are the same. Moreover, even if a particle is released at an interior point E on any chord, then the period of oscillation will still be the same, Figure 33. In addition, a satellite skimming the surface of the planet has the same period of revolution.[17]

Proof/explanation. Gravitational field inside a homogeneous ball with constant density is a linear central field given by $-k\mathbf{r}$ (with a constant k depending on the density); proof of this statement uses the result of Problem 2.39, according to which a spherical shell creates zero gravitational field inside; the details are left to the reader.

[17]Under some idealizing assumptions including the absence of friction with the air and the perfect sphericity of the planet.

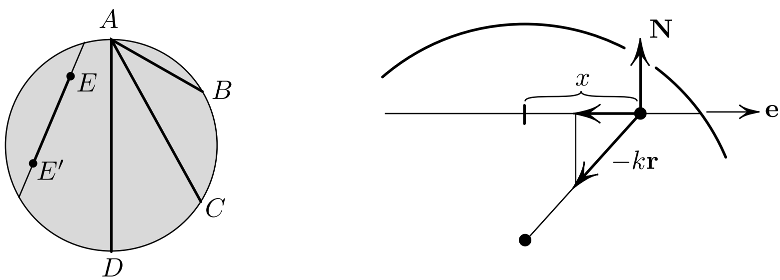

Figure 33. Periods of oscillations in the channels $T_{AB} = T_{AC} = T_{AD} = T_{EE'}$. Moreover, these periods are equal to the orbiting period of a satellite skimming the planet's surface!

We conclude that the particle constrained to the chord is subject to gravity and the reaction from the constraint:

$$\mathbf{F} = -k\mathbf{r} + \mathbf{N}.$$

Let us project this force onto the unit direction vector $\mathbf{e}$ of the chord by taking the dot product with $\mathbf{e}$:

$$F_{\text{chord}} = \mathbf{F} \cdot \mathbf{e} = (-k\mathbf{r} + \mathbf{N}) \cdot \mathbf{e} = -\lambda\mathbf{r} \cdot \mathbf{e} = -kx;$$

here $x = \mathbf{r} \cdot \mathbf{e}$ is the distance to the midpoint of the chord. By Newton's second law we have $m\ddot{x} = -kx$, a harmonic oscillator, whose period $2\pi\sqrt{m/k}$ depends neither on the choice of the chord nor on the amplitude. It remains to explain why the satellite's period of revolution is the same. To that end, note that the force on the satellite $\mathbf{F} = -k\mathbf{r}$ is still given by the same formula, since we assume that $\mathbf{r}$ is negligibly close to the surface of the planet. Therefore for the satellite we have $m\ddot{\mathbf{r}} = -k\mathbf{r}$ and the period of motion is the same as for the particles in channels. ♢

2. Gravitational force = visible size. According to the next theorem, the gravitational pull at O from a mass distribution on a spherical cap in Figure 34 depends on the visible (from O) size of the cap, but not on the cap's actual size! More precisely, we have the following.

Theorem 2.17. *Consider two concentric spherical caps subtending the same cone with the vertex at the shared center O of the spheres,*

Figure 34. Each cap carries a uniform mass distribution of the same area density (i.e., of the same mass per unit area in both caps). Under these conditions, the caps exert the same gravitational pull upon a particle at O.

Proof. Consider two patches subtending the same small solid angle, Figure 34. Let r, R be the radii of the two caps, and let dm, dM be the masses of the patches. Denoting by df, dF the gravitational pulls exerted by these patches upon a unit mass at O, we obtain, treating the patches as point masses:

$$dF = G\frac{dM}{R^2} \qquad df = G\frac{dm}{r^2}.$$

But $dM/dm = (R/r)^2$, i.e., $dM/R^2 = dm/r^2$. We conclude that $dF = df$, which implies that $F = f$. $\diamondsuit$

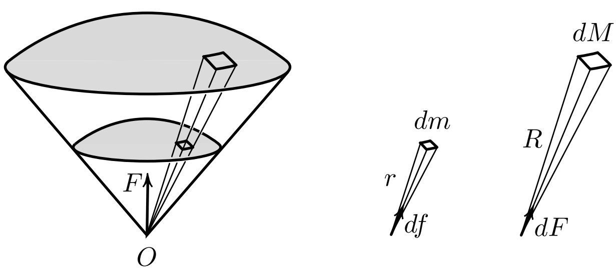

Figure 34. The force of attraction by a spherical cap of fixed thickness depends only on the solid angle at which it is seen.

3. A pendulum on a Hookean spring. The reader may have seen this toy: a heavy rubber ball the size of a ping-pong is attached to a long, lax rubber band. My son could swing the ball back and forth in a horizontal path with swing of about 15–20 feet, as shown in Figure 35. The ball skimmed above the floor, never touching it, in a path that appeared nearly straight, and almost touching the opposite walls of a large room. Let us show that if the spring is a zero length Hookean spring then in fact a straight horizontal path is possible (assuming no friction, etc.).

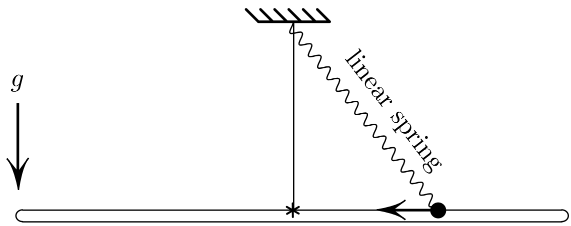

Figure 35. Mass suspended on a linear spring can oscillate in a horizontal straight line!

Consider a mass suspended by a spring whose force of tension is in direct proportion to its length: $T = kL$; note that such a spring has zero length when relaxed.[18] Restoring force of the spring is thus $-k\mathbf{r}$, provided we choose the origin $\mathbf{r} = \mathbf{0}$ at the suspension point, Figure 35. According to Newton's second law,

$$m\ddot{\mathbf{r}} = -k\mathbf{r} + m\mathbf{g},$$

where $\mathbf{g}$ is the vector of gravitational acceleration.

Dividing by m and abbreviating $k/m = \lambda$, we get

$$\ddot{\mathbf{r}} = -\lambda\Big(\mathbf{r} - k^{-1}m\mathbf{g}\Big).$$

Introducing $\mathbf{x} = \mathbf{r} + k^{-1}m\mathbf{g}$, we get

$$\ddot{\mathbf{x}} = -\lambda\mathbf{x};$$

note that $\mathbf{x}$ is the displacement from the equilibrium $-k^{-1}m\mathbf{g}$.

All trajectories of this system are ellipses, as we proved earlier; see page 100. Among these ellipses are the collapsed ones, i.e., segments, as in Figure 35. One of these segments approximates the trajectory of the ball in Eric's toy.

Remark 2.4. Although the assumption of zero relaxed length of a spring may seem unrealistic, it works reasonably well for a *pre-stressed* spring; such a spring satisfies $T = kL$ reasonably well, as long as $L > L_0$, where L_0 is the shortest length. Many springs, e.g.,

[18]Such an idealized spring is not too bad an approximation of the rubber band in Eric's toy, since the relaxed length of the band was small compared to its operating length.

the ones for garage doors, are manufactured in a pre-stressed state so that they are approximately linear for a range of L.

24. Problems

Center of mass.

2.1. Find the center of mass of a cubical box with the top removed. The box is made of uniform thin material.

Human dynamics.

The following five problems help us learn mechanics from our bodies.

2.2. A person walking on a beam, when losing balance, will bend the body so as to lean into the direction of the fall. Why?

2.3. A person standing on a rail and facing the direction perpendicular to the railroad track begins to fall forward, and, wishing to recover balance, begins to rotate his arms. Which way, and why?

2.4. Sometimes when a person stumbles, he begins to run forward, in some priceless cases running for 15 feet or more before either recovering, or falling (this can be quite embarrassing or entertaining, depending on who runs). Can you explain the reason for running?

2.5. A person slips on ice, with the feet going forward. How and why does he bend the body to minimize the impact?

2.6. A biker, going fast in a straight line, wishes to make a quick right to follow the sudden turn of the road. Describe what he does with the handlebars to accomplish this.

Answer. Written backwards: ...thgir ekib eht nael ot tfel leehw eht nrut ylfeirB

Minimization; equilibria.

2.7. Several springs are tied together at a point $A(x, y)$, Figure 36; the other ends of springs are attached to given points $A_i(x_i, y_i)$, $i = 1, \dots, n$ in the plane.

1. Consider the potential energy $U(x, y)$ of the system. Prove that an equilibrium position $A(x_e, y_e)$ is a critical point of $U(x, y)$, whether or not the springs are linear.

2. Assume now that each spring is a linear zero length spring with Hooke's constant k_i for the ith spring. Prove that the equilibrium position (x_e, y_e) of A coincides with the center of mass of the collection of particles

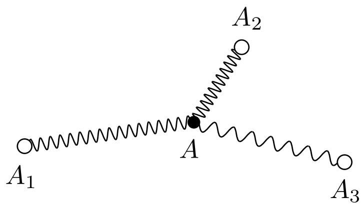

Figure 36. Zero length springs tied together at A.

placed at the points A_i, $i = 1, \ldots, n$ and endowed with masses $m_i = k_i$, $i = 1, \ldots, n$. Is the equilibrium position unique?

3. Let $I(x, y)$ be the sum of moments of inertia relative to the point (x, y) of the collection of masses $m_i = k_i$ positioned at the points A_i. Consider also the potential energy $U(x, y)$ of the system of springs in Figure 36, as defined above. Show that

$$I(x, y) = 2U(x, y). \tag{2.76}$$

In other words, *the moment of inertia can be interpreted as the potential energy of a set of linear zero length springs.*

2.8. The parallel axes theorem states: Let I_A denote the moment of inertia of a collection of masses relative to point A. Then

$$I_A = m(A_cA)^2 + I_{A_c}, \tag{2.77}$$

where A_c is the center of mass of the collection of masses. Interpret the parallel axes theorem as a statement on potential energy of the set of springs.

Solution (and an alternative proof of (2.77)). Note two facts: (i) the equilibrium A_c of A (Figure 36) coincides with the center of mass of the collection of masses $m_i = k_i$, and (ii) the effective Hooke's constant of the collection of springs is $k = \sum k_i = \sum m_i$. Thus, the work it takes to move the point (x, y) from the equilibrium A_c to A equals $\frac{1}{2}k(A_cA)^2$. This work goes into the increase of potential energy of the system:

$$\frac{1}{2}k(A_cA)^2 = U(A) - U(A_c).$$

But this is exactly our claim (2.77), due to (2.76)!

2.9. Figure 37 shows a mass connected by Hookean springs with zero relaxed lengths to fixed points A_1, A_2. There is no gravity. If pulled to the left and released, the mass will execute oscillations in the line A_1A_2. If pulled straight up instead, the mass will oscillate up–down. Which oscillation will have the higher frequency?

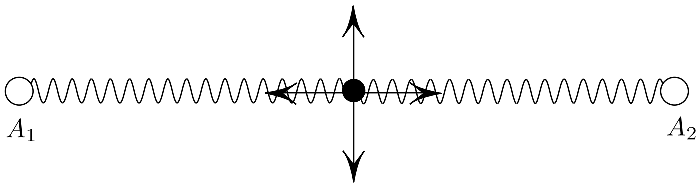

Figure 37. These are zero length springs : tension $T = kL$, where L is the spring's length. Which of the two oscillations has higher frequency?

Answer. The two frequencies are the same. Indeed, the force $\mathbf{F}$ upon the mass located at $\mathbf{x} \in \mathbb{R}^2$ from the ith spring is $-k_i(\mathbf{x} - \mathbf{x}_i)$, where $\mathbf{x}_i$ is the position vector of the point of attachment A_i. The total force is

$$\mathbf{F}(\mathbf{x}) = -\sum k_i(\mathbf{x} - \mathbf{x}_i) = -k(\mathbf{x} - \mathbf{x}_0),$$

where $k = \Sigma k_i$ and $\mathbf{x}_0 = k^{-1}\sum(k_i\mathbf{x}_i)$ (the equilibrium position). We conclude: The resultant force $\mathbf{F}$ is of the same form as the force created by a single linear spring attached at the equlibrium $\mathbf{x}_0$. The proof applies verbatim to any number of springs. In short, any collection of linear springs is equivalent to a single spring whose Hooke's constant is equal to the sum of Hooke's constants.

Vector fields.

2.10. Consider a linear vector field $\mathbf{F}(\mathbf{r}) = A\mathbf{r}$, where A is an $n \times n$ antisymmetric matrix: $A^T = -A$. Show that

(1) $\mathbf{F}(\mathbf{r}) \perp \mathbf{r}$, i.e., the force always acts at the right angle to the radius-vector.

(2) If $n = 2$, show that the work done by $\mathbf{F}$ around a closed path is in direct linear proportion to the area of the path.

(3) If $n = 3$, show that the work done by $\mathbf{F}$ around a closed path in $\mathbb{R}^3$ is in direct linear proportion to the area of the projection of the path onto the plane perpendicular to the zero eigenvalue of A.

2.11. Consider the motion of a particle in the central field with the force $F(r) = \frac{1}{r^a}$. Are there values of a for which a particle with a *nonzero angular momentum* can collide with the center in finite time, executing infinitely many turns around the center in the process?

2.12. Find the expression for the force $F(r)$ of the central force field with the following strange property: all circular orbits have the same angular momentum.

Answer. $F(r) = \frac{k}{r^3}$.

2.13. Consider the motion of a unit point mass in the plane in the force field $\mathbf{F} = -\nabla U(\mathbf{r}) + (y, -x)$:

$$\begin{cases} \ddot{x} = \ \ y - U_x(x, y), \\ \ddot{y} = -x - U_y(x, y), \end{cases} \tag{2.78}$$

where $U(x, y)$ is a smooth function. Show that *every periodic trajectory of* (2.78) *encloses zero area*, where the area A enclosed by C is counted with a sign in the sense of Figure 38, i.e., $A = \oint_C x dy$.

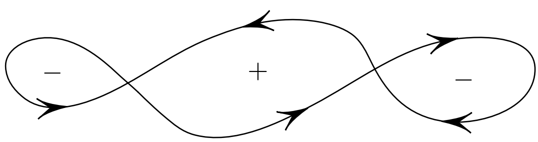

Figure 38. Area enclosed by the trajectory is counted with a sign, as shown. Any periodic trajectory of (2.78) encloses zero area.

Celestial mechanics.

2.14. Two gravitational masses m_1 and m_2 revolve around the common center of mass in circles. Show that the period of their revolution ("the year") depends only on the total mass $m_1 + m_2$ and the distance.

2.15. An asteroid comes from infinity as shown in Figure 39 with speed v_∞ far away from Earth. What is the smallest safe size of the distance d? The radius of Earth is R, and the gravitational acceleration on the surface of Earth is g. Nothing else is given.

Solution. To find the distance r_c of closest approach to the center of the Earth, we first note that at the point of closest approach (assuming the comet misses the surface) the angular momentum is the same as it was initially:

$$r_c v_c = d v_\infty. \tag{2.79}$$

The energy at the closest approach is the same as it was at infinity:

$$\frac{v_c^2}{2} - \frac{k}{r_c} = \frac{v_\infty^2}{2}, \quad k = gR^2,$$

where we divided by the mass of the asteroid. Solving for v_c and substituting into (2.79) we obtain

$$d = R\sqrt{1 + \frac{2k}{r_c v_\infty^2}} = R\sqrt{1 + \frac{2gR^2}{r_c v_\infty^2}}.$$

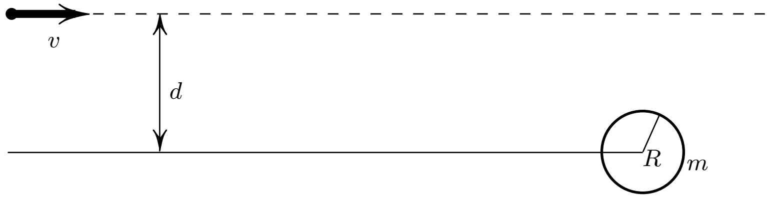

Figure 39. Will the asteroid hit the Earth?

The smallest safe value of d is the one for which the asteroid just grazes the surface of the Earth: $r_c = R$. Substituting this into the expression for d we get

$$d_{\text{critical}} = R\sqrt{1 + \frac{2gR}{v_\infty^2}}.$$

Here is an interesting note: the term inside the square root:

$$\frac{2gR}{v_\infty^2} = \frac{gR}{\frac{v_\infty^2}{2}} = \frac{P.E.}{K.E.},$$

where $P.E.$ is the potential energy of a unit mass on the Earth's surface (the work required to remove it to infinity), while of $K.E.$ is the kinetic energy of the mass at infinity!

The following two problems add to our discussion of Kepler's problem.

2.16. Let r and d be the distances from a point $P(x, y)$ to a given point F and a given line, as shown in Figure 14. Show that the set of all points P satisfying

$$r/d = e, \tag{2.80}$$

where e is a given constant is a conic with a focus at F. The given line is called the *directrix* of the conic.

Proof for the case of $0 < e < 1$ is outlined in Figure 40. Let us position the plane of the alleged ellipse as shown in Figure 40; namely, with the directrix lying in the horizontal plane and at the angle θ to the horizontal, where $\sin\theta = e$.

From amongst all spheres tangent to the tilted plane at F choose the one whose equator lies in the horizontal plane. Now construct the vertical cylinder sharing the equator with this sphere. The intersection of the slanted plane with the cylinder is an ellipse.[19] Now for any P on this ellipse (2.80) holds, as the figure explains (note that $PF = PQ$ as two tangent

[19] Although we take this fact for granted, it is easy to prove geometrically by adding another sphere inside the cylinder, tangent to the tilted plane; the tangency point F_1 is another focus of the ellipse. I omit the details.

segments from a point P tangent to the sphere); if we move P off of the ellipse, then (2.80) fails (proof is left as an exercise). This shows that (2.80) defines an ellipse.

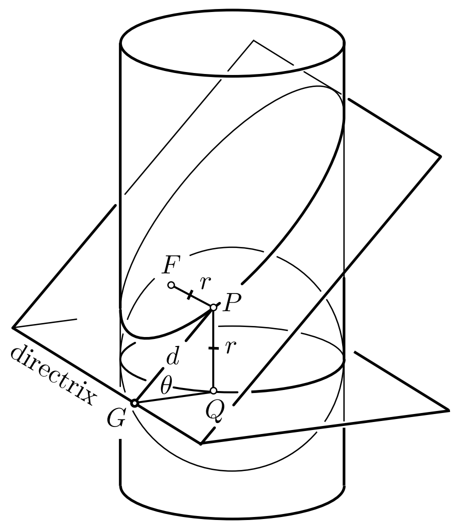

Figure 40. Proof of the facts that (i) section of a cylinder by a plane is an ellipse and (ii) equation of the ellipse is given by $r/d = e$, where $e = \sin\theta$.

Rotation.

2.17. A regulator shown in Figure 41(A) is rotating with a prescribed angular velocity ω. (1) Find the angle θ. (2) Assume that, instead of rotating with a prescribed ω, the regulator rotates without friction around the vertical axis. Given the angular momentum L (instead of ω), find θ.

2.18. A pipe bent as shown in Figure 41(B) is rotating with a prescribed angular velocity ω. Find the distance of the mass m from the bend. Also, find this distance if the angular momentum L is prescribed instead of ω. The masses are assumed to be in equilibrium in the rotating frame.

2.19. A mass on a perfectly slippery table is launched so as to travel in a circle, as shown in Figure 41(C). Find the radius of the circle, given the angular velocity of the mass. Also, write the differential equation for this distance for an arbitrary initial condition.

The following two problems offer small surprises.

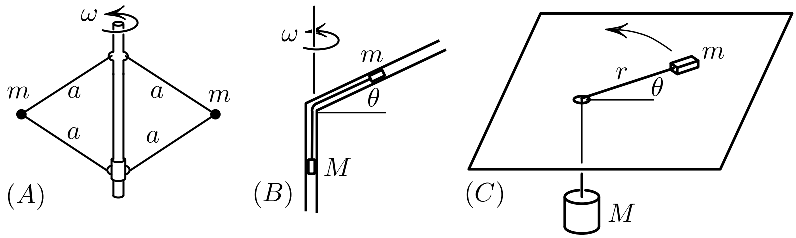

Figure 41. Towards Problems 2.17, 2.18 and 2.19.

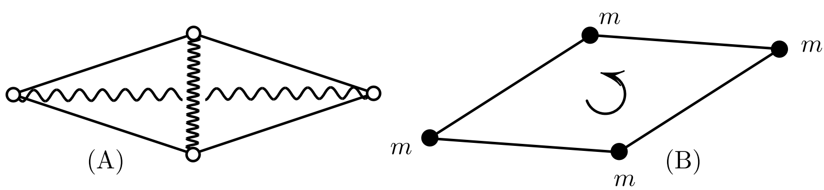

Figure 42. (A): What are all the equilibrium shapes of the rhombus? (B): What shape of the rotating rhombus is preferred?

2.20. A rhombus in Figure 42(A) is made out of four weightless sticks connected by frictionless hinges. The opposite vertices are connected by linear springs (that is, the tension of the spring varies as its length) of equal Hooke's constants. What are the equilibrium shapes of this mechanical system?

2.21. A rhombus in Figure 42(B) is made of weightless sticks connected by frictionless hinges, with four equal masses placed at each vertex. The rhombus is then spun in its plane around its center. Find all the shapes which will remain unchanged while rotating. Clearly, the square is such a shape, as well as the rhombus collapsed to a segment. Are there any others?

The following problem answers the question: Can one build a harmonic oscillator without springs, using only rods and hinges?

2.22. Describe all possible motions of the rhombus in Figure 42(B). Assume that the diagonally opposite masses can pass through each other or, essentially equivalently, undergo perfectly elastic collisions.

Solution. The length x of a diagonal behaves as a harmonic oscillator: $\ddot{x} = -k$, as we now show. We have $x^2 + y^2 = L^2$, where x and y are half-diagonals of the rhombus and where L is the length of each rod. Moreover,

$\dot{x}^2 + \dot{y}^2 = \text{const.} \stackrel{\text{def}}{=} v^2$ (if the rhombus is not rotating this is just conservation of energy. If it is rotating, angular momentum conservation must be used, but we leave the proof as an exercise.). Now let's interpret x and y instead as the coordinates of a point in the plane. The first relationship says that (x, y) lies on a circle; the second relationship says that (x, y) moves with constant speed v! Hence, up to a shift in time, $x = L \cos \omega t$, $y = L \sin \omega t$, where $\omega = v/L$. Both masses execute harmonic oscillation — without any springs!

2.23. Two identical coins spin in weightlessness with the same angular velocity, one around its diameter and the other in its own plane. What is the ratio of kinetic energies of these coins?

2.24. Figure 43 shows a cone rolling on the horizontal plane without sliding. What is the greatest angular velocity of the contact line OC for which the vertex of the cone will not try to lift off the plane? The cone's center of mass is at the distance D from the vertex; The cone's altitude is H and the radius of its base is R.

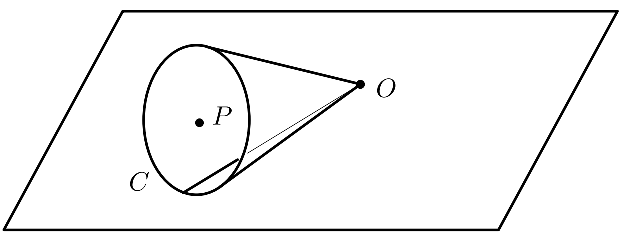

Figure 43. How fast can the cone roll without lifting off?

2.25. Resolve the following paradox. A free particle is at rest in the ground frame. When viewed in the inertial frame, the particle is subject to the centrifugal force. Why doesn't the particle then fly away from the origin?

Solution. One should not overlook the Coriolis force. To a rotating observer, the particle travels in a circle, and the centrifugal force is precisely balanced with the Coriolis force.

2.26. Find the trajectory of a free particle launched with speed v from the origin of a rotating frame, from the point of view of an observer in that frame. The angular velocity of the frame $\omega = \text{const.}$

2.27. Find the magnitude of the Coriolis force acting on a passenger in an airliner flying over the North Pole. The airliner speed is $250m/s$, and the passenger's mass is $70kg$. The answer might surprise you.

Answer. Written backwards: .retaw fo ssalg a flah tuoba fo thgiew etT

2.28. The weight of the airplane resting on the ground on the equator is 300 tons. The plane takes off and heads east along the equator, at the speed of $250m/s$. By how much does the plane's weight change due to its motion? The answer may surprise you.

.sregnessap 02 tuoba fo ro onihr a fo thgiew ehT

Modeling.

2.29. Show that the mass-spring system in Figure 44 is described by (2.51) with

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}, \tag{2.81}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the vector of displacements from the equilibrium.

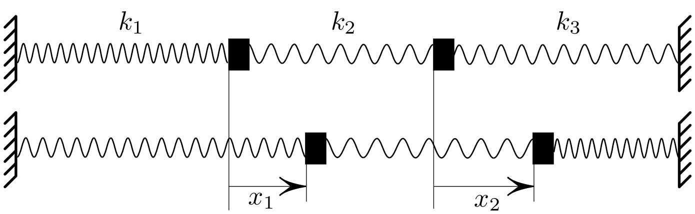

Figure 44. The masses are constrained to a line; the springs satisfy the linear Hooke's law; there is no friction. x_1 and x_2 are the displacements from the equilibrium.

Solution. Let us displace the two masses from the equilibrium by amounts x_1, x_2, as shown. As the result, the springs change lengths as follows:

$$\Delta L_1 = x_1, \quad \Delta L_2 = -x_1 + x_2, \quad \Delta L_3 = -x_2.$$

Therefore, the force upon the mass m_1 changes from zero at equilibrium to

$$F_1 = -k_1 \Delta L_1 - k_2 \Delta L_2 = -(k_1 + k_2)x_1 + k_2 x_2$$

in the displaced state. Similarly, the force upon m_2 caused by displacements $x_1, \ x_2$ is

$$F_2 = -k_2 \Delta L_2 - k_3 \Delta L_3 = k_2 x_1 - (k_2 + k_3)x_2.$$

Newton's law $m_i \ddot{x}_i = F_i$ $(i = 1, 2)$ then reads

$$\begin{cases} m_1 \ddot{x}_1 = -(k_1 + k_2)\, x_1 + k_2 x_2, \\ m_2 \ddot{x}_2 = \quad k_2 x_1 - (k_2 + k_3) x_2. \end{cases}$$

2.30. Show that the kinetic energy of any mechanical system is quadratic in generalized velocities, and that the matrix of the quadratic form is positive definite.

Solution. The kinetic energy is the sum of kinetic energies $m_i\dot{\mathbf{q}}_i^2/2$ of the constituent particles, where $\mathbf{q}_i$ is the position vector of the particle in $\mathbb{R}^n$. Since each $\mathbf{q}_i$ is a function of generalized coordinates $\mathbf{x}$, i.e., $\mathbf{q}_i = \mathbf{f}_i(\mathbf{x})$, we obtain

$$\dot{\mathbf{q}}_i = \mathbf{f}_i'(\mathbf{x})\dot{\mathbf{x}},$$

where $\mathbf{f}_i'$ is the Jacobi matrix of $\mathbf{f}_i$.[20] We conclude that $\dot{\mathbf{q}}_i^2$ is a quadratic form in terms of $\dot{x}_j$, and the same therefore holds for the total kinetic energy. But any quadratic form in $\dot{x}_i$ can be written as

$$\mathcal{K} = \frac{1}{2}(M(\mathbf{x})\dot{\mathbf{x}}, \dot{\mathbf{x}}),$$

where $M(\mathbf{x})$ is a square matrix depending on $\mathbf{x}$. Positive definiteness of $M(\mathbf{x})$ follows from the fact that $\mathcal{K} > 0$ if anything moves, i.e. if $\dot{\mathbf{x}} \neq \mathbf{0}$. $\diamondsuit$

To write the Euler–Lagrange equation for the Lagrangian

$$\frac{1}{2}(M\dot{\mathbf{x}}, \mathbf{x}) - \frac{1}{2}(K\mathbf{x}, \mathbf{x}),$$

we had to take gradients of quadratic forms. The following problem addresses this question.

2.31. Show that if A is a symmetric matrix, then $\nabla_{\mathbf{x}}(A\mathbf{x}, \mathbf{x}) = 2A\mathbf{x}$.

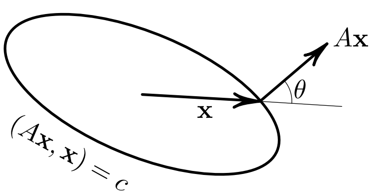

Figure 45. Gradient of the quadratic form $\frac{1}{2}(A\mathbf{x}, \mathbf{x})$ defines the vector field $A\mathbf{x}$. Matrix A is positive iff $|\theta| < \pi/2$.

Solution. Recall that the gradient $\nabla f(\mathbf{x})$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as the vector which satisfies

$$\frac{d}{ds} f(\mathbf{x} + s\mathbf{u})_{s=0} = (\nabla f(\mathbf{x}), \mathbf{u}), \quad \forall \mathbf{u} \in \mathbb{R}^n, \quad \mathbf{u} \neq 0. \tag{2.82}$$

---

[20] As a brief reminder, if $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, then $\mathbf{f}_{\mathbf{x}}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$; note that the row vectors are the gradients of f_k.

Substituting $f(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$ into the left-hand side of this definition gives

$$\frac{d}{ds}(A(\mathbf{x} + s\mathbf{u}), (\mathbf{x} + s\mathbf{u}))_{s=0} = (A\mathbf{u}, \mathbf{x}) + (A\mathbf{x}, \mathbf{u}),$$

and since $(A\mathbf{u}, \mathbf{x}) = (\mathbf{u}, A^T\mathbf{x}) = (\mathbf{u}, A\mathbf{x})$, this simplifies to $2(A\mathbf{x}, \mathbf{u})$. Comparison with the right-hand side of the definition (2.82) gives

$$2(A\mathbf{x}, \mathbf{u}) = (\nabla f(\mathbf{x}), \mathbf{u}),$$

which proves the claim since $\mathbf{u}$ is arbitrary. ◇

2.32. Prove that the Chebyshev polynomials are related by the recursion relation $T_{m+1} = 2xT_m - T_{m-1}$.

Miscellaneous.

The following problem shows that Newtonian mechanics is not deterministic (full information about the present does not determine the future).

2.33. Consider the particle on the line subject to the force with the potential $U(x) = -x^{4/3}$. Show that the equilibrium solution $x(t) \equiv 0$ is not unique, in the sense that the particle in such a potential can sit at $x = 0$, and then start moving spontaneously at an arbitrarily prescribed time. In other words, the Newtonian world is not deterministic if the force field is not sufficiently smooth (note that $F'(0) = -U''(0)$ is undefined).

2.34. A bead slides on a straight line which passes through the origin and rotates in the plane with a constant angular velocity ω. There is no friction between the bead and the line, and no forces other than the constraint force from the line act on the bead. Derive the equation of motion of the bead in terms of s, the bead's distance to the origin.

2.35. A bead slides without friction on the surface of a bowl, subject to gravity acting in the direction of the negative the z-axis. The bowl is a smooth surface of revolution $z = f(x^2 + y^2)$.

1. Write the differential equation for the coordinates (x, y) of the bead.

2. Does there exist a smooth surface of revolution such that the bead starting with initial conditions $x(0) = \dot{y}(0) = 1, \dot{x}(0) = y(0) = 0$ will cross the z-axis? (See Figure 46.)

2.36. Prove the following remarkable fact: If a pendulum rotates with a given angular velocity around the vertical, Figure 47, then the pendulum's height does not depend on the length L of the string (as long as L is greater than a certain critical value). In other words, referring to the figure, H does not depend on L.

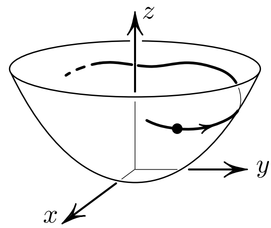

Figure 46. A bead sliding in a bowl without friction.

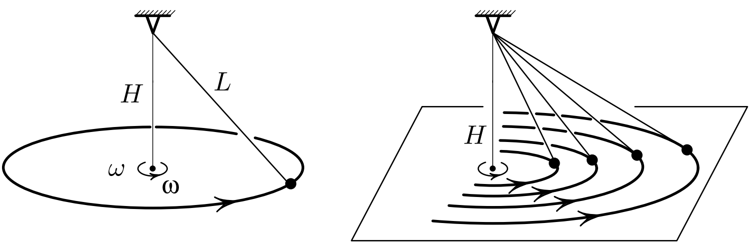

Figure 47. With a given ω, all these masses are at the same height!

Gravitational fields.

2.37. According to the law of gravitational attraction ($\mathbf{F} = -k\frac{\mathbf{r}}{r^3}$, $r = |\mathbf{r}|$), if $r \to 0$, then $F = |\mathbf{F}| \to \infty$. When our feet touch the ground, the distance is near-zero. Why then don't we get stuck to the ground?

2.38. Let $\mathbf{F}(\mathbf{r})$ be a vector field created by a continuous mass distribution, with a continuous mass density $\rho = \rho(\mathbf{r})$. Show that $\operatorname{div}\mathbf{F}(\mathbf{r}) = k\rho(\mathbf{r})$, where k is a constant.

2.39. Show that the gravitational field inside a homogeneous spherical shell is zero (Newton).

Proof #1. Consider a concentric sphere S inside the shell, Figure 48. Divergence theorem applied in S gives:

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dA = \int\int\int_B \operatorname{div}\mathbf{F}\,dV = 0,$$

where B is the ball bounded by the sphere S. Here we used the fact that $\operatorname{div}\mathbf{F} = 0$ in a vacuum. By symmetry, $\mathbf{F}$ must be radial, and thus $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = F$. Thus the integral is just $4\pi r^2 F = 0$, and thus $F = 0$. $\diamondsuit$

Proof #2, not using the divergence theorem. Let P be any point inside the sphere, Figure 49. A narrow cone through P cuts out two small patches

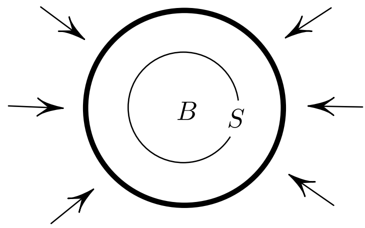

Figure 48. Proving that the field vanishes inside a hollow sphere. Gravitational field is nonzero outside the shell and is zero inside.

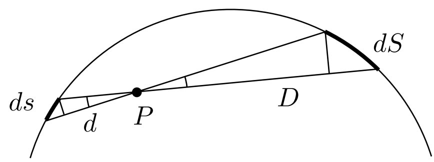

Figure 49. Newton's proof of the vanishing of gravitational field inside a hollow sphere.

ds and dS on the surface of the sphere. We will prove that the gravitational attractions from the two patches cancel each other. Since we can divide the sphere into such pairs of infinitesimal patches, it would follow that the total force upon P is zero. The patches dS and ds have masses proportional to their areas, since the sphere is homogeneous. The gravitational forces exerted upon P by these masses are therefore

$$F = k\frac{dS}{D^2} + o(dS) \quad \text{and} \quad f = k\frac{ds}{d^2} + o(ds). \tag{2.83}$$

The two forces act in opposite directions, and it only remains to show that they are equal. Indeed, by a similarity argument

$$\frac{dS}{D^2} = \frac{ds}{d^2};$$

we leave a geometrical proof of this as an exercise. In view of (2.83) this shows that $F = f + o(ds)$, as claimed. We showed that the forces upon P cancel, up to a small percentage error. By taking finer and finer subdivisions of the sphere we conclude that this error is arbitrarily small, and hence must be zero. $\diamondsuit$

2.40. Show that the gravitational field inside a solid ball of uniform density is linear in the distance to the center of the ball: $\mathbf{F}(\mathbf{x}) = -k\mathbf{x}$, where k is the coefficient depending on the density ρ.

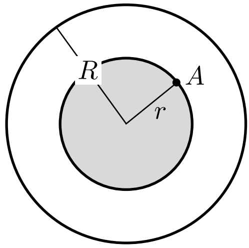

Figure 50. Gravitational force at A is due entirely to the ball of radius r; the spherical shell contributes nothing.

Solution. The force at A in Figure 50 is due only to the ball of radius r, since the shell outside that ball creates zero force inside by Problem 2.39. Now this force is in direct proportion to the mass of the ball, i.e., to r^3, and in inverse proportion to r^2; this explains the linear dependence claimed in the theorem. More explicitly, the gravitational force at A in Figure 50 is caused only by the ball of radius r of mass $M_r = \rho\frac{3\pi r^3}{4}$, so that the force at A is

$$F = \frac{GM_r}{r^2} = \underbrace{\frac{3\pi\rho G}{4}}_{k}\frac{r^3}{r^2} = kr,$$

as claimed.

Chapter 3

Rigid Body Motion

If you have ever watched in slow motion a football wobbling in its flight, or a tennis racket tossed up, you may have noticed that these objects do not simply rotate around some fixed axis, like a wheel; a certain wobbling motion is also present. A dramatic example of tumbling motion in weightlessness can be seen on YouTube: `https://www.youtube.com/watch?v=L2o9eBl_Gzw`: A rigid bracket spins around a certain axis in space, punctuating these spins by abrupt tumbles by 180° at regular intervals. A tennis racket tossed up and given an appropriate spin shows the same behavior. More such videos can be found by Googling "Dzhanibekov Effect". A popular science TV program explains the tumbling thus: "...obviously, there is a zone in space where forces act on the body causing it to tumble." An explanation which does not rely on the mythical forces is given on page 151.

This chapter describes the theory of free motion of rigid bodies. This elegant theory, started by Euler, culminates with a beautiful and simple picture due to Poinsot. Among other items in this chapter is an explanation of the tennis racket paradox (same as the Dzhanibekov effect mentioned in the last paragraph), as well as an observation that the problem is described by a Hamiltonian flow on the sphere. The remaining three sections of the chapter describe the gyroscopic effect, precession of a spinning wheel, and the gyrocompass which uses Earth's rotation to find north.

The highlights of this chapter:

1. How is the angular momentum related to the angular velocity via the tensor of inertia.
2. The kinetic energy is a quadratic form in the angular momentum.
3. The phase portrait of the free rigid body motion.
4. Poinsot's description of the free rigid body motion.
5. Euler's equations of the rigid body motion.
6. Hamiltonian character of Euler's equations.
7. The tennis racket puzzle and its explanation.
8. Gyroscopic effect explained heuristically.
9. Sperry's gyrocompass.

1. Reference frames, angular velocity

A rigid body is a collection of point masses with all mutual distances fixed, or a limit of such discrete bodies producing a continuous mass distribution.

We will use two reference frames: an inertial one, referred to as the *space frame*, and a noninertial frame affixed to the rigid body. The body appears stationary in the body frame, which is the reason for using this frame. Vectors expressed in the space frame will be labeled by the lower case letters ($\boldsymbol{\ell}$, $\mathbf{r}$, $\boldsymbol{\omega}$), while the vectors expressed in the body frame will be denoted by capitals ($\mathbf{L}$, $\mathbf{R}$, $\boldsymbol{\Omega}$); Figure 1 illustrates two representations of one vector. Two representations $\mathbf{x}$, $\mathbf{X}$ of any vector are related via an orthogonal transformation T,

$$\mathbf{x} = T\mathbf{X}, \quad T^T T = id, \tag{3.1}$$

where id is the identity matrix (for the proof, see Problem 3.8 on page 162).

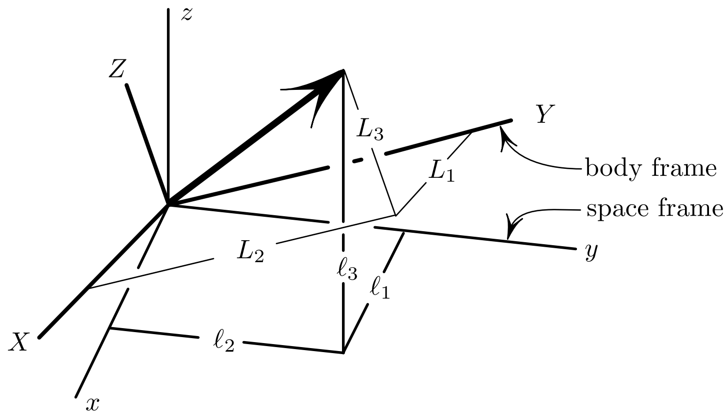

Figure 1. The same vector expressed in the body frame: $\boldsymbol{\ell} = (l_1, l_2, l_3)$ and in the space frame: $\mathbf{L} = (L_1, L_2, L_3)$.

Exercise 3.1. Find the mistake in the following reasoning: "Since the body appears to be at rest to the observer in the body frame, the body's angular momentum $\mathbf{L}$ in the body frame must be zero."

Answer. $\mathbf{L}$ is merely *expressed* in the body frame (in the sense of Figure 1), but it is computed based on the motion of the body relative to the inertial space.

In the next section we define the angular velocity and state its connection with the angular momentum.

2. The tensor of inertia

Recall that the angular velocity of a rigid body is, by definition, the vector $\boldsymbol{\omega}$ parallel to the instantaneous axis of rotation, directed according to the right-hand rule, and whose magnitude is the instantaneous angular speed of rotation around the axis. This verbal definition implies the formula for the velocity $\mathbf{v}$ of a particle $\mathbf{r}$ of the body (see Figure 2):

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{3.2}$$

for every point $\mathbf{r}$ of the body. The key result of this section is the following.

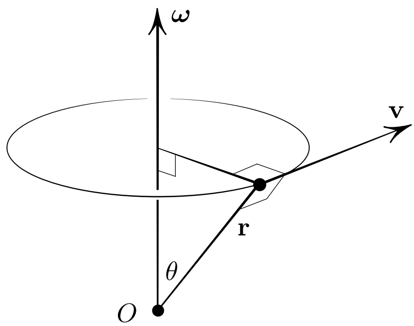

Figure 2. Angular velocity of a rigid body.

Theorem 3.1. *The angular momentum* $\boldsymbol{\ell}$ *of a rigid body is a linear function of its angular velocity* $\boldsymbol{\omega}$*:*

$$\boxed{\boldsymbol{\ell} = I_s \boldsymbol{\omega}} \tag{3.3}$$

where I_s *is a symmetric nonnegative definite matrix given by*

$$I_s = \int_{B_s} \begin{pmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_2 r_1 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_3 r_1 & -r_3 r_2 & r_1^2 + r_2^2 \end{pmatrix} dm. \tag{3.4}$$

Here all the vectors are expressed in the space frame: $dm = \rho(\mathbf{r})\, dr_1 dr_2 dr_3$ *is the mass element of the body, and* B_s *is the region occupied by the body. The matrix* I_s *is called the tensor of inertia.*

The same result holds in the body frame

$$\boxed{\mathbf{L} = I_b \boldsymbol{\Omega}} \tag{3.5}$$

where I_b *is given by*

$$I_b = \int_{B} \begin{pmatrix} R_2^2 + R_3^2 & -R_1 R_2 & -R_1 R_3 \\ -R_2 R_1 & R_1^2 + R_3^2 & -R_2 R_3 \\ -R_3 R_1 & -R_3 R_2 & R_1^2 + R_2^2 \end{pmatrix} dm, \tag{3.6}$$

where $dm = \rho(\mathbf{R})\, dR_1 dR_2 dR_3$*. Since the body is fixed relative to the body frame,* I_b *is a constant matrix.*

Remark 3.1. The relationship (3.3) shows that $\boldsymbol{\omega}$ and $\boldsymbol{\ell}$ need not be aligned — unless $\boldsymbol{\omega}$ is an eigenvector of I_s. Figure 3 explains this misalignment for the rigid body consisting of a single particle connected to the origin by a massless rod. The alignment of $\boldsymbol{\omega}$ and $\boldsymbol{\ell}$ happens for planar rotations, when both vectors are orthogonal to the plane.

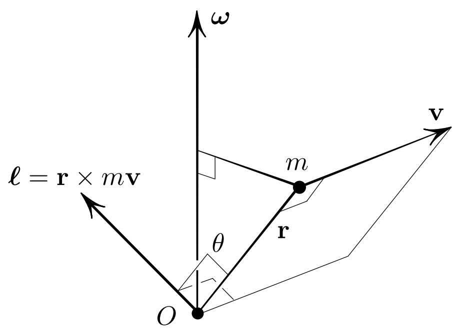

Figure 3. Even for a single particle, $\boldsymbol{\ell}$ and $\boldsymbol{\omega}$ need not line up.

