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An Introduction to Continuum Mechanics

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CHAPTER

I

Tensor Algebra

1. POINTS. VECTORS. TENSORS

The space under consideration will always be a **three-dimensional euclidean point space** $\mathscr{E}$. The term **point** will be reserved for elements of $\mathscr{E}$, the term **vector** for elements of the associated vector space $\mathscr{V}$. The difference

$$\mathbf{v} = \mathbf{y} - \mathbf{x}$$

of two points is a vector (Fig. 1); the sum

$$\mathbf{y} = \mathbf{x} + \mathbf{v}$$

of a point $\mathbf{x}$ and a vector $\mathbf{v}$ is a point. The sum of two points is *not* a meaningful concept.

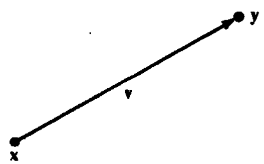

Figure 1

The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ will be designated by $\mathbf{u} \cdot \mathbf{v}$, and we define

$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}, \qquad \mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}.$$

We use the symbol $\mathbb{R}$ for the reals, $\mathbb{R}^+$ for the strictly positive reals.

Representation Theorem for Linear Forms.[1] *Let $\psi: \mathscr{V} \to \mathbb{R}$ be linear. Then there exists a unique vector* **a** *such that*

$$\psi(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$$

for every vector **v**.

A **cartesian coordinate frame** consists of an orthonormal basis $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ together with a point **o** called the **origin**. We assume once and for all that a single, fixed cartesian coordinate frame is given. The (cartesian) components of a vector **u** are given by

$$u_i = \mathbf{u} \cdot \mathbf{e}_i,$$

so that

$$\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i.$$

Similarly, the coordinates of a point **x** are

$$x_i = (\mathbf{x} - \mathbf{o}) \cdot \mathbf{e}_i.$$

The **span** $\mathrm{sp}\{\mathbf{u}, \mathbf{v}, \ldots, \mathbf{w}\}$ of a set $\{\mathbf{u}, \mathbf{v}, \ldots, \mathbf{w}\}$ of vectors is the *subspace* of $\mathscr{V}$ consisting of all linear combinations of these vectors:

$$\mathrm{sp}\{\mathbf{u}, \mathbf{v}, \ldots, \mathbf{w}\} = \{\alpha\mathbf{u} + \beta\mathbf{v} + \cdots + \gamma\mathbf{w} \,|\, \alpha, \beta, \ldots, \gamma \in \mathbb{R}\}.$$

(We will also use this notation for vector spaces other than $\mathscr{V}$.)

Given a vector **v**, we write

$$\{\mathbf{v}\}^\perp = \{\mathbf{u} \,|\, \mathbf{u} \cdot \mathbf{v} = 0\}$$

for the subspace of $\mathscr{V}$ consisting of all vectors perpendicular to **v**.

We use the term **tensor** as a synonym for "linear transformation from $\mathscr{V}$ into $\mathscr{V}$." Thus a tensor **S** is a linear map that assigns to each vector **u** a vector

$$\mathbf{v} = \mathbf{S}\mathbf{u}.$$

The set of all tensors forms a vector space if **addition** and **scalar multiplication** are defined pointwise; that is, $\mathbf{S} + \mathbf{T}$ and $\alpha\mathbf{S}$ ($\alpha \in \mathbb{R}$) are the tensors defined by

$$(\mathbf{S} + \mathbf{T})\mathbf{v} = \mathbf{S}\mathbf{v} + \mathbf{T}\mathbf{v},$$

$$(\alpha\mathbf{S})\mathbf{v} = \alpha(\mathbf{S}\mathbf{v}).$$

The zero element in this space is the **zero tensor 0** which maps every vector **v** into the zero vector:

$$\mathbf{0}\mathbf{v} = \mathbf{0}.$$

[1] Cf., e.g., Halmos [1, §67].

Another important tensor is the **identity I** defined by

$$\mathbf{I}\mathbf{v} = \mathbf{v}$$

for every vector **v**.

The **product ST** of two tensors is the tensor

$$\mathbf{S}\mathbf{T} = \mathbf{S} \circ \mathbf{T};$$

that is,

$$(\mathbf{S}\mathbf{T})\mathbf{v} = \mathbf{S}(\mathbf{T}\mathbf{v})$$

for all **v**. We use the standard notation

$$\mathbf{S}^2 = \mathbf{S}\mathbf{S}, \quad \text{etc.}$$

Generally, $\mathbf{S}\mathbf{T} \neq \mathbf{T}\mathbf{S}$. If $\mathbf{S}\mathbf{T} = \mathbf{T}\mathbf{S}$, we say that **S** and **T** **commute.**

We write $\mathbf{S}^\mathrm{T}$ for the **transpose** of **S**; $\mathbf{S}^\mathrm{T}$ is the unique tensor with the property

$$\mathbf{S}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{S}^\mathrm{T}\mathbf{v}$$

for all vectors **u** and **v**. It then follows that

$$(\mathbf{S} + \mathbf{T})^\mathrm{T} = \mathbf{S}^\mathrm{T} + \mathbf{T}^\mathrm{T},$$

$$(\mathbf{S}\mathbf{T})^\mathrm{T} = \mathbf{T}^\mathrm{T}\mathbf{S}^\mathrm{T}, \tag{1}$$

$$(\mathbf{S}^\mathrm{T})^\mathrm{T} = \mathbf{S}.$$

A tensor **S** is **symmetric** if

$$\mathbf{S} = \mathbf{S}^\mathrm{T},$$

skew if

$$\mathbf{S} = -\mathbf{S}^\mathrm{T}.$$

Every tensor **S** can be expressed uniquely as the sum of a symmetric tensor **E** and a skew tensor **W**:

$$\mathbf{S} = \mathbf{E} + \mathbf{W};$$

in fact,

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{S} + \mathbf{S}^\mathrm{T}),$$

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{S} - \mathbf{S}^\mathrm{T}).$$

We call **E** the **symmetric part** of **S**, **W** the **skew part** of **S**.

The **tensor product** $\mathbf{a} \otimes \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the *tensor* that assigns to each vector $\mathbf{v}$ the vector $(\mathbf{b} \cdot \mathbf{v})\mathbf{a}$:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}.$$

Then

$$\begin{aligned} (\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} &= (\mathbf{b} \otimes \mathbf{a}), \\ (\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d}, \\ (\mathbf{e}_i \otimes \mathbf{e}_i)(\mathbf{e}_j \otimes \mathbf{e}_j) &= \begin{cases} \mathbf{0}, & i \neq j \\ \mathbf{e}_i \otimes \mathbf{e}_i, & i = j, \end{cases} \\ \sum_i \mathbf{e}_i \otimes \mathbf{e}_i &= \mathbf{I}. \end{aligned} \tag{2}$$

Let $\mathbf{e}$ be a unit vector. Then $\mathbf{e} \otimes \mathbf{e}$ applied to a vector $\mathbf{v}$ gives

$$(\mathbf{v} \cdot \mathbf{e})\mathbf{e},$$

which is the projection of $\mathbf{v}$ in the direction of $\mathbf{e}$, while $\mathbf{I} - \mathbf{e} \otimes \mathbf{e}$ applied to $\mathbf{v}$ gives

$$\mathbf{v} - (\mathbf{v} \cdot \mathbf{e})\mathbf{e},$$

which is the projection of $\mathbf{v}$ onto the plane perpendicular to $\mathbf{e}$ (Fig. 2).

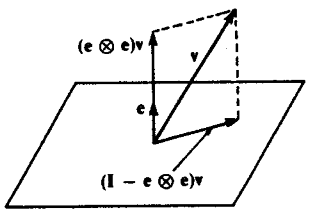

Figure 2

The components S_{ij} of a tensor $\mathbf{S}$ are defined by

$$S_{ij} = \mathbf{e}_i \cdot \mathbf{S}\mathbf{e}_j.$$

With this definition $\mathbf{v} = \mathbf{S}\mathbf{u}$ is equivalent to

$$v_i = \sum_j S_{ij} u_j.$$

Further,

$$\mathbf{S} = \sum_{i,j} S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \tag{3}$$

and

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j.$$

We write $[\mathbf{S}]$ for the matrix

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}.$$

It then follows that

$$\begin{aligned} [\mathbf{S}^{\mathrm{T}}] &= [\mathbf{S}]^{\mathrm{T}}, \\ [\mathbf{S}\mathbf{T}] &= [\mathbf{S}][\mathbf{T}], \end{aligned}$$

and

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The **trace** is the linear operation that assigns to each tensor $\mathbf{S}$ a scalar $\operatorname{tr} \mathbf{S}$ and satisfies

$$\operatorname{tr}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. By (3) and the linearity of tr,

$$\begin{aligned} \operatorname{tr} \mathbf{S} = \operatorname{tr}\Bigl(\sum_{i,j} S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j\Bigr) &= \sum_{i,j} S_{ij} \operatorname{tr}(\mathbf{e}_i \otimes \mathbf{e}_j) \\ &= \sum_{i,j} S_{ij} \mathbf{e}_i \cdot \mathbf{e}_j = \sum_i S_{ii}. \end{aligned}$$

Thus the trace is well defined:

$$\operatorname{tr} \mathbf{S} = \sum_i S_{ii}.$$

This operation has the following properties:

$$\begin{aligned} \operatorname{tr} \mathbf{S}^{\mathrm{T}} &= \operatorname{tr} \mathbf{S}, \\ \operatorname{tr}(\mathbf{S}\mathbf{T}) &= \operatorname{tr}(\mathbf{T}\mathbf{S}). \end{aligned} \tag{4}$$

The space of all tensors has a natural **inner product**

$$\mathbf{S} \cdot \mathbf{T} = \operatorname{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}),$$

which in components has the form

$$\mathbf{S} \cdot \mathbf{T} = \sum_{i,j} S_{ij} T_{ij}.$$

Then

$$\begin{aligned} \mathbf{I}\cdot\mathbf{S} &= \operatorname{tr}\mathbf{S},\\ \mathbf{R}\cdot(\mathbf{ST}) &= (\mathbf{S}^{\mathrm{T}}\mathbf{R})\cdot\mathbf{T} = (\mathbf{RT}^{\mathrm{T}})\cdot\mathbf{S},\\ \mathbf{u}\cdot\mathbf{Sv} &= \mathbf{S}\cdot(\mathbf{u}\otimes\mathbf{v}),\\ (\mathbf{a}\otimes\mathbf{b})\cdot(\mathbf{u}\otimes\mathbf{v}) &= (\mathbf{a}\cdot\mathbf{u})(\mathbf{b}\cdot\mathbf{v}). \end{aligned} \tag{5}$$

More important is the following

Proposition

(a) *If* $\mathbf{S}$ *is symmetric,*

$$\mathbf{S}\cdot\mathbf{T} = \mathbf{S}\cdot\mathbf{T}^{\mathrm{T}} = \mathbf{S}\cdot\{\tfrac{1}{2}(\mathbf{T}+\mathbf{T}^{\mathrm{T}})\}.$$

(b) *If* $\mathbf{W}$ *is skew,*

$$\mathbf{W}\cdot\mathbf{T} = -\mathbf{W}\cdot\mathbf{T}^{\mathrm{T}} = \mathbf{W}\cdot\{\tfrac{1}{2}(\mathbf{T}-\mathbf{T}^{\mathrm{T}})\}.$$

(c) *If* $\mathbf{S}$ *is symmetric and* $\mathbf{W}$ *skew,*

$$\mathbf{S}\cdot\mathbf{W} = 0.$$

(d) *If* $\mathbf{T}\cdot\mathbf{S} = 0$ *for every tensor* $\mathbf{S}$, *then* $\mathbf{T} = \mathbf{0}$.
(e) *If* $\mathbf{T}\cdot\mathbf{S} = 0$ *for every symmetric* $\mathbf{S}$, *then* $\mathbf{T}$ *is skew.*
(f) *If* $\mathbf{T}\cdot\mathbf{W} = 0$ *for every skew* $\mathbf{W}$, *then* $\mathbf{T}$ *is symmetric.*

We define the **determinant** of a tensor $\mathbf{S}$ to be the determinant of the matrix $[\mathbf{S}]$:

$$\det\mathbf{S} = \det[\mathbf{S}].$$

This definition is independent of our choice of basis $\{\mathbf{e}_i\}$.

A tensor $\mathbf{S}$ is **invertible** if there exists a tensor $\mathbf{S}^{-1}$, called the inverse of $\mathbf{S}$, such that

$$\mathbf{SS}^{-1} = \mathbf{S}^{-1}\mathbf{S} = \mathbf{I}.$$

It follows that $\mathbf{S}$ is invertible if and only if $\det\mathbf{S} \neq 0$.

The identities

$$\begin{aligned} \det(\mathbf{ST}) &= (\det\mathbf{S})(\det\mathbf{T}),\\ \det\mathbf{S}^{\mathrm{T}} &= \det\mathbf{S},\\ \det(\mathbf{S}^{-1}) &= (\det\mathbf{S})^{-1},\\ (\mathbf{ST})^{-1} &= \mathbf{T}^{-1}\mathbf{S}^{-1},\\ (\mathbf{S}^{-1})^{\mathrm{T}} &= (\mathbf{S}^{\mathrm{T}})^{-1} \end{aligned} \tag{6}$$

will be useful. For convenience, we use the abbreviation

$$\mathbf{S}^{-\mathrm{T}} = (\mathbf{S}^{-1})^{\mathrm{T}}.$$

A tensor $\mathbf{Q}$ is **orthogonal** if it preserves inner products:

$$\mathbf{Qu}\cdot\mathbf{Qv} = \mathbf{u}\cdot\mathbf{v}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. A necessary and sufficient condition that $\mathbf{Q}$ be orthogonal is that

$$\mathbf{QQ}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I},$$

or equivalently,

$$\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}.$$

An orthogonal tensor with positive determinant is called a **rotation.** (Rotations are sometimes called proper orthogonal tensors.) Every orthogonal tensor is either a rotation or the product of a rotation with $-\mathbf{I}$. If $\mathbf{R} \neq \mathbf{I}$ is a rotation, then the set of all vectors $\mathbf{v}$ such that

$$\mathbf{Rv} = \mathbf{v}$$

forms a one-dimensional subspace of $\mathscr{V}$ called the **axis** of $\mathbf{R}$.

A tensor $\mathbf{S}$ is **positive definite** provided

$$\mathbf{v}\cdot\mathbf{Sv} > 0$$

for all vectors $\mathbf{v} \neq \mathbf{0}$.

Throughout this book we will use the following notation:

Lin = the set of all tensors;
Lin^+ = the set of all tensors $\mathbf{S}$ with $\det\mathbf{S} > 0$;
Sym = the set of all symmetric tensors;
Skw = the set of all skew tensors;
Psym = the set of all symmetric, positive definite tensors;
Orth = the set of all orthogonal tensors;
Orth^+ = the set of all rotations.

The sets Lin^+, Orth, and Orth^+ are *groups* under multiplication; in fact, Orth^+ is a subgroup of both Orth and Lin^+. Orth is the *orthogonal group*; Orth^+ is the *rotation group* (proper orthogonal group).

On any three-dimensional vector space there are exactly two cross products, and one is the negative of the other. We assume that one such cross product, written

$$\mathbf{u}\times\mathbf{v}$$

for all **u** and **v**, has been singled out. Intuitively, $\mathbf{u} \times \mathbf{v}$ will represent the right-handed cross product of **u** and **v**; thus if

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2,$$

then the basis $\{\mathbf{e}_i\}$ is right handed and the components of $\mathbf{u} \times \mathbf{v}$ relative to $\{\mathbf{e}_i\}$ are

$$u_2 v_3 - u_3 v_2, \qquad u_3 v_1 - u_1 v_3, \qquad u_1 v_2 - u_2 v_1.$$

Further,

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u},$$

$$\mathbf{u} \times \mathbf{u} = \mathbf{0},$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}).$$

When **u**, **v**, and **w** are linearly independent, the magnitude of the scalar

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

represents the volume of the parallelepiped $\mathscr{P}$ determined by **u**, **v**, **w**. Further,[1]

$$\det \mathbf{S} = \frac{\mathbf{Su} \cdot (\mathbf{Sv} \times \mathbf{Sw})}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}$$

and hence

$$|\det \mathbf{S}| = \frac{\mathrm{vol}(\mathbf{S}(\mathscr{P}))}{\mathrm{vol}(\mathscr{P})},$$

which gives a geometrical interpretation of the determinant (Fig. 3). Here $\mathbf{S}(\mathscr{P})$ is the image of $\mathscr{P}$ under **S**, and vol designates the volume.

There is a one-to-one correspondence between vectors and skew tensors: given any skew tensor **W** there exists a unique vector **w** such that

$$\mathbf{Wv} = \mathbf{w} \times \mathbf{v} \tag{7}$$

for every **v**, and conversely; indeed,

$$[\mathbf{W}] = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}$$

corresponds to

$$w_1 = \alpha, \qquad w_2 = \beta, \qquad w_3 = \gamma.$$

[1] Cf., e.g., Nickerson, Spencer, and Steenrod [1, §5.2].

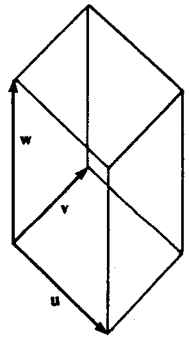

Figure 3

We call **w** the **axial vector** corresponding to **W**. It follows from (7) that (for $\mathbf{W} \neq \mathbf{0}$) the null space of **W**, that is the set of all **v** such that

$$\mathbf{Wv} = \mathbf{0},$$

is equal to the *one-dimensional* subspace spanned by **w**. This subspace is called **axis** of **W**.

We will frequently use the facts that $\mathscr{V}$ and Lin are normed vector spaces and that the standard operations of tensor analysis are continuous. In particular, on $\mathscr{V}$ and Lin, the sum, inner product, and scalar product are continuous, as are the tensor product on $\mathscr{V}$ and the product, trace, transpose, and determinant on suitable subsets of Lin.

EXERCISES

1. Choose $\mathbf{a} \in \mathscr{V}$ and let $\psi: \mathscr{V} \to \mathbb{R}$ be defined by $\psi(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$. Show that $\mathbf{a} = \sum_i \psi(\mathbf{e}_i)\mathbf{e}_i$.
2. Prove the representation theorem for linear forms (page 1).
3. Show that the sum $\mathbf{S} + \mathbf{T}$ and product **ST** are tensors.
4. Establish the existence and uniqueness of the transpose $\mathbf{S}^T$ of **S**.
5. Show that the tensor product $\mathbf{a} \otimes \mathbf{b}$ is a tensor.
6. Prove that
   (a) $\mathbf{S}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{Sa}) \otimes \mathbf{b}$,
   (b) $(\mathbf{a} \otimes \mathbf{b})\mathbf{S} = \mathbf{a} \otimes (\mathbf{S}^T\mathbf{b})$.
   (c) $\sum_i (\mathbf{Se}_i) \otimes \mathbf{e}_i = \mathbf{S}$.
7. Establish (1), (2), (4), and (5).

8. Show that
   (a) $\mathbf{v} = \mathbf{S}\mathbf{u}$ is equivalent to $v_i = \sum_j S_{ij}u_j$,
   (b) $(\mathbf{S}^T)_{ij} = S_{ji}$,
   (c) $(\mathbf{S}\mathbf{T})_{ij} = \sum_k S_{ik}T_{kj}$,
   (d) $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$,
   (e) $\mathbf{S} \cdot \mathbf{T} = \sum_{i,j} S_{ij}T_{ij}$.
9. Prove that the operation $\mathbf{S} \cdot \mathbf{T}$ is indeed an inner product; that is, show that
   (a) $\mathbf{S} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{S}$,
   (b) $\mathbf{S} \cdot \mathbf{T}$ is linear in $\mathbf{T}$ for $\mathbf{S}$ fixed,
   (c) $\mathbf{S} \cdot \mathbf{S} \geq 0$,
   (d) $\mathbf{S} \cdot \mathbf{S} = 0$ only when $\mathbf{S} = \mathbf{0}$.
10. Establish the proposition on p. 6.
11. Show that the trace of a tensor equals the trace of its symmetric part, so that, in particular, the trace of a skew tensor is zero.
12. Prove that $\mathbf{Q}$ is orthogonal if and only if $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$.
13. Show that $\mathbf{Q}$ is orthogonal if and only if $\mathbf{H} = \mathbf{Q} - \mathbf{I}$ satisfies

$$\mathbf{H} + \mathbf{H}^T + \mathbf{H}\mathbf{H}^T = \mathbf{0}, \qquad \mathbf{H}\mathbf{H}^T = \mathbf{H}^T\mathbf{H}.$$

14. Let $\varphi: \mathscr{V} \times \mathscr{V} \times \mathscr{V} \to \mathbb{R}$ be trilinear and skew symmetric; that is, φ is linear in each argument and

$$\varphi(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -\varphi(\mathbf{v}, \mathbf{u}, \mathbf{w}) = -\varphi(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -\varphi(\mathbf{w}, \mathbf{v}, \mathbf{u})$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{V}$. Let $\mathbf{S} \in \mathrm{Lin}$. Show that

$$\varphi(\mathbf{S}\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) + \varphi(\mathbf{e}_1, \mathbf{S}\mathbf{e}_2, \mathbf{e}_3) + \varphi(\mathbf{e}_1, \mathbf{e}_2, \mathbf{S}\mathbf{e}_3) = (\mathrm{tr}\,\mathbf{S})\varphi(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3).$$

15. Let $\mathbf{Q}$ be an orthogonal tensor, and let $\mathbf{e}$ be a vector with

$$\mathbf{Q}\mathbf{e} = \mathbf{e}.$$

   (a) Show that

$$\mathbf{Q}^T\mathbf{e} = \mathbf{e}.$$

   (b) Let $\mathbf{w}$ be the axial vector corresponding to the skew part of $\mathbf{Q}$. Show that $\mathbf{w}$ is parallel to $\mathbf{e}$.
16. Show that if $\mathbf{w}$ is the axial vector of $\mathbf{W} \in \mathrm{Skw}$, then

$$|\mathbf{w}| = \frac{1}{\sqrt{2}}|\mathbf{W}|.$$

17. Let $\mathbf{D} \in \mathrm{Psym}$, $\mathbf{Q} \in \mathrm{Orth}$. Show that $\mathbf{Q}\mathbf{D}\mathbf{Q}^T \in \mathrm{Psym}$.

2. SPECTRAL THEOREM. CAYLEY–HAMILTON THEOREM. POLAR DECOMPOSITION THEOREM

A scalar ω is an **eigenvalue** of a tensor $\mathbf{S}$ if there exists a unit vector $\mathbf{e}$ such that

$$\mathbf{S}\mathbf{e} = \omega\mathbf{e},$$

in which case $\mathbf{e}$ is an **eigenvector.** The **characteristic space** for $\mathbf{S}$ corresponding to ω is the subspace of $\mathscr{V}$ consisting of *all* vectors $\mathbf{v}$ that satisfy the equation

$$\mathbf{S}\mathbf{v} = \omega\mathbf{v}.$$

If this space has dimension n, then ω is said to have **multiplicity** n. The **spectrum** of $\mathbf{S}$ is the list $(\omega_1, \omega_2, \ldots)$, where $\omega_1 \leq \omega_2 \leq \cdots$ are the eigenvalues of $\mathbf{S}$ with each eigenvalue repeated a number of times equal to its multiplicity.

Proposition

(a) *The eigenvalues of a positive definite tensor are strictly positive.*
(b) *The characteristic spaces of a symmetric tensor are mutually orthogonal.*

Proof. Let ω be an eigenvalue of a positive definite tensor $\mathbf{S}$, and let $\mathbf{e}$ be a corresponding eigenvector. Then, since $\mathbf{S}\mathbf{e} = \omega\mathbf{e}$ and $|\mathbf{e}| = 1$, $\omega = \mathbf{e} \cdot \mathbf{S}\mathbf{e} > 0$.

To prove (b) let ω and λ be distinct eigenvalues of a symmetric tensor $\mathbf{S}$, and let

$$\mathbf{S}\mathbf{u} = \omega\mathbf{u}, \qquad \mathbf{S}\mathbf{v} = \lambda\mathbf{v},$$

so that $\mathbf{u}$ belongs to the characteristic space of ω, $\mathbf{v}$ to the characteristic space of λ. Then

$$\omega\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{S}\mathbf{u} = \mathbf{u} \cdot \mathbf{S}\mathbf{v} = \lambda\mathbf{u} \cdot \mathbf{v},$$

and, since $\omega \neq \lambda$, $\mathbf{u} \cdot \mathbf{v} = 0$. □

The next result[1] is one of the central theorems of linear algebra.

Spectral Theorem. *Let* $\mathbf{S}$ *be symmetric. Then there is an orthonormal basis for* $\mathscr{V}$ *consisting entirely of eigenvectors of* $\mathbf{S}$. *Moreover, for any such basis* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *the corresponding eigenvalues* $\omega_1, \omega_2, \omega_3$, *when ordered, form the entire spectrum of* $\mathbf{S}$ *and*

$$\mathbf{S} = \sum_i \omega_i \mathbf{e}_i \otimes \mathbf{e}_i. \tag{1}$$

[1] Cf., e.g., Bowen and Wang [1, §27]; Halmos [1, §79]; Stewart [1, §37]. The first two references state the theorem in terms of perpendicular projections onto characteristic spaces (cf. Exercise 4).

Conversely, if S *has the form* (1) *with* $\{\mathbf{e}_i\}$ *orthonormal, then* $\omega_1, \omega_2, \omega_3$ *are eigenvalues of* S *with* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *corresponding eigenvectors. Further*:

(a) S *has exactly three distinct eigenvalues if and only if the characteristic spaces of* S *are three mutually perpendicular lines through* **0**.

(b) S *has exactly two distinct eigenvalues if and only if* S *admits the representation*

$$\mathbf{S} = \omega_1 \mathbf{e} \otimes \mathbf{e} + \omega_2(\mathbf{I} - \mathbf{e} \otimes \mathbf{e}), \qquad |\mathbf{e}| = 1, \quad \omega_1 \neq \omega_2. \tag{2}$$

In this case ω_1 *and* ω_2 *are the two distinct eigenvalues and the corresponding characteristic spaces are* sp{e} *and* $\{\mathbf{e}\}^\perp$, *respectively. Conversely, if* sp{e} *and* $\{\mathbf{e}\}^\perp$ ($|\mathbf{e}| = 1$) *are the characteristic spaces for* S, *then* S *must have the form* (2).

(c) S *has exactly one eigenvalue if and only if*

$$\mathbf{S} = \omega \mathbf{I}. \tag{3}$$

In this case ω *is the eigenvalue and* $\mathscr{V}$ *the corresponding characteristic space. Conversely, if* $\mathscr{V}$ *is a characteristic space for* S, *then* S *has the form* (3).

The relation (1) is called a **spectral decomposition** of S. Note that, by (1), the matrix of S relative to a basis $\{\mathbf{e}_i\}$ of eigenvectors is diagonal:

$$[\mathbf{S}] = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix}.$$

Further, in view of (a)–(c), the characteristic spaces of a symmetric tensor S are three mutually perpendicular lines through **0**; or a line l through **0** and the plane through **0** perpendicular to l; or S has only one characteristic space, $\mathscr{V}$ itself. Thus if $\mathscr{U}_\alpha$ ($\alpha = 1, \ldots, n \leq 3$) denote the characteristic spaces of S, then any vector v can be written in the form

$$\mathbf{v} = \sum_\alpha \mathbf{v}_\alpha, \qquad \mathbf{v}_\alpha \in \mathscr{U}_\alpha. \tag{4}$$

Commutation Theorem. *Suppose that two tensors* S *and* T *commute. Then* T *leaves each characteristic space of* S *invariant; that is, if* v *belongs to a characteristic space of* S, *then* Tv *belongs to the same characteristic space. Conversely, if* T *leaves each characteristic space of a symmetric tensor* S *invariant, then* S *and* T *commute.*

Proof. Let S and T commute. Suppose that $\mathbf{Sv} = \omega\mathbf{v}$. Then

$$\mathbf{S}(\mathbf{Tv}) = \mathbf{T}(\mathbf{Sv}) = \omega(\mathbf{Tv}),$$

so that **Tv** belongs to the same characteristic space as **v**.

To prove the converse assertion choose a vector v and decompose v as in (4). If T leaves each characteristic space $\mathscr{U}_\alpha$ of S invariant, then $\mathbf{Tv}_\alpha \in \mathscr{U}_\alpha$ and

$$\mathbf{S}(\mathbf{Tv}_\alpha) = \omega_\alpha(\mathbf{Tv}_\alpha) = \mathbf{T}(\omega_\alpha \mathbf{v}_\alpha) = \mathbf{T}(\mathbf{Sv}_\alpha),$$

where ω_α is the eigenvalue corresponding to $\mathscr{U}_\alpha$. We therefore conclude, with the aid of (4), that

$$\mathbf{STv} = \sum_\alpha \mathbf{STv}_\alpha = \sum_\alpha \mathbf{TSv}_\alpha = \mathbf{TSv}.$$

Thus, since v was chosen arbitrarily, ST = TS. □

There is only one subspace of $\mathscr{V}$ that every *rotation* leaves invariant; namely $\mathscr{V}$ itself. We therefore have the following corollary of the spectral theorem.

Corollary. *A symmetric tensor* S *commutes with every rotation if and only if*

$$\mathbf{S} = \omega \mathbf{I}.$$

Crucial to our development of continuum mechanics is the polar decomposition theorem; our proof of this theorem is based on the

Square-Root Theorem. *Let* C *be symmetric and positive definite. Then there is a unique positive definite, symmetric tensor* U *such that*

$$\mathbf{U}^2 = \mathbf{C}.$$

We write $\sqrt{\mathbf{C}}$ *for* U.

Proof. (Existence) Let

$$\mathbf{C} = \sum_i \omega_i \mathbf{e}_i \otimes \mathbf{e}_i$$

be a spectral decomposition of $\mathbf{C} \in$ Psym and define $\mathbf{U} \in$ Psym by

$$\mathbf{U} = \sum_i \sqrt{\omega_i}\, \mathbf{e}_i \otimes \mathbf{e}_i.$$

(Since $\omega_i > 0$, this definition makes sense.) Then $\mathbf{U}^2 = \mathbf{C}$ is a direct consequence of $(1.2)_3$.

(Uniqueness[1]) Suppose

$$\mathbf{U}^2 = \mathbf{V}^2 = \mathbf{C}$$

with $\mathbf{U}, \mathbf{V} \in$ Psym. Let **e** be an eigenvector of **C** with $\omega > 0$ the corresponding eigenvalue. Then, letting $\lambda = \sqrt{\omega}$,

$$\mathbf{0} = (\mathbf{U}^2 - \omega\mathbf{I})\mathbf{e} = (\mathbf{U} + \lambda\mathbf{I})(\mathbf{U} - \lambda\mathbf{I})\mathbf{e}.$$

[1] Stephenson [1].

Let

$$\mathbf{v} = (\mathbf{U} - \lambda\mathbf{I})\mathbf{e}.$$

Then

$$\mathbf{U}\mathbf{v} = -\lambda\mathbf{v}$$

and $\mathbf{v}$ must vanish, for otherwise $-\lambda$ would be an eigenvalue of $\mathbf{U}$, an impossibility since $\mathbf{U}$ is positive definite and $\lambda > 0$. Hence

$$\mathbf{U}\mathbf{e} = \lambda\mathbf{e}.$$

Similarly,

$$\mathbf{V}\mathbf{e} = \lambda\mathbf{e},$$

and $\mathbf{U}\mathbf{e} = \mathbf{V}\mathbf{e}$ for every eigenvector $\mathbf{e}$ of $\mathbf{C}$. Since we can form a basis of eigenvectors of $\mathbf{C}$ (cf. the spectral theorem), $\mathbf{U}$ and $\mathbf{V}$ must coincide. □

Polar Decomposition Theorem. *Let* $\mathbf{F} \in \mathrm{Lin}^+$. *Then there exist positive definite, symmetric tensors* $\mathbf{U}, \mathbf{V}$ *and a rotation* $\mathbf{R}$ *such that*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{5}$$

Moreover, each of these decompositions is unique; in fact,

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}, \qquad \mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}. \tag{6}$$

We call the representation $\mathbf{F} = \mathbf{R}\mathbf{U}$ (*resp.* $\mathbf{F} = \mathbf{V}\mathbf{R}$) *the* **right** (*resp.* **left**) **polar decomposition** *of* $\mathbf{F}$.

Proof. Our first step will be to show that $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$ belong to Psym. Both tensors are clearly symmetric. Moreover,

$$\mathbf{v} \cdot \mathbf{F}^T\mathbf{F}\mathbf{v} = (\mathbf{F}\mathbf{v}) \cdot (\mathbf{F}\mathbf{v}) \geq 0,$$

and this inner product can equal zero only if $\mathbf{F}\mathbf{v} = \mathbf{0}$, or equivalently, since $\mathbf{F}$ is invertible, only if $\mathbf{v} = \mathbf{0}$. Thus $\mathbf{F}^T\mathbf{F} \in \mathrm{Psym}$. Similarly, $\mathbf{F}\mathbf{F}^T \in \mathrm{Psym}$.

(Uniqueness) Let $\mathbf{F} = \mathbf{R}\mathbf{U}$ be a right polar decomposition of $\mathbf{F}$. Then since $\mathbf{R}$ is a rotation,

$$\mathbf{F}^T\mathbf{F} = \mathbf{U}\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^2.$$

But by the square-root theorem there can be at most one $\mathbf{U} \in \mathrm{Psym}$ whose square is $\mathbf{F}^T\mathbf{F}$. Thus $(6)_1$ holds and $\mathbf{U}$ is unique; since $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$, $\mathbf{R}$ is also unique. On the other hand, if $\mathbf{F} = \mathbf{V}\mathbf{R}$ is a left polar decomposition, then

$$\mathbf{F}\mathbf{F}^T = \mathbf{V}^2,$$

and $\mathbf{V}$ is determined uniquely by $(6)_2$, $\mathbf{R}$ by the relation $\mathbf{R} = \mathbf{V}^{-1}\mathbf{F}$.

(Existence) Define $\mathbf{U} \in \mathrm{Psym}$ by $(6)_1$ and let

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}.$$

To verify $\mathbf{F} = \mathbf{R}\mathbf{U}$ is a right polar decomposition we have only to show that $\mathbf{R} \in \mathrm{Orth}^+$. Since $\det \mathbf{F} > 0$ and $\det \mathbf{U} > 0$ (the latter because all eigenvalues of $\mathbf{U}$ are strictly positive), $\det \mathbf{R} > 0$. Thus we have only to show that $\mathbf{R} \in \mathrm{Orth}$. But this follows from the calculation

$$\mathbf{R}^T\mathbf{R} = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I}.$$

Finally, define

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T.$$

Then $\mathbf{V} \in \mathrm{Psym}$ (Exercise 1.17) and

$$\mathbf{V}\mathbf{R} = \mathbf{R}\mathbf{U}\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{U} = \mathbf{F}. \quad \square$$

Given a tensor $\mathbf{S}$, the determinant of $\mathbf{S} - \omega\mathbf{I}$ admits the representation

$$\det(\mathbf{S} - \omega\mathbf{I}) = -\omega^3 + \iota_1(\mathbf{S})\omega^2 - \iota_2(\mathbf{S})\omega + \iota_3(\mathbf{S}) \tag{7}$$

for every $\omega \in \mathbb{R}$, where

$$\begin{aligned} \iota_1(\mathbf{S}) &= \operatorname{tr} \mathbf{S}, \\ \iota_2(\mathbf{S}) &= \tfrac{1}{2}[(\operatorname{tr} \mathbf{S})^2 - \operatorname{tr}(\mathbf{S}^2)], \\ \iota_3(\mathbf{S}) &= \det \mathbf{S}. \end{aligned} \tag{8}$$

We call

$$\mathscr{I}_\mathbf{S} = (\iota_1(\mathbf{S}), \iota_2(\mathbf{S}), \iota_3(\mathbf{S})) \tag{9}$$

the **list of principal invariants**[1] of $\mathbf{S}$. When $\mathbf{S}$ is symmetric $\mathscr{I}_\mathbf{S}$ is completely characterized by the spectrum $(\omega_1, \omega_2, \omega_3)$ of $\mathbf{S}$. Indeed, a simple calculation shows that

$$\begin{aligned} \iota_1(\mathbf{S}) &= \omega_1 + \omega_2 + \omega_3, \\ \iota_2(\mathbf{S}) &= \omega_1\omega_2 + \omega_2\omega_3 + \omega_1\omega_3, \\ \iota_3(\mathbf{S}) &= \omega_1\omega_2\omega_3. \end{aligned} \tag{10}$$

Moreover, the above characterization is a one-to-one correspondence. To see this note first that $\omega \in \mathbb{R}$ is an eigenvalue of a tensor $\mathbf{S}$ if and only if ω satisfies the characteristic equation

$$\det(\mathbf{S} - \omega\mathbf{I}) = 0,$$

[1] $\iota_k(\mathbf{S})$ are called invariants of $\mathbf{S}$ because of the way they transform under the orthogonal group: $\iota_k(\mathbf{Q}\mathbf{S}\mathbf{Q}^T) = \iota_k(\mathbf{S})$ for all $\mathbf{Q} \in \mathrm{Orth}$ [cf. (37.3)].

or equivalently,

$$\omega^3 - \iota_1(\mathbf{S})\omega^2 + \iota_2(\mathbf{S})\omega - \iota_3(\mathbf{S}) = 0. \quad (11)$$

Further, when S is symmetric the multiplicity of an eigenvalue ω is equal to its multiplicity as a root of (11).[1] Thus we have the following

Proposition. *Let* **S** *and* **T** *be symmetric and suppose that*

$$\mathscr{I}_{\mathbf{S}} = \mathscr{I}_{\mathbf{T}}.$$

Then **S** *and* **T** *have the same spectrum.*

More important is the

Cayley–Hamilton Theorem.[2] *Every tensor* **S** *satisfies its own characteristic equation*:

$$\mathbf{S}^3 - \iota_1(\mathbf{S})\mathbf{S}^2 + \iota_2(\mathbf{S})\mathbf{S} - \iota_3(\mathbf{S})\mathbf{I} = \mathbf{0}. \quad (12)$$

Exercises

1. Determine the spectrum, the characteristic spaces, and a spectral decomposition for each of the following tensors:

$$\mathbf{A} = \alpha\mathbf{I} + \beta\mathbf{m} \otimes \mathbf{m},$$

$$\mathbf{B} = \mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}.$$

Here α and β are scalars, while **m** and **n** are orthogonal unit vectors.

2. Granted the validity of the portion of the spectral theorem ending with (1), prove the remainder of the theorem.
3. Let $\mathbf{D} \in \text{Sym}$, $\mathbf{Q} \in \text{Orth}$. Show that the spectrum of **D** equals the spectrum of $\mathbf{QDQ}^T$. Show further that if **e** is an eigenvector of **D**, then **Qe** is an eigenvector of $\mathbf{QDQ}^T$ corresponding to the same eigenvalue.
4. A tensor **P** is a perpendicular projection if **P** is symmetric and $\mathbf{P}^2 = \mathbf{P}$.
   (a) Let **n** be a unit vector. Show that each of the following tensors is a perpendicular projection:

$$\mathbf{I}, \quad \mathbf{0}, \quad \mathbf{n} \otimes \mathbf{n}, \quad \mathbf{I} - \mathbf{n} \otimes \mathbf{n}. \quad (13)$$

   (b) Show that, conversely, if **P** is a perpendicular projection, then **P** admits one of the representations (13).

[1] Cf., e.g., Bowen and Wang [1, Theorem 27.4]; Halmos [1, §78, Theorem 6].
[2] Cf. e.g., Bowen and Wang [1, Theorem 26.1]; Halmos [1, §58].

5. Show that a (not necessarily symmetric) tensor **S** commutes with every skew tensor **W** if and only if

$$\mathbf{S} = \omega\mathbf{I}. \quad (14)$$

6. Let $\mathbf{F} = \mathbf{RU}$ and $\mathbf{F} = \mathbf{VR}$ denote the right and left polar decompositions of $\mathbf{F} \in \text{Lin}^+$.
   (a) Show that **U** and **V** have the same spectrum $(\omega_1, \omega_2, \omega_3)$.
   (b) Show that **F** and **R** admit the representations

$$\mathbf{F} = \sum_i \omega_i \mathbf{f}_i \otimes \mathbf{e}_i,$$

$$\mathbf{R} = \sum_i \mathbf{f}_i \otimes \mathbf{e}_i,$$

   where $\mathbf{e}_i$ and $\mathbf{f}_i$ are, respectively, the eigenvectors of **U** and **V** corresponding to ω_i.
7. Let **R** be the rotation corresponding to the polar decomposition of $\mathbf{F} \in \text{Lin}^+$. Show that **R** is the closest rotation to **F** in the sense that

$$|\mathbf{F} - \mathbf{R}| < |\mathbf{F} - \mathbf{Q}| \quad (15)$$

   for all rotations $\mathbf{Q} \neq \mathbf{R}$.

The result (15) suggests that an alternate proof[1] of the polar decomposition can be based on the following variational problem: Find a rotation **R** that minimizes $|\mathbf{F} - \mathbf{Q}|$ over all rotations **Q**.

Selected References

Bowen and Wang [1].
Brinkman and Klotz [1].
Chadwick [1, Chapter 1].
Halmos [1].
Martins and Podio-Guidugli [1].
Nickerson, Spencer, and Steenrod [1, Chapters 1–5].
Stewart [1].

[1] Such a proof is given by Martins and Podio-Guidugli [1].

CHAPTER

II

Tensor Analysis

3. DIFFERENTIATION

In this section we introduce a notion of differentiation sufficiently general to include scalar, point, vector, or tensor functions whose arguments are scalars, points, vectors, or tensors. To accomplish this we use the fact that $\mathbb{R}$, $\mathscr{V}$, and Lin are normed vector spaces, and where necessary phrase our definitions in terms of such spaces.

Let $\mathscr{U}$ and $\mathscr{W}$ denote normed vector spaces, and let $\mathbf{f}$ be defined in a neighborhood of zero in $\mathscr{U}$ and have values in $\mathscr{W}$. We say that **$\mathbf{f}(\mathbf{u})$ approaches zero faster than $\mathbf{u}$**, and write

$$\mathbf{f}(\mathbf{u}) = o(\mathbf{u}) \qquad \text{as} \quad \mathbf{u} \to \mathbf{0},$$

or, more simply,

$$\mathbf{f}(\mathbf{u}) = o(\mathbf{u}),$$

if

$$\lim_{\substack{\mathbf{u} \to \mathbf{0} \\ \mathbf{u} \neq \mathbf{0}}} \frac{\|\mathbf{f}(\mathbf{u})\|}{\|\mathbf{u}\|} = 0.$$

[Here $\|\mathbf{f}(\mathbf{u})\|$ is the norm of $\mathbf{f}(\mathbf{u})$ on $\mathscr{W}$, while $\|\mathbf{u}\|$ is the norm of $\mathbf{u}$ on $\mathscr{U}$.] Similarly,

$$\mathbf{f}(\mathbf{u}) = \mathbf{g}(\mathbf{u}) + o(\mathbf{u})$$

signifies that

$$\mathbf{f}(\mathbf{u}) - \mathbf{g}(\mathbf{u}) = o(\mathbf{u}).$$

Note that this latter definition also has meaning when $\mathbf{f}$ and $\mathbf{g}$ have values in $\mathscr{E}$, since in this instance $\mathbf{f} - \mathbf{g}$ has values in the vector space $\mathscr{V}$.

As an example consider the function $\varphi(t) = t^\alpha$. Then

$$\varphi(t) = o(t)$$

if and only if $\alpha > 1$.

Let $\mathbf{g}$ be a function whose values are scalars, vectors, tensors, or points, and whose domain is an open interval $\mathscr{D}$ of $\mathbb{R}$. The **derivative** $\dot{\mathbf{g}}(t)$ of $\mathbf{g}$ at t, if it exists, is defined by

$$\dot{\mathbf{g}}(t) = \frac{d}{dt}\mathbf{g}(t) = \lim_{\alpha\to 0}\frac{1}{\alpha}[\mathbf{g}(t+\alpha) - \mathbf{g}(t)]. \tag{1}$$

If the values of $\mathbf{g}$ are points, then $\mathbf{g}(t+\alpha) - \mathbf{g}(t)$ is a point difference and hence a vector. Thus the derivative of a point function is a vector. Similarly, the derivative of a vector function is a vector, and the derivative of a tensor function is a tensor. We say that $\mathbf{g}$ is **smooth** if $\dot{\mathbf{g}}(t)$ exists at each $t \in \mathscr{D}$, and if the function $\dot{\mathbf{g}}$ is continuous on $\mathscr{D}$.

Let $\mathbf{g}$ be differentiable at t. Then (1) implies that

$$\lim_{\alpha\to 0}\frac{1}{\alpha}[\mathbf{g}(t+\alpha) - \mathbf{g}(t) - \alpha\dot{\mathbf{g}}(t)] = \mathbf{0},$$

or equivalently,

$$\mathbf{g}(t+\alpha) = \mathbf{g}(t) + \alpha\dot{\mathbf{g}}(t) + o(\alpha). \tag{2}$$

Clearly,

$$\alpha\dot{\mathbf{g}}(t)$$

is linear in α; thus

$$\mathbf{g}(t+\alpha) - \mathbf{g}(t)$$

is equal to a term linear in α plus a term that approaches zero faster than α. In dealing with functions whose domains lie in spaces of dimension greater than one the most useful definition of a derivative is based on this result; we define the derivative to be the linear map which approximates $\mathbf{g}(t+\alpha) - \mathbf{g}(t)$ for small α.

Thus let $\mathscr{U}$ and $\mathscr{W}$ be finite-dimensional normed vector spaces, let $\mathscr{D}$ be an open subset of $\mathscr{U}$, and let

$$\mathbf{g}: \mathscr{D} \to \mathscr{W}.$$

We say that $\mathbf{g}$ is differentiable at $\mathbf{x} \in \mathscr{D}$ if the difference

$$\mathbf{g}(\mathbf{x}+\mathbf{u}) - \mathbf{g}(\mathbf{x})$$

is equal to a linear function of $\mathbf{u}$ plus a term that approaches zero faster than $\mathbf{u}$. More precisely, $\mathbf{g}$ is **differentiable** at $\mathbf{x}$ if there exists a linear transformation

$$D\mathbf{g}(\mathbf{x}): \mathscr{U} \to \mathscr{W} \tag{3}$$

such that

$$\mathbf{g}(\mathbf{x}+\mathbf{u}) = \mathbf{g}(\mathbf{x}) + D\mathbf{g}(\mathbf{x})[\mathbf{u}] + o(\mathbf{u}) \tag{4}$$

as $\mathbf{u} \to \mathbf{0}$. If $D\mathbf{g}(\mathbf{x})$ exists, it is unique; in fact, for each $\mathbf{u}$,

$$D\mathbf{g}(\mathbf{x})[\mathbf{u}] = \lim_{\substack{\alpha\to 0\\ \alpha\in\mathbb{R}}}\frac{1}{\alpha}[\mathbf{g}(\mathbf{x}+\alpha\mathbf{u}) - \mathbf{g}(\mathbf{x})] = \frac{d}{d\alpha}\mathbf{g}(\mathbf{x}+\alpha\mathbf{u})\big|_{\alpha=0}.$$

We call $D\mathbf{g}(\mathbf{x})$ the **derivative of g at x**. Since any two norms on a finite-dimensional vector space are equivalent, $D\mathbf{g}(\mathbf{x})$ is independent of the choice of norms on $\mathscr{U}$ and $\mathscr{W}$. If $\mathbf{g}$ is differentiable at each $\mathbf{x} \in \mathscr{D}$, then $D\mathbf{g}$ denotes the map $\mathbf{x} \mapsto D\mathbf{g}(\mathbf{x})$ whose domain is $\mathscr{D}$ and whose codomain is the space of linear transformations from $\mathscr{U}$ into $\mathscr{W}$. This space is finite dimensional and can be normed in a natural manner; thus it makes sense to talk about the continuity and differentiability of $D\mathbf{g}$. In particular, we say that $\mathbf{g}$ is of class C^1, or **smooth**, if $\mathbf{g}$ is differentiable at each point of $\mathscr{D}$ and $D\mathbf{g}$ is continuous. Continuing in this manner, we say $\mathbf{g}$ is of **class** C^2 if $\mathbf{g}$ is of class C^1 and $D\mathbf{g}$ is smooth, and so forth.

Of course, the three-dimensional euclidean point space $\mathscr{E}$ is not a normed vector space. However, when the domain $\mathscr{D}$ of $\mathbf{g}$ is contained in $\mathscr{E}$ the above definition remains valid provided we replace $\mathscr{U}$ in (3) by the vector space $\mathscr{V}$ associated with $\mathscr{E}$. Similarly, when $\mathbf{g}$ has values in $\mathscr{E}$ we need only replace $\mathscr{W}$ in (3) by $\mathscr{V}$.

Note that by (2), when the domain $\mathscr{D}$ of $\mathbf{g}$ is contained in $\mathbb{R}$,

$$D\mathbf{g}(t)[\alpha] = \alpha\dot{\mathbf{g}}(t) \tag{5}$$

for every $\alpha \in \mathbb{R}$.

Smooth-Inverse Theorem.[1] *Let $\mathscr{D}$ be an open subset of a finite-dimensional normed vector space $\mathscr{U}$. Further, let* $\mathbf{g}: \mathscr{D} \to \mathscr{U}$ *be a one-to-one function of class* C^n $(n \geq 1)$ *and assume that the linear transformation* $D\mathbf{g}(\mathbf{x}): \mathscr{U} \to \mathscr{U}$ *is invertible at each* $\mathbf{x} \in \mathscr{D}$*. Then* $\mathbf{g}^{-1}$ *is of class* C^n.

Often, the easiest method of computing derivatives is to appeal directly to the definition. We demonstrate this procedure with some examples. Consider first the function $\varphi: \mathscr{V} \to \mathbb{R}$ defined by

$$\varphi(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}.$$

Then

$$\varphi(\mathbf{v} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} = \varphi(\mathbf{v}) + 2\mathbf{v} \cdot \mathbf{u} + o(\mathbf{u}),$$

so that

$$D\varphi(\mathbf{v})[\mathbf{u}] = 2\mathbf{v} \cdot \mathbf{u}.$$

Similarly, for $\mathbf{G}$: Lin $\to$ Lin defined by

$$\mathbf{G}(\mathbf{A}) = \mathbf{A}^2$$

we have

$$\mathbf{G}(\mathbf{A} + \mathbf{U}) = \mathbf{A}^2 + \mathbf{AU} + \mathbf{UA} + \mathbf{U}^2 = \mathbf{G}(\mathbf{A}) + \mathbf{AU} + \mathbf{UA} + o(\mathbf{U}),$$

so that

$$D\mathbf{G}(\mathbf{A})[\mathbf{U}] = \mathbf{AU} + \mathbf{UA}. \tag{6}$$

Next, let $\mathbf{G}$: Lin $\to$ Lin be defined by

$$\mathbf{G}(\mathbf{A}) = \mathbf{A}^3.$$

Then

$$\begin{aligned} \mathbf{G}(\mathbf{A} + \mathbf{U}) &= (\mathbf{A} + \mathbf{U})^3 = \mathbf{A}^3 + \mathbf{A}^2\mathbf{U} + \mathbf{AUA} + \mathbf{UA}^2 + o(\mathbf{U}) \\ &= \mathbf{G}(\mathbf{A}) + \mathbf{A}^2\mathbf{U} + \mathbf{AUA} + \mathbf{UA}^2 + o(\mathbf{U}), \end{aligned}$$

and hence

$$D\mathbf{G}(\mathbf{A})[\mathbf{U}] = \mathbf{A}^2\mathbf{U} + \mathbf{AUA} + \mathbf{UA}^2.$$

As our last example, let $\mathbf{L}: \mathscr{U} \to \mathscr{W}$ be *linear*. Then

$$\mathbf{L}(\mathbf{x} + \mathbf{u}) = \mathbf{L}(\mathbf{x}) + \mathbf{L}(\mathbf{u}),$$

[1] This result is a direct consequence of the inverse function theorem (cf. Dieudonné [1, Theorem 10.2.5]).

so that, trivially,

$$D\mathbf{L}(\mathbf{x}) = \mathbf{L}. \tag{7}$$

The next two results involve far less trivial computations.

Theorem (Derivative of the determinant). *Let φ be defined on the set of all invertible tensors* $\mathbf{A}$ *by*

$$\varphi(\mathbf{A}) = \det \mathbf{A}.$$

Then φ is smooth. In fact,

$$D\varphi(\mathbf{A})[\mathbf{U}] = (\det \mathbf{A}) \operatorname{tr}(\mathbf{UA}^{-1}) \tag{8}$$

for every tensor $\mathbf{U}$.

Proof. By (2.7) with $\omega = -1$,

$$\det(\mathbf{I} + \mathbf{S}) = 1 + \operatorname{tr} \mathbf{S} + o(\mathbf{S})$$

as $\mathbf{S} \to \mathbf{0}$. Thus, for $\mathbf{A}$ invertible and $\mathbf{U} \in$ Lin,

$$\begin{aligned} \det(\mathbf{A} + \mathbf{U}) &= \det[(\mathbf{I} + \mathbf{UA}^{-1})\mathbf{A}] = (\det \mathbf{A}) \det(\mathbf{I} + \mathbf{UA}^{-1}) \\ &= (\det \mathbf{A})[1 + \operatorname{tr}(\mathbf{UA}^{-1}) + o(\mathbf{U})] \\ &= \det \mathbf{A} + (\det \mathbf{A}) \operatorname{tr}(\mathbf{UA}^{-1}) + o(\mathbf{U}) \end{aligned}$$

as $\mathbf{U} \to \mathbf{0}$. Therefore, since the map

$$\mathbf{U} \mapsto (\det \mathbf{A}) \operatorname{tr}(\mathbf{UA}^{-1})$$

is linear, (8) must be valid. The proof that $D\varphi$ is continuous follows from the continuity of the determinant, trace, and inverse operations. □

Theorem (Derivative of the square root). *The function* $\mathbf{H}$: Psym $\to$ Psym *defined by*

$$\mathbf{H}(\mathbf{C}) = \sqrt{\mathbf{C}}$$

is smooth.

Proof. By the square-root theorem (page 13), the function $\mathbf{H}$ is one-to-one with inverse $\mathbf{G}$: Psym $\to$ Psym defined by

$$\mathbf{G}(\mathbf{A}) = \mathbf{A}^2.$$

Clearly, $\mathbf{G}$ is smooth with derivative $D\mathbf{G}(\mathbf{A})$: Sym $\to$ Sym given by (6). Thus, in view of the smooth-inverse theorem, to complete the proof it suffices to show that $D\mathbf{G}(\mathbf{A})$ is invertible at each $\mathbf{A} \in$ Psym, or equivalently, that

$$D\mathbf{G}(\mathbf{A})[\mathbf{U}] = \mathbf{0} \quad \text{implies} \quad \mathbf{U} = \mathbf{0}.$$

Assume the former holds, so that, by (6),

$$AU + UA = 0.$$

Let λ be an eigenvalue of A with e a corresponding eigenvector. Then

$$AUe + UAe = AUe + \lambda Ue = 0$$

and

$$A(Ue) = -\lambda(Ue).$$

If $Ue \neq 0$, then $-\lambda$ is an eigenvalue of A. But A is positive definite, so that λ and $-\lambda$ cannot both be eigenvalues. Thus $Ue = 0$, and this must hold for every eigenvector e of A. By the spectral theorem, there is a basis for $\mathscr{V}$ of eigenvectors of A. Thus $U = 0$. □

It will frequently be necessary to compute the derivative of a product $\pi(f, g)$ of two functions f and g. In tensor analysis there are many different products available; for example, the product of a scalar φ and a vector v

$$\pi(\varphi, v) = \varphi v,$$

the inner product of two vectors

$$\pi(u, v) = u \cdot v,$$

the tensor product of two vectors

$$\pi(u, v) = u \otimes v,$$

the action of a tensor S on a vector v

$$\pi(S, v) = Sv,$$

and so forth. The above operations have one property in common, bilinearity. Therefore, in order to establish a product rule valid in all cases of interest, we consider the general product operation

$$\pi: \mathscr{F} \times \mathscr{G} \to \mathscr{W},$$

which assigns to each $f_0 \in \mathscr{F}$ and $g_0 \in \mathscr{G}$ the product $\pi(f_0, g_0) \in \mathscr{W}$. Here $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{W}$ are finite-dimensional normed spaces and π is a *bilinear* map. Within this general framework the product $h = \pi(f, g)$ of two functions

$$f: \mathscr{D} \to \mathscr{F}, \qquad g: \mathscr{D} \to \mathscr{G}$$

is the function

$$h: \mathscr{D} \to \mathscr{W}$$

defined by

$$h(x) = \pi(f(x), g(x))$$

for all $x \in \mathscr{D}$. Assume now that the common domain $\mathscr{D}$ of f and g is an open subset of a finite-dimensional normed space $\mathscr{U}$ (or of our euclidean point space). We then have the

Product Rule. *Let* f *and* g *be differentiable at* $x \in \mathscr{D}$. *Then their product* $h = \pi(f, g)$ *is differentiable at* x *and*

$$Dh(x)[u] = \pi(f(x), Dg(x)[u]) + \pi(Df(x)[u], g(x)) \tag{9}$$

for all $u \in \mathscr{U}$.

The proof of this result will be given at the end of the section.

Note that when f is constant (9) reduces to

$$Dh(x)[u] = \pi(f, Dg(x)[u]),$$

with a similar result when g is constant. We can use this fact to interpret (9). Thus let x_0 be a given point of $\mathscr{D}$, let f_0 and g_0 denote the constant functions on $\mathscr{D}$ with values $f(x_0)$ and $g(x_0)$, and let

$$h_1 = \pi(f_0, g), \qquad h_2 = \pi(f, g_0).$$

Then (9) becomes

$$Dh(x_0) = Dh_1(x_0) + Dh_2(x_0); \tag{10}$$

that is, *the derivative of* $\pi(f, g)$ at x_0 *is the derivative of the product holding* f *constant at its value* $f(x_0)$ *plus the derivative of the product holding* g *constant at its value* $g(x_0)$.

When the common domain of f and g is an open subset of $\mathbb{R}$, then (5), (9) (with x replaced by t), and the bilinearity of π imply that

$$\dot{h}(t) = \pi(\dot{f}(t), g(t)) + \pi(f(t), \dot{g}(t)).$$

Thus we have the following

Proposition. *Let* φ, v, w, S, *and* T *be smooth functions on an open subset of* $\mathbb{R}$ *with* φ *scalar valued,* v *and* w *vector valued, and* S *and* T *tensor valued. Then*

$$(\varphi v)^{\cdot} = \varphi\dot{v} + \dot{\varphi}v,$$

$$(v \cdot w)^{\cdot} = v \cdot \dot{w} + \dot{v} \cdot w,$$

$$(TS)^{\cdot} = T\dot{S} + \dot{T}S,$$

$$(T \cdot S)^{\cdot} = T \cdot \dot{S} + \dot{T} \cdot S,$$

$$(Su)^{\cdot} = S\dot{u} + \dot{S}u.$$

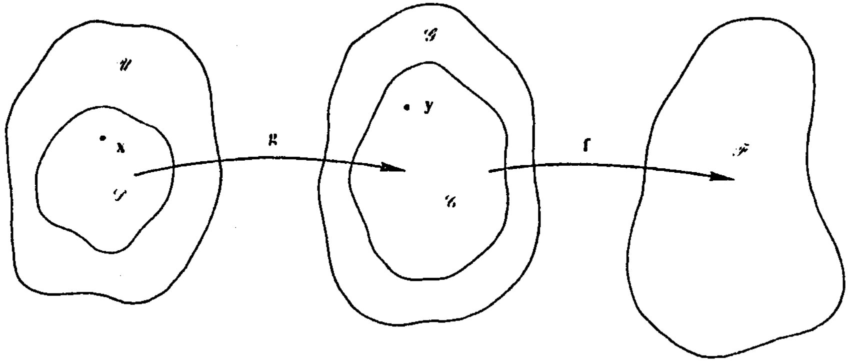

Figure 1

Another theorem which we will use frequently is the chain rule. In order to state this result in sufficient generality, let $\mathscr{U}$, $\mathscr{G}$, and $\mathscr{F}$ denote finite-dimensional normed linear (or euclidean point) spaces, let $\mathscr{C}$ and $\mathscr{D}$ denote open subsets of $\mathscr{G}$ and $\mathscr{U}$, respectively, and let

$$\mathbf{g}: \mathscr{D} \to \mathscr{G}, \qquad \mathbf{f}: \mathscr{C} \to \mathscr{F},$$

with the range of $\mathbf{g}$ contained in $\mathscr{C}$ (Fig. 1).

Chain Rule.[1] *Let* $\mathbf{g}$ *be differentiable at* $\mathbf{x} \in \mathscr{D}$, *and let* $\mathbf{f}$ *be differentiable at* $\mathbf{y} = \mathbf{g}(\mathbf{x})$. *Then the composition*

$$\mathbf{h} = \mathbf{f} \circ \mathbf{g}$$

is differentiable at $\mathbf{x}$ *and*

$$D\mathbf{h}(\mathbf{x}) = D\mathbf{f}(\mathbf{y}) \circ D\mathbf{g}(\mathbf{x}). \tag{11}$$

The relation (11), in less abbreviated form, reads

$$D\mathbf{h}(\mathbf{x})\,[\mathbf{u}] = D\mathbf{f}(\mathbf{g}(\mathbf{x}))\,[D\mathbf{g}(\mathbf{x})\,[\mathbf{u}]]$$

for every $\mathbf{u} \in \mathscr{U}$. In the case $\mathscr{U} = \mathbb{R}$, $\mathbf{g}$, and hence $\mathbf{h}$, is a function of a real variable; writing t in place of $\mathbf{x}$,

$$D\mathbf{h}(t)\,[\alpha] = \alpha\dot{\mathbf{h}}(t), \qquad D\mathbf{g}(t)\,[\alpha] = \alpha\dot{\mathbf{g}}(t)$$

for every $\alpha \in \mathbb{R}$. Thus, since $\mathbf{h}(t) = \mathbf{f}(\mathbf{g}(t))$, we have the important relation

$$\frac{d}{dt}\mathbf{f}(\mathbf{g}(t)) = D\mathbf{f}(\mathbf{g}(t))\,[\dot{\mathbf{g}}(t)]. \tag{12}$$

[1] See, e.g., Bartle [1, Theorem 40.2].

Proposition. *Let* $\mathbf{S}$ *be a smooth tensor-valued function on an open subset* $\mathscr{T}$ *of* $\mathbb{R}$. *Then*

$$(\mathbf{S}^{\mathsf{T}})^{\cdot} = (\dot{\mathbf{S}})^{\mathsf{T}} \equiv \dot{\mathbf{S}}^{\mathsf{T}}, \tag{13}$$

and if $\mathbf{S}(t)$ *is invertible at each* $t \in \mathscr{T}$,

$$(\det \mathbf{S})^{\cdot} = (\det \mathbf{S})\,\mathrm{tr}(\dot{\mathbf{S}}\mathbf{S}^{-1}). \tag{14}$$

Proof. Let $\mathbf{L}$: Lin $\to$ Lin be defined by

$$\mathbf{L}(\mathbf{A}) = \mathbf{A}^{\mathsf{T}}.$$

Then $\mathbf{L}$ is linear, so that (7) and (12) imply

$$(\mathbf{S}^{\mathsf{T}})^{\cdot} = [\mathbf{L}(\mathbf{S})]^{\cdot} = \mathbf{L}(\dot{\mathbf{S}}) = (\dot{\mathbf{S}})^{\mathsf{T}},$$

which is (13). The result (14) is a direct consequence of (8) and (12). □

We close this section with the

Proof of the Product Rule. Let κ_0 denote the maximum of $\|\pi(\mathbf{e}, \mathbf{k})\|$ over all *unit* vectors $\mathbf{e}$ and $\mathbf{k}$. This maximum certainly exists, since the set of all unit vectors is compact. (Recall that the underlying spaces $\mathscr{F}$ and $\mathscr{G}$ are finite dimensional.) Assume $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, and let $\mathbf{e}$ and $\mathbf{k}$ denote the unit vectors

$$\mathbf{e} = \mathbf{a}/\|\mathbf{a}\|, \qquad \mathbf{k} = \mathbf{b}/\|\mathbf{b}\|.$$

Since π is bilinear,

$$\pi(\mathbf{a}, \mathbf{b}) = \|\mathbf{a}\|\,\|\mathbf{b}\|\pi(\mathbf{e}, \mathbf{k})$$

and

$$\|\pi(\mathbf{a}, \mathbf{b})\| \le \kappa_0\|\mathbf{a}\|\,\|\mathbf{b}\|.$$

This inequality is also satisfied when $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, as is clear from the bilinearity of π. A similar argument yields the existence of constants κ_1 and κ_2 such that

$$\|D\mathbf{f}(\mathbf{x})\,[\mathbf{u}]\| \le \kappa_1\|\mathbf{u}\|, \qquad \|D\mathbf{g}(\mathbf{x})\,[\mathbf{u}]\| \le \kappa_2\|\mathbf{u}\|$$

for all $\mathbf{u} \in \mathscr{U}$.

To establish (9) we note first that

$$\mathbf{f}(\mathbf{x} + \mathbf{u}) = \mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})\,[\mathbf{u}] + o(\mathbf{u}),$$

$$\mathbf{g}(\mathbf{x} + \mathbf{u}) = \mathbf{g}(\mathbf{x}) + D\mathbf{g}(\mathbf{x})\,[\mathbf{u}] + o(\mathbf{u}),$$

and

$$\|\pi(Df(x)[u], Dg(x)[u])\| \le \kappa_0\kappa_1\kappa_2\|u\|^2,$$
$$\|\pi(Df(x)[u], o(u))\| \le \kappa_0\kappa_1\|u\|o(u),$$
$$\|\pi(o(u), Dg(x)[u])\| \le \kappa_0\kappa_2\|u\|o(u).$$

Thus, since all of the above terms are $o(\mathbf{u})$, we conclude from the bilinearity of π that

$$\begin{aligned} h(x+u) &= \pi(f(x+u), g(x+u)) \\ &= \pi(f(x), g(x)) + \pi(f(x), Dg(x)[u]) \\ &\quad + \pi(Df(x)[u], g(x)) + o(u), \end{aligned}$$

which yields the desired result, because the first term on the right side is $\mathbf{h}(\mathbf{x})$, while the second and third terms are linear in $\mathbf{u}$. □

Exercises

1. Compute $DG(A)$ for each of the following functions G: Lin → Lin.
   (a) $G(A) = (\mathrm{tr}\, A)A$,
   (b) $G(A) = ABA$ (B a given tensor),
   (c) $G(A) = A^TA$,
   (d) $G(A) = (u \cdot Au)A$ (u a given vector).
2. Let G be defined on the set of all invertible tensors by $G(A) = A^{-1}$. Assuming that G is differentiable, show that
$$DG(A)[H] = -A^{-1}HA^{-1}.$$
3. Let φ be defined on the set of all invertible tensors by $\varphi(A) = \det(A^2)$. Compute $D\varphi(A)$,
4. Let $\varphi(v) = e^{v^2}$ for all $v \in \mathscr{V}$. Compute $D\varphi(v)$.
5. Let G: Lin → Lin be defined by
$$G(A) = K(A)A^T,$$
where K: Lin → Lin is differentiable. Show that if G(A) is symmetric for each A and if K(I) = 0, then $DK(I)$ has symmetric values (i.e., $DK(I)[H] = DK(I)[H]^T$ for every H ∈ Lin).
6. Let Q: ℝ → Orth be differentiable. Show that $Q(t)\dot{Q}(t)^T$ is skew at each $t \in \mathbb{R}$.
7. Let G: Lin → Lin be differentiable and satisfy
$$QG(A)Q^T = G(QA)$$

for all A ∈ Lin and Q ∈ Orth. Show that
$$G(A)W^T + WG(A) = DG(A)[WA]$$
for all A ∈ Lin and W ∈ Skw.
8. Let S: Lin → Lin be smooth.
   (a) Define G: Lin → Lin by
$$G(A) = S(A_0 + \alpha A),$$
   where $\alpha \in \mathbb{R}$ and $A_0 \in$ Lin. Show that
$$DG(A)[U] = \alpha\, DS(A_0 + \alpha A)[U]$$
   for every U ∈ Lin.
   (b) Define φ: Lin → ℝ by
$$\varphi(A) = \int_0^1 A \cdot S(A_0 + \alpha A)\, d\alpha.$$
   Compute $D\varphi$. (It is permissible to differentiate under the integral.)
   (c) Assume that $DS(A)$ is symmetric at each A in the sense that
$$B \cdot DS(A)[C] = C \cdot DS(A)[B]$$
   for all tensors B and C. Show that
$$D\varphi(A)[U] = S(A_0 + A) \cdot U$$
   for every U ∈ Lin.
9. Compute the derivatives of the principal invariants $\iota_1, \iota_2, \iota_3$: Lin → ℝ.

4. GRADIENT. DIVERGENCE. CURL

We now consider functions defined over an open set $\mathscr{R}$ in euclidean space. A function on $\mathscr{R}$ is called a **scalar, vector, tensor,** or **point field** according as its values are scalars, vectors, tensors, or points.

Let φ be a smooth scalar field on $\mathscr{R}$. Then for each $x \in \mathscr{R}$, $D\varphi(x)$ is a *linear* mapping of $\mathscr{V}$ into ℝ, and by the representation theorem for linear forms there exists a vector $a(x)$ such that $D\varphi(x)[u]$ is the inner product of $a(x)$ with $\mathbf{u}$. We write $\nabla\varphi(x)$ for the *vector* $a(x)$, so that
$$D\varphi(x)[u] = \nabla\varphi(x) \cdot u,$$
and we call $\nabla\varphi(x)$ the **gradient** of φ at x. In this case the expansion (3.4) has the form
$$\varphi(x+u) = \varphi(x) + \nabla\varphi(x) \cdot u + o(u).$$

Similarly, if $\mathbf{v}$ is a smooth vector or point field on $\mathscr{R}$, then $D\mathbf{v}(\mathbf{x})$ is a linear transformation from $\mathscr{V}$ into $\mathscr{V}$ and hence a *tensor*. In this case we use the standard notation $\nabla\mathbf{v}(\mathbf{x})$ for $D\mathbf{v}(\mathbf{x})$ and write

$$\nabla\mathbf{v}(\mathbf{x})\,\mathbf{u} = D\mathbf{v}(\mathbf{x})\,[\mathbf{u}].$$

The tensor $\nabla\mathbf{v}(\mathbf{x})$ is the **gradient** of $\mathbf{v}$ at $\mathbf{x}$.

Given a smooth vector field $\mathbf{v}$ on $\mathscr{R}$, the *scalar field*

$$\boxed{\operatorname{div}\mathbf{v} = \operatorname{tr}\nabla\mathbf{v}}$$

is called the **divergence** of $\mathbf{v}$. We can use this operator to define the divergence, $\operatorname{div}\mathbf{S}$, of a smooth tensor field $\mathbf{S}$. Indeed, $\operatorname{div}\mathbf{S}$ is the unique *vector field* with the following property:

$$\boxed{(\operatorname{div}\mathbf{S})\cdot\mathbf{a} = \operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{a})} \tag{1}$$

for every vector $\mathbf{a}$. The reason for defining $\operatorname{div}\mathbf{S}$ in this manner will become clear when we establish the divergence theorem for tensor fields.

Proposition. *Let φ, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{S}$ be smooth fields with φ scalar valued, $\mathbf{v}$ and $\mathbf{w}$ vector valued, and $\mathbf{S}$ tensor valued. Then*

$$\begin{aligned}
\nabla(\varphi\mathbf{v}) &= \varphi\,\nabla\mathbf{v} + \mathbf{v}\otimes\nabla\varphi,\\
\operatorname{div}(\varphi\mathbf{v}) &= \varphi\operatorname{div}\mathbf{v} + \mathbf{v}\cdot\nabla\varphi,\\
\nabla(\mathbf{v}\cdot\mathbf{w}) &= (\nabla\mathbf{w})^{\mathrm{T}}\,\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\,\mathbf{w},\\
\operatorname{div}(\mathbf{v}\otimes\mathbf{w}) &= \mathbf{v}\operatorname{div}\mathbf{w} + (\nabla\mathbf{v})\mathbf{w},\\
\operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{v}) &= \mathbf{S}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\operatorname{div}\mathbf{S},\\
\operatorname{div}(\varphi\mathbf{S}) &= \varphi\operatorname{div}\mathbf{S} + \mathbf{S}\,\nabla\varphi.
\end{aligned} \tag{2}$$

Proof. Let $\mathbf{h} = \varphi\mathbf{v}$. Then by the general product rule (3.9),

$$\begin{aligned}
\nabla\mathbf{h}(\mathbf{x})\,\mathbf{u} &= \varphi(\mathbf{x})\,\nabla\mathbf{v}(\mathbf{x})\,\mathbf{u} + (\nabla\varphi(\mathbf{x})\cdot\mathbf{u})\mathbf{v}(\mathbf{x})\\
&= [\varphi(\mathbf{x})\,\nabla\mathbf{v}(\mathbf{x}) + \mathbf{v}(\mathbf{x})\otimes\nabla\varphi(\mathbf{x})]\mathbf{u},
\end{aligned}$$

which implies $(2)_1$. Taking the trace of $(2)_1$, we arrive at $(2)_2$.

Next, let $\psi = \mathbf{v}\cdot\mathbf{w}$. Then by (3.9),

$$\begin{aligned}
\nabla\psi(\mathbf{x})\cdot\mathbf{u} &= \mathbf{v}(\mathbf{x})\cdot\nabla\mathbf{w}(\mathbf{x})\,\mathbf{u} + \mathbf{w}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x})\,\mathbf{u}\\
&= [\nabla\mathbf{w}(\mathbf{x})^{\mathrm{T}}\,\mathbf{v}(\mathbf{x}) + \nabla\mathbf{v}(\mathbf{x})^{\mathrm{T}}\,\mathbf{w}(\mathbf{x})]\cdot\mathbf{u},
\end{aligned}$$

and we have $(2)_3$.

To prove $(2)_4$ note that, by (1) and $(2)_2$,

$$\mathbf{a}\cdot\operatorname{div}(\mathbf{v}\otimes\mathbf{w}) = \operatorname{div}[(\mathbf{w}\otimes\mathbf{v})\mathbf{a}] = \operatorname{div}[(\mathbf{v}\cdot\mathbf{a})\mathbf{w}] = (\mathbf{v}\cdot\mathbf{a})\operatorname{div}\mathbf{w} + \mathbf{w}\cdot\nabla(\mathbf{v}\cdot\mathbf{a}).$$

But by $(2)_3$,

$$\nabla(\mathbf{v}\cdot\mathbf{a}) = \nabla\mathbf{v}^{\mathrm{T}}\mathbf{a}.$$

Thus

$$\mathbf{a}\cdot\operatorname{div}(\mathbf{v}\otimes\mathbf{w}) = (\mathbf{v}\cdot\mathbf{a})\operatorname{div}\mathbf{w} + \mathbf{w}\cdot(\nabla\mathbf{v}^{\mathrm{T}}\mathbf{a}) = [\mathbf{v}\operatorname{div}\mathbf{w} + (\nabla\mathbf{v})\mathbf{w}]\cdot\mathbf{a},$$

which implies $(2)_4$.

Before proving $(2)_5$, we note that

$$\nabla(\mathbf{A}\mathbf{v}) = \mathbf{A}\,\nabla\mathbf{v} \tag{3}$$

for any (constant) tensor $\mathbf{A}$ and any smooth vector field $\mathbf{v}$. Indeed,

$$\mathbf{v}(\mathbf{x}+\mathbf{u}) = \mathbf{v}(\mathbf{x}) + \nabla\mathbf{v}(\mathbf{x})\,\mathbf{u} + o(\mathbf{u})$$

and hence

$$\mathbf{A}\mathbf{v}(\mathbf{x}+\mathbf{u}) = \mathbf{A}\mathbf{v}(\mathbf{x}) + \mathbf{A}\,\nabla\mathbf{v}(\mathbf{x})\,\mathbf{u} + o(\mathbf{u}),$$

which implies (3). [This result is also a direct consequence of (3.7) and the chain rule (3.11).] Taking the trace of (3), we arrive at the identity

$$\operatorname{div}(\mathbf{A}\mathbf{v}) = \mathbf{A}^{\mathrm{T}}\cdot\nabla\mathbf{v}. \tag{4}$$

We are now in a position to prove $(2)_5$. Clearly,

$$\operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{v}) = \operatorname{tr}\nabla(\mathbf{S}^{\mathrm{T}}\mathbf{v}),$$

and by the product rule in the form (3.10), $\nabla(\mathbf{S}^{\mathrm{T}}\mathbf{v})(\mathbf{x}_0)$ is the sum of two terms: the gradient holding $\mathbf{S}$ constant at the value $\mathbf{S}_0 = \mathbf{S}(\mathbf{x}_0)$ plus the gradient holding $\mathbf{v}$ constant at $\mathbf{v}_0 = \mathbf{v}(\mathbf{x}_0)$. Therefore

$$\begin{aligned}
\operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{v})(\mathbf{x}_0) &= \operatorname{tr}[\nabla(\mathbf{S}_0^{\mathrm{T}}\mathbf{v})(\mathbf{x}_0) + \nabla(\mathbf{S}^{\mathrm{T}}\mathbf{v}_0)(\mathbf{x}_0)]\\
&= \operatorname{div}(\mathbf{S}_0^{\mathrm{T}}\mathbf{v})(\mathbf{x}_0) + \operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{v}_0)(\mathbf{x}_0).
\end{aligned}$$

By (4) with $\mathbf{A} = \mathbf{S}_0^{\mathrm{T}}$,

$$\operatorname{div}(\mathbf{S}_0^{\mathrm{T}}\mathbf{v}) = \mathbf{S}_0\cdot\nabla\mathbf{v}.$$

On the other hand, since $\mathbf{v}_0$ is constant, (1) implies

$$\operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{v}_0) = \mathbf{v}_0\cdot\operatorname{div}\mathbf{S}.$$

Thus

$$\operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{v})(\mathbf{x}_0) = \mathbf{S}_0\cdot\nabla\mathbf{v}(\mathbf{x}_0) + \mathbf{v}_0\cdot\operatorname{div}\mathbf{S}(\mathbf{x}_0),$$

which implies $(2)_5$.

The proof of $(2)_6$ is similar. Using the above argument it is clear that $\operatorname{div}(\varphi\mathbf{S})(\mathbf{x}_0)$ is the divergence holding φ constant at $\varphi_0 = \varphi(\mathbf{x}_0)$ plus the divergence holding $\mathbf{S}$ constant at $\mathbf{S}_0 = \mathbf{S}(\mathbf{x}_0)$:

$$\operatorname{div}(\varphi\mathbf{S})(\mathbf{x}_0) = \operatorname{div}(\varphi_0\mathbf{S})(\mathbf{x}_0) + \operatorname{div}(\varphi\mathbf{S}_0)(\mathbf{x}_0).$$

Clearly,

$$\operatorname{div}(\varphi_0\mathbf{S}) = \varphi_0 \operatorname{div}\mathbf{S}.$$

Also, for any vector $\mathbf{a}$,

$$\mathbf{a}\cdot\operatorname{div}(\varphi\mathbf{S}_0) = \operatorname{div}(\varphi\mathbf{S}_0^T\mathbf{a}),$$

and by $(2)_2$ with $\mathbf{v} = \mathbf{S}_0^T\mathbf{a}$ this becomes

$$\nabla\varphi\cdot\mathbf{S}_0^T\mathbf{a} = \mathbf{a}\cdot\mathbf{S}_0\nabla\varphi;$$

hence

$$\operatorname{div}(\varphi\mathbf{S}_0) = \mathbf{S}_0\nabla\varphi.$$

Therefore

$$\operatorname{div}(\varphi\mathbf{S})(\mathbf{x}_0) = \varphi_0 \operatorname{div}\mathbf{S}(\mathbf{x}_0) + \mathbf{S}_0\nabla\varphi(\mathbf{x}_0),$$

which implies $(2)_6$. □

Another important identity, for a class C^2 vector field $\mathbf{v}$, is

$$\operatorname{div}(\nabla\mathbf{v}^T) = \nabla(\operatorname{div}\mathbf{v}). \tag{5}$$

We postpone the proof of (5) until later.