Proof of Theorem 3.1. Consider first a single particle; its angular momentum is

$$d\boldsymbol{\ell} = \mathbf{r} \times dm\,\mathbf{v} = dm\,\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{3.7}$$

This shows that the output $d\boldsymbol{\ell}$ is a linear function of the input $\boldsymbol{\omega}$; note that $\mathbf{r}$ is fixed. We denote this linear function by dI_s, so that $d\boldsymbol{\ell} = dI_s\boldsymbol{\omega}$; we must characterize dI_s. According to (3.7), this function consists of two consecutive applications of cross product with $\mathbf{r}$. But crossing a vector with $\mathbf{r}$ amounts to multiplication by a matrix $\widehat{\mathbf{r}}$ determined by $\mathbf{r}$, namely

$$\boldsymbol{\omega} \times \mathbf{r} = \widehat{\mathbf{r}}\,\boldsymbol{\omega}, \quad \text{where} \quad \widehat{\mathbf{r}} = \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix}, \tag{3.8}$$

as can be verified by direct multiplication. We then have for $\mathbf{x} \in \mathbb{R}^3$:

$$\mathbf{r} \times \mathbf{x} = -\mathbf{x} \times \mathbf{r} = -\widehat{\mathbf{r}}\,\mathbf{x} = \widehat{\mathbf{r}}^T\,\mathbf{x}.$$

Thus (3.7) becomes

$$d\boldsymbol{\ell} = \underbrace{dm\,\widehat{\mathbf{r}}^T\,\widehat{\mathbf{r}}}_{dI_s}\boldsymbol{\omega}. \tag{3.9}$$

Now $\widehat{\mathbf{r}}^T\,\widehat{\mathbf{r}}$ is symmetric and nonnegative, since for any $\mathbf{x}\in\mathbb{R}^3$ the dot product

$$(\widehat{\mathbf{r}}^T\,\widehat{\mathbf{r}}\mathbf{x},\mathbf{x}) = (\widehat{\mathbf{r}}\mathbf{x},\widehat{\mathbf{r}}\mathbf{x}) \geq 0.$$

A direct multiplication shows that $\widehat{\mathbf{r}}^T\,\widehat{\mathbf{r}}$ is indeed of the form given in (3.4). We proved (3.4), and thus the theorem, for a single-particle body. The general case follows by integration of (3.9):

$$\boldsymbol{\ell} = \int_B d\boldsymbol{\ell} = \int_B \widehat{\mathbf{r}}^T\,\widehat{\mathbf{r}}\,\boldsymbol{\omega}\,dm = \left(\int_B \widehat{\mathbf{r}}^T\,\widehat{\mathbf{r}}\,dm\right)\boldsymbol{\omega}.$$

Note that $\boldsymbol{\omega}$ factors out of the integral. This proof, which we carried out in the space frame, applies to the body frame verbatim. ♢

Remark 3.2. For a point mass m the tensor of inertia is $I = m\,\widehat{\mathbf{r}}^T\,\widehat{\mathbf{r}}$, according to (3.9). This is the generalization of the scalar formula $I = mr^2$ for the moment of inertia of the point mass; see (2.16) on page 85. Recall that in fact we had defined the scalar moment of inertia as the coefficient of proportionality between the scalar angular velocity and the scalar angular momentum. Similarly, tensor of inertia is the "matrix of proportionality" between $\boldsymbol{\ell}$ and $\boldsymbol{\omega}$.

Principal moments of inertia. The eigenvalues I_k $(k = 1,2,3)$ of the tensor of inertia I_b are called the *principal moments of inertia*; these do not depend on the choice of the frame, according to (3.11), and therefore are invariants of the mass distribution in the body. To see that the term "moment of inertia" agrees with the scalar definition, we let $\boldsymbol{\Omega}_k$ $(k = 1, 2$ or $3)$ be an eigenvector of I_b:

$$I_b\boldsymbol{\Omega}_k = I_k\boldsymbol{\Omega}_k.$$

In other words, the angular velocity aligns with the angular momentum, and I_k is the coefficient of proportionality between the two. This indeed fits precisely with the earlier definition of the moment of inertia for planar rotations (see page 86).

Dependence of I_b on the frame; principal axes of inertia. Consider the second body frame, with the same origin as the first frame. Vectors expressed in the two frames are related via

$$\mathbf{L} = U\mathbf{L}', \quad \boldsymbol{\Omega} = U\boldsymbol{\Omega}', \tag{3.10}$$

where U is an orthogonal matrix: $U^{-1} = U^T$, and where the primes refer to the vectors expressed in the new frame. Substituting this into $\mathbf{L} = I_b\mathbf{\Omega}$, we get

$$\mathbf{L}' = I_b'\mathbf{\Omega}',$$

where

$$I_b' = \mathbf{U}^{-1} I_b \mathbf{U}. \tag{3.11}$$

In other words, the tensor of inertia undergoes a similarity transformation under the change of the frame. It is convenient to choose U so as to make the new matrix I' diagonal: $I_b' = \mathrm{diag}(I_1', I_2', I_3')$. This choice amounts to orienting the coordinate axes of the second frame along the eigendirections of the tensor of inertia I_b. These eigendirections are called the *principal axes of inertia* of the body.

3. The kinetic energy

In addition to the expression for the angular momentum, we need one for the kinetic energy. This expression is a generalization of the familiar formula $K = \frac{1}{2}I\omega^2$ discussed earlier for planar rotations.

Theorem 3.2. *Consider a rigid body whose one point is fixed at the origin. The kinetic energy of the body is the dot product*

$$K = \frac{1}{2}(\boldsymbol{\ell}, \boldsymbol{\omega}) \overset{(3.3)}{=} \frac{1}{2}(I_s\boldsymbol{\omega}, \boldsymbol{\omega}), \tag{3.12}$$

where $\boldsymbol{\omega}$ is the angular velocity, $\boldsymbol{\ell}$ is the angular momentum relative to the origin, and I_s is the tensor of inertia, all expressed in the space frame. A similar formula holds in the body frame:

$$K = \frac{1}{2}(\mathbf{L}, \mathbf{\Omega}) = \frac{1}{2}(I_b\mathbf{\Omega}, \mathbf{\Omega}). \tag{3.13}$$

Proof. For a point mass dm positioned at $\mathbf{r}$ we have $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \widehat{\mathbf{r}}\boldsymbol{\omega}$, so that

$$dK = \frac{1}{2}dm\,\mathbf{v}^2 = \frac{dm}{2}(\widehat{\mathbf{r}}\boldsymbol{\omega}, \widehat{\mathbf{r}}\boldsymbol{\omega}) = \frac{1}{2}(dm\,\widehat{\mathbf{r}}^T\widehat{\mathbf{r}}\boldsymbol{\omega}, \boldsymbol{\omega}) \overset{(3.9)}{=} \frac{1}{2}(d\boldsymbol{\ell}, \boldsymbol{\omega}).$$

For a rigid body consisting of many particles, (3.12) follows by summation (or by integration for continuous mass distributions). The particular choice of frame never entered the proof, and therefore the statement of the theorem holds for a body frame as well. $\diamondsuit$

The ellipsoid of inertia. We showed that kinetic energy is a quadratic form in terms of the angular velocity. Associated with this quadratic form is an ellipsoid, called the *ellipsoid of inertia,* defined as follows:

$$\mathcal{E}_s = \{\mathbf{x} : \ (\mathbf{I}_s\mathbf{x}, \mathbf{x}) = 1\} \tag{3.14}$$

in the space frame or

$$\mathcal{E}_b = \{\mathbf{X} : \ (\mathbf{I}_b\mathbf{X}, \mathbf{X}) = 1\} \tag{3.15}$$

in the body frame.

According to (3.14) or (3.15), the ellipsoid of inertia *consists of the angular velocity vectors* $\boldsymbol{\omega}$ *corresponding to a fixed value of kinetic energy* K (namely, $K = \frac{1}{2}$). In other words, $\mathcal{E}$ *is an "equikinetic" surface in the space of angular velocities.* The ellipsoid of inertia is rigidly attached to the body.

The ellipsoid of inertia mimics the shape of the body in the sense that both the body and the ellipsoid tend to be elongated or flattened in the same direction. To be more precise, take the body frame in which I_b is diagonal; the ellipsoid is then given by

$$I_1x_1^2 + I_2x_2^2 + I_3x_3^2 = 1,$$

and the semi-axes are $1/\sqrt{I_k}$, $k = 1, 2, 3$. This shows that the largest I_k corresponds to the shortest axis. For a thin rod, such as a pencil, the two largest moments of inertia I_k are in the directions perpendicular to the pencil. These are therefore the directions of the shortest axes, and thus the ellipsoid is indeed elongated along the pencil. In a similar way, for a rectangular plate, the the ellipsoid's axes reflect the shape of the plate. For future reference, note that the intermediate axis of the ellipsoid is parallel to the shorter sides of the rectangle.

4. Dynamics in the body frame

The angular momentum $\boldsymbol{\ell}$, although fixed in the space frame, appears to be moving to an observer sitting in the body frame. If we can describe this motion, we will gain a crucial insight into the dynamics of the body since we will then know how a line fixed in space moves relative to the body. Let us consider all possible motions with a given $|\boldsymbol{\ell}| = \lambda = \text{const}$. For all such motions

$$|\mathbf{L}(t)| = \lambda, \tag{3.16}$$

and our goal is to describe the time evolution of $\mathbf{L}$ as the body moves without external forces. The set of $\mathbf{L}$ with a fixed value $|\mathbf{L}| = \lambda$ is called the *momentum sphere*. In this section we will show that the tip of $\mathbf{L}$ moves as shown by arrows in Figure 4, and will derive Euler's equations of motions for $\mathbf{L}$ in the next section.

For any free motion of the body, the kinetic energy is conserved:

$$\frac{1}{2}(\mathbf{L}, I_b^{-1}\mathbf{L}) = \text{const.}, \tag{3.17}$$

where we used (3.13) and $\mathbf{L} = I_b \mathbf{\Omega}$ to express kinetic energy in terms of $\mathbf{L}$. The trajectories therefore stay on the ellipsoid (3.17). Summarizing, every path $\mathbf{L}(t)$ stays on the intersection of the momentum sphere (3.16) and some ellipsoid (3.17); these intersections, i.e., the trajectories of $\mathbf{L}$, are shown in Figure 4.

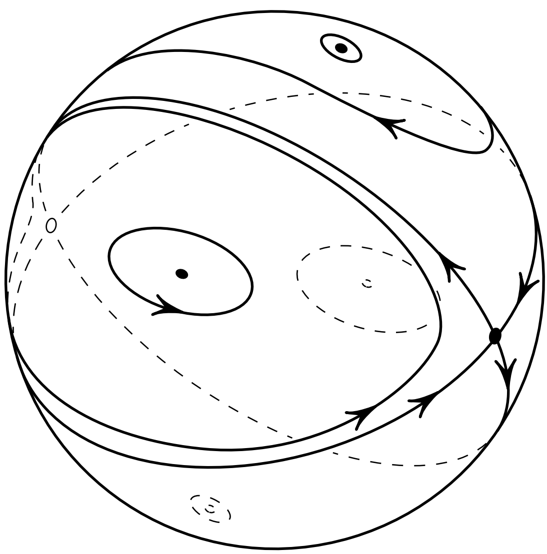

Figure 4. Phase flow on the momentum sphere. The trajectories are the level curves of the energy $\langle \mathbf{L}, I^{-1}\mathbf{L}\rangle$ on the momentum sphere $|\mathbf{L}| = \lambda$.

Exercise 3.2. What motions of the rigid body, as viewed from the space frame, correspond to the rest points in Figure 4?

Answer. The motions with $\mathbf{L}$ = const. correspond to steady rotations around the principal axes of inertia.

5. Euler's equations of motion

So far we managed to understand the qualitative behavior of $\mathbf{L}$ without using differential equations. If, however, we want to know more than the shape of trajectories, but also the value of $\mathbf{L}$ at any time, we need differential equations governing $\mathbf{L}$. These equations were derived by Euler.

Theorem 3.3. *The evolution of the angular momentum of a free rigid body in the body frame is governed by*

$$\boxed{\dot{\mathbf{L}} = \mathbf{L} \times (I_b^{-1}\mathbf{L})} \tag{3.18}$$

Since $\mathbf{L} = I\mathbf{\Omega}$, *the angular velocity obeys*

$$\boxed{\mathbf{I}\dot{\mathbf{\Omega}} = I_b\mathbf{\Omega} \times \mathbf{\Omega}} \tag{3.19}$$

or in coordinates, assuming $I_b = \mathrm{diag}(I_1, I_2, I_3)$*:*

$$\boxed{\begin{aligned} \dot{\Omega}_1 &= \tfrac{I_2-I_3}{I_1}\,\Omega_2\Omega_3 \\ \dot{\Omega}_2 &= \tfrac{I_3-I_1}{I_2}\,\Omega_3\Omega_1 \\ \dot{\Omega}_3 &= \tfrac{I_1-I_2}{I_3}\,\Omega_1\Omega_2 \end{aligned}} \tag{3.20}$$

The proof of Euler's equations is remarkably simple via a creative application of the formula $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

To an observer sitting in the body frame, the surrounding universe can be thought of as a huge rigid body which rotates with angular velocity $-\mathbf{\Omega}$ (all vectors are expressed in the body frame). Every particle of the surrounding space has velocity $\mathbf{V} = (-\mathbf{\Omega}) \times \mathbf{R}$, where $\mathbf{R}$ is the particle's position (in the body frame). But the tip of the angular momentum vector is also fixed in space; its position is $\mathbf{R} = \mathbf{L}$. Thus,

$$\dot{\mathbf{L}} = (-\mathbf{\Omega}) \times \mathbf{L},$$

which proves (3.18) once we use $\mathbf{\Omega} = I_b^{-1}\mathbf{L}$. $\diamondsuit$

6. The tennis racket paradox

The paradox. Some time ago, when tossing up a tennis racket and catching it, I noticed a strange effect. During one head-over-handle tumble the racket also made a flip around its long axis; surprisingly, this flip was indistinguishable from 180° in almost all tosses; when launched with its plane horizontal, the racket's plane was again horizontal when caught. Figure 5 shows the same effect for a book tossed with a spin around the left-right axis[1]. Only in one of these 20 tosses the head was not horizontal when I caught the racket. By further experimentation with different heights of tossing, one observes that the flip may be another integer multiple of 180°. Below is an explanation of the surprising appearance of a half-integer turn.

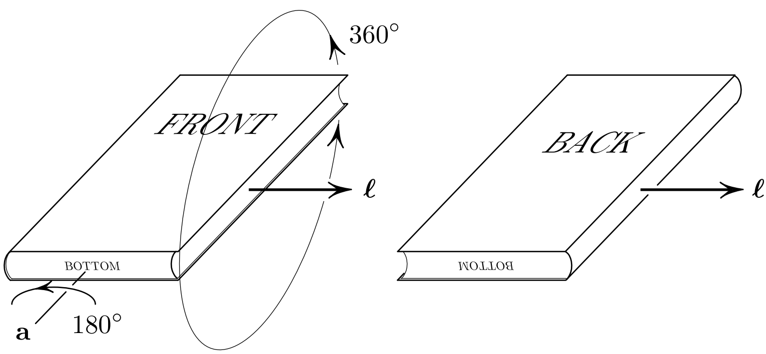

Figure 5. The book is tossed up with a spin around $\boldsymbol{\ell}$ and makes a 360° flip as shown. Surprisingly, in mid-flight, the book makes an additional unexpected and rather abrupt half-flip around another axis (**a**), thus landing face down rather than up. The explanation is provided by the motion of the angular momentum **L** in Figure 4.

An explanation. The tennis racket is launched with the initial angular momentum $\mathbf{L}(0)$ nearly aligned with the intermediate axis of inertia, Figure 6. Since the flow is slow in the vicinity of the saddle points, **L** spends most of its time nearly aligned with the saddles, and only a small proportion of time in transition, see the equally timed

[1] Enclosing the book in a rubber band is a good idea. A long narrow book works better than a squarish one.

positions of $\mathbf{L}$ in the figure. Therefore, if we sample $\mathbf{L}(t)$ at a random time, we are likely to find $\mathbf{L}$ near one of the saddles. Thus the body's intermediate axis of inertia (nearly) lines up with the direction of the angular momentum most of the time, and so most of the time during its flight the book will be in one of the two orientations relative to $\boldsymbol{\ell}$ as shown in Figure 5. This explains the tennis racket paradox.

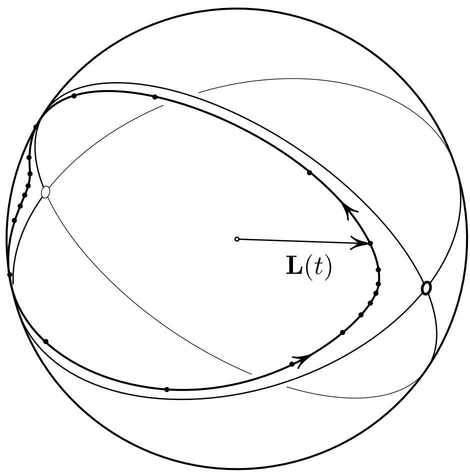

Figure 6. An explanation of the tennis racket paradox. The dots show snapshots of $\mathbf{L}(t)$ at equally spaced time intervals. Since $\mathbf{L}(t)$ spends most of its time near the saddles, the intermediate axis of inertia of the racket aligns with the angular momentum vector most of the time.

7. Poinsot's description of free rigid body motion

The evolution of the angular momentum $\mathbf{L}$ in the body frame (Figure 4) does not yet give us the full picture of the motion of the body in the space frame. Such a picture is given by the beautiful result of Poinsot, stated and proven next.

Theorem 3.4. *The free rigid body, viewed in the inertial frame of its center of mass, moves so that its ellipsoid of inertia rolls without sliding on the fixed plane P perpendicular to the angular momentum*

$\boldsymbol{\ell}$ and lying at the distance $\frac{2\sqrt{E}}{|\boldsymbol{\ell}|}$ from the body's center of mass, Figure 7.

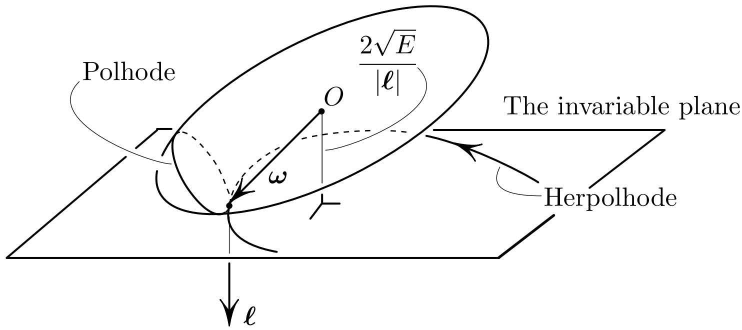

Figure 7. The Poinsot description of the free body motion.

This theorem gives a virtually complete description of the motion; only the speed is not specified, but that can be easily recovered.

Proof of Poinsot's theorem. Instead of the inertia ellipsoid $(I_s\mathbf{x}, \mathbf{x}) = 1$ (everything in the proof refers to the space frame) we first consider its scaled version

$$\{\mathbf{x} : \ \frac{1}{2}(\mathbf{I_s}\mathbf{x}, \mathbf{x}) = E\}, \tag{3.21}$$

referred to as the energy ellipsoid. We will show that this ellipsoid, which is also rigidly attached to the body, rolls on the fixed plane $(\boldsymbol{\ell}, \mathbf{x}) = 2E$ without sliding. The statement about the ellipsoid of inertia (3.14) then follows by scaling $\mathbf{y} = \mathbf{x}/\sqrt{2E}$, designed to dilate the energy ellipsoid (3.21) into the ellipsoid of inertia.

Proof of the theorem amounts to showing that

(i) the ellipsoid (3.21) and the plane $(\boldsymbol{\ell}, \mathbf{x}) = 2E$ intersect at $\boldsymbol{\omega}$ (the tip of the space angular velocity vector),

(ii) $\boldsymbol{\omega}$ is a tangency point and

(iii) there is no sliding, i.e., that the material point on the ellipsoid has zero velocity at the point of contact. By "material point" I mean a point which moves with the rigid body, i.e., is fixed in the body frame. The proof of (i)–(iii) is very short:

(i) Since $(\boldsymbol{\ell}, \boldsymbol{\omega}) = (\mathbf{I_s}\boldsymbol{\omega}, \boldsymbol{\omega}) = 2E$, we conclude that the tip of the vector $\boldsymbol{\omega}$ belongs to the plane $(\boldsymbol{\ell}, \mathbf{x}) = 2E$ and to the ellipsoid (3.21).

(ii) Normal to the ellipsoid (3.21) is given by $\nabla(\mathbf{I_s}\boldsymbol{\omega}, \boldsymbol{\omega}) = 2\mathbf{I_s}\boldsymbol{\omega} = 2\boldsymbol{\ell}$; since $\boldsymbol{\ell}$ is also normal to the plane $(\boldsymbol{\ell}, \mathbf{x}) = 2E$, we conclude that the ellipsoid and the plane are indeed tangent to each other.

(iii) The velocity of any material point $\mathbf{x}$ of the body is $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\omega}$. The material point $\mathbf{x}$ of the space frame which is in contact with the plane at the moment in question has velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} = \boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$. (This is the well-familiar fact: the points on the instantaneous axis of rotation have zero velocity.) The contact is nonslipping; this completes the proof of Poinsot's theorem. ♢

8. The gyroscopic effect — an intuitive explanation

For the rest of this chapter we consider the body which is no longer free but is subject to external torques.

How do we explain the counterintuitive ability of the spinning top to defy gravity? A simple qualitative answer is given in this section. In the next section this answer is quantified: we compute the torque required to reorient a gyroscope. This answer is also used to compute the speed of precession of a spinning wheel in the section that follows. Finally, the last section of the chapter describes Sperry's gyrocompass and explains how it uses Earth's rotation to find north.

The gyroscopic effect and its explanation. Figure 8 sketches a bike wheel held by its axle. The wheel is spinning (towards the observer). You are holding the axle, reorienting its axle steadily as shown by the arrows. Surprisingly, such reorientation requires that you apply the forces ($\mathbf{F}$ and $-\mathbf{F}$) perpendicular to the motion of your hands, as shown in the figure. The next figure explains this counterintuitive effect.

Consider the path $1-2-3$ of a particle of the rim near the top of the wheel in Figure 9. This path is curved due to the axle's precession. Since the particle wants to go by inertia as straight as possible, it "protests" with centrifugal force shown in the figure. Similarly, the antipodal particles apply equal and opposite centrifugal forces. These

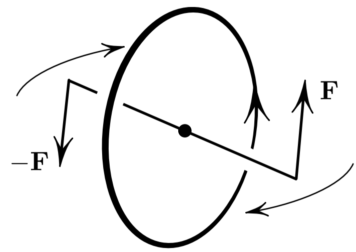

Figure 8. The forces required to reorient a spinning wheel must be perpendicular to the direction of motion of the axle's ends!

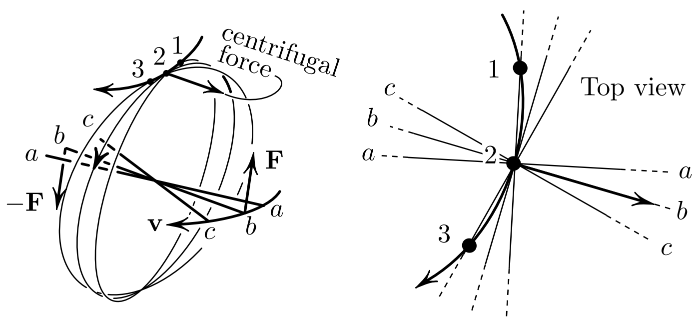

Figure 9. An explanation of the gyroscopic effect: (i) the axle precesses, (ii) causing a particle of the rim to travel in a curved path $1-2-3$; (iii) the particle exerts the centrifugal force (iv) which must be compensated by the forces $\mathbf{F}$, $-\mathbf{F}$ of your hands.

forces exert torque which tries to tip the wheel; we must therefore apply the forces $\mathbf{F}$ and $-\mathbf{F}$ to counteract this tipping, to keep the axle in the horizontal plane (of course, $\mathbf{F}$ must compensate the combined effect of all the particles, not just the ones near the top and bottom.) This completes our heuristic explanation of the gyroscopic effect.

Remark 3.3. One can actually find the magnitude of torque exerted by the couple $\mathbf{F}$, $-\mathbf{F}$ by integrating the torques of all centrifugal forces over the rigid body. This is left as a not-so-easy challenge to the reader. A simpler way to find the gyroscopic torque is discussed in Section 9.

Energy considerations. The fact that $\mathbf{F} \perp \mathbf{v}$ in Figure 9, i.e., that the axle reacts with a force perpendicular to imposed motion, can be explained by conservation of energy, as follows. If you move the ends of axle with constant speed, keeping the axle in the same plane, as in Figure 9, then you do not change the gyroscope's kinetic energy[2]. Therefore, you do zero work, and hence the forces $\mathbf{F}$, $-\mathbf{F}$ *must* be perpendicular to the velocity vectors $\mathbf{v}$, $-\mathbf{v}$.

9. The gyroscopic torque

The arguments of the preceding paragraph, while predicting that $\mathbf{F} \perp \mathbf{v}$, do not give the magnitude of $\mathbf{F}$; in this section we find $\mathbf{F}$, or rather, the torque $\mathbf{T}$ of the couple $\mathbf{F}$, $-\mathbf{F}$.

Assume that the axle is being reoriented, as in Figure 9, at a constant angular velocity ω_{prec} around the vertical axis (we will refer to this motion as the precession). In addition, the wheel is spinning around its axis with a given angular velocity ω_{ax}.

The required torque $\mathbf{T}$ is given by the rotational version of Newton's second law which, we recall, says that the torque $\mathbf{T}$ applied to a system is responsible for the change of its angular momentum $\mathbf{L}$:

$$\mathbf{T} = \frac{d}{dt}\mathbf{L};$$

this was derived on page 81. Now the angular momentum is the sum of the precessional and the axial parts:

$$\mathbf{L} = \mathbf{L}_{\text{prec}} + \mathbf{L}_{\text{ax}},$$

(Figure 10), of which the first one is constant.

Differentiating $\mathbf{L}$, we get

$$\mathbf{T} = \frac{d}{dt}(\mathbf{L}_{\text{prec}} + \mathbf{L}_{\text{ax}}) = \frac{d}{dt}\mathbf{L}_{\text{ax}}. \tag{3.22}$$

But the tip of the vector $\mathbf{L}_{\text{ax}}$ moves in a circle (Figure 10) of radius

$$r = |\mathbf{L}_{\text{ax}}| = I_{\text{ax}}\omega_{\text{ax}}, \tag{3.23}$$

where I_{ax} is the moment of inertia of the wheel around its axis, and the angular velocity of this circular motion is ω_{prec}. By the formula

[2]Assuming that the wheel is perfectly balanced.

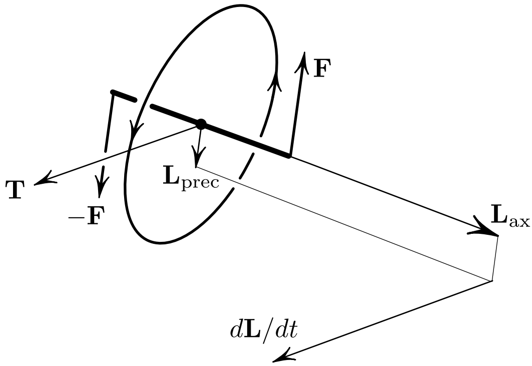

Figure 10. Torque $\mathbf{T}$ applied to the wheel causes the change of the angular momentum.

$v = \omega r$ we get the speed of the tip of $\mathbf{L}_{\text{ax}}$:

$$|\dot{\mathbf{L}}_{\text{ax}}| = \omega_{\text{prec}} r \overset{(3.23)}{=} \omega_{\text{prec}} I_{\text{ax}} \omega_{\text{ax}}.$$

Substituting this into (3.22) and taking the magnitude of both sides, we finally get

$$\boxed{T = \omega_{\text{prec}} I_{\text{ax}} \omega_{\text{ax}}} \tag{3.24}$$

10. Speed of precession

The setting. Let us suspend a rapidly spinning bike wheel on a very long string attached to *one end* of an axle, Figure 11. With the other end of the axle unsupported, the axle may remain horizontal while precessing around the vertical — we explained this using Figure 9. The goal now is to find the speed of this precession. Let us assume that the string is very long, so that it can be treated as vertical at all times; therefore, no horizontal forces act on the wheel and we assume that its center of mass remains fixed.

The desired precession rate ω_{prec} is given by (3.24). The wheel in Figure 11 is subject to the torque $T = mgd$ exerted by the gravity and by the string. Since the center of mass is at rest, our reference frame is inertial, and equation (3.24) applies, giving

$$\omega_{\text{prec}} = \frac{T}{I_{\text{ax}} \omega_{\text{ax}}} = \frac{mgd}{I_{\text{ax}} \omega_{\text{ax}}}.$$

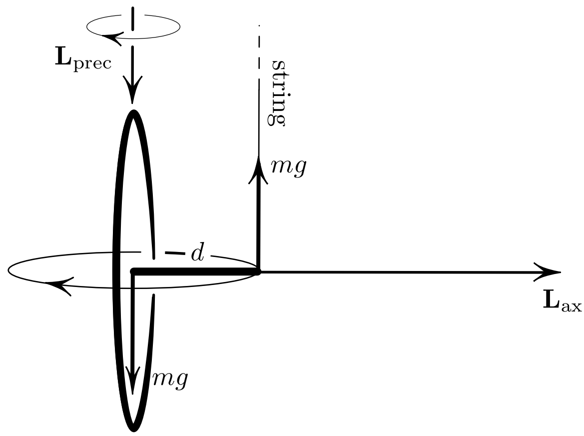

Figure 11. Precessing bike wheel.

Let us now treat the bike wheel as a thin ring of radius r with all its mass m concentrated in the rim. In that case, $I_{\text{ax}} = mr^2$, and substitution in the above gives

$$\omega_{\text{prec}} = \frac{gd}{r^2 \omega_{\text{ax}}}.$$

This formula confirms the intuitive sense that, to get slow precession, it helps to have large ω_{ax} and large r, and/or small d and small g. A 29 inch diameter bike wheel is about 12% larger than the standard 26 inch wheel, but it will precess about 24% slower, thanks to the fact that r is squared. Doubling the wheel's radius will slow its precession by the factor of 4.

How does a spinning top stay up. Before answering this question, let us restate the problem in a more convenient form. Let's build the spinning top by affixing one end of a wheel's axle to the ground by a ball joint, Figure 12. As we established, any time I move the free end P with velocity $\mathbf{v}$ in an arc of great circle on the sphere, the axle applies force $\mathbf{F} \perp \mathbf{v}$ to my hand. Thus I can forget the spinning top and think instead of a particle P constrained to the sphere and

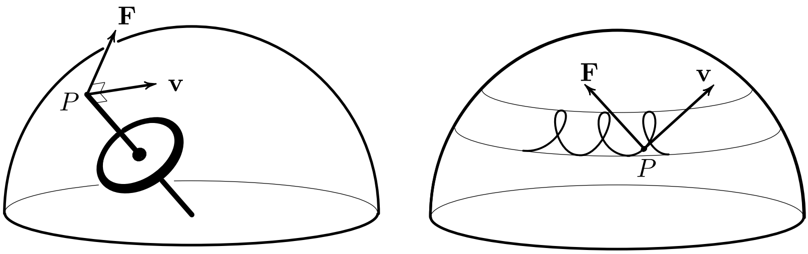

Figure 12. The "magnetic" force $\mathbf{F} \perp \mathbf{v}$ (left) and a typical path of P (right).

subject to the "magnetic" force $\mathbf{F} \perp \mathbf{v}$ whose magnitude varies as the magnitude of $\mathbf{v}$, in addition to the gravitational potential force. The particle P avoids falling down, not because it resists gravity, but by deflection created by the gyroscopic force on P. The constant action of this deflecting force results in a path of the kind illustrated in Figure 12. One could call this mechanism "stability by deflection".

11. The gyrocompass

Let us place the gyroscope, its axis mounted horizontally, on a floating platform; see Figure 13. Surprisingly, the floating platform will slowly turn so that the gyroscope's axis will point exactly along the meridian! And the north is distinguished from the south by the right-hand rule: if the axis were a screw, it would turn so as to advance north. In summary, *the gyroscope tries to align itself with the Earth's rotation as much as the constraint on its axis allows.* The last sentence suggests an even better alternative than floating on a platform: the gyroscope should be given full freedom to reorient itself; this can be done either by immersion in fluid, or by mounting it on a set of gimbals. Such a gyroscope will gradually align itself with the Earth's axis! As an added benefit, the angle of the gyro's axis with the horizontal axis will give the latitude.

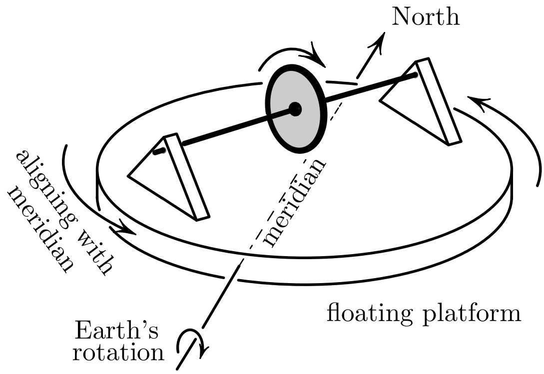

Figure 13. The gyrocompass: a gyroscope on a floating platform.

How does the gyrocompass work. For simplicity, let us place the mechanism on the equator, and align its axis with the equator, Figure 14.

The gyroscope spins as shown by arrow (A). Due to Earth's rotation, the platform turns (arrows (B)) around the north–south line; one end of the axis is being pushed up, the other down. The gyroscope, just like the bike's wheel in Figure 11, responds by reorienting itself in the direction of arrows (C): the axle starts turning north. Eventually, the axis of the floating gyro orients itself along the meridian. Now if we are *not* on the equator, the gyroscope still tries to align itself with Earth's axis and comes as close to this alignment as the floating constraint allows, straining against the constraint.

The gyrocompass has at least two advantages over the magnetic compass: it is immune to magnetic anomalies (unavoidable in steel ships) and it finds the geographic, rather than the magnetic, north.

Exercise 3.3. A gyroscope is suspended in fluid and is neutrally buoyant; its axis is free to orient in any direction. Using the idea of Figure 13, describe qualitatively the path of the gyroscope's axis on its way to aligning with the Earth's axis.

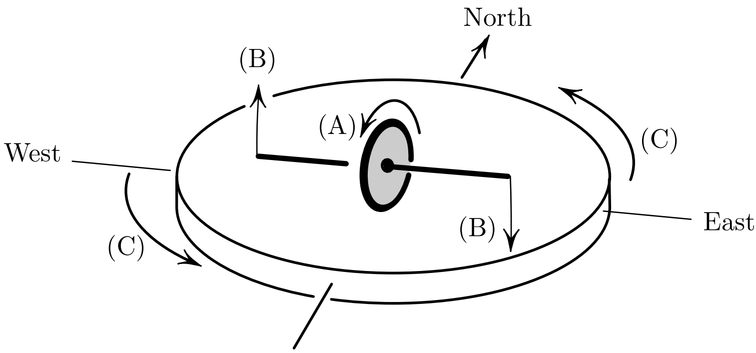

Figure 14. How does the gyrocompass work.

12. Problems

3.1. Compute the tensor of inertia I for a single particle of mass m at $\mathbf{R} = (r_1, r_2, r_3)$.

Answer. Using (3.8), we get

$$I = -m\widehat{\mathbf{R}}^2 = -m \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix}^2 = m \begin{pmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_2 r_1 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_3 r_1 & -r_3 r_2 & r_1^2 + r_2^2 \end{pmatrix}.$$

In particular, for a point mass on the r_3-axis with $r_3 = r$ we have

$$I = \operatorname{diag}(mr^2, mr^2, 0).$$

3.2. Find the principal inertial axes and the principal moments of inertia for a planar disk of mass m and of radius r.

3.3. Show that for any plate, i.e., for any flat body lying in the (x, y)-plane we have $I_x + I_y = I_z$ where the subscript denotes the moments of inertia with respect to the coordinate axes.

3.4. A plate lies in the (x, y)-plane, and is symmetric with respect to the x-axis. Show that the tensor of inertia is then diagonal: $I = \operatorname{diag}(I_x, I_y, I_z)$ and, moreover, that $I_z = I_x + I_y$.

3.5. Prove that the principal moments of inertia I_x, I_y, I_z satisfy the triangle inequality: none of them exceeds the sum of the other two.

3.6. Consider a homogeneous solid cube centered at the origin and with the faces parallel to the coordinate axes. Which moment of inertia is greater: the one around the x-axis, or the one around one of the diagonals of the cube? Can you get the answer without calculation?

3.7. Prove that if a rigid body with one point fixed in space is reoriented in any way, then some points on this body end up in their starting positions [Euler's theorem].

Hint. The change of the body's orientation is given by an orthogonal 3×3 matrix with determinant $+1$. The problem then reduces to showing that any such matrix has an eigenvalue 1.

3.8. Show that if $\mathbf{x}$ and $\mathbf{X}$ are two expressions of the same vector in the space and in the body reference frames, then there exists an orthogonal matrix T with $\det T = +1$ such that

$$\mathbf{x} = T\mathbf{X}.$$

In particular, the column vectors of T are the unit vectors of the body reference frame, expressed in the space frame.

Hint. Show that the map $\mathbf{x} \mapsto \mathbf{X}$ is linear and that it preserves the dot product.

3.9. Consider the rigid body consisting of a point mass affixed to the origin by a rigid rod. (i) What is the set of all angular velocities $\mathbf{\Omega}$ which produce a given angular momentum $\mathbf{L}$? (ii) Find all $\mathbf{\Omega}$ producing $\mathbf{L} = \mathbf{0}$. (iii) Let us treat $\mathbf{L} = \mathbf{L}(\mathbf{\Omega})$ as a vector field in the space of $\mathbf{\Omega}$, i.e., let us attach the arrow $\mathbf{L}$ to the tip of $\mathbf{\Omega}$. Describe this vector field geometrically.

Answers, referring to Figure 15. (i) Every vector $\mathbf{\Omega}$ with the tip on the line AB parallel to $\mathbf{R}$ produces the same $\mathbf{L}$. (ii) Every vector $\mathbf{\Omega}$ along the line A_0B_0 gives $\mathbf{L} = \mathbf{0}$. In other words, the line A_0B_0 is the kernel of the map $\mathbf{\Omega} \mapsto \mathbf{L}$. (iii) Vector field $\mathbf{L}$ is perpendicular to the cylinders with the axis aligned with $\mathbf{R}$; the magnitude of $\mathbf{L}$ is directly proportional to the cylinder's radius, with the coefficient of proportionality mR^2, where $R = |\mathbf{R}|$.

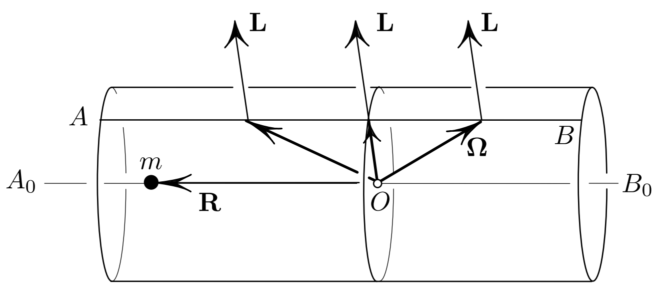

Figure 15. The map $\mathbf{\Omega} \mapsto \mathbf{L}$ for the single particle (Problem 3.9).

3.10. (a) Show directly that the flow of (3.18) conserves the energy and the momentum. In particluar, every sphere $|\mathbf{L}| = \lambda$ is invariant under the flow of this equation. (b) Prove that (3.18) defines an area-preserving flow on the sphere (and thus is a Hamiltonian flow on the sphere).

3.11. Find the Hamiltonian form of Euler's equation (3.18) for $\mathbf{L}$.

Answer. $\dot{\mathbf{L}} = J_{\mathbf{L}} \nabla H(\mathbf{L})$ where $J_{\mathbf{L}}$ is the $\frac{\pi}{2}$-rotation of the tangent plane to the sphere at $\mathbf{L}$.

3.12. Consider the scalar function V defined in $\mathbb{R}^3$ as $V(\mathbf{x}) = \frac{(A\mathbf{x},\mathbf{x})}{(\mathbf{x},\mathbf{x})}$ with $A = \operatorname{diag}(a, b, c)$. Here $(\mathbf{x}, \mathbf{y})$ denotes the dot product of two vectors. In other words, $V(x_1, x_2, x_3) = \frac{ax_1^2+bx_2^2+cx_3^2}{x_1^2+x_2^2+x_3^2}$. Assume that $a < b < c$.

a. Show that the gradient vector field $\mathbf{f}(\mathbf{x}) = \nabla V(\mathbf{x})$ is tangent to any sphere $|\mathbf{x}| = \text{const}$.

b. Find all critical points of $V(\mathbf{x})$ restricted to the unit sphere $|\mathbf{x}| = 1$ and determine their type (max, min, saddle).

c. Sketch the phase portrait of the gradient flow on the unit sphere.

3.13. For a free rigid body, find the range of all possible values of energy E given the magnitude of the angular momentum.

3.14. Consider a flat disk with no external forces. For a prescribed value of the angular momentum, around which axis should the disk be spinning if its kinetic energy is least possible? Largest possible? What is the answer to the same question for an arbitrary rigid body with the principal moments of inertia $I_1 < I_2 < I_3$?

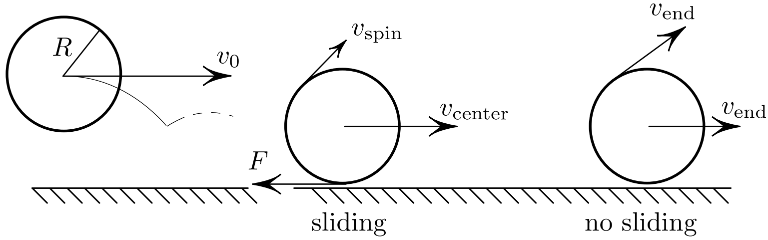

Figure 16. What is the ratio of the final speed to the initial speed?

3.15. I throw a wheel forward with initial forward speed v_0 and with zero spin. The wheel will bounce, skid, and eventually will begin to roll without sliding. Some energy will be lost to heat in skidding and bouncing. Assume that no energy is lost in flying or rolling. What is the eventual speed of the wheel? What proportion of the wheel's kinetic energy will be lost to heat? No information about the sliding friction is given. Only the radius R of the wheel, its mass m, and its moment of inertia I are known.

Solution. The remarkable fact is that the answer does not depend on the nature of friction, or on on the nature of bouncing (e.g., how elastic it is). It doesn't even matter whether I threw the wheel horizontally or not; only the horizontal component v_0 of the initial velocity matters. After the skidding stopped, the wheel will roll with speed

$$\frac{v_{\text{end}}}{v_0} = \frac{1}{1+r},$$

where $r = I/mR^2$. As an example, for the hoop $I = mR^2$, and thus $v_{\text{end}} = v_0/2$. If, on the other hand, all the mass of the wheel is concentrated in its axle, i.e., if $I = 0$, then the formula gives $v_{\text{end}} = v_0$, as expected.

Here is an intuitive idea of the solution; the next paragraph gives full details. As the wheel is rubbing against the ground, its center slows down, but its spin grows. Now both of these effects are directly caused by the same force F of sliding friction, and thus $\Delta v_{\text{center}} = -k\Delta v_{\text{spin}}$ (Figure 16) for some constant k (determined in the next paragraph). This proportionality implies that the two speeds meet at a certain value independent of how each velocity changed over time. The moment these speeds become equal, the sliding stops and the pure rolling begins.