The **curl of v**, denoted curl $\mathbf{v}$, is the unique vector field with the property

$$(\nabla\mathbf{v} - \nabla\mathbf{v}^T)\mathbf{a} = (\operatorname{curl}\mathbf{v})\times\mathbf{a} \tag{6}$$

for every vector $\mathbf{a}$. Thus curl $\mathbf{v}(\mathbf{x})$ is the axial vector corresponding to the skew tensor $\nabla\mathbf{v}(\mathbf{x}) - \nabla\mathbf{v}(\mathbf{x})^T$.

Let Φ be a scalar or vector field of class C^2. Then **the laplacian** $\Delta\Phi$ **of** Φ is defined by

$$\Delta\Phi = \operatorname{div}\nabla\Phi.$$

If

$$\Delta\Phi = 0,$$

then Φ is **harmonic.**

Proposition. *Let* $\mathbf{v}$ *be a class* C^2 *vector field with*

$$\operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = \mathbf{0}.$$

Then $\mathbf{v}$ *is harmonic.*

Proof. Since curl $\mathbf{v} = \mathbf{0}$, (6) implies

$$\nabla\mathbf{v} - \nabla\mathbf{v}^T = \mathbf{0}.$$

Therefore, by (5),

$$\mathbf{0} = \operatorname{div}(\nabla\mathbf{v} - \nabla\mathbf{v}^T) = \Delta\mathbf{v} - \nabla\operatorname{div}\mathbf{v} = \Delta\mathbf{v}. \quad □$$

Let Φ be a smooth scalar, vector, or tensor field. Then

$$D\Phi(\mathbf{x})[\mathbf{e}_i] = \lim_{\alpha\to 0}\frac{1}{\alpha}\{\Phi(\mathbf{x} + \alpha\mathbf{e}_i) - \Phi(\mathbf{x})\}.$$

If $\mathbf{x}$ has components (x_1, x_2, x_3), then $\mathbf{x} + \alpha\mathbf{e}_1$, say, has components $(x_1 + \alpha, x_2, x_3)$. Thus the limit on the right is simply the partial derivative of Φ with respect to x_i:

$$D\Phi(\mathbf{x})[\mathbf{e}_i] = \frac{\partial\Phi(\mathbf{x})}{\partial x_i}.$$

This fact can be used to establish the following component representations:

$$\begin{aligned} (\nabla\varphi)_i &= \frac{\partial\varphi}{\partial x_i}, & (\nabla\mathbf{v})_{ij} &= \frac{\partial v_i}{\partial x_j}, \\ \operatorname{div}\mathbf{v} &= \sum_i \frac{\partial v_i}{\partial x_i}, & (\operatorname{div}\mathbf{S})_i &= \sum_j \frac{\partial S_{ij}}{\partial x_j}, \\ \Delta\varphi &= \sum_i \frac{\partial^2\varphi}{\partial x_i^2}, & (\Delta\mathbf{v})_j &= \Delta v_j. \end{aligned} \tag{7}$$

Further, curl $\mathbf{v}$ has components (α, β, γ) with

$$\alpha = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \qquad \beta = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \qquad \gamma = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},$$

and if $\mathbf{W}$ is the skew part of $\nabla\mathbf{v}$, then

$$[\mathbf{W}] = \frac{1}{2}\begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}.$$

The verification of (5) furnishes an excellent example of the use of the above identities. The components of $\nabla\mathbf{v}^T$ are

$$(\nabla\mathbf{v})_{ij}^T = \frac{\partial v_j}{\partial x_i};$$

hence

$$[\operatorname{div}(\nabla \mathbf{v}^{\mathrm{T}})]_i = \sum_j \frac{\partial}{\partial x_j}\frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}\sum_j \frac{\partial v_j}{\partial x_j} = \frac{\partial}{\partial x_i}(\operatorname{div}\mathbf{v}) = [\nabla(\operatorname{div}\mathbf{v})]_i,$$

which implies (5).

We now list some obvious consequences of the chain rule. First let

$$\mathbf{h} = \mathbf{f} \circ \mathbf{g}$$

with **f** and **g** smooth point fields. Then (3.11) takes the form

$$\nabla \mathbf{h}(\mathbf{x}) = \nabla \mathbf{f}(\mathbf{y})\, \nabla \mathbf{g}(\mathbf{x}),$$

where $\mathbf{y} = \mathbf{g}(\mathbf{x})$. On the other hand, for a smooth scalar field φ, a smooth vector field **v**, and a smooth point-valued function **g** of a real variable, (3.12) reads

$$\frac{d}{dt}\varphi(\mathbf{g}(t)) = \nabla\varphi(\mathbf{g}(t)) \cdot \dot{\mathbf{g}}(t),$$

$$\frac{d}{dt}\mathbf{v}(\mathbf{g}(t)) = \nabla\mathbf{v}(\mathbf{g}(t))\, \dot{\mathbf{g}}(t).$$

A **curve c** in $\mathscr{R}$ is a smooth map

$$\mathbf{c}\colon [0, 1] \to \mathscr{R}$$

with $\dot{\mathbf{c}}$ never zero; **c** is **closed** if $\mathbf{c}(0) = \mathbf{c}(1)$; the **length** of **c** is the number

$$\operatorname{length}(\mathbf{c}) = \int_0^1 |\dot{\mathbf{c}}(\sigma)|\, d\sigma$$

(see Fig. 2). Let **v** be a continuous vector field on $\mathscr{R}$. Then the **integral of v around c** is defined by

$$\int_{\mathbf{c}} \mathbf{v}(\mathbf{x}) \cdot d\mathbf{x} = \int_0^1 \mathbf{v}(\mathbf{c}(\sigma)) \cdot \dot{\mathbf{c}}(\sigma)\, d\sigma.$$

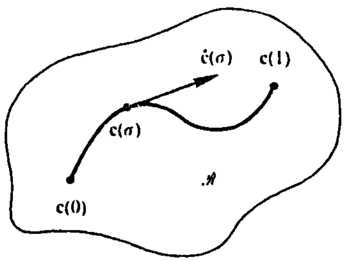

Figure 2

Similarly, for a continuous tensor field **S** on $\mathscr{R}$,

$$\int_{\mathbf{c}} \mathbf{S}(\mathbf{x})\, d\mathbf{x} = \int_0^1 \mathbf{S}(\mathbf{c}(\sigma))\dot{\mathbf{c}}(\sigma)\, d\sigma.$$

Note that if φ is a smooth scalar field on $\mathscr{R}$,

$$\int_{\mathbf{c}} \nabla\varphi(\mathbf{x}) \cdot d\mathbf{x} = \int_0^1 \nabla\varphi(\mathbf{c}(\sigma)) \cdot \dot{\mathbf{c}}(\sigma)\, d\sigma = \int_0^1 \frac{d}{d\sigma}\varphi(\mathbf{c}(\sigma))\, d\sigma$$
$$= \varphi(\mathbf{c}(1)) - \varphi(\mathbf{c}(0));$$

thus

$$\int_{\mathbf{c}} \nabla\varphi(\mathbf{x}) \cdot d\mathbf{x} = 0 \tag{8}$$

whenever **c** is closed.

Given a subset $\mathscr{R}$ of $\mathscr{E}$, we write $\partial\mathscr{R}$ for its **boundary** and $\mathring{\mathscr{R}}$ for its **interior**. We say that $\mathscr{R}$ is **connected** if any two points in $\mathscr{R}$ can be connected by a curve in $\mathscr{R}$; **simply connected** if any closed curve in $\mathscr{R}$ can be continuously deformed to a point without leaving $\mathscr{R}$ (i.e., if given any closed curve **c** in $\mathscr{R}$ there exists a smooth function $\mathbf{f}\colon [0, 1] \times [0, 1] \to \mathscr{R}$ and a point $\mathbf{y} \in \mathscr{R}$ such that, for all $\sigma \in [0, 1]$, $\mathbf{f}(\sigma, 0) = \mathbf{c}(\sigma)$, $\mathbf{f}(\sigma, 1) = \mathbf{y}$, and $\mathbf{f}(0, \sigma) = \mathbf{f}(1, \sigma)$). An **open region** is a connected, open set in $\mathscr{E}$; the closure of an open region is called a **closed region.**

Let $\mathscr{R}$ be a *closed* region. A field Φ is smooth on $\mathscr{R}$ if Φ is smooth on $\mathring{\mathscr{R}}$, and if Φ and $\nabla\Phi$ have continuous extensions to all of $\mathscr{R}$; in this case we also write Φ and $\nabla\Phi$ for the extended functions. An analogous interpretation applies to the statement "Φ is of class C^N on $\mathscr{R}$"; note that (for $\mathscr{R}$ open or closed) this will be true if and only if the components of Φ have continuous partial derivatives of all orders $\leq N$ on $\mathscr{R}$.

A vector field of the form $\mathbf{v} = \nabla\varphi$ satisfies $\operatorname{curl}\mathbf{v} = \mathbf{0}$ (Exercise 5a). Our next theorem, which we state without proof, gives the converse of this result.

Potential Theorem.[1] *Let* **v** *be a smooth point field on an open or closed, simply connected region* $\mathscr{R}$, *and assume that*

$$\operatorname{curl}\mathbf{v} = \mathbf{0}.$$

Then there is a class C^2 *scalar field* φ *on* $\mathscr{R}$ *such that*

$$\mathbf{v} = \nabla\varphi.$$

We close this section by proving that vector fields with constant gradients are affine.

[1] Cf., e.g., Fleming [1, Corollary 2, p. 279].

Proposition. *Let* **f** *be a smooth point- or vector-valued field on an open or closed region* $\mathscr{R}$, *and assume that* $\mathbf{F} = \nabla \mathbf{f}$ *is constant on* $\mathscr{R}$. *Then*

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y}) + \mathbf{F}(\mathbf{x} - \mathbf{y}) \tag{9}$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{R}$.

Proof. Choose $\mathbf{x}, \mathbf{y} \in \mathscr{R}$. Since $\mathscr{R}$ is connected there is a curve **c** in $\mathscr{R}$ from **y** to **x**. Thus

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) = \int_0^1 \frac{d}{d\sigma}\mathbf{f}(\mathbf{c}(\sigma))\,d\sigma = \int_0^1 \nabla\mathbf{f}(\mathbf{c}(\sigma))\dot{\mathbf{c}}(\sigma)\,d\sigma$$

$$= \mathbf{F}\int_0^1 \dot{\mathbf{c}}(\sigma)\,d\sigma = \mathbf{F}(\mathbf{x} - \mathbf{y}). \quad \square$$

By (9) any vector field **f** with *constant* gradient **F** can be written in the form

$$\mathbf{f}(\mathbf{x}) = \mathbf{a} + \mathbf{F}(\mathbf{x} - \mathbf{x}_0)$$

with $\mathbf{a} \in \mathscr{V}$ and $\mathbf{x}_0 \in \mathscr{E}$. Moreover, the point $\mathbf{x}_0$ can be arbitrarily chosen. (Of course, **a** depends on the choice of $\mathbf{x}_0$.)

Exercises

1. Let $\alpha, \varphi, \mathbf{u}, \mathbf{v}, \mathbf{w}$, and **S** be smooth fields with α and φ scalar valued; **u**, **v**, and **w** vector valued; and **S** tensor valued. Establish identities, similar to (2), for
   (a) $\nabla(\alpha\varphi)$,
   (b) $\nabla[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}]$,
   (c) $\operatorname{div}(\varphi\mathbf{S}\mathbf{v})$,
   (d) $\Delta(\mathbf{v}\cdot\mathbf{w})$ (with **v** and **w** of class C^2).
2. Establish the existence and uniqueness of the divergence, div **S**, of a smooth tensor field **S**.
3. Establish the component representations (7).
4. Use (7) to deduce $(2)_4$ and $(2)_5$.
5. Let φ and **v** be class C^2. Show that
   (a) $\operatorname{curl}\nabla\varphi = \mathbf{0}$,
   (b) $\operatorname{div}\operatorname{curl}\mathbf{v} = 0$.

In Exercises 6–8,

$$\mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{o}.$$

6. (a) Show that $\nabla\mathbf{r} = \mathbf{I}$.
   (b) Let $\mathbf{e} = \mathbf{r}/|\mathbf{r}|$. Compute $(\nabla\mathbf{e})\mathbf{e}$.
7. Let $\mathbf{a} \in \mathscr{V}$, $\mathbf{S} \in \mathrm{Lin}$, and define $\varphi: \mathscr{E} \to \mathbb{R}$ by
$$\varphi = \mathbf{a}\cdot(\mathbf{r} \times \mathbf{S}\mathbf{r}).$$
   Compute $\nabla\varphi$.
8. Let **u** be the vector field on $\mathscr{E} - \{\mathbf{o}\}$ defined by
$$\mathbf{u} = \mathbf{r}/|\mathbf{r}|^3.$$
   Show that **u** is harmonic. Find a scalar field whose gradient is **u**.
9. Let **u** be a class C^2 vector field. Show that
   (a) $\operatorname{div}\{(\nabla\mathbf{u})\mathbf{u}\} = \nabla\mathbf{u}\cdot\nabla\mathbf{u}^T + \mathbf{u}\cdot(\nabla\operatorname{div}\mathbf{u})$,
   (b) $\nabla\mathbf{u}\cdot\nabla\mathbf{u}^T = \operatorname{div}\{(\nabla\mathbf{u})\mathbf{u} - (\operatorname{div}\mathbf{u})\mathbf{u}\} + (\operatorname{div}\mathbf{u})^2$.
10. Let **u** and **v** be smooth. Show that
$$\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v}.$$

5. THE DIVERGENCE THEOREM. STOKES' THEOREM

We use the term **regular region** in the sense of Kellogg [1].[1] Roughly speaking, a regular region is a closed region $\mathscr{R}$ with piecewise smooth boundary $\partial\mathscr{R}$. It is important to note that $\mathscr{R}$ may be bounded or unbounded. In the former case we write

$$\operatorname{vol}(\mathscr{R})$$

for the **volume** of $\mathscr{R}$.

Divergence Theorem. *Let* $\mathscr{R}$ *be a bounded regular region, and let* $\varphi: \mathscr{R} \to \mathbb{R}$, $\mathbf{v}: \mathscr{R} \to \mathscr{V}$, *and* $\mathbf{S}: \mathscr{R} \to \mathrm{Lin}$ *be smooth fields. Then*

$$\int_{\partial\mathscr{R}} \varphi\mathbf{n}\,dA = \int_{\mathscr{R}} \nabla\varphi\,dV,$$

$$\int_{\partial\mathscr{R}} \mathbf{v}\cdot\mathbf{n}\,dA = \int_{\mathscr{R}} \operatorname{div}\mathbf{v}\,dV,$$

$$\int_{\partial\mathscr{R}} \mathbf{S}\mathbf{n}\,dA = \int_{\mathscr{R}} \operatorname{div}\mathbf{S}\,dV,$$

where **n** *is the outward unit normal field on* $\partial\mathscr{R}$.

[1] See also Gurtin [1, §5].

Proof. The results concerning φ and $\mathbf{v}$ are classical[1] and will not be verified here. To establish the last relation, let $\mathbf{a}$ be a vector. Then

$$\mathbf{a}\cdot\int_{\partial\mathscr{R}}\mathbf{Sn}\,dA=\int_{\partial\mathscr{R}}\mathbf{a}\cdot\mathbf{Sn}\,dA=\int_{\partial\mathscr{R}}(\mathbf{S}^{\mathrm{T}}\mathbf{a})\cdot\mathbf{n}\,dA$$

$$=\int_{\mathscr{R}}\operatorname{div}(\mathbf{S}^{\mathrm{T}}\mathbf{a})\,dV=\int_{\mathscr{R}}(\operatorname{div}\mathbf{S})\cdot\mathbf{a}\,dV=\mathbf{a}\cdot\int_{\mathscr{R}}\operatorname{div}\mathbf{S}\,dV,$$

which implies the desired result, since $\mathbf{a}$ is arbitrary. ☐

Our reason for defining div $\mathbf{S}$ the way we did should be clear from the foregoing proof.

We will often find it important to deduce local field equations from global formulations of balance laws. This procedure is greatly facilitated by the

Localization Theorem. *Let Φ be a continuous scalar or vector field on an open set $\mathscr{R}$ in $\mathscr{E}$. Then given any $\mathbf{x}_0\in\mathscr{R}$,*

$$\Phi(\mathbf{x}_0)=\lim_{\delta\to 0}\frac{1}{\operatorname{vol}(\Omega_\delta)}\int_{\Omega_\delta}\Phi\,dV,\tag{1}$$

where Ω_δ $(\delta>0)$ is the closed ball of radius δ centered at $\mathbf{x}_0$. Therefore, if

$$\int_\Omega\Phi\,dV=0$$

for every closed ball $\Omega\subset\mathscr{R}$, then

$$\Phi=0.$$

Proof. Let

$$I_\delta=\left|\Phi(\mathbf{x}_0)-\frac{1}{v_\delta}\int_{\Omega_\delta}\Phi\,dV\right|,$$

where $v_\delta=\operatorname{vol}(\Omega_\delta)$. Then

$$I_\delta\le\frac{1}{v_\delta}\int_{\Omega_\delta}|\Phi(\mathbf{x}_0)-\Phi(\mathbf{x})|\,dV_{\mathbf{x}}\le\sup_{\mathbf{x}\in\Omega_\delta}|\Phi(\mathbf{x}_0)-\Phi(\mathbf{x})|$$

which tends to zero as $\delta\to 0$, since Φ is continuous. ☐

[1] See, e.g., Kellogg [1, Chapter 4].

Note that the localization theorem and the divergence theorem together yield the following interesting interpretation of the divergence:

$$\operatorname{div}\mathbf{v}(\mathbf{x})=\lim_{\delta\to 0}\frac{1}{\operatorname{vol}(\Omega_\delta)}\int_{\partial\Omega_\delta}\mathbf{v}\cdot\mathbf{n}\,dA,$$

$$\operatorname{div}\mathbf{S}(\mathbf{x})=\lim_{\delta\to 0}\frac{1}{\operatorname{vol}(\Omega_\delta)}\int_{\partial\Omega_\delta}\mathbf{Sn}\,dA.$$

We will make use of Stokes' theorem, but only in the following very special form.

Stokes' Theorem.[1] *Let $\mathbf{v}$ be a smooth vector field on an open set $\mathscr{R}$ in $\mathscr{E}$. Further, let Ω be a disc in $\mathscr{R}$ (Fig. 3), let $\mathbf{n}$ be a unit normal to Ω, and let the bounding circle $\mathbf{c}(\sigma)$, $0\le\sigma\le 1$, be oriented so that*

$$[\dot{\mathbf{c}}(0)\times\dot{\mathbf{c}}(\sigma)]\cdot\mathbf{n}>0,\qquad 0<\sigma<1.$$

Then

$$\int_\Omega(\operatorname{curl}\mathbf{v})\cdot\mathbf{n}\,dA=\int_{\mathbf{c}}\mathbf{v}\cdot d\mathbf{x}.$$

The integral

$$\int_{\mathbf{c}}\mathbf{v}\cdot d\mathbf{x}$$

represents the circulation of $\mathbf{v}$ around $\mathbf{c}$; it sums the tangential component of $\mathbf{v}$ around the curve $\mathbf{c}$ and can be used to give a physical interpretation of curl $\mathbf{v}$. Indeed, if we let $\mathbf{x}_0$ denote the center and δ the radius of Ω, and write $\Omega=\Omega_\delta$, $\mathbf{c}=\mathbf{c}_\delta$, then by Stokes' theorem and an argument similar to that used to derive (1),

$$\mathbf{n}\cdot\operatorname{curl}\mathbf{v}(\mathbf{x}_0)=\lim_{\delta\to 0}\frac{\int_{\mathbf{c}_\delta}\mathbf{v}\cdot d\mathbf{x}}{A(\Omega_\delta)},$$

where $A(\Omega_\delta)$ is the area of Ω_δ. Considering different choices for $\mathbf{n}$, we conclude that curl $\mathbf{v}$ lies perpendicular to the plane in which the circulation per unit

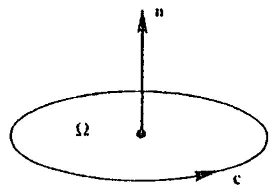

Figure 3

[1] See, e.g., Kellogg [1, Chapter 4, §4].

area is greatest, and that the magnitude of curl $\mathbf{v}$ is the circulation per unit area in that plane.

Exercises

1. Let $\mathscr{R}$ be a bounded regular region, and let $\mathbf{v}, \mathbf{w}: \mathscr{R} \to \mathscr{V}$ and $\mathbf{S}: \mathscr{R} \to \text{Lin}$ be smooth. Show that:
   (a) $\int_{\partial\mathscr{R}} \mathbf{v} \otimes \mathbf{n}\, dA = \int_{\mathscr{R}} \nabla\mathbf{v}\, dV$,
   (b) $\int_{\partial\mathscr{R}} (\mathbf{Sn}) \otimes \mathbf{v}\, dA = \int_{\mathscr{R}} [(\text{div}\, \mathbf{S}) \otimes \mathbf{v} + \mathbf{S}\nabla\mathbf{v}^T]\, dV$,
   (c) $\int_{\partial\mathscr{R}} \mathbf{v} \cdot \mathbf{Sn}\, dA = \int_{\mathscr{R}} (\mathbf{v} \cdot \text{div}\, \mathbf{S} + \mathbf{S} \cdot \nabla\mathbf{v})\, dV$, FORMULA DI GREEN
   (d) $\int_{\partial\mathscr{R}} \mathbf{v}(\mathbf{w} \cdot \mathbf{n})\, dA = \int_{\mathscr{R}} [\mathbf{v}\, \text{div}\, \mathbf{w} + (\nabla\mathbf{v})\mathbf{w}]\, dV$.
2. Let $\mathbf{v}$ be a smooth vector field on an open region $\mathscr{R}$. Show that

$$\int_{\partial\mathscr{P}} \mathbf{v} \cdot \mathbf{n}\, dA = 0$$

   for every regular region $\mathscr{P} \subset \mathscr{R}$ if and only if $\text{div}\, \mathbf{v} = 0$.

Selected References

Bartle [1, Chapter 7].
Chadwick [1, Chapter 1].
Dieudonné [1, Chapters 8, 10].
Gurtin [1, §§5, 6].
Kellogg [1, Chapter 4].
Nickerson, Spencer, and Steenrod [1, Chapters 6–8, 10].
Truesdell and Toupin [1, Appendix by J. L. Ericksen].

CHAPTER

III

Kinematics

6. BODIES. DEFORMATIONS. STRAIN

Bodies have one distinct physical property: they occupy regions of euclidean space $\mathscr{E}$. Although a given body will occupy different regions at different times, and no one of these regions can be intrinsically associated with the body, we will find it convenient to choose one such region, $\mathscr{B}$ say, as reference, and to identify points of the body with their positions in $\mathscr{B}$. Formally, then, a **body** $\mathscr{B}$ is a (possibly unbounded) regular region in $\mathscr{E}$. We will sometimes refer to $\mathscr{B}$ as the **reference configuration.** Points $\mathbf{p} \in \mathscr{B}$ are called **material points**; bounded regular subregions of $\mathscr{B}$ are called **parts.**

Continuum mechanics is, for the most part, a study of *deforming* bodies. Mathematically, a body is deformed via a mapping $\mathbf{f}$ that carries each material point $\mathbf{p}$ into a point

$$\mathbf{x} = \mathbf{f}(\mathbf{p}).$$

The requirement that the body not penetrate itself is expressed by the assumption that $\mathbf{f}$ be one-to-one. As we shall see later, $\det \nabla\mathbf{f}$ represents, locally, the volume after deformation per unit original volume; it is therefore reasonable to assume that $\det \nabla\mathbf{f} \neq 0$. Further, a deformation with $\det \nabla\mathbf{f} < 0$ cannot be reached by a continuous process starting in the reference configuration; that is, by a continuous one-parameter family $\mathbf{f}_\sigma$ $(0 \le \sigma \le 1)$

of deformations with $\mathbf{f}_0$ the identity, $\mathbf{f}_1 = \mathbf{f}$, and det $\nabla\mathbf{f}_\sigma$ never zero. Indeed, since det $\nabla\mathbf{f}_\sigma$ is strictly positive at $\sigma = 0$, it must be strictly positive for all σ. We therefore require that

$$\det \nabla\mathbf{f} > 0. \tag{1}$$

The above discussion should motivate the following definition. By a **deformation** of $\mathscr{B}$ we mean a smooth, one-to-one mapping $\mathbf{f}$ which maps $\mathscr{B}$ onto a closed region in $\mathscr{E}$, and which satisfies (1). The vector

$$\mathbf{u}(\mathbf{p}) = \mathbf{f}(\mathbf{p}) - \mathbf{p} \tag{2}$$

represents the **displacement** of $\mathbf{p}$ (Fig. 1). When $\mathbf{u}$ is a constant, $\mathbf{f}$ is a **translation**; in this case

$$\mathbf{f}(\mathbf{p}) = \mathbf{p} + \mathbf{u}.$$

The tensor

$$\mathbf{F}(\mathbf{p}) = \nabla\mathbf{f}(\mathbf{p}) \tag{3}$$

is called the **deformation gradient** and by (1) belongs to Lin$^+$. A deformation with $\mathbf{F}$ *constant* is **homogeneous.** In view of (4.9), every homogeneous deformation admits the representation

$$\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{q}) + \mathbf{F}(\mathbf{p} - \mathbf{q}) \tag{4}$$

for all $\mathbf{p}, \mathbf{q} \in \mathscr{B}$, and conversely, a point field $\mathbf{f}$ on $\mathscr{B}$ that satisfies (4) with $\mathbf{F} \in$ Lin$^+$ is a homogeneous deformation.

For any given value of $\mathbf{q}$ the right side of (4) is well defined for all $\mathbf{p} \in \mathscr{E}$. Thus any homogeneous deformation of $\mathscr{B}$ can be extended to form a homogeneous deformation of $\mathscr{E}$. We therefore consider homogeneous deformations as defined on all of $\mathscr{E}$.

For future use we note the following *properties of homogeneous deformations*:

(i) Given a point $\mathbf{q}$ and a tensor $\mathbf{F} \in$ Lin$^+$, there is exactly one homogeneous deformation $\mathbf{f}$ with $\nabla\mathbf{f} = \mathbf{F}$ and $\mathbf{q}$ **fixed** [i.e., $\mathbf{f}(\mathbf{q}) = \mathbf{q}$].

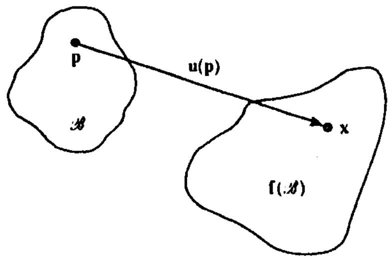

Figure 1

(ii) If $\mathbf{f}$ and $\mathbf{g}$ are homogeneous deformations, then so also is $\mathbf{f} \circ \mathbf{g}$ and

$$\nabla(\mathbf{f} \circ \mathbf{g}) = (\nabla\mathbf{f})(\nabla\mathbf{g}).$$

Moreover, if $\mathbf{f}$ and $\mathbf{g}$ have $\mathbf{q}$ fixed, then so does $\mathbf{f} \circ \mathbf{g}$.

Proposition. *Let* $\mathbf{f}$ *be a homogeneous deformation. Then given any point* $\mathbf{q}$ *we can decompose* $\mathbf{f}$ *as follows:*

$$\mathbf{f} = \mathbf{d}_1 \circ \mathbf{g} = \mathbf{g} \circ \mathbf{d}_2,$$

where $\mathbf{g}$ *is a homogeneous deformation with* $\mathbf{q}$ *fixed, while* $\mathbf{d}_1$ *and* $\mathbf{d}_2$ *are translations. Further, each of these decompositions is unique.*

Proof. (Uniqueness) Assume that the first decomposition $\mathbf{f} = \mathbf{d}_1 \circ \mathbf{g}$ holds. Then $\nabla\mathbf{f} = (\nabla\mathbf{d}_1)(\nabla\mathbf{g})$ and $\nabla\mathbf{d}_1 = \mathbf{I}$ (because $\mathbf{d}_1$ is a translation), so that $\nabla\mathbf{f} = \nabla\mathbf{g}$. Therefore, by property (i) above, $\mathbf{g}$ is uniquely determined. Moreover, since $\mathbf{d}_1 = \mathbf{f} \circ \mathbf{g}^{-1}$, this implies that $\mathbf{d}_1$ is uniquely determined. That $\mathbf{f} = \mathbf{g} \circ \mathbf{d}_2$ is also unique has an analogous proof.

(Existence) By hypothesis,

$$\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{q}) + \mathbf{F}(\mathbf{p} - \mathbf{q}).$$

Since $\mathbf{g}$ must fix $\mathbf{q}$ and have $\nabla\mathbf{g} = \nabla\mathbf{f}\ (= \mathbf{F})$ (cf. the previous paragraph),

$$\mathbf{g}(\mathbf{p}) = \mathbf{q} + \mathbf{F}(\mathbf{p} - \mathbf{q}).$$

Define

$$\mathbf{d}_1 = \mathbf{f} \circ \mathbf{g}^{-1}, \qquad \mathbf{d}_2 = \mathbf{g}^{-1} \circ \mathbf{f}.$$

To complete the proof we must show that $\mathbf{d}_1$ and $\mathbf{d}_2$ are translations. Let

$$\mathbf{u}_0 = \mathbf{f}(\mathbf{q}) - \mathbf{q}.$$

Then, since

$$\mathbf{g}^{-1}(\mathbf{p}) = \mathbf{q} + \mathbf{F}^{-1}(\mathbf{p} - \mathbf{q}),$$

we have

$$\mathbf{d}_1(\mathbf{p}) = \mathbf{f}(\mathbf{q}) + \mathbf{F}(\mathbf{q} + \mathbf{F}^{-1}(\mathbf{p} - \mathbf{q}) - \mathbf{q}) = \mathbf{p} + \mathbf{u}_0,$$
$$\mathbf{d}_2(\mathbf{p}) = \mathbf{q} + \mathbf{F}^{-1}(\mathbf{f}(\mathbf{q}) + \mathbf{F}(\mathbf{p} - \mathbf{q}) - \mathbf{q}) = \mathbf{p} + \mathbf{F}^{-1}\mathbf{u}_0. \quad \square$$

The last proposition allows us to concentrate on homogeneous deformations with a point fixed. An important example of this type of deformation is a **rotation** about $\mathbf{q}$:

$$\mathbf{f}(\mathbf{p}) = \mathbf{q} + \mathbf{R}(\mathbf{p} - \mathbf{q})$$

with $\mathbf{R}$ a rotation.

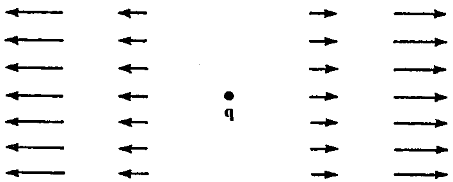

Figure 2

A second example is a **stretch** from **q**, for which

$$\mathbf{f}(\mathbf{p}) = \mathbf{q} + \mathbf{U}(\mathbf{p} - \mathbf{q})$$

with **U** symmetric and positive definite. If, in particular,

$$\mathbf{U} = \mathbf{I} + (\lambda - 1)\mathbf{e} \otimes \mathbf{e}$$

with $\lambda > 0$ and $|\mathbf{e}| = 1$, then **f** is an **extension** of amount λ in the direction **e**. Here the matrix of **U** relative to a coordinate frame with $\mathbf{e} = \mathbf{e}_1$ has the simple form

$$[\mathbf{U}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the corresponding displacement, shown in Fig. 2, has components $(u_1, 0, 0)$ with

$$u_1(\mathbf{p}) = (\lambda - 1)(p_1 - q_1).$$

Properties (i) and (ii) of homogeneous deformations, when used in conjunction with the polar decomposition theorem, yield the following

Proposition. *Let* **f** *be a homogeneous deformation with* **q** *fixed. Then* **f** *admits the decompositions*

$$\mathbf{f} = \mathbf{g} \circ \mathbf{s}_1 = \mathbf{s}_2 \circ \mathbf{g},$$

where **g** *is a rotation about* **q**, *while* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *are stretches from* **q**. *Further, each of these decompositions is unique. In fact, if* $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ *is the polar decomposition of* $\mathbf{F} = \nabla\mathbf{f}$, *then*

$$\nabla\mathbf{g} = \mathbf{R}, \qquad \nabla\mathbf{s}_1 = \mathbf{U}, \qquad \nabla\mathbf{s}_2 = \mathbf{V}.$$

Thus any homogeneous deformation (with a fixed point) can be decomposed into a stretch followed by a rotation, or into a rotation followed by a stretch. The next theorem yields a further decomposition of either of these stretches into a succession of three mutually orthogonal extensions.

Proposition. *Every stretch* **f** *from* **q** *can be decomposed into a succession of three extensions from* **q** *in mutually orthogonal directions. The amounts and directions of the extensions are eigenvalues and eigenvectors of* $\mathbf{U} = \nabla\mathbf{f}$, *and the extensions may be performed in any order.*

Proof. Since $\mathbf{U} \in$ Psym, we conclude from the spectral theorem that

$$\mathbf{U} = \sum_i \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$$

with $\{\mathbf{e}_i\}$ an orthonormal basis of eigenvectors and $\lambda_i > 0$ the eigenvalue associated with $\mathbf{e}_i$ ($\lambda_i > 0$ since **U** is positive definite). In view of the identities $(1.2)_{3,4}$,

$$\mathbf{U} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3,$$

where

$$\mathbf{U}_i = \mathbf{I} + (\lambda_i - 1)\mathbf{e}_i \otimes \mathbf{e}_i.$$

Let $\mathbf{f}_i$ $(i = 1, 2, 3)$ be the extension from **q** of amount λ_i in the direction $\mathbf{e}_i$. By property (ii) of homogeneous deformations, $\mathbf{f}_1 \circ \mathbf{f}_2 \circ \mathbf{f}_3$ is a homogeneous deformation with **q** fixed and deformation gradient $\mathbf{U}_1\mathbf{U}_2\mathbf{U}_3 = \mathbf{U}$. But **f** also has **q** fixed and $\nabla\mathbf{f} = \mathbf{U}$. Thus by property (i) of homogeneous deformations,

$$\mathbf{f} = \mathbf{f}_1 \circ \mathbf{f}_2 \circ \mathbf{f}_3.$$

As a matter of fact, it is clear that

$$\mathbf{f} = \mathbf{f}_{\sigma(1)} \circ \mathbf{f}_{\sigma(2)} \circ \mathbf{f}_{\sigma(3)}$$

for any permutation σ of $\{1, 2, 3\}$. □

In view of the last proposition every stretch can be decomposed into a succession of extensions, the amounts of the extensions being the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of **U**. For this reason we refer to the λ_i as **principal stretches.** Note that, since the stretch tensors **U** and **V** have the same spectrum, the stretch $\mathbf{s}_1$ (of the proposition on p. 44) has the same principal stretches as the stretch $\mathbf{s}_2$. Note also that by (2.10) the principal invariants of **U** take the form

$$\iota_1(\mathbf{U}) = \lambda_1 + \lambda_2 + \lambda_3,$$
$$\iota_2(\mathbf{U}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1,$$
$$\iota_3(\mathbf{U}) = \lambda_1\lambda_2\lambda_3.$$

We now turn to a study of general deformations of $\mathscr{B}$. To avoid repeated hypotheses we assume for the remainder of the section that **f** is a deformation of $\mathscr{B}$. Since **f** is one-to-one its inverse $\mathbf{f}^{-1}: \mathbf{f}(\mathscr{B}) \to \mathscr{B}$ exists. Moreover, by (1),

$\nabla\mathbf{f}(\mathbf{p})$ is invertible at each point $\mathbf{p}$ of $\mathscr{B}$, and we conclude from the smooth-inverse theorem (page 22) that $\mathbf{f}^{-1}$ is a smooth map. Two other important properties of $\mathbf{f}$ are

$$\mathbf{f}(\mathring{\mathscr{B}}) = \mathbf{f}(\mathscr{B})^{\circ}, \qquad \mathbf{f}(\partial\mathscr{B}) = \partial\mathbf{f}(\mathscr{B}), \tag{5}$$

where $\mathbf{f}(\mathscr{B})^{\circ}$ denotes the interior of $\mathbf{f}(\mathscr{B})$. We leave the proof of (5) as an exercise.

The concept of strain is most easily introduced by expanding the deformation $\mathbf{f}$ about an arbitrary point $\mathbf{q} \in \mathscr{B}$:

$$\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{q}) + \mathbf{F}(\mathbf{q})(\mathbf{p} - \mathbf{q}) + o(\mathbf{p} - \mathbf{q}),$$

where $\mathbf{F}$ is the deformation gradient (3). Thus *in a neighborhood of a point* $\mathbf{q}$ *and to within an error of* $o(\mathbf{p} - \mathbf{q})$ *a deformation behaves like a homogeneous deformation.* This motivates the following terminology: let

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

be the pointwise polar decomposition of $\mathbf{F}$; then $\mathbf{R}$ is the **rotation tensor**, $\mathbf{U}$ the **right stretch tensor**, and $\mathbf{V}$ the **left stretch tensor** for the deformation $\mathbf{f}$. $\mathbf{R}(\mathbf{p})$ measures the local rigid rotation of points near $\mathbf{p}$, while $\mathbf{U}(\mathbf{p})$ and $\mathbf{V}(\mathbf{p})$ measure local stretching from $\mathbf{p}$. Since $\mathbf{U}$ and $\mathbf{V}$ involve the square roots of $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$, their computation is often difficult. For this reason we introduce the **right and left Cauchy–Green strain tensors, C and B,** defined by

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \qquad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T; \tag{6}$$

in components,

$$C_{ij} = \sum_m \frac{\partial f_m}{\partial p_i}\frac{\partial f_m}{\partial p_j}, \qquad B_{ij} = \sum_m \frac{\partial f_i}{\partial p_m}\frac{\partial f_j}{\partial p_m}.$$

Note that

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T, \qquad \mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T; \tag{7}$$

thus, since $\mathbf{R}$ is a rotation,

$$\mathscr{I}_{\mathbf{V}} = \mathscr{I}_{\mathbf{U}}, \qquad \mathscr{I}_{\mathbf{B}} = \mathscr{I}_{\mathbf{C}}$$

(cf. (2.10) and Exercise 2.3).

Recall that the angle θ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \qquad (0 \le \theta \le \pi).$$

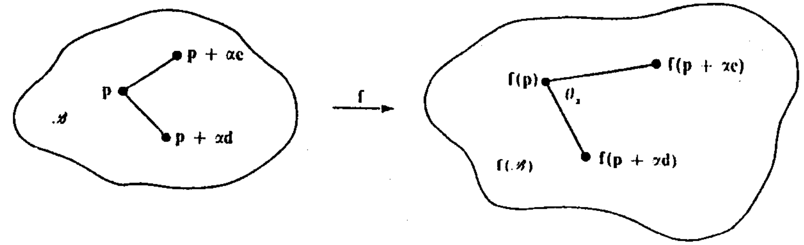

Figure 3

Proposition. *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be unit vectors and let* $\mathbf{p} \in \mathring{\mathscr{B}}$. *Then as* $\alpha \to 0$,

$$\frac{|\mathbf{f}(\mathbf{p} + \alpha\mathbf{e}) - \mathbf{f}(\mathbf{p})|}{|\alpha|} \to |\mathbf{U}(\mathbf{p})\mathbf{e}|, \tag{8}$$

and the angle between $\mathbf{f}(\mathbf{p} + \alpha\mathbf{d}) - \mathbf{f}(\mathbf{p})$ *and* $\mathbf{f}(\mathbf{p} + \alpha\mathbf{e}) - \mathbf{f}(\mathbf{p})$ *tends to the angle between* $\mathbf{U}(\mathbf{p})\mathbf{d}$ *and* $\mathbf{U}(\mathbf{p})\mathbf{e}$ (*Fig.* 3).

Proof. For convenience, we write $\mathbf{F}$ for $\mathbf{F}(\mathbf{p})$ and $\mathbf{U}$ for $\mathbf{U}(\mathbf{p})$. Then given any vector $\mathbf{u}$,

$$\mathbf{f}(\mathbf{p} + \alpha\mathbf{u}) = \mathbf{f}(\mathbf{p}) + \alpha\mathbf{F}\mathbf{u} + o(\alpha)$$

as $\alpha \to 0$. Let

$$\mathbf{d}_\alpha = \mathbf{f}(\mathbf{p} + \alpha\mathbf{d}) - \mathbf{f}(\mathbf{p}), \qquad \mathbf{e}_\alpha = \mathbf{f}(\mathbf{p} + \alpha\mathbf{e}) - \mathbf{f}(\mathbf{p}).$$

Then

$$\mathbf{d}_\alpha = \alpha\mathbf{F}\mathbf{d} + o(\alpha), \qquad \mathbf{e}_\alpha = \alpha\mathbf{F}\mathbf{e} + o(\alpha),$$

and

$$\frac{\mathbf{d}_\alpha\cdot\mathbf{e}_\alpha}{\alpha^2} \to \mathbf{F}\mathbf{d}\cdot\mathbf{F}\mathbf{e} = \mathbf{R}\mathbf{U}\mathbf{d}\cdot\mathbf{R}\mathbf{U}\mathbf{e} = \mathbf{U}\mathbf{d}\cdot\mathbf{U}\mathbf{e},$$

since the rotation tensor $\mathbf{R}$ is orthogonal. Taking $\mathbf{d} = \mathbf{e}$ leads us to (8).

Next, let θ_α designate the angle between $\mathbf{d}_\alpha$ and $\mathbf{e}_\alpha$. Then

$$\cos\theta_\alpha = \frac{\mathbf{d}_\alpha\cdot\mathbf{e}_\alpha}{|\mathbf{d}_\alpha||\mathbf{e}_\alpha|} = \frac{\mathbf{d}_\alpha\cdot\mathbf{e}_\alpha}{\alpha^2}\cdot\frac{\alpha}{|\mathbf{d}_\alpha|}\cdot\frac{\alpha}{|\mathbf{e}_\alpha|} \to \frac{\mathbf{U}\mathbf{d}\cdot\mathbf{U}\mathbf{e}}{|\mathbf{U}\mathbf{d}||\mathbf{U}\mathbf{e}|},$$

which is the cosine of the angle between $\mathbf{U}\mathbf{d}$ and $\mathbf{U}\mathbf{e}$. (Note that since $\mathbf{U}$ is invertible, $\mathbf{U}\mathbf{d}, \mathbf{U}\mathbf{e} \neq \mathbf{0}$, and since $\mathbf{f}$ is one-to-one, $\mathbf{d}_\alpha, \mathbf{e}_\alpha \neq \mathbf{0}$ for $\alpha \neq 0$. Hence the last computation makes sense.) Finally, since the cosine has a continuous inverse on $[0, \pi]$, θ_α tends to the angle between $\mathbf{U}\mathbf{d}$ and $\mathbf{U}\mathbf{e}$. □

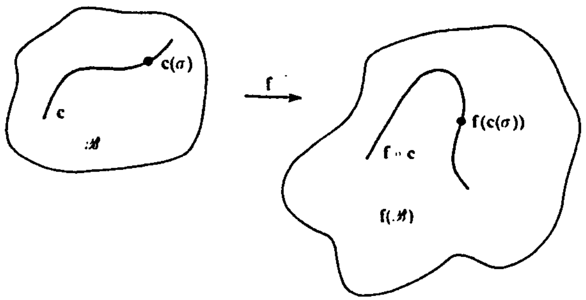

Figure 4

The above proposition shows that the stretch tensor **U** measures local distance and angle changes under the deformation. In particular, since $|\alpha|$ is the distance between **p** and $\mathbf{q} = \mathbf{p} + \alpha\mathbf{e}$, (8) asserts that to within an error that vanishes as α approaches zero, $|\mathbf{U}(\mathbf{p})\mathbf{e}|$ is the distance between **p** and **q** after the deformation per unit original distance. The next result shows that **U** also determines the deformed length of curves in $\mathscr{B}$. In this regard, note that given any curve **c** in $\mathscr{B}$, $\mathbf{f} \circ \mathbf{c}$ is the deformed curve $\mathbf{f}(\mathbf{c}(\sigma))$, $0 \le \sigma \le 1$ (Fig. 4).

Proposition. *Given any curve* **c** *in* $\mathscr{B}$,

$$\text{length}(\mathbf{f} \circ \mathbf{c}) = \int_0^1 |\mathbf{U}(\mathbf{c}(\sigma))\dot{\mathbf{c}}(\sigma)|\, d\sigma. \tag{9}$$

Proof. By definition,

$$\text{length}(\mathbf{f} \circ \mathbf{c}) = \int_0^1 \left|\frac{d}{d\sigma}\mathbf{f}(\mathbf{c}(\sigma))\right| d\sigma.$$

But by the chain rule,

$$\frac{d}{d\sigma}\mathbf{f}(\mathbf{c}(\sigma)) = \mathbf{F}(\mathbf{c}(\sigma))\dot{\mathbf{c}}(\sigma) = \mathbf{R}(\mathbf{c}(\sigma))\mathbf{U}(\mathbf{c}(\sigma))\dot{\mathbf{c}}(\sigma),$$

where $\mathbf{R}(\mathbf{p})$ is the rotation tensor. Thus, since **R** is orthogonal,

$$\left|\frac{d}{d\sigma}\mathbf{f}(\mathbf{c}(\sigma))\right| = |\mathbf{U}(\mathbf{c}(\sigma))\dot{\mathbf{c}}(\sigma)|. \quad \square$$

A deformation that preserves distance is said to be rigid. More precisely, **f** is **rigid** if

$$|\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})| = |\mathbf{p} - \mathbf{q}| \tag{10}$$

for all $\mathbf{p}, \mathbf{q} \in \mathscr{B}$. This condition imposes severe restrictions; indeed, as the next theorem shows, a deformation **f** is rigid if and only if: (i) **f** is homogeneous and (ii) $\nabla\mathbf{f}$ is a rotation.

Theorem (Characterization of rigid deformations). *The following are equivalent:*

(a) **f** *is a rigid deformation.*
(b) **f** *admits a representation of the form*

$$\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{q}) + \mathbf{R}(\mathbf{p} - \mathbf{q})$$

for all $\mathbf{p}, \mathbf{q} \in \mathscr{B}$ *with* **R** *a rotation.*

(c) $\mathbf{F}(\mathbf{p})$ *is a rotation for each* $\mathbf{p} \in \mathscr{B}$.
(d) $\mathbf{U}(\mathbf{p}) = \mathbf{I}$ *for each* $\mathbf{p} \in \mathscr{B}$.
(e) *For any curve* **c** *in* $\mathscr{B}$, $\text{length}(\mathbf{c}) = \text{length}(\mathbf{f} \circ \mathbf{c})$.

Proof. We will show that (a) ⇔ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (b).

(a) ⇔ (b). Let **f** be a rigid deformation. If we use $(4.2)_3$ and (4.3) to differentiate

$$[\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})] \cdot [\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})] = (\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p} - \mathbf{q}),$$

first with respect to **q** and then with respect to **p**, we find that

$$\begin{aligned} \nabla\mathbf{f}(\mathbf{q})^T[\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})] &= \mathbf{p} - \mathbf{q}, \\ \nabla\mathbf{f}(\mathbf{q})^T\nabla\mathbf{f}(\mathbf{p}) &= \mathbf{I}. \end{aligned} \tag{11}$$

Taking $\mathbf{q} = \mathbf{p}$ in $(11)_2$ we see that $\nabla\mathbf{f}(\mathbf{p})$ is orthogonal at each **p**; hence $(11)_2$ implies that

$$\nabla\mathbf{f}(\mathbf{p}) = \nabla\mathbf{f}(\mathbf{q})$$

for all **p** and **q**, so that $\nabla\mathbf{f}$ is constant. Finally, since $\det\nabla\mathbf{f} > 0$, $\nabla\mathbf{f}$ is a rotation. Thus **f** is a homogeneous deformation with $\mathbf{R} = \nabla\mathbf{f}$ a rotation. Conversely, assume that (b) holds. Then, since **R** is orthogonal,

$$[\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})] \cdot [\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})] = \mathbf{R}(\mathbf{p} - \mathbf{q}) \cdot \mathbf{R}(\mathbf{p} - \mathbf{q}) = (\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p} - \mathbf{q})$$

and (a) follows.

(b) ⇒ (c) ⇒ (d). If (b) holds, then $\nabla\mathbf{f} = \mathbf{R}$, so that (c) is satisfied. Assume next that $\mathbf{F}(\mathbf{p})$ is a rotation. Then $\mathbf{C}(\mathbf{p}) = \mathbf{F}(\mathbf{p})^T\mathbf{F}(\mathbf{p}) = \mathbf{I}$. But $\mathbf{U}(\mathbf{p})^2 = \mathbf{C}(\mathbf{p})$, and by the square-root theorem (page 13) the tensor $\mathbf{U}(\mathbf{p}) \in \text{Psym}$ which satisfies this equation is unique; hence $\mathbf{U}(\mathbf{p}) = \mathbf{I}$.

(d) ⇒ (e). This is an immediate consequence of (9).

(e) ⇒ (b). This is the most delicate portion of the proof. Assume that (e) holds. It clearly suffices to show that

$$\mathbf{F} \quad \text{is a constant rotation.} \tag{12}$$

Thus choose $\mathbf{p}_0 \in \mathring{\mathscr{B}}$ and let Ω be an open ball centered at $\mathbf{p}_0$ and sufficiently small that $\mathbf{f}(\Omega)$ is contained in an open ball Γ in $\mathbf{f}(\mathscr{B})$ (cf. the discussion at the end of the proof). Let $\mathbf{p}, \mathbf{q} \in \Omega$ $(\mathbf{p} \neq \mathbf{q})$, and let $\mathbf{c}$ be the *straight line* from $\mathbf{q}$ to $\mathbf{p}$:

$$\mathbf{c}(\sigma) = \mathbf{q} + \sigma(\mathbf{p} - \mathbf{q}), \qquad 0 \leq \sigma \leq 1.$$

Then, trivially,

$$|\mathbf{p} - \mathbf{q}| = \text{length}(\mathbf{c}),$$

and since the end points of $\mathbf{f} \circ \mathbf{c}$ are $\mathbf{f}(\mathbf{q})$ and $\mathbf{f}(\mathbf{p})$,

$$\text{length}(\mathbf{f} \circ \mathbf{c}) \geq |\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})|;$$

hence (c) implies that

$$|\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})| \leq |\mathbf{p} - \mathbf{q}|. \tag{13}$$

Next, since $\mathbf{f}(\mathbf{q})$ and $\mathbf{f}(\mathbf{p})$ lie in the open ball $\Gamma \subset \mathbf{f}(\mathscr{B})$, the *straight line* $\mathbf{h}$ from $\mathbf{f}(\mathbf{q})$ to $\mathbf{f}(\mathbf{p})$ lies in $\mathbf{f}(\mathscr{B})$. Consider the curve $\mathbf{c}$ in $\mathscr{B}$ that maps into $\mathbf{h}$:

$$\mathbf{c}(\sigma) = \mathbf{f}^{-1}(\mathbf{h}(\sigma)), \qquad 0 \leq \sigma \leq 1.$$

Then the argument used to derive (13) now yields the opposite inequality

$$|\mathbf{f}(\mathbf{p}) - \mathbf{f}(\mathbf{q})| \geq |\mathbf{p} - \mathbf{q}|.$$

Thus (10) holds for all $\mathbf{p}, \mathbf{q} \in \Omega$, and $\mathbf{f}$ restricted to Ω is a rigid deformation. The argument given previously [in the proof of the assertion (a) ⇒ (b)] therefore tells us that (12) holds on Ω.

We have shown that (12) holds in some neighborhood of each point of $\mathring{\mathscr{B}}$. Thus, in particular, the derivative of $\mathbf{F}$ exists and is zero on $\mathring{\mathscr{B}}$; since $\mathscr{B}$ is connected, this means that $\mathbf{F}$ is constant on $\mathscr{B}$. Thus (12) holds on $\mathscr{B}$. □

We remark that $\mathbf{U}$ in (d) can be replaced by $\mathbf{C}$, $\mathbf{V}$, or $\mathbf{B}$ without impairing the validity of the theorem.

We now construct the open ball Ω used in the above proof. By $(5)_1$, $\mathbf{f}(\mathscr{B})$ contains an open ball Γ centered at $\mathbf{f}(\mathbf{p}_0)$. Moreover, since $\mathbf{f}$ is continuous, $\mathbf{f}^{-1}(\Gamma)$ is an open neighborhood of $\mathbf{p}_0$ and hence contains an open ball Ω centered at $\mathbf{p}_0$. Trivially, $\mathbf{f}(\Omega) \subset \Gamma$, so that Ω has the requisite properties.

It follows from (b) of the last theorem that every rigid deformation is a translation followed by a rotation or a rotation followed by a translation, and conversely; thus (as a consequence of the first two propositions of this section) every homogeneous deformation can be expressed as a rigid deformation followed by a stretch, or as a stretch followed by a rigid deformation.

It is often important to convert integrals over $\mathbf{f}(\mathscr{P})$ to integrals over $\mathscr{P}$. The next proposition, which we state without proof, gives the corresponding transformation law.[1]

Proposition. *Let $\mathbf{f}$ be a deformation of $\mathscr{B}$, and let φ be a continuous scalar field on $\mathbf{f}(\mathscr{B})$. Then given any part $\mathscr{P}$ of $\mathscr{B}$,*

$$\begin{aligned} \int_{\mathbf{f}(\mathscr{P})} \varphi(\mathbf{x})\, dV_x &= \int_{\mathscr{P}} \varphi(\mathbf{f}(\mathbf{p})) \det \mathbf{F}(\mathbf{p})\, dV_p, \\ \int_{\partial \mathbf{f}(\mathscr{P})} \varphi(\mathbf{x})\mathbf{m}(\mathbf{x})\, dA_x &= \int_{\partial\mathscr{P}} \varphi(\mathbf{f}(\mathbf{p}))\mathbf{G}(\mathbf{p})\mathbf{n}(\mathbf{p})\, dA_p, \end{aligned} \tag{14}$$

where

$$\mathbf{G} = (\det \mathbf{F})\mathbf{F}^{-T},$$

while $\mathbf{m}$ *and* $\mathbf{n}$, *respectively, are the outward unit normal fields on* $\partial\mathbf{f}(\mathscr{P})$ *and* $\partial\mathscr{P}$.

Given a part $\mathscr{P}$,

$$\text{vol}(\mathbf{f}(\mathscr{P})) = \int_{\mathbf{f}(\mathscr{P})} dV$$

represents the volume of $\mathscr{P}$ after it is deformed under $\mathbf{f}$. In view of $(14)_1$,

$$\text{vol}(\mathbf{f}(\mathscr{P})) = \int_{\mathscr{P}} \det \mathbf{F}\, dV, \tag{15}$$

and therefore, by the localization theorem (5.1),

$$\det \mathbf{F}(p) = \lim_{\delta \to 0} \frac{\text{vol}(\mathbf{f}(\Omega_\delta))}{\text{vol}(\Omega_\delta)}, \tag{16}$$

where Ω_δ is the closed ball of radius δ and center at $\mathbf{p}$. Thus $\det \mathbf{F}$ gives the volume after deformation per unit original volume.

We say that $\mathbf{f}$ is **isochoric** (volume preserving) if given any part $\mathscr{P}$,

$$\text{vol}(\mathbf{f}(\mathscr{P})) = \text{vol}(\mathscr{P}).$$

An immediate consequence of this definition is the following

Proposition. *A deformation is isochoric if and only if*

$$\det \mathbf{F} = 1. \tag{17}$$

[1] For $(14)_1$, cf., e.g., Bartle [1, Theorem 45.9]; for $(14)_2$, cf., e.g., Truesdell and Toupin [1, Eq. (20.8)].

EXERCISES

1. Establish properties (i) and (ii) of homogeneous deformations.
2. A homogeneous deformation of the form

$$x_1 = p_1 + \gamma p_2,$$
$$x_2 = p_2,$$
$$x_3 = p_3$$

is called a *pure shear*. For this deformation compute:

(a) the matrices of $\mathbf{F}$, $\mathbf{C}$, and $\mathbf{B}$;
(b) the list $\mathscr{I}_{\mathbf{C}}$ of principal invariants of $\mathbf{C}$ (or $\mathbf{B}$);
(c) the principal stretches.

3. Compute $\mathbf{C}$, $\mathbf{B}$, and $\mathscr{I}_{\mathbf{C}}$ for an extension of amount λ in the direction $\mathbf{e}$.
4. Show that a deformation is isochoric if and only if $\det \mathbf{C} = 1$.
5. Show that

$$\mathbf{C} = \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}} + \nabla\mathbf{u}^{\mathrm{T}}\nabla\mathbf{u}.$$

6. Show that a deformation is rigid if and only if $\mathscr{I}_{\mathbf{C}} = (3, 3, 1)$.
7. Show that the principal invariants of $\mathbf{C}$ are given by

$$\iota_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$
$$\iota_2(\mathbf{C}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2,$$
$$\iota_3(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2$$

with λ_i the principal stretches.

8. A deformation of the form

$$x_1 = f_1(p_1, p_2),$$
$$x_2 = f_2(p_1, p_2),$$
$$x_3 = p_3$$

is called a *plane strain*. Show that for such a deformation the principal stretch λ_3 (in the p_3 direction) is unity. Show further that the deformation is isochoric if and only if the other two principal stretches, λ_α and λ_β, satisfy

$$\lambda_\alpha = \frac{1}{\lambda_\beta}.$$

9. Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be deformations of $\mathscr{B}$ with the same right Cauchy–Green strain tensors. Show that there exists a rigid deformation $\mathbf{g}$ such that

$$\mathbf{f}_2 = \mathbf{g} \circ \mathbf{f}_1.$$

10. Establish the following analogs of $(14)_2$:

$$\int_{\partial\mathbf{f}(\mathscr{P})} \mathbf{v}(\mathbf{x}) \cdot \mathbf{m}(\mathbf{x})\, dA_{\mathbf{x}} = \int_{\partial\mathscr{P}} \mathbf{v}(\mathbf{f}(\mathbf{p})) \cdot \mathbf{G}(\mathbf{p})\mathbf{n}(\mathbf{p})\, dA_{\mathbf{p}},$$

$$\int_{\partial\mathbf{f}(\mathscr{P})} \mathbf{T}(\mathbf{x})\mathbf{m}(\mathbf{x})\, dA_{\mathbf{x}} = \int_{\partial\mathscr{P}} \mathbf{T}(\mathbf{f}(\mathbf{p}))\mathbf{G}(\mathbf{p})\mathbf{n}(\mathbf{p})\, dA_{\mathbf{p}}, \qquad (18)$$

$$\int_{\partial\mathbf{f}(\mathscr{P})} (\mathbf{x} - \mathbf{o}) \times \mathbf{T}(\mathbf{x})\mathbf{m}(\mathbf{x})\, dA_{\mathbf{x}} = \int_{\partial\mathscr{P}} (\mathbf{f}(\mathbf{p}) - \mathbf{o}) \times \mathbf{T}(\mathbf{f}(\mathbf{p}))\mathbf{G}(\mathbf{p})\mathbf{n}(\mathbf{p})\, dA_{\mathbf{p}}.$$

Here $\mathbf{v}$ and $\mathbf{T}$ are continuous fields on $\mathbf{f}(\mathscr{B})$ with $\mathbf{v}$ vector valued and $\mathbf{T}$ tensor valued.

11. Consider the hypothesis and notation of the proposition on page 47. The number

$$\frac{|\mathbf{d}_\alpha \times \mathbf{e}_\alpha|}{|\alpha\mathbf{d} \times \alpha\mathbf{e}|}$$

represents the ratio of the area $\Delta A_{\mathbf{x}}$ at $\mathbf{x} = \mathbf{f}(\mathbf{p})$ spanned by $\mathbf{d}_\alpha = \mathbf{f}(\mathbf{p} + \alpha\mathbf{d}) - \mathbf{f}(\mathbf{p})$ and $\mathbf{e}_\alpha = \mathbf{f}(\mathbf{p} + \alpha\mathbf{e}) - \mathbf{f}(\mathbf{p})$ to the area $\Delta A_{\mathbf{p}}$ spanned by $\alpha\mathbf{e}$ and $\alpha\mathbf{d}$ (Fig. 5). Define

$$\frac{dA_{\mathbf{x}}}{dA_{\mathbf{p}}} = \lim_{\alpha\to 0} \frac{|\mathbf{d}_\alpha \times \mathbf{e}_\alpha|}{|\alpha\mathbf{d} \times \alpha\mathbf{e}|}.$$

Use the identity

$$(\mathbf{S}\mathbf{a}) \times (\mathbf{S}\mathbf{b}) = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}(\mathbf{a} \times \mathbf{b})$$

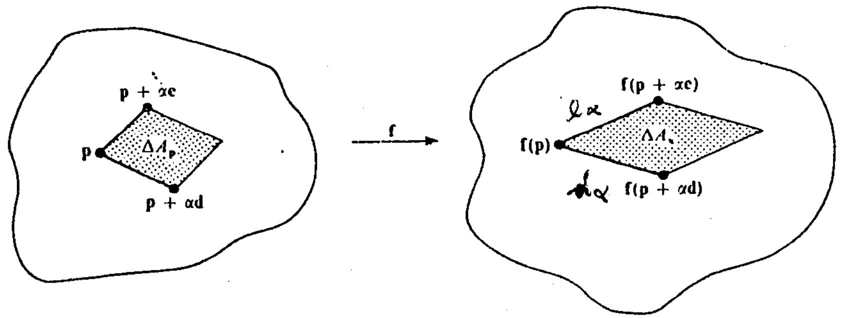

Figure 5

to show that

$$\mathbf{m}(\mathbf{x})\frac{dA_{\mathbf{x}}}{dA_{\mathbf{p}}} = \mathbf{G}(\mathbf{p})\mathbf{n}(\mathbf{p}),$$

where $\mathbf{m}(\mathbf{x})$ and $\mathbf{n}(\mathbf{p})$ are the unit normals

$$\mathbf{m}(\mathbf{x}) = \lim_{\alpha \to 0} \frac{\mathbf{d}_\alpha \times \mathbf{e}_\alpha}{|\mathbf{d}_\alpha \times \mathbf{e}_\alpha|},$$

$$\mathbf{n}(\mathbf{p}) = \frac{\mathbf{d} \times \mathbf{e}}{|\mathbf{d} \times \mathbf{e}|}.$$

[Cf. $(14)_2$.]

12. Establish (5).
13. Let $\mathscr{B}$ be the closed half-space

$$\mathscr{B} = \{\mathbf{p} \mid 0 \le p_1 < \infty\}$$

and consider the mapping $\mathbf{f}$ on $\mathscr{B}$ defined by

$$x_1 = -\frac{1}{p_1 + 1},$$

$$x_2 = p_2,$$

$$x_3 = p_3.$$

(a) Verify that $\mathbf{f}$ is one-to-one and $\det \nabla\mathbf{f} > 0$.
(b) Compute $\mathbf{f}(\mathscr{B})$ and use this result to demonstrate that $\mathbf{f}$ is *not* a deformation.
(c) Show that $(5)_2$ is not satisfied.

7. SMALL DEFORMATIONS

We now study the behavior of the various kinematical fields when the displacement gradient $\nabla\mathbf{u}$ is small. Since

$$\mathbf{f}(\mathbf{p}) = \mathbf{p} + \mathbf{u}(\mathbf{p}),$$

it follows that

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}; \tag{1}$$

hence the Cauchy–Green strain tensors $\mathbf{C}$ and $\mathbf{B}$, defined by (6.6), obey the relations

$$\begin{aligned}\mathbf{C} &= \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u},\\ \mathbf{B} &= \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}\nabla\mathbf{u}^T.\end{aligned} \tag{2}$$

When the deformation is rigid, $\mathbf{C} = \mathbf{B} = \mathbf{I}$ and

$$\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}\nabla\mathbf{u}^T = \mathbf{0}. \tag{3}$$

Moreover, in this case $\nabla\mathbf{u}$ is *constant*, because $\mathbf{F}$ is.

The tensor field

$$\mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \tag{4}$$

is called the **infinitesimal strain**; clearly

$$\begin{aligned}\mathbf{C} &= \mathbf{I} + 2\mathbf{E} + \nabla\mathbf{u}^T\nabla\mathbf{u},\\ \mathbf{B} &= \mathbf{I} + 2\mathbf{E} + \nabla\mathbf{u}\nabla\mathbf{u}^T.\end{aligned} \tag{5}$$

Proposition. *Let* $\mathbf{f}_\varepsilon$ $(0 < \varepsilon < \varepsilon_0)$ *be a one-parameter family of deformations with*

$$|\nabla\mathbf{u}_\varepsilon| = \varepsilon.$$

Then

$$2\mathbf{E}_\varepsilon = \mathbf{C}_\varepsilon - \mathbf{I} + o(\varepsilon) = \mathbf{B}_\varepsilon - \mathbf{I} + o(\varepsilon) \tag{6}$$

as $\varepsilon \to 0$. *Further, if each* $\mathbf{f}_\varepsilon$ *is rigid, then*

$$\nabla\mathbf{u}_\varepsilon = -\nabla\mathbf{u}_\varepsilon^T + o(\varepsilon). \tag{7}$$

Proof. The result (6) is a trivial consequence of (2), while (7) follows from (3). □

This proposition asserts that to within an error of order $o(\varepsilon)$ the tensors $2\mathbf{E}_\varepsilon$, $\mathbf{C}_\varepsilon - \mathbf{I}$, and $\mathbf{B}_\varepsilon - \mathbf{I}$ coincide. It asserts, in addition, that to within the same error the displacement gradient corresponding to a rigid deformation is *skew*.

The above discussion should motivate the following definition: An **infinitesimal rigid displacement** of $\mathscr{B}$ is a vector field $\mathbf{u}$ on $\mathscr{B}$ with $\nabla\mathbf{u}$ *constant* and *skew*; or equivalently, a vector field $\mathbf{u}$ that admits the representation

$$\mathbf{u}(\mathbf{p}) = \mathbf{u}(\mathbf{q}) + \mathbf{W}(\mathbf{p} - \mathbf{q}) \tag{8}$$

for all $\mathbf{p}, \mathbf{q} \in \mathscr{B}$, where $\mathbf{W}$ is skew (cf. the proposition on page 36). Of course, using the relation between skew tensors and vectors, we can also write $\mathbf{u}$ in the form

$$\mathbf{u}(\mathbf{p}) = \mathbf{u}(\mathbf{q}) + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{q})$$

with $\boldsymbol{\omega}$ the axial vector corresponding to $\mathbf{W}$.

Theorem (Characterization of infinitesimal rigid displacements). *Let* $\mathbf{u}$ *be a smooth vector field on* $\mathscr{B}$. *Then the following are equivalent:*

(a) $\mathbf{u}$ *is an infinitesimal rigid displacement.*
(b) $\mathbf{u}$ *has the* **projection property**: *for all* $\mathbf{p}, \mathbf{q} \in \mathscr{B}$,

$$(\mathbf{p} - \mathbf{q}) \cdot [\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})] = 0.$$

(c) $\nabla\mathbf{u}(\mathbf{p})$ *is skew at each* $\mathbf{p} \in \mathscr{B}$.
(d) *The infinitesimal strain* $\mathbf{E}(\mathbf{p}) = \mathbf{0}$ *at each* $\mathbf{p} \in \mathscr{B}$.

Proof. (a) ⇒ (b). Let $\mathbf{u}$ be rigid. Then (8) implies

$$(\mathbf{p} - \mathbf{q}) \cdot [\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})] = (\mathbf{p} - \mathbf{q}) \cdot \mathbf{W}(\mathbf{p} - \mathbf{q}) = 0,$$

since $\mathbf{W}$ is skew.

(b) ⇒ (a). If we differentiate the expression in (b) with respect to $\mathbf{p}$, we arrive at

$$\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q}) + \nabla\mathbf{u}(\mathbf{p})^{\mathrm{T}}(\mathbf{p} - \mathbf{q}) = \mathbf{0},$$

and this result, when differentiated with respect to $\mathbf{q}$, yields

$$-\nabla\mathbf{u}(\mathbf{q}) - \nabla\mathbf{u}(\mathbf{p})^{\mathrm{T}} = \mathbf{0}. \tag{9}$$

Taking $\mathbf{p} = \mathbf{q}$ tells us that $\nabla\mathbf{u}(\mathbf{p})$ is skew; hence (9) implies that

$$\nabla\mathbf{u}(\mathbf{p}) = \nabla\mathbf{u}(\mathbf{q})$$

for all $\mathbf{p}, \mathbf{q} \in \mathscr{B}$, and $\nabla\mathbf{u}$ is constant. Thus (a) holds.

(a) ⇔ (c). Trivially, (a) implies (c). To prove the converse assertion assume that $\mathbf{H}(\mathbf{p}) = \nabla\mathbf{u}(\mathbf{p})$ is skew at each $\mathbf{p} \in \mathscr{B}$. We must show that $\mathbf{H}$ is constant. Let Ω be an open ball in $\mathscr{B}$. Choose $\mathbf{p}, \mathbf{q} \in \Omega$ and let

$$\mathbf{c}(\sigma) = \mathbf{q} + \sigma(\mathbf{p} - \mathbf{q}), \qquad 0 \le \sigma \le 1,$$

so that $\mathbf{c}$ describes the straight line from $\mathbf{q}$ to $\mathbf{p}$. Then

$$\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q}) = \int_{\mathbf{c}} \nabla\mathbf{u}(\mathbf{x})\, d\mathbf{x} = \int_0^1 \mathbf{H}(\mathbf{c}(\sigma))\dot{\mathbf{c}}(\sigma)\, d\sigma = \int_0^1 \mathbf{H}(\mathbf{c}(\sigma))(\mathbf{p} - \mathbf{q})\, d\sigma,$$

so that

$$(\mathbf{p} - \mathbf{q}) \cdot [\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})] = \int_0^1 (\mathbf{p} - \mathbf{q}) \cdot \mathbf{H}(\mathbf{c}(\sigma))(\mathbf{p} - \mathbf{q})\, d\sigma = 0,$$

since $\mathbf{H}$ is skew. Thus $\mathbf{u}$ has the projection property on Ω, and the argument given previously [in the proof of the assertion (b) ⇒ (a)] tells us that $\mathbf{H}$ is constant on Ω. But Ω is an arbitrary open ball in $\mathscr{B}$; thus $\mathbf{H}$ is constant on $\mathscr{B}$.

(c) ⇔ (d). This is a trivial consequence of (4). □

Exercises

1. Under the hypotheses of the proposition containing (6) show that

$$\mathbf{E}_\varepsilon = \mathbf{U}_\varepsilon - \mathbf{I} + o(\varepsilon) = \mathbf{V}_\varepsilon - \mathbf{I} + o(\varepsilon),$$
$$\det \mathbf{F}_\varepsilon - 1 = \operatorname{div} \mathbf{u}_\varepsilon + o(\varepsilon).$$

Give a physical interpretation of $\det \mathbf{F}_\varepsilon - 1$ in terms of the volume change in the deformation $\mathbf{f}_\varepsilon$.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be smooth vector fields on $\mathscr{B}$ and suppose that $\mathbf{u}$ and $\mathbf{v}$ correspond to the same infinitesimal strain. Show that $\mathbf{u} - \mathbf{v}$ is an infinitesimal rigid displacement.

For the remaining exercises $\mathbf{u}$ is a smooth vector field on $\mathscr{B}$ and $\mathbf{E}$ is the corresponding infinitesimal strain. Also, in 3 and 5, $\mathscr{B}$ is bounded.

3. Define the *mean strain* $\bar{\mathbf{E}}$ by

$$\operatorname{vol}(\mathscr{B})\bar{\mathbf{E}} = \int_{\mathscr{B}} \mathbf{E}\, dV.$$

Show that

$$\operatorname{vol}(\mathscr{B})\bar{\mathbf{E}} = \frac{1}{2}\int_{\partial\mathscr{B}} (\mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u})\, dA,$$

so that $\bar{\mathbf{E}}$ depends only on the boundary values of $\mathbf{u}$.

4. Let

$$\mathbf{W} = \tfrac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}}).$$

Show that

$$|\mathbf{E}|^2 + |\mathbf{W}|^2 = |\nabla\mathbf{u}|^2,$$
$$|\mathbf{E}|^2 - |\mathbf{W}|^2 = \nabla\mathbf{u} \cdot \nabla\mathbf{u}^{\mathrm{T}}.$$

5. (*Korn's inequality*) Let $\mathbf{u}$ be of class C^2 and suppose that

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \quad \partial\mathscr{B}.$$

Show that

$$\int_{\mathscr{B}} |\nabla\mathbf{u}|^2\, dV \le 2\int_{\mathscr{B}} |\mathbf{E}|^2\, dV.$$

6. Consider the deformation defined in cylindrical coordinates

$$x_1 = r\cos\theta, \qquad p_1 = R\cos\Theta,$$
$$x_2 = r\sin\theta, \qquad p_2 = R\sin\Theta,$$
$$x_3 = z \qquad\quad p_3 = Z,$$

by

$$r = R, \qquad \theta = \Theta + \alpha Z, \qquad z = Z.$$

A deformation of this type is called a *pure torsion*; it describes a situation in which a cylinder with generators parallel to the Z-axis is twisted uniformly along its length with cross sections remaining parallel and plane. The constant α represents the *angle of twist* per unit length. Show that the corresponding displacement is given by

$$u_1(\mathbf{p}) = p_1(\cos\beta - 1) - p_2\sin\beta, \qquad \beta = \alpha p_3,$$
$$u_2(\mathbf{p}) = p_2(\cos\beta - 1) + p_1\sin\beta,$$
$$u_3(\mathbf{p}) = 0.$$

Show further that both $\nabla\mathbf{u}$ and $\mathbf{u}$ approach zero as $\alpha \to 0$, and that, in fact,

$$u_1(\mathbf{p}) = -\alpha p_2 p_3 + o(\alpha), \qquad (10)$$
$$u_2(\mathbf{p}) = \alpha p_1 p_3 + o(\alpha)$$

as $\alpha \to 0$.

8. MOTIONS

Let $\mathscr{B}$ be a body. A **motion** of $\mathscr{B}$ is a class C^3 function

$$\mathbf{x}: \mathscr{B} \times \mathbb{R} \to \mathscr{E}$$

with $\mathbf{x}(\cdot, t)$, for each fixed t, a deformation of $\mathscr{B}$ (Fig. 6). Thus a motion is a smooth one-parameter family of deformations, the **time** t being the parameter. We refer to[1]

$$\mathbf{x} = \mathbf{x}(\mathbf{p}, t)$$

as the **place** occupied by the material point $\mathbf{p}$ at time t, and write

$$\mathscr{B}_t = \mathbf{x}(\mathscr{B}, t)$$

[1] We carefully distinguish between the motion $\mathbf{x}$ and its values $\mathbf{x}$, and between the reference map $\mathbf{p}$ and material points $\mathbf{p}$.

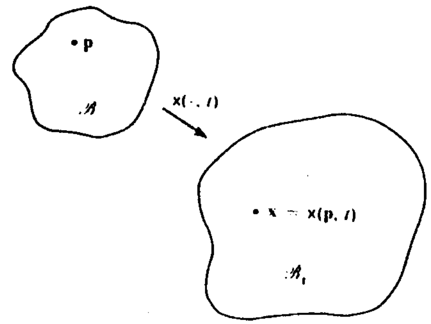

Figure 6

for the region of space occupied by the body at t. It is often more convenient to work with places and times rather than with material points and times, and for this reason we introduce the **trajectory**

$$\mathscr{T} = \{(\mathbf{x}, t) \mid \mathbf{x} \in \mathscr{B}_t, t \in \mathbb{R}\}.$$

At each t, $\mathbf{x}(\cdot, t)$ is a one-to-one mapping of $\mathscr{B}$ onto $\mathscr{B}_t$; hence it has an inverse

$$\mathbf{p}(\cdot, t): \mathscr{B}_t \to \mathscr{B}$$

such that

$$\mathbf{x}(\mathbf{p}(\mathbf{x}, t), t) = \mathbf{x}, \qquad \mathbf{p}(\mathbf{x}(\mathbf{p}, t), t) = \mathbf{p}.$$

Given $(\mathbf{x}, t) \in \mathscr{T}$,

$$\mathbf{p} = \mathbf{p}(\mathbf{x}, t)$$

is the material point that occupies the place $\mathbf{x}$ at time t. The map

$$\mathbf{p}: \mathscr{T} \to \mathscr{B}$$

so defined is called the **reference map** of the motion.

We call

$$\dot{\mathbf{x}}(\mathbf{p}, t) = \frac{\partial}{\partial t}\mathbf{x}(\mathbf{p}, t)$$

the **velocity** and

$$\ddot{\mathbf{x}}(\mathbf{p}, t) = \frac{\partial^2}{\partial t^2}\mathbf{x}(\mathbf{p}, t)$$

the **acceleration.** Using the reference map p we can describe the velocity $\dot{\mathbf{x}}(\mathbf{p}, t)$ as a function $\mathbf{v}(\mathbf{x}, t)$ of the place x and time t. Specifically,

$$\mathbf{v}: \mathscr{T} \to \mathscr{V}$$

is defined by

$$\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}(\mathbf{p}(\mathbf{x}, t), t)$$

and is called the **spatial description of the velocity.** The vector $\mathbf{v}(\mathbf{x}, t)$ is the velocity of the material point which at time t occupies the place x.

More generally, any field associated with the motion can be expressed as a function of the *material point* and time with domain $\mathscr{B} \times \mathbb{R}$, or as a function of the *place* and time with domain $\mathscr{T}$. We therefore introduce the following terminology: a **material field** is a function with domain $\mathscr{B} \times \mathbb{R}$; a **spatial field** is a function with domain $\mathscr{T}$. The field $\dot{\mathbf{x}}$ is material, the field v is spatial. It is a simple matter to transform a material field into a spatial field, and vice versa. We define the **spatial description** Φ_s of a material field $(\mathbf{p}, t) \mapsto \Phi(\mathbf{p}, t)$ by

$$\Phi_s(\mathbf{x}, t) = \Phi(\mathbf{p}(\mathbf{x}, t), t),$$

and the **material description** Ω_m of a spatial field $(\mathbf{x}, t) \mapsto \Omega(\mathbf{x}, t)$ by

$$\Omega_m(\mathbf{p}, t) = \Omega(\mathbf{x}(\mathbf{p}, t), t).$$

Clearly,

$$(\Phi_s)_m = \Phi, \qquad (\Omega_m)_s = \Omega.$$

Smoothness Lemma. *The reference map* p *is of class C^3. Thus a material field is of class C^n ($n \le 3$) if and only if its spatial description is of class C^n.*

The proof of this lemma will be given at the end of the section.

Given a material field Φ we write

$$\dot{\Phi}(\mathbf{p}, t) = \frac{\partial}{\partial t}\Phi(\mathbf{p}, t)$$

for the derivative with respect to time t holding the material point p fixed, and

$$\nabla\Phi(\mathbf{p}, t) = \nabla_{\mathbf{p}}\Phi(\mathbf{p}, t)$$

for the gradient with respect to p holding t fixed. $\dot{\Phi}$ is called the **material time derivative** of Φ, $\nabla\Phi$ the **material gradient** of Φ. In particular, the material field

$$\mathbf{F} = \nabla\mathbf{x}$$

is the **deformation gradient** in the motion x. Since the mapping $\mathbf{p} \mapsto \mathbf{x}(\mathbf{p}, t)$ is a deformation of $\mathscr{B}$,

$$\det \mathbf{F} > 0. \tag{1}$$

Similarly, given a spatial field Ω we write

$$\Omega'(\mathbf{x}, t) = \frac{\partial}{\partial t}\Omega(\mathbf{x}, t)$$

for the derivative with respect to t holding the place x fixed, and

$$\operatorname{grad}\Omega(\mathbf{x}, t) = \nabla_{\mathbf{x}}\Omega(\mathbf{x}, t)$$

for the gradient with respect to x holding t fixed. Ω' is called the **spatial time derivative** of Ω, grad Ω the **spatial gradient** of Ω.

We define the **spatial divergence** and the **spatial curl**, div and curl, to be the divergence and curl operations for spatial fields, so that grad is the underlying gradient. Similarly, Div and Curl designate the **material divergence** and the **material curl** computed using the material gradient ∇.

The notation introduced above is summarized in Table 1.

It is also convenient to define the **material time derivative** $\dot{\Omega}$ of a *spatial* field Ω. Roughly speaking, $\dot{\Omega}$ represents the time derivative of Ω holding the material point fixed. Thus to compute $\dot{\Omega}$ we transform Ω to the material description, take the material time derivative, and then transform back to the spatial description:

$$\dot{\Omega} = ((\Omega_m)^{\cdot})_s; \tag{2}$$

that is,

$$\dot{\Omega}(\mathbf{x}, t) = \frac{\partial}{\partial t}\Omega(\mathbf{x}(\mathbf{p}, t), t)\big|_{\mathbf{p}=\mathbf{p}(\mathbf{x},t)}.$$

The next proposition shows that the material time derivative commutes with both the material and spatial transformations.

Table 1

	Material field Φ	Spatial field Ω
Domain	$\mathscr{B} \times \mathbb{R}$	$\mathscr{T}$
Arguments	Material point p and time t	Place x and time t
Gradient with respect to first argument	$\nabla\Phi$	grad Ω
Derivative with respect to second argument (time)	$\dot{\Phi}$	Ω'
Divergence	Div Φ	div Ω
Curl	Curl Φ	curl Ω

Proposition. *Let Φ be a smooth material field, Ω a smooth spatial field. Then*

$$(\dot{\Phi})_s = (\Phi_s)' \equiv \dot{\Phi}_s,$$
$$(\dot{\Omega})_m = (\Omega_m)^{\cdot} \equiv \dot{\Omega}_m. \tag{3}$$

Proof. If we take the material description of (2), we arrive at $(3)_2$. Also, by definition,

$$(\Phi_s)' = ((\Phi_s)_m^{\cdot})_s = (\dot{\Phi})_s. \quad \square$$

Note that, by $(3)_2$ with $\Omega = \mathbf{v}$,

$$(\dot{\mathbf{v}})_m = (\mathbf{v}_m)^{\cdot} = \ddot{\mathbf{x}},$$

so that $\dot{\mathbf{v}}$ *is the spatial description of the acceleration.*

The relation between the material and spatial time derivatives is brought out by the following

Proposition. *Let φ and $\mathbf{u}$ be smooth spatial fields, with φ scalar valued and $\mathbf{u}$ vector valued. Then*

$$\dot{\varphi} = \varphi' + \mathbf{v} \cdot \operatorname{grad} \varphi,$$
$$\dot{\mathbf{u}} = \mathbf{u}' + (\operatorname{grad} \mathbf{u})\mathbf{v}. \tag{4}$$

Thus, in particular,

$$\dot{\mathbf{v}} = \mathbf{v}' + (\operatorname{grad} \mathbf{v})\mathbf{v}. \tag{5}$$

Proof. By the chain rule,

$$\begin{aligned}\dot{\varphi}(\mathbf{x}, t) &= \frac{\partial}{\partial t} \varphi(\mathbf{x}(\mathbf{p}, t), t)\big|_{\mathbf{p}=\mathbf{p}(\mathbf{x},t)} \\ &= [\operatorname{grad} \varphi(\mathbf{x}, t)] \cdot \dot{\mathbf{x}}(\mathbf{p}(\mathbf{x}, t), t) + \varphi'(\mathbf{x}, t) \\ &= \mathbf{v}(\mathbf{x}, t) \cdot \operatorname{grad} \varphi(\mathbf{x}, t) + \varphi'(\mathbf{x}, t),\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{u}}(\mathbf{x}, t) &= \frac{\partial}{\partial t} \mathbf{u}(\mathbf{x}(\mathbf{p}, t), t)\big|_{\mathbf{p}=\mathbf{p}(\mathbf{x},t)} \\ &= [\operatorname{grad} \mathbf{u}(\mathbf{x}, t)]\dot{\mathbf{x}}(\mathbf{p}(\mathbf{x}, t), t) + \mathbf{u}'(\mathbf{x}, t) \\ &= [\operatorname{grad} \mathbf{u}(\mathbf{x}, t)]\mathbf{v}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t). \quad \square\end{aligned}$$

A simple application of (4) is expressed in the next result, which gives the material time derivative of the **position vector** $\mathbf{r}: \mathscr{E} \to \mathscr{V}$ defined by

$$\mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{o}.$$

Proposition. *Consider the position vector as a spatial field by defining $\mathbf{r}(\mathbf{x}, t) = \mathbf{r}(\mathbf{x})$ for every $(\mathbf{x}, t) \in \mathscr{T}$. Then*

$$\dot{\mathbf{r}} = \mathbf{v}. \tag{6}$$

Proof. Since $\mathbf{r}' = \mathbf{0}$ and $\operatorname{grad} \mathbf{r} = \mathbf{I}$, $(4)_2$ with $\mathbf{u} = \mathbf{r}$ yields (6). [This result can also be arrived at directly by noting that $\mathbf{r}_m(\mathbf{p}, t) = \mathbf{x}(\mathbf{p}, t) - \mathbf{o}$.] $\square$

Proposition. *Let $\mathbf{u}$ be a smooth spatial vector field. Then*

$$\nabla(\mathbf{u}_m) = (\operatorname{grad} \mathbf{u})_m \mathbf{F}, \tag{7}$$

where $\mathbf{F}$ is the deformation gradient.

Proof. By definition,

$$\mathbf{u}_m(\mathbf{p}, t) = \mathbf{u}(\mathbf{x}(\mathbf{p}, t), t);$$

that is,

$$\mathbf{u}_m(\cdot, t) = \mathbf{u}(\cdot, t) \circ \mathbf{x}(\cdot, t).$$

Thus the chain rule (3.11) tells us that $\nabla(\mathbf{u}_m)$ is the gradient $\operatorname{grad} \mathbf{u}$ of $\mathbf{u}$ times the gradient $\mathbf{F} = \nabla \mathbf{x}$ of $\mathbf{x}$. $\square$

The spatial field

$$\mathbf{L} = \operatorname{grad} \mathbf{v}$$

is called the **velocity gradient.**

Proposition

$$\dot{\mathbf{F}} = \mathbf{L}_m \mathbf{F},$$
$$\ddot{\mathbf{F}} = (\operatorname{grad} \dot{\mathbf{v}})_m \mathbf{F}. \tag{8}$$

Proof. Since $\mathbf{x}$ is by definition C^3,

$$\dot{\mathbf{F}}(\mathbf{p}, t) = \frac{\partial}{\partial t} \nabla \mathbf{x}(\mathbf{p}, t) = \nabla \dot{\mathbf{x}}(\mathbf{p}, t) = \nabla \mathbf{v}_m(\mathbf{p}, t),$$

and $(8)_1$ follows from (7) with $\mathbf{u} = \mathbf{v}$. Similarly,

$$\ddot{\mathbf{F}}(\mathbf{p}, t) = \nabla \ddot{\mathbf{x}}(\mathbf{p}, t) = \nabla \dot{\mathbf{v}}_m(\mathbf{p}, t),$$

and taking $\mathbf{u} = \dot{\mathbf{v}}$ in (7) we are led to $(8)_2$. $\square$

Given a material point $\mathbf{p}$, the function $\mathbf{s}: \mathbb{R} \to \mathscr{E}$ defined by

$$\mathbf{s}(t) = \mathbf{x}(\mathbf{p}, t)$$

is called the **path line** of $\mathbf{p}$. Clearly, $\mathbf{s}$ is a solution of the differential equation

$$\dot{\mathbf{s}}(t) = \mathbf{v}(\mathbf{s}(t), t),$$

and conversely every maximal solution of this equation is a path line. (A solution is maximal provided it is not a portion of another solution.) On the other hand, if we freeze the time at $t = \tau$ and look at solution curves of the vector field $\mathbf{v}(\cdot, \tau)$, we get the **streamlines** of the motion at time τ. Thus each streamline is a maximal solution $\mathbf{s}$ of the differential equation

$$\dot{\mathbf{s}}(\lambda) = \mathbf{v}(\mathbf{s}(\lambda), \tau).$$

An example of a motion $\mathbf{x}$ (of $\mathscr{E}$) is furnished by the mapping defined in cartesian components by

$$x_1 = p_1 e^{t^2},$$
$$x_2 = p_2 e^t,$$
$$x_3 = p_3.$$

The deformation gradient $\mathbf{F}$ is given by

$$[\mathbf{F}(\mathbf{p}, t)] = \begin{bmatrix} e^{t^2} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

while the velocity $\dot{\mathbf{x}}$ has components

$$2p_1 t e^{t^2}, \qquad p_2 e^t, \qquad 0.$$

Thus, since the reference map $\mathbf{p}$ is given by

$$p_1 = x_1 e^{-t^2},$$
$$p_2 = x_2 e^{-t},$$
$$p_3 = x_3,$$

the spatial description of the velocity has components

$$v_1(\mathbf{x}, t) = 2x_1 t,$$
$$v_2(\mathbf{x}, t) = x_2,$$
$$v_3(\mathbf{x}, t) = 0,$$

and the velocity gradient $\mathbf{L}$ has the matrix

$$[\mathbf{L}(\mathbf{x}, t)] = \begin{bmatrix} 2t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The streamlines of the motion at time t are solutions of the system

$$\dot{s}_1(\lambda) = 2ts_1(\lambda),$$
$$\dot{s}_2(\lambda) = s_2(\lambda),$$
$$\dot{s}_3(\lambda) = 0,$$

so that

$$s_1(\lambda) = y_1 e^{2t\lambda},$$
$$s_2(\lambda) = y_2 e^{\lambda},$$
$$s_3(\lambda) = y_3$$

is the streamline passing through (y_1, y_2, y_3) at $\lambda = 0$.

We close this section by giving the

Proof of the Smoothness Lemma. It suffices to show that $\mathbf{p}$ is of class C^3, for then the remaining assertion in the lemma follows trivially.

Consider the mapping

$$\mathbf{\Psi}: \mathscr{B} \times \mathbb{R} \to \mathscr{T}$$

defined by

$$\mathbf{\Psi}(\mathbf{p}, t) = (\mathbf{x}(\mathbf{p}, t), t).$$

It follows from the properties of $\mathbf{x}$ that $\mathbf{\Psi}$ is class C^3 and one-to-one; in fact,

$$\mathbf{\Psi}^{-1}(\mathbf{x}, t) = (\mathbf{p}(\mathbf{x}, t), t).$$

Thus to complete the proof it suffices to show that the derivative

$$D\mathbf{\Psi}(\mathbf{p}, t): \mathscr{V} \times \mathbb{R} \to \mathscr{V} \times \mathbb{R}$$

is invertible at each $(\mathbf{p}, t)$, for then the smooth-inverse theorem (page 22) tells us that $\mathbf{\Psi}^{-1}$ is as smooth as $\mathbf{\Psi}$, and hence that $\mathbf{p}$ is of class C^3.

Since

$$\mathbf{x}(\mathbf{p} + \mathbf{h}, t + \tau) = \mathbf{x}(\mathbf{p}, t) + \mathbf{F}(\mathbf{p}, t)\mathbf{h} + \dot{\mathbf{x}}(\mathbf{p}, t)\tau + o(\varepsilon)$$

as $\varepsilon = (\mathbf{h}^2 + \tau^2)^{1/2} \to 0$, it follows that

$$\mathbf{\Psi}(\mathbf{p} + \mathbf{h}, t + \tau) = \mathbf{\Psi}(\mathbf{p}, t) + (\mathbf{F}(\mathbf{p}, t)\mathbf{h} + \dot{\mathbf{x}}(\mathbf{p}, t)\tau, \tau) + o(\varepsilon).$$

Thus

$$D\mathbf{\Psi}(\mathbf{p}, t)\,[\mathbf{h}, \tau] = (\mathbf{F}(\mathbf{p}, t)\mathbf{h} + \dot{\mathbf{x}}(\mathbf{p}, t)\tau, \tau) \qquad (9)$$

for all $\mathbf{h} \in \mathscr{V}$ and $\tau \in \mathbb{R}$.

To show that $D\Psi(\mathbf{p}, t)$ is invertible, it suffices to show that

$$D\Psi(\mathbf{p}, t)\,[\mathbf{h}, \tau] = \mathbf{0} \tag{10}$$

implies

$$\mathbf{h} = \mathbf{0}, \qquad \tau = 0.$$

Thus assume that (10) holds. Then by (9),

$$\tau = 0, \qquad \mathbf{F}(\mathbf{p}, t)\mathbf{h} = \mathbf{0},$$

and, since $\mathbf{F}(\mathbf{p}, t)$ is invertible, $\mathbf{h} = \mathbf{0}$. □

Exercises

1. A motion is a *simple shear* if the velocity field has the form

$$\mathbf{v}(\mathbf{x}, t) = v_1(x_2)\mathbf{e}_1$$

in some cartesian frame. Show that for a simple shear

$$\operatorname{div} \mathbf{v} = 0, \qquad (\operatorname{grad} \mathbf{v})\mathbf{v} = \mathbf{0}, \qquad \dot{\mathbf{v}} = \mathbf{v}'.$$

In the next two exercises $\mathbf{D}$ and $\mathbf{W}$, respectively, are the symmetric and skew parts of grad $\mathbf{v}$.

2. Prove that

$$\dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{D}_m\mathbf{F}.$$

3. Let $\mathbf{v}$ be a class C^2 velocity field. Show that

$$\operatorname{div} \dot{\mathbf{v}} = (\operatorname{div} \mathbf{v})^{\cdot} + |\mathbf{D}|^2 - |\mathbf{W}|^2.$$

4. Consider the motion of $\mathscr{E}$ defined by

$$x_1 = p_1 e^t,$$
$$x_2 = p_2 + t,$$
$$x_3 = p_3,$$

in some cartesian frame. Compute the spatial velocity field $\mathbf{v}$ and determine the streamlines.

5. Consider the motion $\mathbf{x}$ defined by

$$\mathbf{x}(\mathbf{p}, t) = \mathbf{p}_0 + \mathbf{U}(t)[\mathbf{p} - \mathbf{p}_0],$$

where

$$\mathbf{U}(t) = \sum_{i=1}^{3} \alpha_i(t)\mathbf{e}_i \otimes \mathbf{e}_i$$

with $\alpha_i > 0$ smooth. (Here $\{\mathbf{e}_i\}$ is an orthonormal basis.) Compute $\mathbf{p}$, $\mathbf{v}$, and $\mathbf{L}$, and determine the streamlines.

6. Define the spatial gradient and spatial time derivative of a material field and show that

$$\mathbf{x}' = \mathbf{0}, \qquad \operatorname{grad} \mathbf{x} = \mathbf{I}.$$

7. Consider a surface $\mathscr{S}$ in $\mathscr{B}$ of the form

$$\mathscr{S} = \{\mathbf{p} \in \mathscr{D} \,|\, \varphi(\mathbf{p}) = 0\},$$

where $\mathscr{D}$ is an open subset of $\mathscr{B}$ and φ is a smooth scalar field on $\mathscr{D}$ with $\nabla\varphi$ never zero on $\mathscr{S}$. Let $\mathbf{x}$ be a motion of $\mathscr{B}$. Then, at time t, $\mathscr{S}$ occupies the surface

$$\mathscr{S}_t = \{\mathbf{x} \in \mathscr{D}_t \,|\, \psi(\mathbf{x}, t) = 0\},$$

where $\mathscr{D}_t = \mathbf{x}(\mathscr{D}, t)$ and

$$\psi(\mathbf{x}, t) = \varphi(\mathbf{p}(\mathbf{x}, t)).$$

Show that:

(a) $\nabla\varphi(\mathbf{p})$ is normal to $\mathscr{S}$ at $\mathbf{p} \in \mathscr{S}$;
(b) grad $\psi(\mathbf{x}, t)$ is normal to $\mathscr{S}_t$ at $\mathbf{x} \in \mathscr{S}_t$;
(c) $\nabla\varphi = \mathbf{F}^T(\operatorname{grad}\psi)_m$, and hence grad $\psi(\mathbf{x}, t)$ never vanishes on $\mathscr{S}_t$;
(d) $|\nabla\varphi|^2 = (\operatorname{grad}\psi)_m \cdot \mathbf{B}(\operatorname{grad}\psi)_m$, where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy Green strain tensor;
(e) $\psi' = -\mathbf{v} \cdot \operatorname{grad} \psi$.

9. TYPES OF MOTIONS. SPIN. RATE OF STRETCHING

A motion $\mathbf{x}$ is **steady** if

$$\mathscr{B}_t = \mathscr{B}_0$$

for all time t and

$$\mathbf{v}' = \mathbf{0}$$

everywhere on the trajectory $\mathscr{T}$. Note that $\mathscr{T} = \mathscr{B}_0 \times \mathbb{R}$, because the body occupies the same region $\mathscr{B}_0$ for all time. Also, since the velocity field $\mathbf{v}$ is independent of time, we may consider $\mathbf{v}$ as a function $\mathbf{x} \mapsto \mathbf{v}(\mathbf{x})$ on $\mathscr{B}_0$. Thus in a steady motion the particles that cross a given place $\mathbf{x}$ all cross $\mathbf{x}$ with the same velocity $\mathbf{v}(\mathbf{x})$. Of course, for a given material point $\mathbf{p}$ the velocity will generally change with time, since $\dot{\mathbf{x}}(\mathbf{p}, t) = \mathbf{v}(\mathbf{x}(\mathbf{p}, t))$.

Consider now a (not necessarily steady) motion $\mathbf{x}$ and choose a point $\mathbf{p}$ with $\mathbf{x}(\mathbf{p}, \tau) \in \partial\mathscr{B}_\tau$ at some time τ. Then $(6.5)_2$ with $\mathbf{f} = \mathbf{x}(\cdot, \tau)$ implies that

$\mathbf{p} \in \partial\mathcal{B}$ and a second application of $(6.5)_2$, this time with $\mathbf{f} = \mathbf{x}(\cdot, t)$, tells us that $\mathbf{x}(\mathbf{p}, t) \in \partial\mathcal{B}_t$ for all t. Thus *a material point once on the boundary lies on the boundary for all time.* If the boundary is independent of time ($\partial\mathcal{B}_t = \partial\mathcal{B}_0$ for all t), as is the case in a steady motion, then $\mathbf{x}(\mathbf{p}, t)$, as a function of t, describes a curve on $\partial\mathcal{B}_0$, and $\dot{\mathbf{x}}(\mathbf{p}, t)$ is *tangent* to $\partial\mathcal{B}_0$. Thus we have the following

Proposition. *In a steady motion the velocity field is tangent to the boundary; i.e.,* $\mathbf{v}(\mathbf{x})$ *is tangent to* $\partial\mathcal{B}_0$ *at each* $\mathbf{x} \in \partial\mathcal{B}_0$.

In a steady motion path lines and streamlines satisfy the same autonomous differential equation

$$\dot{\mathbf{s}}(t) = \mathbf{v}(\mathbf{s}(t)).$$

Thus, as a consequence of the uniqueness theorem for ordinary differential equations, we have the following

Proposition. *In a steady motion every path line is a streamline and every streamline is a path line.*

Let Φ be a smooth field on the trajectory of a steady motion. Then Φ is **steady** if

$$\Phi' = 0 \tag{1}$$

[in which case we consider Φ as a function $\mathbf{x} \mapsto \Phi(\mathbf{x})$ on $\mathcal{B}_0$].

Proposition. *Let φ be a smooth, steady scalar field on the trajectory of a steady motion. Then the following are equivalent:*

(a) φ *is* **constant on streamlines**; *that is, given any streamline* $\mathbf{s}$,

$$\frac{d}{dt}\varphi(\mathbf{s}(t)) = 0$$

for all t.

(b) $\dot{\varphi} = 0$.

(c) $\mathbf{v} \cdot \operatorname{grad} \varphi = 0$.

Proof. Note first that, by $(8.4)_1$ and (1),

$$\dot{\varphi} = \mathbf{v} \cdot \operatorname{grad} \varphi;$$

thus (b) and (c) are equivalent. Next, for any streamline $\mathbf{s}$,

$$\frac{d}{dt}\varphi(\mathbf{s}(t)) = \dot{\mathbf{s}}(t) \cdot \operatorname{grad} \varphi(\mathbf{s}(t)) = \mathbf{v}(\mathbf{s}(t)) \cdot \operatorname{grad} \varphi(\mathbf{s}(t)), \tag{2}$$

so that (c) implies (a). On the other hand, since every point of $\mathcal{B}_0$ has a streamline passing through it, (a) and (2) imply (c). $\square$

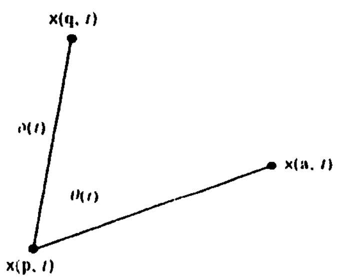

Figure 7

The number

$$\delta(t) = |\mathbf{x}(\mathbf{p}, t) - \mathbf{x}(\mathbf{q}, t)| \tag{3}$$

represents the **distance at time** t between the material points $\mathbf{p}$ and $\mathbf{q}$. Similarly, the **angle** $\theta(t)$ **at time** t subtended by the material points $\mathbf{a}$, $\mathbf{p}$, $\mathbf{q}$ is the angle between the vectors $\mathbf{x}(\mathbf{a}, t) - \mathbf{x}(\mathbf{p}, t)$ and $\mathbf{x}(\mathbf{q}, t) - \mathbf{x}(\mathbf{p}, t)$. (See Fig. 7.)

A motion $\mathbf{x}$ is **rigid** if

$$\frac{\partial}{\partial t}|\mathbf{x}(\mathbf{p}, t) - \mathbf{x}(\mathbf{q}, t)| = 0 \tag{4}$$

for all materials points $\mathbf{p}$ and $\mathbf{q}$ and each time t. Thus a motion is rigid if the distance between any two material points remains constant in time.

Theorem (Characterization of rigid motions). *Let* $\mathbf{x}$ *be a motion, and let* $\mathbf{v}$ *be the corresponding velocity field. Then the following are equivalent:*

(a) $\mathbf{x}$ *is rigid.*

(b) *At each time* t, $\mathbf{v}(\cdot, t)$ *has the form of an infinitesimal rigid displacement of* $\mathcal{B}_t$; *that is,* $\mathbf{v}(\cdot, t)$ *admits the representation*

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{y}, t) + \mathbf{W}(t)(\mathbf{x} - \mathbf{y}) \tag{5}$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{B}_t$, *where* $\mathbf{W}(t)$ *is a skew tensor.*

(c) *The velocity gradient* $\mathbf{L}(\mathbf{x}, t)$ *is skew at each* $(\mathbf{x}, t) \in \mathcal{T}$.

Proof. If we use (3) to differentiate $\delta(t)^2$, we find that

$$\delta(t)\,\dot{\delta}(t) = [\mathbf{x}(\mathbf{p}, t) - \mathbf{x}(\mathbf{q}, t)] \cdot [\dot{\mathbf{x}}(\mathbf{p}, t) - \dot{\mathbf{x}}(\mathbf{q}, t)],$$

or equivalently, letting $\mathbf{x}$ and $\mathbf{y}$ denote the places occupied by $\mathbf{p}$ and $\mathbf{q}$ at time t,

$$\delta(t)\,\dot{\delta}(t) = (\mathbf{x} - \mathbf{y}) \cdot [\mathbf{v}(\mathbf{x}, t) - \mathbf{v}(\mathbf{y}, t)]. \tag{6}$$

By (6), (4), and the fact that $\delta(t) \neq 0$ for $\mathbf{p} \neq \mathbf{q}$, $\mathbf{x}$ is rigid if and only if $\mathbf{v}(\cdot, t)$ has the projection property at each time t. The equivalence of (a), (b), and (c) is therefore a direct consequence of the theorem characterizing infinitesimal rigid displacements (page 56). $\square$

Let $\boldsymbol{\omega}(t)$ be the axial vector corresponding to $\mathbf{W}(t)$; then (5) becomes

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{y}, t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{y}),$$

which is the classical formula for the velocity field of a rigid motion. The vector function $\boldsymbol{\omega}$ is called the **angular velocity**. Note that

$$\operatorname{curl} \mathbf{v} = 2\boldsymbol{\omega},$$

which gives a physical interpretation of curl $\mathbf{v}$, at least for rigid motions.

For convenience, we suppress the argument t and write

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y}) + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{y}).$$

Assume $\boldsymbol{\omega} \neq \mathbf{0}$. Then for fixed $\mathbf{y}$ the velocity field

$$\mathbf{x} \mapsto \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{y})$$

vanishes for $\mathbf{x}$ on the line

$$\{\mathbf{y} + \alpha\boldsymbol{\omega} \mid \alpha \in \mathbb{R}\}$$

and represents a rigid rotation about this line. Thus given any fixed $\mathbf{y}$, $\mathbf{v}$ is the sum of a uniform velocity field with constant value

$$\mathbf{v}(\mathbf{y})$$

and a rigid rotation about the line through $\mathbf{y}$ spanned by $\boldsymbol{\omega}$. For this reason we call $l = \operatorname{sp}\{\boldsymbol{\omega}\}$ the spin axis. For future use, we note that $l = l(t)$ can also be specified as the set of all vectors $\mathbf{e}$ such that

$$\mathbf{W}\mathbf{e} = \mathbf{0}.$$

As we have seen, a rigid motion is characterized (at each time) by a velocity gradient which is both constant and skew. We now study the case in which the gradient is still constant, but is symmetric rather than skew. Thus consider a velocity field of the form

$$\mathbf{v}(\mathbf{x}) = \mathbf{D}(\mathbf{x} - \mathbf{y})$$

with $\mathbf{D}$ a symmetric tensor. By the spectral theorem $\mathbf{D}$ is the sum of three tensors of the form

$$\alpha\mathbf{e} \otimes \mathbf{e}, \qquad |\mathbf{e}| = 1,$$

with corresponding $\mathbf{e}$'s mutually orthogonal. It therefore suffices to limit our discussion to the velocity field

$$\mathbf{v}(\mathbf{x}) = \alpha(\mathbf{e} \otimes \mathbf{e})(\mathbf{x} - \mathbf{y}). \tag{7}$$

Relative to a coordinate frame with $\mathbf{e} = \mathbf{e}_1$, $\mathbf{v}$ has components

$$(v_1, 0, 0), \qquad v_1(\mathbf{x}) = \alpha(x_1 - y_1),$$

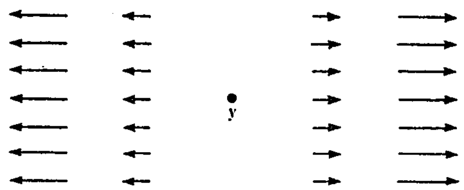

Figure 8

and is described in Fig. 8. What we have shown is that every velocity field with gradient symmetric and constant is (modulo a constant field) the sum of three velocity fields of the form (7) with "axes" $\mathbf{e}$ mutually perpendicular.

Now consider a general velocity field $\mathbf{v}$. Since $\mathbf{L} = \operatorname{grad} \mathbf{v}$, it follows that

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y}) + \mathbf{L}(\mathbf{y})(\mathbf{x} - \mathbf{y}) + o(\mathbf{x} - \mathbf{y})$$

as $\mathbf{x} \to \mathbf{y}$, where $\mathbf{y}$ is a given point, and where we have suppressed the argument t. Let $\mathbf{D}$ and $\mathbf{W}$, respectively, denote the symmetric and skew parts of $\mathbf{L}$:

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \tfrac{1}{2}(\operatorname{grad} \mathbf{v} + \operatorname{grad} \mathbf{v}^T),$$

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^T) = \tfrac{1}{2}(\operatorname{grad} \mathbf{v} - \operatorname{grad} \mathbf{v}^T).$$

Then

$$\mathbf{L} = \mathbf{D} + \mathbf{W}$$

and

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y}) + \mathbf{W}(\mathbf{y})(\mathbf{x} - \mathbf{y}) + \mathbf{D}(\mathbf{y})(\mathbf{x} - \mathbf{y}) + o(\mathbf{x} - \mathbf{y}).$$

Thus *in a neighborhood of a given point* $\mathbf{y}$ *and to within an error of* $o(\mathbf{x} - \mathbf{y})$ *a general velocity field is the sum of a rigid velocity field*

$$\mathbf{x} \mapsto \mathbf{v}(\mathbf{y}) + \mathbf{W}(\mathbf{y})(\mathbf{x} - \mathbf{y})$$

and a velocity field of the form

$$\mathbf{x} \mapsto \mathbf{D}(\mathbf{y})(\mathbf{x} - \mathbf{y}).$$

For this reason we call $\mathbf{W}(\mathbf{y}, t)$ and $\mathbf{D}(\mathbf{y}, t)$, respectively, the **spin** and the **stretching**, and we use the term **spin axis** at $(\mathbf{y}, t)$ for the subspace l of $\mathscr{V}$ consisting of all vectors $\mathbf{e}$ for which

$$\mathbf{W}(\mathbf{y}, t)\mathbf{e} = \mathbf{0}.$$

[Of course, l has dimension one when $\mathbf{W}(\mathbf{y}, t) \neq \mathbf{0}$.] The term stretching is further motivated by the following

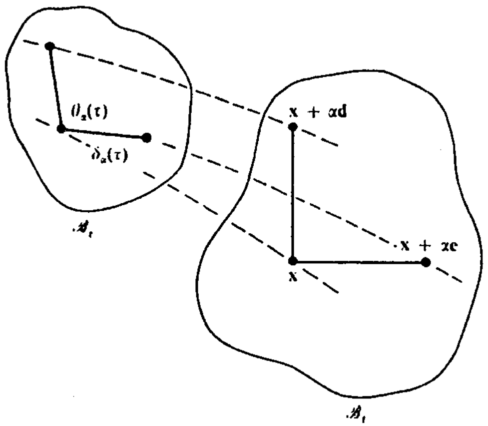

Figure 9

Proposition. *Let* $\mathbf{x} \in \mathring{\mathscr{B}}_t$ *with t a fixed time, let* $\mathbf{e}$ *be a unit vector, and let* $\delta_\alpha(\tau)$ *(for* α *sufficiently small) denote the distance at time* τ *between the material points that occupy the places* $\mathbf{x}$ *and* $\mathbf{x} + \alpha\mathbf{e}$ *at time t (Fig. 9). Then*

$$\lim_{\alpha\to 0} \frac{\dot{\delta}_\alpha(t)}{\delta_\alpha(t)} = \mathbf{e}\cdot\mathbf{D}(\mathbf{x}, t)\mathbf{e}. \tag{8}$$

Further, if $\mathbf{d}$ *is a unit vector perpendicular to* $\mathbf{e}$*, and if* $\theta_\alpha(\tau)$ *is the angle at time* τ *subtended by the material points that occupy the places* $\mathbf{x} + \alpha\mathbf{e}, \mathbf{x}, \mathbf{x} + \alpha\mathbf{d}$ *at time t, then*

$$\lim_{\alpha\to 0} \dot{\theta}_\alpha(t) = -2\mathbf{d}\cdot\mathbf{D}(\mathbf{x}, t)\mathbf{e}.$$

Proof. Since $\delta_\alpha(t)$ is the distance between $\mathbf{x}$ and $\mathbf{x} + \alpha\mathbf{e}$, which is α, (6) with $\mathbf{y} = \mathbf{x} + \alpha\mathbf{e}$ implies

$$\frac{\dot{\delta}_\alpha(t)}{\delta_\alpha(t)} = \frac{\mathbf{e}\cdot[\mathbf{v}(\mathbf{x} + \alpha\mathbf{e}, t) - \mathbf{v}(\mathbf{x}, t)]}{\alpha}.$$

But

$$\lim_{\alpha\to 0} \frac{1}{\alpha}[\mathbf{v}(\mathbf{x} + \alpha\mathbf{e}, t) - \mathbf{v}(\mathbf{x}, t)] = \mathbf{L}(\mathbf{x}, t)\mathbf{e} \tag{9}$$

and

$$\mathbf{e}\cdot\mathbf{L}(\mathbf{x}, t)\mathbf{e} = \mathbf{e}\cdot\mathbf{D}(\mathbf{x}, t)\mathbf{e}$$, MA L ≠ D

[since $\mathbf{W}(\mathbf{x}, t)$ is skew]. Thus (8) holds.

Next, let $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{a}$ denote the material points that occupy the places $\mathbf{x}$, $\mathbf{y} = \mathbf{x} + \alpha\mathbf{e}$, and $\mathbf{z} = \mathbf{x} + \alpha\mathbf{d}$, respectively, at time t. Further, let

$$\mathbf{u}(\tau) = \mathbf{x}(\mathbf{q}, \tau) - \mathbf{x}(\mathbf{p}, \tau),$$
$$\mathbf{w}(\tau) = \mathbf{x}(\mathbf{a}, \tau) - \mathbf{x}(\mathbf{p}, \tau),$$

so that

$$\mathbf{u}(t) = \alpha\mathbf{e}, \qquad \dot{\mathbf{u}}(t) = \mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t),$$
$$\mathbf{w}(t) = \alpha\mathbf{d}, \qquad \dot{\mathbf{w}}(t) = \mathbf{v}(\mathbf{z}, t) - \mathbf{v}(\mathbf{x}, t).$$

Thus

$$\frac{1}{\alpha}(\mathbf{u}\cdot\mathbf{w})^{\cdot}(t) = \mathbf{e}\cdot[\mathbf{v}(\mathbf{z}, t) - \mathbf{v}(\mathbf{x}, t)] + \mathbf{d}\cdot[\mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t)].$$

Further,

$$\cos\theta_\alpha = \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}||\mathbf{w}|},$$

and, as $\mathbf{u}$ and $\mathbf{w}$ are orthogonal at time t,

$$(\cos\theta_\alpha)^{\cdot}(t) = \frac{(\mathbf{u}\cdot\mathbf{w})^{\cdot}(t)}{|\mathbf{u}(t)||\mathbf{w}(t)|}.$$

On the other hand, since $\sin\theta_\alpha(t) = 1$,

$$(\cos\theta_\alpha)^{\cdot}(t) = -\dot{\theta}_\alpha(t),$$

and the above relations imply that

$$-\alpha\dot{\theta}_\alpha(t) = \mathbf{e}\cdot[\mathbf{v}(\mathbf{x} + \alpha\mathbf{d}, t) - \mathbf{v}(\mathbf{x}, t)] + \mathbf{d}\cdot[\mathbf{v}(\mathbf{x} + \alpha\mathbf{e}, t) - \mathbf{v}(\mathbf{x}, t)].$$

If we divide by α and let $\alpha \to 0$, we conclude, with the aid of (9) and its counterpart for $\mathbf{d}$, that

$$\begin{aligned}\lim_{\alpha\to 0} \dot{\theta}_\alpha(t) &= -\mathbf{e}\cdot\mathbf{L}(\mathbf{x}, t)\mathbf{d} - \mathbf{d}\cdot\mathbf{L}(\mathbf{x}, t)\mathbf{e} \\ &= -\mathbf{d}\cdot[\mathbf{L}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)^T]\mathbf{e} \\ &= -2\mathbf{d}\cdot\mathbf{D}(\mathbf{x}, t)\mathbf{e}. \quad \square\end{aligned}$$

Using the spin $\mathbf{W}$ we can establish the following important relations for the acceleration $\dot{\mathbf{v}}$.

Proposition

$$\begin{aligned}\dot{\mathbf{v}} &= \mathbf{v}' + \tfrac{1}{2}\,\mathrm{grad}(v^2) + 2\mathbf{W}\mathbf{v}, \\ \dot{\mathbf{v}} &= \mathbf{v}' + \tfrac{1}{2}\,\mathrm{grad}(v^2) + (\mathrm{curl}\,\mathbf{v}) \times \mathbf{v}.\end{aligned} \tag{10}$$

$\dot{v} = v' + (\text{grad } v)v$
$\dot{v} = v' + \frac{1}{2}\text{grad}(v^2) + 2Wv$

Proof. Since

$$2\mathbf{W}\mathbf{v} = (\text{grad } \mathbf{v} - \text{grad } \mathbf{v}^T)\mathbf{v} = (\text{grad } \mathbf{v})\mathbf{v} - \tfrac{1}{2}\,\text{grad}(\mathbf{v}^2), \qquad (11)$$

$(10)_1$ follows from (8.5). The result $(10)_2$ follows from $(10)_1$ and the fact that curl $\mathbf{v}$ is twice the axial vector corresponding to $\mathbf{W}$. □

A motion is **plane** if the velocity field has the form

$$\mathbf{v}(\mathbf{x}, t) = v_1(x_1, x_2, t)\mathbf{e}_1 + v_2(x_1, x_2, t)\mathbf{e}_2$$

in some cartesian frame.

Proposition. *In a plane motion*

$$\mathbf{W}\mathbf{D} + \mathbf{D}\mathbf{W} = (\text{div } \mathbf{v})\mathbf{W}. \qquad (12)$$

Proof. Clearly, $\mathbf{D}$ and $\mathbf{W}$ have matrices of the form

$$[\mathbf{D}] = \begin{bmatrix} \alpha & \lambda & 0 \\ \lambda & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\mathbf{W}] = \begin{bmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(relative to the above frame), and a trivial computation shows that

$$[\mathbf{W}\mathbf{D} + \mathbf{D}\mathbf{W}] = \begin{bmatrix} 0 & \gamma(\alpha + \beta) & 0 \\ -\gamma(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (\alpha + \beta)[\mathbf{W}].$$

But

$$\text{div } \mathbf{v} = \text{tr } \mathbf{L} = \text{tr } \mathbf{D} = \alpha + \beta,$$

and the proof is complete. □

EXERCISES

1. It is often convenient to label material points by their positions at a given time τ. Suppose that a material point $\mathbf{p}$ occupies the place $\mathbf{y}$ at τ and $\mathbf{x}$ at an arbitrary time t (Fig. 10):

$$\mathbf{y} = \mathbf{x}(\mathbf{p}, \tau), \qquad \mathbf{x} = \mathbf{x}(\mathbf{p}, t).$$

Roughly speaking, we want $\mathbf{x}$ as a function of $\mathbf{y}$. Thus, since

$$\mathbf{p} = \mathbf{p}(\mathbf{y}, \tau),$$

we have

$$\mathbf{x} = \mathbf{x}(\mathbf{p}(\mathbf{y}, \tau), t).$$

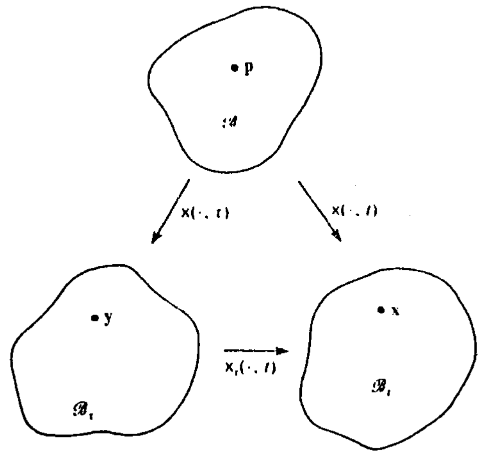

Figure 10

We call the function

$$\mathbf{x}_\tau : \mathcal{B}_\tau \times \mathbb{R} \to \mathcal{E}$$

defined by

$$\mathbf{x}_\tau(\mathbf{y}, t) = \mathbf{x}(\mathbf{p}(\mathbf{y}, \tau), t)$$

the *motion relative to time* τ; $\mathbf{x}_\tau(\mathbf{y}, t)$ is the place occupied at time t by the material point that occupies $\mathbf{y}$ at time τ. Let

$$\mathbf{F}_\tau(\mathbf{y}, t) = \nabla_y \mathbf{x}_\tau(\mathbf{y}, t),$$

where ∇_y is the gradient with respect to $\mathbf{y}$ holding t fixed. Also let

$$\mathbf{F}_\tau = \mathbf{R}_\tau \mathbf{U}_\tau$$

denote the right polar decomposition of $\mathbf{F}_\tau$, and define

$$\mathbf{C}_\tau = (\mathbf{U}_\tau)^2.$$

(a) Show that

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial}{\partial t}\mathbf{x}_\tau(\mathbf{y}, t) \qquad (13)$$

provided $\mathbf{x} = \mathbf{x}_\tau(\mathbf{y}, t)$.

(b) Use the relation $\mathbf{x}(\cdot, t) = \mathbf{x}_\tau(\cdot, t) \circ \mathbf{x}(\cdot, \tau)$ to show that

$$\mathbf{F}_\tau(\mathbf{y}, t)\mathbf{F}(\mathbf{p}, \tau) = \mathbf{F}(\mathbf{p}, t),$$

where $\mathbf{y} = \mathbf{x}(\mathbf{p}, \tau)$, and then appeal to the uniqueness of the polar decomposition to prove that

$$\mathbf{F}_\tau(\mathbf{y}, \tau) = \mathbf{U}_\tau(\mathbf{y}, \tau) = \mathbf{R}_\tau(\mathbf{y}, \tau) = \mathbf{I}.$$

(c) Show that

$$\mathbf{C}(\mathbf{p}, t) = \mathbf{F}(\mathbf{p}, \tau)^{\mathrm{T}}\mathbf{C}_\tau(\mathbf{y}, t)\mathbf{F}(\mathbf{p}, \tau). \tag{14}$$

(d) Show that

$$\mathbf{L}(\mathbf{y}, \tau) = \frac{\partial}{\partial t}\mathbf{F}_\tau(\mathbf{y}, t)|_{t=\tau}$$

$$= \left[\frac{\partial}{\partial t}\mathbf{U}_\tau(\mathbf{y}, t) + \frac{\partial}{\partial t}\mathbf{R}_\tau(\mathbf{y}, t)\right]_{t=\tau}, \tag{15}$$

$$\mathbf{D}(\mathbf{y}, \tau) = \frac{\partial}{\partial t}\mathbf{U}_\tau(\mathbf{y}, t)|_{t=\tau}, \quad \mathbf{W}(\mathbf{y}, \tau) = \frac{\partial}{\partial t}\mathbf{R}_\tau(\mathbf{y}, t)|_{t=\tau}. \tag{16}$$

(e) Show that

$$\frac{\partial^{n+2}}{\partial t^{n+2}}\mathbf{F}_\tau(\mathbf{y}, t)|_{t=\tau} = \operatorname{grad} \mathbf{a}^{(n)}(\mathbf{y}, \tau),$$

where $\mathbf{a}^{(n)}$ is the spatial description of the material time derivative of x of order $n + 2$.

2. Let x be a motion and suppose that for some fixed τ,

$$\mathbf{x}_\tau(\mathbf{y}, t) = \mathbf{q}(t) + \mathbf{Q}(t)(\mathbf{y} - \mathbf{z}), \tag{17}$$

with z and $\mathbf{q}(t)$ points and $\mathbf{Q}(t) \in \text{Orth}^+$. Show that x is rigid.

3. Let x be a rigid motion. Show that $\mathbf{x}_\tau$ has the form (17).

Exercises 2 and 3 assert that, given a motion x and a time τ, $\mathbf{x}_\tau(\cdot, t)$ is a rigid deformation at each t if and only if x is a rigid motion.

4. Let x be a C^∞ motion. The tensors

$$\mathbf{A}_n(\mathbf{y}, \tau) = \frac{\partial^n}{\partial t^n}\mathbf{C}_\tau(\mathbf{y}, t)|_{t=\tau} \qquad (n = 1, 2, \ldots) \tag{18}$$

are called the *Rivlin–Ericksen tensors.*

(a) Show that $\mathbf{A}_1 = 2\mathbf{D}$.

(b) Show, by differentiating (14) with respect to t, that

$$\mathbf{C}^{(n)} = \mathbf{F}^{\mathrm{T}}\mathbf{A}_n\mathbf{F}, \tag{19}$$

where $\mathbf{C}^{(n)}$ is the nth material time derivative of C, and where we have omitted the subscript o from C and F.

(c) Verify that

$$\mathbf{A}_{n+1} = \dot{\mathbf{A}}_n + \mathbf{A}_n\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_n.$$

5. Show that the acceleration field of a rigid motion has the form

$$\dot{\mathbf{v}}(\mathbf{x}, t) = \dot{\mathbf{v}}(\mathbf{y}, t) + \dot{\boldsymbol{\omega}}(t) \times (\mathbf{x} - \mathbf{y}) + \boldsymbol{\omega}(t) \times [\boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{y})]$$

with $\boldsymbol{\omega}$ the angular velocity.

10. TRANSPORT THEOREMS. VOLUME. ISOCHORIC MOTIONS

Let x be a motion of $\mathscr{B}$. Given a part $\mathscr{P}$, we write

$$\mathscr{P}_t = \mathbf{x}(\mathscr{P}, t)$$

for the region of space occupied by $\mathscr{P}$ at time t. Thus

$$\text{vol}(\mathscr{P}_t) = \int_{\mathscr{P}_t} dV$$

represents the volume of $\mathscr{P}$ at time t. Using the deformation gradient $\mathbf{F} = \nabla \mathbf{x}$ we can also express $\text{vol}(\mathscr{P}_t)$ as an integral over $\mathscr{P}$ itself [cf. (6.15)]:

$$\text{vol}(\mathscr{P}_t) = \int_{\mathscr{P}} \det \mathbf{F}\, dV.$$

Thus

$$\frac{d}{dt}\text{vol}(\mathscr{P}_t) = \int_{\mathscr{P}} (\det \mathbf{F})^{\cdot}\, dV. \tag{1}$$

Next, (3.14) and $(8.8)_1$ imply that

$$(\det \mathbf{F})^{\cdot} = (\det \mathbf{F}) \operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = (\det \mathbf{F}) \operatorname{tr} \mathbf{L}_m.$$

But

$$\operatorname{tr} \mathbf{L} = \operatorname{tr} \operatorname{grad} \mathbf{v} = \operatorname{div} \mathbf{v},$$

so that

$$(\det \mathbf{F})^{\cdot} = (\det \mathbf{F})(\operatorname{div} \mathbf{v})_m \tag{2}$$

and (1) becomes

$$\frac{d}{dt}\text{vol}(\mathscr{P}_t) = \int_{\mathscr{P}} (\operatorname{div} \mathbf{v})_m \det \mathbf{F}\, dV = \int_{\mathscr{P}_t} \operatorname{div} \mathbf{v}\, dV.$$

Thus we have the following

Theorem (Transport of volume). *For any part $\mathscr{P}$ and time t,*

$$\frac{d}{dt}\operatorname{vol}(\mathscr{P}_t) = \int_{\mathscr{P}} (\det \mathbf{F})^{\boldsymbol{\cdot}}\, dV = \int_{\mathscr{P}_t} \operatorname{div} \mathbf{v}\, dV = \int_{\partial\mathscr{P}_t} \mathbf{v}\cdot\mathbf{n}\, dA. \tag{3}$$

Thus $(\det \mathbf{F})^{\boldsymbol{\cdot}}$ and $\operatorname{div} \mathbf{v}$ represent rates of change of volume per unit volume: $(\det \mathbf{F})^{\boldsymbol{\cdot}}$ is measured per unit volume in the reference configuration; $\operatorname{div} \mathbf{v}$ is measured per unit volume in the current configuration.

We say that $\mathbf{x}$ is **isochoric** if

$$\frac{d}{dt}\operatorname{vol}(\mathscr{P}_t) = 0 \tag{4}$$

for every part $\mathscr{P}$ and time t. As a direct consequence of (3) and (4) we have the following

Theorem (Characterization of isochoric motions). *The following are equivalent:*

(a) $\mathbf{x}$ *is isochoric.*
(b) $(\det \mathbf{F})^{\boldsymbol{\cdot}} = 0$.
(c) $\operatorname{div} \mathbf{v} = 0$.
(d) *For every part $\mathscr{P}$ and time t,*

$$\int_{\partial\mathscr{P}_t} \mathbf{v}\cdot\mathbf{n}\, dA = 0.$$

Warning: For a motion to be isochoric the volume of each part must be constant throughout the motion; it is *not* necessary that the volume of a part during the motion be equal to its volume in the reference configuration.

Note that rigid motions are isochoric, a fact which follows from (c) of the above theorem and (c) of the theorem on page 69.

The computations leading to (3) are easily generalized; the result is

Reynolds' Transport Theorem. *Let Φ be a smooth spatial field, and assume that Φ is either scalar valued or vector valued. Then for any part $\mathscr{P}$ and time t,*

$$\begin{aligned}\frac{d}{dt}\int_{\mathscr{P}_t} \Phi\, dV &= \int_{\mathscr{P}_t} (\dot\Phi + \Phi \operatorname{div} \mathbf{v})\, dV,\\ \frac{d}{dt}\int_{\mathscr{P}_t} \Phi\, dV &= \int_{\mathscr{P}_t} \Phi'\, dV + \int_{\partial\mathscr{P}_t} \Phi \mathbf{v}\cdot\mathbf{n}\, dA.\end{aligned} \tag{5}$$

Proof. In view of (2),

$$\frac{d}{dt}\int_{\mathscr{P}_t} \Phi\, dV = \frac{d}{dt}\int_{\mathscr{P}} \Phi_m \det \mathbf{F}\, dV = \int_{\mathscr{P}} (\Phi_m \det \mathbf{F})^{\boldsymbol{\cdot}}\, dV$$

$$= \int_{\mathscr{P}} (\dot\Phi + \Phi \operatorname{div} \mathbf{v})_m \det \mathbf{F}\, dV = \int_{\mathscr{P}_t} (\dot\Phi + \Phi \operatorname{div} \mathbf{v})\, dV,$$

which is $(5)_1$. To derive $(5)_2$ assume that Φ is scalar valued. Then $(8.4)_1$ yields the identity

$$\dot\Phi + \Phi \operatorname{div} \mathbf{v} = \Phi' + \operatorname{div}(\Phi\mathbf{v}),$$

and $(5)_2$ follows from $(5)_1$ and the divergence theorem. The proof for Φ vector valued is exactly the same. □

Note that

$$\int_{\mathscr{P}_t} \Phi'\, dV = \int_{\mathscr{P}_t} \frac{\partial}{\partial t}\Phi(\mathbf{x}, t)\, dV_{\mathbf{x}} = \left[\frac{d}{d\tau}\int_{\mathscr{P}_t} \Phi(\mathbf{x}, \tau)\, dV_{\mathbf{x}}\right]_{\tau=t}.$$

Thus $(5)_2$ asserts that the rate at which the integral of Φ over $\mathscr{P}_t$ is changing is equal to the rate computed as if $\mathscr{P}_t$ were *fixed* in its current position plus the rate at which Φ is carried out of this region across its boundary.

Exercises

1. Let β be a smooth spatial scalar field with $\dot\beta = 0$ and define $\varphi = \beta/(\det \mathbf{F})_s$. Show that

$$\varphi' + \operatorname{div}(\varphi\mathbf{v}) = 0.$$

2. Prove that (for $\mathscr{B}$ bounded)

$$\int_{\mathscr{B}_t} (2\mathbf{W}\mathbf{v} + \mathbf{v} \operatorname{div} \mathbf{v})\, dV = \int_{\partial\mathscr{B}_t} [\mathbf{v}(\mathbf{v}\cdot\mathbf{n}) - \tfrac{1}{2}v^2\mathbf{n}]\, dA,$$

where $\mathbf{n}$ is the outward unit normal to $\partial\mathscr{B}_t$, so that in an isochoric motion with $\mathbf{v} = \mathbf{0}$ on $\partial\mathscr{B}_t$,

$$\int_{\mathscr{B}_t} \mathbf{W}\mathbf{v}\, dV = \mathbf{0}.$$

3. Derive $(5)_2$ for Φ vector valued.

11. SPIN. CIRCULATION. VORTICITY

As we have seen, a rigid motion is characterized by a skew velocity gradient; that is, the velocity field is determined (up to a spatially uniform vector field) by its spin

$$\mathbf{W} = \tfrac{1}{2}(\operatorname{grad} \mathbf{v} - \operatorname{grad} \mathbf{v}^T).$$

More generally, given any motion the spin $\mathbf{W}(\mathbf{x}, t)$ describes the local rigid rotation of material points currently near $\mathbf{x}$. As our next theorem shows, the evolution of $\mathbf{W}$ with time is governed by the field

$$\mathbf{J} = \tfrac{1}{2}(\operatorname{grad} \dot{\mathbf{v}} - \operatorname{grad} \dot{\mathbf{v}}^T),$$

which represents the skew part of the acceleration gradient. The statement and proof of this theorem are greatly facilitated if we introduce the notation

$$\mathbf{G}_\mathbf{F} = \mathbf{F}^T\mathbf{G}_m\mathbf{F} \tag{1}$$

for any spatial tensor field $\mathbf{G}$, where $\mathbf{F}$ is the deformation gradient.

Theorem (Transport of spin). *The spin $\mathbf{W}$ satisfies the differential equations*

$$(\mathbf{W}_\mathbf{F})^{\cdot} = \mathbf{J}_\mathbf{F},$$
$$\dot{\mathbf{W}} + \mathbf{D}\mathbf{W} + \mathbf{W}\mathbf{D} = \mathbf{J}, \tag{2}$$

where $\mathbf{D}$ is the stretching.

Proof. Recall (8.8):

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \qquad \ddot{\mathbf{F}} = (\operatorname{grad} \dot{\mathbf{v}})\mathbf{F},$$

where for convenience we have omitted the subscript m from $\mathbf{L}$ and $\operatorname{grad} \dot{\mathbf{v}}$. Since $2\mathbf{W} = \mathbf{L} - \mathbf{L}^T$,

$$2\mathbf{W}_\mathbf{F} = \mathbf{F}^T\dot{\mathbf{F}} - \dot{\mathbf{F}}^T\mathbf{F},$$

and hence

$$2\dot{\mathbf{W}}_\mathbf{F} = \mathbf{F}^T\ddot{\mathbf{F}} - \ddot{\mathbf{F}}^T\mathbf{F} = \mathbf{F}^T \operatorname{grad} \dot{\mathbf{v}}\mathbf{F} - \mathbf{F}^T \operatorname{grad} \dot{\mathbf{v}}^T\mathbf{F},$$

which implies $(2)_1$. Next, by (1) and $(2)_1$,

$$\mathbf{F}^T\dot{\mathbf{W}}\mathbf{F} + \dot{\mathbf{F}}^T\mathbf{W}\mathbf{F} + \mathbf{F}^T\mathbf{W}\dot{\mathbf{F}} = \mathbf{F}^T\mathbf{J}\mathbf{F}.$$

Thus

$$\dot{\mathbf{W}} + \mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{W} + \mathbf{W}\dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{J},$$

and, since $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and $\mathbf{L}^T = \mathbf{F}^{-T}\dot{\mathbf{F}}^T$, this equation implies

$$\dot{\mathbf{W}} + \mathbf{L}^T\mathbf{W} + \mathbf{W}\mathbf{L} = \mathbf{J}.$$

Thus as

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \qquad \mathbf{D} = \mathbf{D}^T, \qquad \mathbf{W} = -\mathbf{W}^T,$$

it follows that

$$\dot{\mathbf{W}} + (\mathbf{D} - \mathbf{W})\mathbf{W} + \mathbf{W}(\mathbf{D} + \mathbf{W}) = \mathbf{J},$$

which implies $(2)_2$. □

We now introduce two important (and somewhat related) definitions. A motion is **irrotational** if

$$\mathbf{W} = \mathbf{0},$$

or equivalently, if

$$\operatorname{curl} \mathbf{v} = \mathbf{0}.$$

A spatial vector field $\mathbf{g}$ is the **gradient of a potential** if there exists a spatial scalar field α such that

$$\mathbf{g}(\mathbf{x}, t) = \operatorname{grad} \alpha(\mathbf{x}, t)$$

for all $(\mathbf{x}, t)$ on the trajectory of the motion.

For a large class of fluids, in particular, inviscid fluids under a conservative body force, the acceleration $\dot{\mathbf{v}}$ is the gradient of a potential. When this is the case the potential, α say, is C^2 in $\mathbf{x}$, because $\dot{\mathbf{v}}$ is C^1, and thus grad grad α is symmetric; therefore

$$\mathbf{J} = \tfrac{1}{2}[\operatorname{grad}\operatorname{grad} \alpha - (\operatorname{grad}\operatorname{grad} \alpha)^T] = \mathbf{0},$$

and we have the following consequence of $(2)_1$.

Lagrange–Cauchy Theorem. *A motion with acceleration the gradient of a potential is irrotational if it is irrotational at one time.*

Proof. Since $\mathbf{J} = \mathbf{0}$, we conclude from $(2)_1$ that

$$(\mathbf{W}_\mathbf{F})^{\cdot} = \mathbf{0}, \tag{3}$$

so that $\mathbf{W}_\mathbf{F}(\mathbf{p}, t)$ is independent of t. But at some time τ, $\mathbf{W}(\mathbf{x}, \tau) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathscr{B}_\tau$, and hence $\mathbf{W}_\mathbf{F}(\mathbf{p}, \tau) = \mathbf{0}$ for all $\mathbf{p}$ in $\mathscr{B}$. Thus $\mathbf{W}_\mathbf{F}(\mathbf{p}, t) = \mathbf{0}$ for all $\mathbf{p}$ and t, and, since $\mathbf{F}$ is invertible, (1) implies that $\mathbf{W} = \mathbf{0}$. Hence the motion is irrotational. □

As is clear from (9.12) and $(2)_2$, for plane, isochoric motions a result stronger than (3) holds.

Proposition. *For a plane, isochoric motion with acceleration the gradient of a potential,*

$$\dot{\mathbf{W}} = \mathbf{0}.$$

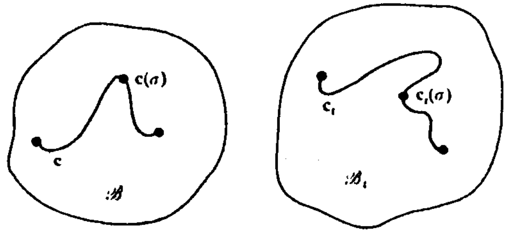

Figure 11

Let x be a motion of $\mathscr{B}$. By a **material curve** we mean a curve c in $\mathscr{B}$. Choose an arbitrary material point $\mathbf{c}(\sigma)$ on **c**. At time t, $\mathbf{c}(\sigma)$ will occupy the place

$$\mathbf{x}(\mathbf{c}(\sigma), t).$$

Thus the material points of the curve form a curve

$$\mathbf{c}_t(\sigma) = \mathbf{x}(\mathbf{c}(\sigma), t), \qquad 0 \le \sigma \le 1 \tag{4}$$

in $\mathscr{B}_t$ (Fig. 11). When **c** (and hence $\mathbf{c}_t$) is closed,

$$\int_{\mathbf{c}_t} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x}$$

gives the **circulation** around **c** at time t; this integral sums the tangential component of the velocity around the curve $\mathbf{c}_t$.

Theorem (Transport of circulation). *Let* **c** *be a closed material curve. Then*

$$\frac{d}{dt} \int_{\mathbf{c}_t} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} = \int_{\mathbf{c}_t} \dot{\mathbf{v}}(\mathbf{x}, t) \cdot d\mathbf{x}. \tag{5}$$

Proof. By definition,

$$\int_{\mathbf{c}_t} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} = \int_0^1 \mathbf{v}(\mathbf{c}_t(\sigma), t) \cdot \frac{\partial}{\partial \sigma} \mathbf{c}_t(\sigma)\, d\sigma;$$

our proof will involve differentiating the right side under the integral. We therefore begin by investigating the partial derivatives $(\partial/\partial t)\mathbf{v}(\mathbf{c}_t(\sigma), t)$ and $(\partial^2/\partial t\, \partial \sigma)\mathbf{c}_t(\sigma)$. Note that $\mathbf{c}_t(\sigma)$ has derivatives with respect to σ and t which are jointly continuous in (σ, t). In particular, by (4),

$$\frac{\partial}{\partial t} \mathbf{c}_t(\sigma) = \dot{\mathbf{x}}(\mathbf{c}(\sigma), t) = \mathbf{v}(\mathbf{c}_t(\sigma), t). \tag{6}$$

Clearly, this relation has a derivative with respect to σ which is jointly continuous in (σ, t). Thus we can switch the order of differentiation; that is,

$$\frac{\partial^2}{\partial t\, \partial \sigma} \mathbf{c}_t(\sigma)$$

exists and by (6)

$$\frac{\partial^2}{\partial t\, \partial \sigma} \mathbf{c}_t(\sigma) = \frac{\partial^2}{\partial \sigma\, \partial t} \mathbf{c}_t(\sigma) = \frac{\partial}{\partial \sigma} \mathbf{v}(\mathbf{c}_t(\sigma), t).$$

Note also that (6) implies

$$\frac{\partial}{\partial t} \mathbf{v}(\mathbf{c}_t(\sigma), t) = \ddot{\mathbf{x}}(\mathbf{c}(\sigma), t) = \dot{\mathbf{v}}(\mathbf{c}_t(\sigma), t).$$

These identities can be used to transform the left side of (5) as follows:

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbf{c}_t} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} &= \frac{d}{dt} \int_0^1 \mathbf{v}(\mathbf{c}_t(\sigma), t) \cdot \frac{\partial \mathbf{c}_t(\sigma)}{\partial \sigma}\, d\sigma \\
&= \int_{\mathbf{c}_t} \dot{\mathbf{v}}(\mathbf{x}, t) \cdot d\mathbf{x} + \int_0^1 \mathbf{v}(\mathbf{c}_t(\sigma), t) \cdot \frac{\partial}{\partial \sigma} \mathbf{v}(\mathbf{c}_t(\sigma), t)\, d\sigma \\
&= \int_{\mathbf{c}_t} \dot{\mathbf{v}}(\mathbf{x}, t) \cdot d\mathbf{x} + \tfrac{1}{2}\{\mathbf{v}^2(\mathbf{c}_t(1), t) - \mathbf{v}^2(\mathbf{c}_t(0), t)\}.
\end{aligned}$$

But **c** (and hence $\mathbf{c}_t$) is closed; thus $\mathbf{c}_t(1) = \mathbf{c}_t(0)$ and the last term vanishes. □

We say that the motion **preserves circulation** if

$$\frac{d}{dt} \int_{\mathbf{c}_t} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} = 0$$

for every closed material curve **c** and all time t. When $\dot{\mathbf{v}} = \operatorname{grad} \alpha$ the right side of (5) vanishes, as $\mathbf{c}_t$ is closed [cf. (4.8)], and we have

Kelvin's Theorem. *Assume that the acceleration is the gradient of a potential. Then the motion preserves circulation.*

A curve **h** in $\mathscr{B}_t$ is a **vortex line** at time t if the tangent to **h** at each point **x** on **h** lies on the spin axis of the motion at $(\mathbf{x}, t)$. Since the spin axis at $(\mathbf{x}, t)$ is the set of all vectors **e** such that

$$\mathbf{W}(\mathbf{x}, t)\mathbf{e} = \mathbf{0},$$

h is a vortex line if and only if

$$\mathbf{W}(\mathbf{h}(\sigma), t) \frac{d\mathbf{h}(\sigma)}{d\sigma} = \mathbf{0}$$

for $0 \le \sigma \le 1$.

Theorem (Transport of vorticity). *Assume that the acceleration is the gradient of a potential. Then* **vortex lines are transported with the motion**; *that is, for any material curve* $\mathbf{c}$, *if* $\mathbf{c}_t$ *is a vortex line at some time* $t = \tau$, *then* $\mathbf{c}_t$ *is a vortex line for all time* t.

Proof. Let $\mathbf{c}$ be a material curve. Then by (4),

$$\frac{d}{d\sigma}\mathbf{c}_t(\sigma) = \mathbf{F}(\mathbf{c}(\sigma), t)\mathbf{k}(\sigma), \qquad \mathbf{k}(\sigma) = \frac{d\mathbf{c}(\sigma)}{d\sigma}. \tag{7}$$

If $\mathbf{c}_\tau$ is a vortex line, then

$$0 = \mathbf{W}(\mathbf{c}_\tau(\sigma), \tau)\frac{d}{d\sigma}\mathbf{c}_\tau(\sigma) = \mathbf{W}(\mathbf{c}_\tau(\sigma), \tau)\mathbf{F}(\mathbf{c}(\sigma), \tau)\mathbf{k}(\sigma),$$

so that trivially

$$\mathbf{W}_\mathbf{F}(\mathbf{c}(\sigma), \tau)\mathbf{k}(\sigma) = \mathbf{0}. \tag{8}$$

Moreover, by (3), $\mathbf{W}_\mathbf{F}$ is constant in time, so that (8) is valid for *any* t. Therefore, multiplying by $\mathbf{F}(\mathbf{c}(\sigma), t)^{-\mathrm{T}}$ and using (1),

$$\mathbf{W}(\mathbf{c}_t(\sigma), t)\mathbf{F}(\mathbf{c}(\sigma), t)\mathbf{k}(\sigma) = \mathbf{0},$$

and this relation, with (7), implies that $\mathbf{c}_t$ is a vortex line for all t. □

We close this section by listing an important property of irrotational, isochoric motions; this result follows from (c) of the theorem on page 78 and the proposition on page 32.

Theorem. *The velocity field of an irrotational, isochoric motion is harmonic:*

$$\Delta\mathbf{v} = \mathbf{0}.$$

Exercises

We use the notation

$$\mathbf{w} = \operatorname{curl}\mathbf{v}, \qquad \upsilon = (\det\mathbf{F})_\kappa.$$

1. Show that in a plane motion

$$(\upsilon\mathbf{W})^{\cdot} = \upsilon\mathbf{J},$$

so that when $\dot{\mathbf{v}}$ is the gradient of a potential,

$$(\upsilon\mathbf{W})^{\cdot} = \mathbf{0}.$$

2. Establish the identity

$$(\upsilon\mathbf{w})^{\cdot} = \upsilon\mathbf{L}\mathbf{w} + \upsilon\operatorname{curl}\dot{\mathbf{v}}.$$

Note that when $\dot{\mathbf{v}}$ is the gradient of a potential this reduces to

$$(\upsilon\mathbf{w})^{\cdot} = \upsilon\mathbf{L}\mathbf{w}. \tag{9}$$

3. Assume that $\dot{\mathbf{v}}$ is the gradient of a potential. Show, as a consequence of (9), that

$$\mathbf{w}(\mathbf{x}, t) = [\det\mathbf{F}_\tau(\mathbf{y}, t)]^{-1}\mathbf{F}_\tau(\mathbf{y}, t)\mathbf{w}(\mathbf{y}, \tau),$$

where $\mathbf{x}$ is the place occupied at time t by the material point which occupies $\mathbf{y}$ at time τ (cf. Exercise 9.1).

4. Let $\mathbf{u}$ be a smooth spatial vector field and $\mathbf{c}$ a material curve. Show that

$$\frac{d}{dt}\int_{\mathbf{c}_t}\mathbf{u}\cdot d\mathbf{x} = \int_{\mathbf{c}_t}(\dot{\mathbf{u}} + \mathbf{L}^{\mathrm{T}}\mathbf{u})\cdot d\mathbf{x}.$$

5. Let $\mathbf{v}$ be the gradient of a potential φ. Show that

$$\dot{\mathbf{v}} = \operatorname{grad}\left(\varphi' + \frac{\mathbf{v}^2}{2}\right), \tag{10}$$

so that the acceleration is also the gradient of a potential.

Selected References

Chadwick [1, Chapter 2].
Eringen [1, Chapter 2].
Germain [1, Chapters 1, 5].
Serrin [1, §§11, 17, 21–23, 25–29].
Truesdell [1, Chapter 2].
Truesdell and Noll [1, §§21–25].
Truesdell and Toupin [1, §§13–149].

CHAPTER

IV

Mass. Momentum

12. CONSERVATION OF MASS

One of the most important properties of bodies is that they possess mass. We here consider bodies whose mass is distributed continuously. No matter how severely such a body is deformed, its mass is the integral of a density field; that is, given any deformation **f** there is a density field $\rho_{\mathbf{f}}$ on $\mathbf{f}(\mathscr{B})$ such that the **mass** $m(\mathscr{P})$ of any part $\mathscr{P}$ is given by

$$m(\mathscr{P}) = \int_{\mathbf{f}(\mathscr{P})} \rho_{\mathbf{f}} \, dV.$$

Since the mass of a part cannot be altered by deforming the part, $m(\mathscr{P})$ is independent of the deformation **f**.

We now make the above ideas precise. A **mass distribution** for $\mathscr{B}$ is a family of smooth density fields

$$\rho_{\mathbf{f}}: \mathbf{f}(\mathscr{B}) \to \mathbb{R}^+,$$

one for each deformation **f**, such that

$$\int_{\mathbf{f}(\mathscr{P})} \rho_{\mathbf{f}} \, dV = \int_{\mathbf{g}(\mathscr{P})} \rho_{\mathbf{g}} \, dV \equiv m(\mathscr{P}) \tag{1}$$

for any part $\mathscr{P}$ and all deformations $\mathbf{f}$ and $\mathbf{g}$. The number $\rho_\mathbf{f}(\mathbf{x})$ represents the **density** at the place $\mathbf{x} \in \mathbf{f}(\mathscr{B})$ in the deformation $\mathbf{f}$, and equation (1) expresses **conservation of mass.**

We denote by ρ_0 the density field $\rho_\mathbf{f}$ when $\mathbf{f}(\mathbf{p}) = \mathbf{p}$ for all $\mathbf{p} \in \mathscr{B}$. Thus $\rho_0(\mathbf{p})$ gives the density at $\mathbf{p}$ when the body is in the reference position. Note that, by the localization theorem,

$$\rho_0(\mathbf{p}) = \lim_{\delta \to 0} \frac{m(\Omega_\delta)}{\operatorname{vol}(\Omega_\delta)},$$

where Ω_δ is the ball of radius δ centered at $\mathbf{p}$. As our next result shows, the **reference density** ρ_0 determines the density in all deformations.

Proposition. *Let* $\mathbf{f}$ *be a deformation of* $\mathscr{B}$ *and let* $\mathbf{F} = \nabla\mathbf{f}$. *Then*

$$\rho_\mathbf{f}(\mathbf{x}) \det \mathbf{F}(\mathbf{p}) = \rho_0(\mathbf{p}) \tag{2}$$

provided $\mathbf{x} = \mathbf{f}(\mathbf{p})$.

Proof. By (1) and the definition of ρ_0,

$$\int_{\mathbf{f}(\mathscr{P})} \rho_\mathbf{f}(\mathbf{x})\, dV_\mathbf{x} = \int_{\mathscr{P}} \rho_0(\mathbf{p})\, dV_\mathbf{p}.$$

On the other hand, if we change the variable of integration on the left side from $\mathbf{x}$ to $\mathbf{p}$, we arrive at

$$\int_{\mathscr{P}} \rho_\mathbf{f}(\mathbf{f}(\mathbf{p})) \det \mathbf{F}(\mathbf{p})\, dV_\mathbf{p}.$$

Thus

$$\int_{\mathscr{P}} [\rho_\mathbf{f}(\mathbf{f}(\mathbf{p})) \det \mathbf{F}(\mathbf{p}) - \rho_0(\mathbf{p})]\, dV_\mathbf{p} = 0$$

for every part $\mathscr{P}$, and (2) follows from the localization theorem. □

Given a motion $\mathbf{x}$ of $\mathscr{B}$ we will always write $\rho(\mathbf{x}, t)$ for the density at the place $\mathbf{x} \in \mathscr{B}_t$ in the deformation $\mathbf{x}(\cdot, t)$. Thus

$$\rho : \mathscr{T} \to \mathbb{R}^+$$

is defined by

$$\rho(\mathbf{x}, t) = \rho_{\mathbf{x}(\cdot, t)}(\mathbf{x});$$

we will refer to ρ as the **density in the motion** $\mathbf{x}$. In view of (1),

$$m(\mathscr{P}) = \int_{\mathscr{P}_t} \rho(\mathbf{x}, t)\, dV_\mathbf{x} \equiv \int_{\mathscr{P}_t} \rho\, dV,$$

and we have the following

Theorem. *For every part* $\mathscr{P}$ *and time* t,

$$\frac{d}{dt} \int_{\mathscr{P}_t} \rho\, dV = 0. \tag{3}$$

If $\mathbf{F} = \nabla\mathbf{x}$ is the deformation gradient in the motion, then (2) takes the form

$$\rho(\mathbf{x}, t) \det \mathbf{F}(\mathbf{p}, t) = \rho_0(\mathbf{p}) \tag{4}$$

provided $\mathbf{x} = \mathbf{x}(\mathbf{p}, t)$. Thus ρ is the spatial description of $\rho_0/\det \mathbf{F}$, and we conclude from the smoothness lemma that ρ is smooth on $\mathscr{T}$.

Theorem (Local conservation of mass)

$$\begin{gathered} \dot\rho + \rho \operatorname{div} \mathbf{v} = 0, \\ \rho' + \operatorname{div}(\rho\mathbf{v}) = 0. \end{gathered} \tag{5}$$

Proof. By (4),

$$\rho(\det \mathbf{F})^{\cdot}_s + \dot\rho(\det \mathbf{F})_s = 0,$$

which with (10.2) yields $(5)_1$. Next, by $(8.4)_1$,

$$\dot\rho = \rho' + \mathbf{v} \cdot \operatorname{grad} \rho,$$

and $(5)_1$, when combined with this relation, implies $(5)_2$. □

Since a motion is isochoric if and only if $\operatorname{div} \mathbf{v} = 0$, (5) has the following

Corollary. *A motion is isochoric if and only if*

$$\dot\rho = 0.$$

Equation (3) expresses conservation of mass for a region $\mathscr{P}_t$ that moves with the body. It is often more convenient, however, to work with a *fixed* region $\mathscr{R}$ called a control volume. Of course, material will generally flow into and out of $\mathscr{R}$, a fact which will become clear in the derivation of the resulting conservation law.

By a **control volume** at time t we mean a bounded regular region $\mathscr{R}$ with

$$\mathscr{R} \subset \mathscr{B}_\tau$$

for all τ in some neighborhood of t. Thus for δ sufficiently small we have the situation shown in Fig. 1. Let $\mathbf{n}$ denote the outward unit normal to $\partial\mathscr{R}$. By the divergence theorem,

$$\int_{\mathscr{R}} \operatorname{div}(\rho\mathbf{v})\, dV = \int_{\partial\mathscr{R}} \rho\mathbf{v} \cdot \mathbf{n}\, dA.$$

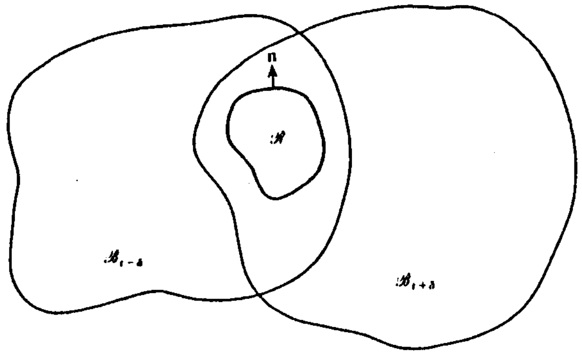

Figure 1

Further,

$$\int_{\mathscr{R}} \rho'(\mathbf{x}, t)\, dV_{\mathbf{x}} = \int_{\mathscr{R}} \frac{\partial}{\partial t} \rho(\mathbf{x}, t)\, dV_{\mathbf{x}} = \frac{d}{dt} \int_{\mathscr{R}} \rho(\mathbf{x}, t)\, dV_{\mathbf{x}},$$

where the last step uses the fact that $\mathscr{R}$ is independent of t. Conservation of mass $(5)_2$, when combined with the relations above, yields the following

Theorem (Conservation of mass for a control volume). *Let $\mathscr{R}$ be a control volume at time t. Then*

$$\frac{d}{dt} \int_{\mathscr{R}} \rho(\mathbf{x}, t)\, dV_{\mathbf{x}} = -\int_{\partial\mathscr{R}} \rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})\, dA_{\mathbf{x}}. \tag{6}$$

Since $\mathbf{n}$ is the outward unit normal to $\partial\mathscr{R}$, $\rho\mathbf{v} \cdot \mathbf{n}$ represents the mass flow, per unit area, out of $\mathscr{R}$ across its boundary. Thus (6) asserts that *the rate of increase of mass in $\mathscr{R}$ is equal to the mass flow into $\mathscr{R}$ across its boundary.*

Lemma. *Let Φ be a continuous spatial field. Then given any part $\mathscr{P}$,*

$$\int_{\mathscr{P}_t} \Phi(\mathbf{x}, t)\rho(\mathbf{x}, t)\, dV_{\mathbf{x}} = \int_{\mathscr{P}} \Phi_m(\mathbf{p}, t)\rho_0(\mathbf{p})\, dV_{\mathbf{p}}. \tag{7}$$

Proof. We simply change the variable of integration from $\mathbf{x}$ to $\mathbf{p}$ in the left side of (7); the result is

$$\int_{\mathscr{P}_t} \Phi(\mathbf{x}, t)\rho(\mathbf{x}, t)\, dV_{\mathbf{x}} = \int_{\mathscr{P}} \Phi_m(\mathbf{p}, t)\rho_m(\mathbf{p}, t) \det \mathbf{F}(\mathbf{p}, t)\, dV_{\mathbf{p}},$$

and, in view of (4), this relation implies (7). □

If we differentiate (7) with respect to t we arrive at the integral of $\dot{\Phi}_m\rho_0$ over $\mathscr{P}$, or equivalently [by (7) with Φ replaced by $\dot{\Phi}$] the integral of $\dot{\Phi}\rho$ over $\mathscr{P}_t$. Thus we have the following important

Theorem. *Let Φ be a smooth spatial field. Then given any part $\mathscr{P}$,*

$$\frac{d}{dt} \int_{\mathscr{P}_t} \Phi\rho\, dV = \int_{\mathscr{P}_t} \dot{\Phi}\rho\, dV. \tag{8}$$

In other words, to differentiate

$$\int_{\mathscr{P}_t} \Phi\rho\, dV$$

with respect to time we simply differentiate under the integral sign treating the "mass measure" $\rho\, dV$ as a constant.

Note that if we take $\Phi \equiv 1$ in (8) we recover (3).

Exercises

1. Derive $(5)_1$ and (6) using (3) and Reynold's transport theorem (10.5).
2. Show that a deformation $\mathbf{f}$ is isochoric if and only if

$$\rho_{\mathbf{f}}(\mathbf{x}) = \rho_0(\mathbf{p})$$

whenever $\mathbf{x} = \mathbf{f}(\mathbf{p})$.

3. Let Φ be a smooth spatial field. Show that for any part $\mathscr{P}$,

$$\int_{\mathscr{P}_t} \Phi(\mathbf{x}, t)\rho(\mathbf{x}, t)\, dV_{\mathbf{x}} = \int_{\mathscr{P}_\tau} \Phi(\mathbf{x}_\tau(\mathbf{y}, t), t)\rho(\mathbf{y}, \tau)\, dV_{\mathbf{y}},$$

where $\mathbf{x}_\tau$, defined in Exercise 9.1, is the motion relative to time τ.

4. Prove *Kelvin's theorem*: Of all motions of a body which, at a given time t,
   (a) correspond to a given steady density ρ,
   (b) have $\mathscr{B}_t$ assigned,
   (c) have $\mathbf{v} \cdot \mathbf{n}$ prescribed on $\partial\mathscr{B}_t$,

   one with velocity the gradient of a potential yields the least kinetic energy at time t. More precisely, let $\mathscr{R}$ be a bounded regular region of space, let $\rho > 0$ be a smooth scalar field on $\mathscr{R}$, let λ be a scalar field on $\partial\mathscr{R}$, let $\mathscr{A}$ be the set of all smooth vector fields $\mathbf{v}$ on $\mathscr{R}$ that satisfy

$$\operatorname{div}(\rho\mathbf{v}) = 0 \quad \text{in} \quad \mathscr{R}, \qquad \mathbf{v} \cdot \mathbf{n} = \lambda \quad \text{on} \quad \partial\mathscr{R},$$

and define the kinetic energy $\mathscr{K}$ on $\mathscr{A}$ by

$$\mathscr{K}\{\mathbf{v}\} = \int_{\mathscr{R}} \frac{\mathbf{v}^2}{2} \rho\, dV.$$

Show that if $\mathbf{v} \in \mathscr{A}$ has the form

$$\mathbf{v} = \operatorname{grad} \varphi,$$

then

$$\mathscr{K}\{\mathbf{v}\} \leq \mathscr{K}\{\mathbf{g}\}$$

for all $\mathbf{g} \in \mathscr{A}$, with equality holding only when $\mathbf{v} = \mathbf{g}$.