To make the preceding paragraph more specific, Newton's second law gives

$$m\dot{v}_{\text{center}} = -F; \quad I\dot{\omega} = FR, \tag{3.25}$$

where ω is the wheel's angular velocity. Wishing to refer to v_{center} throughout instead of ω, substitute $\omega = v_{\text{center}}/R$ into (3.25), so that the equations become

$$\dot{v}_{\text{center}} = -\frac{1}{m}F, \quad \dot{v}_{\text{spin}} = \frac{R^2}{I}F.$$

Integration gives us the speeds at any time t:

$$v_{\text{center}}(t) = v_0 - \frac{1}{m}\int_0^t F(s)\,ds, \quad v_{\text{spin}}(t) = \frac{R^2}{I}\int_0^t F(s)\,ds.$$

At some time T the sliding stops and the rolling begins; we then have $v_{\text{center}}(T) = v_{\text{spin}}(T)$ or

$$v_0 - \frac{1}{m}\int_0^T F(s)\,ds = \frac{R^2}{I}\int_0^T F(s)\,ds.$$

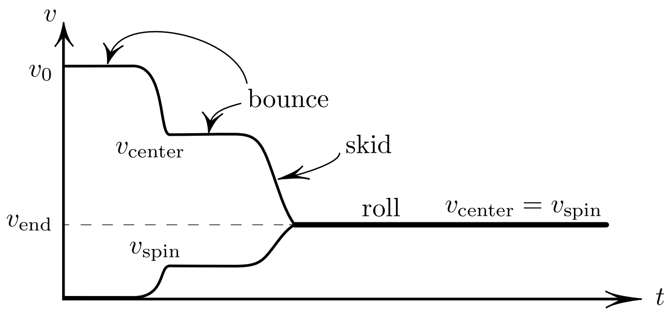

Figure 17. Speeds of the wheel's center and of the wheel's rim (relative to the center). Skidding on ice takes longer to settle into rolling, but the final speed is still the same.

3.16. A cylinder is resting on a sheet of paper on the table; see Figure 18. The paper is yanked away in the horizontal direction from under the cylinder to the right, as shown in the figure. As the result, the cylinder starts spinning counterclockwise and also moving to the right, thus sliding relative to the table. Show that the center of the cylinder stops moving the moment the sliding stops, i.e., that the cylinder comes to rest, i.e., that the cylinder cannot end up rolling. Just as in the preceding problem, the nature of sliding friction is irrelevant.

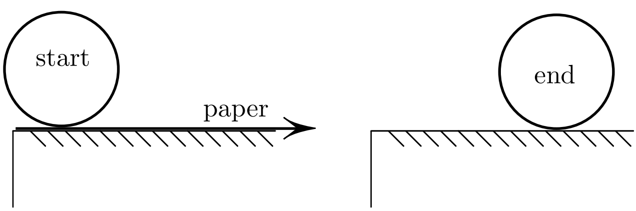

Figure 18. After the sheet is pulled from under the cylinder, the rolling and sliding will stop simultaneously.

Much more on this and related problems can be found in T. Tokieda's American Mathematical Monthly article [**19**].

3.17 (An automatic regulator)**.** Two equal point masses m are connected by a linear zero length spring. The system is launched in weightlessness with each mass given its own arbitrary initial velocity; the angular momentum of the pair relative to its center of mass is nonzero. Assume that the spring is slightly viscous, i.e., that Hooke's law for the spring's tension has an additional term: $F = -kx - \varepsilon\dot{x}$ with a small constant ε. Show that the dumbbell will approach angular velocity $\omega = \sqrt{2k/m}$ as $t \to \infty$, *independent of the initial conditions*(!)

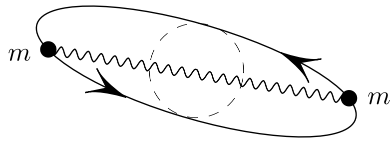

Figure 19. The ellipse is the trajectory in the spring is undamped. Actual trajectories spiral towards the dashed circle. The limiting angular velocity is independent of initial data.

Does the same hold if the masses m_1, m_2 are different, and if so, what is the limiting angular velocity?

Answer. Yes, $\omega = \sqrt{k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$.

Chapter 4

Variational Principles of Mechanics

Lagrange published his equations of mechanics in 1788, 101 years after Newton's *Principia* came out. Although Lagrange's equations are equivalent to Newton's laws, they were a major step along the way that led to quantum mechanics. The advantages of Lagrange's formulation, and of the related Hamilton's principle are listed on page 173.

1. The setting

In this section we formulate Lagrange's equations for systems with several degrees of freedom, extending what was done for the one-degree-of-freedom systems on page 18. The first step is to specify the class of systems for which Lagrange's formulation applies.

We consider a collection of particles whose configuration, i.e., the position of each particle of the system, is defined by an n-tuple of numbers $\mathbf{x} = (x_1, \ldots, x_n)$. The components x_k are called the *generalized coordinates* of the system. Here are some examples of the generalized coordinates:

(1) For a particle in space, the triple of Cartesian coordinates: $\mathbf{x} = (x, y, z)$.

(2) For the same particle, the triple of its spherical coordinates: $\mathbf{x} = (\varphi, \theta, r)$.

(3) For the double pendulum in Figure 1, the angles $\mathbf{x} = (\theta_1, \theta_2)$.

(4) For a rigid body in space, $\mathbf{x} = (x_1, x_2, x_3, \varphi, \theta, \psi)$, where x_1, x_2, x_3 are the coordinates of the center of mass and where φ, ψ, θ are the three Euler angles (these are the angles which specify the body's orientation; see [**10**]).

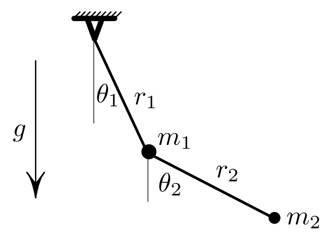

Figure 1. The double pendulum.

In all of the examples above, there are no constraints on infinitesimal displacements of $\mathbf{x}$. In the double pendulum, for instance, any infinitesimal change of θ_i is allowed (although the system does involve constraints: the lengths of the rods are fixed). Such systems are called *holonomic*. By contrast, Chaplygin's sleigh (page 106) has generalized coordinates x, y, θ, with no constraints on themselves, but with a constraint on their velocities; these constraints come from the fact that the skate cannot be displaced "sideways". Systems with *constraints on velocity not derivable from constraints on positions* are referred to as *nonholonomic*, or nonintegrable; other examples include rolling balls, pebbles, or coins. In addtion to our brief discussion on page 106, more on nonholonomic systems can be found in the book [**13**]. All systems considered in this book are holonomic, with the one exception of Chaplygin's sleigh.

2. Lagrange's equations

Consider a mechanical system whose potential energy U is a function of generalized coordinates $U = U(\mathbf{x})$, while its kinetic energy $K =$

$K(\mathbf{x}, \dot{\mathbf{x}})$. Just as in the one-degree-of-freedom case (see page 18), the Lagrangian of the system is defined as the difference:

$$L(\mathbf{x}, \dot{\mathbf{x}}) = K(\mathbf{x}, \dot{\mathbf{x}}) - U(\mathbf{x}). \tag{4.1}$$

The equations of motion of the system are then given by the vector system of *Lagrange's equations*

$$\boxed{\frac{d}{dt} L_{\dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}) - L_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}} \tag{4.2}$$

where

$$L_{\dot{\mathbf{x}}} = \nabla_{\dot{\mathbf{x}}} L = (L_{\dot{x}_1}, \cdots, L_{\dot{x}_n})$$

denotes the gradient of L with respect to the generalized velocity $\dot{\mathbf{x}}$; subscripts stand for partial derivatives. As in the one-dimensional case, when $L_{\dot{x}_i}$ is computed, $\dot{x}_i$ is treated as an independent variable.

3. Examples

1. Two degrees of freedom. Let us write (4.2) more explicitly for a two-degree-of-freedom system. Recall that the kinetic energy of any mechanical system is a quadratic form in the generalized velocities:

$$K = \frac{1}{2} \langle M(\mathbf{x}) \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle,$$

where M is an $n \times n$ positive definite matrix, possibly dependent on $\mathbf{x}$; see Problem 2.30 on page 135 for the proof. With this expression for K we have

$$L_{\dot{\mathbf{x}}} = M(\mathbf{x}) \dot{\mathbf{x}}, \quad L_{x_i} = (M_{x_i} \dot{\mathbf{x}}, \dot{\mathbf{x}}), \quad i = 1, 2,$$

so that the Euler–Lagrange equations (4.2) become

$$\begin{cases} \frac{d}{dt}(M_{11}\dot{x}_1 + M_{12}\dot{x}_2) - (M_{x_1}\dot{\mathbf{x}}, \dot{\mathbf{x}}) + U_{x_1} = 0, \\ \frac{d}{dt}(M_{12}\dot{x}_1 + M_{22}\dot{x}_2) - (M_{x_2}\dot{\mathbf{x}}, \dot{\mathbf{x}}) + U_{x_2} = 0, \end{cases}$$

where M_{ij} is the ijth element of M and M_{x_k} is the derivative of M with respect to x_k. In particular, if $M(\mathbf{x}) = M$ is a constant matrix, then the middle term vanishes and the system becomes

$$\begin{cases} M_{11}\ddot{x}_1 + M_{12}\ddot{x}_2 + U_{x_1} = 0, \\ M_{12}\ddot{x}_1 + M_{22}\ddot{x}_2 + U_{x_2} = 0, \end{cases}$$

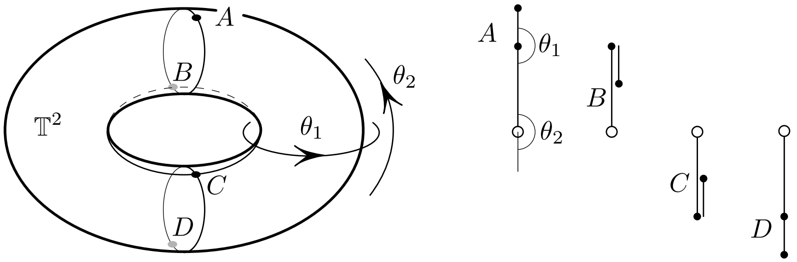

Figure 2. The double pendulum viewed as a point constrained to a 2-torus in $\mathbb{R}^4$. The configurations A, B, C, D are represented by points A, B, C, D on the torus.

or more elegantly,

$$M\ddot{\mathbf{x}} = -\nabla U(\mathbf{x}). \tag{4.3}$$

Exercise 4.1. Write down Lagrange's equations for the double pendulum (Figure 1) in the small angle approximation, i.e., by using approximation $\sin\theta = \theta + O(\theta^3)$ and $\cos\theta = 1 - \theta^2/2 + O(\theta^4)$.

Solution (an outline). It is convenient to use the complex notation: For the position z_1 of the first mass we have $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$; similarly, $z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$. The resulting linearized equation (that is, the equation obtained by deleting higher order terms in θ_i) is of the form $M\ddot{\theta} = -A\theta$ with constant matrices

$$M = \begin{pmatrix} (m_1+m_2)r_1^2 & m_2 r_1 r_2 \\ m_2 r_1 r_2 & m_2 r_2^2 \end{pmatrix}, \quad A = \begin{pmatrix} (m_1+m_2)g r_1 & 0 \\ 0 & m_2 g r_2 \end{pmatrix}.$$

Such linear systems were studied in Section 18, page 108.

2. Geodesics on a surface. Let $\mathbf{q} = (q_1, \dots, q_N)$ be the list of all Cartesian coordinates of all the particles in a system. The constraints on $\mathbf{q}$, if any, define a submanifold $S \subset \mathbb{R}^N$. Thus the system of particles can be thought of as a single particle constrained to a submanifold in $\mathbb{R}^N$; generalized coordinates parametrize S. For an example of the double pendulum, $\mathbf{q} = (x_1, y_1, x_2, y_2) \in \mathbb{R}^4$ is the list of coordinates of the two masses, and the submanifold S is a 2-torus in $\mathbb{R}^4$, parametrized by the generalized coordinates (θ_1, θ_2); see Figure 2. It turns out that any motion of the particle is a geodesic curve, i.e., a locally shortest curve on S, if "shortest" is defined in an appropriate way; this fact is referred to as the Maupertuis' principle, a special case of which is stated in Section 9. In this sense, mechanics is a branch of differential geometry.

4. Hamilton's principle

The setting is the same as in the previous section: a mechanical system with the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}})$ is given. Hamilton's principle is a variational restatement of the Euler–Lagrange equations, and is a central principle of classical mechanics.

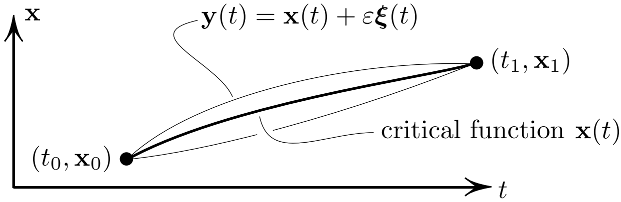

Figure 3. The action is stationary under variations of space-time curves with ends kept fixed.

Hamilton's principle states that the motions of holonomic mechanical systems are critical functions of the action (just as stated for the scalar case on page 20). More precisely, if $\mathbf{x} = \mathbf{x}(t)$ is a critical function of the action functional

$$\mathcal{S}[\mathbf{y}] = \int_{t_0}^{t_1} L(\mathbf{y}, \dot{\mathbf{y}})\, dt, \quad \mathbf{y}(t_0) = \mathbf{x}_0, \quad \mathbf{y}(t_1) = \mathbf{x}_1 \tag{4.4}$$

with fixed ends $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$, then $\mathbf{x}(t)$ satisfies Lagrange's equation

$$\frac{d}{dt} L_{\dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}) - L_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0} \tag{4.5}$$

and, conversely, any solution $\mathbf{x}$ of (4.5) with $\mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1$ is a critical function of the action (4.4). We assume throughout that L has continuous partial derivatives in its variables. In brief, Hamilton's principle states:

$$\delta \mathcal{S}[x] = 0 \quad \Leftrightarrow \quad \frac{d}{dt} L_{\dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}) - L_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}.$$

The proof of Hamilton's principle, i.e., of the above equivalence, is the the same as in the scalar case, and is given in the next section.

5. Hamilton's principle ⇔ Euler–Lagrange equations

The proof of the equivalence follows the proof of the scalar version of this result (page 22) almost verbatim; here are the details. Assume that $\mathbf{x}(t)$ is a critical function of S, meaning that

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} \int_{t_0}^{t_1} L(\mathbf{x}(t) + \varepsilon\boldsymbol{\xi}(t), \dot{\mathbf{x}}(t) + \varepsilon\dot{\boldsymbol{\xi}}(t))\, dt = 0$$

for any differentiable perturbation function $\boldsymbol{\xi} : [t_0, t_1] \to \mathbb{R}^n$ satisfying $\boldsymbol{\xi}(t_0) = \mathbf{0}$, $\boldsymbol{\xi}(t_1) = \mathbf{0}$ (so as to keep the ends of the perturbed path fixed). Now the $d/d\varepsilon$ can be applied to the integrand directly, and we obtain, using the chain rule[1]:

$$\int_{t_0}^{t_1} (L_{\mathbf{x}} \cdot \boldsymbol{\xi} + L_{\dot{\mathbf{x}}} \cdot \dot{\boldsymbol{\xi}})\, dt = 0,$$

where the subscript denotes the gradient with respect to the variable in the subscript, and "$\cdot$" denotes the dot product in $\mathbb{R}^n$. Integrating the second term by parts results in

$$\int_{t_0}^{t_1} (L_{\mathbf{x}} - \frac{d}{dt} L_{\dot{\mathbf{x}}}) \cdot \boldsymbol{\xi}\, dt = 0;$$

the boundary terms vanish since $\boldsymbol{\xi} = \mathbf{0}$ at $t = t_0,\ t_1$. Using the arbitrariness of $\boldsymbol{\xi}$, just as we did on page 22, we conclude that $L_{\mathbf{x}} - \frac{d}{dt} L_{\dot{\mathbf{x}}} = \mathbf{0}$. The proof of the converse goes by retracing the steps in the above proof. ♢

Invariance under a change of variables. Euler–Lagrange equations are invariant under coordinate transformations. To decipher the last sentence, let $l(\mathbf{x}, \dot{\mathbf{x}})$ be a Lagrangian, and let $\mathbf{f}$ be a smooth change of variables: $\mathbf{x} = \mathbf{f}(\mathbf{X})$. To define the Lagrangian in the new variables X, note that

$$\dot{\mathbf{x}} = \mathbf{f}'(\mathbf{X})\dot{\mathbf{X}},$$

where $\mathbf{f}'(\mathbf{X})$ is the Jacobi derivative matrix (see footnote on page 135 for the definition of the Jacobi derivative matrix of $\mathbf{f}$). Substituting this into the old Lagrangian l we get the Lagrangian in terms of $\mathbf{X}$:

$$l(\mathbf{x}, \dot{\mathbf{x}}) = l(\mathbf{f}(\mathbf{X}), \mathbf{f}'(\mathbf{X})\dot{\mathbf{X}}) \stackrel{\text{def}}{=} L(\mathbf{X}, \dot{\mathbf{X}}).$$

[1] which states: $\frac{d}{dt} f(x_1(t), x_2(t), \dots, x_N(t)) = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_N}\frac{dx_N}{dt} = \nabla f \cdot \dot{\mathbf{x}}$.

By the invariance of the Euler–Lagrange equation with respect to the change $\mathbf{x} = \mathbf{f}(\mathbf{X})$ we mean that

$$\frac{d}{dt} l_{\dot{\mathbf{x}}} - l_{\mathbf{x}} = 0 \text{ is equivalent to } \frac{d}{dt} L_{\dot{\mathbf{X}}} - L_{\mathbf{X}} = 0.$$

An immediate proof of this equivalence amounts to the observation that a critical function of a functional remains critical under smooth changes of variables. As a plodding alternative, one can substitute $\mathbf{x} = \mathbf{f}(\mathbf{X})$ into the Euler–Lagrange equation for $\mathbf{x}$ and massage the resulting equation into the desired form. This exercise is still useful, however, and is left to the reader.

6. Advantages of Hamilton's principle

Although Hamilton's principle is equivalent to Newton's laws (in the case of holonomic conservative systems), it offers major advantages over Newton's formulation. On the practical side, Hamilton's principle, in the form of Euler–Lagrange equations, makes it *much* easier in practice to write the equations of motion in most cases — a huge advantage for engineers; see Problem 4.6 for an example. On the theoretical side, Hamilton's principle has the following advantages over Newton's equations:

(1) Hamilton's principle (HP) leads naturally to Hamilton's equations (see Chapter 8).

(2) HP explains *why* the phase volume is conserved — a property which seems like a miraculous coincidence if explained via Newton's laws (see Chapter 8).

(3) HP makes Noether's theorem obvious (see Chapter 8).

(4) HP leads to the Schrödinger equation of quantum mechanics via Feynman's approach (see Chapter 8).

(5) HP makes it clear why the Euler–Lagrange equations are invariant under changes of coordinates (as was explained at the end of the last section).

(6) HP makes the Poincaré integral invariants obvious (see Chapter 8).

7. Maupertuis' principle — some history

Well over a century before Hamilton's principle was introduced, its special case was discovered. This case is of independent interest, and is described in this section.

Some history. The so-called Maupertuis principle, while being a consequence of Hamilton's principle, historically precedes it. Leibnitz knew Maupetuis' principle in 1707 (when Maupertuis was 9 years old), although Leibnitz's discovery remained largely unknown. Euler published the principle in 1744, in a beautifully written paper. In the same year Maupertuis' paper came out.[2] Maupertuis' paper is also very clear, but, unlike Euler's, it seems (to me, at least) to contain a crucial mistake, misstating Fermat's principle by misplacing the speed of light from the denominator into the numerator in the action integral.

Having read these two papers I was drawn to the conclusion that the credit for the least action principle belongs to Euler. Subsequently I learned that Caratheodory arrived at the same conclusion much earlier, after considerable research.[3] It is in fact unclear why the principle took on Maupertuis' name, since the main original part in Maupertuis' 1744 paper seems to be the mistake mentioned above; perhaps references to (perceived) philosophical/theological significance of the least action principle played a role in exciting general interest.

Seven years after Maupertuis' publication, in 1751, Samuel König, a mathematician in the employ of King Frederick of Prussia, claimed that Leibnitz discovered this principle in 1707, showing a copy of the letter from Leibnitz; the original seems to have been lost. In the ensuing controversy, the King of Prussia and Euler took Maupertuis' side, while Voltaire defended König. Some accused König of forgery. Some 150 years later Leibnitz's letters confirming König's claim were

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[2] "Accord between different laws of Nature that seemed incompatible"; it can be found on Wikipedia. According to Maupertuis, "Nature is thrifty in all its actions."

[3] As pointed out by H. H. Goldstine in *A History of the Calculus of Variations from the 17th Through the 19th Century*, Springer-Verlag, New York, Heidelberg, Berlin, 1980. For a discussion of the same topic, see also W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory*, 3rd Ed., Dover, New York, 1968, as well as
`http://en.wikipedia.org/wiki/Principle_of_least_action#cite_note-mau44-1`.

discovered in Bernoulli's archive. More details can be found in the references mentioned in the footnote.

Before giving a general statement of Maupertuis' principle, let us consider an example.

8. Maupertuis' principle on an example

The setting. Figure 4 shows the path AB of a projectile in the gravitational field with constant gravitational acceleration $-g$ along the y-axis. The following is crucial: *We limit our attention to motions with a fixed value of total energy:* $mv^2/2 + mgy = E$. With E fixed once and for all, the speed $v = \sqrt{2(E/m - gy)}$ becomes a function of position.[4] Given any curve γ in the allowed region $y \le E/mg$, we consider the "weighted length" of γ, namely

$$\mathcal{A}[\gamma] = \int_\gamma v\, ds, \tag{4.6}$$

where $v = v(y) = \sqrt{2(E/m - gy)}$ and where ds is an element of arc length. We will call this integral the *action*. Note that v is well defined, since γ is in the allowed region.

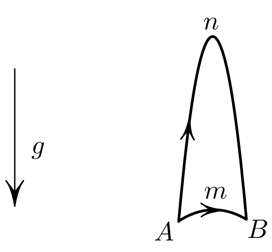

Figure 4. Two trajectories with the same energy connecting A and B. The action is minimal along AmB and of saddle type along AnB.

Maupertuis' principle (in this example) states: *Any projectile trajectory connecting points A and B and having a fixed energy E is a*

[4]Note that v is defined only if the height $y \le E/mg$; the region above $y = E/mg$ is forbidden lest the expression under the square root becomes negative. Physically, fixing E puts a limit on how high a projectile can climb.

critical curve of the action (4.6)*, and conversely, any critical curve of* (4.6) *with fixed endpoints is a projectile trajectory with energy* E*.*

Figure 4 shows two possible trajectories AmB and AnB with the same prescribed energy (these are the only possible trajectories with these data). Both paths are critical curves of the action, but only AmB is the minimizer. The Hessian of the action along AnB has one negative eigenvalue; in other words, the Morse index of AnB is 1.

9. Maupertuis' principle — a more general statement

The setting. We consider the motion of a point mass in a potential $U(\mathbf{x})$ in $\mathbb{R}^n$. From now on, we fix a value E of energy,

$$\frac{m\dot{\mathbf{x}}^2}{2} + U(\mathbf{x}) = E,$$

which slaves the speed to the position via

$$v = |\dot{\mathbf{x}}| = \sqrt{2(E - U(\mathbf{x}))/m};$$

note that this does not restrict the direction of $\dot{\mathbf{x}}$. Just as in the example with the projectile, for any curve γ connecting two given points A and B in $\mathbb{R}^n$ (this curve need not be a trajectory) we define the action

$$\mathcal{A}[\gamma] = \int_\gamma v\, ds. \tag{4.7}$$

For the action to be well defined we must assume in addition that v is well defined, i.e., that γ lies in the allowed set $\{\mathbf{x} : U(\mathbf{x}) \le E\}$ (this set is referred to as *Hill's region*).

Theorem 4.1 (Maupertuis' principle)**.** *Critical curves of the action* (4.7) *defined for smooth curves* γ *with prescribed ends* A*,* B *are trajectories of the Newtonian particle obeying* $m\ddot{x} = -\nabla U(\mathbf{x})$ *with energy* E*, and vice versa.*

Proof. Let $\mathbf{x} = \mathbf{x}(t)$ be a parametrization of γ by a parameter t, arbitrary so far but later to be specified as the time. Using $ds = |\dot{\mathbf{x}}|dt$ we get

$$\int_{t_0}^{t_1} \sqrt{2(E - U(\mathbf{x}))}\; |\dot{\mathbf{x}}|\; dt, \quad \mathbf{x}(t_0) = A, \quad \mathbf{x}(t_1) = B.$$

Now if γ is a critical curve, then $\mathbf{x}$ satisfies the Euler–Lagrange equation. To write this equation we must differentiate the above integrand by $\dot{\mathbf{x}}$; using the identity $\frac{\partial}{\partial \mathbf{v}}|\mathbf{v}| = \mathbf{v}/|\mathbf{v}|$ for $\mathbf{v} \in \mathbb{R}^n$, we get

$$\frac{d}{dt}\left(\sqrt{2(E-U(\mathbf{x}))}\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|}\right) - \frac{\partial}{\partial \mathbf{x}}\left(\sqrt{2(E-U(\mathbf{x}))}\ |\dot{\mathbf{x}}|\right) = 0. \tag{4.8}$$

So far t has been arbitrary; now we specify t so that as to satisfy the energy condition $\dot{\mathbf{x}}^2/2 + U(\mathbf{x}) = E$, i.e.,

$$|\dot{\mathbf{x}}| = \sqrt{2(E-U(\mathbf{x})}. \tag{4.9}$$

With this choice, there is cancellation in the first term in (4.8); this term becomes simply $\ddot{\mathbf{x}}$. In the second term in (4.8), we take the gradient $\frac{\partial}{\partial \mathbf{x}}$ and then substitute (4.9). Cancellation happens again and the term simplifies to $\nabla U(\mathbf{x})$. In summary, (4.8) reduces to $\ddot{\mathbf{x}} + \nabla U(\mathbf{x}) = 0$, so that $\mathbf{x}(t)$ is a trajectory. Furthermore, the energy of this trajectory is E, according to (4.9). This completes the proof. $\diamondsuit$

10. Discussion of the Maupertuis principle

How do trajectories "know" to minimize action? One answer would be to refer to the proof of Theorem 4.1. Still, doesn't the ability to choose the shortest path imply some knowledge of other paths? How can a trajectory "know" other paths? It is this kind of question that led Feynman to some of his discoveries; see page xix for a brief discussion.

How to tell whether the action is minimal or merely critical. This question is answered here for the example of a projectile; a general answer is given in Chapter 7. Figure 5 shows a set of projectile trajectories launched from A with the same speed. The envelope of the family is a curve to which each trajectory is tangent.[5] The point C of tangency of a trajectory with the envelope is said to be *conjugate* to A. The beautiful criterion of minimality is simply this: *an arc AX minimizes the action if and only if this arc does not contain a point*

[5] One can show that the envelope of this family is the parabola $y = a - bx^2$ with $a = v_0^2/2g$ and $b = g/2v_0^2$.

conjugate to A. For example, the arc AB in Figure 5 minimizes the action, while the arc ACD does not. In fact, more can be said: The number of negative eigenvalues of the Hessian of $\mathcal{A}$ defined on an arc AX equals the number of points conjugate to A on the arc. This number is called the *Morse index*; see [**12**]. The beautiful theory of the Morse index of critical functions is described in Chapter 7.

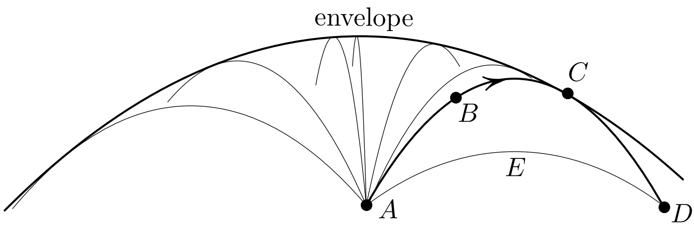

Figure 5. AB minimizes the action, since the point C conjugate to A does not lie on AB. Arc $ABCD$ is not minimal since C lies on it. Arc AED is minimal. The Morse index of $ABCD$ is 1 since this trajectory contains one point (C) conjugate to A.

Concavity of trajectories. Maupertuis' principle "explains" why the trajectory of a projectile is bent in an arc. Indeed, at greater heights the speed v is less since the total energy is fixed. Therefore, to minimize $\int v\, ds$ it pays to bend the path in the direction of lesser v, i.e., upwards. To be a little more precise, let us see why the curve AqB in Figure 6 is "better" than the straight line ApB. The bending of the straight line ApB has two competing effects on the action $\int v\, ds$. On the one hand, the bent curve AqB is longer; on the other hand, $\mathbf{v}$ along AqB is smaller. Which effect wins? The length increases by $O(\varepsilon^2)$ (the proof is left to the reader), but v decreases by $O(\varepsilon)$ on the average; the decrease wins and thus $\mathcal{A}(AqB) < \mathcal{A}(ApB)$.

Hamilton and Maupertuis principles juxtaposed. Hamilton's principle fixes both ends of the curve in *space-time:* the arrival and departure times and locations are prescribed. Maupertuis' principle, by contrast, fixes the energy and the endpoints in space, but not the times (the travel time can be found afterwards by computing the speed v from the known energy and by integrating $dt = ds/v$).

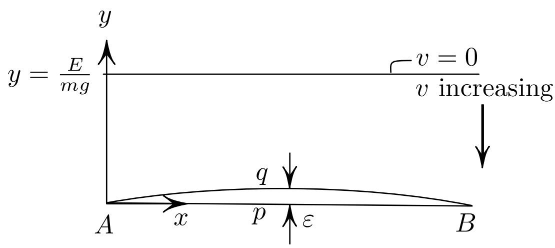

Figure 6. Minimization of $\int v\, ds$ "explains" why the trajectories are bent upwards.

Hamilton's principle gives the path in space-time; Maupertuis' principle gives only the projection of this path onto the space. This is a bit like prescribing the path of the train, but not the times of arrival/departure (unfortunately, what Amtrak often does in practice).

11. Problems

4.1. Consider a conservative force field $\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x})$ in $\mathbb{R}^2$ whose equipotential lines are straight. Consider motions of a particle in this field, with a fixed value of total energy. Prove that along any trajectory one has $v \sin\theta = \text{const.}$, where θ is the angle between the normal to the equipotential and the trajectory and $v = |\dot{\mathbf{x}}|$. Does $v \sin\theta = \text{const.}$ still hold if the equipotential lines are curved?

4.2. Maupertuis principle follows from Newton's law, but not the other way around. Where in the derivation of the former from the latter does one lose information?

4.3. Find the function $y = f(x)$ defined on $[0, 1]$ with $f(0) = f(1) = 0$ which minimizes the integral

$$\int_0^1 \sqrt{H - f(x)}\sqrt{1 + f'(x)^2}\, dx. \tag{4.10}$$

Can you do this without appealing to the Euler–Lagrange equation?

Hint. This integral represents $\int v\, ds$ for a projectile in a constant gravitational field.

4.4. What does the Maupertuis' principle (MP) say about a particle moving on a line?

Answer. Nothing. MP is useful only in dimensions $n \geq 2$. By contrast, Hamilton's principle will give full information on the motion $x = x(t)$.

4.5. Write down the equations of motion for a bead sliding on the surface $z = f(x, y)$ in $\mathbb{R}^3$ under the influence of gravity pointing down the z-axis. To better appreciate the labor-saving advantage of Lagrange's equations, try doing this also using Newton's laws.

4.6. Write the equations of motion for the double pendulum in Figure 1 using the Lagrangian approach. Reproduce the result using Newton's second and third laws.

4.7. Show that the configuration space of the double pendulum is the two-dimensional torus $\mathbb{T}^2$. In other words, show that there exists a continuous one-to-one correspondence between the shapes of the double pendulum and the points on $\mathbb{T}^2$.

4.8. Write down the Lagrangian of a particle confined to the surface of revolution $x^2 + y^2 = R^2(z)$ where $R(z)$ is a positive function.

Hint. Should be read backwards: .setanidrooc dezilareneg elbatius esoohC

4.9. Consider a particle constrained to the unit sphere centered at the origin, with no other forces, including gravity, applied to the particle. Write down the Lagrangian of this particle using (i) spherical coordinates, and (ii) the (x, y)-coordinates, assuming in the latter case that the particle is in the upper hemisphere.

4.10. A weightless wheel with a point mass m attached to its rim is placed on the table (with the wheel's plane vertical). The wheel rolls without sliding. Write down the equations of motion of the wheel assuming that the wheel does not lose contact with the table and that the wheel has a nonzero moment of inertia.

4.11. In the situation of Problem 4.10, assume that the contact with the table is perfectly frictionless, i.e., that the table is perfectly slippery. Write the equation of motion of the wheel.

4.12. Figure 7 is a schematic rear view of a biker traveling away from the observer. The bike+rider can be treated as a rigid rod with a homogeneous mass distribution. If the surface is perfectly slippery, then the wheels begin to slide sideways. Write the differential equation governing the change of angle θ valid for as long as the tires are in contact with the ice. Find the angle at which the wheels lose contact with the ice, assuming that the rod started in a nearly vertical position, nearly at rest, and that the contact is frictionless.

4.13. A pendulum of length ℓ and of mass m is mounted on the cart of mass M. The cart rolls on the table without friction. Write the equations of motion of the system. Verify that in the limit of large M/m the equation of the simple pendulum emerges.

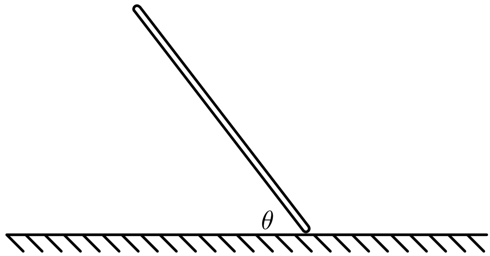

Figure 7. A biker on ice.

For the next problem, consider a smooth curve γ on a smooth surface S in $\mathbb{R}^3$. *Geodesic curvature* of γ at a point $p \in \gamma$ is defined as the curvature at p of the planar curve obtained by projecting γ onto the plane tangent to S at p. Intuitively, geodesic curvature is what the two-dimensional inhabitant of the surface who thinks the surface is flat perceives as a planar curvature.

4.14. Show that the shortest curve on S connecting two given points $A, B \in S$ has zero geodesic curvature. This means that a car driving on S along a geodesic must point its wheels straight, without turning left or right.

4.15. (a) Prove that the shortest paths between two points on the sphere are arcs of great circles. (b) Prove that the shortest path between two points on a cylinder is a helix.

4.16. Find all shortest paths along the surface of a cube between two diagonally opposite vertices.

Chapter 5

Classical Problems of Calculus of Variations

1. Introduction and an overview

Calculus of variations deals with the problem of minimizing scalar-valued functions of curves, or of functions. Hamilton's principle is a prime example where this problem comes up — we recall from the last chapter that, according to Hamilton's principle, the motion $x(t)$ of a particle makes the integral

$$\int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, dt$$

critical; here $L = \frac{m\dot{x}^2}{2} - U(x)$ is the Lagrangian of the particle. In other words, the nature is an analog computer which can find a minimum, or at least a critical function, of the above integral. Here are some particular examples of such problems:

1. The brachistochrone problem: Find the shape of the curve connecting two points A and B in the vertical plane such that a bead released from A and sliding along the curve will reach B in least time.

2. The catenary: For a chain with fixed endpoints, find the shape which minimizes potential energy (this is the equilibrium shape).

3. The soap film: Of all the surfaces with a given boundary, find the one with the least area.

4. The beehive problem: How to build the walls in a beehive with a minimum of wax.[1]

The first three of these problems, among others, are described in this chapter.

Calculus of variations is one of the richest and most fascinating mathematical subjects. One reason for this may be the fact that many fundamental physical laws — not only in mechanics, as we saw, but also in electromagnetism (Maxwell's equations), can be formulated as variational principles. Physical principles play a central role in physics, and therefore in mathematics, and in particular, in ordinary and partial differential equations and in differential geometry. The theory of *optimal control* is a more recently developed branch of calculus of variations, and is important in engineering applications. Just to give an example, here is a simple (but still nontrivial) question: How do we stop a harmonic oscillator in shortest time using a force which cannot exceed a prescribed value? This problem is solved in the next chapter on page 246.

2. Dido's problem — a historical note

Although much of classical mechanics is built on Hamilton's variational principle, historical origins of calculus of variations lie not in dynamics but in geometry, in a problem that is over 2,800 years old. The famous Dido's problem asks us to *find the curve of given length which encloses maximum area of land adjacent to the shore*, Figure 1.

A more modern equivalent problem is to find the path of an airplane bounding maximal possible area of land, given the plane's speed and the time of flight.

[1] This problem was considered by Pappus of Alexandria, who remarks in his work that bees know only as much geometry as they need. In the words of Lord Kelvin, Pappus then "proceeds to apply what he calls his own superior human intelligence to investigation of useless knowledge".

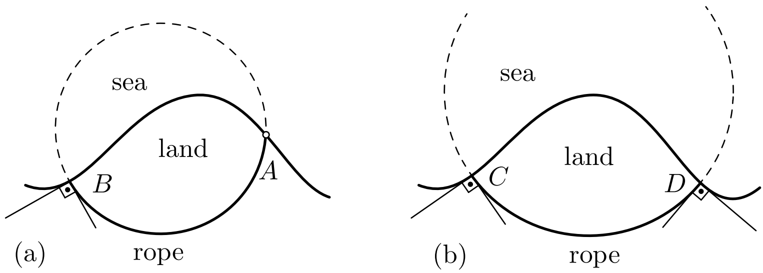

Figure 1. Dido's problem. The point A is prescribed; B, C and D are free.

Some history. Queen Dido of Tyre — a city in the present day Lebanon — fled her home after her husband, who was simultaneously her uncle, was murdered for fiscal reasons at the instigation of Dido's brother Pygmalion. Fleeing her dysfunctional and dangerous family, Dido arrived on the coast on North Africa circa 825 BC in the company of her servants and in possession of her wealth. Dido obtained a grant from the local Berber chief to a piece of land along the coast — a piece as large as she could enclose by an ox hide. Dido had a hide cut into an extremely long thin strip which was used to bound a piece of land; on this land she built Carthage (presently a suburb of Tunis).

The area-maximizing shape of the rope is a circular arc (Figure 1), whether the ends of the rope are prescribed or free. At a free end the arc must be perpendicular to the shore, as in Figure 1. Incidentally, almost all calculus students go through a baby version of Dido's problem — that of finding the proportions of a *rectangular* lot of maximal area bounded by a (straight) river on one side and by a fence of prescribed length on the remaining three sides.

3. A special class of Lagrangians

All the problems discussed in this chapter lead to functionals of a special form:

$$\int_a^b F(y)\sqrt{1+y'^2}\,dx = \int_a^b F(y)\,ds. \tag{5.1}$$

Note that ds is the element of arc length of the curve $y = y(x)$, since

$$\sqrt{1+y'^2}\,dx = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{dx^2+dy^2} = ds.$$

The main result of this section is the following.

Theorem 5.1. *If $y = y(x)$ minimizes* (5.1) *subject to fixed end conditions*

$$y(a) = A, \quad y(b) = B, \tag{5.2}$$

then y satisfies

$$\frac{F(y)}{\sqrt{1+y'^2}} = \text{const.}, \tag{5.3}$$

or its equivalent version

$$F(y)\sin\theta = \text{const.}, \tag{5.4}$$

where the meaning of θ is explained by Figure 2.

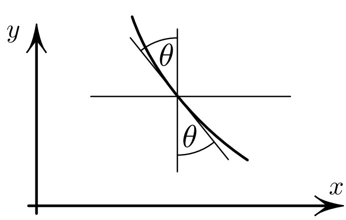

Figure 2. θ is the angle between the tangent and the y-axis.

Remark 5.1. The integral (5.1) is the weighted sum of Euclidean lengths ds with weights $F(y)$, and is referred to as the *length of γ in the metric $F(y)\,ds = d\rho$.* The metric is said to be *conformal* because an infinitesimal circle in the sense of the metric ρ is also a Euclidean circle. This is so because $\mathbf{F}$ does not depend on the direction of the curve.[2]

[2]This is so even if F depends on x as well: $F = F(x, y)$, and so this more general case is also conformal.

Proof. Since the integrand in (5.1) does not depend on x, Noether's theorem (page 23) applies, and thus the minimizer satisfies

$$y' L_{y'} - L = c = \text{const.}$$

With $L = F(y)\sqrt{1 + y'^2}$ this gives (5.3), after a short simplification. To obtain (5.4), note that $y' = \cot\theta$; substituting into (5.3) and simplifying gives (5.4). $\diamondsuit$

This proof is short, but it lacks an "aha!" moment. The elementary solution given in the next section does have such a moment. This solution is based on a mechanical analogy, in the spirit of Archimedes.

Boundary conditions. (5.3) is a first order differential equation for the unknown function $y = y(x)$; the solution depends on one constant of integration, for a total of two arbitrary constants (the other one being the right-hand side). Varying these two parameters should allow us to satisfy the two boundary conditions – provided they can be satisfied.

4. The shortest way to the smallest integral

Here is a remarkably short shortcut to (5.4), bypassing the Euler–Lagrange equation and Noether's theorem. This method is based on endowing (5.1) with the mechanical interpretation as the potential energy of a system of springs (see Figure 3); the solution (5.4) simply expresses an equilibrium condition. Here are the details.

Step 1: discretization. Let us divide the (x, y)-plane into many thin strips $y_k \le y < y_{k+1}$, approximate $F(y)$ by the step function with the constant value $F_k = F(y_k)$ in each strip, and replace the curve γ by a broken line. The integral (5.1) then is approximated by the sum

$$\sum F_k \Delta s_k, \tag{5.5}$$

where Δs_k is the length of the kth segment in Figure 3. Note that the weights F_k are fixed, and that we can vary s_k by sliding the break points of the broken line along the lines $y = y_k$. The goal is to slide these points to their optimal positions, the ones that minimize (5.5). A mechanical analogy will tell us how to do this.

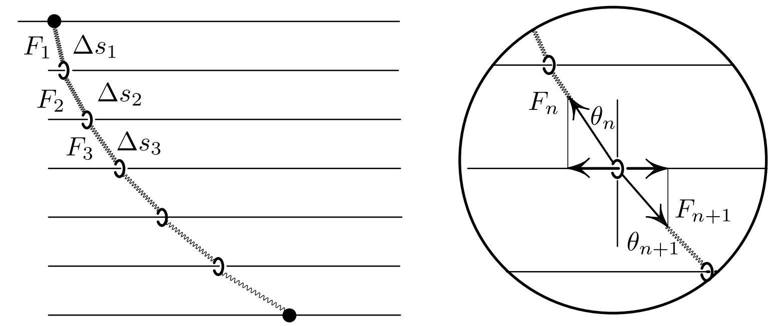

Figure 3. Minimizing $\int F(y)\, ds$ by a mechanical "analog computer".

Step 2: an "analog computer". As a thought experiment sketched in Figure 3, let frictionless rings slide without friction along the fixed rods $y = y_k$; connect each pair of neighboring rings by a *constant tension* spring[3] with tension F_k. The end rings A and B are held fixed. Now (5.5) is the potential energy of our a system; indeed, potential energy of the kth spring is the work required to stretch it from zero length to its current length Δs_k, i.e. $F_k \Delta s_k$.

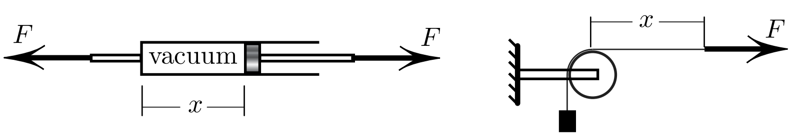

Figure 4. Two ways to construct a constant tension "spring". The potential energy of each "spring" is xF.

Step 3: the conclusion. If the potential energy (5.5) is minimal, then each ring is in equilibrium and so the forces in the x-direction are in balance:

$$F_n \sin\theta_n = F_{n+1} \sin\theta_{n+1},$$

or

$$F_k \sin\theta_k = \text{const.};$$

in the continuous limit this becomes (5.4). ♢

[3]Such a spring can be realized by hanging a weight as in Figure 4, or as a piston in a cylinder with vacuum in the same figure.