13. LINEAR AND ANGULAR MOMENTUM. CENTER OF MASS

Let $\mathbf{x}$ be a motion of $\mathscr{B}$. Given a part $\mathscr{P}$ of $\mathscr{B}$, the **linear momentum** $\mathbf{l}(\mathscr{P}, t)$ and the **angular momentum** $\mathbf{a}(\mathscr{P}, t)$ (about the origin $\mathbf{o}$) of $\mathscr{P}$ at time t are defined by

$$\begin{aligned} \mathbf{l}(\mathscr{P}, t) &= \int_{\mathscr{P}_t} \mathbf{v}\rho \, dV, \\ \mathbf{a}(\mathscr{P}, t) &= \int_{\mathscr{P}_t} \mathbf{r} \times \mathbf{v}\rho \, dV, \end{aligned} \tag{1}$$

where $\mathbf{r}: \mathscr{E} \to \mathscr{V}$ is the position vector

$$\mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{o}. \tag{2}$$

Proposition. *For every part $\mathscr{P}$ and time t,*

$$\begin{aligned} \dot{\mathbf{l}}(\mathscr{P}, t) &= \int_{\mathscr{P}_t} \dot{\mathbf{v}}\rho \, dV, \\ \dot{\mathbf{a}}(\mathscr{P}, t) &= \int_{\mathscr{P}_t} \mathbf{r} \times \dot{\mathbf{v}}\rho \, dV. \end{aligned} \tag{3}$$

Proof. The identity (12.8) yields $(3)_1$ and

$$\dot{\mathbf{a}}(\mathscr{P}, t) = \int_{\mathscr{P}_t} (\mathbf{r} \times \mathbf{v})^{\cdot} \rho \, dV.$$

But by (8.6),

$$(\mathbf{r} \times \mathbf{v})^{\cdot} = \mathbf{r} \times \dot{\mathbf{v}} + \mathbf{v} \times \mathbf{v},$$

and, since $\mathbf{v} \times \mathbf{v} = \mathbf{0}$, $(3)_2$ follows. □

Assume now that $\mathscr{B}$ is bounded, so that $m(\mathscr{B})$ is finite. Then the **center of mass** $\boldsymbol{\alpha}(t)$ at time t is the point of space defined by

$$\boldsymbol{\alpha}(t) - \mathbf{o} = \frac{1}{m(\mathscr{B})} \int_{\mathscr{B}_t} \mathbf{r}\rho \, dV. \tag{4}$$

(It is not difficult to show that this definition is independent of the choice of origin $\mathbf{o}$.) If we differentiate (4) with respect to t and use (12.8) and (8.6), we find that

$$\dot{\boldsymbol{\alpha}}(t) = \frac{1}{m(\mathscr{B})} \int_{\mathscr{B}_t} \mathbf{v}\rho \, dV,$$

so that $\dot{\boldsymbol{\alpha}}$ represents the average velocity of the body. This result and $(1)_1$ imply that

$$\mathbf{l}(\mathscr{B}, t) = m(\mathscr{B})\dot{\boldsymbol{\alpha}}(t). \tag{5}$$

Thus *the linear momentum of a body $\mathscr{B}$ is the same as that of a particle of mass $m(\mathscr{B})$ attached to the center of mass of $\mathscr{B}$.*

EXERCISES

In these exercises $\mathscr{B}$ is bounded.

1. Show that

$$\boldsymbol{\alpha}(t) - \mathbf{z} = \frac{1}{m(\mathscr{B})} \int_{\mathscr{B}_t} (\mathbf{x} - \mathbf{z})\rho(\mathbf{x}, t) \, dV_{\mathbf{x}}, \tag{6}$$

for any point $\mathbf{z}$, so that $\boldsymbol{\alpha}(t)$ is independent of the choice of origin.

2. Other types of momenta of interest are the angular momentum $\mathbf{a}_z(t)$ relative to a moving point $\mathbf{z}(t)$ and the *spin angular momentum* $\mathbf{a}_{spin}(t)$:

$$\begin{aligned} \mathbf{a}_z(t) &= \int_{\mathscr{B}_t} \mathbf{r}_z \times \mathbf{v}\rho \, dV, \\ \mathbf{a}_{spin}(t) &= \int_{\mathscr{B}_t} \mathbf{r}_\alpha \times \mathbf{v}_\alpha \rho \, dV. \end{aligned} \tag{7}$$

Here

$$\mathbf{r}_z(\mathbf{x}, t) = \mathbf{x} - \mathbf{z}(t) \tag{8}$$

is the position vector from $\mathbf{z}(t)$, $\mathbf{r}_\alpha(\mathbf{x}, t)$ is the position vector from $\boldsymbol{\alpha}(t)$, and

$$\mathbf{v}_\alpha = \dot{\mathbf{r}}_\alpha = \mathbf{v} - \dot{\boldsymbol{\alpha}}$$

is the velocity relative to $\boldsymbol{\alpha}$. For convenience, let $\mathbf{l}(t) = \mathbf{l}(\mathscr{B}, t)$ and $\mathbf{a}(t) = \mathbf{a}(\mathscr{B}, t)$. Show that

$$\begin{aligned} \mathbf{a}_z &= \mathbf{a}_{spin} + (\boldsymbol{\alpha} - \mathbf{z}) \times \mathbf{l}, \\ \dot{\mathbf{a}} &= \dot{\mathbf{a}}_{spin} + (\boldsymbol{\alpha} - \mathbf{o}) \times \dot{\mathbf{l}}. \end{aligned}$$

The term $(\alpha - z) \times l$ is usually referred to as the orbital angular momentum about z; it represents the angular momentum $\mathscr{B}$ would have if all of its mass were concentrated at the center of mass.

3. Consider a rigid motion. By (9.17),

$$x_0(y, t) = q(t) + Q(t)(y - z) \tag{9}$$

for all $y \in \mathscr{B}_0$ and all t. (Here x_0 is x_τ at $\tau = 0$; i.e., x_0 is the motion taking the configuration at time $\tau = 0$ as reference.)

(a) Show that

$$\alpha(t) = q(t) + Q(t)[\alpha(0) - z] \tag{10}$$

and hence [noting that $x_0(y, t)$ is well defined for all $y \in \mathscr{E}$]

$$\alpha(t) = x_0(\alpha(0), t).$$

What is the meaning of this result?

(b) Show that the angular velocity $\omega(t)$ is the axial vector of $\dot{Q}(t)Q(t)^T$ and that

$$v_\alpha = \omega \times r_\alpha. \tag{11}$$

(c) A vector function k on $\mathbb{R}$ *rotates with the body* if

$$k(t) = Q(t)k(0) \tag{12}$$

for all t. [Note that $Q(0) = I$, since $x_0(y, 0) = y$ for all y.] Show that k rotates with the body if and only if

$$\dot{k} = \omega \times k. \tag{13}$$

(d) Use the identity

$$f \times (d \times f) = (f^2 I - f \otimes f)d$$

to show that

$$a_{spin} = J\omega,$$

where

$$J(t) = \int_{\mathscr{B}_t} (r_\alpha^2 I - r_\alpha \otimes r_\alpha)\rho \, dV \tag{14}$$

is the *inertia tensor* of $\mathscr{B}_t$ relative to the center of mass.

(e) Show that

$$J(t) = Q(t)J(0)Q(t)^T, \tag{15}$$

and use this fact to prove that the matrix $[J(t)]$ of $J(t)$—relative to any orthonormal basis $\{e_i(t)\}$ that rotates with the body—is independent of t.

(f) Construct an orthonormal basis $\{e_i(t)\}$ that rotates with the body and has

$$[J] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}.$$

In this case $\{e_i(t)\}$ is called a *principal basis* and the corresponding numbers J_i are called *moments of inertia*. Let $\omega_i(t)$ denote the components of $\omega(t)$ with respect to $\{e_i(t)\}$. Show that the components of $\dot{a}_{spin}(t)$ relative to this basis are given by

$$(\dot{a}_{spin})_1 = J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3,$$
$$(\dot{a}_{spin})_2 = J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3,$$
$$(\dot{a}_{spin})_3 = J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2.$$

4. Define the *kinetic energy* $\mathscr{K}(t)$ and the *relative kinetic energy* $\mathscr{K}_\alpha(t)$ by

$$\mathscr{K}(t) = \tfrac{1}{2}\int_{\mathscr{B}_t} v^2 \rho \, dV, \qquad \mathscr{K}_\alpha(t) = \tfrac{1}{2}\int_{\mathscr{B}_t} v_\alpha^2 \, \rho \, dV.$$

(a) Prove *König's theorem*:

$$\mathscr{K} = \mathscr{K}_\alpha + \tfrac{1}{2}m(\mathscr{B})\dot{\alpha}^2.$$

(b) Consider a rigid motion. Use the identity

$$(d \times f)^2 = d \cdot (f^2 I - f \otimes f)d$$

to prove that

$$\mathscr{K}_\alpha = \tfrac{1}{2}\omega \cdot J\omega.$$

5. Show that

$$J = (\operatorname{tr} M)I - M,$$

where

$$M(t) = \int_{\mathscr{B}_t} r_\alpha \otimes r_\alpha \rho \, dV$$

is called the *Euler tensor*.

Selected Reference

Truesdell and Toupin [1, §§155–171].

CHAPTER

V

Force

14. FORCE. STRESS. BALANCE OF MOMENTUM

During a motion mechanical interactions between parts of a body or between a body and its environment are described by forces. Here we shall be concerned with three types of forces: (i) contact forces between separate parts of a body; (ii) contact forces exerted on the boundary of a body by its environment; (iii) body forces exerted on the interior points of a body by the environment.

One of the most important and far reaching axioms in continuum mechanics is *Cauchy's hypothesis* concerning the form of the contact forces. Cauchy assumed[1] the existence of a surface force density $\mathbf{s}(\mathbf{n}, \mathbf{x}, t)$ defined for each unit vector $\mathbf{n}$ and every $(\mathbf{x}, t)$ in the trajectory $\mathscr{T}$ of the motion (Fig. 1). This field has the following property: Let $\mathscr{S}$ be an oriented surface in $\mathscr{B}_t$ with positive unit normal $\mathbf{n}$ at $\mathbf{x}$. Then $\mathbf{s}(\mathbf{n}, \mathbf{x}, t)$ is the force, per unit area, exerted across $\mathscr{S}$ *upon* the material on the negative side of $\mathscr{S}$ *by* the material on the positive side. Cauchy's hypothesis is quite strong; indeed, if $\mathscr{C}$ is an oriented surface tangent to $\mathscr{S}$ at $\mathbf{x}$ and having the same positive unit normal there, then the force per unit area at $\mathbf{x}$ is the same on $\mathscr{C}$ as on $\mathscr{S}$ (Fig. 2).

[1] Noll [3] has shown that Cauchy's hypothesis actually follows from balance of linear momentum under very general assumptions concerning the form of the surface force $\mathbf{s}$. (Cf. also Truesdell [1, p. 136]; Gurtin and Williams [1].)

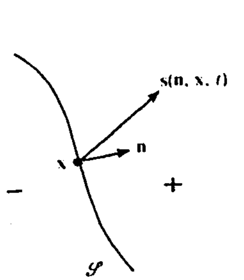

Figure 1

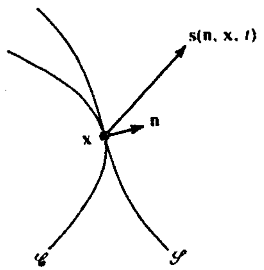

Figure 2

To determine the contact force between two separate parts $\mathscr{P}$ and $\mathscr{D}$ at time t (Fig. 3) one simply integrates s over the surface of contact

$$\mathscr{S}_t = \mathscr{P}_t \cap \mathscr{D}_t;$$

thus

$$\int_{\mathscr{S}_t} \mathbf{s}(\mathbf{n}_\mathbf{x}, \mathbf{x}, t)\, dA_\mathbf{x} \equiv \int_{\mathscr{S}_t} \mathbf{s}(\mathbf{n})\, dA$$

gives the force exerted on $\mathscr{P}$ by $\mathscr{D}$ at time t. Here $\mathbf{n}_\mathbf{x}$ is the outward unit normal to $\partial\mathscr{P}_t$ at x. For points on the boundary of $\mathscr{B}_t$, $\mathbf{s}(\mathbf{n}, \mathbf{x}, t)$—with n the outward unit normal to $\partial\mathscr{B}_t$ at x—gives the surface force, per unit area, applied to the body by the environment. This force is usually referred to as the **surface traction**. In any case, given a part $\mathscr{P}$,

$$\int_{\partial\mathscr{P}_t} \mathbf{s}(\mathbf{n})\, dA$$

represents[1] the *total* contact force exerted on $\mathscr{P}$ at time t (Fig. 4).

The environment can also exert forces on *interior* points of $\mathscr{B}$, a classical example being the force field due to gravity. Such forces are determined by a vector field b on $\mathscr{T}$; $\mathbf{b}(\mathbf{x}, t)$ gives the force, per unit volume, exerted by the environment on x. Thus for any part $\mathscr{P}$ the integral

$$\int_{\mathscr{P}_t} \mathbf{b}(\mathbf{x}, t)\, dV_\mathbf{x} \equiv \int_{\mathscr{P}_t} \mathbf{b}\, dV$$

gives that part of the environmental force on $\mathscr{P}$ not due to contact.

[1] Here it is tacit that, given any part $\mathscr{P}$ and time t, $\mathbf{s}(\mathbf{n}_\mathbf{x}, \mathbf{x}, t)$ is an integrable function of x on $\partial\mathscr{P}_t$. This assumption of integrability actually follows from the momentum balance laws (1) and our hypotheses (i) and (ii) concerning force systems. Indeed, (i) trivially implies integrability on surfaces of polyhedra, and this is all that is needed to establish the existence of the stress T; (9) and (11) then yield integrability on $\partial\mathscr{P}_t$ for any part $\mathscr{P}$.

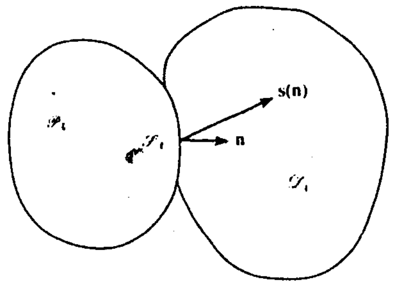

Figure 3

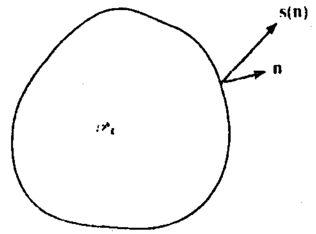

Figure 4

The above discussion should motivate the following definitions. Let $\mathscr{N}$ be the set of all unit vectors. By a **system of forces** for $\mathscr{B}$ during a motion (with trajectory $\mathscr{T}$) we mean a pair (s, b) of functions

$$\mathbf{s}: \mathscr{N} \times \mathscr{T} \to \mathscr{V}, \qquad \mathbf{b}: \mathscr{T} \to \mathscr{V},$$

with

(i) $\mathbf{s}(\mathbf{n}, \mathbf{x}, t)$, for each $\mathbf{n} \in \mathscr{N}$ and t, a smooth function of x on $\mathscr{B}_t$;
(ii) $\mathbf{b}(\mathbf{x}, t)$, for each t, a continuous function of x on $\mathscr{B}_t$.

We call s the **surface force** and b the **body force**; and we define the **force** $\mathbf{f}(\mathscr{P}, t)$ and **moment** $\mathbf{m}(\mathscr{P}, t)$ (about o) on a part $\mathscr{P}$ at time t by

$$\mathbf{f}(\mathscr{P}, t) = \int_{\partial\mathscr{P}_t} \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{P}_t} \mathbf{b}\, dV,$$

$$\mathbf{m}(\mathscr{P}, t) = \int_{\partial\mathscr{P}_t} \mathbf{r} \times \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{P}_t} \mathbf{r} \times \mathbf{b}\, dV.$$

Here n is the outward unit normal to $\partial\mathscr{P}_t$ and r is the position vector (13.2).

The basic axioms connecting motion and force are the **momentum balance laws**. These laws assert that for every part $\mathscr{P}$ and time t,

$$\boxed{\begin{aligned} \mathbf{f}(\mathscr{P}, t) &= \dot{\mathbf{l}}(\mathscr{P}, t), \\ \mathbf{m}(\mathscr{P}, t) &= \dot{\mathbf{a}}(\mathscr{P}, t); \end{aligned}} \tag{1}$$

they express, respectively, balance of linear momentum and balance of angular momentum [cf. (13.1)].

Tacit in the statement of axiom (1) is the existence of an observer (frame of reference) relative to which the motion and forces are measured. The existence of such an observer is nontrivial, since the momentum balance laws are generally not invariant under changes in observer (cf. Exercise 21.3). Observers relative to which (1) hold are called inertial; in practice the fixed stars are often used to define the class of inertial observers.

An obvious consequence of $(1)_1$ and (13.5) is that

$$\mathbf{f}(\mathscr{B}, t) = m(\mathscr{B})\ddot{\boldsymbol{\alpha}}(t),$$

provided $\mathscr{B}$ is bounded. Thus *the total force on a finite body is equal to the mass of the body times the acceleration of its mass center.*

By virtue of (13.3) the laws of momentum balance can be written as follows:

$$\int_{\partial\mathscr{P}_t} \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{P}_t} \mathbf{b}\, dV = \int_{\mathscr{P}_t} \dot{\mathbf{v}}\rho\, dV,$$
$$\int_{\partial\mathscr{P}_t} \mathbf{r} \times \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{P}_t} \mathbf{r} \times \mathbf{b}\, dV = \int_{\mathscr{P}_t} \mathbf{r} \times \dot{\mathbf{v}}\rho\, dV. \tag{2}$$

If we introduce the **total body force**

$$\mathbf{b}_* = \mathbf{b} - \rho\dot{\mathbf{v}}, \tag{3}$$

which includes the *inertial body force* $-\rho\dot{\mathbf{v}}$, and define

$$\mathbf{f}_*(\mathscr{P}, t) = \int_{\partial\mathscr{P}_t} \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{P}_t} \mathbf{b}_*\, dV,$$
$$\mathbf{m}_*(\mathscr{P}, t) = \int_{\partial\mathscr{P}_t} \mathbf{r} \times \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{P}_t} \mathbf{r} \times \mathbf{b}_*\, dV, \tag{4}$$

then (1) takes the simple form

$$\mathbf{f}_*(\mathscr{P}, t) = \mathbf{0}, \qquad \mathbf{m}_*(\mathscr{P}, t) = \mathbf{0}. \tag{5}$$

Our next result gives a far less trivial characterization of the momentum balance laws. Recall that an *infinitesimal rigid displacement* (of $\mathscr{E}$) is a mapping $\mathbf{w}: \mathscr{E} \to \mathscr{V}$ of the form

$$\mathbf{w}(\mathbf{x}) = \mathbf{w}_0 + \mathbf{W}(\mathbf{x} - \mathbf{o}) \tag{6}$$

with $\mathbf{w}_0$ a vector and $\mathbf{W}$ a skew tensor [cf. (7.8)].

Theorem of Virtual Work. *Let* $(\mathbf{s}, \mathbf{b})$ *be a system of forces for* $\mathscr{B}$ *during a motion. Then a necessary and sufficient condition that the momentum balance laws be satisfied is that given any part* $\mathscr{P}$ *and time* t,

$$\int_{\partial\mathscr{P}_t} \mathbf{s}(\mathbf{n}) \cdot \mathbf{w}\, dA + \int_{\mathscr{P}_t} \mathbf{b}_* \cdot \mathbf{w}\, dV = 0 \tag{7}$$

for every infinitesimal rigid displacement $\mathbf{w}$.

Proof. Note first that by (13.2) we can write (6) in the form

$$\mathbf{w} = \mathbf{w}_0 + \boldsymbol{\omega} \times \mathbf{r}, \tag{8}$$

with $\boldsymbol{\omega}$ the axial vector corresponding to $\mathbf{W}$. Let $\varphi(\mathbf{w}_0, \boldsymbol{\omega})$ denote the left side of (7) with $\mathbf{w}$ given by (8). Then $\varphi(\mathbf{w}_0, \boldsymbol{\omega})$ involves integrals of the fields

$$\mathbf{s} \cdot \mathbf{w} = \mathbf{s} \cdot \mathbf{w}_0 + \mathbf{s} \cdot (\boldsymbol{\omega} \times \mathbf{r}), \qquad \mathbf{b}_* \cdot \mathbf{w} = \mathbf{b}_* \cdot \mathbf{w}_0 + \mathbf{b}_* \cdot (\boldsymbol{\omega} \times \mathbf{r}),$$

where we have written $\mathbf{s}$ for $\mathbf{s}(\mathbf{n})$, and since

$$\mathbf{k} \cdot (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega} \cdot (\mathbf{r} \times \mathbf{k}),$$

it follows that

$$\mathbf{s} \cdot \mathbf{w} = \mathbf{w}_0 \cdot \mathbf{s} + \boldsymbol{\omega} \cdot (\mathbf{r} \times \mathbf{s}),$$
$$\mathbf{b}_* \cdot \mathbf{w} = \mathbf{w}_0 \cdot \mathbf{b}_* + \boldsymbol{\omega} \cdot (\mathbf{r} \times \mathbf{b}_*).$$

Thus by (4),

$$\varphi(\mathbf{w}_0, \boldsymbol{\omega}) = \mathbf{w}_0 \cdot \mathbf{f}_*(\mathscr{P}, t) + \boldsymbol{\omega} \cdot \mathbf{m}_*(\mathscr{P}, t),$$

and $\varphi(\mathbf{w}_0, \boldsymbol{\omega}) = 0$ for all vectors $\mathbf{w}_0$ and $\boldsymbol{\omega}$ if and only if (5) hold. □

The next theorem is one of the central results of continuum mechanics. Its main assertion is that $\mathbf{s}(\mathbf{n})$ is *linear* in $\mathbf{n}$.

Cauchy's Theorem (Existence of stress). *Let* $(\mathbf{s}, \mathbf{b})$ *be a system of forces for* $\mathscr{B}$ *during a motion. Then a necessary and sufficient condition that the momentum balance laws be satisfied is that there exist a spatial tensor field* $\mathbf{T}$ *(called the* **Cauchy stress**) *such that*

(a) *for each unit vector* $\mathbf{n}$,

$$\mathbf{s}(\mathbf{n}) = \mathbf{T}\mathbf{n}; \tag{9}$$

(b) $\mathbf{T}$ *is symmetric;*

(c) $\mathbf{T}$ *satisfies the* **equation of motion**

$$\operatorname{div} \mathbf{T} + \mathbf{b} = \rho\dot{\mathbf{v}}. \tag{10}$$

Proof. In stating condition (c) we do not assume explicitly that

$$\mathbf{T}(\mathbf{x}, t) \text{ is smooth in } \mathbf{x}, \tag{11}$$

so that, a priori, it is not clear whether or not (c) makes sense. We therefore begin our proof by showing that (11) is a direct consequence of (9). Indeed, by (9), for an orthonormal basis $\{\mathbf{e}_i\}$,

$$\sum_i \mathbf{s}(\mathbf{e}_i) \otimes \mathbf{e}_i = \sum_i (\mathbf{T}\mathbf{e}_i) \otimes \mathbf{e}_i;$$

thus, in view of Exercise 1.6c,

$$\mathbf{T}(\mathbf{x}, t) = \sum_i \mathbf{s}(\mathbf{e}_i, \mathbf{x}, t) \otimes \mathbf{e}_i, \tag{12}$$

and (11) follows from (12) and property (i) of force systems.

We are now in a position to establish the necessity and sufficiency of (a)–(c).

Necessity. Assume that (1) are satisfied. For convenience, we fix the time t and suppress it as an argument in most of what follows. The proof will proceed in a number of steps.

Assertion 1. Given any $\mathbf{x} \in \mathscr{B}_t$, any orthonormal basis $\{\mathbf{e}_i\}$, and any unit vector $\mathbf{k}$ with

$$\mathbf{k} \cdot \mathbf{e}_i > 0 \qquad (i = 1, 2, 3), \tag{13}$$

it follows that

$$\mathbf{s}(\mathbf{k}, \mathbf{x}) = -\sum_i (\mathbf{k} \cdot \mathbf{e}_i)\mathbf{s}(-\mathbf{e}_i, \mathbf{x}). \tag{14}$$

Proof. First let $\mathbf{x}$ belong to the interior of $\mathscr{B}_t$. Let $\delta > 0$ and consider the tetrahedron $\mathscr{G}_\delta$ with the following properties: the faces of $\mathscr{G}_\delta$ are $\mathscr{S}_\delta$, $\mathscr{S}_{1\delta}$, $\mathscr{S}_{2\delta}$, and $\mathscr{S}_{3\delta}$, where $\mathbf{k}$ and $-\mathbf{e}_i$ are the outward unit normals to $\partial\mathscr{G}_\delta$ on $\mathscr{S}_\delta$ and $\mathscr{S}_{i\delta}$, respectively; the vertex opposite to $\mathscr{S}_\delta$ is $\mathbf{x}$; the distance from $\mathbf{x}$ to $\mathscr{S}_\delta$ is δ (Fig. 5). Clearly, $\mathscr{G}_\delta$ is contained in the interior of $\mathscr{B}_t$ for all sufficiently small δ, say $\delta \leq \delta_0$.

Next, $\mathbf{b}_*(\mathbf{x}, t)$, defined by (3), is continuous in $\mathbf{x}$, since $\mathbf{b}(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$, and $\dot{\mathbf{v}}(\mathbf{x}, t)$ have this property; hence $\mathbf{b}_*(\cdot, t)$ is bounded on $\mathscr{G}_{\delta_0}$ (for t fixed). Thus if we apply $(5)_1$ to the part $\mathscr{P}$ which occupies the region $\mathscr{G}_\delta$ at time t, and use $(4)_1$, we conclude that

$$\left| \int_{\partial\mathscr{G}_\delta} \mathbf{s}(\mathbf{n})\, dA \right| \leq \kappa \operatorname{vol}(\mathscr{G}_\delta) \tag{15}$$

for all $\delta \leq \delta_0$, where κ is independent of δ.

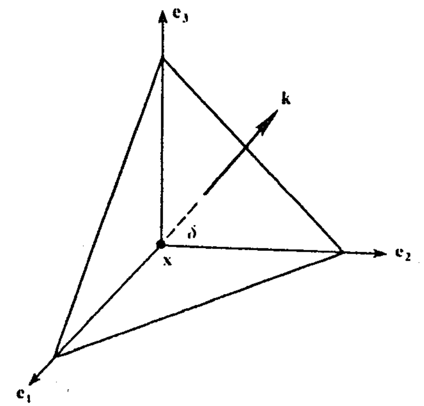

Figure 5

Let $A(\delta)$ denote the area of $\mathscr{S}_\delta$. Since $A(\delta)$ is a constant times δ^2, while $\operatorname{vol}(\mathscr{G}_\delta)$ is a constant times δ^3, we may conclude from (15) that

$$\frac{1}{A(\delta)} \int_{\partial\mathscr{G}_\delta} \mathbf{s}(\mathbf{n})\, dA \to \mathbf{0}$$

as $\delta \to 0$. But

$$\int_{\partial\mathscr{G}_\delta} \mathbf{s}(\mathbf{n})\, dA = \int_{\mathscr{S}_\delta} \mathbf{s}(\mathbf{k})\, dA + \sum_i \int_{\mathscr{S}_{i\delta}} \mathbf{s}(-\mathbf{e}_i)\, dA,$$

and, since $\mathbf{s}(\mathbf{n}, \mathbf{x})$ is continuous in $\mathbf{x}$ for each fixed $\mathbf{n}$,

$$\frac{1}{A(\delta)} \int_{\mathscr{S}_\delta} \mathbf{s}(\mathbf{k})\, dA \to \mathbf{s}(\mathbf{k}, \mathbf{x}),$$

$$\frac{1}{A(\delta)} \int_{\mathscr{S}_{i\delta}} \mathbf{s}(-\mathbf{e}_i)\, dA \to (\mathbf{k} \cdot \mathbf{e}_i)\mathbf{s}(-\mathbf{e}_i, \mathbf{x}),$$

where we have used the fact that the area of $\mathscr{S}_{i\delta}$ is $A(\delta)\mathbf{k} \cdot \mathbf{e}_i$. Combining the above relations we conclude that (14) holds as long as $\mathbf{x}$ lies in the interior of $\mathscr{B}_t$. But $\mathbf{x} \mapsto \mathbf{s}(\mathbf{n}, \mathbf{x})$ is continuous on $\mathscr{B}_t$. Thus, by continuity, (14) must hold everywhere on $\mathscr{B}_t$.

Assertion 2 (Newton's law of action and reaction). For each $\mathbf{x} \in \mathscr{B}_t$ the map $\mathbf{n} \mapsto \mathbf{s}(\mathbf{n}, \mathbf{x})$ is continuous on $\mathscr{N}$ and satisfies

$$\mathbf{s}(\mathbf{n}, \mathbf{x}) = -\mathbf{s}(-\mathbf{n}, \mathbf{x}). \tag{16}$$

Proof. Since the right side of (14) is a continuous function of $\mathbf{k}$, $\mathbf{s}(\mathbf{k}, \mathbf{x})$ is continuous on the set of all unit vectors $\mathbf{k}$ consistent with (13). Thus, since this is true for every choice of orthonormal basis $\{\mathbf{e}_i\}$, $\mathbf{s}(\mathbf{k}, \mathbf{x})$ must be a continuous function of $\mathbf{k}$ everywhere on $\mathscr{N}$. Therefore, if in (14) we let $\mathbf{k} \to \mathbf{e}_1$, we arrive at $\mathbf{s}(\mathbf{e}_1, \mathbf{x}) = -\mathbf{s}(-\mathbf{e}_1, \mathbf{x})$, and again, since the basis $\{e_i\}$ is arbitrary, (16) must hold.

Assertion 3. Given any $\mathbf{x} \in \mathscr{B}_t$ and any orthonormal basis $\{\mathbf{e}_i\}$,

$$\mathbf{s}(\mathbf{k}, \mathbf{x}) = \sum_i (\mathbf{k} \cdot \mathbf{e}_i)\mathbf{s}(\mathbf{e}_i, \mathbf{x}) \tag{17}$$

for *all* unit vectors $\mathbf{k}$.

Proof. Choose an orthonormal basis $\{\mathbf{e}_i\}$ and a unit vector $\mathbf{k}$ that does not lie in a coordinate plane (that is, in a plane spanned by two $\mathbf{e}_i$). Then there is no i such that $\mathbf{k} \cdot \mathbf{e}_i = 0$, and we can define a new orthonormal basis $\{\bar{\mathbf{e}}_i\}$ by

$$\bar{\mathbf{e}}_i = [\operatorname{sgn}(\mathbf{k} \cdot \mathbf{e}_i)]\mathbf{e}_i.$$

Then $\mathbf{k}\cdot\bar{\mathbf{e}}_i > 0$ for $i = 1, 2, 3$, and Assertion 1 together with (16) applied to the basis $\{\bar{\mathbf{e}}_i\}$ yields

$$\mathbf{s}(\mathbf{k}, \mathbf{x}) = -\sum_i (\mathbf{k}\cdot\bar{\mathbf{e}}_i)\mathbf{s}(-\bar{\mathbf{e}}_i, \mathbf{x}) = \sum_i (\mathbf{k}\cdot\bar{\mathbf{e}}_i)\mathbf{s}(\bar{\mathbf{e}}_i, \mathbf{x}) = \sum_i (\mathbf{k}\cdot\mathbf{e}_i)\mathbf{s}(\mathbf{e}_i, \mathbf{x}).$$

Thus (17) holds as long as $\mathbf{k}$ does not lie in a coordinate plane. But the map $\mathbf{n}\mapsto\mathbf{s}(\mathbf{n}, \mathbf{x})$ is continuous on $\mathscr{U}$. Thus (17) holds for all unit vectors $\mathbf{k}$, and Assertion 3 is proved.

Choose an orthonormal basis $\{\mathbf{e}_i\}$ and let $\mathbf{T}(\mathbf{x}, t)$ be defined by (12). Then $\mathbf{T}(\mathbf{x}, t)$ is smooth in $\mathbf{x}$, and, in view of (17), (9) holds. By (9), balance of linear momentum $(2)_1$ takes the form

$$\int_{\partial\mathscr{P}_t} \mathbf{T}\mathbf{n}\,dA + \int_{\mathscr{P}_t} \mathbf{b}\,dV = \int_{\mathscr{P}_t} \dot{\mathbf{v}}\rho\,dV,$$

or equivalently, using the divergence theorem,

$$\int_{\mathscr{P}_t} (\operatorname{div}\mathbf{T} + \mathbf{b} - \rho\dot{\mathbf{v}})\,dV = \mathbf{0}.$$

By the localization theorem, this relation can hold for every part $\mathscr{P}$ and time t only if the equation of motion (10) is satisfied.

To complete the proof of necessity we have only to establish the symmetry of $\mathbf{T}$. Let $\mathbf{w}$ be any smooth vector field on $\mathscr{B}_t$. We then conclude, with the aid of the divergence theorem and $(4.2)_5$, that

$$\int_{\partial\mathscr{P}_t} \mathbf{T}\mathbf{n}\cdot\mathbf{w}\,dA = \int_{\partial\mathscr{P}_t} (\mathbf{T}^{\mathrm{T}}\mathbf{w})\cdot\mathbf{n}\,dA = \int_{\mathscr{P}_t} \operatorname{div}(\mathbf{T}^{\mathrm{T}}\mathbf{w})\,dV$$
$$= \int_{\mathscr{P}_t} (\mathbf{w}\cdot\operatorname{div}\mathbf{T} + \mathbf{T}\cdot\operatorname{grad}\mathbf{w})\,dV.$$

Thus, by (3), (9), and (10),

$$\int_{\partial\mathscr{P}_t} \mathbf{s}(\mathbf{n})\cdot\mathbf{w}\,dA + \int_{\mathscr{P}_t} \mathbf{b}_*\cdot\mathbf{w}\,dV = \int_{\mathscr{P}_t} \mathbf{T}\cdot\operatorname{grad}\mathbf{w}\,dV. \tag{18}$$

In particular, if we take $\mathbf{w}$ equal to the infinitesimal rigid displacement (6), then $\operatorname{grad}\mathbf{w} = \mathbf{W}$ and (7) implies

$$\int_{\mathscr{P}_t} \mathbf{T}\cdot\mathbf{W}\,dV = 0$$

for every part $\mathscr{P}$, so that

$$\mathbf{T}\cdot\mathbf{W} = 0.$$

Since this result must hold for every skew tensor $\mathbf{W}$, (f) of the proposition on page 6 yields the symmetry of $\mathbf{T}$.

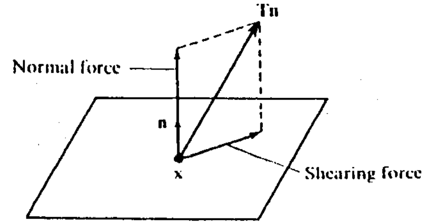

Figure 6

Sufficiency. Assume that there exists a symmetric spatial tensor field $\mathbf{T}$ consistent with (9) and (10). Clearly, (18) holds in the present circumstances. Since $\mathbf{T}$ is symmetric, and since the gradient of an infinitesimal rigid displacement is skew, when $\mathbf{w}$ is such a field the right side of (18) vanishes. We therefore conclude from the theorem of virtual work that the momentum balance laws hold. □

Actually, one can show that (a) and (c) are equivalent to balance of linear momentum $(1)_1$; and that granted $(1)_1$, the symmetry of $\mathbf{T}$ is equivalent to balance of angular momentum $(1)_2$.

Let $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ be the stress at a particular place and time. If

$$\mathbf{T}\mathbf{n} = \sigma\mathbf{n}, \qquad |\mathbf{n}| = 1,$$

then σ is a ***principal stress*** and $\mathbf{n}$ is a ***principal direction***, so that principal stresses and principal directions are eigenvalues and eigenvectors of $\mathbf{T}$. Since $\mathbf{T}$ is symmetric, there exist three mutually perpendicular principal directions and three corresponding principal stresses.

Consider an arbitrary oriented plane surface with positive unit normal $\mathbf{n}$ at $\mathbf{x}$ (Fig. 6). Then the surface force $\mathbf{T}\mathbf{n}$ can be decomposed into the sum of a **normal force**

$$(\mathbf{n}\cdot\mathbf{T}\mathbf{n})\mathbf{n} = (\mathbf{n}\otimes\mathbf{n})\mathbf{T}\mathbf{n}$$

and a **shearing force**

$$(\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\mathbf{T}\mathbf{n},$$

and it follows that $\mathbf{n}$ is a principal direction if and only if the corresponding shearing force vanishes.

A fluid at rest is incapable of exerting shearing forces. In this instance $\mathbf{T}\mathbf{n}$ is parallel to $\mathbf{n}$ for each unit vector $\mathbf{n}$, and every such vector is an eigenvector of $\mathbf{T}$. In view of the discussion given in the paragraph following the spectral theorem, $\mathbf{T}$ has only one characteristic space, $\mathscr{V}$ itself, and (c) of the spectral theorem implies that

$$\mathbf{T} = -\pi\mathbf{I}$$

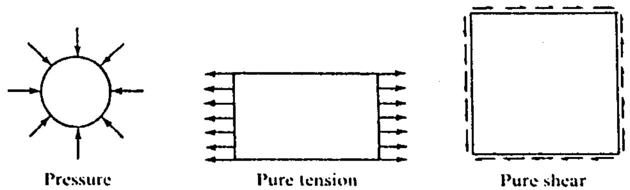

Figure 7

with π a scalar. π is called the **pressure** of the fluid. Note that in this case the force (per unit area) on any surface in the fluid is $-\pi\mathbf{n}$.

Two other important states of stress are:

(a) **Pure tension** (or compression) with tensile stress σ in the direction $\mathbf{e}$, where $|\mathbf{e}| = 1$:

$$\mathbf{T} = \sigma(\mathbf{e} \otimes \mathbf{e}).$$

(b) **Pure shear** with shear stress τ relative to the direction pair $(\mathbf{k}, \mathbf{n})$, where $\mathbf{k}$ and $\mathbf{n}$ are orthogonal unit vectors:

$$\mathbf{T} = \tau(\mathbf{k} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{k}).$$

The surface force fields corresponding to the above examples (with $\mathbf{T}$ constant) are shown in Fig. 7.

Exercises

In Exercises 1, 4, and 8, $\mathscr{B}$ is bounded.

1. The moment $\mathbf{m}_z(t)$ about a moving point $\mathbf{z}(t)$ is defined by

$$\mathbf{m}_z(t) = \int_{\partial\mathscr{B}_t} \mathbf{r}_z \times \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{B}_t} \mathbf{r}_z \times \mathbf{b}\, dV,$$

where $\mathbf{r}_z$ is the position vector from $\mathbf{z}$ [cf. (13.8)].

(a) Let $\mathbf{f}(t) = \mathbf{f}(\mathscr{B}, t)$. Show that, for $\mathbf{y}: \mathbb{R} \to \mathscr{E}$,

$$\mathbf{m}_z = \mathbf{m}_y + (\mathbf{y} - \mathbf{z}) \times \mathbf{f}.$$

(b) Let $\mathbf{l}(t) = \mathbf{l}(\mathscr{B}, t)$. Show that [cf. (13.7)]

$$\mathbf{m}_z = \dot{\mathbf{a}}_z + \dot{\mathbf{z}} \times \mathbf{l},$$

$$\mathbf{m}_\alpha = \dot{\mathbf{a}}_{\text{spin}}.$$

2. Prove that the momentum balance laws (1) hold for every choice of origin (constant in time) if they hold for one such choice.

3. Show that the Cauchy stress is *uniquely* determined by the force system.
4. Consider a rigid motion with angular velocity $\boldsymbol{\omega}$. Let $\{\mathbf{e}_i(t)\}$ denote a principal basis of inertia and let J_i denote the corresponding moments of inertia (relative to $\boldsymbol{\alpha}$) (cf. Exercise 13.3f). Further, let $\omega_i(t)$ and $m_i(t)$ denote the components of $\boldsymbol{\omega}(t)$ and $\mathbf{m}_\alpha(t)$ relative to $\{\mathbf{e}_i(t)\}$. Derive *Euler's equations*

$$m_1 = J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3,$$
$$m_2 = J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3,$$
$$m_3 = J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2.$$

These relations supplemented by

$$\mathbf{f} = m(\mathscr{B})\ddot{\boldsymbol{\alpha}}$$

constitute the basic equations of rigid body mechanics. When $\mathbf{f}$ and $\mathbf{m}_\alpha$ are known they provide a system of nonlinear ordinary differential equations for $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$.

5. Let $\mathbf{T}$ be the stress at a particular place and time. Suppose that the corresponding surface force on a given plane $\mathscr{S}$ is perpendicular to $\mathscr{S}$, while the surface force on any plane perpendicular to $\mathscr{S}$ vanishes. Show that $\mathbf{T}$ is a pure tension.
6. Let $\mathbf{T}$ be a pure shear. Compute the corresponding principal stresses and directions.
7. Prove directly that $\mathbf{s}(\mathbf{n}, \mathbf{x}) = -\mathbf{s}(-\mathbf{n}, \mathbf{x})$ by applying $(2)_1$ to the part $\mathscr{P}$ that occupies the region $\mathscr{R}_\delta$ at time t. Here $\mathscr{R}_\delta$ is the rectangular region which is centered at $\mathbf{x}$, which has dimensions $\delta \times \delta \times \delta^2$, and which has $\mathbf{n}$ normal to the $\delta \times \delta$ faces (Fig. 8).

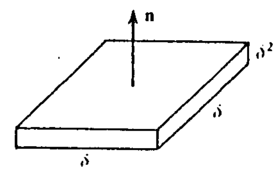

Figure 8

8. Prove *Da Silva's theorem*: At a given time the total moment on a body can be made to vanish by subjecting the surface and body forces to a suitable rotation; that is, there exists an orthogonal tensor $\mathbf{Q}$ such that

$$\int_{\partial\mathscr{B}_t} \mathbf{r} \times \mathbf{Q}\mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{B}_t} \mathbf{r} \times \mathbf{Q}\mathbf{b}\, dV = \mathbf{0}.$$

Show in addition that

$$\int_{\partial \mathscr{B}_t} (\mathbf{Q}^T\mathbf{r}) \times \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{B}_t} (\mathbf{Q}^T\mathbf{r}) \times \mathbf{b}\, dV = \mathbf{0}.$$

What is the meaning of this relation?

9. Suppose that at time t the surface traction vanishes on the boundary $\partial\mathscr{B}_t$. Show that at any $\mathbf{x} \in \partial\mathscr{B}_t$ the stress vector on any plane perpendicular to $\partial\mathscr{B}_t$ is tangent to the boundary.

15. CONSEQUENCES OF MOMENTUM BALANCE

By Cauchy's theorem (and Exercise 14.3), to each force system consistent with the momentum balance laws there corresponds exactly one symmetric tensor field $\mathbf{T}$ consistent with (14.9) and (14.10). Conversely, the force system $(\mathbf{s}, \mathbf{b})$ is completely determined by the stress $\mathbf{T}$ and the motion $\mathbf{x}$. Indeed, the surface force $\mathbf{s}$ is given by (14.9), while the body force $\mathbf{b}$ is easily computed using the equation of motion (14.10) [with ρ computed using (12.4)]:

$$\mathbf{s}(\mathbf{n}) = \mathbf{T}\mathbf{n}, \qquad \mathbf{b} = \rho\dot{\mathbf{v}} - \operatorname{div} \mathbf{T}.$$

This observation motivates the following definition. By a **dynamical process** we mean a pair $(\mathbf{x}, \mathbf{T})$ with

(a) $\mathbf{x}$ a motion,
(b) $\mathbf{T}$ a symmetric tensor field on the trajectory $\mathscr{T}$ of $\mathbf{x}$, and
(c) $\mathbf{T}(\mathbf{x}, t)$ a smooth function of $\mathbf{x}$ on $\mathscr{B}_t$.

Further, if $\mathbf{v}$ and ρ are the velocity field and density corresponding to $\mathbf{x}$, then the list $(\mathbf{v}, \rho, \mathbf{T})$ is called a **flow**. In view of the above discussion, to each force system consistent with the momentum balance laws there corresponds exactly one dynamical process (or equivalently, exactly one flow), and conversely.

For the remainder of this section $(\mathbf{v}, \rho, \mathbf{T})$ is a flow and $(\mathbf{s}, \mathbf{b})$ is the associated force system.

Theorem (Balance of momentum for a control volume). *Let $\mathscr{R}$ be a control volume at time t. Then at that time*

$$\int_{\partial\mathscr{R}} \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{R}} \mathbf{b}\, dV = \frac{d}{dt}\int_{\mathscr{R}} \mathbf{v}\rho\, dV + \int_{\partial\mathscr{R}} (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}\, dA,$$
$$\int_{\partial\mathscr{R}} \mathbf{r} \times \mathbf{s}(\mathbf{n})\, dA + \int_{\mathscr{R}} \mathbf{r} \times \mathbf{b}\, dV = \frac{d}{dt}\int_{\mathscr{R}} \mathbf{r} \times \mathbf{v}\rho\, dV + \int_{\partial\mathscr{R}} \mathbf{r} \times (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}\, dA. \qquad (1)$$

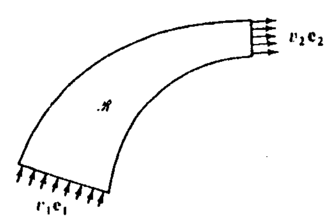

Figure 9

Proof. We will prove only $(1)_1$. First of all, by $(4.2)_4$, (8.5), and $(12.5)_2$,

$$\rho\dot{\mathbf{v}} = \rho\mathbf{v}' + \rho(\operatorname{grad}\mathbf{v})\mathbf{v} = (\rho\mathbf{v})' - \rho'\mathbf{v} + (\operatorname{grad}\mathbf{v})(\rho\mathbf{v})$$
$$= (\rho\mathbf{v})' + \mathbf{v}\operatorname{div}(\rho\mathbf{v}) + (\operatorname{grad}\mathbf{v})(\rho\mathbf{v}) = (\rho\mathbf{v})' + \operatorname{div}(\mathbf{v}\otimes\rho\mathbf{v}).$$

Thus, since

$$\int_{\mathscr{R}} \operatorname{div}\mathbf{T}\, dV = \int_{\partial\mathscr{R}} \mathbf{T}\mathbf{n}\, dA = \int_{\partial\mathscr{R}} \mathbf{s}(\mathbf{n})\, dA,$$
$$\int_{\mathscr{R}} (\rho\mathbf{v})'\, dV = \frac{d}{dt}\int_{\mathscr{R}} \rho\mathbf{v}\, dV,$$
$$\int_{\mathscr{R}} \operatorname{div}(\mathbf{v}\otimes\rho\mathbf{v})\, dV = \int_{\partial\mathscr{R}} (\mathbf{v}\otimes\rho\mathbf{v})\mathbf{n}\, dA = \int_{\partial\mathscr{R}} (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}\, dA,$$

if we integrate the equation of motion (14.10) over $\mathscr{R}$ we arrive at $(1)_1$. □

Equation $(1)_1$ asserts that *the total force on the control volume $\mathscr{R}$ is equal to the rate at which the linear momentum of the material in $\mathscr{R}$ is increasing plus the rate of outflow of momentum across $\partial\mathscr{R}$.* Equation $(1)_2$ has an analogous interpretation.

A flow $(\mathbf{v}, \rho, \mathbf{T})$ is **steady** if $\mathscr{B}_t = \mathscr{B}_0$ for all t and

$$\mathbf{v}' = \mathbf{0}, \qquad \rho' = 0, \qquad \mathbf{T}' = \mathbf{0}.$$

In this case we call $\mathscr{B}_0$ the **flow region**. Note that our definitions are consistent: in a steady flow the underlying motion is a steady motion (cf. Section 9).

As an example of the last theorem consider the steady flow of a fluid through a curved pipe (Fig. 9), and let $\mathscr{R}$ be the control volume bounded by the pipe walls and the cross sections marking the ends of the pipe. Assume that the stress is a pressure and that the velocity, density, and pressure at the entrance and exit have the constant values

$$v_1\mathbf{e}_1, \rho_1, \pi_1 \qquad \text{and} \qquad v_2\mathbf{e}_2, \rho_2, \pi_2,$$

respectively, with $\mathbf{e}_1$ perpendicular to the entrance cross section and $\mathbf{e}_2$ perpendicular to the exit cross section. Since the flow is steady,

$$\frac{d}{dt}\int_{\mathscr{R}} \mathbf{v}\rho\, dV = \mathbf{0}.$$

Further, $\mathbf{v}\cdot\mathbf{n}$ must vanish at the pipe walls; therefore

$$\int_{\partial\mathcal{R}} (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}\,dA = \rho_2 v_2^2 A_2 \mathbf{e}_2 - \rho_1 v_1^2 A_1 \mathbf{e}_1,$$

where A_1 and A_2 are the areas of the entrance and exit cross sections, respectively. Let $\mathbf{f}$ denote the total force exerted *by* the fluid *on* the pipe walls. Then the total force on the control volume $\mathcal{R}$ is

$$-\mathbf{f} + \pi_1 A_1 \mathbf{e}_1 - \pi_2 A_2 \mathbf{e}_2,$$

and $(1)_1$ yields

$$\mathbf{f} = (\pi_1 + \rho_1 v_1^2)A_1\mathbf{e}_1 - (\pi_2 + \rho_2 v_2^2)A_2\mathbf{e}_2.$$

A similar analysis, based on conservation of mass (12.6), tells us that

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \equiv m,$$

and hence

$$\mathbf{f} = (\pi_1 A_1 + m v_1)\mathbf{e}_1 - (\pi_2 A_2 + m v_2)\mathbf{e}_2.$$

Thus we have a simple expression for the force on the pipe walls in terms of conditions at the entrance and exit. Although the assumptions underlying this example are restrictive, they are, in fact, good approximations for a large class of physical situations.

Theorem of Power Expended. *For every part $\mathcal{P}$ and time t,*

$$\int_{\partial\mathcal{P}_t} \mathbf{s}(\mathbf{n})\cdot\mathbf{v}\,dA + \int_{\mathcal{P}_t} \mathbf{b}\cdot\mathbf{v}\,dV = \int_{\mathcal{P}_t} \mathbf{T}\cdot\mathbf{D}\,dV + \frac{d}{dt}\int_{\mathcal{P}_t} \frac{\mathbf{v}^2}{2}\rho\,dV, \tag{2}$$

where

$$\mathbf{D} = \tfrac{1}{2}(\operatorname{grad}\mathbf{v} + \operatorname{grad}\mathbf{v}^{\mathrm{T}})$$

is the stretching.

Proof. Since $\mathbf{T}$ is symmetric, (a) of the proposition on page 6 implies

$$\mathbf{T}\cdot\operatorname{grad}\mathbf{v} = \mathbf{T}\cdot\mathbf{D}.$$

Also, by (12.8),

$$\frac{d}{dt}\int_{\mathcal{P}_t} \frac{\mathbf{v}^2}{2}\rho\,dV = \int_{\mathcal{P}_t} \mathbf{v}\cdot\dot{\mathbf{v}}\rho\,dV,$$

so that (14.3) yields

$$\int_{\mathcal{P}_t} \mathbf{b}_*\cdot\mathbf{v}\,dV = \int_{\mathcal{P}_t} \mathbf{b}\cdot\mathbf{v}\,dV - \frac{d}{dt}\int_{\mathcal{P}_t} \frac{\mathbf{v}^2}{2}\rho\,dV.$$

Equation (14.18) (with $\mathbf{w} = \mathbf{v}$) and the above relations yield the desired identity. □

The terms

$$\int_{\mathcal{P}_t} \frac{\mathbf{v}^2}{2}\rho\,dV \quad \text{and} \quad \int_{\mathcal{P}_t} \mathbf{T}\cdot\mathbf{D}\,dV$$

are called, respectively, the **kinetic energy** and the **stress power** of $\mathcal{P}$ at time t. The theorem of power expended asserts that *the power expended on $\mathcal{P}$ by the surface and body forces is equal to the stress power plus the rate of change of kinetic energy.*

A flow is **potential** if the velocity is the gradient of a potential:

$$\mathbf{v} = \operatorname{grad}\varphi.$$

Note that φ is a spatial field of class C^2 (because $\mathbf{v}$ is C^2); hence

$$\operatorname{curl}\mathbf{v} = \operatorname{curl}\operatorname{grad}\varphi = \mathbf{0}$$

and potential flows are irrotational. Conversely, by the potential theorem (page 35), if a flow is irrotational, and if $\mathcal{B}_t$ is simply connected at some (and hence every) time t, then the flow is potential.

For a body force $\mathbf{b}$, the field $\mathbf{b}/\rho$ represents the body force per unit mass. We say that the body force is **conservative** with potential β if

$$\mathbf{b}/\rho = -\operatorname{grad}\beta. \tag{3}$$

If the flow is also steady the equation of motion (14.10) implies that $\mathbf{b}' = \mathbf{0}$; in this case we will require that

$$\beta' = 0.$$

Bernoulli's Theorem. *Consider a flow* $(\mathbf{v}, \rho, \mathbf{T})$ *with stress a pressure* $-\pi\mathbf{1}$ *and body force conservative with potential* β.

(a) *If the flow is potential,*

$$\operatorname{grad}\left(\varphi' + \frac{\mathbf{v}^2}{2} + \beta\right) + \frac{1}{\rho}\operatorname{grad}\pi = \mathbf{0}. \tag{4}$$

(b) *If the flow is steady,*

$$\mathbf{v}\cdot\operatorname{grad}\left(\frac{\mathbf{v}^2}{2} + \beta\right) + \frac{1}{\rho}\mathbf{v}\cdot\operatorname{grad}\pi = 0. \tag{5}$$

(c) *If the flow is steady and irrotational,*

$$\operatorname{grad}\left(\frac{\mathbf{v}^2}{2} + \beta\right) + \frac{1}{\rho}\operatorname{grad}\pi = \mathbf{0}. \tag{6}$$

Proof. By hypothesis,

$$\mathbf{T} = -\pi\mathbf{I},$$

so that

$$\operatorname{div}\mathbf{T} = -\operatorname{grad}\pi;$$

thus the equation of motion (14.10) takes the form

$$-\operatorname{grad}\pi + \mathbf{b} = \rho\dot{\mathbf{v}}, \tag{7}$$

or equivalently, by (3),

$$\dot{\mathbf{v}} = -\frac{1}{\rho}\operatorname{grad}\pi - \operatorname{grad}\beta. \tag{8}$$

On the other hand, by $(9.10)_1$, for a potential (and consequently irrotational) flow

$$\dot{\mathbf{v}} = \mathbf{v}' + \tfrac{1}{2}\operatorname{grad}(\mathbf{v}^2) = \operatorname{grad}\left(\varphi' + \frac{\mathbf{v}^2}{2}\right); \tag{9}$$

for a steady flow

$$\mathbf{v}\cdot\dot{\mathbf{v}} = \mathbf{v}\cdot\operatorname{grad}\left(\frac{\mathbf{v}^2}{2}\right) \tag{10}$$

(where we have used the fact that $\mathbf{v}\cdot\mathbf{W}\mathbf{v} = 0$, since $\mathbf{W}$ is skew); for a steady, irrotational flow

$$\dot{\mathbf{v}} = \operatorname{grad}\left(\frac{\mathbf{v}^2}{2}\right). \tag{11}$$

The relations (9)–(11), when combined with (8), yield the desired results (4)–(6). □

Exercises

1. Consider a statical situation in which a (bounded) body occupies the region $\mathscr{B}_0$ for all time. Let $\mathbf{b}: \mathscr{B}_0 \to \mathscr{V}$ and $\mathbf{T}: \mathscr{B}_0 \to \text{Sym}$ with $\mathbf{T}$ smooth satisfy

$$\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0}.$$

Define the mean stress $\bar{\mathbf{T}}$ through

$$\text{vol}(\mathscr{B}_0)\bar{\mathbf{T}} = \int_{\mathscr{B}_0} \mathbf{T}\, dV.$$

(a) (*Signorini's theorem*) Show that $\bar{\mathbf{T}}$ is completely determined by the surface traction $\mathbf{Tn}$ and the body force $\mathbf{b}$ as follows:

$$\text{vol}(\mathscr{B}_0)\bar{\mathbf{T}} = \int_{\partial\mathscr{B}_0} (\mathbf{Tn}\otimes\mathbf{r})\, dA + \int_{\mathscr{B}_0} \mathbf{b}\otimes\mathbf{r}\, dV.$$

(b) Assume that $\mathbf{b} = \mathbf{0}$ and that $\partial\mathscr{B}_0$ consists of two closed surfaces $\mathscr{S}_0$ and $\mathscr{S}_1$ with $\mathscr{S}_1$ enclosing $\mathscr{S}_0$ (Fig. 10). Assume further that $\mathscr{S}_0$ and $\mathscr{S}_1$ are acted on by uniform pressures π_0 and π_1, so that

$$\mathbf{s}(\mathbf{n}) = -\pi_0\mathbf{n} \quad \text{on} \quad \mathscr{S}_0,$$
$$\mathbf{s}(\mathbf{n}) = -\pi_1\mathbf{n} \quad \text{on} \quad \mathscr{S}_1,$$

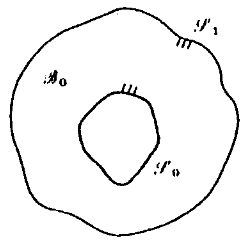

Figure 10

with π_0 and π_1 constants. Show that $\bar{\mathbf{T}}$ is a pressure of amount

$$\frac{\pi_1 v_1 - \pi_0 v_0}{v_1 - v_0},$$

where v_0 and v_1 are, respectively, the volumes enclosed by $\mathscr{S}_0$ and $\mathscr{S}_1$.

2. Prove $(1)_2$.

3. Derive the following counterpart of (2) for a control volume $\mathscr{R}$:

$$\int_{\partial\mathscr{R}} \mathbf{s}(\mathbf{n})\cdot\mathbf{v}\, dA + \int_{\mathscr{R}} \mathbf{b}\cdot\mathbf{v}\, dV$$
$$= \int_{\mathscr{R}} \mathbf{T}\cdot\mathbf{D}\, dV + \frac{d}{dt}\int_{\mathscr{R}} \frac{\mathbf{v}^2}{2}\rho\, dV + \int_{\partial\mathscr{R}} \frac{\rho\mathbf{v}^2}{2}(\mathbf{v}\cdot\mathbf{n})\, dA.$$

Selected References

Gurtin and Martins [1].
Noll [3, 4].
Truesdell [1, Chapter 3].
Truesdell and Toupin [1, §§195–238].

CHAPTER

VI

Constitutive Assumptions. Inviscid Fluids

16. CONSTITUTIVE ASSUMPTIONS

The axioms laid down for force—namely the laws of momentum balance—are common to most bodies in nature. These laws, however, are insufficient to fully characterize the behavior of bodies because they do not distinguish between different types of materials. Physical experience has shown that two bodies of the same size and shape subjected to the same motion will generally not have the same resulting force distribution. For example, two thin wires of the same length and diameter, one of steel and one of copper, will require different forces to produce the same elongation. We therefore introduce additional hypotheses, called *constitutive assumptions*, which serve to distinguish different types of material behavior.

Here we will consider three types of constitutive assumptions.

(i) *Constraints on the possible deformations the body may undergo.* The simplest constraint is that only rigid motions be possible, a constraint which forms the basis of rigid body mechanics. Another example is the assumption of incompressibility, under which only isochoric deformations are permissible. This assumption is realistic for liquids such as water under normal flow conditions and even describes the behavior of gases like air provided the velocities are not too large.

(ii) *Assumptions on the form of the stress tensor.* The most widely used assumption of this type is that the stress be a pressure, an assumption valid for most fluids when viscous effects are negligible.

(iii) *Constitutive equations relating the stress to the motion.* Here a classical example is the equation of state of a gas giving the pressure as a function of the density.

We now make these ideas precise.

A **material body** is a body $\mathscr{B}$ together with a mass distribution and a family $\mathscr{C}$ of dynamical processes. $\mathscr{C}$ is called the **constitutive class** of the body; it consists of those dynamical processes consistent with the constitutive assumptions of the body.

A dynamical process $(\mathbf{x}, \mathbf{T})$ is **isochoric** if

$$\mathbf{x}(\cdot, t) \quad \text{is an isochoric deformation} \tag{1}$$

at each t; a material body is **incompressible** if each $(\mathbf{x}, \mathbf{T}) \in \mathscr{C}$ is isochoric. By (1), given any part $\mathscr{P}$,

$$\text{vol}(\mathscr{P}_t) = \text{vol}(\mathscr{P})$$

for all t, which implies (10.4). Thus every motion of an incompressible body is isochoric. The restriction (1), however, implies more: not only does each motion preserve volume, but the volume of any part throughout the motion must be the same as its volume in the reference configuration. Further, in view of (6.17) and (c) of the theorem characterizing isochoric motions (page 78), every flow $(\mathbf{v}, \rho, \mathbf{T})$ of an incompressible body must have

$$\det \mathbf{F} = 1, \qquad \operatorname{div} \mathbf{v} = 0. \tag{2}$$

Also, by $(2)_1$ and (12.4), $\rho(\mathbf{x}, t) = \rho_0(\mathbf{p})$ for $\mathbf{x} = \mathbf{x}(\mathbf{p}, t)$. Thus, in particular, when ρ_0 is *constant*,

$$\rho \equiv \rho_0;$$

in this case we refer to ρ_0 as the **density of the body.**

A dynamical process $(\mathbf{x}, \mathbf{T})$ is **Eulerian** if the stress $\mathbf{T}$ is a pressure; that is, if

$$\mathbf{T} = -\pi \mathbf{I}$$

with π a scalar field on the trajectory of $\mathbf{x}$.

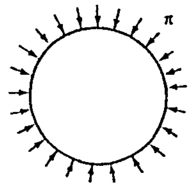

Figure 1

An **ideal fluid** is a material body consistent with the following two constitutive assumptions:

(a) The constitutive class is the set of *all* isochoric, Eulerian dynamical processes.

(b) The density ρ_0 is constant.

Thus an ideal fluid is an incompressible material body for which the stress is a pressure in every flow.

Note that for an ideal fluid the pressure is not determined uniquely by the motion: an infinite number of pressure fields correspond to each motion. This degree of flexibility in the pressure is common to all incompressible materials. That such a property is physically reasonable can be inferred from the following example. Consider a ball composed of an ideal fluid under a time-independent uniform pressure π (Fig. 1), and assume, for the moment, that all body forces, including gravity, are absent. Then the ball should remain in the same configuration for all time. Moreover, since the ball is incompressible, an increase or decrease in pressure should not result in a deformation. Thus the *same* "motion" should correspond to *all* uniform pressure fields. This argument is easily extended to nonuniform pressure fields and arbitrary motions, but for consistency the requirement of null body forces must be dropped to insure satisfaction of the equation of motion.

We next introduce a material body, called an **elastic fluid**, in which compressibility effects are not ignored, and for which the pressure is completely specified by the motion. Here the constitutive class is defined by a smooth *response function*

$$\hat{\pi} : \mathbb{R}^+ \to \mathbb{R}$$

giving the pressure when the density is known:

$$\pi = \hat{\pi}(\rho). \tag{3}$$

That is, the constitutive class is the set of *all* Eulerian dynamical processes $(\mathbf{x}, -\pi\mathbf{I})$ which obey the *constitutive equation* (3).

We will consistently write

$$(\mathbf{v}, \rho, \pi) \quad \text{in place of} \quad (\mathbf{v}, \rho, -\pi\mathbf{I})$$

for the flow of an ideal fluid or of an elastic fluid. (Of course, $\rho = \rho_0$ in the former case.)

An inviscid fluid is often characterized by the requirement that it be incapable of exerting shearing forces; in this sense both ideal and elastic fluids are inviscid.

Exercise

1. Consider a material body $\mathscr{B}$ with constitutive class $\mathscr{C}$. A *simple constraint* for $\mathscr{B}$ is a function

$$\gamma: \mathrm{Lin}^+ \to \mathbb{R}$$

such that each dynamical process $(\mathbf{x}, \mathbf{T}) \in \mathscr{C}$ satisfies

$$\gamma(\mathbf{F}) = 0. \tag{4}$$

For such a material one generally lays down the following constraint axiom: the stress is determined by the motion only to within a stress $\mathbf{N}$ that does no work in any motion consistent with the constraint. [The rate at which a stress $\mathbf{N}$ does work is given by the stress power, per unit volume,

$$\mathbf{N} \cdot \mathbf{D}$$

(cf. page 111), where $\mathbf{D}$ is the stretching.]

We now make this idea precise. Let $\mathscr{D}$ be the set of all possible stretching tensors; that is, $\mathscr{D}$ is the set of all tensors $\mathbf{D}$ with the following property: for some C^2 function $\mathbf{F}: \mathbb{R} \to \mathrm{Lin}^+$ consistent with (4), $\mathbf{D}$ is the symmetric part of

$$\mathbf{L} = \dot{\mathbf{F}}(t)\mathbf{F}(t)^{-1}$$

at some fixed time t. Let

$$\mathscr{R} = \mathscr{D}^{\perp};$$

that is,

$$\mathscr{R} = \{\mathbf{N} \in \mathrm{Sym} \mid \mathbf{N} \cdot \mathbf{D} = 0 \quad \text{for all} \quad \mathbf{D} \in \mathscr{D}\}.$$

Then the *constraint axiom* can be stated as follows: If $(\mathbf{x}, \mathbf{T})$ belongs to $\mathscr{C}$, then so also does every dynamical process of the form $(\mathbf{x}, \mathbf{T} + \mathbf{N})$ with

$$\mathbf{N}(\mathbf{x}, t) \in \mathscr{R}$$

for all $(\mathbf{x}, t)$ in the trajectory of $\mathbf{x}$. We call $\mathscr{R}$ the *reaction space*.

An incompressible material can be defined by the simple constraint

$$\gamma(\mathbf{F}) = \det \mathbf{F} - 1. \tag{5}$$

Show that this constraint satisfies the constraint axiom if and only if the corresponding reaction space is the set of all tensors of the form $-\pi\mathbf{I}$, $\pi \in \mathbb{R}$; i.e., if and only if the stress is determined by the motion at most to within an arbitrary pressure field. Show further that the constitutive assumptions of an ideal fluid are consistent with the constraint axiom.

17. IDEAL FLUIDS

Consider an ideal fluid with density ρ_0. The basic equations are the equation of motion (15.7) and the constraint equation $(16.2)_2$:

$$\begin{aligned} -\operatorname{grad} \pi + \mathbf{b} &= \rho_0 \dot{\mathbf{v}}, \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned} \tag{1}$$

When the body force is conservative with potential β, (15.8) implies

$$\dot{\mathbf{v}} = -\operatorname{grad}\left(\frac{\pi}{\rho_0} + \beta\right),$$

and *the acceleration is the gradient of a potential*. We therefore have the following corollary of the results established in Section 11.

Theorem (Properties of ideal fluid motion). *The flow of an ideal fluid under a conservative body force has the following properties:*

(a) *If the flow is irrotational at one time, it is irrotational at all times.*
(b) *The flow preserves circulation.*
(c) *Vortex lines are transported with the fluid.*

As we shall see, the equations of motion of a fluid are greatly simplified when the flow is assumed to be irrotational. The above theorem implies that the motion of an ideal fluid under a conservative body force is irrotational if the flow starts from a state of rest. This furnishes the usual justification for the assumption of irrotationality. For a plane flow in an infinite region a much stronger assertion can be made. Indeed, if the flow is steady and in a uniform state at infinity [i.e., if $\operatorname{grad} \mathbf{v}(\mathbf{x}) \to \mathbf{0}$ as $\mathbf{x} \to \infty$], and if each streamline begins or ends at infinity, then, since $\mathbf{W}$ is constant on streamlines (cf. the proposition on page 81), the flow must necessarily be irrotational.

Bernoulli's Theorem for Ideal Fluids. *Let* $(\mathbf{v}, \rho_0, \pi)$ *be a flow of an ideal fluid under a conservative body force with potential* β.

(a) *If the flow is potential* ($\mathbf{v} = \text{grad}\,\varphi$),

$$\text{grad}\left(\varphi' + \frac{\mathbf{v}^2}{2} + \frac{\pi}{\rho_0} + \beta\right) = \mathbf{0}.$$

(b) *If the flow is steady,*

$$\left(\frac{\mathbf{v}^2}{2} + \frac{\pi}{\rho_0} + \beta\right)^{\cdot} = 0, \tag{2}$$

so that $(\mathbf{v}^2/2) + (\pi/\rho_0) + \beta$ *is constant on streamlines.*

(c) *If the flow is steady and irrotational,*

$$\frac{\mathbf{v}^2}{2} + \frac{\pi}{\rho_0} + \beta = \text{const} \tag{3}$$

everywhere.

Proof. Conclusion (a) follows from (15.4) (with $\rho = \rho_0$). To establish (b) and (c) let

$$\eta = \frac{\mathbf{v}^2}{2} + \frac{\pi}{\rho_0} + \beta.$$

For a steady flow

$$\eta' = 0,$$

and by (15.5),

$$\mathbf{v} \cdot \text{grad}\,\eta = 0;$$

hence

$$\dot{\eta} = \eta' + \mathbf{v} \cdot \text{grad}\,\eta = 0,$$

which is (2). Finally, for a steady, irrotational flow (15.6) yields

$$\text{grad}\,\eta = \mathbf{0},$$

which, with $\eta' = 0$, implies (3). □

By Bernoulli's theorem, for a steady, irrotational flow under a conservative body force equations (1) reduce to

$$\text{div}\,\mathbf{v} = 0, \qquad \text{curl}\,\mathbf{v} = \mathbf{0},$$
$$\frac{\mathbf{v}^2}{2} + \frac{\pi}{\rho_0} + \beta = \text{const}. \tag{4}$$

These equations are also sufficient to characterize this type of flow. Indeed, suppose that $(\mathbf{v}, \pi)$ is a steady solution of (4) on a region of space $\mathscr{B}_0$. Then, since $\mathbf{W} = \mathbf{0}$, $(9.10)_1$ implies

$$\dot{\mathbf{v}} = \mathbf{v}' + \text{grad}\left(\frac{\mathbf{v}^2}{2}\right) = \text{grad}\left(\frac{\mathbf{v}^2}{2}\right) = -\text{grad}\left(\frac{\pi}{\rho_0} + \beta\right),$$

and the basic equations (1) (with $\mathbf{b}/\rho_0 = -\text{grad}\,\beta$) are satisfied.

In a steady motion the velocity is tangent to the boundary (cf. the proposition on page 68); thus (4) should be supplemented by the *boundary condition*

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathscr{B}_0. \tag{5}$$

When the flow is nonsteady we are left with the system (1) to solve. The basic nonlinearity of these equations is obscured by the material time derivative in $(1)_1$. Indeed, in terms of spatial operators these equations take the form

$$\mathbf{v}' + (\text{grad}\,\mathbf{v})\mathbf{v} = -\text{grad}\,\pi + \mathbf{b},$$
$$\text{div}\,\mathbf{v} = 0, \tag{6}$$

where, for convenience, we have written π for π/ρ_0 and $\mathbf{b}$ for $\mathbf{b}/\rho_0$. Equations (6) are usually referred to as *Euler's equations.*

EXERCISES

1. Show that in the flow of an ideal fluid the stress power is zero (cf. page 111).
2. Consider the flow of a bounded ideal fluid under a conservative body force, and suppose that for each t and each $\mathbf{x} \in \partial\mathscr{B}_t$, $\mathbf{v}(\mathbf{x}, t)$ is tangent to $\partial\mathscr{B}_t$. Show that

$$\frac{d}{dt}\int_{\mathscr{B}_t} \mathbf{v}^2\, dV = 0,$$

so that the kinetic energy is constant.

3. Consider a homogeneous motion of the form

$$\mathbf{x}(\mathbf{p}, t) = \mathbf{p}_0 + \mathbf{F}(t)[\mathbf{p} - \mathbf{p}_0].$$

Show that if $\mathbf{x}$ represents a motion of an ideal fluid under conservative body forces, then $\ddot{\mathbf{F}}(t)\mathbf{F}(t)^{-1}$ must be symmetric at each t.

4. Consider a steady, irrotational flow of an ideal fluid over an obstacle $\mathscr{R}$ (Fig. 2); that is, $\mathscr{R}$ is a bounded regular region whose interior lies outside the flow region $\mathscr{B}_0$ and whose boundary is a subset of $\partial\mathscr{B}_0$. Assume that

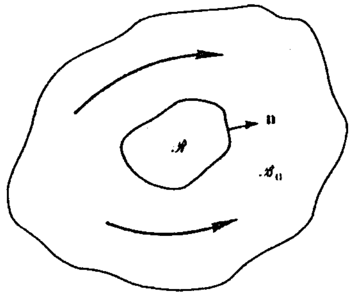

Figure 2

the body force is zero. Show that the total force exerted on $\mathscr{R}$ by the fluid is equal to

$$\frac{\rho_0}{2}\int_{\partial\mathscr{R}} \mathbf{v}^2\mathbf{n}\, dA.$$

18. STEADY, PLANE, IRROTATIONAL FLOW OF AN IDEAL FLUID

For a steady, plane flow the velocity field has the form

$$\mathbf{v}(\mathbf{x}) = v_1(x_1, x_2)\mathbf{e}_1 + v_2(x_1, x_2)\mathbf{e}_2, \tag{1}$$

and the flow region can be identified with a region $\mathscr{R}$ in $\mathbb{R}^2$. For convenience, we identify $\mathbf{v}$ and $\mathbf{x}$ with vectors in $\mathbb{R}^2$ and write, in place of (1),

$$\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x})), \qquad \mathbf{x} = (x_1, x_2).$$

Then the basic equations (17.4) reduce to

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0,$$

$$\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = 0, \tag{2}$$

$$\tfrac{1}{2}(v_1^2 + v_2^2) + \frac{\pi}{\rho_0} = \text{const}$$

with π also a field on $\mathscr{R}$. Here, for convenience, we have assumed that the *body force is zero.*

For the moment let

$$\alpha = v_1, \qquad \beta = -v_2.$$

Then the first two relations in (2) are the Cauchy–Riemann equations:

$$\frac{\partial \alpha}{\partial x_1} = \frac{\partial \beta}{\partial x_2}, \qquad \frac{\partial \alpha}{\partial x_2} = -\frac{\partial \beta}{\partial x_1};$$

these are necessary and sufficient conditions that

$$g(z) = \alpha(x_1, x_2) + i\beta(x_1, x_2)$$

be an analytic function of the complex variable

$$z = x_1 + ix_2$$

on $\mathscr{R}$ (considered as a region in the complex plane $\mathbb{C}$). Thus we have the following

Theorem. *Consider a steady, plane, irrotational flow of an ideal fluid. Let $g: \mathscr{R} \to \mathbb{C}$ be defined by*

$$g(z) = v_1(x_1, x_2) - iv_2(x_1, x_2). \tag{3}$$

Then g is an analytic function. Conversely, any analytic function g generates, in the sense of (3), a solution of $(2)_{1,2}$.

The function g is called the **complex velocity**.

Let $\mathbf{c} = (c_1, c_2)$ be a curve in $\mathscr{R}$. Then $\mathbf{c}$ can also be considered as a curve

$$c(\sigma) = c_1(\sigma) + ic_2(\sigma), \qquad 0 \le \sigma \le 1$$

in $\mathbb{C}$. Given any complex function h on $\mathscr{R}$ we write

$$\int_{\mathbf{c}} h(z)\, dz = \int_0^1 h(c(\sigma))\dot{c}(\sigma)\, d\sigma;$$

in particular, we define

$$\Gamma(\mathbf{c}) = \int_{\mathbf{c}} g(z)\, dz. \tag{4}$$

To interpret $\Gamma(\mathbf{c})$, note first that

$$\mathbf{k}(\sigma) = (\dot{c}_2(\sigma), -\dot{c}_1(\sigma)) \tag{5}$$

is normal to $\mathbf{c}$ at $\mathbf{c}(\sigma)$. Therefore when $\mathbf{c}$ is the boundary curve of $\mathscr{R}$ the requirement that $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathscr{R}$ [cf. (17.5)] becomes

$$v_1(\mathbf{c}(\sigma))\dot{c}_2(\sigma) = v_2(\mathbf{c}(\sigma))\dot{c}_1(\sigma). \tag{6}$$

Thus in this instance

$$\int_{\mathbf{c}} g(z)\, dz = \int_0^1 (v_1 - iv_2)(\dot{c}_1 + i\dot{c}_2)\, d\sigma = \int_0^1 (v_1\dot{c}_1 + v_2\dot{c}_2)\, d\sigma,$$

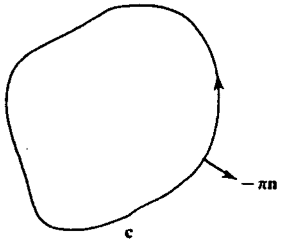

Figure 3

and

$$\Gamma(\mathbf{c}) = \int_{\mathbf{c}} \mathbf{v}(\mathbf{x}) \cdot d\mathbf{x}$$

is the *circulation around* **c**.

Next,

$$|\mathbf{k}| = \sqrt{\dot{c}_1^2 + \dot{c}_2^2},$$

so that $|\mathbf{k}(\sigma)|\, d\sigma$ is the element of arc length along **c**. The vector

$$\mathbf{f}(\mathbf{c}) = -\int_0^1 \pi(\mathbf{c}(\sigma))\mathbf{k}(\sigma)\, d\sigma \tag{7}$$

therefore represents the integral along **c** of the surface force $-\pi\mathbf{n}$ ($\mathbf{n} = \mathbf{k}/|\mathbf{k}|$) with respect to arc length. In fact, for a simple closed curve oriented *counter-clockwise* as shown in Fig. 3, $\mathbf{f}(\mathbf{c})$ gives the total force (per unit length in the x_3-direction) on the material inside **c**.

Blasius–Kutta–Joukowski Theorem. *Consider a steady, plane, irrotational flow of an ideal fluid in a region* $\mathscr{R}$ *whose boundary is a simple closed curve* **c**. *Let* g *be the complex velocity of the flow. Then*

$$f_1(\mathbf{c}) - if_2(\mathbf{c}) = \frac{i\rho_0}{2} \int_{\mathbf{c}} g(z)^2\, dz. \tag{8}$$

If in addition $\mathscr{R}$ *is the region exterior to* **c** *and the velocity is uniform at infinity in the sense that*

$$g(z) \to u \qquad \text{as} \quad z \to \infty \tag{9}$$

with u *real, then*

$$f_1(\mathbf{c}) = 0, \qquad f_2(\mathbf{c}) = -\rho_0 u \Gamma(\mathbf{c}). \tag{10}$$

Proof. Since **c** is closed,

$$\int_0^1 \mathbf{k}\, d\sigma = \mathbf{0},$$

and we conclude from (7) and the Bernoulli equation $(2)_3$ that

$$\mathbf{f}(\mathbf{c}) = \frac{\rho_0}{2} \int_0^1 \mathbf{v}^2\mathbf{k}\, d\sigma.$$

Thus

$$f_1(\mathbf{c}) - if_2(\mathbf{c}) = \frac{\rho_0}{2} \int_0^1 \mathbf{v}^2(\dot{c}_2 + i\dot{c}_1)\, d\sigma = \frac{i\rho_0}{2} \int_0^1 \mathbf{v}^2(\dot{c}_1 - i\dot{c}_2)\, d\sigma.$$

On the other hand,

$$\int_{\mathbf{c}} g^2\, dz = \int_0^1 (v_1 - iv_2)^2(\dot{c}_1 + i\dot{c}_2)\, d\sigma,$$

and a simple calculation using (6) yields

$$(v_1^2 + v_2^2)(\dot{c}_1 - i\dot{c}_2) = (v_1 - iv_2)^2(\dot{c}_1 + i\dot{c}_2).$$

Thus (8) holds.