The following exercise gives a nice interpretation of the momentum and the Hamiltonian for the Lagrangian in (5.1).

Exercise 5.1. Using Figure 3, find a mechanical interpretation of the momentum $p = L_{y'}$ and of the Hamiltonian $H = y'L_{y'} - L$ associated with the Lagrangian $L = 2\pi y\sqrt{1+y'^2}$ in (5.1).

Answer. The Hamiltonian $H = F\sin\theta$ is the horizontal component of the spring's tension; $p = F\cos\theta$ is the vertical component of the spring's tension. Remarkably, H and p are simply the coordinates of the vector of tension: $\mathbf{F} = (H, p)$!

In the following few sections we will apply (5.4) to three classical problems.

5. The brachistochrone problem

This problem was proposed by Johann Bernoulli in June of 1696:

Find the shape of a trough connecting two points A and B in the vertical plane (Figure 5) such that a particle released from rest at A and sliding down this trough without friction will reach B in least time.

Of course, the height of B must not exceed that of A for the solution to exist.

Some intuition. Figure 5 illustrates some crude guesses. Straight line 4 may seem like a good answer until one realizes that 3 is probably better: it pays to gain speed as early as possible. This suggests that the tangent to the solution at A should be vertical, and that the solution must be a concave curve. Beyond that, it is difficult to say more without further analysis, which is given next.

The descent time. Let us introduce the coordinate axes as shown in Figure 5, with the y-axis pointing straight down. We will show that the time of travel along the curve AB is given by the line integral

$$T = k\int_{AB}\frac{ds}{\sqrt{y}}, \quad \text{where} \quad k = \frac{1}{\sqrt{2g}}. \tag{5.6}$$

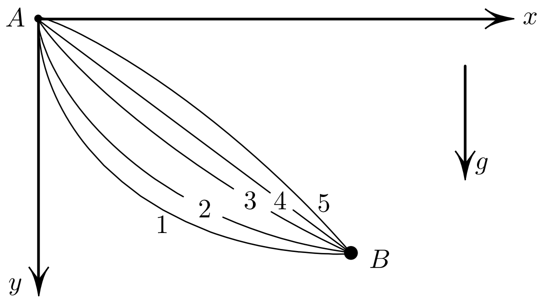

Figure 5. Which path takes least time?

Indeed,

$$T = \int_{AB} dt = \int_{AB} \frac{ds}{v}, \tag{5.7}$$

where the speed $v = v(y)$ is determined by the height from the energy conservation:

$$\frac{v^2}{2} - mgy = E = \text{const};$$

the minus is due to the fact that the y-axis points down. Now $E = 0$ since $v = 0$ at $y = 0$ — the bead is released from rest. Solving for v and substituting in (5.7) proves (5.6).

Theorem 5.2. *Assume that $y_A \geq y_B$. The brachistochrone (i.e., the critical function of* (5.6)*) is an arc AB of a cycloid generated by the point on a circle rolling without sliding on the horizontal line through A, Figure* 6*.*

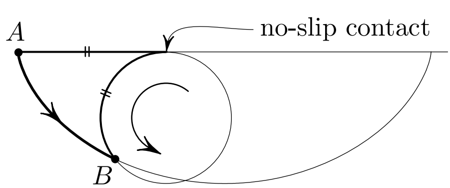

Figure 6. The brachistochrone is an arc of the cycloid with a cusp at A and passing through B.

Remark 5.2. Figure 7 illustrates different possibilities depending on the relative positions of the starting and ending points. The brachistochrone dips below B if and only if the line AB has slope $|s| < 2/\pi$.

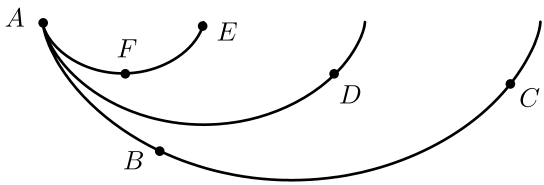

Figure 7. Quickest paths from A to various points in the vertical plane. Note that the quickest path may dip below the destination point.

Proof of the theorem. Our integral (5.6) is of special form $\int F(y)\,ds$, where $F = 1/\sqrt{y}$. As we showed before, minimizing curves satisfiy $F(y)\sin\theta = \text{const.}$, or

$$\frac{\sin\theta}{\sqrt{y}} = c = \text{const.}, \tag{5.8}$$

where θ is the angle between the tangent to the curve and the vertical.

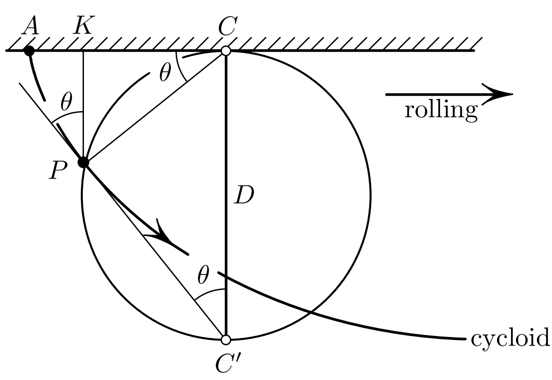

Figure 8. Proving that the cycloid satisfies (5.8).

Figure 8 encapsulates the proof that (5.8) describes a cycloid. Consider a circle rolling on the "ceiling" (the horizontal line through A) without sliding; a point P on the rolling circle traces out a cycloid.

Let A be the initial position of P on the "ceiling". Now the chord PC' where C' is the diametric opposite of the contact point C, is tangent to the cycloid (a quick proof is in the next paragraph). But then $\theta = \angle PCK = \angle PC'C$, so that

$$y \stackrel{\text{def}}{=} PK = PC\sin\theta = D\sin^2\theta,$$

which can be written in the form (5.10):

$$\frac{\sin\theta}{\sqrt{y}} = D^{-1/2}.$$

We also see how k defines the diameter of the circle: $k = D^{-1/2}$.

It remains to prove that PC' is tangent to the cycloid. Now the velocity of P, i.e., tangent to the cycloid, is perpendicular to the position vector CP since C is the instantaneous center of rotation and since the wheel is a rigid body. But also $PC' \perp CP$ since $\angle CC'$ is a diameter. This proves that PC' is indeed tangent to the cycloid.

We showed that cycloids satisfy the ODE (5.8). Since any initial condition can be satisfied by a properly chosen cycloid, we conclude from the uniqueness theorem that no solutions other than cycloids exist. $\diamondsuit$

Remark 5.3. Note that we only proved that the cycloid is a critical function; however, by using the Jacobi criterion of Chapter 6 one can show that it is indeed a minimizer.

6. Johann Bernoulli's solution of the brachistochrone problem

In section 4 we used a mechanical analogy to show that curve minimizing $\int_\gamma F(y)\,ds$ satisfies $F(y)\sin\theta = \text{const}$. Bernoulli's beautiful idea leads to the same result, but via an optical analogy. I reproduce Bernoulli's solution because of its historical interest, although the solution on page 187 is shorter and more self-contained, since it does not appeal to Fermat's principle or Snell's law.

Before describing Bernoulli's solution, let us recall Snell's law. Consider an optical medium in the (x, y)-plane with speed of light $c(y)$ depending on y only. Snell's law states that for each ray the sine

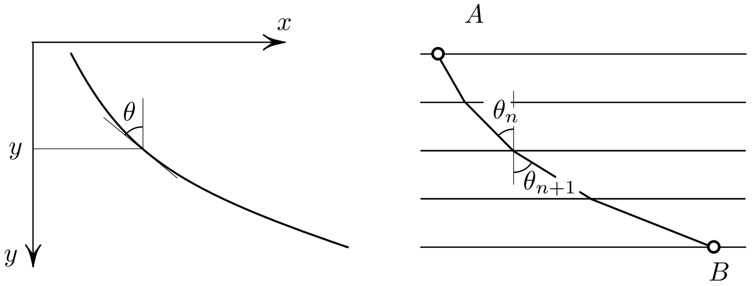

Figure 9. Johann Bernoulli's solution of the brachistochrone problem.

of the angle θ with the y-direction (Figure 9) varies as the speed of light:

$$\frac{\sin\theta}{c} = \text{const.} \tag{5.9}$$

This can be derived from Snell's law

$$\frac{\sin\theta_n}{c_n} = \frac{\sin\theta_{n+1}}{c_{n+1}}$$

for a layered medium in Figure 9, with constant speed c_n in the nth layer.

Bernoulli's idea for finding the brachistochrone was the following. As a thought experiment, superimpose an optical medium over Figure 5, choosing the speed of light $c = v = \sqrt{2gy}$, the same as the speed of the sliding bead. By Fermat's principle, the shape of the ray between A and B minimizes the time of travel

$$\int \frac{ds}{c},$$

which is also the time of travel of the bead because of our choice $c = v$. Therefore the ray "finds" the shape of the brachistochrone, and since the ray satisfies Snell's law (5.9), so does the brachistochrone:

$$\frac{\sin\theta}{\sqrt{2gy}} = \text{const.} \tag{5.10}$$

This essentially solves the problem by reducing it to a differential equation which we already discussed in the preceding section.

7. Geodesics in Poincaré's metric

By the Poincaré's length of a planar curve γ one simply means the modified "length"

$$L[\gamma] = \int_\gamma \frac{ds}{y},$$

where $ds = \sqrt{dx^2 + dy^2}$. For the integral to be well defined and positive we restrict attention to curves in the upper half-plane. The modified length $d\rho = ds/y$ is called the Poincaré metric. As an example, all the horizontal segments in Figure 11 have the same Poincaré lengths.

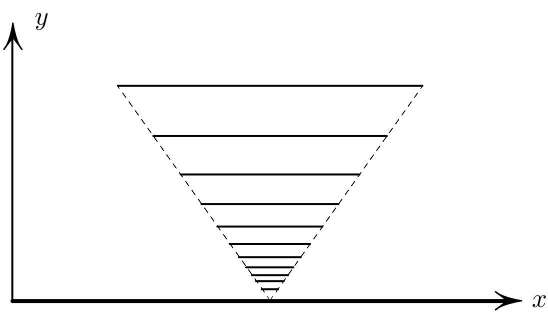

Figure 10. All these horizontal segments have the same length in Poincaré's metric.

A *geodesic* in the Poincaré metric is, by definition, a curve which makes the above integral stationary for variations of the curve with ends fixed. Since the above integral is of the conformal type $\int F(y)\,ds$ discussed in sections 3 and 4, geodesics satisfy $F(y)\sin\theta = c$ (see (5.4)), with $F(y) = 1/y$, or

$$\frac{\sin\theta}{y} = \text{const.} \tag{5.11}$$

Theorem 5.3. *Every geodesic of Poincaré's metric is a semicircle in $y > 0$ with the center on the x-axis or, as a limiting case, a vertical ray $x = c$, $y > 0$. Conversely, every such semicircle or ray is a geodesic in Poincaré's metric.*

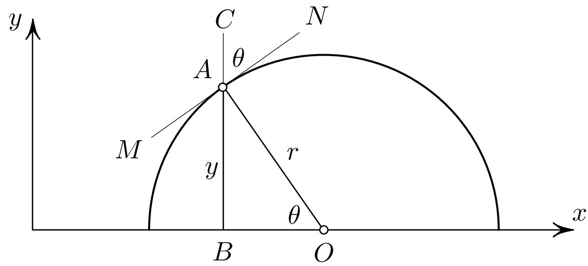

Figure 11. Geodesics in Poincaré's metric are semicircles or vertical rays.

Proof. Referring to Figure 11, consider a semicircle with the center O on the x-axis. Since

$$\theta \equiv \angle CAN = \angle BOA,$$

we conclude from ΔBOA that

$$\sin\theta = \frac{y}{r},$$

showing that (5.11) holds for such semicircles. Vertical rays satisfiy (5.11) with $\theta \equiv 0$ and the zero constant. This proves the first half of the theorem. To prove the converse, let us show that any solution of (5.11) is a semicircle with the center on the x-axis, or else a vertical ray. Indeed, (5.11) is a first order ODE . Given a point A and a slope y' at A, there is a unique solution with these initial data. But the semicircle in Figure 11 already satisfies this ODE (as we showed), and hence there is no other solution by the uniqueness theorem. It remains to consider the case when the initial slope is infinite. But in that case the geodesic must be a vertical ray. Indeed, otherwise there is a point on the geodesic where the slope is finite, but we already showed that such a geodesic is a semicircle whose slope is finite everywhere. The contradiction shows that the geodesic whose slope is vertical at one point is indeed a vertical ray. $\diamondsuit$

The Poincaré metric is the simplest imaginable non-Euclidean metric in the sense explained by the following exercise.

Exercise 5.2. Assume that the speed of light in an optical medium in the upper half-plane varies linearly with y, namely, $c(y) = y$. Using Fermat's principle, show that the rays are the Poincaré geodesics, i.e., semicircles perpendicular to the x-axis.

Solution. According to Fermat's principle, rays minimize (or make critical) the travel time

$$\int_A^B \frac{ds}{c} = \int_A^B ds/y. \qquad \diamondsuit$$

8. The soap film, or the minimal surface of revolution

This classical problem asks us to *find the the surface of least possible area spanning two parallel circular hoops sharing a common axis, Figure* 12*.* Physically, such a minimal surface is realized by a soap film spanning two hoops.[4] Soap film is an "analog computer" which finds the minimal surface. In fact, one can derive the differential equation for the desired shape as the equilibrium condition. Instead, we solve the problem directly by minimizing the area. To that end, we formulate the problem analytically.

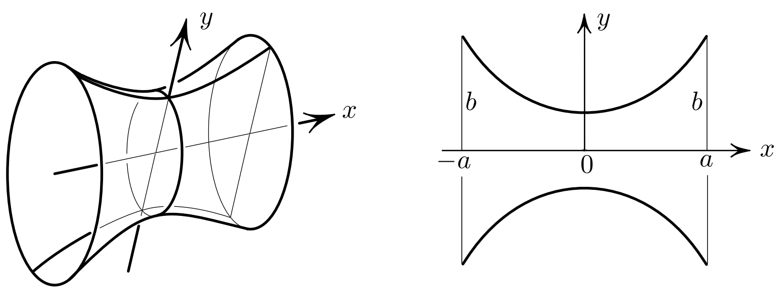

Figure 12. Surface of revolution spanning two hoops.

Analytic formulation. Figure 12 shows the axial cross-section of the surface of revolution spanning the two hoops at $x = \pm a$ of radius b. Revolving the graph of $y = f(x)$ around the x-axis yields the surface area

$$A[f] = 2\pi \int_{-a}^{a} y\sqrt{1+(y')^2}\,dx = 2\pi \int y\,ds; \tag{5.12}$$

[4]If we ignore gravity and other relatively small effects, such as the variation of surface tension with temperature, etc.

here f satisfies the boundary conditions

$$f(-a) = f(a) = b. \tag{5.13}$$

The soap film "finds" f which minimizes the above integral subject to (5.13), and our goal is to do the same.

Finding the minimizing f. The integral (5.12) is again of the conformal type $\int F(y)\,ds$ (see (5.1), page 185), and thus the minimizers satisfy $F(y)\sin\theta = \text{const.}$, or

$$\frac{y}{\sqrt{1+(y')^2}} = \text{const.} \tag{5.14}$$

(this equation has a simple physical interpretation, see Exercise 5.3 on page 199). The general solution of (5.14) is

$$y = \frac{1}{c}\cosh c(x - c_1), \tag{5.15}$$

where c and c_1 are constants, as can be checked by substitution.

Hyperbolic cosine has a nice incarnation as the shape of a soap film stretched on two hoops!

It still remains to satisfy the boundary conditions (5.13) (using the freedom of choice of c, c_1). Intuition suggests that this is not always possible: if we spread the hoops sufficiently far apart, the soap film will snap and form two disjoint flat disks. Let us find the critical distance a, or rather the critical ratio b/a which separates existence from nonexistence.

Note first that $c_1 = 0$, as expected by symmetry; this is confirmed by substituting (5.15) into (5.13) resulting in $\cosh(-a - c_1) = \cosh(a - c_1)$, which implies the claim. The critical function of our area functional, if it exists, is even:

$$y = \frac{1}{c}\cosh(cx), \tag{5.16}$$

and it remains to find for which a, b can we satisfy the boundary condition

$$\frac{1}{c}\cosh(ca) = b \tag{5.17}$$

by a choice of c.

For which (a, b) can (5.17) be satisfied? All the graphs of (5.16) are dilations of one another; see Figure 13.[5] As we change c from 0

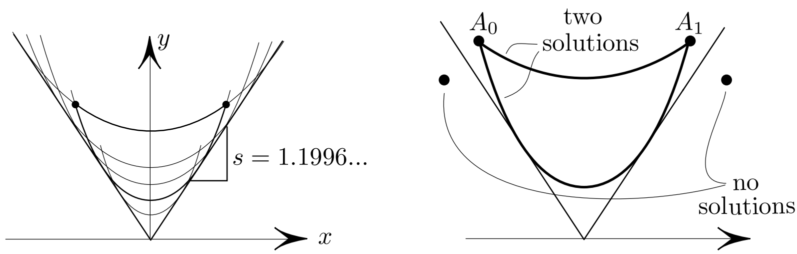

Figure 13. For $b/a > s = 1.1996...$ there are two equilibrium shapes of the soap film. For $b/a < s$ no solutions satisfying $y(\pm a) = b$ exist.

to ∞, these dilations fill out the sector $y \geq s|x|$, where s is the slope of the tangent to $y = \cosh x$ passing through the origin:

$$s = \min_{x>0} \frac{\cosh x}{x} = 1.1996....$$

We conclude that the soap film connecting the two hoops snaps if b/a gets close to the critical value $s \approx 1.2$. Figure 13 shows three possibilities: (i) if $b/a < s$ then *two* curves from the family (5.16) connect $(-a, b)$ with (a, b); (ii) when $b/a = s$, there is precisely one curve connecting the pair, and (iii) if $b/a > s$ then no curves from the family connect the pair.

A minimality criterion. Figure 14(b) shows two solutions for the same boundary conditions. Which of these solutions (if any) minimizes the area? This quesiton is physically significant since the area-minimizing solution is also energy-minimizing and thus stable, i.e., physically observable. To answer this question, we need the concept of *conjugate points*. Fix a critical function of a functional (the thick curve in Figure 14(a)), and consider a narrow fan of other solutions of the Euler–Lagrange equation passing through C_0.

Two points C_0 and C_1 on the graph of the chosen function are said to be conjugate of one another if, loosely speaking, this fan refocuses

[5]This should be clear at the outset, with no calculation, since any dilation of a minimal surface is a minimal surface.

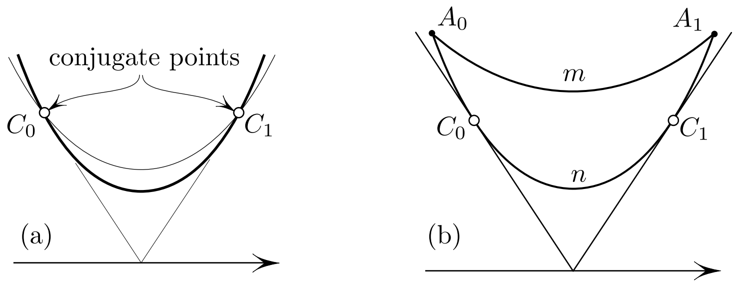

Figure 14. (a): Conjugate points C_0 and C_1. (b): Segment A_0mA_1 is a minimizer since it does not contain a conjugate pair. Segment A_0nA_1 is not a minimizer since it contains a conjugate pair.

at he point C_1 (for a rigorous definition, see Section 1, page 212). Figure 14 suggests that the tangency points C_0 and C_1 are conjugate to each other; this is a reasonable suggestion since two infinitesimally close curves crossing each other near C_0 recross again near C_1. This idea can be turned into a rigorous proof. Now the general minimality criterion states: *a critical function of* (5.12) *is a minimizer if and only if the graph of the function does not contain pairs of conjugate points.*[6]

Which solutions are minimizers? The arc A_0mA_1 in Figure 14 is a minimizer, since it does not contain a conjugate pair. On the other hand, the arc A_0nA_1 is not a minimizer, since it contains a conjugate pair C_0, C_1. We conclude from this that the soap film corresponding to A_0mA_1 is stable, since its area, and hence potential energy of surface tension, is minimal. On the other hand, the shape corresponding to A_0nA_1 is an equilibrium, i.e., is theoretically possible, but will never be realized in practice since it is unstable.

Exercise 5.3. What is a physical interpretation of (5.14)?

Answer. Equation (5.14) states that the x-component $H(x)$ of the surface tension[7] at any cutting plane $x =$ const. does not depend on x. Indeed,

[6] Full details on this are given in Chapter 6.

[7] We denote the tension by H as a reminder that is it the Hamiltonian associated with the Lagrangian in (5.12).

note that

$$H(x) = (2\pi y) \cdot \cos\theta = \frac{2\pi y}{\sqrt{1+y'}}$$

and consider the tube lying between the planes $x = x_0$ and $x = x_1$. This tube is under the action of external tensions $-H(x_0)$ and $H(x_1)$ from the remaining part of the film, and the equilibrium condition $-H(x_0)+H(x_1) = 0$ implies that $H(x) =$ const. since x_0, x_1 are arbitrary.

Exercise 5.4. (See the closely related Exercise 5.1 on page 189.) Find a mechanical interpretation of the momentum $p = L_{y'}$ associated with the minimal surface Lagrangian in (5.12), and of the Hamiltonian $H = y'L_{y'} - L$ associated with this Lagrangian.

Answer. The momentum $p = L_{y'}$ is the integral of the *radial* component of the surface tension at a fixed circle $x =$ const. The hamiltonian H is the integral of the x-component of the surface tension at a fixed circle $x =$ const., as discussed in the preceding exercise.

9. The catenary: formulating the problem

The equilibrium shape of a chain suspended by both ends is referred to as the *catenary*, Figure 15 (*Catena* in Latin means "chain"). Our idealized chain is a perfectly flexible and unstretchable curve with zero thickness but possessing mass, equidistributed over the chain's length. Our goal is to find the shape of the chain which gives it the least potential energy; this shape corresponds to an equilibrium.

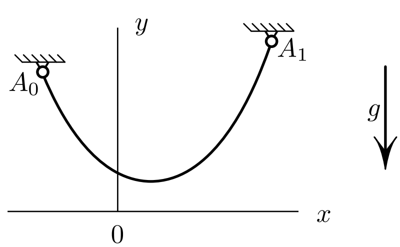

Figure 15. The catenary problem.

Why does minimal potential energy correspond to an equilibrium. Assume that the chain is in the shape of least potential energy and is initially at rest; I claim that it will remain at rest, i.e., that it is in equilibrium. Assume the contrary, i.e., that the chain stars moving spontaneously. Then its kinetic energy becomes positive; the potential energy, on the other hand, does not decrease, and hence the total energy increases from the initial state, which is a contradiction with energy conservation. For full disclosure, we proved conservation of energy only for finite systems of particles, and not for continua, such as our chain, so that the argument in this paragraph has a gap in rigor.

The catenary functional. Consider the chain whose shape is given by $y = f(x)$; so far f is arbitrary. The potential energy of an infinitesimal element of the chain is $dm\, gy = \rho\, ds\, gy = \rho g f(x)\sqrt{1+f'(x)^2}\, dx$, where ρ is the linear density (mass per unit length); the total potential energy is the integral:

$$U[f] = \rho g \int_{x_0}^{x_1} f(x)\sqrt{1+f'(x)^2}\, dx. \tag{5.18}$$

Let us fix the ends of the chain at (x_0, y_0) and (x_1, y_1), thus imposing the boundary conditions on f:

$$f(x_0) = y_0, \quad f(x_1) = y_1; \tag{5.19}$$

and since the length L of the chain is prescribed, we further restrict attention to the functions satisfying the length constraint

$$\int_{x_0}^{x_1} \sqrt{1+(y')^2}\, dx = L. \tag{5.20}$$

The problem is to minimize the energy functional (5.18) subject to the boundary conditions (5.19) and to the length constraint (5.20). To that end we review Lagrange's method of dealing with constraints.

10. Minimizing with constraints — Lagrange multipliers

The catenary problem is an example of the general problem of minimizing a functional

$$\mathcal{F}[x] = \int_{t_0}^{t_1} F(x, \dot{x})dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1 \tag{5.21}$$

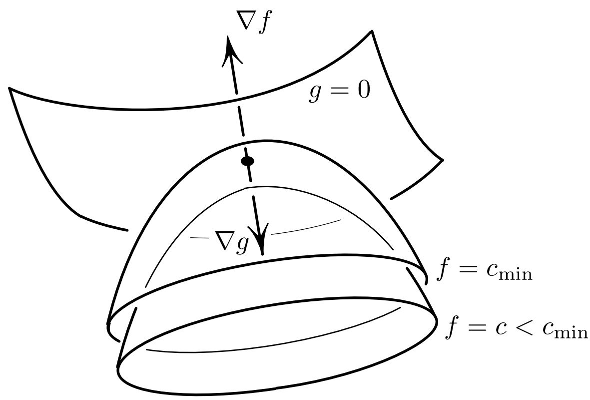

Figure 16. A geometrical interpretation of the method of Lagrange multipliers .

subject to a constraint

$$\mathcal{G}[x] = \int_{t_0}^{t_1} G(x, \dot{x})dt = c. \tag{5.22}$$

Theorem 5.4. *If $x = x(t)$ is a critical function of the functional* (5.21) *subject to the constraint* (5.22)*, then there exists a constant λ such that x satisfies the Euler–Lagrange equation corresponding to the integrand $L = F - \lambda G$:*

$$\frac{d}{dt}((F - \lambda G)_{\dot{x}})) - (F - \lambda G)_x = 0. \tag{5.23}$$

Before proving the theorem, let us review its finite-dimensional analog. One seeks to minimize $f(\mathbf{x})$ subject to $g(\mathbf{x}) = 0$, where $\mathbf{x} \in \mathbb{R}^n$, and where f and g take values in $\mathbb{R}$. Figure 16 explains that if $\mathbf{x}$ is a minimium of f subject to $g = 0$, then the level surfaces of f and of g passing through $\mathbf{x}$ are tangent at $\mathbf{x}$. Thus $\nabla f \parallel \nabla g$, i.e., $\nabla(f - \lambda g) = 0$ for some λ. To make this argument rigorous, let us consider a curve $\mathbf{r} = \mathbf{r}(s)$ lying on the surface $g = 0$, with $\mathbf{r}(0) = \mathbf{x}$; differentiating the identity $g(\mathbf{r}(s)) = 0$ by s at $s = 0$ we get, using the chain rule:

$$\frac{d}{ds} g(\mathbf{r}(s)) = \nabla g(\mathbf{x}) \cdot \dot{\mathbf{r}}(0) = \nabla g(\mathbf{x}) \cdot \mathbf{v} = 0; \tag{5.24}$$

and since $\mathbf{x}$ is a critical point, we have

$$\frac{d}{ds} f(\mathbf{r}(s))_{s=0} = \nabla f(\mathbf{r}(0)) \cdot \mathbf{r}'(0) = \nabla f(\mathbf{x}) \cdot \mathbf{v} = 0. \tag{5.25}$$

Summarizing, we showed that if $\mathbf{v} \perp \nabla g(\mathbf{x})$, then $\mathbf{v} \perp \nabla f(\mathbf{x})$. This implies that $\nabla f(\mathbf{x}) \| \nabla g(\mathbf{x})$, and thus $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ (see Problem 5.1 on page 205 for rigorous details).

Proof of Theorem 5.4 is the infinite-dimensional version of the above proof and follows it verbatim if we replace $\mathbf{x} \in \mathbb{R}^n$ by $x \in C^2[t_0, t_1]$ (the space of twice continuously differentiable functions on $[t_0, t_1]$; f by $\mathcal{F}$; ∇f by $\nabla \mathcal{F} = \frac{d}{dt} F_{\dot{x}} - F_x$, and the dot product $\mathbf{x} \cdot \mathbf{y}$ by $x \cdot y = \int_{t_0}^{t_1} x(t) y(t)\, dt$; the orthogonal decomposition of a function $(\nabla \mathcal{F})$ is to be understood in the sense of orthogonality with respect to this dot product. $\diamondsuit$

11. Catenary — the solution

In this section we apply the method of Lagrange multipliers (Theorem 5.4) to solve the catenary problem. As dictated by the theorem, we form the difference

$$L(y, y') = y\sqrt{1 + (y')^2} - \lambda\sqrt{1 + (y')^2} = (y - \lambda)\sqrt{1 + (y')^2};$$

note that we can drop the factor ρg. Now rather than solving the Euler–Lagrange equation (5.23), we use Noether's theorem, which applies since L does not contain x explicitly. According to this theorem, any solution of the Euler–Lagrange equation satisfies

$$y' L_{y'} - L = (y - \lambda)\frac{1}{\sqrt{1 + (y')^2}} = k = \text{const}. \tag{5.26}$$

The general solution of this first order ODE is

$$y = \lambda + k \cosh \frac{x - x_0}{k}, \tag{5.27}$$

as can be checked by direct substitution (or discovered by separation of variables). This solution depends on three parameters: λ, k and x_0, matching the number of conditions to be satisfied: two for the boundary plus one for the length constraint.

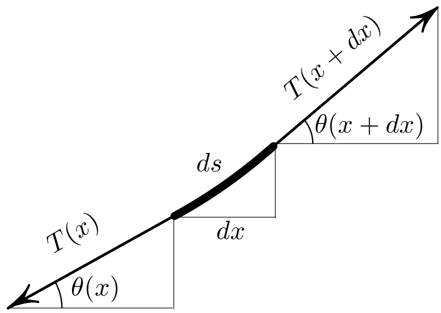

Figure 17. Deriving the equation of the catenary.

12. An elementary solution for the catenary

In this section I present an elementary solution of the problem which uses no theory beyond basic calculus. In fact, this elementary solution provides an alternative derivation of the Euler–Lagrange equation for the present example, based on a physical idea.

Deriving the ODE for the catenary. The key idea is that the minimum of the functional is an equilibrium; and an equilibrium condition — Newton's first law — is a differential equation in disguise. This differential equation, which we will now produce, is the same as the Euler–Lagrange equation given by the general theorem, but obtained by elementary means. We must write the equilibrium condition for an infinitesimal arc of the chain, Figure 17. The arc is subject to three forces: the tensions on each end and the gravity $dm\, g$. The balance of forces in the x- and y-directions gives, in the notations of the figure:

$$\begin{aligned} &T(x)\cos\theta(x) = T(x+ds)\cos\theta(x+dx), \\ &T(x+ds)\sin\theta(x+dx) - T(x)\sin\theta(x) = dm\, g. \end{aligned} \tag{5.28}$$

By the first of these equations, $T(x)\cos\theta(x) = T_0 = \text{const}$. Dividing the second line in (5.28) term-by-term by $T(x+ds)\cos\theta(x+dx) = T(x)\cos\theta(x) = T_0$, we get

$$\tan\theta(x+dx) - \tan\theta(x) = \frac{g}{T_0} dm.$$

Since $\tan\theta(x) = f'(x)$ and $dm = \rho ds = \rho\sqrt{1+(f'(x))^2}dx$, we have

$$\frac{f'(x+dx) - f'(x)}{dx} = \frac{g}{T_0}\sqrt{1+(f')^2}.$$

In the limit of $dx \to 0$ we obtain (writing $y = f(x)$):

$$y'' = k\sqrt{1+(y')^2}, \quad k = \frac{g}{T_0},$$

the differential equation for a catenary.

The general solution of this equation is exactly the hyperbolic cosine (5.27). In fact, there are *two* solutions (assuming the distance between the endpoints is less than the chain's length); one of these is an inverted arc of the hyperbolic cosine, like Gateway to the West of St. Louis, Missouri.[8] This second solution corresponds to $k = g/T_0 < 0$; negative T_0 means that the chain is under compression rather than tension. Such a chain would crumple under compression, and the mathematical solution cannot be realized physically (unless the chain is prevented from crumpling; but this would alter the mathematical model).

Remark 5.4. Note that T_0 is simply the tension of the chain at its lowest point where the tangent is horizontal.

Remark 5.5. The tension T of a hanging chain is a linear function of the height: for any two points on the chain, $T_2 - T_1 = \rho g(y_2 - y_1)$, see Problem 1.10 on page 52.

13. Problems

5.1. Let $\mathbf{a}$ and $\mathbf{b}$ ($\mathbf{b} \neq \mathbf{0}$) be two vectors in $\mathbb{R}^n$ such that if $\mathbf{b}\cdot\mathbf{v} = 0$ then $\mathbf{a}\cdot\mathbf{v} = 0$. Prove: $\mathbf{a} \parallel \mathbf{b}$, i.e., $\mathbf{a} = \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$.

Solution. Consider the orthogonal decomposition of $\mathbf{a}$ in the part parallel to $\mathbf{b}$ and a part perpendicular to $\mathbf{b}$:

$$\mathbf{a} = \lambda\mathbf{b} + \mathbf{r}, \quad \lambda \in \mathbb{R}, \quad \mathbf{r} \perp \mathbf{b}.$$

Now since $\mathbf{b}\cdot\mathbf{r} = 0$, we have $\mathbf{a}\cdot\mathbf{r} = 0$ (by assumption, anything orthogonal to $\mathbf{b}$ is also orthogonal to $\mathbf{a}$). Taking the dot product of the above decomposition with $\mathbf{r}$ gives $\mathbf{a}\cdot\mathbf{r} = \lambda\mathbf{b}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{r}$. Thus $\mathbf{r}\cdot\mathbf{r} = 0$, implying that $\mathbf{a} = \lambda\mathbf{b}$. ♢

[8] The analogy is not quite perfect because the thickness of that arc is variable.

5.2. Recall the method of Lagrange multipliers : if the function $f : \mathbb{R}^n \to \mathbb{R}$ constrained to a surface $g(\mathbf{x}) = 0$ has a critical point $\mathbf{x}$ on the surface, then the gradients are parallel at x:

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \tag{5.29}$$

Find a mechanical interpretation of this statement.

Answer. In a nutshell, Lagrange's theorem (5.29) amounts to the statement that a particle in a force field constrained frictionlessly to a surface is in equilibrium iff the force field is perpendicular to the surface. Here are the details. Interpret f as the potential energy of a particle in a force field $\mathbf{F} = -\nabla f$ in $\mathbb{R}^n$. The particle is constrained to the surface $g = 0$. If the potential energy f restricted to $g = 0$ is minimal at $\mathbf{x}$, then $\mathbf{x}$ is an equilibrium, i.e., the field force $-\nabla f$ is balanced by reaction force $\mathbf{R}$ of the constraint. And since $\mathbf{R}$ is normal to the surface, we have $\mathbf{R} = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$. This explains (5.29).

The following problem requires some background on Snell's law and on Fermat's principle; these are described in the present paragraph. When a ray of light crosses an interface between two media, e.g., air and water, it refracts, i.e. changes the direction, according to Snell's law:

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}, \tag{5.30}$$

where $\theta_1,\ \theta_2$ are the angles between the ray and the normal to the interface, and $c_1,\ c_2$ are the speeds of light in the media; see Figure 18. Fermat's principle, on the other hand, states that the ray between two points chooses the path of minimal (or rather critical) time.

5.3. Consider a ray of light traveling from a point A in the air to a point B in the water. Prove that Snell's law (5.30) follows from Fermat's principle.

5.4. Find a geometrical interpretation of the ratio $\sin\theta/c$ in Snell's law (5.30).

Solution. Figure 18 shows two fronts at times t and $t+h$. Let us find the speed of the break point A_t in the front as it slides along the horizontal line (the water surface). In time h the point moves the distance A_tA_{t+h}. But A_tA_{t+h} is a shared hypotenuse of two right triangles in the figure, and so

$$A_tA_{t+h} = \frac{c_1 h}{\sin\theta_1} = \frac{c_2 h}{\sin\theta_2}.$$

Dividing by h, we get the speed of A_t:

$$v = \frac{A_tA_{t+h}}{h} = \frac{c_1}{\sin\theta_1} = \frac{c_2}{\sin\theta_2}.$$

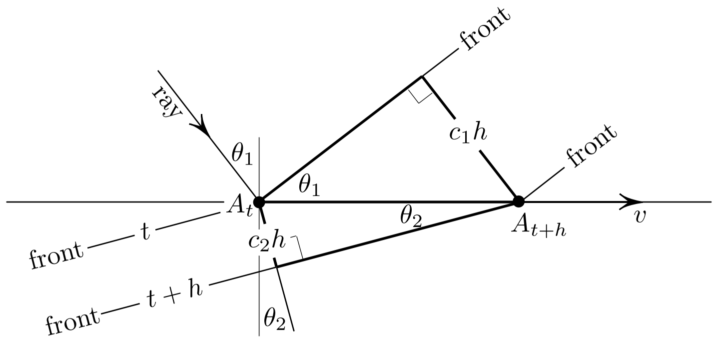

Figure 18. A kinematic interpretation of $\sin\theta/c$ as the speed at which the break point of the front (the fat dot) slides along the interface.

5.5. Find the continuous analog of Snell's law in the case when $c = c(x, y)$ depends on both coordinates.

Answer. Let $\mathbf{x} = \mathbf{x}(s)$ be the ray parametrized by arc length s. Then

$$(n\mathbf{x}')' = \nabla n,$$

where $' = \frac{d}{ds}$. Note that $\mathbf{p} = n\mathbf{x}'$ is referred to as the *vector of normal slowness* since, as it turns out, $\mathbf{p}$ is normal to the front, and since $|\mathbf{p}| = 1/c$ is indeed the slowness, c being the speed of light.

5.6. Figure 6 shows the family of trajectories of a projectile shot in different directions in the vertical plane with a given speed v. (i) Find the envelope of the family. (ii) Show that if the point B lies below this envelope, then there are precisely two trajectories reaching B. One of these is a minimum of the Maupertuis integral, while the other is a critical function of Morse index 1.

5.7. Consider a chain suspended at points $\mathbf{x}_0$ and $\mathbf{x}_1$ in $\mathbb{R}^3$, Figure 19. Let $\mathbf{p}_0$ be the force with which the chain pulls on the support at $\mathbf{x}_0$, and let $\mathbf{p}_1$ be the force with which the support at $\mathbf{x}_1$ pulls on the chain. Assuming that the pair $(\mathbf{x}_0, \mathbf{p}_0)$ defines uniquely the pair $(\mathbf{x}_1, \mathbf{p}_1)$ (provided $|\mathbf{x}_1 - \mathbf{x}_0| < L$, the length of the chain), prove that the mapping

$$\varphi = (\mathbf{x}_0, \mathbf{p}_0) \mapsto (\mathbf{x}_1, \mathbf{p}_1)$$

preserves the integral invariant of Poincaré:

$$\oint_\gamma \mathbf{p}\, d\mathbf{x} = \oint_{\varphi(\gamma)} \mathbf{p}\, d\mathbf{x} \tag{5.31}$$

for any closed smooth curve γ in $\mathbb{R}^6$.

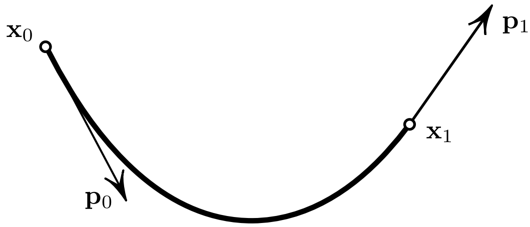

Figure 19. A symplectic mapping $\varphi : \mathbb{R}^6 \mapsto \mathbb{R}^6$ given by $\varphi(\mathbf{x}_0, \mathbf{p}_0) = (\mathbf{x}_1, \mathbf{p}_1)$ associated with a hanging chain.

Hint. Let $U(\mathbf{x}_0, \mathbf{x}_1)$ be the potential energy of the chain hanging in equilibrium between two points $\mathbf{x}_0$, $\mathbf{x}_1$, and note that $U_{\mathbf{x}_0} = -\mathbf{p}_0$, $U_{\mathbf{x}_1} = \mathbf{p}_1$. The map φ satisfying (5.31) is called *symplectic*; $U(\mathbf{x}_0, \mathbf{x}_1)$ is called a *generating function* of φ.

5.8. Find *all* locally shortest paths[9] on the cylinder $x^2 + y^2 = 1$ connecting the points $A(1, 0, 0)$ and $B(1, 0, H)$ (do this in two different ways: (1) geometrically, by unrolling the cylinder and (2) analytically, by writing down the length functional in cylindrical coordinates.

5.9 (Clairault integral)**.** Let S be a surface of revolution $x^2 + y^2 = R^2(z)$ in $\mathbb{R}^3$. Let γ be a geodesic connecting two given points A, B on S (a geodesic γ is a curve on S for which the length functional $\int_\gamma ds$ is critical for the class of smooth curves connecting A and B.) Write down the functional for the length of a curve on S. The associated Euler–Lagrange equation has a conserved quantity due to rotational symmetry of the problem. Show that this quantity, called the Clairault integral, is given by $r \sin \theta$, where r is the distance to the symmetry axis and θ is the angle between the geodesic and the meridian.

5.10. Referring to the preceding problem, a geodesic can be realized by mechanical means, as an equilibrium shape of a constant tension spring confined to the surface S and with ends held fixed (we assume that the constraint to the surface is frictionless). (i) Explain why this is so, and (ii) show that the Clairault integral equals the torque around the symmetry axis applied to the ends of the spring to hold it in place. Explain the preservation of the Clairault integral by the torque balance.

5.11. Instead of a constant tension spring as in the preceding problem, consider a zero length Hookean spring constrained frictionlessly to the surface $x^2 + y^2 = R^2(z)$, with two ends of the spring held fixed. No external forces act upon the spring. Write the expression for the potential energy

[9] A path is said to be locally shortest, or locally minimal if it is shorter than any nearby path with the same ends.

of such a spring, and show that this expression is the same as the integral of the Lagrangian of a point mass constrained to the surface. Explore the dynamics-statics analogy from page 45 in this case.

5.12. Show that the Clairault integral can be interpreted as the angular momentum of a particle sliding on the surface of revolution without friction.

Chapter 6

The Conditions of Legendre and Jacobi for a Minimum

In the 1750s Euler and Lagrange showed that their equation is a *necessary* condition for a function to be a minimizer of the functional

$$\mathcal{S}[x] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, dt \tag{6.1}$$

with fixed ends

$$x(t_0) = x_0,\ x(t_1) = x_1. \tag{6.2}$$

In his 1786 paper Legendre showed that for a critical function to be a minimizer it is necessary that $L_{\dot{x}\dot{x}}(x, \dot{x}) \geq 0$ be satisfied for the critical function (subscripts denote partial differentiation). Legendre also claimed, erroneously, that a strict inequality $L_{\dot{x}\dot{x}} > 0$, now called the Legendre condition, is sufficient for a minimizer. It took another 50 years until a sufficient condition was published by Jacobi in 1836. In retrospect it may seem surprising that it took so long, since the simplest nontrivial example of $L = \dot{x}^2 - x^2$ (corresponding to the harmonic oscillator!) shows that the Legendre condition is not sufficient, also suggesting the sufficient one. An intuitive explanation is given below (starting on page 226).

This chapter consists of three parts. Sufficient conditions for a minimum are given in the first part (Sections 1–4). Necessary conditions for a minimum are contained in the second part (Sections 5 and 6). And an intuitive discussion of some of the main ideas concludes the chapter.

1. Conjugate points

An informal definition. Let $x_c(t)$ be a critical function of (6.1)–(6.2). Imagine varying the initial slope of x_c as shown in Figure 1, creating a "fan" of curves, each satisfying the Euler–Lagrange equation. A point $C(t^*, x_c(t^*))$ on the graph of x_c is said to be conjugate to the initial point $O(t_0, x(t_0))$ with respect to the functional (6.1) if C is a "focal point" of this fan. That is, as we vary the initial slope s, the height $x(t^*, s)$ at $t = t^*$ changes with zero speed. Here is a precise definition.