Assume now that $\mathscr{R}$ is exterior to **c** and that (9) holds. Assume further that the origin **o** lies inside **c**. Then, since g is analytic and (9) holds, the Laurent expansion of g about the origin has the form

$$g(z) = u + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots$$

for any $z \in \mathscr{R}$. Hence (4) and Cauchy's theorem of residues imply

$$\Gamma(\mathbf{c}) = 2\pi i\alpha_1.$$

Next, since g is analytic, we can compute g^2 by termwise multiplication. Thus

$$g(z)^2 = u^2 + \frac{2u\alpha_1}{z} + \frac{\alpha_1^2 + 2u\alpha_2}{z^2} + \cdots,$$

and a second application of Cauchy's theorem tells us that

$$\int_{\mathbf{c}} g^2\, dz = 4\pi i u\alpha_1 = 2u\Gamma(\mathbf{c}).$$

Therefore, in view of (8), (10) holds. □

If $\mathscr{R}$ is simply connected, then g can be written as the derivative of an analytic function w:

$$g = \frac{dw}{dz}.$$

(This result extends to multiply connected regions provided one is willing to admit multivalued functions.) The function $w: \mathscr{R} \to \mathbb{C}$ is called the **complex potential**; the real and imaginary parts of w, denoted φ and ψ, respectively, generate the velocity $\mathbf{v}$ through

$$v_1 = \frac{\partial\varphi}{\partial x_1} = \frac{\partial\psi}{\partial x_2}, \qquad v_2 = \frac{\partial\varphi}{\partial x_2} = -\frac{\partial\psi}{\partial x_1}.$$

Let $\mathbf{s} = (s_1, s_2)$ be a streamline so that

$$\dot{\mathbf{s}}(t) = \mathbf{v}(\mathbf{s}(t)). \tag{11}$$

Then

$$\frac{d}{dt}\psi(\mathbf{s}(t)) = \frac{\partial\psi}{\partial x_1}\dot{s}_1 + \frac{\partial\psi}{\partial x_2}\dot{s}_2 = -v_2 v_1 + v_1 v_2 = 0, \tag{12}$$

and ψ is *constant on streamlines*. For this reason ψ is called the **stream function**.

Let $\mathbf{c}$ be a curve. Then $\mathbf{c}$ is **essentially a segment of a streamline** $\mathbf{s}$ if there is a smooth one-to-one mapping τ of $[0, 1]$ onto a closed interval of $\mathbb{R}$ such that

$$\mathbf{c}(\sigma) = \mathbf{s}(\tau(\sigma)), \qquad 0 \le \sigma \le 1. \tag{13}$$

Proposition. *Consider a flow with a complex potential. Let* $\mathbf{c}$ *be a curve in the flow region and assume that* $\mathbf{v}$ *is nowhere zero on* $\mathbf{c}$. *Then the following are equivalent:*

(a) $v_1(\mathbf{c}(\sigma))\dot{c}_2(\sigma) = v_2(\mathbf{c}(\sigma))\dot{c}_1(\sigma)$ *for* $0 \le \sigma \le 1$.
(b) $\mathbf{v}(\mathbf{c}(\sigma))$ *is parallel to* $\dot{\mathbf{c}}(\sigma)$ *for* $0 \le \sigma \le 1$.
(c) $\mathbf{c}$ *is essentially a segment of a streamline.*
(d) ψ *is constant on* $\mathbf{c}$; *that is, for* $0 \le \sigma \le 1$,

$$\frac{d}{d\sigma}\psi(\mathbf{c}(\sigma)) = 0.$$

Proof. We will show that (d) $\Leftrightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(d) $\Leftrightarrow$ (a). This follows from the identity [cf. (12)]

$$\frac{d}{d\sigma}\psi(\mathbf{c}(\sigma)) = -v_2(\mathbf{c}(\sigma))\dot{c}_1(\sigma) + v_1(\mathbf{c}(\sigma))\dot{c}_2(\sigma).$$

(a) $\Leftrightarrow$ (b). Here we use the fact that (a) and (b) are each equivalent to the assertion that $\mathbf{k}(\sigma)\cdot\mathbf{v}(\mathbf{c}(\sigma)) = 0$ for $0 \le \sigma \le 1$, where $\mathbf{k}$, defined by (5), is normal to the curve $\mathbf{c}$.

(c) $\Rightarrow$ (b). If $\mathbf{c}$ is essentially a segment of a streamline $\mathbf{s}$, then (11), (13), and the chain rule yield

$$\dot{\mathbf{c}}(\sigma) = \dot{\mathbf{s}}(\tau(\sigma))\dot{\tau}(\sigma) = \mathbf{v}(\mathbf{s}(\tau(\sigma)))\dot{\tau}(\sigma) = \dot{\tau}(\sigma)\mathbf{v}(\mathbf{c}(\sigma)), \tag{14}$$

and (b) is satisfied.

(b) $\Rightarrow$ (c). Assume that (b) holds. Then, since $\mathbf{v}(\mathbf{c}(\sigma))$ and $\dot{\mathbf{c}}(\sigma)$ never vanish,

$$\dot{\mathbf{c}}(\sigma) = \alpha(\sigma)\mathbf{v}(\mathbf{c}(\sigma)), \qquad 0 \le \sigma \le 1,$$

with α a continuous function on $[0, 1]$ which never vanishes. Let

$$\tau(\sigma) = \int_0^\sigma \alpha(\lambda)\, d\lambda, \qquad 0 \le \sigma \le 1;$$

let $\mathbf{s}$ be the streamline which passes through $\mathbf{c}(0)$ at time $t = 0$, so that $\mathbf{s}$ satisfies (11) and

$$\mathbf{s}(0) = \mathbf{c}(0);$$

and let $\mathbf{d}$ be the curve in $\mathbb{R}^2$ defined by

$$\mathbf{d}(\sigma) = \mathbf{s}(\tau(\sigma)), \qquad 0 \le \sigma \le 1. \tag{15}$$

Then

$$\mathbf{d}(0) = \mathbf{c}(0)$$

and, using the argument (14),

$$\dot{\mathbf{d}}(\sigma) = \alpha(\sigma)\mathbf{v}(\mathbf{d}(\sigma)), \qquad 0 \le \sigma \le 1.$$

Thus $\mathbf{c}$ and $\mathbf{d}$ satisfy the same differential equation and the same initial condition. We therefore conclude from the uniqueness theorem for ordinary differential equations that $\mathbf{c} = \mathbf{d}$, so that, by (15), $\mathbf{c}$ is essentially a segment of a streamline. ☐

For problems involving flows about a stationary region one usually assumes, as the boundary condition, that the boundary of the region be a streamline. In view of the last proposition, this insures that the velocity field be tangent to the boundary, or equivalently, that the boundary be impenetrable to the fluid (cf. the proposition on page 68).

As an example consider the region exterior to the unit circle and assume that the velocity is uniform and in the x_1-direction at infinity, so that (9) is satisfied. Consider the flow generated by the complex potential

$$w(z) = u\left(z + \frac{1}{z}\right).$$

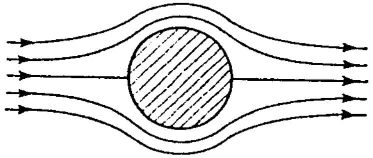

Figure 4

In terms of cylindrical coordinates $z = re^{i\theta}$ the stream function ψ has the form

$$\psi(r, \theta) = a\left(r - \frac{1}{r}\right)\sin\theta,$$

so that ψ has the *constant* value zero on the unit circle. Further, the complex velocity

$$g(z) = a\left(1 - \frac{1}{z^2}\right)$$

satisfies (9). The streamlines are shown in Fig. 4. For this flow

$$\Gamma(\text{unit circle}) = 0,$$

and by the Blasius–Kutta–Joukowski theorem there is no net force on the boundary.

Another flow with the desired properties is generated by

$$w(z) = a\left(z + \frac{1}{z} + i\gamma \log z\right).$$

This potential has a stream function

$$\psi(r, \theta) = a\left[\left(r - \frac{1}{r}\right)\sin\theta + \gamma \log r\right],$$

which again vanishes on the circle $r = 1$. Further, w generates a complex velocity

$$g(z) = a\left(1 - \frac{1}{z^2} + \frac{i\gamma}{z}\right),$$

which satisfies (9).

A point at which $g(z) = 0$ is called a *stagnation point*. Points z on the unit circle are given by the equation $z = e^{i\theta}$. A necessary and sufficient condition that there exist a stagnation point at $z = e^{i\theta}$ is that

$$\sin\theta = -\frac{\gamma}{2},$$

and hence that $-2 \leq \gamma \leq 2$. Assume $\gamma \geq 0$. Then: (a) for $0 \leq \gamma < 2$ there are two stagnation points on the cylinder; (b) for $\gamma = 2$ there is one stagnation

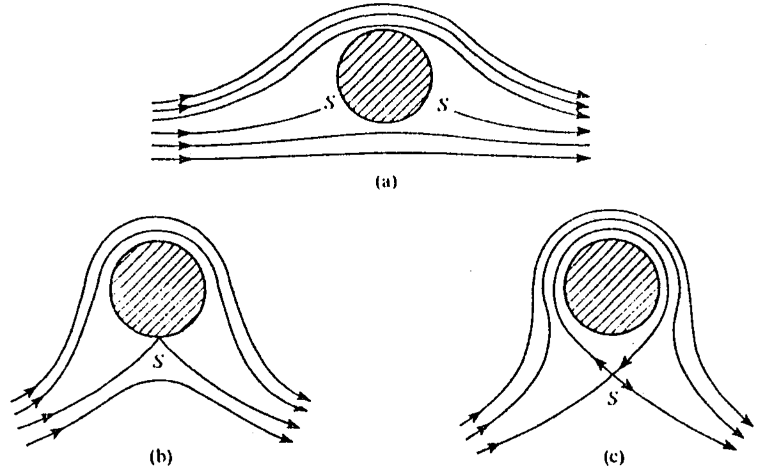

Figure 5

point on the cylinder; (c) for $\gamma > 2$ there are no stagnation points on the cylinder. These three cases are shown in Fig. 5 with the stagnation points denoted by S. For these flows

$$\Gamma(\text{unit circle}) = -2\pi a\gamma$$

provided the unit circle has a counter-clockwise orientation; thus there is no drag (force in the x_1-direction), but there is a lift (force in the x_2-direction) equal to

$$2\pi\rho_0 a^2\gamma.$$

The problem of flow about an airfoil is far more complicated than the simple flow presented in Fig. 5, but the results are qualitatively the same. There the circulation is adjusted as shown in Fig. 6 to insure two stagnation points with the aft stagnation point coincident with the (sharp) trailing edge (Kutta–Joukowski condition).

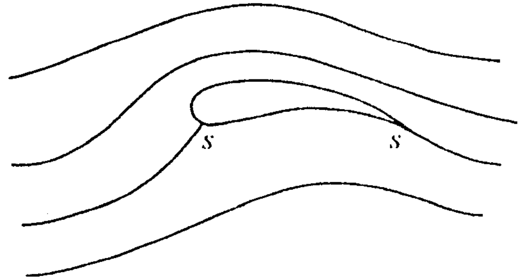

Figure 6

Exercises

1. Show that the complex potential

$$w(z) = -Ce^{-i\pi\beta} z^{\beta+1}$$

represents the flow past a wedge (Fig. 7) of angle 2α, where C and β ($\beta > -1$) are real constants and

$$\alpha = \frac{\beta\pi}{1+\beta}.$$

Determine the stagnation points (if any).

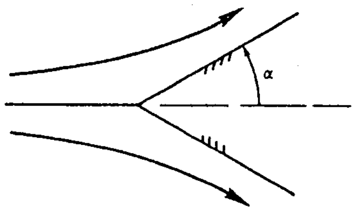

Figure 7

2. Sketch the streamlines of the complex potential

$$w(z) = Cz^2,$$

where C is a real constant, and show that the absolute value of the velocity is proportional to the distance from the origin.

3. Consider a steady, plane, irrotational flow of an ideal fluid. Let **c** be a section of a streamline with

$$\mathbf{c}(0) = (x_A, y_A), \qquad \mathbf{c}(1) = (x_B, y_B).$$

Let π_0 be the pressure at a point at which the velocity is $\mathbf{v}_0$. Show that the force (8) on **c** is given by

$$f_1(\mathbf{c}) - if_2(\mathbf{c}) = \left(\pi_0 + \frac{\rho_0 \mathbf{v}_0^2}{2}\right)[(y_A - y_B) + i(x_A - x_B)] + \frac{i\rho_0}{2}\int_{\mathbf{c}} g^2\, dz.$$

19. ELASTIC FLUIDS

The basic equations for the flow of an elastic fluid are the equation of motion

$$-\operatorname{grad}\pi + \mathbf{b} = \rho\dot{\mathbf{v}}, \tag{1}$$

conservation of mass

$$\rho' + \operatorname{div}(\rho\mathbf{v}) = 0, \tag{2}$$

and the constitutive equation

$$\pi = \hat{\pi}(\rho). \tag{3}$$

We assume that $\hat{\pi}$ has a strictly positive derivative, and we define functions $\kappa > 0$ and ε on $\mathbb{R}^1$ by

$$\kappa^2(\rho) = \frac{d\hat{\pi}(\rho)}{d\rho}, \qquad \varepsilon(\rho) = \int_{\rho_*}^{\rho} \frac{\kappa^2(\zeta)}{\zeta}\, d\zeta, \tag{4}$$

where ρ_* is an arbitrarily chosen value of the density. The function $\kappa(\rho)$ is called the **sound speed**; the reason for this terminology will become apparent when we discuss the acoustic equations.

In a flow $\varepsilon(\rho)$ may be interpreted as a spatial field. We may therefore conclude from the chain rule that

$$\operatorname{grad}\varepsilon(\rho) = \frac{\kappa^2(\rho)}{\rho}\operatorname{grad}\rho = \frac{1}{\rho}\frac{d\hat{\pi}(\rho)}{d\rho}\operatorname{grad}\rho = \frac{1}{\rho}\operatorname{grad}\pi. \tag{5}$$

Thus when the body force is conservative with potential β, (1) takes the form

$$\dot{\mathbf{v}} = -\operatorname{grad}(\varepsilon(\rho) + \beta),$$

and *the acceleration is the gradient of a potential*. We therefore have the following corollary of the results established in Section 11.

Theorem (Properties of elastic fluid motion). *The flow of an elastic fluid under a conservative body force has the following properties:*

(a) *If the flow is irrotational at one time, it is irrotational at all times.*
(b) *The flow preserves circulation.*
(c) *Vortex lines are transported with the fluid.*

Thus, in particular, the flow of an elastic fluid under a conservative body force is irrotational provided the flow starts from rest.

Bernoulli's Theorem for Elastic Fluids. *Let $(\mathbf{v}, \rho, \pi)$ be a flow of an elastic fluid under a conservative body force with potential β. Let ε be defined by (4).*

(a) *If the flow is potential,*

$$\operatorname{grad}\left(\varphi' + \frac{\mathbf{v}^2}{2} + \varepsilon(\rho) + \beta\right) = 0.$$

(b) *If the flow is steady,*

$$\left(\frac{\mathbf{v}^2}{2} + \varepsilon(\rho) + \beta\right)^{\cdot} = 0, \tag{6}$$

so that $(\mathbf{v}^2/2) + \varepsilon(\rho) + \beta$ *is constant on streamlines.*

(c) *If the flow is steady and irrotational,*

$$\frac{\mathbf{v}^2}{2} + \varepsilon(\rho) + \beta = \text{const} \tag{7}$$

everywhere.

Proof. Conclusion (a) follows from (15.4) and (5). Let

$$\eta = \frac{\mathbf{v}^2}{2} + \varepsilon(\rho) + \beta.$$

For a steady flow

$$\eta' = 0, \qquad \mathbf{v}\cdot\operatorname{grad}\eta = 0,$$

where we have used (15.5) and (5); hence

$$\dot{\eta} = \eta' + \mathbf{v}\cdot\operatorname{grad}\eta = 0,$$

which is (6). Finally, for a steady, irrotational flow, (5) and (15.6) yield $\operatorname{grad}\eta = \mathbf{0}$, which with $\eta' = 0$ implies (7). □

If we define

$$\alpha(\rho) = \kappa^2(\rho)/\rho,$$

then (1)–(3) take the form

$$\begin{aligned} \mathbf{v}' + (\operatorname{grad}\mathbf{v})\mathbf{v} + \alpha(\rho)\operatorname{grad}\rho &= \mathbf{b}/\rho, \\ \rho' + \operatorname{div}(\rho\mathbf{v}) &= 0. \end{aligned} \tag{8}$$

Equations (8) constitute a *nonlinear* system for ρ and $\mathbf{v}$ and are generally quite difficult to solve. Further, for many problems of interest the solution will not be smooth because of the appearance of shock waves (surfaces across which the velocity suffers a jump discontinuity), and the notion of a weak solution must be introduced. A careful discussion of these matters, however, is beyond the scope of this book.

We now consider flows which are close to a given rest state with *constant* density ρ_0. We therefore assume that $|\rho - \rho_0|$, $|\operatorname{grad}\rho|$, $|\mathbf{v}|$, and $|\operatorname{grad}\mathbf{v}|$ are small.[1] Let δ be an upper bound for these fields. Then, since

$$\alpha(\rho)\operatorname{grad}\rho = \alpha(\rho_0)\operatorname{grad}\rho + [\alpha(\rho) - \alpha(\rho_0)]\operatorname{grad}\rho,$$

[1] Since our interest lies only in deriving the asymptotic form of the field equations in the *limit* as $\delta \to 0$, it is not necessary to work with dimensionless quantities.

and since α is continuous, we have, formally,

$$\alpha(\rho)\operatorname{grad}\rho = \alpha(\rho_0)\operatorname{grad}\rho + o(\delta)$$

as $\delta \to 0$. Similarly,

$$\operatorname{div}(\rho\mathbf{v}) = \rho_0 \operatorname{div}\mathbf{v} + o(\delta),$$
$$(\operatorname{grad}\mathbf{v})\mathbf{v} = o(\delta).$$

Thus if we assume $\mathbf{b} = \mathbf{0}$ and neglect terms of $o(\delta)$ in (8), we arrive at the *linear* system

$$\mathbf{v}' + \alpha(\rho_0)\operatorname{grad}\rho = \mathbf{0},$$
$$\rho' + \rho_0 \operatorname{div}\mathbf{v} = 0.$$

These equations are usually called the *acoustic equations*; they constitute an approximate system of field equations appropriate to small departures from a state of rest. If we take the divergence of the first equation and the spatial time derivative of the second, and then eliminate the velocity, we obtain the *classical wave equation*

$$\rho'' = \kappa^2(\rho_0)\,\Delta\rho,$$

where $\Delta = \operatorname{div}\operatorname{grad}$ is the spatial laplacian. Thus within the framework of the linearized acoustic equations disturbances about a rest state with density ρ_0 propagate with speed $\kappa(\rho_0)$. This should motivate our use of the term "sound speed" for $\kappa(\rho)$.

We now return to the general nonlinear system (8). The ratio

$$m = \frac{v}{\kappa(\rho)}$$

of fluid speed

$$v = |\mathbf{v}|$$

to sound speed is called the **Mach number**, and a flow is **subsonic**, **sonic**, or **supersonic** at $(\mathbf{x}, t)$ according as $m(\mathbf{x}, t)$ is < 1, $= 1$, or > 1.

Consider a steady flow with $\mathbf{b} = \mathbf{0}$. Since $\rho' = 0$,

$$\dot{\rho} = \mathbf{v}\cdot\operatorname{grad}\rho,$$

and $(8)_1$ implies that

$$\mathbf{v}\cdot\dot{\mathbf{v}} = -\frac{\kappa^2(\rho)}{\rho}\mathbf{v}\cdot\operatorname{grad}\rho = -\frac{\kappa^2(\rho)}{\rho}\dot{\rho}.$$

Thus

$$\mathbf{v}\cdot(\rho\mathbf{v})^{\cdot} = \mathbf{v}\cdot(\rho\dot{\mathbf{v}} + \dot{\rho}\mathbf{v}) = \rho(\mathbf{v}\cdot\dot{\mathbf{v}})(1 - m^2).$$

Further, if we differentiate $v^2 = \mathbf{v} \cdot \mathbf{v}$, we find that

$$v\dot{v} = \mathbf{v} \cdot \dot{\mathbf{v}}.$$

Similarly,

$$v(\rho v)^{\cdot} = \mathbf{v} \cdot (\rho \mathbf{v})^{\cdot}.$$

The last three identities yield the following

Proposition. *In a steady flow of an elastic fluid under vanishing body forces*

$$(\rho v)^{\cdot} = \rho(1 - m^2)\dot{v} \tag{9}$$

whenever $\mathbf{v} \neq \mathbf{0}$.

Equation (9) demonstrates one of the chief qualitative differences between subsonic and supersonic flow. Consider an arbitrary streamline s (Fig. 8) and a point $\mathbf{x}$ on s. Then the quantity $\rho(\mathbf{x})v(\mathbf{x})$ represents the mass flow, per unit area, across the plane through $\mathbf{x}$ perpendicular to s. For subsonic flow an increase in v along s is associated with an *increase* in mass flow; for supersonic flow an increase in v corresponds to a *decrease* in mass flow. Thus if v increases along s from subsonic to supersonic values, then the mass flow increases in the subsonic range to a maximum at $m = 1$, and subsequently decreases along the remainder of s, which is supersonic. This explains, at least qualitatively, the difference between subsonic and supersonic nozzles. Consider the nozzles shown in Fig. 9. Assume that the density and velocity are constant over each cross section, so that the total mass flow across each such section is $\rho v A$, with A the corresponding area. (Although this assumption can be satisfied exactly only when A is constant, it is reasonable when A varies slowly with length.) By conservation of mass (cf. the anlysis on page 110), $\rho v A$ is independent of position along the nozzle, and hence ρv is inversely proportional to A. In the subsonic nozzle the flow is always in the subsonic range, and the decreasing area from entrance to exit results in an increasing mass flow and a concomitant increase in velocity. A nozzle which increases the velocity from subsonic to supersonic values cannot have this design; indeed, in the supersonic region a decreasing area would result in a decreasing velocity. A nozzle with the desired properties is the de Laval nozzle. There the velocity is subsonic at the entrance and increases with decreasing area to a value of

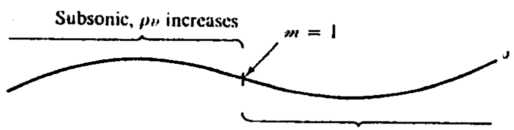

Figure 8. Streamline with v increasing.

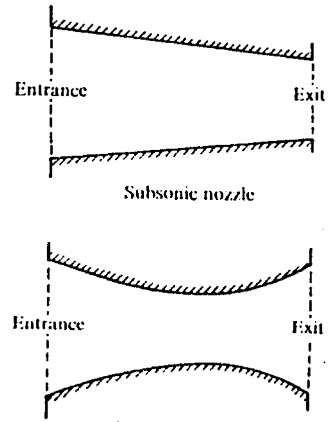

Figure 9

$m = 1$ at the throat (point of minimum area); the subsequent increasing area then accelerates the fluid into the supersonic range.

By Bernoulli's theorem, when the flow is steady and irrotational, and when the body force is conservative,

$$\begin{gathered} \operatorname{div}(\rho \mathbf{v}) = 0, \qquad \operatorname{curl} \mathbf{v} = \mathbf{0}, \\ \frac{\mathbf{v}^2}{2} + \varepsilon(\rho) + \beta = C = \text{const.} \end{gathered} \tag{10}$$

Of course, as for an ideal fluid, (10) should be supplemented by the boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial \mathscr{B}_0. \tag{11}$$

Equations (10) also *suffice* to characterize this type of flow. Indeed, $(10)_1$ implies (2), and if π is defined by (3), then $(10)_3$ and $(9.10)_1$ imply

$$\dot{\mathbf{v}} = \operatorname{grad}\left(\frac{\mathbf{v}^2}{2}\right) = -\operatorname{grad}(\varepsilon(\rho) + \beta),$$

which, by (5), implies (1).

For a steady potential flow the basic equations reduce to a single nonlinear second-order equation for the velocity potential φ. To derive this equation note first that, by $(10)_1$,

$$\mathbf{v} \cdot \operatorname{grad} \rho + \rho \operatorname{div} \mathbf{v} = 0, \tag{12}$$

and if we take the inner product of $(8)_1$ with $\mathbf{v}$ and use (12) to eliminate the term involving grad ρ, we arrive at

$$\mathbf{v} \cdot (\operatorname{grad} \mathbf{v})\mathbf{v} = \kappa^2(\rho) \operatorname{div} \mathbf{v}.$$

Here, for convenience, we have assumed that $\mathbf{b} = \mathbf{0}$. Thus, if

$$\mathbf{v} = \operatorname{grad} \varphi, \tag{13}$$

then φ satisfies the partial differential equation

$$\operatorname{grad} \varphi \cdot (\operatorname{grad}^2 \varphi) \operatorname{grad} \varphi = \kappa^2(\rho)\, \Delta\varphi, \tag{14}$$

where $\operatorname{grad}^2 = \operatorname{grad}\operatorname{grad}$, or equivalently, in components,

$$\sum_{i,j} \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} = \kappa^2(\rho)\sum_i \frac{\partial^2\varphi}{\partial x_i^2}.$$

Further, by (11) and (13), the corresponding boundary condition is

$$\frac{\partial\varphi}{\partial \mathbf{n}} = 0 \qquad \text{on} \quad \partial\mathscr{B}_0,$$

where

$$\frac{\partial\varphi}{\partial \mathbf{n}} = \mathbf{n}\cdot\operatorname{grad}\varphi$$

is the normal derivative on $\partial\mathscr{B}_0$.

The sound speed $\kappa(\rho)$ in (14) can also be expressed as a function of $\operatorname{grad}\varphi$ by solving the Bernoulli equation

$$\varepsilon(\rho) = C - \frac{|\operatorname{grad}\varphi|^2}{2}$$

[cf. $(10)_3$ with $\beta = 0$] for ρ as a function of $\operatorname{grad}\varphi$ as follows. Since $\kappa > 0$, $(4)_2$ implies that $d\varepsilon/d\rho > 0$, so that $\varepsilon(\rho)$ is an invertible function of ρ; we may therefore write

$$\rho = \varepsilon^{-1}\left(C - \frac{|\operatorname{grad}\varphi|^2}{2}\right)$$

and use this relation to express $\kappa(\rho)$ as a function

$$\hat{\kappa}(\operatorname{grad}\varphi) = \kappa\left(\varepsilon^{-1}\left(C - \frac{|\operatorname{grad}\varphi|^2}{2}\right)\right) \tag{15}$$

of $\operatorname{grad}\varphi$. Of course C and hence $\hat{\kappa}$ will generally vary from flow to flow.

Exercises

1. An *ideal gas* is an elastic fluid defined by a constitutive equation of the form

$$\pi = \lambda\rho^\gamma, \tag{16}$$

where $\lambda > 0$ and $\gamma > 1$ are strictly positive constants. [Note that an ideal gas is *not* an ideal fluid. Actually, (16) represents the isentropic behavior of an ideal gas with constant specific heats.]

(a) Show that the sound speed $\kappa = \kappa(\rho)$ is given by

$$\kappa^2 = \gamma\pi/\rho.$$

(b) Show that the relation (15) is given by

$$\hat{\kappa}^2(\operatorname{grad}\varphi) = \frac{\gamma - 1}{2}(v^2 - |\operatorname{grad}\varphi|^2)$$

with v a constant.

2. Consider the ideal gas (16) in equilibrium ($\dot{\mathbf{v}} = \mathbf{0}$) under the gravitational body force

$$\mathbf{b} = -\rho g \mathbf{e}_3,$$

where g is the gravitational constant. Assume that

$$\pi(\mathbf{x}) = \pi_0 = \text{constant at } x_3 = 0.$$

Determine the pressure distribution as a function of height x_3.

3. Show that for an elastic fluid the stress power of a part $\mathscr{P}$ at time t is

$$\int_{\mathscr{P}_t} \mathbf{T}\cdot\mathbf{D}\, dV = -\int_{\mathscr{P}_t} \pi\dot{v}\rho\, dV$$

(cf. page 111), where

$$v = \frac{1}{\rho}$$

is the specific volume (i.e., the volume per unit mass).

Selected References

Courant and Friedrichs [1].
Hughes and Marsden [1, §§9–11, 13, 14].
Lamb [1, Chapters 1–10].
Meyer [1, §6].
Serrin [1, §§15–18, 35, 45, 46].

CHAPTER

VII

Change in Observer. Invariance of Material Response

20. CHANGE IN OBSERVER

Two observers viewing a moving body will generally record different motions for the body; in fact, the two recorded motions will differ by the rigid motion which represents the movement of one observer relative to the other. We now make this idea precise.

Let $\mathbf{x}$ and $\mathbf{x}^*$ be motions of $\mathscr{B}$. Then $\mathbf{x}$ and $\mathbf{x}^*$ are *related by a change in observer* if

$$\mathbf{x}^*(\mathbf{p}, t) = \mathbf{q}(t) + \mathbf{Q}(t)[\mathbf{x}(\mathbf{p}, t) - \mathbf{o}] \tag{1}$$

for every material point $\mathbf{p}$ and time t, where $\mathbf{q}(t)$ is a point of space and $\mathbf{Q}(t)$ is a rotation. That is, writing

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{q}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{o}), \tag{2}$$

then at each time t the deformation $\mathbf{x}^*(\cdot, t)$ is simply the deformation $\mathbf{x}(\cdot, t)$ followed by the *rigid* deformation $\mathbf{f}(\cdot, t)$:

$$\mathbf{x}^*(\cdot, t) = \mathbf{f}(\cdot, t) \circ \mathbf{x}(\cdot, t). \tag{3}$$

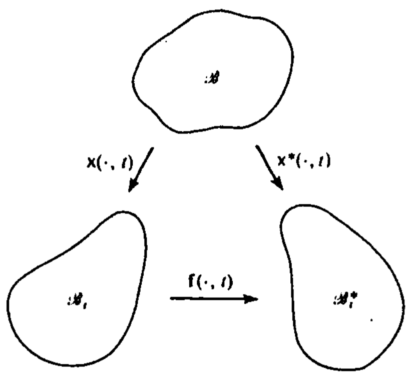

Figure 1

For convenience, we write $\mathscr{B}_t^*$ for the region of space occupied by $\mathscr{B}$ at time t in the motion $\mathbf{x}^*$:

$$\mathscr{B}_t^* = \mathbf{x}^*(\mathscr{B}, t);$$

then the relation between $\mathbf{x}$ and $\mathbf{x}^*$ is illustrated by Fig. 1.

We now determine the manner in which the various kinematical quantities transform under a change in observer. Letting

$$\mathbf{F} = \nabla\mathbf{x}, \qquad \mathbf{F}^* = \nabla\mathbf{x}^*$$

and differentiating (1) with respect to $\mathbf{p}$, we arrive at

$$\mathbf{F}^*(\mathbf{p}, t) = \mathbf{Q}(t)\mathbf{F}(\mathbf{p}, t), \tag{4}$$

which is the transformation law for the deformation gradient. Note that, since $\det \mathbf{Q} = 1$, (4) implies

$$\det \mathbf{F}^* = \det \mathbf{F}. \tag{5}$$

Next, let

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \qquad \mathbf{F}^* = \mathbf{R}^*\mathbf{U}^* = \mathbf{V}^*\mathbf{R}^*$$

be the polar decompositions of $\mathbf{F}$ and $\mathbf{F}^*$. Then (4) implies

$$\mathbf{F}^* = \mathbf{R}^*\mathbf{U}^* = \mathbf{QRU},$$

and, since $\mathbf{QR}$ is a rotation, we conclude from the uniqueness of the polar decomposition that

$$\mathbf{R}^* = \mathbf{QR}, \qquad \mathbf{U}^* = \mathbf{U}.$$

Also,

$$\mathbf{V}^* = \mathbf{R}^*\mathbf{U}^*\mathbf{R}^{*\mathrm{T}} = \mathbf{QRUR}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}$$

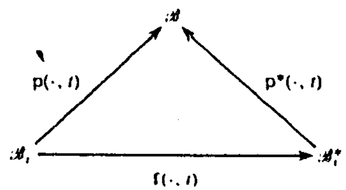

Figure 2

and therefore, since $\mathbf{V} = \mathbf{RUR}^{\mathrm{T}}$,

$$\mathbf{V}^* = \mathbf{QVQ}^{\mathrm{T}}.$$

From these relations it follows that the Cauchy-Green strain tensors

$$\mathbf{C} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{V}^2, \qquad \mathbf{C}^* = \mathbf{U}^{*2}, \qquad \mathbf{B}^* = \mathbf{V}^{*2}$$

transform according to

$$\mathbf{C}^* = \mathbf{C}, \qquad \mathbf{B}^* = \mathbf{QBQ}^{\mathrm{T}}.$$

Intuitively it is clear that if $\mathbf{x} \in \mathscr{B}_t$ and $\mathbf{x}^* \in \mathscr{B}_t^*$ are related through

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}, t), \tag{6}$$

then $\mathbf{x}$ and $\mathbf{x}^*$ *must correspond to the same material point*. To verify that this is indeed the case, note that, by (3),

$$\mathbf{x}(\cdot, t) = \mathbf{f}(\cdot, t)^{-1} \circ \mathbf{x}^*(\cdot, t),$$

which is easily inverted to give

$$\mathbf{p}(\cdot, t) = \mathbf{p}^*(\cdot, t) \circ \mathbf{f}(\cdot, t),$$

where $\mathbf{p}$ and $\mathbf{p}^*$ are the reference maps corresponding to the motions $\mathbf{x}$ and $\mathbf{x}^*$, respectively (Fig. 2). Thus

$$\mathbf{p}(\mathbf{x}, t) = \mathbf{p}^*(\mathbf{f}(\mathbf{x}, t), t),$$

or equivalently, by (6),

$$\mathbf{p}(\mathbf{x}, t) = \mathbf{p}^*(\mathbf{x}^*, t), \tag{7}$$

so that $\mathbf{x}$ and $\mathbf{x}^*$ correspond to the same material point.

As a consequence of (1),

$$\dot{\mathbf{x}}^*(\mathbf{p}, t) = \dot{\mathbf{q}}(t) + \mathbf{Q}(t)\dot{\mathbf{x}}(\mathbf{p}, t) + \dot{\mathbf{Q}}(t)[\mathbf{x}(\mathbf{p}, t) - \mathbf{o}].$$

If we take $\mathbf{p} = \mathbf{p}^*(\mathbf{x}^*, t)$ in the left side and $\mathbf{p} = \mathbf{p}(\mathbf{x}, t)$ in the right side of this relation—a substitution justified by (7) provided $\mathbf{x}^*$ and $\mathbf{x}$ are related by (6)—we conclude that

$$\mathbf{v}^*(\mathbf{x}^*, t) = \dot{\mathbf{q}}(t) + \mathbf{Q}(t)\mathbf{v}(\mathbf{x}, t) + \dot{\mathbf{Q}}(t)(\mathbf{x} - \mathbf{o}), \tag{8}$$

where

$$v^* = (\dot{x}^*)_s, \qquad v = (\dot{x})_s$$

are the spatial descriptions of the velocity in the two motions. Equation (8) is the transformation law for the velocity.

Now let

$$L = \text{grad } v, \qquad L^* = \text{grad } v^*.$$

If we take $x^* = f(x, t)$ in (8) and differentiate with respect to x using the chain rule and the fact that grad $f = Q$, we arrive at

$$L^*(x^*, t)Q(t) = Q(t)L(x, t) + \dot{Q}(t),$$

so that the velocity gradient transforms according to

$$L^*(x^*, t) = Q(t)L(x, t)Q(t)^T + \dot{Q}(t)Q(t)^T. \tag{9}$$

Since $QQ^T = I$, it follows that

$$\dot{Q}Q^T + Q\dot{Q}^T = 0,$$

or equivalently that

$$\dot{Q}Q^T = -(\dot{Q}Q^T)^T,$$

and the tensor $\dot{Q}Q^T$ is *skew*. Thus the symmetric part of (9) is the following transformation law for the stretching:

$$D^*(x^*, t) = Q(t)D(x, t)Q(t)^T, \tag{10}$$

where

$$D^* = \tfrac{1}{2}(L^* + L^{*T}), \qquad D = \tfrac{1}{2}(L + L^T).$$

Exercises

1. Show that the spin W transforms according to

$$W^*(x^*, t) = Q(t)W(x, t)Q(t)^T + \dot{Q}(t)Q(t)^T.$$

2. A spatial tensor field A is *indifferent* if, during any change in observer, A transforms according to

$$A^*(x^*, t) = Q(t)A(x, t)Q(t)^T.$$

Assume that A is smooth and indifferent.

(a) Show that $\dot{A}$ is indifferent if and only if $A(x, t) = \beta(x, t)I$ for all x and t.

(b) Show that the tensor fields

$$\overset{\circ}{A} = \dot{A} - WA + AW,$$
$$\hat{A} = \dot{A} + L^TA + AL$$

are indifferent. In the case of the stress T [cf. (21.1)], $\overset{\circ}{T}$ and $\hat{T}$ are called, respectively, the *corotational* and *convected stress rates*.

(c) Show that

$$\overset{\circ}{W} = \dot{W},$$
$$\hat{W} = \dot{W} + DW + WD. \tag{11}$$

3. Use (9.14) to show that

$$C_t^*(y^*, t) = Q(\tau)C_t(y, t)Q(\tau)^T,$$

and use this result to show that the Rivlin–Ericksen tensors (9.18) are indifferent.

21. INVARIANCE UNDER A CHANGE IN OBSERVER

One of the main axioms of mechanics is the requirement that *material response be independent of the observer*. This axiom is never stated explicitly in elementary books on mechanics, probably because it seems so obvious; nonetheless it is instructive to look at a simple example from that subject. A thin elastic string weighted at the end is spun with constant angular velocity and suffers an elongation of amount δ. The axiom above asserts that the force in the string which produces this extension is the same as the force required to produce the elongation δ when the string is held in one place. Indeed, the two motions of the string differ only by a change in observer; that is, an observer spinning with the string sees the string undergoing the same motion as a "still observer" sees when he extends the string by hand.

We now give a precise statement of this axiom within our general framework. To do this we must first deduce the manner in which we would expect the stress to transform under a change in observer. Thus let (x, T) and (x*, T*) be dynamical processes with x and x* related through (20.1). If T and T* are also related through this change in observer, then the corresponding surface force fields should transform as shown in Fig. 3. Thus if

$$n^* = Qn, \qquad Q = Q(t)$$

we would expect that

$$s^*(n^*) = Qs(n).$$

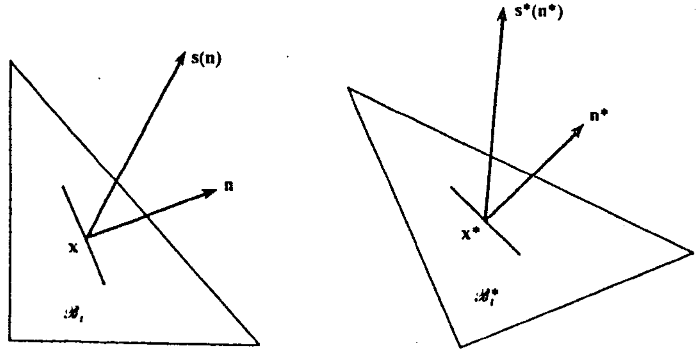

Figure 3

But

$$\mathbf{s}(\mathbf{n}) = \mathbf{T}\mathbf{n}, \qquad \mathbf{s}^*(\mathbf{n}^*) = \mathbf{T}^*\mathbf{n}^*,$$

and hence

$$\mathbf{T}^*\mathbf{n}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\mathbf{n}^*.$$

Since this relation should hold for every unit vector $\mathbf{n}^*$, we would expect the following transformation law for the stress tensor:

$$\mathbf{T}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\mathbf{T}(\mathbf{x}, t)\mathbf{Q}(t)^{\mathrm{T}}. \tag{1}$$

Of course, in (1) it is understood that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}, t)$ with $\mathbf{f}$ given by (20.2).

This discussion should motivate the following definition: $(\mathbf{x}, \mathbf{T})$ and $(\mathbf{x}^*, \mathbf{T}^*)$ **are related by a change in observer** if there exist C^3 functions

$$\mathbf{q}: \mathbb{R} \to \mathscr{E}, \qquad \mathbf{Q}: \mathbb{R} \to \mathrm{Orth}^+$$

such that

(a) the transformation law (20.1) holds for all $\mathbf{p} \in \mathscr{B}$ and $t \in \mathbb{R}$;
(b) the transformation law (1) holds for all $(\mathbf{x}, t)$ in the trajectory of $\mathbf{x}$.

We say that **the response of a material body is independent of the observer** provided its constitutive class $\mathscr{C}$ has the following property: if a process $(\mathbf{x}, \mathbf{T})$ belongs to $\mathscr{C}$, then so does every dynamical process related to $(\mathbf{x}, \mathbf{T})$ by a change in observer. The content of this definition is, of course, the axiom discussed previously.

Theorem. *The response of ideal fluids and elastic fluids are independent of the observer.*

The proof of this result makes use of the following lemma, which is a direct consequence of (16.2) and (20.5).

Lemma. *A dynamical process related to an isochoric dynamical process by a change in observer is itself isochoric.*

Proof of the Theorem. Let $\mathscr{C}$ be the constitutive class of an ideal fluid, let $(\mathbf{x}, \mathbf{T}) \in \mathscr{C}$, and let $(\mathbf{x}^*, \mathbf{T}^*)$ be related to $(\mathbf{x}, \mathbf{T})$ by a change in observer. To show that $(\mathbf{x}^*, \mathbf{T}^*) \in \mathscr{C}$ we must show that $(\mathbf{x}^*, \mathbf{T}^*)$ is isochoric and Eulerian. The former assertion follows from the lemma; to verify the latter note that, since $\mathbf{T} = -\pi\mathbf{I}$, (1) yields

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} = -\pi\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = -\pi\mathbf{I}.$$

Therefore ideal fluids are independent of the observer. The proof that elastic fluids also have this property proceeds in the same manner and is left as an exercise. □

Exercises

1. Prove that the response of an elastic fluid is independent of the observer.

In the next two exercises we consider the change in observer defined by (20.1).

2. Show that

$$\operatorname{div}_{\mathbf{x}^*} \mathbf{T}^*(\mathbf{x}^*, t) = \mathbf{Q}(t) \operatorname{div}_{\mathbf{x}} \mathbf{T}(\mathbf{x}, t).$$

3. Show that the acceleration transforms according to

$$\dot{\mathbf{v}}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\dot{\mathbf{v}}(\mathbf{x}, t) + \ddot{\mathbf{q}}(t) + 2\dot{\mathbf{Q}}(t)\mathbf{v}(\mathbf{x}, t) + \ddot{\mathbf{Q}}(t)(\mathbf{x} - \mathbf{o}).$$

Show further that if the body force transforms according to

$$\mathbf{b}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\mathbf{b}(\mathbf{x}, t),$$

then

$$\operatorname{div} \mathbf{T}^* + \mathbf{b}^* + \mathbf{k} = \rho^*\dot{\mathbf{v}}^*,$$

where

$$\mathbf{k}(\mathbf{x}^*, t) = \rho(\mathbf{x}, t)[\ddot{\mathbf{q}}(t) + 2\dot{\mathbf{Q}}(t)\mathbf{v}(\mathbf{x}, t) + \ddot{\mathbf{Q}}(t)(\mathbf{x} - \mathbf{o})].$$

Because of the additional term $\mathbf{k}$, the equation of motion is *not invariant* under all changes in observer. The observers for whom this equation transforms "properly" are exactly those with $\mathbf{Q}$ and $\dot{\mathbf{q}}$ constant. Such observers are called *Galilean* and have the property of being accelerationless with respect to the underlying inertial observer.

Selected References

Noll [1, 2].
Truesdell and Noll [1, §§17–19].

CHAPTER

VIII

Newtonian Fluids. The Navier–Stokes Equations

22. NEWTONIAN FLUIDS

Friction in fluids generally manifests itself through shearing forces which retard the relative motion of fluid particles. The fluids discussed thus far never exhibit shearing stress, and for this reason are incapable of describing frictional forces of this type. A measure of the relative motion of fluid particles is furnished by the velocity gradient

$$\mathbf{L} = \operatorname{grad} \mathbf{v},$$

a fact which motivates our considering constitutive equations of the form

$$\mathbf{T} = -\pi \mathbf{I} + \mathsf{C}[\mathbf{L}]. \tag{1}$$

Materials defined by relations of this type with C linear are called Newtonian fluids; they furnish the simplest—and most useful—model of viscous fluid behavior. Here we will use the term Newtonian fluid to mean *incompressible* Newtonian fluid; thus, since $\operatorname{tr} \mathbf{L} = \operatorname{div} \mathbf{v}$, we limit our discussion to fields $\mathbf{L}$ with

$$\operatorname{tr} \mathbf{L} = 0.$$

Since C is linear, $\mathrm{C}[\mathbf{0}] = \mathbf{0}$; thus

$$\mathbf{T} = -\pi\mathbf{I}$$

when $\mathbf{L} = \mathbf{0}$, and a Newtonian fluid at rest behaves like an ideal fluid. Also, as for an ideal fluid, the pressure π is arbitrary (i.e., not determined by the motion). This flexibility in π leads to a certain ambiguity regarding the response function C; given any linear, scalar-valued function $\beta(\mathbf{L})$ of $\mathbf{L}$, we can rewrite (1) in the form

$$\mathbf{T} = -\{\pi + \beta(\mathbf{L})\}\mathbf{I} + \mathrm{C}_\beta[\mathbf{L}],$$

where

$$\mathrm{C}_\beta[\mathbf{L}] = \mathrm{C}[\mathbf{L}] + \beta(\mathbf{L})\mathbf{I}.$$

Since π is arbitrary, π can absorb the term $\beta(\mathbf{L})$. Thus our constitutive equation (1) is essentially unaltered by replacing the term $\mathrm{C}[\mathbf{L}]$ by $\mathrm{C}_\beta[\mathbf{L}]$. To remove this ambiguity, we normalize C by demanding that

$$\operatorname{tr} \mathrm{C}[\mathbf{L}] = 0. \tag{2}$$

(This is equivalent to replacing C in (1) by C_β with $\beta(\mathbf{L}) = -\frac{1}{3}\operatorname{tr}\mathrm{C}[\mathbf{L}]$.) Then C has domain

$$\mathrm{Lin}_0 = \{\mathbf{L} \in \mathrm{Lin} \mid \operatorname{tr}\mathbf{L} = 0\}$$

and codomain

$$\mathrm{Sym}_0 = \{\mathbf{T} \in \mathrm{Sym} \mid \operatorname{tr}\mathbf{T} = 0\}.$$

In view of the restriction (2), the pressure π is uniquely determined by the stress. In fact, if we take the trace of (1) and use (2), we conclude that

$$\pi = -\tfrac{1}{3}\operatorname{tr}\mathbf{T}.$$

We define the **extra stress** $\mathbf{T}_0$ by

$$\mathbf{T}_0 = \mathbf{T} + \pi\mathbf{I} = \mathbf{T} - \tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I},$$

so that $\mathbf{T}_0$ is the traceless part of the stress tensor. Our constitutive equation (1) then takes the simple form

$$\mathbf{T}_0 = \mathrm{C}[\mathbf{L}]. \tag{3}$$

The above discussion should motivate our next definition.

A **Newtonian fluid** is an incompressible material body consistent with the following constitutive assumptions:

(a) There exists a linear *response function*

$$\mathrm{C}: \mathrm{Lin}_0 \to \mathrm{Sym}_0$$

such that the constitutive class is the set of all isochoric dynamical processes $(\mathrm{x}, \mathbf{T})$ which obey the constitutive equation (3).

(b) The density ρ_0 is constant.

The dimensions of the spaces Lin_0 and Sym_0 are 8 and 5, respectively, so that the matrix of C relative to a coordinate frame has 40 entries. Thus at first sight it would appear that it takes 40 independent constants to specify a Newtonian fluid. The next theorem shows that, in actuality, the response is determined by a *single constant*.

Theorem. *A necessary and sufficient condition that the response of a Newtonian fluid be independent of the observer is that its response function* C *have the form*

$$\mathrm{C}[\mathbf{L}] = 2\mu\mathbf{D} \tag{4}$$

for every $\mathbf{L} \in \mathrm{Lin}_0$, *where*

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}).$$

The scalar constant μ *is called the* **viscosity** *of the fluid.*

Proof. (Sufficiency) Assume that (4) holds. Let $(\mathrm{x}, \mathbf{T})$ belong to the constitutive class $\mathscr{C}$ of the fluid. Then $(\mathrm{x}, \mathbf{T})$ is isochoric and

$$\mathbf{T}_0 = 2\mu\mathbf{D}.$$

Let $(\mathrm{x}^*, \mathbf{T}^*)$ be related to $(\mathrm{x}, \mathbf{T})$ by a change in observer. Then $(\mathrm{x}^*, \mathbf{T}^*)$ is isochoric (cf. the lemma on page 145). Moreover, by (20.10) and (21.1),

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}},$$

and $(1.4)_2$ yields

$$\operatorname{tr}\mathbf{T}^* = \operatorname{tr}(\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}) = \operatorname{tr}(\mathbf{T}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}) = \operatorname{tr}\mathbf{T}.$$

Therefore

$$\begin{aligned}\mathbf{T}_0^* &= \mathbf{T}^* - \tfrac{1}{3}(\operatorname{tr}\mathbf{T}^*)\mathbf{I} = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} - \tfrac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{T}_0\mathbf{Q}^{\mathrm{T}} \\ &= \mathbf{Q}(2\mu\mathbf{D})\mathbf{Q}^{\mathrm{T}} = 2\mu\mathbf{D}^*.\end{aligned} \tag{5}$$

Thus $(\mathrm{x}^*, \mathbf{T}^*) \in \mathscr{C}$ and the response is independent of the observer.

The proof of necessity is facilitated by the following

Lemma. *Let* $\mathbf{L}_0 \in \mathrm{Lin}_0$. *Then there exists a motion* x *with velocity gradient*

$$\mathbf{L} = \mathbf{L}_0 \tag{6}$$

and with $\mathrm{x}(\cdot, t)$ *isochoric at each time* t.

Proof. Take

$$\mathbf{F}(t) = e^{\mathbf{L}_0 t}$$

so that $\mathbf{F}$ is the unique solution of

$$\dot{\mathbf{F}} = \mathbf{L}_0\mathbf{F}, \qquad \mathbf{F}(0) = \mathbf{I}. \tag{7}$$

By (36.2),

$$\det \mathbf{F} = e^{(\operatorname{tr}\mathbf{L}_0)t} = 1.$$

Thus

$$x(\mathbf{p}, t) = \mathbf{q} + \mathbf{F}(t)[\mathbf{p} - \mathbf{q}]$$

defines a motion with deformation gradient $\mathbf{F}$ and with $x(\cdot, t)$ isochoric at each t. Further, (6) follows from $(7)_1$ and the identity $(8.8)_1$.

We now return to the proof of the theorem. To establish the *necessity* of (4) we assume that

the response is independent of the observer. (8)

Let $\mathbf{L}_0 \in \text{Lin}_0$ be arbitrary, let x be the motion constructed in the lemma, and let $\mathbf{T} = \mathbf{T}_0$ be the constant field defined by (3). Then, clearly, $(x, \mathbf{T}) \in \mathscr{C}$. Let $(x^*, \mathbf{T}^*)$ be related to $(x, \mathbf{T})$ by a change in observer. Then by (8), $(x^*, \mathbf{T}^*) \in \mathscr{C}$ and

$$\mathbf{T}_0^* = \mathsf{C}[\mathbf{L}^*]. \tag{9}$$

But

$$\mathbf{T}_0^* = \mathbf{Q}\mathbf{T}_0\mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{L}^* = \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$$

[cf. (20.9), (21.1), and (5)]; hence (9) yields

$$\mathbf{Q}\mathbf{T}_0\mathbf{Q}^{\mathrm{T}} = \mathsf{C}[\mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}],$$

and we conclude from (3) and (6) that

$$\mathbf{Q}\mathsf{C}[\mathbf{L}_0]\mathbf{Q}^{\mathrm{T}} = \mathsf{C}[\mathbf{Q}\mathbf{L}_0\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}]. \tag{10}$$

Clearly, this relation holds for every $\mathbf{L}_0 \in \text{Lin}_0$ (the domain of C) and every C^3 function $\mathbf{Q}: \mathbb{R} \to \text{Orth}^+$. Fix $\mathbf{L}_0$ and take

$$\mathbf{Q}(t) = e^{-\mathbf{W}_0 t},$$

where

$$\mathbf{W}_0 = \tfrac{1}{2}(\mathbf{L}_0 - \mathbf{L}_0^{\mathrm{T}}).$$

Then $\mathbf{Q}(t)$ is a rotation, since $\mathbf{W}_0$ is skew, and

$$\mathbf{Q}(0) = \mathbf{I}, \qquad \dot{\mathbf{Q}}(0) = -\mathbf{W}_0$$

(cf. the discussion in Section 36). Using this function $\mathbf{Q}$ in (10) at $t = 0$ yields

$$\mathsf{C}[\mathbf{L}_0] = \mathsf{C}[\mathbf{L}_0 - \mathbf{W}_0] = \mathsf{C}[\mathbf{D}_0], \tag{11}$$

where

$$\mathbf{D}_0 = \tfrac{1}{2}(\mathbf{L}_0 + \mathbf{L}_0^{\mathrm{T}}).$$

Thus C is completely determined by its restriction to Sym_0.

Next, let $\mathbf{Q}$ be a constant function with value in Orth^+. Then (10) with $\mathbf{L}_0 = \mathbf{D}_0$ ($\mathbf{D}_0 \in \text{Sym}_0$) implies that

$$\mathbf{Q}\mathsf{C}[\mathbf{D}_0]\mathbf{Q}^{\mathrm{T}} = \mathsf{C}[\mathbf{Q}\mathbf{D}_0\mathbf{Q}^{\mathrm{T}}]. \tag{12}$$

Since this relation must hold for every $\mathbf{D}_0 \in \text{Sym}_0$ and every $\mathbf{Q} \in \text{Orth}^+$, the restriction of C to Sym_0 is isotropic; we therefore conclude from the representation (37.26) that

$$\mathsf{C}[\mathbf{D}_0] = 2\mu\mathbf{D}_0$$

for all $\mathbf{D}_0 \in \text{Sym}_0$. □

Note that the results (11) and (12) are valid also for *nonlinear* functions C; we appealed to the linearity of C only in the last step of the argument.

By (4) the constitutive equation (1) takes the form

$$\mathbf{T} = -\pi\mathbf{I} + 2\mu\mathbf{D}. \tag{13}$$

This equation must be supplemented by the equation of motion

$$\rho_0[\mathbf{v}' + (\operatorname{grad}\mathbf{v})\mathbf{v}] = \operatorname{div}\mathbf{T} + \mathbf{b}$$

and the incompressibility condition

$$\operatorname{div}\mathbf{v} = 0.$$

In view of (4.5),

$$2\operatorname{div}\mathbf{D} = \operatorname{div}(\operatorname{grad}\mathbf{v} + \operatorname{grad}\mathbf{v}^{\mathrm{T}}) = \Delta\mathbf{v} + \operatorname{grad}\operatorname{div}\mathbf{v} = \Delta\mathbf{v},$$

where $\Delta = \operatorname{div}\operatorname{grad}$ is the spatial laplacian. Thus the above equations reduce to

$$\begin{aligned} \rho_0[\mathbf{v}' + (\operatorname{grad}\mathbf{v})\mathbf{v}] &= \mu\Delta\mathbf{v} - \operatorname{grad}\pi + \mathbf{b}, \\ \operatorname{div}\mathbf{v} &= 0. \end{aligned} \tag{14}$$

These relations are the **Navier–Stokes equations**; given μ, ρ_0, and $\mathbf{b}$ they constitute a nonlinear system of partial differential equations for the velocity $\mathbf{v}$ and the pressure π. If we define the **kinematical viscosity** ν by

$$\nu = \mu/\rho_0$$

and write

$$\pi_0 = \pi/\rho_0, \qquad \mathbf{b}_0 = \mathbf{b}/\rho_0,$$

we can rewrite the Navier–Stokes equations in the form

$$\begin{aligned} \mathbf{v}' + (\operatorname{grad} \mathbf{v})\mathbf{v} &= \nu\,\Delta\mathbf{v} - \operatorname{grad} \pi_0 + \mathbf{b}_0, \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned} \tag{15}$$

If the flow is steady, and if we neglect the nonlinear term $(\operatorname{grad} \mathbf{v})\mathbf{v}$, then (15) reduce to

$$\begin{aligned} \nu\,\Delta\mathbf{v} &= \operatorname{grad} \pi_0 - \mathbf{b}_0, \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned}$$

Solutions of this equation are called *Stokes flows* and are presumed to describe slow or *creeping* flows of Newtonian fluids.

We now return to the general Navier–Stokes equations (15). One of the basic differences between flows described by these equations and flows of ideal or elastic fluids is the way circulation and spin are transported.

Theorem (Transport of spin and circulation). *Consider the flow of a Newtonian fluid under a conservative body force. Then*

$$\dot{\mathbf{W}} + \mathbf{D}\mathbf{W} + \mathbf{W}\mathbf{D} = \nu\,\Delta\mathbf{W}, \tag{16}$$

and given any closed material curve $\mathbf{c}$,

$$\frac{d}{dt}\int_{\mathbf{c}_t} \mathbf{v}(\mathbf{x}, t)\cdot d\mathbf{x} = \nu \int_{\mathbf{c}_t} \Delta\mathbf{v}(\mathbf{x}, t)\cdot d\mathbf{x}. \tag{17}$$

If in addition the flow is plane,

$$\dot{\mathbf{W}} = \nu\,\Delta\mathbf{W}. \tag{18}$$

Proof. Since the left side of $(15)_1$ represents $\dot{\mathbf{v}}$, and since $\mathbf{b}_0 = -\operatorname{grad}\beta$,

$$\dot{\mathbf{v}} = \nu\,\Delta\mathbf{v} - \operatorname{grad}(\pi_0 + \beta).$$

If we substitute this equation into (11.5), we recover (17), since $\mathbf{c}$ is closed. Further,

$$\operatorname{grad}\dot{\mathbf{v}} = \nu\,\Delta \operatorname{grad}\mathbf{v} - \operatorname{grad}\operatorname{grad}(\pi_0 + \beta),$$

because $\Delta \operatorname{grad} = \operatorname{grad} \Delta$. (Here and in what follows we need the additional assumption that, as functions of position, $\mathbf{v}$ is of class C^3, while π_0 and β are of class C^2.) If we take the skew part of this equation and use the fact that $\operatorname{grad}\operatorname{grad}(\pi_0 + \beta)$ is symmetric, we arrive at

$$\tfrac{1}{2}(\operatorname{grad}\dot{\mathbf{v}} - \operatorname{grad}\dot{\mathbf{v}}^{\mathrm{T}}) = \nu\,\Delta\mathbf{W}.$$

This equation and $(11.2)_2$ imply (16). Finally, since $\operatorname{div}\mathbf{v} = 0$, (16) and (9.12) yield (18). □

Both ideal and elastic fluids preserve circulation. In contrast, (17) tells us that *circulation in a Newtonian fluid generally varies with time.*

Equation (18) asserts that in plane flow the spin satisfies a diffusion equation. Further, if in (16) we neglect terms of second order in $\operatorname{grad}\mathbf{v}$, we again arrive at (18), but here for flows which are not necessarily plane. This motivates the standard assertion that *spin diffuses in a viscous fluid.* Finally, note that by $(20.11)_2$ we can rewrite (16) as

$$\overset{\circ}{\mathbf{W}} = \nu\,\Delta\mathbf{W}.$$

Suppose that the flow takes place in a region $\mathscr{R}$. For the *inviscid* fluids considered previously the boundary condition was that $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathscr{R}$, which is simply the requirement that the boundary be impenetrable to the fluid. For a *Newtonian* fluid we add the restriction that the fluid adhere, without slipping, to the boundary. For a stationary boundary this means that $\mathbf{v} = \mathbf{0}$ on $\partial\mathscr{R}$. If the boundary moves, then at each point on the boundary the fluid velocity must coincide with the velocity of the boundary.

The theorem of power expended has an interesting consequence for Newtonian fluids. Indeed, by (13),

$$\mathbf{T}\cdot\mathbf{D} = -\pi\mathbf{I}\cdot\mathbf{D} + 2\mu|\mathbf{D}|^2 = 2\mu|\mathbf{D}|^2,$$

since $\mathbf{I}\cdot\mathbf{D} = \operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v} = 0$, and (15.2) yields the following

Theorem (Balance of energy for a viscous fluid). *Consider the flow of a Newtonian fluid. Then*

$$\int_{\partial\mathscr{P}_t} \mathbf{s}(\mathbf{n})\cdot\mathbf{v}\,dA + \int_{\mathscr{P}_t} \mathbf{b}\cdot\mathbf{v}\,dV = \frac{d}{dt}\int_{\mathscr{P}_t} \frac{\mathbf{v}^2}{2}\rho_0\,dV + 2\mu\int_{\mathscr{P}_t} |\mathbf{D}|^2\,dV \tag{19}$$

for every part $\mathscr{P}$.

The term

$$2\mu\int_{\mathscr{P}_t} |\mathbf{D}|^2\,dV$$

represents the rate at which the fluid in $\mathscr{P}$ dissipates energy. The energy equation (19) asserts that *the total power expended on* $\mathscr{P}$ *must equal the rate of change of kinetic energy plus the rate of energy dissipation.* Note that for a finite body, if $\mathbf{v}$ vanishes on $\partial\mathscr{B}_t$ for all time, and if $\mathbf{b} \equiv \mathbf{0}$, then (assuming $\mu > 0$)

$$\frac{d}{dt}\int_{\mathscr{B}_t} \frac{\mathbf{v}^2}{2}\rho_0\,dV \le 0,$$

so that the kinetic energy decreases with time. This result represents a type of stability inherent in viscous fluids; in Section 24 we will establish a stronger form of stability: we will show that under these conditions the kinetic energy actually decreases exponentially with time.

It is instructive to write the Navier–Stokes equations in dimensionless form. Consider a solution of (15) with

$$\mathbf{b} = \mathbf{0}.$$

Let l and υ be numbers with l a typical length (such as the diameter of a body) and υ a typical velocity (such as a mainstream velocity). Further, identify points $\mathbf{x}$ with their position vectors $\mathbf{x} - \mathbf{o}$ from a given origin $\mathbf{o}$, and define the dimensionless position vector

$$\bar{\mathbf{x}} = \frac{\mathbf{x}}{l},$$

the dimensionless time

$$\bar{t} = \frac{t\upsilon}{l},$$

the dimensionless velocity

$$\bar{\mathbf{v}}(\bar{\mathbf{x}}, \bar{t}) = \frac{1}{\upsilon}\,\mathbf{v}(\mathbf{x}, t),$$

and the dimensionless pressure

$$\bar{\pi}_0(\bar{\mathbf{x}}, \bar{t}) = \frac{1}{\upsilon^2}\,\pi_0(\mathbf{x}, t).$$

Then

$$\operatorname{grad}\bar{\mathbf{v}} = \frac{l}{\upsilon}\operatorname{grad}\mathbf{v}, \qquad \bar{\mathbf{v}}' = \frac{l}{\upsilon^2}\mathbf{v}', \qquad \operatorname{grad}\bar{\pi}_0 = \frac{l}{\upsilon^2}\operatorname{grad}\pi_0$$

[where $\operatorname{grad}\bar{\mathbf{v}}(\bar{\mathbf{x}}, \bar{t}) = \nabla_{\bar{\mathbf{x}}}\bar{\mathbf{v}}(\bar{\mathbf{x}}, \bar{t})$, etc.], so that (15) reduces to

$$\begin{aligned} \bar{\mathbf{v}}' + (\operatorname{grad}\bar{\mathbf{v}})\bar{\mathbf{v}} &= \frac{1}{\mathrm{Re}}\Delta\bar{\mathbf{v}} - \operatorname{grad}\bar{\pi}_0, \\ \operatorname{div}\bar{\mathbf{v}} &= 0, \end{aligned} \tag{20}$$

where

$$\mathrm{Re} = \frac{l\upsilon}{\nu}$$

is a dimensionless quantity called the *Reynolds number* of the flow.

Equations (20) show that a solution of the Navier–Stokes equations with a given Reynolds number can be used to generate solutions which have different length and velocity scales, but the same Reynolds number. This fact allows one to model a given flow situation in the laboratory by adjusting the length and velocity scales and the viscosity to give an experimentally tractable problem with the same Reynolds number.

Some typical values for the kinematical viscosity are[1]:

$$\textit{water:}\quad \nu = 1.004 \times 10^{-2}\ \ \mathrm{cm^2/sec},$$

$$\textit{air:}\quad \nu = 15.05 \times 10^{-2}\ \ \mathrm{cm^2/sec}.$$

Exercises

1. A *Reiner–Rivlin fluid*[2] is defined by the constitutive assumptions of the Newtonian fluid with the assumption of linearity removed. Use (37.15) to show that the response of a Reiner–Rivlin fluid is independent of the observer if and only if the constitutive equation has the form

$$\mathbf{T} = -\pi\mathbf{I} + \alpha_0(\mathscr{I}_{\mathbf{D}})\mathbf{D} + \alpha_1(\mathscr{I}_{\mathbf{D}})\mathbf{D}^2$$

with $\alpha_0(\mathscr{I}_{\mathbf{D}})$ and $\alpha_1(\mathscr{I}_{\mathbf{D}})$ scalar functions of the list $\mathscr{I}_{\mathbf{D}} = (0, \iota_2(\mathbf{D}), \iota_3(\mathbf{D}))$ of principal invariants of $\mathbf{D}$.

2. Consider a Newtonian fluid in a fixed, bounded region $\mathscr{R}$ of space and assume that

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\mathscr{R}$$

for all time.

(a) Show that the rate at which the fluid dissipates energy, i.e.,

$$2\mu\int_{\mathscr{R}} |\mathbf{D}|^2\,dV,$$

can be written in the alternative forms

$$2\mu\int_{\mathscr{R}} |\mathbf{W}|^2\,dV, \qquad \mu\int_{\mathscr{R}} |\operatorname{curl}\mathbf{v}|^2\,dV,$$

indicating that all of the energy dissipation is due to spin.

[1] At 20°C and atmospheric pressure. Cf., e.g., Bird, Stewart, and Lightfoot [1, p. 8].

[2] There are better models of non-Newtonian fluids (cf. Truesdell and Noll [1, Chapter E] and Coleman, Markovitz, and Noll [1]).

(b) The surface force exerted on $\partial\mathscr{R}$ by the fluid is given by the simple expression

$$\mathbf{s} = \pi\mathbf{n} - 2\mu\mathbf{W}\mathbf{n} = \pi\mathbf{n} + \mu\mathbf{n} \times \text{curl}\,\mathbf{v},$$

where $\mathbf{n}$ is the outward unit normal to $\partial\mathscr{R}$. Establish this relation for a plane portion of the boundary.

23. SOME SIMPLE SOLUTIONS FOR PLANE STEADY FLOW

For steady flow in the absence of body forces the Navier–Stokes equations (22.14) reduce to

$$\begin{aligned} \rho_0(\text{grad}\,\mathbf{v})\mathbf{v} &= \mu\,\Delta\mathbf{v} - \text{grad}\,\pi, \\ \text{div}\,\mathbf{v} &= 0. \end{aligned} \tag{1}$$

Consider the plane velocity field

$$\mathbf{v}(\mathbf{x}) = v_1(x_1, x_2)\mathbf{e}_1. \tag{2}$$

Then $(1)_2$ implies that

$$\frac{\partial v_1}{\partial x_1} = 0,$$

so that $v_1 = v_1(x_2)$ and $\mathbf{v}$ is a *simple shear* (see Fig. 1). Further, the matrix of grad $\mathbf{v}$ is

$$[\text{grad}\,\mathbf{v}] = \begin{bmatrix} 0 & \dfrac{\partial v_1}{\partial x_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

and

$$\text{div}\,\mathbf{v} = 0, \qquad (\text{grad}\,\mathbf{v})\mathbf{v} = \mathbf{0}.$$

Therefore (1) reduces to

$$\mu\frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial \pi}{\partial x_1}, \qquad \frac{\partial \pi}{\partial x_2} = \frac{\partial \pi}{\partial x_3} = 0, \tag{4}$$

and $\pi = \pi(x_1)$.

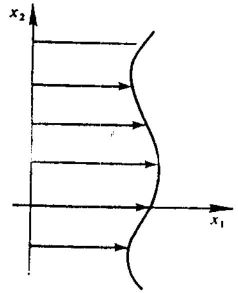

Figure 1

We now consider two specific problems consistent with the above flow.

Problem 1 (Flow between two plates). Consider the flow between two infinite flat plates, one at $x_2 = 0$ and one at $x_2 = h$, with the bottom plate held stationary and the top plate moving in the x_1 direction with velocity v (Fig. 2). Then the boundary conditions are

$$v_1(0) = 0, \qquad v_1(h) = v. \tag{5}$$

We assume in addition that there is no pressure drop in the x_1-direction, so that

$$\pi \equiv \text{const}.$$

Then $(4)_1$ yields

$$v_1 = \alpha + \beta x_2,$$

and this relation satisfies the boundary conditions (5) if and only if $\alpha = 0$ and $\beta = v/h$. Thus the solution of our problem is

$$v_1 = vx_2/h. \tag{6}$$

In view of (22.13), (3), and (6), the stress $\mathbf{T}$ is the constant field

$$[\mathbf{T}] = -\pi\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\mu v}{h}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

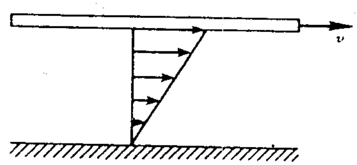

Figure 2

that is,

$$T_{11} = T_{22} = T_{33} = -\pi,$$

$$T_{12} = T_{21} = \frac{\mu v}{h}.$$

Thus the force per unit area exerted by the fluid on the top plate has a normal component π and a tangential (shearing) component $-\mu v/h$. This fact furnishes a method of determining the viscosity of the fluid.

Problem 2 (Flow between two fixed plates under a pressure gradient). Again we consider the above configuration, but now we hold the two plates fixed in space (Fig. 3). Then the relevant boundary conditions are

$$v_1(0) = v_1(h) = 0. \tag{7}$$

If the pressure were constant, then the solution of the first problem would tell us that the velocity is identically zero. We therefore allow a pressure drop; in particular, since $v = v(x_2)$, $(4)_1$ implies that

$$\frac{\partial \pi}{\partial x_1} = -\delta$$

with δ (the pressure drop per unit length) constant. For this case $(4)_1$ has the solution

$$v_1 = -\frac{\delta x_2^2}{2\mu} + \alpha + \beta x_2,$$

which satisfies (7) if and only if $\alpha = 0$ and $\beta = \delta h/2\mu$. Thus

$$v_1 = \frac{\delta x_2}{2\mu}(h - x_2)$$

and the velocity distribution is parabolic. Further, we conclude from (22.13) that the nonzero components of the stress are

$$T_{11} = T_{22} = T_{33} = -\pi,$$

$$T_{12} = T_{21} = \delta\left(\frac{h}{2} - x_2\right).$$

Thus the shearing stress is a maximum at the walls and vanishes at the center.

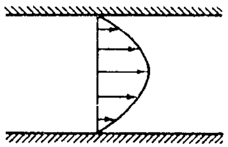

Figure 3

The volume discharge Q per unit time through a cross section one unit deep is obtained by integrating $v_1(x_2)$ from $x_2 = 0$ to $x_2 = h$; the result is

$$Q = h^3 \delta/12\mu.$$

Since the discharge Q is easily measured, as is the pressure drop δ, this formula yields a convenient method of determining the viscosity.

Exercises

1. Consider the flow of a Reiner–Rivlin fluid (cf. Exercise 1 of Section 22) between two flat plates. Show that the linear velocity profile (2), (6) remains a solution of the underlying equations with π_0 constant. Show further that, in contrast to the linear theory, the normal stresses are no longer equal.
2. Consider the plane, steady flow of a Newtonian fluid of depth h down a flat surface inclined at an angle α to the horizontal (Fig. 4). Thus the fluid flows under the influence of the gravitational force

$$\mathbf{b} = \rho_0 g(\mathbf{e}_1 \sin\alpha - \mathbf{e}_2 \cos\alpha),$$

and the boundary conditions are

$$\mathbf{v} = \mathbf{0} \quad \text{at} \quad x_2 = 0, \qquad \mathbf{T}\mathbf{e}_2 = -C\mathbf{e}_2 \quad \text{at} \quad x_2 = h,$$

where C is the (constant) atmospheric pressure. Assume that the flow has the form (2). Determine the velocity and stress fields in the fluid.

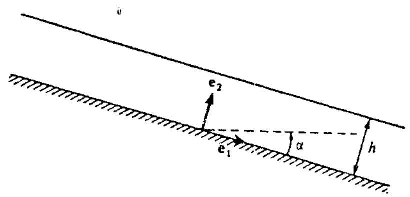

Figure 4

24. UNIQUENESS AND STABILITY

In this section we establish theorems of uniqueness and stability for the Navier–Stokes equations. Aside from their intrinsic interest, the proofs of these theorems utilize techniques which are important in their own right.

The classical **viscous flow problem** for the Navier–Stokes equations can be posed as follows:

Given: a bounded regular region $\mathscr{R}$, a kinematic viscosity $\nu > 0$, a body force field $\mathbf{b}$ on $\mathscr{R} \times [0, \infty)$, an initial velocity distribution $\mathbf{v}_0$ on $\mathscr{R}$, and a boundary velocity distribution $\hat{\mathbf{v}}$ on $\partial\mathscr{R} \times [0, \infty)$.

Find: a class C^2 velocity field $\mathbf{v}$ and a smooth pressure field π on $\mathscr{R} \times [0, \infty)$ that satisfy the Navier–Stokes equations

$$\mathbf{v}' + (\operatorname{grad} \mathbf{v})\mathbf{v} = \nu \Delta \mathbf{v} - \operatorname{grad} \pi + \mathbf{b}, \qquad \operatorname{div} \mathbf{v} = 0, \tag{1}$$

the *initial condition*

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$$

for every $\mathbf{x} \in \mathscr{R}$, and the *boundary condition*

$$\mathbf{v} = \hat{\mathbf{v}} \quad \text{on} \quad \partial\mathscr{R} \times [0, \infty).$$

A pair $(\mathbf{v}, \pi)$ with these properties will be called a **solution**. (Here, for convenience, we have written π and $\mathbf{b}$ for what we earlier called π_0 and $\mathbf{b}_0$.)

Uniqueness Theorem. *Let $(\mathbf{v}_1, \pi_1)$ and $(\mathbf{v}_2, \pi_2)$ be solutions of the (same) viscous flow problem. Then*

$$\mathbf{v}_1 = \mathbf{v}_2, \qquad \pi_1 = \pi_2 + \alpha$$

with α spatially constant:

$$\operatorname{grad} \alpha = \mathbf{0}.$$

The proof of this theorem is based on the following

Lemma. *Let $\mathbf{w}$ be a smooth vector field on $\mathscr{R}$, let β be a smooth scalar field on $\mathscr{R}$, and assume that*

$$\operatorname{div} \mathbf{w} = 0$$

and that at each point of $\partial\mathscr{R}$ either $\mathbf{w} = \mathbf{0}$ or $\beta = 0$. Then

$$\int_{\mathscr{R}} \mathbf{w} \cdot \operatorname{grad} \beta \, dV = 0.$$

Proof. The proof follows from the identity

$$\operatorname{div}(\beta \mathbf{w}) = \beta \operatorname{div} \mathbf{w} + \mathbf{w} \cdot \operatorname{grad} \beta,$$

the divergence theorem, and the hypotheses. □

Proof of Theorem. Let

$$\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2, \qquad \alpha = \pi_1 - \pi_2.$$

Then, since $(\mathbf{v}_1, \pi_1)$ and $(\mathbf{v}_2, \pi_2)$ are solutions,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \qquad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\mathscr{R} \times [0, \infty), \qquad \operatorname{div} \mathbf{u} = 0. \tag{2}$$

Further, subtracting $(1)_1$ with $\mathbf{v} = \mathbf{v}_2, \pi = \pi_2$ from $(1)_1$ with $\mathbf{v} = \mathbf{v}_1, \pi = \pi_1$, we obtain the relation

$$\mathbf{u}' + (\operatorname{grad} \mathbf{v}_1)\mathbf{v}_1 - (\operatorname{grad} \mathbf{v}_2)\mathbf{v}_2 = \nu \Delta \mathbf{u} - \operatorname{grad} \alpha,$$

and, since

$$(\operatorname{grad} \mathbf{v}_1)\mathbf{v}_1 = (\operatorname{grad} \mathbf{u})\mathbf{v}_1 + (\operatorname{grad} \mathbf{v}_2)\mathbf{v}_1,$$

we have

$$\mathbf{u}' + (\operatorname{grad} \mathbf{u})\mathbf{v}_1 + (\operatorname{grad} \mathbf{v}_2)\mathbf{u} = \nu \Delta \mathbf{u} - \operatorname{grad} \alpha. \tag{3}$$

If we take the inner product of (3) with $\mathbf{u}$, and use the identities

$$\begin{aligned} \mathbf{u} \cdot \Delta \mathbf{u} &= \operatorname{div}[(\operatorname{grad} \mathbf{u})^{\mathrm{T}}\mathbf{u}] - |\operatorname{grad} \mathbf{u}|^2, \\ \mathbf{u} \cdot (\operatorname{grad} \mathbf{u})\mathbf{v}_1 &= \mathbf{v}_1 \cdot (\operatorname{grad} \mathbf{u})^{\mathrm{T}}\mathbf{u} = \mathbf{v}_1 \cdot \operatorname{grad}\left(\frac{\mathbf{u}^2}{2}\right), \\ \mathbf{u} \cdot (\operatorname{grad} \mathbf{v}_2)\mathbf{u} &= \mathbf{u} \cdot \mathbf{D}\mathbf{u}, \end{aligned} \tag{4}$$

where

$$\mathbf{D} = \tfrac{1}{2}(\operatorname{grad} \mathbf{v}_2 + \operatorname{grad} \mathbf{v}_2^{\mathrm{T}}),$$

we conclude that

$$\tfrac{1}{2}(\mathbf{u}^2)' + \mathbf{v}_1 \cdot \operatorname{grad}\left(\frac{\mathbf{u}^2}{2}\right) + \mathbf{u} \cdot \mathbf{D}\mathbf{u} = \nu \operatorname{div}[(\operatorname{grad} \mathbf{u})^{\mathrm{T}}\mathbf{u}] - \nu|\operatorname{grad} \mathbf{u}|^2 - \mathbf{u} \cdot \operatorname{grad} \alpha.$$

If we integrate this relation over $\mathscr{R}$, use the lemma twice (first with $\mathbf{w} = \mathbf{v}_1$, $\beta = \mathbf{u}^2/2$, then with $\mathbf{w} = \mathbf{u}, \beta = \alpha$), and appeal to the divergence theorem, we conclude, with the aid of (2), that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \int_{\mathscr{R}} \mathbf{u} \cdot \mathbf{D}\mathbf{u} \, dV \le 0, \tag{5}$$

where

$$\|\mathbf{u}\|^2(t) = \int_{\mathscr{R}} \mathbf{u}^2(\mathbf{x}, t) \, dV_{\mathbf{x}}.$$

Since $\operatorname{div} \mathbf{v} = 0$, $\operatorname{tr} \mathbf{D} = 0$; thus the lowest proper value of the symmetric tensor $\mathbf{D}(\mathbf{x}, t)$ is ≤ 0. Let $-\gamma(\mathbf{x}, t)$ denote this proper value, so that $\gamma \geq 0$ and

$$\mathbf{u} \cdot \mathbf{D}\mathbf{u} \geq -\gamma \mathbf{u}^2.$$

Now choose $\tau > 0$ and let

$$\lambda = 2 \sup_{\substack{\mathbf{x} \in \mathscr{R} \\ 0 \leq t \leq \tau}} \gamma(\mathbf{x}, t).$$

(Since $\mathbf{v}_2$ is smooth, $\mathbf{D}$ is continuous and $0 \leq \lambda < \infty$.) Thus

$$\mathbf{u} \cdot \mathbf{D}\mathbf{u} \geq -\frac{\lambda}{2}\mathbf{u}^2 \qquad \text{on} \quad \mathscr{R} \times [0, \tau]$$

and (5) becomes

$$\frac{d}{dt}\|\mathbf{u}\|^2 - \lambda\|\mathbf{u}\|^2 \leq 0 \qquad \text{on} \quad [0, \tau].$$

Therefore

$$\frac{d}{dt}\{\|\mathbf{u}\|^2 e^{-\lambda t}\} \leq 0 \qquad \text{on} \quad [0, \tau],$$

and hence

$$\|\mathbf{u}\|^2(\tau) \leq \|\mathbf{u}\|^2(0)e^{\lambda\tau}.$$

But by $(2)_1$,

$$\|\mathbf{u}\|^2(0) = 0;$$

thus $\|\mathbf{u}\|^2(\tau) = 0$ and

$$\mathbf{u}(\mathbf{x}, \tau) = \mathbf{0}$$

for every $\mathbf{x} \in \mathscr{R}$. Since τ was arbitrarily chosen, $\mathbf{u} \equiv \mathbf{0}$ and $\mathbf{v}_1 = \mathbf{v}_2$. Finally, (3) implies that $\operatorname{grad} \alpha = \mathbf{0}$. □

Stability Theorem. *Consider the viscous flow problem with vanishing boundary data and conservative body forces:*

$$\hat{\mathbf{v}} = \mathbf{0} \qquad \text{on} \quad \partial\mathscr{R} \times [0, \infty),$$
$$\mathbf{b} = -\operatorname{grad} \beta. \tag{6}$$

Let $(\mathbf{v}, \pi)$ *be the solution (if it exists) of this problem. Then there exists a constant* $\lambda > 0$ *such that*

$$\|\mathbf{v}\|(t) \leq \|\mathbf{v}_0\|e^{-\lambda t}. \tag{7}$$

Proof. In view of (6), $(\mathbf{v}, \pi)$ satisfies the system

$$\begin{aligned} \mathbf{v}' + (\operatorname{grad} \mathbf{v})\mathbf{v} &= \nu\,\Delta\mathbf{v} - \operatorname{grad} \alpha, \\ \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{v} = \mathbf{0} \quad &\text{on} \quad \partial\mathscr{R} \times [0, \infty), \\ \mathbf{v}(\mathbf{x}, 0) &= \mathbf{v}_0(\mathbf{x}), \end{aligned} \tag{8}$$

where $\alpha = \pi + \beta$. If we take the inner product of $(8)_1$ with $\mathbf{v}$ and use the identities $(4)_{1,2}$, we arrive at the relation

$$\tfrac{1}{2}(\mathbf{v}^2)' + \mathbf{v} \cdot \operatorname{grad}(\alpha + \tfrac{1}{2}\mathbf{v}^2) = \nu \operatorname{div}\{(\operatorname{grad} \mathbf{v})^T\mathbf{v}\} - \nu|\operatorname{grad} \mathbf{v}|^2.$$

If we integrate this relation over $\mathscr{R}$, we conclude, with the aid of the divergence theorem, $(8)_{2,3}$, and the lemma, that

$$\frac{d}{dt}\|\mathbf{v}\|^2 + 2\nu\|\operatorname{grad} \mathbf{v}\|^2 = 0,$$

where

$$\|\operatorname{grad} \mathbf{v}\|^2 = \int_{\mathscr{R}} |\operatorname{grad} \mathbf{v}|^2\, dV.$$

By the Poincaré inequality (Exercise 1) there exists a constant $\lambda_0 > 0$ such that

$$\|\operatorname{grad} \mathbf{v}\|^2 \geq \lambda_0\|\mathbf{v}\|^2.$$

Thus

$$\frac{d}{dt}\|\mathbf{v}\|^2 + 2\lambda\|\mathbf{v}\|^2 \leq 0,$$

where $\lambda = \nu\lambda_0$, and hence

$$\frac{d}{dt}\{e^{2\lambda t}\|\mathbf{v}\|^2\} \leq 0,$$

which integrates to give

$$\|\mathbf{v}\|^2(t) \leq \|\mathbf{v}\|^2(0)e^{-2\lambda t},$$

or equivalently,

$$\|\mathbf{v}\|(t) \leq \|\mathbf{v}\|(0)e^{-\lambda t}.$$

In view of $(8)_4$, this inequality clearly implies (7). □

Remark. It should be emphasized that the preceding two theorems require that the solution be *smooth* for all time and therefore may be misleading, since there is no global regularity theorem for the Navier–Stokes equations.[1]

[1] Cf. Ladyzhenskaya [1, §6]; Temam [1, Chapter III].

Exercises

1. Supply the details of the following proof of the Poincaré inequality. Let φ be a smooth scalar field on $\mathscr{R}$ with $\varphi = 0$ on $\partial\mathscr{R}$. Since $\mathscr{R}$ is bounded, φ can be extended to a box $\mathscr{D}$ (with sides parallel to coordinate planes) by defining $\varphi = 0$ on $\mathscr{D} - \mathscr{R}$. Suppose that for $\mathbf{x} \in \mathscr{D}$, x_1 varies between α and β. Define $\varphi_1 = \partial\varphi/\partial x_1$. Then
$$\varphi^2(\mathbf{x}) = 2\int_\alpha^{x_1} \varphi(\xi, x_2, x_3)\varphi_1(\xi, x_2, x_3)\, d\xi,$$
and hence
$$\begin{aligned}\int_{\mathscr{R}} \varphi^2\, dV &= 2\int_{\mathscr{D}}\int_\alpha^{x_1} \varphi(\xi, x_2, x_3)\varphi_1(\xi, x_2, x_3)\, d\xi\, dV_{\mathbf{x}} \\ &\le 2\left(\int_{\mathscr{D}}\int_\alpha^{\beta} \varphi^2(\xi, x_2, x_3)\, d\xi\, dV_{\mathbf{x}}\right)^{1/2} \\ &\quad\times \left(\int_{\mathscr{D}}\int_\alpha^{\beta} \varphi_1^2(\xi, x_2, x_3)\, d\xi\, dV_{\mathbf{x}}\right)^{1/2} \\ &\le 2(\beta - \alpha)\left(\int_{\mathscr{R}} \varphi^2\, dV\right)^{1/2}\left(\int_{\mathscr{R}} |\mathrm{grad}\,\varphi|^2\, dV\right)^{1/2},\end{aligned}$$
which implies the Poincaré inequality
$$\int_{\mathscr{R}} \varphi^2\, dV \le 4(\beta - \alpha)^2 \int_{\mathscr{R}} |\mathrm{grad}\,\varphi|^2\, dV.$$
Show that this implies an analogous result for a smooth vector field $\mathbf{v}$ on $\mathscr{R}$ with $\mathbf{v} = \mathbf{0}$ on $\partial\mathscr{R}$.
2. Establish (4).

Selected References

Bird, Stewart, and Lightfoot [1, Part 1].
Hughes and Marsden [1, §§16, 17].
Ladyzhenskaya [1].
Lamb [1, Chapter 1].
Noll [1].
Serrin [1, Chapter G].
Temam [1].

CHAPTER

IX

Finite Elasticity

25. ELASTIC BODIES

In classical mechanics the force on an elastic spring depends only on the change in length of the spring; this force is independent of the past history of the length as well as the rate at which the length is changing with time. For a body the deformation gradient $\mathbf{F}$ measures local distance changes, and it therefore seems natural to define an elastic body as one whose constitutive equation gives the stress $\mathbf{T}(\mathbf{x}, t)$ at $\mathbf{x} = \mathbf{x}(\mathbf{p}, t)$ when the deformation gradient $\mathbf{F}(\mathbf{p}, t)$ is known:
$$\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{p}, t), \mathbf{p}). \tag{1}$$
Formally, then, an **elastic body** is a material body whose constitutive class $\mathscr{C}$ is defined by a smooth response function
$$\hat{\mathbf{T}}: \mathrm{Lin}^+ \times \mathscr{B} \to \mathrm{Sym}$$
as follows: $\mathscr{C}$ is the set of all dynamical processes $(\mathbf{x}, \mathbf{T})$ consistent with (1).

Note that for an elastic body the function $\hat{\mathbf{T}}$ is completely determined by the response of the material to time-independent homogeneous motions of the form
$$\mathbf{x}(\mathbf{p}, t) = \mathbf{q} + \mathbf{F}(\mathbf{p} - \mathbf{p}_0); \tag{2}$$
that is, to homogeneous deformations.

When convenient, and when there is no danger of confusion, we will write

$\hat{\mathbf{T}}(\mathbf{F})$ in place of $\hat{\mathbf{T}}(\mathbf{F}, \mathbf{p})$.

Proposition. *A necessary and sufficient condition that the response of an elastic body be independent of the observer is that the response function* $\hat{\mathbf{T}}$ *satisfy*

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T = \hat{\mathbf{T}}(\mathbf{QF}) \tag{3}$$

for every $\mathbf{F} \in \text{Lin}^+$ *and every* $\mathbf{Q} \in \text{Orth}^+$.

Proof. Choose $\mathbf{F} \in \text{Lin}^+$ arbitrarily, let $\mathbf{x}$ be the motion (2), and let $\mathbf{T}$ be defined by (1), so that $(\mathbf{x}, \mathbf{T}) \in \mathscr{C}$, the constitutive class of the body. Assume that the response is independent of the observer. Then every $(\mathbf{x}^*, \mathbf{T}^*)$ related to $(\mathbf{x}, \mathbf{T})$ by a change in observer must also belong to $\mathscr{C}$. By (20.4) and (21.1), this can happen only if, given any rotation $\mathbf{Q}$,

$$\mathbf{QTQ}^T = \hat{\mathbf{T}}(\mathbf{QF}).$$

Since $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F})$, this clearly implies (3). Conversely, if (3) is satisfied, then, in view of (20.4) and (21.1), the response is independent of the observer. □

We assume henceforth that the response is independent of the observer, so that (3) *holds.*

The importance of the strain tensors $\mathbf{U}$ and $\mathbf{C}$ is brought out by the next result, which gives alternative forms for the constitutive equation (1).

Corollary (Reduced constitutive equations). *The response function* $\hat{\mathbf{T}}$ *is completely determined by its restriction to* Psym; *in fact,*

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\hat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T \tag{4}$$

for every $\mathbf{F} \in \text{Lin}^+$, *where* $\mathbf{R} \in \text{Orth}^+$ *is the rotation tensor and* $\mathbf{U} \in \text{Psym}$ *the right stretch tensor corresponding to* $\mathbf{F}$; *that is,* $\mathbf{F} = \mathbf{RU}$ *is the right polar decomposition of* $\mathbf{F}$. *Further, there exist smooth response functions* $\overset{*}{\mathbf{T}}$, $\tilde{\mathbf{T}}$, *and* $\bar{\mathbf{T}}$ *from* Psym *into* Sym *such that*

$$\begin{aligned} \hat{\mathbf{T}}(\mathbf{F}) &= \mathbf{F}\overset{*}{\mathbf{T}}(\mathbf{U})\mathbf{F}^T, \\ \hat{\mathbf{T}}(\mathbf{F}) &= \mathbf{R}\tilde{\mathbf{T}}(\mathbf{C})\mathbf{R}^T, \\ \hat{\mathbf{T}}(\mathbf{F}) &= \mathbf{F}\bar{\mathbf{T}}(\mathbf{C})\mathbf{F}^T, \end{aligned} \tag{5}$$

with $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ *the right Cauchy–Green strain tensor corresponding to* $\mathbf{F}$.

Proof. To derive (4) we simply choose $\mathbf{Q}$ in (3) equal to $\mathbf{R}^T$. Next, since $\mathbf{F} = \mathbf{RU}$, (4) can be written in the form

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{FU}^{-1}\hat{\mathbf{T}}(\mathbf{U})\mathbf{U}^{-1}\mathbf{F}^T.$$

Thus if we define

$$\overset{*}{\mathbf{T}}(\mathbf{U}) = \mathbf{U}^{-1}\hat{\mathbf{T}}(\mathbf{U})\mathbf{U}^{-1}, \tag{6}$$

we are led to $(5)_1$. On the other hand, if we let

$$\begin{aligned} \tilde{\mathbf{T}}(\mathbf{C}) &= \hat{\mathbf{T}}(\mathbf{C}^{1/2}), \\ \bar{\mathbf{T}}(\mathbf{C}) &= \overset{*}{\mathbf{T}}(\mathbf{C}^{1/2}), \end{aligned} \tag{7}$$

for $\mathbf{C} \in \text{Psym}$, then (4) and $(5)_1$ imply $(5)_{2,3}$. Finally, by (6), $\overset{*}{\mathbf{T}}$ is smooth, while (7) and the smoothness of the square root (page 23) yield the smoothness of $\tilde{\mathbf{T}}$ and $\bar{\mathbf{T}}$. □

The converse is also true: each of the constitutive equations in (4) and (5) is independent of the observer. We leave the proof of this assertion as an exercise.

It is important to note that the response functions $\overset{*}{\mathbf{T}}$, $\tilde{\mathbf{T}}$, and $\bar{\mathbf{T}}$ depend also on the material point $\mathbf{p}$ under consideration.

Suppose that we rotate a specimen of material and then perform an experiment upon it. If the outcome is the same as if the specimen had not been rotated, then the rotation is called a symmetry transformation. To fix ideas consider the two-dimensional example shown in Fig. 1; there an elastic ring is connected by identical mutually perpendicular elastic springs which meet at the center of the ring. The forces required to produce any given deformation are the same for any prerotation of the ring by a multiple of 90°, so that such rotations are symmetry transformations. Moreover, they are the only symmetry transformations, since any other prerotation can be detected by some subsequent deformation. We now apply these ideas to the general elastic body $\mathscr{B}$.

Choose a point $\mathbf{p}$ of $\mathscr{B}$ and let $\mathbf{f_F}$ denote the homogeneous deformation from $\mathbf{p}$ with deformation gradient $\mathbf{F}$:

$$\mathbf{f_F}(\mathbf{q}) = \mathbf{p} + \mathbf{F}(\mathbf{q} - \mathbf{p})$$

for every $\mathbf{q} \in \mathscr{B}$. Consider two experiments:

(1) Deform $\mathscr{B}$ with the homogeneous deformation $\mathbf{f_F}$.

(2) Rotate $\mathscr{B}$ with the rotation $\mathbf{f_Q}$ ($\mathbf{Q} \in \text{Orth}^+$) about $\mathbf{p}$ and then deform the rotated body with the homogeneous deformation $\mathbf{f_F}$ (Fig. 2). In this case the total deformation is $\mathbf{f_F} \circ \mathbf{f_Q}$ and the deformation gradient is $\mathbf{FQ}$.

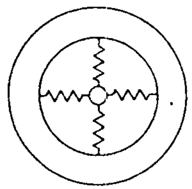

Figure 1

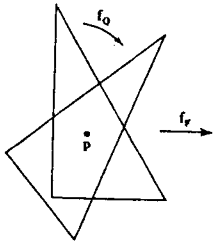

Figure 2

The stress at $\mathbf{p}$ is

$$\hat{\mathbf{T}}(\mathbf{F}, \mathbf{p}) \quad \text{in Experiment 1}$$

and

$$\hat{\mathbf{T}}(\mathbf{FQ}, \mathbf{p}) \quad \text{in Experiment 2.}$$

If for $\mathbf{Q}$ fixed these two stress tensors coincide for each $\mathbf{F}$, then $\mathbf{Q}$ is called a symmetry transformation; $\mathbf{Q}$ has the property that the response of the material at $\mathbf{p}$ is the same before and after the rotation $\mathbf{f_Q}$. Thus, more succinctly, a **symmetry transformation** at $\mathbf{p}$ is a tensor $\mathbf{Q} \in \text{Orth}^+$ such that

$$\hat{\mathbf{T}}(\mathbf{F}, \mathbf{p}) = \hat{\mathbf{T}}(\mathbf{FQ}, \mathbf{p}) \tag{8}$$

for every $\mathbf{F} \in \text{Lin}^+$. We write $\mathscr{G}_\mathbf{p}$ for the set of all symmetry transformations at $\mathbf{p}$ and call $\mathscr{G}_\mathbf{p}$ the **symmetry group** at $\mathbf{p}$. The next proposition shows that this terminology is consistent.

Proposition. *$\mathscr{G}_\mathbf{p}$ is a subgroup of* Orth^+.

Proof. Let us agree to write

$$\mathscr{G} \quad \text{in place of} \quad \mathscr{G}_\mathbf{p}$$

whenever there is no danger of confusion. [Recall our previous agreement to write $\hat{\mathbf{T}}(\mathbf{F})$ for $\hat{\mathbf{T}}(\mathbf{F}, \mathbf{p})$.] Clearly, $\mathscr{G}$ is a subset of Orth^+ and $\mathbf{I} \in \mathscr{G}$. To show that $\mathscr{G}$ is a subgroup of Orth^+ it therefore suffices to show that $\mathscr{G}$ is closed under both inversion and multiplication:

$$\mathbf{Q} \in \mathscr{G} \Rightarrow \mathbf{Q}^{-1} \in \mathscr{G},$$

$$\mathbf{Q}, \mathbf{H} \in \mathscr{G} \Rightarrow \mathbf{QH} \in \mathscr{G}.$$

Choose $\mathbf{Q} \in \mathscr{G}$ and $\mathbf{F} \in \text{Lin}^+$ arbitrarily. Then (8) with $\mathbf{F}$ replaced by $\mathbf{FQ}^{-1}$ yields

$$\hat{\mathbf{T}}(\mathbf{FQ}^{-1}) = \hat{\mathbf{T}}((\mathbf{FQ}^{-1})\mathbf{Q}) = \hat{\mathbf{T}}(\mathbf{F})$$

so that $\mathbf{Q}^{-1} \in \mathscr{G}$. Next, choose $\mathbf{F} \in \text{Lin}^+$ and $\mathbf{Q}, \mathbf{H} \in \mathscr{G}$. Then

$$\hat{\mathbf{T}}(\mathbf{F}) = \hat{\mathbf{T}}(\mathbf{FQ}) = \hat{\mathbf{T}}((\mathbf{FQ})\mathbf{H}) = \hat{\mathbf{T}}(\mathbf{FQH})$$

so that $\mathbf{QH} \in \mathscr{G}$. $\square$

Our next result shows that (3) (which expresses invariance under observer changes) and (8) together imply the invariance of $\hat{\mathbf{T}}$ under $\mathscr{G}$ (cf. Section 37).

Proposition. *The response functions $\hat{\mathbf{T}}$, $\bar{\mathbf{T}}$, $\overset{*}{\mathbf{T}}$, and $\tilde{\mathbf{T}}$ are invariant under $\mathscr{G}$. Thus, in particular,*

$$\begin{aligned} \mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T &= \hat{\mathbf{T}}(\mathbf{QFQ}^T), \\ \mathbf{Q}\bar{\mathbf{T}}(\mathbf{C})\mathbf{Q}^T &= \bar{\mathbf{T}}(\mathbf{QCQ}^T), \end{aligned} \tag{9}$$

for every $\mathbf{Q} \in \mathscr{G}$, $\mathbf{F} \in \text{Lin}^+$, and $\mathbf{C} \in \text{Psym}$.

Proof. Choose $\mathbf{Q} \in \mathscr{G}$. Then $\mathbf{Q}^T \in \mathscr{G}$ and (8) implies

$$\hat{\mathbf{T}}(\mathbf{QF}) = \hat{\mathbf{T}}(\mathbf{QFQ}^T).$$

We therefore conclude, with the aid of (3), that $(9)_1$ holds.

To prove $(9)_2$ choose $\mathbf{C} \in \text{Psym}$ arbitrarily and let $\mathbf{U}$ be the unique square root of $\mathbf{C}$:

$$\mathbf{C} = \mathbf{U}^2.$$

By (6) and $(7)_2$,

$$\bar{\mathbf{T}}(\mathbf{C}) = \mathbf{U}^{-1}\hat{\mathbf{T}}(\mathbf{U})\mathbf{U}^{-1}.$$

Thus, since

$$(\mathbf{QUQ}^T)^2 = \mathbf{QU}^2\mathbf{Q}^T = \mathbf{QCQ}^T,$$

$$(\mathbf{QUQ}^T)^{-1} = \mathbf{QU}^{-1}\mathbf{Q}^T,$$

we conclude from $(9)_1$ that

$$\begin{aligned} \mathbf{Q}\bar{\mathbf{T}}(\mathbf{C})\mathbf{Q}^T &= \mathbf{QU}^{-1}\mathbf{Q}^T\mathbf{Q}\hat{\mathbf{T}}(\mathbf{U})\mathbf{Q}^T\mathbf{QU}^{-1}\mathbf{Q}^T \\ &= (\mathbf{QUQ}^T)^{-1}\hat{\mathbf{T}}(\mathbf{QUQ}^T)(\mathbf{QUQ}^T)^{-1} = \bar{\mathbf{T}}((\mathbf{QUQ}^T)^2) = \bar{\mathbf{T}}(\mathbf{QCQ}^T). \end{aligned}$$

We leave as an exercise the proof of the assertions concerning $\overset{*}{\mathbf{T}}$ and $\tilde{\mathbf{T}}$. $\square$

We say that the material at $\mathbf{p}$ is **isotropic** if

$$\mathscr{G}_\mathbf{p} = \text{Orth}^+$$

(so that every rotation is a symmetry transformation), **anisotropic** if

$$\mathscr{G}_\mathbf{p} \neq \text{Orth}^+.$$

This definition, the last proposition, and the proposition on page 230 imply the following

Proposition. *Assume that the material at* $\mathbf{p}$ *is isotropic. Then each of the response functions* $\hat{\mathbf{T}}$, $\overset{\dagger}{\mathbf{T}}$, $\tilde{\mathbf{T}}$, *and* $\bar{\mathbf{T}}$ *(at* $\mathbf{p}$*) is an isotropic function.*

The stress

$$\mathbf{T}_R = \hat{\mathbf{T}}(\mathbf{I})$$

is called the **residual stress** at $\mathbf{p}$; $\mathbf{T}_R$ is the stress at $\mathbf{p}$ when the body is undeformed. Since $\mathbf{C} = \mathbf{U} = \mathbf{R} = \mathbf{I}$ when $\mathbf{F} = \mathbf{I}$, (5) implies that

$$\mathbf{T}_R = \overset{\dagger}{\mathbf{T}}(\mathbf{I}) = \tilde{\mathbf{T}}(\mathbf{I}) = \bar{\mathbf{T}}(\mathbf{I}). \tag{10}$$

Proposition. *If the material at* $\mathbf{p}$ *is isotropic, then* $\mathbf{T}_R$ *is a pressure.*

Proof. Let $\mathbf{Q} \in$ Orth. Then, since $\hat{\mathbf{T}}$ is isotropic,

$$\mathbf{Q}\mathbf{T}_R\mathbf{Q}^T = \mathbf{Q}\hat{\mathbf{T}}(\mathbf{I})\mathbf{Q}^T = \hat{\mathbf{T}}(\mathbf{Q}\mathbf{I}\mathbf{Q}^T) = \hat{\mathbf{T}}(\mathbf{I}) = \mathbf{T}_R,$$

or, equivalently, $\mathbf{Q}\mathbf{T}_R = \mathbf{T}_R\mathbf{Q}$, so that $\mathbf{T}_R$ commutes with every orthogonal tensor. We therefore conclude from the corollary on page 13 that $\mathbf{T}_R = -\pi\mathbf{I}$ with π a scalar. □

By (6.7) the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$ and the right and left Cauchy–Green strain tensors $\mathbf{C}$ and $\mathbf{B}$ are related by

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T, \qquad \mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T,$$

where $\mathbf{R} \in \mathrm{Orth}^+$ is the corresponding rotation tensor. Therefore when the material is isotropic, (4), $(5)_2$, and the proposition containing (9) yield

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\hat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T = \hat{\mathbf{T}}(\mathbf{R}\mathbf{U}\mathbf{R}^T) = \hat{\mathbf{T}}(\mathbf{V}),$$

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\tilde{\mathbf{T}}(\mathbf{C})\mathbf{R}^T = \tilde{\mathbf{T}}(\mathbf{R}\mathbf{C}\mathbf{R}^T) = \tilde{\mathbf{T}}(\mathbf{B}),$$

and thus the constitutive relation $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F})$ can be written in the alternative forms

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{V}), \qquad \mathbf{T} = \tilde{\mathbf{T}}(\mathbf{B}), \tag{11}$$

with $\hat{\mathbf{T}}$ and $\tilde{\mathbf{T}}$ isotropic functions. In view of (37.21), these comments have the following consequence:

Theorem (Constitutive equation for an isotropic material). *Assume that the material at* $\mathbf{p}$ *is isotropic. Then the constitutive relation can be written in the form*

$$\mathbf{T} = \beta_0(\mathscr{I}_\mathbf{B})\mathbf{I} + \beta_1(\mathscr{I}_\mathbf{B})\mathbf{B} + \beta_2(\mathscr{I}_\mathbf{B})\mathbf{B}^{-1}, \tag{12}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ *is the left Cauchy–Green strain tensor and* β_0, β_1, *and* β_2 *are scalar functions of the list* $\mathscr{I}_\mathbf{B}$ *of principal invariants of* $\mathbf{B}$.

Further, as is clear from (11) and (37.15), the constitutive equation (12) for an isotropic material point can also be expressed in the forms

$$\mathbf{T} = \alpha_0(\mathscr{I}_\mathbf{B})\mathbf{I} + \alpha_1(\mathscr{I}_\mathbf{B})\mathbf{B} + \alpha_2(\mathscr{I}_\mathbf{B})\mathbf{B}^2,$$

$$\mathbf{T} = \kappa_0(\mathscr{I}_\mathbf{V})\mathbf{I} + \kappa_1(\mathscr{I}_\mathbf{V})\mathbf{V} + \kappa_2(\mathscr{I}_\mathbf{V})\mathbf{V}^{-1},$$

$$\mathbf{T} = \delta_0(\mathscr{I}_\mathbf{V})\mathbf{I} + \delta_1(\mathscr{I}_\mathbf{V})\mathbf{V} + \delta_2(\mathscr{I}_\mathbf{V})\mathbf{V}^2,$$

where $\mathbf{V} = \mathbf{B}^{1/2}$ is the left stretch tensor.

We now return to the general theory and list the complete system of field equations; this consists of the constitutive equation

$$\mathbf{T} = \mathbf{F}\tilde{\mathbf{T}}(\mathbf{C})\mathbf{F}^T, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \tag{13}$$

the equation of motion

$$\operatorname{div}\mathbf{T} + \mathbf{b} = \rho\dot{\mathbf{v}}, \tag{14}$$

and balance of mass

$$\rho \det\mathbf{F} = \rho_0, \tag{15}$$

where ρ_0 is the density in the reference configuration. When the body is isotropic (13) may be replaced by

$$\mathbf{T} = \beta_0(\mathscr{I}_\mathbf{B})\mathbf{I} + \beta_1(\mathscr{I}_\mathbf{B})\mathbf{B} + \beta_2(\mathscr{I}_\mathbf{B})\mathbf{B}^{-1},$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T.$$

We say that the body is **homogeneous** provided both $\rho_0(\mathbf{p})$ and $\hat{\mathbf{T}}(\mathbf{F}, \mathbf{p})$ are independent of the material point $\mathbf{p}$. In this case each of the response functions, as well as the symmetry group $\mathscr{G}_\mathbf{p}$, is independent of $\mathbf{p}$.

Consider the homogeneous deformation (2). For a homogeneous body the corresponding stress $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F})$ is constant, because $\mathbf{F}$ is constant. Therefore $\mathbf{T}$ satisfies the equation of equilibrium

$$\operatorname{div}\mathbf{T} = \mathbf{0}$$

and $(\mathbf{x}, \mathbf{T})$ is a solution of (13)–(15) with $\mathbf{b} = \mathbf{0}$. Thus *a homogeneous body can be deformed homogeneously without body force.*

Exercises

1. Show that each of the constitutive equations in (5) is independent of the observer.
2. Show that the response functions $\overset{\dagger}{\mathbf{T}}$ and $\tilde{\mathbf{T}}$ are invariant under $\mathscr{G}$.

3. Show that an elastic fluid is an isotropic elastic body.
4. Define the *extended symmetry group* $\mathscr{H}_{\mathbf{p}}$ to be the set of all tensors $\mathbf{H} \in \text{Lin}^+$ such that

$$\hat{\mathbf{T}}(\mathbf{F}, \mathbf{p}) = \hat{\mathbf{T}}(\mathbf{FH}, \mathbf{p}) \tag{16}$$

for all $\mathbf{F} \in \text{Lin}^+$.

(a) Show that $\mathscr{H}_{\mathbf{p}}$ is a subgroup of Lin^+.
(b) Give a physical interpretation of $\mathscr{H}_{\mathbf{p}}$.
(c) Let Unim denote the *proper unimodular group*; that is,

$$\text{Unim} = \{\mathbf{H} \in \text{Lin}^+ \mid \det \mathbf{H} = 1\}.$$

Show that if $\mathscr{H}_{\mathbf{p}}$ is not contained in Unim, then there exist sequences $\{\mathbf{F}_n\}$ and $\{\mathbf{G}_n\}$ with $\mathbf{F}_n, \mathbf{G}_n \in \text{Lin}^+$ such that

(i) $\det \mathbf{F}_n \to \infty$ but $\hat{\mathbf{T}}(\mathbf{F}_n, \mathbf{p})$ is the same for all n;
(ii) $\det \mathbf{G}_n \to 0$ but $\hat{\mathbf{T}}(\mathbf{G}_n, \mathbf{p})$ is the same for all n.

Why is this physically unreasonable?

For the remainder of this section we assume that

$$\mathscr{H}_{\mathbf{p}} \subset \text{Unim}. \tag{17}$$

(d) Show that an elastic fluid has

$$\mathscr{H}_{\mathbf{p}} = \text{Unim} \tag{18}$$

for every $\mathbf{p} \in \mathscr{B}$.

(e) Show that, conversely, an elastic body whose response function obeys (18) is an elastic fluid. [For bodies exhibiting more general types of behavior, such as viscoelasticity, the condition (18) furnishes a useful definition of a fluid.]

5. It is often convenient to use a deformed configuration as reference (see Fig. 3). With this in mind, let $\mathbf{g}$ be a deformation of $\mathscr{B}$, let $\mathbf{f}$ be a deformation of $\mathbf{g}(\mathscr{B})$, and let

$$\mathbf{G} = \nabla\mathbf{g}(\mathbf{p}), \qquad \mathbf{F} = \nabla\mathbf{f}(\mathbf{q}), \qquad \mathbf{q} = \mathbf{g}(\mathbf{p}).$$

Then (under $\mathbf{f} \circ \mathbf{g}$) the material point $\mathbf{p}$ experiences the stress

$$\mathbf{T} = \hat{\mathbf{T}}(\nabla(\mathbf{f} \circ \mathbf{g})(\mathbf{p}), \mathbf{p}) = \hat{\mathbf{T}}(\mathbf{FG}, \mathbf{p}).$$

Let $\hat{\mathbf{T}}_{\mathbf{g}}$ be defined by

$$\hat{\mathbf{T}}_{\mathbf{g}}(\mathbf{F}, \mathbf{p}) = \hat{\mathbf{T}}(\mathbf{FG}, \mathbf{p})$$

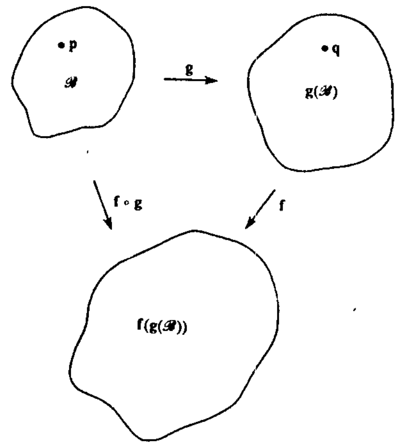

Figure 3

for every $\mathbf{F} \in \text{Lin}^+$; $\hat{\mathbf{T}}_{\mathbf{g}}(\cdot, \mathbf{p})$ is the response function for $\mathbf{p}$ when the deformed configuration (under the deformation $\mathbf{g}$) is used as reference.

(a) Show that $\mathbf{G} \in \text{Unim}$ belongs to $\mathscr{H}_{\mathbf{p}}$ if and only if, for every $\mathbf{F} \in \text{Lin}^+$,

$$\hat{\mathbf{T}}_{\mathbf{g}}(\mathbf{F}, \mathbf{p}) = \hat{\mathbf{T}}(\mathbf{F}, \mathbf{p})$$

for every deformation $\mathbf{g}$ of $\mathscr{B}$ with $\nabla\mathbf{g}(\mathbf{p}) = \mathbf{G}$.

(b) Define $\mathscr{H}_{\mathbf{p}}(\mathbf{g})$ to be the extended symmetry group taking $\mathbf{g}$ as reference; that is, $\mathscr{H}_{\mathbf{p}}(\mathbf{g})$ is the set of all $\mathbf{H} \in \text{Lin}^+$ such that

$$\hat{\mathbf{T}}_{\mathbf{g}}(\mathbf{F}, \mathbf{p}) = \hat{\mathbf{T}}_{\mathbf{g}}(\mathbf{FH}, \mathbf{p}) \tag{19}$$

for all $\mathbf{F} \in \text{Lin}^+$. Show that

$$\mathscr{H}_{\mathbf{p}}(\mathbf{g}) = \mathbf{G}\mathscr{H}_{\mathbf{p}}\mathbf{G}^{-1}, \tag{20}$$

where $\mathbf{G} = \nabla\mathbf{g}(\mathbf{p})$. (Here $\mathbf{G}\mathscr{H}_{\mathbf{p}}\mathbf{G}^{-1}$ is the set of all tensors of the form $\mathbf{GHG}^{-1}$, $\mathbf{H} \in \mathscr{H}_{\mathbf{p}}$.)

(c) Show that

$$\mathscr{H}_{\mathbf{p}}(\mathbf{g}) \subset \text{Unim}.$$

Show further that if

$$\mathscr{H}_{\mathbf{p}}(\mathbf{g}) = \text{Unim} \tag{21}$$

for some deformation $\mathbf{g}$, then (21) holds for every deformation $\mathbf{g}$.

(d) A material is a solid if there exists a configuration from which any change in shape (i.e., any nonrigid deformation) is detectable by some subsequent experiment. More precisely, the material at $\mathbf{p}$ is *solid* if there exists a deformation $\mathbf{g}$ such that

$$\mathscr{H}_{\mathbf{p}}(\mathbf{g}) \subset \text{Orth}^{+},$$

in which case $\mathbf{g}$ is *undistorting* at $\mathbf{p}$. Assume that $\mathbf{p}$ is solid and that the identity map on $\mathscr{B}$ is undistorting at $\mathbf{p}$ (so that $\mathscr{H}_{\mathbf{p}} \subset \text{Orth}^{+}$). Let $\mathbf{g}$ be any other deformation which is undistorting at $\mathbf{p}$. Show that

$$\mathscr{H}_{\mathbf{p}}(\mathbf{g}) = \mathbf{R}\mathscr{H}_{\mathbf{p}}\mathbf{R}^{-1},$$

where $\mathbf{R}$ is the rotation in the polar decomposition of $\mathbf{G} = \nabla\mathbf{g}(\mathbf{p})$. Show further that if, in addition, the material at $\mathbf{p}$ is isotropic, then

$$\mathscr{H}_{\mathbf{p}}(\mathbf{g}) = \mathscr{H}_{\mathbf{p}}(= \text{Orth}^{+}),$$

and $\mathbf{G}$ is a similarity transformation:

$$\mathbf{G} = \lambda\mathbf{Q}$$

with $\lambda > 0$ and $\mathbf{Q} \in \text{Orth}^{+}$.

6. An *incompressible elastic body* is an incompressible material body defined by a constitutive equation of the form

$$\mathbf{T} = -\pi\mathbf{I} + \hat{\mathbf{T}}(\mathbf{F}), \tag{22}$$

or, more precisely, for $\mathbf{x} = \mathbf{x}(\mathbf{p}, t)$,

$$\mathbf{T}(\mathbf{x}, t) = -\pi(\mathbf{p}, t)\mathbf{I} + \hat{\mathbf{T}}(\mathbf{F}(\mathbf{p}, t), \mathbf{p})$$

with

$$\hat{\mathbf{T}}: \text{Unim} \times \mathscr{B} \to \text{Sym}$$

smooth. (As in the case of an incompressible fluid the pressure π is not uniquely determined by the motion.) For such a body:

(a) Determine necessary and sufficient conditions that the response be independent of the observer.
(b) Define the notions of material symmetry and isotropy.
(c) Show that in the case of isotropy the constitutive equation can be written in the form

$$\mathbf{T} = -\pi\mathbf{I} + \beta_0(\mathscr{I}_{\mathbf{B}})\mathbf{B} + \beta_1(\mathscr{I}_{\mathbf{B}})\mathbf{B}^{-1} \tag{23}$$

with $\mathscr{I}_{\mathbf{B}} = (\iota_1(\mathbf{B}), \iota_2(\mathbf{B}), 1)$. [The quantities π appearing in (22) and (23) are not necessarily the same.]

Two important examples of (23) are the *Mooney–Rivlin material* for which

$$\mathbf{T} = -\pi\mathbf{I} + \beta_0\mathbf{B} + \beta_1\mathbf{B}^{-1}$$

with β_0 and β_1 constants, and the *neo-Hookean material* for which

$$\mathbf{T} = -\pi\mathbf{I} + \beta_0\mathbf{B}$$

with β_0 constant.

26. SIMPLE SHEAR OF A HOMOGENEOUS AND ISOTROPIC ELASTIC BODY

Let $\mathscr{B}$ be a homogeneous, isotropic body in the shape of a cube. Consider the deformation $\mathbf{x} = \mathbf{x}(\mathbf{p})$ defined (in cartesian components) by

$$x_1 = p_1 + \gamma p_2,$$
$$x_2 = p_2,$$
$$x_3 = p_3,$$

where

$$\gamma = \tan\theta$$

is the *shearing strain* (Fig. 4). Since this deformation is homogeneous, the corresponding stress $\mathbf{T}$ is constant. Thus $(\mathbf{x}, \mathbf{T})$ represents a solution of the basic equations with body force $\mathbf{b} = \mathbf{0}$. (Cf. the discussion given at the end of the last section.) The matrix corresponding to the deformation gradient $\mathbf{F}$ is given by

$$[\mathbf{F}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

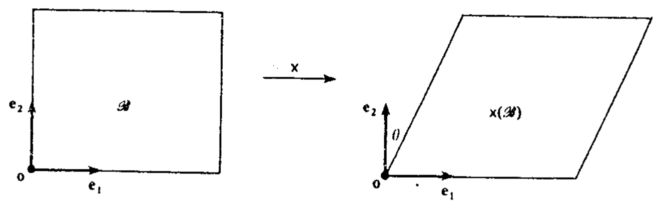

Figure 4

and the matrix for the left Cauchy–Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is

$$[\mathbf{B}] = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A simple calculation shows that

$$[\mathbf{B}]^{-1} = \begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\det(\mathbf{B} - \omega\mathbf{I}) = -\omega^3 + (3+\gamma^2)\omega^2 - (3+\gamma^2)\omega + 1,$$

and hence the list of principal invariants of $\mathbf{B}$ is

$$\mathscr{I}_{\mathbf{B}} = (3+\gamma^2, 3+\gamma^2, 1).$$

We therefore conclude from (25.12) that

$$\mathbf{T} = \beta_0(\gamma^2)\mathbf{I} + \beta_1(\gamma^2)\mathbf{B} + \beta_2(\gamma^2)\mathbf{B}^{-1},$$

or, equivalently,

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \beta_0(\gamma^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta_1(\gamma^2)\begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$+ \beta_2(\gamma^2)\begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $T_{13} = T_{23} = 0$ and

$$T_{12}/\gamma = \mu(\gamma^2), \tag{1}$$

where

$$\mu(\gamma^2) = \beta_1(\gamma^2) - \beta_2(\gamma^2) \tag{2}$$

is the *generalized shear modulus*. The result (1) asserts that the shear stress T_{12} is an *odd* function of the shear strain γ.

In linear elasticity theory (cf. page 202) the normal stresses T_{11}, T_{22}, and T_{33} in simple shear are zero. Here

$$\begin{aligned} T_{11} &= \gamma^2\beta_1 + \tau, \\ T_{22} &= \gamma^2\beta_2 + \tau, \\ T_{33} &= \tau = \tau(\gamma^2) = \beta_0 + \beta_1 + \beta_2. \end{aligned} \tag{3}$$

Thus for the normal stresses to vanish both β_1 and β_2 would have to be zero, and this, in turn, would imply that $\mu = 0$. Thus if

$$\mu(\gamma^2) \neq 0 \qquad \text{for} \quad \gamma \neq 0, \tag{4}$$

which is a reasonable assumption, then it is impossible to produce a simple shear by applying shear stresses alone. Further (2)–(4) imply that, for $\gamma \neq 0$,

$$T_{11} \neq T_{22},$$

which is the *Poynting effect*. In fact, (1)–(3) yield the important result

$$T_{11} - T_{22} = \gamma T_{12}.$$

This relation is independent of the material properties of the body; it is satisfied by every isotropic elastic body in simple shear.

Exercises

1. Show that the components of the unit normals on the slanted faces of the deformed cube are given by

$$\begin{aligned} n_1 &= \pm(1+\gamma^2)^{-1/2} \\ n_2 &= \mp\gamma(1+\gamma^2)^{-1/2}, \\ n_3 &= 0, \end{aligned}$$

and use this fact to show that

$$T_{22} - \frac{\gamma T_{12}}{1+\gamma^2} \quad \text{and} \quad \frac{T_{12}}{1+\gamma^2}$$

represent, respectively, normal and tangential components of the surface force $\mathbf{Tn}$ on these faces.

2. Consider the *uniform extension* defined by

$$\begin{aligned} x_1 &= \lambda p_1, \\ x_2 &= \omega p_2, \\ x_3 &= \omega p_3. \end{aligned}$$

Show that the corresponding stress $\mathbf{T}$ is a pure tension in the direction $\mathbf{e}_1$; i.e.,

$$\mathbf{T} = \sigma(\mathbf{e}_1 \otimes \mathbf{e}_1),$$

if and only if

$$\sigma = \beta_0 + \beta_1\lambda^2 + \beta_2\lambda^{-2},$$
$$0 = \beta_0 + \beta_1\omega^2 + \beta_2\omega^{-2},$$

where the scalar functions β_0, β_1, and β_2 are each functions of λ^2 and ω^2 (through the invariants $\mathscr{I}_{\mathbf{B}}$).

27. THE PIOLA–KIRCHHOFF STRESS

The Cauchy stress **T** measures the contact force per unit area in the *deformed* configuration. In many problems of interest—especially those involving solids—it is not convenient to work with **T**, since the deformed configuration is not known in advance. For this reason we introduce a stress tensor which gives the force measured per unit area in the *reference* configuration.

Let (x, T) be a dynamical process. Then given a part $\mathscr{P}$, we can use $(6.18)_2$ to write the total surface force on $\mathscr{P}$ at time t (see Fig. 5) as an integral over $\partial\mathscr{P}$; the result is

$$\int_{\partial\mathscr{P}_t} \mathbf{Tm}\, dA = \int_{\partial\mathscr{P}} (\det \mathbf{F})\mathbf{T}_m\mathbf{F}^{-\mathrm{T}}\mathbf{n}\, dA,$$

where **m** and **n**, respectively, are the outward unit normal fields on $\partial\mathscr{P}_t$ and $\partial\mathscr{P}$, while $\mathbf{T}_m$ is the material description of **T**. Thus if we let

$$\mathbf{S} = (\det \mathbf{F})\mathbf{T}_m\mathbf{F}^{-\mathrm{T}}, \tag{1}$$

then

$$\int_{\partial\mathscr{P}_t} \mathbf{Tm}\, dA = \int_{\partial\mathscr{P}} \mathbf{Sn}\, dA. \tag{2}$$

We call the field

$$\mathbf{S}: \mathscr{B} \times \mathbb{R} \to \mathrm{Lin}$$

defined by (1) the **Piola–Kirchhoff stress.**[1] By (2), **Sn** is the surface force measured per unit area in the reference configuration.

Similarly, if **b** is the body force corresponding to (x, T), then

$$\int_{\mathscr{P}_t} \mathbf{b}\, dV = \int_{\mathscr{P}} \mathbf{b}_m(\det \mathbf{F})\, dV = \int_{\mathscr{P}} \mathbf{b}_0\, dV, \tag{3}$$

[1] In the literature S is often referred to as the *first* Piola–Kirchhoff stress.

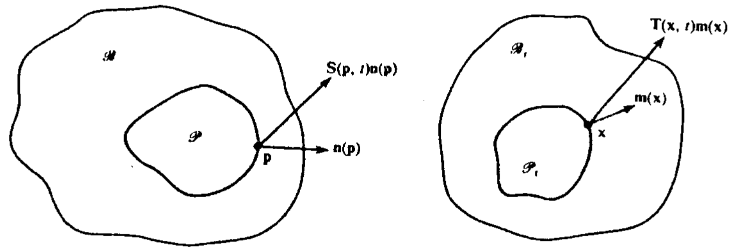

Figure 5

where

$$\mathbf{b}_0 = (\det \mathbf{F})\mathbf{b}_m. \tag{4}$$

We call $\mathbf{b}_0$ the **reference body force**; $\mathbf{b}_0$ gives the body force measured per unit volume in the reference configuration.

Proposition. *The Piola–Kirchhoff stress* **S** *satisfies the balance equations*

$$\int_{\partial\mathscr{P}} \mathbf{Sn}\, dA + \int_{\mathscr{P}} \mathbf{b}_0\, dV = \int_{\mathscr{P}} \ddot{\mathbf{x}}\rho_0\, dV,$$
$$\int_{\partial\mathscr{P}} (\mathbf{x} - \mathbf{o}) \times \mathbf{Sn}\, dA + \int_{\mathscr{P}} (\mathbf{x} - \mathbf{o}) \times \mathbf{b}_0\, dV = \int_{\mathscr{P}} (\mathbf{x} - \mathbf{o}) \times \ddot{\mathbf{x}}\rho_0\, dV \tag{5}$$

for every part $\mathscr{P}$.

Proof. By (12.7),

$$\int_{\mathscr{P}_t} \dot{\mathbf{v}}\rho\, dV = \int_{\mathscr{P}} \ddot{\mathbf{x}}\rho_0\, dV.$$

This relation, (2), and (3), when combined with balance of linear momentum $(14.2)_1$, imply $(5)_1$. Similarly, $(6.18)_3$, (12.7), $(14.2)_2$, and an analogous argument yields $(5)_2$. □

Proposition. **S** *satisfies the field equations*

$$\mathrm{Div}\,\mathbf{S} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{x}},$$
$$\mathbf{S}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{S}^{\mathrm{T}}. \tag{6}$$

Proof. By $(5)_1$ and the divergence theorem,

$$\int_{\mathscr{P}} (\mathrm{Div}\,\mathbf{S} + \mathbf{b}_0 - \rho_0\ddot{\mathbf{x}})\, dV = \mathbf{0},$$

and, since this relation must be satisfied for every part $\mathscr{P}$, $(6)_1$ must hold. Further, (1) implies

$$\mathbf{T}_m = (\det \mathbf{F})^{-1}\mathbf{S}\mathbf{F}^{\mathrm{T}},$$

and $(6)_2$ follows from the symmetry of **T**. □

It is important to note that, by $(6)_2$, **S** *generally is not symmetric.*

Next, by the symmetry of **T**, (a) of the proposition on page 6, $(8.8)_1$, and $(1.5)_2$,

$$\mathbf{T}\cdot\mathbf{D} = \mathbf{T}\cdot\mathbf{L} = \mathbf{T}\cdot(\dot{\mathbf{F}}_o\mathbf{F}_o^{-1}) = (\mathbf{T}\mathbf{F}_o^{-\mathrm{T}})\cdot\dot{\mathbf{F}}_o,$$

and therefore

$$\int_{\mathscr{P}_t}\mathbf{T}\cdot\mathbf{D}\,dV = \int_{\mathscr{P}}(\det\mathbf{F})(\mathbf{T}_m\mathbf{F}^{-\mathrm{T}})\cdot\dot{\mathbf{F}}\,dV = \int_{\mathscr{P}}\mathbf{S}\cdot\dot{\mathbf{F}}\,dV.$$

Thus the *stress power* of $\mathscr{P}$ at time t is given by

$$\int_{\mathscr{P}}\mathbf{S}\cdot\dot{\mathbf{F}}\,dV.$$

Also, (1), (4), $(6.18)_1$, and $(6.14)_1$ imply

$$\begin{aligned}\int_{\partial\mathscr{P}_t}\mathbf{T}\mathbf{m}\cdot\mathbf{v}\,dA &= \int_{\partial\mathscr{P}_t}(\mathbf{T}\mathbf{v})\cdot\mathbf{m}\,dA = \int_{\partial\mathscr{P}}\mathbf{S}\mathbf{n}\cdot\dot{\mathbf{x}}\,dA,\\ \int_{\mathscr{P}_t}\mathbf{b}\cdot\mathbf{v}\,dV &= \int_{\mathscr{P}}\mathbf{b}_0\cdot\dot{\mathbf{x}}\,dV,\end{aligned}\tag{7}$$

while (12.7) yields

$$\int_{\mathscr{P}_t}\frac{\mathbf{v}^2}{2}\rho\,dV = \int_{\mathscr{P}}\frac{\dot{\mathbf{x}}^2}{2}\rho_0\,dV.$$

We therefore have the following alternative version of (15.2).

Theorem of Power Expended. *Given any part* $\mathscr{P}$,

$$\boxed{\int_{\partial\mathscr{P}}\mathbf{S}\mathbf{n}\cdot\dot{\mathbf{x}}\,dA + \int_{\mathscr{P}}\mathbf{b}_0\cdot\dot{\mathbf{x}}\,dV = \int_{\mathscr{P}}\mathbf{S}\cdot\dot{\mathbf{F}}\,dV + \frac{d}{dt}\int_{\mathscr{P}}\frac{\dot{\mathbf{x}}^2}{2}\rho_0\,dV.}\tag{8}$$

The results established thus far are consequences of balance of momentum and are independent of the particular constitution of the body.

Assume now that the body is elastic, so that by (1) and (25.1), **S** is given by a constitutive equation of the form

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F})\tag{9}$$

with

$$\hat{\mathbf{S}}(\mathbf{F}) = (\det\mathbf{F})\hat{\mathbf{T}}(\mathbf{F})\mathbf{F}^{-\mathrm{T}}.\tag{10}$$

(As before, we do not mention the explicit dependence of $\hat{\mathbf{S}}$ on the material point **p**.) Choose $\mathbf{Q}\in\mathrm{Orth}^+$. Then

$$\det(\mathbf{QF}) = \det\mathbf{F},\qquad (\mathbf{QF})^{-\mathrm{T}} = \mathbf{Q}\mathbf{F}^{-\mathrm{T}},$$

so that (25.3) and (10) imply

$$\hat{\mathbf{S}}(\mathbf{QF}) = \det(\mathbf{QF})\hat{\mathbf{T}}(\mathbf{QF})(\mathbf{QF})^{-\mathrm{T}} = (\det\mathbf{F})\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{F}^{-\mathrm{T}} = \mathbf{Q}\hat{\mathbf{S}}(\mathbf{F}).$$

Thus

$$\hat{\mathbf{S}}(\mathbf{QF}) = \mathbf{Q}\hat{\mathbf{S}}(\mathbf{F})\tag{11}$$

for every $\mathbf{F}\in\mathrm{Lin}^+$ and $\mathbf{Q}\in\mathrm{Orth}^+$. This relation expresses the requirement that the response be independent of the observer; it can be used to deduce constitutive equations for **S** analogous to (25.5). It is easier, however, to proceed directly from $(25.5)_3$. Indeed, $(25.5)_3$, (9), and (10) imply that

$$\mathbf{S} = (\det\mathbf{F})\mathbf{F}\overline{\mathbf{T}}(\mathbf{C}),\tag{12}$$

and if we define

$$\bar{\mathbf{S}}(\mathbf{C}) = \sqrt{\det\mathbf{C}}\,\overline{\mathbf{T}}(\mathbf{C}),\tag{13}$$

then, since

$$\det\mathbf{C} = \det(\mathbf{F}^{\mathrm{T}}\mathbf{F}) = (\det\mathbf{F})^2,$$

(12) reduces to

$$\boxed{\mathbf{S} = \mathbf{F}\bar{\mathbf{S}}(\mathbf{C}).}\tag{14}$$

We leave it as an exercise to show that this constitutive equation is independent of the observer.

Note that, by (13) and the smoothness of $\overline{\mathbf{T}}$ (cf. the corollary on page 166), $\bar{\mathbf{S}}$ *is a smooth mapping of Psym into Sym.*

In view of $(6)_1$ and (14), we can rewrite the basic system of field equations (25.13)–(25.15) in the form

$$\boxed{\begin{gathered}\mathbf{S} = \mathbf{F}\bar{\mathbf{S}}(\mathbf{C}),\\ \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F},\qquad \mathbf{F} = \nabla\mathbf{x}\\ \mathrm{Div}\,\mathbf{S} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{x}}.\end{gathered}}\tag{15}$$

The relation $(6)_2$ follows automatically from the fact that $\bar{\mathbf{S}}$ has symmetric values. Also, since the density enters (15) only through its reference value ρ_0, which is assumed known a priori, balance of mass (25.15) need not be included in the list of field equations.

All of the fields in (15) have $\mathscr{B} \times \mathbb{R}$ as their domain, and the operator Div is with respect to the material point **p** in $\mathscr{B}$. In contrast, some of the fields in (25.13)–(25.15) are defined on the trajectory $\mathscr{T}$, and, more importantly, the operator div is with respect to the place **x** in the current configuration. For this reason the formulation (15) is more convenient than (25.13)–(25.15) in problems for which the trajectory is not known in advance.

The boundary-value problems of finite elasticity are obtained by adjoining to (15) suitable initial and boundary conditions. As *initial conditions* one usually specifies the initial motion and velocity:

$$\mathrm{x}(\mathbf{p}, 0) = \mathrm{x}_0(\mathbf{p}), \qquad \dot{\mathrm{x}}(\mathbf{p}, 0) = \mathbf{v}_0(\mathbf{p}) \tag{16}$$

with x_0 and $\mathbf{v}_0$ prescribed functions on $\mathscr{B}$. To specify the *boundary conditions* one considers complementary regular[1] subsets $\mathscr{S}_1$ and $\mathscr{S}_2$ of $\partial\mathscr{B}$ (so that

$$\partial\mathscr{B} = \mathscr{S}_1 \cup \mathscr{S}_2, \qquad \mathring{\mathscr{S}}_1 \cap \mathring{\mathscr{S}}_2 = \varnothing,$$

where $\mathring{\mathscr{S}}_\alpha$ is the relative interior of $\mathscr{S}_\alpha$) and then prescribes the motion on $\mathscr{S}_1$, the surface traction on $\mathscr{S}_2$:

$$\mathrm{x} = \hat{\mathrm{x}} \quad \text{on} \quad \mathscr{S}_1 \times [0, \infty), \qquad \mathbf{Sn} = \hat{\mathbf{s}} \quad \text{on} \quad \mathscr{S}_2 \times [0, \infty) \tag{17}$$

with $\hat{\mathrm{x}}$ and $\hat{\mathbf{s}}$ prescribed vector fields on $\mathscr{S}_1 \times [0, \infty)$ and $\mathscr{S}_2 \times [0, \infty)$, respectively.

Another form of boundary condition arises when one specifies the surface traction **Tm** on the deformed surface $\mathrm{x}(\mathscr{S}_2, t)$. A simple example of this type of condition arises when one considers the effects of a uniform pressure π_0. Here

$$\mathbf{T}(\mathrm{x}, t)\mathbf{m}(\mathrm{x}) = -\pi_0 \mathbf{m}(\mathrm{x})$$

for all $\mathrm{x} \in \mathrm{x}(\mathscr{S}_2, t)$ and all $t \geq 0$. We can also write this condition as a restriction on the Piola–Kirchhoff stress S; the result is (cf. Exercise 6)

$$\mathbf{Sn} = -\pi_0(\det \mathbf{F})\mathbf{F}^{-\mathrm{T}}\mathbf{n} \qquad \text{on} \quad \mathscr{S}_2 \times [0, \infty). \tag{18}$$

Clearly, this is not a special case of (17) because $\mathbf{F}(\mathbf{p}, t)$ is not known in advance. We can, of course, generalize (17) to include this case by allowing $\hat{\mathbf{s}}$ to be a function $\hat{\mathbf{s}}(\mathbf{F}, \mathbf{p}, t)$ of **F**, **p**, and t, rather than a function of **p** and t only. [Actually, one can show that the combination $(\det \mathbf{F})\mathbf{F}^{-\mathrm{T}}\mathbf{n}$ depends only on the tangential gradient of x on $\mathscr{S}_2$.]

[1] Cf. Kellogg [1]; Gurtin [1, p. 14]. Roughly speaking, each $\mathscr{S}_\alpha$ is a relatively closed, piecewise smooth surface with boundary a piecewise smooth closed curve.

In the *statical* theory all of the fields are independent of time and the underlying boundary-value problem consists in finding a deformation **f** that satisfies the field equations

$$\begin{gathered} \mathbf{S} = \mathbf{F}\tilde{\mathbf{S}}(\mathbf{C}), \\ \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \qquad \mathbf{F} = \nabla \mathbf{f}, \\ \operatorname{Div} \mathbf{S} + \mathbf{b}_0 = \mathbf{0}, \end{gathered} \tag{19}$$

and the boundary conditions

$$\mathbf{f} = \hat{\mathbf{f}} \quad \text{on} \quad \mathscr{S}_1, \qquad \mathbf{Sn} = \hat{\mathbf{s}} \quad \text{on} \quad \mathscr{S}_2 \tag{20}$$

with $\hat{\mathbf{f}}$ and $\hat{\mathbf{s}}$ prescribed functions on $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively.

When tractions are prescribed over the entire boundary, that is, when (20) has the form

$$\mathbf{Sn} = \hat{\mathbf{s}} \qquad \text{on} \quad \partial\mathscr{B}, \tag{21}$$

$(5)_1$ implies that

$$\int_{\partial\mathscr{B}} \hat{\mathbf{s}}\, dA + \int_{\mathscr{B}} \mathbf{b}_0\, dV = \mathbf{0}.$$

This relation involves only the data and furnishes a *necessary condition* for the existence of a solution. On the other hand, $(5)_2$ yields

$$\int_{\partial\mathscr{B}} (\mathbf{f} - \mathbf{o}) \times \hat{\mathbf{s}}\, dA + \int_{\mathscr{B}} (\mathbf{f} - \mathbf{o}) \times \mathbf{b}_0\, dV = \mathbf{0},$$

and, because of the presence of the deformation **f**, is not a restriction on the data, but rather a compatibility condition which will automatically be satisfied by any solution of the boundary-value problem (19), (21). This difference between force and moment balance is illustrated in Fig. 6. As long

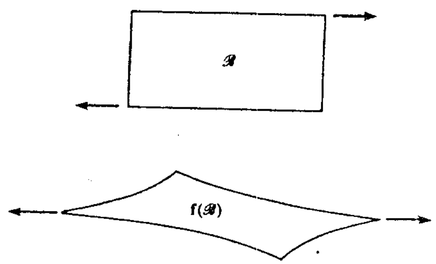

Figure 6

as the forces acting on $\mathscr{B}$ obey balance of forces, we expect a solution, even though moments are not balanced in the reference configuration; the body will simply deform to insure balance of moments in the *deformed* configuration.

Exercises

1. Give the details of the proof of $(5)_2$.
2. Show that

$$\hat{\mathbf{S}}(\mathbf{FQ}) = \hat{\mathbf{S}}(\mathbf{F})\mathbf{Q} \tag{22}$$

for every $\mathbf{Q}$ in the symmetry group $\mathscr{G}$ and every $\mathbf{F} \in \mathrm{Lin}^+$. Show further that $\hat{\mathbf{S}}$ and $\bar{\mathbf{S}}$ are invariant under $\mathscr{G}$.

3. Use (11) to show that

$$\hat{\mathbf{S}}(\mathbf{Q}) \cdot \dot{\mathbf{Q}} = 0 \tag{23}$$

for any smooth function (of time) $\mathbf{Q}$ with values in Orth^+.

4. Let $\mathscr{B}$ be bounded. Consider a dynamical process and define

$$\varphi = \frac{1}{2}\int_{\mathscr{B}} \mathbf{u}^2 \rho_0 \, dV,$$

where $\mathbf{u}(\mathbf{p}, t) = \mathbf{x}(\mathbf{p}, t) - \mathbf{p}$ is the displacement. Show that

$$\ddot{\varphi} = \int_{\mathscr{B}} \dot{\mathbf{u}}^2 \rho_0 \, dV - \int_{\mathscr{B}} \mathbf{S} \cdot \nabla \mathbf{u} \, dV + \int_{\partial\mathscr{B}} \mathbf{u} \cdot \mathbf{Sn} \, dA.$$

5. Show that the constitutive equation (14) is independent of the observer.
6. Equation $(6.18)_2$ holds for any sufficiently regular subsurface $\mathscr{S}$ of $\partial\mathscr{P}$:

$$\int_{\mathbf{f}(\mathscr{S})} \mathbf{T}(\mathbf{x})\mathbf{m}(\mathbf{x}) \, dA_{\mathbf{x}} = \int_{\mathscr{S}} \mathbf{T}(\mathbf{f}(\mathbf{p}))\mathbf{G}(\mathbf{p})\mathbf{n}(\mathbf{p}) \, dA_{\mathbf{p}}.$$

Use this fact to establish (18).

28. HYPERELASTIC BODIES

Thus far the only general restriction we have placed on constitutive equations is the requirement that material response be independent of the observer. Thermodynamics also serves to constrain constitutive behavior, and a standard thermodynamic axiom (consistent with a purely mechanical theory) is the requirement of nonnegative work in closed processes. We now study the implications of this axiom for elastic bodies.

Let $(\mathbf{x}, \mathbf{T})$ be a dynamical process, and let $\mathbf{b}$ be the corresponding body force. Then given any part $\mathscr{P}$, the **work** on $\mathscr{P}$ during a time interval $[t_0, t_1]$ is given by

$$\int_{t_0}^{t_1}\left\{\int_{\partial\mathscr{P}_t} \mathbf{Tn} \cdot \mathbf{v} \, dA + \int_{\mathscr{P}_t} \mathbf{b} \cdot \mathbf{v} \, dV\right\} dt,$$

or equivalently, in view of (27.7), by

$$\int_{t_0}^{t_1}\left\{\int_{\partial\mathscr{P}} \mathbf{Sn} \cdot \dot{\mathbf{x}} \, dA + \int_{\mathscr{P}} \mathbf{b}_0 \cdot \dot{\mathbf{x}} \, dV\right\} dt \tag{1}$$

with $\mathbf{S}$ the Piola–Kirchhoff stress and $\mathbf{b}_0$ the reference body force. Let us agree to call $(\mathbf{x}, \mathbf{T})$ **closed** during $[t_0, t_1]$ provided

$$\mathbf{x}(\mathbf{p}, t_0) = \mathbf{x}(\mathbf{p}, t_1), \qquad \dot{\mathbf{x}}(\mathbf{p}, t_0) = \dot{\mathbf{x}}(\mathbf{p}, t_1) \tag{2}$$

for all $\mathbf{p} \in \mathscr{B}$. If we integrate (27.8) between t_0 and t_1 and use $(2)_2$, we see that for processes of this type (1) reduces to

$$\int_{t_0}^{t_1}\int_{\mathscr{P}} \mathbf{S} \cdot \dot{\mathbf{F}} \, dV \, dt.$$

We consider now an *elastic body* and restrict our attention to dynamical processes belonging to the constitutive class $\mathscr{C}$ of the body. For convenience, we will use the term **process** as a synonym for "dynamical process in $\mathscr{C}$." We say that the **work is nonnegative in closed processes** if given any part $\mathscr{P}$ and any time interval $[t_0, t_1]$,

$$\int_{t_0}^{t_1}\int_{\mathscr{P}} \mathbf{S} \cdot \dot{\mathbf{F}} \, dV \, dt \geq 0 \tag{3}$$

in any process which is closed during $[t_0, t_1]$.

Proposition. *The work is nonnegative in closed processes if and only if given any* $\mathbf{p} \in \mathscr{B}$ *and any time interval* $[t_0, t_1]$,

$$\int_{t_0}^{t_1} \mathbf{S}(\mathbf{p}, t) \cdot \dot{\mathbf{F}}(\mathbf{p}, t) \, dt \geq 0 \tag{4}$$

in any process which is closed during $[t_0, t_1]$.

Proof. Let $\lambda(\mathbf{p})$ denote the left side of (4). Clearly, $\lambda(\mathbf{p}) \geq 0$ for all $\mathbf{p}$ implies (3). Conversely, assume that (3) holds for every part $\mathscr{P}$. Then, by interchanging the integrations in (3), we may conclude that

$$\int_{\mathscr{P}} \lambda \, dV \geq 0$$

for every part $\mathscr{P}$. In view of the localization theorem (5.1), this implies that $\lambda(\mathbf{p}) \geq 0$ for all $\mathbf{p}$. □

Our next step will be to describe the class of elastic materials for which the work is nonnegative in closed processes. Anticipating the result, we introduce the following definition: An elastic body is **hyperelastic** if the Piola–Kirchhoff stress $\hat{\mathbf{S}}(\mathbf{F}, \mathbf{p})$ is the derivative of a scalar function $\hat{\sigma}(\mathbf{F}, \mathbf{p})$; that is,

$$\hat{\mathbf{S}}(\mathbf{F}, \mathbf{p}) = D\hat{\sigma}(\mathbf{F}, \mathbf{p}), \tag{5}$$

where the derivative is with respect to $\mathbf{F}$ holding $\mathbf{p}$ fixed. The scalar function

$$\hat{\sigma}: \mathrm{Lin}^+ \times \mathscr{B} \to \mathbb{R}$$

is called the **strain-energy density**. (Note that $\hat{\mathbf{S}}$ determines $\hat{\sigma}$ only up to an arbitrary function of $\mathbf{p}$ alone.)

For (5) to make sense we must interpret the derivative $D\hat{\sigma}$ as a tensor. By definition $D\hat{\sigma}(\mathbf{F}, \mathbf{p})$ is a linear mapping of Lin into $\mathbb{R}$, and hence $D\hat{\sigma}(\mathbf{F}, \mathbf{p})[\mathbf{A}]$ can be written as the inner product[1] of a tensor $\mathbf{K}$ with $\mathbf{A}$. We here identify $D\hat{\sigma}(\mathbf{F}, \mathbf{p})$ with $\mathbf{K}$ and write

$$D\hat{\sigma}(\mathbf{F}, \mathbf{p}) \cdot \mathbf{A} \quad \text{in place of} \quad D\hat{\sigma}(\mathbf{F}, \mathbf{p})[\mathbf{A}].$$

With this interpretation $D\hat{\sigma}(\mathbf{F}, \mathbf{p})$ is a *tensor* and (5) has meaning. The components of this tensor are

$$D\hat{\sigma}(\mathbf{F}, \mathbf{p})_{ij} = \frac{\partial}{\partial F_{ij}} \hat{\sigma}(\mathbf{F}, \mathbf{p}).$$

Theorem. *An elastic body is hyperelastic if and only if the work is nonnegative in closed processes.*

Proof. Assume that $\mathscr{B}$ is hyperelastic. Consider a closed process during $[t_0, t_1]$. Then

$$\frac{d}{dt} \hat{\sigma}(\mathbf{F}(\mathbf{p}, t), \mathbf{p}) = D\hat{\sigma}(\mathbf{F}(\mathbf{p}, t), \mathbf{p}) \cdot \dot{\mathbf{F}}(\mathbf{p}, t) = \hat{\mathbf{S}}(\mathbf{F}(\mathbf{p}, t), \mathbf{p}) \cdot \dot{\mathbf{F}}(\mathbf{p}, t), \tag{6}$$

and, since

$$\mathbf{F}(\mathbf{p}, t_0) = \mathbf{F}(\mathbf{p}, t_1)$$

[cf. $(2)_1$],

$$\begin{aligned} \int_{t_0}^{t_1} \mathbf{S}(\mathbf{p}, t) \cdot \dot{\mathbf{F}}(\mathbf{p}, t)\, dt &= \int_{t_0}^{t_1} \frac{d}{dt} \hat{\sigma}(\mathbf{F}(\mathbf{p}, t), \mathbf{p})\, dt \\ &= \hat{\sigma}(\mathbf{F}(\mathbf{p}, t_1), \mathbf{p}) - \hat{\sigma}(\mathbf{F}(\mathbf{p}, t_0), \mathbf{p}) = 0. \end{aligned} \tag{7}$$

Thus the work is nonnegative (in fact, zero) in closed processes.

Conversely, assume that the work is nonnegative in closed processes.

[1] The representation theorem for linear forms (page 2) is valid on any finite-dimensional inner-product space.

Assertion 1. Let $\mathbf{F}: \mathbb{R} \to \mathrm{Lin}^+$ be a C^2 function with

$$\mathbf{F}(t_0) = \mathbf{F}(t_1), \qquad \dot{\mathbf{F}}(t_0) = \dot{\mathbf{F}}(t_1). \tag{8}$$

Then

$$\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}) \cdot \dot{\mathbf{F}}\, dt = 0 \tag{9}$$

(Here and in what follows we suppress the dependence of $\hat{\mathbf{S}}$ on $\mathbf{p}$.)

Proof. Let $\mathscr{F} = \mathscr{F}(t_0, t_1)$ denote the set of all C^2 functions $\mathbf{F}: \mathbb{R} \to \mathrm{Lin}^+$ which satisfy (8). As our first step we observe that given any $\mathbf{F} \in \mathscr{F}$,

$$\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}) \cdot \dot{\mathbf{F}}\, dt \geq 0, \tag{10}$$

for if $\mathbf{x}$ is the motion

$$\mathbf{x}(\mathbf{p}, t) = \mathbf{p}_0 + \mathbf{F}(t)(\mathbf{p} - \mathbf{p}_0),$$

then the corresponding process is closed during $[t_0, t_1]$ and (4) implies (10).

Choose $\mathbf{F} \in \mathscr{F}$ and let $\mathbf{F}^*: \mathbb{R} \to \mathrm{Lin}^+$ be defined by

$$\mathbf{F}^*(t) = \mathbf{F}(t_0 + t_1 - t),$$

so that $\mathbf{F}^*$ represents the reversal (in time) of $\mathbf{F}$. Then, by (8),

$$\mathbf{F}^*(t_0) = \mathbf{F}(t_1) = \mathbf{F}(t_0) = \mathbf{F}^*(t_1), \tag{11}$$

and, since

$$\dot{\mathbf{F}}^*(t) = \frac{d}{dt} \mathbf{F}(t_0 + t_1 - t) = -\dot{\mathbf{F}}(t_0 + t_1 - t), \tag{12}$$

it follows that

$$\dot{\mathbf{F}}^*(t_0) = -\dot{\mathbf{F}}(t_1) = -\dot{\mathbf{F}}(t_0) = \dot{\mathbf{F}}^*(t_1). \tag{13}$$

By (11) and (13), $\mathbf{F}^* \in \mathscr{F}$; thus we may use (10) with $\mathbf{F}$ replaced by $\mathbf{F}^*$ and (12) as follows:

$$\begin{aligned} 0 \leq \int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}^*) \cdot \dot{\mathbf{F}}^*\, dt &= -\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}(t_0 + t_1 - t)) \cdot \dot{\mathbf{F}}(t_0 + t_1 - t)\, dt \\ &= -\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}(\tau)) \cdot \dot{\mathbf{F}}(\tau)\, d\tau. \end{aligned}$$

This inequality is clearly compatible with (10) only if (9) holds.

Our next step will be to show that we can drop condition $(8)_2$ without affecting the validity of (9). We do this by applying (9) to a family of functions which satisfy both of (8), but whose limit satisfies only $(8)_1$. This family is constructed in

Assertion 2. Let $\mathbf{F}$ be a piecewise smooth,[1] closed curve in Lin^+; that is, let $\mathbf{F}: [0, 1] \to \text{Lin}^+$ be piecewise smooth and satisfy

$$\mathbf{F}(0) = \mathbf{F}(1) \equiv \mathbf{A}.$$

Then there exists a one-parameter family $\mathbf{F}_\delta$ $(0 < \delta < \delta_0)$ of C^∞ functions $\mathbf{F}_\delta: \mathbb{R} \to \text{Lin}^+$ such that:

(a) $\mathbf{F}_\delta = \mathbf{A}$ on $(-\infty, -\delta] \cup [1 + \delta, \infty)$;
(b) $|\mathbf{F}_\delta|$ and $|\dot{\mathbf{F}}_\delta|$ are bounded on $\mathbb{R}$ with bounds independent of δ;
(c) as $\delta \to 0$, $\mathbf{F}_\delta \to \mathbf{F}$ everywhere on $[0, 1]$, while $\dot{\mathbf{F}}_\delta \to \dot{\mathbf{F}}$ at points of continuity of $\dot{\mathbf{F}}$ on $(0, 1)$.

Proof. We use the Friedrichs–Sobolev method of mollifiers to construct the family $\mathbf{F}_\delta$. For each $\delta > 0$ let $\rho_\delta: \mathbb{R} \to \mathbb{R}$ be a C^∞ function with the following properties:

(i) $\rho_\delta \geq 0$;
(ii) $\rho_\delta(t) = 0$ whenever $|t| \geq \delta$;
(iii) $\displaystyle\int_{-\infty}^{\infty} \rho_\delta(t)\, dt = 1.$

An example of such a family of functions is furnished by

$$\rho_\delta(t) = \begin{cases} \kappa_\delta \exp\left(-\dfrac{1}{\delta^2 - t^2}\right), & |t| < \delta \\ 0, & |t| \geq \delta \end{cases}$$

with κ_δ chosen to insure satisfaction of (iii).

Using ρ_δ we define a C^∞ family $\mathbf{F}_\delta$ as

$$\mathbf{F}_\delta(t) = \int_{-\infty}^{\infty} \rho_\delta(t - \tau)\mathbf{F}(\tau)\, d\tau \tag{14}$$

for all $t \in \mathbb{R}$, where we have extended the domain of $\mathbf{F}$ to $\mathbb{R}$ by defining

$$\mathbf{F} = \mathbf{A} \quad \text{on} \quad (-\infty, 0) \cup (1, \infty).$$

By (ii) and (iii), for $t \leq -\delta$ or $t \geq 1 + \delta$,

$$\mathbf{F}_\delta(t) = \int_{t-\delta}^{t+\delta} \rho_\delta(t - \tau)\mathbf{A}\, d\tau = \mathbf{A},$$

and (a) follows.

[1] $\mathbf{F}$ is continuous on [0, 1], while $\dot{\mathbf{F}}$ exists and is continuous at all but a finite number of points where it suffers, at most, jump discontinuities.

Next, (14) and an integration by parts using (ii) yield

$$\dot{\mathbf{F}}_\delta(t) = \int_{-\infty}^{\infty} \dot{\rho}_\delta(t - \tau)\mathbf{F}(\tau)\, d\tau = \int_{-\infty}^{\infty} \left[-\frac{\partial}{\partial\tau}\rho_\delta(t - \tau)\right]\mathbf{F}(\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} \rho_\delta(t - \tau)\dot{\mathbf{F}}(\tau)\, d\tau. \tag{15}$$

In view of (i) and (iii), we conclude from (14) and (15) that

$$|\mathbf{F}_\delta(t)| \leq \sup_{[0,1]} |\mathbf{F}| \int_{-\infty}^{\infty} \rho_\delta(t - \tau)\, d\tau = \sup_{[0,1]} |\mathbf{F}|,$$
$$|\dot{\mathbf{F}}_\delta(t)| \leq \sup_{[0,1]} |\dot{\mathbf{F}}|,$$

and these bounds yield (b).

Finally, by (14), (ii), and (iii),

$$\mathbf{F}_\delta(t) - \mathbf{F}(t) = \int_{-\infty}^{\infty} \rho_\delta(t - \tau)\mathbf{F}(\tau)\, d\tau - \mathbf{F}(t)\int_{-\infty}^{\infty} \rho_\delta(t - \tau)\, d\tau$$
$$= \int_{t-\delta}^{t+\delta} \rho_\delta(t - \tau)[\mathbf{F}(\tau) - \mathbf{F}(t)]\, d\tau;$$

hence (ii) and (iii) imply that

$$|\mathbf{F}_\delta(t) - \mathbf{F}(t)| \leq \sup_{\tau \in (t-\delta, t+\delta)} |\mathbf{F}(\tau) - \mathbf{F}(t)|.$$

Similarly, using (15),

$$|\dot{\mathbf{F}}_\delta(t) - \dot{\mathbf{F}}(t)| \leq \sup_{\tau \in (t-\delta, t+\delta)} |\dot{\mathbf{F}}(\tau) - \dot{\mathbf{F}}(t)|,$$

and the last two inequalities imply (c). Moreover, since $\mathbf{F}$ is uniformly continuous on $\mathbb{R}$ ($\mathbf{F}$ is continuous on $\mathbb{R}$ and constant outside a compact interval), $\mathbf{F}_\delta \to \mathbf{F}$ uniformly on $\mathbb{R}$; thus, since $\det \mathbf{F} > 0$, there must exist a $\delta_0 > 0$ such that $\det \mathbf{F}_\delta > 0$ on $\mathbb{R}$ for $0 < \delta < \delta_0$. For this range of δ, $\mathbf{F}_\delta$ can be considered as a mapping of $\mathbb{R}$ into Lin^+. This completes the proof of Assertion 2.

Our last step is

Assertion 3. The body is hyperelastic.

Proof. Let $\mathbf{F}$ be an arbitrary piecewise smooth, closed curve in Lin^+, and let $\mathbf{F}_\delta$ $(0 < \delta < \delta_0)$ be the family established in Assertion 2. By (a) of Assertion 2,

$$\mathbf{F}_\delta(-1) = \mathbf{F}_\delta(2), \qquad \dot{\mathbf{F}}_\delta(-1) = \dot{\mathbf{F}}_\delta(2)$$

as long as $\delta < 1$. Let $\delta_1 = \min(1, \delta_0)$. Then for $0 < \delta < \delta_1$, each $\mathbf{F}_\delta$ satisfies the hypotheses of Assertion 1 (with $t_0 = -1$ and $t_1 = 2$); hence

$$\int_{-1}^{2} \varphi_\delta(t)\, dt = 0, \tag{16}$$

where

$$\varphi_\delta = \hat{\mathbf{S}}(\mathbf{F}_\delta) \cdot \dot{\mathbf{F}}_\delta. \tag{17}$$

By (a) of Assertion 2 the integral from -1 to $-\delta$ and from $1 + \delta$ to 2 vanishes; we can therefore rewrite (16) as

$$\int_0^1 \varphi_\delta\, dt + \int_{-\delta}^{0} \varphi_\delta\, dt + \int_1^{1+\delta} \varphi_\delta\, dt = 0. \tag{18}$$

By (b) and (c) of Assertion 2 in conjunction with Lebesgue's dominated convergence theorem[1] (recall that $\dot{\mathbf{F}}$ has at most a finite number of discontinuities),

$$\int_0^1 \varphi_\delta\, dt \to \int_0^1 \hat{\mathbf{S}}(\mathbf{F}) \cdot \dot{\mathbf{F}}\, dt.$$

On the other hand, (b) of Assertion 2 and (17) imply that the remaining integrals in (18) converge to zero as $\delta \to 0$. Thus

$$\int_0^1 \hat{\mathbf{S}}(\mathbf{F}) \cdot \dot{\mathbf{F}}\, dt = 0.$$

What we have shown is that the integral of $\hat{\mathbf{S}}$ over any piecewise smooth, closed curve in Lin^+ vanishes. Since Lin^+ is an open, connected subset of (the vector space) Lin, a standard theorem[2] in vector analysis tells us that $\hat{\mathbf{S}}$ is the derivative of a smooth scalar function $\hat{\sigma}$ on Lin^+. Thus (5) holds and the proof is complete. □

Hyperelastic materials have several interesting properties; an example is the following direct consequence of (7).

Proposition. *For a hyperelastic material the work is zero in closed processes.*

If we write

$$\sigma(\mathbf{p}, t) = \hat{\sigma}(\mathbf{F}(\mathbf{p}, t), t)$$

for the values of $\hat{\sigma}$ in a process, then (6) implies

$$\dot{\sigma} = \mathbf{S} \cdot \dot{\mathbf{F}},$$

[1] Cf., e.g., Natanson [1, p. 161].

[2] Cf., e.g., Nickerson, Spencer, and Steenrod [1, Theorem 8.4].

and the theorem of power expended (27.8) has the following important corollary.

Theorem (Balance of energy for hyperelastic materials). *Each dynamical process for a hyperelastic body satisfies the energy equation*

$$\int_{\partial\mathscr{P}} \mathbf{Sn} \cdot \dot{\mathbf{x}}\, dA + \int_{\mathscr{P}} \mathbf{b}_0 \cdot \dot{\mathbf{x}}\, dV = \frac{d}{dt}\int_{\mathscr{P}} \left(\sigma + \rho_0 \frac{\dot{\mathbf{x}}^2}{2}\right) dV$$

for each part $\mathscr{P}$. Here $\mathbf{S}$ *is the Piola–Kirchhoff stress, ρ_0 is the reference density, and* $\mathbf{b}_0$ *is the reference body force* (27.4).