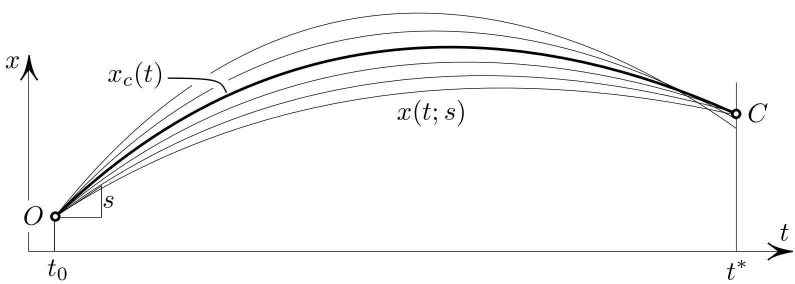

Figure 1. Definition of the conjugate point.

Definition 6.1. Referring to Figure 1, consider a one-parameter family $x(t; s)$ of solutions of the Euler–Lagrange equation parametrized by the initial slope s:

$$x(t_0, s) = x_0 \quad \text{and} \quad \dot{x}(t_0, s) = s, \tag{6.3}$$

for all s in a small neighborhood of $s_0 \stackrel{\text{def}}{=} \dot{x}_c(t_0)$. The point $(t^*, x_c(t^*))$ on the graph of x_c of a critical function of (6.1)–(6.2) is said to be *conjugate to* (t_0, x_0) *with respect to the Lagrangian* L if

$$\partial_s x(t^*; s_0) = 0; \tag{6.4}$$

here ∂_s denotes the partial derivative with respect to s.

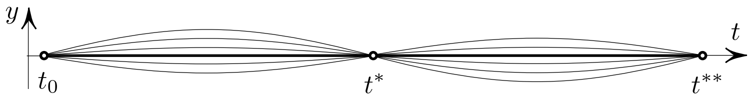

Figure 2. Definition of the conjugate point.

An equivalent definition of conjugate points. Figure 2 shows infinitesimal deviations $x(t;s)-x_c(t)$ between two solutions from Figure 1, to the leading order in $\Delta s = s - s_0$. More precisely, consider the scaled deviation

$$y(t) = \lim_{s\to s_0} \frac{x(t;s)-x_c(t)}{s-s_0} = \frac{\partial}{\partial s}x(t;s_0). \tag{6.5}$$

Since $x(t;s)$ satisfies the Euler–Lagrange equation

$$\frac{d}{dt}L_{\dot{x}}(x(t;s),\dot{x}(t;s)) - L_x(x(t;s),\dot{x}(t;s)) = 0$$

for an interval of s-values, i.e., is an identity in s, we can differentiate by s. Using the chain rule, substituting $s = s_0$, using (6.5) and simplifying, we end up with the so-called *linearized equation* for y:

$$\boxed{(P\dot{y})^{\bullet} + Qy = 0} \tag{6.6}$$

where

$$P = P(t) = L_{\dot{x}\dot{x}}, \quad Q = Q(t) = \frac{d}{dt}L_{x\dot{x}} - L_{xx}, \tag{6.7}$$

and where the subscripts denote partial derivatives, all evaluated at $(x_c(t),\dot{x}_c(t))$. Note also that $y(t_0)=0$ and $\dot{y}(t_0)=1$, as follows from (6.5) and (6.3). All solutions of (6.6) vanishing at t_0 are multiples of $y(t)$; these solutions are illustrated in Figure 2, for the simple example of constant P and Q (both positive).

Now the conjugate point on the graph of x_c corresponds to the first root of the solution to the linearized equation, as the following theorem states.

Theorem 6.1 (An equivalent definition of a conjugate point)**.** *Consider the linearization* (6.6) *of the Euler–Lagrange equation around a critical function* $x_c(t)$ *of the functional* (6.1), *and let* $y(t)$ *be a solution of* (6.6) *satisfying* $y(t_0)=0$, $\dot{y}(t_0)\neq 0$. *If* $y(t^*)=0$ *for some* $t^*\neq t_0$, *then* $(t^*,x_c(t^*))$ *is a point conjugate to* $(t_0,x_c(t_0))$ *with* *respect to L. The converse is also true: If* $(t^*, x_c(t^*))$ *is a point conjugate to* $(t_0, x_c(t_0))$*, then any solution* y *of the linearized equation satisfying* $y(t_0) = 0$ *vanishes at* $t = t^*$.

Proof. The definition (6.4) and the statement $y(t^*) = 0$ are equivalent by the definition of the linearized solution y, see (6.5). ♢

Corollary 6.1. *The conjugacy is a reflexive relation: If* $C^* = (t^*, x_c(t^*))$ *is a point conjugate to* $C_0(t_0, x_c(t_0))$*, then, vice versa,* C_0 *is a point conjugate to* C^*.

Proof. By the preceding theorem, two points on the graph of x_c are conjugates of one another iff the corresponding t-values are zeros of a nontrivial solution of the linearized equation. This condition treats the two points symmetrically. ♢

Conjugate points for quadratic functionals. Especially important is the special case of the Lagrangian quadratic in both variables x, $\dot{x}$:

$$\mathcal{Q}[x] = \int_{t_0}^{t_1} (P(t)\dot{x}^2 + Q(t)x^2)\, dt, \tag{6.8}$$

with zero boundary values:

$$x(t_0) = x(t_1) = 0; \tag{6.9}$$

here P and Q are given (sufficiently smooth) functions of t. For any critical function $x_c(t)$ of this functional, the conjugate points of $(t_0, 0)$, must lie on the t-axis. This is so because the Euler–Lagrange equation for quadratic functional is linear and coincides with its linearization (6.6). But the solution y of (6.6) with $y(t_0) = 0$ vanishes at the conjugate time: $y(t^*) = 0$ by the last theorem, and we conclude that $x_c(t^*) = 0$ as well.

Since $x_c(t^*) = 0$ is automatic for quadratic functionals, we will simply say that t^* *is conjugate to* t_0 *with respect to* (6.8), without mentioning the x-coordinates of the two conjugate points.

Reducing (6.8) to a simpler form. If $P > 0$ on $[t_0, t_1]$ and is continuous, then the functional (6.8) can be reduced to a simpler form

$$\int_0^T (y')^2 + q(\tau)y^2 \, d\tau, \quad y(0) = y(T) = 0,$$

via a change of variables $d\tau = dt/P(t)$, $y(\tau) = x(t)$, $q(\tau) = P(t)Q(t)$. For more details and for related questions, see Problem 6.1 on page 229. Had we assumed that $P > 0$ throughout this section, we could have discussed this simpler functional all along, without loss of generality.

2. The Legendre and the Jacobi conditions

The Legendre condition.

Definition 6.2. A Lagrangian L is said to satisfy the Legendre condition at a point (x, v) if

$$L_{\dot{x}\dot{x}}(x, v) > 0. \tag{6.10}$$

If the inequality in (6.10) is not strict, one refers to the *weak* Legendre condition.

For the special case of the quadratic functional (6.8)–(6.9), we have $L_{\dot{x}\dot{x}} = P(t)$, and the Legendre condition amounts to

$$P(t) > 0, \quad t \in [t_0, t_1],$$

and does not depend on (x, v).

The Legendre condition will be referred to in the strong sense from now on, unless stated otherwise.

The Legendre condition as a triangle inequality. Figure 3 shows a "triangle" APB; the "sides" of this triangle are the graphs of functions which minimize $\int L \, dt$ among all curves in the (t, x)-plane connecting the endpoints. Define the "length" of the side AB as $|AB| = \int_A^B L \, dt$, where the integration takes place along the minimizer of the integral connecting A and B. If the Legendre condition

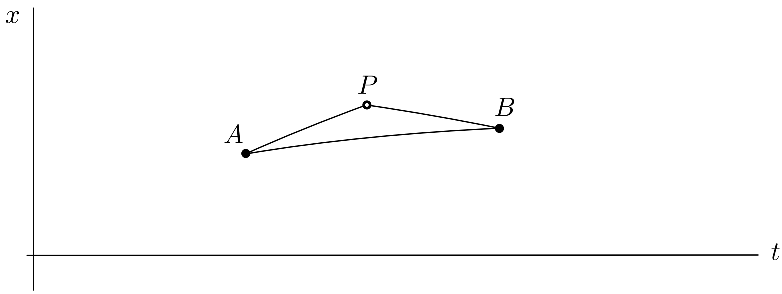

Figure 3. Legendre condition as a triangle inequality.

$L_{\dot{x}\dot{x}} > 0$ is satisfied along the side AB, then we have the triangle inequality

$$|AB| \leq |AP| + |PB|, \tag{6.11}$$

provided that A, B are sufficiently close to each other, and that the slopes of AP and PB are close to the slope of AB. For the proof, see Problem 6.3.

Besides giving a geometrical significance to the Legendre condition, (6.11) also explains why *the minimizers cannot have corners* if the Legendre condition is satisfied. For more details, see Problem 6.2 on page 230.

Having defined the Legendre condition, we now state the Jacobi condition.

Definition 6.3. A critical function $x_c = x_c(t)$ of the functional (6.1)–(6.2) is said to satisfy the *Jacobi condition* if the graph of x_c contains no points conjugate to (t_0, x_0) with respect to this functional, on the interval $[t_0, t_1]$.

Note that, unlike the Legendre condition, the Jacobi condition is not local.

In the special case of the quadratic functional (6.8)– (6.9) the Jacobi condition amounts to the nonvanishing of the solution of the linearized Euler–Lagrange equation (6.6) with $y(t_0) = 0,\ \dot{y}(t_0) \neq 0$ on $(t_0, t_1]$.

3. Quadratic functionals: the fundamental theorem

The main problem of this chapter is to determine whether a critical function of a rather general functional (6.1)–(6.2) is a minimizer. This problem quickly reduces to determining positivity of quadratic functionals (via an infinite-dimensional version of the second derivative test, as we shall see later on page 220). Let us therefore first consider quadratic functionals (6.8)–(6.9); the substance of the problem is there.

Theorem 6.2. *If the functional* (6.8)–(6.9) *satisfies the Jacobi condition, as well as the Legendre condition : $P > 0$ on $[t_0, t_1]$, then the functional is strictly positive: $\mathcal{Q}[x] > 0$ for any admissible $x(t)$ not identically zero.*[1] *In particular, then $\mathcal{Q}$ is minimal for $x \equiv 0$.*

Outline of the proof. A beautiful proof of this lemma was invented by Legendre and is described in Gelfand–Fomin [**8**]. However, since this proof is algebraic rather than visual, I give a sketch (but not full details) of a more geometric proof.[2]

Step 1. Reduction to the proof of positivity of the eigenvalues of the operator

$$Dx = -(P\dot{x})^{\boldsymbol{\cdot}} + Qx, \quad x(t_0) = x(t_1) = 0.$$

To prove that $\mathcal{Q}$ is positive it suffices to show that it has a positive minimum over the unit sphere in the function space:

$$\min_{(x,x)=1} \mathcal{Q}[x] > 0, \quad x(t_0) = x(t_1) = 0, \tag{6.12}$$

where (x, y) is defined by

$$(x, y) = \int_{t_0}^{t_1} x(t)y(t)dt, \tag{6.13}$$

[1] Admissible means twice continuously differentiable and satisfying the boundary conditions (6.9).

[2] As mentioned before, Legendre (seems to have) believed that $L_{\dot{x}\dot{x}} > 0$ is sufficient for the minimum, and tried to use his method without further assumptions. His method relied on the solvability on the entire interval $[t_0, t_1]$ of a certain Ricatti equation; presumably, Legendre assumed that this solvability is not really necessary. We now know that this solvability amounts exactly to the absence of conjugate points on the interval.

and where minimization is taken over all continuously differentiable functions not identically zero. The minimizing function x_m exists and is twice continuously differentiable.[3] Note that this minimization problem is an infinite-dimensional analog of Rayleigh's quotient; see page 117. Indeed, integrating the derivative term in $\mathcal{Q}(x)$ by parts, we get

$$\mathcal{Q}[x] = \int_{t_0}^{t_1} (P\dot{x}^2 + Qx^2)\,dt = \int_{t_0}^{t_1} (-\dot{P\dot{x}} + Qx)x\,dt = (Dx, x),$$

where D was defined above; this is the form analogous to the finite-dimensional quadratic form $(K\mathbf{x}, \mathbf{x})$, and D is the analog of the matrix K. By the verbatim repetition of the proof of Rayleigh's criterion (Theorem 2.16), treating (Dx, x) as we treated $(K\mathbf{x}, \mathbf{x})$, we conclude that the minimizer x_m is an eigenfunction of D:

$$-(P\dot{x}_m)^{\bullet} + Qx_m = \lambda_m x_m, \quad x_m(t_0) = x_m(t_1) = 0, \tag{6.14}$$

and that the eigenvalue λ_m is the value of the minimum (6.12).

Therefore, it suffices to prove that all eigenvalues of D are positive.

Step 2 (Sturm's theorem)**.** The idea of the rest of the proof is simple: the Jacobi condition tells us that the solution of

$$Dx = 0, \quad x(t_0) = 0, \quad \dot{x}(t_0) = 1 \tag{6.15}$$

has no zeros in (t_0, t_1) (note that only the left boundary condition in (6.15) is prescribed, as opposed to (6.14)). If we now change (6.15) to

$$Dx = \lambda x \tag{6.16}$$

(with the same boundary condition), and start decreasing λ starting from the value $\lambda = 0$, then no roots of x can appear in $(t_0, t_1]$ by Sturm's theorem (discussed next), Figure 4; this shows that λ_m in (6.14) cannot be negative (thus completing the proof), since x_m *does* have a root at t_1. ♢

To state Sturm's theorem used in the preceding paragraph, let $x(t;\lambda)$ denote the solution of (6.16), Figure 4. Sturm's theorem states that *the roots of* $x(t;\lambda)$ *are decreasing functions of* λ, as illustrated in

[3]This is where $P > 0$ is used. For details, see Evans [**4**] or Mikhlin [**11**].

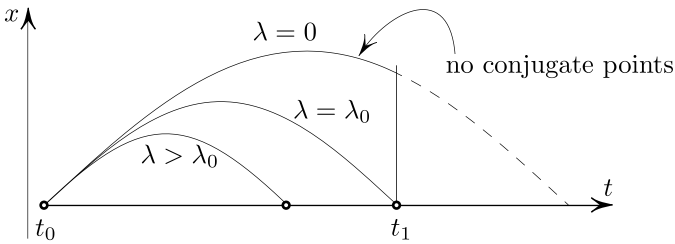

Figure 4. Solutions $x(t;\lambda)$ of (6.16) for different values of λ.

the figure. I omit a rigorous proof which can be found, for instance, in [**3**]. Physically, Sturm's theorem says that making the spring in a mass-spring system stiffer decreases the return time to the origin. A rigorous geometrical proof goes by writing (6.16) as a system in the phase plane, considering the angular velocity $\omega = \dot{\theta}$ of the phase vector $\mathbf{z} = (x, P\dot{x})$ and observing that ω is a monotone function of λ: the greater is λ, the faster the phase vectors turn. Now the root of $x(t;\lambda)$ corresponds to the moment of $\mathbf{z}$ crossing the line $x = 0$ in the phase plane; and since a greater λ corresponds to faster turning, the next moment of crossing $x = 0$ comes sooner if λ is increased.

4. Sufficient conditions for a minimum for a general functional

In the previous section we found sufficient conditions for minimality for quadratic functionals. In this section we extend the result to general functionals, but first we need to define the minimum precisely.

Definition 6.4. A function $x_c = x_c(t)$ is said to be a *weak minimizer* of the functional (6.1) if there exists $\varepsilon > 0$ such that

$$\mathcal{S}[x] \geq \mathcal{S}[x_c] \tag{6.17}$$

for any differentiable function x on $[t_0, t_1]$ close to x_c with its derivative:

$$|x(t) - x_c(t)| < \varepsilon, \quad |\dot{x}(t) - \dot{x}_c(t)| < \varepsilon, \tag{6.18}$$

and satisfying the same boundary conditions as x_c.

From now on, let us refer to the *weak minimizer* simply as a minimizer. We can now state the main theorem.

Theorem 6.3. *If the graph of a critical function x_c of* (6.1)–(6.9) *satisfies the Jacobi condition (that is, there are no conjugate points to $O(t_0, x_0)$ on the graph), and if the Legendre condition $L_{\dot{x}\dot{x}}(x_c(t), \dot{x}_c(t)) > 0$ holds for all $t \in [t_0, t_1]$, then x_c is a local minimizer.*

This theorem is the analog of the second derivative test in vector calculus, where one shows that if the quadratic part of a function at the critical point is positive, then the function has a minimum.

Outline of the proof.

Step 1. Wishing to prove (6.17), consider a perturbation $x = x_c + \varepsilon\xi$, where $\xi : [t_0, t_1] \to \mathbb{R}$ satisfies $\xi(t_0) = \xi(t_1) = 0$ and $|\xi(t)| \leq 1$, $|\dot{\xi}(t)| \leq 1$ for all $t \in [t_0, t_1]$.

Recall Taylor's formula with the remainder in Lagrange's form. This formula, written to the second order, states that

$$f(\varepsilon) = f(0) + f'(0)\varepsilon + \frac{1}{2}f''(\theta\varepsilon)\varepsilon^2,$$

for some $\theta \in [0, 1]$. Applying this to the function $f(\varepsilon) = \mathcal{S}[x_c + \varepsilon\xi]$ we get

$$\mathcal{S}[x_c + \varepsilon\xi] = \mathcal{S}[x_c] + \underbrace{\left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{S}[x_c + \lambda\xi]}_{=0} + \frac{1}{2}\frac{d^2}{d\lambda^2}\mathcal{S}[x_c + \lambda\xi]_{\lambda=\theta\varepsilon},$$

for some $0 \leq \theta \leq 1$ (note that θ may depend on the function ξ, but not on t). The middle term on the right vanishes because x_c is a critical function. To prove that the last term in the last expression is positive it suffices to show that

$$\frac{d^2}{d\lambda^2}\mathcal{S}[x_c + \lambda\xi] > 0 \tag{6.19}$$

for all sufficiently small ε and for all ξ with $|\xi(t)| \leq 1$, $|\dot{\xi}(t)| \leq 1$ on $[t_0, t_1]$.

Step 2. Computing the derivative (6.19) and using the chain rule we get:

$$\frac{d^2}{d\varepsilon^2}\mathcal{S}[x_c+\varepsilon\xi] = \int_{t_0}^{t_1} (\overline{L}_{\dot{x}\dot{x}}\dot{\xi}^2 + 2\overline{L}_{x\dot{x}}\xi\dot{\xi} + \overline{L}_{xx}\xi^2)\, dt,$$

where the bar indicates that the evaluation takes place at $x = x_c + \varepsilon\xi$, $\dot{x} = \dot{x}_c + \varepsilon\dot{\xi}$. Integrating the middle term by parts and using the boundary condition $\xi(t_0) = \xi(t_1) = 0$, we get

$$\frac{d^2}{d\varepsilon^2}\mathcal{S}[x_c+\varepsilon\xi] = \int_{t_0}^{t_1} (\overline{P}\dot{\xi}^2 + \overline{Q}\xi^2)\, dt, \tag{6.20}$$

where

$$\overline{P} = \overline{P}(t) = \overline{L}_{\dot{x}\dot{x}}, \quad \overline{Q} = \overline{Q}(t) = \frac{d}{dt}\overline{L}_{x\dot{x}} - \overline{L}_{xx}. \tag{6.21}$$

Note that these expressions coincide with the coefficients of the linearized equation (6.7) on page 213.[4]

Step 3. By the assumption of the theorem, the linearized system around x_c:

$$(P\dot{y})^{\bullet} + Qy = 0$$

has no points on $[t_0, t_1]$ conjugate to $t = t_0$. Provided ε is small enough, the same holds for

$$(\overline{P}\dot{y})^{\bullet} + \overline{Q}y = 0,$$

since $|\xi| \le 1$, $|\dot{\xi}| \le 1$ (the proof is left as an exercise)[5]. By the main theorem (6.2) on quadratic functionals, (6.20) is nonnegative. This concludes the proof of Theorem 6.3. $\diamondsuit$

Remark 6.1. The quadratic functional

$$\mathcal{Q}[\xi] = \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0}\mathcal{S}[x_c+\varepsilon\xi] = \int_{t_0}^{t_1} (L_{\dot{x}\dot{x}}\dot{\xi}^2 + 2L_{x\dot{x}}\xi\dot{\xi} + L_{xx}\xi^2)\, dt$$

is called the *second variation* of the functional $\mathcal{S}$ at x_c (here the integrand is evaluated along x_c), and is often denoted by $\mathcal{Q} = \delta^2\mathcal{S}$.

[4] This is not surprising: indeed, $d\mathcal{S}/d\varepsilon$ contains the Euler–Lagrange expression (call it (EL)) inside the integral — this is how (EL) arose in the first place; taking $d/d\varepsilon$ again amounts to differentiating (EL). But the linearized equation is also obtained by the differentiation of (EL)$= 0$ with respect to the parameter!

[5] The proof follows from the argument of Sturm's theorem, outlined briefly on page 218.

5. Necessity of the Legendre condition for a minimum

In this section we show that without the Legendre condition minimality is impossible. In fact, this result should be expected since the Legendre condition is an analog of the triangle inequality, page 216.

Theorem 6.4. *Let x_c be a critical function of* (6.1)–(6.2)*, where L is a smooth Lagrangian. If the Legendre condition fails, i.e., if*

$$P(t) = L_{\dot{x}\dot{x}}(x_c(t), \dot{x}_c(t)) < 0 \tag{6.22}$$

for some $t \in [t_0, t_1]$, then x_c is not a minimizer.

Proof. Consider the second variation of $\mathcal{S}$:

$$\mathcal{Q}[\xi] = \int_{t_0}^{t_1} (P\dot{\xi}^2 + Q\xi^2)\, dt, \tag{6.23}$$

$\xi(t_0) = \xi(t_1) = 0$, and show that $\mathcal{Q}[\xi] < 0$ for some ξ. Now (6.22) implies, by continuity, that

$$P(t) \leq -\alpha$$

for some $\alpha > 0$ and for all t on some interval $[a, b] \subset [t_0, t_1]$. We will choose ξ as in Figure 5 to make $\mathcal{Q}[\xi] < 0$. The "crinkled" nature of ξ will make $\int_a^b P\dot{\xi}^2$ large negative (since P is negative), causing this integral to dominate $\int_a^b Q\xi^2$ which stays bounded no matter how crinkled ξ is. To be specific, let

$$\xi(t) = f(t) \sin \omega t,$$

where ω is large and where f is as shown in Figure 5, namely, (i) f is smooth; (ii) $f(t) = 0$ outside $[a, b]$; (iii) $0 \leq f(t) \leq 1$ inside (a, b), and (iv) $f(t) = 1$ on some subinterval (a', b') of (a, b). Now

$$\mathcal{Q}[\xi] = \underbrace{\int_a^b P\dot{\xi}^2(t)\, dt}_{A} + \underbrace{\int_a^b Q\xi^2(t)\, dt}_{B}.$$

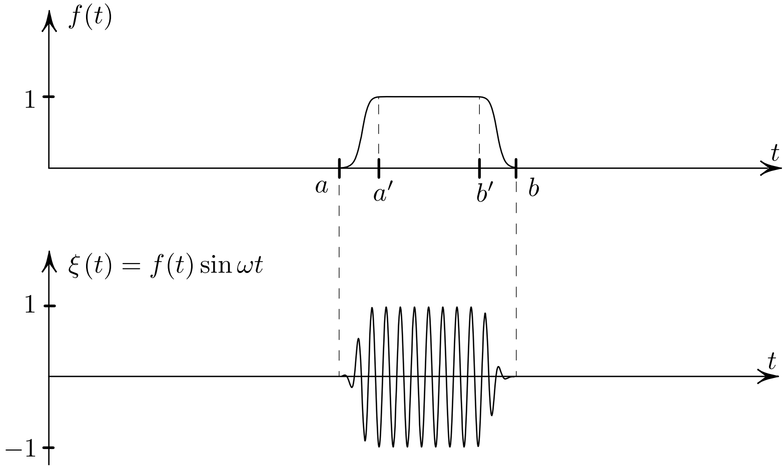

Figure 5. A choice of $\xi_\omega(t)$ for which $\mathcal{Q}[\xi_\omega] \to -\infty$ as $\omega \to \infty$, in case $P < 0$ on (a, b).

We show now that $\lim_{\omega\to\infty} A = -\infty$, and that B stays bounded as $\omega \to \infty$. That B is bounded is seen from the boundedness of each term in its integrand. To see that $A \to -\infty$ we write

$$A = \int_a^b P\dot{\xi}^2\, dt \overset{\text{(i)}}{<} \int_{a'}^{b'} P\dot{\xi}^2\, dt \overset{\text{(ii)}}{<} -\alpha \int_{a'}^{b'} \dot{\xi}^2\, dt \overset{\text{(iii)}}{=} -\alpha\omega^2 \int_{a'}^{b'} \cos^2 \omega t\, dt,$$

where (i) follows from $P\dot{\xi}^2 \le 0$ in $[a, b]$, (ii) follows from $P < -\alpha$ on $[a', b']$, and (iii) follows from $\xi = f(t)\sin\omega t = \sin\omega t$ on $[a', b']$ since $f = 1$ on $[a', b']$. We showed that $\mathcal{Q}[\xi] < 0$, which completes the proof. ♢

6. Necessity of the Jacobi condition for a minimum

Having shown that the Legendre condition $L_{\dot{x}\dot{x}} > 0$ is necessary for a minimum, we show now that the Jacobi condition is necessary as well: any minimizer is free of conjugate points. More precisely, we have the following.

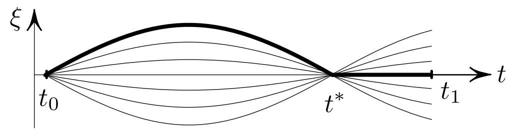

Figure 6. Solutions of the linearized equation vanish at $t = t^*$.

Theorem 6.5. *If x_c is a minimizer of* (6.1)–(6.2)*, then the graph of x_c contains no points conjugate to $(t_0, x_c(t_0))$ in the interval $[t_0, t_1)$; that is, the Jacobi condition holds.*

Proof. Assume that a conjugate point with $t^* \in [t_0, t_1)$ exists on the graph of the minimizer x_c; it suffices to produce a function ξ_- for which $\mathcal{Q}[\xi_-] < 0$, where $\mathcal{Q}$ is the second variation (6.23) of (6.1) along x_c. By the definition of the conjugate point, the solution of the linearized equation

$$(P\dot{\xi})^{\bullet} - Q\xi = 0, \quad \xi(t_0) = 0, \ \dot{\xi}(t_0) \neq 0. \tag{6.24}$$

vanishes at $t = t^*$.[6] We then define a concatenated function $\widehat{\xi}$ (the thick line in Figure 6):

$$\widehat{\xi}(t) = \begin{cases} \xi(t), & t \in [t_0, t^*], \\ 0, & t \in [t^*, t_1], \end{cases} \tag{6.25}$$

where ξ is a nonzero solution of (6.24).[7] The the rest of the proof is shown in Figure 7: First, we will show that $\mathcal{Q}[\widehat{\xi}] = 0$, and second, that by straightening the corner in the graph we decrease $\mathcal{Q}$, thus making it negative and completing the proof. Here are the details.

Observation 1. All the the graphs in Figure 7(i) have the same "length":

$$\int_{t_0}^{t^*} (P\dot{\xi}^2 + Q\xi^2)\, dt = 0 \tag{6.26}$$

[6]Since any two solutions of (6.24) differ by a constant factor, it doesn't matter which solution of (6.24) we speak of.

[7]The symbol $\widehat{\ }$ suggests a corner in the graph.

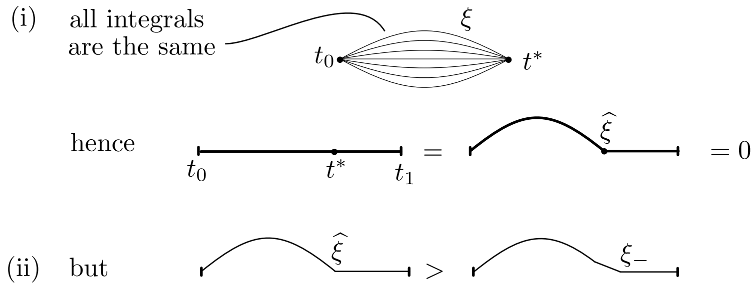

Figure 7. An outline of the proof of Theorem 6.5: $\mathcal{Q}[\xi_-] < 0$.

where ξ is any solution of (6.24). Indeed, every such ξ is a critical function of the functional on $[t_0, t^*]$ with zero boundary conditions:

$$\mathcal{Q}_*[\xi] = \int_{t_0}^{t^*} (P\dot{\xi}^2 + Q\xi^2)\, dt, \quad \xi(t_0) = \xi(t^*) = 0.$$

Moreover, these critical functions form a *continuous family*, since $s\xi$ is a solution for of (6.24) for any $s \in \mathbb{R}$. Now it is a general fact that if critical points form a one-parameter family, then the functional is constant along this family. Indeed, the directional derivative at a critical point vanishes in any direction, in particular in the direction along the family of critical points.

This proves (6.26). Now since $\widehat{\xi}(t) = 0$ for $t^* \leq t \leq t_1$,

(6.27) $$\mathcal{Q}[\widehat{\xi}] = \int_{t_0}^{t_1} (P\dot{\widehat{\xi}}^2 + Q\widehat{\xi}^2)\, dt = 0,$$

as also noted in Figure 7, second line.

Observation 2. We show that straightening the corner of the graph of $\widehat{\xi}$ on a short interval $I_\varepsilon = [t^* - \varepsilon, t^* + \varepsilon]$ as in Figure 7(ii) and Figure 8 (leaving $\widehat{\xi}$ unchanged elsewhere) decreases $\mathcal{Q}$:

(6.28) $$\mathcal{Q}[\xi_-] < \mathcal{Q}[\widehat{\xi}] = 0$$

(this is where the Legendre condition will be used). This would complete the proof, except that ξ_- is not differentiable. But by smoothing ξ_- in a small neighborhood of each corner A and C (Figure 8) we can change $\mathcal{Q}[\xi_-]$ arbitrarily little so as to preserve its negativity. It thus

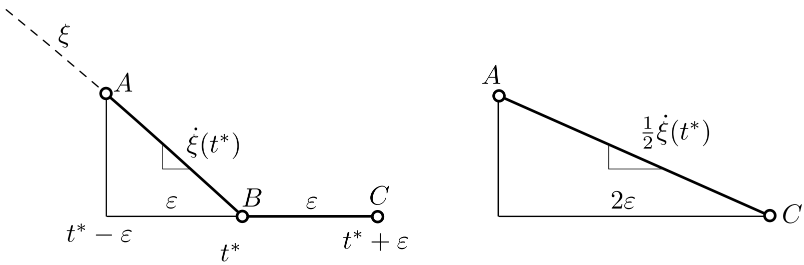

Figure 8. Straightening a corner decreases the quadratic functional.

remains to prove (6.28), which amounts to

$$\int_{t^*-\varepsilon}^{t^*+\varepsilon} P\dot{\xi}_-^{\,2} + Q\xi_-^2 < \int_{t^*-\varepsilon}^{t^*} P\dot{\widehat{\xi}}^{2} + Q\widehat{\xi}^2, \tag{6.29}$$

since $\widehat{\xi}$ has been altered only on I_ε, and since $\widehat{\xi} = 0$ for $t \geq t^*$. Since $\widehat{\xi}(t) = O(\varepsilon)$ and $\dot{\widehat{\xi}}(t) = \dot\xi(t^*) + O(\varepsilon)$ for $t^* - \varepsilon \leq t < t^*$, substitution into the right-hand side of (6.29) gives

$$\int_{t^*-\varepsilon}^{t^*} P\dot{\widehat{\xi}}^{2} + Q\widehat{\xi}^2 = \varepsilon P(t^*)\dot\xi(t^*)^2 + O(\varepsilon^3).$$

Note that $\dot\xi_- = \frac{1}{2}\dot\xi(t^*) + O(\varepsilon)$ on $[t^* - \varepsilon, t^* + \varepsilon]$; substitution into the left-hand side of (6.29) shows that

$$\int_{t^*-\varepsilon}^{t^*+\varepsilon} P\dot{\xi}_-^{\,2} + Q\xi_-^2 = 2\varepsilon \cdot \left(\frac{1}{2}\dot\xi^2(t^*)\right) + O(\varepsilon^3) = \frac{1}{2}\varepsilon P(t^*)\dot\xi(t^*)^2 + O(\varepsilon^3).$$

Comparing this with the last equation and using $P(t^*) > 0$ proves the validity of (6.29) (and hence of (6.28)) for sufficiently small ε. $\diamondsuit$

7. Some intuition on positivity of functionals

This subsection discusses some intuitive insights into the question of positivity of quadratic functionals.

A simple example. Let us consider the simplest interesting example

$$\mathcal{Q}[x] = \int_0^T (\dot x^2 - x^2)dt \tag{6.30}$$

with boundary conditions $x(0) = x(T) = 0$. Because of the minus sign, there is a competition between $\dot{x}^2$ and x^2, and it is not clear a priori whether $\mathcal{Q}[x]$ is nonnegative. Although the general theory described earlier tells us whether $\mathcal{Q}$ is positive (namely, for $T < \pi$,[8] the following few remarks may help one's intuition.

Remark 6.2. Every quadratic functional (6.23) satisfying the Legendre condition $P > 0$ (smoothness of P is assumed throughout) can be reduced to the simpler form

$$\int_0^T (\dot{x}^2 + q(t)x^2)dt,$$

see Problem 6.1 on page 229. Our example (6.30) corresponds to the choice of $q = -1$.

A preliminary look at (6.30). Figure 9 illustrates the role of T in determining positivity of $\mathcal{Q}$. Let $x(t)$ be a piecewise linear function as in Figure 9 (we can smooth x in a tiny neighborhood of the corner so as to satisfy the differentiability requirement). For T large, $\dot{x}^2$ is small, while the average of x^2 is not, so that $\mathcal{Q}[x] < 0$. For T small, on the other hand, we have $\mathcal{Q}[x] > 0$ since $\dot{x}^2$ is large, while $x^2 \leq 1$. This suggests, but does not prove, that $\mathcal{Q}$ is a positive functional.

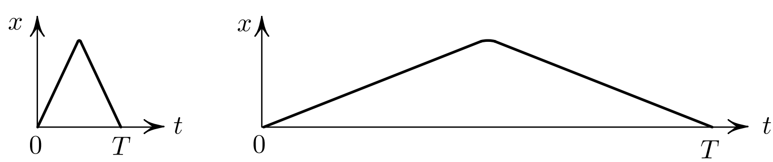

Figure 9. For T small $\int \dot{x}^2$ "dominates" $\int x^2$ and $\mathcal{Q} = \int_0^T (\dot{x}^2 - x^2)\ dt$ has a minimum at $x \equiv 0$. For large T this is no longer the case.

A Fourier series proof of positivity. The crude idea of the last paragraph actually turns into a rigorous proof if instead of a "hat" function in Figure 9 we use the Fourier harmonics of x. Expanding $x(t)$ in the Fourier series in sines (which can be done thanks to the

[8] Since $t^* = \pi$ is the first root of the solutions of the Euler–Lagrange equation $\ddot{x} + x = 0$ satisfying $x(0) = 0$

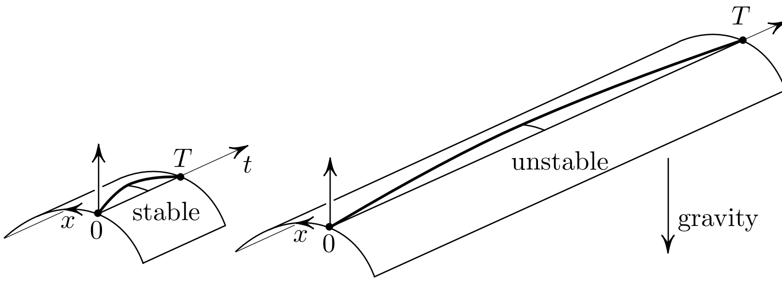

Figure 10. For small T the position $x(t) \equiv 0$ is stable; for large T it is not.

zero boundary conditions), we get

$$x(t) = \sum_{n=1}^{\infty} x_n \sin \frac{\pi n}{T} t, \quad \dot{x}(t) = \frac{\pi}{T} \sum_{n=1}^{\infty} n x_n \cos \frac{\pi n}{T} t.$$

After substituting x and $\dot{x}$ into $\mathcal{Q}$, multiplying out the series and integrating, we are left with

$$\frac{2}{T} \mathcal{Q}[x] = \sum_{n=1}^{\infty} \left(\left(\frac{\pi}{T} \right)^2 n^2 - 1 \right) x_n^2.$$

This expression is positive for any x iff the smallest coefficient is positive:

$$\left(\frac{\pi}{T} \right)^2 - 1 > 0,$$

i.e., iff

$$T < \pi,$$

in agreement with the Jacobi criterion.

A mechanical interpretation of the Jacobi condition. The functional $\mathcal{Q}[x] = \int_0^T (\dot{x}^2 - x^2)\, dt$ can be interpreted as the potential energy of an Eulerian string in the (t, x)-plane[9] with the ends fixed at $(0, 0)$ and $(0, T)$ and subject to the potential $U = -x^2$ which repels the string from the t-axis; Figure 10 illustrates this situation loosely: the repelling potential can be realized by placing the string on the inverted slippery trough.

[9]Euler's model of the string assumes that the displacements are purely in the x-direction and that the x-component of the tension varies as $\dot{x}$.

A string deflected as shown in Figure 10 wants to snap back to the the ridge $x \equiv 0$ due to the tension but is repelled from the ridge due to the gravity; the figure shows a principal vibrational mode of two strings with the same average deflection. Now for a long string, the gravity wins over the restoring effect of tension (this is so because the curvature for the same deflection is less in a long string; and the curvature determines the resultant tension in the x-direction per unit length of the string).

We conclude that for a critical length $T = T^*$ the straight string will be neutrally stable, and thus (since the functional is quadratic) there is a family of neutrally stable equilibria, meaning that $t = 0$ and $t = T^*$ are conjugate times. At the same time, the loss of stability means that the potential energy lost its minimality! This explains why appearance of conjugate points accompanies the loss of minimality, illustrating the main point of this chapter.

8. Problems

6.1. Assume that $P(t) \neq 0$ for all $t \in [t_0, t_1]$ and is continuous.

(1) Show that the quadratic functional $\mathcal{Q}$ can be reduced to a simpler form:

$$\int_{t_0}^{t_1} (P(t)\dot{x}^2 + Q(t)x^2)\, dt = \int_0^T ((y')^2 + q(\tau)y^2) d\tau,$$

where $\tau,\ y(\tau)$ are related to $t, x(t)$ via

$$\tau = \int_{t_0}^{t} \frac{ds}{P(s)}, \quad y(\tau) = x(t), \quad q(\tau) = P(t)Q(t).$$

(2) Where in the above reduction is the condition $P \neq 0$ used?

(3) Show how the transformation in (1) also converts $(P\dot{x})^{\bullet} + Qx = 0$ into $y'' + q(\tau)y = 0$, provided that P is continuously differentiable.

(4) Assume that $P < 0$ on the entire interval, and take $Q = 0$. According to (1),

$$\int_{t_0}^{t_1} P\dot{x}^2\, dt = \int_0^T (y')^2\, d\tau.$$

But the two integrands have opposite signs; is there a mistake?

Remark. The above transformation can be found by rescaling the time: $d\tau = \alpha(t)dt$ and by seeking α so as to simplify the coefficient in front of the derivative inside the integral.

In preparation for the next problem, Figure 11 shows two nearby points $A = (a, x_a)$ and $A_1 = (b, x_b)$, $a < b$ in the (t, x)-plane with $b - a$ small. We fix $a < \bar{t} < b$ and consider the point $P = (\bar{t}, x)$ where x is variable. Let $L = L(x, \dot{x})$ be a Lagrangian, and consider

$$S(x) = \underbrace{\min \int_a^{\bar{t}} L\, dt}_{S_a(x)} + \underbrace{\min \int_{\bar{t}}^b L\, dt}_{S_b(x)}. \tag{6.31}$$

Minimization is taken with A, B and $\bar{t}$ fixed. Since x is the only variable, S is a function of x alone.

6.2. Assume that $L_{\dot{x}\dot{x}} \neq 0$. Show that if the sum S defined by (6.31) has a critical point $x = x_c$, then the slopes of the critical graphs AP and PB match at $P = (\bar{t}, x_c)$, Figure 11.

Hint. Let $X_a(t; x)$ be a critical function of the first integral in (6.31), and similarly, let $X_b(t; x)$ be a critical funciton of the second integral. One can show that

$$S_a'(x) = L_{\dot{x}}(x, \dot{X}_a(\bar{t}, x)) \quad \text{and} \quad S_b'(x) = -L_{\dot{x}}(x, \dot{X}_b(\bar{t}, x)). \tag{6.32}$$

(see (8.6), page 260). Then $S'(x_c) = S_a'(x_c) + S_b'(x_c) = 0$ shows that the values of $L_{\dot{x}}$ at P match from the left and from the right; and since $L_{\dot{x}}$ is a monotone function of $\dot{x}$ by the assumption $L_{\dot{x}\dot{x}} \neq 0$, we conclude that the slopes match as well, proving the claim.

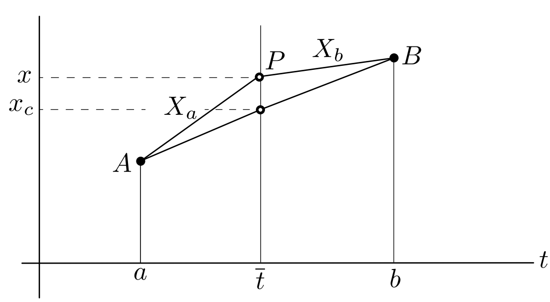

Figure 11. Illustrating problems 6.2 and 6.3.

6.3 (Another geometrical interpretation of Legendre's condition)**.** Show that the Legendre condition $L_{\dot{x}\dot{x}} > 0$ (for all $x, \dot{x}$) implies that the action $S(x)$ defined in (6.31) is convex at the critical point $x = x_c$: $S''(x_c) > 0$, provided $b - a$ is small enough. Since $S'(x_c) = 0$, this shows that $S(x_c)$ is a minimum.

Solution (an outline). Differentiating (6.32) by x gives

$$S_a''(x) = L_{x\dot{x}} + L_{\dot{x}\dot{x}}\ \partial_x \dot{X}_a(\bar{t}, x),$$

where L is evaluated along X_a, and

$$S_b''(x) = -L_{x\dot{x}} - L_{\dot{x}\dot{x}}\ \partial_x \dot{X}_b(\bar{t}, x),$$

where L is evaluated along X_b. Upon the addition of these two equations, the $L_{x\dot{x}}$ terms cancel since $\dot{X}_a(\bar{t}, x_c) = \dot{X}_a(\bar{t}, x_c)$ (according to the preceding problem), and we obtain

$$S''(x_c) = L_{\dot{x}\dot{x}}\ (\partial_x \dot{X}_a(\bar{t}, x_c) - \partial_x \dot{X}_a(\bar{t}, x_c)).$$

But $\partial_x \dot{X}_a(\bar{t}, x_c) > 0$ and $\partial_x \dot{X}_a(\bar{t}, x_c)) < 0$ if $b - a$ is sufficiently small (this follows from the assumption that $L_{\dot{x}\dot{x}} \neq 0$ which allows us to rewrite the Euler–Lagrange equation in the form $\ddot{x} = f(x, \dot{x})$), and we conclude that $S''(x_c) > 0$ since $L_{\dot{x}\dot{x}} > 0$.

6.4. Give a mechanical interpretation of the Legendre condition $P > 0$ for the quadratic functional $\mathcal{Q}$.

Hint. P can be interpreted as the tension of an Eulerian string in the (t, x)-plane (Figure 10) subject to the potential $\frac{1}{2}Q(t)x^2$. Recall that in Euler's model the particles are constrained to move in the x-direction only, so that the string's tension $P(t)$ can be prescribed as a function of the coordinate t.[10] Then $\mathcal{Q} = \frac{1}{2}\int (P\dot{x}^2 + Qx^2)\ dt$ is the potential energy of such a string. It it intuitively obvious that the positive tension $P > 0$ causes the string to want to avoid corners (see (6.11) on page 216), while the negative tension favors crinkling — just as we proved in Section 5.

[10] The string can be constructed as suggested by Figure 3 on page 188.

Chapter 7

Optimal Control

This chapter contains a simple heuristic derivation of Pontryagin's Maximum Principle, for the particular problem of time-optimal control.

The Maximum Principle in a nutshell. This principle is a combination of two main ideas, one of which is essentially Huygens's principle, and amounts to the following: "To be the first in a crowd of runners to reach the goal, one must stay on the front/boundary of the crowd, and to that end *one must maximize one's velocity in the direction of the normal to the front.*" The second idea answers the question on how to keep track of the normal to the front mentioned in the preceding sentence. To be more precise and honest, the front may be nonsmooth, and thus the normal vector may be undefined. Instead, the front (or rather a part of it generated "small" perturbations of the optimal control) has a supporting vector. All this is made more precise after the statement of the Maximum Principle.

1. Formulation of the problem

Optimal control theory deals with finding the "best" way to drive a system from one state to another. Many such problems are modeled by systems of ordinary differential equations with a parameter:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x} \in \mathbb{R}^n; \tag{7.1}$$

the parameter $\mathbf{u} \in U \subset \mathbb{R}^m$, called the *control*, may be either vector or scalar.

The time-optimal control problem *is to find a control function* $\mathbf{u} = \mathbf{v}(t)$ *in* (7.1) *such that the solution of the resulting system passes from* $\mathbf{x}_0$ *to* $\mathbf{x}_1$ *in shortest possible time.*

Example 1. A river flows with speed $V(\mathbf{x})$ depending on the location $\mathbf{x}$. A boat can travel with any speed up to $v_{\max}$ (measured relative to water). The goal is to get from point $\mathbf{x}_0$ to point $\mathbf{x}_1$ in *shortest possible time.* The equation of motion for the boat is of the form (7.1) with

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{V}(\mathbf{x}) + \mathbf{u};$$

the range of $\mathbf{u}$ is the disk $U = \{|\mathbf{u}| \leq v_{\max}\}$. An optimal control may not exist. For instance, getting from $\mathbf{x}_0$ to $\mathbf{x}_1$ may be impossible unless the flow satisfies certain conditions.

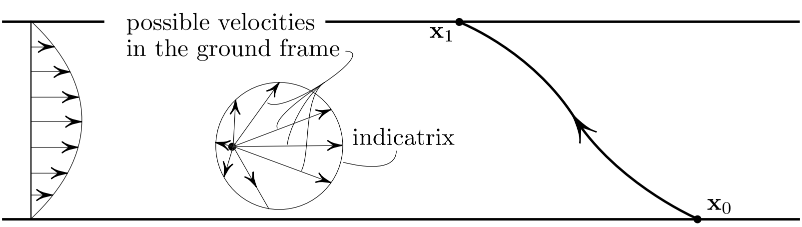

Figure 1. A boat in the river: an optimal control problem.

Example 2. A particle on the line is subject to force u of magnitude $|u| \leq 1$. Given the initial position and velocity, find the control $u = v(t)$ which brings the particle to rest at the origin in least time. This problem falls in the general framework just described. Indeed, Newton's law $\ddot{x} = u$ can be written in vector form (7.1) with

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} y \\ u \end{pmatrix};$$

the initial point is $\mathbf{x}_0 = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$, and the destination point $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the origin in the phase plane. The control parameter $\mathbf{u} = u$ is scalar in this example.

Example 3. A basic airplane has five controls: the throttle and four angles of control surfaces (two for the ailerons, one for the stabilizers, and one for the rudder). The range of each control u_k, $k = 1, \ldots, 5$ is an interval, so that the set $U \subset \mathbb{R}^5$ is a closed cube. Newton's law for the airplane's motion can be written, with some simplifying assumptions, as a first order system (7.1) in $\mathbb{R}^{12}$, provided we use local coordinates (say, Euler's angles) for the airplane's orientation.

Further examples of the above problems include: (i) guiding a rocket into a prescribed orbit with a minimal expenditure of fuel, or (ii) in least time; (iii) reorienting a satellite, i.e., a rigid body in weightlessness, using internal gyroscopes in least time, etc.

A more general problem is to find, amongst all controls which take the solution of (7.1) from $\mathbf{x}_0$ to $\mathbf{x}_1$, the one which minimizes the integral

$$F(u(\cdot),\ \mathbf{x}_0, \mathbf{x}_1) = \int_0^T g(\mathbf{x}(\tau) u(\tau)) d\tau, \tag{7.2}$$

where T is the time of travel from $\mathbf{x}_0$ to $\mathbf{x}_1$, not given but depending on the choice of $u = u(t)$. In the special case of $g \equiv 1$ this problem reduces to the time-optimal control, since for $g = 1$ we have $F(u(\cdot),\ \mathbf{x}_0, \mathbf{x}_1) = T$. Since the main idea can be illustrated on the problem of time-optimization, we limit our attention to that problem, referring the reader to [**15**] for more a much more extensive treatment.

2. The Maximum Principle

I will only formulate the Maximum Principle for the time-optimal controls since it captures all the main ideas of the more general case (7.2).

The Maximum Principle is a necessary (but not sufficient) condition for the control to be optimal. Here is a precise statement.

Theorem 7.1. *Let* $\mathbf{v}(t)$, $0 \le t \le T$ *be a time-optimal control of the control system* (7.1), *and let* $\mathbf{x}(t)$ *be the corresponding optimal trajectory, with* $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}(T) = \mathbf{x}_1$. *Consider an auxiliary linear* *system,*

$$\dot{\mathbf{p}} = -A(t)^T\mathbf{p} \quad \textit{with} \quad A(t) = \mathbf{f_x}(\mathbf{x}(t), \mathbf{v}(t)), \tag{7.3}$$

where $\mathbf{f_x}$ *is the Jacobi derivative matrix*[1] *of* $\mathbf{f}$*. There exists a nonzero solution* $\mathbf{p}(t)$ *of* (7.3) *such that for all* $t \in [0, T]$ *the optimal control maximizes the dot product* $\mathbf{f} \cdot \mathbf{p}$*:*

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) \cdot \mathbf{p} = \max_{\mathbf{u} \in U} \mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{p} \tag{7.4}$$

and, moreover,

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) \cdot \mathbf{p} \geq 0 \quad \textit{at} \quad t = T, \tag{7.5}$$

see Figure 2.

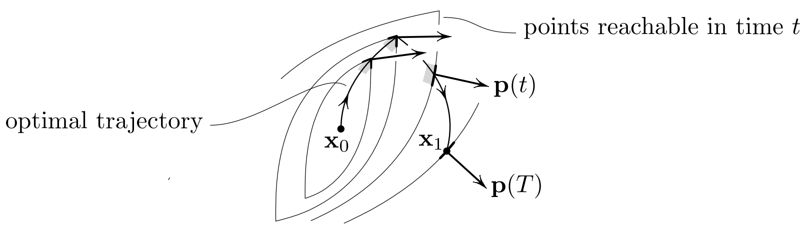

Figure 2. A geometrical significance of $\mathbf{p}(t)$: it is a supporting vector to the shaded set of points reachable in time $\leq t$ by certain small perturbations of $\mathbf{v}$.

Remark 7.1. The pair of equations consisting of (7.1)–(7.3) with $\mathbf{u} = \mathbf{v}(t)$ is a Hamiltonian system with the Hamiltonian

$$H(\mathbf{x}, \mathbf{p}, t) = \mathbf{f}(\mathbf{x}, \mathbf{v}(t)) \cdot \mathbf{p}.$$

Proof. $H_{\mathbf{p}} = \mathbf{f} = \dot{\mathbf{x}}$, $-H_{\mathbf{x}} = -\mathbf{f}_{\mathbf{x}}^T\mathbf{p} = \dot{\mathbf{p}}$. $\diamondsuit$

A heuristic outline of the proof of the Maximum Principle is given in the next section; a full proof, consisting of many pages, can be found in [**2**] or [**15**]. It is hoped, however, that the discussion here gives a clear intuitive insight into the Maximum Principle and is sufficiently convincing.

[1] Defined in the footnote on page 135.

Remark 7.2. Note that the setting of the time-optimal control problem is almost identical to that of geometrical optics. In anisotropic optical media, such as some crystals, the speed of light depends both on the point $\mathbf{x}$ and on the direction $\mathbf{u}$ of the ray, where $\mathbf{u}$ ranges over the sphere of directions $U = \mathbb{S}^3$. Our control system (7.1) can be interpreted as giving possible velocities of light, in terms of the directions $\mathbf{u}$. The light rays traveling between two points make a time-optimal choice of $\mathbf{u}$, according to Fermat's principle. In other words, the nature is an analog computer which solves the time-optimal problem in the optical case. The difference between the optical setting and ours here is that the control set U here need not be a "smooth" set.

Remark 7.3. Maximizing $\mathbf{f} \cdot \mathbf{p}$ (see (7.4)) is closely related to Huygens's principle, according to which the rays maximize their normal velocity to the front (the latter statement is proven on page 250).

3. A geometrical explanation of the Maximum Principle

A pictorial preview.

Definition 7.1. A *reachable set* $R_t(\mathbf{x}_0)$ associated with the control system (7.1) is the set of points in the phase space $\{\mathbf{x}\}$ reachable by solutions of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$ in time $\leq t$ via all admissible choices of $\mathbf{u} = \mathbf{u}(t)$.[2]

Figure 2 shows two examples of reachable sets. To separate the difficulties, we will consider a smooth boundary at first, but next we will deal with corners since they are typical even in simplest problems such as the one mentioned earlier on stopping a harmonic oscillator with a bounded force.

The figure also shows optimal trajectories, along with shaded patches which consist of points reachable by "sub-optimal" controls,

[2] Here "admissible" means that $\mathbf{u}(t)$ is piecewise continuous with the range in U. With such $\mathbf{u}(t)$, $\mathbf{f}(\mathbf{x}, \mathbf{u}(t))$ is still smooth in $\mathbf{x}$, by assumption, but only piecewise continuous as a function of t. Fortunately, the theorem of existence, uniqueness, and smooth dependence on initial data applies under such conditions (in fact, measurable t-dependence suffices), see, e.g., [**3**].

i.e., by small variations of the optimal control $\mathbf{v}(t)$; the meaning of "small" is explained later; see Figure 7. Note that the patches lie on one side of the front: small variations of an optimal control cause the trajectories to lag behind the front.

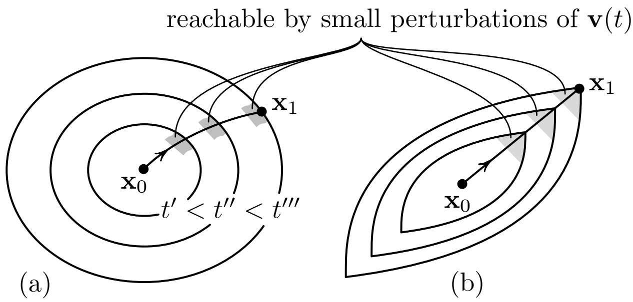

Figure 3. Reachable sets for different values of t; sets reachable by special small step variations are shaded.

The next two paragraphs give a quick nonrigorous explanation of the Maximum Principle for the special case of smooth fronts. This is then followed by an explanation with no special assumptions.

Explaining (7.4) for smooth fronts. If the boundary of $R_t(\mathbf{x}_0)$ is smooth for all $t \in (0, T]$,[3] then we can speak of the outward normal vector $\mathbf{p}(t)$ at $\mathbf{x}(t)$ for each t. Figure 4 illustrates the key point: the time-optimal trajectory, in order to be the first to reach the goal $\mathbf{x}_1$, must stay on the front for all time, and to that end *the component of its velocity in the outward normal direction to the front must be maximal possible;* this is precisely (7.4) of the Maximum Principle. It now remains to explain the origin of the ODE (7.3) for $\mathbf{p}(t)$.

Explaining (7.3) for smooth fronts. Consider a time-dependent vector function $\boldsymbol{\xi}$ evolving according to the linearized system $\dot{\boldsymbol{\xi}} = A(t)\boldsymbol{\xi}$, $A = \mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{v})$. *If $\boldsymbol{\xi}$ is tangent to the front at some time t, then it remains tangent to the front.*[4] This is expected because a tangent vector can be viewed as an infinitesimal difference between two nearby solutions, and such a difference evolves according to the

[3] This assumption often fails; see Figure 2(b)

[4] A quick discussion of linearized equations is given on page 252.

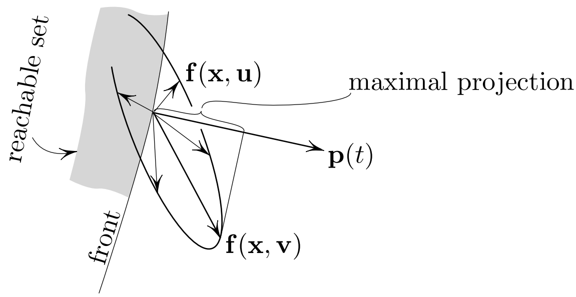

Figure 4. The geometrical meaning of (7.4): the optimal control $\mathbf{v}$ maximizes the normal velocity to the front. In this example, the "indicatrix", i.e., the set of velocities $\{\mathbf{f}(\mathbf{x},\mathbf{u})\,|\,\mathbf{u}\in U\}$ is a smooth curve; the optimal choice of control $\mathbf{v}$ maximizes the component of the velocity $\mathbf{f}(\mathbf{x},\mathbf{u})$ in the direction $\mathbf{p}$ normal to the front.

linearized equation. Having established that the tangent vectors to the front evolve according to the linearized equation, we now show that normal vectors evolve according to the adjoint equation.

Indeed, let $\mathbf{p}(t)$ satisfy the adjoint equation $\dot{\mathbf{p}} = -A^T(t)\mathbf{p}$; then $\boldsymbol{\xi}\cdot\mathbf{p} = \text{const.}$, as one can check by differentiation (see Theorem 7.3 on page 253). We conclude: If $\mathbf{p}(t)$ evolves as stated, it remains orthogonal to the front for all times if it is orthogonal at some time. This explains the choice of the adjoint equation for $\mathbf{p}(t)$ in the statement of the Maximum Principle, and completes the explanation for smooth fronts.

We now proceed to explain the Maximum Principle for fronts with corners; the explanation consists of three steps. In a nutshell, we will show that the part of R_t reached by small perturbations of $\mathbf{v}$ is (approximately) a *convex* cone (shaded in Figure 5) which is not the entire space, i.e., which lies to one side of a hyperplane; a supporting vector of this cone will adopt the role of the normal vector to the front.

Step 1: The displacement formula. We will compute infinitesimal displacements of the endpoint $\mathbf{x}(T)$ of the optimal trajectory caused by small perturbations of $\mathbf{v}$, Figure 5; the result will be the displacement formula (7.6).

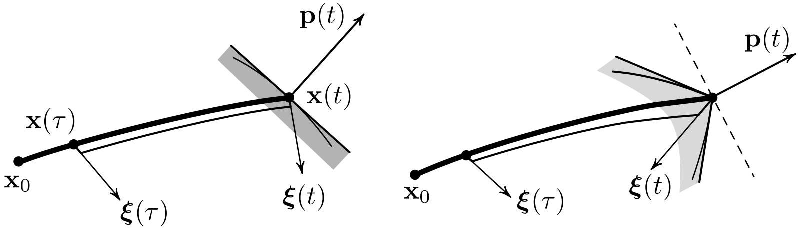

Figure 5. Infinitesimal reachable cones.

Step 2. Infinitesimal displacements computed in Step 1 form a convex cone K_T with the vertex at $\mathbf{x}(T)$, Figure 5. If $\mathbf{x}(t)$ is optimal, then the cone K_T does not coincide with the whole space (as we shall see) and hence possesses a supporting vector $\mathbf{p}_T$, Figure 6.

Step 3. We will define $\mathbf{p}(t)$ as the solution of the adjoint linearized equation (7.3) with $\mathbf{p}(T) = \mathbf{p}_T$ and will prove that (7.4) and (7.5) hold.

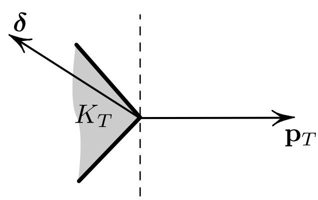

Figure 6. A supporting vector $\mathbf{p}$ of the cone K: $\boldsymbol{\eta} \cdot \mathbf{p} \leq 0$ for any $\boldsymbol{\eta} \in K$.

We now carry out Steps 1–3.

Step 1. Let us perturb $\mathbf{v}(t)$ by replacing it by constants $\mathbf{c}_k \in U$ on a finite number N of (short) intervals $[\tau_k, \tau_k + \varepsilon]$, Figure 7.[5] Let $\mathbf{x}^*(t)$ denote the solution corresponding to the perturbed control, with the same initial condition $\mathbf{x}^*(0) = \mathbf{x}_0$ as $\mathbf{x}(t)$. We assume throughout that $\mathbf{v}$ is continuous at $t = T$, that all τ_k are in the interior of $[0, T]$ and

[5]The variations of $\mathbf{v}(t)$ we are considering are therefore small in the sense that they give rise to small displacements of the trajectory. This is in the spirit of the classical calculus of variations, where we considered only small perturbations.

are distinct and that each τ_k is a point of continuity of $\mathbf{v}$ (although $\mathbf{v}$ may have jumps, we insist that τ_k avoids them).

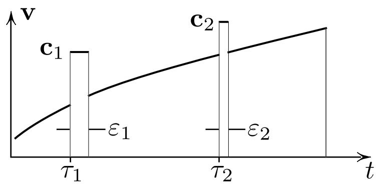

Figure 7. Variations of $\mathbf{v}$ consist of finitely many constant "bumps".

Theorem 7.2. *When the control is perturbed as described in the preceding paragraph, and the travel time is perturbed as well, then the endpoint of the trajectory displaces by*

$$\boxed{\mathbf{x}^*(T-\Delta t)-\mathbf{x}(T)=\underbrace{-\mathbf{f}\,\Delta t+\sum\varepsilon_k\boldsymbol{\xi}_k(T)}_{\textit{linear part}}+o(\varepsilon+\Delta t)} \tag{7.6}$$

where $\Delta t\geq 0,\ \varepsilon=\sum\varepsilon_k,\ \mathbf{f}=\mathbf{f}(\mathbf{x}(T),\mathbf{v}(T))$, *and where each* $\boldsymbol{\xi}_k(t)$ *is a solution of the linearized equation*

$$\dot{\boldsymbol{\xi}}=\mathbf{f}_{\mathbf{x}}(\mathbf{x},\mathbf{v})\boldsymbol{\xi}\quad \textit{for}\quad t\neq\tau_k,\quad t\in[0,T], \tag{7.7}$$

with the zero initial condition

$$\boldsymbol{\xi}(0)=\mathbf{0} \tag{7.8}$$

and with a jump at $t=\tau$*:*

$$\boldsymbol{\xi}(\tau+0)-\boldsymbol{\xi}(\tau-0)=\mathbf{f}(\mathbf{x},\mathbf{c}_k)-\mathbf{f}(\mathbf{x},\mathbf{v}(\tau)). \tag{7.9}$$

Proof of the displacement formula — an outline. Consider at first the perturbation of $\mathbf{v}$ with one bump. Since the perturbed control $\mathbf{v}^*(t)=\mathbf{v}(t)$ for t outside $[\tau,\tau+\varepsilon]$, $\mathbf{x}^*(t)$ satisfies the same ODE as does $\mathbf{x}(t)$ – *this is the main reason for using localized perturbations.* Subtracting the two ODEs, we get

$$\frac{d}{dt}(\mathbf{x}^*-\mathbf{x})=\mathbf{f}(\mathbf{x}^*,\mathbf{v})-\mathbf{f}(\mathbf{x},\mathbf{v})=\mathbf{f}_{\mathbf{x}}(\mathbf{x},\mathbf{v})(\mathbf{x}^*-\mathbf{x})+o(\varepsilon),$$

where $o(\varepsilon)/\varepsilon\to 0$ as $\varepsilon\to 0$, and where $t\notin[\tau,\tau+\varepsilon]$. Dividing by ε and denoting the limit

$$\lim_{\varepsilon\to 0}(\mathbf{x}^*(t)-\mathbf{x}(t))/\varepsilon=\boldsymbol{\xi}(t), \tag{7.10}$$

we conclude that $\boldsymbol{\xi}$ satisfies the linearized ODE (7.7). Since $\mathbf{x}^*(t) = \mathbf{x}(t)$ for $t < \tau$, $\boldsymbol{\xi}(0) = \mathbf{0}$ is immediate. The jump condition (7.9) follows by integrating

$$\frac{1}{\varepsilon}(\dot{\mathbf{x}}^* - \dot{\mathbf{x}}) = \frac{1}{\varepsilon}\mathbf{f}(\mathbf{x}(\tau), \mathbf{c}) - \mathbf{f}(\mathbf{x}(\tau), \mathbf{v}(\tau))$$

over $[\tau, \tau + \varepsilon]$ and by using the continuity of $\mathbf{v}$ at τ.

Now (7.10) implies that

$$\mathbf{x}^*(t) = \mathbf{x}(t) + \varepsilon\boldsymbol{\xi}(t) + o(\varepsilon), \tag{7.11}$$

and this essentially proves the displacement formula for the case of one bump.

When $\mathbf{v}(t)$ is perturbed on N disjoint intervals $[\tau_k, \tau_k + \varepsilon_k]$ $(k = 1, \ldots, N)$, the effects of these perturbations add to the leading order in $\varepsilon = \max(\varepsilon_1, \ldots, \varepsilon_N)$:

$$\mathbf{x}^*(T) = \mathbf{x}(T) + \sum \varepsilon_k \boldsymbol{\xi}_k(T) + o(\varepsilon); \tag{7.12}$$

as before, each $\boldsymbol{\xi}_k(t)$ satisfies (7.7)–(7.9) with $\tau = \tau_k$.

It remains to consider the variation of T. Observe that our perturbations leave $\mathbf{v}(t)$ unchanged near $t = T$ if ε_k are small enough. Then the velocity

$$\dot{\mathbf{x}}^*(T) = \mathbf{f}(\mathbf{x}^*(T), \mathbf{v}(T)) = \mathbf{f}(\mathbf{x}(T), \mathbf{v}(T)) + o(\varepsilon),$$

so that the position

$$\mathbf{x}^*(T - \Delta t) = \mathbf{x}^*(T) - \mathbf{f}(\mathbf{x}, \mathbf{v})_{t=T}\Delta t + o(\varepsilon + \Delta t).$$

Substituting this into (7.12) gives the displacement formula (7.6).

Step 2. The infinitesimal reachable cone. Linear parts of displacements (7.6) form a cone

$$K_T \stackrel{\text{def}}{=} \left\{-\mathbf{f}\Delta t + \sum \varepsilon_k \boldsymbol{\xi}_k(T) \,\middle|\, \Delta t \geq 0,\ \varepsilon_k \geq 0\right\}, \tag{7.13}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x}(T), \mathbf{v}(T))$. Near its vertex, this cone is a good approximation to set reachable by small perturbations of $\mathbf{v}$. It is not hard to show that K_T is convex cone (a detailed proof can be found in [**2**]).

A key property of K_T: *If $\mathbf{v}(t)$ is time-optimal, then K_T does not coincide with the whole space.*[6] This is very plausible intuitively, since the contrary ($K_T = \mathbb{R}^n$) would suggest that an open neighborhood of $\mathbf{x}_1$ is reachable by perturbations of $\mathbf{v}(t)$ and of T (with $\Delta T \leq 0$); such perturbations would allow us to "overshoot" $\mathbf{x}_1$ in time $\leq T$,

[6] This is a formal expression of the loose principle given earlier: "To be the first, one has to be on the front."

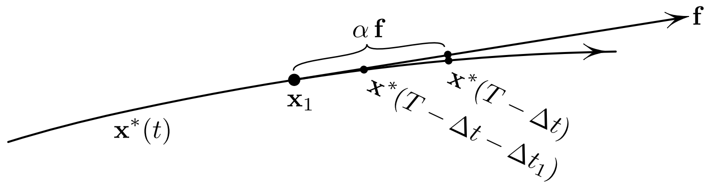

Figure 8. If $K_T = \mathbb{R}^n$, then $\mathbf{v}(t)$ $\mathbf{x}_1$ can be reached before $t = T$.

suggesting that we can reach $\mathbf{x}_1$ in time $t < T$, contradicting the optimality.

Here is an outline of the rigorous proof. Assume the contrary: $K_T = \mathbb{R}^n$. Then for any real α we have $\alpha\,\mathbf{f}(\mathbf{x}_1, \mathbf{v}(T)) \in K_T$; but any vector in K_T approximates an infinitesimal displacement $\mathbf{x}^*(T - \Delta t) - \mathbf{x}_1$, and we conclude that

$$\mathbf{x}^*(T - \Delta t) - \mathbf{x}_1 = \alpha\mathbf{f} + o(\varepsilon + \Delta T), \tag{7.14}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x}_1, \mathbf{v}(T))$, provided α is small enough. Figure 8 illustrates (7.14): the solution "overshoots" the destination $\mathbf{x}_1$ before T; this suggests that we can reach $\mathbf{x}_1$ at an earlier time $T - \Delta t$. Indeed, (7.14) gives

$$\mathbf{x}^*(T - \Delta t - \Delta t_1) = \mathbf{x}_1 + (\alpha - \Delta t_1)\mathbf{f} + o(\varepsilon + \Delta T),$$

and for $\Delta t_1 = \alpha$ we have

$$\mathbf{x}^*(T - 2\Delta t) = \mathbf{x}_1,$$

up to an error $o(\varepsilon + \Delta T)$; an implicit function argument (see [**2**]) shows that this error can be reduced to zero by a proper adjustment of the perturbation.

Step 3. Defining $\mathbf{p}(t)$ and the conclusion. Since the cone K_T is convex and is not the whole space, it possesses a supporting vector $\mathbf{p}_T$, Figure 6, i.e., a vector such that

$$\boldsymbol{\delta} \cdot \mathbf{p}_T \le 0 \tag{7.15}$$

for all infinitesimal displacements

$$\boldsymbol{\delta} = -\mathbf{f}\Delta t + \sum \varepsilon_k \boldsymbol{\xi}_k(T).$$

In particular, for the simplest perturbations $\boldsymbol{\delta} = \boldsymbol{\xi}(T)$ (7.15) gives

$$\boldsymbol{\xi}(T) \cdot \mathbf{p}_T \le 0 \tag{7.16}$$

where $\boldsymbol{\xi}(t)$ is the solution of the linearized equation $\dot{\boldsymbol{\xi}} = \mathbf{f_x}\,\boldsymbol{\xi}$ on $[\tau, T]$ with

$$\boldsymbol{\xi}(\tau) = \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}_c) - \mathbf{f}(\mathbf{x}(\tau), \mathbf{v}(\tau)), \tag{7.17}$$

and where $\tau \in (0, T)$ is arbitrary; see (7.8) and (7.9). Now the appearance of $\boldsymbol{\xi}(T) \cdot \mathbf{p}_T$ suggests, in light of the remark on adjoint systems (Theorem 7.3, page 252) that we define $\mathbf{p}(t)$ as the solution of the adjoint linearized system

$$\dot{\mathbf{p}} = -\mathbf{f}_{\mathbf{x}}^T \mathbf{p}$$

with the initial (or rather terminal) condition $\mathbf{p}(T) = \mathbf{p}_T$. Since $\boldsymbol{\xi}(t) \cdot \mathbf{p}(t) = \text{const.}$, we conclude that

$$\boldsymbol{\xi}(T) \cdot \mathbf{p}_T = \boldsymbol{\xi}(\tau) \cdot \mathbf{p}(\tau) = (\mathbf{f}(\mathbf{x}(\tau), \mathbf{c}) - \mathbf{f}(\mathbf{x}(\tau), \mathbf{v}(\tau))) \cdot \mathbf{p}(\tau) \overset{(7.16)}{\leq} 0.$$

This amounts to (7.4). Finally, setting all $\varepsilon_k = 0$, (7.15) gives

$$\mathbf{f} \cdot \mathbf{p}_T \geq 0, \quad \text{where} \ \ \mathbf{f} = \mathbf{f}(\mathbf{x}(T), \mathbf{v}(T)); \tag{7.18}$$

this explains the second statement (7.5) of the Maximum Principle and completes the discussion.

4. Example 1: a smooth landing

In this section we analyze an example mentioned earlier: Given a bounded control force, bring the particle moving on line with no additional forces to rest at the origin in least time. The motion of the particle is given by

$$\ddot{x} = u, \quad |u| \leq 1, \tag{7.19}$$

and we are seeking a recipe for choosing an optimal control $u = v(t)$ for getting to the origin in the phase plane in shortest time from any state $(x, \dot{x} = y)$.

"Smooth landing" in the title of this section refers to the requirement of zero arrival speed; we want to get to the origin without crashing into it.

Figure 9(a) shows the final answer: a certain "switching" curve separates the phase plane into two regions; the optimal control takes values $u = -1$ or $u = 1$ depending on which side of the switching curve the phase point $(x, \dot{x})$ lies. Figure 9(b) shows two optimal

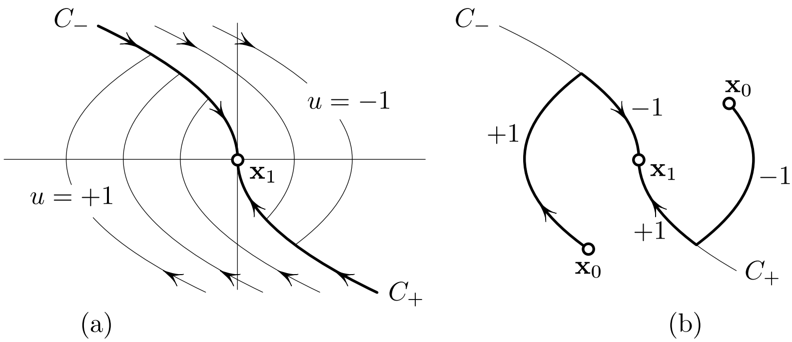

Figure 9. C_- is the trajectory leading into the origin with $u = -1$; similarly, C_+ leads into $\mathbf{0}$ with $u = 1$.

trajectories starting on the different sides of the switching curve; the optimal control switches once, either from 1 to -1, or from -1 to 1, depending on the initial condition. Just like in these two examples, every optimal trajectory makes only one switch, as we will see shortly. To derive the solution just described, we rewrite (7.19) as a system:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix}, \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u), \quad \text{with} \quad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} y \\ u \end{bmatrix}.$$

According to the Maximum Principle, the optimal control $v(t)$ maximizes

$$\mathbf{f}(\mathbf{x}, u) \cdot \mathbf{p} = \begin{bmatrix} y \\ u \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = y(t)p_1(t) + up_2(t),$$

for some solution $\mathbf{p}$ of the adjoint linearized system. Now this expression is maximized by $u = +1$ if $p_2(t) > 0$ and by $u = -1$ if $p_2(t) < 0$; in short, the optimal control is $v(t) = \operatorname{sign} p_2(t)$. To compute p_2 we must solve $\dot{\mathbf{p}} = -\mathbf{f}_{\mathbf{x}}^T \mathbf{p}$, where

$$\mathbf{f}_{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{so that} \quad -\mathbf{f}_{\mathbf{x}}^T = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix};$$

the equation for $\mathbf{p}$ is therefore

$$\frac{d}{dt}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \end{bmatrix}.$$

All solutions are of the form $p_1 = c_1 = \text{const.}$, $p_2(t) = -c_1 t + c_2$, so that $v(t) = \operatorname{sign} p_2(t)$ changes sign at most once.

Summarizing, *the optimal control takes values* $v(t) = \pm 1$ *and* v *switches sign at most once.*

This is all we need to come up with Figure 9. Indeed, let us trace any optimal trajectory backwards in time from its destination $\mathbf{x}_1 = \mathbf{0}$. Traced backward, the trajectory must come out of the origin either with $v = -1$ or with $v = +1$ (as we have established); this is shown in Figure 9: either we follow first along C_- and then along one of the parabolas with $u = +1$, or first along C_+ and then along one of the parabolas with $u = -1$. Any point in $\mathbb{R}^2$ can be reached by such motion, which shows that we can reach the destination $\mathbf{x}_1 = \mathbf{0}$ from any starting point $\mathbf{x}_0$ in $\mathbb{R}^2$.

Note that our choice of control at every moment depends only on the current position (x, y); we do not need to know where we started. To make this important point explicit, define the function V by setting $V(x, y) = -1$ for (x, y) to the left of the switching curve $C_- \cup C_+$ in Figure 9(a), and $V(x, y) = 1$ to the right of this curve. Now let the control u be a function of (x, y), via $u = V(x, y)$; our control system becomes an ordinary differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, V(\mathbf{x}))$. The solutions of this ODE are optimal trajectories. This function V is an example of an optimal *feedback* control, also called a *synthesis* function. The term "feedback" refers to the fact that we "feed" the state (x, y) of the system back into the equation by setting $u = V(x, y)$.

Remark 7.4. Maximum Principle is a necessary, but not a sufficient condition for an optimal control. In particular, we did not prove that such a control actually exists.

5. Example 2: stopping a harmonic oscillator

We now consider a harmonic oscillator with applied force:

$$\ddot{x} + x = u, \quad |u| \leq 1, \tag{7.20}$$

and answer the earlier question of how to bring the oscillator to rest at the origin in shortest time.

The answer is summarized by Figure 5: the switching curve C consists of concatenated arcs of unit semicircles; the optimal control is obtained by choosing $u = -1$ for the points $(x, \dot{x})$ above C and $u = 1$ for the points below C. With such a choice, the motion in the phase plane consists of circular arcs, the last arc terminating at the origin.

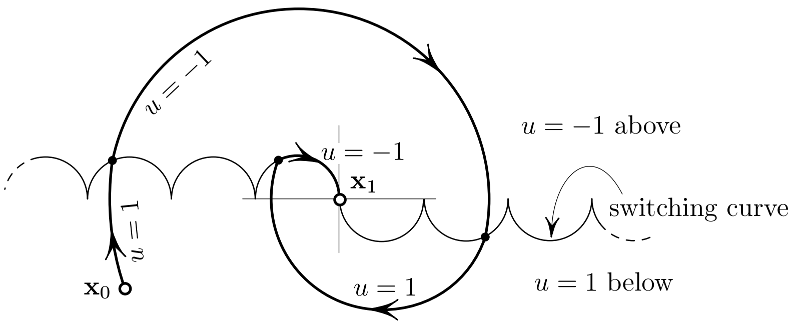

Figure 10. Stopping the harmonic oscillator in least time. The switching curve separates the plane into the region with $u = 1$ and $u = -1$. Time between two switches is π.

To justify the answer given in the last paragraph, we write the system in vector form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u, \end{aligned} \tag{7.21}$$

along with the linearized adjoint system

$$\begin{aligned} \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -p_1. \end{aligned} \tag{7.22}$$

The optimal control maximizes

$$\mathbf{f}(\mathbf{x}, u) \cdot \mathbf{p} = x_2 p_1 + (-x_1 + u) p_2 = -u\, p_2 + \dots, \tag{7.23}$$

where ... denotes terms independent of u; we do not care about these since they do not contain u over which we must maximize. As in the previous example, the maximum is achieved by choosing $v(t) = -\text{sign}\, p_2$. Now $p_2 = A \cos(t - c)$ according to (7.22), and we conclude that $v(t)$ is a step function with steps of length π (Figure 11), taking values $u = -1$ and $u = 1$.

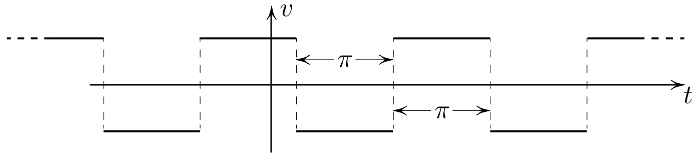

Figure 11. Optimal control $v(t)$ for (7.20).

Figure 12 shows that the trajectories of (7.21) with $u = \pm 1$ are circles centered at $(1, 0)$ and at $(-1, 0)$, traversed clockwise with angular velocity 1: in time t each solution turns around the center through the angle $-t$.

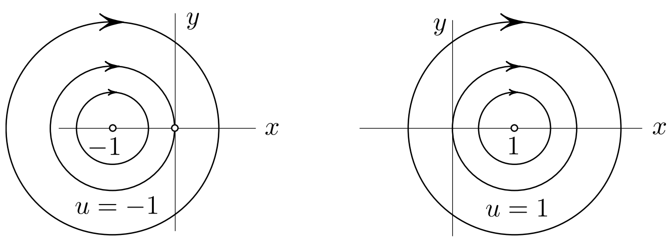

Figure 12. Trajectories for $u = \pm 1$ are circles centered at $(\pm 1, 0)$. Angular velocity of motion is -1.

The information we gathered in Figures 11 and 12 now suffices to solve the problem. Just as we did in the first example, let us trace a typical optimal trajectory backwards from the origin. If $u = -1$ near the end of its trip $t = T$ then the trajectory, traced backwards in time, follows the left circle in Figure 13, since this is the only trajectory entering the origin with $u = -1$. Now it takes time π to travel one semicircle, and since the jumps in $v(t)$ occur at intervals π apart, we conclude that the jump preceding $t = T$ happens at some point A on the upper semicircle, Figure 14. We now trace A backwards for time π along the flow with $u = +1$, arriving at the point $A_{-\pi}$, Figure 14. The point $A_{-\pi}$ is obtained by rotating A counterclockwise through angle π around $(1, 0)$. Since A is an arbitrary point on the upper

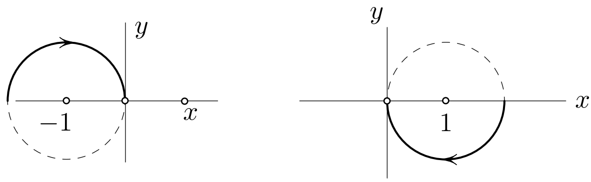

Figure 13. Travel time along each semicircle equals π. It takes time $> \pi$ to reach the origin starting on the dotted parts.

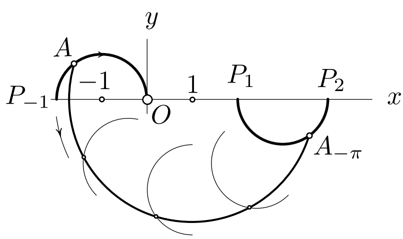

Figure 14. Travel time along each semicircle equals π. When starting on the dotted parts, it takes time $> \pi$ to reach the origin.

semicircle, we can rotate the whole semicircle $P_{-1}O$ obtaining the semicircle P_1P_2. We conclude that u must jump when an optimal trajectory crosses P_1P_2.

The construction now repeats in Figure 15: tracing the semicircle P_1P_2 backwards for duration π, we obtain the next semicircle $P_{-3}P_{-2}$. When our optimal trajectories cross this semicircle, the control switches from -1 to $+1$ (traveling backwards in time). This construction is continued ad infinitum. We established that any optimal trajectory entering the origin with $u = -1$ must switch control when crossing any of these semicircles. In exactly the same way we can build up switching semicircles for those trajectories which enter the origin along the right semicircle with $u = +1$; see Figure 13. These semicircles fill in the gaps in Figure 15, and the union of all switching semicircles, together with the semicircles in Figure 15. The result is a scalloped curve in Figure 16. For (x, y) above this curve the optimal control is $u = -1$, while for (x, y) below $u = 1$. In other

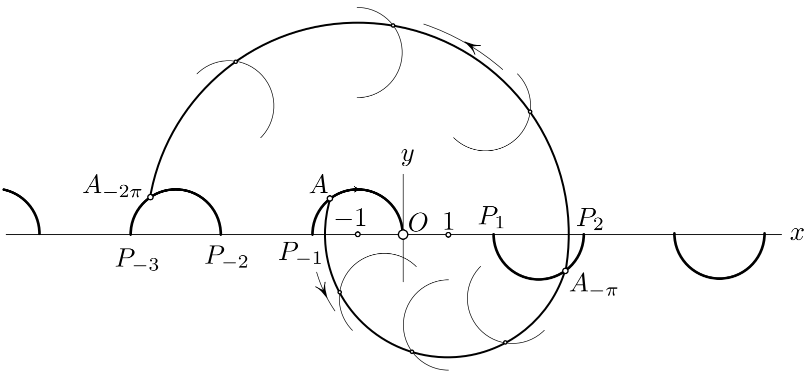

Figure 15. Building up the switching curve from the original "seed" O by tracing this seed backwards at time intervals π along flows with alternating values of $u = \pm 1$.

words, the function $V(x, y)$ taking values ± 1 as shown in Figure 16 is the optimal control feedback for our problem.

This completes our discussion.

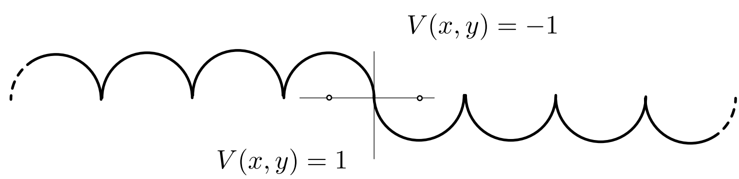

Figure 16. The feedback control V jumps from 1 to -1 as we cross the switching curve from below to above.

6. Huygens's principle vs. Maximum Principle

Recall that one half of Pontryagin's Maximum Principle states that the time-optimal trajectories maximize their normal velocity to the front (see (7.4)); this maximization amounts to Huygens's principle, as we point out in this section. Namely, we will show that Huygens's principle implies that *optical rays maximize their normal velocity to the front.*

To state Huygens's principle, consider an optical medium with speed of light possibly depending on the direction.[7] Figure 17 depicts an indicatrix at $\mathbf{x}$, i.e., the set of tips of velocity vectors at $\mathbf{x}$. The set $w_{\Delta t}(\mathbf{x})$ denotes the wave front at time Δt resulting from an initial disturbance at $\mathbf{x}$. For small Δt, this front is approximated by a dilation of the indicatrix by the factor Δt, Figure 17.

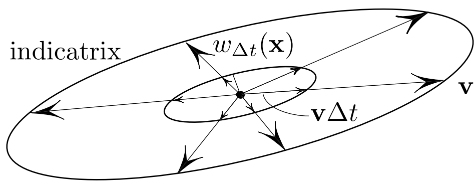

Figure 17. The indicatrix in an optical medium is the set of tips of velocity vectors of light rays at a point.

Consider now a wave front W_t propagating in such a medium, Figure 18. Huygens's principle states that the new front $W_{t+\Delta t}$ is the envelope of infinitesimal fronts $w_{\Delta t}(\mathbf{x})$ with $\mathbf{x} \in W_t$ generated by disturbances at the points of the old front (we assume that all the fronts in question are smooth so that we can speak of tangency).

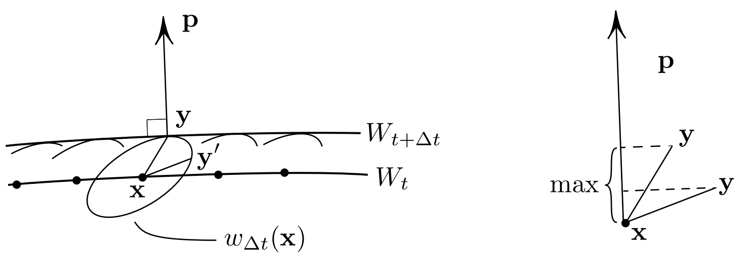

Figure 18. Huygens's principle and its implication: rays maximize their normal velocity to the front.

We now state the main claim of this section: *Huygens's principle implies that the propagating rays in optical media maximize their velocity in the direction of the normal to the front.* We will prove this

[7]Some crystals, e.g., calcite, as well as some plastics under stress, are optically anisotropic.

claim under the (unnecessarily restrictive) assumption that $w_{\Delta t}(\mathbf{x})$ is smooth and convex.