The term

$$\int_{\mathscr{P}} \sigma\, dV$$

represents the *strain energy* of $\mathscr{P}$. The energy equation asserts that *the power expended on $\mathscr{P}$ must equal the rate at which the total energy of $\mathscr{P}$ is changing.*

As a direct consequence of the above theorem we have the following important

Corollary (Conservation of energy). *Assume that the body is finite and hyperelastic. Consider a dynamical process for the body corresponding to body force* $\mathbf{b} = \mathbf{0}$, *and suppose that*

$$\mathbf{Sn} \cdot \dot{\mathbf{x}} = 0 \quad \text{on} \quad \partial\mathscr{B}$$

for all time. Then the total energy is constant:

$$\int_{\mathscr{B}} \left(\sigma + \rho_0 \frac{\dot{\mathbf{x}}^2}{2}\right) dV = \text{const.}$$

Exercises

1. Consider a hyperelastic body with strain-energy density $\hat{\sigma}$. (For convenience, we suppress dependence on the material point $\mathbf{p}$.)
   (a) Use (27.23) and (5) to show that
   $$\hat{\sigma}(\mathbf{Q}) = \hat{\sigma}(\mathbf{I}) \tag{19}$$
   for every $\mathbf{Q} \in \mathrm{Orth}^+$. (Here you may use the connectivity of Orth^+ to insure the existence of a smooth curve $\mathbf{R}: [0, 1] \to \mathrm{Orth}^+$ with $\mathbf{R}(0) = \mathbf{I}$ and $\mathbf{R}(1) = \mathbf{Q}$.)
   (b) Use (27.11) and (5) to show that
   $$\hat{\sigma}(\mathbf{QF}) = \hat{\sigma}(\mathbf{F})$$
   for every $\mathbf{Q} \in \mathrm{Orth}^+$ and $\mathbf{F} \in \mathrm{Lin}^+$.

(c) Use (27.22) and (5) to show that

$$\hat{\sigma}(\mathbf{FQ}) = \hat{\sigma}(\mathbf{F})$$

for every $\mathbf{Q}$ in the symmetry group and every $\mathbf{F} \in \mathrm{Lin}^+$.

(d) Show that

$$\hat{\sigma}(\mathbf{F}) = \hat{\sigma}(\mathbf{U})$$

and that there exists a function $\tilde{\sigma}$ such that

$$\hat{\sigma}(\mathbf{F}) = \tilde{\sigma}(\mathbf{C}).$$

Here $\mathbf{U}$ is the right stretch tensor, $\mathbf{C}$ the right Cauchy–Green strain tensor.

(e) Show that

$$\bar{\mathbf{S}} = 2\, D\tilde{\sigma},$$

where $D\tilde{\sigma}(\mathbf{C}) \in \mathrm{Sym}$ with components

$$\frac{\partial}{\partial C_{ij}} \tilde{\sigma}(\mathbf{C})$$

has an interpretation analogous to $D\hat{\sigma}(\mathbf{F})$.

(f) Assume that the material at $\mathbf{p}$ is isotropic. Show that

$$\tilde{\sigma}(\mathbf{C}) = \tilde{\sigma}(\mathbf{B})$$

with $\mathbf{B}$ the left Cauchy–Green strain tensor, and that

$$\tilde{\sigma}(\mathbf{B}) = \tilde{\sigma}(\imath_1(\mathbf{B}), \imath_2(\mathbf{B}), \imath_3(\mathbf{B}))$$

[cf. (37.4)]. Let

$$\tilde{\sigma}_k = \frac{\partial \tilde{\sigma}}{\partial \imath_k(\mathbf{B})}.$$

Show that the Piola–Kirchhoff and Cauchy stress tensors are given by

$$\mathbf{S} = 2\{\tilde{\sigma}_1 \mathbf{F} + \tilde{\sigma}_2[(\mathrm{tr}\,\mathbf{B})\mathbf{I} - \mathbf{B}]\mathbf{F} + (\det \mathbf{B})\tilde{\sigma}_3 \mathbf{F}^{-\mathrm{T}}\},$$
$$\mathbf{T} = 2(\det \mathbf{F})^{-1}\{(\det \mathbf{B})\tilde{\sigma}_3 \mathbf{I} + [\tilde{\sigma}_1 + (\mathrm{tr}\,\mathbf{B})\tilde{\sigma}_2]\mathbf{B} - \tilde{\sigma}_2 \mathbf{B}^2\},$$

where we have omitted the arguments of $\tilde{\sigma}_k$.

2. Extend the results of this section to incompressible elastic bodies (cf. Exercise 25.6).
3. Consider a Newtonian fluid. Show that the work is nonnegative in closed processes if and only if the viscosity μ is nonnegative.

4. Consider the statical (time-independent) behavior of a homogeneous hyperelastic body without body forces. Show that

$$\mathrm{Div}[\hat{\sigma}(\mathbf{F})\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{S}] = \mathbf{0}$$

and hence that

$$\int_{\partial \mathscr{P}} [\hat{\sigma}(\mathbf{F})\mathbf{n} - \mathbf{F}^{\mathrm{T}}\mathbf{S}\mathbf{n}]\, dA = \mathbf{0}$$

for every part $\mathscr{P}$ of $\mathscr{B}$. (This result is of importance in fracture mechanics.)

5. (*Principle of stationary potential energy*) For a hyperelastic body the (statical) *mixed problem* discussed in Section 27 can be stated as follows: find a C^2 deformation $\mathbf{f}$ such that

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}) = D\hat{\sigma}(\mathbf{F}), \qquad \mathbf{F} = \nabla \mathbf{f},$$
$$\mathrm{Div}\,\mathbf{S} + \mathbf{b}_0 = \mathbf{0}$$

on $\mathscr{B}$ and

$$\mathbf{f} = \hat{\mathbf{f}} \quad \text{on} \quad \mathscr{S}_1, \qquad \mathbf{S}\mathbf{n} = \hat{\mathbf{s}} \quad \text{on} \quad \mathscr{S}_2. \tag{20}$$

Let us agree to call a C^2 function $\mathbf{f}: \mathscr{B} \to \mathscr{E}$ with $\det \nabla \mathbf{f} > 0$ *kinematically admissible* if $\mathbf{f}$ satisfies the boundary condition $(20)_1$. Assume that $\mathscr{B}$ is bounded. Define the *potential energy* Φ on the set of kinematically admissible functions by

$$\Phi\{\mathbf{f}\} = \int_{\mathscr{B}} \hat{\sigma}(\nabla \mathbf{f})\, dV - \int_{\mathscr{B}} \mathbf{b}_0 \cdot \mathbf{u}\, dV - \int_{\mathscr{S}_2} \hat{\mathbf{s}} \cdot \mathbf{u}\, dA,$$

where $\mathbf{u}(\mathbf{p}) = \mathbf{f}(\mathbf{p}) - \mathbf{p}$. We say that the *variation of* Φ *is zero at* $\mathbf{f}$, and write

$$\delta\Phi\{\mathbf{f}\} = 0,$$

if

$$\frac{d}{d\alpha} \Phi\{\mathbf{f} + \alpha \mathbf{g}\}\big|_{\alpha=0} = 0$$

for every $\mathbf{g}$ with $\mathbf{f} + \alpha\mathbf{g}$ in the domain of Φ for all sufficiently small α (that is, for every C^2 function $\mathbf{g}: \mathscr{B} \to \mathscr{V}$ with $\mathbf{g} = \mathbf{0}$ on $\mathscr{S}_1$). Show that

$$\delta\Phi\{\mathbf{f}\} = 0$$

provided $\mathbf{f}$ is a solution of the mixed problem.

29. THE ELASTICITY TENSOR

The behavior of the constitutive equation

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F})$$

[cf. (27.9)] near $\mathbf{F} = \mathbf{I}$ is governed by the linear transformation

$$\mathsf{C}: \text{Lin} \to \text{Lin}$$

defined by

$$\mathsf{C} = D\hat{\mathbf{S}}(\mathbf{I}). \tag{1}$$

(As before we have suppressed mention of the dependence on the material point **p**.) C is called the **elasticity tensor** for the material point **p**; roughly speaking, C is the derivative of the Piola–Kirchhoff stress with respect to **F** at $\mathbf{F} = \mathbf{I}$. The importance of this tensor will become apparent in the next section, where we deduce the linearized theory appropriate to small deformations from the reference configuration.

For convenience, *we assume throughout this section that the residual stress vanishes*:

$$\hat{\mathbf{S}}(\mathbf{I}) = \hat{\mathbf{T}}(\mathbf{I}) = \mathbf{0}. \tag{2}$$

Our first result shows that, because of (2), C could also have been defined as the derivative of $\hat{\mathbf{T}}$, the response function for the Cauchy stress.

Proposition

$$\mathsf{C} = D\hat{\mathbf{T}}(\mathbf{I}). \tag{3}$$

Proof. By (27.10),

$$\hat{\mathbf{S}}(\mathbf{F})\mathbf{F}^{\mathsf{T}} = \varphi(\mathbf{F})\hat{\mathbf{T}}(\mathbf{F}),$$

where $\varphi(\mathbf{F}) = \det \mathbf{F}$. If we differentiate this relation with respect to **F**, using the product rule, we find that

$$\hat{\mathbf{S}}(\mathbf{F})\mathbf{H}^{\mathsf{T}} + D\hat{\mathbf{S}}(\mathbf{F})[\mathbf{H}]\mathbf{F}^{\mathsf{T}} = \varphi(\mathbf{F})\, D\hat{\mathbf{T}}(\mathbf{F})\,[\mathbf{H}] + D\varphi(\mathbf{F})\,[\mathbf{H}]\hat{\mathbf{T}}(\mathbf{F})$$

for every $\mathbf{H} \in \text{Lin}$. Evaluating this expression at $\mathbf{F} = \mathbf{I}$, we conclude, with the aid of (2), that

$$D\hat{\mathbf{S}}(\mathbf{I})\,[\mathbf{H}] = D\hat{\mathbf{T}}(\mathbf{I})\,[\mathbf{H}],$$

which implies (3). □

Proposition (Properties of the elasticity tensor)

(a) $\mathsf{C}[\mathbf{H}] \in \text{Sym}$ *for every* $\mathbf{H} \in \text{Lin}$;
(b) $\mathsf{C}[\mathbf{W}] = \mathbf{0}$ *for every* $\mathbf{W} \in \text{Skw}$.

Proof. Assertion (a) follows from the relation

$$\mathsf{C}[\mathbf{H}] = \frac{d}{d\alpha}\hat{\mathbf{T}}(\mathbf{I} + \alpha\mathbf{H})\Big|_{\alpha=0}$$

and the fact that $\hat{\mathbf{T}}$ has symmetric values. To prove (b) choose $\mathbf{W} \in \text{Skw}$ and take

$$\mathbf{Q}(t) = e^{\mathbf{W}t},$$

so that $\mathbf{Q}(t) \in \text{Orth}^+$ (cf. Section 36). Then (25.3) with $\mathbf{F} = \mathbf{I}$ and (2) imply

$$\hat{\mathbf{T}}(\mathbf{Q}(t)) = \mathbf{0},$$

and this relation, when differentiated with respect to t, yields

$$D\hat{\mathbf{T}}(\mathbf{Q}(t))\,[\dot{\mathbf{Q}}(t)] = \mathbf{0}.$$

Since $\mathbf{Q}(0) = \mathbf{I}$ and $\dot{\mathbf{Q}}(0) = \mathbf{W}$, if we evaluate this expression at $t = 0$ and use (3), we are led to (b). □

Let

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathsf{T}})$$

denote the symmetric part of $\mathbf{H} \in \text{Lin}$. Then $\mathbf{H} = \mathbf{E} + \mathbf{W}$, with **W** skew, and (b) and the linearity of C imply that

$$\mathsf{C}[\mathbf{H}] = \mathsf{C}[\mathbf{E}]; \tag{4}$$

hence C is completely determined by its restriction to Sym.

Our next step is to establish the invariance properties of C.

Proposition. C *is invariant under the symmetry group* $\mathscr{G}$ *for the material at* **p**.

Proof. In view of $(25.9)_1$, the last theorem in Section 37, and (3), for $\mathbf{H} \in \text{Lin}$ and $\mathbf{Q} \in \mathscr{G}$,

$$\begin{aligned}\mathbf{Q}\mathsf{C}[\mathbf{H}]\mathbf{Q}^{\mathsf{T}} &= \mathbf{Q}D\hat{\mathbf{T}}(\mathbf{I})\,[\mathbf{H}]\mathbf{Q}^{\mathsf{T}} = D\hat{\mathbf{T}}(\mathbf{Q}\mathbf{I}\mathbf{Q}^{\mathsf{T}})\,[\mathbf{Q}\mathbf{H}\mathbf{Q}^{\mathsf{T}}]\\ &= D\hat{\mathbf{T}}(\mathbf{I})\,[\mathbf{Q}\mathbf{H}\mathbf{Q}^{\mathsf{T}}] = \mathsf{C}[\mathbf{Q}\mathbf{H}\mathbf{Q}^{\mathsf{T}}],\end{aligned}$$

and C is invariant under $\mathscr{G}$. □

Since C has values in Sym [cf. (a) above], this proposition and (37.22) have the following obvious but important consequence.

Theorem. *Assume that the material at* **p** *is isotropic. Then there exist scalars* μ *and* λ *such that*

$$\mathsf{C}[\mathbf{E}] = 2\mu\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} \tag{5}$$

for every symmetric tensor **E**. *The scalars* $\mu = \mu(\mathbf{p})$ *and* $\lambda = \lambda(\mathbf{p})$ *are called the* **Lamé moduli** *at* **p**.

We say that C is **symmetric** if

$$\mathbf{H} \cdot \mathsf{C}[\mathbf{G}] = \mathbf{G} \cdot \mathsf{C}[\mathbf{H}]$$

for all tensors **H** and **G**.

Since $\mathsf{C}[\mathbf{W}] = \mathbf{0}$ for all skew **W**, C can never be positive definite in the usual sense. There are, however, important situations in which C restricted to Sym has this property. Thus let us agree to call C **positive definite** if

$$\mathbf{E} \cdot \mathsf{C}[\mathbf{E}] > 0$$

for all *symmetric* tensors $\mathbf{E} \neq \mathbf{0}$.

Proposition. *Assume that the material at* **p** *is isotropic. Then* C *is symmetric. Moreover,* C *is positive definite if and only if the Lamé moduli obey the inequalities*:

$$\mu > 0, \qquad 2\mu + 3\lambda > 0. \tag{6}$$

Proof. Let **H** and **G** be tensors with symmetric parts $\mathbf{H}_s$ and $\mathbf{G}_s$, respectively. Then, by (a) and (b) of the proposition on page 195, and since $\mathbf{I} \cdot \mathbf{H}_s = \operatorname{tr}\mathbf{H}$, (5) implies

$$\mathbf{H} \cdot \mathsf{C}[\mathbf{G}] = \mathbf{H}_s \cdot \mathsf{C}[\mathbf{G}_s] = 2\mu\mathbf{H}_s \cdot \mathbf{G}_s + \lambda(\operatorname{tr}\mathbf{H})(\operatorname{tr}\mathbf{G}) = \mathbf{G} \cdot \mathsf{C}[\mathbf{H}],$$

so that C is symmetric. Next, choose a symmetric tensor **H** and let

$$\alpha = \tfrac{1}{3}\operatorname{tr}\mathbf{H}, \qquad \mathbf{H}_0 = \mathbf{H} - \alpha\mathbf{I}$$

so that

$$\mathbf{H} = \mathbf{H}_0 + \alpha\mathbf{I}, \qquad \operatorname{tr}\mathbf{H}_0 = 0.$$

Then, since $\mathbf{I} \cdot \mathbf{H}_0 = 0$,

$$\mathbf{H} \cdot \mathsf{C}[\mathbf{H}] = 2\mu(|\mathbf{H}_0|^2 + 3\alpha^2) + 9\lambda\alpha^2 = 2\mu|\mathbf{H}_0|^2 + 3\alpha^2(2\mu + 3\lambda).$$

Trivially, (6) implies that C is positive definite. Conversely, if C is positive definite, then by choosing $\mathbf{H} = \alpha\mathbf{I}$ we conclude that $2\mu + 3\lambda > 0$, and by choosing **H** with $\operatorname{tr}\mathbf{H} = 0$ we see that $\mu > 0$. □

Proposition. *Assume that the material at p is hyperelastic. Then* C *is symmetric.*

Proof. Since

$$\mathsf{C} = D\hat{\mathbf{S}}(\mathbf{I}) = D^2\hat{\sigma}(\mathbf{I})$$

[cf. (28.5)], the proposition follows from the symmetry of the second derivative. To see this note that

$$\frac{\partial}{\partial\beta}\hat{\sigma}(\mathbf{I} + \alpha\mathbf{H} + \beta\mathbf{G}) = \hat{\mathbf{S}}(\mathbf{I} + \alpha\mathbf{H} + \beta\mathbf{G}) \cdot \mathbf{G},$$

and hence

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\hat{\sigma}(\mathbf{I} + \alpha\mathbf{H} + \beta\mathbf{G})|_{\alpha=\beta=0} = D\hat{\mathbf{S}}(\mathbf{I})[\mathbf{H}] \cdot \mathbf{G} = \mathsf{C}[\mathbf{H}] \cdot \mathbf{G};$$

by switching the order of differentiation we see that this expression equals

$$\frac{\partial^2}{\partial\beta\,\partial\alpha}\hat{\sigma}(\mathbf{I} + \alpha\mathbf{H} + \beta\mathbf{G})|_{\alpha=\beta=0} = \mathsf{C}[\mathbf{G}] \cdot \mathbf{H},$$

and the symmetry of C follows. □

EXERCISES

1. Show that, as a consequence of (27.11),

$$D\hat{\mathbf{S}}(\mathbf{F})[\mathbf{W}\mathbf{F}] = \mathbf{W}\hat{\mathbf{S}}(\mathbf{F}) \tag{7}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{W} \in \mathrm{Skw}$.

2. Relate $\mathsf{C} = D\hat{\mathbf{S}}(\mathbf{I})$ and $\mathsf{L} = D\hat{\mathbf{T}}(\mathbf{I})$ for the case in which (2) is not satisfied.
3. C is *strongly elliptic* if

$$\mathbf{A} \cdot \mathsf{C}[\mathbf{A}] > 0$$

whenever **A** has the form $\mathbf{A} = \mathbf{a} \otimes \mathbf{c}$, $\mathbf{a} \neq \mathbf{0}$, $\mathbf{c} \neq \mathbf{0}$.

(a) Show that if C is positive definite, then C is strongly elliptic.
(b) Show that for an isotropic material C is strongly elliptic if and only if

$$\mu > 0, \qquad 2\mu + \lambda > 0.$$

SELECTED REFERENCES

Coleman and Noll [1].
Green and Zerna [1].
Sternberg and Knowles [1].
Truesdell and Noll [1, Chapter D].
Wang and Truesdell [1].

CHAPTER

X

Linear Elasticity

30. DERIVATION OF THE LINEAR THEORY

We now deduce the linearized theory appropriate to situations in which the displacement gradient $\nabla \mathbf{u}$ is small. The crucial step is the linearization of the general constitutive equation

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}) \tag{1}$$

for the Piola–Kirchhoff stress [cf. (27.9)]. In order to discuss the behavior of this equation as

$$\mathbf{H} = \nabla \mathbf{u}$$

tends to zero, we consider $\hat{\mathbf{S}}(\mathbf{F})$ as a function of $\mathbf{H}$ using the relation

$$\mathbf{F} = \mathbf{I} + \mathbf{H}$$

[cf. (7.1)].

Theorem (Asymptotic form of the constitutive relation). *Assume that the residual stress vanishes. Then*

$$\hat{\mathbf{S}}(\mathbf{F}) = \mathsf{C}[\mathbf{E}] + o(\mathbf{H}) \tag{2}$$

as $\mathbf{H} \to 0$, *where* C *is the elasticity tensor* (29.1) *and*

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T) \tag{3}$$

is the infinitesimal strain (7.4).

Proof. Since the residual stress vanishes, we may conclude from (29.1), (29.2), and (29.4) that

$$\begin{aligned}\hat{S}(F) = \hat{S}(I + H) &= \hat{S}(I) + D\hat{S}(I)[H] + o(H)\\ &= C[H] + o(H)\\ &= C[E] + o(H). \quad \square\end{aligned}$$

Using (2) we can write the asymptotic form of the constitutive equation (1) as

$$S = C[E] + o(\nabla u). \tag{4}$$

Therefore, *if the residual stress in the reference configuration vanishes, then to within terms of* $o(\nabla u)$ *as* $\nabla u \to 0$ *the stress* S *is a linear function of the infinitesimal strain* E. *Also, since* C *has symmetric values* [cf. (a) of the proposition on page 195], *to within the same error* S *is symmetric.*

The linear theory of elasticity is based on the stress-strain law (4) with the terms of order $o(\nabla u)$ neglected, the strain-displacement relation (3), and the equation of motion $(27.6)_1$:

$$\begin{aligned} S &= C[E],\\ E &= \tfrac{1}{2}(\nabla u + \nabla u^T),\\ \operatorname{Div} S + b_0 &= \rho_0 \ddot{u}. \end{aligned} \tag{5}$$

Note that these equations are expressed in terms of the displacement

$$u(p, t) = x(p, t) - p,$$

rather than the motion x.

It is important to emphasize that the formal derivation of the linearized constitutive equation $(5)_1$ was based on the following two assumptions:

(a) The residual stress in the reference configuration vanishes.
(b) The displacement gradient is small.

Note that, by $(5)_1$ and the theorem on page 56, $E = S = 0$ in an infinitesimal rigid displacement. This is an important property of the linearized theory.

Given C, ρ_0, and b_0, (5) is a linear system of partial differential equations for the fields u, E, and S. By (29.5), when the body is *isotropic* $(5)_1$ may be replaced by

$$S = 2\mu E + \lambda(\operatorname{tr} E)I. \tag{6}$$

Moreover, when the body is homogeneous, ρ_0, μ, and λ are constants.

Assume now that $\mathscr{B}$ is homogeneous and isotropic. Then, since

$$\operatorname{Div}(\nabla u + \nabla u^T) = \Delta u + \nabla \operatorname{Div} u$$

and

$$\operatorname{tr} E = \operatorname{Div} u,$$

the equations $(5)_{2,3}$ and (6) are easily combined to give the *displacement equation of motion*

$$\mu \Delta u + (\lambda + \mu)\nabla \operatorname{Div} u + b_0 = \rho_0 \ddot{u}. \tag{7}$$

In the statical theory $\ddot{u} = 0$ and we have the *displacement equation of equilibrium*

$$\mu \Delta u + (\lambda + \mu)\nabla \operatorname{Div} u + b_0 = 0. \tag{8}$$

Exercises

1. Let u be a C^4 solution of (8) with $b_0 = 0$. Show that Div u and Curl u are harmonic functions. Show further that u is biharmonic; that is,
$$\Delta\Delta u = 0.$$
2. Let u be a C^4 solution of (7) with $b_0 = 0$. Show that Div u and Curl u satisfy wave equations.
3. (*Boussinesq–Papkovitch–Neuber solution*) Let φ and g be *harmonic* fields on $\mathscr{B}$ (with φ scalar valued and g vector valued) and define
$$u = g - \alpha\nabla(r \cdot g + \varphi),$$
$$\alpha = \frac{\mu + \lambda}{2(2\mu + \lambda)},$$
where $r(p) = p - o$. Show that u is a solution of (8) with $b_0 = 0$.
4. Consider the case in which the residual stress $\hat{S}(I) \neq 0$. Show that
$$\hat{S}(F) = \hat{S}(I) + W\hat{S}(I) + C[E] + o(H).$$

31. SOME SIMPLE SOLUTIONS

Assume now that the body is homogeneous and isotropic. Then any statical (i.e., time-independent) displacement field u with E constant generates a solution of the field equations $(30.5)_{2,3}$ and (30.6) with S constant and

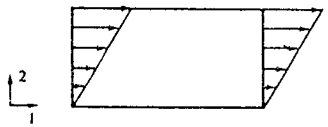

Figure 1

$\mathbf{b}_0 = \mathbf{0}$. We now discuss some particular solutions of this type, using cartesian coordinates where convenient.

(a) *Pure shear.* Let

$$\mathbf{u}(\mathbf{p}) = \gamma p_2 \mathbf{e}_1$$

(Fig. 1), so that the matrices of **E** and S are

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\mathrm{S}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with

$$\tau = \mu\gamma.$$

Thus μ determines the response of the body in shear, at least within the linear theory, and for this reason is called the **shear modulus**. Note that, in contrast to the general nonlinear theory (cf. Section 26), the normal stresses are all zero; the only stress present is the shear stress τ.

(b) *Uniform compression or expansion.* Here

$$\mathbf{u}(\mathbf{p}) = \varepsilon(\mathbf{p} - \mathbf{o}).$$

Then

$$\mathbf{E} = \varepsilon\mathbf{I},$$
$$\mathrm{S} = -\pi\mathbf{I},$$
$$\pi = -3\kappa\varepsilon,$$

where

$$\kappa = \tfrac{2}{3}\mu + \lambda$$

is the **modulus of compression.**

For our third solution it is simpler to work with the stress-strain law (30.6) inverted to give **E** as a function of S. This inversion is easily accomplished upon noting that

$$\operatorname{tr} \mathrm{S} = (2\mu + 3\lambda) \operatorname{tr} \mathbf{E},$$

and hence

$$\mathbf{E} = \frac{1}{2\mu}\left[\mathrm{S} - \frac{\lambda}{2\mu + 3\lambda}(\operatorname{tr} \mathrm{S})\mathbf{I}\right]. \qquad (1)$$

(c) *Pure tension.* Here we want the stress to have the form

$$[\mathrm{S}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding strain tensor is then given by

$$[\mathbf{E}] = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{bmatrix}$$

with

$$\varepsilon = \frac{1}{E}\sigma, \qquad l = -\nu\varepsilon,$$

and

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}, \qquad \nu = \frac{\lambda}{2(\mu + \lambda)}.$$

Note that the strain **E** corresponds to a displacement field of the form

$$\mathbf{u}(\mathbf{p}) = \varepsilon p_1 \mathbf{e}_1 + l p_2 \mathbf{e}_2 + l p_3 \mathbf{e}_3.$$

The modulus E is obtained by dividing the tensile stress σ by the longitudinal strain ε produced by it. It is known as **Young's modulus.** The modulus ν is the ratio of the lateral contraction to the longitudinal strain of a bar under pure tension. It is known as **Poisson's ratio.**

If we write $\mathbf{E}_0$ and S_0 for the traceless parts of **E** and S, that is,

$$\mathbf{E}_0 = \mathbf{E} - \tfrac{1}{3}(\operatorname{tr} \mathbf{E})\mathbf{I}, \qquad \mathrm{S}_0 = \mathrm{S} - \tfrac{1}{3}(\operatorname{tr} \mathrm{S})\mathbf{I}, \qquad (2)$$

then the isotropic constitutive relation (30.6) is equivalent to the following pair of relations:

$$\mathrm{S}_0 = 2\mu\mathbf{E}_0,$$
$$\operatorname{tr} \mathrm{S} = 3\kappa \operatorname{tr} \mathbf{E}.$$

Another important form for this stress-strain relation is the one taken by the inverted relation (1) when Young's modulus and Poisson's ratio are used:

$$\mathbf{E} = \frac{1}{E}[(1 + \nu)\mathrm{S} - \nu(\operatorname{tr} \mathrm{S})\mathbf{I}]. \qquad (3)$$

Since an elastic solid should increase its length when pulled, should decrease its volume when acted on by a pure pressure, and should respond to a positive shearing strain by a positive shearing stress, we would expect that

$$E > 0, \qquad \kappa > 0, \qquad \mu > 0.$$

Also, a pure tensile stress should produce a contraction in the direction perpendicular to it; thus

$$\nu > 0.$$

Even though these inequalities are physically well motivated, we will not assume that they hold, for in many circumstances other (somewhat weaker) assumptions are more natural. In particular, we will usually assume that C is positive definite. A simple computation, based on (29.6), shows that this restriction is equivalent to *either* of the following two sets of inequalities:

(i) $\mu > 0, \quad \kappa > 0;$
(ii) $E > 0, \quad -1 < \nu < \frac{1}{2}.$

Some typical values for Young's modulus E and Poisson's ratio ν are[1]

carbon steel:	$E = 2.1 \times 10^{11}\ \text{N/m}^2$,	$\nu = 0.29$,
copper:	$E = 10^{11}\ \text{N/m}^2$,	$\nu = 0.33$,
glass:	$E = 0.55 \times 10^{11}\ \text{N/m}^2$,	$\nu = 0.25$.

Exercises

1. Assume that C is symmetric and define

$$\hat{\varepsilon}(\mathbf{E}) = \tfrac{1}{2}\mathbf{E} \cdot \mathsf{C}[\mathbf{E}]$$

for every $\mathbf{E} \in \text{Sym}$.

(a) Show that the stress-strain law $\mathbf{S} = \mathsf{C}[\mathbf{E}]$ can be written as

$$\mathbf{S} = D\hat{\varepsilon}(\mathbf{E}).$$

(b) Show that, for an isotropic material,

$$\hat{\varepsilon}(\mathbf{E}) = \mu|\mathbf{E}|^2 + \frac{\lambda}{2}(\text{tr}\,\mathbf{E})^2,$$

$$\hat{\varepsilon}(\mathbf{E}) = \mu|\mathbf{E}_0|^2 + \frac{\kappa}{2}(\text{tr}\,\mathbf{E})^2,$$

$$\alpha|\mathbf{E}|^2 \le \hat{\varepsilon}(\mathbf{E}) \le \beta|\mathbf{E}|^2,$$

where α is the smaller and β the larger of the numbers μ and $3\kappa/2$.

[1] Cf., e.g., Sokolnikoff [1, p. 70].

2. Show that C is positive definite if and only if either of the two sets of inequalities (i) or (ii) hold.

32. LINEAR ELASTOSTATICS

The system of field equations for the statical behavior of an elastic body—within the framework of the linear theory—consists of the *strain-displacement relation*

$$\mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}), \tag{1}$$

the *stress-strain relation*

$$\mathbf{S} = \mathsf{C}[\mathbf{E}], \tag{2}$$

and the *equation of equilibrium*

$$\text{Div}\,\mathbf{S} + \mathbf{b} = \mathbf{0} \tag{3}$$

(where we have written $\mathbf{b}$ for $\mathbf{b}_0$). The elasticity tensor C, which is a linear mapping of tensors into symmetric tensors, will generally depend on position $\mathbf{p}$ in $\mathscr{B}$; writing $\mathsf{C}_{\mathbf{p}}$ to emphasize this dependence, we assume henceforth that $\mathsf{C}_{\mathbf{p}}$ is a smooth function of $\mathbf{p}$ on $\mathscr{B}$.

A list $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ of fields which are smooth on $\mathscr{B}$ and satisfy (1)–(3) for a given body force $\mathbf{b}$ will be called an **elastic state** corresponding to $\mathbf{b}$. Note that by (1), (2), and the properties of C, the fields $\mathbf{E}$ and $\mathbf{S}$ are *symmetric.*

We assume throughout this section that $\mathscr{B}$ is bounded.

Theorem of Work and Energy. *Let $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be an elastic state corresponding to the body force $\mathbf{b}$. Then*

$$\int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\mathbf{u}\,dA + \int_{\mathscr{B}} \mathbf{b}\cdot\mathbf{u}\,dV = 2\mathscr{U}\{\mathbf{E}\}, \tag{4}$$

where

$$\mathscr{U}\{\mathbf{E}\} = \frac{1}{2}\int_{\mathscr{B}} \mathbf{E}\cdot\mathsf{C}[\mathbf{E}]\,dV \tag{5}$$

is the **strain energy**.

The proof of this theorem is an immediate consequence of the following

Lemma. *Let $\mathbf{S}$ be a smooth symmetric tensor field on $\mathscr{B}$, let $\tilde{\mathbf{u}}$ be a smooth vector field on $\mathscr{B}$, and let*

$$\text{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0}, \qquad \tilde{\mathbf{E}} = \tfrac{1}{2}(\nabla\tilde{\mathbf{u}} + \nabla\tilde{\mathbf{u}}^{\mathrm{T}}).$$

Then

$$\int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\tilde{\mathbf{u}}\,dA + \int_{\mathscr{B}} \mathbf{b}\cdot\tilde{\mathbf{u}}\,dV = \int_{\mathscr{B}} \mathbf{S}\cdot\tilde{\mathbf{E}}\,dV. \tag{6}$$

Proof. By the symmetry of S, the divergence theorem, $(4.2)_5$, and (a) of the proposition on page 6,

$$\int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\tilde{\mathbf{u}}\,dA = \int_{\partial\mathscr{B}} (\mathbf{S}\tilde{\mathbf{u}})\cdot\mathbf{n}\,dA = \int_{\mathscr{B}} \mathrm{Div}(\mathbf{S}\tilde{\mathbf{u}})\,dV = \int_{\mathscr{B}} (\tilde{\mathbf{u}}\cdot\mathrm{Div}\,\mathbf{S} + \mathbf{S}\cdot\nabla\tilde{\mathbf{u}})\,dV,$$

$$\mathbf{S}\cdot\nabla\tilde{\mathbf{u}} = \mathbf{S}\cdot\{\tfrac{1}{2}(\nabla\tilde{\mathbf{u}} + \nabla\tilde{\mathbf{u}}^{\mathrm{T}})\} = \mathbf{S}\cdot\tilde{\mathbf{E}}.$$

These relations clearly imply the desired result (6). □

We can interpret the left side of (4) as the work done by the external forces; (4) asserts that this work is equal to twice the strain energy. Note that when C is positive definite, $\mathscr{U}\{\mathbf{E}\} \geq 0$, and the work is nonnegative.

The next theorem is one of the major results of elastostatics; in essence, it expresses the fact that the underlying system of field equations is self-adjoint when C is symmetric.

Betti's Reciprocal Theorem. *Assume that* C *is symmetric. Let* [**u**, **E**, **S**] *and* [$\tilde{\mathbf{u}}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}$] *be elastic states corresponding to body force fields* **b** *and* $\tilde{\mathbf{b}}$, *respectively. Then*

$$\int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\tilde{\mathbf{u}}\,dA + \int_{\mathscr{B}} \mathbf{b}\cdot\tilde{\mathbf{u}}\,dV = \int_{\partial\mathscr{B}} \tilde{\mathbf{S}}\mathbf{n}\cdot\mathbf{u}\,dA + \int_{\mathscr{B}} \tilde{\mathbf{b}}\cdot\mathbf{u}\,dV$$

$$= \int_{\mathscr{B}} \mathbf{S}\cdot\tilde{\mathbf{E}}\,dV = \int_{\mathscr{B}} \tilde{\mathbf{S}}\cdot\mathbf{E}\,dV. \tag{7}$$

Proof. Since C is symmetric, we conclude from the stress-strain relation that

$$\mathbf{S}\cdot\tilde{\mathbf{E}} = \mathrm{C}[\mathbf{E}]\cdot\tilde{\mathbf{E}} = \mathrm{C}[\tilde{\mathbf{E}}]\cdot\mathbf{E} = \tilde{\mathbf{S}}\cdot\mathbf{E},$$

and (7) follows from (6) and the analogous relation with the roles of the two states reversed. □

Betti's theorem asserts that given two elastic states, the work done by the external forces of the first over the displacement of the second equals the work done by the second over the displacement of the first.

Let $\mathscr{S}_1$ and $\mathscr{S}_2$ denote complementary regular subsets of the boundary of $\mathscr{B}$, so that

$$\partial\mathscr{B} = \mathscr{S}_1 \cup \mathscr{S}_2, \qquad \mathring{\mathscr{S}}_1 \cap \mathring{\mathscr{S}}_2 = \emptyset$$

with $\mathring{\mathscr{S}}_\alpha$ the relative interior of $\mathscr{S}_\alpha$. Then the **mixed problem** of elastostatics can be stated as follows:

Given: $\mathscr{B}, \mathscr{S}_1, \mathscr{S}_2$, an elasticity tensor C on $\mathscr{B}$, a body force field **b** on $\mathscr{B}$, surface displacements $\hat{\mathbf{u}}$ on $\mathscr{S}_1$, surface tractions $\hat{\mathbf{s}}$ on $\mathscr{S}_2$.

Find: An elastic state [**u**, **E**, **S**] that corresponds to **b** and satisfies the *boundary conditions*

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \quad \mathscr{S}_1, \qquad \mathbf{Sn} = \hat{\mathbf{s}} \quad \text{on} \quad \mathscr{S}_2. \tag{8}$$

An elastic state with these properties will be called a **solution**.

Uniqueness Theorem. *Assume that* C *is positive definite. Let* [$\mathbf{u}_1, \mathbf{E}_1, \mathbf{S}_1$] *and* [$\mathbf{u}_2, \mathbf{E}_2, \mathbf{S}_2$] *be solutions of the (same) mixed problem. Then*

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{w}, \qquad \mathbf{E}_1 = \mathbf{E}_2, \qquad \mathbf{S}_1 = \mathbf{S}_2,$$

where **w** *is an infinitesimal rigid displacement of* $\mathscr{B}$.

Proof. Let

$$\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2, \qquad \mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2, \qquad \mathbf{S} = \mathbf{S}_1 - \mathbf{S}_2.$$

Then [**w**, **E**, **S**] is an elastic state that corresponds to null body forces and satisfies the boundary conditions

$$\mathbf{w} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_1, \qquad \mathbf{Sn} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_2.$$

Thus

$$\mathbf{Sn}\cdot\mathbf{w} = 0 \qquad \text{on} \quad \partial\mathscr{B},$$

and we conclude from (4) that

$$\int_{\mathscr{B}} \mathbf{E}\cdot\mathrm{C}[\mathbf{E}]\,dV = 0.$$

Since C is positive definite, this relation can hold only if $\mathbf{E} = \mathbf{0}$; this in turn implies that $\mathbf{S} = \mathbf{0}$ and that **w** is an infinitesimal rigid displacement of $\mathscr{B}$ (cf. the theorem on page 56). □

When $\mathscr{S}_1 = \partial\mathscr{B}$ ($\mathscr{S}_2 = \emptyset$) the boundary condition (8) takes the form

$$\mathbf{u} = \hat{\mathbf{u}} \qquad \text{on} \quad \partial\mathscr{B},$$

and the mixed problem is referred to as the **displacement problem**. On the other hand, when $\mathscr{S}_2 = \partial\mathscr{B}$ ($\mathscr{S}_1 = \emptyset$), so that

$$\mathbf{Sn} = \hat{\mathbf{s}} \qquad \text{on} \quad \partial\mathscr{B},$$

we have the **traction problem**. This last definition, (3), and Cauchy's theorem (page 101) have the following immediate consequence:

Proposition. *A necessary condition that the traction problem have a solution is that*

$$\int_{\partial\mathscr{B}} \hat{\mathbf{s}}\, dA + \int_{\mathscr{B}} \mathbf{b}\, dV = \mathbf{0},$$

$$\int_{\partial\mathscr{B}} \mathbf{r}\times\hat{\mathbf{s}}\, dA + \int_{\mathscr{B}} \mathbf{r}\times\mathbf{b}\, dV = \mathbf{0}, \tag{9}$$

where $\mathbf{r}(\mathbf{p}) = \mathbf{p} - \mathbf{o}$ *is the position vector.*

Equations (9) insure equilibrium of the external forces applied to $\mathscr{B}$. It is interesting to compare these conditions with the analogous restrictions of the general nonlinear theory discussed at the end of Section 27. There the applied forces need not obey balance of moments in the reference configuration [cf. the discussion following (27.21)], as in the nonlinear theory the body is allowed to undergo the possibly large deformation needed to insure moment balance in the deformed configuration.

We now show that the solution of the mixed problem, if it exists, can be characterized as the minimum value of a certain functional.

By a **kinematically admissible state** we mean a list $\mathfrak{s} = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ with $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{S}$ smooth fields on $\mathscr{B}$ that satisfy the field equations

$$\mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}), \qquad \mathbf{S} = \mathsf{C}[\mathbf{E}],$$

and the boundary condition

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \quad \mathscr{S}_1.$$

Let Φ be the functional defined on the set of kinematically admissible states by

$$\Phi\{\mathfrak{s}\} = \mathscr{U}\{\mathbf{E}\} - \int_{\mathscr{B}} \mathbf{b}\cdot\mathbf{u}\, dV - \int_{\mathscr{S}_2} \hat{\mathbf{s}}\cdot\mathbf{u}\, dA. \tag{10}$$

Principle of Minimum Potential Energy. *Assume that* C *is symmetric and positive definite. Let* $\mathfrak{s} = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ *be a solution of the mixed problem. Then*

$$\Phi\{\mathfrak{s}\} \le \Phi\{\tilde{\mathfrak{s}}\}$$

for every kinematically admissible state $\tilde{\mathfrak{s}} = [\tilde{\mathbf{u}}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}]$, *and equality holds only if* $\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{w}$ *with* $\mathbf{w}$ *an infinitesimal rigid displacement of* $\mathscr{B}$.

Proof. Let

$$\mathbf{w} = \tilde{\mathbf{u}} - \mathbf{u}, \qquad \bar{\mathbf{E}} = \tilde{\mathbf{E}} - \mathbf{E}.$$

Then, since $\mathfrak{s}$ is a solution and $\tilde{\mathfrak{s}}$ is kinematically admissible,

$$\bar{\mathbf{E}} = \tfrac{1}{2}(\nabla\mathbf{w} + \nabla\mathbf{w}^{\mathrm{T}}),$$

$$\mathbf{w} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_1. \tag{11}$$

Further, since C is symmetric and $\mathbf{S} = \mathsf{C}[\mathbf{E}]$,

$$\begin{aligned}\tilde{\mathbf{E}}\cdot\mathsf{C}[\tilde{\mathbf{E}}] &= \mathbf{E}\cdot\mathsf{C}[\mathbf{E}] + \bar{\mathbf{E}}\cdot\mathsf{C}[\bar{\mathbf{E}}] + \mathbf{E}\cdot\mathsf{C}[\bar{\mathbf{E}}] + \bar{\mathbf{E}}\cdot\mathsf{C}[\mathbf{E}]\\ &= \mathbf{E}\cdot\mathsf{C}[\mathbf{E}] + \bar{\mathbf{E}}\cdot\mathsf{C}[\bar{\mathbf{E}}] + 2\mathbf{S}\cdot\bar{\mathbf{E}};\end{aligned}$$

hence

$$\mathscr{U}\{\tilde{\mathbf{E}}\} - \mathscr{U}\{\mathbf{E}\} = \mathscr{U}\{\bar{\mathbf{E}}\} + \int_{\mathscr{B}} \mathbf{S}\cdot\bar{\mathbf{E}}\, dV.$$

Because $\mathfrak{s}$ is a solution, we conclude from $(11)_2$ and (6) with $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{E}}$ replaced by $\mathbf{w}$ and $\bar{\mathbf{E}}$ that

$$\int_{\mathscr{B}} \mathbf{S}\cdot\bar{\mathbf{E}}\, dV = \int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\mathbf{w}\, dA + \int_{\mathscr{B}} \mathbf{b}\cdot\mathbf{w}\, dV = \int_{\mathscr{S}_2} \hat{\mathbf{s}}\cdot\mathbf{w}\, dA + \int_{\mathscr{B}} \mathbf{b}\cdot\mathbf{w}\, dV.$$

In view of (10) and the last two relations,

$$\Phi\{\tilde{\mathfrak{s}}\} - \Phi\{\mathfrak{s}\} = \mathscr{U}\{\bar{\mathbf{E}}\}.$$

Thus, since C is positive definite,

$$\Phi\{\mathfrak{s}\} \le \Phi\{\tilde{\mathfrak{s}}\}$$

and

$$\Phi\{\mathfrak{s}\} = \Phi\{\tilde{\mathfrak{s}}\} \quad \text{only when} \quad \bar{\mathbf{E}} = \mathbf{0};$$

that is, only when $\mathbf{w} = \tilde{\mathbf{u}} - \mathbf{u}$ is an infinitesimal rigid displacement. □

In words, the principle of minimum potential energy asserts that the difference between the strain energy and the work done by the body force and prescribed surface traction assumes a smaller value for the solution of the mixed problem than for any other kinematically admissible state.

A standard technique for obtaining approximate solutions is based on the principle of minimum potential energy. We now demonstrate this method, but for convenience limit our attention to the *traction problem*. Consider an approximate solution of the form

$$\mathbf{u}(\mathbf{p}) = \sum_{n=1}^{N} \alpha_n \mathbf{g}_n(\mathbf{p}), \tag{12}$$

where $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_N$ are *given* vector fields on $\mathscr{B}$, and where the scalar constants $\alpha_1, \alpha_2, \ldots, \alpha_N$ are chosen to make $\Phi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ a minimum. Here

$\Phi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the function obtained by substituting (12) into the right side of (10):

$$\Phi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \frac{1}{2} \sum_{m,n=1}^{N} K_{mn}\alpha_m\alpha_n - \sum_{n=1}^{N} \varphi_n\alpha_n,$$

where

$$K_{mn} = \int_{\mathscr{B}} \mathbf{G}_m \cdot \mathsf{C}[\mathbf{G}_n]\, dV, \qquad \mathbf{G}_n = \tfrac{1}{2}(\nabla \mathbf{g}_n + \nabla \mathbf{g}_n^T),$$

$$\varphi_n = \int_{\mathscr{B}} \mathbf{b} \cdot \mathbf{g}_n\, dV + \int_{\partial\mathscr{B}} \hat{\mathbf{s}} \cdot \mathbf{g}_n\, dA.$$

Let $\mathbf{K}$ be the matrix with entries K_{mn}. If C is symmetric and positive definite, $\mathbf{K}$ will be symmetric and positive semidefinite; thus $\Phi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ will be a minimum at a "vector" $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ if and only if the vector is a solution of the equation

$$\mathbf{K}\boldsymbol{\alpha} = \boldsymbol{\varphi},$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\varphi}$ are the column vectors with entries α_n and φ_n, respectively. The problem is therefore reduced to solving a matrix equation; the corresponding solution represents the displacement field of the form (12) which is "nearest in energy" to the true solution. The matrix $\mathbf{K}$, which characterizes the response of the system, is usually called the *stiffness matrix*.

The crucial part of this approximation technique lies in the choice of the functions $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_N$ (one specific family of choices generates what is commonly called the finite-element method); in particular, an important consideration is convergence as $N \to \infty$. This matter, however, is beyond the scope of this book.

EXERCISES

1. Show that if the traction problem has a solution $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$, then $[\mathbf{u} + \mathbf{w}, \mathbf{E}, \mathbf{S}]$ is a solution for every infinitesimal rigid displacement $\mathbf{w}$. Does the analogous result hold for the mixed problem in general?
2. Show that for a homogeneous, isotropic elastic body the "infinitesimal volume change"

$$\int_{\partial\mathscr{B}} \mathbf{u} \cdot \mathbf{n}\, dA = \int_{\mathscr{B}} \operatorname{div} \mathbf{u}\, dV$$

in an elastic state is given by

$$\frac{1}{3\kappa}\left\{\int_{\partial\mathscr{B}} \mathbf{r} \cdot \mathbf{s}\, dA + \int_{\mathscr{B}} \mathbf{r} \cdot \mathbf{b}\, dV\right\},$$

where $\mathbf{r}(\mathbf{p}) = \mathbf{p} - \mathbf{o}$.

For the remaining exercises C is symmetric.

3. Assume that C is positive definite. Then C restricted to symmetric tensors is invertible. Let K denote the corresponding inverse, so that the stress-strain relation $\mathbf{S} = \mathsf{C}[\mathbf{E}]$ can be inverted to give

$$\mathbf{E} = \mathsf{K}[\mathbf{S}].$$

(a) Show that K is symmetric.
(b) Show that the strain energy can be written in the form

$$\mathscr{U}_{\mathsf{K}}\{\mathbf{S}\} = \frac{1}{2}\int_{\mathscr{B}} \mathbf{S} \cdot \mathsf{K}[\mathbf{S}]\, dV.$$

(c) (*Principle of minimum complementary energy*) A *statically admissible stress field* $\mathbf{S}$ is a smooth symmetric tensor field that satisfies the equilibrium equation

$$\operatorname{Div} \mathbf{S} + \mathbf{b} = \mathbf{0}$$

and the boundary condition

$$\mathbf{S}\mathbf{n} = \hat{\mathbf{s}} \quad \text{on } \mathscr{S}_2.$$

Let Ψ be the functional defined on the set of statically admissible stress fields by

$$\Psi\{\mathbf{S}\} = \mathscr{U}_{\mathsf{K}}\{\mathbf{S}\} - \int_{\mathscr{S}_1} \mathbf{S}\mathbf{n} \cdot \hat{\mathbf{u}}\, dA.$$

Show that if $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is a solution of the mixed problem, then

$$\Psi\{\mathbf{S}\} \le \Psi\{\tilde{\mathbf{S}}\}$$

for every statically admissible stress field $\tilde{\mathbf{S}}$.
(d) Show that if $\mathfrak{s} = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is a solution of the mixed problem, then

$$\Phi\{\mathfrak{s}\} + \Psi\{\mathbf{S}\} = 0.$$

4. (*Hu–Washizu principle*) An *admissible state* is a list $\mathscr{s} = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ of smooth fields on $\mathscr{B}$ with $\mathbf{E}$ and $\mathbf{S}$ symmetric. Let Λ be the functional defined on the space of admissible states by

$$\Lambda\{\mathscr{s}\} = \mathscr{U}\{\mathbf{E}\} - \int_{\mathscr{B}} \mathbf{S}\cdot\mathbf{E}\,dV - \int_{\mathscr{B}} (\operatorname{Div}\mathbf{S} + \mathbf{b})\cdot\mathbf{u}\,dV + \int_{\mathscr{S}_1} \mathbf{Sn}\cdot\hat{\mathbf{u}}\,dA + \int_{\mathscr{S}_2} (\mathbf{Sn} - \hat{\mathbf{s}})\cdot\mathbf{u}\,dA.$$

Show that

$$\delta\Lambda\{\mathscr{s}\} = 0,$$

provided $\mathscr{s}$ is a solution of the mixed problem (cf. Exercise 28.5).

5. (*Hellinger–Prange–Reissner principle*) Assume that C restricted to symmetric tensors is invertible with inverse K. Let $\mathscr{A}$ be the set of all admissible states that satisfy the strain-displacement relation (1) and define Θ on $\mathscr{A}$ by

$$\Theta\{\mathscr{s}\} = \mathscr{U}_{\mathsf{K}}\{\mathbf{S}\} - \int_{\mathscr{B}} \mathbf{S}\cdot\mathbf{E}\,dV + \int_{\mathscr{B}} \mathbf{b}\cdot\mathbf{u}\,dV + \int_{\mathscr{S}_1} \mathbf{Sn}\cdot(\mathbf{u} - \hat{\mathbf{u}})\,dA + \int_{\mathscr{S}_2} \hat{\mathbf{s}}\cdot\mathbf{u}\,dA.$$

Show that

$$\delta\Theta\{\mathscr{s}\} = 0$$

provided $\mathscr{s}$ is a solution of the mixed problem.

6. Show that for $\mathscr{s}$ kinematically admissible,

$$\Lambda\{\mathscr{s}\} = \Phi\{\mathscr{s}\}.$$

7. Consider a one-parameter family of (homogeneous and isotropic) elastic states $[\mathbf{u}_\nu, \mathbf{E}_\nu, \mathbf{S}_\nu]$ with Poisson's ratio ν as parameter. These states all correspond to the same shear modulus and to vanishing body forces.
   (a) Assuming that $[\mathbf{u}_\nu, \mathbf{E}_\nu, \mathbf{S}_\nu]$ tends to a limit $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ as $\nu \to \frac{1}{2}$, show that

$$\mathbf{S} = -\pi\mathbf{I} + 2\mu\mathbf{E}, \qquad \operatorname{tr}\mathbf{E} = 0,$$

   where $\pi = -\frac{1}{3}\operatorname{tr}\mathbf{S}$. Give an argument to support the claim that $\nu = \frac{1}{2}$ corresponds to *incompressibility*. (Note that, as in the case of ideal and Newtonian fluids, the "pressure" π is not uniquely determined by the deformation; that is, E does not uniquely determine π.)

   (b) Assuming that the limit state $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is approached in a manner which is sufficiently regular to justify interchanges of limits and derivatives, show that

$$\mu\,\Delta\mathbf{u} - \nabla\pi = \mathbf{0},$$
$$\operatorname{Div}\mathbf{u} = 0.$$

   (Cf. the equations on page 152 describing Stokes flows.)

8. An elastic state $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is a *state of plane strain* if $\mathbf{u}$ has the form

$$\mathbf{u}(\mathbf{p}) = u_1(p_1, p_2)\mathbf{e}_1 + u_2(p_1, p_2)\mathbf{e}_2$$

in some cartesian frame. Consider such a state, and assume that the body is isotropic and that $\mathbf{b} = \mathbf{0}$.
   (a) Show that $\mathbf{E}$ and $\mathbf{S}$ are functions of (p_1, p_2), and that

$$E_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}),$$
$$S_{\alpha\beta} = 2\mu E_{\alpha\beta} + \lambda\delta_{\alpha\beta}(E_{11} + E_{22}), \tag{13}$$
$$\sum_{\beta=1}^{2} S_{\alpha\beta,\beta} = 0,$$

   while

$$E_{13} = E_{23} = E_{33} = 0,$$
$$S_{13} = S_{23} = 0, \qquad S_{33} = \lambda(E_{11} + E_{22}) = \nu(S_{11} + S_{22}).$$

   Here α and β have the range of the integers $(1, 2)$, $\delta_{\alpha\beta} = 1$ when $\alpha = \beta$ and zero otherwise, and

$$u_{\alpha,\beta} = \frac{\partial u_\alpha}{\partial p_\beta}, \quad \text{etc.}$$

   (b) Show that $\mathbf{E}$ (when C^2) satisfies the *compatibility equation*

$$2E_{12,12} = E_{11,22} + E_{22,11}. \tag{14}$$

For the remaining exercises $\mathscr{R}$ is an open *simply connected* region in $\mathbb{R}^2$.

9. Let $E_{\alpha\beta}\ (= E_{\beta\alpha})$ be a C^2 solution of (14) on $\mathscr{R}$. Show that there exist displacements u_α such that $(13)_1$ hold.

10. Let $E_{\alpha\beta}$ and $S_{\alpha\beta}$ be C^2 functions on $\mathscr{R}$ consistent with $(13)_{2,3}$. Show that (14) is satisfied if and only if

$$\Delta(S_{11} + S_{22}) = 0. \tag{15}$$

[Thus for a simply connected region plane elastic states are completely characterized by $(13)_3$ and (15). Indeed, when these equations are satisfied, $E_{\alpha\beta}$ can be defined by $(13)_2$ (assuming invertibility) and, since (14) holds, there exist u_α that satisfy $(13)_1$.]

11. Let φ be a scalar field of class C^4 on $\mathscr{R}$. Define

$$S_{11} = \varphi_{,22}, \qquad S_{22} = \varphi_{,11}, \qquad S_{12} = -\varphi_{,12}. \tag{16}$$

Show that $S_{\alpha\beta}$ satisfies $(13)_3$. Show further that (15) holds if and only if φ is *biharmonic*:

$$\Delta\Delta\varphi = 0.$$

The field φ is called an *Airy stress function.*

12. Let $S_{\alpha\beta}\,(=S_{\beta\alpha})$ be a C^2 solution of $(13)_3$ and (15) on $\mathscr{R}$. Show that there exists a biharmonic function φ on $\mathscr{R}$ such that (16) holds.

33. BENDING AND TORSION

A. Bending of a Bar

Consider a (homogeneous, isotropic) cylindrical bar with generators parallel to the p_3-axis (Fig. 2). Let the end faces $\mathscr{S}_0$ and $\mathscr{S}_l$ be located at $p_3 = 0$ and $p_3 = l$, respectively, with the origin at the centroid of $\mathscr{S}_0$ and with the p_1- and p_2-axes principal axes of inertia:

$$\int_{\mathscr{S}_0} p_1\, dA = \int_{\mathscr{S}_0} p_2\, dA = \int_{\mathscr{S}_0} p_1 p_2\, dA = 0. \tag{1}$$

We assume that the bar is loaded only on the end faces, and by opposing couples about the p_2-axis. More precisely, we assume that

(a) body forces are zero;
(b) the lateral surface $\mathscr{L}$ is traction-free; that is,

$$\mathbf{Sn} = \mathbf{0} \qquad \text{on} \quad \mathscr{L},$$

or equivalently, since $n_3 = 0$ on $\mathscr{L}$,

$$S_{11}n_1 + S_{12}n_2 = S_{12}n_1 + S_{22}n_2 = S_{13}n_1 + S_{23}n_2 = 0 \qquad \text{on} \quad \mathscr{L}; \tag{2}$$

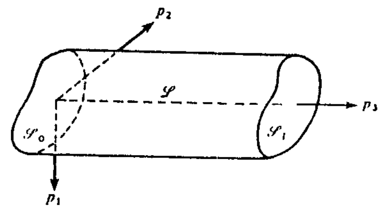

Figure 2

(c) the total force on $\mathscr{S}_0$ vanishes and the total moment on $\mathscr{S}_0$ is equipollent to a moment of magnitude m about the negative p_2-axis; that is,

$$\int_{\mathscr{S}_0} \mathbf{Sn}\, dA = \mathbf{0},$$

$$\int_{\mathscr{S}_0} \mathbf{r} \times \mathbf{Sn}\, dA = -m\mathbf{e}_2,$$

or equivalently, since $\mathbf{n} = -\mathbf{e}_3$ and $\mathbf{r}(\mathbf{p}) = \mathbf{p} - \mathbf{o}$,

$$\begin{gathered}\int_{\mathscr{S}_0} S_{13}\, dA = \int_{\mathscr{S}_0} S_{23}\, dA = \int_{\mathscr{S}_0} S_{33}\, dA = 0,\\ \int_{\mathscr{S}_0} p_2 S_{33}\, dA = \int_{\mathscr{S}_0} (p_1 S_{23} - p_2 S_{13})\, dA = 0,\\ \int_{\mathscr{S}_0} p_1 S_{33}\, dA = -m.\end{gathered} \tag{3}$$

We do not specify the loading on $\mathscr{S}_l$, since balance of forces and moments require that (Exercise 1)

(d) the total force on $\mathscr{S}_l$ vanish and the total moment on $\mathscr{S}_l$ equal $m\mathbf{e}_2$; in fact, (3) hold with $\mathscr{S}_0$ replaced by $\mathscr{S}_l$.

There are many sets of boundary conditions consistent with (c) and (d). We take the simplest possible and assume that

$$S_{13}(\mathbf{p}) = S_{23}(\mathbf{p}) = 0, \qquad S_{33}(\mathbf{p}) = \kappa_0 + \kappa_1 p_1 + \kappa_2 p_2 \qquad \text{at} \quad p_3 = 0, l,$$

with κ_0, κ_1, and κ_2 constants. Then (1) and (3) imply that

$$\kappa_0 = \kappa_2 = 0, \qquad \kappa_1 = -m/I,$$

where

$$I = \int_{\mathscr{S}_0} p_1^2\, dA$$

is the *moment of inertia* of $\mathscr{S}_0$ about the p_2-axis. Our boundary conditions therefore consist of (2) and

$$S_{13}(\mathbf{p}) = S_{23}(\mathbf{p}) = 0, \qquad S_{33}(\mathbf{p}) = -\frac{mp_1}{I} \quad \text{at} \quad p_3 = 0, l. \tag{4}$$

A stress field compatible with (2) and (4) is obtained by taking

$$S_{11}(\mathbf{p}) = S_{22}(\mathbf{p}) = S_{12}(\mathbf{p}) = S_{13}(\mathbf{p}) = S_{23}(\mathbf{p}) = 0,$$

$$S_{33}(\mathbf{p}) = -\frac{mp_1}{I}$$

for all $\mathbf{p}$ in the body. This field clearly satisfies

$$\operatorname{Div} \mathbf{S} = \mathbf{0}.$$

Thus to show that S is actually a solution of our problem we need only construct the corresponding displacement field.

Using the stress–strain law in the form (31.3), we see that

$$E_{12}(\mathbf{p}) = E_{13}(\mathbf{p}) = E_{23}(\mathbf{p}) = 0,$$

$$E_{11}(\mathbf{p}) = E_{22}(\mathbf{p}) = \frac{\nu m p_1}{EI}, \qquad E_{33}(\mathbf{p}) = -\frac{mp_1}{EI},$$

and the strain–displacement relation (32.1) is easily integrated to give (Exercise 2)

$$u_1(\mathbf{p}) = \frac{m}{2EI}[p_3^2 + \nu(p_1^2 - p_2^2)] + w_1(\mathbf{p}),$$

$$u_2(\mathbf{p}) = \frac{m}{EI}\nu p_1 p_2 + w_2(\mathbf{p}), \tag{5}$$

$$u_3(\mathbf{p}) = -\frac{m}{EI}p_1 p_3 + w_3(\mathbf{p})$$

with $\mathbf{w}(\mathbf{p})$ an arbitrary infinitesimal rigid displacement.

To compare our results with classical beam theory we "fix" $\mathscr{S}_0$ at $p_1 = p_2 = 0$ by requiring that

$$\mathbf{u}(\mathbf{o}) = \nabla\mathbf{u}(\mathbf{o}) = \mathbf{0}.$$

Then $\mathbf{w}(\mathbf{p}) \equiv \mathbf{0}$ and the displacement of the *centroidal axis* (i.e., the p_3-axis) takes the form

$$u_1(0, 0, p_3) = \frac{mp_3^2}{2EI},$$

$$u_2(0, 0, p_3) = u_3(0, 0, p_3) = 0.$$

[Actually, to insure that $\mathbf{w}(\mathbf{p}) \equiv \mathbf{0}$ it suffices to assume that $\mathbf{u}(\mathbf{o}) = \mathbf{0}$ and $\nabla\mathbf{u}(\mathbf{o}) = \nabla\mathbf{u}(\mathbf{o})^{\mathrm{T}}$.]

In this solution the maximum stress occurs at points of $\mathscr{L}$ for which $|p_1|$ attains its largest value, c say, and this maximal stress has magnitude

$$mc/I.$$

Moreover, the maximum deflection of the centroidal axis occurs at $p_3 = l$ and is given by

$$\frac{ml^2}{2EI}.$$

B. Torsion of a Circular Cylinder

Consider a circular cylinder of length l and radius a (Fig. 3). As before, the axis of the cylinder coincides with the p_3-axis, while $\mathscr{S}_0$ and $\mathscr{S}_l$ correspond to $p_3 = 0$ and $p_3 = l$. We assume that the end face $\mathscr{S}_0$ is held fixed, while the end face $\mathscr{S}_l$ is rigidly rotated about the p_3-axis through an angle β. Thus, assuming that the lateral surface $\mathscr{L}$ is traction-free, our boundary conditions consist of (2) and the requirement that

$$\mathbf{u}(\mathbf{p}) = \mathbf{0} \quad \text{at} \quad p_3 = 0,$$

$$u_1(\mathbf{p}) = -\alpha l p_2, \qquad u_2(\mathbf{p}) = \alpha l p_1, \qquad u_3(\mathbf{p}) = 0 \quad \text{at} \quad p_3 = l,$$

with $\alpha = \beta/l$ the *angle of twist* per unit length. (The boundary condition at $p_3 = l$ represents an infinitesimal rigid rotation of $\mathscr{S}_l$ about the p_3-axis [cf. (7.10) with $p_3 = l$]).

It seems reasonable to expect that under this type of loading the cylinder will twist uniformly along its length; thus, in view of (7.10), we consider a displacement field of the form

$$u_1(\mathbf{p}) = -\alpha p_2 p_3,$$

$$u_2(\mathbf{p}) = \alpha p_1 p_3, \tag{6}$$

$$u_3(\mathbf{p}) = 0.$$

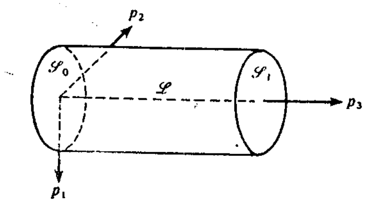

Figure 3

The corresponding stress field, computed using the strain–displacement relation (32.1) and the stress–strain relation (30.6), is given by

$$S_{11}(\mathbf{p}) = S_{22}(\mathbf{p}) = S_{33}(\mathbf{p}) = S_{12}(\mathbf{p}) = 0, \qquad S_{23}(\mathbf{p}) = \mu\alpha p_1, \qquad S_{13}(\mathbf{p}) = -\mu\alpha p_2, \tag{7}$$

and clearly satisfies

$$\operatorname{Div} \mathbf{S} = \mathbf{0}.$$

Further, S satisfies the boundary condition (2); indeed $(2)_{1,2}$ are satisfied trivially, and, since $\mathbf{n}(\mathbf{p})$ on $\mathscr{L}$ is proportional to $(p_1, p_2, 0)$, it follows that

$$S_{13}n_1 + S_{23}n_2 = 0 \quad \text{on} \quad \mathscr{L}, \tag{8}$$

which is $(2)_3$. Thus (6) and (7) comprise the solution of our problem.

Next, by (1) and (7),

$$\int_{\mathscr{S}_0} S_{13}\, dA = \int_{\mathscr{S}_0} S_{23}\, dA = \int_{\mathscr{S}_0} S_{33}\, dA = 0,$$

so that

$$\int_{\mathscr{S}_0} \mathbf{S}\mathbf{n}\, dA = \mathbf{0}, \tag{9}$$

and the total force on each end face vanishes. In addition,

$$\int_{\mathscr{S}_0} p_1 S_{33}\, dA = \int_{\mathscr{S}_0} p_2 S_{33}\, dA = 0,$$

$$\int_{\mathscr{S}_0} (p_1 S_{23} - p_2 S_{13})\, dA = \mu\alpha \int_{\mathscr{S}_0} (p_1^2 + p_2^2)\, dA = \mu\alpha I_0,$$

where $I_0 = \pi a^4/2$ is the *polar moment of inertia* of the cross section. Therefore

$$\int_{\mathscr{S}_0} \mathbf{r} \times \mathbf{S}\mathbf{n}\, dA = -m\mathbf{e}_3 \tag{10}$$

with

$$m = \mu\alpha I_0.$$

Further, by balance of moments,

$$\int_{\mathscr{S}_l} \mathbf{r} \times \mathbf{S}\mathbf{n}\, dA = m\mathbf{e}_3.$$

Thus the cylinder is twisted by equal and opposite couples about the p_3-axis.

Exercises

1. Establish (d).
2. Derive (5).
3. Show that $\mathbf{w}(\mathbf{p}) \equiv \mathbf{0}$ in (5) provided $\mathbf{u}(\mathbf{o}) = \mathbf{0}$ and $\nabla\mathbf{u}(\mathbf{o}) = \nabla\mathbf{u}(\mathbf{o})^{\mathrm{T}}$.

In the next two exercises we consider the torsion of an arbitrary cylindrical bar (cf. Fig. 2).

4. Show that (7) is compatible with (8) only if the cross section is circular.
5. Consider the displacement field

$$u_1(\mathbf{p}) = -\alpha p_2 p_3, \qquad u_2(\mathbf{p}) = \alpha p_1 p_3, \qquad u_3(\mathbf{p}) = \alpha\varphi(p_1, p_2);$$

φ is called the *warping function*.

(a) Compute the corresponding stress field S and show that $\operatorname{Div} \mathbf{S} = \mathbf{0}$ is equivalent to

$$\Delta\varphi = 0.$$

Here

$$\Delta\varphi = \varphi_{,11} + \varphi_{,22}, \qquad \varphi_{,11} = \frac{\partial^2\varphi}{\partial p_1^2}, \quad \text{etc.}$$

(b) Show that (2) is equivalent to

$$\frac{\partial\varphi(\mathbf{p})}{\partial\mathbf{n}} = p_2 n_1(\mathbf{p}) - p_1 n_2(\mathbf{p})$$

for all $\mathbf{p} \in \mathscr{L}$, where

$$\frac{\partial\varphi}{\partial\mathbf{n}} = \varphi_{,1}n_1 + \varphi_{,2}n_2 \tag{11}$$

is the normal derivative of φ on $\mathscr{L}$.

(c) Show that (9) and (10) hold, where

$$m = \kappa\alpha,$$

$$\kappa = \mu \int_{\mathscr{S}_0} (p_1^2 + p_2^2 + p_1\varphi_{,2} - p_2\varphi_{,1})\, dA;$$

κ is called the *torsional rigidity* of the cross section.

34. LINEAR ELASTODYNAMICS

In this section we will establish some of the basic results of linear elastodynamics. Here the basic equations, as derived previously, are

$$\begin{aligned} \mathbf{E} &= \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \\ \mathbf{S} &= \mathbf{C}[\mathbf{E}], \\ \operatorname{Div}\mathbf{S} + \mathbf{b} &= \rho\ddot{\mathbf{u}} \end{aligned} \tag{1}$$

(where we have written ρ for ρ_0 and $\mathbf{b}$ for $\mathbf{b}_0$).

We assume that $\mathscr{B}$ *is bounded and that* ρ *is continuous.*

Let $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be a list of fields on $\mathscr{B} \times [0, \infty)$ with $\mathbf{u}$ of class C^2 and $\mathbf{E}$ and $\mathbf{S}$ smooth, and suppose that (1) holds with $\mathbf{b}$ a given body force field on $\mathscr{B} \times [0, \infty)$. Then $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is called an **elastic process** corresponding to $\mathbf{b}$. Since $\mathbf{E}$ is time-dependent, the strain energy

$$\mathscr{U}\{\mathbf{E}\} = \tfrac{1}{2}\int_{\mathscr{B}} \mathbf{E}\cdot\mathbf{C}[\mathbf{E}]\,dV$$

depends on time. In fact, when $\mathbf{C}$ is symmetric,

$$\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{C}[\mathbf{E}])^{\cdot} = \tfrac{1}{2}(\dot{\mathbf{E}}\cdot\mathbf{C}[\mathbf{E}] + \mathbf{E}\cdot\mathbf{C}[\dot{\mathbf{E}}]) = \dot{\mathbf{E}}\cdot\mathbf{C}[\mathbf{E}] = \mathbf{S}\cdot\dot{\mathbf{E}}.$$

Thus

$$(\mathscr{U}\{\mathbf{E}\})^{\cdot} = \int_{\mathscr{B}} \mathbf{S}\cdot\dot{\mathbf{E}}\,dV, \tag{2}$$

so that the rate of change of strain energy is equal to the stress power.

Theorem of Power and Energy. *Assume that* $\mathbf{C}$ *is symmetric. Let* $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ *be an elastic process corresponding to the body force* $\mathbf{b}$. *Then*

$$\int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\dot{\mathbf{u}}\,dA + \int_{\mathscr{B}} \mathbf{b}\cdot\dot{\mathbf{u}}\,dV = (\mathscr{U}\{\mathbf{E}\} + \mathscr{K}\{\dot{\mathbf{u}}\})^{\cdot}, \tag{3}$$

where

$$\mathscr{K}\{\dot{\mathbf{u}}\} = \tfrac{1}{2}\int_{\mathscr{B}} \rho\dot{\mathbf{u}}^2\,dV$$

is the **kinetic energy**.

Proof. In view of the lemma on page 205 with $\tilde{\mathbf{u}}$ replaced by $\dot{\mathbf{u}}$ and $\mathbf{b}$ by $\mathbf{b} - \rho\ddot{\mathbf{u}}$,

$$\int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\dot{\mathbf{u}}\,dA + \int_{\mathscr{B}} \mathbf{b}\cdot\dot{\mathbf{u}}\,dV = \int_{\mathscr{B}} \mathbf{S}\cdot\dot{\mathbf{E}}\,dV + \int_{\mathscr{B}} \rho\ddot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,dV.$$

This relation, (2), and the identity

$$(\mathscr{K}\{\dot{\mathbf{u}}\})^{\cdot} = \int_{\mathscr{B}} \rho\ddot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,dV$$

imply (3). □

This theorem asserts that the power expended by the external forces equals the rate at which the total energy is changing.

Corollary. *Assume that* $\mathbf{C}$ *is symmetric. Let* $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ *be an elastic process corresponding to body force* $\mathbf{b} = \mathbf{0}$, *and suppose that*

$$\mathbf{Sn}\cdot\dot{\mathbf{u}} = 0 \quad \text{on} \quad \partial\mathscr{B}. \tag{4}$$

Then the total energy is constant:

$$\mathscr{U}\{\mathbf{E}\} + \mathscr{K}\{\dot{\mathbf{u}}\} = \text{const.} \tag{5}$$

Another interesting corollary may be deduced as follows. Consider an elastic process which starts from an unstrained rest state. Then

$$\mathbf{E}(\cdot, 0) = \mathbf{0}, \qquad \dot{\mathbf{u}}(\cdot, 0) = \mathbf{0},$$

and the total energy is initially zero. Assume that $\mathbf{C}$ is positive definite and $\rho > 0$, so that the total energy is always nonnegative. Then, if we integrate (3) with respect to time from 0 to τ, we arrive at

$$\int_0^\tau\int_{\partial\mathscr{B}} \mathbf{Sn}\cdot\dot{\mathbf{u}}\,dA\,dt + \int_0^\tau\int_{\mathscr{B}} \mathbf{b}\cdot\dot{\mathbf{u}}\,dV\,dt \geq 0. \tag{6}$$

The left side of (6) represents the work done by the external forces in the time interval $[0, \tau]$; (6) asserts that for an elastic process starting from an unstrained rest state this work is always nonnegative.

The **mixed problem** of elastodynamics can be stated as follows:

Given: $\mathscr{B}$, complementary regular subsets $\mathscr{S}_1$ and $\mathscr{S}_2$ of $\partial\mathscr{B}$, an elasticity tensor $\mathbf{C}$ on $\mathscr{B}$, a density field ρ on $\mathscr{B}$, a body force field $\mathbf{b}$ on $\mathscr{B} \times [0, \infty)$, surface displacements $\hat{\mathbf{u}}$ on $\mathscr{S}_1 \times [0, \infty)$, surface tractions $\hat{\mathbf{s}}$ on $\mathscr{S}_2 \times [0, \infty)$, an initial displacement field $\mathbf{u}_0$ on $\mathscr{B}$, an initial velocity field $\mathbf{v}_0$ on $\mathscr{B}$.

Find: An elastic process $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ that corresponds to $\mathbf{b}$, satisfies the initial conditions

$$\mathbf{u}(\mathbf{p}, 0) = \mathbf{u}_0(\mathbf{p}), \qquad \dot{\mathbf{u}}(\mathbf{p}, 0) = \mathbf{v}_0(\mathbf{p})$$

for every $\mathbf{p} \in \mathscr{B}$, and satisfies the boundary conditions

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \quad \mathscr{S}_1 \times [0, \infty), \qquad \mathbf{Sn} = \hat{\mathbf{s}} \quad \text{on} \quad \mathscr{S}_2 \times [0, \infty).$$

An elastic process with these properties will be called a **solution**.

Uniqueness Theorem. *The mixed problem has at most one solution provided* C *is symmetric and positive definite and* $\rho > 0$.

Proof. Let $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ denote the difference between two solutions. Then $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is an elastic process corresponding to $\mathbf{b} = \mathbf{0}$ and satisfies

$$\mathbf{u}(\cdot, 0) = \dot{\mathbf{u}}(\cdot, 0) = \mathbf{0},$$
$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_1 \times [0, \infty), \qquad \mathbf{Sn} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_2 \times [0, \infty). \tag{7}$$

Thus (4) holds and we conclude from (5) that

$$\mathscr{U}\{\mathbf{E}\} + \mathscr{K}\{\dot{\mathbf{u}}\} = 0 \tag{8}$$

for all time. Here we have used the fact that, by $(7)_{1,2}$, the total energy is initially zero. But since C is positive definite and $\rho > 0$, both the strain energy and kinetic energy are nonnegative; hence (8) implies that $\mathscr{K}\{\dot{\mathbf{u}}\} = 0$, and this in turn yields $\dot{\mathbf{u}} = \mathbf{0}$ on $\mathscr{B} \times [0, \infty)$. This fact and $(7)_1$ imply that

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \mathscr{B} \times [0, \infty),$$

and the proof is complete. □

Exercises

Throughout these exercises C is symmetric and $\mathscr{B}$ bounded.

1. Consider an elastic process and define

$$\varepsilon = \tfrac{1}{2}\mathbf{E} \cdot \mathsf{C}[\mathbf{E}] + \tfrac{1}{2}\rho\dot{\mathbf{u}}^2,$$
$$\mathbf{q} = -\mathbf{S}\dot{\mathbf{u}}.$$

Show that

$$\dot{\varepsilon} = -\operatorname{Div} \mathbf{q} + \mathbf{b} \cdot \dot{\mathbf{u}}.$$

2. (*Brun's theorem*) Consider an elastic process with *null initial data*, i.e., with

$$\mathbf{u}(\mathbf{p}, 0) = \mathbf{0}, \qquad \dot{\mathbf{u}}(\mathbf{p}, 0) = \mathbf{0} \tag{9}$$

for all $\mathbf{p} \in \mathscr{B}$. Show that

$$\mathscr{U}\{\mathbf{E}\} - \mathscr{K}\{\dot{\mathbf{u}}\} = \Phi, \tag{10}$$

where

$$\Phi(t) = \frac{1}{2}\int_0^t [\varphi(t + \tau, t - \tau) - \varphi(t - \tau, t + \tau)]\, d\tau,$$
$$\varphi(\alpha, \beta) = \int_{\partial\mathscr{B}} \mathbf{s}(\mathbf{p}, \alpha) \cdot \dot{\mathbf{u}}(\mathbf{p}, \beta)\, dA_{\mathbf{p}} + \int_{\mathscr{B}} \mathbf{b}(\mathbf{p}, \alpha) \cdot \dot{\mathbf{u}}(\mathbf{p}, \beta)\, dV_{\mathbf{p}},$$

with $\mathbf{s} = \mathbf{Sn}$.

3. Use (8) and (10) to establish uniqueness for the mixed problem when C is symmetric (but not necessarily positive definite) and $\rho > 0$.
4. Show that under the null initial data (9) the equation of motion

$$\operatorname{Div} \mathbf{S} + \mathbf{b} = \rho\ddot{\mathbf{u}}$$

can be written in the form

$$\lambda * (\operatorname{Div} \mathbf{S} + \mathbf{b}) = \rho\mathbf{u},$$

where $\lambda(t) = t$ and $*$ denotes convolution; i.e., if Λ and Ψ are functions on $\mathscr{B} \times [0, \infty)$,

$$(\Lambda * \Psi)(\mathbf{p}, t) = \int_0^t \Lambda(\mathbf{p}, t - \tau)\Psi(\mathbf{p}, \tau)\, d\tau. \tag{11}$$

5. (*Graffi's reciprocal theorem*) Let us agree to use the notation (11) when Λ and Ψ are *vector* fields, but here the product in the integrand is the inner product $\Lambda(\mathbf{p}, t - \tau) \cdot \Psi(\mathbf{p}, \tau)$. Show that if $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ and $[\tilde{\mathbf{u}}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}]$ are elastic processes corresponding to body forces $\mathbf{b}$ and $\tilde{\mathbf{b}}$, respectively, and to null initial data, then

$$\int_{\partial\mathscr{B}} \mathbf{s} * \tilde{\mathbf{u}}\, dA + \int_{\mathscr{B}} \mathbf{b} * \tilde{\mathbf{u}}\, dV = \int_{\partial\mathscr{B}} \tilde{\mathbf{s}} * \mathbf{u}\, dA + \int_{\mathscr{B}} \tilde{\mathbf{b}} * \mathbf{u}\, dV, \tag{12}$$

where $\mathbf{s} = \mathbf{Sn}$ and $\tilde{\mathbf{s}} = \tilde{\mathbf{S}}\mathbf{n}$.

35. PROGRESSIVE WAVES

Sinusoidal progressive waves form an important class of solutions to the equations of linear elastodynamics. We will study these waves under the assumptions of homogeneity and isotropy, and in the absence of body forces. The underlying field equation is then the displacement equation of motion

$$\mu\,\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{Div} \mathbf{u} = \rho\ddot{\mathbf{u}}. \tag{1}$$

A vector field $\mathbf{u}$ of the form

$$\mathbf{u}(\mathbf{p}, t) = \mathbf{a}\sin(\mathbf{r} \cdot \mathbf{m} - ct), \qquad \mathbf{r} = \mathbf{p} - \mathbf{o} \tag{2}$$

with $|\mathbf{m}| = 1$ is called a **sinusoidal progressive wave** with **amplitude a**, **direction m**, and **velocity** c. We say that $\mathbf{u}$ is **longitudinal** if $\mathbf{a}$ and $\mathbf{m}$ are parallel, **transverse** if $\mathbf{a}$ and $\mathbf{m}$ are perpendicular.