By Huygens's principle $w_{\Delta t}(\mathbf{x})$ and $W_{t+\Delta t}$ are tangent at some $\mathbf{y}$, and thus share the same normal vector $\mathbf{p}$; and since $w_{\Delta t}(\mathbf{x})$ is convex, we have $(\mathbf{y}' - \mathbf{y}) \cdot \mathbf{p} \leq 0$, Figure 18, or

$$((\mathbf{y}' - \mathbf{x}) - (\mathbf{y} - \mathbf{x}) \cdot \mathbf{p} \leq 0. \tag{7.24}$$

But $\mathbf{y}' - \mathbf{x} = \mathbf{v}_{\mathbf{xy}'}\Delta t + o(\Delta t)$ and $\mathbf{y} - \mathbf{x} = \mathbf{v}_{\mathbf{xy}}\Delta t + o(\Delta t)$, where $\mathbf{v}_{\mathbf{xy}}$ is the velocity of the ray in the direction $\mathbf{xy}$. Substituting these expressions into (7.24), dividing by Δt and setting $\Delta t \to 0$ results in

$$\mathbf{v}_{\mathbf{xy}} \cdot \mathbf{p} \geq \mathbf{v}_{\mathbf{xy}'} \cdot \mathbf{p}.$$

This shows that light rays maximize their speed in the direction of the normal to the front. $\diamondsuit$

7. Background on linearized and adjoint equations

The ideas of this section were used in the earlier explanation of the Maximum Principle. We first give a quick explanation of how linearized equations arise, and then state a geometric relationship between a linear equation and its adjoint.

Linearized equation. To give a formal definition, let $\mathbf{x}(t)$ be a solution of a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

in $\mathbb{R}^n$. The *linearization* of this system around $\mathbf{x}(t)$ is, by the definition, the linear system

$$\dot{\boldsymbol{\xi}} = A(t)\boldsymbol{\xi}, \tag{7.25}$$

where $A(t) = \mathbf{f}_{\mathbf{x}}(\mathbf{x}(t))$ is the Jacobi derivative matrix of the vector function $\mathbf{f}$.

The linearized system is of interest because *it describes the infinitesimal difference* $\boldsymbol{\xi}$ *between two nearby solutions*, see Figure 19. In another interpretation, linearized equation describes the *propagation* *of an error in initial condition.* To see why, let $\mathbf{y}(t)$ is a solution of our system near $\mathbf{x}(t)$; subtracting and using Taylor's formula we get

$$\frac{d}{dt}(\mathbf{y}-\mathbf{x}) = \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}_{\mathbf{x}}(\mathbf{x})(\mathbf{y}-\mathbf{x}) + o(|\mathbf{y}-\mathbf{x}|).$$

By dropping the last term and renaming $\mathbf{y}-\mathbf{x} = \boldsymbol{\xi}$, we get (7.25). It should be pointed out that since (7.25) is linear, any scalar multiple $c\boldsymbol{\xi}$ is a solution, and therefore $\boldsymbol{\xi}$ need not be thought of as small, Figure 19.

Formally, we can obtain (7.25) by considering a one-parameter family $\mathbf{x} = \mathbf{x}(t;\varepsilon)$ of solutions. Differentiation of the identity $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with respect to ε then yields (7.25), where $\boldsymbol{\xi} = \frac{\partial}{\partial\varepsilon}\mathbf{x}(t;\varepsilon)$.

This completes our introduction of the linearized equation, and we now proceed to discuss the adjoint systems.

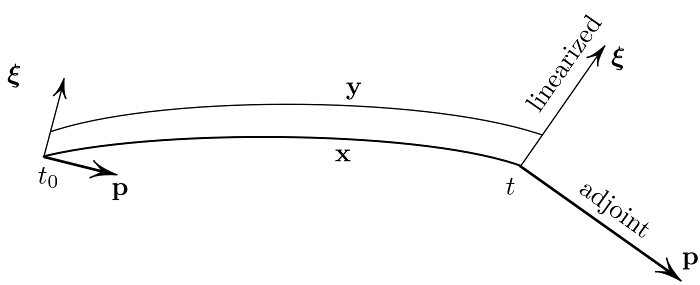

Figure 19. Linearized equation governs the evolution of infinitesimal difference between nearby solutions.

Adjoint equation. Given a linear ODE (7.25), its *adjoint* system is defined as

$$\dot{\mathbf{p}} = -A^T(t)\mathbf{p}; \tag{7.26}$$

here A^T is the transpose of the matrix A. The adjoint system is interesting for the following reason.

Theorem 7.3. *For any solution* $\boldsymbol{\xi}$ *a linear system* (7.25) *and for any solution* $\mathbf{p}$ *of the adjoint system* (7.26)*, the dot product is constant:*

$$\boldsymbol{\xi}\cdot\mathbf{p} = \text{const.} \tag{7.27}$$

Proof. Differentiating the dot product we get

$$\frac{d}{dt}\boldsymbol{\xi}\cdot\mathbf{p} = (A\boldsymbol{\xi})\cdot\mathbf{p} + \boldsymbol{\xi}\cdot(-A^T\mathbf{p}) = (A\boldsymbol{\xi})\cdot\mathbf{p} - (A\boldsymbol{\xi})\cdot\mathbf{p} = 0. \qquad \diamondsuit$$

Corollary 7.1. In the notations of the last theorem, if $\boldsymbol{\xi}(t) \perp \mathbf{p}(t)$ holds for some t, then it holds for all t.

Remark 7.5. The system and its adjoint, viewed as a pair:

$$\begin{cases} \dot{\boldsymbol{\xi}} = A(t)\boldsymbol{\xi}, \\ \dot{\mathbf{p}} = -A(t)^T\mathbf{p} \end{cases} \tag{7.28}$$

is a Hamiltonian system in $\mathbb{R}^{2n}$ with the Hamiltonian function $H(\boldsymbol{\xi}, \mathbf{p}) = (A\boldsymbol{\xi})\cdot\mathbf{p}$. This shows that *any linear system is a subsystem of a Hamiltonian system.* Actually, more is true: *any* system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ can be extended to a Hamiltonian system by doubling the dimension of the phase space; see Problem 7.7.

8. Problems

Optimal Control.

7.1. Sketch some reachable sets for the free particle with a bounded control force: $\ddot{x} = u$, $|u| \le 1$ (Example 1 on page 244).

Answer. See Figure 2(b).

7.2. For the system $\ddot{x} = u$, $|u| \le 1$ considered in the preceding problem, show directly that the linearized equation along an optimal trajectory $\mathbf{x}(t)$ governs the evolution of tangent vectors to the front, as long as the front is smooth.

Hint. See Figure 20.

7.3. The set R_T in Figure 2(b) on page 236 consists of all points obtained by flowing from the origin with $\dot{x} = y$, $\dot{y} = u$ with $u = \pm 1$ for the total time $t \le T$, where only one jump of $u(t)$ is allowed. Prove directly (without appealing to the Maximum Principle) that the set R_T thus constructed is indeed the reachable set, i.e., that one cannot get outside R_T in time $\le T$ by any other control $u(t)$ with $|u(t)| \le 1$.

7.4. Find the time-optimal feedback function for reaching the origin according to the following system:

$$\begin{cases} \dot{x} = \phantom{-x+{}} y + u_1, \\ \dot{y} = -x + u_2 \end{cases}$$

with $|u_1| \le 1,\ |u_2| \le 1$.

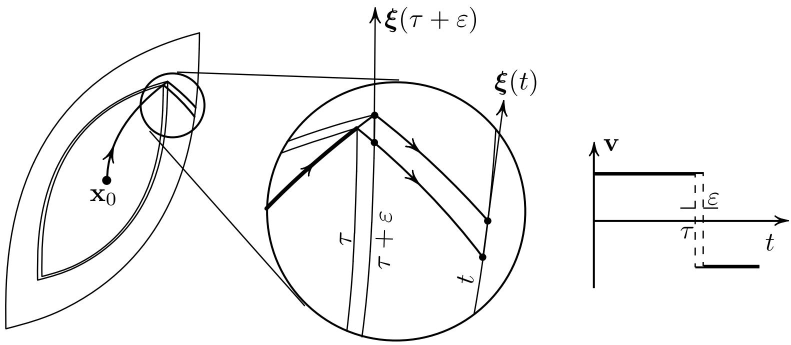

Figure 20. Illustrating Problem 7.2.

Miscellaneous

The following calculus problem from Putnam Competition can be best solved using the idea of reachable set.

7.5. Find the straight line segment connecting a given point A to a point on a given circle C in the vertical plane such that the bead released from A at rest and sliding along the segment will reach C in least time.

Hint. The "brute force" solution would be to write the time as a function of the point on C; this leads into an algebraic morass. A more enlightening approach is to consider all lines through A and to release beads from A, one bead per line, simultaneously, Figure 21. The beads form a "front" W_t which at some time $t = T$ touches the circle C. The point P of first contact is the one we are looking for. It turns out that W_t is a circle (of diameter $gt^2/2$) (proving this is subject of the next problem). The geometric recipe for finding P just obtained solves the problem.

7.6. Prove that the sets W_t in Figure 21 (defined in Problem 7.5) are indeed circles.

7.7. Show that any system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$ can be extended to a Hamiltonian system in $\mathbb{R}^{2n}$.

Hint. Consider the Hamiltonian $H(\mathbf{x}, \mathbf{p}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{p}$.

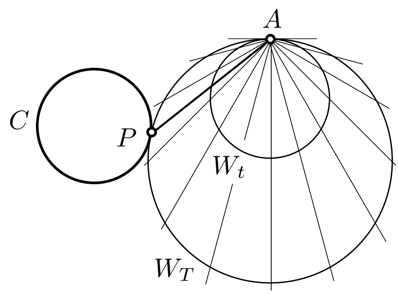

Figure 21. A toy example of the wave front.

Chapter 8

Heuristic Foundations of Hamiltonian Mechanics

1. Some fundamental questions

This chapter answers some questions on heuristic motivation I had while in school (and even after), but was not able to find answers to in books. By an "answer" I don't mean a formal proof but rather a mental picture which makes the fact *intuitively* obvious from first principles. Here are some of these questions:

1. *Why* does a symmetry of the Lagrangian imply the existence of conserved quantities (Noether's theorem)? Is there a simple way to understand this without calculation (unlike the derivation in [**1**] which uses Euler–Lagrange equations)? (See pages 276 and 267).

2. *Why* are Poincaré integral invariants $\oint_\gamma p\, dq$ conserved? What is an intuitive reason for this conservation? (See Sections 6, 7, 11).

3. What is a physical interpretation of the symplectic form? (See page 277).

4. *Why* is the phase volume preserved in a Hamiltonian system (Liouville's theorem)? Our proof relied on the divergence-free nature of Hamiltonian vector fields which was proven by a (very simple) computation. But is there something deeper and simpler than a formal

calculation, and do we really need Hamilton's equations to explain Liouville's theorem?

5. What motivates the Hamilton–Jacobi equation? What is its geometrical meaning? (see page 266).

6. What is a geometrical meaning of the Legendre condition $L_{\dot{x}\dot{x}} > 0$? (see page 216 in Chapter 6).

7. *Why* is the Hamiltonian defined via the Legendre transform $H(x,p) = \dot{x}L_{\dot{x}} - L$, with $\dot{x}$ defined from $L_{\dot{x}} = p$? (explanation is in the heuristic derivation of the main theorem, page 260).

Answers to these questions almost fall into our laps from a single picture and a single idea captured in the main theorem on page 260; a simple heuristic derivation of this theorem is on page 260.

To this list one can add one more question:

8. How to discover the Schrödinger equation from Hamilton's principle? (See page 286).

My goal in this chapter is to sketch the answers to these questions. The next section contains a brief overview of this brief chapter.

2. The main idea

Our starting point is Hamilton's principle which states that any motion $x = x(t)$ of a system with a Lagrangian L is a critical function of the functional $\int_{t_0}^{t_1} L(y, \dot{y})\, dt$, in the space of functions with fixed ends. The value of this functional evaluated at this critical function will be denoted by

$$\boxed{S(A_0, A_1) = S(t_0, x_0; t_1, x_1) = \int_{t_0}^{t_1} L(x, \dot{x})\, dt} \tag{8.1}$$

where

$$x(t_0) = x_0,\ x(t_1) = x_1 \tag{8.2}$$

and called the *action*. We assume throughout that the critical function

$$x(t) = X(t; t_0, x_0, t_1, x_1) \tag{8.3}$$

depends on the ends smoothly, so that S is a smooth function of its arguments as well.[1]

If I were allowed only one statement to make on the foundations of mechanics beyond Hamilton's principle, it would be this:

> *"Treat the action $S(A_0, A_1)$ as an analog of the potential energy of a heavy spring in the (t, x)-space, subject to some potential force, hanging in equilibrium, with the ends held at A_0, A_1; consider the counterparts of forces required to hold the ends of the spring in place."*

This principle leads easily and *automatically* to all the concepts of Hamiltonian mechanics, as shown in this chapter. For instance, the Hamiltonian and the momentum turn out to be *verbatim* analogs of the forces holding the end of the spring. The table on page 280 gives a "dictionary" between Hamiltonian dynamics and statics of springs. In this sense, Hamiltonian dynamics could be referred to as a kind of "spring theory".

In the main theorem stated next we will compute the gradients of $S(A_0, A_1)$ with respect to each end; Figure 1 shows the gradient with respect to A_1 with fixed A_0. The coordinates of this gradient turn out to be the Hamiltonian and the momentum, as the figure anticipates.

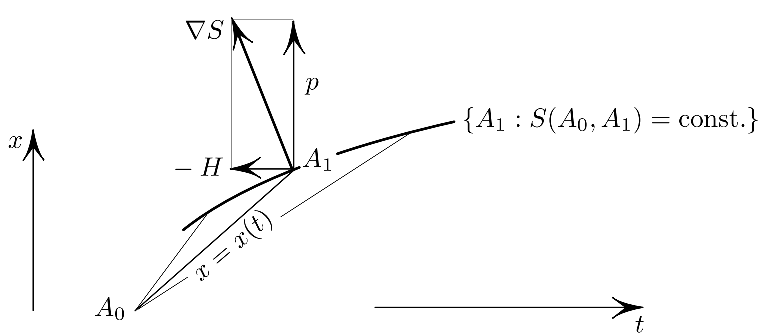

Figure 1. The gradient ∇S is taken with respect to A_1.

[1] For this smoothness assumption to hold it suffices for L to satisfy the Legendre condition, and for the points A_0 and A_1 not to be conjugate to each other.

Theorem 8.1 (Main theorem)**.** *Assume that $S(A_0, A_1)$ is a smooth function of the endpoints $A_0(t_0, x_0)$ and $A_1(t_1, x_1)$. Then*

$$\boxed{\nabla_{A_1} S(A_0, A_1) \equiv \langle S_{t_1}, S_{x_1} \rangle = \langle L - \dot{x} L_{\dot{x}} \ , \ L_{\dot{x}} \rangle_{t=t_1}} \tag{8.4}$$

where $x(t)$ is the critical function (8.3) *in the definition of S and where the subscripts denote partial differentiation. Similarly, at the left end the result is the same, modulo a sign:*

$$\nabla_{A_0} S(A_0, A_1) \equiv \langle S_{t_0}, S_{x_0} \rangle = -\langle L - \dot{x} L_{\dot{x}} \ , \ L_{\dot{x}} \rangle_{t=t_0}. \tag{8.5}$$

Componentwise, this amounts to the expressions for partial derivatives

$$S_{t_1} = L - \dot{x} L_{\dot{x}}|_{t=t_1}, \quad S_{x_1} = L_{\dot{x}}|_{t=t_1}, \tag{8.6}$$

and similarly for S_{t_0}, S_{x_0}, up to a sign.

All of the above applies to the higher dimensional case of $\mathbf{x}_0, \ \mathbf{x}_1 \in \mathbb{R}^n$ if we replace partial derivatives S_x by the gradients $S_{\mathbf{x}} = \nabla_{\mathbf{x}} S$.

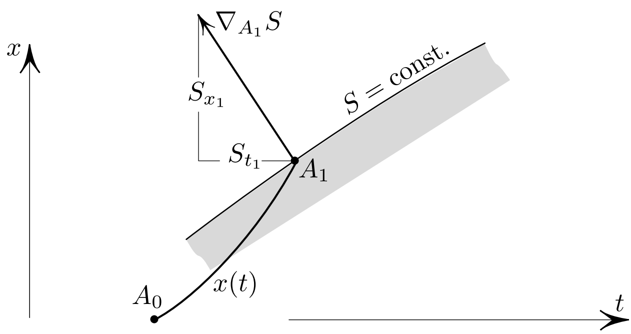

Figure 2. Illustration of the main theorem. Shaded is the set $\{A' : S(A_0, A') \le S(A_0, A_1)\}$ with A_0 fixed. Gradient is taken with respect to A_1.

A heuristic derivation of (8.4) is very simple if we observe that for any point A_0' lying on the graph of the critical function connecting A_0 and A_1 (see Figure 3), one has

$$\nabla_{A_1} S(A_0, A_1) = \nabla_{A_1} S(A_0', A_1). \tag{8.7}$$

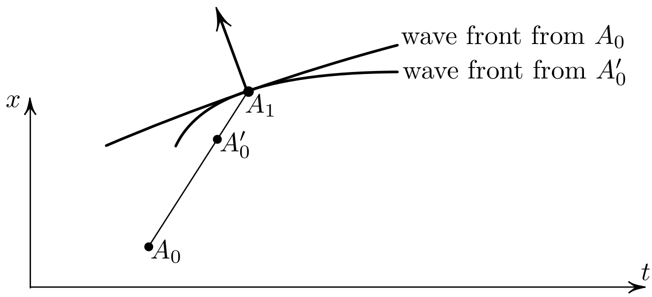

Figure 3. Independence of the gradient on the starting point.

Thanks to this relation (proven after Problem 8.3 on page 288), we can choose $A_0' = (a, b)$ arbitrarily close to A_1, so that the integral is approximated by an algebraic expression (this is the key idea):

$$S(A_0', A_1)) = L\left(a, \frac{x_1 - b}{t_1 - a}\right)(t_1 - a) + \dots, \tag{8.8}$$

where ... stands for higher order terms in $t_1 - a$. Differentiating by t_1 gives

$$S_{t_1} = L - L_{\dot{x}}\,\frac{x_1 - b}{t_1 - a} + \dots = L - \dot{x}L_{\dot{x}}|_{t=t_1} + \dots, \tag{8.9}$$

since $\frac{x_1 - b}{t_1 - a} \to \dot{x}(t_1)$ as $a \to t_1$; the terms denoted by ... are small when a is close to t_1. Taking $a \to t_1$ in the (8.9) yields the first part of (8.6). The second part of (8.6) is justified identically.

To prove (8.5) for the left endpoint $A_0(t_0, x_0)$, we apply the previous result by treating A_0 as the right endpoint of another interval. To that end let us extend the critical curve A_0A_1 to the left to some point $A_-(t_-, x_-)$ where $t_- < t_0$, and form a function

$$F(Z) = S(A_-, Z) + S(Z, A_1), \tag{8.10}$$

where A_- and A_1 are fixed and Z is variable. Since A_0 lies on the critical curve A_-A_1, we have

$$\nabla_Z F(Z)|_{Z=A_0} = 0.$$

In view of (8.10) this gives

$$\nabla_Z S(Z, A_1)|_{Z=A_0} = -\nabla_Z S(A_-, Z) \overset{(8.4)}{=} -\langle L - \dot{x}L_{\dot{x}}, L_{\dot{x}}\rangle|_{t=t_0}.$$

Note that A_0 is the right end of the curve A_-A_0, so that (8.4) is applicable to A_0. The heuristic "proof" of the main theorem (8.1) is complete. $\diamondsuit$

Remark 8.1. (8.7) has the following analog in optics. Let (t, x)-plane be an optical medium and let $S(A_0, A_1)$ be the optical distance, i.e. the time it takes the ray to travel from A_0 to A_1. Then $\nabla_{A_1} S(A_0, A_1)$ gives the vector of normal slowness to the front created by the disturbance at A_0, (see Problem 8.13 on page 292). In this optical interpretation, (8.7) simply says that *if two fronts passing through A_1 originate from two points (A_0 and $A_0{}'$) on the same ray, then the fronts are tangent to each other and move with the same speed at A_1; see Figure* 3. As a side remark, the last statement can be derived this from Huygens's principle.

Rather than turning the above heuristic argument into a proof, we use an alternative approach.

A formal proof of Theorem 8.1. To prove (8.4), we can fix the left endpoint (t_0, x_0) and drop it from the notation until further notice, so that the critical function (8.3) $x(t) = X(t; t_1, x_1)$ and

$$S(t_1, x_1) = \int_{t_0}^{t_1} L(X, X_t)\, dt, \tag{8.11}$$

where $X = X(t; t_1, x_1)$ and $X_t = \frac{\partial X}{\partial t}$. Differentiating by x_1 yields

$$S_{x_1} = \int_{t_0}^{t_1} (L_x X_{x_1} + L_{\dot{x}} X_{tx_1})\, dt, \tag{8.12}$$

with subscripts denoting partial differentiation. Integrating the second term by parts results in

$$S_{x_1} = \int_{t_0}^{t_1} \underbrace{\left(L_x - \frac{d}{dt}(L_{\dot{x}})\right)}_{0} X_{x_1}\, dt + L_{\dot{x}}\, X_{x_1}\big|_{t=t_0}^{t=t_1}\ .$$

The integral vanishes since X satisfies the Euler–Lagrange equation. To show that the last term gives thedesired expression, note that

$X_{x_1}(t_1; t_1, x_1) = 1$ and $X_{x_1}(t_0; t_1, x_1) = 0$, as follows by taking the x_1-derivative of the boundary conditions $X(t_1; t_1, x_1) = x_1$ and $X(t_0; t_1, x_1) = x_0$. We proved that $S_{x_1} = L_{\dot{x}}|_{t=t_1}$, one part of the claim (8.6). To prove the remaining claim for S_{t_1} in (8.6) we could repeat the above argument, but give a shortcut instead. Differentiating $\int_{t_0}^{t} L(x(\tau), \dot{x}(\tau)\, dt$ by t gives

$$\frac{d}{dt} S\,(t, x(t)) = L(x(t), \dot{x}(t)); \tag{8.13}$$

here (t_0, x_0) is fixed and suppressed as before. But $\frac{d}{dt} S\,(t, x(t)) = S_t + S_x \dot{x}$; solving for S_t gives $S_t = L - \dot{x} S_x = L = \dot{x} L_{\dot{x}}$, as claimed.

3. The Legendre transform, the Hamiltonian, the momentum

The main theorem (8.4) leads at once to the three concepts listed in the title of this section.

Note that the gradient

$$\nabla_{(t_1, x_1)} S(t_0, x_0; t_1, x_1) = \langle L - \dot{x} L_{\dot{x}}, L_{\dot{x}} \rangle|_{t=t_1} \tag{8.14}$$

is determined by the slope $\dot{x} = \dot{x}(t_1)$. A glance at (8.14) makes it clear that $L_{\dot{x}}$ is a more convenient parameter than $\dot{x}$. Guided by this idea, we use

$$L_{\dot{x}}(x, \dot{x}) \stackrel{\text{def}}{=} p, \tag{8.15}$$

rather than $\dot{x}$, to parametrize ∇S. Assume that (8.15) can be solved for $v = \dot{x}$ (the slope of the critical function) as a function of p: $v = v(x, p)$, with x playing the role of a parameter. Substituting $\dot{x} = v(x, p)$ into (8.5) we turn ∇S into a function of x and p; the first coordinate S_t is denoted by $-H$:

$$L - vL \stackrel{\text{def}}{=} -H(x, p), \quad v = v(x, p), \tag{8.16}$$

where the minus sign is dictated by the tradition. H is called the *Hamiltonian* associated with the Lagrangian L.

We were quickly led to three fundamental concepts: the momentum (8.15) and the Hamiltonian (8.16), and to the *Legendre transform*, i.e. to the passage from $v, L(x,v)$ to $p, H(x,p)$ via (8.15) – (8.16). The precise definition and properties of the Legendre transform are given in the next section.

To conclude, the Hamiltonian, the momentum, and the Legendre transform fell into our lap automatically once we asked "what is ∇S?"; we did not have to pull any of these concepts "out of thin air".

Remark 8.2. We are considering the scalar case: $x \in \mathbb{R}$ in most of this chapter; for the higher dimension of $x \in \mathbb{R}^n$ all discussions carry over almost verbatim.

Exercise 8.1. Let $L = m\dot{x}^2/2 - U(x)$. Show that (8.15) and (8.16) give the momentum $m\dot{x}$ and the energy $m\dot{x}^2/2 + U$. It is remarkable that the palpable physical quantities arise also as geometrical objects – namely, as coordinates of the gradient of the action.

4. Properties of the Legendre transform

In the preceding section the Legendre transform fell into our laps. In this section we digress from the main story to describe some interesting properties of this transform, whose definition we first state precisely.

Assume that a function $L : \mathbb{R} \to \mathbb{R}$ is such that the derivative function $L'(v)$ is invertible, i.e., $L'(v) = p$ has a unique solution v for any $p \in \mathbb{R}$, Figure 4[2]. Then the function

$$H(p) = vp - L(v) \quad \text{where} \quad L'(v) = p \tag{8.17}$$

is called the Legendre transform of L.[3] Figure 4 gives two geometrical interpretations of the Legendre transform.

[2] In the preceding section L depended on a parameter x, but here this is not relevant; we concentrate on the "true" variables v and p.

[3] All this extends to the case of $v, p \in \mathbb{R}^n$ verbatim; in that case, $vp = v \cdot p$ is to be understood as the dot product.

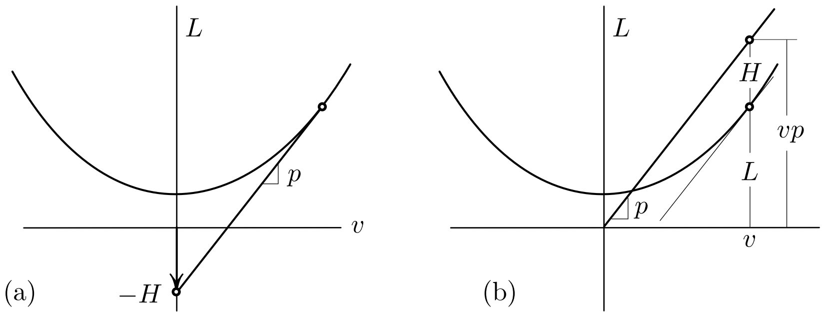

Figure 4. Two geometrical interpretations of the Legendre transform.

Two geometrical interpretations. Referring to Figure 4(a), $-H(p)$ is the vertical intercept of the tangent line of slope p to the graph of $L(v)$ in the (v, L)-plane. Alternatively, according to Figure 4(b), $H(p)$ is the critical vertical distance between the graph of $L(v)$ and the line of slope p passing through the origin. All of these interpretations make it clear that for the Legendre transform to be well defined the graph of L must be convex.

The symmetric form of the relation $L(v) + H(p) = vp$ suggests the following theorem.

Theorem 8.2. *The Legendre transform is involutive: If H is a Legendre transform of L, then L is the Legendre transform of H. In particular, treating v as the independent variable, we have*

$$L(v) = v \cdot p(v) - H(p) \quad \textit{where} \quad H'(p) = v. \tag{8.18}$$

Proof. Differentiating (8.17) by p gives

$$H'(p) = v + \cancel{v'_p}\,p - \cancel{L'v'_p} = v.$$

Now p in (8.18) is an independent variable, but since the correspondence $v \leftrightarrow p$ is one-to-one, we can treat v as an independent variable; this now proves the second statement in (8.18), and the first part of (8.18) follows at once from (8.17) with v treated as an independent variable. ♢

The above proof relies on a short calculation; the following integral interpretation relies on a picture instead.

An integral interpretation of the Legendre transform. Referring to Figure 5, let us interpret $L(v)$ as the area in the (v, p)-plane under a curve, over the interval $[0, v]$. This curve is therefore given by $p = L'(v)$. Assume that this curve passes through the origin, for the purposes of this discussion. Let $H(p)$ be the complementary area shown in the figure, so that the two areas make up a rectangle of area $L(v)+H(p) = vp$. The relations $H'(p) = v$, $L'(v) = p$ are simply the statements of the fundamental theorem of calculus. These relations make the involutive character of the Legendre transform obvious.

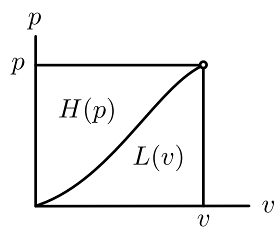

Figure 5. One more geometrical interpretation of the Legendre transform.

5. The Hamilton–Jacobi equation

The fact that S was defined in terms of L suggests that S must be special in some way. That special property is the fact that S satisfies the Hamilton–Jacobi equation, as this short section explains.

Let us fix the left point (t_0, x_0) in $S(t_0, x_0; t_1, x_1)$, treating S as the function of the right point (t_1, x_1) only; dropping the subscripts, we study $S = S(t, x)$. The fact that S came from L is captured by the following characteristic property of S.

Theorem 8.3. *If $S(t, x)$ is the action associated with the Lagrangian L (see the preceding paragraph), then S satisfies the Hamilton–Jacobi equation*

$$S_t + H(x, S_x) = 0, \tag{8.19}$$

where H is the Legendre transform of L and where the subscripts denote partial derivatives.

Proof. From the main theorem (8.4) and the definition of H we get $S_t(t,x) = -H(x,p)$ and $S_x = p$. $\diamondsuit$

The eikonal equation as the analog of the Hamilton–Jacobi equation. Consider an optical medium in the plane, with the speed of light $c = c(\mathbf{x})$ depending on the point $\mathbf{x}$. Define $T(\mathbf{x})$ as the time it takes for a ray starting at (say) the origin to reach $\mathbf{x} \in \mathbb{R}^2$. One can show that T must satisfy

$$|\nabla T| = c(x)^{-1}; \tag{8.20}$$

this equation, called the *eikonal* equation, simply restates the fact that the speed of light at x is $c(x)$; see Problem 8.13 on page 292.[4] (8.20) makes perfect intuitive sense: Large $|\nabla T|$ suggests that that the times between nearby points are long, i.e., that the speed is slow.

Just like the eikonal equation, the Hamilton–Jacobi equation restricts the gradient of the "cost function" S to a curve.

6. Noether's theorem

Noether's theorem is a fundamentally important consequence of the main theorem — and an almost obvious one. The theorem relates the symmetry of the Lagrangian with the existence of conserved quantities. To state the theorem, we must first define the symmetry. For Noether's theorem to be really meaningful we must consider several degrees of freedom: $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, and so we now depart from the scalar case.

Let $h^s : \mathbb{R}^n \to \mathbb{R}^n$ be a one-parameter family of transformations of the configuration space $\mathbb{R}^n$. We assume throughout that $h^s(\mathbf{x})$ depends smoothly on both s and $\mathbf{x}$ and that $h^0\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Common examples of h^s in $\mathbb{R}^3$ arising in mechanics include translations $h^s\mathbf{x} = \mathbf{x} + \mathbf{e}s$ in $\mathbb{R}^n$ in the direction of a given unit vector $\mathbf{e}$ by distance s, rotations $h^s\mathbf{x} = R_{\mathbf{e}}(s)\mathbf{x}$ in $\mathbb{R}^3$ around the line through the origin defined by the unit vector $\mathbf{e}$, and corkscrew motions $h^s\mathbf{x} = a\mathbf{e}s + R_{\mathbf{e}}(s)\mathbf{x}$ ($a = \text{const.}$) consisting of a combination of the preceding two transformations.

[4] The term *eikonal* comes from the Greek $\varepsilon\iota\kappa\omega\nu$ (image), sharing this origin with the words *icon* and *iconic*.

Definition 8.1. A Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}})$ is said to be invariant under the action of a family h^s of transformations of $\mathbb{R}^n$ if $L(\mathbf{x}, \dot{\mathbf{x}})$ does not change when $\mathbf{x}(t)$ is replaced by $h^s\mathbf{x}(t)$, i.e., if for any function $\mathbf{x}(t)$ we have

$$L(h^s\mathbf{x}(t), \frac{d}{dt}h^s\mathbf{x}(t)) = L(\mathbf{x}(t), \dot{\mathbf{x}}(t)). \tag{8.21}$$

The definition (8.21) is equivalent to the requirement that

$$L(h^s\mathbf{x}, (dh^s)\dot{\mathbf{x}}) = L(\mathbf{x}, \dot{\mathbf{x}}) \tag{8.22}$$

for all $\mathbf{x},\ \dot{\mathbf{x}} \in \mathbb{R}^n$; here $dh^s = dh^s(\mathbf{x})$ is the Jacobi derivative matrix of h^s at $\mathbf{x}$ (for the definition, see page 135).

Theorem 8.4 (Noether's theorem)**.** *If the Lagrangian L is invariant under the action of a one-parameter family of diffeomorphisms h^s, then the quantity*

$$I(\mathbf{x}, \dot{\mathbf{x}}) \stackrel{\text{def}}{=} L_{\dot{\mathbf{x}}} \cdot \frac{d}{ds}\bigg|_{s=0} h^s\mathbf{x} \tag{8.23}$$

is constant along any solution $\mathbf{x} = \mathbf{x}(t)$ of the Euler–Lagrange equation $\frac{d}{dt}L_{\dot{\mathbf{x}}} - L_{\mathbf{x}} = 0$. Such a constant quantity is called an integral of motion.

Proof. Let us fix any solution $\mathbf{x} = \mathbf{x}(t)$ of the Euler–Lagrange equation, and consider the action $S(t_0, \mathbf{x}_0; t_1, \mathbf{x}_1)$ between two points on the graph of $\mathbf{x}$. By the assumption,

$$S(t_0, h^s\mathbf{x}_0; t_1, h^s\mathbf{x}_1) = S(t_0, \mathbf{x}_0; t_1, \mathbf{x}_1) \ \text{ for all } \ s, \tag{8.24}$$

Figure 6. Differentiating by s at $s = 0$ gives

$$S_{\mathbf{x}_0} \cdot \frac{d}{ds}\bigg|_{s=0} h^s\mathbf{x}_0 + S_{\mathbf{x}_1}\frac{d}{ds}\bigg|_{s=0} h^s\mathbf{x}_1 = 0.$$

By the Theorem 8.4, $S_{\mathbf{x}_1} = L_{\dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}})|_{t=t_1}$ and $S_{\mathbf{x}_0} = -L_{\dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}})|_{t=t_0}$. Since the time t_1 can be chosen arbitrarily, this proves that (8.23) is indeed constant along the solution $\mathbf{x}(t)$. ♢

Remark 8.3. As the proof shows, Noether's integral (8.23) is simply the directional derivative of the action S in the direction of motion under h^s, i.e., in the direction $\frac{d}{ds}h^s(\mathbf{x})$.

Remark 8.4. The above proof explains where $I(\mathbf{x}, \dot{\mathbf{x}})$ came from.

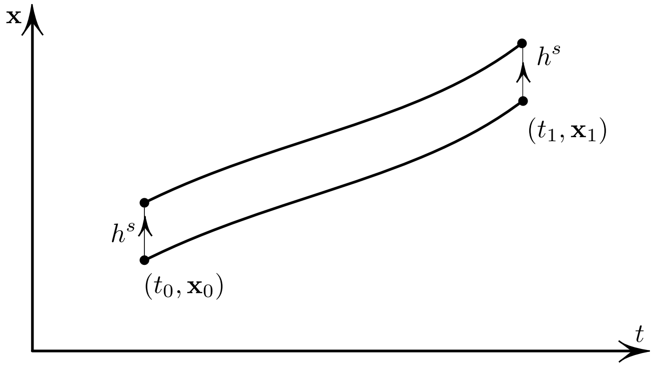

Figure 6. Proof of Noether's theorem.

Examples (see Problem 8.8).

1. If h^s is a translation in some direction $\mathbf{e} \in \mathbb{R}^n$, then the integral (8.23) is the component of the linear momentum in the direction $\mathbf{e}$.

2. If $\mathbf{x} \in \mathbb{R}^3$, and if h^s is a rotation around the line of unit vector $\mathbf{e}$ through angle s, then (8.23) is the component of the angular momentum in the direction $\mathbf{e}$.

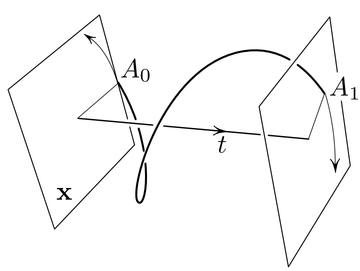

Figure 7. The torques applied to the ends A_0 and A_1 of a spring held in equilibrium in a rotationally symmetric potential are equal and opposite. This is a verbatim analog of the conservation of the angular momentum for a particle in a centrally symmetric potential; see Section 11.

Remark 8.5. Conservation of the angular momentum of a point mass in a rotationally symmetric potential in $\mathbb{R}^2$ has a static analog: it is *the torque applied to the ends of the spring* in Figure 7. For more on this analogy with statics, see Section 11. Mathematical expressions for the angular momentum of a particle and for the torque on the spring's end are identical.

7. Conservation of energy

Conservation of energy is another immediate consequence of the main theorem; the proof is the same as for Noether's theorem, except that translation is in time rather than in space.

Theorem 8.5. *If the Lagrangian $L = L(\mathbf{x}, \dot{\mathbf{x}})$ has no explicit time dependence (as has been the case in all of our discussions so far), then*

$$\frac{d}{dt}(\dot{\mathbf{x}} L_{\dot{\mathbf{x}}} - L) = 0 \tag{8.25}$$

along any solution $\mathbf{x} = \mathbf{x}(t)$ of the Euler–Lagrange equation.

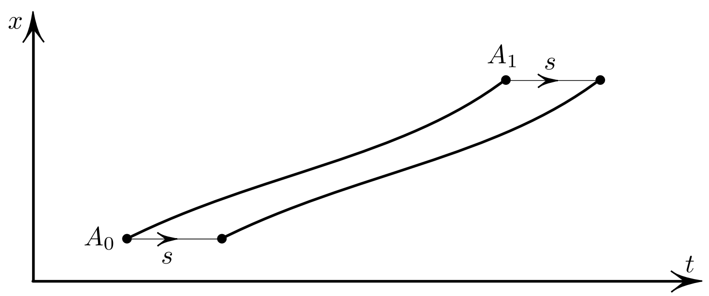

Figure 8. Noether's theorem for the time-invariant L.

Proof. Let $A_0(t_0, \mathbf{x}_0)$ and $A_1(t_1, \mathbf{x}_1)$ be two points on the graph of a solution $\mathbf{x}(t)$ of the Euler–Lagrange equation. Since L has no explicit dependence on t, the action is invariant under t-translations of both ends (see Figure 8):

$$S(t_0 + s, \mathbf{x}_0; t_1 + s, \mathbf{x}_1) = S(t_0, \mathbf{x}_0; t_1, \mathbf{x}_1)$$

for any s. Differentiating by s and setting $s = 0$ gives

$$S_{t_0} + S_{t_1} = 0,$$

which by (8.4) and (8.5) amounts to

$$-(L - \dot{\mathbf{x}}L_{\dot{\mathbf{x}}})|_{t=t_0} + (L - \dot{\mathbf{x}}L_{\dot{\mathbf{x}}})|_{t=t_1} = 0,$$

proving that $L - \dot{\mathbf{x}}L_{\dot{\mathbf{x}}}$ is constant along the solution $\mathbf{x}$. ♢

Remark 8.6. The energy conservation is a verbatim analog of *Newton's first law for springs* (see the table on page 280, second to the last line, and the preceding explanations.).

8. Poincaré's integral invariants

Here is another immediate consequence of the main theorem.

Theorem 8.6. *Let $\mathbf{x}(t;s)$ be a one-parameter family of solutions of the Euler–Lagrange equation depending on the parameter $s \in [0,1]$ cyclically:*

$$\mathbf{x}(t;0) = \mathbf{x}(t;1);$$

for any fixed t consider the closed curve $\gamma_t(s) = (\mathbf{x}(t;s), \mathbf{p}(t;s)) \in \mathbb{R}^{2n}$, where s is the parameter along the curve, and where $\mathbf{p} = L_{\dot{\mathbf{x}}}$. Then

$$\int_{\gamma_t} \mathbf{p}\, d\mathbf{x} = \int_{\gamma_0} \mathbf{p}\, d\mathbf{x}, \tag{8.26}$$

where $\mathbf{p}\, d\mathbf{x} = \sum p_k dx_k$ denotes the dot (scalar) product.

As we will show in the next section, $(\mathbf{x}(t), \mathbf{p}(t))$ evolves according to Hamilton's equations (8.29) (page 273), and with that advance notice the theorem can be reformulated as follows:

Theorem 8.7. *The flow φ^t of Hamilton's equations (8.29) preserves the integral $\int_\gamma p\, dx$ for any smooth closed curve γ in $\mathbb{R}^{2n} = \{(\mathbf{x}, \mathbf{p})\}$:*

$$\int_{\gamma} \mathbf{p}\, d\mathbf{x} = \int_{\varphi^t\gamma} \mathbf{p}\, d\mathbf{x}. \tag{8.27}$$

Proof of (8.26)-(8.27). Consider the action $S(0, \mathbf{x}(0,s); t, \mathbf{x}(t,s))$ between two points on the graph of the solution $\mathbf{x}(t;s)$. Since S is cyclic in s, we have

$$0 = \int_0^1 \frac{d}{ds} S\, ds = \int_0^1 (S_1 \cdot \mathbf{x}_s(0;s) + S_2 \cdot \mathbf{x}_s(t,s))\, ds,$$

where S_1, S_2 are the gradients of S with respect to the two spatial variables. But $S_1 = -\mathbf{p}(0,s)$ and $S_2 = \mathbf{p}(t,s)$ according to the main theorem, and the above turns into

$$\int_0^1 \mathbf{p}(0,s) \cdot \mathbf{x}_s(0;s)\, ds = \int_0^1 \mathbf{p}(t,s) \cdot \mathbf{x}_s(t,s))\, ds,$$

which amounts to (8.26) or (8.27). ♢

Remark 8.7. Although the theorem mentions Hamilton's equations, the proof does not use them, but rather comes from the main theorem, almost immediately. The point here is that using Hamilton's equations to prove (8.27) (as is sometimes done) obscures the "real" reason for Poincaré's integral invariance.

Remark 8.8. Geometrically, $\int_\gamma \mathbf{p}\ d\mathbf{x} = \sum \int_\gamma p_k\ dx_k$ is the sum of signed areas[5] of the projections of γ onto the n two-dimensional planes (x_k, p_k), $k=1,\dots,n$. For $n=1$ this integral is just the signed area enclosed by γ, and *we recover Liouville's theorem* for Hamiltonian systems. As opposed to the proof presented in Chapter 2, the preceding proof shows the "real" reason why Liouville's theorem holds.

Poincaré's integral invariance is thus intimately related to the nonexistence of a perpetual motion machine.[6] For details, see Figures 9 and 12 and the surrounding discussion.

Remark 8.9. Poincaré's integral invariants and Noether's theorem are really just two aspects of the same idea: "S does not change". For the Poincaré invariants, S does not change under a cyclic change of $\mathbf{x}_0, \mathbf{x}_1$ (this is, of course, true automatically, as long as S is single-valued). For Noether's theorem, the key is that S does not change, by the assumption, under simultaneous transformations of $\mathbf{x}_0$ and $\mathbf{x}_1$.

[5] The sign of the area is negative if the region's boundary is traced in the counterclockwise direction.

[6] The nonexistence of the perpetual motion machine can be used to deduce the Pythagorean Theorem and many other theorems from Euclidean geometry (see [**14**]), where one can also find a discussion of the legitimate question: How can a physical principle lead to a rigorous mathematical proof?

9. The generating function

In this section we consider the action with the times t_0, t_1 fixed. Dropping these times from the notation, we will write $S(t_0, \mathbf{x}_0; t_1, \mathbf{x}_1) = S(\mathbf{x}_0, \mathbf{x}_1)$. By the main theorem we have

$$\begin{cases} S_1(\mathbf{x}_0, \mathbf{x}_1) = -\mathbf{p}_0, \\ S_2(\mathbf{x}_0, \mathbf{x}_1) = \quad \mathbf{p}_1, \end{cases} \tag{8.28}$$

where S_1 stands for the gradient with respect to the first variable $\mathbf{x}_0$, with a similar meaning for S_2. If S satisfies certain conditions (e.g., $\det S_{12} \neq 0$), then the equations (8.28) define a map

$$\varphi := (\mathbf{x}_0, \mathbf{p}_0) \mapsto (\mathbf{x}_1, \mathbf{p}_1),$$

at least locally. In this case S is called the *generating function* of φ. We already showed that (8.28) implies that φ is a symplectic map, i.e., that (8.27) holds. It should be noted that not every symplectic map can be given by a generating function via (8.28); for more details on this see, e.g., Arnold's Mechanics [**1**].

10. Hamilton's equations

We have not introduced Hamilton's equations so far; this was deliberate, since too early an introduction would make the path from Hamilton's principle to the results obtained so far too circuitous.

Theorem 8.8. *If $\mathbf{x}(t)$ is a critical function of $\int L\,dt$ then the vector $(\mathbf{x}(t), \mathbf{p}(t))$, where $\mathbf{p} = L_{\dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}})$, satisfies Hamilton's equations*

$$\begin{cases} \dot{\mathbf{x}} = \quad H_{\mathbf{p}}(\mathbf{x}, \mathbf{p}), \\ \dot{\mathbf{p}} = -H_{\mathbf{x}}(\mathbf{x}, \mathbf{p}), \end{cases} \tag{8.29}$$

where H is the Legendre trasform of L and where $H_{\mathbf{x}}, H_{\mathbf{p}}$ are the gradients with respect to $\mathbf{x}$ and $\mathbf{p}$.

Proof. We already showed that $\dot{\mathbf{x}} = \mathbf{v} = H_{\mathbf{p}}$, see (8.18); the first of Hamilton's equations is therefore simply a restatement of the definition $\mathbf{p} = L_{\dot{\mathbf{x}}}$. For the second equation, let $S(t, \mathbf{x})$ be the action associated with a central family of critical functions[7] of which $\mathbf{x}(t)$

[7] That is, a family of critical functions satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$.

is a member. By the fundamental theorem (8.4), $\mathbf{p} = S_{\mathbf{x}}(t, \mathbf{x}(t))$. Differentiating this identity by t we get

$$\dot{\mathbf{p}} = S_{t\mathbf{x}} + S_{\mathbf{xx}}\dot{\mathbf{x}} = S_{t\mathbf{x}} + S_{\mathbf{xx}}H_{\mathbf{p}}, \tag{8.30}$$

where we substituted $\dot{\mathbf{x}} = H_{\mathbf{p}}$. But the right-hand side of (8.30) is simply $-H_{\mathbf{x}}$, as seen by differentiating the Hamilton–Jacobi equation $S_t + H(\mathbf{x}, S_{\mathbf{x}}) = 0$ by $\mathbf{x}$:

$$S_{t\mathbf{x}} + H_{\mathbf{x}} + S_{\mathbf{xx}}H_{\mathbf{p}} = 0, \quad \text{or} \quad S_{t\mathbf{x}} + S_{\mathbf{xx}}H_{\mathbf{p}} = -H_{\mathbf{x}}.$$

Substituting this into (8.30) gives $\dot{\mathbf{p}} = -H_{\mathbf{x}}$. ♢

11. Hamiltonian mechanics as the "spring theory"

In this section I describe a static analog of the main dynamical concepts discussed so far in this chapter; the table on page 280 summarizes this analogy. This analogy gives a palpable interpretation of most of the ideas of the preceding sections.

It should be noted that the analogy described here is different from the dynamics-statics *equivalence* from page 42 in that the time t here will be a geometrical coordinate, whereas in the earlier discussion the time was mathematically equivalent to mass.

Figure 9 shows a heavy elastic band, or a spring, in gravitational field, held *in equilibrium* by two external forces $\mathbf{F}_0$, $\mathbf{F}_1$ applied to the ends. Note that the two spatial coordinates (in the plane of the spring) are denoted by t, x. The precise properties of the spring do not matter in this discussion; we only assume that the potential energy U of the spring (consisting of the gravitational and the stretching energies) is a function of the endpoints: $U = U(A_0, A_1)$.

Let us denote the two components of the forces as shown in Figure 10:

$$\mathbf{F}_1 = \langle -H_1, p_1 \rangle, \quad -\mathbf{F}_0 = \langle -H_0, p_0 \rangle; \tag{8.31}$$

we wrote $-\mathbf{F}_0$ rather than $\mathbf{F}_0$ so as to have the vectors $-\mathbf{F}_0$, $\mathbf{F}_1$ point in the same direction along the spring. I chose the notations to make the analogy with Hamiltonian mechanics verbatim, as we shall see shortly. Our choice of $-H$ instead of H in $\langle -H, p\rangle$ is the price to pay for the traditional definition of the Hamiltonian.

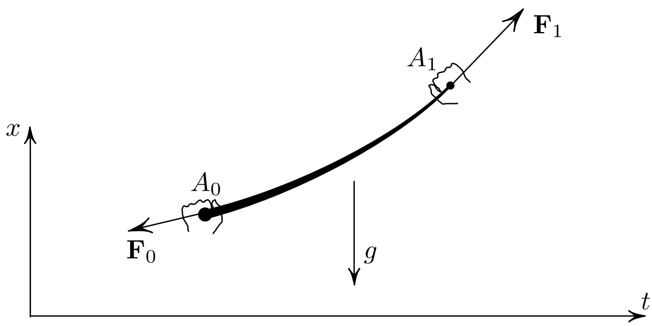

Figure 9. An illustrative problem: forces $\mathbf{F}_0$, $\mathbf{F}_1$ hold the spring in equilibrium.

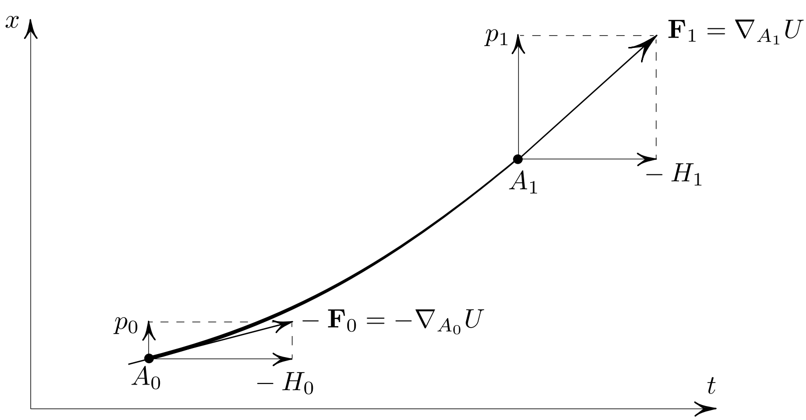

Figure 10. The two components of the spring's tension are the analogs of of H and of p. Noether's theorem, the symplectic 1-form $\mathbf{p}\, d\mathbf{x}$, and the Poincaré integral invariant have their analogs as well, as explained in the text and summarized in the table on page 280.

"The generating function". From the definition of the potential energy as the work required to bring the spring from some initial equilibrium state to the current one, we have

$$\mathbf{F}_0 = \nabla_{A_0} U \quad \text{and} \quad \mathbf{F}_1 = \nabla_{A_1} U; \tag{8.32}$$

the subscripts A_0, A_1 indicate the variables with respect to which the gradients are taken.[8] Reading off the x-components of (8.32) and recalling the notation (8.31) we get

$$\begin{aligned} p_0 &= -U_{x_0}(x_0, x_1), \\ p_1 &= U_{x_1}(x_0, x_1). \end{aligned} \tag{8.33}$$

These equations look the same as (8.28) on page 273. Therefore, just as in the case of (8.28), our equations (8.33) define the symplectic map $\varphi = (x_0, p_0) \mapsto (x_1, p_1)$; here the generating function is U.[9] It is remarkable that the hanging spring gives rise to a symplectic map. And the generating function of this map is the spring's potential energy.

Exercise 8.2. Find the map φ corresponding to a Hookean spring held by two ends $A_0 = (0, x_0)$ and $A_1(1, x_1)$, with no external forces applied to it besides the ones at the two ends.

Solution. The potential energy $U = \frac{1}{2}k(\text{length})^2$, or

$$U(x_0, x_1) = \frac{k}{2}((x_1 - x_0)^2 + 1).$$

Substituting this U into (8.33) and solving for x_1, p_1 gives the solution, written here in vector form:

$$\begin{pmatrix} x_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 & k^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}.$$

Thus φ is the shear of strength k^{-1} in the x-direction.

"Energy conservation". Since the spring is in equilibrium, the horizontal components of external forces are in balance: $\mathrm{proj}_t \mathbf{F}_0 = -\mathrm{proj}_t \mathbf{F}_1$, or

$$H_0 = H_1. \tag{8.34}$$

[8] See (2.22) on page 88 for the discussion of the potential energy for a single particle; we leave the extension of the concept for the spring as an exercise.

[9] For φ to be well defined, it suffices to assume that the first equation in (8.33) determines x_1 uniquely.

Although this is clear from Newton's first law, a formal proof of (8.34) would lead us to a verbatim repetition of the proof of Noether's theorem on page 267. So we can think of *Noether's theorem as a twin sister of Newton's first law.*

"The symplectic 1-form". If hand A_1 in Figure 9 moves by $d\mathbf{r}_1 = (dt_1, dx_1)$, then it does work

$$dW_1 = \mathbf{F}_1 \cdot d\mathbf{r}_1 = p_1\ dx_1 - H_1 dt_1. \tag{8.35}$$

Similarly, when hand A_0 moves by $\mathbf{r}_0 = (dt_0, dx_0)$, it does work $dW_0 = \mathbf{F}_0 \cdot d\mathbf{r}_0 = -p_0\ dx_0 + H_0 dt_0$. This is an interpretation of the 1-form $\mathbf{p}\, dx - H\, dt$ as the infinitesimal work.[10] In particular, $p_1\, dx_1$ is the infinitesimal work required to displace the right end of the spring in the x-direction by dx_1.

"Poincaré's integral invariants". Speaking somewhat loosely, by moving both ends A_0 and A_1 in Figure 9 arbitrarily and returning them to the beginning state I do zero work on the spring:[11] $\oint \mathbf{F}_0 d\mathbf{r}_0 + \oint \mathbf{F}_1 d\mathbf{r}_1 = 0$. Equivalently,

$$-\oint_{\gamma_0} p\ dx - H dt + \oint_{\gamma_1} p\ dx - H\ dt = 0, \tag{8.36}$$

where γ_0, γ_1 are closed curves traced out by the cyclically varying points $(t_0, x_0,\ p_0)$ and $(t_1, x_1,\ p_1)$. In the special case, when t_0 and t_1 remain fixed, i.e., when A_0, A_1 move only in the x-direction, the $H dt$-terms drop out, and we get

$$\oint p_0 dx_0 = \oint p_1 dx_1.$$

Formalizing this intuitively plausible discussion gives a verbatim repetition of the earlier proof of Poincaré's integral invariance; see page 271.

[10] This form is called Poincaré's 1-form; one can find more on it in [**1**].

[11] Assuming that the motion is quasi-static: no oscillations are excited and the spring acquires no kinetic energy.

Exercise 8.3. If hands A_0 and A_1 move in a cyclic fashion, must *each hand individually* do zero work?

Answer. No. For details, see Problem 8.6 on page 289.

Area-preservation. Consider the special case of the cyclic motion of the preceding paragraph where both t_0 and t_1 remain fixed, i.e., where the motion happens in the x-direction only. Assume that (x_0, p_0) determines (x_1, p_1), so that we have a mapping $\varphi := (x_0, p_0) \mapsto (x_1, p_1)$. During the cyclic motion the point and its image describe closed curves $C_0,\ C_1$ with $C_1 = \varphi(C_0)$. From (8.36) we get

$$\text{area}(C_0) = \text{area}(C_1), \tag{8.37}$$

thus showing that φ is area-preserving.

We thus repeated the proof of Liouville's theorem in $\mathbb{R}^2$, essentially the same proof as already given on page 271. This time, however, we discover a simple mechanical interpretation: the work done during a cyclic motion of the ends $A_0,\ A_1$ is zero.

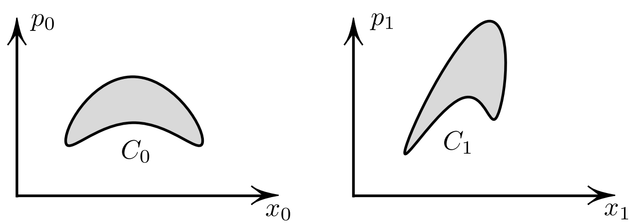

Figure 11. A geometrical interpretation of the fact that zero work done in a (vertical) cyclic motion of the ends A_0 and A_1.

Poincaré's integral invariant in higher dimension. All of the above extends to higher dimensions of $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n$ for $n > 1$ as well;[12] Figure 12 illustrates the case of $n = 2$. When moving the ends of

[12] One says that $\mathbf{p}$ lies in the *cotangent space* to $\mathbb{R}^n$ (which is then identified with $\mathbb{R}^n$). The reason for such language is the fact that the gradient $\nabla_{\mathbf{v}} L = \mathbf{p}$ is defined as an *operator* on $\mathbb{R}^n$, via $\frac{d}{d\varepsilon} L(\mathbf{v}+\varepsilon\mathbf{w}) \stackrel{\text{def}}{=} \nabla_{\mathbf{v}} L \cdot \mathbf{w}$. Thus $\nabla_{\mathbf{v}} L$ indeed operates on tangent vectors $\mathbf{w} \in \mathbb{R}^n$, assigning to each $\mathbf{w}$ the directional derivative in the direction $\mathbf{w}$.

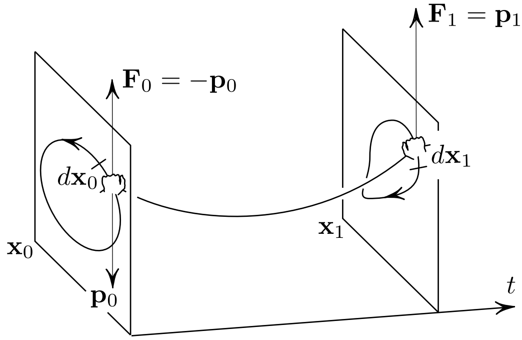

Figure 12. Analog of Poincaré integral invariant for $\mathbf{x} \in \mathbb{R}^2$.

the hanging spring around two closed paths I return the spring to the original state and thus do zero net work:

$$\oint (-\mathbf{p}_0) d\mathbf{x}_0 + \oint \mathbf{p}_1 d\mathbf{x}_1 = 0, \tag{8.38}$$

where the minus sign is due to the fact that the left-hand applies force $-\mathbf{p}_0$. Here the $\mathbf{p}\, d\mathbf{x}$ denotes the dot product in $\mathbb{R}^2$.

Lagrangian manifolds. Let us fix the left end $\mathbf{x}_0$ in Figure 12 (so that we are still speaking of $\mathbf{x} \in \mathbb{R}^2$). For every $\mathbf{x}_1$ we have an associated force vector $\mathbf{p}_1 = U_{\mathbf{x}_1}(\mathbf{x}_1)$. In other words, vectors $(\mathbf{x}_1, \mathbf{p}_1) = (\mathbf{x}_1, U_{\mathbf{x}_1}(\mathbf{x}_1)) \in \mathbb{R}^4$ form a two-dimensional manifold in $\mathbb{R}^4$ (the manifold is parametrized by $\mathbf{x}_1 \in \mathbb{R}^2$). This is an example of the *Lagrangian manifold.* The characteristic property of such a manifold is that $\oint_C \mathbf{p}\, d\mathbf{x} = 0$ for any closed path C on the manifold. Physically, this means that the work done over the path is zero. A more detailed discussion of Lagrangian manifolds can be found in [**1**].

The following table summarizes some of the discussion of this section. In this table t_0, t_1 are fixed, as they are in most of this section.

Hamiltonian dynamics	**Statics of a spring**
$S(A_0, A_1)$ – action	$U(A_0, A_1)$ – potential energy
$\nabla_{A_0} S$, $\nabla_{A_1} S$	$\nabla_{A_0} U = \mathbf{F}_0$, $\nabla_{A_1} U = \mathbf{F}_1$
p – momentum	p – force in the x-direction
H – Hamiltonian	H – force in the $(-t)$-direction
$p\,dx$ symplectic 1-form	$p_1\,dx_1$ infinitesimal work done in displacing A_1
S is a generating function of the map $\varphi = (x_0, p_0) \mapsto (x_1, p_1)$ via $p_0 = -S_{x_0},\ p_1 = S_{x_1}$	U is a generating function of the map $\varphi = (x_0, p_0) \mapsto (x_1, p_1)$ via $p_0 = -U_{x_0},\ p_1 = U_{x_1}$
$\int_{\varphi(\gamma)} p\,dx = \int_\gamma p\,dx$, that is, φ is symplectic	$\int_{\varphi(\gamma)} p\,dx = \int_\gamma p\,dx$, i.e., $\oint p_1 dx_1 + \oint(-p_0)dx_0 = 0$ — work done on the spring in a cyclic motion is zero
Energy conservation $H_0 = H_1$	Newton's first law: $H_0 = H_1$
Conservation of angular momentum	Torque balance (Figure 7)

12. The optical-mechanical analogy

The dynamics-statics analogy of the preceding section seems to be very recent, apparently pedagogically original. Historically, it was another analogy — the one with optics — that illuminated the path for mechanics since the times of Leibnitz, Euler and Maupertuis. Optical analogy motivated at least two developments: the Maupertuis principle and Hamilton's theory. That the optical-mechanical analogy exists in the first place is not surprising in the hindsight, since classical mechanics is the limiting case of wave mechanics just like geometrical optics is the limiting case of wave optics. Before summarizing the optical analogy (on page 283) let us describe the setting.

Optical objects. Figure 13(a) shows an optical medium side-by-side with space-time. Let $T(\mathbf{x})$ be the time it takes a ray to reach point $\mathbf{x}$ from the origin A_0. The wave front, defined as the set of points $\mathbf{x}$ with $T(\mathbf{x}) = t = \text{const.}$, Figure 13(a), gives the location of the optical disturbance at time t which originated at A_0, and $\nabla T(\mathbf{x})$ is the normal vector to the front.[13]

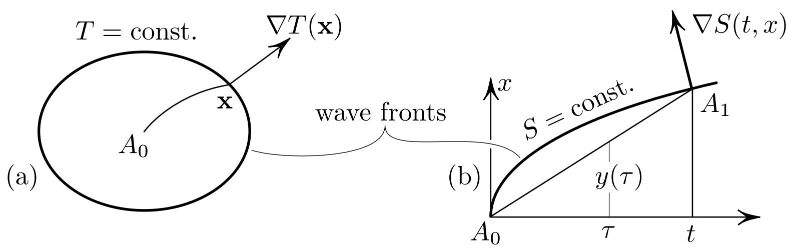

Figure 13. Analogy between optics (a) and mechanics (b).

Analogs of optics in mechanics. Mechanical space-time (t, x) is the analog of an optical medium. Optical distance, i.e., the time $T(\mathbf{x})$, is the counterpart of the action

$$S(t, x) = \int_{A_0}^{A_1} L \, d\tau, \tag{8.39}$$

where the integrand involves the critical function with boundary conditions $A_0 = (t_0, x_0)$ and $A_1 = (t, x)$; we fix A_0 in this discussion,

[13] ∇T is called the vector of *normal slowness*, since $|\nabla T|$ is the reciprocal of the speed of light; see Problem 8.13 on page 292.

although in general it should be allowed to vary. In order for S to be smooth we assume that the critical function depends smoothly on the endpoint (t, x). This critical function in the (t, x)-plane is the analog of the optical ray.

Exercise 8.4. Show that for the free particle Lagrangian $L = \dot{x}^2/2$, and for $A_0 = (0, 0)$ the action (8.39) takes the form $S(t, x) = \frac{x^2}{t}$.

Solution. The minimizer of the action is a linear function $y(\tau) = \frac{x}{t}\tau$, Figure 13(b).[14] Thus $y'(\tau) = x/t$, and

$$S(t, x) = \int_0^t \frac{1}{2}\left(\frac{x}{t}\right)^2 d\tau = \frac{x^2}{t},$$

as claimed. Level curves of $S(t, x)$ are parabolas in Figure 14(b); note that they all pass through A_0; in this respect optics is different: the wave fronts move away from A_0 as T increases.[15] The picture in Figure 13(b) near A_0 is typical not just for our simple example, but for a general Lagrangian, as long as the Legendre condition $L_{\dot{x}\dot{x}} > 0$ is satisfied.

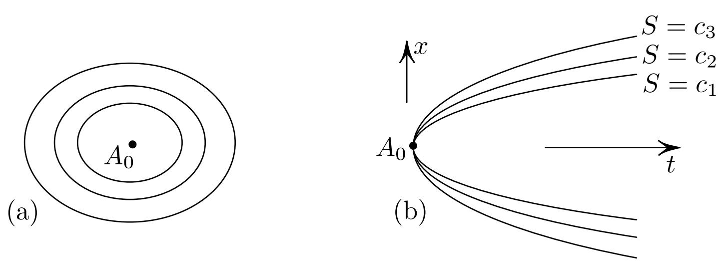

Figure 14. Propagating fronts in optics and in mechanics. Here $c_1 < c_2 < c_3$.

[14]We use τ, y for the coordinates along the trajectory since t and x have been reserved for the endpoint.

[15]Strictly speaking, A_0 does not belong to any of the fronts in the figure, so that the front is really a parabola with A_0 deleted.

Optics	**Mechanics**
$\mathbf{x}$ – space	(t, x) – space-time
rays	critical functions of $\int L\,dt$
$T(\mathbf{x})$ – travel time from A to x	$S(t, x)$ – action
$T(\mathbf{x}) = \text{const.}$ – wave front	$S(t, x) = \text{const.}$
$\nabla T(\mathbf{x})$ – normal to the front	$\nabla S(t, x)$
$\nabla T = c^{-1}$ (eikonal equation)	$S_t + H(x, S_x) = 0$ (Hamilton–Jacobi equation)

The last line in the table was explained on page 267.

Huygens's principle in space-time. As an instructive side remark, let us see what Huygens's principle looks like in mechanics; Figure 15 shows the similarities and the differences between the traditional optical picture and the nontraditional mechanical one. In both cases the new front is an envelope of fronts created by disturbances at the points on the old one. For instance, the new front $S = c+\varepsilon$ is the envelope of the fronts $S(B, X) = \varepsilon$ "emanating" from points B on the old front $S = c$.

Interestingly, Huygens's principle suggests the following simple geometrical observation.

Exercise 8.5. Consider the family of parabolas obtained by moving the parabola $y = ax^2$, with $a > 1$, parallel to itself so that its vertex traces out another parabola $y = bx^2$, where $b \neq a$, and where a, b have the same sign. Prove that the envelope of the family of these translates is also a parabola of the form $y = cx^2$.

Solution. Although we can verify this by a direct computation, Huygens's principle explains this fact with no computation besides the one we already did in Exercise 8.4 on page 282. Moreover, Huygens's principle even lets us find c without further computation!

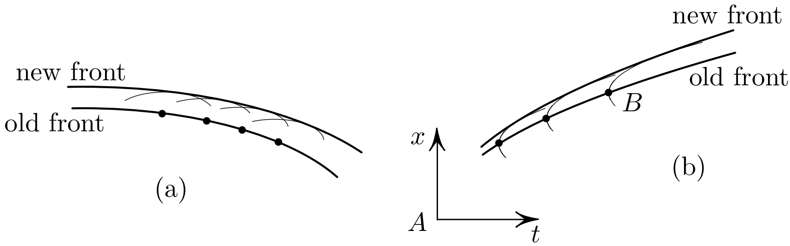

Figure 15. Huygens's principle in optics (a) and in space-time (b).

13. Hamilton–Jacobi equation leading to the Schrödinger equation

The analogy between Hamiltonian mechanics and geometrical optics guided the development of the former. As it became clear in the 20th century, this old analogy is a symptom of a deeper connection, the one between wave mechanics and wave optics. This connection explains why Hamiltonian mechanics turned out to be a key tool in quantum mechanics.

This section describes one small aspect of this relationship, by showing how to guess the Schrödinger equation. The key idea, due to Feynman, is this: the Hamilton–Jacobi equation relates to the Schrödinger equation like the eikonal equation relates to Maxwell's equation; see Figure 17, last two rows.

Consider a classical particle — an idealized point mass m — in a potential $V(x)$ on the line. For very light particles it does not make sense to speak of the precise position $x(t)$, since it is in principle unknowable: identical experiments give different measurements of x. In other words, all experimental evidence gives us no option but to conclude that precise knowledge of $x(t)$ is impossible *in principle.*

The one remaining option then is to speak of the probability density, i.e., of the function $p(t, x) \geq 0$ which gives the probability of finding the particle in the interval $[x, x + dx]$ at time t as $p(t, x)\, dx$. In Newtonian mechanics, p is the delta function centered at the position $x(t)$ of the particle, but in quantum mechanics p is smeared around the Newtonian position $x(t)$.

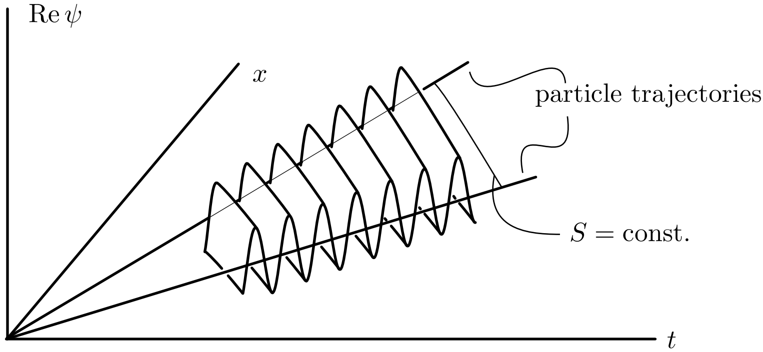

Figure 16. Trajectories of classical particles are the "rays" corresponding to the quantum mechanical wave ψ.

Now there are physical reasons (namely, the interference in experiments) which lead one to write p in the form $p = |\psi|^2$, where ψ is a complex function, called complex probability amplitude, a more basic object than p. As described in [**5**], ψ satisfies the principle of quantum superposition (while p doesn't), and this makes it analogous to the electromagnetic field for which the principle of superposition holds. This suggests treating ψ as an analog of the electric field in wave optics, the case we briefly consider before returning to ψ.

Let us take the simplest case: a standing high frequency electromagnetic wave whose electric field E is perpendicular to the (x, y)-plane. The highly oscillatory nature of E can be expressed by

$$E(x,y) = a(x,y)\cos\frac{T(x,y)}{\omega} = \operatorname{Re} ae^{i\frac{T(x)}{\omega}}, \tag{8.40}$$

where $\omega >> 1$, where $T(x, y)$ is the time it takes the light rays to reach (x, y) from some initial wave front and where Re denotes the real part. Now guided by (8.40) we seek ψ in the form

$$\psi(t,x) = a(t,x)e^{\frac{i}{\hbar}S(t,x)}, \tag{8.41}$$

where $\hbar$ is very small, and where $S(t, x) = S(t_0, x_0; t, x)$ with (t_0, x_0) fixed. We placed S in the exponent since S is the analog of T, as was mentioned earlier and as summarized in the table on page 283. But S satisfies the Hamilton–Jacobi equation $S_t + H(x, S_x) = 0$. Since $H = p^2/2m + V$, we conclude that S satisfies

$$S_t + \frac{S_x^2}{2m} + V = 0. \tag{8.42}$$

Now S and ψ are related via (8.41); what does the statement (8.42) on S say about ψ? To find out, differentiate (8.41) and keep the leading terms only:

$$\psi_t = \frac{i}{\hbar}\psi S_t + \dots, \quad \psi_{xx} = \left(\frac{i}{\hbar}\right)^2 \psi S_x^2 + \dots,$$

where ... denote the higher order terms, i.e., the ones small compared to the ones written. The reason we differentiated by x twice is to get S_x^2 which appears in (8.42). It is only a matter of simple algebra to express S_t and S_x^2 from the last equations and plug them into (8.42) to get the Schrödinger equation. Here are the details. We have

$$S_t = \frac{\hbar}{i}\psi^{-1}\psi_t, \quad S_{xx} = \left(\frac{\hbar}{i}\right)^2 \psi^{-1}\psi_{xx},$$

where the lower order terms were deleted; substituting into (8.42) gives

$$\frac{\hbar}{i}\psi^{-1}\psi_t + \frac{1}{2m}\left(\frac{\hbar}{i}\right)^2 \psi^{-1}\psi_{xx} + V = 0.$$

Multiplying by ψ gives the Schrödinger equation

$$\boxed{i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V\psi}$$

Figure 17 summarizes the parallel history of two subjects, leading to Maxwell's equation and to the Schrödinger equation.

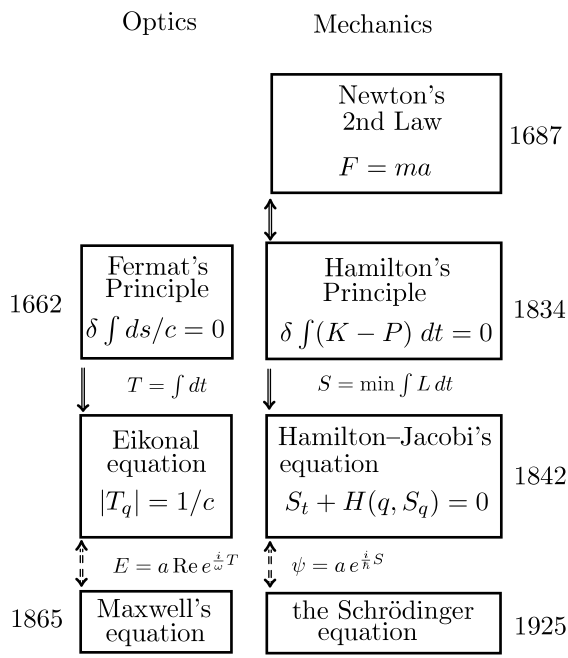

Figure 17. When read down, this diagram shows the historical development. Solid arrows stand for theorems. Dashed arrows indicate either an inspired guess or that one statement is the limiting case of the other.

14. Examples and Problems

Action.

8.1. Find the action $S(t, x) \equiv S(0, 0; t, x)$ for each of the following:

1. The free particle Lagrangian $L = m\dot{x}^2/2$.
2. The harmonic oscillator: $L = \dot{x}^2/2 - x^2/2$.
3. A particle in a repelling linear potential: $L = \dot{x}^2/2 + x^2/2$.

Answer for (1): $S(t, x) = m\frac{x^2}{2t}$. The level sets of S are shown in Figure 14, page 282.

The following problem is the same as the one before except that the left end is allowed to be free.

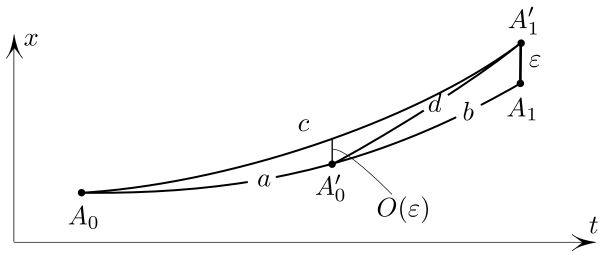

Figure 18. Proving (8.43).

8.2. Find the action $S(t_0, x_0; t_1, x_1)$ for the free particle of mass m moving on the x-axis.

Solution. The motion is given by $y(t) = x_0 + vt$, where

$$v = \frac{x_1 - x_0}{t_1 - t_0},$$

Substituting into (1.22) gives the answer:

$$S(t_0, x_0; t_1, x_1) = m\frac{(x_1 - x_0)^2}{2(t_1 - t_0)}.$$

8.3. Consider three points A_0, A_0', A_1 on the graph of a critical function corresponding to the Lagrangian L, Figure 3, page 261. Prove that

$$\nabla_{A_1} S(A_0, A_1) = \nabla_{A_1} S(A_0', A_1), \tag{8.43}$$

where $S(A, B)$ is the integral of L along the critical function whose graph connects A and B in the (t, x)-plane. Assume that the critical function depends smoothly on the endpoints.

Solution. Let us displace A_1 by a small distance ε to a new position A_1', Figure 18. Since the critical function depends smoothly on the endpoints, A_0' is $O(\varepsilon)$-close to the new critical graph A_0A_1'; and since A_0A_1' is a critical curve, we have

$$c = (a + d) + O(\varepsilon^2), \tag{8.44}$$

where a, b, c, d stand for the actions as labeled in the figure. Subtracting $a + b = S(A_0, A_1)$ from both sides, we get

$$\underbrace{c - (a + b)}_{\nabla_{A_1} S(A_0 A_1)\cdot \Delta A_1 + O(\varepsilon^2)} = \underbrace{d - b}_{\nabla_{A_1} S(A_0' A_1)\cdot \Delta A_1 + O(\varepsilon^2)} + O(\varepsilon^2),$$

where the definition of the gradient was used. This proves (8.43). ◊

The classical mechanical uncertainty principle.

8.4. Extend the "uncertainty principle" of classical mechanics (see page 36) for the case of a particle in $\mathbb{R}^3$.

8.5. What is the analog of the "uncertainty principle" of classical mechanics for the light rays passing through an optical device such as a telescope or a microscope? (The answer can be found in [**14**]).

Poincaré's integral invariant

8.6. This problem uses notations of Section 11, page 274. Let the spring's ends A_0, A_1 in Figure 9 (page 275) execute a cyclic motion; assume for simplicity that the motion is in the x-direction only. Show by an example that, although $\int p_0\, dx_0 + \int p_1\, dx_1 = 0$, the individual summands need not vanish; that is, although the total work done by both "hands" in moving the ends of the spring in the x-direction is zero, each hand may do nonzero work.

Solution. 1-by calculation. Take the simplest case of a Hookean spring without gravity, with Hooke's constant $k = 1$; potential energy of such a spring is $U = \frac{1}{2}\text{length}^2$, and since t_0, t_1 are fixed, we have $U = \frac{1}{2}(x_1 - x_0)^2 + c$, where $c = \frac{1}{2}(t_1 - t_0)^2 = \text{const}$. Define the cyclic change $x_0 = \sin s$, $x_1 = \cos s$, where $0 \leq s \leq 2\pi$ (the idea is that the "hands" move out of phase). From $p_0 = -U_{x_0}$, $p_1 = U_{x_1}$ we get $p_0 = p_1 = x_1 - x_0$, and thus

$$\oint p_0\, dx_0 = \int_0^{2\pi} (\cos s - \sin s)\, \cos s\, ds = \pi \neq 0.$$

2-without calculation. Figure 19 shows the same spring, with the ends A_0, A_1 moving in four stages (in a way different from the one described above). Whenever the right hand moves it does positive work, since $p_1\, dx_1 > 0$ as p_1 and dx_1 have the same sign. Similarly, whenever the left hand moves it does negative work, since p_0 and dx_0 have opposite signs. This shows that $\oint p_0\, dx_0 < 0 < \oint p_1\, dx_1$.

The next problem refers to the concept of *brake orbits* for a particle in the potential U:

$$\ddot{\mathbf{x}} = -\nabla U(\mathbf{x}).$$

A nonconstant solution $\mathbf{x}(t)$ is said to be a *brake orbit* if there exist times $t_1 < t_2$ such that $\dot{\mathbf{x}}(t_1) = \dot{\mathbf{x}}(t_2) = \mathbf{0}$. A normal mode of a linear system $\ddot{\mathbf{x}} = -A\mathbf{x}$ (where A is a positive definite matrix) is an example of a brake orbit.

8.7. Consider a centrally symmetric potential $U_0 : \mathbb{R}^2 \to \mathbb{R}$ monotonically increasing with the distance from the origin, with $\nabla U_0(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \neq 0$. Let $E > \min U_0$. Show that if U is sufficiently close (with two derivatives) to U_0, then the system $\ddot{\mathbf{x}} = -\nabla U(\mathbf{x})$ has a brake orbit with energy E.

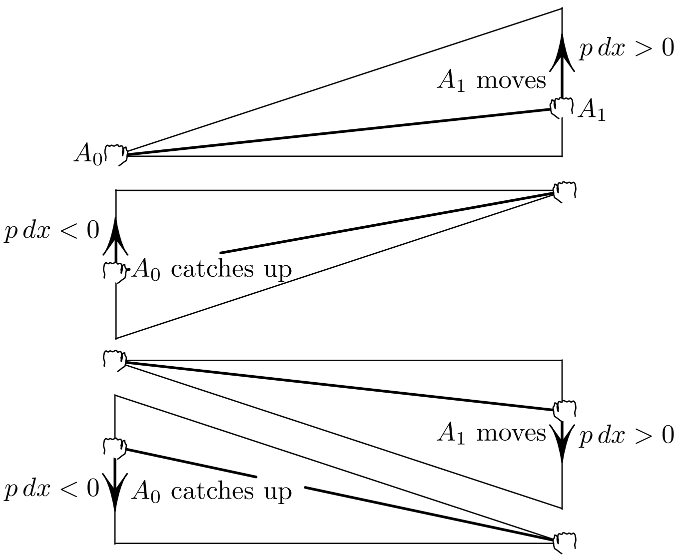

Figure 19. Although $\int p_0\,dx_0 + \int p_1\,dx_1 = 0$ when the ends A_0, A_1 execute a cyclic motion, the individual integrals need not vanish. Each time A_1 moves, positive work is done. Each time A_0 moves, negative work is done. In short, "hand A_1 transfers energy to hand A_0".

Hint. Use the Poincaré's integral invariant for the curve of initial data which start on the zero velocity curve $\{\mathbf{x} : U(\mathbf{x}) = E\}$.

Noether's theorem

8.8. (Noether's theorem – applications). For each of the following families of transformations, find the explicit form of the conserved quantity given by Noether's theorem.

1. Space translation in the direction of a vector $\mathbf{e}$: $h^s\mathbf{x} = \mathbf{x} + s\mathbf{e}$.
2. Rotation through angle s around the vector $\mathbf{e} \in \mathbb{R}^3$: $h^s\mathbf{x} = R_{\mathbf{e}}(s)\mathbf{x}$.
3. Helical symmetry $h^s\mathbf{x} = R_{\mathbf{e}}(s) + as\mathbf{e}$, where $a = \text{const}$.

8.9. Find a conserved quantity for a particle in a rotating potential in $\mathbb{R}^2$:

$$\ddot{\mathbf{x}} = -\nabla U(R_{\omega t}\mathbf{x}),$$

where R_s is the counterclockwise rotation through angle s around the origin.

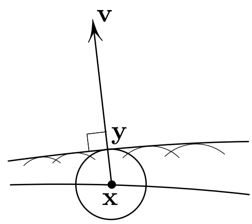

Figure 20. Problem 8.11.

Hint. Extend Noether's theorem to translations of space-time given by $(t, \mathbf{x}) \mapsto (t+s, R_{\omega s}\mathbf{x})$. Alternatively, consider the system in a rotating frame.

8.10. (Conserved quantity for charged particles in magnetic fields). Consider the motion of a charged particle in the magnetic field $B(x, y)$ perpendicular to the plane:

$$\ddot{z} + iB(z)\dot{z} + \nabla V(z) = 0.$$

Here the complex notation is used: $z = (x, y) = x + iy$, and B and V are real-valued smooth functions.

1. Show that the total energy $\dot{z}^2 + V(z)$ of the particle is conserved.

2. Find another conserved quantity (besides the energy), given that B and V are rotationally symmetric, i.e., if both depend only on $r = |z|$.

3. Find the conserved quantity (besides the energy) given that B and V are invariant under translations.

Hint. Find the Lagrangian and use Noether's theorem.

4. In the special case of $B =$ const. and V rotationally symmetric, give a physical interpretation of this conserved quantity.

Hint (written backwards to discourage peeking)**:**
.emarf gnitator a ni mutnemom ralugnA

Wave fronts

8.11. Show that if the optical medium in $\mathbb{R}^2$ is isotropic (meaning that the speed of light $c(\mathbf{x})$ does not depend on the direction of the ray), then the rays are normal to the wavefronts.

Solution. Consider two infinitesimally close fronts, as in Figure 20. Huygens's principle says in effect that the direction of the normal $\mathbf{n}$ to the front is conjugate to the direction $\mathbf{v}$ of the ray in the sense of the Figure 20. More precisely, $\mathbf{n}$ is normal to the indicatrix at $\mathbf{x}$ (the point of tangency between the indicatrix and the new front), while $\mathbf{v}$ is the radius-vector of the point $\mathbf{x}$. But in an isotropic medium the indicatrices are spherical (unlike those in the Figure 20), and thus $\mathbf{n}$ and $\mathbf{v}$ are parallel. $\diamondsuit$

8.12. Show that the infinitesimal distance between two nearby level sets $f(\mathbf{x}) = h$ and $f(\mathbf{y}) = h+\varepsilon$ of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ (take of $n = 2$ without the loss of generality) is approximately $\varepsilon/|\nabla f|$. More precisely, consider a smooth curve perpendicular to every level curve $f =$ const., and let $\mathbf{x}$ and $\mathbf{y}$ be the points of intersection of this curve with the two surfaces. Prove that the arc length $|\mathbf{xy}| = \varepsilon/|\nabla f(\mathbf{x})| + o(\varepsilon)$.

Hint. Use the definition of the gradient

$$f(\mathbf{x} + \Delta\mathbf{x}) - f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \Delta\mathbf{x} + o(\Delta\mathbf{x}).$$

8.13. Consider an isotropic optical medium with speed of light $c(\mathbf{x})$. Show that the optical distance $T(\mathbf{x})$, i.e., the time it takes the ray to reach $\mathbf{x}$ from a given point A_0, satisfies the eikonal equation $|\nabla T| = c(\mathbf{x})^{-1}$.

Remark. The vector ∇T is called the *normal slowness* of the front — a very reasonable name, given that $|\nabla T|$ is the reciprocal $c(\mathbf{x})^{-1}$ of the speed.

Solution. Consider two infinitesimally close wave fronts $T = T_0$ and $T = T_0+\varepsilon$, and a ray normal to these, Figure 21. Since the medium is isotropic, the rays which carry the fronts are perpendicular to these fronts (according to Problem 8.11), and thus the length of the arc $\mathbf{xy}$ gives, to the leading order, the distance between the two curves. According to the preceding problem this distance is

$$|\mathbf{xy}| = \frac{\varepsilon}{|\nabla T(\mathbf{x})|},$$

where the terms of higher order in ε have been dropped. On the other hand, the distance traveled is the product of the average speed $= c(\mathbf{x})+O(\varepsilon)$ and the time:

$$|\mathbf{xy}| = c(\mathbf{x})\varepsilon + O(\varepsilon^2).$$

Comparing the two expressions and taking $\varepsilon \to 0$ gives $|\nabla T| = c^{-1}$.

8.14. Consider a wave front formed by a pencil of optical rays emanating from a point in $\mathbb{R}^2$. Given that the speed of light at $\mathbf{x}$ satisfies a quadratic relation $\langle A(\mathbf{x})\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle = 1$ (the set of such velocities is called the *indicatrix* at $\mathbf{x}$), where $A(\mathbf{x})$ is a symmetric matrix, find the relationship between the normal $\mathbf{n}$ to the front and the tangent to the ray.

Hint. Use Huygens's principle, observing two facts: (i) the normal to the front is also normal to the indicatrix (the infinitesimal front), and (ii) the ray goes from the center of the indicatrix to the tangency point of the indicatrix with the advanced position of the front.

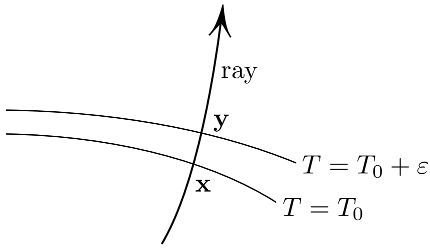

Figure 21. Derivation of the eikonal equation (8.20).

Answer. $\mathbf{n} = A\dot{\mathbf{x}}$.

8.15. In the setting of the preceding problem, replace the indicatrix $\langle A\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle = 1$ by a more general curve $f(\dot{\mathbf{x}}) = 0$. Find the relationship between the normal $\mathbf{n}$ to the front and the direction of the ray.

Answer. $\mathbf{n} = \nabla f(\dot{\mathbf{x}})$.

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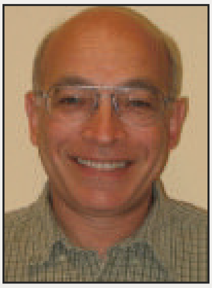

It is hard to imagine a more original and insightful approach to classical mechanics. Most physicists would regard this as a well-worn and settled subject. But Mark Levi's treatment sparkles with freshness in the many examples he treats and his unexpected analogies, as well as the new approach he brings to the principles. This is inspired pedagogy at the highest level.

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