We now determine conditions which are necessary and sufficient that (2) solve (1). By the chain rule,

$$\nabla\mathbf{u} = (\mathbf{a} \otimes \mathbf{m}) \cos\varphi,$$

where

$$\varphi(\mathbf{p}, t) = \mathbf{r} \cdot \mathbf{m} - ct,$$

and hence

$$\operatorname{Div} \mathbf{u} = (\mathbf{a} \cdot \mathbf{m}) \cos \varphi,$$
$$\operatorname{Curl} \mathbf{u} = (\mathbf{m} \times \mathbf{a}) \cos \varphi.$$

Thus the wave is

$$\text{longitudinal} \Leftrightarrow \operatorname{Curl} \mathbf{u} = \mathbf{0},$$
$$\text{transverse} \Leftrightarrow \operatorname{Div} \mathbf{u} = 0.$$

Next,

$$\Delta \mathbf{u} = -\mathbf{a} \sin \varphi,$$
$$\nabla \operatorname{Div} \mathbf{u} = -(\mathbf{a} \cdot \mathbf{m})\mathbf{m} \sin \varphi,$$
$$\ddot{\mathbf{u}} = -c^2 \mathbf{a} \sin \varphi,$$

and **u** satisfies (1) if and only if

$$\mu \mathbf{a} + (\lambda + \mu)(\mathbf{a} \cdot \mathbf{m})\mathbf{m} = \rho c^2 \mathbf{a}. \tag{3}$$

We call the tensor

$$\mathbf{A}(\mathbf{m}) = \frac{1}{\rho}[\mu \mathbf{I} + (\lambda + \mu)\mathbf{m} \otimes \mathbf{m}]$$

the **acoustic tensor**; with this definition (3) takes the simple form

$$\mathbf{A}(\mathbf{m})\mathbf{a} = c^2 \mathbf{a}. \tag{4}$$

Thus a necessary and sufficient condition that **u** satisfy (1) is that c^2 be an eigenvalue and **a** a corresponding eigenvector of the acoustic tensor **A(m)**.

A simple computation shows that

$$\mathbf{A}(\mathbf{m}) = \left(\frac{\lambda + 2\mu}{\rho}\right)\mathbf{m} \otimes \mathbf{m} + \frac{\mu}{\rho}(\mathbf{I} - \mathbf{m} \otimes \mathbf{m}).$$

But this is simply the spectral decomposition of **A(m)**, and we conclude from (b) of the spectral theorem (page 11) that

$$(\lambda + 2\mu)/\rho \qquad \text{and} \qquad \mu/\rho$$

are the eigenvalues of **A(m)**, while the line spanned by **m** and the plane (through **0**) perpendicular to **m** are the corresponding characteristic spaces. Thus we have the following

Theorem. *A sinusoidal progressive wave with velocity c will be a solution of* (1) *if and only if either*

(a) $c^2 = (2\mu + \lambda)/\rho$ *and the wave is longitudinal, or*
(b) $c^2 = \mu/\rho$ *and the wave is transverse.*

This theorem asserts that for a homogeneous and isotropic material only two types of sinusoidal progressive waves are possible: longitudinal and transverse. The two corresponding speeds

$$\sqrt{(2\mu + \lambda)/\rho} \qquad \text{and} \qquad \sqrt{\mu/\rho}$$

are called, respectively, the *longitudinal* and *transverse sound speeds* of the material. (Note that these speeds are real when the elasticity tensor is positive definite.) For an anisotropic body the situation is far more complicated. A propagation condition of the form (4) still holds (Exercise 1), but the waves will generally be neither longitudinal nor transverse, and they will generally propagate with different speeds in different directions.

Exercises

1. For an anisotropic but homogeneous body the displacement equation of motion has the form

$$\operatorname{Div} \mathsf{C}[\nabla \mathbf{u}] = \rho \ddot{\mathbf{u}} \tag{5}$$

with C and ρ independent of position. (Here we have assumed zero body forces.)

(a) Show that the sinusoidal progressive wave (2) solves (5) if and only if the propagation condition (4) holds, where now **A(m)** is the tensor defined by

$$\mathbf{A}(\mathbf{m})\mathbf{k} = \frac{1}{\rho}\mathsf{C}[\mathbf{k} \otimes \mathbf{m}]\mathbf{m}$$

for every vector **k**.

(b) Show that **A(m)** is positive definite for each direction **m** if and only if C is strongly elliptic (cf. Exercise 29.3).

(c) Show that

$$\mathbf{Q}\mathbf{A}(\mathbf{m})\mathbf{Q}^T = \mathbf{A}(\mathbf{Q}\mathbf{m})$$

for any symmetry transformation **Q**.

SELECTED REFERENCES

Fichera [1].
Gurtin [1].
Love [1].
Sokolnikoff [1].
Villaggio [1].

Appendix

36. THE EXPONENTIAL FUNCTION

The exponential of a tensor can be defined either in terms of its series representation[1] or in terms of a solution to an ordinary differential equation. For our purposes the latter is more convenient. Thus let $\mathbf{A}$ be a tensor and consider the initial-value problem

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t), \qquad t > 0, \qquad \mathbf{X}(0) = \mathbf{I}, \tag{1}$$

for a tensor function $\mathbf{X}(t)$, $0 \le t < \infty$. The existence theorem for linear differential equations tells us that this problem has exactly one solution $\mathbf{X}: [0, \infty) \to \mathrm{Lin}$, which we write in the form

$$\mathbf{X}(t) = e^{\mathbf{A}t}.$$

Proposition. *For each* $t \ge 0$, $e^{\mathbf{A}t}$ *belongs to* Lin^+ *and*

$$\det(e^{\mathbf{A}t}) = e^{(\mathrm{tr}\,\mathbf{A})t}. \tag{2}$$

Proof. Let $\mathbf{X}(t) = e^{\mathbf{A}t}$. Since $\mathbf{X}$ is continuous and

$$\det \mathbf{X}(0) = 1, \tag{3}$$

[1] See, e.g., Hirsch and Smale [1, Chapter 5, §3].

$\det \mathbf{X}(t) > 0$ in some nonempty interval $[0, \tau)$. Let τ be as large as possible; that is, let

$$\tau = \sup\{\lambda \mid \det \mathbf{X}(t) > 0 \quad \text{for} \quad 0 \le t \le \lambda\}.$$

To show that $\mathbf{X}(t) \in \text{Lin}^+$ for all $t \geq 0$ we must show that $\tau = \infty$. Suppose, to the contrary, that τ is finite. Then, since $\mathbf{X}$ is continuous,

$$\det \mathbf{X}(\tau) = 0. \tag{4}$$

We will show that this leads to a contradiction. Since $\mathbf{X}(t)$ is invertible for all $t \in [0, \tau)$, $(1)_1$ implies

$$\mathbf{A} = \dot{\mathbf{X}}(t)\mathbf{X}(t)^{-1}$$

on $(0, \tau)$, and we conclude from (3.14) that

$$(\det \mathbf{X})^{\cdot} = (\text{tr}\, \mathbf{A}) \det \mathbf{X}$$

on $(0, \tau)$. This equation with the initial condition (3) has the unique solution

$$\det \mathbf{X}(t) = e^{(\text{tr}\, \mathbf{A})t} \tag{5}$$

for $0 \leq t < \tau$. In view of the continuity of $\mathbf{X}$, this result clearly implies that $\det \mathbf{X}(\tau) > 0$, which contradicts (4). Thus $\tau = \infty$ and (5) implies (2). □

Proposition. *Let* $\mathbf{W}$ *be a skew tensor. Then* $e^{\mathbf{W}t}$ *is a rotation for each* $t \geq 0$.

Proof. Let $\mathbf{X}(t) = e^{\mathbf{W}t}$ for $t \geq 0$. Then, by definition,

$$\dot{\mathbf{X}} = \mathbf{W}\mathbf{X}, \qquad \mathbf{X}(0) = \mathbf{I}.$$

Let

$$\mathbf{Z} = \mathbf{X}\mathbf{X}^{\mathsf{T}}.$$

Then, since $\mathbf{W}$ is skew,

$$\dot{\mathbf{Z}} = \dot{\mathbf{X}}\mathbf{X}^{\mathsf{T}} + \mathbf{X}\dot{\mathbf{X}}^{\mathsf{T}} = \mathbf{W}\mathbf{X}\mathbf{X}^{\mathsf{T}} + \mathbf{X}\mathbf{X}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}} = \mathbf{W}\mathbf{X}\mathbf{X}^{\mathsf{T}} - \mathbf{X}\mathbf{X}^{\mathsf{T}}\mathbf{W},$$

and $\mathbf{Z}$ satisfies

$$\dot{\mathbf{Z}} = \mathbf{W}\mathbf{Z} - \mathbf{Z}\mathbf{W}, \qquad \mathbf{Z}(0) = \mathbf{I}.$$

This initial-value problem has the unique solution $\mathbf{Z}(t) = \mathbf{I}$ for all $t \geq 0$. Thus

$$\mathbf{X}(t)\mathbf{X}(t)^{\mathsf{T}} = \mathbf{I}$$

and $\mathbf{X}(t)$ is orthogonal. But from the preceding proposition $\mathbf{X}(t) \in \text{Lin}^+$. Thus $\mathbf{X}(t)$ is a rotation. □

Exercise

1. Let $\mathbf{A} \in \text{Lin}$ and $\mathbf{v}_0 \in \mathscr{V}$. Show that the function

$$\mathbf{v}(t) = e^{\mathbf{A}t}\mathbf{v}_0$$

satisfies the initial-value problem

$$\dot{\mathbf{v}}(t) = \mathbf{A}\mathbf{v}(t), \qquad t > 0,$$
$$\mathbf{v}(0) = \mathbf{v}_0.$$

Selected Reference

Hirsch and Smale [1].

37. ISOTROPIC FUNCTIONS

Let $\mathscr{G} \subset \text{Orth}$. A set $\mathscr{A} \subset \text{Lin}$ is **invariant under** $\mathscr{G}$ if $\mathbf{QAQ}^{\mathsf{T}} \in \mathscr{A}$ whenever $\mathbf{A} \in \mathscr{A}$ and $\mathbf{Q} \in \mathscr{G}$.

Proposition. *The following sets are invariant under* Orth:

$$\text{Lin}, \quad \text{Lin}^+, \quad \text{Orth}, \quad \text{Orth}^+, \quad \text{Sym}, \quad \text{Skw}, \quad \text{Psym}.$$

Proof. We will give the proof only for Lin^+. Choose $\mathbf{A} \in \text{Lin}$ and $\mathbf{Q} \in \text{Orth}$. Then

$$\det(\mathbf{QAQ}^{\mathsf{T}}) = (\det \mathbf{A})(\det \mathbf{Q})^2 = \det \mathbf{A}, \tag{1}$$

since $|\det \mathbf{Q}| = 1$. Thus $\mathbf{A} \in \text{Lin}^+$ implies $\mathbf{QAQ}^{\mathsf{T}} \in \text{Lin}^+$. □

Let $\mathscr{A} \subset \text{Lin}$. A **scalar function**

$$\varphi : \mathscr{A} \to \mathbb{R}$$

is invariant under $\mathscr{G}$ if $\mathscr{A}$ is invariant under $\mathscr{G}$ and

$$\varphi(\mathbf{A}) = \varphi(\mathbf{QAQ}^{\mathsf{T}}) \tag{2}$$

for every $\mathbf{A} \in \mathscr{A}$ and $\mathbf{Q} \in \mathscr{G}$. Similarly, a **tensor function**

$$\mathbf{G} : \mathscr{A} \to \text{Lin}$$

is invariant under $\mathscr{G}$ if $\mathscr{A}$ is invariant under $\mathscr{G}$ and

$$\mathbf{QG(A)Q}^{\mathsf{T}} = \mathbf{G}(\mathbf{QAQ}^{\mathsf{T}})$$

for every $\mathbf{A} \in \mathscr{A}$ and $\mathbf{Q} \in \mathscr{G}$. An **isotropic function** is a function invariant under Orth.

Proposition. *Let Φ be a scalar or tensor function with domain in* Lin. *Then Φ is isotropic if Φ is invariant under* Orth^+.

Proof. The proof follows from the identity

$$(-\mathbf{Q})\mathbf{A}(-\mathbf{Q})^T = \mathbf{QAQ}^T$$

and the fact that for $\mathbf{Q} \in \text{Orth}$ either $\mathbf{Q}$ or $-\mathbf{Q}$ belongs to Orth^+. □

Theorem

(a) det *and* tr, *considered as functions on* Lin, *are isotropic.*
(b) *The list* (2.9) *of principal invariants is isotropic; that is,*

$$\mathscr{I}_{\mathbf{A}} = \mathscr{I}_{\mathbf{QAQ}^T} \tag{3}$$

for every $\mathbf{A} \in \text{Lin}$ *and* $\mathbf{Q} \in \text{Orth}$.

Proof. That det is isotropic follows from (1). Next, for $\mathbf{Q} \in \text{Orth}$,

$$\text{tr}(\mathbf{QAQ}^T) = \text{tr}(\mathbf{AQ}^T\mathbf{Q}) = \text{tr}\,\mathbf{A},$$

so that tr is isotropic. By (2.8) and (2.9), since $\mathbf{A} \mapsto (\text{tr}\,\mathbf{A})^2$ is isotropic, to establish (b) we have only to show that $\mathbf{A} \mapsto \text{tr}(\mathbf{A}^2)$ is isotropic. Choose $\mathbf{A} \in \text{Lin}$ and $\mathbf{Q} \in \text{Orth}$. Then

$$(\mathbf{QAQ}^T)^2 = \mathbf{QAQ}^T\mathbf{QAQ}^T = \mathbf{QA}^2\mathbf{Q}^T,$$

and hence

$$\text{tr}[(\mathbf{QAQ}^T)^2] = \text{tr}(\mathbf{QA}^2\mathbf{Q}^T) = \text{tr}\,\mathbf{A}^2. \quad \square$$

For convenience, we write

$$\mathscr{I}(\mathscr{A}) = \{\mathscr{I}_{\mathbf{A}} \mid \mathbf{A} \in \mathscr{A}\}$$

for the set of all possible lists $\mathscr{I}_{\mathbf{A}}$ as $\mathbf{A}$ ranges through the set $\mathscr{A}$. Of course, $\mathscr{I}(\mathscr{A}) \subset \mathbb{R}^3$.

Next we establish several important representation theorems for functions with domain $\mathscr{A}$ in Sym. *We assume for the remainder of this section that $\mathscr{A}$ is invariant under* Orth.

Representation Theorem for Isotropic Scalar Functions. *A function*

$$\varphi: \mathscr{A} \to \mathbb{R} \qquad (\mathscr{A} \subset \text{Sym})$$

is isotropic if and only if there exists a funcion $\tilde{\varphi}: \mathscr{I}(\mathscr{A}) \to \mathbb{R}$ such that

$$\varphi(\mathbf{A}) = \tilde{\varphi}(\mathscr{I}_{\mathbf{A}}) \tag{4}$$

for every $\mathbf{A} \in \mathscr{A}$.

Proof. Assume that φ is isotropic. To establish the representation (4) it suffices to show that

$$\varphi(\mathbf{A}) = \varphi(\mathbf{B}) \tag{5}$$

whenever

$$\mathscr{I}_{\mathbf{A}} = \mathscr{I}_{\mathbf{B}}. \tag{6}$$

Thus let $\mathbf{A}, \mathbf{B} \in \text{Sym}$ and assume that (6) holds. By (6) and the proposition on page 16, $\mathbf{A}$ and $\mathbf{B}$ have the same spectrum. Thus by the spectral theorem there exist orthonormal bases $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_i\}$ such that

$$\mathbf{A} = \sum_i \omega_i \mathbf{e}_i \otimes \mathbf{e}_i, \qquad \mathbf{B} = \sum_i \omega_i \mathbf{f}_i \otimes \mathbf{f}_i.$$

Let $\mathbf{Q}$ be the orthogonal tensor carrying the basis $\{\mathbf{f}_i\}$ into the basis $\{\mathbf{e}_i\}$:

$$\mathbf{Qf}_i = \mathbf{e}_i.$$

Then, since

$$\mathbf{Q}(\mathbf{f}_i \otimes \mathbf{f}_i)\mathbf{Q}^T = (\mathbf{Qf}_i) \otimes (\mathbf{Qf}_i),$$

it follows that

$$\mathbf{QBQ}^T = \mathbf{A}.$$

But since φ is isotropic, $\varphi(\mathbf{QBQ}^T) = \varphi(\mathbf{B})$; thus (5) holds and φ admits the representation (4).

The converse assertion, that (4) defines an isotropic function, is a trivial consequence of (3). □

Transfer Theorem. *Let*

$$\mathbf{G}: \mathscr{A} \to \text{Lin} \qquad (\mathscr{A} \subset \text{Sym})$$

be isotropic. Then every eigenvector of $\mathbf{A} \in \mathscr{A}$ *is an eigenvector of* $\mathbf{G}(\mathbf{A})$.

Proof. Let $\mathbf{e}$ be an eigenvector of $\mathbf{A} \in \mathscr{A}$, and let $\mathbf{Q} \in \text{Orth}$ be the reflection across the plane perpendicular to $\mathbf{e}$:

$$\mathbf{Qe} = -\mathbf{e}, \qquad \mathbf{Qf} = \mathbf{f} \quad \text{if} \quad \mathbf{f}\cdot\mathbf{e} = 0. \tag{7}$$

Then by the spectral theorem, $\mathbf{Q}$ leaves invariant the characteristic spaces of $\mathbf{A}$; hence we may conclude from the commutation theorem that

$$\mathbf{QAQ}^T = \mathbf{A}.$$

Thus, since $\mathbf{G}$ is isotropic,

$$\mathbf{QG}(\mathbf{A})\mathbf{Q}^T = \mathbf{G}(\mathbf{QAQ}^T) = \mathbf{G}(\mathbf{A}),$$

so that $\mathbf{Q}$ commutes with $\mathbf{G}(\mathbf{A})$. Therefore

$$\mathbf{QG}(\mathbf{A})\mathbf{e} = \mathbf{G}(\mathbf{A})\mathbf{Qe} = -\mathbf{G}(\mathbf{A})\mathbf{e},$$

and $\mathbf{Q}$ transforms $\mathbf{G}(\mathbf{A})\mathbf{e}$ into its negative. But by (7) this can happen only if $\mathbf{G}(\mathbf{A})\mathbf{e}$ is parallel to $\mathbf{e}$:

$$\mathbf{G}(\mathbf{A})\mathbf{e} = \omega\mathbf{e}.$$

Thus $\mathbf{e}$ is an eigenvector of $\mathbf{G}(\mathbf{A})$. □

The following lemma, whose statement is based on the spectral theorem, will be extremely useful in what follows.

Wang's Lemma. *Let* $\mathbf{A} \in \text{Sym}$.

(a) *Consider the spectral decomposition*

$$\mathbf{A} = \sum_i \omega_i \mathbf{e}_i \otimes \mathbf{e}_i, \tag{8}$$

and assume that the eigenvalues ω_i *are distinct. Then the set* $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\}$ *is linearly independent and*

$$\text{sp}\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\} = \text{sp}\{\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3\}. \tag{9}$$

(b) *Assume that* $\mathbf{A}$ *has exactly two distinct eigenvalues, so that*

$$\mathbf{A} = \omega_1 \mathbf{e} \otimes \mathbf{e} + \omega_2(\mathbf{I} - \mathbf{e} \otimes \mathbf{e}), \qquad |\mathbf{e}| = 1. \tag{10}$$

Then $\{\mathbf{I}, \mathbf{A}\}$ *is linearly independent and*

$$\text{sp}\{\mathbf{I}, \mathbf{A}\} = \text{sp}\{\mathbf{e} \otimes \mathbf{e}, \mathbf{I} - \mathbf{e} \otimes \mathbf{e}\}. \tag{11}$$

Proof. We prove only (a). To establish the linear independence of $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\}$ we must show that

$$\alpha\mathbf{A}^2 + \beta\mathbf{A} + \gamma\mathbf{I} = \mathbf{0} \tag{12}$$

implies

$$\alpha = \beta = \gamma = 0. \tag{13}$$

Assume (12) holds. Acting with (12) on the eigenvector $\mathbf{e}_i$ leads to the relation

$$(\alpha\omega_i^2 + \beta\omega_i + \gamma)\mathbf{e}_i = \mathbf{0},$$

so that

$$\alpha\omega_i^2 + \beta\omega_i + \gamma = 0,$$

and the ω_i are roots of a quadratic equation; since the ω_i are distinct, this is possible only if (13) hold. Thus $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\}$ is linearly independent.

Next, the subspace

$$\mathscr{H} = \text{sp}\{\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3\}$$

of Lin has dimension 3. On the other hand, since

$$\mathbf{A}^2 = \sum_i \omega_i^2 \mathbf{e}_i \otimes \mathbf{e}_i, \tag{14}$$

as is clear from $(1.2)_3$ and (8), we may conclude from $(1.2)_4$ and (8) that $\mathbf{I}, \mathbf{A}, \mathbf{A}^2 \in \mathscr{H}$. But 3 linearly independent vectors in a vector space of dimension 3 must necessarily span the space. Thus $\mathscr{H} = \text{sp}\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\}$ and the proof of (a) is complete. We leave the remainder of the proof as an exercise. □

We are now in a position to establish several important representation theorems for isotropic tensor functions.

First Representation Theorem for Isotropic Tensor Functions. *A function*

$$\mathbf{G}: \mathscr{A} \to \text{Sym} \qquad (\mathscr{A} \subset \text{Sym})$$

is isotropic if and only if there exist scalar functions $\varphi_0, \varphi_1, \varphi_2: \mathscr{I}(\mathscr{A}) \to \mathbb{R}$ *such that*

$$\mathbf{G}(\mathbf{A}) = \varphi_0(\mathscr{I}_\mathbf{A})\mathbf{I} + \varphi_1(\mathscr{I}_\mathbf{A})\mathbf{A} + \varphi_2(\mathscr{I}_\mathbf{A})\mathbf{A}^2 \tag{15}$$

for every $\mathbf{A} \in \mathscr{A}$.

Proof. Assume first that $\mathbf{G}$ admits the representation (15). Choose $\mathbf{A} \in \mathscr{A}$ and $\mathbf{Q} \in \text{Orth}$. Then by (3),

$$\begin{aligned}\mathbf{G}(\mathbf{QAQ}^T) &= \varphi_0(\mathscr{I}_{\mathbf{QAQ}^T})\mathbf{I} + \varphi_1(\mathscr{I}_{\mathbf{QAQ}^T})\mathbf{QAQ}^T + \varphi_2(\mathscr{I}_{\mathbf{QAQ}^T})\mathbf{QAQ}^T\mathbf{QAQ}^T \\ &= \varphi_0(\mathscr{I}_\mathbf{A})\mathbf{QQ}^T + \varphi_1(\mathscr{I}_\mathbf{A})\mathbf{QAQ}^T + \varphi_2(\mathscr{I}_\mathbf{A})\mathbf{QA}^2\mathbf{Q}^T = \mathbf{QG}(\mathbf{A})\mathbf{Q}^T,\end{aligned}$$

so that $\mathbf{G}$ is isotropic.

To prove the converse assertion assume that $\mathbf{G}$ is isotropic. Choose $\mathbf{A} \in \mathscr{A}$.

Case 1. $\mathbf{A}$ has exactly three distinct eigenvalues. Let (8) be the spectral decomposition of $\mathbf{A}$. By the transfer theorem

$$\mathbf{G}(\mathbf{A}) = \sum_i \beta_i \mathbf{e}_i \otimes \mathbf{e}_i,$$

and we conclude from (9) that there exist scalars $\alpha_0(\mathbf{A})$, $\alpha_1(\mathbf{A})$, and $\alpha_2(\mathbf{A})$ such that

$$\mathbf{G}(\mathbf{A}) = \alpha_0(\mathbf{A})\mathbf{I} + \alpha_1(\mathbf{A})\mathbf{A} + \alpha_2(\mathbf{A})\mathbf{A}^2. \tag{16}$$

Case 2. A has exactly two distinct eigenvalues. Then A admits the representation (10), and the characteristic spaces for A are sp{e} and $\{e\}^\perp$. By the transfer theorem each of these two subspaces must be contained in a characteristic space for G(A). This is possible only if either: (i) G(A) has as characteristic spaces sp{e} and $\{e\}^\perp$, or (ii) $\mathscr{V}$ is the only characteristic space for G(A). By (b) and (c) of the spectral theorem,

$$\mathbf{G}(\mathbf{A}) = \beta_1 \mathbf{e} \otimes \mathbf{e} + \beta_2(\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) \tag{17}$$

in either case [with $\beta_1 \neq \beta_2$ in case (i), $\beta_1 = \beta_2$ in case (ii)]. In view of (11), (17) can be written in the form (16) with

$$\alpha_2(\mathbf{A}) = 0. \tag{18}$$

Case 3. A has exactly one distinct eigenvalue. Then by (c) of the spectral theorem in conjunction with the transfer theorem, $\mathscr{V}$ is the characteristic space for A and G(A), so that $\mathbf{G}(\mathbf{A}) = \beta\mathbf{I}$, and G(A) again admits the representation (16) with $\alpha_0(\mathbf{A}) = \beta$ and $\alpha_1(\mathbf{A}) = \alpha_2(\mathbf{A}) = 0$.

Since these three cases are exhaustive, we have shown that G, when isotropic, admits the representation (16). In view of the representation theorem for isotropic scalar functions, to complete the proof we have only to show that α_0, α_1, and α_2 are isotropic:

$$\alpha_k(\mathbf{A}) = \alpha_k(\mathbf{QAQ}^T) \qquad (k = 0, 1, 2) \tag{19}$$

for every $\mathbf{A} \in \mathscr{A}$ and $\mathbf{Q} \in$ Orth. Thus choose $\mathbf{A} \in \mathscr{A}$, $\mathbf{Q} \in$ Orth. Since G is isotropic,

$$\mathbf{QG}(\mathbf{A})\mathbf{Q}^T - \mathbf{G}(\mathbf{QAQ}^T) = \mathbf{0},$$

or equivalently,

$$\mathbf{G}(\mathbf{A}) - \mathbf{Q}^T\mathbf{G}(\mathbf{QAQ}^T)\mathbf{Q} = \mathbf{0},$$

and, since

$$\mathbf{Q}^T(\mathbf{QAQ}^T)^2\mathbf{Q} = \mathbf{Q}^T\mathbf{QAQ}^T\mathbf{QAQ}^T\mathbf{Q} = \mathbf{A}^2,$$

it follows that

$$[\alpha_0(\mathbf{A}) - \alpha_0(\mathbf{QAQ}^T)]\mathbf{I} + [\alpha_1(\mathbf{A}) - \alpha_1(\mathbf{QAQ}^T)]\mathbf{A} + [\alpha_2(\mathbf{A}) - \alpha_2(\mathbf{QAQ}^T)]\mathbf{A}^2 = \mathbf{0}. \tag{20}$$

We consider again the three cases studied previously.

Case 1. By (a) of Wang's lemma, $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\}$ is linearly independent, so that (20) implies (19).

Case 2. By (3) and the proposition on page 16, A and $\mathbf{QAQ}^T$ have the same spectrum. Thus A and $\mathbf{QAQ}^T$ each have exactly two distinct eigenvalues and (18) yields $\alpha_2(\mathbf{A}) = \alpha_2(\mathbf{QAQ}^T) = 0$. Moreover, by (b) of Wang's lemma, {I, A} is linearly independent, and (20) again yields (19).

Case 3. Here $\mathbf{A} = \lambda\mathbf{I}$, so that $\mathbf{A} = \mathbf{QAQ}^T$ and (19) holds trivially. □

By the Cayley–Hamilton theorem (2.12),

$$\mathbf{A}^2 = \iota_1(\mathbf{A})\mathbf{A} - \iota_2(\mathbf{A})\mathbf{I} + \iota_3(\mathbf{A})\mathbf{A}^{-1},$$

provided, of course, A is invertible. Thus, by (2.9), the previous theorem has the following corollary:

Second Representation Theorem for Isotropic Tensor Functions. *Let $\mathscr{A}$ be a subset of the set of all invertible, symmetric tensors. Then*

$$\mathbf{G}: \mathscr{A} \to \text{Sym}$$

is isotropic if and only if there exist scalar functions $\beta_0, \beta_1, \beta_2: \mathscr{I}(\mathscr{A}) \to \mathbb{R}$ such that

$$\mathbf{G}(\mathbf{A}) = \beta_0(\mathscr{I}_\mathbf{A})\mathbf{I} + \beta_1(\mathscr{I}_\mathbf{A})\mathbf{A} + \beta_2(\mathscr{I}_\mathbf{A})\mathbf{A}^{-1} \tag{21}$$

for every $\mathbf{A} \in \mathscr{A}$.

For linear functions there is a far simpler result.

Representation Theorem for Isotropic Linear Tensor Functions. *A linear function*

$$\mathbf{G}: \text{Sym} \to \text{Sym}$$

is isotropic if and only if there exist scalars μ and λ such that

$$\mathbf{G}(\mathbf{A}) = 2\mu\mathbf{A} + \lambda(\text{tr } \mathbf{A})\mathbf{I} \tag{22}$$

for every $\mathbf{A} \in$ Sym.

Proof. Let $\mathscr{N}$ be the set of all unit vectors. For $\mathbf{e} \in \mathscr{N}$ the tensor $\mathbf{e} \otimes \mathbf{e}$ has spectrum {0, 0, 1} and characteristic spaces sp{e} and $\{e\}^\perp$. Thus the same argument used to arrive at (17) now leads to the conclusion that there exist functions $\mu, \lambda: \mathscr{N} \to \mathbb{R}$ such that

$$\mathbf{G}(\mathbf{e} \otimes \mathbf{e}) = 2\mu(\mathbf{e})\mathbf{e} \otimes \mathbf{e} + \lambda(\mathbf{e})\mathbf{I} \tag{23}$$

for every $\mathbf{e} \in \mathcal{N}$. Choose $\mathbf{e}, \mathbf{f} \in \mathcal{N}$, and let $\mathbf{Q}$ be any orthogonal tensor such that $\mathbf{Qe} = \mathbf{f}$. (Clearly, at least one such $\mathbf{Q}$ exists.) Since

$$\mathbf{Q}(\mathbf{e} \otimes \mathbf{e})\mathbf{Q}^T = \mathbf{f} \otimes \mathbf{f},$$

and since $\mathbf{G}$ is isotropic,

$$\mathbf{0} = \mathbf{QG}(\mathbf{e} \otimes \mathbf{e})\mathbf{Q}^T - \mathbf{G}(\mathbf{f} \otimes \mathbf{f}) = 2[\mu(\mathbf{e}) - \mu(\mathbf{f})]\mathbf{f} \otimes \mathbf{f} + [\lambda(\mathbf{e}) - \lambda(\mathbf{f})]\mathbf{I}.$$

But $\{\mathbf{I}, \mathbf{f} \otimes \mathbf{f}\}$ is linearly independent; thus

$$\mu(\mathbf{e}) = \mu(\mathbf{f}), \qquad \lambda(\mathbf{e}) = \lambda(\mathbf{f}).$$

Therefore μ and λ must be scalar constants, and we conclude from (23) that

$$\mathbf{G}(\mathbf{e} \otimes \mathbf{e}) = 2\mu\mathbf{e} \otimes \mathbf{e} + \lambda\mathbf{I}. \tag{24}$$

Next, choose $\mathbf{A} \in \mathrm{Sym}$ arbitrarily. By the spectral theorem $\mathbf{A}$ admits the representation

$$\mathbf{A} = \sum_i \omega_i \mathbf{e}_i \otimes \mathbf{e}_i$$

with $\{\mathbf{e}_i\}$ orthonormal; therefore, in view of (24) and the linearity of $\mathbf{G}$,

$$\mathbf{G}(\mathbf{A}) = \sum_i \omega_i \mathbf{G}(\mathbf{e}_i \otimes \mathbf{e}_i) = 2\mu\mathbf{A} + \lambda(\omega_1 + \omega_2 + \omega_3)\mathbf{I}. \tag{25}$$

Since

$$\mathrm{tr}\,\mathbf{A} = \omega_1 + \omega_2 + \omega_3,$$

(25) implies the desired result (22). The converse assertion, that (22) delivers an isotropic function, is left as an exercise. □

Corollary. *Let*

$$\mathrm{Sym}_0 = \{\mathbf{A} \in \mathrm{Sym} \mid \mathrm{tr}\,\mathbf{A} = 0\},$$

and let

$$\mathbf{G}: \mathrm{Sym}_0 \to \mathrm{Sym}$$

be linear. Then $\mathbf{G}$ *is isotropic if and only if there exists a scalar* μ *such that*

$$G(\mathbf{A}) = 2\mu\mathbf{A} \tag{26}$$

for every $\mathbf{A} \in \mathrm{Sym}_0$.

Proof. Clearly (26) defines an isotropic function. To prove the converse assertion assume that $\mathbf{G}$ is isotropic. For $\mathbf{A} \in \mathrm{Sym}$, $\mathbf{A} - \frac{1}{3}(\mathrm{tr}\,\mathbf{A})\mathbf{I}$ belongs to Sym_0. Thus we can extend $\mathbf{G}$ from Sym_0 to Sym by defining

$$\hat{\mathbf{G}}(\mathbf{A}) = \mathbf{G}(\mathbf{A} - \tfrac{1}{3}(\mathrm{tr}\,\mathbf{A})\mathbf{I})$$

for every $\mathbf{A} \in \mathrm{Sym}$. Since $\mathbf{G}$ is isotropic, for every $\mathbf{A} \in \mathrm{Sym}$ and $\mathbf{Q} \in \mathrm{Orth}$,

$$\begin{aligned} \mathbf{Q}\hat{\mathbf{G}}(\mathbf{A})\mathbf{Q}^T &= \mathbf{G}(\mathbf{QAQ}^T - \tfrac{1}{3}(\mathrm{tr}\,\mathbf{A})\mathbf{I}) \\ &\ \mathbf{G}(\mathbf{QAQ}^T - \tfrac{1}{3}\mathrm{tr}(\mathbf{QAQ}^T)\mathbf{I}) \\ &= \hat{\mathbf{G}}(\mathbf{QAQ}^T), \end{aligned}$$

where we have used the fact that tr is isotropic. Thus $\hat{\mathbf{G}}$ is isotropic, and we conclude from (22) and the fact that $\hat{\mathbf{G}}$ and $\mathbf{G}$ coincide on Sym_0 that

$$\mathbf{G}(\mathbf{A}) = \hat{\mathbf{G}}(\mathbf{A}) = 2\mu\mathbf{A} + \lambda(\mathrm{tr}\,\mathbf{A})\mathbf{I} = 2\mu\mathbf{A}$$

for every $\mathbf{A} \in \mathrm{Sym}_0$. □

Our next step is to discuss the invariance properties of the derivative of a tensor function. Thus let $\mathcal{A}$ be an open subset of a subspace $\mathcal{U}$ of Lin, and let $\mathbf{G}: \mathcal{A} \to \mathrm{Lin}$ be smooth.

Theorem (Invariance of the derivative). *Let* $\mathbf{G}$ *be invariant under* $\mathcal{G}$. *Then*

$$\mathbf{Q}D\mathbf{G}(\mathbf{A})[\mathbf{U}]\mathbf{Q}^T = D\mathbf{G}(\mathbf{QAQ}^T)[\mathbf{QUQ}^T] \tag{27}$$

for every $\mathbf{A} \in \mathcal{A}$, $\mathbf{U} \in \mathcal{U}$, *and* $\mathbf{Q} \in \mathcal{G}$.

Proof. For (27) to make sense we must show that both $\mathcal{A}$ and $\mathcal{U}$ are invariant under $\mathcal{G}$. Since $\mathcal{A}$ is the domain of $\mathbf{G}$ and $\mathbf{G}$ is invariant, $\mathcal{A}$ is (by definition) invariant under $\mathcal{G}$. To see that $\mathcal{U}$ has this invariance, choose $\mathbf{U} \in \mathcal{U}$, $\mathbf{A} \in \mathcal{A}$, and $\mathbf{Q} \in \mathcal{G}$. Then for all sufficiently small α, $(\mathbf{A} + \alpha\mathbf{U}) \in \mathcal{A}$, so that $\mathbf{Q}(\mathbf{A} + \alpha\mathbf{U})\mathbf{Q}^T \in \mathcal{A}$ and hence $(\mathbf{QAQ}^T + \alpha\mathbf{QUQ}^T) \in \mathcal{A} \subset \mathcal{U}$. Thus, since $\mathcal{U}$ is a subspace, $\mathbf{QUQ}^T \in \mathcal{U}$, and $\mathcal{U}$ is invariant under $\mathcal{G}$.

We have only to show that (27) is satisfied. For $\mathbf{A} \in \mathcal{A}$, $\mathbf{U} \in \mathcal{U}$, and $\mathbf{Q} \in \mathcal{G}$,

$$\begin{aligned} \mathbf{G}(\mathbf{Q}(\mathbf{A} + \mathbf{U})\mathbf{Q}^T) &= \mathbf{G}(\mathbf{QAQ}^T + \mathbf{QUQ}^T) \\ &= \mathbf{G}(\mathbf{QAQ}^T) + D\mathbf{G}(\mathbf{QAQ}^T)[\mathbf{QUQ}^T] + o(\mathbf{U}) \end{aligned}$$

as $\mathbf{U} \to \mathbf{0}$. But since $\mathbf{G}$ is invariant under $\mathcal{G}$,

$$\begin{aligned} \mathbf{G}(\mathbf{Q}(\mathbf{A} + \mathbf{U})\mathbf{Q}^T) &= \mathbf{QG}(\mathbf{A} + \mathbf{U})\mathbf{Q}^T \\ &= \mathbf{QG}(\mathbf{A})\mathbf{Q}^T + \mathbf{Q}D\mathbf{G}(\mathbf{A})[\mathbf{U}]\mathbf{Q}^T + o(\mathbf{U}), \\ &= \mathbf{G}(\mathbf{QAQ}^T) + \mathbf{Q}D\mathbf{G}(\mathbf{A})[\mathbf{U}]\mathbf{Q}^T + o(\mathbf{U}) \end{aligned}$$

as $\mathbf{U} \to \mathbf{0}$. Thus by the uniqueness of the derivative,

$$\mathbf{Q}D\mathbf{G}(\mathbf{A})[\mathbf{U}]\mathbf{Q}^T = D\mathbf{G}(\mathbf{QAQ}^T)[\mathbf{QUQ}^T],$$

which is (27). □

EXERCISES

1. Show that the sets Orth, Orth$^+$, Sym, Skw, and Psym are invariant under Orth.
2. A scalar function $\varphi: \mathscr{V} \rightarrow \mathbb{R}$ is isotropic if
$$\varphi(\mathbf{v}) = \varphi(\mathbf{Q}\mathbf{v})$$
for all $\mathbf{v} \in \mathscr{V}$ and $\mathbf{Q} \in$ Orth. Show that φ is isotropic if and only if there exists a function $\hat{\varphi}: [0, \infty) \rightarrow \mathbb{R}$ such that
$$\varphi(\mathbf{v}) = \hat{\varphi}(|\mathbf{v}|)$$
for all $\mathbf{v} \in \mathscr{V}$.
3. A vector function $\mathbf{q}: \mathscr{V} \rightarrow \mathscr{V}$ is isotropic if
$$\mathbf{Q}\mathbf{q}(\mathbf{v}) = \mathbf{q}(\mathbf{Q}\mathbf{v})$$
for all $\mathbf{v} \in \mathscr{V}$ and $\mathbf{Q} \in$ Orth. Show that $\mathbf{q}$ is isotropic if and only if there exists a function $\varphi: [0, \infty) \rightarrow \mathbb{R}$ such that
$$\mathbf{q}(\mathbf{v}) = \varphi(|\mathbf{v}|)\mathbf{v}$$
for all $\mathbf{v} \in \mathscr{V}$.
4. Let $\mathbf{G}$: Lin $\rightarrow$ Lin be defined by
$$\mathbf{G}(\mathbf{A}) = \mathbf{A}^n \qquad (n \geq 1 \quad \text{an integer}).$$
Show that $\mathbf{G}$ is isotropic.
5. Show that the mapping
$$\mathbf{A} \mapsto \mathbf{A}^{-1}$$
from Lin$^+$ into Lin$^+$ is isotropic.
6. Show that the function $\mathbf{G}$: Sym $\rightarrow$ Sym defined by (22) is isotropic.
7. Establish (b) of Wang's lemma.

SELECTED REFERENCES

Gurtin [2].
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Serrin [1, §59].
Truesdell and Noll [1, §§10–13].

38. GENERAL SCHEME OF NOTATION

INDEX OF FREQUENTLY USED SYMBOLS

Symbol	*Name*	*Place of definition or first occurrence*
$\mathbf{a}$	angular momentum	92
$\mathscr{B}$	body	41
$\mathscr{B}_t$	region occupied by body at time t	58
$\mathbf{b}$	body force	98
$\mathbf{b}_0$	reference body force	179
$\mathbf{B}$	left Cauchy–Green strain tensor	46
$\mathbf{C}$	right Cauchy–Green strain tensor	46
$\mathbf{c}$	curve	34
C	elasticity tensor	194
curl	curl, spatial curl	32, 61
div	divergence, spatial divergence	30, 61
Div	material divergence	61
D	derivative	21
$\mathbf{D}$	stretching	71
det	determinant	6
E	Young's modulus	203
$\mathbf{E}$	infinitesimal strain	55
$\mathscr{E}$	euclidean point space	1
$\mathbf{f}$	deformation	42
$\mathbf{F}$	deformation gradient	42
g	complex velocity	123
$\mathscr{G}$	symmetry group	168
$\mathbf{H}$	displacement gradient	199
$\mathscr{I}_S$	list of principal invariants of a tensor S	15
$\mathbf{I}$	identity tensor	3
$\mathscr{K}\{\dot{\mathbf{u}}\}$	kinetic energy	220
$\mathbf{L}$	velocity gradient	63
$\mathbf{l}$	linear momentum	92
Lin	set of all tensors	7
Lin$^+$	set of all tensors S with det $S > 0$	7
$m(\mathscr{P})$	mass of $\mathscr{P}$	87
m	subscript indicating material description, also mach number	60, 133
$\mathbf{n}$	outward unit normal to boundary	37
$o(\mathbf{u})$	symbol for "small order $\mathbf{u}$"	19
$\mathbf{o}$	origin	2
Orth	set of all orthogonal tensors	7
Orth$^+$	set of all rotations (proper orthogonal tensors)	7
$\mathbf{p}$	material point	41
p	reference map	59
$\mathscr{P}$	part of $\mathscr{B}$	41
Psym	set of all symmetric, positive definite tensors	7
$\mathbf{Q}$	orthogonal tensor	7
$\mathbf{R}$	rotation tensor	46
$\mathbf{r}$	position vector	92

Symbol	*Name*	*Place of definition or first occurrence*
$\mathbb{R}$	reals	1
$\mathbb{R}^+$	strictly-positive reals	1
$\mathbf{S}$	Piola-Kirchhoff stress	178
$\mathbf{s}$	stress vector	97
Sym	set of all symmetric tensors	7
Skw	set of all skew tensors	7
$\mathscr{s}$	subscript indicating spatial description, also kinematically admissible state	60, 208
sp	span	2
tr	trace	5
t	time	58
$\mathbf{T}$	Cauchy stress	101
$\mathscr{T}$	trajectory	59
$\mathbf{u}$	displacement	42
$\mathscr{U}\{\mathbf{E}\}$	strain energy	205
$\mathbf{U}$	right stretch tensor	46
$\mathbf{V}$	left stretch tensor	46
$\mathbf{v}$	spatial description of the velocity	60
v	speed	133
$\mathscr{V}$	vector space associated with $\mathscr{E}$	1
vol	volume	37
$\mathbf{W}$	spin	71
w	complex potential	126
$\mathbf{x}$	place	58
$\mathbf{x}$	motion	58
$\boldsymbol{\alpha}$	center of mass	92
β	potential of body force	111
φ	potential of flow	111
$\Phi\{\mathscr{s}\}$	potential energy	208
$\iota_k(\mathbf{S})$	principal invariants of a tensor $\mathbf{S}$	15
$\kappa(\rho)$	sound speed	131
λ_k	principal stretches	45
λ	Lamé modulus	196
π	pressure	106
ρ_0	reference density	88
ρ	density in motion	88
$\sigma(\mathbf{F})$	strain-energy density	186
μ	viscosity, also shear modulus	149, 196
ν	kinematic viscosity, also Poisson's ratio	151, 203
$\boldsymbol{\omega}$	angular velocity	70
∇	gradient, material gradient	29, 60
Δ	laplacian	32
$\otimes$	tensor product	4
$\times$	cross product	7
$\dot{\varphi}$	material time derivative of φ	61
φ'	spatial time derivative of φ	61

General Notation

Expression	*Meaning*
$\mathring{\mathscr{R}}$	interior of $\mathscr{R}$
$\partial\mathscr{R}$	boundary of $\mathscr{R}$
$\bar{\mathscr{R}}$	closure of $\mathscr{R}$
$\mathscr{R} \cup \mathscr{F}$	union of $\mathscr{R}$ and $\mathscr{F}$
$\mathscr{R} \cap \mathscr{F}$	intersection of $\mathscr{R}$ and $\mathscr{F}$
$\mathscr{R} \subset \mathscr{F}$	$\mathscr{R}$ is a subset of $\mathscr{F}$
$x \in \mathscr{R}$	x is an element of the set $\mathscr{R}$
$f: \mathscr{R} \to \mathscr{F}$	f maps the set $\mathscr{R}$ into the set $\mathscr{F}$; $\mathscr{R}$ is the domain, $\mathscr{F}$ the codomain
$x \mapsto f(x)$	the mapping that carries x into $f(x)$; e.g., $x \mapsto x^2$ is the mapping that carries every real number x into its square
$f \circ g$	composition of the mappings f and g; that is, $(f \circ g)(x) = f(g(x))$
$\{x \mid R(x) \text{ holds}\}$	the set of all x such that $R(x)$ holds; e.g., $\{x \mid 0 \le x \le 1\}$ is the interval $[0, 1]$

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Hints for Selected Exercises

SECTION 1

1. Consider $\psi(\mathbf{v})$ with $\mathbf{v} = \sum_i v_i \mathbf{e}_i$.

2. Take $\mathbf{a} = \sum_i \psi(\mathbf{e}_i)\mathbf{e}_i$.

3. Show that $\mathbf{F} = \mathbf{S} + \mathbf{T}$ is linear:

$$\mathbf{F}(\mathbf{u} + \mathbf{v}) = \mathbf{F}\mathbf{u} + \mathbf{F}\mathbf{v},$$
$$\mathbf{F}(\alpha\mathbf{u}) = \alpha\mathbf{F}\mathbf{u}$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ and scalars α.

4. Fix $\mathbf{v}$ and consider $\mathbf{S}\mathbf{u} \cdot \mathbf{v}$ as a function of $\mathbf{u}$. Show that this function is linear and hence can be written as the inner product of a vector $\mathbf{a}_\mathbf{v}$ (which depends on $\mathbf{v}$) and $\mathbf{u}$:

$$\mathbf{S}\mathbf{u} \cdot \mathbf{v} = \mathbf{a}_\mathbf{v} \cdot \mathbf{u}.$$

Show that $\mathbf{a}_\mathbf{v}$ is a linear function of $\mathbf{v}$ and hence can be written as $\mathbf{a}_\mathbf{v} = \mathbf{A}\mathbf{v}$ with $\mathbf{A}$ a tensor. Define $\mathbf{S}^T = \mathbf{A}$.

6. Apply each side of the identity to an arbitrary vector $\mathbf{v}$.

7. To verify $(1)_1$ prove that

$$\mathbf{u}\cdot(\mathbf{S}+\mathbf{T})\mathbf{v} = (\mathbf{S}^T+\mathbf{T}^T)\mathbf{u}\cdot\mathbf{v},$$

which establishes $\mathbf{S}^T+\mathbf{T}^T$ as the transpose of $\mathbf{S}+\mathbf{T}$. To verify $(2)_2$ apply each side of the identity to an arbitrary vector $\mathbf{v}$.

14. Use the identity

$$\mathbf{S}\mathbf{e}_1 = \sum_j S_{j1}\mathbf{e}_j$$

to show that

$$\varphi(\mathbf{S}\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = S_{11}\varphi(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3).$$

SECTION 2

2. For part (a) use (b) of the first proposition. To derive (2) and (3) use (1) and $(1.2)_3$.

3. Show that $\mathbf{QDQ}^T \in \text{Sym}$ and $\mathscr{I}_\mathbf{D} = \mathscr{I}_{\mathbf{QDQ}^T}$.

4b. Prove that each eigenvalue of $\mathbf{P}$ is either 0 or 1 and then (with the aid of the spectral theorem) consider separately the following four possibilities for the spectrum of $\mathbf{P}$: (0, 0, 0), (1, 1, 1), (0, 1, 1), (0, 0, 1).

5. (only if) Let S commute with every $\mathbf{W}\in$ Skw. Choose a vector $\mathbf{w}$ and let $\mathbf{W}$ be the skew tensor whose axial vector is $\mathbf{w}$. Show that

$$\mathbf{W}(\mathbf{S}\mathbf{w}) = \mathbf{S}\mathbf{W}\mathbf{w} = \mathbf{0},$$

so that $\mathbf{S}\mathbf{w}$ belongs to the axis of $\mathbf{W}$ and is hence parallel to $\mathbf{w}$. Thus

$$\mathbf{S}\mathbf{w} = \sigma\mathbf{w}.$$

(Of course, σ may depend on the choice of $\mathbf{w}$.) For $\mathbf{S}\in$ Sym use the spectral theorem and the fact that $\mathbf{w}$ is arbitrary to establish (14). For $\mathbf{S}\in$ Skw,

$$0 = \mathbf{w}\cdot\mathbf{S}\mathbf{w} = \sigma\mathbf{w}^2 \Rightarrow \sigma = 0$$

and (14) holds trivially. We therefore have the desired result for $\mathbf{S}\in$ Sym and $\mathbf{S}\in$ Skw. Complete the proof by showing that if S commutes with every $\mathbf{W}\in$ Skw, then so also do the symmetric and skew parts of S.

7. Choose $\mathbf{Q}\in\text{Orth}^+$, $\mathbf{Q}\neq\mathbf{R}$. Use the identity

$$|\mathbf{F}-\mathbf{Q}|^2 = |\mathbf{F}|^2 - 2\mathbf{F}\cdot\mathbf{Q} + 3$$

to show that

$$|\mathbf{F}-\mathbf{Q}|^2 - |\mathbf{F}-\mathbf{R}|^2 = 2\mathbf{U}\cdot(\mathbf{I}-\mathbf{Q}_0),$$

where $\mathbf{F}=\mathbf{RU}$ is the right polar decomposition of $\mathbf{F}$ and

$$\mathbf{Q}_0 = \mathbf{R}^T\mathbf{Q} \neq \mathbf{I}.$$

Next, establish the identity

$$2\mathbf{U}\cdot(\mathbf{I}-\mathbf{Q}_0) = \mathbf{U}\cdot(\mathbf{Q}_0-\mathbf{I})(\mathbf{Q}_0-\mathbf{I})^T = \text{tr}\{(\mathbf{Q}_0-\mathbf{I})^T\mathbf{U}(\mathbf{Q}_0-\mathbf{I})\},$$

and show that $\mathbf{U}\cdot(\mathbf{I}-\mathbf{Q}_0) > 0$ by showing that $(\mathbf{Q}_0-\mathbf{I})^T\mathbf{U}(\mathbf{Q}_0-\mathbf{I})\in$ Psym.

SECTION 3

2. Differentiate $\mathbf{AA}^{-1} = \mathbf{I}$.
3. Use the chain rule.
6. Differentiate $\mathbf{QQ}^T = \mathbf{I}$.
7. Take $\mathbf{Q}(t) = e^{\mathbf{W}t}$.

8c. Show that

$$D\varphi(\mathbf{A})[\mathbf{U}] = \mathbf{U}\cdot\int_0^1\{\mathbf{S}(\mathbf{A}_0+\alpha\mathbf{A}) + \alpha D\mathbf{S}(\mathbf{A}_0+\alpha\mathbf{A})[\mathbf{A}]\}\,d\alpha,$$

and show further that the integrand is equal to

$$\frac{d}{d\alpha}[\alpha\mathbf{S}(\mathbf{A}_0+\alpha\mathbf{A})].$$

SECTION 4

2. Show that $\text{div}(\mathbf{S}^T\mathbf{a})$ is linear in $\mathbf{a}$ and use the representation theorem for linear forms.

SECTION 5

1a. Use the fact that, given any vector $\mathbf{a}$,

$$\left(\int_{\partial\mathscr{R}}\mathbf{v}\otimes\mathbf{n}\,dA\right)\mathbf{a} = \int_{\partial\mathscr{R}}(\mathbf{v}\otimes\mathbf{n})\mathbf{a}\,dA = \int_{\partial\mathscr{R}}(\mathbf{v}\otimes\mathbf{a})\mathbf{n}\,dA.$$

SECTION 6

6. Use (2.7) and (2.10) to show that $\mathscr{I}_C = (3, 3, 1)$ if and only if the spectrum of $\mathbf{C}$ is $(1, 1, 1)$.

9. Let

$$\mathbf{F}_1 = \mathbf{R}_1\mathbf{U}_1 \quad \text{and} \quad \mathbf{F}_2 = \mathbf{R}_2\mathbf{U}_2$$

be right polar decompositions of $\mathbf{F}_1 = \nabla\mathbf{f}_1$ and $\mathbf{F}_2 = \nabla\mathbf{f}_2$, and note that $\mathbf{U}_1(\mathbf{p}) = \mathbf{U}_2(\mathbf{p})$ for all $\mathbf{p} \in \mathscr{B}$. Define $\mathbf{g} = \mathbf{f}_2 \circ \mathbf{f}_1^{-1}$. Choose $\mathbf{q} \in \mathbf{f}_1(\mathscr{B})$ arbitrarily and let $\mathbf{p} = \mathbf{f}_1^{-1}(\mathbf{q})$. Show that

$$\nabla\mathbf{g}(\mathbf{q}) = \mathbf{R}_2(\mathbf{p})\mathbf{R}_1(\mathbf{p})^T$$

and appeal to the characterization theorem for rigid deformations.

10. To establish $(18)_1$ let $\mathbf{v} = \sum_i v_i\mathbf{e}_i$, write

$$\int_{\partial f(\mathscr{P})} \mathbf{v}\cdot\mathbf{m}\,dA = \sum_i \mathbf{e}_i \cdot \int_{\partial f(\mathscr{P})} v_i\mathbf{m}\,dA,$$

and use (14.2). For $(18)_2$ let $\mathbf{a}$ be a vector, write

$$\mathbf{a}\cdot\int_{\partial f(\mathscr{P})} \mathbf{T}\mathbf{m}\,dA = \int_{\partial f(\mathscr{P})} (\mathbf{T}^T\mathbf{a})\cdot\mathbf{m}\,dA,$$

and use $(18)_1$. For $(18)_3$ use the identity

$$\mathbf{a}\cdot(\mathbf{r}\times\mathbf{T}\mathbf{m}) = (\mathbf{T}\mathbf{m})\cdot(\mathbf{a}\times\mathbf{r}) = \mathbf{m}\cdot\mathbf{T}^T(\mathbf{a}\times\mathbf{r}) \qquad [\mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{o}]$$

to write

$$\mathbf{a}\cdot\int_{\partial f(\mathscr{P})} \mathbf{r}\times\mathbf{T}\mathbf{m}\,dA = \int_{\partial f(\mathscr{P})} \mathbf{m}\cdot\mathbf{T}^T(\mathbf{a}\times\mathbf{r})\,dA$$

and appeal to $(18)_1$ again.

12. To verify $(5)_1$ use the continuity of $\mathbf{f}$ and $\mathbf{f}^{-1}$ together with the fact that for a continuous function the inverse image of an open set is open. To prove $(5)_2$ note that, since $\mathscr{B}$ is closed,

$$\mathbf{f}(\mathscr{B}) = \mathbf{f}(\mathring{\mathscr{B}}) \cup \mathbf{f}(\partial\mathscr{B}),$$

and since $\mathbf{f}(\mathscr{B})$ is closed,

$$\mathbf{f}(\mathscr{B}) = \mathbf{f}(\mathscr{B})^\circ \cup \partial\mathbf{f}(\mathscr{B}).$$

Use these identities with $(5)_1$ to verify $(5)_2$.

13b. Show that $\mathbf{f}(\mathscr{B})$ is not closed.

SECTION 7

1. Consider $\mathbf{C}$ and $\mathbf{U}$ as functions of $\mathbf{H} = \nabla\mathbf{u}$:

$$\mathbf{C} = \hat{\mathbf{C}}(\mathbf{H}), \qquad \mathbf{U} = \hat{\mathbf{U}}(\mathbf{H}).$$

Use the chain rule and the identity $\hat{\mathbf{C}} = \hat{\mathbf{U}}^2$ to conclude that

$$D\hat{\mathbf{C}}(\mathbf{0})\,[\mathbf{S}] = 2\,D\hat{\mathbf{U}}(\mathbf{0})\,[\mathbf{S}]$$

for all tensors $\mathbf{S}$, and then conclude from $(5)_1$ that

$$\hat{\mathbf{U}}(\mathbf{H}) = \mathbf{I} + \mathbf{E} + o(\mathbf{H}).$$

5. Use Exercise 4 in conjunction with Exercise 4.9b.

6. Use the trigonometric identities

$$\cos(\Theta + \beta) = \cos\Theta\cos\beta - \sin\Theta\sin\beta,$$
$$\sin(\Theta + \beta) = \sin\Theta\cos\beta + \cos\Theta\sin\beta,$$

together with the estimates

$$\cos\beta = 1 + o(\beta),$$
$$\sin\beta = \beta + o(\beta).$$

[It is a simple matter to verify that $\nabla\mathbf{u} \to \mathbf{0}$ as $\alpha \to 0$ without explicitly computing the gradient. Indeed, one simply notes that $\mathbf{u}$, as a function of α and $\mathbf{p}$, is smooth for $(\alpha, \mathbf{p}) \in \mathbb{R}\times\mathscr{E}$; one then obtains the above limit by first evaluating $\mathbf{u}$ at $\alpha = 0$ and then taking its gradient.]

SECTION 8

3. Compute div $\mathbf{v}$ using (5) and Exercise 4.9a.

7a. Choose $\mathbf{p} \in \mathscr{S}$ and let $\mathbf{c}(\sigma)$, $0 \le \sigma \le 1$, be a smooth curve in $\mathscr{S}$ with $\mathbf{c}(0) = \mathbf{p}$. Use the relation

$$\frac{d}{d\sigma}\varphi(\mathbf{c}(\sigma)) = 0$$

to show that $\nabla\varphi(\mathbf{p})$ is normal to $\dot{\mathbf{c}}(0)$.

SECTION 9

1d. Show that the two terms in the right side of (15) are symmetric and skew, respectively, and use the uniqueness of the expansion $\mathbf{L} = \mathbf{D} + \mathbf{W}$ of $\mathbf{L}$ into symmetric and skew parts.

3. Show that

$$|\mathbf{x}_\tau(\mathbf{y}, t) - \mathbf{x}_\tau(\mathbf{z}, t)| = |\mathbf{y} - \mathbf{z}|,$$

so that $\mathbf{x}_\tau(\cdot, t)$ is a rigid deformation.

4c. By (19),

$$\overset{(n+1)}{\mathbf{C}} = \mathbf{F}^T\mathbf{A}_{n+1}\mathbf{F} = (\overset{(n)}{\mathbf{C}})^{\cdot} = (\mathbf{F}^T\mathbf{A}_n\mathbf{F})^{\cdot}.$$

SECTION 10

2. Use $(4.2)_4$ and (9.11).

SECTION 11

1. Use (9.12), (10.2), and $(2)_2$.
2. Take the curl of $(9.10)_2$ and use the identity

$$\operatorname{curl}(\mathbf{w} \times \mathbf{v}) = (\operatorname{grad} \mathbf{w})\mathbf{v} - (\operatorname{grad} \mathbf{v})\mathbf{w} + (\operatorname{div} \mathbf{v})\mathbf{w} - (\operatorname{div} \mathbf{w})\mathbf{v}$$

and (10.2).

3. Fix $\mathbf{y}$ and τ, and let

$$\mathbf{g}(t) = \upsilon(\mathbf{x}_\tau(\mathbf{y}, t), t)\mathbf{w}(\mathbf{x}_\tau(\mathbf{y}, t), t).$$

It suffices to show that

$$\mathbf{g}(t) = \mathbf{F}_\tau(\mathbf{y}, t)\mathbf{g}(\tau).$$

By (9), $\mathbf{g}$ satisfies a certain ordinary differential equation. Show that

$$\mathbf{f}(t) = \mathbf{F}_\tau(\mathbf{y}, t)\mathbf{g}(\tau)$$

satisfies the same differential equation and $\mathbf{f}(\tau) = \mathbf{g}(\tau)$. Appeal to the uniqueness theorem for ordinary differential equations.

5. Use (9.10).

SECTION 12

3. Show that

$$\rho(\mathbf{y}, \tau) = \rho(\mathbf{x}, t) \det \mathbf{F}_\tau(\mathbf{y}, t)$$

and then proceed as in the proof of (7).

4. Let $\mathbf{f} = \mathbf{g} - \mathbf{v}$ and use the divergence theorem to prove that $\mathscr{K}\{\mathbf{g}\} = \mathscr{K}\{\mathbf{v}\} + \mathscr{K}\{\mathbf{f}\}$.

SECTION 13

3a. Use (9) and (12.7) to convert (4) to an integral over $\mathscr{B}_0$.

3b. Use (10) to eliminate $\mathbf{q}(t)$ from (9) and derive

$$\mathbf{x}_0(\mathbf{y}, t) = \boldsymbol{\alpha}(t) + \mathbf{Q}(t)[\mathbf{y} - \boldsymbol{\alpha}(0)].$$

By (9.13),

$$\mathbf{v}(\mathbf{x}_0(\mathbf{y}, t), t) = \dot{\mathbf{x}}_0(\mathbf{y}, t) = \dot{\boldsymbol{\alpha}}(t) + \dot{\mathbf{Q}}(t)[\mathbf{y} - \boldsymbol{\alpha}(0)].$$

Choose $\mathbf{x} \in \mathscr{B}_t$ arbitrarily, and let $\mathbf{y}$ be such that $\mathbf{x} = \mathbf{x}_0(\mathbf{y}, t)$, i.e.,

$$\mathbf{y} - \boldsymbol{\alpha}(0) = \mathbf{Q}(t)^T[\mathbf{x} - \boldsymbol{\alpha}(t)].$$

Then

$$\mathbf{v}(\mathbf{x}, t) = \dot{\boldsymbol{\alpha}}(t) + \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T[\mathbf{x} - \boldsymbol{\alpha}(t)].$$

Compare the gradients (with respect to $\mathbf{x}$) of the above equation and (9.7).

3c. (If) Define $\mathbf{g}(t) = \mathbf{Q}(t)\mathbf{k}(0)$ and show that $\mathbf{g}$ and $\mathbf{k}$ satisfy the same differential equation and $\mathbf{g}(0) = \mathbf{k}(0)$. Appeal to the uniqueness theorem for ordinary differential equations.

3e. To establish (15) transform (14) to an integral over $\mathscr{B}_0$ using the same steps as in (3a). Then use (12) (with $\mathbf{k} = \mathbf{e}$) and (15) to show that

$$\mathbf{e}_i(t) \cdot \mathbf{J}(t)\mathbf{e}_j(t) = \mathbf{e}_i(0) \cdot \mathbf{J}(0)\mathbf{e}_j(0).$$

3f. Choose $\{\mathbf{e}_i(0)\}$ to be an orthonormal basis of eigenvectors. Since

$$\mathbf{a}_{spin}(t) = \sum_{i,j} J_{ij}\omega_j(t)\mathbf{e}_i(t),$$

where J_{ij} are the components of $\mathbf{J}(t)$ relative to $\{\mathbf{e}_i(t)\}$, we may conclude, with the aid of (13), that

$$\dot{\mathbf{a}}_{spin} = \sum_{i,j} [J_{ij}\dot{\omega}_j\mathbf{e}_i + J_{ij}\omega_j(\boldsymbol{\omega} \times \mathbf{e}_i)].$$

4a. Show that

$$\int_{\partial\mathcal{B}_t} \mathbf{v}_\alpha \rho \, dV = \mathbf{0}$$

and then compute $\mathscr{K}$ with $\mathbf{v} = \mathbf{v}_\alpha + \dot{\boldsymbol{\alpha}}$.

4b. Use (11).

SECTION 14

8. Prove that it suffices to find a tensor $\mathbf{Q}$ such that

$$\mathbf{G}(\mathbf{Q}) = \int_{\partial\mathcal{B}_t} \mathbf{r} \otimes \mathbf{Q}\mathbf{s}(\mathbf{n}) \, dA + \int_{\mathcal{B}_t} \mathbf{r} \otimes \mathbf{Q}\mathbf{b} \, dV$$

is symmetric. Show that $\mathbf{G}(\mathbf{Q}) = \mathbf{G}(\mathbf{I})\mathbf{Q}^T$, take the left polar decomposition of $\mathbf{G}(\mathbf{I})$, and choose $\mathbf{Q}^T$ to make $\mathbf{G}(\mathbf{I})\mathbf{Q}^T$ symmetric. Here you will need the following extended version of the polar decomposition theorem[1]: given $\mathbf{F} \in \text{Lin}$ there exist unique positive semi-definite, symmetric tensors $\mathbf{U}$, $\mathbf{V}$ and a (not necessarily unique) orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}.$$

9. We must show that

$$\mathbf{n} \cdot \mathbf{T}(\mathbf{x}, t)\mathbf{k} = 0,$$

whenever $\mathbf{n}$ is normal to—and $\mathbf{k}$ tangent to—$\partial\mathcal{B}_t$ at $\mathbf{x}$. Use the symmetry of $\mathbf{T}$ and the fact that $\partial\mathcal{B}_t$ is traction-free.

SECTION 15

1a. Use Exercise 5.1b.

1b. Use Exercise 5.1a.

2. Use components.

[1] Cf., e.g., Halmos [1, §83].

SECTION 16

1. Show that

$$\mathscr{D} = \{\mathbf{D} \in \text{Sym} \mid \text{tr}\, \mathbf{D} = 0\}.$$

Given any tensor $\mathbf{N} \in \text{Sym}$, write

$$\mathbf{N} = -\pi\mathbf{I} + \mathbf{N}_0,$$

where $\pi = -\frac{1}{3}\text{tr}\,\mathbf{N}$, so that $\text{tr}\,\mathbf{N}_0 = 0$. Show that $\mathbf{N} \cdot \mathbf{D} = 0$ for every $\mathbf{D} \in \mathscr{D}$ if and only if $\mathbf{N}_0 = \mathbf{0}$.

SECTION 17

2. Use the divergence theorem to express the term involving $\mathbf{b}$ in (15.2) as an integral over $\partial\mathcal{B}_t$.

3. Compute grad $\dot{\mathbf{v}}$.

4. Use Bernoulli's theorem.

SECTION 20

2a. Use Exercise 2.5.

SECTION 21

2. Use (20.1), $(6.18)_2$, and (1) to show that

$$\int_{\partial\mathscr{P}_t^*} \mathbf{T}^*\mathbf{n}^* \, dA = \mathbf{Q}(t) \int_{\partial\mathscr{P}_t} \mathbf{T}\mathbf{n} \, dA.$$

Apply the divergence theorem to both sides and then use $(6.14)_1$ to convert the integral over $\mathscr{P}_t^*$ to an integral over $\mathscr{P}_t$.

SECTION 22

2a. Use Exercises 4.9b and 7.4.

2b. Use coordinates with, say, $\mathbf{e}_3 = \mathbf{n}$ together with the relation $\text{div}\,\mathbf{v} = 0$ to show that $(\text{grad}\,\mathbf{v})^T\mathbf{n} = \mathbf{0}$.

SECTION 25

4c. For $\mathbf{H} \in \mathscr{H}_\mathbf{p}$ with $\det \mathbf{H} \neq 1$ take $\mathbf{F}_n = \mathbf{H}^n$ and $\mathbf{G}_n = \mathbf{H}^{-n}$ (or vice versa).

4e. Choose $\mathbf{F} \in \mathrm{Lin}^+$ arbitrarily and take $\mathbf{H} = (\det \mathbf{F})^{1/3}\mathbf{F}^{-1}$ to reduce $\hat{\mathbf{T}}$ to a function of the density. Show next that $\hat{\mathbf{T}}$ is isotropic and use this isotropy to reduce the stress to a pressure (cf. the corollary on page 13).

5b. Choose $\mathbf{H} \in \mathscr{H}_\mathbf{p}$. By (16) and (19),

$$\hat{\mathbf{T}}_\mathbf{g}(\mathbf{F}\mathbf{G}^{-1}) = \hat{\mathbf{T}}(\mathbf{F}) = \hat{\mathbf{T}}(\mathbf{F}\mathbf{H}) = \hat{\mathbf{T}}_\mathbf{g}(\mathbf{F}\mathbf{H}\mathbf{G}^{-1})$$

for every $\mathbf{F} \in \mathrm{Lin}^+$, so that

$$\hat{\mathbf{T}}_\mathbf{g}(\mathbf{F}) = \hat{\mathbf{T}}_\mathbf{g}(\mathbf{F}\mathbf{G}\mathbf{H}\mathbf{G}^{-1})$$

for every $\mathbf{F} \in \mathrm{Lin}^+$. Thus $\mathbf{H} \in \mathscr{H}_\mathbf{p}$ implies $\mathbf{G}\mathbf{H}\mathbf{G}^{-1} \in \mathscr{H}_\mathbf{p}(\mathbf{g})$ and $\mathbf{G}\mathscr{H}_\mathbf{p}\mathbf{G}^{-1} \subset \mathscr{H}_\mathbf{p}(\mathbf{g})$. Similarly, $\mathscr{H}_\mathbf{p}(\mathbf{g}) \subset \mathbf{G}\mathscr{H}_\mathbf{p}\mathbf{G}^{-1}$.

5c. Use (17) and (20).

5d. Let $\mathbf{G} = \mathbf{R}\mathbf{U}$ be a right polar decomposition of $\mathbf{G}$. Then

$$\mathscr{H}_\mathbf{p}(\mathbf{g}) = \mathbf{R}\mathbf{U}\mathscr{H}_\mathbf{p}\mathbf{U}^{-1}\mathbf{R}^{-1}.$$

Choose $\mathbf{Q} \in \mathscr{H}_\mathbf{p}$. Then

$$\hat{\mathbf{Q}} = \mathbf{R}\mathbf{U}\mathbf{Q}\mathbf{U}^{-1}\mathbf{R}^{-1}$$

belongs to $\mathscr{H}_\mathbf{p}(\mathbf{g})$. Thus, since $\mathbf{Q}, \hat{\mathbf{Q}} \in \mathrm{Orth}^+$ (why?),

$$(\hat{\mathbf{Q}}\mathbf{R})\mathbf{U} = (\mathbf{R}\mathbf{Q})(\mathbf{Q}^\mathrm{T}\mathbf{U}\mathbf{Q}) \qquad (\alpha)$$

represent polar decompositions of the same tensor; hence $\hat{\mathbf{Q}} = \mathbf{R}\mathbf{Q}\mathbf{R}^\mathrm{T}$. Thus $\mathbf{Q} \in \mathscr{H}_\mathbf{p}$ implies $\mathbf{R}\mathbf{Q}\mathbf{R}^\mathrm{T} \in \mathscr{H}_\mathbf{p}(\mathbf{g})$, and $\mathbf{R}\mathscr{H}_\mathbf{p}\mathbf{R}^\mathrm{T} \subset \mathscr{H}_\mathbf{p}(\mathbf{g})$. Similarly, $\mathscr{H}_\mathbf{p}(\mathbf{g}) \subset \mathbf{R}\mathscr{H}_\mathbf{p}\mathbf{R}^\mathrm{T}$. Equation ($\alpha$) also implies that $\mathbf{U} = \mathbf{Q}^\mathrm{T}\mathbf{U}\mathbf{Q}$. When $\mathbf{p}$ is isotropic this result and the corollary on page 13 tell us that $\mathbf{U} = \lambda\mathbf{I}$, so that $\mathbf{G} = \lambda\mathbf{R}$.

6a. Without loss in generality add the constraint

$$\mathrm{tr}\, \hat{\mathbf{T}}(\mathbf{F}) = 0$$

for all $\mathbf{F} \in \mathrm{Unim}$ (cf. the remarks on page 148).

6c. Define

$$\tilde{\mathbf{T}}(\mathbf{F}) = \hat{\mathbf{T}}((\det \mathbf{F})^{-1/3}\mathbf{F}) \qquad (\beta)$$

for all $\mathbf{F} \in \mathrm{Lin}^+$. Then $\tilde{\mathbf{T}}(\mathbf{F}) = \hat{\mathbf{T}}(\mathbf{F})$ for all $\mathbf{F} \in \mathrm{Unim}$, so that $\tilde{\mathbf{T}}$ provides an extension of $\hat{\mathbf{T}}$ to Lin^+. Show that $\tilde{\mathbf{T}}(\mathbf{F})$ must reduce to an isotropic function of $\mathbf{B} = \mathbf{F}\mathbf{F}^\mathrm{T}$ and use (37.21).

SECTION 27

6. Transform

$$\int_{\mathbf{x}(\mathscr{S},t)} (\mathbf{T} + \pi_0\mathbf{I})\mathbf{m}\, dA$$

to an integral over $\mathscr{S}$ ($\subset \mathscr{S}_2$) and use the fact that $\mathscr{S}$ is arbitrary.

SECTION 28

1a. Integrate $\hat{\sigma}(\mathbf{R})^{\cdot}$ along a path from $\mathbf{R} = \mathbf{I}$ to $\mathbf{R} = \mathbf{Q}$.

1b. Substitute (5) into (27.11) and use the chain rule to verify that

$$D_\mathbf{F}[\hat{\sigma}(\mathbf{Q}\mathbf{F}) - \hat{\sigma}(\mathbf{F})] = \mathbf{0}.$$

Integrate this from $\mathbf{F} = \mathbf{I}$ [using (19)].

1e. Use the chain rule to differentiate

$$\hat{\sigma}(\mathbf{F}) = \tilde{\sigma}(\mathbf{F}^\mathrm{T}\mathbf{F})$$

with respect to $\mathbf{F}$ and appeal to (5) and (27.14).

2. Use the incompressibility of the body to show that

$$(\pi\mathbf{I}) \cdot \mathbf{D} = 0,$$

so that the term involving the pressure π does not enter the expression for the work. Then use the extension (β) to reduce the problem to the one studied in this section.

3. Show that the work done on a part $\mathscr{P}$ during $[t_0, t_1]$ is given by

$$\int_{t_0}^{t_1}\int_{\mathscr{P}_t} \mathbf{T} \cdot \mathbf{D}\, dV\, dt$$

provided the process is closed during this time interval.

4. Use components.

SECTION 29

1. Extend the proof of (b) of the theorem on page 195.

SECTION 30

4. Use (29.7) with $\mathbf{F} = \mathbf{I}$.

SECTION 32

2. Use Exercise 15.1.
9. The following theorem is needed: if w_α are class C^N $(N \geq 1)$ fields on $\mathscr{R}$ with

$$w_{1,2} = w_{2,1},$$

then there exists a class C^{N+1} field ψ on $\mathscr{R}$ such that

$$w_\alpha = \psi_{,\alpha}.$$

The compatibility equation in the form

$$(E_{11,2} - E_{12,1})_{,2} = (E_{12,2} - E_{22,1})_{,1}$$

establishes the existence of a field ψ such that

$$E_{11,2} = (E_{12} + \psi)_{,1}, \qquad E_{22,1} = (E_{12} - \psi)_{,2},$$

and hence fields u_α such that

$$E_{11} = u_{1,1}, \qquad E_{12} + \psi = u_{1,2},$$
$$E_{22} = u_{2,2}, \qquad E_{12} - \psi = u_{2,1}.$$

11. Equation $(13)_3$, written in the form

$$S_{11,1} = -S_{12,2}, \qquad S_{22,2} = -S_{12,1},$$

implies the existence of fields ψ and θ such that

$$S_{11} = \psi_{,2}, \qquad S_{12} = -\psi_{,1}, \qquad S_{22} = \theta_{,1}, \qquad S_{12} = -\theta_{,2}.$$

Hence

$$\psi_{,1} = \theta_{,2}$$

and there exists a φ such that

$$\psi = \varphi_{,2}, \qquad \theta = \varphi_{,1}.$$

SECTION 33

4. Assume that the boundary of the cross section is parametrized by $p_1(\sigma)$ and $p_2(\sigma)$ with σ the arc length. Then the vector with components

$$n_1(\sigma) = \frac{dp_2(\sigma)}{d\sigma}, \qquad n_2(\sigma) = -\frac{dp_1(\sigma)}{d\sigma}$$

is normal to this boundary at each σ. Use this fact, (7), and (8) to show that

$$p_1(\sigma)^2 + p_2(\sigma)^2 = \text{const}.$$

5c. Show that

$$S_{13} = \mu\alpha\{[p_1(\varphi_{,1} - p_2)]_{,1} + [p_1(\varphi_{,2} + p_1)]_{,2}\},$$

and use the divergence theorem on $\mathscr{S}_0$ (as a region in $\mathbb{R}^2$) and (11) to verify that

$$\int_{\mathscr{S}_0} S_{13}\, dA = 0.$$

Similarly,

$$\int_{\mathscr{S}_0} S_{23}\, dA = 0.$$

SECTION 34

2. Show that

$$\int_{\mathscr{B}} \int_0^t \{\dot{\mathbf{E}}(t-\tau)\cdot \mathbf{C}[\mathbf{E}(t+\tau)] - \dot{\mathbf{E}}(t+\tau)\cdot \mathbf{C}[\mathbf{E}(t-\tau)]\}\, d\tau\, dV = 2\mathscr{U}\{\mathbf{E}\}(t),$$

$$\int_{\mathscr{B}} \int_0^t \rho\{\dot{\mathbf{u}}(t-\tau)\cdot \ddot{\mathbf{u}}(t+\tau) - \dot{\mathbf{u}}(t+\tau)\cdot \ddot{\mathbf{u}}(t-\tau)\}\, d\tau\, dV = -2\mathscr{K}\{\dot{\mathbf{u}}\}(t)$$

(where we have suppressed the argument $\mathbf{p}$), and that the left sides sum to $\Phi(t)$. To derive these identities write the integrands on the left sides as derivatives with respect to τ.

5. Show that the convolution of the left side of (12) with $\lambda(t) = t$ is equal to

$$\lambda * \int_{\mathscr{B}} \mathbf{S} * \tilde{\mathbf{E}}\, dV + \int_{\mathscr{B}} \rho \mathbf{u} * \tilde{\mathbf{u}}\, dV,$$

where the convolution of two tensor fields is defined using the inner product of tensors in the integrand in (11). Show further, using the commutativity of the convolution operation and the symmetry of C, that

$$\mathbf{S} * \tilde{\mathbf{E}} = \tilde{\mathbf{S}} * \mathbf{E}.$$

In view of the above remarks we have the identity obtained by taking the convolution of (12) with λ, and if we differentiate this relation twice with respect to time, we arrive at (12).

SECTION 37

2. Choose $\mathbf{u}$ and $\mathbf{v}$ with $|\mathbf{u}| = |\mathbf{v}|$, let $\mathbf{Q}$ be the orthogonal tensor carrying $\mathbf{u}$ into $\mathbf{v}$, and show that $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$.
3. Choose $\mathbf{v}$ and let $\mathbf{Q}$ be any rotation about $\mathbf{v}$. Show that for this choice of $\mathbf{Q}$, $\mathbf{Q}\mathbf{q}(\mathbf{v}) = \mathbf{q}(\mathbf{v})$, and use this fact to prove that $\mathbf{q}(\mathbf{v})$ is parallel to $\mathbf{v}$. Thus $\mathbf{q}(\mathbf{v}) = \varphi(\mathbf{v})\mathbf{v}$. Show that φ is isotropic and use Exercise 1.
4. Use induction.

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